COMPLEX ANALYSIS Spring 2024

Lecture 1 2024-01-15

Goal: Study functions $f : D \to \mathbb{C}$, for $D \subset \mathbb{C}$ subset.

Reason: Such functions have very nice properties. For example:

1- If f is differentiable in an open set $D \subset \mathbb{C}$, then it is automatically infinitely differentiable in D. More: it is analytic, which means: if $D(p,r) \subset D \subset \mathbb{C}$ is the open disk of radius r centered at p, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n \quad \text{for every } z \in D(p,r).$$

(ie, f is given by its Taylor series centered at p).

Note: The book uses the word "analytic" a bit differently. We will say more about that later.

2- Fundamental theorem of Algebra: given a non-constant polynomial

$$p(z) = a_0 + a_1 z + \dots + a_n z^n, \quad \text{with } a_0, \dots, a_n \in \mathbb{C},$$

there is (at least one) $z \in \mathbb{C}$ such that $p(z) = 0$.

Note: This is not true if we replace $\mathbb{C}$ with $\mathbb{R}$

Ex: if $p(x) = 1 + x^2$, there is no $x \in \mathbb{R}$ s.t. $p(x) = 0$.

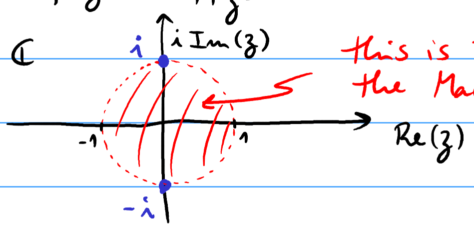
3- The domains of power series behave in a more natural way than in $\mathbb{R}$.

Ex: $f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for all $-1 < x < 1$

the radius of convergence of the Maclaurin series is 1.

Why?

$f(z) = \frac{1}{1+z^2}$ is not defined when $z = \pm i$:



this is the disk of convergence of the Maclaurin series. It is as large as possible, since $f(z)$ is not defined at $\pm i$.

4- Some (especially improper) integrals become easier to calculate using tools from complex analysis

Ex: $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{\cos(x)}{1+x^2}}_{\text{no "nice" antiderivative}} dx = \frac{\pi}{e}$

Important background from Vector Calculus in $\mathbb{R}^2$:

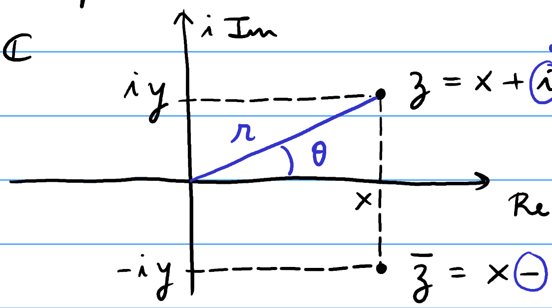
We will need at least the following content. Please review as needed

- limits and continuity of scalar functions $\phi : \mathbb{R}^2 \to \mathbb{R}$
- partial derivatives of $\phi : \mathbb{R}^2 \to \mathbb{R}$
- differentiable vector functions $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and the Jacobian matrix of $\vec{F}$
- inverse function theorem for $\vec{F}$
- line integral of $\vec{F}$ along a path γ in $\mathbb{R}^2$
- fundamental theorem of calculus for line integrals of conservative fields $\vec{F} = \nabla\phi$
- Green's theorem for domains in $\mathbb{R}^2$.

Let us now review some facts about $\mathbb{C}$ and study some important complex functions

Complex numbers

$\mathbb{C}$



$i^2 = -1$

$z = x + iy = r e^{i\theta} = \underbrace{\underbrace{r\cos\theta}_{x} + i\,\underbrace{r\sin\theta}_{y}}_{\text{polar form of } z}$ Euler's formula

$\bar{z} = x - iy$ complex conjugate

- sum: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
  like addition of vectors in $\mathbb{R}^2$

- **product**: $z_1 \cdot z_2 = (x_1 + i y_1) \cdot (x_2 + i y_2) =$

$$= x_1 x_2 + i x_1 y_2 + i y_1 \cdot x_2 + \overset{=-1}{i^2} y_1 \cdot y_2 =$$

$$= (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + y_1 x_2)$$

if $\begin{cases} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{cases}$ $= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Ultimately, the nice properties of complex functions come from this new way of multiplying vectors in $\mathbb{R}^2$.

- **modulus**: $r = |z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$ is the **modulus** of z

same as the norm of the vector (x, y) in $\mathbb{R}^2$

Note: $e^{i(x+y)} \overset{\text{Euler}}{=} \cos(x+y) + i \sin(x+y)$

$e^{a+b} = e^a \cdot e^b$

$$e^{ix} \cdot e^{iy} = (\cos(x) + i \sin(x)) \cdot (\cos(y) + i \sin(y)) =$$

$$\overset{\text{Euler}}{=} (\cos x \cdot \cos y - \sin x \cdot \sin y) + i\,(\cos x \cdot \sin y + \sin x \cdot \cos y)$$

$$\Rightarrow \begin{cases} \cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y \\ \sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y \end{cases}$$

Complex exponential

Define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y)).$$

Note: Later, we will see that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for every } z \in \mathbb{C}.$$

Important properties: a) $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$.

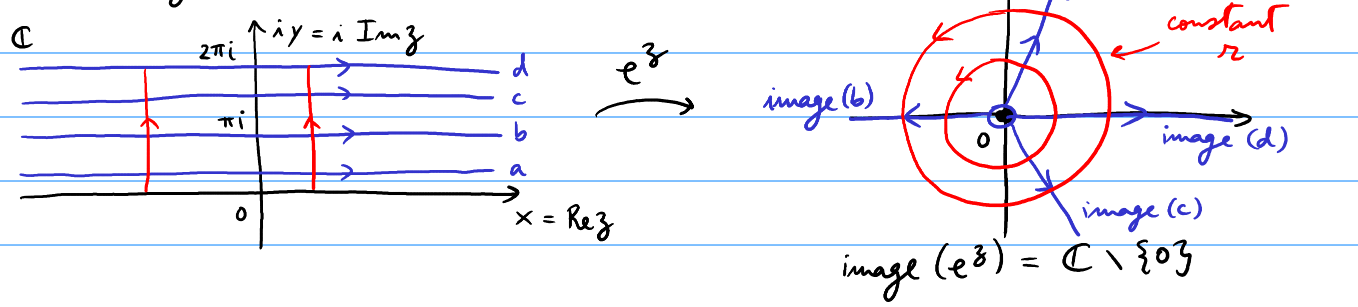
b) $e^{z + 2\pi i} = e^z$: e^z has period $2\pi i$, so it is not injective.

Proof: a) Exercise

b) $e^{z + 2\pi i} \overset{a)}{=} e^z \cdot e^{2\pi i}$ and

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1.$$

Visualization :



Logarithm

The inverse of the function e^z is denoted $\underline{\log z}$, and defined by

$$e^w = z \iff \log z = w$$

Since e^z is not injective, $\log z$ is <u>not uniquely defined</u>.
We say that it is a <u>multivalued function</u> from $\mathbb{C}\setminus\{0\}$ to $\mathbb{C}$. This means that for each $z \in \mathbb{C}\setminus\{0\}$, $\log(z)$ an (infinite) set of numbers.

If $z = re^{i\theta} = e^{\log r} e^{i\theta} = e^{\log r + i\theta}$, then

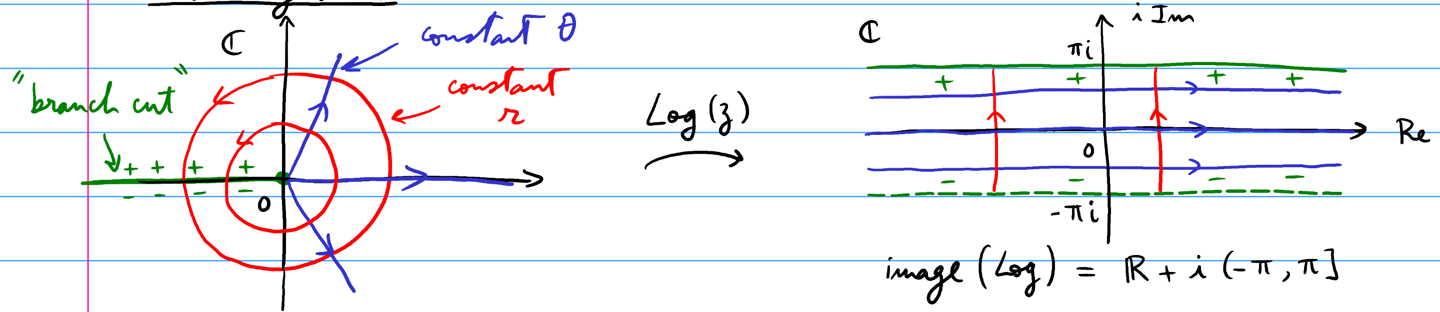
$$\begin{aligned} \log z &= \log r + i\theta + 2\pi i m, \quad m \in \mathbb{Z} \\ &= \log|z| + i\,\mathrm{Arg}(z) + 2\pi i m, \quad m \in \mathbb{Z} \\ &= \log|z| + i \arg(z), \quad \text{where} \end{aligned}$$

- $\mathrm{Arg}(z) \in (-\pi, \pi]$ is such that $z = |z| e^{i\,\mathrm{Arg}(z)}$
- $\arg(z) = \mathrm{Arg}(z) + 2\pi m$, $m \in \mathbb{Z}$ is a <u>multivalued function</u> from $\mathbb{C}\setminus\{0\}$ to $\mathbb{R}$.

Call
- $\mathrm{Arg}(z) \in (-\pi, \pi]$ the <u>principal argument</u> of z
- $\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$ the <u>principal value</u> of $\log(z)$.

<u>Exercise</u>: Compute $\log(1 - i\sqrt{3})$.

Visualization :



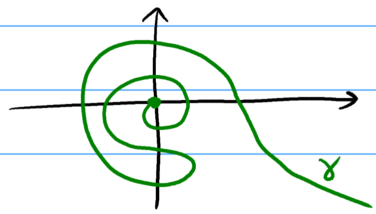
Branches

In $U = \mathbb{C} \setminus \overbrace{(-\infty, 0]}^{\text{branch cut}}$, $\mathrm{Log}(z) : U \to \mathbb{C}$ defines a **branch** of the (multivalued function) $\log(z)$. For any $m \in \mathbb{Z}$,

$$\mathrm{Log}(z) + 2\pi i m : U \to \mathbb{C} \quad \text{is also a branch.}$$

For every curve $\gamma : [0, \infty) \to \mathbb{C}$ such that

- γ is continuous and injective,
- $\gamma(0) = 0$,
- $\lim_{t\to\infty} |\gamma(t)| = \infty$,



there is a branch of $\log(z)$ defined on $\mathbb{C} \setminus \mathrm{image}(\gamma)$. Call such γ a **branch cut**.

(Very technical note: • the 3 conditions above, esp. $\lim_{t\to\infty} |\gamma(t)| = \infty$, imply γ is proper.

- Munkres Thm 63.2 implies that $\mathbb{C} \setminus \mathrm{image}(\gamma)$ is connected.
- Ahlfors Section 4.4.2 defines $U \subset \mathbb{C}$ open connected to be simply connected if $\mathbb{CP}^1 \setminus U$ is connected. The Riemann mapping theorem then implies that U is homeomorphic to an open disk, hence $\pi_1(U, *) \cong \{1\}$. Since $\mathbb{C} \setminus \mathrm{image}(\gamma)$ is open and connected and
$$\mathbb{CP}^1 \setminus (\mathbb{C} \setminus \mathrm{image}(\gamma)) = \mathrm{image}(\gamma) \cup \{\infty\}$$
is connected, we get that $\mathbb{C} \setminus \mathrm{image}(\gamma)$ is simply connected.
- As we'll see, $\mathbb{C} \setminus \mathrm{image}(\gamma)$ simply connected implies that the holomorphic function $\frac{1}{z}$ has an antiderivative in that set. This gives a branch of $\log(z)$ on $\mathbb{C} \setminus \mathrm{image}(\gamma)$, as wanted.)

Lecture 2 01-16

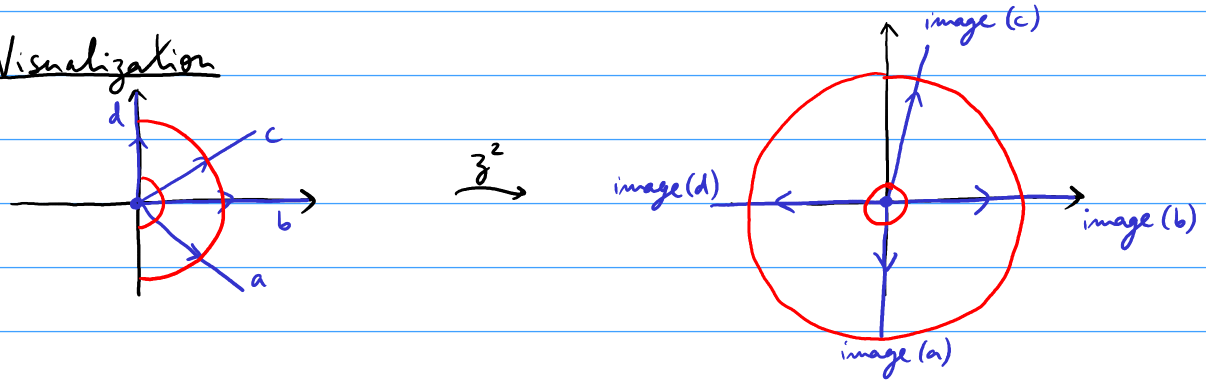
Complex powers

Let $f(z) = z^2$. Writing $z = re^{i\theta}$,

$f(re^{i\theta}) = r^2 e^{i2\theta}$

Since $f(-z) = f(z)$, the function f is not injective.

Visualization



Since $f(z) = z^2$ is not injective, its inverse is the multivalued function square root, given by

$\sqrt{z} = z^{1/2} = (re^{i\theta})^{1/2} = \{\sqrt{r}\, e^{i\theta/2}, \underbrace{-\sqrt{r}\, e^{i\theta/2}}_{= \sqrt{r}\, e^{i(\theta/2 + \pi)}}\}$

If $z \neq 0$, then $\sqrt{z}$ takes two values.

More generally, given $w \in \mathbb{C}$, define a multivalued function

$z^w = e^{w \log(z)}$, for $z \neq 0$

Note: If we specify a branch of $\log(z)$, we get a branch of z^w.

Ex: • if $w = n \in \mathbb{Z}$, then given two values $\log|z| + i\theta$ and $\log|z| + i(\theta + 2\pi m)$ for $\log(z)$,

$e^{n(\log|z| + i(\theta + 2\pi m))} = e^{n(\log|z| + i\theta)} \cdot \underbrace{e^{i2\pi mn}}_{= 1 \text{ since } m, n \in \mathbb{Z}}$

$\Rightarrow z^n$ is a function, not a multivalued function (because for every $z \in \mathbb{C}$, z^n is a single number).

• if $w = \frac{1}{2}$, then

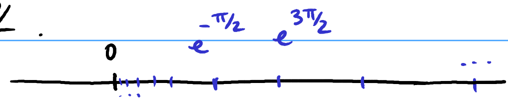
$e^{\frac{1}{2}(\log|z| + i(\theta + 2\pi m))} = e^{\frac{1}{2}(\log|z| + i\theta)} \cdot \overbrace{e^{i\pi m}}^{\in \{\pm 1\}}$

which again shows that $z^{1/2} = \sqrt{z}$ takes two values for every $z \neq 0$.

