COMPLETE BIPARTITE GRAPHS FLEXIBLE IN THE PLANE

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Abstract. A complete bipartite graph $K_{3,3}$, considered as a linkage with joints at the vertices and with rods as edges, in general admits only motions as a whole, i.e., is inflexible. Two types of its paradoxical mobility were found by Dixon in 1899. Later on, in a series of papers by different authors, the question of flexibility of $K_{m,n}$ was solved for almost all pairs (m, n). In the present paper, we solve it for all complete bipartite graphs. We give independent self-contained proofs without extensive computations.

1. Introduction. Main results

We find necessary and sufficient conditions (Theorem 1 and Remarks 1 and 2) for the flexibility of frameworks corresponding to complete bipartite graphs $K_{m,n}$. We define a *planar framework corresponding to the complete bipartite graph* $K_{m,n}$ (*(m, n)-framework* for short) as a collection of points in the Euclidean plane $\mathbf{p} = (p_1, \dots, p_m;\ q_1, \dots, q_n)$ such that $p_i \neq q_j$ for all i, j. The *parts* (of the bipartite graph) are $(p_1, \dots)$ and $(q_1, \dots)$. Speaking of (m, n)-frameworks, we call the points p_i and q_j *joints* and pairs of points (p_i, q_j) from different parts *rods*. We say that an (m, n)-framework $\mathbf{p}$ is *non-overlapping* if all its joints are pairwise distinct. Finally, we say that an (m, n)-framework is *flexible* if it admits a *flex*, that is a continuous non-constant motion of its joints $\mathbf{p}(t) = \big(p_1(t), \dots, q_n(t)\big)$, $t \in [0, 1]$, such that $\mathbf{p}(0) = \mathbf{p}$, the lengths of the rods are constant, i. e. $|p_i(t) - q_j(t)|$ does not depend on t for each i, j, and some two joints from different parts do not move: $p_{i_0}(t) = p_{i_0}$ and $q_{i_0}(t) = q_{i_0}$. These definitions evidently can be extended to all connected graphs but we do not need it.[1]

Theorem 1. *Let* $\min(m, n) \geq 3$. *Then a non-overlapping (m, n)-framework is flexible if and only if one of the following conditions holds.*

(D1) The points $p_1, \dots, p_m$ lie on a line P, the points $q_1, \dots, q_n$ lie on another line Q, and these two lines are orthogonal to each other.

(D2) One can choose an orthogonal coordinate system and two rectangles with sides parallel to the axes and with the common center of symmetry at the origin so that $p_1, \dots, p_m$ are at the vertices of one of these rectangles and $q_1, \dots, q_n$ are at the vertices of the other one. Since all points are distinct, we have in this case $m \leq 4$ and $n \leq 4$.

Remark 1. It is easy to see that any $(1, n)$-framework is flexible (and has $n - 1$ degrees of freedom), and any non-overlapping $(2, n)$-framework is flexible if and only if it does not contain a quadruple of joints p_i, q_j, p_k, q_l placed in this order on some straight line (cf. Lemma 4). All non-overlapping flexible (m, n)-frameworks with $\min(m, n) \geq 2$ have one degree of freedom.

Remark 2. It is evident that an (m, n)-framework $\mathbf{p}$ with overlapping joints is flexible if and only if so is the non-overlapping bipartite framework $\overline{\mathbf{p}}$, obtained by identifying each pair of overlapping joints. It is also clear that the number of degrees of freedom $d(\mathbf{p})$ of $\mathbf{p}$ is

[1] In the literature on mechanics, rigid frameworks are usually called *trusses*, and flexible ones are called, depending on the context, *mechanisms* or *states* of a mechanism.

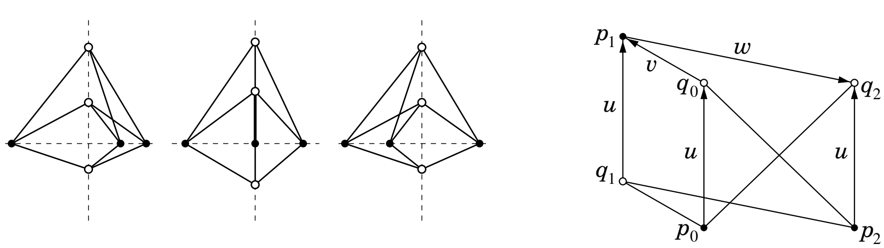
equal to $\max_{\mathbf{q}} d(\overline{\mathbf{q}})$ where the maximum is taken over all frameworks $\mathbf{q}$ obtained from $\mathbf{p}$ by small flexes. Thus $d(\mathbf{p}) = d(\overline{\mathbf{p}}) \le 1$ unless $p_1 = \dots = p_m$ (in which case $d(\mathbf{p}) = n-1$) or, symmetrically, $q_1 = \dots = q_n$, $d(\mathbf{p}) = m-1$.

In the case $m = n = 3$, the flexible frameworks (D1) and (D2) were discovered by Dixon (see [1, §27(d), §28(n)]). They are called *Dixon mechanisms of the first and second kind* respectively. We shall use these names for any $m, n \ge 3$. The Dixon mechanism of the second kind for $m = n = 4$ apparently was first described by Bottema in [2] (see also [9]). One can equivalently reformulate (D1) and (D2) in terms of the rod lengths. In the case of (D2) we do it for $(m, n) = (3, 3)$ only, but analogous conditions for $(3, 4)$ and $(4, 4)$ can be easily derived.

Proposition 1. *(a). A non-overlapping (m, n)-framework is a Dixon mechanism of the first kind (see Fig. 1(a)) if and only if, for each cycle $p_i q_j p_k q_l$, the sums of squared lengths of the opposite sides are equal. The number of these conditions is $\binom{m}{2}\binom{n}{2} = \frac{1}{4}(m^2-m)(n^2-n)$ but it is easily seen that only $(m-1)(n-1)$ of them are independent; one can choose, for example, only the conditions corresponding to the cycles $p_i q_j p_k q_l$ with fixed i and j (in particular, four conditions are independent among the nine ones when $m = n = 3$).*

(b). A flexible non-overlapping $(3, 3)$-framework $(p_0, p_1, p_2; q_0, q_1, q_2)$ is a Dixon mechanism of the second kind if and only if, up to renumbering of the vertices in the parts, $|q_0p_0| = |q_1p_1| = |q_2p_2| = a$, $|q_0p_1| = |q_1p_0| = b$, $|q_0p_2| = |q_2p_0| = c$, $|q_1p_2| = |q_2p_1| = d$ (see Fig. 2) and the relation $a^2 + c^2 = b^2 + d^2$ holds. In this case, all the 4-cycles twice including a are parallelogrammatic, i.e., have opposite sides of equal lengths.

