6

Techniques for Calculating the Efficient Frontier

In Chapters 4 and 5 we discussed the properties of the efficient frontier under alternative assumptions about lending and borrowing and alternative assumptions about short sales. In this chapter we describe and illustrate methods that can be used to calculate efficient portfolios. By necessity, this chapter is more mathematically complex than those that preceded it and most of those that follow. The reader who is concerned only with a conceptual approach to portfolio management can skip this chapter and still understand later ones. However, we believe that knowledge of the solution techniques to portfolio problems outlined here yields a better understanding and appreciation of portfolio management.

We have not followed the same order in presenting solution techniques for portfolio problems as was followed in describing the properties of the efficient set (Chapter 5). Rather, we have rearranged the order so that solution techniques are presented from the simplest to the most complex. The first four sections of this chapter discuss the solution to the portfolio problem when it is assumed in turn that

1. short sales are allowed and riskless lending and borrowing is possible

2. short sales are allowed but riskless lending or borrowing is not permitted

3. short sales are disallowed but riskless lending and borrowing exists

4. neither short sales nor riskless lending and borrowing is allowed

A fifth section shows how additional constraints, such as the need for a minimum dividend yield, can be incorporated into the portfolio problem. The solution techniques discussed here are the ones used in actual applications. For most problems, the calculations are lengthy enough that computers are used. Indeed, computer programs exist for each of the techniques discussed. In addition, in Chapter 9, we present simplifications of the procedures discussed in the present chapter that are useful in solving most real problems. This chapter is necessary for an understanding of the computer programs and an appreciation of the simple rules discussed later. Thus, although this chapter is more demanding than some others, it is well worth the effort needed to understand it.

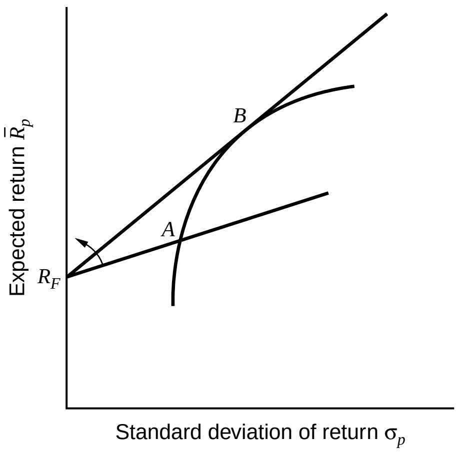

Figure 6.1 Combinations of the riskless asset in a risky portfolio.

SHORT SALES ALLOWED WITH RISKLESS LENDING AND BORROWING

The derivation of the efficient set when short sales are allowed and there is a riskless lending and borrowing rate is the simplest case we can consider. From Chapter 5 we know that the existence of a riskless lending and borrowing rate implies that there is a single portfolio of risky assets that is preferred to all other portfolios. Furthermore, in return standard deviation space, this portfolio plots on the ray connecting the riskless asset and a risky portfolio that lies furthest in the counterclockwise direction. For example, in Figure 6.1, the portfolio on the ray R_F—B is preferred to all other portfolios of risky assets. The efficient frontier is the entire length of the ray extending through R_F and B. Different points along the ray R_F—B represent different amounts of borrowing and/or lending in combination with the optimum portfolio of risky assets B.

An equivalent way of identifying the ray R_F—B is to recognize that it is the ray with the greatest slope. Recall that the slope of the line connecting a riskless asset and a risky portfolio is the expected return on the portfolio minus the risk-free rate divided by the standard deviation of the return on the portfolio. Thus the efficient set is determined by finding that portfolio with the greatest ratio of excess return (expected return minus risk-free rate) to standard deviation that satisfies the constraint that the sum of the proportions invested in the assets equals 1. In equation form we have the following: maximize the objective function

$$\theta = \frac{\bar{R}_P - R_F}{\sigma_P}$$

subject to the constraint[1]

$$\sum_{i=1}^{N} X_i = 1$$

[1]Lintner (1965) has advocated an alternative definition of short sales, one that is more realistic. He assumes correctly that when an investor sells stock short, cash is not received but rather is held as collateral. Furthermore, the investor must put up an additional amount of cash equal to the amount of stock he or she sells short. The investor generally does not receive any compensation (interest) on these funds. However, if the investor is a broker-dealer, interest can be earned on both the money put up and the money received from the short sale of securities. As shown in Appendix A, this leads to the constraint $\Sigma|X_i| = 1$ and leaves all other equations unchanged.

This is a constrained maximization problem. There are standard solution techniques available for solving it. For example, it can be solved by the method of Lagrangian multipliers. There is an alternative. The constraint could be substituted into the objective function and the objective function maximized as in an unconstrained problem. This latter procedure will be followed subsequently. We can write R_F as R_F times 1. Thus we have

$$R_F = 1R_F = \left(\sum_{i=1}^{N} X_i\right) R_F = \sum_{i=1}^{N} (X_i R_F)$$

Making this substitution in the objective function and stating the expected return and standard deviation of return in the general form, derived in Chapter 4, yields

$$\theta = \frac{\sum_{i=1}^{N} X_i\left(\overline{R}_i - R_F\right)}{\left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_{ij}\right]^{1/2}}$$

The problem stated previously is a very simple maximization problem and as such can be solved using the standard methods of basic calculus. In calculus it is shown that to find the maximum of a function, you take the derivative with respect to each variable and set it equal to zero.[2] Thus the solution to the maximization problem just presented involves finding the solution to the following system of simultaneous equations:

1. $\dfrac{d\theta}{dX_1} = 0$
2. $\dfrac{d\theta}{dX_2} = 0$
3. $\dfrac{d\theta}{dX_3} = 0$

$\vdots$

N. $\dfrac{d\theta}{dX_N} = 0$

In Appendix B at the end of this chapter we show that

$$\frac{d\theta}{dX_i} = -\left(\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \lambda X_3 \sigma_{3i} + \cdots + \lambda X_i \sigma_i^2 + \cdots + \lambda X_{N-1} \sigma_{N-1i} + \lambda X_N \sigma_{Ni}\right) + \overline{R}_i - R_F = 0$$

---

[2]Solving the problem without constraining the solution by

$$\sum_{i=1}^{N} X_i = 1$$

does not work in every maximization problem. It works here because the equations are homogeneous of degree zero.

where λ is a constant.[3] A mathematical trick allows a useful modification of the derivative. Note that each X_k is multiplied by a constant λ. Define a new variable $Z_k = \lambda X_k$. The X_k are the fraction to invest in each security, and the Z_k are proportional to this fraction. Substituting Z_k for the λX_k simplifies the formulation. To solve for the X_k after obtaining the Z_k, one divides each Z_k by the sum of the Z_k. Substituting Z_k for $\lambda_k X_k$ and moving the variance covariance terms to the right-hand side of the equality yields

$$\bar{R}_i - R_F = Z_1\sigma_{1i} + Z_2\sigma_{2i} + \cdots + Z_i\sigma_i^2 + \cdots + Z_{N-1}\sigma_{N-1i} + Z_N\sigma_N$$

We have one equation like this for each value of i. Thus the solution involves solving the following system of simultaneous equations:

$$\begin{aligned}
\bar{R}_1 - R_F &= Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + Z_N\sigma_{1N} \\
\bar{R}_2 - R_F &= Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + Z_N\sigma_{2N} \\
\bar{R}_3 - R_F &= Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^2 + \cdots + Z_N\sigma_{3N} \\
&\vdots \\
\bar{R}_N - R_F &= Z_1\sigma_{1N} + Z_2\sigma_{2N} + Z_3\sigma_{3N} + \cdots + Z_N\sigma_N^2
\end{aligned} \tag{6.1}$$

