UPPSALA UNIVERSITET

Föreläsningsateckningar

Finasiella Derivat

Rami Abou Zahra

Inlämningsdatum
December 3, 2024

Contents

1. Options 2
2. Continous time & Brownian Motion 4
2.1. Simple Random Walk 4
2.2. Stochastic integration 6
2.3. Properties of the stochastic integral 7
3. Martingales 8
4. Itos formula 9
4.1. Taylor Expansion 9
4.2. Multi-dimensional Ito formula 11
5. Correlated Brownian Motions 12
6. Stochastic Differential Equations 13
7. Geometric Brownian Motion 13
8. Partial Differential Equtions 14
9. Portfolio Dynamics 17
10. Arbitrage Pricing 18
10.1. Drift estimation 23
11. Volatility 23
11.1. Historic volatility 23
11.2. Implied volatility 23
12. Completeness and Hedging 23
13. Volatility Mis-specification 26
14. Asian Options 26
14.1. Completeness vs Absence of Arbitrage 28

1. Options

Motivating Discussion:
Say a Swedish company has signed a contract to buy a machine from a US company for 100000USD to be paid at delivery 6 months from now. $T = \frac{1}{2}$ years.
Current exchange rate is 11SEK/USD. The buyer is suject to currency risk. There are 3 possible strategies to implement:

1. Buy 100000USD today and deposit in the bank.
The risk is eliminated but money is tied up for a long time and the company may not have access to this money

2. Buy a *forward contract* from a bank, i.e the bank delivers the sum you need at $T = \frac{1}{2} = t$, in return, the company payes some constant $K \cdot 100000USD$ at $T = t$, where K is chosen at $t = 0$ such that no transfer of money is needed at $t = 0$. Here, the bank takes all of the risk, but if the exchange rate drops below K then we would have preffered to do nothing.

3. Buy a *European call option* on 100000USD, with strike price K and exercise date T. I.e, it gives the right but not the obligation to buy 100000USD at price $K \cdot 100000USD$ at time $T = t$. If exchange rate at T is $> K$, then we use the option. If its below at $t = T$ thne we do not use the option (right, not obligation)

The last one is a good choice, but not free. This leads to the 2 main problems in the course:

- How much is a fair price for an option?
- If you are the seller of an option, how to protect (hedge) from risk of exchange rate not going up?

Motivating Example in discrete time
At $t = 0$, we can trade in a market with 2 assets:

- *Bank account* (risk-free/non-risky asset)
  At $t = 0$ the value is 1 and at $t = 1$ the value is 1
- *Stock* (risky asset)
  At $t = 0$, $S_0 = 100$ then it either grows ($S_1 = 120$) or declines ($S_1 = 80$) with probability $p = 0.6$ and $p = 0.4$ respectively

Definition 1.1 Call option

A *call option* is a contract that gives its holder the right but not the obligation to buy one share of a stock at time T with predetermined price K. Thus, at time $t = 1$, the option is worth $S_1 - K$ if $S_1 > K$ and 0 else

What is a fair price of the option? The sensible thing to pay would be $p(S_1 - K)$. Assuming $K = 110$ in the above example, then $0.6(120 - 110) = 6$. But this is not the best price!

The idea is to replicate the option by finding a trading stategy using both the risk-free (B) and the risky asset (S) such that the value of the stock at $t = 1$ coincides with the value of hte option.
Is that possible? Yes. Let x = amount in the bank at $t = 0$ and y be the number of shares of stock. We want to pick x, y such that regardles if stock goes up or down we have increase.
At $t = 1$

$$\left.\begin{aligned} &x + S_1 y = S_1 - K \\ &x + S_1 y = 0 \end{aligned}\right\}$$

If $K = 110$ and $S_1 = \{120, 80\}$, then $x = -20$ and $y = \frac{1}{4}$ since

$$\begin{cases} x + 120y = 10 \\ x + 80y = 0 \end{cases}$$

At $t = 0$. Our strategy is therefore to borrow 20 from the bank and buy $\frac{1}{4}$ of a share. The cost is $25 - 20 = 5$ which is less than 6.

At time $t=1$ our holdings are worth $\frac{1}{4}S_1 - 20 = \begin{cases} 10 & \text{if } S_1 = 120 \\ 0 & \text{if } S_1 = 80 \end{cases}$ which is exactly the same as the option.

Conclusion:
By the APT (Arbitrage pricing theory), the price of the call must be equal to the cost of setting up this portfolio.

Remark:
The probabilities do not influence the option value. They were never used in the calculation of the price.

Remark:
Let us change p into q such that $\mathbb{E}(S_1) = S_0 = 100$ in the example, which value of q satisfies this? It is symmetric in the example, so let $p = q = \frac{1}{2}$
Then $\mathbb{E}(\max\{S_1 - k, 0\}) = 10 \cdot \frac{1}{2} + 0 \cdot \frac{1}{5} = 5$
In general, the option price is $\mathbb{E}^Q\left(\frac{B_0}{B_1}\max\{S_1 - k, 0\}\right)$ where Q is chosen such that $\mathbb{E}^Q\left(\frac{B_0 S_1}{B_1}\right) = \frac{S_0}{B_0}$

Notation:
$a^+ = \max\{a, 0\}$. In particular,

$$(s-K)^+ = \begin{cases} s - K & \text{if } s \geq K \\ 0 & \text{if } s < K \end{cases}$$

Exercise:

- In the above example, find a replicating strategy for a put option (right but not obligated to sell one share) at price $K = 110$
- Find the vlaue of the option at $t = 0$

Answer:

$$\left.\begin{aligned} x &= 90 \\ y &= \frac{-3}{4} \end{aligned}\right\} \quad \text{option value of } 15$$

2. Continous time & Brownian Motion

2.1. Simple Random Walk.

Let X_i be i.i.d.r.v with $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \dfrac{1}{2}$
Let $S_n = \sum_{i=1}^n X_i$, then this is a stochastic process, still in discrete time. Do note that the expectation is 0 for the r.v. and that:

$$\mathbb{E}(S_n) = \sum_{k=1}^n \mathbb{E}(X_i) = 0$$
$$\mathrm{Var}\,(S_n) = \mathbb{E}(S_n^2) - \underbrace{(\mathbb{E}(S_n))^2}_{=0} = \sum_{k=1}^n \mathrm{Var}\,(X_i) = \sum_{k=1}^n 1 = n$$

Note that this was discrete time, how do we proceed to make this continuous?
We do this by scaling to finer time. Frist, fix a time interval:
Stage 1
Let $X_0^1 = 0$
At $t = 0$, toss a coin, $X_T^1 = \begin{cases} \sqrt{T} & \text{heads} \\ -\sqrt{T} & \text{tails} \end{cases}$.
Here $\mathbb{E}(X_T^1) = 0$ and $\mathrm{Var}\,\left(X_T^1\right) = T =$ elapsed time.

Stage 2

Add another time step. Let $X_0^2 = 0$, toss a coin, $X_{T/2}^2 = \begin{cases} \sqrt{\dfrac{T}{2}} & \text{heads} \\ -\sqrt{\dfrac{T}{2}} & \text{tails} \end{cases}$

Repeat at $t = \dfrac{T}{2}$, adding/subtracting $\sqrt{\dfrac{T}{2}}$

Stage n
Let $X_0^n = 0$, at each time $t_k = \dfrac{k}{n}T$, toss a coin.
Define $X_{t_{k+1}}^n = X_{t_k}^n + Y_k$ where $Y_k = \pm\sqrt{\dfrac{T}{2}}$ with prob. 1/2. Simulating our coin tosses.
Here

$$\mathbb{E}(X_{t_k}^n) = \mathbb{E}\left(\sum_{i=1}^{k-1} Y_i\right) = \sum_{i=1}^{k-1} \mathbb{E}(Y_i) = 0$$
$$\mathrm{Var}\,\left(X_{t_k}^n\right) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) \overset{\text{indep}}{=} \sum_{i=1}^{k} = \frac{T}{n}k = t_k$$

Now the question becomes, what happens when $n \to \infty$? We obtain *Brownian Motion*, aka Weiner process.

Definition 2.2 Brownian Motion

Brownian Motion is a stochastic process W if:
- $W_0 = 0$
- Independent increments, i.e $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent (as long as they are not overlapping)
- $W_t - W_s \sim N(0, t-s)$
- $t \mapsto W_t$ is continuous

This is a nice definition and all, but does there even exists something which satsifies our definition?