Ex: $i^i = e^{i \log(i)} = e^{i\left(\frac{\pi}{2} i + 2\pi i m\right)}$, $m \in \mathbb{Z}$

$= e^{-\left(\frac{\pi}{2} + 2\pi m\right)}$, $m \in \mathbb{Z}$.

this is an infinite subset of $\mathbb{R}$



Trigonometric and hyperbolic functions

Know: given $x \in \mathbb{R}$,

$$\begin{cases} e^{ix} = \cos x + i \sin x \\ e^{-ix} = \cos x - i \sin x \end{cases} \Rightarrow \begin{cases} \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \\ \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \end{cases}$$

Hence, it is natural to define:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \text{ for all } z \in \mathbb{C}$$

Also, define

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^{z} + e^{-z}}{2}, \text{ for all } z \in \mathbb{C}$$

Many properties of the real functions also hold for the complex extensions, for example:

- $\cos^2 z + \sin^2 z = 1$ & $\cosh^2 z - \sinh^2 z = 1$
- $\sin(z_1 + z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2$
  $\cos(z_1 + z_2) = \cos z_1 \cdot \cos z_2 - \sin z_1 \cdot \sin z_2$
- $\sin(z)$ and $\cos(z)$ have period 2π.
- new: $\sinh(z)$ and $\cosh(z)$ have period $2\pi i$.

Note: $\sin(iy) = i \sinh(y)$ and $\cos(iy) = \cosh(y)$

Using the formulas for sin and cos of a sum (also true in $\mathbb{C}$):

$$\sin(x+iy) = \sin x \cdot \cos(iy) + \cos x \cdot \sin(iy) =$$
$$= \sin x \cdot \cosh y + i \cos x \cdot \sinh y$$

$$\cos(x+iy) = \cos x \cdot \cos(iy) - \sin x \cdot \sin(iy) =$$
$$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$

So, the complex trigonometric functions are related to the hyperbolic functions.

Ex: Find the solutions of $\sin z = 5 \quad (*)$

$$\frac{e^{iz} - e^{-iz}}{2i} = 5 \iff e^{iz} - e^{-iz} = 10i \iff$$ multiply by e^{iz}

$$\iff e^{2iz} - 10i\,e^{iz} - 1 = 0$$ quadratic eq. in e^{iz}

$$\iff e^{iz} = 5i \pm \sqrt{-25+1} = 5i \pm \sqrt{-24} = 5i \pm 2\sqrt{6}\,i = (5 \pm 2\sqrt{6})\,i$$

$$\iff iz = \log\big(\underbrace{(5\pm 2\sqrt{6})}_{\in \mathbb{R}_{>0}}\, i\big) = \log(5\pm 2\sqrt{6}) + \log i =$$

$$= \log(5\pm 2\sqrt{6}) + i\left(\tfrac{\pi}{2} + 2\pi m\right), \quad m \in \mathbb{Z}$$

Note: $\log(5+2\sqrt{6}) + \log(5-2\sqrt{6}) = \log\big((5+2\sqrt{6})(5-2\sqrt{6})\big) =$

$$= \log(25-24) = \log 1 = 0 \Rightarrow \log(5-2\sqrt{6}) = -\log(5+2\sqrt{6})$$

Therefore, $\underbrace{z = \left(\tfrac{\pi}{2} + 2\pi m\right) \pm i \log(5+2\sqrt{6})}_{\text{the infinitely many solutions to } (*)}, \quad m \in \mathbb{Z}$

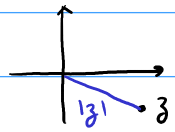
Lecture 3 — 01-22

Topology of $\mathbb{C}$

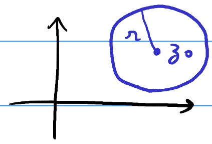
Notions like "open set" and "distance" work the same way for $\mathbb{C}$ as for $\mathbb{R}^2$, under the identification $\mathbb{C} \longrightarrow \mathbb{R}^2$, $x+iy \longmapsto (x,y)$.

In particular:

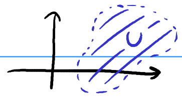
- $\underbrace{|z|}_{\text{in } \mathbb{C}} = \sqrt{x^2+y^2} = \underbrace{\|(x,y)\|}_{\text{in } \mathbb{R}^2}$



- $D_r(z_0) = D(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$
  open disk of radius r centered at z_0.



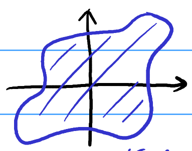
- a subset $U \subset \mathbb{C}$ is open if it is a union of (possibly infinitely many) open disks



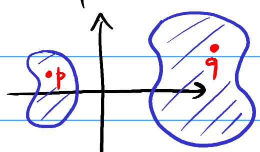
the boundary is ~~not~~ not included

- a subset $A \subset \mathbb{C}$ is closed if $A^c = \mathbb{C} \setminus A$ (complement) is open.
- given a subset $A \subset \mathbb{C}$, $z \in \mathbb{C}$ is a boundary point of A if every disk $D_r(z)$ contains points in A and points in A^c.
  Write $\partial A = \{\text{boundary points of } A\}$.
- $U \subset \mathbb{C}$ open is connected if for every $p, q \in A$ there is a continuous function $\gamma : [0,1] \to A$ s.t. $\gamma(0) = p$ and $\gamma(1) = q$.

Idea: U has "only one piece".



connected



not connected

Note: This is the definition of U being *path-connected*.

For $U \subset \mathbb{C}$ open, this is equivalent to U being *connected*. In general, U is connected if it is not disconnected. And $U \subset \mathbb{C}$ is *disconnected* if there are open sets $U_1, U_2 \subset \mathbb{C}$ such that, if $\tilde{U}_1 = U_1 \cap U$ and $\tilde{U}_2 = U_2 \cap U$, we have: $\tilde{U}_1 \neq \emptyset$, $\tilde{U}_2 \neq \emptyset$, $\tilde{U}_1 \cup \tilde{U}_2 = U$ and $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$.

- $D \subset \mathbb{C}$ is a *domain* if it is open and connected.

Limits and continuous functions

Limits and continuous functions work the same way for functions $\mathbb{C} \to \mathbb{C}$ as they do for functions $\mathbb{R}^2 \to \mathbb{R}^2$, under the usual identification
$$\mathbb{C} \xrightarrow{\cong} \mathbb{R}^2$$
$$z = x+iy \mapsto (x,y)$$

Recall: limits in $\mathbb{R}^2$ are subtle.

Ex: If $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is given by $f(x,y) = \frac{xy}{x^2+y^2}$, then:
- along the line $y=0$: $f(x,0) = \frac{0}{x+0} = 0$ and $\lim_{x\to 0} f(x,0) = 0$ $\neq$
- along the line $y=x$: $f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$ and $\lim_{x \to 0} f(x,x) = \frac{1}{2}$

Conclusion: $\lim_{(x,y)\to(0,0)} f(x,y)$ *does not exist*.

Def: Let $f : D \to \mathbb{C}$ for some $D \subset \mathbb{C}$. Given $z_0 \in \mathbb{C}$, write $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ ("the limit of $f(z)$ as z tends to z_0 is L") if

$$\underset{\text{"for all } \varepsilon>0\text{"}}{\forall_{\varepsilon>0}}\ \underset{\text{"exists } \delta>0\text{" s.t.}}{\exists_{\delta>0}}\ \underset{\text{"for all } x\in D\text{"}}{\forall_{z\in D}}\ \underbrace{0<|z-z_0|<\delta \Rightarrow |f(z)-L|<\varepsilon}$$

"if $z \neq z_0$ and their distance is $<\delta$, then the distance between $f(z)$ and L is $<\varepsilon$"

This is *equivalent* to each of the following two conditions:

a) writing $f(x+iy) = \underbrace{u(x,y)}_{\text{Re } f} + i\, \underbrace{v(x,y)}_{\text{Im } f}$ for functions $u, v : D \to \mathbb{R}$,
$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \mathrm{Re}(L) \quad \underline{\text{and}} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \mathrm{Im}(L).$$

b) for every sequence $z_1, z_2, \ldots$ in $D \setminus \{z_0\}$ (denote the sequence by (z_n)), if $\lim_{n\to\infty} z_n = z_0$ then $\lim_{n\to\infty} f(z_n) = L$.

<u>Note</u>: analogously to sequences of real numbers, a sequence (z_n) in $\mathbb{C}$ converges to z_0 if:
$\forall_{\varepsilon>0}\ \exists_{n_0\in\mathbb{N}}\ \forall_{n>n_0}\ |z_n - z_0| < \varepsilon$.

<u>Note</u>: The usual properties of limits (with respect to sum, difference, product, quotient, composition) also hold in $\mathbb{C}$.

<u>Def</u>: $f: D \to \mathbb{C}$, with $D \subset \mathbb{C}$, is <u>continuous at $z_0 \in D$</u> if $\lim_{z\to z_0} f(z) = f(z_0)$. f is <u>continuous</u> if it is continuous at <u>every</u> $z_0 \in D$.

<u>Ex</u>: If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial with complex coefficients, then it defines a function
$$P: \mathbb{R}^2 \to \mathbb{R}^2$$
by $P(x,y) = (u(x,y), v(x,y))$, with $p(x+iy) = u(x,y) + i\,v(x,y)$.
It is easy to check that u and v are polynomials in x, y.
Hence, $P(x,y)$ is continuous, and so is $p(z)$.

<u>Derivatives and holomorphic functions</u>

As we saw, limits in $\mathbb{C}$ work like limits in $\mathbb{R}^2$.
We will now see that derivatives in $\mathbb{C}$ are new, because their definition uses <u>multiplication</u> in $\mathbb{C}$ (by taking quotients).

<u>Def</u>: $f: D \to \mathbb{C}$ is <u>differentiable at $z_0 \in \mathbb{C}$</u> if the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. Call this limit the <u>derivative</u> of f at z_0, and denote it by $f'(z_0)$ or $\frac{df}{dz}(z_0)$. Also write $f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$.
$\Delta z = z - z_0$

Say that f is <u>holomorphic</u> on an <u>open</u> set $U \subset D$ if it is differentiable at <u>every</u> $z_0 \in U$.

<u>Note</u>: Gamelin's book uses the word "analytic" instead of "holomorphic". It also asks the derivative function $f'(z)$ to be <u>continuous</u>, but we will see later that (surprisingly) the existence of $f'(z)$ in U implies the continuity of $f'(z)$ in U.

<u>Note</u>: As with real functions, f differentiable at z_0 implies f continuous at z_0. Also have usual formulas for derivatives of sums, differences, products, quotients and compositions.

<u>Exs</u>: • $f(z) = z^2$ is holomorphic in $\mathbb{C}$ (call it <u>entire</u>):

For every $z_0 \in \mathbb{C}$,

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{z_0^2} + 2 z_0 \cdot \cancel{\Delta z} + (\Delta z)^{\cancel{2}} - \cancel{z_0^2}}{\cancel{\Delta z}} =$$

$$= \lim_{\Delta z \to 0} 2 z_0 + \Delta z = 2 z_0 \Rightarrow f'(z_0) = 2 z_0 \quad \text{(as expected)}.$$

• $f(z) = \bar{z}$ is <u>not differentiable anywhere</u>!

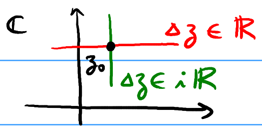
For every $z_0 \in \mathbb{C}$,

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overbrace{\overline{z_0 + \Delta z}}^{\cancel{\bar{z}_0} + \overline{\Delta z}} - \cancel{\bar{z}_0}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Very important: the limit must be the same whichever way that $\Delta z \to 0$ in $\mathbb{C} \cong \mathbb{R}^2$. But:

these are different, so the limit does <u>not</u> exist

* if $\Delta z \in \mathbb{R}$ is real, then $\dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{\Delta z}{\Delta z} = 1$
* if $\Delta z \in i\mathbb{R}$ is imaginary, then $\dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{-\Delta z}{\Delta z} = -1$



Conclusion: $f'(z_0) = \lim_{\Delta z \to 0} \dfrac{\overline{\Delta z}}{\Delta z}$ does not exist for any $z_0 \in \mathbb{C}$.

Lecture 4 01-25

Cauchy - Riemann equations

<u>Thm</u>: Let $f: D \to \mathbb{C}$, for $D \subset \mathbb{C}$ open. Write $f = u + iv$.

If f is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$, then

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \leftarrow \text{Cauchy - Riemann equations.}$$

at the point (x_0, y_0)

Also, $\boxed{f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0)}$ $(= u_x - i u_y)$.

<u>Note</u>: Many of the nice properties of holomorphic functions will follow from these equations.

<u>Proof</u>: Recall that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \quad \Delta z = \Delta x + i \Delta y$$

$$= \lim_{\Delta z \to 0} \frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]}{\Delta x + i \Delta y}$$

- if $\Delta z = \Delta x \in \mathbb{R}$: then,

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$
$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0)$$

- if $\Delta z = i\Delta y \in i\mathbb{R}$: then,

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}$$
$$= \frac{1}{i} u_y(x_0,y_0) + v_y(x_0,y_0) = v_y(x_0,y_0) - i\, u_y(x_0,y_0).$$

Since the two limits are equal, we get

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad v_x(x_0,y_0) = -u_y(x_0,y_0). \qquad \square$$

The previous theorem has the following <u>converse</u>:

<u>Thm</u>: Let $f: D \to \mathbb{C}$, for $D \subset \mathbb{C}$ open. Write $f = u + iv$.

Looman-Menchoff Thm: this assumption could be removed with a lot more work

Suppose that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist in U and are continuous at $(x_0,y_0) \in U$. If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ <u>and</u> $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x_0,y_0), then f is differentiable at $x_0 + iy_0$.

Proof: Use results from Calculus of several variables.

Since u_x and u_y are continuous at (x_0, y_0), we know that $u : D \to \mathbb{R}$ is differentiable at (x_0, y_0), with

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - u_x(x_0,y_0).\Delta x - u_y(x_0,y_0).\Delta y|}{\underbrace{\|(\Delta x, \Delta y)\|}_{\to \sqrt{(\Delta x)^2 + (\Delta y)^2}}} = 0$$

Equivalently:

$$u(x_0+\Delta x, y_0+\Delta y) = u(x_0,y_0) + u_x(x_0,y_0)\cdot\Delta x + u_y(x_0,y_0)\cdot\Delta y + \underbrace{R(\Delta x, \Delta y)}_{\text{"the rest"}},$$

with $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{|R(\Delta x,\Delta y)|}{\|(\Delta x,\Delta y)\|} = 0$

Similarly, $v : D \to \mathbb{R}$ is differentiable at (x_0, y_0) and

$$v(x_0+\Delta x, y_0+\Delta y) = v(x_0,y_0) + v_x(x_0,y_0)\cdot\Delta x + v_y(x_0,y_0)\cdot\Delta y + \underbrace{S(\Delta x, \Delta y)}_{\text{"the rest"}},$$

with $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{|S(\Delta x,\Delta y)|}{\|(\Delta x,\Delta y)\|} = 0$

Therefore, writing $z_0 = x_0 + iy$ and $\Delta z = \Delta x + i\Delta y$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} =$$

$$= \frac{u(x_0+\Delta x, y_0+\Delta y) + i\, v(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - i\, v(x_0,y_0)}{\Delta z} =$$

$$= \frac{u_x(x_0,y_0).\Delta x + \overbrace{u_y(x_0,y_0).\Delta y}^{-v_x(x_0,y_0).\Delta y} + i\, v_x(x_0,y_0).\Delta x + \overbrace{i\, v_y(x_0,y_0).\Delta y}^{u_x(x_0,y_0)\, i\Delta y} + R(\Delta x,\Delta y) + iS(\Delta x,\Delta y)}{\Delta z}$$

$\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0,y_0).\Delta z}{\Delta z} + i\,\frac{v_x(x_0,y_0).\Delta z}{\Delta z} + \frac{R(\Delta x,\Delta y)}{\Delta z} + i\,\frac{S(\Delta x,\Delta y)}{\Delta z}$$

$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0) + \frac{R(\Delta x,\Delta y)}{\Delta z} + i\,\frac{S(\Delta x,\Delta y)}{\Delta z} \xrightarrow{\Delta z \to 0} u_x(x_0,y_0) + i\, v_x(x_0,y_0).$$

Hence, $f'(z_0) = \displaystyle\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists and is given by $u_x(x_0,y_0) + i\, v_x(x_0,y_0)$. ∎

Exs: • $f(z) = e^z \overset{z = x+iy}{=} \underbrace{(e^x \cos y)}_{u(x,y)} + i\,\underbrace{(e^x \sin y)}_{v(x,y)}$.

