Recall the total variation of a function f:

$$TV_f([a,b]) = \sup_{\pi} \left\{ \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| : \pi = \{a = t_0 < \dots < t_n = b\} \right\}$$

(Section 3.3)

If f continuous then

$$TV_f[a,b] = \lim_{|\pi| \to 0} \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| ,$$

$$|\pi| = \max_{1 \le k \le n} |\Delta t_k| .$$

For Brownian motion, put

$$V_t^{(p)} = \lim_{|\pi| \to 0} \sum_{t_k \le t} |B_{t_k} - B_{t_{k-1}}|^p$$

$p=2$, $$V_t^{(2)} = \lim_{|\pi| \to 0} \sum_{t_k \le t} (\Delta B_{t_k})^2$$

Proposition $\quad V_t^2 = t \quad$ a.s.

Proof $$\sum_{t_k \le t} (\Delta B_{t_k})^2 - t = \sum_{t_k \le t} \{ (\Delta B_{t_k})^2 - \Delta t_k \}$$

This is a sum $\sum_{t_k \le t} Z_k$ of independent random variables with mean

$$\mathbb{E}(Z_k) = \mathbb{E}[(\Delta B_{t_k})^2 - \Delta t_k] = \Delta t_k - \Delta t_k = 0$$

The variance is

$$\mathbb{E}\left[\left(\sum_{t_k \le t} (\Delta B_{t_k})^2 - t\right)^2\right] = Var\left(\sum_{t_k \le t} Z_k\right)$$

$$= \sum_{t_k \le t} Var(Z_k)$$

Here, $Var(Z_k) = Var((\Delta B_{t_k})^2)$

$$= \mathbb{E}((\Delta B_{t_k})^4) - \mathbb{E}((\Delta B_{t_k})^2)^2$$

$$= 3\cdot(\Delta t_k)^2 - (\Delta t_k)^2 = 2(\Delta t_k)^2$$

so $\mathbb{E}\left[\left(\sum_{t_k \le t} (\Delta B_{t_k})^2 - t\right)^2\right] = 2\sum_{t_k \le t} (\Delta t_k)^2$

$$\le 2 \max_k |\Delta t_k| \cdot \underbrace{\sum_{t_k \le t} \Delta t_k}_{= t}$$

$$= 2t \cdot |\pi|$$

and therefore $V_t^{(2)} = t$ in L^2. This is Prop. 3.3.12

Moreover, by Chebyshev inequality

$$\mathbb{P}\left(\left|\sum_{t_k \le t} (\Delta B_{t_k})^2 - t\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2} \cdot 2t|\pi|$$

$$\longrightarrow 0 \text{ as } |\pi| \to 0$$

so we have $V_t^{(2)} = t$ with convergence in probability.

Now we take the special partition

$$t_k^{(n)} = \frac{k}{2^n} \cdot t \;, k = 0, \ldots, 2^n \;, \; n \geq 1$$

Then, with $A_n = \left\{ \left| \sum_{t_k^{(n)} \leq t} (\Delta B_{t_k})^2 - t \right| > \varepsilon \right\}$,

we have $\mathbb{P}(A_n) \leq \frac{1}{\varepsilon^2} \cdot 2t \cdot \frac{1}{2^n}$

and hence

$$\sum_n \mathbb{P}(A_n) \leq \frac{2t}{\varepsilon^2} \sum_n \frac{1}{2^n} < \infty$$

Thus, by the Borel-Cantelli Lemma

$$V_t^{(2)} = t \quad a.s. \qquad \square$$

We also note

$$\sum_{t_k \leq t} (\Delta B_{t_k})^2 \leq \max_k |\Delta B_{t_k}| \cdot \sum_{t_k \leq t} |\Delta B_{t_k}| -$$

Hence $V_t^{(2)} \leq \underbrace{\max_k |\Delta B_{t_k}|}_{\to 0 \text{ a.s. } \sim\sim\text{in probability}\sim\sim} \cdot V_t^{(1)}$

where $\underbrace{\quad}_{\to t \text{ a.s.}}$

$[\{B_t\}_{[0,t]}$ cont. $\Rightarrow \{B_s\}_{[0,t]}$ unif cont.$]$

Therefore $V_t^{(1)} = +\infty$ ~~in probability~~ a.s. !

In general, the quadratic variation of a random process Y is defined as

$$[Y]_t = \lim_{|\pi| \to 0} \sum_{t_k \le t} (\Delta Y_{t_k})^2$$

with convergence in probability (or L^1)

If Y has finite variation, then

$$[Y]_t = \sum_{s \le t} (\Delta Y_s)^2$$

Example • $\{N_t\}_{t \ge 0}$ Poisson process

$$[N]_t = \sum_{s \le t} (\Delta N_s)^2 = \sum_{s \le t} \Delta N_s = N_t$$

• $\{M_t\}_{t \ge 0}$ $M_t = N_t - \lambda t$

$$[M]_t = \sum_{s \le t} (\Delta M_s)^2$$

$$= \sum_{s \le t} (\Delta N_s)^2 = N_t$$

Prop a, If $\{M_t\}_{t \ge 0}$ is a martingale then $\{[M]_t\}_{t \ge 0}$ exists and is right continuous

b, If $\{M_t\}_{t \ge 0}$ is square integrable the limit exists in L^1, ~~then~~ $M_t^2 - [M]_t$, $t \ge 0$, is also a martingale and $\mathbb{E}[M_t^2] = \mathbb{E}[[M]_t]$