Remark 3. The following example shows that Statement (b) of Proposition 1 is wrong without the flexibility assumption: $p_0 = (b, 0)$, $p_1 = (0, a)$, $p_2 = (d, 0)$, $q_0 = (b, a)$, $q_1 = (0, 0)$, $q_2 = (d, a)$, where a, b, d are positive, $b \ne d$, and $bd = a^2$. Indeed, all the conditions on the rod lengths are satisfied in this case, but the framework is not a Dixon 2nd kind mechanism. This is also an example of two non-overlapping $(3, 3)$-frameworks with equal lengths of the respective rods, one of whom is flexible (a Dixon mechanism of the 2nd kind) and the other one is rigid by Proposition 1. This example is a particular case of the example in Fig. 1(b).



(a) (b)

FIGURE 1. (a) Dixon mechanism of the first kind in motion. (b) Rigid $(3, 3)$-framework with lengths as in Dixon mechanism of the second kind. Here the vectors u, v, and w satisfy the relations $u^2 + vw = u(v + w) = 0$.

The proof of Proposition 1 is not difficult and it is given at the end of this section. Notice that Theorem 1 is proven in [6] for $m \ge 3$ and $n \ge 5$. Also, as proven in [7], the lengths of the rods of flexible non-overlapping $(3, 3)$-frameworks are as in Proposition 1. This fact

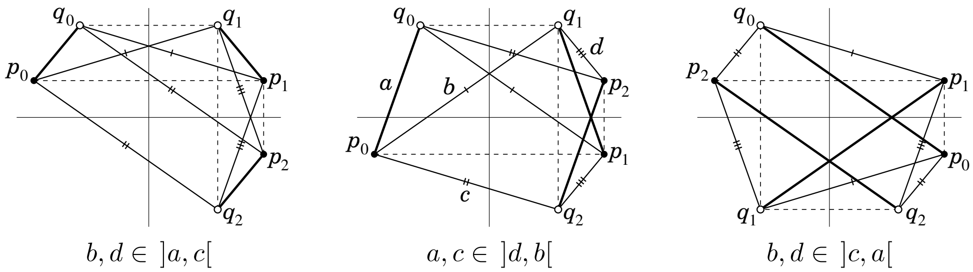


FIGURE 2. Dixon mechanism of the second kind. The indicated conditions on the lengths are determined (up to exchange of b and d) by the fact, whether p_1 and q_1 are in the same quadrant, in the adjacent ones, or in the opposite ones.

combined with Proposition 1 yields Theorem 1 for $m = n = 3$. The reduction of the general case to the case $m = n = 3$ is very simple. It is as follows.

Proof of Theorem 1 under the assumption that it holds for $m = n = 3$. Consider a non-overlapping (m, n)-framework with $n \geq m \geq 3$. The points p_1, p_2, p_3 and q_1, q_2, q_3 satisfy one of the conditions (D1) or (D2).

Let they satisfy (D1). Then p_1, p_2, p_3 and q_1, q_2, q_j, $j \geq 3$, do not satisfy (D2). Hence, since the subgraph spanned by them is flexible, they satisfy (D1), i.e., q_j is on the line Q. Thus $q_1, \ldots, q_n$ are all on Q. By the same reason, $p_1, \ldots, p_m$ are all on P.

Now suppose that p_1, p_2, p_3 and q_1, q_2, q_3 satisfy (D2). Consider the $(3, 3)$-framework $(p_1, p_2, p_3;\, q_1, q_2, q_j)$, $j \geq 3$. It also satisfies (D2) because (D1) cannot hold (for p_1, p_2, p_3 are not collinear). A priori (D2) could hold for another choice of the coordinate axes, however, the triangle $p_1p_2p_3$ has a single pair of mutually orthogonal sides, which uniquely determines the rectangle, and hence, it determines the axes. Notice also that a rectangle which is symmetric with respect to the origin and which has sides parallel to the axes, is determined by any of its vertices. Therefore q_1, q_2, q_3, and q_j are at the vertices of the same rectangle. Analogously, $p_1, \ldots, p_m$ are at the vertices of the same rectangle. The theorem is proven. □

One could finish the paper here, but we give a complete self-contained proof for the case $m = n = 3$. Thanks to the usage of simplest properties of branched coverings of Riemann surfaces, our proof requires incomparably smaller volume of computations than in [7] (see the discussion in the end of §2). The rest of this section is devoted to the proof of Proposition 1, while all the other sections of the paper are devoted to the self-contained proof of Theorem 1 for the case $m = n = 3$.

Lemma 1. *Let $m, n \geq 2$. Then any flex of a non-overlapping (m, n)-framework (see the definition above) leaves unmovable two joints only.*

Proof. The statement follows from the fact that the immobility of any two joints of one part implies the immobility of all joints of the other part. □

Lemma 2. *The diagonals of a quadrangle (maybe, self-crossing) are orthogonal if and only if the sums of the squared lengths of the opposite sides are equal.*

Proof. Let u, v, w be the vectors of three consecutive sides of the quadrangle. Then twice the dot product of the diagonals is $2(u + v)(v + w) = v^2 + (u + v + w)^2 - u^2 - w^2$. □

It is clear that any parallelogrammatic non-overlapping quadrangle is either a *parallelogram* (when its opposite sides are parallel) or an *antiparallelogram* (when its diagonal are parallel). It is both simultaneously if and only if it is *degenerate*, i.e., all its verices are collinear.

Proof of Proposition 1. (a). The statement follows from Lemma 2.