It is clear that u and v have continuous partial derivatives.

$$u_x = e^x \cos y = v_y \quad \text{and} \quad u_y = -e^x \sin y = -v_x.$$

Therefore, f is entire (meaning: holomorphic in $\mathbb{C}$).

$$\frac{d}{dz}(e^z) = f'(z) = u_x + i v_x = e^z.$$

- similar arguments imply that polynomials, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$ are entire, and have the expected derivatives.

Harmonic functions

Def: Let $\phi : U \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}^2$. If ϕ is C^2, then it is harmonic if

$$\boxed{\Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}. \quad \leftarrow \text{Laplace equation}$$

Say that $f = u + iv$ is harmonic if u and v are harmonic.

Note: Electrostatic and gravitational potentials in the vacuum are harmonic functions. More generally, if there is electric charge or mass distributed according to a density function ρ, then there is a constant C s.t.

$$\boxed{\Delta \phi = C \cdot \rho} \quad \leftarrow \text{Poisson equation}$$

later, we'll see that if f is holomorphic then this follows

Thm: If $f : U \to \mathbb{C}$ is holomorphic $\boxed{\text{and } u, v \text{ are } C^2}$, then u and v are harmonic. Hence, f is also harmonic.

Proof: $$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\underbrace{\frac{\partial u}{\partial y}}_{\overset{CR}{=} -\frac{\partial v}{\partial x}}\right) \underset{v \text{ is } C^2}{\overset{\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}}{=}} \frac{\partial}{\partial x}\Big(\overbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}^{\overset{CR}{=} 0}\Big) = 0.$$

The proof that $\Delta v = 0$ is similar.

$\Rightarrow \Delta f = \Delta u + i \Delta v = 0$. ∎

Q: If $\Delta u = 0$, is u the real part of a holomorphic f?

Def: If $u : U \to \mathbb{R}$ is harmonic on $U \subset \mathbb{C}$ open, and if $v : U \to \mathbb{R}$ is also harmonic and $f = u + iv$ is holomorphic, then v is called a harmonic conjugate of u.

Ex: Let $u(x,y) = 2x - y - 2xy$

Show that u is harmonic and find its harmonic conjugates.

$$\Delta u = u_{xx} + u_{yy} = \frac{\partial}{\partial x}\underbrace{\left(2(1-y)\right)}_{u_x} + \frac{\partial}{\partial y}\underbrace{(-1-2x)}_{u_y} = 0.$$

Want: $v(x,y)$ s.t. $u + iv$ is holomorphic.

CR eqs: $v_x = -u_y = 1+2x$, $v_y = u_x = 2-2y$

$\Rightarrow v(x,y) = \int v_x \, dx = \int (1+2x)\, dx = x + x^2 + C(y)$

$\Rightarrow v_y = C'(y) = 2 - 2y \Rightarrow C(y) = 2y - y^2 + C$

$\Rightarrow v(x,y) = x + x^2 + 2y - y^2 + C$, $C \in \mathbb{R}$

Note: $f(z) = u + iv = (2x - y - 2xy) + i(x + 2y + x^2 - y^2 + C)$

$= (2+i)(x+iy) + i(x+iy)^2 + iC = (2+i)z + iz^2 + C$, $C \in \mathbb{R}$ ∎

We'll study the following result later, giving "local existence of harmonic conjugates".

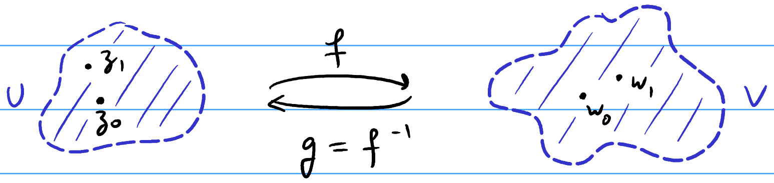
Thm: If $D = D_r(z_0) \subset \mathbb{C}$ is an open disk and $u: D \to \mathbb{R}$ is C^2 and harmonic (i.e. $\Delta u = 0$), then u has a harmonic conjugate $v: D \to \mathbb{R}$ (so that $f = u + iv$ is holomorphic).
Actually, it is enough for D to be a domain that is simply connected (intuitively: "D has no holes").

Lecture 5 01-30

Inverse functions

Thm: Let $f : U \to V$ be a continuous bijective map, for $U, V \subset \mathbb{C}$ open sets. If f is differentiable at z_0, with $f'(z_0) \neq 0$, then $f^{-1} : V \to U$ is differentiable at $w_0 = f(z_0)$, with $(f^{-1})'(w_0) = \dfrac{1}{f'(z_0)}$ $\left(= \dfrac{1}{f'(g(w_0))} \right)$.

Proof: Since f is continuous, so is $g = f^{-1}$.



Let $w_1 \in V$, with $w_1 \neq w_0$. Since f is a bijection, there is a unique $z_1 \in U$ s.t. $f(z_1) = w_1$. Note that $g(w_0) = z_0$ and $g(w_1) = z_1$

$$\Rightarrow \frac{g(w_1) - g(w_0)}{w_1 - w_0} = \frac{z_1 - z_0}{f(z_1) - f(z_0)} = \frac{1}{\frac{f(z_1) - f(z_0)}{z_1 - z_0}} \xrightarrow[\boxed{z_1 \to z_0}]{w_1 \to w_0} \frac{1}{f'(z_0)} . \qquad \blacksquare$$

numerator and denominator both $\neq 0$

g continuous

Ex: Let $U \subset \mathbb{C}$ be s.t. e^z is injective on U. Let $V = \{ e^z : z \in U \}$. Then, there is a branch of $\log(z)$ defined on V. Denote it by $g(z)$. By the theorem above, and using the fact that $\frac{d}{dz}(e^z) = e^z$, gives

$$g'(z) = \frac{1}{e^{g(z)}} = \frac{1}{z} .$$

therefore, we write $\dfrac{d}{dz}(\log(z)) = \dfrac{1}{z}$ (true for <u>every branch</u> of $\log(z)$).

Important note: Let $f : U \to \mathbb{C}$, for some $U \subset \mathbb{C}$ open.
Identifying $\mathbb{C}$ with $\mathbb{R}^2$, and writing $f = u + iv$, we get a map

$$\vec{F} : U \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (u(x,y), v(x,y))$$

If f is holomorphic in U and u, v are C^1, then $\vec{F}$ is <u>differentiable</u> as a map $U \to \mathbb{R}^2$, with <u>Jacobian matrix</u>

$$\vec{F}'(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} (x,y) \underset{\text{eqs}}{\overset{\text{CR}}{=}} \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} (x,y)$$

The Jacobian is $\det \vec{F}'(x,y) = (u_x(x,y))^2 + (u_y(x,y))^2 = |f'(z)|^2$.
The Inverse Function Theorem from Vector Calculus now implies the following: if f is holomorphic and u,v are C^1, and if $f'(z_0) \neq 0$ for some $z_0 \in U$, then there is an open set $\tilde{U} \subset U$ containing z_0 s.t. $\vec{F}$ is injective on $\tilde{U}$. Furthermore, if $\tilde{V} = \vec{F}(\tilde{U}) = \{\vec{F}(x,y) : (x,y) \in \tilde{U}\}$, then the inverse $\vec{G} : \tilde{V} \to \tilde{U}$ is differentiable, with

$$\vec{G}'(x,y) = \underbrace{(\vec{F}')^{-1}(\vec{G}(x,y))}_{= \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}} = \frac{1}{\det \vec{F}'} \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} = \begin{bmatrix} \overset{= r_x}{\frac{u_x}{|f'|^2}} & \overset{= r_y}{-\frac{u_y}{|f'|^2}} \\ \underset{= s_x}{\frac{u_y}{|f'|^2}} & \underset{= s_y}{\frac{u_x}{|f'|^2}} \end{bmatrix} \quad (*)$$

Writing $\vec{G}(x,y) = (r(x,y), s(x,y))$, and defining a complex function $g : \tilde{V} \to \tilde{U}$ by $g(x+iy) = r(x,y) + i\, s(x,y)$, $(*)$ above implies that g satisfies the CR-equations, and that r and s are C^1 (because u and v are C^1). Hence, g is differentiable and

$$\boxed{g'(z) = r_x - i\, r_y = \frac{u_x}{|f'|^2} + i \frac{u_y}{|f'|^2} = \frac{\cancel{u_x + i u_y}}{(\cancel{u_x + i u_y})(u_x - i u_y)} = \frac{1}{f'(g(z))}}.$$

This implies:

Thm: Let $f : U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open, with $f = u + iv$ and u, v C^1. Then, if $f'(z_0) \neq 0$ for some $z_0 \in U$, there is an open set $\tilde{U} \subset U$ containing z_0 s.t.:

a) f is injective on $\tilde{U}$. Write $\tilde{V} = f(\tilde{U})$.

b) the inverse $g : \tilde{V} \to \tilde{U}$ of f is holomorphic, with
$$g'(z) = \frac{1}{f'(g(z))} \quad \text{for all } z \in \tilde{V}$$

c) if $g = r + is$, then r and s are C^1 on $\tilde{V}$.

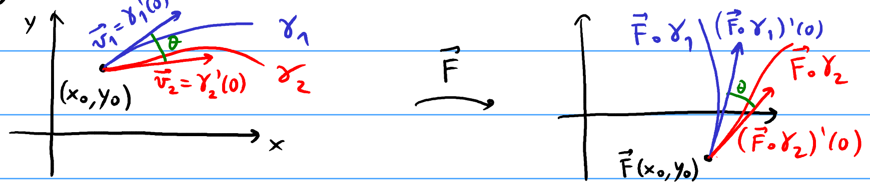
Conformal mappings

As mentioned before, $f = u + iv : U \to \mathbb{C}$, $U \subset \mathbb{C}$ open, gives $\vec{F} : U \to \mathbb{R}^2$ by thinking of $U \subset \mathbb{R}^2 \cong \mathbb{C}$ and taking
$$\vec{F}(x,y) = (u(x,y), v(x,y)).$$

Def: If $\vec{F} = (u,v) : U \to \mathbb{R}^2$ and u and v are C^1, then

- $\vec{F}$ is <u>conformal at $(x_0,y_0) \in U$</u> if $\vec{F}$ <u>preserves angles</u> at (x_0,y_0).

  this means that, given two differentiable paths
  $$\gamma_1, \gamma_2 : [0,1] \to U \quad \text{with} \quad \gamma_1(0) = (x_0,y_0) = \gamma_2(0),$$
  the <u>angle</u> between $\gamma_1'(0)$ and $\gamma_2'(0)$ is the same as the angle between $(\vec{F}\circ\gamma_1)'(0)$ and $(\vec{F}\circ\gamma_2)'(0)$



- $f = u + iv$ is <u>conformal</u> at $x_0 + iy_0$ if $\vec{F} = (u,v)$ is conformal at (x_0,y_0).
- $\vec{F}$ (and f) is <u>conformal</u> if $\vec{F}$ is conformal at <u>every</u> $(x_0,y_0) \in U$.

Lemma: $\vec{F}$ is conformal at (x_0,y_0) iff the matrix $A = \vec{F}'(x_0,y_0)$ is s.t. the angle between $\vec{v}_1$ and $\vec{v}_2$ is the same as the angle between $A\vec{v}_1$ and $A\vec{v}_2$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.

Proof: As in the discussion of the Inverse Function theorem above, since u and v are C^1 we have that $\vec{F} = (u,v)$ is differentiable, with Jacobian matrix $\vec{F}' = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$.

Given a differentiable path $\gamma : [0,1] \to U$ with $\gamma(0) = (x_0,y_0)$ as in the figure above, the tangent vector given by $\vec{F}\circ\gamma$ is
$$(\vec{F}\circ\gamma)'(0) \overset{\text{chain rule}}{=} \underbrace{\vec{F}'(x_0,y_0)}_{A} \cdot \gamma'(0).$$
Therefore, $\vec{F}$ is conformal at (x_0,y_0) iff the angle between any two vectors $\vec{v}_1 = \gamma_1'(0)$ and $\vec{v}_2 = \gamma_2'(0)$ is the same as the angle between $(\vec{F}\circ\gamma_1)'(0) = A\vec{v}_1$ and $(\vec{F}\circ\gamma_2)'(0) = A\vec{v}_2$, as wanted. ∎

later: follows from f holo.

Thm: If $f : U \to \mathbb{C}$ is holomorphic $\boxed{\text{and } u \text{ and } v \text{ are } C^1}$, then $f'(z_0) \neq 0 \Rightarrow f$ is conformal at z_0.

Proof: Applying the previous lemma, we need to check that if $\vec{F}(x,y) = (u(x,y), v(x,y))$ then $\vec{F}'(x_0,y_0)$ preserves angles. Since f is holomorphic, we can write

$f'(z_0) = R e^{i\alpha} \neq 0$, for some $R>0$ and $\alpha \in \mathbb{R}$.

$\Rightarrow f'(z_0) = R\cos\alpha + i R \sin\alpha$

$= u_x + i v_x \overset{CR}{=} v_y - i u_y$

$\Rightarrow \vec{F}'(x_0,y_0) = \begin{bmatrix} R\cos\alpha & -R\sin\alpha \\ R\sin\alpha & R\cos\alpha \end{bmatrix} = R \cdot \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{\text{rotation matrix by angle } \alpha \text{ in } \mathbb{R}^2}$

Since rotation matrices preserve angles, so does the scaled matrix $R\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. This finishes the proof. ∎

Ex: Recall the visualizations of the holomorphic functions e^z, $\mathrm{Log}(z)$, z^2 from before. Lines and curves meeting at $90°$ angles were mapped to lines and curves meeting at $90°$ angles. This is due to these functions being holomorphic, hence conformal where their derivative does not vanish.

Note: In Gamelin's book, a map $f: D \to V$ is conformal if f is conformal at every $z \in D$ and f is a bijection. We do not impose the second condition.

Lecture 6 01-31

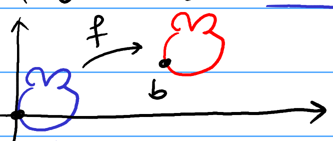
Fractional linear transformations (a.k.a. Möbius transformations)

Goal: Study functions given by

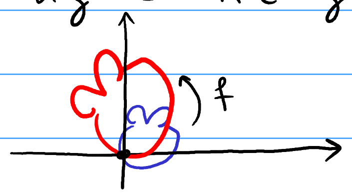
$$\boxed{f(z) = \frac{az+b}{cz+d}}, \quad \text{for } a,b,c,d \in \mathbb{C} \text{ constants.}$$

Why: they are nice examples of holomorphic functions, with interesting geometric properties.

Ex: • $f_1(z) = z + b$ is a translation by $b \in \mathbb{C}$

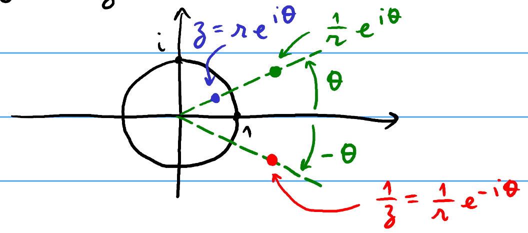


• $f_2(z) = az \overset{a = Re^{i\alpha}}{=} Re^{i\alpha} z$ is a scaling (by R) + rotation (by α).



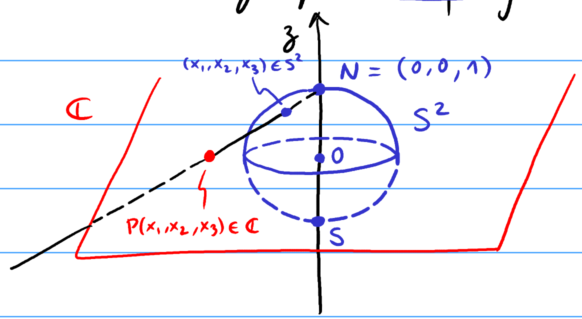
$(f_1 \circ f_2)(z) = az + b$ is an affine map

- $f_3(z) = \frac{1}{z} \overset{z = re^{i\theta}}{=} \frac{1}{r} e^{-i\theta}$ is <u>inversion</u>.



