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Boundary Value Problems

Ref

Brian Bradie, "A friendly introduction to Numerical Analysis, New Age International Publications

Two point Boundary Value Problems

1-d second order problems: Boundary is just two points $x = a, \quad x = b$

$$y'' = f(x, y, y')$$

If $f(x,y,y') = p(x)\, y + q(x)\, y' + r(x)$

$\longrightarrow$ Linear BVP

with two conditions -- one at x=a, and other at x=b --- Three types of BC

1. Robin or mixed Bounadry Conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$$

$$\beta_1 y(b) + \beta_2 y'(b) = \beta_3$$

If $\alpha_2 = \beta_2 = 0$

i.e. $y(a) = \alpha$ & $y(b) = \beta$ $\rightarrow$ Dirichlet B.C.

If $\alpha_1 = \beta_1 = 0$

$y'(a) = \alpha$, $y'(b) = \beta$

$\rightarrow$ Neumann B.C.

periodic B.C.

Numerical MEthods -- Shooting, Finite difference, finite element, etc

Theorem Dirichlet's BC

Suppose the function f in the boundary-value problem

$$y'' = f(x, y, y') \quad \text{for } a \le x \le b \text{ with } y(a) = \alpha \;\&\; y(b) = \beta$$

is continuous on the set

$$D = \{(x, y, y') \mid a \le x \le b,\ -\infty < y < \infty \text{ and } -\infty < y' < \infty\}$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

(i) $f(x, y, y') > 0 \ \forall\ x, y, y' \in D$ and

(ii) a constant M exists, with

$$|f(x, y, y')| < M. \ \forall\ x, y, y' \in D$$

then the boundary-value problem has a unique solution.

Shooting Method

The key idea of the shooting method is to transform the boundary value ODE into a system of first-order ODEs and solve as an initial value problem. Only boundary condition on one side is used as one of th einitial conditions. The additional initial condition is assumed.

Then an iterative approach is used to vary the assumed initial condition till the boundary condition on the otherside is satisfied

Consider BVP with Dirichlet Boundary Condition

$$y'' = f(x, y, y') \qquad a \leq x \leq b$$

$$y(a) = \alpha \quad , \quad y(b) = \beta$$

Basic Philospphy

If we wish to solve this by converting it into initial value problem, we need to know value of y' at x= a so make an assumption for it

Let $y'(a) = s$

We solve the IVP

$$y'' = f(x, y, y') \qquad y(a) = \alpha, \quad y'(a) = s$$

upto x=b using any numerical method for IVP. Let the solution to this IVP be $y(x, s)$

The solution of this problem should satisfy $y(b, s) = \beta$

$$\text{If} \quad \phi(s) = y(b, s) - \beta$$

Thus, the problem reduces to finding $s = s^*$ such that $\phi(s^*) = 0$ which is in general a non-linear equation which may be solved iteratively by Newton-Raphson or secant method

Iteratively new values of s are taken, Each iteration involves solving one IVP also

Thus, we approximate the solution to the boundary-value problem by using the solutions to a sequence of initial-value problems involving a parameter s

THese initial value problems are

$$y'' = f(x, y, y') \quad \text{for } a \le x \le b \quad \text{with } y(a) = \alpha \quad \text{and } y'(a) = s$$

Choose the parameters $s = s_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, s_k) = y(b) = \beta$$