(b). The condition on the lengths is derived from (D2) by a direct computation. Let us prove the inverse implication. Since $a^2 + c^2 = b^2 + d^2$, Lemma 2 implies that the diagonals of the quadrangle $p_0q_1p_1q_2$ are mutually orthogonal. The same is true for the diagonals of $q_0p_1q_1p_2$ (see Fig. 2), i.e., $p_0p_1 \perp q_1q_2$ and $q_0q_1 \perp p_1p_2$. By hypothesis, the cycles $\Pi_{ij} = p_iq_ip_jq_j$, $i < j$, are parallelogrammatic.

Suppose that both Π_{01} and Π_{12} are non-degenerate parallelograms (see Fig. 1(b)). Then $p_0q_0q_2p_2$ also is a parallelogram and, since its both diagonals are of length c, it is a rectangle with sides a and $\sqrt{c^2 - a^2}$. Hence any flex fixing p_0 and q_0 fixes p_2 and q_2 as well, which contradicts Lemma 1.

The obtained contradiction shows that Π_{01} or Π_{12} is an antiparallelogram (maybe, degenerate). Let it be Π_{01} (the case of Π_{12} is analogous). Then $q_1q_2 \perp p_0p_1 || q_0q_1 \perp p_1p_2$, hence $q_1q_2 \,||\, p_1p_2$, i.e., Π_{12} also is an antiparallelogram. Hence q_0 and q_2 are symmetric to q_1 with respect to the mutually orthogonal symmetry axes of these antiparallelograms (see Fig. 2). The same is true for p_0, p_2, and p_1. The proposition is proven.

2. A general scheme of the proof of Theorem 1 for $m = n = 3$.

Consider a flex of a non-overlapping $(3, 3)$-framework $\mathbf{p} = (p_0, p_1, p_2;\ q_0, q_1, q_2)$ such that the joints p_0 and q_0 are fixed. Then p_1, p_2, q_1, q_2 move along the circles which we denote by P_1, P_2, Q_1, Q_2 respectively. Forget for a while the joint p_2. Then generically (when the segment q_1p_1 is not orthogonal to P_1) the displacement of q_1 uniquely determines the displacement of p_1, which, in its turn, generically determines the displacement of q_2. We obtain a dependence $q_2 = \mathcal{F}_1(q_1)$ (see Fig. 3).[2] Analogously, p_2 ensures a dependence $q_2 = \mathcal{F}_2(q_1)$. In order for our $(3, 3)$-framework not to be jammed, the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ should coincide. The point $(q_1, \mathcal{F}_i(q_1))$, $i = 1, 2$, moves along a certain real algebraic curve C_i on the torus $Q_1 \times Q_2$. The flexibility of $\mathbf{p}$ requires that C_1 and C_2 have an irreducible component in common.

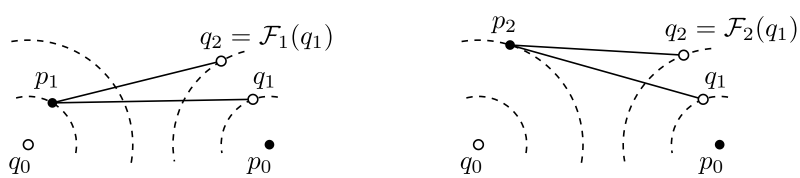


Figure 3. The transmission functions $\mathcal{F}_1$ and $\mathcal{F}_2$

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Let us proceed to a more formal exposition. Fix two points $p_0, q_0 \in \mathbb{R}^2$. Without loss of generality we may set $q_0 = (0, 0)$ and $p_0 = (1, 0)$. Fix real positive numbers r_{ij}, $i, j \in \{0, 1, 2\}$, $r_{00} = 1$. Denote also $R_i = r_{i0}$ and $r_i = r_{0i}$, $i = 1, 2$. Let M be the set of all quadruples $(p_1, p_2;\ q_1, q_2)$ such that $|p_iq_j| = r_{ij}$, $i, j \in \{0, 1, 2\}$. It is natural to consider M as the moduli space of $(3, 3)$-frameworks with a given matrix of the lengths. Abusing the language, we shall

[2]In engineering, this dependence is called *zero order transmission function* or *position function* (see, e.g., [5, §41]).

call the elements of M also $(3,3)$-frameworks implicitly assuming them to include p_0 and q_0. As above, we define the circles

$$P_k = \{p_k \in \mathbb{R}^2 \ : \ |q_0 p_k| = R_k\}, \qquad Q_k = \{q_k \in \mathbb{R}^2 \ : \ |p_0 q_k| = r_k\}, \qquad k = 1,2,$$

and set $Q = Q_1 \times Q_2$. For $k = 1,2$, consider the space of $(2,3)$-frameworks $(p_0, p_k;\, q_0, q_1, q_2)$ with these lengths:

$$M_k = \{(p_k, q_1, q_2) \in P_k \times Q \ : \ |p_k q_1| = r_{k1},\ |p_k q_2| = r_{k2}\}.$$

Set $C_k = \tau_k(M_k)$ where $\tau_k : P_k \times Q \to Q$, $k = 1,2$, are the standard projections (these are the curves appeared in the above discussion of transmission functions). It is clear that generically C_1 and C_2 are algebraic curves on Q (though if, for example, M has an element such that $p_0 = p_k$, then $C_k = Q$). Let us find the defining equations for C_1 and C_2. As in [7], we parametrize the circle Q_j, $j = 1,2$, by a complex number t_j, running over the circle $|t_j| = r_j$ in the complex plane. The coordinates of the vector $p_0 q_j$ are $(\operatorname{Re} t_j, \operatorname{Im} t_j)$, in other words, the parameter of q_j is the image of the vector $p_0 q_j$ under the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$. Analogously we choose parameters T_i on the circles P_i. In these coordinates, the conditions $|p_i q_j| = r_{ij}$, $i,j = 1,2$, take the form $f_{ij}(T_i, t_j) = 0$ where f_{ij} is the numerator of the rational function obtained from the expression $(1 + T_i - t_j)(1 + \overline{T}_i - \overline{t}_j) - r_{ij}^2$ by the replacement $\overline{T}_i = R_i^2/T_i$, $\overline{t}_j = r_j^2/t_j$, i.e., (cf. [7, eqs. (6)–(9)])

$$f_{ij}(T_i, t_j) = (1 + T_i - t_j)(T_i t_j + R_i^2 t_j - r_j^2 T_i) - r_{ij}^2 T_i t_j.$$

C_i is the set of solutions of the system of equations $f_{i1} = f_{i2} = 0$, hence it is given by the equation $F_i(t_1, t_2) = 0$ where

$$F_i(t_1, t_2) = R_i^{-2} \operatorname{Res}_{T_i}(f_{i1}, f_{i2}) \tag{2.1}$$

(see Remark 4 below). The expression for F_i (as a polynomial in t_1, t_2 and in all the r_{ij}'s) has 126 monomials and we have $\deg_{t_j} F_i = 4$. In the case of a non-overlapping flex, the images of M in Q_j are not discrete by Lemma 1, which implies the following fact.