Def: The <u>extended complex plane</u> is $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
"point at infinity"

Can identify $\hat{\mathbb{C}}$ with the sphere
$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$
via <u>stereographic projection</u>:



$$P : S^2 \setminus N \longrightarrow \mathbb{C}$$
$$(x_1, x_2, x_3) \longmapsto \frac{x_1 + i x_2}{1 - x_3}$$

Extend this to the <u>bijection</u>
$$\hat{P} : S^2 \to \hat{\mathbb{C}}$$
$$N \longmapsto \infty$$

$\hat{\mathbb{C}}$ is also called <u>Riemann sphere</u>.

Def: A <u>fractional linear transformation</u>, or <u>Möbius transformation</u> is a holomorphic map $T : U \to \mathbb{C}$, $U \subset \mathbb{C}$ open, given by
$$\boxed{T(z) = \frac{az + b}{cz + d}}, \quad \text{for } a, b, c, d \in \mathbb{C} \text{ constants s.t. } \boxed{ad - bc \neq 0}.$$

Note: $U = \begin{cases} \mathbb{C} & , \text{ if } c = 0 \\ \mathbb{C} \setminus \{-\frac{d}{c}\}, & \text{if } c \neq 0 \end{cases}$

where denominator is zero

<u>Useful intuition</u>: given any $\alpha \in \mathbb{C} \setminus \{0\}$, $\frac{az+b}{cz+d} = \frac{\alpha a z + \alpha b}{\alpha c z + \alpha d}$.
Hence, $\frac{az+b}{cz+d}$ "only has 3 independent complex parameters" (not 4).

<u>Proposition</u>: Every Möbius transformation extends to a <u>bijection</u> $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
injective and surjective

<u>Proof</u>: If $c=0$, then T is affine and hence $T: \mathbb{C} \to \mathbb{C}$ is a bijection. Extend T by defining $T(\infty) = \infty$.

If $c \neq 0$, we can find the inverse of T by:

$$\frac{az+b}{cz+d} = w \iff czw + dw = az + b \iff cwz - az = -dw + b$$

$$\iff z = \frac{-dw+b}{cw-a} \qquad \text{(have all these equivalences if } z \neq -\tfrac{d}{c}\text{)}$$

So, $T^{-1}(z) = \dfrac{-dz+b}{cz-a}$, which is also a Möbius Transformation.

The domain of T^{-1} is $\mathbb{C} \setminus \{\frac{a}{c}\}$. Have a bijection

$$\mathbb{C} \setminus \{-\tfrac{d}{c}\} \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \mathbb{C} \setminus \{\tfrac{a}{c}\}$$

Extend T to $\hat{\mathbb{C}}$ by defining $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$. ∎

Lecture 7   02-02

<u>Thm</u>: Compositions of Möbius transformations are Möbius transfs. Moreover, the set of Möbius transformations forms a <u>group</u> G:

a) $T_1, T_2, T_3 \in G \Rightarrow T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$ (associativity)

b) $I(z) = z$ is in G and $T \in G \Rightarrow T \circ I = T = I \circ T$ (identity)

c) $T \in G \Rightarrow$ there is $T^{-1} \in G$ s.t. $T \circ T^{-1} = I = T^{-1} \circ T$ (inverses)

<u>Proof</u>: <u>Composition</u>: let $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ and $T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$.

$$(T_2 \circ T_1)(z) = \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \cdots = \frac{(a_2 a_1 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1) z + (c_2 b_1 + d_2 d_1)}.$$

This is also a Möbius transformation.

<u>Important observation</u>: composition of Möbius transformations behaves like <u>matrix multiplication</u>:

$$\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{bmatrix}$$

<u>Group axioms</u>:

a) True in general for composition of functions

b) True in general for functions

c) Saw in previous proof that if $T(z) = \dfrac{az+b}{cz+d}$, then $T^{-1}(z) = \dfrac{-dz+b}{cz-a}$, which is also a Möbius transformation.

Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightsquigarrow \dfrac{\frac{d}{\cancel{ad-bc}}z - \frac{b}{\cancel{ad-bc}}}{\frac{-c}{\cancel{ad-bc}}z + \frac{a}{\cancel{ad-bc}}} = \frac{-dz+b}{cz-a}$, compatibly w/ composition of Möbius transfs. behaving like product of matrices. ∎

Note: this group G is sometimes called Möbius group and denoted $PGL(2,\mathbb{C})$ ("projective general linear groups of 2x2 matrices over $\mathbb{C}$").

Thm: Every Möbius transformation can be obtained by composing the following types of maps:

- affine maps $z \mapsto az+b$,
- inversion $z \mapsto \frac{1}{z}$.

Proof: Let $T(z) = \frac{az+b}{cz+d}$. If $c=0$, then T is affine.
If $c \neq 0$, then $T(z) \overset{\text{calculation}}{=} \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}} = (T_3 \circ T_2 \circ T_1)(z)$, with

$$\underset{\text{affine}}{\underset{\uparrow}{T_1(z) = z + \frac{d}{c}}}, \quad \underset{\text{inversion}}{\underset{\uparrow}{T_2(z) = \frac{1}{z}}}, \quad \underset{\text{affine}}{\underset{\uparrow}{T_3(z) = -\frac{ad-bc}{c^2}z + \frac{a}{c}}}.$$

∎

Thm: Let T be a Möbius transformation. Given any line or circle in $\mathbb{C}$, its image under T is a line or a circle.

Proof: Since T is a composition of affine maps and inversions, we only need to check for those types of maps.
Clearly, affine maps take lines to lines and circles to circles.
Need: Show that inversion $T(z) = \frac{1}{z} = \frac{1}{x+iy} = \overset{u=}{\frac{x}{x^2+y^2}} \underbrace{- i\frac{y}{x^2+y^2}}_{=iv}$ takes lines and circles to lines and circles.

$$u^2+y^2 = \frac{1}{x^2+y^2}$$

We use the fact that lines and circles in $\mathbb{R}^2$ are the solution sets of equations of the form

(∗) $A(x^2+y^2) + Cx + Dy + E = 0$, for $A, C, D, E \in \mathbb{R}$ constants

(If $A=0$, get the equation of a line; if $A \neq 0$, complete the squares to get a circle.)

$$(*) \iff A + C\frac{x}{x^2+y^2} + D\frac{y}{x^2+y^2} + \frac{E}{x^2+y^2} = 0$$

$$\iff E(u^2+v^2) + Cu - Dv + A = 0 \quad (**).$$

Conclusion: When we apply T to the points that solve $(*)$ (which form a circle or a line), we get the points that solve $(**)$ (which also form a circle or a line). $\square$

Ex: The Cayley map is the Möbius transformation

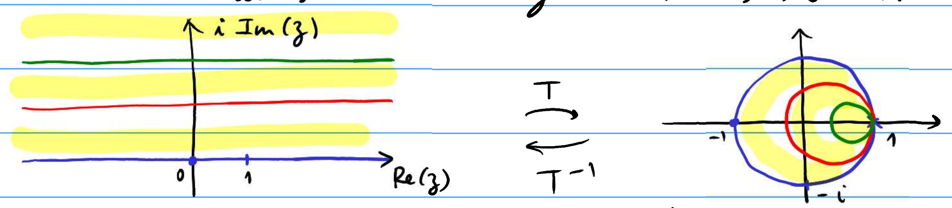
$$T(z) = \frac{z-i}{z+i}.$$

Visualization: The image of the line $y=0$ is a line or a circle. It suffices to find 3 points in it:

$$T(0) = -1, \quad T(\infty) = 1, \quad T(1) = \frac{1-i}{1+i} = \frac{1-i}{\sqrt{2}} \Big/ \frac{1+i}{\sqrt{2}} = \frac{e^{-i\frac{\pi}{4}}}{e^{i\frac{\pi}{4}}} = e^{-i\frac{\pi}{2}} = -i$$

$\Rightarrow$ T takes the line $y=0$ to the unit circle.

It also takes other horizontal lines to circles containing 1:



The inverse of T is $T^{-1}(z) = \frac{z+1}{iz-i}$:

$$w = \frac{z-i}{z+i} \iff wz + iw = z - i \iff z = \frac{-i-iw}{w-1} = \frac{w+1}{iw-i}$$

Thm: Let z_0, z_1, z_2 be a collection of different points in $\hat{\mathbb{C}}$ and let w_0, w_1, w_2 also be a collection of different points in $\hat{\mathbb{C}}$. There exists a unique Möbius transformation T s.t.

$$T(z_0) = w_0, \quad T(z_1) = w_1 \quad \text{and} \quad T(z_2) = w_2.$$

Proof: • Existence: if $w_0 = 0$, $w_1 = 1$ and $w_2 = \infty$, take

$$\boxed{T_1(z) = \frac{z-z_0}{z-z_2}\,\frac{z_1-z_2}{z_1-z_0}}$$

(this is called the "cross ratio" of z, z_0, z_1, z_2).

For other values of w_0, w_1, w_2, take the corresponding

$$T_2(z) = \frac{z-w_0}{z-w_2}\,\frac{w_1-w_2}{w_1-w_0}.$$

Then, $\boxed{T_2^{-1} \circ T_1}$ is the desired Möbius transformation:

$$T_2^{-1}(T_1(z_0)) = T_2^{-1}(0) = w_0, \quad T_2^{-1}(T_1(z_1)) = T_2^{-1}(1) = w_1, \quad T_2^{-1}(T_1(z_2)) = T_2^{-1}(\infty) = w_2.$$

- <u>Uniqueness</u>: if $z_0 = w_0 = 0$, $z_1 = w_1 = 1$ and $z_2 = w_2 = \infty$, then there is a unique $T(z) = \frac{az+b}{cz+d}$:

$$T(\infty) = \infty \Rightarrow c = 0 \Rightarrow T(z) = \frac{a}{d} z + \frac{b}{d}.$$
$$T(0) = 0 \Rightarrow \frac{b}{d} = 0 \Rightarrow T(z) = \frac{a}{d} z$$
$$T(1) = 1 \Rightarrow \frac{a}{d} = 1 \Rightarrow \boxed{T(z) = z}.$$

For other values of the z_i and w_i: suppose that there are Möbius transformations T_1 and T_2 that take

z_0 to w_0, z_1 to w_1 and z_2 to w_2.

<u>Want</u>: To show that $T_1 = T_2$.

Let T_3 be a Möbius transformation such that

$T_3(z_0) = 0$, $T_3(z_1) = 1$, $T_3(z_2) = \infty$ and

(there is such a Möbius transformation, by the proof of existance.)

then, $T(z) = (T_3 \circ T_2^{-1} \circ T_1 \circ T_3^{-1})(z)$ is such that

$$T(0) = 0, \quad T(1) = 1 \quad \text{and} \quad T(\infty) = \infty.$$

By the uniqueness above, we get

$$T(z) = z \Rightarrow (T_2^{-1} \circ T_1)(z) = (T_3^{-1} \circ T_3)(z) = z \Rightarrow T_1 = T_2. \quad \blacksquare$$

<u>Note</u>: Recall the intuition that $\frac{az+b}{cz+d}$ "only has 3 independent parameters". This is compatible with the previous thm, that can be summarized by saying that $\frac{az+b}{cz+d}$ "is uniquely determined by the images of 3 different points".

<u>Note</u>: To find an explicit formula for T in the previous theorem, note that in the proof we had

$$T = T_2^{-1} \circ T_1, \text{ with } T_1(z) = \frac{z - z_0}{z - z_2} \frac{z_1 - z_2}{z_1 - z_0} \text{ and } T_2(z) = \frac{z - w_0}{z - w_2} \frac{w_1 - w_2}{w_1 - w_0}.$$

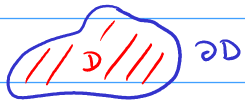
Hence, writing $w = T(z) = (T_2^{-1} \circ T_1)(z)$, we get $T_2(w) = T_1(z)$

$$\Rightarrow \boxed{\frac{w - w_0}{w - w_2} \frac{w_1 - w_2}{w_1 - w_0} = \frac{z - z_0}{z - z_2} \frac{z_1 - z_2}{z_1 - z_0}}$$

We can get $w = T(z)$ by solving the equation above for w.

Lecture 8 02-07

The Dirichlet problem. Finding solutions with conformal mappings



$\bar{D} = D \cup \partial D$

Suppose that
- $D \subset \mathbb{C}$ is a domain (open and connected), $\bar{D}$ = closure of D $(= D \cup \partial D)$
- $\tilde{\phi} : S \to \mathbb{C}$ continuous, for some subset $S \subset \partial D$

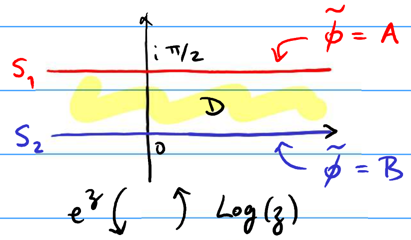
  Usually: $S = \partial D$ or the closure $\bar{S} = \partial D$

Want: Find $\phi : D \cup S \to \mathbb{C}$ continuous st
- ϕ is harmonic on D (so, ϕ is C^2 and $\Delta\phi = 0$)
- ϕ extends $\tilde{\phi}$ (so, $\phi(z) = \tilde{\phi}(z)$ for all $z \in S$)

Such ϕ is said to solve the Dirichlet problem with boundary condition given by $\tilde{\phi}$ on S.

Goal: use holomorphic functions to solve Dirichlet problems.
Recall: if $f = u + iv$ and u, v are C^2, then

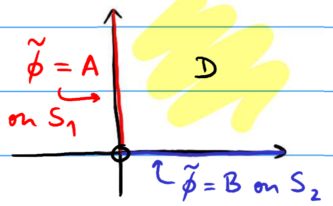
$$\Delta u = \Delta v = \Delta f = 0.$$

Exs:
- 

  $\partial D = S_1 \cup S_2 = S$. Can take

  $$\phi(x,y) = \frac{2}{\pi}(A-B)\,y + B$$

  $z = x+iy$: $= \operatorname{Im}\Big(\underbrace{\tfrac{2}{\pi}(A-B)\,z + iB}_{\text{a holomorphic function}}\Big)$

- 

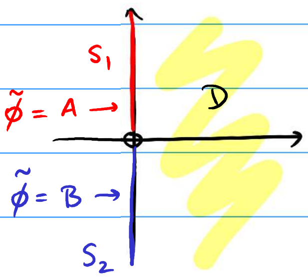
  $\partial D = S_1 \cup S_2 \cup \{0\} = S \cup \{0\}$.

  take $\phi(x,y) = \operatorname{Im}\Big(\overbrace{\tfrac{2}{\pi}(A-B)\,\mathrm{Log}(z) + iB}^{\text{holomorphic}}\Big)$

  $$= \frac{2}{\pi}(A-B)\,\mathrm{Arg}(z) + B$$

  $$= \frac{2}{\pi}(A-B)\arctan\left(\frac{y}{x}\right) + B$$

- Similarly,

  $\partial D = S_1 \cup S_2 \cup \{0\} = S \cup \{0\}$

  Take $\phi(x,y) = \frac{A-B}{\pi}\,\mathrm{Arg}(z) + \frac{A+B}{2}$

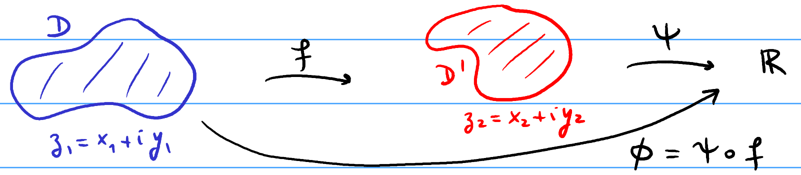
  $$= \frac{A-B}{\pi}\arctan\left(\frac{y}{x}\right) + \frac{A+B}{2}$$

One can sometimes use <u>composition</u> of functions to solve Dirichlet problems:

