Richardson Extrapolation

Interpolation is to estimate a value between a given set of known values. Extrapolation is to use known values to project a value outside of the intended range of the previous values. Using the concept of Richardson Extrapolation, very higher order integration can be achieved using only a series of values from Trapezoidal Rule. Similarly, accurate values of derivatives could be obtained using low-order central difference derivatives.

Generally, when an approximate formula is developed, for example, the Trapezoidal Rule, the formula could be written as

$$\int_a^b f(x)\,dx \approx \frac{h}{2}f(a) + \frac{h}{2}f(b) \qquad ,$$

in which h is the increment between the sample points. The approximation, however, could be replaced by an equation as

$$\int_a^b f(x)\,dx = \frac{h}{2}f(a) + \frac{h}{2}f(b) + O(h^2) \qquad .$$

The expression $O(h^2)$ is an estimate to the error resulting from the approximation, it means the "order" of the error is of h^2. If the increment is 1/2 as large, then the error should be of the order 1/4 times smaller. Simpson's Rule has an error term of $O(h^4)$, therefore, if the increment is 1/2 as large, the error should be of the order 1/16 times smaller. Bode's Rule has an error of order (h^6).

Concider an equally spaced approximate integration formula of the form:

$$\int_a^b f(x)\,dx = \sum_i w_i f(x_i) + O(h^n) \qquad .$$

Richardson extrapolation assumes the term $O(h^n)$ could written as Ch^n, in which C is a constant. The higher order terms of h are ignored. Rewrite now

$$A_h = \sum_i w_i f(x_i) \qquad ,$$

and

$$A_0 = \int_a^b f(x)\,dx \qquad .$$

A_h indicates the solution was obtained using the increment h and A_0 indicates the solution was obtained using an infinitesimally small h and that it could be considered to be the exact solution. The Richardson Extrapolation approximation could then be written as

$$A_0 = A_h + Ch^n \qquad .$$

If another approximate solution could be obtained using an increment of $2h$, then A_0 could be estimated as

$$A_0 = A_{2h} + C(2h)^n \qquad .$$

The value of C can be obtained by subtracting the two above alegraic equations as

$$C = \frac{A_h - A_{2h}}{(2^n - 1)h^n}$$

Using Richarson Extrapolation, the best value can be extrapolated to be

$$A_0 = A_h + \frac{1}{(2^n - 1)}\left(A_h - A_{2h}\right) \qquad .$$

Richardson Extrapolation for Trapezoidal Rule

With an order term of $O(h^2)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{3}\left(A_h - A_{2h}\right) \qquad .$$

Richardson Extrapolation for Simpson's Rule

With an order term of $O(h^4)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{15}\left(A_h - A_{2h}\right) \qquad .$$

Richardson Extrapolation for Bode's Rule

With an order term of $O(h^6)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{63}\left(A_h - A_{2h}\right) \qquad .$$

Some Applications of Richardson Extrapolation

Trapezoidal Rule

Using 3 sample points, x_1, x_2, x_3 and an increment of h, the estimate for A_h is

$$A_h = \frac{h}{2}f(x_1) + hf(x_2) + \frac{h}{2}f(x_3) \qquad .$$

Using only 2 sample points, x_1, x_3 and an increment of $2h$, the estimate for A_{2h} is

$$A_{2h} = \frac{(2h)}{2}f(x_1) + \frac{(2h)}{2}f(x_3) \qquad .$$

Using the Richardson Extrapolation formula for Trapezoidal Rule:

$$A_0 = A_h + \frac{1}{3}\left(A_h - A_{2h}\right) = \frac{4}{3}A_h - \frac{1}{3}A_{2h} \qquad ,$$

The best estimate for A_0 is

$$\begin{aligned} A_0 &= \frac{4}{3}\left(\frac{h}{2}f(x_1) + hf(x_2) + \frac{h}{2}f(x_3)\right) - \frac{1}{3}\left(\frac{(2h)}{2}f(x_1) + \frac{(2h)}{2}f(x_3)\right) \\ &= \frac{h}{3}f(x_1) + \frac{4h}{3}f(x_2) + \frac{h}{3}f(x_3) \qquad . \end{aligned}$$

The result is the Simpson's Rule. Amazing!