Lemma 3. *If M contains a flexible non-overlapping framework, then*

$$\operatorname{Res}_{t_1}(F_1, F_2) = \operatorname{Res}_{t_2}(F_1, F_2) = 0. \tag{2.2}$$

Thus the search of all flexible $(3,3)$-frameworks is reduced to a computation of the resultant of F_1 and F_2 and a solution of the system of equations obtained by equating all its coefficients to zero. This is the way Walter and Husty have obtained in [7] the result (mentioned in the introduction) that the lengths of the rods of flexible non-overlapping $(3,3)$-frameworks are always as in Dixon's mechanisms. According to [7], $\operatorname{Res}(F_1, F_2)$ has 4.900.722 monomials, and it is said in [7] that "the computations are very extensive with respect to time and memory". Also, as far as we understood from [7], one needs to do some programming to interpret the solutions obtained with Maple or Singular.

When we started working on flexible $(3,3)$-frameworks (not knowing about the paper [7]), we also tried to solve this system of equations. However, we did not succeed to overcome the computational difficulties and looked for how to avoid them.[3] So we found the proof exposed below. The longest computation in our proof is that of the resultant (7.1), which takes 25 ms of CPU time. It should be pointed out that the choice of the parameters T_i and t_j borrowed from [7] further simplified the computations in Lemma 6 (initially, we used the standard parametrization of the circle by the tangent half-angle).

[3]Probably, we would not do it, if we were acquainted that time with the paper [7].

The outline of our proof is as follows. If M contains a flexible non-overlapping $(3,3)$-framework, then the curves C_1 and C_2 have a common component, i.e., the polynomials F_1 and F_2 have a common divisor. If one of F_1, F_2 is irreducible, they are proportional. This gives equations, which are easy to solve.

If F_1 and F_2 have a common divisor not being proportional, we look how the complexifications of the curves M_i, C_i, $C_{ij} = \{f_{ij} = 0\}$, and P_i are mapped to each other under the projections. A not difficult study shows that, for each $i = 1, 2$, either one of C_{ij} is reducible, or the projections $C_{i1} \to P_i$ and $C_{i2} \to P_i$ are ramified over the same points. Both conditions lead to equations which allow us to conclude that the framework contains either a parallelogrammatic cycle or a *deltoid* (a 4-cycle symmetric with respect to a diagonal) arranged in a certain way with respect to p_0 and q_0. Varying the choice of the fixed joints we arrive either to Dixon-1 or to a framework which contains three parallelogrammatic cycles adjacent to each other as in Dixon-2. In the latter case, the resultant of F_1 and F_2 is easy to compute.

Remark 4. Even when the coefficients of T_i^2 in f_{ij} vanish, the resultant in (2.1) is understood as the resultant of quadratic polynomials ($R_{2,2}$ in the notation of [3, Ch. 12], that is the determinant of the 4×4 Sylvester matrix). Similarly, the resultants in (2.2) and the discriminants D_j and $\Delta_j^{\pm}$ in §6 always correspond to $R_{4,4}$ and D_2 from [3, Ch. 12].

3. Preliminary lemmas

Lemma 4. (Immediate from Lemma 1.) *If an (m,n)-framework, $m, n \geq 2$, contains a 4-cycle with a rod whose length is equal to the sum of the lengths of the three other rods of the cycle, then the framework is not flexible.*

Lemma 5. *Let $\mathbf{p} = (p_0, p_1, p_2;\, q_0, q_1, q_2)$ be a flexible non-overlapping $(3,3)$-framework. Suppose that $|q_0p_j| = |q_1p_j|$ for all $j = 1, 2, 3$, i.e., the joints q_0 and q_1 are equidistant from p_0, p_1, p_2. Then $\mathbf{p}$ is a Dixon mechanism of the first kind.*

Since flexible frameworks are infinitesimally flexible, this lemma follows from Whiteley Theorem[4] [8] according to which *a non-overlapping (m,n)-framework with $\min(m,n) \geq 3$ is infinitesimally flexible if and only if either all joints lie on a second order curve, or all joints of one part and at least one joint of the other part are collinear* (for $m = n = 3$, the second condition is a particular case of the first one). However, since we are giving a self-contained proof of Theorem 1, let us prove Lemma 5 directly.

Proof. Denote the rod lengths by $R_i = |p_iq_0| = |p_iq_1|$, $r_i = |p_iq_2|$, $i = 0, 1, 2$. Consider a continuous deformation of $\mathbf{p}$. The equidistance condition implies that the points p_j rest collinear and $q_0q_1 \perp p_0p_1$ during the deformation. Hence, without loss of generality, we may assume that the p_j's remain on the axis $y = 0$, whereas q_0 and q_1 remain on the axis $x = 0$. Set $p_i = (x_i, 0)$, $i = 0, 1, 2$, and denote the x-coordinate of q_2 by a. Then

$$R_0^2 - x_0^2 = R_j^2 - x_j^2, \qquad j = 1, 2, \tag{3.1}$$

$$r_0^2 - (x_0 - a)^2 = r_j^2 - (x_j - a)^2, \qquad j = 1, 2. \tag{3.2}$$

Subtracting (3.1) from (3.2) and setting $b_j = r_j^2 - R_j^2$, we obtain

$$b_0 + 2ax_0 = b_j + 2ax_j, \qquad j = 1, 2. \tag{3.3}$$

Let us show that q_2 remains on the axis $x = 0$. Suppose that q_2 is not on the axis $x = 0$. Then $a \neq 0$ and we can express x_1 and x_2 via x_0 from (3.3). Plugging the result into (3.1),

[4] It was essentially used in [6] in the proof of Theorem 1 for $m \geq 3$ and $n \geq 5$.