<u>Thm</u>: If $f = u + iv : D \to D'$ is holomorphic for domains $D, D' \subset \mathbb{C}$, $\boxed{\text{if } u \text{ and } v \text{ are } C^2}$ and $\psi : D' \to \mathbb{R}$ is harmonic, then

later: follows from f holo

$$\phi(x,y) = \psi(u(x,y), v(x,y)) \quad \text{is harmonic in } D.$$



<u>Proof 1</u>: <u>Calculation</u> using $\underbrace{\text{CR}}_{\text{chain rule}}$ and $\underbrace{\text{CR}}_{\text{Cauchy-Riemann eqs.}}$ (not very illuminating):

$$\Delta\phi = \partial^2_{x_1x_1}\psi + \partial^2_{y_1y_1}\psi = \frac{\partial}{\partial x_1}\left(\frac{\partial\psi}{\partial x_2}u_{x_1} + \frac{\partial\psi}{\partial y_2}v_{x_1}\right) + \frac{\partial}{\partial y_1}\left(\frac{\partial\psi}{\partial x_2}u_{y_1} + \frac{\partial\psi}{\partial y_2}v_{y_1}\right)$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial\psi}{\partial x_2}\right)u_{x_1} + \frac{\partial\psi}{\partial x_2}u_{x_1x_1} + \frac{\partial}{\partial x_1}\left(\frac{\partial\psi}{\partial y_2}\right)v_{x_1} + \frac{\partial\psi}{\partial y_2}v_{x_1x_1} +$$

$$+ \frac{\partial}{\partial y_1}\left(\frac{\partial\psi}{\partial x_2}\right)u_{y_1} + \frac{\partial\psi}{\partial x_2}u_{y_1y_1} + \frac{\partial}{\partial y_1}\left(\frac{\partial\psi}{\partial y_2}\right)v_{y_1} + \frac{\partial\psi}{\partial y_2}v_{y_1y_1}$$

$$= \frac{\partial^2\psi}{\partial x_2^2}u_{x_1}^2 + \frac{\partial^2\psi}{\partial y_2\partial x_2}v_{x_1}u_{x_1} + \frac{\partial\psi}{\partial x_2}u_{x_1x_1} + \frac{\partial^2\psi}{\partial x_2\partial y_2}u_{x_1}v_{x_1} + \frac{\partial^2\psi}{\partial y_2^2}v_{x_1}^2 + \frac{\partial\psi}{\partial y_2}v_{x_1x_1}$$

$$\frac{\partial^2\psi}{\partial x_2^2}u_{y_1}^2 + \frac{\partial^2\psi}{\partial y_2\partial x_2}v_{y_1}u_{y_1} + \frac{\partial\psi}{\partial x_2}u_{y_1y_1} + \frac{\partial^2\psi}{\partial x_2\partial y_2}u_{y_1}v_{y_1} + \frac{\partial^2\psi}{\partial y_2^2}v_{y_1}^2 + \frac{\partial\psi}{\partial y_2}v_{y_1y_1}$$

+ cancel, by CR   + cancel   + cancel

+ cancel $v_{x_1} = -u_{y_1}$

+ cancel $v_{y_1} = u_{x_1}$

<u>Proof 2</u>: Nice <u>trick</u>: It is enough to prove the result for every open disk $B \subset D$. We'll see:

<u>Thm</u>: If $u : B \to \mathbb{R}$ is harmonic on an open disk $B \subset \mathbb{C}$, then there exists a <u>harmonic conjugate</u> $v : B \to \mathbb{R}$ (so that $u + iv$ is holomorphic).

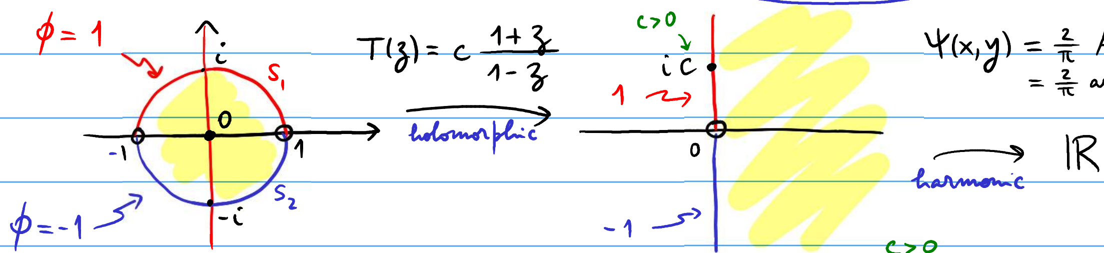
Apply this Thm to ψ, and call its harmonic conjugate $\tilde{\psi}$. Then, $g = \psi + i\tilde{\psi}$ is holomorphic on B. Therefore,

$$\phi = \psi \circ f = \mathrm{Re}(\underbrace{g \circ f}_{\text{holomorphic}}) \quad \text{is harmonic}$$

∎

Ex : • Find ϕ harmonic on the unit disk $D = \{|z|<1\}$

s.t. $\phi(z) = \begin{cases} 1, & \text{if } z \in \partial D \cap \{Re(z)>0\} \leftarrow S_1 \\ -1, & \text{if } z \in \partial D \cap \{Re(z)<0\} \leftarrow S_2 \end{cases}$



$\Psi(x,y) = \frac{2}{\pi} Arg(z)$
$= \frac{2}{\pi} \arctan\left(\frac{y}{x}\right)$

(To find T: it satisfies $T(-1)=0$, $T(i) = ic$ ($c>0$), $T(1)=\infty$, so

$$\frac{z+1}{z-1}\,\frac{i-1}{i+1} = \frac{w-0}{\cancel{w-\infty}}\,\frac{\cancel{ic-\infty}}{ic-0} = \frac{w}{ic} \Leftrightarrow w = \frac{z+1}{z-1}\, ic\, \frac{i-1}{i+1} = -c\,\frac{z+1}{z-1} = c\,\frac{1+z}{1-z}\ )$$

Important : Note that T takes S_1 and S_2 to the S_1 and S_2 in previous example. By the theorem, the following solves the Dirichlet problem:

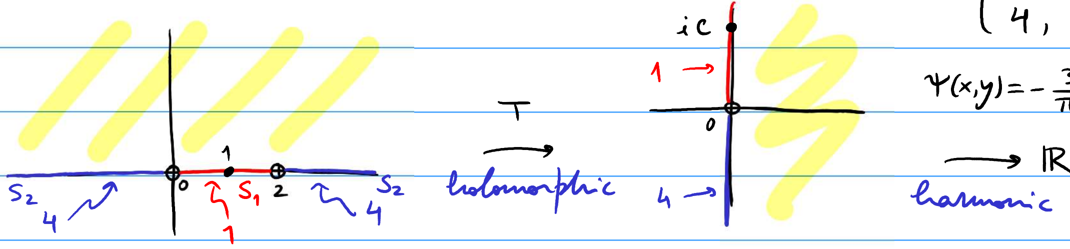
$$\phi(x,y) = \Psi(u(x,y), v(x,y)) = \frac{2}{\pi} Arg\left(c\frac{1+z}{1-z}\right) = \frac{2}{\pi}\arctan\left(\frac{v}{u}\right), \text{ where } c\frac{1+z}{1-z} = u+iv.$$

$$c\,\frac{1+z}{1-z} = c\,\frac{(1+z)(1-\bar z)}{|1-z|^2} = c\,\frac{1-\bar z + z - |z|^2}{|1-z|^2} \overset{z=x+iy}{=} c\,\frac{1+2iy-x^2-y^2}{(1-x)^2+y^2} = \underbrace{c\,\frac{1-x^2-y^2}{(1-x)^2+y^2}}_{u} + i\,\underbrace{c\,\frac{2y}{(1-x)^2+y^2}}_{v}$$

$$\Rightarrow \frac{v}{u} = \frac{2y}{1-x^2-y^2} \quad (\text{note: this is independent of } c).$$

$$\Rightarrow \boxed{\phi(x,y) = \frac{2}{\pi}\arctan\left(\frac{2y}{1-x^2-y^2}\right)}$$

• Find ϕ harmonic on the upper half plane, s.t. $\phi(x) = \begin{cases} 1, & x \in (0,2) \quad S_1 \\ 4, & x \in \mathbb{R}\setminus[0,2] \quad S_2 \end{cases}$



$\Psi(x,y) = -\frac{3}{\pi} Arg(z) + \frac{5}{2}$

Find T: $T(0) \overset{(*)}{=} \infty$, $T(2) \overset{(**)}{=} 0$, $T(1) \overset{(***)}{=} ic$, for some $c>0$

$(*)$ and $(**)$ $\Rightarrow T(z) = \alpha\,\frac{z-2}{z}$, for suitable $\alpha \in \mathbb{C}$

$(***)$ $\Rightarrow T(1) = \alpha\,\frac{1-2}{1} = -\alpha = ic \Rightarrow \alpha = -ic \Rightarrow T(z) = -ic\,\frac{z-2}{z}$

In a manner similar to the previous example, we have a solution to Dirichlet's problem of the form

$$\phi(x,y) = \Psi(u,v) = -\frac{3}{\pi} Arg\big(\underbrace{-ic\,\tfrac{z-2}{z}}_{u+iv}\big) + \frac{5}{2} = -\frac{3}{\pi}\arctan\left(\frac{v}{u}\right) + \frac{5}{2}$$

$$-ic\,\frac{z-2}{z} = -ic\,\frac{(z-2)\bar z}{|z|^2} = -ic\,\frac{|z|^2 - 2\bar z}{|z|^2} = -ic\,\frac{|z|^2 - 2x + 2iy}{|z|^2} =$$

$$= \underbrace{c\,\frac{2y}{x^2+y^2}}_{u} + i\,\underbrace{c\,\frac{2x-x^2-y^2}{x^2+y^2}}_{v} \Rightarrow \frac{v}{u} = \frac{2x-x^2-y^2}{2y}$$

$$\Rightarrow \boxed{\phi(x,y) = -\frac{3}{\pi}\arctan\left(\frac{2x-x^2-y^2}{2y}\right) + \frac{5}{2}}$$

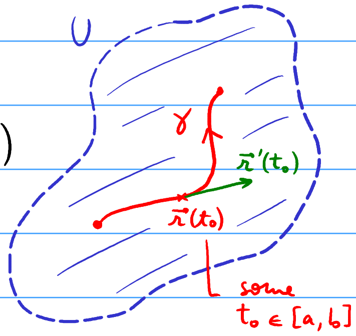
Lecture 9 02-12

Complex line integrals

Some types of integrals

- $f = u + iv : [a,b] \longrightarrow \mathbb{R}$ continuous function
  $$\rightsquigarrow \int_a^b f(x)\,dx = \underbrace{\int_a^b u(x)\,dx}_{\in \mathbb{R}} + i \underbrace{\int_a^b v(x)\,dx}_{\in \mathbb{R}} \in \mathbb{C}$$

- $\vec{F} : U \to \mathbb{R}^2$ continuous, $U \subset \mathbb{R}^2$ open and $\gamma \subset U$ curve parametrized by $\vec{r} : [a,b] \to U$ (the parametrization induces an orientation on U)
  the line integral of $\vec{F}$ along γ is
  $$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \in \mathbb{R}$$
  dot product in $\mathbb{R}^2$



Want: Given $f : U \to \mathbb{C}$ continuous, $U \subset \mathbb{C}$ open and $\gamma \subset U$ curve, define $\int_\gamma f(z)\,dz \in \mathbb{C}$.
this new type of integral will relate nicely to the two types of integrals above.

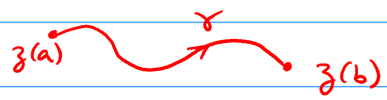
First: describe the curves we will integrate over

Curves in $\mathbb{C}$

Def: a) $\gamma \subset \mathbb{C}$ is a differentiable arc if it is the image of a C^1-function $z : [a,b] \to \mathbb{C}$, such that:

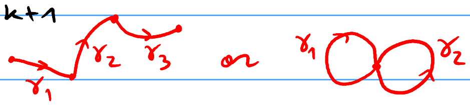
meaning: has continuous derivative

- $z'(t) \neq 0$ for every $t \in [a,b]$
- z is injective on $[a,b)$ (open at b to allow arcs to "close up")

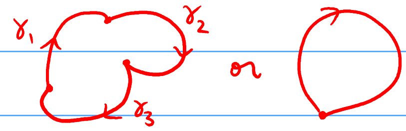


Call the function z a parametrization of γ. Note: z orients γ.

b) $\gamma \subset \mathbb{C}$ is a piecewise differentiable arc if it is a union of finitely many differentiable arcs $\gamma_1, \dots, \gamma_n$ with
terminal point of γ_k = initial point of γ_{k+1}
for all $1 \leq k < n$



c) $\gamma \subset \mathbb{C}$ is a <u>closed piecewise differentiable arc</u>
if $\gamma = \gamma_1 \cup \ldots \cup \gamma_m$ and
terminal point of γ_m = initial point of γ_1.



If these are the only intersections between the γ_i, say that γ is <u>simple</u>.

d) $\gamma \subset \mathbb{C}$ is a <u>curve</u> if it is a union of finitely many diff. arcs.

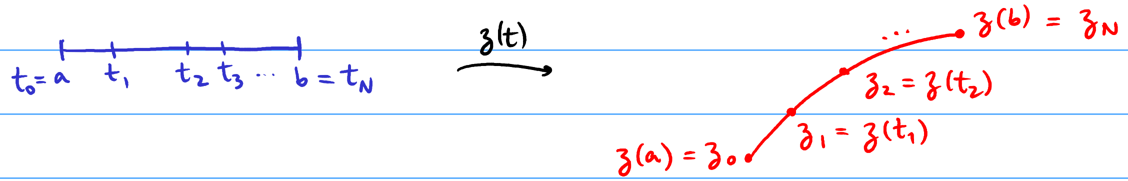
e) $\gamma \subset \mathbb{C}$ is a <u>closed curve</u> if is a union of finitely many closed piecewise differentiable arcs.

<u>Note</u>: A curve is sometimes also called a <u>contour</u>.

<u>Complex line integrals</u>

Let $f: U \to \mathbb{C}$, $U \subset \mathbb{C}$ open and $\gamma \subset U$ a differentiable arc, parametrized by $z: [a,b] \to U$.

Let $P = \{t_0 = a, t_1, \ldots, t_N = b\}$ be a set of increasing numbers in $[a,b]$ (call it a <u>partition</u> of $[a,b]$). Write $z_k = z(t_k)$.



Define $\mu(P) = \max\limits_{1 \leq k \leq N} |t_k - t_{k-1}|$.

Intuition: when $\mu(P)$ is small, t_k is close to t_{k-1} for all $1 \leq k \leq N$.

Also, $|z_k - z_{k-1}| = \left| \int_{t_k}^{t_{k+1}} z'(t)\, dt \right| \leq \int_{t_{k-1}}^{t_k} |z'(t)|\, dt \leq C.(t_k - t_{k-1})$,

where $C = \max\limits_{t \in [a,b]} \{|z'(t)|\} < \infty$. Hence z_k is also close to z_{k-1}.

The <u>Riemann sum</u> $\sum\limits_{k=1}^{N} f(z_k) \cdot \underbrace{(z_k - z_{k-1})}_{\Delta z_k}$ should give

product in $\mathbb{C}$

a good approximation of the integral $\int_\gamma f(z)\, dz$, when $\mu(P)$ is small. This motivates the following.

<u>Def</u>: Let $f: U \to \mathbb{C}$ be a function on $U \subset \mathbb{C}$ open and let $\gamma \subset U$ be a differentiable arc, parametrized by $z: [a,b] \to U$.

Say that $L \in \mathbb{C}$ is the <u>integral of f along γ</u> if

$$\forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{P \text{ partition of } [a,b]} \quad \mu(P) < \delta \Rightarrow \left| L - \sum_{k=1}^{N} f(z_k) \cdot \Delta z_k \right| < \varepsilon.$$

Intuitively: " $\lim_{\mu(P)\to 0} \sum_{k=1}^{N} f(z_k)\cdot \Delta z_k = L$ ".

