# Financial Theory – Lecture 6

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Markets with a risk-free asset.
- The Capital Asset Pricing Model.

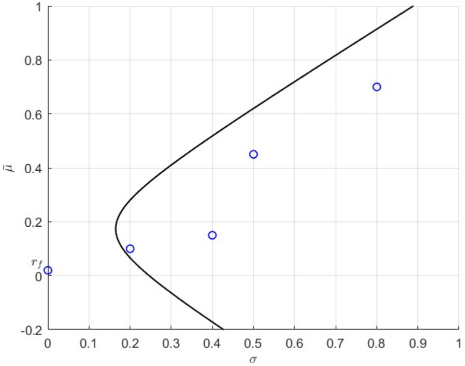
The lecture is based on

- Chapters 7 and 10 in the course book.

# Mean-variance analysis with a risk-free asset

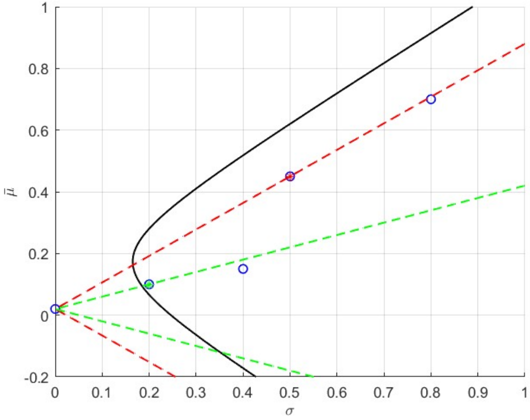
We now assume that there are $N$ risky assets represented by the vector $\mathbf{r}$ and additionally a risk-free asset with rate of return $r_f$.

A risk-free asset has standard deviation equal to 0, so it is represented by a point on the $\bar{\mu}$-axis.



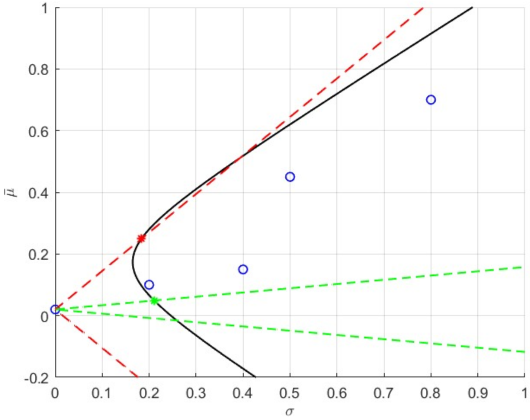
# Mean-variance analysis with a risk-free asset

The combination of an investment in the risk-free asset and any risky asset will result in straight lines in the $(\sigma, \bar{\mu})$-plane.



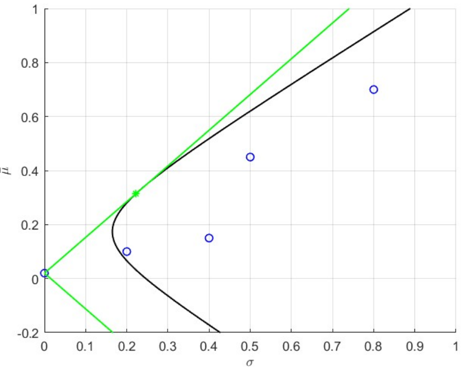
# Mean-variance analysis with a risk-free asset

But we can also combine the risk-free asset with a portfolio on the mean-variance frontier.



# Mean-variance analysis with a risk-free asset

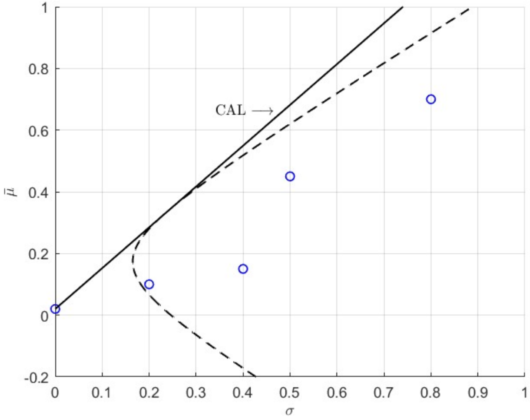
The best frontier when there is a risk-free is achieved if we combine the risk-free asset with the portfolio on the portfolio frontier that makes the straight line tangent to the mean-variance frontier.




