We have discussed Non-linear Shooting method for solving second order BVP with Dirichlet's boundary condition

$$y'' = f(x, y, y') \quad \text{for } a \le x \le b$$

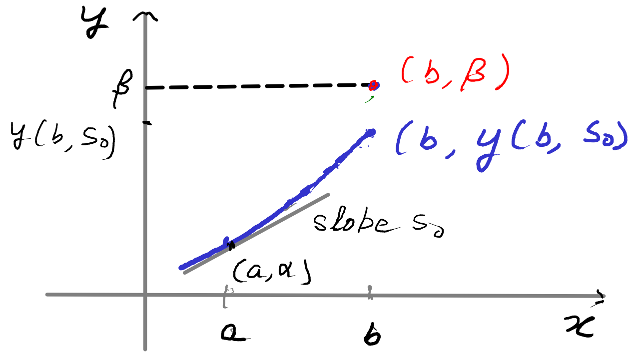
$$\text{with} \quad y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

The solution to the boundary-value problem is approximated by using the solutions to a sequence of initial-value problems involving a parameter s having the form

$$y'' = f(x, y, y') \qquad \text{for } a \le x \le b$$

$$\frac{\beta - \alpha}{b - a} = s_0$$

$$\text{with} \quad y(a) = \alpha \quad \text{and} \quad y'(a) = s$$



This is done by choosing the parameters $s = s_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, s_k) = y(b) = \beta$$

where y(x, s_k) denotes the solution to the initial-value problem with s=s_k ,
while y(x) denotes the solution to the boundary-value problem

-- is called a "shooting" method by analogy to the procedure of firing objects at a stationary target.

If y(b, s_0) is not sufficiently close to β, the approximation is corrected by choosing s = s_1, s_2, and so on until y(b, s_k) is sufficiently close to "hitting" β

$$y(b, s_k) - \beta = 0$$

This is a non-linear equation that is solved by the Newton-Raphson or Secant method

Neumann Boundary Conditions

$$y'(a) = \alpha, \quad y'(b) = \beta$$

Now y(a) is approximated and then improved in each iteration

Use the initial conditions

$$y(a) = s, \quad y'(a) = \alpha$$

s is chosen such that $F(s) = y'(b,s) - y'(b) = y'(b,s) - \beta = 0$

$y(x) \rightarrow$ sol^n of BVP

$y(x,s) \rightarrow$ sol^n of IVP with $y(a) = s$

Robin Boundary Condition at x=b

$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$

$y(a) = s$

$\alpha_1 s + \alpha_2 y'(a) = \alpha_3$

$y'(a) = \dfrac{\alpha_3 - \alpha_1 s}{\alpha_2}$

$y(a) = \alpha$

$\beta_1 y(b) + \beta_2 y'(b) = \beta_3$

$y'(a) = s$

OR

$y'(a) = \alpha$

$\beta_1 y(b) + \beta_2 y(b) = \beta_3$

$y(a) = s$

Now the objective function whose roots are to be determined is

$$F(s) = \beta_3 - \beta_1 y(b, s) - \beta_2 y'(b, s)$$

Example

$$2xy'' + (y')^2 - 4y = 4x \quad \text{for} \quad 1 \le x \le 3$$

$$y'(1) = 4, \quad y(3) + 2y'(3) = 32$$

Neumann BC   Robin BC

Analytical Solution $\quad y(x) = (x+1)^2$

Objective function $\quad F(s) = 32 - y(3, s) - 2y'(3, s)$

Using RK 4 with step size 0.2

and secant method with initial guess $s_0 = 0$, $s_1 = 1$

Tolerance 0.5×10^{-6} i.e. $|F(s)| < 0.5 \times 10^{-6}$ is condition for termination

s	y(3,s)	y'(3,s)	F(s)
0	8.303482	5.781270	12.133978
1	10.361601	6.445059	8.748282
3.583894	15.254059	7.811455	1.123031
3.964445	15.936660	7.984070	0.0952008
3.999693	15.999486	7.999773	9.677×10^{-4}
4.000054	16.000131	7.999934	8.265×10^{-7}
4.000055	16.000132	7.999934	7.168×10^{-12}

with s= s_6 $= 4.000055$

analytical

x_i	$y(x_i; p_6)$ $y(x_i, s_6)$	$y(x_i)$ $y(x_i)$	Absolute Error $\lvert y(x_i) - y(x_i, s)\rvert$
1.00	4.000055	4.00	0.000055
1.20	4.840102	4.84	0.000102
1.40	5.760118	5.76	0.000118
1.60	6.760124	6.76	0.000124
1.80	7.840125	7.84	0.000125
2.00	9.000125	9.00	0.000125
2.20	10.240126	10.24	0.000126
2.40	11.560127	11.56	0.000127
2.60	12.960128	12.96	0.000128
2.80	14.440130	14.44	0.000130
3.00	16.000132	16.00	0.000132