Sats 2.1

$t \mapsto W_t$ is of infinite variation and nowhere differentiable
By infinite variation, it is meant

$$\lim_{n\to\infty} \sum_k \left| W_{t_{k+1}} - W_{t_k} \right| = \infty$$

A regular differentiable function has bounded variation. The next goal is to define the stochastic integral $\int_0^t g_s dW_s$, where g_t is a stochastic process determined by the Brownian motion W

Definition 2.3 Measurable w.r.t σ-algebra

Let X_t be a stochastic process. An event A is $\mathcal{F}_t^X$ measurable (denoted $A \in \mathcal{F}_t^X$) if it is possible to determine whether A has happened or not based on observations of $\{X_s : 0 \leq s \leq t\}$

Example:
$A = \{X_s \leq 7 : \forall s \leq 9\} \in \mathcal{F}_9^X$

Definition 2.4

If a random variable Z can be determined by observations of $\{X_s : 0 \leq s \leq t\}$, then $Z \in \mathcal{F}_t^X$

Example:

$$Z = \int_0^5 X_s d_s \in \mathcal{F}_5^X$$

If you only know X_5 up to 4, then you cannot determine Z

Definition 2.5

A stochastic process Y_t with $Y_t \in \mathcal{F}_t^X \quad \forall t$ is *adapted to the filtration* $\mathcal{F}_t^X$

Example:
$Y_t = \sup_{0\leq s\leq t} W_s$ is adapted to $\mathcal{F}_t^W$

Definition 2.6

The process $g_t \in \mathcal{L}^2$ if

- g is adapted to $\mathcal{F}_t^W$
- $\int_0^t \mathbb{E}(g_s^2)ds < \infty$

Example:

Brownian motion $\in \mathcal{L}^2$, its adapted to $\mathcal{F}_t^W$ and $\int_0^t \mathbb{E}(\overbrace{W_s^2}^{\sim N(0,\sqrt{s})})ds = \int_0^t sds = \dfrac{t^2}{2} < \infty$

2.2. Stochastic integration.
Assume $g \in \mathcal{L}^2$. If g is simple (i.e $g_s = g_{t_k}$ for $s \in [t_k, t_{k+1}]$), then we define

$$\int_0^t g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})$$

For egeneral $g \in \mathcal{L}^2$, we can approximate g using step functions which are simple such that

$$\int_0^t \mathbb{E}((g_s - g_s^n)^2) ds \to 0 \quad \text{as } n \to \infty$$

Then, one defines the stochastic integral as

$$\int_0^t g_s dW_s = \lim_{n \to \infty} g_s^n dW_s$$

Remark
One can show that the limit indeed exists and does not depend on the sequence used for approximation.

Remark:
Forward increments are used! The integrand is fixed at t_k, and we look at forward movements of the Brownian motion.

Remark:
Steiltjes integration si not possible since paths are not of unbounded variation.

Proposition:
Assume $g \in \mathcal{L}^2$ and adapted to a filtration, then:
1. $\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0$
2. $\mathbb{E}\left(\left(\int_0^t g_s dW_s\right)^2\right) = 0 = \int_0^t \mathbb{E}(g_s^2) dW_s$ (Ito isometry)
3. $X_t = \int_0^t g_s dW_s$, then X_t is $\mathcal{F}^W$-adapted

Bevis 2.1

Assume g is simple (if it was not, then approximate using step functions).

1.

$$\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0 = \mathbb{E}\left(\sum_{k=1}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right) = \sum_{k=0}^{n-1} \mathbb{E}\left(\underbrace{g_{t_k}}_{\text{indep.}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep.}}\right)$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}) \mathbb{E}\underbrace{(W_{t_{k+1}} - W_{t_k})}_{\sim N(0,\sigma^2)} = 0$$

2. This is the variance of a stochastic integral:

$$\mathbb{E}\left(\left(\sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right)^2\right) = \mathbb{E}\left(\sum_{k=0}^{n-1} g_{t_k}^2 (W_{t_{k+1}} - W_{t_j})\right)^2 + 2\sum_{j<k} \underbrace{g_{t_k} g_{t_j}}_{\in \mathcal{F}_{t_k}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep. of } \mathcal{F}_{t_k}} \underbrace{(W_{t_{j+1}} W_{t_j})}_{\in \mathcal{F}_{t_k}}$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left(g_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2\right) + 2\sum_{j<k} \mathbb{E}\left(g_{t_k} g_{t_j} (W_{t_{k+1}} - W_{t_k})(W_{t_{j+1}} - W_{t_j})\right)$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}^2) \mathbb{E}\left(\underbrace{(W_{t_{k+1}} - W_{t_k})^2}_{t_{k+1} - t_k}\right) + 2\sum_{j<k} \mathbb{E}(\cdots) \underbrace{\mathbb{E}(W_{t_k+1} - W_{t_k})}_{=0}$$

$$= \int_0^t \mathbb{E}(g_{t_k}^2) dW_s$$

$\square$

2.3. **Properties of the stochastic integral.**
Examples:
$\int_0^t 1 dW_s = W_t - W_0 = W_t$, but that is $\int_0^t W_s dW_s$? W_s is not piecewise constant, but we may approximate it by letting $g_t^n = W_{t_k}$ for $t \in [t_k, t_{k+1})$. What happens here is essentially discretisation but for finer and finer time.
This yields the approximation

$$\int_0^t \mathbb{E}\left((g_s^n - W_s)^2\right) ds = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \underbrace{\mathbb{E}\left((W_s - W_{t_k})^2\right)}_{s-t_k} \leftarrow \quad \text{variance of increment of BM}$$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds = \sum_{k=0}^{n-1} \frac{1}{2}(t_{k+1} - t_k)^2 = \sum_{k=0}^{n-1} \frac{1}{2} \Delta t$$

$$\Delta t = \frac{t}{n} \Rightarrow \frac{1}{2}(\Delta t)^2 \frac{t}{\Delta t} = \frac{\Delta t}{2} t \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}} - W_{t_k}) = \frac{1}{2} \sum_{k=0}^{n-1} \left(W_{t_{k+1}}^2 - W_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2\right) = \frac{1}{2} W_{t_n} - \underbrace{\frac{1}{2} \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2}_{I_n}$$

We claim $I_n \to t$ as $n \to \infty$:

$$\mathbb{E}(I_n) = \underbrace{\mathbb{E}\left(\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2\right)}_{\text{2nd moment}} = \sum_{k=0}^{n-1} (t_{k+1} - t_k) = t_n = t$$

Need to check $\mathbb{E}((I_n - t)^2) = 0$:

$$\mathbb{E}\left((\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2 - \overbrace{(t_{k+1} - t_k)}^{\Delta t}) \right)^2$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left(\left((W_{t_{k+1}} - W_{t_k})^2 - \Delta t\right)^2 \right) + \sum_{j \neq k} \mathbb{E}\left(((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)((W_{t_{j+1}} - W_{t_j}) - \Delta t)\right)$$

$$= \sum_{j \neq k} \mathbb{E}\left((W_{t_{k+1}} W_{t_k})^4\right) - (\Delta t)^2 = \sum_{k=0}^{n-1} 2(\Delta t)^2 \sim \Delta t \to 0$$

hus, $I_n \to t$ as $n \to \infty$, so

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}$$

Remark:
Lets prove if $X \sim N(0, \sigma)$, then $\mathbb{E}(X^4) = 3\sigma^2$

$$\mathbb{E}(X^4) = \int z^4 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-z^2}{2\sigma^2}\right\} \overset{\text{parts}}{\Rightarrow} - \left[z^3 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}\right]_{-\infty}^{\infty} - \int 3z^2 \frac{\sigma^2}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\pi\sigma^3\right\} dz$$

$$= 3\sigma^2 \cdot \underbrace{\int z^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}}_{\sigma^2} = 3\sigma^4$$

3. Martingales

Let $\mathcal{F}_t$ be a filtration, "information generated by B; up to a time t".
If Y is a random variable, then $\mathbb{E}(Y \mid \mathcal{F}_t)$ is the conditional expectation given all information up to time t

Example:

$$\mathbb{E}(W_s \mid \mathcal{F}_t) = W_t$$

Definition 3.7 Martingale

A process X is a martingale if X is $\mathcal{F}_t$-adapted. X_t integrable, i.e

- $\mathbb{E}(|X_t|) < \infty \quad \forall t$
- $\mathbb{E}(X_s \mid \mathcal{F}_t) = X_t$ for $s > t$