Simpson's Rule

Using 5 sample points, x_1, x_2, x_3, x_4, x_5, and an increment of h, the estimate for A_h is

$$A_h = \frac{h}{3}f(x_1) + \frac{4h}{3}hf(x_2) + \frac{2h}{3}hf(x_3) + \frac{4h}{3}hf(x_4) + \frac{h}{3}f(x_5) \qquad .$$

Using only 3 sample points, the minimum, x_1, x_3, x_5 and an increment of $2h$, the estimate for A_{2h} is

$$A_{2h} = \frac{2h}{3}f(x_1) + \frac{4(2h)}{3}hf(x_3) + \frac{2h}{3}f(x_5) \qquad .$$

Using the Richardson Extrapolation formula for Simpson's Rule:

$$A_0 = A_h + \frac{1}{15}\left(A_h - A_{2h}\right) = \frac{16}{15}A_h - \frac{1}{15}A_{2h} \qquad ,$$

The best estimate for A_0 is

$$\begin{aligned} A_0 &= \frac{16}{15}\left(\frac{h}{3}f(x_1) + \frac{4h}{3}hf(x_2) + \frac{2h}{3}hf(x_3) + \frac{4h}{3}hf(x_4) + \frac{h}{3}f(x_5)\right) \\ &\qquad - \frac{1}{15}\left(\frac{2h}{3}f(x_1) + \frac{4(2h)}{3}hf(x_3) + \frac{2h}{3}f(x_5)\right) \\ &= \frac{14h}{45}f(x_1) + \frac{64h}{45}f(x_2) + \frac{24h}{45}f(x_3) + \frac{64h}{45}f(x_4) + \frac{14h}{45}f(x_5) \qquad . \end{aligned}$$

The result is the Bode's Rule. Amazing!

Central Difference, First Derivative

Using 3 sample points, x_{-1}, x_0, x_1 and an increment of h, the estimate for A_h, central difference for first derivative, is

$$A_h = \frac{f_1 - f_{-1}}{2h} \qquad .$$

Using 3 sample points, but a wider interval, x_{-2}, x_0, x_2 and an increment of $2h$, the estimate for A_{2h} is

$$A_{2h} = \frac{f_2 - f_{-2}}{2(2h)} \qquad .$$

Using the Richardson Extrapolation formula for $O(h^2)$:

$$A_0 = A_h + \frac{1}{3}\left(A_h - A_{2h}\right) = \frac{4}{3}A_h - \frac{1}{3}A_{2h} \qquad ,$$

The best estimate for A_0 is

$$A_0 = \frac{4}{3}\left(\frac{f_1 - f_{-1}}{2h}\right) - \frac{1}{3}\left(\frac{f_2 - f_{-2}}{2(2h)}\right) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} \qquad .$$

The result is that of a 4-th order polynomial fitted to 5 sample points, at x_{-2}, x_{-1}, x_0, x_1, x_2.

Central Difference, Second Derivative

Using 3 sample points, x_{-1}, x_0, x_1 and an increment of h, the estimate for A_h, central difference for second derivative, is

$$A_h = \frac{f_1 - 2f_0 + f_{-1}}{h^2} \qquad .$$

Using 3 sample points, but a wider interval, x_{-2}, x_0, x_2 and an increment of $2h$, the estimate for A_{2h} is

$$A_{2h} = \frac{f_2 - 2f_0 + f_{-2}}{(2h)^2} \qquad .$$

Using the Richardson Extrapolation formula for $O(h^2)$:

$$A_0 = A_h + \frac{1}{3}\left(A_h - A_{2h}\right) = \frac{4}{3}A_h - \frac{1}{3}A_{2h} \qquad ,$$

The best estimate for A_0 is

$$A_0 = \frac{4}{3}\left(\frac{f_1 - 2f_0 + f_{-1}}{h^2}\right) - \frac{1}{3}\left(\frac{f_2 - 2f_0 + f_{-2}}{4h^2}\right) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} \ .$$

The result is that of a 4-th order polynomial fitted to 5 sample points, at x_{-2}, x_{-1}, x_0, x_1, x_2.