Le 18

More about integral calculation

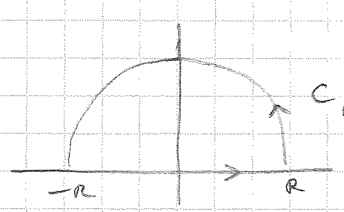
Recall: Last time we computed

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i3x}}{x^2+4}\, dx$$

by considering the integral

$$\int_{\Gamma_R} \frac{e^{i3z}}{z^2+4}\, dx$$

along the contour $\Gamma_R = [-R, R] \cup C_R^+$:



in the limit as $R \to +\infty$.

Suppose we want to calculate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\, dx \right).$$

using the same method. Then we want to show that

$$\lim_{R \to +\infty} \int_{C_R^+} \frac{z e^{iz}}{z^2+1}\, dz = 0$$

The previous rough estimate, using $|e^{iz}| = e^{-y} \le 1$, only shows that

$$\left| \int_{C_R^+} \frac{z e^{iz}}{z^2+1}\, dz \right| \le \frac{R}{R^2-1} \pi R \le \text{const}$$

A more accurate estimate shows the following:

m or ω?

<u>Thm</u> (Jordan's lemma)

Suppose $m > 0$, and that P and Q are polynomials such that $\deg Q \geq \deg P + 1$. (*)

Then,

$$\lim_{R \to +\infty} \int_{C_R^+} e^{imz} \frac{P(z)}{Q(z)} \, dz = 0,$$

where C_R^+ is the upper half-circle of radius R centered at 0.

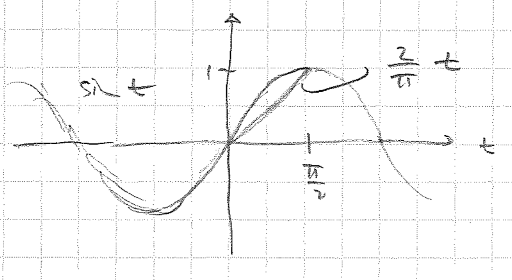
<u>Proof</u> Parametrizing C_R^+ by $z(t) = Re^{it}$, $0 \leq t \leq \pi$, we obtain

$$\left| \int_{C_R^+} e^{imz} \frac{P(z)}{Q(z)} \, dz \right| = \left| \int_0^\pi e^{im(Re^{it})} \frac{P(Re^{it})}{Q(Re^{it})} iRe^{it} \, dt \right|$$

$$\leq \int_0^\pi \left| e^{imR(\cos t + i \sin t)} \frac{P(Re^{it})}{Q(Re^{it})} iRe^{it} \right| dt \leq$$

$$\leq \Big/ (*) \Rightarrow \left| \frac{P(Re^{it})}{Q(Re^{it})} \right| \leq \frac{K}{R} \text{ for some constant } K \text{ if } R \text{ is large} \Big/ \leq$$

$$\leq \int_0^\pi e^{-mR\sin t} \frac{K}{\cancel{R}} \cancel{R} \, dt = K \int_0^\pi e^{-mR \sin t} \, dt$$



From the figure we see that

$$\int_0^{\pi} e^{-mR\sin t}\, dt = 2\int_0^{\pi/2} e^{-mR\sin t}\, dt \le$$

$$\le 2\int_0^{\pi/2} e^{-mR\frac{2}{\pi}t}\, dt = \cancel{2}\left[-\frac{\pi}{\cancel{2}mR} e^{-mR\frac{2}{\pi}t}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{mR}\left(1-e^{-mR}\right) \le \frac{\pi}{mR}$$

This gives that $\left|\int_{C_R^+} e^{imz}\frac{P(z)}{Q(z)}\,dz\right| \le \frac{K\pi}{mR} \to 0,\ R\to\infty$

$\boxed{}$

__Remark:__ 1) $\frac{P(z)}{Q(z)}$ could be replaced by a fcn $f(z)$ such that $f(Re^{i\theta}) \to 0$ (unif w.r.t $\theta\in[0,\pi]$) as $R\to+\infty$.

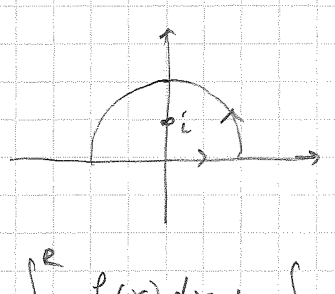
2) The inequality $\int_0^{\pi} e^{-R\sin\theta}\,d\theta \le \frac{\pi}{R}$ is called __Jordan's inequality__.

__Ex.__ Compute $I = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx$

__Sol.__ $I = \operatorname{Im}\left(\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+1}\,dx\right)$

Put $f(z) = \frac{ze^{iz}}{z^2+1}$ and consider $\int_{\Gamma_R} f(z)\,dz$,

where



$(R>1)$

$$\int_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz \overset{!}{=} 2\pi i\cdot\frac{ie^{i\cdot i}}{2i}$$

Now let $R \to +\infty$. According to Jordan's lemma

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\, dx + 0 = i\pi e^{-1},$$

so $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx = \mathrm{Im}\,(i\pi e^{-1}) = \underline{\pi e^{-1}}$$

$\blacksquare$

<u>Principal values</u>

Recall from Calculus the following definition:

<u>Def.</u> Suppose that f is continuous on $\mathbb{R}$.

