180111 Solutions/Answers

(1) $\begin{cases} dY(t) = a\,dt + \sqrt{Y(t)}\,dW(t) \\ Y(0) = y \end{cases}$

$$E[Y(t)] = E\left[y + at + \int_0^t \sqrt{Y(s)}\,dW(s)\right] = \underline{y + at}$$

$$Var(Y(t)) = E\left[\left(Y(t) - E[Y(t)]\right)^2\right] = E\left[\left(\int_0^t \sqrt{Y(s)}\,dW(s)\right)^2\right]$$

$$= \int_0^t E[Y(s)]\,ds = \int_0^t (y + as)\,ds = \underline{yt + a\frac{t^2}{2}}$$

Ito isometry

(2) $\begin{cases} u_t + \frac{1}{2}u_{xx} - 2u = 0 \\ u(T,x) = \cos x \end{cases}$

Let $Y_t = \cos(x + W_t). = \cos X_t$

Then $dY_t = -\sin(x + W_t)\,dW - \frac{1}{2}\cos(x + W_t)\,dt$

so $E[Y_T] = (\cos x)\,e^{-\frac{T}{2}}$

Feynman-Kac $\Rightarrow$ $u(t,x) = E_{t,x}\left[e^{-2(T-t)}\cos X(T)\right]$

$$= \underline{e^{-\frac{5}{2}(T-t)}\cos x}$$

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(3) Price is $\dfrac{s^2 e^{-rT}}{(2r + \sigma^2)T}\left(e^{(2r+\sigma^2)T} - 1\right)$

(5) Answer: 18

(6) $800\, e^{-0.005} N(-d_2) + 820\, e^{-0.01} N(d_1)$

where $\begin{cases} d_1 = \dfrac{\ln \frac{41}{40} + 0.035}{\sqrt{0.08}} \\ d_2 = \dfrac{\ln \frac{41}{40} - 0.045}{\sqrt{0.08}} \end{cases}$

(7) $p(0,T) = e^{\frac{\sigma^2}{2a^2}\left(T + \frac{1}{2a}\left(1-e^{-2aT}\right) - \frac{2}{a}\left(1-e^{-aT}\right)\right) - \frac{1}{a}\left(1-e^{-aT}\right) r_0}$

(8) a) $(1-\delta) S$

b) At $t=0$: Buy $1-\delta$ shares of S.

At $t=T_0$: Receive $(1-\delta)\delta S(T_0-)$ in dividends.

The new stock price is $(1-\delta) S(T_0-)$.

Use the dividends to buy δ new shares. You then have

$1-\delta + \delta = 1$ shares.

At $t=T$: You have replicated $\mathcal{X}$!