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Today

Chap. 5. Testing hypotheses (continued)

- Example: Binomial test
- Uniformly most powerful tests
- Monotone likelihood ratio

Example: Binomial test

- Assume that $X \sim \mathrm{Bin}(5, p)$ where $p \in [1/2, 1]$.
- Test $H_0 : p = 1/2$ vs $H_1 : p = 3/4$ on level 0.05.
- Derive the most powerful test.

Uniformly most powerful tests

- Suppose we observe $\mathbf{X} = (X_1, ..., X_n)$ where the X_i are independent, distributed as X with parameter θ.
- The parameter space is $\Theta = \Theta_0 \cup \Theta_1$ where $\Theta_0 \cap \Theta_1 = \varnothing$.
- Test H_0: $\theta \in \Theta_0$ vs H_1: $\theta \in \Theta_1$.
- We want to find a test which is most powerful for a given size (significance level).
- Today: generalize the Neyman-Pearson lemma, where we had H_0: $\theta = \theta_0$, H_1: $\theta = \theta_1$, to
  - H_0: $\theta \leq \theta_0$, $H_1 : \theta > \theta_0$
  - H_0: $\theta \geq \theta_0$, H_1: $\theta < \theta_0$.
- At first, we need to define the test size!

Uniformly most powerful tests

Definition (5.8)

A test φ is called a **test of size α** if.f.

$$\alpha = \sup_{\theta \in \Theta_0} \mathrm{E}_\theta\{\varphi(\mathbf{X})\}.$$

Example 1:

- Suppose $X \sim \mathrm{Bin}(5, p)$.
- Test $H_0 : p \leq 1/2$ vs $H_1 : p > 1/2$ on level 0.05.
- Consider the Binomial test example:

$$\varphi(x) = \begin{cases} 1 & \text{if } x = 5, \\ \gamma = 0.12 & \text{if } x = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the maximum of the power function lies on the border to the alternative.

Uniformly most powerful tests

A uniformly most powerful test is the one with highest power among those that have (at most) the same size.

Definition (5.9)

A test φ^* is called **uniformly most powerful (UMP) of size α** if.f.

$$\sup_{\theta \in \Theta_0} \mathrm{E}_\theta\{\varphi^*(\mathbf{X})\} = \alpha$$

and

$$\mathrm{E}_\theta\{\varphi^*(\mathbf{X})\} \geq \mathrm{E}_\theta\{\varphi(\mathbf{X})\}$$

for all $\theta \in \Theta_1$ and for all tests φ of size at most α.

Monotone likelihood ratio

Definition (5.10)

A model $\{\mathrm{P}_\theta, \theta \in \Theta\}$ with $\Theta \subseteq \mathcal{R}$ is said to have a **monotone likelihood ratio (MLR)** in the statistic T if.f. for all θ, θ' with $\theta > \theta'$ there exists a nondecreasing function $F_{\theta,\theta'}$ such that

$$\frac{L(\theta; \mathbf{x})}{L(\theta'; \mathbf{x})} = F_{\theta,\theta'}\{T(\mathbf{x})\}.$$

Example 2:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be a i.i.d. sample from $N(\mu, \sigma^2)$ with known σ^2.
- Show that the model of the sample has a MLR in $T(\mathbf{x}) = \sum_{i=1}^n x_i$.

Monotone likelihood ratio

Theorem (5.3)

Let $\mathbf{X} = (X_1, ..., X_n)$ *be a i.i.d. sample from* P_θ. *If the distribution* P_θ *belongs to a one-parameter exponential family with*

$$p(x; \theta) = A(\theta)h(x) \exp\{\zeta(\theta)R(x)\},$$

where $\zeta(\cdot)$ *is monotone nondecreasing, then the model of the sample has a MLR in* $T(\mathbf{x}) = \sum_{i=1}^n R(x_i)$.

Monotone likelihood ratio

Theorem (5.4)

Blackwell *Suppose that we have a MLR family in T.*

(a) *For the test problem H_0: $\theta \geq \theta_0$ vs H_1: $\theta < \theta_0$ the test*

$$\varphi\{T(\mathbf{x})\} = \begin{cases} 1 & \text{if} \quad T(\mathbf{x}) < k, \\ \gamma & \text{if} \quad T(\mathbf{x}) = k, \\ 0 & \text{if} \quad T(\mathbf{x}) > k, \end{cases}$$

with γ, k such that $\mathrm{E}_{\theta_0}\{\varphi(\mathbf{X})\} = \alpha$ is a UMP test of size α.

(b) *For the test problem H_0: $\theta \leq \theta_0$ vs H_1: $\theta > \theta_0$ the test*

$$\varphi\{T(\mathbf{x})\} = \begin{cases} 1 & \text{if} \quad T(\mathbf{x}) > k, \\ \gamma & \text{if} \quad T(\mathbf{x}) = k, \\ 0 & \text{if} \quad T(\mathbf{x}) < k, \end{cases}$$

with γ, k such that $\mathrm{E}_{\theta_0}\{\varphi(\mathbf{X})\} = \alpha$ is a UMP test of size α.

(c) *For any θ of the respective alternatives the tests in (a) and (b) have maximal power among all tests ψ with $\mathrm{E}_{\theta_0}\{\psi(\mathbf{X})\} = \alpha$.*

Monotone likelihood ratio

Corollary 5.2: The exponential families with a MLR satisfy theorem 5.4.

Example 3: Derive UMP tests on level $\alpha = 0.05$ for the following situations.

1. Let $\mathbf{X} = (X_1, ..., X_n)$ be an i.i.d. sample from $N(\mu, \sigma^2)$ with known σ^2. Test H_0: $\mu \geq \mu_0$ vs H_1: $\mu < \mu_0$.
2. $X \sim \mathrm{Bin}(5, p)$. Test $H_0 : p \leq 1/2$ vs $H_1 : p > 1/2$.

News of today

Generalization to one-sided tests:

- Test size
- UMP test
- MLR
- MLR in the exponential family
- Blackwell: When LR tests are UMP.
- It works for the exponential family.