Example:
W_t is a martingale, $W_t^2 - t$ is a martingale since

$$\begin{gathered} Y_t := W_t^2 - t \qquad \mathbb{E}(Y_t \mid \mathcal{F}_s) = \mathbb{E}(W_t^2 - t \mid \mathcal{F}_s) \\ = \mathbb{E}((W_t - W_s)^2 + 2W_sW_t - W_s^2 \mid \mathcal{F}_s) - t \\ = t - s + 2\mathbb{E}(W_sW_t \mid \mathcal{F}_s) - \mathbb{E}(W_s^2 \mid \mathcal{F}_s) - t = 2W_s \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} W_s^2 - s \\ = W_s^2 - s = Y_s \end{gathered}$$

$Y_t = \int_0^t g_u dW_u$ is a martingale since:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = \mathbb{E}\left(\int_0^s g_u dW_u \mid \mathcal{F}_s\right) + \mathbb{E}\left(\int_s^t g_u dW_u \mid \mathcal{F}_s\right) = \int_0^s g_u dW_u = Y_s$$

However, W_t^3 is *not* a martingale:

$$\begin{gathered} \mathbb{E}(W_t^3 \mid \mathcal{F}_s) = \mathbb{E}(W_s^3 + (W_t - W_s)^3 - 3W_tW_s^2 + 3W_t^2W_s \mid \mathcal{F}_s) \\ = W_s^3 + 0 - 3W_s^2 \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} + 3W_s \underbrace{\mathbb{E}(W_t^2 \mid \mathcal{F}_s)}_{t-s+W_s^2} \\ = W_s^3 + 3(t-s)W_s \neq W_s^3 \end{gathered}$$

Remark: A martingale is a "fair game"

4. Itos formula

Assume
$$X_t = a + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$
for some adapted process μ_t and σ_t. Short-hand notation $\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t \\ X_0 = a \end{cases}$

Let $f(t,x)$ be a $C^{1,2}$-function and define $Z_t = f(t, X_t)$, what does dZ_t look like?

Recall:
$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$$
so $W_t^2 = t + 2\int_0^t W_s dW_s$, thus
$$d(W_t^2) = dt + 2W_t dW_t$$

Fix n and let $t_k = \frac{k}{n} t$
Let $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$ and consider
$$S_n = \sum_{k=0}^{n-1} (\Delta W_{t_k})^2$$
We have
$$\mathbb{E}(S_n) = \sum_{k=0}^{n-1} \mathbb{E}\left((\Delta W_{t_k})^2\right) = \sum_{k=0}^{n-1} \frac{t}{n} = t$$
and
$$\operatorname{Var}(S_n) \overset{\text{indep.}}{=} \sum_{k=0}^{n-1} \operatorname{Var}\left((\Delta W_{t_k})^2\right) = n\operatorname{Var}\left((\Delta W_{t_0})^2\right) = n \cdot 2\frac{t^2}{n^2} \to 0 \qquad \text{as } n \to \infty$$
Thus $S_n \to t$ as $n \to \infty$ (in $\mathcal{L}^2$). This motivates to write
$$\int_0^t (dW_s^2) = t$$
$$\Leftrightarrow dW_t^2 = dt$$

4.1. Taylor Expansion.

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} + \frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x} dt dX_t + \text{ higher order terms} \\ &= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW + \text{ higher order terms} \end{aligned}$$

Sats 4.2: Itos formula

If $dX_t = \mu_t dt + \sigma_t dW_t$ and $Z_t = f(t, X_t)$, then
$$dZ_t = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

Here $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(t, X_t)$ and similarly for other derivatives of f

Alternative formulation:
$$dZ_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$
Where $(dX_t)^2$ is calculated using

- $(dt)^2 = 0$

- $dtdW_t = 0$
- $(dW_t)^2 = dt$

Example:
Compute $\int_0^t W_s dW_s$. Let $Z_t = W_t^2$, then by Itos formula

$$\begin{aligned} dZ_t &= 2W_t dW_t + \frac{1}{2} \cdot 2(dW_t)^2 \\ &= dt + 2W_t dW_t \end{aligned}$$

Thus $W_t^2 = Z_t = t + 2\int_0^t W_s dW_s$, so $\int_0^t W_s dW_s = \dfrac{W_t^2}{2} - \dfrac{t}{2}$

Example:
Compute $\mathbb{E}(W_t^4)$
Let $Z_t = W_t^4$, then by Itos formula

$$\begin{aligned} dZ_t &= 4W_t^3 dW_t + \frac{1}{2} \cdot 12W_t^2 (dW_t)^2 \\ &= 6W_t^2 dt + 4W_t^3 dW_t \end{aligned}$$

Thus

$$W_t^4 = Z_t = 6\int_0^t W_s^2 ds + 4\int_0^t W_s^3 dW_s$$

Taking expectation yields

$$\begin{aligned} \mathbb{E}(W_t^4) &= 6\int_0^t \underbrace{\mathbb{E}(W_s^2)}_{s} ds + 4\underbrace{\mathbb{E}\left(\int_0^t W_s^3 dW_s\right)}_{=0} \\ &= 6\int_0^t s ds = 3t^2 \end{aligned}$$

Alternatively, without using Itos formula

$$\begin{aligned} \mathbb{E}(W_t^4) = \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx &\overset{\text{parts.}}{=} \left[x^3 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)}\right]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \\ &= 3t\mathrm{Var}\,(W_t) = 3t^2 \end{aligned}$$

Example:
Compute $\mathbb{E}(e^{\alpha W_t})$
Let $Z_t = e^{\alpha W_t}$. Itos formula yields

$$\begin{aligned} dZ_t &= \alpha e^{\alpha W_t} dW_t + \frac{1}{2}\alpha^2 e^{\alpha W_t} (dW_t)^2 \\ &= \frac{\alpha^2}{2} e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t \\ &= \frac{\alpha^2}{2} Z_t dt + \alpha Z_t dW_t \end{aligned}$$

Integration yields

$$Z_t = 1 + \frac{\alpha^2}{2}\int_0^t Z_s ds + \alpha\int_0^t Z_s dW_s$$

So

$$\begin{aligned} \mathbb{E}(Z_t) &= 1 + \mathbb{E}\left(\frac{\alpha^2}{2}\int_0^t Z_s ds\right) + \underbrace{\mathbb{E}\left(\alpha\int_0^t Z_s dW_s\right)}_{=0} \\ &= 1 + \frac{\alpha^2}{2}\int_0^t \mathbb{E}(Z_s) ds \end{aligned}$$

Let $m(t) = \mathbb{E}(Z_t)$, then

$$\begin{cases} \dfrac{dm}{dt} = \dfrac{\alpha^2}{2} m(t) \\ m(0) = 1 \end{cases}$$

Which has the solution $m(t) = e^{\frac{\alpha^2}{2}t}$

4.2. **Multi-dimensional Ito formula.**
Assume $dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$ where W^i are d independent Brownian motions.
On a matrix form:

$$\underbrace{dX_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1} dt + \underbrace{\sigma_t}_{n\times d} \underbrace{dW_t}_{d\times 1}$$

Let $Z_t = f(t, X_t)$ where $f : [0, \infty] \times \mathbb{R}^2 \to \mathbb{R}$ is $C^{1,2}$

Sats 4.3: Itos multi-dimensional formula

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

Where

- $dW_t^i dW_t^j = 0$ if $i \neq j$
- $\left(dW_t^i\right) = dt$
- $(dt)^2 = dt dW_t = 0$

Alternatively

$$dZ_t = \left(\frac{\partial f}{\partial t} + \sum_{i=1}^n \mu_t^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_t^{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \sigma_t^i dW_t$$

Where $C = \sigma\sigma^*$ and σ^i is the i:th row of σ
Indeend,

$$\begin{aligned} dX_t^i dX_t^j &= \left(\sum_{j)1}^d \sigma^{ik} dW^k \right) \left(\sum_{l=1}^d \sigma^{jl} dWl \right) \\ &= \left(\sum_{k=1}^d \sigma^{ik} \sigma^{jl} \right) dt \\ &= (\sigma\sigma^*)^{ij} dt \end{aligned}$$

Example:
If $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ dY_t = \gamma Y_t dt + \delta Y_t dV_t \end{cases}$ and $Z_t = X_t Y_t$; find dZ_t

Itos formula yields

$$\begin{aligned} dZ_t &= Y_t dX_t + X_t dY_t + \frac{1}{2} \cdot 2 dX_t dY_t \\ &= (\alpha + \gamma) Z_t dt + Z_t(\sigma dW_t + \delta dV_t) \end{aligned}$$

Setting $\overline{W}_t = \dfrac{1}{\sqrt{\sigma^2 + \delta^2}}(\sigma W_t + \delta V_t)$, then $\overline{W}$ is a Brownian Motion and

$$dZ_t = (\alpha + \gamma) Z_t dt + \sqrt{\sigma^2 + \delta^2} Z_t d\overline{W}_t$$

5. Correlated Brownian Motions

Let $\overline{W} = \begin{bmatrix} \overline{W}^1 \\ \vdots \\ \overline{W}^d \end{bmatrix}$ where $\overline{W}^1, \cdots, \overline{W}^d$ are independent

Consider $W = \delta \overline{W}$ where

$$\delta = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1d} \\ \vdots & \vdots & \vdots \\ \delta_{d1} & \cdots & \delta_{dd} \end{bmatrix} = \underbrace{\begin{bmatrix} \delta_1 \\ \vdots \\ \delta_d \end{bmatrix}}_{\text{Row vectors with } ||\delta_i|| = 1}$$

Here $||\delta_i|| = \sqrt{\delta_{i1}^2 + \cdots + \delta_{id}^2}$.
So W^i is a Brownian motion.