This is tangent portfolio with weights $\boldsymbol{\pi}_{\text{tan}}$.

# Mean-variance analysis with a risk-free asset

An investor will only choose a portfolio on the upper straight line. This is called the capital allocation line (CAL).



# Mean-variance analysis with a risk-free asset

Again we have two-fund separation: Every portfolio on the mean-variance frontier can be written as a combined portfolio of the risk-free asset and the tangent portfolio.

Actually, this requires that $B - Cr_f \neq 0$ as we will soon see.

# Mean-variance analysis with a risk-free asset

How do we find the tangent portfolio?

Recall the Sharpe ratio:

$$SR = \frac{\bar{\mu} - r_f}{\sigma}.$$

The Sharpe ratio for a portfolio with weights $\boldsymbol{\pi}$ is

$$SR = \frac{\boldsymbol{\pi} \cdot \boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}}}.$$

Since we know that the tangent portfolio is on the mean-variance frontier, we can write

$$SR = \frac{\bar{\mu} - r_f}{\sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}}$$

and maximise this over $\bar{\mu}$.

This is how it is done in the book. I'll take a different route.

# Mean-variance analysis with a risk-free asset

Again we want to find the portfolio with mean $\bar{\mu}$ that has the smallest variance.

Let $\boldsymbol{\pi}$ be the portfolio weights in the risky assets and let $\pi_0$ be the portfolio weight in the risk-free asset. Thus

$$\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1.$$

The return on this portfolio can be written

$$r_p = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}.$$

Then

$$E\left[r_p\right] = E\left[\pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}\right] = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu}$$

and

$$\mathrm{Var}[r_p] = \mathrm{Var}[\pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}] = \mathrm{Var}[\boldsymbol{\pi} \cdot \mathbf{r}] = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

# Mean-variance analysis with a risk-free asset

Now

$$\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1 \quad \Leftrightarrow \quad \pi_0 = 1 - \boldsymbol{\pi} \cdot \mathbf{1}$$

and we can write

$$E\left[r_p\right] = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu} = (1 - \boldsymbol{\pi} \cdot \mathbf{1}) r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu} = r_f + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1}).$$

This leads to the problem

$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & r_f + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1}) = \overline{\mu}. \end{array}$$

**Note:** We have replaced $\pi_0$ with $1 - \boldsymbol{\pi} \cdot \mathbf{1}$, so the condition $\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1$ is already in place.

# Mean-variance analysis with a risk-free asset

Lagrangian:

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda\big(\bar{\mu} - r_f - \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1})\big).$$

FOC:

$$\frac{\partial L}{\partial \boldsymbol{\pi}} = 2\Sigma\boldsymbol{\pi} - \lambda(\boldsymbol{\mu} - r_f \mathbf{1}) = 0,$$

or

$$\Sigma\boldsymbol{\pi} = \frac{\lambda}{2}(\boldsymbol{\mu} - r_f \mathbf{1}) \quad \Rightarrow \quad \boldsymbol{\pi} = \frac{\lambda}{2}\Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}).$$

At this point we are only interested in the tangent portfolio.

The tangent portfolio $\boldsymbol{\pi}_{\text{tan}}$ satisfies $\mathbf{1} \cdot \boldsymbol{\pi}_{\text{tan}} = 1 \longrightarrow$

$$1 = \mathbf{1} \cdot \boldsymbol{\pi}_{\text{tan}} = \frac{\lambda_{\text{tan}}}{2}\mathbf{1} \cdot \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}) \quad \Rightarrow \quad \frac{\lambda_{\text{tan}}}{2} = \frac{1}{\mathbf{1} \cdot \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1})}.$$

# Mean-variance analysis with a risk-free asset

Using this we can write

$$\begin{aligned}
\boldsymbol{\pi}_{\text{tan}} &= \frac{1}{\mathbf{1} \cdot \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1})} \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}) \\
&= \frac{1}{\mathbf{1} \cdot \Sigma^{-1}\boldsymbol{\mu} - r_f \mathbf{1} \cdot \Sigma^{-1}\mathbf{1}} \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}) \\
&= \frac{1}{B - C r_f} \Sigma^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}).
\end{aligned}$$

We see that if $B - Cr_f = 0$, then there is division by zero.