we obtain

$$R_0^2 = R_j^2 - \frac{b_0 - b_j}{2a}\,x_0 - \frac{(b_0 - b_j)^2}{4a^2}, \qquad j = 1, 2. \tag{3.4}$$

The numbers x_i are pairwise distinct. Then (3.3) implies that the numbers b_i are pairwise distinct as well. In particular, the coefficient of x_0 in each equation (3.4) is nonzero. Eliminating x_0 from (3.4), we obtain an equation for a of the form $Ca^2 = (b_0 - b_1)(b_1 - b_2)(b_2 - b_0)$ where C is a polynomial in R_j, r_j, $j = 0, 1, 2$. Since all b_j are distinct, it has at most two solutions. Hence the parameter a is constant during the deformation. By (3.4) and (3.3) it determines x_0, x_1, x_2, thus all joints are fixed. This contradicts the assumption that the framework is flexible. The lemma is proven. □

4. General case: F_1 and F_2 are proportional

Let the notation be as in §2. Suppose that M contains a flexible non-overlapping $(3,3)$-framework $\mathbf{p} = (p_0, p_1, p_2; q_0, q_1, q_2)$.

Lemma 6. *If $F_1 = \lambda F_2$ for some number λ, then $\mathbf{p}$ is a Dixon mechanism of the first kind.*

Proof. Set $F = F_1 - \lambda F_2$. This is a polynomial of the form $\sum_{k,l=0}^{4} c_{kl}\, t_1^k t_2^l$ where c_{kl} are polynomials in r_{ij}^2 and we have $c_{00} = c_{01} = c_{43} = c_{44} = 0$. By hypothesis c_{kl} must vanish. There is a symmetry $c_{4-k,4-l} = r_1^{2k-4} r_2^{2l-4} c_{k,l}$, hence only 11 of these 21 equations are distinct. A computation shows that

$$c_{04} = r_1^4(R_1^2 - 1 - \lambda R_2^2 + \lambda),$$

$$c_{20} = r_2^4\big(r_1^2(\lambda - 1) - \lambda r_{21}^2 + r_{11}^2\big), \qquad c_{02} = r_1^4\big(r_2^2(\lambda - 1) - \lambda r_{22}^2 + r_{12}^2\big).$$

Case 1. $\lambda = 0$. Then the equations $c_{04} = c_{20} = c_{02} = 0$ yield $R_1 = 1$, $r_1 = r_{11}$, and $r_2 = r_{12}$, hence the joints p_0 and p_1 are equidistant from all the q_j and the result follows from Lemma 5.

Case 2. $\lambda = 1$. Then the equations $c_{04} = c_{20} = c_{02} = 0$ yield $R_1 = R_2$, $r_{11} = r_{21}$, and $r_{12} = r_{22}$, hence the joints p_1 and p_2 are equidistant from all the q_j and again the result follows from Lemma 5.

Case 3. $R_2 = 1$ and $\lambda(1 - \lambda) \neq 0$. Then the equation $c_{04} = 0$ implies $R_1 = 1$. Find r_{11}^2 and r_{12}^2 from the equations $c_{20} = c_{02} = 0$ and plug the result into $c_{12} + c_{13} = 0$. We obtain the equation

$$\lambda(\lambda - 1) r_1^2 (r_{22}^2 - r_2^2)^2 = 0,$$

whence $r_{22} = r_2$, and the equation $c_{21} = 0$ takes the form

$$\lambda(\lambda - 1) r_2^2 (r_{21}^2 - r_1^2)^2 = 0.$$

Thus $R_2 = 1$, $r_{21} = r_1$ and $r_{22} = r_2$, hence the joints p_0 and p_2 are equidistant from all the q_j and once again the result follows from Lemma 5.

Case 4. $R_2 \neq 1$ and $\lambda(1 - \lambda) \neq 0$. From $c_{04} = 0$ we find $\lambda = (R_1^2 - 1)/(R_2^2 - 1)$. Then the conditions $\lambda \neq 0$ and $\lambda \neq 1$ imply that $R_1 \neq 1$ and $R_1 \neq R_2$.

Find r_{11}^2 and r_{12}^2 from the equations $c_{20} = 0$ and $c_{02} = 0$ respectively and substitute the result (and the found expression for λ) in the equations $c_{12} + c_{13} = 0$ and $c_{21} = 0$. We obtain, respectively, $r_1^2 \rho (A + B)^2 = 0$ and $r_2^2 \rho AB = 0$ where

$$\rho = r_1^2 (R_1^2 - R_2^2)(R_1^2 - 1)(R_2^2 - 1)^{-2},$$

$$A = 1 + r_{21}^2 - r_1^2 - R_2^2, \quad B = r_1^2 + r_{22}^2 - r_2^2 - r_{21}^2, \quad A + B = 1 + r_{22}^2 - r_2^2 - R_2^2.$$

Since $\rho \neq 0$, we have $AB = A + B = 0$ whence $A = B = 0$. Put the expression for a into c_{20} and c_{02}, and then replace $R_2^2 = 1 + r_{21}^2 - r_1^2$ (in c_{20}) and $R_2^2 = 1 + r_{22}^2 - r_2^2$ (in c_{02}). We obtain, respectively, $1+r_{11}^2=r_1^2+R_1^2$ and $1+r_{12}^2=r_2^2+R_1^2$. These conditions together with $A=0$ and $B=0$ span all the conditions on the rod lengths in Proposition 1(a). $\square$

5. Complexification and compactification of the considered curves

Instead of the affine coordinates T_i and t_j (see §2), it will be more convenient for us to use the projective (homogeneous) coordinates $(S_i:T_i)$ and $(s_j:t_j)$ running over the circles $\{T_i\overline{T}_i=R_i^2S_i\overline{S}_i\}$ and $\{t_j\overline{t}_j=r_j^2s_j\overline{s}_j\}$ in the complex projective line $\mathbb{CP}^1$.