We say that the (improper) integral $\int_{-\infty}^{\infty} f(x)\,dx$ is <u>convergent</u> if $\lim\limits_{a\to-\infty} \int_a^0 f(x)\,dx$ and $\lim\limits_{b\to+\infty} \int_0^b f(x)\,dx$ both exist. Then

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{a\to-\infty} \int_0^a f(x)\,dx + \lim_{b\to+\infty} \int_0^b f(x)\,dx \overset{!}{=} \lim_{R\to+\infty} \int_{-R}^{R} f(x)\,dx$$

One also makes the following:

<u>Def.</u> Suppose that f is continuous on $\mathbb{R}$.

The <u>principal value</u> of $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{R\to+\infty} \int_{-R}^{R} f(x)\,dx,$$

provided that the limit exists.

<u>Remark:</u> 1) $\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,dx$ may exist even if $\int_{-\infty}^{\infty} f(x)\,dx$ is not convergent. E.g., let $f(x) = x$.

2) One can show that $\int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} dx$

is convergent for $m \neq 0$ provided that

$\deg Q \geq \deg P + 1$, $Q(x) \neq 0$ on $\mathbb{R}$.

For this reason, I didn't write PV.

Analogously, one makes the following:

<u>Def</u> Let $a < x_0 < b$ and suppose that

f is continuous for $a \leq x < x_0$ and for

$x_0 < x \leq b$. The <u>principal value</u> of $\int_a^b f(x)\,dx$

is defined as

$$PV \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx \right)$$

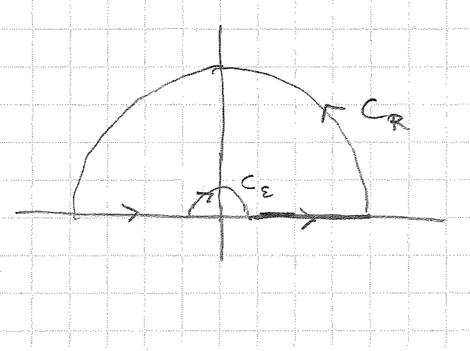
provided that the limit exists.

<u>Ex</u> We want to compute

$$I = PV \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \lim_{\substack{R \to +\infty \\ \varepsilon \to 0+}} \left(\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx \right)$$

It seems natural to let $f(z) = \frac{e^{iz}}{z}$,

and consider $\int_{\Gamma_{\varepsilon,R}} f(z)\,dz$ with $\Gamma_{\varepsilon,R}$ as below:

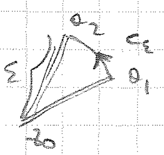


We then want to compute $\lim\limits_{\varepsilon \to 0+} \int_{C_\varepsilon} f(z)\,dz$.

The following holds:

<u>Thm</u> (Fractional residue thm)

Suppose that z_0 is a simple pole of $f(z)$, and that C_ε is the circular arc



$$C_\varepsilon : \quad z = z_0 + \varepsilon e^{i\theta}, \quad \theta_1 \le \theta \le \theta_2 .$$

Then,

$$\lim_{\varepsilon \to 0+} \int_{C_\varepsilon} f(z)\,dz = i(\theta_2 - \theta_1)\,\mathrm{Res}(f, z_0).$$

<u>Proof</u> As f has a simple pole at z_0,

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z),$$

where g is analytic in a punctured disk about z_0.

$$\int_{C_\varepsilon} f(z)\,dz = a_{-1} \int_{C_\varepsilon} \frac{dz}{z - z_0} + \int_{C_\varepsilon} g(z)\,dz .$$

1) $$\int_{C_\varepsilon} \frac{dz}{z - z_0} = \Big/\ z(\theta) := z_0 + \varepsilon e^{i\theta},\ \theta_1 \le \theta \le \theta_2 \Big/ =$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\cancel{\varepsilon e^{i\theta}}}\, i \cancel{\varepsilon e^{i\theta}}\, d\theta = i(\theta_2 - \theta_1)$$

2) $|g(z)| \le M$ for $z \in C_\varepsilon$ if ε is small

$$\Rightarrow \left| \int_{C_\varepsilon} g(z)\,dz \right| \overset{ML}{\le} M\,L(C_\varepsilon) = M\varepsilon(\theta_2 - \theta_1) \to 0, \quad \varepsilon \to 0+ .$$

Thus, $$\lim_{\varepsilon \to 0+} \int_{C_\varepsilon} f(z)\,dz = i(\theta_2 - \theta_1)\,\mathrm{Res}(f, z_0)$$