The Zs are proportional to the optimum amount to invest in each security. To determine the optimum amount to invest, we first solve the equations for the Zs. Note that this does not present a problem. There are N equations (one for each security) and N unknowns (the Z_k for each security). Then the optimum proportion to invest in stock k is X_k, where

$$X_k = Z_k \Big/ \sum_{i=1}^{N} Z_i$$

Let us solve an example. Consider three securities: Colonel Motors with expected return of 14% and standard deviation of return of 6%, Separated Edison with average return of 8% and standard deviation of return of 3%, and Unique Oil with mean return of 20% and standard deviation of return of 15%. Furthermore, assume that the correlation coefficient between Colonel Motors and Separated Edison is 0.5, between Colonel Motors and Unique Oil is 0.2, and between Separated Edison and Unique Oil is 0.4. Finally, assume that the riskless lending and borrowing rate is 5%. Equation (6.1) for three securities is

$$\begin{aligned}
\bar{R}_1 - R_F &= Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} \\
\bar{R}_2 - R_F &= Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23} \\
\bar{R}_3 - R_F &= Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^2
\end{aligned}$$

Substituting in the assumed values, we get the following system of simultaneous equations:

$$\begin{aligned}
14 - 5 &= 36Z_1 + (0.5)(6)(3)Z_2 + (0.2)(6)(15)Z_3 \\
8 - 5 &= (0.5)(6)(3)Z_1 + 9Z_2 + (0.4)(3)(15)Z_3 \\
20 - 5 &= (0.2)(6)(15)Z_1 + (0.4)(3)(15)Z_2 + 225Z_3
\end{aligned}$$

[3]The constant is equal to $(\bar{R}_P - R_F)$ divided by σ_P^2.

Simplifying,

$$\begin{aligned} 9 &= 36Z_1 + 9Z_2 + 18Z_3 \\ 3 &= 9Z_1 + 9Z_2 + 18Z_3 \\ 15 &= 18Z_1 + 18Z_2 + 225Z_3 \end{aligned}$$

Further simplifying,

$$\begin{aligned} 1 &= 4Z_1 + Z_2 + 2Z_3 \\ 1 &= 3Z_1 + 3Z_2 + 6Z_3 \\ 5 &= 6Z_1 + 6Z_2 + 75Z_3 \end{aligned}$$

The solution to this system of simultaneous equations is

$$Z_1 = \frac{14}{63}, \qquad Z_2 = \frac{1}{63}, \qquad \text{and} \qquad Z_3 = \frac{3}{63}$$

The reader can verify this solution by substituting these values of Z_k into the foregoing equations.[4] The proportion to invest in each security is easy to determine. We know that each Z_k is proportional to X_k. Consequently, all we have to do to determine X_k is to scale the Z_k so that they add to 1.[5] For the foregoing problem,

$$\sum_{i=1}^{3} Z_i = \frac{18}{63}$$

Thus the proportion to invest in each security is

$$X_1 = \frac{14}{18}, \qquad X_2 = \frac{1}{18}, \qquad \text{and} \qquad X_3 = \frac{3}{18}$$

The expected return on the portfolio is

$$\overline{R}_P = \frac{14}{18}(14) + \frac{1}{18}(8) + \frac{3}{18}(20) = 14\tfrac{2}{3}\%$$

The variance of the return on the portfolio is[6]

$$\begin{aligned} \sigma_P^2 = {} & \left(\frac{14}{18}\right)^2 (36) + \left(\frac{1}{18}\right)^2 9 + \left(\frac{3}{18}\right)^2 (225) + 2\left(\frac{14}{18}\right)\left(\frac{1}{18}\right)(6)(3)(0.5) \\ & + 2\left(\frac{14}{18}\right)\left(\frac{3}{18}\right)(6)(15)(0.2) + 2\left(\frac{1}{18}\right)\left(\frac{3}{18}\right)(3)(15)(0.4) = \frac{203}{6} = 33\tfrac{5}{6} \end{aligned}$$

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[4]See Appendix C at the end of this chapter for a description of solution techniques for systems of simultaneous equations.

[5]In the case of Lintnerian short sales, simply scale so that

$$\sum_{i=1}^{3} |X_i| = 1$$

[6]The variance of the portfolio could have been determined in another way. Recall that λ is the ratio of the excess return on the optimum portfolio divided by the variance of the optimum portfolio. Thus

$$\lambda = \frac{\overline{R}_P - R_F}{\sigma_P^2} = \frac{14\frac{2}{3} - 5}{\sigma_P^2}$$

Also recall that $Z_i = \lambda X_i$ so that $\Sigma Z_i = \lambda \Sigma X_i = \lambda$. Earlier we determined that $\Sigma Z_i = \lambda = 18/63$. Equating these two equations and solving for σ_P^2 yields the value presented earlier.

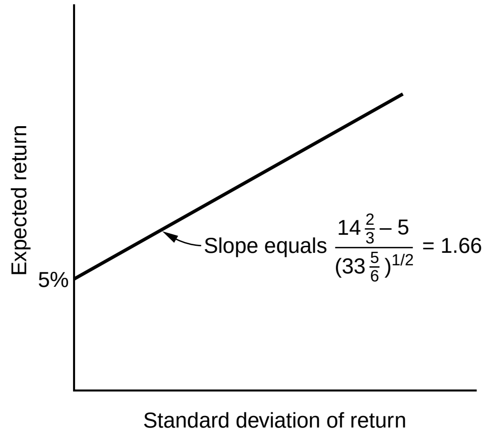

Figure 6.2 The efficient set with riskless lending and borrowing.

The efficient set is a straight line with an intercept at the risk-free rate of 5% and a slope equal to the ratio of excess return to standard deviation (see Figure 6.2). There are standard computer packages for the solution of a system of simultaneous equations. Appendix C at the end of this chapter presents some methods of solving them when the number of securities involved is limited so that hand calculations are reasonable.

SHORT SALES ALLOWED: NO RISKLESS LENDING AND BORROWING

When the investor does not wish to make the assumption that she can borrow and lend at the riskless rate of interest, the solution developed in the last section must be modified. However, much of the analysis can still be utilized. Consider Figure 6.3. The riskless lending and borrowing rate of 5% led to the selection of portfolio *B*. If the riskless lending and borrowing rate had been 4%, the investor would invest in portfolio *A*. If the investor's lending and borrowing rate was 6%, the investor would select portfolio *C*. These observations suggest the following procedure. Assume that a riskless lending and borrowing rate exists and find the optimum portfolio. Then assume that a different riskless lending and borrowing rate exists and find the optimum portfolio that corresponds to this second rate. Continue changing the assumed riskless rate until the full efficient frontier is determined.[7]

In Appendix D we present a general solution to this problem. We show that the optimal proportion to invest in any security is simply a linear function of R_F. Furthermore, because the entire efficient frontier can be constructed as a combination of any two portfolios that lie along it, the identification of the characteristics of the optimal portfolio for any two arbitrary values of R_F is sufficient to trace out the total efficient frontier.

RISKLESS LENDING AND BORROWING WITH SHORT SALES NOT ALLOWED

This problem is analogous to the case of riskless lending and borrowing with short sales allowed. One portfolio is optimal. Once again, it is the one that maximizes the slope of the line connecting the riskless asset and a risky portfolio. However, the set of portfolios that

[7]This works only for the standard definition of short sales. The Lintner definition of short sales assumes riskless lending and borrowing at a particular rate for each point on the original (curved) efficient frontier.