If the limit L exists, denote it by $\int_\gamma f(z)\,dz = L$.

Given a curve $\gamma = \gamma_1 \cup \ldots \cup \gamma_m$, let

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z)\,dz .$$

(Technical note: the definition of $\int_\gamma f(z)\,dz$ above would be equivalent if we had defined $\mu(P) = \max_{1\le k\le N} \underbrace{\int_{t_{k-1}}^{t_k} |z'(t)|\,dt}_{\text{length of the piece of } \gamma \text{ between } z_{k-1} \text{ and } z_k}$.)

<u>Important properties</u>:

- If we take two different parametrizations for γ (inducing the same orientation), then they give the same value for $\int_\gamma f(z)\,dz$ (the integral is <u>independent of parametrization</u>).
- If f is <u>continuous</u>, then $\int_\gamma f(z)\,dz$ exists.
- $\int_\gamma a f(z) + b g(z)\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz$ (<u>linearity</u>)

<u>Calculating integrals</u>

<u>Thm</u>: If $f: U \to \mathbb{C}$ is continuous on $U \subset \mathbb{C}$ open and $\gamma \subset U$ is a differentiable arc parametrized by $z:[a,b]\to U$, then

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(z(t))\cdot z'(t)\,dt}$$

<u>Note</u>: The right side is the integral of $g(t) = f(z(t))\cdot z'(t) : [a,b] \to \mathbb{C}$, so that expression is a type of integral we knew before.

<u>Proof idea</u>: Let P be a partition of $[a,b]$ with $\mu(P)$ small. Then,

$$\int_\gamma f(z)\,dz \approx \sum_{k=1}^{N} f(z_k)\cdot(z_k - z_{k-1}) =$$

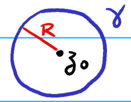
because $\mu(P)$ is small (in the def. of $\int_\gamma f(z)\,dz$)

$$= \sum_{k=1}^{N} f(z(t_k))\cdot \underbrace{\frac{z(t_k)-z(t_{k-1})}{t_k - t_{k-1}}}_{\approx z'(t_k)} \cdot \underbrace{(t_k - t_{k-1})}_{\Delta t}$$

because $\mu(P)$ is small (in the definition of $\int_a^b g(t)\,dt$)

$$\approx \int_a^b f(z(t))\cdot z'(t)\,dt$$ ∎

<u>Ex 1</u>: Fix $z_0 \in \mathbb{C}$ and $R > 0$ and let $\gamma = \{ |z - z_0| = R \} = \partial D(z_0, R)$.



Calculate $\int_\gamma (z - z_0)^m \, dz$, for $m \in \mathbb{Z}$.

Parametrize γ with $z : [0, 2\pi] \to \mathbb{C}$, $z(t) = R e^{it} + z_0 \Rightarrow z'(t) = R i e^{it}$.

$$\Rightarrow \int_\gamma f(z) \, dz = \int_0^{2\pi} f(z(t)) \cdot z'(t) \, dt =$$

$$= \int_0^{2\pi} (R e^{it} + \cancel{z_0} - \cancel{z_0})^m \cdot R i e^{it} \, dt =$$

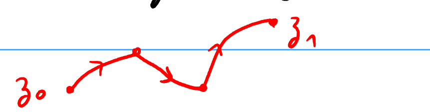
$$= R^{m+1} i \int_0^{2\pi} e^{i(m+1)t} \, dt = \begin{cases} R^{m+1} i \left. \dfrac{e^{i(m+1)t}}{i(m+1)} \right]_0^{2\pi}, & \text{if } m \neq -1 \\ i z \Big]_0^{2\pi}, & \text{if } m = -1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } m \neq -1 \\ 2\pi i, & \text{if } m = -1 \end{cases}.$$

<u>Ex 2</u>: If $\gamma \subset \mathbb{R}$ is the interval $[a, b]$ and we take the parametrization $z : [a, b] \to [a, b]$ with $z(t) = t$, then the theorem gives $\int_\gamma f(z) \, dz = \int_a^b f(t) \, dt$.
This means that the definition of complex line integral generalizes the integrals of the type $\int_a^b f(t) \, dt$.

<u>Fundamental theorem of Calculus</u>: Let $f : U \to \mathbb{C}$ be continuous on $U \subset \mathbb{C}$ open, and let $F : U \to \mathbb{C}$ be such that $F' = f$.
Let $\gamma \subset \mathbb{C}$ be a piecewise differentiable arc starting at z_0 and ending at z_1. Then,

$$\boxed{\int_\gamma f(z) \, dz = F(z_1) - F(z_0)}.$$



<u>Proof</u>: It is enough to consider differentiable arcs. Suppose that γ is parametrized by $z : [a, b] \to U$. Then,

$$\int_\gamma f(z) \, dz \underset{\text{thm above}}{=} \int_a^b \underbrace{f}_{= F'}(z(t)) \cdot z'(t) \, dt \underset{\text{chain rule}}{=} \int_0^1 \frac{d}{dt}(F(z(t))) \, dt \underset{\text{FTC for } g = F \circ z \, : \, [a,b] \to \mathbb{C}}{=} F(z(b)) - F(z(a)).$$

∎

<u>Back to Ex 1</u>: If $m \neq -1$, then $f(z) = (z - z_0)^m = F'(z)$, for $F(z) = \dfrac{(z - z_0)^{m+1}}{m+1}$ on $\mathbb{C} \setminus \{z_0\}$. So, FTC $\Rightarrow \int_\gamma z^m \, dz = F(1) - F(1) = 0$.

<u>Note</u>: If $m = -1$, $f(z) = \dfrac{1}{z - z_0}$, which <u>does not</u> have an antiderivative function on all of $\mathbb{C} \setminus \{z_0\}$.

Lecture 10 02-13

Complex line integrals vs line integrals of vector fields

Let $f = u + iv : U \to \mathbb{C}$ be continuous on $U \subset \mathbb{C}$ and let $\gamma \subset U$ be a differentiable arc parametrized by $z : [a,b] \to U$.

Write $z(t) = x(t) + i\,y(t)$. Then,

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt =$$

$$= \int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big) \cdot \big(x'(t) + i\,y'(t)\big)\,dt =$$

$$= \int_a^b \underbrace{\big((u,-v)(x(t),y(t))\big)}_{\in \mathbb{R}^2} \underset{\text{dot product}}{\cdot} \underbrace{(x'(t),y'(t))}_{\in \mathbb{R}^2}\,dt \; +$$

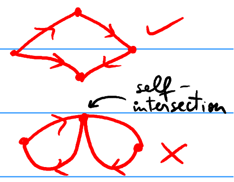
$$+ \; i \int_a^b \underbrace{\big((v,u)(x(t),y(t))\big)}_{\in \mathbb{R}^2} \underset{\text{dot product}}{\cdot} \underbrace{(x'(t),y'(t))}_{\in \mathbb{R}^2}\,dt \; +$$

Define vector fields in U by $\vec{F}_1(x,y) = (u(x,y), -v(x,y))$ and
$\vec{F}_2(x,y) = (v(x,y), u(x,y))$.

Then, the above calculation gives

$$\int_\gamma f(z)\,dz = \underbrace{\int_\gamma \vec{F}_1 \cdot d\vec{r} \; + \; i\int_\gamma \vec{F}_2 \cdot d\vec{r}}_{\text{line integrals of vector fields in } \mathbb{R}^2} \in \mathbb{C}.$$

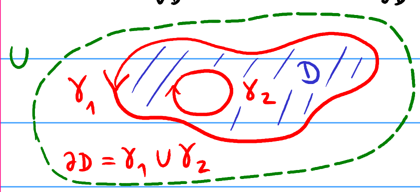
Before recalling Green's theorem, we need one more definition



Def: A bounded domain $D \subset \mathbb{C}$ has nice boundary if ∂D is a disjoint union of simple closed piecewise differentiable arcs (each such arc is a union $\gamma_1 \cup \ldots \cup \gamma_m$ of differentiable arcs that only intersect at initial and terminal points of consecutive arcs).

Green's theorem: If $\vec{F} = (P,Q) : U \to \mathbb{R}^2$ is a continuous vector function on $U \subset \mathbb{R}^2$ open, if $D \subset U$ is a bounded domain with nice boundary $\partial D \subset U$, then

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_{\partial D} P\,dx + Q\,dy = \int_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{= \nabla \times \vec{F}} dx\,dy,$$



where ∂D has the "boundary orientation" (travel along γ keeping D on the left).

The following is one of the most important results in this course.

Cauchy's Integral Theorem I (for boundaries of domains) CTI

Let $f: U \to \mathbb{C}$ be holomorphic $\boxed{\text{and } C^1}$ on $U \subset \mathbb{C}$ open.
Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$.
Then, $\boxed{\int_{\partial D} f(z)\, dz = 0.}$

eventually, we will remove this assumption

Proof: As before, write $f(x+iy) = u(x,y) + i\, v(x,y)$ and define C^1 vector fields $\vec{F}_1, \vec{F}_2 : U \to \mathbb{R}^2$ by

$$\vec{F}_1(x,y) = (u(x,y), -v(x,y)), \quad \vec{F}_2(x,y) = (v(x,y), u(x,y))$$

$$\Rightarrow \nabla \times \vec{F}_1 = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \quad \nabla \times \vec{F}_2 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$

by the CR equations.

Therefore,

$$\int_{\partial D} f(z)\, dz = \int_{\partial D} \vec{F}_1 \cdot d\vec{r} + i \int_{\partial D} \vec{F}_2 \cdot d\vec{r} \overset{\text{Green (can use because } \vec{F}_1, \vec{F}_2 \text{ are } C^1)}{=}$$

$$= \int_D \underbrace{(\nabla \times \vec{F}_1)}_{=0}\, dx\, dy + i \int_D \underbrace{(\nabla \times \vec{F}_2)}_{=0}\, dx\, dy = 0$$ ∎

Ex: The earlier calculation

$$\int_{\{|z-z_0|=R\}} (z-z_0)^n\, dz = 0, \quad \text{if } n \geq 0 \text{ integer}$$

follows immediately, taking $U = \mathbb{C}$, $f(z) = (z-z_0)^n$ and $D = D(z_0, R)$

Soon: we'll see a related calculation of the integral for $n < 0$
(note that $(z-z_0)^n$ is not holomorphic at z_0 if $n < 0$).

Lecture 11 02-21

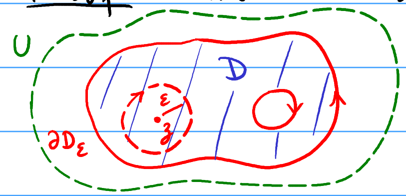
Cauchy's Integral Formula I (for boundaries of domains) CFI

Let $f: U \to \mathbb{C}$ be holomorphic $\boxed{\text{and } C^1}$ on $U \subset \mathbb{C}$ open.
Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$.
Then, for all $z \in D$

eventually, we will remove this assumption

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\, dw}.$$

Proof: Take $\varepsilon > 0$ small so that the disk $D(z,\varepsilon) \subset D$.



Let $D_\varepsilon = D \setminus D(z,\varepsilon)$ and $g(w) = \frac{1}{2\pi i}\frac{f(w)}{w-z}$.

This function is holomorphic and C^1 on D_ε.

By CTI,

oriented as boundary of $D(z,\varepsilon)$, not of D_ε

$$\int_{\partial D_\varepsilon} g(w)\,dw = 0 \iff \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dz = \frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)} \frac{f(w)}{w-z}\,dz.$$

Parametrize $\partial D(z,\varepsilon)$ by $\gamma : [0,1] \to D$ s.t. $\gamma(t) = z + \varepsilon e^{2\pi i t}$.

Then, $\gamma'(t) = 2\pi i \varepsilon e^{2\pi i t}$ and so

$$\frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)} \frac{f(w)}{w-z}\,dt = \frac{1}{\cancel{2\pi i}}\int_0^1 \frac{f(z+\varepsilon e^{2\pi i t})}{\cancel{\varepsilon e^{2\pi i t}}}\,\cancel{2\pi i}\,\cancel{\varepsilon e^{2\pi i t}}\,dt =$$

$$= \int_0^1 f(z+\varepsilon e^{2\pi i t})\,dt$$

Since f is continuous,

the integral is an average of the values of f (continuous) on a circle close to z

$$\lim_{\varepsilon \to 0} \int_0^1 f(z+\varepsilon e^{2\pi i t})\,dt = f(z).$$

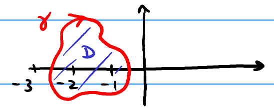
Hence, $$\boxed{\frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dz = \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)} \frac{f(w)}{w-z}\,dz = f(z)}$$ ∎

Exs: • Let $U = \mathbb{C}$, $f(z) = 1$, $D = D(z_0, R)$. Then,

CFI $\Rightarrow \frac{1}{2\pi i}\int_{\partial D} \frac{1}{z-z_0}\,dz = f(z_0) = 1 \Rightarrow \int_{\{|z-z_0|=R\}} (z-z_0)^{-1}\,dz = 2\pi i$,

as we had seen before.

• Calculate $\int_\gamma \frac{e^{4z}}{z(z+2)}\,dz$, where γ is the curve



Let $f(z) = \frac{e^{4z}}{z}$. Let D be the bounded domain s.t. $\gamma = \partial D$.

f is holomorphic on the open set $\mathbb{C}\setminus\{0\}$, which contains $D \cup \gamma$.

CFI $\Rightarrow \int_\gamma \frac{f(z)}{z+2}\,dz = -2\pi i\, f(-2) = -2\pi i \frac{e^{-8}}{-2} = \pi e^{-8} i$.

γ has the orientation opposite to the boundary of D

Applications of the Cauchy integral formula

The CFI has several important consequences:

Thm: (Cauchy integral formula for derivatives):

Let $f : U \to \mathbb{C}$ be holomorphic $\boxed{\text{and } C^1}$ on $U \subset \mathbb{C}$ open.

Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$.

eventually, we will remove this assumption

Then, for all $z \in D$ and all $m \geq 0$

$$\boxed{f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dz}$$

Proof: By induction on m.

Induction base: $m=0$. True by CFI.

Induction hypothesis (IH): assume that the result is true for $m-1$.

Induction step: we'll prove the formula for m: using the limit definition of derivative, let z_0 and $z_0+\Delta z \in D$.

$$\frac{f^{(m-1)}(z_0+\Delta z)-f^{(m-1)}(z_0)}{\Delta z} = \frac{\frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-(z_0+\Delta z))^m}dw - \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z_0)^m}dw}{\Delta z} =$$

$$= \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{\Delta z}\left(\frac{1}{(w-(z_0+\Delta z))^m} - \frac{1}{(w-z_0)^m}\right)dw =$$

$$= \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{\Delta z}\left(\frac{(w-z_0)^m-(w-(z_0+\Delta z))^m}{(w-(z_0+\Delta z))^m\cdot(w-z_0)^m}\right)dw$$

binomial formula

$$\text{Now, } (w-(z_0+\Delta z))^m = ((w-z_0)-\Delta z)^m = \sum_{k=0}^{m}\binom{m}{k}(w-z_0)^{m-k}(-\Delta z)^k =$$

$$= \cancel{(w-z_0)^m} - m(w-z_0)^{m-1}\cdot\Delta z + \underbrace{g(w,\Delta z)}_{\text{a polynomial in } w,\ \Delta z \ (z_0 \text{ is constant in this argument})}\cdot(\Delta z)^2$$

$$\Rightarrow \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{\Delta z}\left(\frac{\cancel{(w-z_0)^m}-(w-(z_0+\Delta z))^m}{(w-(z_0+\Delta z))^m\cdot(w-z_0)^m}\right)dw$$

$$= \frac{(m-1)!}{2\pi i}\int_{\partial D}\frac{f(w)}{\cancel{\Delta z}}\left(\frac{m(w-z_0)^{m-1}\cdot\cancel{\Delta z}+g\cdot(\Delta z)^{\cancel{2}}}{(w-(z_0+\Delta z))^m\cdot(w-z_0)^m}\right)dw$$

$$= \frac{(m-1)!}{2\pi i}\int_{\partial D}\underbrace{\frac{m f(w)\cdot(w-z_0)^{m-1}+f(w)\cdot g\cdot(\Delta z)}{(w-(z_0+\Delta z))^m\cdot(w-z_0)^m}}_{=h(w,\Delta z)\xrightarrow{\Delta z\to 0}\frac{m f(w)}{(w-z_0)^{m+1}}}dw \xrightarrow{\Delta z\to 0} \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z_0)^{m+1}}dw.$$

In the last step above, we used $\lim\limits_{\Delta z\to 0}\int_{\partial D}\dots = \int_{\partial D}\lim\limits_{\Delta z\to 0}\dots$

This is not always possible, and in this case it follows from the fact that $h(w,\Delta z)\xrightarrow{\Delta z\to 0}\frac{m f(w)}{(w-z_0)^{m+1}}$ uniformly in w.