In this and the next sections, P_i and Q_j will denote copies of $\mathbb{CP}^1$ endowed with the respective coordinates. Accordingly, M, M_i, and C_i will denote the compactifications of the complexifications of the respective algebraic sets introduced in §2. Namely, $M=\{\hat f_{11}=\dots=\hat f_{22}=0\}\subset P_1\times P_2\times Q$, $M_i=\{\hat f_{i1}=\hat f_{i2}=0\}\subset P_i\times Q$, and $C_i=\{\hat F_i=0\}\subset Q$ where $Q=Q_1\times Q_2=\mathbb{CP}^1\times\mathbb{CP}^1$,

$$\hat f_{ij}(S_i,T_i;\,s_j,t_j)=S_i^2s_j^2f(T_i/S_i,t_j/s_j),\qquad \hat F_i(s_1,t_1;\,s_2,t_2)=s_1^4s_2^4F(t_1/s_1,t_2/s_2).$$

We also define the curves $C_{ij}=\{\hat f_{ij}=0\}\subset P_i\times Q_j$.

Despite the fact that we have extended M, we still reserve the term $(3,3)$-framework for “true $(3,3)$-frameworks” only, i.e., for the elements of M all whose coordinates belong to the circles $\{T_i\overline{T}_i=R_i^2S_i\overline{S}_i\}$ and $\{t_j\overline{t}_j=r_j^2s_j\overline{s}_j\}$; we denote the set of them (i.e., “the old M”) by $\mathbb{R}M$. This is the fixed point set of the antiholomorphic involution which acts on each factor P_i, Q_j as

$$(T_i:S_i)\mapsto(R_i^2\overline{S}_i:\overline{T}_i),\qquad (t_j:s_j)\mapsto(r_j^2\overline{s}_j:\overline{t}_j). \tag{5.1}$$

6. Consequences of the reducibility of $\hat f_{ij}$ and $\hat F_i$.

Introduce the notation as in §5. Assume that M contains a flexible non-overlapping $(3,3)$-framework $\mathbf{p}$. In this section we find necessary conditions for the reducibility of $\hat F_1$. Let us simplify the notation: $T=T_1$, $S=S_1$, $R=R_1$,

$$a_j=r_{1j},\qquad A_0^\pm=R\pm1,\qquad A_j^\pm=a_j\pm r_j,\qquad j=1,2,$$

(i.e., $A_j^\pm=r_{1j}\pm r_{0j}$, $j=0,1,2$). Set $D_j=\operatorname{Discr}_{t_j}(f_{1j})$. A computation shows that

$$D_j=d_j^+d_j^-,\qquad d_j^\pm=T^2+\big(R^2+1-(A_j^\pm)^2\big)T+R^2,\qquad j=1,2, \tag{6.1}$$

and for $\Delta_j^\pm=\operatorname{Discr}_T(d_j^\pm)$ we have

$$\Delta_j^\pm=(A_j^\pm+A_0^+)(A_j^\pm-A_0^+)(A_j^\pm+A_0^-)(A_j^\pm-A_0^-). \tag{6.2}$$

It follows from Lemma 4 that

$$A_j^+\pm A_k^-\ne0,\qquad A_j^++A_k^+\ne0,\qquad j,k=0,1,2. \tag{6.3}$$

Lemma 7. (Proof is obvious.) *If two polynomials $T^2+b_kT+R^2$, $k=1,2$, have a common root, then they coincide.*

Recall that *deltoid* is a 4-cycle symmetric with respect to one of its diagonals, which we call in this case the *axis* of the deltoid.

Lemma 8. *The polynomial $\hat f_{1j}$, $j=1,2$, is reducible over $\mathbb{C}$ if and only if the 4-cycle $p_0q_0p_1q_j$ either is parallelogrammatic or it is a deltoid.*[5]

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[5]This statement is similar but not equivalent to [4, Lemma 4].

Proof. The reducibility in the deltoid case is evident. For a parallelogrammatic cycle which is not a deltoid, it is also easily seen: the irreducible components correspond to parallelograms and antiparallelograms. Let us prove that there are no other cases of reducibility.

Let $\hat f_{1j}$ be reducible. Consider firstly the case when $\hat f_{1j}$ has a non-constant divisor $\hat f_0$ of degree zero in t_j. Write $\hat f_{1j} = c_2t_j^2 + c_1s_jt_j + c_0s_j^2$. Then $\hat f_0$ divides all the coefficients $c_k(S,T)$. We have $c_0 = r_j^2T(S-T)$ and $c_2 = S(T-R^2S)$. Hence $R=1$ and $\hat f_0 = S-T$, i.e., the polynomial c_1 must vanish identically in T after the substitution $R=1$, $S=T$. Performing this substitution, we obtain $c_1 = (r_j^2-a_j^2)T^2$. Hence $R=1$ and $r_j=a_j$, which corresponds to a deltoid.

Now consider the case when $\hat f_{1j}$ does not have non-constant divisors of degree zero in t_j. Then $\hat f_{1j} = \hat f_1\hat f_2$, $\deg_{t_j}\hat f_k = 1$, $k=1,2$. In this case, the discriminant D_j must be a complete square. We have $d_j^+ - d_j^- = 4a_jr_jT$ (see (6.1)), hence d_j^+ and d_j^- do not coincide. This fact combined with Lemma 7 implies that $d_j^\pm$ are also complete squares, i.e., $\Delta_j^+ = \Delta_j^- = 0$. Then, due to (6.2) and (6.3),

$$A_j^+ - A_0^+ = A_j^- - A_0^- = 0 \quad \text{or} \quad A_j^+ - A_0^+ = A_j^- + A_0^- = 0.$$

Solving these systems of equations, we obtain either $a_j = R$ and $r_j = 1$ (deltoid), or $a_j = 1$ and $r_j = R$ (parallelogram). The lemma is proven. □

Lemma 9. *Suppose that $\hat f_{11}$ and $\hat f_{12}$ are irreducible. Then:*

(a) *the projection of M_1 to each of the factors P_1, Q_1, or Q_2 is finite (i.e., the preimage of each point is finite), and hence M_1 is an algebraic curve;*

(b) *the surfaces $\{\hat f_{1j} = 0\} \subset P_1 \times Q$, $j=1,2$, cross transversally everywhere except, maybe, a finite number of points.*

Proof. (a). Denote with $\mathrm{pr}_j : P_1\times Q \to P_1\times Q_j$, $j=1,2$, the standard projections. If $\mathrm{pr}_1^{-1}(p,q) \subset M_1$, then $\{p\}\times Q_2 \subset \mathrm{pr}_2(M_1) = C_{12}$, which contradicts the irreducibility of $\hat f_{12}$. Hence the projection of M_1 to $P_1\times Q_1$ is finite. In the same way we prove the finiteness of the projections of M_1 to $P_1\times Q_2$ and Q. The finiteness of the projection of C_{1j} (and hence of M_1) to P_1 and Q_j is immediate from the irreducibility of $\hat f_{1j}$.