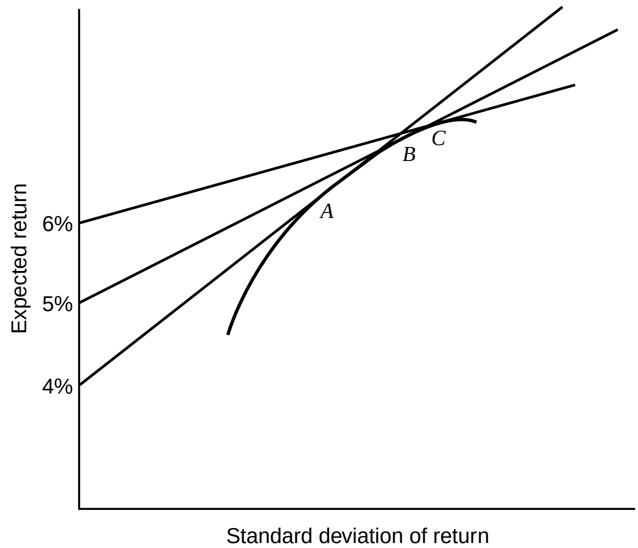

Figure 6.3 Tangency portfolios for different riskless rates.

is available to combine with lending and borrowing is different because a new constraint has been added. Investors cannot hold securities in negative amounts. More formally, the problem can be stated as

$$\text{Maximize } \theta = \frac{\overline{R}_P - R_F}{\sigma_P}$$

subject to

$$(1) \quad \sum_{i=1}^{N} X_i = 1$$

$$(2) \quad X_i \geq 0 \qquad \text{all } i$$

This is a mathematical programming problem because of the inequality restriction on X_i. At first glance, this might look like a linear programming problem. In fact, the constraints (1) and (2) are linear constraints. The problem is that the objective function (the expression we are maximizing) is not linear; σ_P contains terms involving X_i^2 and X_iX_j. Equations involving squared terms and cross-product terms are called quadratic equations. Since this looks like a linear programming problem, except that the objective function is quadratic rather than linear, it is called a *quadratic programming problem*. There are standard computer packages for solving quadratic programming problems, just as there are for linear programming problems, and the reader interested in solving a large-scale problem would utilize one of them. Some discussion of solution techniques is contained in Appendix E at the end of this chapter.

NO SHORT SELLING AND NO RISKLESS LENDING AND BORROWING

Recall that an efficient set is determined by minimizing the risk for any level of expected return. If we specify the return at some level and minimize risk, we have one point on the efficient frontier. Thus, to get one point on the efficient frontier, we minimize risk subject to the return being some level plus the restriction that the sum of the proportions invested in each security is 1 and that all securities have positive or zero investment. This yields the following problem:

$$\text{Minimize } \sum_{i=1}^{N}\left(X_i^2\sigma_i^2\right)+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(X_iX_j\sigma_{ij}\right)$$

subject to

$$(1)\quad \sum_{i=1}^{N} X_i = 1$$

$$(2)\quad \sum_{i=1}^{N}\left(X_i\overline{R}_i\right)=\overline{R}_P$$

$$(3)\quad X_i \geq 0, \qquad i=1,\ldots,N$$

Varying $\overline{R}_P$ between the return on the minimum variance portfolio and the return on the maximum return portfolio traces out the efficient set. Once again, the problem is a quadratic programming problem because of the presence of terms such as X_i^2 and X_iX_j (squared and cross-product terms). However, there are standard packages available that solve this problem.

THE INCORPORATION OF ADDITIONAL CONSTRAINTS

The imposition of short sale constraints has complicated the solution technique, forcing us to use quadratic programming. Once we resort to this technique, however, it is a simple matter to impose other requirements on the solution. Literally any set of requirements that can be formulated as linear functions of the investment weights can be imposed on the solution. For example, some managers wish to select optimum portfolios given that the dividend yield on the optimum portfolios is at least some number (e.g., 2%). If we let D stand for the target dividend yield and d_i stand for the dividend yield on stock i, then we can impose this requirement by adding a fourth constraint to the problem described in the previous section:

$$(4)\quad \sum_{i=1}^{N}\left(X_id_i\right)\geq D$$

If we desire the dividend constraint but want to allow short sales, we simply eliminate the third constraint,

$$(3)\quad X_i \geq 0, \qquad i=1,\ldots,N$$

from the problem.

Note that once we impose inequality constraints such as the one described for dividends, we must solve a quadratic programming problem instead of a system of simultaneous equations, even if short sales are allowed.

Other types of constraints are frequently employed in solving portfolio problems. Perhaps the most frequent constraints are those that place an upper limit on the fraction of the portfolio that can be invested in any stock. Upper limits on the amount that can be

Table 6.1 Input Data for Asset Allocation

	S&P	Bonds	Canadian	Japan	Emerging Market	Pacific	Europe	Small Stock
Expected return	14.00	6.50	11.00	14.00	16.00	18.00	12.00	17.00
Standard deviation	18.50	5.00	16.00	23.00	30.00	26.00	20.00	24.00
			Correlation Coefficients					
S&P	1.00	0.45	0.70	0.20	0.64	0.30	0.61	0.79
Bonds		1.00	0.27	−0.01	0.41	0.01	0.13	0.28
Canadian			1.00	0.14	0.51	0.29	0.48	0.59
Japan				1.00	0.25	0.73	0.56	0.13
Emerging market					1.00	0.28	0.61	0.75
Pacific						1.00	0.54	0.16
Europe							1.00	0.44
Small stock								1.00

invested in any one stock are often part of the charter of mutual funds. Also, upper limits (and occasionally lower limits) are often placed on the fraction of a portfolio that can be invested in any industry. Finally, it is possible to build in constraints on the amount of turnover in a portfolio and to allow the consideration of transaction costs in computing returns.

AN EXAMPLE

This chapter has presented techniques for obtaining the efficient frontier when there are a large number of assets to choose from. Table 6.1 shows the data for the asset allocation problem we will examine. The manager is considering an allocation across three U.S. categories and international stocks. The three U.S. categories are large stocks, small stocks, and bonds. Large stocks are represented by the Standard and Poor's index including dividends, bonds by Barclays Government Credit index, and small stocks by the Center for Research in Security Prices (CRSP) small stock index.[8] The international data were obtained by using returns on international stock mutual funds. The international portfolios are selected to divide the world into as many nonoverlapping segments as possible. Thus there is a Canadian fund, a European fund, a Japanese fund, a Pacific funds, and an emerging market fund. There is some overlap. The Pacific fund and the Japanese fund have stocks in Japan in common. Similarly, the emerging market and Pacific funds have some countries in common. The effect of overlap can be seen by examining the correlation coefficients. The correlation between the Japan fund and the Pacific fund is 0.73, which is the highest correlation between Japan and any other fund. The emerging market is interesting. Before examining the data, one would expect that the correlations would be very low with the major countries. However, the correlations are high with major markets, implying that the performance of emerging markets is very much affected by what happens in major markets.

The correlation matrix initially was calculated by using return data over the prior five years and was calculated for returns expressed in U.S. dollars. Then, security analysts at a

[8]The CRSP small stock index is roughly the smallest quintile of stocks on the New York Stock Exchange plus American Stock Exchange and NASDAQ stocks of similar size. See the footnote to Table 17.1 for a detailed description of the construction of the CRSP small stock index.

major investment banking firm compared the correlations calculated using returns from the most recent five-year period with prior five-year periods. Using these data and their judgment, analysts modified some historic numbers to obtain their best estimate of what the future correlations would be.

The standard deviations are expressed in annual returns. They were also calculated over the prior five years. Once again, however, analysts modified them slightly utilizing both data from earlier periods and their experience to obtain their best subjective estimates for the future. The mean returns are the estimates of a major financial intermediary concerned with the allocation decisions analyzed here. At this time they were fairly pessimistic about U.S. bond markets, Canadian stocks, and European stocks, and this is reflected in their estimates. The final input needed is a riskless rate of interest, which was estimated at 5% for U.S. investors over subsequent years.

