Linear Shooting

The differential equation

$$y'' = f(x, y, y')$$

is linear when functions p(x), q(x), and r(x) exist with

$$f(x, y, y') = p(x)y + q(x)y' + r(x)$$

Suppose the linear boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x) \quad \text{for} \quad a \le x \le b$$

$$\text{with} \quad y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

satisfies

(i) $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$,

(ii) $p(x) > 0$ on $[a, b]$

Then the boundary value problem has a unique solution

$y = y_h + y_p \longrightarrow$ particular sol^n of non-homogeneous eq^n

$\downarrow$

general sol^n of homogeneous eq^n $(r = 0)$

y_{1h} y_{2h} are 2 l.i sol^n

$c_1 y_{1h} + c_2 y_{2h}$

To approximate the unique solution to this linear problem, we first consider the initial-value problems

$$y'' = p(x)y + q(x)y' + r(x) \quad \text{for} \quad a \le x \le b$$

with $y(a) = \alpha$ and $y'(a) = 0$

AND $$y'' = p(x)y + q(x)y' \quad \text{for} \quad a \le x \le b$$

with $y(a) = 0$ and $y'(a) = 1$

Under the assumption

(i) $p(x)$, $q(x)$, $r(x)$ are continuous on $[a, b]$

(ii) $p(x) > 0$ on $[a, b]$

both problems have a unique solution.

IVP1 $\boxed{\begin{array}{l} y'' = p(x)y + q(x)y' + r(x) \qquad a \le x \le b \\ \text{with} \quad y(a) = \alpha \quad ; \quad y'(a) = 0 \end{array}}$

and

IVP 2. $\boxed{\begin{array}{l} y'' = p(x)y + q(x)y' \qquad a \le x \le b \\ y(a) = 0 \quad , \quad y'(a) = 1 \end{array}}$

$y_1(x)$ — sol^n of IVP1

$y_2(x)$ — sol^n of IVP2

$$y(x) = y_1(x) + c\, y_2(x)$$

$$y_1'' = p(x)y_1 + q(x)y_1' + r(x) \qquad y_1(a) = \alpha\ ,\ y_1'(a) = 0$$

$$y_2'' = p(x)y_2 + q(x)y_2' \qquad y_2(a) = 0\ ,\ y_2'(a) = 1$$

$$y''(x) = y_1''(x) + c y_2''(x)$$

$$= p(x)(y_1 + c y_2) + q(x)(y_1' + c y_2') + r(x)$$

$$y''(x) = p(x) y + q(x) y' + r(x)$$

$$y(a) = y_1(a) + c y_2(a) = \alpha$$

If $y(b) = y_1(b) + c y_2(b) = \beta$ then it reduces to the given BVP

$$c = \frac{\beta - y_1(b)}{y_2(b)}$$

$$\Rightarrow \quad y(x) = y_1(x) + \left(\frac{\beta - y_1(b)}{y_2(b)}\right) y_2(x)$$ is the

solⁿ of given BVP

<u>Ques</u>

$$-y'' + \pi^2 y = 2\pi^2 \sin(\pi x)$$

<u>HW</u>

$$0 \leq x \leq 1 \qquad y(0) = y(1) = 0$$

$p(x) = \pi^2 \;, \quad q(x) = 0$

$r(x) = -2\pi^2 \sin \pi x$

IVP 1

$$y'' = \pi^2 y - \underbrace{2\pi^2 \sin(\pi x)}$$

$$y(0) = 0 \;, \quad y'(0) = 0$$

IVP 2

$$y'' = \pi^2 y$$

$$y(0) = 0 \;, \quad y'(0) = 1$$

$$c = \frac{\beta - y_1(b)}{y_2(b)} = \frac{0 - y_1(b)}{y_2(b)} = \frac{y_1(b)}{y_2(b)}$$

$$y(x) = y_1 - \left(\frac{y_1(b)}{y_2(b)}\right) y_2$$

x_i	$y_1(x_i)$	$y_2(x_i)$	$y_n = y_1 + c y_2$	y_a	$\lvert y_n - y_a \rvert$
0.0	0.0	0.0	0.0		0
0.25	-0.157372	0.275702	0.707129		
0.50	-1.290357	0.730213	0.999327		
0.75	-4.490694	1.657343	0.706132		
1.00	-11.466375	3.656793	0.0		0

$c = 3.135637$

RK 4

Plot $y_{ana}(x)$

and $y_{num}(x)$

for no of intervals $= 2, 4, 8, 16$

$$y_{exact} = \sin(\pi x)$$

Make another table to verify order of convergence

Neuman B.C.

$$y'(a) = \alpha$$
$$y'(b) = \beta$$

Two IVP

IVP1

$$y'' = p(x)\,y + q(x)\,y' + r(x)$$
$$y(a) = 0, \quad y'(a) = \alpha$$

sol^n $y_1(x)$

y_1'

IVP2

$$y'' = p(x)y + q(x)y'$$
$$y(a) = 1, \quad y'(a) = 0$$

sol^n $y_2(x)$

$y_2'(x)$

$$y(x) = y_1(x) + c\,y_2(x)$$
$$y'(x) = y_1'(x) + c\,y_2'(x)$$

$$y'(b) = \beta$$

$$y_1'(b) + c\,y_2'(b) = \beta$$

$$c = \frac{\beta - y_1'(b)}{y_2'(b)}$$

Robin B.C.