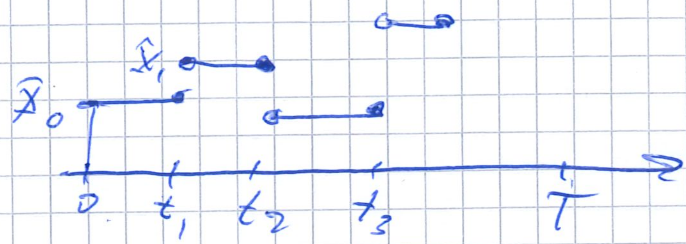
Define a <u>simple</u> stochastic process

F7:5

$$H_t = 1_{\{t=0\}}\,\xi_0 + \sum_{k=0}^{n-1} 1_{(t_k,\,t_{k+1}]}\,\xi_k$$

↑ left continuous

where $\xi_k \in \mathcal{F}_{t_k}$



<u>Note</u> The value of H_t on $(t_k, t_{k+1}]$ is ξ_k and $\xi_k \in \mathcal{F}_{t_k}$ ← leftmost point of the interval

compare predictability in discrete time, $H_{t_k} = \xi_k$

$$\text{Now put } I(H) = \sum_{k=0}^{n-1} \underbrace{\xi_k}_{H_{t_k}} \left(B_{t_{k+1}} - B_{t_k}\right),$$

where $\{B_t\}_{0\le t\le T}$ is BM.

Compare <u>mart. transform</u> of H w.r.t. the martingale $\{B_t\}$

$$\text{"}\,I(H) = \int_0^T H_s\,dB_s\,\text{"}$$

Note $E[I(H)] = 0$

F7:6

$$\mathbb{E}[I(H)^2] = \mathbb{E}\left[\left(\sum_{k=0}^{n-1} X_k (B_{t_{k+1}} - B_{t_k})\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{k=0}^{n-1} X_k^2 (B_{t_{k+1}} - B_{t_k})^2\right]$$

$$+ 2\mathbb{E}\left[\sum\sum_{k<l} X_k X_l (B_{t_{l+1}} - B_{t_l})(B_{t_{k+1}} - B_{t_k})\right]$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left[\mathbb{E}[X_k^2 (B_{t_{k+1}} - B_{t_k})^2 | \mathcal{F}_{t_k}]\right]$$

$$+ 2\sum\sum_{k<l} \mathbb{E}[\quad]$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(X_k^2)\, \underbrace{\mathbb{E}[(B_{t_{k+1}} - B_{t_k})^2]}_{t_{k+1} - t_k = \Delta t_k}$$

$$+ 2\sum\sum_{k<l} \mathbb{E}[X_k \Delta B_{t_k} \mathbb{E}[X_l \Delta B_{t_l} | \mathcal{F}_{t_l}]]$$

$$\mathbb{E}[\Delta B_{t_l}] = 0$$

$$= \sum_{k=0}^{n-1} \mathbb{E}[H_{t_k}^2] \Delta t_k$$

$$= \int_0^T \mathbb{E}[H_t^2]\, dt$$

by def of Leb. integral for simple fnct

We have seen $I(H) \sim L^2(\Omega, \mathcal{F}, \mathbb{P})$

$$\|I(H)\|_2^2 = \mathbb{E}[I(H)^2] = \int_0^T \mathbb{E}(H_t^2)\,dt < \infty$$

Here R.H.S is norm of $L^2([0,T]\times\Omega, dt\otimes\mathbb{P})$
$= L^2_T$

So $\|I(H)\|_{L^2} = \|H\|_{L^2_T}$ <u>isometry</u>

Corresponding scalar product:

$$\langle I(H), I(G)\rangle = \mathbb{E}[I(H)I(G)]$$
$$= \dots = \int_0^T \mathbb{E}(H_s G_s)\,ds = \langle H, G\rangle_{L^2_T}$$

Thus, the set $\mathcal{H}_0$ of simple processes is a subspace of $L^2([0,T]\times\Omega)$, and the map $I: \mathcal{H}_0 \to L^2(\Omega)$ is <u>norm-preserving</u>.

Now suppose for $\Phi \in L^2([0,T]\times\Omega)$ there exists $\{H_n\}$, $H_n \in \mathcal{H}_0$ such that

$$H_n \to \Phi \text{ in } L^2([0,T]\times\Omega)$$

Then $\|H_n - H_m\|_{L^2_T} = \|H_n - \Phi + \Phi - H_m\|_{L^2_T}$

$$\leq \|H_n - \Phi\|_{L^2_T} + \|H_m - \Phi\|_{L^2_T} \to 0 \text{ as } n, m \to \infty.$$