(b). Consider the affine chart (T,t_1,t_2) on $P_1\times Q$ (the arguments for the other affine charts are the same). In this chart, M_1 is defined by the equations $f_{11} = f_{12} = 0$. The gradients have the form $\nabla f_{11} = (a,b,0)$, $\nabla f_{12} = (c,0,d)$. If such vectors are proportional, then $b=0$ or $d=0$, which means that one of the partial derivatives $\partial f_{1j}/\partial t_{1j}$ is equal to zero. This may happen only on a finite number of lines of the form $T = t_j = \text{const}$. Due to (a), each such line crosses M_1 at a finite number of points, which completes the proof. □

Lemma 10. *If $\hat f_{11}$ and $\hat f_{12}$ are irreducible and $\hat F_1$ is a non-zero reducible polynomial which is not a power of an irreducible polynomial, then the 4-cycle $p_0q_1p_1q_2$ either is parallelogrammatic, or it is a deltoid with axis p_0p_1.*

Proof. Recall that $C_{1j} = \{\hat f_{1j} = 0\} \subset P_1\times Q_j$. Let $\tilde\pi_j : M_1 \to C_{1j}$ and $\pi_j : C_{1j} \to P_1$ be the standard projections $P_1\times Q \to P_1\times Q_j \to P_1$ restricted to the respective curves. By hypothesis, the image of M_1 under the projection $P_1\times Q\to Q$ is the reducible curve $C_1 = \{\hat F_1 = 0\}$, hence the curve M_1 is reducible as well. Let M_1' and M_1'' be two distinct irreducible components of M_1. By Lemma 9, none of them can be contracted to a point by the projections $\tilde\pi_j$. Therefore, since these projections are two-fold (recall that the degree of

f_{ij} in each variable is 2), their restrictions to each component of M_1 are bijective. Hence the composition

$$\eta = \pi_2 \circ \tilde{\pi}_2 \circ \left(\tilde{\pi}_1|_{M_1'}\right)^{-1} : C_{11} \to M_1' \to C_{12} \to P_1$$

has the same branching points (the critical values) as π_2. Since $\eta = \pi_1$, we conclude that π_1 and π_2 have the same branching points.

The branching points of π_j are the odd multiplicity zeros of D_j (see (6.1)), hence $D_1 D_2$ is a complete square. Since π_j is a two-fold projection of the irreducible curve C_{1j}, each of π_1, π_2 has branching points. Then D_1 and D_2 have a common root. Hence, by Lemma 7, one of $d_1^\pm$ coincides with one of $d_2^\pm$. Note that $d_1^+ - d_2^- = (A_2^- + A_1^+)(A_2^- - A_1^+)T$, whence $d_1^+ \not\equiv d_2^-$ by (6.3). Similarly, $d_1^- \not\equiv d_2^+$. Hence one of the following cases takes place.

Case 1. $d_1^+ \equiv d_2^+$ and $d_1^- \equiv d_2^-$. Since $d_1^\pm - d_2^\pm = (A_2^\pm + A_1^\pm)(A_2^\pm - A_1^\pm)T$, we derive from (6.3) that

$$A_2^+ - A_1^+ = A_2^- - A_1^- = 0 \quad \text{or} \quad A_2^+ - A_1^+ = A_2^- + A_1^- = 0. \tag{6.4}$$

By solving these systems of equations, we obtain either $a_1 = r_2$ and $a_2 = r_1$ (parallelogram), or $a_1 = a_2$ and $r_1 = r_2$ (deltoid with axis $p_0 p_1$).

Case 2. $d_1^- \equiv d_2^-$, $\Delta_1^+ = \Delta_2^+ = 0$. Due to (6.2) and (6.3), the second condition yields $A_1^+ - A_0^+ = A_2^+ - A_0^+ = 0$. Eliminating A_0^+ and factorizing (as in Case 1) $d_1^- - d_2^-$, we again obtain (6.4).

Case 3. $d_1^+ \equiv d_2^+$, $\Delta_1^- = \Delta_2^- = 0$. Due to (6.2) and (6.3), the second condition yields $(A_1^- - A_0^-)(A_1^- + A_0^-) = (A_2^- - A_0^-)(A_2^- + A_0^-) = 0$, which is equivalent to four systems of linear equations. Eliminating A_0^- from each of them and combining the result with the equation $A_2^+ - A_1^+ = 0$ (which follows from $d_1^+ \equiv d_2^+$), each time we obtain one of the systems of equations in (6.4). The lemma is proven. $\square$

Lemma 11. *Suppose that $\hat{f}_{11}$ and $\hat{f}_{12}$ are irreducible and $\hat{F}_1 = F^m$, $m \geq 1$, where F is either identically zero or an irreducible polynomial. Then* **p** *is a Dixon mechanism of the first kind.*[6]

Proof. If $F = 0$, this is a particular case ($\lambda = 0$) of Lemma 6, so let $F \neq 0$. If $m = 1$ (i.e., $\hat{F}_1$ is irreducible), then, since $\hat{F}_1$ and $\hat{F}_2$ are bihomogeneous polynomials of the same bidegree which have a common divisor, the result follows again from Lemma 6.

Let $m \geq 2$. Let us prove in this case that the projection $\pi : M_1 \to C_1$ is two-fold. Suppose that C_1 contains a smooth point q with a single preimage. Let $\gamma : (\mathbb{C}, 0) \to (Q, q)$ be a holomorphic germ transverse to C_1. By Lemma 9, we may assume that the surfaces $\hat{f}_{1j} = 0$ are smooth and cross transversally over q. Using the expression of the resultant of two polynomials via their roots (see, e.g., [3, Ch. 12, eq. (1.3)]) one can easily derive that $F_1(\gamma(t))$ has a first order zero at $t = 0$. This fact contradicts the condition $m \geq 2$, hence the projection π cannot be one-fold. Since $\deg_T \hat{f}_{ij} = 2$, we conclude that it is two-fold.