The efficient frontier without riskless lending and borrowing but with short sales is the curved figure shown in Figure 6.4. Each asset class as a separate investment is represented by a dot in Figure 6.4. The global minimum variance portfolio has a mean return of 6.41% and a standard deviation of 3.91%. Note that bonds are by far the least risky asset. However, a portfolio of assets is less risky than bonds, even though the next least risky asset has a standard deviation more than 3 times larger than bonds. Alternatively, the optimum portfolio with the same risk as bonds has a mean return of 8.42%, or 1.92% more than bonds. This is an illustration of the power of diversification. Note that all assets are held either long or short. Furthermore, note that for the higher returns (above portfolio 2), the short sales involved are substantial and would involve short selling more than margin requirements would allow. Thus the efficient frontier would terminate after portfolio 2. At low risks, the major long purchase is bonds. As expected return is increased, the S&P, small stocks, and the Pacific fund all are held long in substantial amounts, with Japan held long in a somewhat smaller proportion. These are all relatively

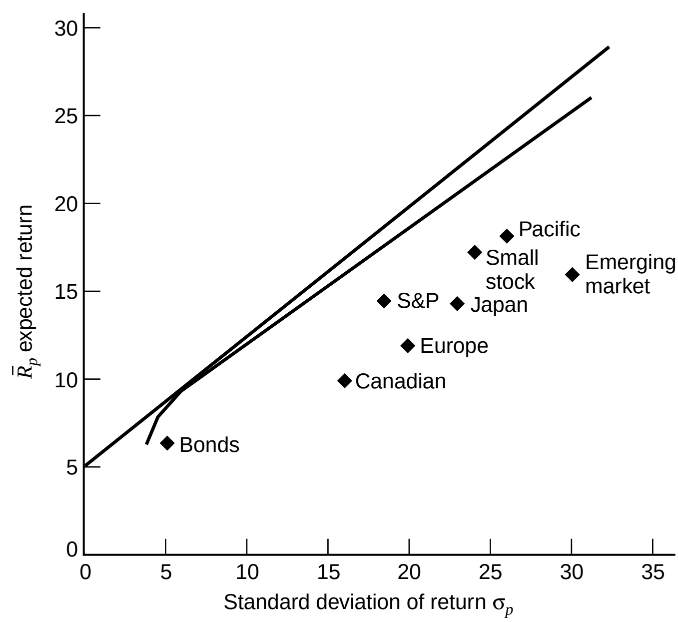

Figure 6.4 The efficient frontier with riskless lending and borrowing and short sales allowed.

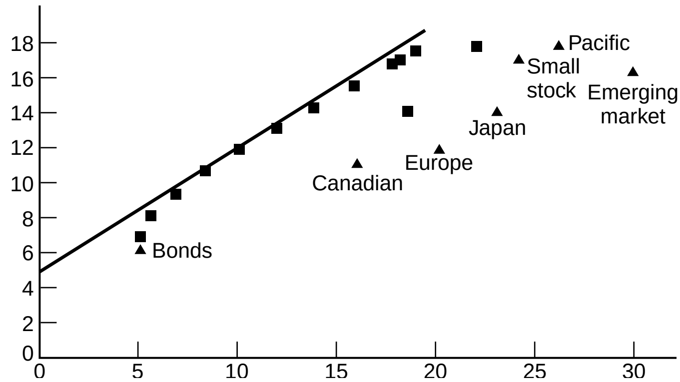

Figure 6.5 The efficient frontier with no riskless lending and borrowing and no short sales.

high-expected-return portfolios. Notice, however, that emerging markets, the other high mean return portfolio, does not enter into the optimum. This is because it has a very high correlation with the other countries and thus does not contribute much to the diversification. Europe and bonds are sold short for portfolios with higher mean returns. These are both low-expected-return assets. In addition, Europe has the advantage of being relatively highly correlated with the assets held long. When an asset is sold short, the covariance term with a long asset is negative, thus reducing risk. It is therefore desirable for a short-sold asset to be highly correlated with an asset held long.

Now consider the solution when short sales are not allowed and there is no riskless lending and borrowing. The efficient frontier is the curved region in Figure 6.5. The composition for a number of portfolios is shown in Table 6.2. The case where short sales are not allowed is probably the realistic case to consider for the pension fund manager whose problem we are analyzing. As shown in Table 6.2, the global minimum variance portfolio has an expected return of 6.89% and a standard deviation of 4.87%. This is of course a higher standard deviation than if short sales were allowed. A comparison of the numbers in Figures 6.1 and 6.2 shows that the efficient frontier with short sales allowed offers a higher mean return for a given risk (either with or without riskless lending and borrowing). This is because short sales offer additional investment opportunities that are used.

As shown in Table 6.2, the minimum-risk portfolio is primarily investment in bonds. Without short sales, the minimum risk is only slightly less than the risk of bonds alone—4.87% compared to 5%—and the expected return is only 0.39% higher. As we increase the risk on the portfolio, the percentage invested in bonds goes down, and we start to invest primarily in small stocks and Pacific. A minor amount is invested in Japan. The highest mean return portfolio is of course 100% in the highest-return asset, Pacific bond.

When riskless lending and borrowing is allowed, the efficient frontier is the straight line shown in Figures 6.4 and 6.5. The equations of the straight lines are

Short sales allowed

$$\overline{R}_P = 5 + 0.714\sigma_P$$

Short sales not allowed

$$\overline{R}_P = 5 + 0.685\sigma_P$$

Table 6.2 Proportions Invested When Short Sales Are Not Allowed

	Global Minimum	1	2	3	4	5
Mean return	6.89	9.36	11.83	14.30	16.77	18.00
Standard deviation	4.88	6.66	10.03	13.86	17.87	26.00
			Proportions			
S&P	0.00	0.00	0.00	0.00	0.63	0.00
Bond	95.16	72.91	50.51	28.12	5.51	0.00
Canadian	0.06	0.00	0.00	0.00	0.00	0.00
Japan	3.96	3.57	3.17	2.77	2.41	0.00
Emerging market	0.00	0.00	0.00	0.00	0.00	0.00
Pacific	0.81	12.42	22.86	33.29	43.62	100.00
Europe	0.00	0.00	0.00	0.00	0.00	0.00
Small stock	0.00	11.10	23.46	35.82	47.82	0.00

Obviously, the efficient frontier with short sales allowed is steeper. The tangency portfolio for short sales not allowed has a mean return of 11.51%. Higher returns involve borrowing at the riskless rate. For the pension manager whose problem is being analyzed, this is likely infeasible. For this manager, the efficient frontier is likely to be the straight line segment from R_F to the tangency point and the curved shape from there to the right. Given the low return of the tangency portfolio, the choice would likely lie on the curve to the right of the tangency portfolio. This would involve bonds, small stocks, Pacific, and a little invested in Japan. It would be important to vary the inputs in a reasonable range to see how this composition would change given reasonable changes in the inputs.

CONCLUSION

In this chapter we discussed and illustrated the use of techniques that can be employed to solve for the set of all possible portfolios that are efficient. All of the solution techniques discussed are feasible and have been used to solve problems. However, the techniques require gigantic amounts of input data and large amounts of computation time. Furthermore, the input data are in a form to which the security analyst and portfolio manager cannot easily relate. For this reason, it is difficult to get estimates of the input data or to get practitioners to relate to the final output.

The next logical step is to simplify the number and type of input requirements for portfolio selection and, in turn, to see if this reduction in data complexity can also be used to simplify the computational procedure. This is the subject of the next three chapters.

APPENDIX A

AN ALTERNATIVE DEFINITION OF SHORT SALES

Modeling short sales from the viewpoint of the broker-dealer, we first note that the broker-dealer has a fixed sum of money to invest. A short sale involves putting up an amount of money equal to the short sale. Thus the short sale is a use rather than a source of funds to the short seller. The total funds the broker-dealer invests short, plus the funds invested long, must add to the original investment. Because for short sales, $X_i < 0$, the proportion of the funds invested in the short sale is $|X|_i$. In addition, the short seller (if a broker-dealer) receives interest on both the money put up against short sales and the money received from the short sale. Thus the expected return from short selling 0.10 of stock i is $-0.1\overline{R}_i + 0.2R_F$. Because X_i is negative for short sales, this can be written as $X_i(\overline{R}_i - 2R_F)$. Assume stocks 1 to k are held long and stocks $k + 1$ to N are sold short. Then

$$\overline{R}_P = \sum_{i=1}^{k} X_i\left(\overline{R}_i\right) + \sum_{i=k+1}^{N} X_i\left(\overline{R}_i - 2R_F\right)$$

$$\overline{R}_P = \sum_{i=1}^{N} X_i\overline{R}_i - 2\sum_{i=k+1}^{N} X_i\left(R_F\right)$$

The constraint with the Lintner definition of short sales is

$$\sum_{i=1}^{N} |X| = 1$$

Substituting this for 1 times R_F yields

$$R_F = \sum_{i=1}^{N} |X_i| R_F = \sum_{i=1}^{k} X_i R_F - \sum_{i=k+1}^{N} X_i R_F \tag{A.1}$$

This is the expression used for R_F. Subtracting R_F from both sides of the equation for $\overline{R}_P$ and using (A.1) for R_F on the right-hand side of the equation yields

$$\overline{R}_P - R_F = \left[\sum_{i=1}^{N} X_i\overline{R}_i - 2\sum_{i=k+1}^{N} X_i R_F\right] - \left[\sum_{i=1}^{k} X_i R_F - \sum_{i=k+1}^{N} X_i R_F\right]$$

$$\overline{R}_P - R_F = \sum_{i=1}^{N} X_i\overline{R}_i - \sum_{i=1}^{N} X_i R_F = \sum_{i=1}^{N} X_i\left(\overline{R}_i - R_F\right)$$

This is identical to the equation given in the text. The reader should note that in the Lintnerian definition of short sales, the final portfolio weights must be scaled so that the sum of the absolute value of the weights, rather than their sum, is 1.