Moreover,

$$\begin{aligned} dW_t^i dW_t^j &= \left(\sum_{k=1}^d \delta_{ik} d\overline{W}_t^k \right) \left(\sum_{l=1}^d \delta_{jl} d\overline{W}_t^l \right) \\ &= \sum_{k=1}^d \delta_{ik} \delta dt = (\delta\delta^*)_{ij} dt \end{aligned}$$

Definition 5.8 Correlated Wiener Process

W_t as constructed above is a d-dimensional *correlated Wiener process* with correlation matrix $\rho = \delta\delta^*$

Sats 5.4: Itos formula, correlated version

If W_t is a correlated Wiener process as above, and

$$\underbrace{dX_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1} dt + \underbrace{\sigma_t}_{n\times d} \underbrace{dW_t}_{d\times 1}$$

satisfies

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

where

- $(dt)^2 = dt dW^i = 0$
- $dW^i dW^j = \rho_{ij} dt$

Example:

Given $\overline{W} = \begin{bmatrix} \overline{W}^1 \\ \overline{W}^2 \end{bmatrix}$ (where $\overline{W}^1, \overline{W}^2$ are independent), construct $W = \begin{bmatrix} W^1 \\ W^2 \end{bmatrix}$ with correlation matrix $\rho = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix}$

Note that $\delta = \begin{bmatrix} 1 & 0 \\ \rho_0 & \sqrt{1-\rho_0^2} \end{bmatrix}$ satisfies $\rho\rho^* = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix} = \rho$

Thus $W = \begin{bmatrix} \overline{W}^1 \\ \rho_0 \overline{W}^1 + \sqrt{1-\rho_0^2} \overline{W}^2 \end{bmatrix}$ is a correlated Wiener process with correlated matrix δ

What other choices for δ are possible?

6. Stochastic Differential Equations

Let

- a d-dimensiona Brownian motion W
- $\mu : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$
- $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$
- $x_0 \in \mathbb{R}^n$

be given. A *stochastic differential equation* is an equation at the form

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{1}$$

Or, equivalently,

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

Sats 6.5

Assume

$$||\mu(t,x) - \mu(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K\,||x - y||$$

and $||\mu(t,x)|| + ||\sigma(t,x)|| \leq K\,||x||$ for some K

Then there exists a unique solution X_t to the SDE (1). Moreover,
1. X is $\mathcal{F}^W$-adapted
2. X_t has continuous trajectories
3. X is a Markov process

7. Geometric Brownian Motion

Consider

$$\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \qquad \alpha, \sigma \text{ constans} \\ X_0 = x \end{cases}$$

Anmärkning:
If $\sigma = 0$, then $dX_t = \alpha X_t dt$ so $X_t = x_0 e^{\alpha t}$
Let $Z_t = \ln(X_t)$. Then

$$dZ_t \overset{\text{Ito}}{=} \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)\,A^2 = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma W_t$$

so $Z_t = \ln(x_0) + \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t$ and $X_t = e^{Z_t} = x_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t}$

Moreover,

$$\mathbb{E}(X_t) = x_0 + \mathbb{E}\left[\int_0^t \alpha X_s ds\right] + \underbrace{\mathbb{E}\left[\int_0^t \sigma X_s dW_s\right]}_{=0}$$

So if $m(t) = \mathbb{E}(X_t)$, we find $\begin{cases} \frac{dm}{dt} = \alpha m(t) \\ m(0) = x_0 \end{cases}$

Thus $m(t) = x_0 e^{\alpha t}$

Results:
The solution of $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ X_0 = x_0 \end{cases}$ is $X_t = x_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$
Moreover, $\mathbb{E}(X_t) = x_0 e^{\alpha t}$

Example:

Consider the SDE $\begin{cases} dX_t = -X_t dt + dW_t \\ X_0 = x \end{cases}$ (this is a mean-reverting Ornstein-Uhlenbeck process)
The trick here is to let $Y_t = e^t X_t$. Then

$$dY_t = e^t X_t dt + e^t dX_t = e^t dW_t$$
$$\Rightarrow Y_t = x + \int_0^t e^s dW_s$$

Thus $X_t = e^{-t} Y_t = xe^{-t} + e^{-t} \int_0^t e^s dW_s$
Moreover $\mathbb{E}(X_t) = xe^{-t}$

Definition 7.9 Diffusion process

The solution X of an SDE

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW \\ X_0 = x_0 \end{cases}$$

is called a *diffusion process*.
μ is called the *drift* and σ is the *diffusion coefficient*

8. Partial Differential Equtions

Consider the following *terminal value problem*:
Given function σ, μ, ϕ, find a function $F(t, x)$ such that

$$\begin{cases} \frac{\partial F}{\partial t}(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2 F}{\partial x^2} F(t, x) + \mu(t, x) \frac{\partial F}{\partial t}(t, x) = 0 \\ F(T, x) = \phi(x) \end{cases} \tag{2}$$

If $F(t, x)$ satisfies (2), define X_s by $\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t = x \end{cases}$ and let $Z_s = F(s, X_s)$. Then

$$dZ_s \overset{\text{Ito}}{=} \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x} dX_s + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dX_s)^2$$
$$= \underbrace{\left(\frac{\partial F}{\partial s} + \mu \frac{\partial F}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} \right)}_{=0} ds + \sigma \frac{\partial F}{\partial x} dW_s$$
$$= \sigma \frac{\partial F}{\partial x} dW_s$$

Integrate:

$$Z_T = Z_t + \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s$$

Take expectation:

$$\mathbb{E}(Z_T) = Z_t = F(t, x) = \mathbb{E}(F(T, X_T)) \overset{*}{=} \mathbb{E}(\phi(X_t))$$

We write $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$ (to indicate that $X_t = x$)

We have thus proved the following:

Sats 8.6: Feynman-Kac

If $F(t, x)$ satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2(t, x)}{2} \frac{\partial^2 F(t, x)}{\partial x^2} + \mu(t, x) \frac{\partial F}{\partial x} = 0 \qquad (t < T) \\ F(t, x) = \phi(x) \end{cases}$$

then $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$ where $\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t = x \end{cases}$

Example:

Solve the PDE $\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2} = 0 \\ F(T,x) = x^2 \end{cases}$

Solution:

Let X_s be the solution of $\begin{cases} dX_s = \sigma dW_s \\ X_t = x \end{cases}$ i.e $X_s = x + \sigma(W_s - W_t)$

By Feynman-Kac:

$$F(t,x) = \mathbb{E}_{t,x}(X_T^2) = \mathbb{E}((x+\sigma(W_T - W_t))^2)$$
$$= x^2 + 2x\sigma\mathbb{E}(W_t - W_t) + \sigma^2\mathbb{E}((W_T - W_t)^2)$$
$$= x^2 + \sigma^2(T-t)$$

$F(t,x) = x^2 + \sigma^2(T-t)$

Sats 8.7: Feynman-Kac in higher dimensions + discounting

Assume that $F : [0,T] \times R^n \to \mathbb{R}$ satisfies

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{1}{2}\sum_{i,j=1}^n C_{i,j}(t,x)\dfrac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i(t,x)\dfrac{\partial F}{\partial x_i} - rF(t,x) = 0 \\ F(T,x) = \phi(x) \end{cases}$$

Where $C(t,x) = \sigma(t,x)\sigma^*(t,x)$ for some matrix σ $(n \times d)$
Then $F(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}(\phi(X_T))$ where

$$\begin{cases} dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \\ X_t = x \end{cases}$$

Bevis 8.1

Let $Z_s = e^{-r(s-t)}F(s,X_s)$. Then

$$dZ_s \overset{\text{Ito}}{=} e^{-r(s-t)}\underbrace{\left(\frac{\partial F}{\partial s} + \frac{1}{2}\sum_{i,j=1}^n C_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i \frac{\partial F}{\partial x_i} - rF\right)}_{=0} ds + e^{-r(s-t)}\sum_{i=1}^n \frac{\partial F}{\partial x_i}\sigma_i dW_s$$