where $y(x, s_k)$ is the solution of IVP with parameter $s = s_k$ while $y(x)$

is the solution of given BVP

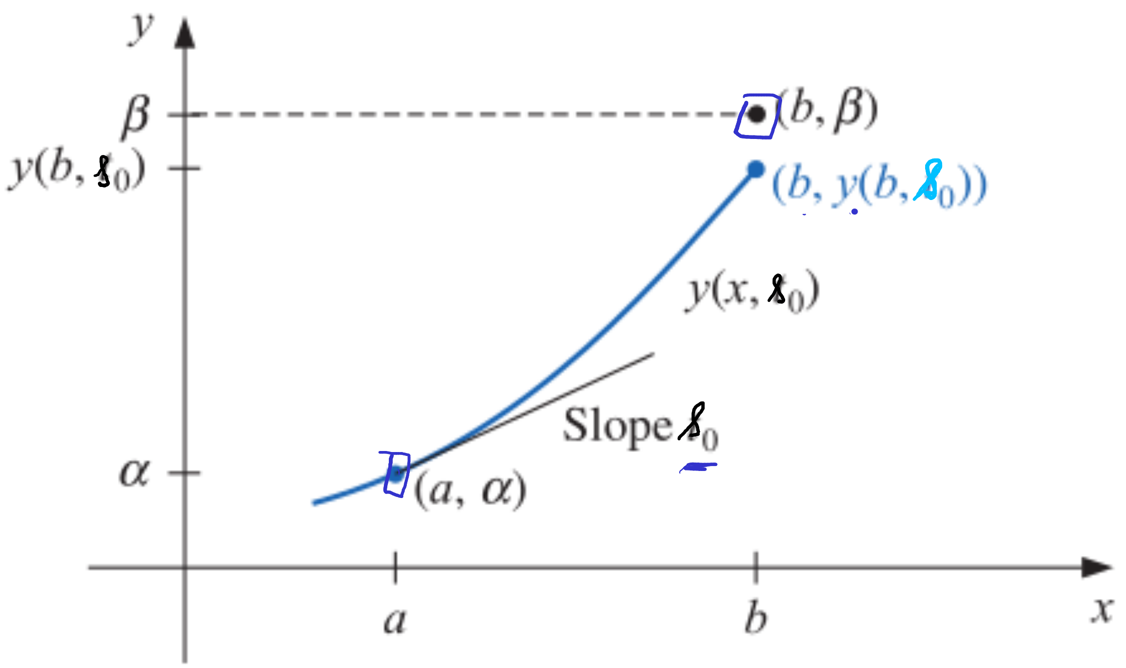
Step 1

This technique is called a "shooting" method by analogy to the procedure of firing objects at a stationary target) We start with a parameter s_0 that determines the initial elevation at which the object is fired from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(x, y, y') \quad a \le x \le b \quad \text{with } y(a) = \alpha \quad \text{and } y'(a) = s_0$$

Step 2 check $|y(b, s_0) - \beta|$

If y(b, s_0) is not sufficiently close to β , we correct our approximation by choosing elevations s_1, s_2, and so on, until y(b, s_ k) is sufficiently close to "hitting" β



The parameters s_k are determined by either Newton-RAphson/Secant method

Secant Method

$$s_2 = s_1 - \frac{(s_1 - s_0)\,(y(b,s_1) - \beta)}{y(b,s_1) - y(b,s_0)}$$

Here we need two initial approximations s_0 & s_1

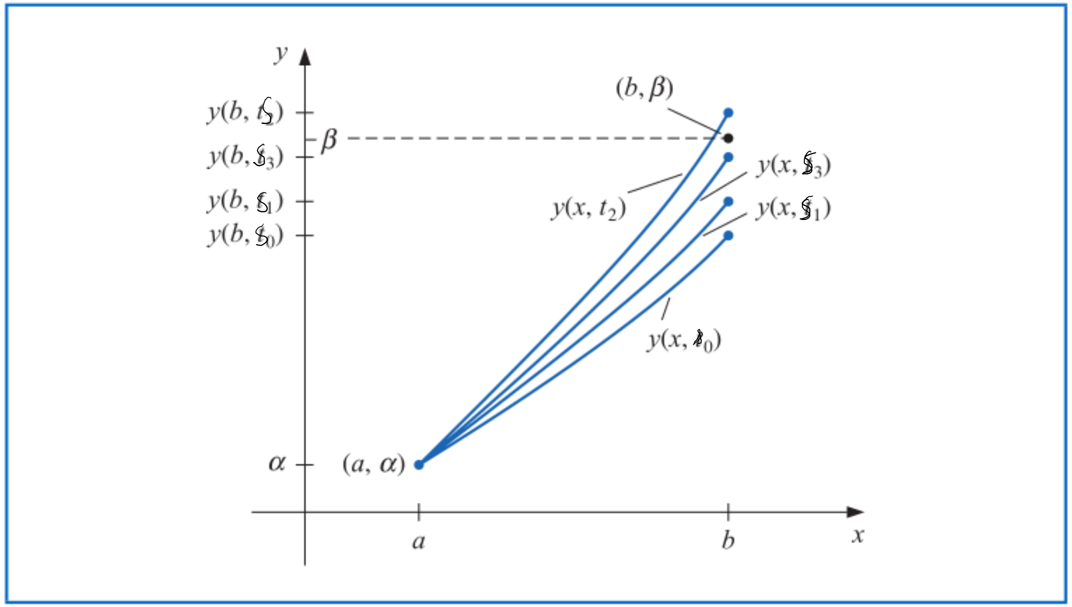
and then the remaining terms of the sequenceare generated by

$$s_k = s_{k-1} - \frac{(s_{k-1} - s_{k-2})\,\phi(s_{k-1})}{\phi(s_{k-1}) - \phi(s_{k-2})}$$