APPENDIX B

DETERMINING THE DERIVATIVE

In the text we discussed that to solve the portfolio problem when short sales are allowed, the derivative of θ with respect to X_k was needed.[9] In the text we presented the value of the derivative. In this appendix we derive its value. To determine the derivative, rewrite the θ shown in the text as

$$\theta = \left[\sum_{i=1}^{N} X_i\left(\overline{R}_i - R_F\right)\right]\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right]^{-1/2}$$

[9]To ensure a maximum, the second derivative should be negative. The structure of this problem guarantees this.

Two rules from calculus are needed:

1. **The product rule:** θ is the product of two functions. The product rule states that the derivative of the product of two functions is the first function times the derivative of the second function plus the second times the derivative of the first. In symbols,

$$\frac{d}{dX}\left[\left[F_1(X)\right]\left[F_2(X)\right]\right]=\left[F_1(X)\right]\frac{dF_2(X)}{dX}+\left[F_2(X)\right]\frac{dF_1(X)}{dX} \tag{B.1}$$

Let

$$F_1(X)=\sum_{i=1}^{N}X_i\left(\bar{R}_i-R_F\right) \tag{B.2}$$

$$F_2(X)=\left(\sum_{i=1}^{N}X_i^2\sigma_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}X_iX_j\sigma_{ij}\right)^{-1/2} \tag{B.3}$$

Consider first the derivative of $F_1(X)$. At first glance, the reader may believe it is difficult. However, it turns out to be trivial. An expression like

$$\sum_{i=1}^{N}X_i\left(\bar{R}_i-R_F\right)$$

involves a lot of terms that do not contain an X_k and one term involving X_k. The derivatives of the terms not involving X_k are zero (they are constants as far as X_k is concerned). The derivative of the term involving X_k is $\bar{R}_k - R_F$. Thus

$$\frac{dF_1(X)}{dX_k}=\bar{R}_k-R_F \tag{B.4}$$

Now consider the derivative of $F_2(X)$. To determine this, a second rule from calculus is needed.

2. **The chain rule:** $F_2(X)$ involves a term in brackets to a power (the power $-\frac{1}{2}$). The chain rule states that its derivative is the power, times the expression in parentheses to the power minus one, times the derivative of what is inside the brackets. Thus

$$\frac{dF_2(X)}{dX_k}=\left(-\frac{1}{2}\right)\left(\sum_{i=1}^{N}X_i^2\sigma_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}X_iX_j\sigma_{ij}\right)^{-3/2}$$
$$\times\left(2X_k\sigma_k^2+2\sum_{\substack{j=1\\ j\neq k}}^{N}X_j\sigma_{jk}\right) \tag{B.5}$$

The only term that requires comment is the last one. The derivative of

$$\sum_{i=1}^{N}X_i^2\sigma_i^2$$

follows the same principles discussed earlier. All terms not involving k are constant as far as k is concerned, and thus their derivative is zero. The term involving k is $X_k^2\sigma_k^2$ and has a derivative of $2X_k\sigma_k^2$. The derivation of the double summation is more complex. Consider the double summation term

$$\left(\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right)$$

We get X_k twice, once when $i = k$ and once when $j = k$. When $i = k$, we have

$$\sum_{\substack{j=1\\ j\neq k}}^{N} X_k X_j \sigma_{kj} = X_k\left[\sum_{\substack{j=1\\ j\neq k}}^{N} X_j \sigma_{kj}\right]$$

The derivative of this is, of course,

$$\sum_{\substack{j=1\\ j\neq k}}^{N} X_j \sigma_{kj}$$

Similarly, when $j = k$, we have

$$\sum_{\substack{i=1\\ i\neq k}}^{N} X_i X_k \sigma_{ik} = X_k\left(\sum_{\substack{i=1\\ i\neq k}}^{N} X_i \sigma_{ik}\right)$$

The derivative of this is also

$$\sum_{\substack{i=1\\ i\neq k}}^{N} X_i \sigma_{ik}$$

where i and j are simply summation indicators. It does not matter which we use. Furthermore, $\sigma_{ik} = \sigma_{ki}$. Thus

$$\sum_{\substack{j=1\\ j\neq k}}^{N} X_j \sigma_{kj} = \sum_{\substack{i=1\\ i\neq k}}^{N} X_i \sigma_{ik}$$

and we have the expression shown in the derivative, namely,

$$2\sum_{\substack{j=1\\ j\neq k}}^{N} X_j \sigma_{kj}$$

Substituting (B.2), (B.3), (B.4), and (B.5) into the product rule, expression (B.1) yields

$$\frac{d\theta}{dX_k} = \left[\sum_{i=1}^{N} X_i\left(\bar{R}_i - R_F\right)\right]\left[\left(-\frac{1}{2}\right)\left(\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right)^{-3/2}\right.$$
$$\left.\times\left(2X_k\sigma_k^2 + 2\sum_{\substack{j=1\\ j\neq k}}^{N} X_j\sigma_{kj}\right)\right] + \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right]^{-1/2}$$
$$\times\left[\left(\bar{R}_k - R_F\right)\right] = 0$$

Multiplying the derivative by

$$\left(\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right)^{1/2}$$

and rearranging yields

$$-\left[\frac{\sum_{i=1}^{N} X_i\left(\bar{R}_i - R_F\right)}{\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}}\right]\left[X_k\sigma_k^2 + \sum_{\substack{j=1\\ j\neq k}}^{N} X_j\sigma_{kj}\right] + \left[\bar{R}_k - R_F\right] = 0$$

Defining λ as

$$\frac{\sum_{i=1}^{N} X_i\left(\bar{R}_i - R_F\right)}{\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j \sigma_{ij}}$$

yields

$$-\lambda\left[X_k\sigma_k^2 + \sum_{\substack{j=1\\ j\neq k}}^{N} X_j\sigma_{kj}\right] + \left(\bar{R}_k - R_F\right) = 0$$

Multiplying the terms in the brackets by λ yields

$$-\left[\lambda X_k \sigma_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda X_j \sigma_{kj}\right] + \left(\bar{R}_k - R_F\right) = 0$$

This is the expression shown in the text.