So

$$Z_T = \underbrace{Z_t}_{F(t,x)} + \int_t^T \cdots dW_s = e^{-r(T-t)}\phi(X_T)$$

Thus $F(t,x) = e^{-r(T-t)}\mathbb{E}(\phi(X_T))$ $\square$

Example:

Solve the PDE $\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\delta^2}{2}\dfrac{\partial^2 F}{\partial y^2} - rF = 0 \\ F(T,x,y) = xy \end{cases}$

Solution:

Here $C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \delta^2 \end{bmatrix}$ so $\sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix}$ satisfies $C = \sigma\sigma^*$

$$d\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} \Rightarrow \begin{cases} X_t = x + \sigma(W_T^1 - W_t^1) \\ Y_T = y + \delta(W_T^2 - W_t^2) \end{cases}$$

Feynman-Kac gives

$$F(t,x,y) = \mathbb{E}_{t,x,y}\left(e^{-r(T-t)}X_T Y_T\right) = e^{-r(T_t)}\mathbb{E}\left(\left(x+\sigma(W_T^1 - W_t^1)\right)\left(y+\delta(W_T^2 - W_t^2)\right)\right)$$
$$\overset{\text{indep}}{=} e^{-r(T-t)}\mathbb{E}\left(x+\sigma(W_T^1 - W_t^1)\right)\mathbb{E}\left(y+\delta(W_T^2 - W_t^2)\right) = e^{-r(T-t)}xy$$

par Answer is therefore $F(t,x,y) = e^{-r(T-t)}xy$

Definition 8.10 Infitesimal Operator

The differential operator

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i \frac{\partial}{\partial x_i}$$

is called the *infitesimal operator* of X

Itos formula:
If $Z_t = f(t, X_t)$, then $dZ_t = \left(\frac{\partial f}{\partial t} + \mathcal{A}f\right)dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sigma_i dW_t$

9. Portfolio Dynamics

Let the time axis be discrete

Definition 9.11

- N = the number of different assets
- S_n^i = the price of one unit of asset i at time n
- h_n^i = the number of units of asset i bought at time n
- $h_n = (h_n^1, h_n^2, \cdots, h_n^N)$ is a *portfolio*
- V_n = the value of a portfolio h_n at time $n = \sum_{i=1}^N h_n^i s_n^i = h_n \cdot S_n$

The interpretation:

- At time $n-$ we have an old portfolio h_{n-1} from the previous period
- At time n, S_n becomes observable
- At time n, after observing S_n, we chose h_n

Definition 9.12 Budget equation

$h_n \cdot S_{n+1} = h_{n+1} \cdot S_{n+1}$

Notation: If $\{x_n\}_{n=0}^{\infty}$ is a sequence of real numbers, let $\Delta x_n = x_{n+1} - x_n$.
The budget equation becomes $S_{n+1} \cdot \Delta h_n = 0$

Recall $Y_n = h_n \cdot S_n$
Since $\Delta V_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_{n+1} + h_n \cdot S_{n+1} - h_n \cdot S_n$
$= S_{n+1} \cdot \Delta h_n + h_n \cdot \Delta S_n$
we have $\Delta V_n = h_n \cdot \Delta S_n$ if the budget equation is fulfilled.

Below we use this relation to *define* what is meant by a self-financing portfolio in continuous time.

Definition 9.13

Let $\{S_t \mid t \geq 0\}$ be an N-dimensional process

- A *portfolio* h is an $\mathcal{F}^s$-adapted N-dimensional process
- h is *Markovian* if $h_t = h(t, S_t)$ for some function h
- The *value process* V^h of h is

$$V_t^h = \sum_{i=1}^N h_t^i S_t^i = h_t \cdot S_t$$

- A portfolio h is *self-financing* if

$$dV_t^h = h_t \cdot dS_t$$

- For a given portfolio h, the corresponding *relative portfolio* w is

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h} \qquad i = 1, \cdots, N$$

Note that $\sum_{i=1}^N w_t^i = 1$.

Also, h is self-financing if and only if $dV_t^h = V_t^h \sum_{i=1}^N \dfrac{\partial w_t^i}{S_t^i} dS_t^i$

10. Arbitrage Pricing

In this chapter, $N = 2$ (two assets):

$$dB_t = rB_t dt$$

This is a risk-free asset, think bank account and r is a constant interest rate, and

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

is a risky asset, think stock price

Remarks:

1. $B_t = B_0 e^{rt}$
2. μ (local mean rate of return) and σ (volatility) are functions of t and current stock price
3. In the Black-Scholes model, μ and σ are constants

The aim is to find a "fair" value of options written on S
Options are also called *financial derivatives*

Definition 10.14 European Call Option

A *European call option* with strike price K and maturity date T on the underlying asset S is a contract such that the holder (owner) at time T has the right, but not the obligation to buy one share of S at price K from the option writer (seller)

Remarks:

- A *European put option* gives the right (but not the obligation) to *sell* one share of S at time T at price K
- An *American call/put* gives the right to buy/sell at *any* time before T

Definition 10.15

A *contingent claim with maturity* T (or a T*-claim*) is a random variable $X \in \mathcal{F}_T^S$
A contingent claim is *simple* is $X = \phi(S_T)$ for some *contract function* (or payoff function) ϕ

Example:
For a European call option, $\phi(x) = (x - K)^+ = \max\{x - K, 0\}$
Indeed, if $S_T \geq K$, then buy at price K and make profit $S_T - K$. If $S_T < K$, do not exercise the option.
For a European put option $\phi(x) = (K - x)^+$
We will determine the price $\pi(t, X)$ of a T-claim X at time t by requiring the market to be *arbitrage-free*.

Definition 10.16

A self-financing portfolio h is an *arbitrage* if $\begin{cases} V_0^h = 0 \\ \mathbb{P}(V_T^h \geq 0) = 1 \\ \mathbb{P}(V_T^h > 0) > 0 \end{cases}$

The market is *arbitrage-free* if no arbitrage exists.

Example:

$$\begin{cases} dS_t^1 = dt + dW_t \\ dS_t^2 = dW_t \\ dB_t = 0 \end{cases} \qquad \text{is not arbitrage free}$$

$$\begin{cases} dS_t^1 = dt + dW_t^1 \\ dS_t^2 = dW_t^2 \\ dB_t = 0 \end{cases} \qquad \text{is arbitrage free (first two lines indep)}$$

Assumption: The price process $\Pi_t(X)$ is such that $(B_t, S_t, \Pi_t(X))$ is arbitrage-free.
We also assume that all assets (including the option) can be sold/bought with no market frictions (no transaction consts, no liquidity constraints)

Idea: Create a self-financing portfolio of options and the sock such that its vlaue process is locally risk-free (has no dW-term). The drift of the valu emust then coincide with the interest rate (otherwise arbitrage). This will give a condition on the price of the option.

Assume $X = \phi(S_T)$ (simple T-claim) and that $\Pi_t(X) = F(t, S_t)$ for some function F.
New Notation: $F_t = \dfrac{\partial F}{\partial t}$, $F_s = \dfrac{\partial F}{\partial s}$, $F_{ss} = \dfrac{\partial^2 F}{\partial s^2}$

Then

$$dF(t,S_t) \overset{\text{Ito}}{=} F_t dt + F_s dS_t + \frac{1}{2}F_{ss}(dS_t)^2$$

$$= \underbrace{\frac{\left(F_t + \frac{\sigma^2 S_t^2}{2}F_{ss} + \mu S_t F_s\right)}{F}}_{=\mu^F} F(t,S_t)dt + \underbrace{\frac{\sigma S_t F_s}{F}}_{=\sigma^F} F dW_t$$

$$= \mu^F F dt + \sigma^F F dW_t$$

Let (w^S, w^F) be a self financing relative portfolio of stocks and options ($w^S + w^F = 1$), and let V be its value process. Then

$$dV_t = V_t\left(\frac{w^S}{S_t}dS_t + \frac{w^F}{F}dF_t\right)$$

$$= \left(\mu w^S + \mu^F w^F\right)V_t dt + (\sigma w^S + \sigma^F w^F)V_t dW_t$$

Let (w^S, w^F) be defined by

$$\left.\begin{aligned} w^S + w^F &= 1 \\ \sigma w^S + \sigma^F w^F &= 0 \end{aligned}\right\} \Leftrightarrow \begin{cases} w^s = \dfrac{\sigma^F}{\sigma^F - \sigma} \\ w^F = \dfrac{-\sigma}{\sigma^F - \sigma} \end{cases}$$