We will say more about uniform limits later, but it is useful to know / remember that "uniform limits commute with integrals".

Note: To prove that the limit is uniform, one would use the fact that $h(w,\Delta z)$ is continuous on a set that is bounded and closed (also called compact), namely $\gamma\times\{|\Delta z|\le\varepsilon\}$ for $\varepsilon>0$ small. ∎

assumption will be removed

Cor: If $f: U\to\mathbb{C}$ is holomorphic $\boxed{\text{and } C^1}$ on $U\subset\mathbb{C}$ open, then f is infinitely differentiable on U.

Proof: For every $z\in U$, apply previous result to $D(z,\varepsilon)$ s.t. $\overline{D(z,\varepsilon)}\subset U$ to conclude that $f^{(m)}(z)$ exists for all $m\ge 1$. ∎

Lecture 12 02-23

The Cauchy integral formulas for derivatives of f have their own important consequences:

assumption will be removed

Thm (Cauchy estimates): Let $f: U \to \mathbb{C}$ be holomorphic $\boxed{\text{and } C^1}$ on $U \subset \mathbb{C}$ open. Suppose that $z_0 \in U$, $R > 0$ and $M > 0$ are s.t. $A = \{z \in \mathbb{C} : |z - z_0| \le R\} \subset U$ and $|f(z)| \le M$ for all $z \in A$. Then,

$$|f^{(m)}(z_0)| \le \frac{m!}{R^m} M \quad \text{for all } m \ge 1.$$

A, R, z_0, $\gamma = \partial A$

Proof: Parametrize $\gamma = \partial A$ by $z(t) = z_0 + Re^{it}$, $t \in [0, 2\pi)$.

Apply the Cauchy integral formula for derivatives to A:

$$|f^{(m)}(z_0)| = \left| \frac{m!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{m+1}} \, dw \right| = \left| \frac{m!}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^{m+1}} \cancel{i} R e^{it} \, dt \right| \le$$

triangle inequality

$$\le \frac{m!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R^m} \, dt \le \frac{m!}{\cancel{2\pi}} \frac{M}{R^m} \cancel{2\pi}.$$ ∎

assumption will be removed

Thm (Liouville's Thm): Let $f: \mathbb{C} \to \mathbb{C}$ be holomorphic $\boxed{\text{and } C^1}$. If f is bounded, then f is constant.

f is called entire

Proof: We will show that $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$. This implies that f is constant.

Let $M > 0$ be s.t. $|f(z)| \le M$ for every $z \in \mathbb{C}$.

Fix $z_0 \in \mathbb{C}$ and let $R > 0$. By the Cauchy estimates,

$$|f'(z_0)| \le \frac{M}{R}.$$

Since this is true for every $R > 0$, by letting $R \to 0$ we get

$f'(z_0) = 0$, as wanted. ∎

Cor: (Fundamental theorem of Algebra): Let

$$p(z) = a_0 + a_1 z + \dots + a_n z^n, \quad a_0, \dots, a_n \in \mathbb{C}$$

be a polynomial s.t. $a_n \neq 0$ and $n \ge 1$ (so, $p(z)$ is not constant). Then, $p(z)$ has at least one zero in $\mathbb{C}$.

Proof: Suppose that $p(z)$ is never zero. Then, $f(z) = \frac{1}{p(z)}$ is an entire C^1 function.

Claim: f is bounded.

Liouville's Thm + Claim $\Rightarrow$ f constant $\Rightarrow$ p constant, which is a contradiction. Hence, p has a zero.

We are only left with proving the Claim:

Idea of the proof of the Claim:

$$\lim_{z\to\infty} f(z) = \frac{1}{\lim_{z\to\infty} p(z)} = 0$$
. Since f is continuous, it must then be bounded.

More detailed proof of the Claim:

- $\lim_{z\to\infty} |p(z)| = \infty$. So, there is $R>0$ s.t. for every $z\in\mathbb{C}$ with $|z|>R$
  $|p(z)|>1$, which is equivalent to $|f(z)| = \frac{1}{|p(z)|} < 1$.
- Let $A_R = \{z\in\mathbb{C} : |z|\leq R\}$. This is a bounded and closed subset of $\mathbb{C}$. Since $|p(z)|$ is continuous, it must have (*) a minimum (and maximum) value, call it $m\geq 0$. Since p is assumed to have no zero, $m>0$. Hence, if $|z|\leq R$,
  $$|f(z)| = \frac{1}{|p(z)|} \leq \frac{1}{m}.$$

Conclusion: for every $z\in\mathbb{C}$, $|f(z)| \leq \begin{cases} 1 & \text{, if } |z|>R \\ \frac{1}{m} & \text{, if } |z|\leq R \end{cases}$.

Hence, f is bounded, which proves the Claim. $\boxtimes$

(*) If $f:[a,b]\to\mathbb{R}$ is a continuous function on a bounded closed interval $[a,b]\subset\mathbb{R}$, then f has a maximum value and a minimum value on $[a,b]$.

The higher-dimensional generalization of this is that if $f: A\to\mathbb{R}$ is a continuous function on $A\subset\mathbb{R}^n$ bounded and closed, then f has a maximum and a minimum value on A. This is what we used in the proof above, with $A = A_R$.

Lecture 13 02-27

The following technical result is often useful:

ML-estimate: Let $\gamma\subset\mathbb{C}$ be a bounded piecewise differentiable arc and $f:\gamma\to\mathbb{C}$ continuous. If $M>0$ is s.t. $|f(z)|\leq M$ for all $z\in\gamma$ and if $L = \text{length}(\gamma)$, then $\left|\int_\gamma f(z)\,dz\right| \leq ML$.

Proof: Since γ is the concatenation of finitely many differentiable arcs, it is enough to assume that γ is a differentiable arc. Suppose γ is parametrized by $z:[a,b]\to\mathbb{C}$.

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\cdot z'(t)\,dt\right| \le$$

triangle inequality

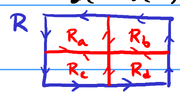
$$\le \int_a^b |f(z(t))|\cdot|z'(t)|\,dt \le M\underbrace{\int_a^b |z'(t)|\,dt}_{L} = ML. \qquad \blacksquare$$

Our next goal is to show that if f is holomorphic on an open set $U\subset\mathbb{C}$, then f is C^1 (by which we mean that $f=u+iv$ and that u and v are C^1). This will follow from a couple of important results.

Goursat's theorem: Let $f:U\to\mathbb{C}$ be holomorphic on $U\subset\mathbb{C}$ open. Then, for every rectangle $R\subset U$, $\boxed{\int_{\partial R} f(z)\,dz = 0}$.

Note: We did not need to assume that f is C^1, as we did in the CTI.

Proof: Let $R\subset U$ be a rectangle. Divide it into four equal subrectangles:



Orient all the rectangles counter-clockwise (an edge shared by two rectangles will get opposite orientations, hence cancelling integrals).

$$\left|\int_{\partial R} f(z)\,dz\right| = \left|\int_{\partial R_a} f(z)\,dz + \int_{\partial R_b} f(z)\,dz + \int_{\partial R_c} f(z)\,dz + \int_{\partial R_d} f(z)\,dz\right|$$
$$\le \left|\int_{\partial R_a} f(z)\,dz\right| + \left|\int_{\partial R_b} f(z)\,dz\right| + \left|\int_{\partial R_c} f(z)\,dz\right| + \left|\int_{\partial R_d} f(z)\,dz\right|$$

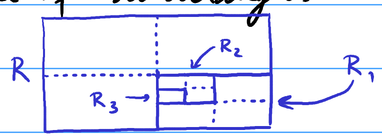
Let R_1 be the subrectangle for which the term on the right is the biggest.

$$\Rightarrow \begin{cases} \bullet\ \left|\int_{\partial R} f(z)\,dz\right| \le 4\left|\int_{\partial R_1} f(z)\,dz\right| \\ \bullet\ \text{length}(\partial R) = 2\cdot\text{length}(\partial R_1) \\ \bullet\ \text{diam}(R) = 2\cdot\text{diam}(R) \end{cases}$$

diameter = length of the diagonal

Repeat the same process to obtain a sequence of subrectangles R_n s.t.

$$\begin{cases} \bullet\ \left|\int_{\partial R} f(z)\,dz\right| \le 4^n\left|\int_{\partial R_n} f(z)\,dz\right| \\ \bullet\ \text{length}(\partial R) = 2^n\cdot\text{length}(\partial R_n) \\ \bullet\ \text{diam}(R) = 2^n\cdot\text{diam}(R) \end{cases}$$



Since the diameters of the R_n tend to zero, there exists a single point z_0 that belongs to <u>every</u> R_n.

Since f is differentiable at z_0,

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{z\in U}\quad |z-z_0|<\delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\varepsilon.$$

Fix $\varepsilon>0$ and take a corresponding $\delta>0$. Let $n>0$ be s.t.

$R_n \subset D_\delta(z_0)$. 

Then,

$$\left|\int_{\partial R} f(z)\,dz\right| \leqslant 4^n\left|\int_{\partial R_n} f(z)\,dz\right| \underset{\uparrow}{=} 4^n\left|\int_{\partial R_n} f(z)-f(z_0)-(z-z_0).f'(z_0)\,dz\right|$$

CTI applied to $g(z)=f(z_0)-(z-z_0).f'(z_0)$ on R_n (note: g is C^1)

Since $|f(z)-f(z_0)-(z-z_0).f'(z_0)|<|z-z_0|\,\varepsilon \leqslant \mathrm{diam}(R_n)\,\varepsilon$, the <u>ML estimate</u> gives

$$4^n\left|\int_{\partial R_n} f(z)-f(z_0)-(z-z_0).f'(z_0)\,dz\right| \leqslant 4^n(\mathrm{diam}(R_n)\,\varepsilon).\,\mathrm{length}(R_n) =$$

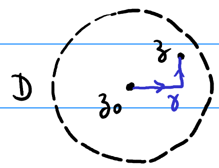
$$= \cancel{4^n}.\ \frac{\mathrm{diam}(R)}{\cancel{2^n}}\ \varepsilon\ \frac{\mathrm{length}(R)}{\cancel{2^n}} = \mathrm{diam}(R).\,\mathrm{length}(R).\,\varepsilon.$$

Since $\varepsilon>0$ can be made arbitrarily small, we get

$\int_{\partial R} f(z)\,dz = 0$, as wanted. ∎

<u>Thm</u> (Existence of antiderivatives on disks): Let $f: D(z_0,r)\to\mathbb{C}$ be holomorphic on the open disk $D=D(z_0,r)\subset\mathbb{C}$. Then, there is $F: D\to\mathbb{C}$ s.t. $F'(z)=f(z)$ for every $z\in D$.

<u>Proof</u>: For every $z\in D$, consider the arc $\gamma\subset D$ connecting z_0 to z, first by a horizontal line segment and then by a vertical line segment.



Define $F(z)=\int_\gamma f(z)\,dz$.

<u>Claim</u>: $F'(z)=f(z)$ for all $z\in D$.

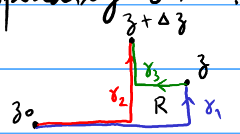
<u>Proof of Claim</u>: Fix $z\in D$. We want to show that $F'(z)=f(z)$.

Since f is continuous at z (recall: f differentiable $\Rightarrow$ f continuous)

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{w\in D}\quad |w-z|<\delta \Rightarrow |f(w)-f(z)|<\varepsilon.$$

Fix $\varepsilon>0$ and take a corresponding $\delta>0$. Let $\Delta z\in D(0,\delta)$.

Consider the following curves:



<u>Note</u>: $\partial R = \gamma_1+\gamma_3-\gamma_2$

$$F(z+\Delta z)-F(z) = \int_{\gamma_2} f(w)\,dw - \int_{\gamma_1} f(w)\,dw = \int_{\gamma_3} f(w)\,dw - \underbrace{\int_{\partial R} f(w)\,dw}_{=0,\text{ by }\underline{\text{Goursat}}}.$$

$$\Rightarrow \left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z}\int_{\gamma_3} f(w)\,dw - f(z) \right| \overset{\substack{\Delta z = \int_{\gamma_3} 1\,dw \\ f(z) = \frac{1}{\Delta z}\int_{\gamma_3} f(z)\,dw}}{=} \left| \frac{1}{\Delta z}\left(\int_{\gamma_3} (f(w)-f(z))\,dw\right)\right| \overset{\text{ML estimate}}{\le}$$

$$\le \frac{1}{|\Delta z|}\,\varepsilon \cdot \text{length}(\gamma_3) \underset{\text{length}(\gamma_3)\le 2|\Delta z|}{\le} \frac{1}{|\Delta z|}\,\varepsilon\, 2|\Delta z| = 2\varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, we conclude that

$$F'(z) = \lim_{\Delta z\to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z), \quad \text{as wanted.} \qquad \blacksquare$$

<u>Cor</u>: Let $f: U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open. Then, f is <u>infinitely differentiable</u> on U. (<u>Note</u>: Don't need to assume that f is C^1!)

<u>Proof</u>: Fix $z_0 \in U$ and let $D = D(z_0, r) \subset U$, for $r>0$ small enough. The previous thm implies the existence of $F: D \to \mathbb{C}$ s.t.

$$F'(z) = f(z) \quad \text{for all } z \in D.$$

Since f is continuous, $\boxed{F \text{ is } C^1}$. As we saw before, the Cauchy integral formula for derivatives implies that F is infinitely differentiable. Therefore, $f = F'$ is also infinitely differentiable. $\blacksquare$

This implies that if $f: U \to \mathbb{C}$ is holomorphic in $U \subset \mathbb{C}$ open, then it is C^k for every $k \geq 0$. Say that f is C^∞. We can remove some assumptions from several results that we studied before and get:

used to need to assume f is C^1

<u>Cauchy's Integral Theorem II</u> (CT II) Let $f: U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open. Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$. Then, $\boxed{\int_{\partial D} f(z)\,dz = 0.}$

<u>Cauchy's Integral Formula II</u> (CF II) Let $f: U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open. Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$. Then, for all $z \in D$, $\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw}$.

<u>Thm</u>: (Cauchy integral formula for <u>derivatives II</u>): Let $f: U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open. Let $D \subset U$ be a bounded domain with nice boundary $\partial D \subset U$. Then, for all $z \in D$ and all $m \geq 0$,

$$\boxed{f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dz}.$$

Also:

needed f C^1 before

Thm: If $f: U \to \mathbb{C}$ is holomorphic on $U \subset \mathbb{C}$ open, then
$f'(z_0) \neq 0 \Rightarrow f$ is conformal at $z_0 \in U$.

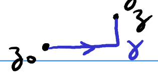
needed f C^2 before

Thm: If $f: U \to \mathbb{C}$ is holomorphic on $U \subset \mathbb{C}$ open, then
u and v are harmonic.

Thm: If $f = u + iv : D \to D'$ is holomorphic for domains $D, D' \subset \mathbb{C}$, and $\psi : D' \to \mathbb{R}$ is harmonic, then
$$\phi(x,y) = \psi(u(x,y), v(x,y)) \text{ is harmonic in } D.$$

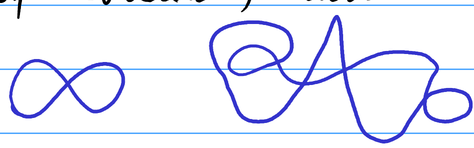
The following result is often useful, and can be thought of as a converse to Goursat's thm. It says that continuous functions that satisfy the conclusion of the CTI must be holomorphic.