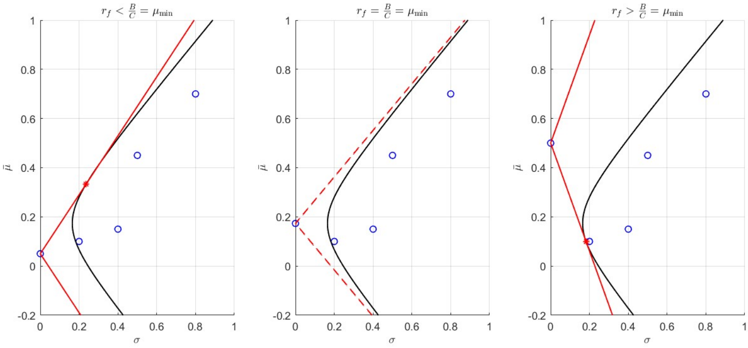
What is the problem?

$$B - Cr_f = 0 \quad \Leftrightarrow \quad B = Cr_f \quad \Leftrightarrow \quad r_f = \frac{B}{C} = \mu_{\text{min}}.$$

# Mean-variance analysis with a risk-free asset

When $r_f = B/C$ there is no tangent to the mean-variance frontier.

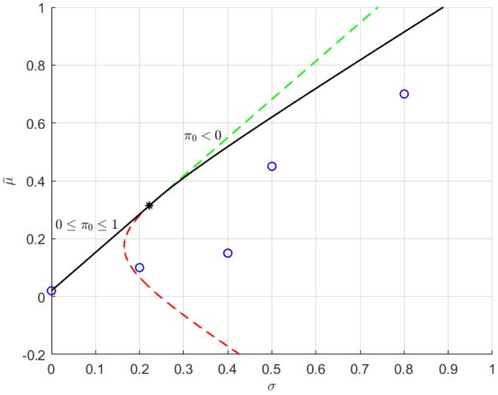
There are three cases:




The first one is the most reasonable from an economic point of view.

# Mean-variance analysis with borrowing constraints

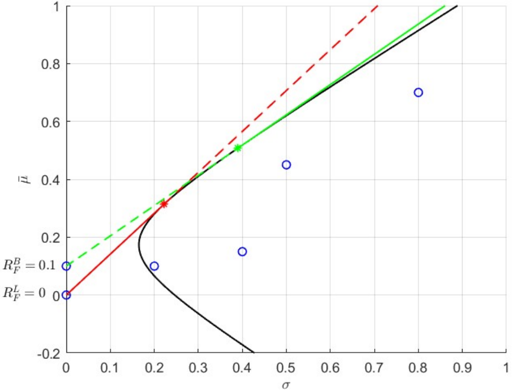
Sometimes an investor is credit constrained in the sense that he or she cannot borrow any money.



# Mean-variance analysis with different borrowing and lending rates

In reality, the lending rate is lower than the borrowing rate.

In this case the mean-variance frontier is given by the following.



# Mean-variance analysis with a risk-free asset

Let $\boldsymbol{\pi}$ be any portfolio weights.