Thus almost all points of C_1 have two preimages in M_1. Since **p** is flexible, we may then assume that $\pi^{-1}(q_1, q_2) = \{(p_1, q_1, q_2), (p_1', q_1, q_2)\}$, $p_1' \neq p_1$. This set itself and one of its elements are invariant under the antiholomorphic involution (5.1), hence the other element is invariant as well. Therefore $(p_1', p_2; q_1, q_2) \in \mathbb{R}M$. Moreover, all this remains true during a deformation of **p**. Hence the $(4,3)$-framework $(p_0, p_1, p_2, p_1'; q_0, q_1, q_2)$ is flexible. Its joints p_1 and p_1' are equidistant from all the q_j's. With help of Lemma 5, it is easy to derive from this fact that **p** is a Dixon mechanism of the first kind. The lemma is proven. $\square$

[6] For Dixon mechanisms of the first kind we have $F_1 = (R_1^2 - 1)F^2$ and $F_2 = (R_2^2 - 1)F^2$ with the same F.

Recall our assumption that M contains a flexible non-overlapping $(3,3)$-framework $\mathbf{p}$. Say that a cycle in $\mathbf{p}$ is *fastened*, if it contains the edge p_0q_0. One can summarize Lemmas 8, 10, and 11 as follows.

Lemma 12. (Main Lemma.) *If $\mathbf{p}$ is not a Dixon mechanism of the first kind, then the $(2,3)$-framework $(p_0, p_1;\, q_0, q_1, q_2)$ contains either a parallelogrammatic cycle, or a fastened deltoid, or a not fastened deltoid with axis p_0p_1.*

7. Completing the proof of Theorem 1

Let $\mathbf{p}$ be a flexible non-overlapping $(3,3)$-framework which is not a Dixon mechanism of the first kind. Let us show that $\mathbf{p}$ is a Dixon mechanism of the second kind.

Lemma 13. *Any deltoid in $\mathbf{p}$ is a rhombus.*

Proof. Suppose that $\mathbf{p}$ contains a deltoid Δ which is not a rhombus. Renumber the joints so that $\Delta = p_0q_1p_1q_2$ and $|p_0q_1| = |p_1q_1| \neq |p_0q_2| = |p_1q_2|$ (see Fig. 4, on the left). By Lemma 12, the $(2,3)$-framework $(p_0, p_1;\, q_0, q_1, q_2)$ must contain a 4-cycle Δ' realizing one of the following cases. In each of them (except the last one) we show that p_0 and p_1 are equidistant from q_0, which contradicts Lemma 5.

Case 1. Parallelogrammatic cycle. Then Δ' contains both p_0, p_1, and also at least one of q_i, $i = 1$ or 2. Since $|p_0q_i| = |p_1q_i|$, we conclude that Δ' is a rhombus. But $\Delta' \neq \Delta$ (since Δ is not a rhombus), hence $q_0 \in \Delta'$. Therefore p_0 and p_1 are equidistant from q_0.

Case 2. Fastened deltoid with axis p_0p_1. We may assume that $\Delta' = p_0q_0p_1q_1$, $|p_0q_0| = |p_0q_1|$ and $|p_1q_1| = |p_1q_0|$. Then $|p_0q_0| = |p_0q_1| = |p_1q_1| = |p_1q_0|$.

Case 3. Fastened deltoid with axis q_0q_j. By definition $|p_0q_0| = |p_1q_0|$.

Case 4. Not fastened deltoid with axis p_0p_1. Then $\Delta' = \Delta$ and this is a deltoid with two axes, that is a rhombus. A contradiction. The lemma is proven. $\square$

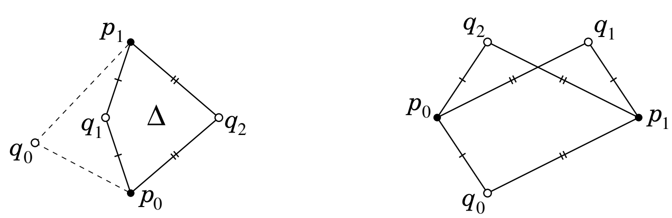

FIGURE 4. Illustration to the proofs of Lemmas 13 (on the left) and 14 (on the right).

Lemma 14. $\mathbf{p}$ *cannot contain two distinct parallelogrammatic cycles with three common vertices.*

Proof. Suppose that $\mathbf{p}$ contains two distinct parallelogrammatic cycles Π_1 and Π_2 with three common vertices. Up to renumbering, we may assume that these are $q_0p_0q_1p_1$ and $p_0q_1p_1q_2$ (see Fig. 4, on the right). Then $q_0p_0q_2p_1$ is a deltoid. By Lemma 13, it must be a rhombus. Hence Π_1 and Π_2 are rhombi as well. It is easy to check that this is impossible. The lemma is proven. $\square$

Lemmas 12 and 13 imply that each $(2,3)$-framework obtained from $\mathbf{p}$ by removal of one joint contains a parallelogrammatic 4-cycle. Using Lemma 14, it is easy to derive from this fact that the joints of $\mathbf{p}$ can be numbered so that the three 4-cycles $\Pi_{ij} = p_i q_i p_j q_j$, $i < j$, become parallelogrammatic. This means that one can denote the lengths of the rods by a, b, c, d as in Proposition 1(b). It remains to prove that the relation $a^2 + c^2 = b^2 + d^2$ holds up to renumbering of the joints. In the notation of §2 we have

$$r_{11} = r_{22} = a = 1, \qquad R_1 = r_1 = b, \qquad R_2 = r_2 = c, \qquad r_{12} = r_{21} = d.$$

Doing these substitutions, we express the coefficients of F_1 and F_2 as polynomials in b, c, d. By Lemma 3, the resultant of F_1 and F_2 with respect to t_1 identically vanishes. Hence the resultant of $F_1(t_1, -c)$ and $F_2(t_1, -c)$ is zero. A computation shows that it is equal to

$$16\, b^8 c^{16} (1+c)^8 (1 + b^2 - c^2 - d^2)^4 (1 + d^2 - b^2 - c^2)^4 (1 + c^2 - b^2 - d^2)^4, \tag{7.1}$$

which completes the proof of Theorem 1.

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