$\square$

Since $f(z) = \frac{e^{iz}}{z}$ is analytic inside and on $T_{\varepsilon,R}$,

we have that $\int_{T_{\varepsilon,R}} f(z)\,dz = 0 \quad \forall\, \varepsilon, R > 0$

If we let $R \to +\infty$ and $\varepsilon \to 0+$, we obtain

$$I + \underbrace{(-i\pi)\,\mathrm{Res}(f,0)}_{\text{Fract. res. thm}} + \underbrace{0}_{\text{Jordan's lemma}} = 0$$

$$\Rightarrow I = i\pi\,\mathrm{Res}(f,0) = i\pi \lim_{z\to 0} z\frac{e^{iz}}{z} = \underline{\underline{i\pi}}$$

Remark: It follows that

$$2\underbrace{\int_0^\infty \frac{\sin x}{x}\,dx}_{\substack{\text{even} \\ \text{(cont. at 0)}}} = (PV)\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \mathrm{Im}\left(PV\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx\right) = \pi$$

i.e. $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Integrands with branch points

Ex Compute $I = \int_0^\infty \frac{x^{-a}}{x+1}\,dx \quad (0 < a < 1)$.

Sol We want to compute

$$I = \lim_{\substack{R\to+\infty \\ \varepsilon\to 0+}} \int_\varepsilon^R \frac{x^{-a}}{x+1}\,dx$$

Let z^{-a} be the branch given by

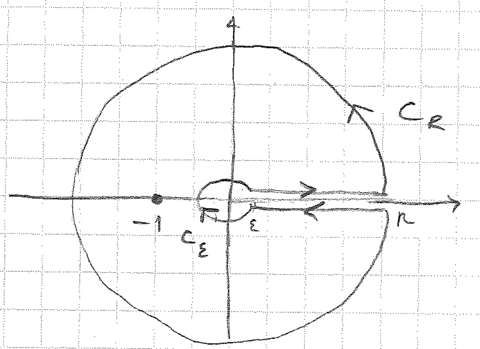
$$z^{-a} = e^{-a(\ln r + i\theta)} = r^{-a}e^{-ia\theta}, \quad 0 < \theta < 2\pi,$$

where $z = re^{i\theta}$. Then z^{-a} is analytic in $\mathbb{C}\setminus[0,\infty)$.

Now let

$$f(z) = \frac{z^{-a}}{z+1} = \frac{r^{-a} e^{-ia\theta}}{re^{i\theta}+1}, \quad z = re^{i\theta},$$

and consider $\displaystyle\int_{\Gamma_{\varepsilon,R}} f(z)\,dz$ with $\Gamma_{\varepsilon,R}$ as below:



$(\varepsilon < 1 < R)$

We use $\theta = 0$ on the "upper" and $\theta = 2\pi$ on the "lower" side of the segment $[\varepsilon, R]$. So we have

$$f(z) = \frac{r^{-a}}{r+1} = \frac{x^{-a}}{x+1}$$

on the "upper segment", and

$$f(z) = \frac{r^{-a} e^{-ia2\pi}}{r+1} = \frac{x^{-a} e^{-ia2\pi}}{x+1}$$

on the "lower segment".

By the residue theorem

$$\int_\varepsilon^R \frac{x^{-a}}{x+1}\,dx + \int_{C_R} f(z)\,dz + \int_R^\varepsilon \frac{x^{-a} e^{-ia2\pi}}{x+1}\,dx + \int_{C_\varepsilon} f(z)\,dz =$$

$$= 2\pi i \cdot \operatorname{Res}(f, -1) \qquad (*)$$

Note that f has a simple pole at $z = -1$, so

$$\operatorname{Res}(f,-1) = \lim_{z \to -1} z^{-a} = e^{-a(\ln 1 + i\pi)} = e^{-ia\pi}$$

So, from (*) it follows that

$$(1-e^{-ia2\pi})\int_{\varepsilon}^{R}\frac{x^{-a}}{x+1}\,dx+\int_{C_R}f(z)\,dz+\int_{C_\varepsilon}f(z)\,dz=$$

$$=2\pi i\cdot e^{-ia\pi} \qquad (**)$$

But

$$\left|\int_{C_R}f(z)\,dz\right| \overset{ML}{\leq} \frac{R^{-a}}{R-1}\cdot 2\pi R \longrightarrow 0,\quad R\to+\infty$$

$$\left|\int_{C_\varepsilon}f(z)\,dz\right| \leq \frac{\varepsilon^{-a}}{1-\varepsilon}\,2\pi\varepsilon \longrightarrow 0,\quad \varepsilon\to 0^+ \quad (0<a<1)$$

If we let $R\to+\infty$ and $\varepsilon\to 0^+$ in (**) we get

$$I=\frac{2\pi i e^{-ia\pi}}{1-e^{-ia2\pi}}=\frac{2\pi i}{e^{ia\pi}-e^{-ia\pi}}=\underline{\underline{\frac{\pi}{\sin a\pi}}}$$

∎