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PROV I MATEMATIK
Markov Processes
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Permitted aids: Calculator

1. Coin 1 comes up heads with probability 0.6 and coin 2 comes up heads with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

   (a) What is the long-run proportion of flips that use coin 1? (3p)

   (b) If we start the process with coin 1, what is the probability that coin 1 is used in the third flip? (2p)

Solution:

Let X_n be the coin used in the nth flip. The sequence $(X_n)_{n=1}^{\infty}$ is an irreducible, aperiodic Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}.$$

(a) The long run proportion of flips that use coin 1 is given by π_1 where $\pi = (\pi_1, \pi_2) = (5/9, 4/9)$ is the unique stationary distribution, i.e. the unique probability vector satisfying the equation $\pi\mathbf{P} = \pi$. Thus he long run proportion of flips that use coin 1 is $5/9$.

(b) $P(X_3 = 1 \mid X_1 = 1)$ is given by the element on row 1 and column 1 in the matrix $\mathbf{P}^2$.

Since

$$\mathbf{P}^2 = \begin{pmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{pmatrix}$$

it thus follows that $P(X_3 = 1 \mid X_1 = 1) = 0.56$.

2. Let $(X_t)_{t\geq 0}$ be a Markov process with state space $S = \{1, 2, 3, 4\}$, and generator

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

starting in state $X_0 = 1$.

   (a) Find the distribution of X_t for any fixed $t \geq 0$. (3p)

   (b) Let $T = \inf\{t \geq 0 : X_t = 4\}$. Calculate $E(T)$. (3p)

Solution:

(a) The matrices of transition probabilities $\mathbf{P}(t)$ are given as the solutions to the forward equation $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$ i.e.

$$\begin{pmatrix} p'_{11}(t) & p'_{12}(t) & p'_{13}(t) & p'_{14}(t) \\ p'_{21}(t) & p'_{22}(t) & p'_{23}(t) & p'_{24}(t) \\ p'_{31}(t) & p'_{32}(t) & p'_{33}(t) & p'_{34}(t) \\ p'_{41}(t) & p'_{42}(t) & p'_{43}(t) & p'_{44}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) & p_{14}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) & p_{24}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) & p_{34}(t) \\ p_{41}(t) & p_{42}(t) & p_{43}(t) & p_{44}(t) \end{pmatrix} \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

In particular, we have

$$p'_{11}(t) = -3p_{11}(t) + p_{12}(t) + p_{13}(t) + p_{14}(t) = -3p_{11}(t) + (1 - p_{11}(t)) = 1 - 4p_{11}(t),$$

i.e.

$$p'_{11}(t) + 4p_{11}(t) = 1.$$

Thus $\frac{d}{dt}(e^{4t}p_{11}(t)) = e^{4t} \Leftrightarrow p_{11}(t) = 1/4 + c_1e^{-4t}$, where c_1 is a constant. Since $p_{11}(0) = 1/4 + c_1 = 1$ it follows that $c_1 = 3/4$ and thus

$$P(X_t = 1) = p_{11}(t) = 1/4 + 3e^{-4t}/4.$$

By symmetry

$$P(X_t = 2) = P(X_t = 3) = P(X_t = 4) = (1 - p_{11}(t))/3 = \frac{1}{4}(1 - e^{-4t}).$$

(b) Let $T_{ij} = \inf\{t \geq 0 : X_t = j | X_0 = i)$. By conditioning on the outcome of the first jump, and using the fact that by symmetry $E(T_{24}) = E(T_{34})$ we get

$$\begin{aligned} E(T_{14}) &= \frac{1}{3} + \frac{2}{3}E(T_{24}) \\ E(T_{24}) &= 1 + E(T_{14}). \end{aligned}$$

Thus $E(T) = E(T_{14}) = \frac{1}{3} + \frac{2}{3}(1 + E(T_{14})) = 1 + \frac{2}{3}E(T_{14})$ i.e. $E(T) = 3$.

3. (a) Give an example of a transition matrix P of a Markov chain with 3 states with more than one stationary distribution. (2p)

(b) Give an example of a a transition matrix P of a periodic Markov chain with 3 states with a unique stationary distribution that is not reversible. (2p)

(c) Give an example of birthrates λ_n, $n \geq 0$ such that a birth process $(X_t)_{t\geq 0}$ on $S = \{0, 1, \ldots\}$ stating in $X_0 = 0$ with birth rates λ_n explodes on average before time $t = 7$. (2p)

Solution:

(a) Any probability vector (π_1, π_2, π_3) is stationary for

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) A Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

has period 3 and unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (1/3, 1/3, 1/3)$. This distribution is not reversible since e.g. $\pi_1 p_{12} \neq \pi_2 p_{21}$.

(c) If $\lambda_n = 2^n$ then a birth process with birthrates λ_n explodes on average at time $t = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

4. A bucket contains totally 2 balls. Some balls are green and the other balls are blue. A ball is chosen at random at time points whose spacings are independent and exponentially distributed with intensity 5 per minute. The chosen ball is immediately replaced by a ball of the other color. Let X_t denote the number of blue balls in the bucket at time t.