APPENDIX C

SOLVING SYSTEMS OF SIMULTANEOUS EQUATIONS

To solve large systems of simultaneous equations, one would use any of the large number of standard computer packages that exist for this purpose. However, small systems can be solved by hand. The simplest way is by repetitive substitution. Consider the following system of simultaneous equations:

$$4X_1 + X_2 = 7 \tag{C.1}$$

$$3X_1 + 2X_2 = 5 \tag{C.2}$$

Equation (C.1) can be rearranged so that X_2 is expressed as a function of X_1. This rearrangement yields

$$X_2 = 7 - 4X_1$$

Substituting this into Equation (C.2) yields

$$\begin{aligned} 3X_1 + 2(7 - 4X_1) &= 5 \\ 3X_1 + 14 - 8X_1 &= 5 \\ -5X_1 &= -9 \\ X_1 &= \tfrac{9}{5} \end{aligned}$$

Substituting the value for X_1 into rearranged Equation (C.1) yields

$$X_2 = 7 - 4\left(\tfrac{9}{5}\right) = 7 - \tfrac{36}{5} = -\tfrac{1}{5}$$

This technique is extremely easy and can be applied to solving any number of simultaneous equations, although with many equations, it becomes extremely time consuming. For a second example, consider the problem analyzed in the section "Short Sales Allowed":

$$1 = 4Z_1 + Z_2 + 2Z_3 \tag{C.3}$$

$$1 = 3Z_1 + 3Z_2 + 6Z_3 \tag{C.4}$$

$$5 = 6Z_1 + 6Z_2 + 75Z_3 \tag{C.5}$$

Solving Equation (C.3) for Z_2 and eliminating Z_2 from Equation (C.4) yields

$$Z_2 = 1 - 4Z_1 - 2Z_3 \tag{C.3$'$}$$

$$1 = 3Z_1 + 3(1 - 4Z_1 - 2Z_3) + 6Z_3 \tag{C.4$'$}$$

Simplifying (C.4′) yields

$$-2 = -9Z_1$$

Following the same procedure for Equation (C.5) yields

$$5 = 6Z_1 + 6(1 - 4Z_1 - 2Z_3) + 75Z_3 \tag{C.5$'$}$$

Simplifying (C.5′) yields

$$-1 = -18Z_1 + 63Z_3$$

Equation (C.4′) gives an immediate solution for Z_1; it is $Z_1 = \frac{2}{9}$. Substituting this into Equation (C.5′) allows us to solve for Z_3:

$$-1 = -18\left(\frac{2}{9}\right) + 63Z_3$$

$$-1 = -4 + 63Z_3$$

$$Z_3 = \frac{3}{63}$$

Substituting the values of Z_3 and Z_1 into (C.3′) yields for Z_2

$$Z_2 = 1 - \frac{8}{9} - \frac{6}{63} = \frac{1}{63}$$

This is the solution stated in the text. When the number of equations and variables becomes large, it is usually more convenient to solve the problem by working on a tableau. A tableau for the last problem is presented here.

Z_1	Z_2	Z_3	= Constant
4	1	2	= 1
3	3	6	= 1
6	6	75	= 5

Under each of the variables is the coefficient shown in the system of Equations (C.3), (C.4), and (C.5). If c_1, c_2, c_3 are arbitrary constants, the solution is reached when the tableau looks as follows:

Z_1	Z_2	Z_3	= Constant
1	0	0	c_1
0	1	0	c_2
0	0	1	c_3

To move from the first tableau to the second, three operations are allowed:

1. You can multiply or divide any row by a constant.

2. You can add or subtract a constant times one row from another row.

3. You can exchange any two rows.

Let us apply this to the problem discussed earlier. Subtracting twice row 2 from row 3 yields

Z_1	Z_2	Z_3	= Constant
4	1	2	1
3	3	6	1
0	0	63	3

Dividing row 3 by 63 yields

Z_1	Z_2	Z_3	= Constant
4	1	2	1
3	3	6	1
0	0	1	$\frac{3}{63}$

Subtracting 2 times row 3 from row 1, and 6 times row 3 from row 2 yields

Z_1	Z_2	Z_3	= Constant
4	1	0	$\frac{57}{63}$
3	3	0	$\frac{45}{63}$
0	0	1	$\frac{3}{63}$

Subtracting $\frac{1}{3}$ of row 2 from row 1 yields

Z_1	Z_2	Z_3	= Constant
3	0	0	$\frac{42}{63}$
3	3	0	$\frac{45}{63}$
0	0	1	$\frac{3}{63}$

Taking $\frac{1}{3}$ of row 1 and $\frac{1}{3}$ of row 2 yields

Z_1	Z_2	Z_3	= Constant
1	0	0	$\frac{14}{63}$
1	1	0	$\frac{15}{63}$
0	0	1	$\frac{3}{63}$

Subtracting row 1 from row 2 yields the final tableau:

Z_1	Z_2	Z_3	= Constant
1	0	0	$\frac{14}{63}$
0	1	0	$\frac{1}{63}$
0	0	1	$\frac{3}{63}$

The now familiar solution can be read directly from this tableau. It is

$$Z_1 = \frac{14}{63}, \qquad Z_2 = \frac{1}{63}, \qquad \text{and} \qquad Z_3 = \frac{3}{63}$$

Either of these methods can be used to solve a system of simultaneous equations.

APPENDIX D

A GENERAL SOLUTION

Although we have just outlined a feasible procedure for identifying the efficient frontier, there is a simpler one. When we assumed a particular riskless lending and borrowing rate, we determined that the optimum portfolio is the one that solves the following system of simultaneous equations:

$$
\begin{aligned}
\bar{R}_1 - R_F &= Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + Z_N\sigma_{1N} \\
\bar{R}_2 - R_F &= Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + Z_N\sigma_{2N} \\
\bar{R}_3 - R_F &= Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^2 + \cdots + Z_N\sigma_{3N} \\
&\vdots \\
\bar{R}_N - R_F &= Z_1\sigma_{1N} + Z_2\sigma_{2N} + Z_3\sigma_{3N} + \cdots + Z_N\sigma_N
\end{aligned}
$$

When we solved this system of simultaneous equations, we substituted, in particular, values of $\bar{R}_i$, R_F, σ_i^2, and σ_{ij}. However, we do not have to substitute in a particular value of R_F. We can simply leave R_F as a general parameter and solve for Z_k in terms of R_F. This results in a solution of the form

$$Z_k = C_{0k} + C_{1k}R_F$$

where C_{0k} and C_{1k} are constants. They have a different value for each security k, but that value does not change with changes in R_F. Once the Z_k are determined as functions of R_F, we could vary R_F to determine the amount to invest in each security at various points along the efficient frontier. Let us apply this to the example following Equation (6.1). The system of simultaneous equations for a general R_F is

$$14 - R_F = 36Z_1 + 9Z_2 + 18Z_3 \tag{D.1}$$

$$8 - R_F = 9Z_1 + 9Z_2 + 18Z_3 \tag{D.2}$$

$$20 - R_F = 18Z_1 + 18Z_2 + 225Z_3 \tag{D.3}$$

The solution to this system of simultaneous equations is

$$Z_1 = \frac{42}{189} \tag{D.4}$$

$$Z_2 = \frac{118}{189} - \frac{23}{189}R_F \tag{D.5}$$

$$Z_3 = \frac{4}{189} + \frac{1}{189}R_F \tag{D.6}$$

This solution can be confirmed by substituting these values into Equations (D.1), (D.2), and (D.3). Also, as a further check, note that the substitution of $R_F = 5$ (which was the value we assumed in the last section) into Equations (D.4), (D.5), and (D.6) yields

$$Z_1 = \frac{42}{189} = \frac{14}{63}$$
$$Z_2 = \frac{118}{189} - \frac{23}{189}(5) = \frac{118 - 115}{189} = \frac{3}{189} = \frac{1}{63}$$
$$Z_3 = \frac{4}{189} + \frac{1}{189}(5) = \frac{9}{189} = \frac{3}{63}$$

the same solution we obtained earlier. The values of Z_k just determined can be scaled to sum to 1 exactly as was done before so that the optimum proportions can be determined.

Determining the General Coefficient from Two Portfolios

In the last section we determined that

$$Z_2 = \frac{118}{189} - \frac{23}{189} R_F$$

Assume that we had not determined this general expression. Rather, we simply solved the system of simultaneous equations for two arbitrary values of R_F. The value of Z_2 corresponding to an R_F of 5 is $\frac{1}{63}$, and the Z_2 corresponding to an R_F of 2 is $\frac{72}{189}$. Can we determine the general expression? The answer is clearly yes. As an example, assume we had solved the equations for an R_F of 2 and 5. We know the general expression has the form

$$Z_2 = C_{02} + C_{12} R_F$$

Furthermore, we know that

$$Z_2 = \tfrac{1}{63} \qquad \text{if } R_F = 5$$
$$Z_2 = \tfrac{72}{189} \qquad \text{if } R_F = 2$$

Utilizing this in the previous equation, we have

$$\tfrac{1}{63} = C_{02} + C_{12}(5)$$
$$\tfrac{72}{189} = C_{02} + C_{12}(2)$$

This is a system of two equations and two unknowns. We can use it to solve for $C_{02} = \frac{118}{189}$ and $C_{12} = -\frac{23}{189}$. Thus, if we have the optimum portfolio for any two values of R_F, we can obtain the value for C_{0k} and C_{1k} and then, by varying R_F, obtain the full efficient frontier.