The covariance between the rate of return $r_i$ and the portfolio rate of return $\boldsymbol{\pi}^T \boldsymbol{r}$ is given by

$$\begin{aligned} \mathrm{Cov}[r_i, \boldsymbol{\pi}^T \boldsymbol{r}] &= \mathrm{Cov}\left[ r_i, \sum_{j=1}^{N} \pi_j r_j \right] \\ &= \sum_{j=1}^{N} \mathrm{Cov}[r_i, r_j] \pi_j \\ &= \sum_{j=1}^{N} \Sigma_{ij} \pi_j \\ &= \left( \Sigma \boldsymbol{\pi} \right)_i \longleftarrow \text{The element of row } i \text{ in } \Sigma \boldsymbol{\pi} \end{aligned}$$

# Mean-variance analysis with a risk-free asset

We will now prove that

$$\mathrm{Cov}[r_i, r_{\tan}] = \left(E\left[r_i\right] - r_f\right) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

Start with

$$\mathrm{Cov}[r_i, r_{\tan}] = \mathrm{Cov}[r_i, \boldsymbol{\pi}_{\tan} \cdot \boldsymbol{r}] = \left(\Sigma \boldsymbol{\pi}_{\tan}\right)_i.$$

Now

$$\Sigma \boldsymbol{\pi}_{\tan} = \frac{1}{B - Cr_f} \Sigma \Sigma^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}) = \frac{1}{B - Cr_f} (\boldsymbol{\mu} - r_f \mathbf{1}),$$

so

$$\mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \cdot \left(E\left[r_i\right] - r_f\right).$$

Multiply this with $\pi_{\tan,i}$:

$$\pi_{\tan,i} \mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \cdot \pi_{\tan,i} \left(E\left[r_i\right] - r_f\right).$$

# Mean-variance analysis with a risk-free asset

Sum this equation over $i$:

$$\sum_{i=1}^{N} \pi_{\tan,i} \mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \sum_{i=1}^{N} \pi_{\tan,i} (\mu_i - r_f)$$

$$\Leftrightarrow$$

$$\mathrm{Cov}\Bigg[\underbrace{\sum_{i=1}^{N} \pi_{i,\tan} r_i}_{r_{\tan}}, r_{\tan}\Bigg] = \frac{1}{B - Cr_f} \Bigg( \underbrace{\sum_{i=1}^{N} \pi_{\tan,i} \mu_i}_{= E[r_{\tan}]} - r_f \underbrace{\sum_{i=1}^{N} \pi_{\tan,i}}_{=1} \Bigg)$$

$$\Leftrightarrow$$

$$\mathrm{Cov}[r_{\tan}, r_{\tan}] = \mathrm{Var}[r_{\tan}] = \frac{1}{B - Cr_f} \cdot \left( E\left[r_{\tan}\right] - r_f \right).$$

# Mean-variance analysis with a risk-free asset

It follows that

$$\frac{1}{B - Cr_f} = \frac{\text{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

Finally,

$$\begin{aligned}
\text{Cov}[r_i, r_{\tan}] &= \left(E\left[r_i\right] - r_f\right) \cdot \frac{1}{B - Cr_f} \\
&= \left(E\left[r_i\right] - r_f\right) \cdot \frac{\text{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.
\end{aligned}$$

We will use this result later today!

# The Capital Asset Pricing Theory

Consider an economy with a number of investors.

The investors all agree that the expected rate of return vector is $\mu$ and that the variance-covariance matrix of the returns is $\Sigma$.

There also exists a risk-free asset with rate of return $r_f$.

The investors are mean-variance optimisers, which means that each invester prefers a high mean to a lower one and a low variance to a higher one.

# The Capital Asset Pricing Theory

The market portfolio is the portfolio of all risky assets in the economy.

In the market portfolio every asset that is traded is present.

It consists of all stocks, bonds, real estate, funds and derivatives as well as anything that investors invest in.

**Example** Consider an economy with the following assets:

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 25 | 25 000 | 6.25% |
| B | 1 750 | 100 | 175 000 | 43.75% |
| C | 1 250 | 80 | 100 000 | 25% |
| D | 500 | 200 | 100 000 | 25% |
| | | | 400 000 | 100% |

$$\text{Market cap} = \text{Units} \cdot \text{Price per unit}.$$

$$\text{Market weight} = \frac{\text{Market cap}}{\text{Total market cap}}.$$

# The Capital Asset Pricing Theory

Every investor will invest in a portfolio on the upper efficient frontier (which is a stright line since there is a risk-free asset in the economy).