Morera's Thm: Let $f: U \to \mathbb{C}$ be continuous on $U \subset \mathbb{C}$ open.
If $\int_{\partial R} f(z)\,dz = 0$ for every rectangle $R \subset U$, then
f is holomorphic on U.

Proof: We show that f is holomorphic on every disk $D = D(z_0, r) \subset U$.
As in the proof of existence of antiderivatives above, we can define
$F: D \to \mathbb{C}$ s.t. $F'(z) = f(z)$ for all $z \in D$, by
$F(z) = \int_\gamma f(w)\,dw$ with 

Since F is holomorphic on D, it is infinitely differentiable.
As a consequence, $f = F'$ is holomorphic as wanted. ∎

Lecture 14 02-28

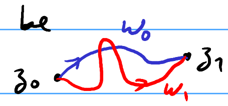
Goal: Get a version of the Cauchy Integral Theorem for curves that are not just boundaries of domains, and that might have self-intersections: 

Homotopy

Idea: we want to study $\int_\gamma f(z)\,dz$ when the curve γ is "deformed in a continuous manner".

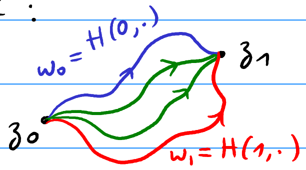
Let $U \subset \mathbb{C}$ be open.

Def 1: Let $z_0, z_1 \in U$ and let $w_0, w_1 : [0,1] \to U$ be continuous paths starting at z_0 and ending at z_1.



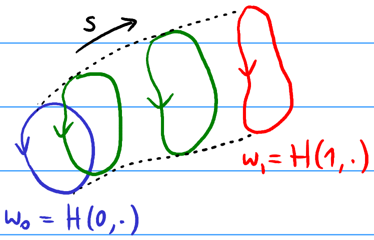
A continuous function $H : [0,1] \times [0,1] \to U$ is a homotopy with fixed endpoints in U from w_0 to w_1 if:

- $H(0,t) = w_0(t)$ for all $t \in [0,1]$
- $H(1,t) = w_1(t)$ for all $t \in [0,1]$
- $H(s,0) = z_0$ and $H(s,1) = z_1$ for all $s \in [0,1]$



Def: Let $w_0, w_1 : [0,1] \to U$ be continuous and closed (so, $w_0(0) = w_0(1)$ and $w_1(0) = w_1(1)$). A continuous function $H : [0,1] \times [0,1] \to U$ is a homotopy of closed curves in U from w_0 to w_1 if:

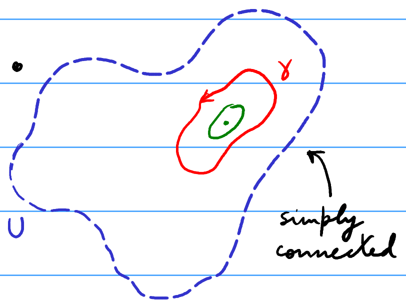
- $H(0,t) = w_0(t)$ for all $t \in [0,1]$
- $H(1,t) = w_1(t)$ for all $t \in [0,1]$
- $H(s,0) = H(s,1)$ for all $s \in [0,1]$



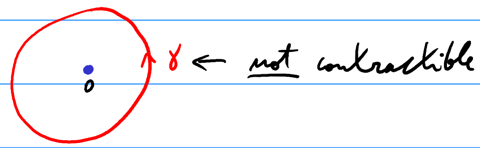
Def: Let $U \subset \mathbb{C}$ be open. $w : [0,1] \to U$ continuous and closed ($w(0) = w(1)$) is contractible in U if it is homotopic in U to a constant map $w_1 : [0,1] \to U$.

$U \subset \mathbb{C}$ open is simply connected if it is connected and every $w : [0,1] \to U$ continuous and closed ($w(0) = w(1)$) is contractible in U.

Exs:

- 

- $\mathbb{C}$ is simply connected
- $U = \mathbb{C} \setminus \{0\}$ is not simply connected



<u>Thm</u> (Deformation theorem): Let $f: U \to \mathbb{C}$ be holomorphic on $U \subset \mathbb{C}$ open.

a) If $w_0, w_1 : [0,1] \to \mathbb{C}$ are continuous, piecewise differentiable and homotopic with fixed endpoints, then $\boxed{\int_{w_0} f(z)dz = \int_{w_1} f(z)dz}$.

b) If $w_0, w_1 : [0,1] \to \mathbb{C}$ are closed, continuous, piecewise differentiable and homotopic as closed curves, then
$$\boxed{\int_{w_0} f(z)\,dz = \int_{w_1} f(z)\,dz}.$$

<u>Note</u>: the integrals are defined via $\int_w f(z)\,dz = \int_0^1 f(w(t)) \cdot w'(t)\,dt$.

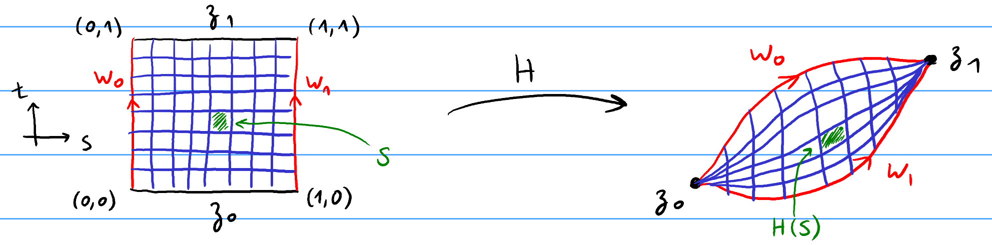
<u>Proof</u>: Let $H: [0,1] \times [0,1] \to \mathbb{C}$ be a homotopy. Assume that H is C^1 To simplify the proof (the result holds even if H is only continuous, but the proof is harder).
The proof use three technical consequences of the fact that $[0,1] \times [0,1] \subset \mathbb{R}^2$ is bounded and closed - also called <u>compact</u>:

i) H is continuous, so the image $H([0,1]\times[0,1]) \subset \mathbb{C}$ is also compact.

ii) Since $H([0,1]\times[0,1])$ is compact (by (i)), there is $\varepsilon > 0$ s.t. for every $z \in H([0,1]\times[0,1])$ the open disk $D(z,\varepsilon) \subset U$.

iii) Since H is continuous on the compact $[0,1]\times[0,1]$, H is <u>uniformly continuous</u> which means:
$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(s,t),(s',t') \in [0,1]\times[0,1]}\ \|(s,t)-(s',t')\| < \delta \Rightarrow |H(s,t) - H(s',t')| < \varepsilon.$$

We present the proof in case (a), since case (b) is similar.
Take $\varepsilon > 0$ as in (ii) and a corresponding $\delta > 0$ as in (iii).
Divide $[0,1]\times[0,1]$ into a grid of small squares, each with diameter $< \delta$. Let S be one such square. We know that the image $H(S)$ is contained in an open disk $D \subset U$ of radius ε.



there is an antiderivative of f in D. Since $H(\partial S)$ is closed in D, the FTC implies that $\int_{H(\partial S)} f(z)\,dz = 0$. Hence,

$$0 = \sum_{\text{all } S} \int_{H(\partial S)} f(z)\,dz \overset{(*)}{=} \int_{H(\partial([0,1]\times[0,1]))} f(z)\,dz =$$

$$= \underbrace{\int_{H([0,1]\times\{0\})} f(z)\,dz}_{=0 \text{ because constant at } z_0} + \underbrace{\int_{H(\{1\}\times[0,1])} f(z)\,dz}_{=\int_{w_1} f(z)\,dz} - \underbrace{\int_{H([0,1]\times\{1\})} f(z)\,dz}_{=0 \text{ because constant at } z_1} - \underbrace{\int_{H(\{0\}\times[0,1])} f(z)\,dz}_{=\int_{w_0} f(z)\,dz}$$

$$\Rightarrow \int_{w_0} f(z)\,dz = \int_{w_1} f(z)\,dz.$$

(In $(*)$, we used the fact that the edges of small squares inside $[0,1]\times[0,1]$ cancel in pairs. Only the edges on the boundary of $[0,1]\times[0,1]$ don't cancel.) ∎

Lecture 15 03-04

Cauchy's Integral Theorem III (CT III): Let $f: U \to \mathbb{C}$ be a holomorphic function on $U \subset \mathbb{C}$ open. For every contractible closed curve $\gamma \subset U$, $\boxed{\int_\gamma f(z)\,dz = 0}$.

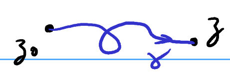
Note: If U is simply connected, then this is true for every closed piecewise differentiable arc $\gamma \subset U$.

Proof: If γ is homotopic to $w: [0,1] \to U$ constant, then $\int_\gamma f(z)\,dz \overset{\text{Deformation Thm}}{=} \int_0^1 f(w(t)) \cdot \underbrace{w'(t)}_{=0}\,dt = 0$. ∎

Theorem (Existence of antiderivatives of holomorphic functions on simply connected domains): Let $f: D \to \mathbb{C}$ be holomorphic on a simply connected domain $D \subset \mathbb{C}$. Then, there is $F: D \to \mathbb{C}$ holomorphic s.t. $F'(z) = f(z)$ for every $z \in D$.

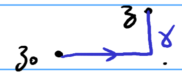
Note: We proved this before when $D = D(z_0, r)$ is a disk.

Proof: Fix a point $z_0 \in D$. For every $z \in D$, take a piecewise differentiable arc $\gamma \subset D$ starting at z_0 and ending at z



Define $F(z) = \int_\gamma f(z)\,dz$.

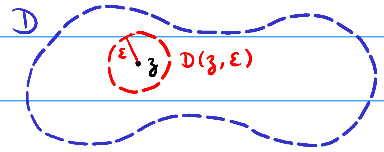
<u>Notes</u>: • The deformation theorem implies that $F(z)$ is independent of the choice of γ.

• When D was a disk $D(z_0, r)$, we used γ like 

But now we see that any choice of γ would work.

<u>Claim</u>: $F'(z) = f(z)$ for every $z \in D$.

<u>Proof of Claim</u>: Fix $z \in D$. Take $\varepsilon > 0$ small s.t. $D(z, \varepsilon) \subset D$.



We saw in the proof of the result for disks that $F'(w) = f(w)$ for every $w \in D(z, \varepsilon)$. In particular, $F'(z) = f(z)$. ∎

(<u>Note</u>: In the previous proof, as elsewhere - like in the proof of Morera's theorem - we construct an antiderivative of f by integrating f along paths. This can be interpreted via <u>vector calculus</u>, as an instance of the fact that a continuous vector field $\vec{F}$ in $U \subset \mathbb{R}^n$ open is <u>conservative</u> (meaning: has a scalar potential ϕ s.t. $\nabla\phi = \vec{F}$) iff its line integrals are path-independent (meaning: $\int_{\gamma_1} \vec{F} \cdot d\vec{r} = \int_{\gamma_2} \vec{F} \cdot d\vec{r}$ if γ_1 and γ_2 in U have the same endpoints). In the proof of this fact, ϕ is also constructed by integrating $\vec{F}$ along paths. To apply this fact to complex analysis, recall that the real and imaginary parts of complex line integrals are line integrals of vector fields in $\mathbb{R}^2$ - and that the Cauchy-Riemann equations imply that these vector fields are irrotational.)

<u>Ex</u>: $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, which is <u>not</u> simply connected.

<u>Important fact</u>: If $\gamma \subset \mathbb{C}$ is a <u>branch cut</u>, then $U = \mathbb{C} \setminus \text{image}(\gamma)$ is simply connected.

The theorem implies that f has an antiderivative on U the <u>branches</u> of $\log(z)$ are examples of such antiderivatives.

Note that if $g(z)$ is a branch of $\log(z)$ on U, then

- $g(z) + C$ is also an antiderivative of $1/z$ on U for all $C \in \mathbb{C}$.
- $g(z) + C$ is also a branch of $\log(z)$ on U if $C = 2\pi i k$ for $k \in \mathbb{Z}$.

<u>Theorem</u> (Existence of harmonic conjugates on simply connected domains): Let $u : D \to \mathbb{R}$ be a <u>harmonic</u> function on a simply connected domain $D \subset \mathbb{C}$ (recall: u is C^2 and $\Delta u = 0$). Then, there is a harmonic function $v : D \to \mathbb{C}$ s.t. $f = u + iv$ is holomorphic on D.

<u>Proof</u>: If there is such a holomorphic f, then
$$f' = u_x - i u_y \quad , \text{ by the } \underline{\text{CR-equations}}.$$
Define $g(z) = g(x+iy) = u_x(x,y) - i u_y(x,y) = \underbrace{r(x,y)}_{u_x(x,y)} + i \underbrace{s(x,y)}_{-u_y(x,y)}$

<u>Claim</u>: g is holomorphic.

<u>Proof of claim</u>: g is C^1 because u is C^2. Also, g satisfies the CR-equations: $\begin{cases} r_x = u_{xx} \overset{\Delta u = 0}{=} -u_{yy} = s_y \\ r_y = u_{yx} \overset{u \text{ is } C^2}{=} u_{xy} = -s_x \end{cases}$. Therefore, g is holomorphic.

Since D is simply connected, g has an antiderivative $G : D \to \mathbb{C}$ s.t. $G' = g$. Write
$$G(z) = G(x+iy) = \alpha(x,y) + i\,\beta(x,y) \Rightarrow \underset{u_x - i u_y}{g} = G' = \alpha_x - i\alpha_y$$
Then, $\left.\begin{aligned} \partial_x(u-\alpha) &= u_x - \alpha_x = u_x - u_x = 0 \\ \partial_y(u-\alpha) &= u_y - \alpha_y = u_y - u_y = 0 \end{aligned}\right\} \Rightarrow u - \alpha = C \in \mathbb{R}$ constant

Therefore, $v(x,y) = \beta(x,y)$ is a harmonic conjugate of u (since it is a harmonic conjugate of $\alpha = u - C$). ∎

<u>Cor</u>: If $u : U \to \mathbb{R}$ is harmonic on $U \subset \mathbb{R}^2$ open, then u is C^∞.

<u>Proof</u>: It is enough to check that u is C^∞ on every open disk $D \subset U$. By the previous result, u is the real part of $f : D \to \mathbb{C}$ holomorphic. This implies that f, and hence also u, is C^∞. ∎

We can sum up several of the results we obtained so far as follows:

"the following are equivalent"

Thm: Let $U \subset \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be a continuous function. Write $f(x+iy) = u(x,y) + i\, v(x,y)$. TFAE:

1) f is holomorphic;
2) u and v are C^1 and: $u_x = v_y$ and $u_y = -v_x$ (CR eqs);
   Looman-Menchoff theorem: this assumption can be replaced with the continuity of f. The proof is (hard and) in Narasimhan-Nievergelt
3) for every rectangle $R \subset U$, $\int_{\partial R} f(z)\, dz = 0$ (Goursat + Morera);
4) for every bounded domain $D \subset U$ with nice boundary $\partial D \subset U$,
   $$\int_{\partial R} f(z)\, dz = 0 \qquad (\text{CT II});$$
5) for every contractible closed curve $\gamma \subset U$, $\int_\gamma f(z)\, dz = 0$ (CT III);
   γ only needs to be null-homologous. The (hard) proof is in Ahlfors
6) for every simply connected domain $D \subset U$, f has an antiderivative in D;
7) f is infinitely differentiable (say C^∞) and for every $z_0 \in U$
   $$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^{m+1}}\, dw, \quad m \geq 0,$$
   for any $D \subset U$ bounded domain with nice boundary $\partial D \subset U$ (CF II).

(We will also see:

8) f is analytic: for every open disk $D(z_0, r) \subset U$
   $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \quad \text{for all } z \in D(z_0, r) \text{ (Taylor series).}$$
   Furthermore, the power series converges uniformly on every compact subset $K \subset D(z_0, r)$.)