Then $dV_t = \dfrac{\mu\sigma^F - \mu^F\sigma}{\sigma^F - \sigma}V_t dt$

By a no-arbitrage argument, we must have $r = \dfrac{\mu\sigma^F - \mu^F\sigma}{\sigma^F - \sigma}$

Here $\underbrace{r\sigma^F - r\sigma}_{=\frac{r\sigma S_t F_s}{F} - r\sigma} = \underbrace{\mu\sigma^F - \mu^F\sigma}_{=\frac{\mu\sigma S_t F_s}{F} - \frac{\sigma(F_t + \mu S_t F_s +) + \frac{-2S_t^2}{2}F_{ss}}{F}}$

$$rS_tF_s - rf = \mu S_t F_s - F_t - \mu S_t F_s + \frac{\sigma^2}{2}S_t^2 F_{ss}$$

$$= -F_t + \frac{\sigma^2}{2}S_t^2 F_{ss}$$

$$F_t + \frac{\sigma^2 S_t^2}{2}F_{ss} + rS_tF_r - rF = 0$$

Since S_t can take any value, F must satisfy the PDE

$$F_t(t,s) + \frac{\sigma^2(t,s)}{2}s^2 F_{ss} + rsF_s(t,s) - rF(t,s) = 0$$

Also, $\Pi_T(X) = F(T, S_T) = \phi(S_T)$, so we also have $F(T,S) = \phi(S_T)$

Sats 10.8: Black-Sholes equation

In the market $\begin{cases} dB_t = rB_t dt \\ dS_t = \mu(t,S_t)S_t dt + \sigma(t,S_t)S_t dW_t \end{cases}$, the only arbitrage-free price of a T-claim $X = \phi(S_T)$ is $F(t,S_t)$, where $F(t,s)$ solves

$$\begin{cases} F_t(t,s) + \dfrac{\sigma^2(t,s)}{2} s^2 F_{ss}(t,s) + rsF_s(t,s) - rF(t,s) = 0 \\ F(T,s) = \phi(s) \end{cases}$$

The solution to the BS-equation is by Feynman-Kac

$$F(t,s) = \mathbb{E}_{t,s}\left(\exp\left\{-r(T-t)\phi(S_T)\right\}\right)$$

where

$$\begin{aligned} dS_u &= rS_u du + \sigma(u,S_u)S_u dW_u \\ S_t &= s \end{aligned} \tag{3}$$

we refer to

$$\begin{cases} dS_u = \mu(u,S_u)S_u du + \sigma(u,S_u)S_u dW_u \\ S_t = s \end{cases} \tag{4}$$

as the *P-dynamics* of S (the specification of S under the "physical measure" P). (3) is refered to as the *Q-dynamics* of S (Q is the *pricing measure*, or the *martingale measure*)

Sats 10.9

The arbitrage-free price of a simple T-claim $X = \phi(S_T)$ is $F(t,S_t)$ where

$$F(t,s) = \mathbb{E}^Q_{t,s}\left(\exp\left\{-r(T-t)\phi(S_T)\right\}\right)$$

and the Q-dynamics of S are as in (3)

Example:
In the standard BS-model (i.e constant σ), what is the arbitrage-free price of the T-claim $X = S_T^2$?
By risk-neutral valuation, $F(t,s) = \exp\left\{-r(T-t)\right\}\mathbb{E}^Q_{t,s}(S_T^2)$
Let $Y_u = S_u^2$, then

$$dY_u = 2S_u dS_u + (dS_u)^2 \overset{dS_u = rS_u du + \sigma S_u dW_u}{=} (2r+\sigma^2)Y_u du + 2\sigma Y_u dW_u$$

Y is a gBm and thus

$$\mathbb{E}^Q_{t,s}(S_T^2) = \mathbb{E}^Q(Y_T) = s^2 \exp\left\{(2r+\sigma^2)(T-t)\right\}$$

Which is the price of X at time t

Example:
What is the price of $X = S_t$?
By risk-neutral valuation

$$F(t,s) = \exp\left\{-r(T-t)\right\}\mathbb{E}^Q_{t,s}(S_T) = s$$

So the price at time t is S_t

Remark:
In time-homogenous models (such as the BS-model), the relevant quantity is time $T-t$ left to maturity.

Example: *Binary option*

In the standard BS-model, find the value of $X = \phi(S_T)$ where $\phi(x) = \begin{cases} 1 & \text{if } x \geq K \\ 0 & \text{if } x < K \end{cases}$

$$
\begin{aligned}
F(0,s) &= \exp\{-rT\}\,\mathbb{E}^Q_{0,s}\left(I_{\{S_T\geq K\}}\right) = \exp\{-rT\}\,Q(S_T\geq K)\\
&= \exp\{-rT\}\,Q(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma W_T\right\}\geq K)\\
&= \exp\{-rT\}\,Q\left(\frac{1}{\sqrt{T}}W_T\geq\frac{\ln\left(\frac{K}{S}\right)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\\
&= \exp\{-rT\}\,Q\left(\frac{1}{\sqrt{T}}W_t\leq\frac{\ln\left(\frac{S}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\\
&= \exp\{-rT\}\,N\left(\frac{\ln\left(\frac{S}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Where $N(x)\sim N(0,1)$, and the last line is the price at time t

Example:
What is the price of a European call option $X=(S_T-K)^+$? In the standard BS-model

$$
F(0,s) = \exp\{-rT\}\,\mathbb{E}^Q_{0,s}\left((S_t-K)^+\right) = \exp\{-rT\}\,\mathbb{E}^Q\left(\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma W_T\right\}-K\right)^+\right)
$$

$$
= \exp\{-rT\}\int_a^\infty\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}-K\right)\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx \qquad a=\frac{\ln\left(\frac{K}{S}\right)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
$$

$$
\begin{aligned}
&s\int_a^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(x-\sigma\sqrt{T})^2}{2}\right\}dx-K\exp\{-rT\}\,N(-a)\\
&= s\int_{a-\sigma\sqrt{T}}^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx-K\exp\{-rT\}\,N(-a)\\
&= sN(\sigma\sqrt{T}-a)-K\exp\{-rT\}\,N(-a)
\end{aligned}
$$

Here we used the fact that the normal-distribution has symmetric tails

Sats 10.10: Black-Scholes formula

In teh standard BS-model, the price of a European call option is $F(t,S_t)$, where

$$
F(t,s)=sN(d_1)-K\exp\{-r(T-t)\}\,N(d_2)
$$

and

$$
\begin{cases}
d_1=\dfrac{\ln\left(\frac{S}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\\
d_2=d_1-\sigma\sqrt{T-t}
\end{cases}
$$

Consider $F(0,s)=sN(d_1)-K\exp\{-rT\}\,N(d_2)$ as above, then we have

$$
F(0,s)=\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)^+\right)\leq\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T)\right)=s
$$

and

$$
F(0,s)=\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)^+\right)\geq\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)\right)=s-K\exp\{-rT\}
$$

We shall see below that $F(0,s)=F(0,s;\sigma)$ is increasing in σ

Remark:

What about the put option?

$$\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}(K-S_T)^+\right) = \text{ similar to above}$$

Alternatively, $(K-s)^+ = K - s + (s-K)^+$. We have priced $(s-K)^+$, and s, so $p(0,s) = K\exp\{-rT\} - s + c(0,s)$ where p is the put price and c is the call price. This relation is called the *put-call parity*
Thus,

$$\begin{aligned} p(0,s) &= K\exp\{-rT\} - s + sN(d_1) - K\exp\{-rT\}N(d_2) \\ &= K\exp\{-rT\}\underbrace{(1-N(d_2))}_{=N(-d_2)} - s\underbrace{(1-N(d_1))}_{=N(-d_1)} \end{aligned}$$

Sats 10.11

Let $F(t,s)$ be the pricing function f a simple T-claim $X = \phi(S_T)$ in the standard BS-model.
If ϕ is convex, then:

1. $F(t,s)$ is convex in s
2. $F(t,s)$ is increasing in σ

Bevis 10.1

$$F(0,s) = \exp\{-rT\}\int_{\mathbb{R}}\phi\left(s\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx$$

1.

$$F_{ss} = \exp\{-rT\}\int_{\mathbb{R}}\phi''\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}\right)\exp2\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx \geq 0$$

2.