   (a) Find the transition matrix of the jump chain of $(X_t)_{t\geq 0}$. (3p)

   (b) Find the limit $\lim_{t\to\infty} P(X_t = 1)$. (3p)

Solution:

(a) If $X_t = i$ then the chosen ball will be blue with probability $i/2$, and if a blue ball is chosen and replaced by a green ball then the number of blue balls will change to $i-1$. Similarly the chosen ball is green with probability $(2-i)/2$, and if a green ball is chosen and replaced by a blue ball then the number of blue balls will change to $i+1$. It follows that $(X_t)_{t\geq 0}$ has jump chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) Since the holding times in each state are exponentially distributed with intensity parameter $5 = -q_{ii}$, $i = 0, 1, 2$, it follows that $(X_t)_{t\geq 0}$ has generator

$$\mathbf{Q} = \begin{pmatrix} -5 & 5 & 0 \\ 5/2 & -5 & 5/2 \\ 0 & 5 & -5 \end{pmatrix},$$

and since $\pi = (\pi_0, \pi_1, \pi_2) = (1/4, 1/2, 1/4)$ is the unique probability vector solving $\pi\mathbf{Q} = \mathbf{0}$, and the Markov process is irreducible with a finite state space it follows from the convergence theorem that $\lim_{t\to\infty} P(X_t = 1) = \pi_1 = 1/2$.

5. Let $(X_n)_{n=0}^{\infty}$, be a Markov chain with state space $S = \{0, 1, 2, ...\}$, with $X_0 = 0$, and transition probabilities $p_{j,j+1} = \gamma_j$ and $p_{j,0} = 1 - \gamma_j$, $j \geq 0$.

   (a) Show that the state 0 is recurrent iff

   $$\gamma_0 \cdots \gamma_n \to 0, \qquad \text{as } n \to \infty.$$

   (3p)

   (b) Show that the state 0 is positive recurrent iff

   $$\sum_{n=0}^{\infty} (\gamma_0 \cdots \gamma_n) < \infty.$$

   (3p)

Solution:

Let $T = \inf\{n \geq 1 : X_n = 0\}$ be the first return time to 0.

(a) If the first n moves contains no returns to 0, then $T > n$ and thus $P(T > n) = P(X_n = n) = \gamma_0\gamma_1 \cdots \gamma_{n-1}$.
State 0 is by definition recurrent iff $P(T < \infty) = 1$.
Since

$$P(T < \infty) = \lim_{n\to\infty} P(T \le n) = \lim_{n\to\infty}(1 - P(T > n)) = 1 - \lim_{n\to\infty} \gamma_0\gamma_1 \cdots \gamma_{n-1},$$

we thus see that 0 is recurrent iff $\lim_{n\to\infty} \gamma_0\gamma_1 \cdots \gamma_n = 0$.

(b) State 0 is, by definition, positive if $E(T) < \infty$.
Since $P(T = 1) = (1 - \gamma_0)$, $P(T = k) = \gamma_0 \cdots \gamma_{k-2}(1 - \gamma_{k-1})$, $k \ge 2$, we have

$$\begin{aligned} E(T) &= \sum_{k=1}^{\infty} kP(T = k) = (1 - \gamma_0) + 2\gamma_0(1 - \gamma_1) + 3\gamma_0\gamma_1(1 - \gamma_2) + \ldots.. \\ &= 1 - \gamma_0 + 2\gamma_0 - 2\gamma_0\gamma_1 + 3\gamma_0\gamma_1 - 3\gamma_0\gamma_1\gamma_2 + \ldots.. = 1 + \sum_{k=0}^{\infty} \gamma_0 \cdots \gamma_k. \end{aligned}$$

Thus 0 is positive iff $\sum_{n=0}^{\infty} \gamma_0 \cdots \gamma_n < \infty$.

6. Let $(X_t)_{t\ge 0}$ be a Brownian motion with variance parameter $\sigma^2 = 9$.

(a) Compute $P(X_2 > X_3 > X_1)$. (3p)

(b) Express $P(X_t > -9,\ \text{for all } 0 < t \le 4)$ in terms of the distribution function of a standard normal random variable. (3p)

Solution:

Let $Y_1 = X_3 - X_2$ and $Y_2 = X_1 - X_2$. Then Y_1 and Y_2 are independent $N(0, \sigma^2)$ distributed random variables. Since $P(Y_1 < 0, Y_2 < 0) = 1/4$ and by symmetry

$$P(0 > Y_1 > Y_2) = P(0 > Y_2 > Y_1),$$

it thus follows that

(a)

$$P(X_2 > X_3 > X_1) = P(0 > X_3 - X_2 > X_1 - X_2) = P(0 > Y_1 > Y_2) = 1/8.$$

(b) Let $B_t = X_t/\sigma$. Since $\max_{0\le t\le 4} B_t$ has the same distribution as $|B_4|$ by the reflection principle, it follows that

$$\begin{aligned} P(X_t > -9,\ \text{for all } 0 < t \le 4) &\underbrace{=}_{sym.} P(X_t < 9,\ \text{for all } 0 < t \le 4) \\ &= P(\max_{0\le t\le 4} X_t < 9) \\ &= P(\max_{0\le t\le 4} B_t < 9/\sigma) \\ &= P(|B_4| < 3) = P(-3/2 < B_4/2 < 3/2) \\ &\underbrace{=}_{B_4/2\sim N(0,1)} \Phi(1.5) - \underbrace{\Phi(-1.5)}_{1-\Phi(1.5)} = 2\Phi(1.5) - 1, \end{aligned}$$

where Φ denotes the distribution function of a standard normal random variable.