17. Currency Derivatives

We only discuss **17.1 Pure Currency Derivatives**

$X(t)$ = exchange rate at t = $\dfrac{\text{units of domestic currency}}{\text{units of foreign currency}}$

$\simeq 8.50$ SEK/USD

Given:
$$\begin{cases} dX = \alpha_X X\,dt + \sigma_X X\,d\bar{W} & \\ dB_d = r_d B_d\,dt & \text{(measured in domestic currency)} \\ dB_f = r_f\,dB_f\,dt & \text{(measured in foreign currency)} \end{cases}$$

($\alpha_X, \sigma_X, r_d, r_f$ are constants)

Problem: Price a currency derivative, i.e a T-claim

$$Z = \Phi(X(T)).$$

Ex: If $\Phi(x) = (x-K)^+$, then the owner of Z has the option to buy 1 unit of the foreign currency at time T at price K.

Assumption: All holdings of foreign currency are invested in the foreign bank account.

<u>Expectations</u>: The foreign bank account is a risky asset if quoted in domestic currency. $M = R = 1$ in the meta-theorem, so we expect a unique price of Z. Moreover, owning foreign currency gives you an interest, which is similar to owning a stock that pays dividends.

B_f units of foreign currency is worth XB_f (in domestic currency). Let $\tilde{B}_f := B_f(t)X(t)$.

$$d\tilde{B}_f(t) = B_f\, dX + X\, dB_f$$

$$= (\alpha_X + r_f)\tilde{B}_f\, dt + \sigma_X \tilde{B}_f\, d\bar{W}$$

Risk-neutral valuation gives

$$\Pi(t; Z) = e^{-r_d(T-t)} E^Q_{t,x}\left[\Phi(X(T))\right]$$

What are the Q-dynamics of X?

<u>Answer</u>: All traded (domestic) assets have drift r under Q. Thus $d\tilde{B}_f = r_d \tilde{B}_f\, dt + \sigma_X \tilde{B}_f\, dW$ under Q. $X = \tilde{B}_f / B_f$ gives

$$dX(t) = (r_d - r_f)X\, dt + \sigma_X X\, dW.$$

Proposition 17.2

$\Pi(t; Z) = F(t, X(t))$ where

$$F(t,x) = e^{-r_d(T-t)} E^Q_{t,x}\left[\Phi(X(T))\right]$$

where $\begin{cases} dX(u) = (r_d - r_f) X(u)\,du + \sigma_X X(u)\,dW(u) \\ X(t) = x \end{cases}$ under Q.

Alternatively, $F(t,x)$ solves

$$\begin{cases} F_t + \frac{\sigma_X^2}{2} x^2 F_{xx} + (r_d - r_f) x F_x - r_d F = 0 \\ F(T,x) = \Phi(x) \end{cases}$$

Prop. 17.3 The price of a currency derivative $\Phi(X(T))$

is $F(t,x) = F_0(t, x e^{-r_f(T-t)})$

$\uparrow$

BS. price of Φ

If $\Phi(x) = (x-K)^+$ then

$$F(t,x) = x e^{-r_f(T-t)} \left(N(d_1) - K e^{-r_d(T-t)} N(d_2)\right)$$

where $\begin{cases} d_1 = \dfrac{\ln \frac{x}{K} + (r_d - r_f + \frac{\sigma_X^2}{2})(T-t)}{\sigma_X \sqrt{T-t}} \\ d_2 = d_1 - \sigma_X \sqrt{T-t} \end{cases}$

Ex: Find a replicating portfolio for $Z = X(T)$.

By Prop. 17.3, the initial value of the portfolio should be $xe^{-r_f T}$.

The replicating portfolio:

At $t=0$, invest the amount $xe^{-r_f T}$ (in domestic currency) in the foreign bank account, i.e. $e^{-r_f T}$ in foreign currency. At $t=T$ this has grown to 1 in foreign currency, i.e. $X(T)$ in domestic currency!

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29. Forwards and Futures

A forward contract on a T-claim $\mathcal{X}$, contracted at t with delivery time T is specified by:

- At time T the hdder receives $\mathcal{X}$ from the seller
- At time T, the holder pays $f(t;T;\mathcal{X})$ to the seller.
- The so-called forward price $f(t,T;\mathcal{X})$ is determined at t so that the price of the forward contract at t is zero.

$$0 = \Pi(t;\mathcal{X} - f(t,T;\mathcal{X})) = \Pi(t;\mathcal{X}) - \Pi(t;f(t,T;\mathcal{X}))$$

$$= \Pi(t;\mathcal{X}) - e^{-r(T-t)} f(t,T;\mathcal{X})$$

so $f(t,T;\mathcal{X}) = e^{r(T-t)} \Pi(t;\mathcal{X})$

Proposition 29.3 The forward price is

$$f(t,T;\mathcal{X}) = e^{r(T-t)} \Pi(t;\mathcal{X}).$$

Exercise 29.1 If $X = S(T)$ (non-dividend paying asset) what is its forward price?

Answer: $f(t, T, S(T)) = \Pi(t; S(T))\, e^{r(T-t)} = e^{r(T-t)} S(t)$

What is the value of a forward contract at time s, where $t < s < T$?

Answer: $\Pi(s; X) - e^{-r(T-s)} f(t, T; X)$ (why?)