$$\begin{aligned} \frac{\partial F}{\partial \sigma} &= \int_{\mathbb{R}}\phi'\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}\right)s\exp\left\{-\frac{\sigma^2 T}{2}+\sigma\sqrt{T}x\right\}\sqrt{T}(x-\sigma\sqrt{T})\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx \\ &= s\sqrt{T}\int_{\mathbb{R}}\phi'\left(\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}\right)(x-\sigma\sqrt{T})\exp\left\{-\frac{(x-\sigma\sqrt{T})^2}{2}\right\}\frac{1}{\sqrt{2\pi}}dx \\ &\stackrel{\text{parts.}}{=} s\sqrt{T}\int_{\mathbb{R}}\phi''()s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}\sigma\sqrt{T}\exp\left\{-\frac{(x-\sigma\sqrt{T})^2}{2}\right\}\frac{1}{\sqrt{2\pi}}dx \geq 0 \end{aligned}$$

$\square$

10.1. **Drift estimation.**
Assume $X_t = \mu_t + \sigma W_t$ and we want a confidence interval for μ. An estimare for μ is $\widehat{\mu} = \dfrac{X_t}{t} \in N\left(\mu, \dfrac{\sigma}{\sqrt{t}}\right)$ and a confidence interval is

$$\left(\widehat{\mu} - \frac{\sigma}{\sqrt{t}} \cdot 1.96, \widehat{\mu} + \frac{\sigma}{\sqrt{t}} \cdot 1.96\right)$$

If one wants a certain precision $\Delta\mu$ so that $\mathbb{P}(\mu \in (\widehat{\mu} - \Delta\mu, \widehat{\mu} + \Delta\mu)) = 0.95$, one needs

$$\frac{2\sigma}{\sqrt{T}} = \Delta\mu \quad \Leftrightarrow \quad t = \frac{4\sigma^2}{(\Delta\mu)^2}$$

Plug in reasonable values $\left.\begin{aligned} \sigma &= 0.3 \\ \Delta\mu &= 0.06 \end{aligned}\right\} \Rightarrow t = 100$ years!

Remark:
When pricing options, the drift of the stock needs not be estimated (since under the pricing measure Q, the drift is r)

11. Volatility

In the BS-formula, s, r, t are observable, T, K are specified in the contract and σ is not directly observable. All are needed.
There are 2 approaches, one using *historic volatility* and one using *implied volatility.*

11.1. **Historic volatility.**
If $dS_t = \mu S_t dt + \sigma S_t dW_t$, then sample S at $n+1$ time points and let

$$\xi_i = \ln\left(\frac{S_{ti}}{S_{t_{i-1}}}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}}) \sim N\left((\mu - \frac{\sigma^2}{2})\Delta t, \sigma\sqrt{\Delta t}\right)$$

An esimate of σ^2 is then $S^2 = \dfrac{\sum_{i=1}^n (\xi_i - \overline{\xi})^2}{(n-1)\Delta t}$ where $\overline{\xi} = \dfrac{1}{n}\sum_{i=1}^n \xi_i$

11.2. **Implied volatility.**
Let p be the price in the market of a certain call option (maturity T, with strike price K). Find σ such that $p = \mathrm{BS}(s, t, T, r, \sigma, K)$ where BS denotes the Black-Scholes formula
This σ is called *implied volatility*

Remark:
Recall that the BS-formula is increasing in σ

If gBm is the correct model (i.e option prices are calculated using the BS-formula), then the *same* implied volatility would be obtained for different K and T

12. Completeness and Hedging

Definition 12.17

A T-claim X can be *replicated* if there exists a self-financing portfolio h with $\mathbb{P}(V_T^h = X) = 1$.
If every T-claim can be replicated then the market is *complete*

Sats 12.12

Assume that a T-claim X can be replicated using h. Then the only possible arbitrage-free price of X is $\Pi_t(X) = V_t^h$

Bevis 12.1

If for example $\Pi_t(X) < V_t^h$ for some t; sell the portfolio and buy the claim $\Rightarrow$ arbitrage $\square$

We now specialize to the model

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \mu(t,S_t)S_t dt + \sigma(t,S_t)S_t dW_t \end{cases} \tag{5}$$

with $\sigma(t,s) > 0$

Sats 12.13

The model (5) is complete

We will prove a simpler result, namely that all *simple* T-claims can be replicated.
Recall that the value $\Pi_t(X)$ of a simple T-claim $X = \phi(S_T)$ is $F(t,S_t)$ where $F(t,s)$ is the pricing function. Thus

$$\begin{aligned} d\Pi_t &= F_t dt + F_s dS_t + \frac{1}{2}F_{ss}(dS_t)^2 \\ &= \left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss}\right)dt + F_s dS_t \end{aligned}$$

Moreover, a portfolio $h = (h^B, h^S)$ is self-financing if $dV_t^h = h_t^B dB_t + h_t^s dS_t$. Choose $h_t^S = F_s(t,S_t)$

Sats 12.14

Let $X = \phi(S_T)$ and define $F(t,s)$ by

$$\begin{cases} F_t + \dfrac{\sigma^2 S^2}{2}F_{ss} + rsF_s - rF = 0 \\ F(T,s)\phi(s) \end{cases}$$

Define $h = (h^B, h^S)$ by

$$\begin{cases} h_t^B = \dfrac{F(t,S_t) - S_t F_s(t,S_t)}{B_t} \\ h_t^S = F_s(t,S_t) \end{cases}$$

Then h replicates X and $\Pi_t(X) = V_t^h = F(t,S_t)$

Bevis 12.2

$V_t^h = h_t^B B_t + h_t^S S_t = F(t,S_t)$, so

$$\begin{aligned} dV_t^h &= F_t dt + F_s dS_t + \frac{1}{2}F_{ss}(dS_t)^2 \\ &= \left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss}\right)dt + F_s dS_t \\ &\overset{\text{BS PDE}}{=} r(F - S_t F_s)dt + F_s dS_t = h_t^B dB_t + h_t^S dS_t \end{aligned}$$

Thus h is self-financing. Since $V_T^h = F(T,S_t) = \phi(S_T) = X$, h replicates X.
By no-arbitrage $\Pi_t(X) = V_t^h = F(t,S_t)$ $\square$

Example:
If $X = S_T$, then $F(t,s) = s$, so $h_t^S = F_s = 1$

Example:

For a call option (in the standard BS-model), $F(0,s) = sN(d_1) - K\exp\{-rT\}N(d_2)$, thus

$$F_S(0,s) = N(d_1) + \frac{1}{\sqrt{2\pi}}\left(s\exp\left\{-\frac{d_1^2}{2}\right\} - K\exp\{-rT\}\exp\left\{-\frac{d_2^2}{2}\right\}\right)\frac{\partial d_1}{\partial s}$$

Moreoever,

$$s\exp\left\{-\frac{d_1^2}{2}\right\} - K\exp\{-rT\}\exp\left\{-\frac{d_2^2}{2}\right\} = \exp\left\{-\frac{d^2}{2}\right\}\left(s - K\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{\sigma\sqrt{T}d_1\right\}\right) = 0$$

so $F_s(0,s) = N(d_1)$

Remark:
The derivative $\Delta = F_s$ is called the *delta*.
In a replicating portfolio one should hold Δ shares of S at each time.

If the pricing function is convex in S, then in order to replicate it then Δ goes up then buy more stock. Conversely, sell off if the opposite.

Example:
For a call option in the standard BS-model

$$F(0,s) = sN(d_1) - K\exp\{-rT\}N(d_2)$$

Where
$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{s}{K}\right) + (r + \dfrac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 = \dfrac{\ln\left(\dfrac{s}{K}\right) + (r - \dfrac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{cases}$$

Thus

$$\begin{aligned}\Delta = F_s(0,s) &= N(d_1) + s\varphi(d_1)\frac{1}{s\sigma\sqrt{T}} - K\exp\{-rT\}\varphi(d_2)\frac{1}{s\sigma\sqrt{T}} \\ &= N(d_1) + \frac{1}{\sigma\sqrt{T}}\left(\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)\right)\end{aligned}$$

Where

$$N(x) = \int_{-\infty}^{x}\varphi(z)dz$$
$$\varphi(z) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}$$

The claim is that we are left with 0 on the second term, we check:

$$\begin{aligned}\sqrt{2\pi}\frac{\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)}{} &\overset{d_2=d_1-\sigma\sqrt{T}}{=} \exp\left\{-\frac{d_1^2}{2}\right\} - \frac{K}{s}\exp\{-rT\}\exp\left\{-\frac{\left(d_1-\sigma\sqrt{T}\right)^2}{2}\right\} \\ &= \exp\left\{-\frac{d_1^2}{2}\right\}\left(1 - \frac{K}{s}\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{d_1\sigma\sqrt{T}\right\}\right) \\ &= \exp\left\{-\frac{d_1^2}{2}\right\}\left(1 - \frac{K}{s}\underbrace{\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{\ln\left(\frac{s}{K}\right) + (r+\sigma^2/2)T\right\}}_{\frac{s}{K}}\right) = 0 \\ &\Rightarrow N(d_1) + \frac{1}{\sigma\sqrt{T}}\left(\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)\right) = N(d_1)\end{aligned}$$

The Δ is simply the first derivative of the pricing function.