From two-fund separation we know that every investor will only invest in a combination of the risk-free asset and the tangent portfolio.

Hence, the only risky assets the investors hold is a fraction of the tangent portfolio.

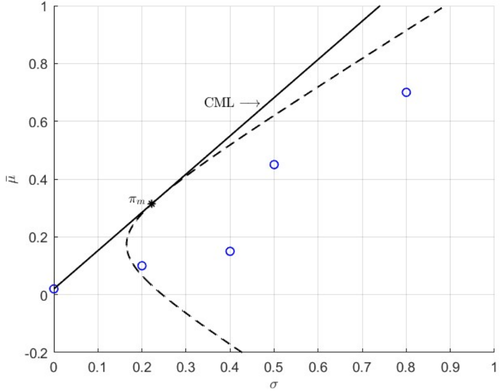
This implies that if the market is in equilibrium, then the market portfolio is equal to the tangent portfolio:

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\text{tan}}$$

From now on, the subindex $m$ stands for quantities connected to the market portfolio.

# The Capital Asset Pricing Theory

In this case the possible portfolios that rational investors choose from is called the capital market line (CML).



# The Capital Asset Pricing Theory

The CML intersects the $\bar{\mu}$-axis at $r_f$ and has slope

$$\frac{E\left[r_m\right]-r_f}{\sigma_m} = \text{The SR of the market portfolio.}$$

The equation is of the CML is

$$\bar{\mu} = r_f + \frac{E\left[r_m\right]-r_f}{\sigma_m}\sigma.$$

This can be written

$$\frac{\bar{\mu}-r_f}{\sigma} = \frac{E\left[r_m\right]-r_f}{\sigma_m};$$

all portfolios on the CML has the same SR.

# The Capital Asset Pricing Theory

Recall that

$$\mathrm{Cov}[r_i, r_{\tan}] = \left(E\left[r_i\right] - r_f\right) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

With $r_{\tan} = r_m$:

$$\left(E\left[r_m\right] - r_f\right)\mathrm{Cov}[r_i, r_m] = \left(E\left[r_i\right] - r_f\right)\mathrm{Var}[r_m]$$

This can be written

$$E\left[r_i\right] - r_f = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}\left(E\left[r_m\right] - r_f\right).$$

This is the Capital Asset Pricing Model (CAPM) equation.

# The Capital Asset Pricing Theory

The CAPM equation can be written:

$$E[r_i] - r_f = \beta_i(E[r_m] - r_f),$$

where

- $r_m$ is the rate of return of the market portfolio, and
- $\beta_i = \dfrac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}$ is the beta-value.

We used:

1) The mathematical formula

$$\mathrm{Cov}[r_i, r_{\mathrm{tan}}] = (E[r_i] - r_f) \cdot \frac{\mathrm{Var}[r_{\mathrm{tan}}]}{E[r_{\mathrm{tan}}] - r_f}.$$

2) The economics result

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\mathrm{tan}}$$

about a market in equilibrium.

# The Capital Asset Pricing Theory

Consider an economy with the following assets:

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 25 | 25 000 | 0.25 |
| B | 3 000 | 25 | 75 000 | 0.75 |
| | | | 100 000 | 1 |

Now assume that the tangent portfolio is $\boldsymbol{\pi}_{\text{tan}} = (0.5, 0.5)^\top$.

The tangent portfolio when all investors use the same $\boldsymbol{\mu}$ and $\Sigma$ is the portfolio of risky assets every investor wants to hold.

Since the price of both assets is the same, but the supply of asset A is bigger than for asset B, this can not be a market in equilibrium.

# The Capital Asset Pricing Theory

What will happen is that the price of asset A will increase to 30, and the price of asset B will decrease to 10 resulting in

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\text{tan}}.$$

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 30 | 30 000 | 0.5 |
| B | 3 000 | 10 | 30 000 | 0.5 |
| | | | 60 000 | 1 |