This is an extremely powerful result. It means that the solution of the system of simultaneous equations for any two values of R_F allows us to trace out the full efficient frontier.

The tracing out of the efficient frontier can be done in two ways. First, we could solve for the general expression for Z_k in terms of R_F by determining Z_k for any two arbitrary values of R_F. Then, by varying R_F over the relevant range, we could trace out the efficient frontier.

A second procedure is suggested by the previous discussion. We showed that solving the system of simultaneous equations for any two values of R_F allowed us to obtain a general expression for Z_k in terms of R_F, thus enabling us to trace out the efficient frontier. This suggests that the efficient frontier can be determined directly simply by calculating any two optimum portfolios rather than indirectly by first determining Z_k as a function of R_F. It can be shown that this direct procedure is appropriate.[10] Thus the entire efficient frontier can be traced out by determining the composition of any two portfolios and then determining all combinations of these two portfolios. This is an extremely powerful result and is the preferred way to determine the efficient set.

In the previous chapter we showed how to trace out all combinations (portfolios) of two assets. Nothing prevents the two assets from being efficient portfolios. Thus, given that the efficient frontier can be traced out by combining two efficient portfolios, if we find two efficient portfolios, we can utilize the procedures discussed in the last chapter to trace out the full efficient frontier. Let us see how this is done.

Tracing Out the Efficient Frontier

The Z_k that correspond to an $R_F = 2$ are from Equations (D.4), (D.5), and (D.6):

$$Z_1 = \frac{42}{189}, \qquad Z_2 = \frac{72}{189}, \qquad Z_3 = \frac{6}{189}$$

The proportions to invest in each security are

$$X_1 = \frac{42}{120} = \frac{7}{20}$$
$$X_2 = \frac{72}{120} = \frac{12}{20}$$
$$X_3 = \frac{6}{120} = \frac{1}{20}$$

The expected return associated with this portfolio is

$$\overline{R}_P = \left(\frac{7}{20}\right)(14) + \left(\frac{12}{20}\right)(8) + \left(\frac{1}{20}\right)(20) = 10\tfrac{7}{10}$$

The variance of return on this portfolio is

$$\sigma_P^2 = \left(\frac{7}{20}\right)^2(36) + \left(\frac{12}{20}\right)^2 9 + \left(\frac{1}{20}\right)^2(225)$$
$$+2\left(\frac{7}{20}\right)\left(\frac{12}{20}\right)9 + 2\left(\frac{7}{20}\right)\left(\frac{1}{20}\right)(18) + 2\left(\frac{12}{20}\right)\left(\frac{1}{20}\right)18 = \frac{5{,}481}{400}$$

If we knew the covariance between the portfolios associated with an $R_F = 5$ and an $R_F = 2$, we could trace out the full efficient frontier by treating each portfolio as an asset and utilizing the method discussed in Chapter 5. The covariance is determined as follows.

[10]See Black (1972) for a rigorous proof that this holds.

Consider a portfolio consisting of $\frac{1}{2}$ of each of the two portfolios already determined. The investment proportions are

$$X_1'' = \frac{1}{2}\frac{7}{20} + \frac{1}{2}\frac{14}{18} = \frac{203}{360}$$
$$X_2'' = \frac{1}{2}\frac{12}{20} + \frac{1}{2}\frac{1}{18} = \frac{118}{360}$$
$$X_3'' = \frac{1}{2}\frac{1}{20} + \frac{1}{2}\frac{3}{18} = \frac{39}{360}$$

Its variance is

$$\begin{aligned}\sigma_P^2 &= \left(\frac{203}{360}\right)^2 36 + \left(\frac{118}{360}\right)^2 9 + \left(\frac{39}{360}\right)^2 225 \\ &+ 2\left(\frac{203}{360}\right)\left(\frac{118}{360}\right)9 + 2\left(\frac{203}{360}\right)\left(\frac{118}{360}\right)18 \\ &+ 2\left(\frac{118}{360}\right)\left(\frac{39}{360}\right)18 = 21.859\end{aligned}$$

But we know that this portfolio is a weighted average of the other two portfolios. In Chapter 5 we showed that the variance of a portfolio composed of two assets or portfolios was

$$\sigma_P^2 + X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}$$

Thus the variance of a portfolio consisting of $\frac{1}{2}$ of portfolio 1 and $\frac{1}{2}$ of portfolio 2 is

$$\sigma^2 = \left(\frac{1}{2}\right)^2\left(\frac{203}{6}\right) + \left(\frac{1}{2}\right)^2\left(\frac{5481}{400}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sigma_{12}$$

We know the variance of this portfolio is 21.859. Thus σ_{12} can be determined from

$$21.859 = \left(\frac{1}{2}\right)^2\left(\frac{203}{6}\right) + \left(\frac{1}{2}\right)^2\left(\frac{5481}{400}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sigma_{12}$$

and

$$\sigma_{12} = 19.95$$

Knowing the expected return variance and covariance, we can trace out the efficient frontier exactly as we did for combinations of two assets in Chapter 5. We have done so in Figure 6.6.

The Number of Securities Included

Before leaving this section, some observations are in order. First, when short sales are allowed, the investor takes a position in almost all securities. Each security will have, in general, one value of R_F for which it is not held, namely, when $C_{0k} + C_{1k}R_F = 0$. But for all other values of R_F, it will be held either long or short. In fact, for all values of R_F above this value, the security will be held only long or short, and vice versa for values of R_F

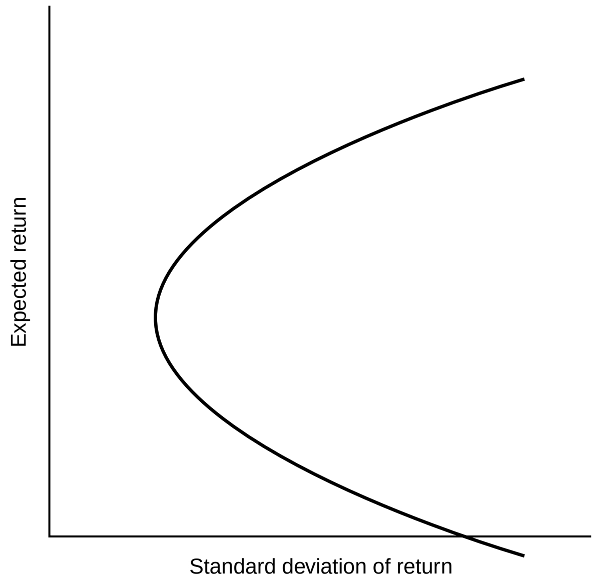

Figure 6.6 The minimum variance frontier.

below the value. Let us examine the expressions for Z_k as a function of R_F from our previous example:

$$Z_1 = \frac{42}{189}$$
$$Z_2 = \frac{118}{189} - \frac{23}{189} R_F$$
$$Z_3 = \frac{4}{189} + \frac{1}{189} R_F$$

Security 1 is always held long. Security 2 is held long if R_F is less than $\frac{118}{23}$ and short for all values of R_F greater than $\frac{118}{23}$. Finally, security 3 is held long if R_F is greater than -4 and short for values of R_F below -4. The various values of Z as a function of R_F are shown in Figure 6.7.

The inclusion of almost all or all securities in the optimum portfolio makes intuitive sense. If a security's characteristics make it undesirable to hold, then the investor should issue it by selling it short. Thus "good" securities are held and "bad" securities are issued to someone else. Of course, for someone else to be willing to take "bad" securities, there has to be a difference of opinion regarding what is good and what is bad.

APPENDIX E

QUADRATIC PROGRAMMING AND KUHN–TUCKER CONDITIONS

These quadratic programming algorithms are based on a technique from advanced calculus called Kuhn–Tucker conditions. For small-scale problems, these conditions may be able to be used directly. Furthermore, an understanding of the nature of the solution to this type of portfolio problem can be gained by understanding the Kuhn–Tucker conditions.