$k = 2, 3, \ldots$

The IVP is initially solved for two values s_0 and s_1

The iteration is stopped when $|\phi(s_k)| <$ given error tolerance



Newton-Raphson Iteration

$$s_k = s_{k-1} - \frac{\phi(s_{k-1})}{\phi'(s_{k-1})} \qquad k = 1, 2, \ldots$$

$$\phi = y(b,s) - \beta$$

it requires the knowledge of derivative $\left.\dfrac{dy}{ds}\right|_{s=s_{k-1}}$

$$\phi'(s) = \frac{d\,y(b,s)}{ds}$$

This presents a difficulty because an explicit representation for y(b, s) is not known;

we know only the values

$$y(b,s_0),\ y(b,s_1),\ \ldots,\ y(b,s_{k-1})$$

we rewrite the initial-value problem, emphasizing that the solution depends on both x and the parameter s

Denote: $y_s = y(x,s)$, $y_s' = y'(x,s)$, $y_s'' = y''(x,s)$

Then the IVP is

$$\boxed{y_s'' = f(x, y_s, y_s') \quad a \le x \le b, \quad y(a,s) = \alpha, \quad y'(a,s) = s}$$

prime indicates differentiation with respect to x.

We need to find $\left.\dfrac{dy}{ds}\right|_{(b, s_{k-1})}$

$$\frac{\partial y''}{\partial s} = \frac{\partial f}{\partial s}\left(x, y(x,s), y'(x,s)\right)$$

$$= \frac{\partial f}{\partial x}\left(x, y(x,s), y'(x,s)\right)\underbrace{\frac{\partial x}{\partial s}} + \frac{\partial f}{\partial y}\left(x, y(x,s), y'(x,s)\right)\frac{\partial y}{\partial s}(x,s)$$

$$+ \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial s}$$

x & s are independent $\therefore \dfrac{\partial x}{\partial s} = 0$

& $\dfrac{\partial y''}{\partial s} = \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial y'}\dfrac{\partial y'}{\partial s}$ for $a \leq x \leq b$.

IC $\dfrac{\partial y}{\partial s}(a,s) = 0$ & $\dfrac{\partial y'}{\partial s}(a,s) = 1$

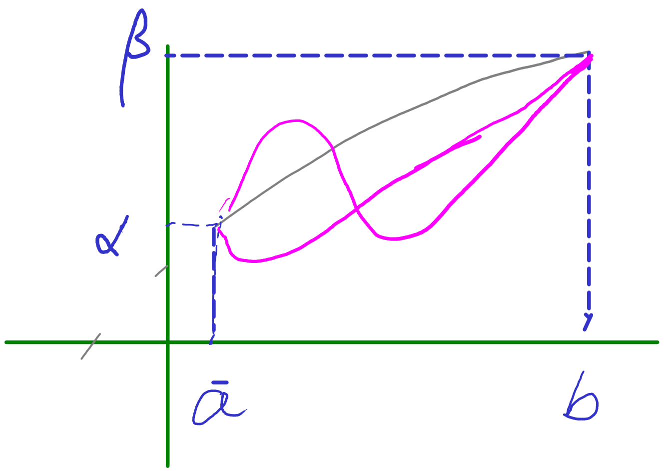
$$\frac{\partial y}{\partial s} = z(x,s)$$

$$z' = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial s}\right) = \frac{\partial}{\partial s}(y')$$

$$\& \quad z'' = \frac{\partial}{\partial s} y''$$

$$\boxed{\begin{gathered} z''(x,s) = \frac{\partial f}{\partial y} z + \frac{\partial f}{\partial y'} z' \qquad a \le x \le b \\ z(a,s) = 0, \quad z'(a,s) = 1 \end{gathered}}$$

$$s_k = s_{k-1} - \left(\frac{y(b, s_{k-1}) - \beta}{z(b, s_{k-1})}\right)$$



① given $s_0 = \dfrac{\beta - \alpha}{b - a}$ ✓ (input)

② solve 2 IVP with $s = s_0$ & find $y(b, s_0)$ & $z(b, s_0)$

③ determine s_1

④ Repeat ② & ③ with $s = s_1$ & find s_2

Examples

my_ivp

$$yy'' + (y')^2 + 1 = 0$$

subject to $y(1) = 1$, $y(2) = 2$

$\phi(s) = -y(b,s) + \beta$

Let $y'(1) = s$, $\phi(s) = 2 - y(2,s)$

Secant Method

$$s_k = s_{k-1} - \phi(s_{k-1}) \left[\frac{s_{k-1} - s_{k-2}}{\phi(s_{k-1}) - \phi(s_{k-2})} \right]$$

