3. Differential equations and optimisation

We solve the problems together in the exercise sessions. Note that these problems are optional and for learning purposes: solving these does not provide extra points. Actual home assignments (giving you extra points) are given separately.

It is advised to take a look of the problems beforehand. Note that some of the problems might be very challenging, so do not feel bad if you are unable to solve them independently: we will go through the solutions together!

Problems for the session

3.1 In which of the following sets D a continuous function f (defined on D) necessarily attains its maximum and minimum values:

- $D = \{(x, y) : |x| \leq 1, |y| \leq 1\}$
- $D = \{(x, y) : x^2 + y^2 \leq 1\}$
- $D = \{(x, y) : x^2 + y^2 \geq 1\}$
- $D = \{(x, y) : xy \geq 1\}$
- $D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$

3.2 Find largest and smallest value of $f(x, y) = \frac{x^2-y^2}{(2+x^2+y^2)^3}$ on $-1 \leq y \leq 1$.

3.3 Let $f(x, y) = (x^2 + xy + y^2)e^{-x-2y}$, where $(x, y) \in \mathbb{R}^2$. Has f largest value? What about the smallest?

3.4 For two times continuously differentiable function f, set $Q(h, k) = f''_{xx}h^2 + 2f''_{xy}hk + f''_{yy}k^2$. Find conditions in terms of f''_{xx}, f''_{xy}, and f''_{yy} under which $Q(h, k) > 0$ for all $(h, k) \in \mathbb{R}^2$. What about $Q(h, k) < 0$?

Problems for individual practice

In addition to the problems below, one can get routine by solving similar exercises from the exercise-book ”övningar i flerdimensionell analys”.

3.1 Find all two times continuously differentiable functions f of one variable such that $u(x, y) = yf(xy)$ solves

$$yu''_{xy} + xu''_{yy} = 0, \quad x, y > 0.$$

3.2 Solve differential equation

$$x^2 f''_{xx} + 2xy f''_{xy} + y^2 f''_{yy} = xy.$$

Tip: try change of variables $u = x$ and $v = \frac{x}{y}$.

3.3 Show that changing to polar coordinates ($x = r\cos\phi, y = r\sin\phi$) leads to

$$\frac{\partial f}{\partial \phi} = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}.$$

3.4 Let $F = (P, Q)$ be a vector field, where $P = -yf(r)$ and $Q = xf(r)$ with $r = |(x, y)|$, and f is a function of one variable. Find all such functions f for which

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial x}, (x, y) \neq (0, 0).$$

3.5 Suppose that a tangent plane for $z = f(x, y)$ at $(0, 0, 0)$ is given by $x + y + z = 1$. Give the equation of the tangent plane for $z = (2 + f(x, y))^2$ at the point $(x, y) = (0, 0)$.

3.6 Find extreme points for $f(x, y) = x^3y^2 + 27xy + 27y$.

3.7 Find largest and smallest value of $f(x, y) = xy + x^2y^2$ on $0 \leq x \leq 1, 0 \leq y \leq 2$.

3.8 Find largest and smallest value of $f(x, y) = x^2 + x(y^2 - 1)$ on the ball $x^2 + y^2 \leq 1$.

3.9 Find largest and smallest value of $f(x, y) = (x^2 + y)e^{-x-y}$ on $0 \leq x, y < \infty$.

3.10 Find largest and smallest value of $f(x, y) = (2x + 3y + 1)^2$ on the circle $x^2 + y^2 = 1$.