13. Volatility Mis-specification

Assume that a trader believes in

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

whereas the stock actually follows

$$d\,\widetilde{S}_t = \widetilde{\mu}\,(t, \widetilde{S}_t)\,\widetilde{S}_t\,dt + \widetilde{\sigma}\,(t, \widetilde{S}_t) d\,\widetilde{W}_t$$

What happens if the trader tries to replicate a simple T-claim $x = \phi(\widetilde{S}_T)$?

The trader solves $\begin{cases} F_t + \dfrac{\sigma^2}{2} s^2 F_{ss} + rsF_s - rF = 0 \\ F(T, s) = \phi(s) \end{cases}$ and constructs a portfolio $h = (h^B, h^S)$ with initial value $V_0^h = F(0, s)$ containing $F_s(t, \widetilde{\S}_t)$ shares of $\widetilde{S}$ at each time (and $V_t^h - \widetilde{S}_t\ F_s(t, S_t))$ in the bank account.

The *tracking error* $Y_t = V_t^h - F(t, \widetilde{S}_t)$ satisfies $Y_0 = 0$ and

$$\begin{aligned} dY_t &= r(V_t^h - \widetilde{S}_t\ F_s)dt + F_s d\,\widetilde{S} - \left(F_t dt + F_s d\,\widetilde{S}_t + \frac{1}{2}\,\widetilde{\sigma}^2\,\widetilde{S}_t^{\,2}\ F_{ss}dt\right) \\ &= rV_t^h dt - \underbrace{\left(F_t + \frac{1}{2}\sigma^2\,\widetilde{S}^2\ F_{ss} + r\,\widetilde{S}_t\ F_s\right)}_{rF} dt + \frac{\sigma^2 - \widetilde{\sigma}^2}{2}\ \widetilde{S}_t^{\,2}\ F_{ss}dt \\ &= rY_t dt + \frac{\sigma^2 - \widetilde{\sigma}^2}{2}\ \widetilde{S}_t^{\,2}\ F_{ss}dt \end{aligned}$$

Thus, if $\sigma^2 \geq \widetilde{\sigma}^2$ and $F_\sigma \geq 0$, then $Y(T) = V(T) - \phi(\widetilde{S}_T) \geq 0$

A trader who overestimates volatility and who uses a model with a convex price will superreplicate the claim!

14. Asian Options

Asian options are option on the *average* of S.

An Asian call option pays $\chi = \left(\dfrac{1}{T}\int_0^T S_t dt - K\right)^+$ at T.

Note, it is not a simple T-claim!

Sats 14.15

Let $\chi = \phi(S_T, Z_T)$, where $Z_t = \int_0^t g(u, S_u)du$ for some function g.
Let $F(t, s, z)$ solve

$$\begin{cases} F_t + \dfrac{\sigma^2 s^2}{2} F_{ss} + rsF_s + g(t,s)F_z - rF = 0 \\ F(T, s, z) = \phi(s, Z) \end{cases}$$

and let $\begin{cases} h_t^B = \dfrac{F(t, S_t, Z_t) - S_t F_s(t, S_t, Z_t)}{B_t} \\ h_t^S = F_s(t, S_t, Z_t) \end{cases}$
Then h is self-financing and it replicates χ, with

$$\Pi_t(\chi) = V_t^h = F(t, S_t, Z_t)$$

Moreover, $F(t, s, Z) = \exp\{-r(T-t)\} \mathbb{E}_{t,s,z}^Q [\phi(S_T, Z_T)]$
where the Q-dynamics are

$$\begin{cases} dS_u = rS_u du + \sigma(u, S_u) S_u dW_u^Q \\ S_t = s \\ dZ_u = g(u, S_u)du \\ Z_t = z \end{cases}$$

Bevis 14.1

$$V_t^h = h_t^B B_t + h_t^S S_t = F(t, S_t, Z_t)$$

In particular, $V_T^h = F(T, S_T, Z_T) = \phi(S_T, Z_T) = \chi$
Moreover,

$$dV_t^h \overset{\text{Ito}}{=} F_t dt + F_s dS_t + \underbrace{F_z dZ_t}_{gdt} + \frac{1}{2} F_{ss}(dS_t)^2 + \underbrace{\frac{1}{2} F_{zz}(dZ)^2}_{=0} + F_{sz} \underbrace{dSdZ}_{=0}$$

$$= \underbrace{\left(F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} + g(t, S_t)F_z \right)}_{=r(F - S_t F_s) \text{ by BS PDE}} dt + F_s dS_t$$

$$= r(F - S_t F_s)dt + F_s dS_t = h_t^B dB_t + h_t^S dS_t$$

So h is self-financing and replicates χ
Therefore, by no arbitrage, $\Pi_t(\chi) = V_t^h = F(t, S_t, Z_t)$
Finally, the stochastic representation follows from Feynman-Kac □

Example:
$\chi = \dfrac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u du$ paid at T_2
What is the value of the T_2-claim χ at time 0?

$$\mathbb{E}_{t,s}^Q \left[\exp\{-r(T_2 - t)\} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u du \right] = \frac{\exp\{-r(T_2 - t)\}}{T_2 - T_1} \int_{T_1}^{T_2} \underbrace{\mathbb{E}_{t,s}[S_u]}_{s\exp\{r(u-t)\}} du$$

$$= \frac{\exp\{-r(T_2 - t)\}}{T_2 - T_1} \frac{s}{r} \left(\exp\{r(T_2 - t)\} - \exp\{r(T_1 - t)\} \right)$$

$$= \frac{s}{r(T_2 - T_1)} \left(1 - \exp\{-r(T_2 - T_1)\} \right)$$

Which yields the answer, i.e the price is $\dfrac{S_t}{r(T_2 - T_1)} \left(1 - \exp\{-r(T_2 - T_1)\} \right)$

Remark:
All T-claims χ are priced as $\mathbb{E}^Q [\exp\{-rT\} \chi]$ (not only simple T-claims and Asian options)

Remark:
What is the value of χ in the previous exercise at $t \in [T_1, T_2]$?

$$\chi = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_u du = \underbrace{\frac{1}{T_2 - T_1}\int_{T_1}^{t} S_u du}_{\text{known at } t} + \underbrace{\frac{1}{T_2 - T_1}\int_{t}^{T_2} S_u du}_{y}$$

Price of y:

$$\begin{aligned}
&\mathbb{E}^Q_{t,s}\left[\exp\{-r(T_2-t)\}\frac{1}{T_2-T_1}\int_t^{T_2} S_u du\right] \\
&= \frac{\exp\{-r(T_2-t)\}}{T_2-T_1}\int_t^{T_2} s\exp\{r(u-t)\}\,du \\
&= \frac{s}{r(T_2-T_1)}\left(1-\exp\{-r(T_2-t)\}\right)
\end{aligned}$$

The answer is $\frac{1}{T_2 - T_1}\left(\exp\{-r(T_2-t)\}\int_{T_1}^t S_u du + \frac{S_t}{r}\left(1-\exp\{-r(T_2-t)\}\right)\right)$

14.1. **Completeness vs Absence of Arbitrage.**

1. The BS-model $\begin{cases} dB_t = rB_t dt \\ dS_t = \mu S_t dt + \sigma S_t dW_t \end{cases}$ is arbitrage-free and complete

2. The model

$$\begin{aligned} dB_t &= rB_t dt \\ dS^1_t &= \mu_1 S^1_t dt + \sigma_1 S^1_t dW_t \\ dS^2_t &= \mu_2 S^1_t dt + \sigma_2 S^2_t dW_t \end{aligned}$$

is complete, but (typically) *not* arbitrage free since one may construct a portfolio in S^1, S^2 with do dW term and with local mean rate of return $\neq r$

3. The model

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t dt + \sigma_1 S_t dW^1_t + \sigma_2 S_t dW^2_t \end{aligned}$$

is arbitrage-free but *not* complete since $\chi = W^1_T$ cannot be replicated

Sats 14.16: Meta-theorem

Let M = the number of traded assets excluding B and R = the number random sources (BMs, Poisson processes) etc. Then:

- Absence of arbitrage $\Leftrightarrow M \leq R$
- Completeness $\Leftrightarrow M \geq R$
- Absence of arbitrage and completeness $\Leftrightarrow M = R$