RK4 with ten steps $\rightarrow$ $y(1.1)$, $y(1.2)$ ... $y(1.9)$, $y(2)$

Let

① $s_0 = 0$ $\quad$ $y'(1) = 0$

use RK4 $y(2,0) \simeq 0.104101 \Rightarrow \phi(0) = 1.895899 > tol$

② choose $s_1 = 1$ & use RK4 to get $y(2,1) \simeq 1.414197$

$\Rightarrow \phi(1) = 0.585803$

(3) Apply secant

$$S_2 = S_1 - \phi(S_1)\left[\frac{S_1 - S_0}{\phi(S_1) - \phi(S_0)}\right]$$

$$= 1.447145$$

use RK4 with $y'(1) = S_2$

$y(2, S_2) \simeq 1.701210$ and $\phi(S_2) = 0.298790$

(4) $S_3 = S_2 - \phi(S_2)\left[\frac{S_2 - S_1}{\phi(S_2) - \phi(S_1)}\right] = 1.912638$

k	S_k	$y(2, S_k)$	$\phi(S_k)$
3	1.912638	1.955692	0.044308
4	1.993685	1.996677	0.003322
5	2.000256	1.999963	3.698×10^{-5}
6	2.000330	1.99999969	$3.088 \times 10^{-8} < \text{Tol}$

$\text{Tol} = 0.5 \times 10^{-6}$

$$y_{anal} = \sqrt{6x - 4 - x^2} = y_A \qquad \underline{\max(E)}$$

$i = 1, \dots M$

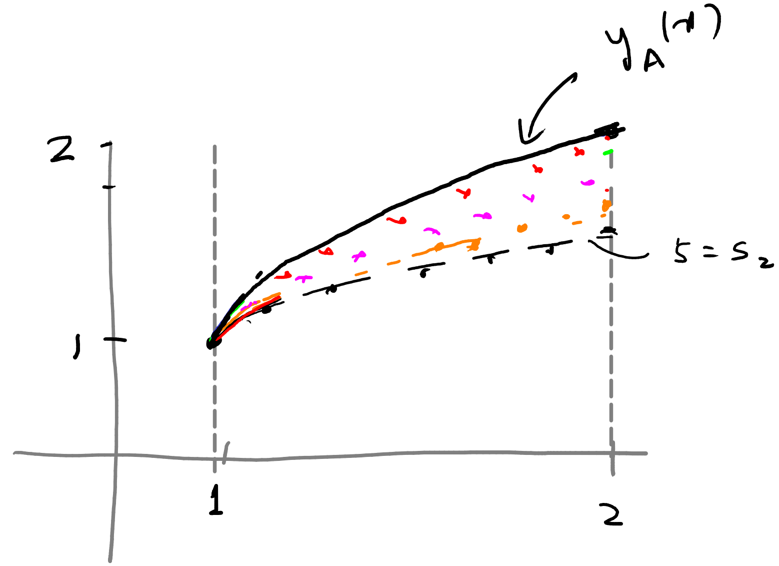
$\underline{\underline{M = 10}}$

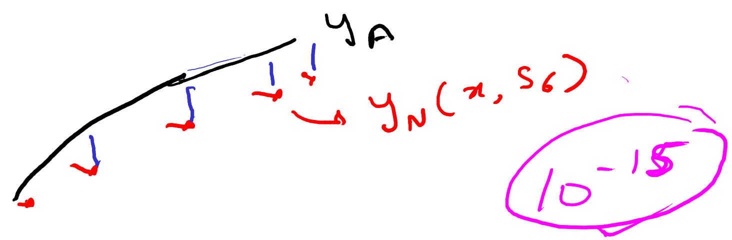
$\underline{h = \frac{2-1}{M}}$

$len(x)$

x_i	$y_N(x_i, s_6)$	$y_A(x_i)$
1.0	1.0	1.0
1.1		
1.2		
1.3		
⋮		
2.0		

$$\underline{E}(x_i) = |y_A(x_i) - y_N(x_i)|$$





check with $Tol = 0.5 \times 10^{-13}$

Input $\underline{Tol}$, $\underline{M}$, $\underline{s_0, s_1}$
from user

B.C

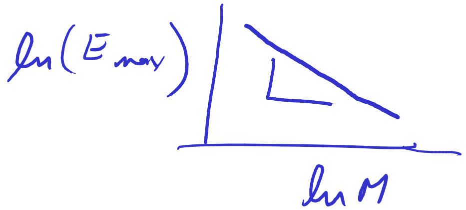
a, b, α, β

$f(x, y, y')$

ORDER OF CONVERGENCE

vary M

$M = 2 * M$



(4)

rms Error $= \sqrt{\dfrac{\sum_{x_i} (y_N - y_A)^2}{N}}$

M	Max absolute error (E_{max})	Error Ratio	rms Error	Error ratio
5	3.048872×10^{-4}		1.85	
		$E_{max}^{M=5} / E_{max}^{M=10}$ = 11.182906		
10	2.726369×10^{-5}			
		$E_{max}^{M=10} / E_{max}^{M=20}$		
20				
		16.23358		
40				
		16.24859		
80				
		16.15256		16.
160				
		16.0847		16.08105
320				

$16 = 2^4$

$$\frac{E(M = M_1)}{E(M = 2M_1)} = 16 = 2^4 \sim M^4$$