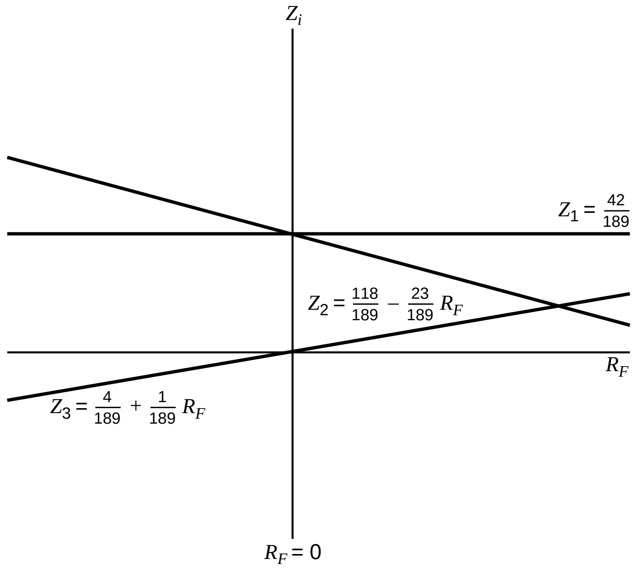

Figure 6.7 Portfolio proportion as a function of the riskless rate.

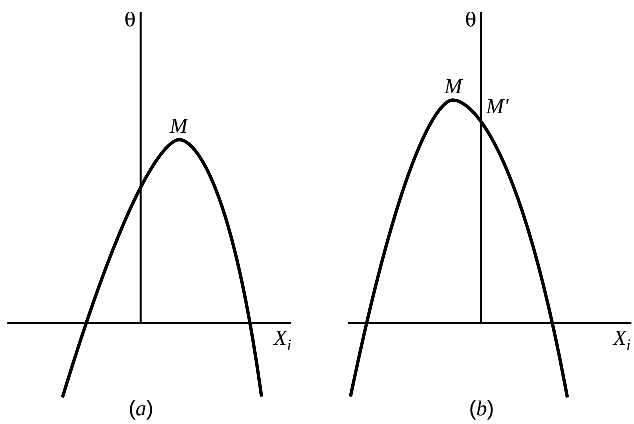

Figure 6.8 Value of the function as X changes.

Earlier we simply took the derivative of θ with respect to each X_i and set it equal to zero to find a maximum value of θ. This maximum is indicated by point M in Figure 6.8*a* or 6.8*b*. When X_i must be nonnegative, a problem can occur because the unconstrained maximum may be at a value of X_i, which is infeasible. Variable θ as a function of X_i might look like Figure 6.8*b* rather than Figure 6.8*a*. In this case (Figure 6.8*b*), the maximum feasible value of θ occurs at point M' rather than M. Notice that if the maximum value for X_i occurs at M', then $d\theta/dX_i < 0$ at the maximum feasible value ($X_i = 0$), whereas if it occurs when X_i is positive, then $d\theta/dX_i = 0$. Thus, in general, with X_i constrained to be larger than or equal to zero, we can write

$$\frac{d\theta}{dX_i} \leq 0$$

We could make this an equality by writing

$$\frac{d\theta}{dX_i} + U_i = 0$$

This is the first Kuhn–Tucker condition for a maximum.

Note two things about U_i. If the optimum occurs when X_i is positive, then the $d\theta/dX_i = 0$ and U_i is zero. Furthermore, if the optimum occurs when the maximum occurs at $X_i = 0$, then $d\theta/dX_i < 0$ and U_i is positive. To summarize, at the optimum we have

$$\begin{aligned} X_i &> 0, \qquad U_i = 0 \\ X_i &= 0, \qquad U_i > 0 \end{aligned}$$

This is the second Kuhn–Tucker condition. It can be written compactly as

$$\begin{aligned} X_i U_i &= 0 \\ X_i &\geq 0 \\ U_i &\geq 0 \end{aligned}$$

The four Kuhn–Tucker conditions are

$$\begin{aligned} &(1) \quad \frac{d\theta}{dX_i} + U_i = 0 \\ &(2) \quad X_i U_i = 0 \\ &(3) \quad X_i \geq 0 \\ &(4) \quad U_i \geq 0 \end{aligned}$$

If someone suggested a solution to us and it satisfied the Kuhn–Tucker conditions, then we could be sure that he had indeed given us the optimum portfolio.[11] For example, assume the lending and borrowing rate was 6% and the securities being considered are the three securities considered throughout this chapter. Furthermore, assume the solution was

$$\begin{aligned} X_1 &= \frac{43}{53}, \qquad U_1 = 0 \\ X_2 &= 0, \qquad U_2 = \frac{5}{8} \\ X_3 &= \frac{10}{53}, \qquad U_3 = 0 \end{aligned}$$

Because this solution meets all the Kuhn–Tucker conditions, it is optimal.

[11]There are conditions on the shape of θ for this to be optimum, but they are always met for the portfolio problem and so can be safely ignored here.

To see that this solution meets the Kuhn–Tucker conditions, consider the following. First, all Xs and Us are positive; thus conditions 3 and 4 are met. U_1, X_2, and $U_3 = 0$; thus either X or U is zero for any pair of securities, and condition 2 is met. Finally, recall that

$$\frac{d\theta}{dX_i} = \bar{R}_i - R_F - \lambda\left[X_i\sigma_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} X_j\sigma_{ij}\right]$$

Adding U_i to this equation and substituting in the returns, variances, and covariances for the various securities, we have

$$8 - \lambda[36X_1 + 9X_2 + 18X_3] + U_1$$
$$2 - \lambda[9X_1 + 9X_2 + 18X_3] + U_2$$
$$14 - \lambda[18X_1 + 18X_2 + 225X_3] + U_3$$

where $\lambda = (\bar{R}_P - R_F)/\sigma_P^2$. A little calculation shows that $\lambda = \frac{53}{216}$.

Substituting for X_1, X_2, and X_3 yields

$$8 - \frac{53}{216}\left[36\left(\frac{43}{53}\right) + 9(0) + 18\left(\frac{10}{53}\right)\right] + 0$$
$$2 - \frac{53}{216}\left[9\left(\frac{43}{53}\right) + 9(0) + 18\left(\frac{10}{53}\right)\right] + \frac{5}{8}$$
$$14 - \frac{53}{216}\left[18\left(\frac{43}{53}\right) + 18(0) + 225\left(\frac{10}{53}\right)\right] + 0$$

Because all three equal zero, the Kuhn–Tucker conditions are met.

QUESTIONS AND PROBLEMS

1. Assume analysts provide the following types of information. Assume (standard definition) short sales are allowed. What is the optimum portfolio if the lending and borrowing rate is 5%?

Security	Mean Return	Standard Deviation	Covariance with A	B	C
A	10	4		20	40
B	12	10			70
C	18	14			

2. Given the following information, what is the optimum portfolio if the lending and borrowing rate is 6%, 8%, or 10%? Assume the Lintner definition of short sales.

Security	Mean Return	Standard Deviation	Covariance with A	Covariance with B	Covariance with C
A	11	2		10	4
B	14	6			30
C	17	9			

3. Assume the information given in Problem 1 but that short sales are not allowed. Set up the formulation necessary to solve the portfolio problem.

4. Consider the following data. What is the optimum portfolio, assuming short sales are allowed (standard definition)? Trace out the efficient frontier.

Number	$\bar{R}_i$	σ_i
1	10	5
2	8	6
3	12	4
4	14	7
5	6	2
6	9	3
7	5	1
8	8	4
9	10	4
10	12	2
	$\rho_{ij} = 0.5$ for all ij	
	$R_F = 4$	

5. Assume that the data below apply to two efficient portfolios. What is the efficient frontier? Assume the standard definition of short sales.

Portfolio	$\bar{R}_i$	σ_i	
A	10	6	
B	8	4	$\sigma_{ij} = 20$

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