$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$$

$$\beta_1 y(a) + \beta_2 y'(a) = \beta_3$$

IVP1

$$y'' = p(x) y + q(x) y' + r(x)$$

$$y(a) = 0 , \quad y'(a) = 0$$

y_1

IVP2

$$y''(x) = p(x) y + q(x) y'$$

$$y(a) = 1, \quad y'(a) = 0$$

y_2

IVP3

$$y''(x) = p(x) y + q(x) y'$$

$$y(a) = 0 , \quad y'(a) = 1$$

y_3

$$y(x) = y_1(x) + C_1 y_2(x) + C_2 y_3(x)$$

$\underline{x=a}$ $$\alpha_1 [y_1(a) + C_1 y_2(a) + C_2 y_3(a)] + \alpha_2 [y_1'(a) + C_1 y_2'(a) + C_2 y_3'(a)] = \alpha_3$$

$$\boxed{\alpha_1 C_1 + \alpha_2 C_2 = \alpha_3} \quad (A)$$

$\underline{x=b}$ $$[\beta_1 y_2(b) + \beta_2 y_2'(b)] C_1 + [\beta_1 y_3(b) + \beta_2 y_3'(b)] C_2$$

$$\Rightarrow \beta_1 y_1(b) - \beta_2 y_1'(b) = \beta_3 \quad (B)$$

Solve (A) & (B) for C_1 & C_2

$0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$

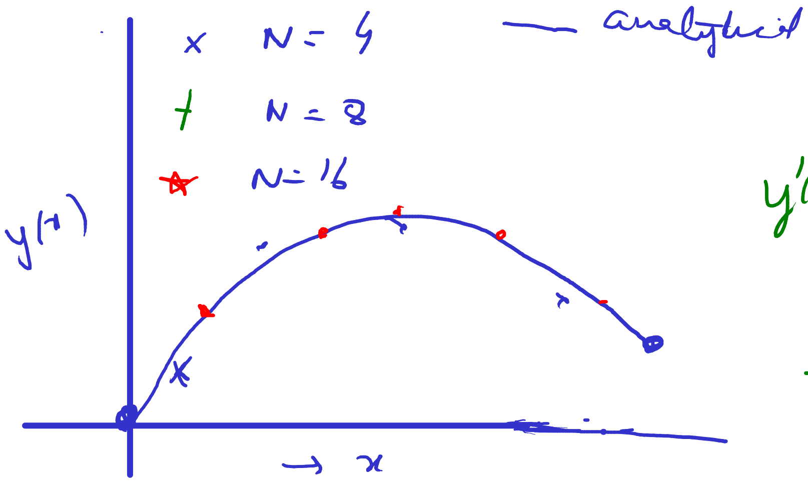
<u>HW</u>

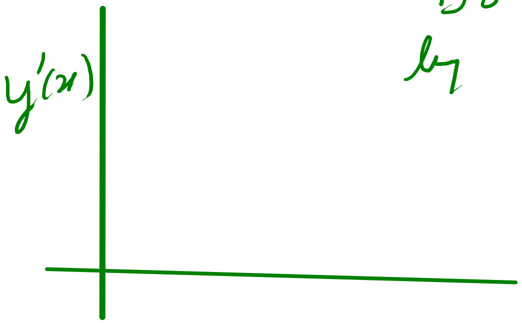
$$y'' + y = \sin(3x) \qquad 0 \le x \le \frac{\pi}{2} \qquad \frac{\pi}{2}$$

$$y(0) + y'(0) = -1$$

$$y'(\pi/2) = 1$$

$$y_{exact} = \frac{3}{8}\sin x - \cos x - \frac{1}{8}\sin(3x)$$





Do numerical calculation by hand for N = 4

y for $0 < x < 1$

$\vec{x}$ $\quad \left| y(N) - y(N/2) \right|_{max} < tol$

$x_0 \qquad\qquad x_0 = a$

$x_0 + \frac{h}{2}$

$x_0 + h \qquad\qquad x_0 + h$

$\vdots$

$x_0 + Nh \qquad\qquad x_0 + \frac{N}{2} h = b$

Q Let u represent the electrostatic potential between two concentric metal spheres of radii R_1 and R_2 (R_1 < R_2). The potential of the inner sphere is kept constant at V_1 volts, and the potential of the outer sphere is 0 volts. The potential in the region between the two spheres is governed by Laplace's equation

a) Show that in this particular application, Laplace Equation reduces to

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0 \qquad R_1 \leq r \leq R_2 \; ; \; u(R_1) = V_1 \, , \; u(R_2) = 0$$

and the solution is given by

$$u(r) = \frac{V_1 R_1}{r}\left(\frac{R_2 - r}{R_2 - R_1}\right)$$

b) Suppose R_1 = 5cm, R_2 = 10cm and V_1 = 110 volts. Approximate the solution by Shooting method and compare with the actual solution