HW

$y'' = 2y^3 \qquad 0 \leq x \leq 1$

a) $3y(0) - 9y'(0) = 2$
$y(1) = \frac{1}{4}$

b) $y(0) = \frac{1}{3}$, $2y(1) + 2y'(1) = 1$

c) $y(0) = \frac{1}{3}$, $y(1) = \frac{1}{4}$

d) $y'(0) = -\frac{1}{9}$, $y'(1) = \frac{1}{4}$

$y_{anal}(x) = \frac{1}{x+3}$

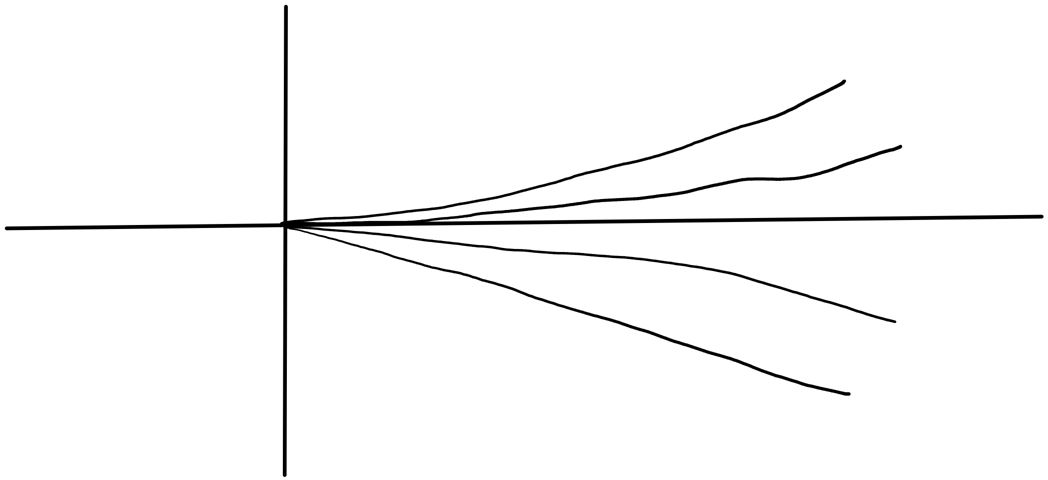
If Use (c) $\quad y(0) = \frac{1}{3}$, $y'(0) = s$] — I

If use (d) $\quad y(0) = s$, $y'(0) = -\frac{1}{9}$] — II

If use (a) (i) $y(0) = s \Rightarrow y'(0) = -\frac{2}{9} + \left(\frac{1}{3}\right)s$]
(ii) $y'(0) = s \Rightarrow y(0) = \frac{2}{3} + 3s$] — III

Plot $y_I(x, s)$ & $y_{III}(x, s)$ with $s = -1, -0.5, 0, 0.5, 1$

Plot $y_{II}(x, s)$ & $y_{III}(x, s)$ with $s = -1, -0.5, 0, 0.5, 1$



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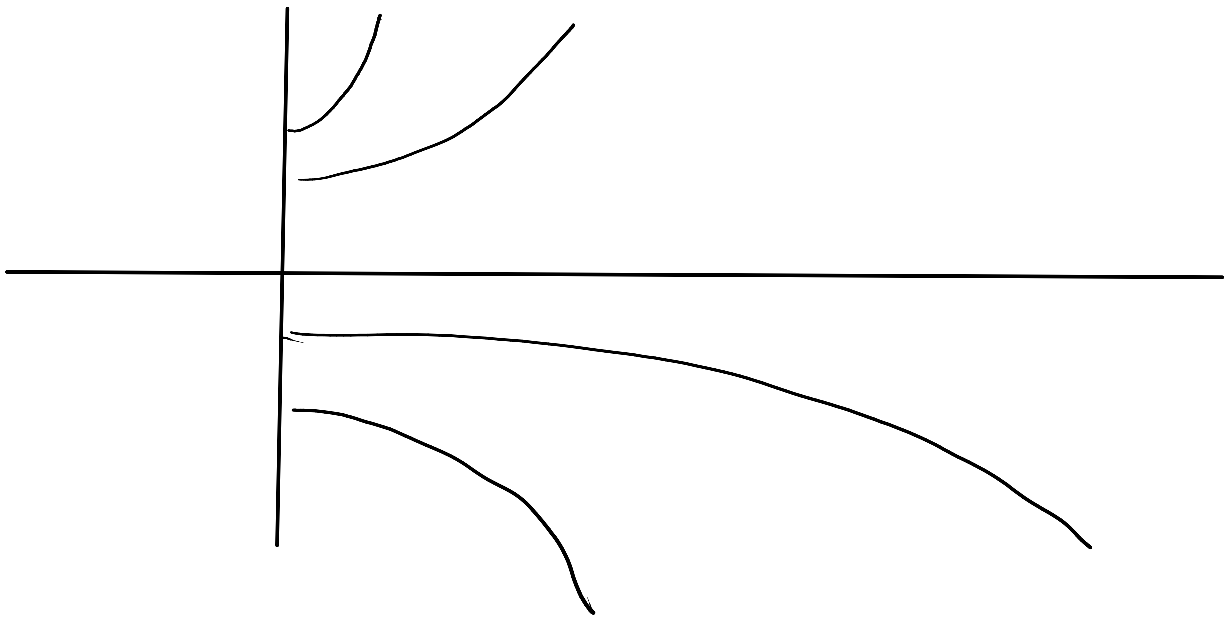

III

$$y'' = 2y^3$$

$$3y(0) - 9y'(0) = 2, \quad y(1) = \frac{1}{4}$$

$z = 1 - x$

$$\frac{d}{dx} = -\frac{d}{dz}$$

B.V.P in terms of z

$$y'' = 2y^3 \qquad y' = \frac{dy}{dz}$$

$$y(0) = \frac{1}{4}, \quad 3y(1) - 9y'(1) = 2$$

Solve using IVP

$$y'' = 2y^3$$

$$y(0) = \frac{1}{4}, \quad y'(0) = s$$