

UPPSALA UNIVERSITET

FÖRELÄSNINGSANTECKNINGAR

# Inferensteori

*Rami Abou Zahra*

Inlämningsdatum  
November 15, 2022

## CONTENTS

1. TODO	2
2. Data Analysis (K6)	3
2.1. Location Measures	3
2.2. Dispersion measures	3
2.3. Graphical illustration	4
2.4. Data materials in several dimensions	5
3. Statistical Inference	7
4. Estimation	7
4.1. Properties of estimates	8
4.2. Asymptotic properties	10
5. Methods of estimation	12
5.1. Methods of moments	12
5.2. Maximum likelihood	13
5.3. Least Squares	15
5.4. Parameter estimation in standard distributions	17
5.5. Estimation of distributions	18
6. Lesson 1	19
6.1. 727	19
6.2. 7.210	19
6.3. 7.2.12	20
6.4. 7.2.14	21
6.5. 702	21

## 1. TODO

- Experiment in r (QQ-plot of exp vs  $n(0,1)$  data)
- Understand .dat files
- Add proof from book of theorem 4.9
- Problems 7.2.2 in the book
- Stora talens lag
- MLE better than methods of moments
- Derivatan av binomial

## 2. DATA ANALYSIS (K6)

Vi kommer undersöka statistisk säkerställd skillnad (Opinion polls example), hypotestestning (räknar sannolikheten att hypotesen är sann).

### Anmärkning:

Vanligtvis antar vi att datan är normalfördelad, men inte i alla fall (såsom stickprov av lön)

### 2.1. Location Measures.

A data set is given by  $x_1, \dots, x_n$

#### Definition/Sats 2.1: Sample mean

Sample mean is given by:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

#### Definition/Sats 2.2: Median

The "middle value" of the sorted data. Different from the mean.

If  $n$  is even, the median is defined as the mean of the two middle values

#### Definition/Sats 2.3: Mode

This doesn't work if it's continuous data but it can be made discrete (such as age/time)

Mode is the most common data value

### Example:

Let our data points be:

32 34 41 44 45 50 50 54 55 57 58 60 63

Find mean, median mode:

*Mean:* 13 data sets  $\Rightarrow n = 13$ :

$$\frac{1}{13}(32 + 34 + 41 + 44 + 45 + 50 + 50 + 54 + 55 + 57 + 58 + 60 + 63) \approx 49.46$$

*Median:* The middle value is 50

*Mode:* 50 is the only data value appearing more than once.

### Anmärkning:

In this example, the median = mode. This is not always the case!

### 2.2. Dispersion measures.

Describes the "spread" of the data, such as the variance. We have the following:

#### Definition/Sats 2.4: Sample variance

The sample variance is given by:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

**Definition/Sats 2.5: Sample standard variance**

Is given by:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

Also called *sample standard deviation*

**Definition/Sats 2.6: Range**

Variationsbredden. The difference between the largest and the smallest values of the data

**Definition/Sats 2.7: Inter quartile range**

Kvartilavståndet is the difference between the upper and lower quartiles.  
If we have an odd amount of data it is including the median!

**Definition/Sats 2.8: Mid-range**

The mean between the biggest and smallest value in the sample

**Definition/Sats 2.9: Lower/Upper quartile**

The *lower quartile* is the median of the lower half of the data material including the median if  $n$  is odd

The *upper quartile* is the median of the upper half of the data material including the median if  $n$  is odd

**Example:**

0 0 1 1 2 2

Here, the mean is given by  $\frac{(1+1+2+2)}{6} = 1$ .

Therefore, the sample variance is given by  $\frac{4}{5}$  and the sample standard deviation  $\sqrt{\frac{4}{5}}$

We can find the inter quartile range by looking at the half, like this:

[0  $\underbrace{0}_{\Delta}$  1] [1  $\underbrace{2}_{\Delta}$  2]

Therefore, the inter quartile range here is  $2 - 0 = 2$

**2.3. Graphical illustration.****Stem av leafplots:**

```
u = c(32,34,...)
stem(u)
```

**Boxplots:**

Uses quartiles, max min, and median. Useful if you want a quick look at the dispersion of data.

**Bar chart:**

Good for illustrating the frequency of each data point, but for large data points the data is hard to read

**Histogram:**

Attempts to fix the readability issues with the bar chart and is easier to compare with probability density functions.

Easier to manipulate data for readability (use bigger/smaller intervals) (one can ask what the optimal width for a histogram would be)

Very often you can ask if the data follow a normal distribution, which can be hard by just looking at the histogram (because the width varies)

**Thoughts:**

Dynamically widths on histograms, the more sparse data the greater the width and the more dense, the less the width

**QQ-plot:**

Is the data normally distributed? You order your data and construct a table with your data and compare it with if it was normally distributed:

$$\Phi(z) = \frac{i - 0.5}{n}$$

If data was perfectly normal on both axis,  $x_i$  would be a linear function of  $z$ , ie. normally distributed  $N(0, 1)$

We plot  $z$  on the  $x$ -axis and  $x_i$  on the  $y$ -axis

The name comes from quantile-quantile-plot (QQ-plot). It is a graphical way of comparing two probability distributions (sannolikhetsfördelning)

**2.4. Data materials in several dimensions.**

We can calculate correlation through sample covariance:

**Definition/Sats 2.10: Sample covariance**

$$c_{xy} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

**Anmärkning:**

Not scale invariant (if you measure  $x$  in meters and go to cm then it is not the same). Therefore we need to norm it with something, which is where the correlation comes in:

**Definition/Sats 2.11: Sample correlation coefficient**

$$r_{xy} = \frac{c_{xy}}{s_x s_y}$$

Where  $s_x$  and  $s_y$  are the sample standard deviations for  $x$  and  $y$

**Definition/Sats 2.12: Sample correlation satisfies**

The sample correlation coefficient satisfies

$$-1 \leq r_{xy} \leq 1$$

If it is 1, then there is a strong positive correlation (the linear regression has a line with positive derivative), similarly for negative.

When it is 0 there is no *linear* relation. There might be other, for example quadratic relation.

### Bevis 2.1: Sample correlation satisfaction

$$\begin{aligned}
0 &\leq \frac{1}{n-1} \sum_{i=1}^n \left( \frac{x_i - \bar{x}}{s_x} - \frac{y_i - \bar{y}}{s_y} \right)^2 \\
&= \underbrace{\frac{1}{s_x^2} \frac{1}{n-1} \sum_i (x_i - \bar{x})^2}_{s_x^2} + \underbrace{\frac{1}{s_y^2} \frac{1}{n-1} \sum_i (y_i - \bar{y})^2}_{s_y^2} - 2 \underbrace{\frac{1}{s_x s_y} \frac{1}{n-1} \sum_i (x_i - \bar{x})(y_i - \bar{y})}_{c_{xy}} \\
&= 2 - 2r_{xy} \Rightarrow r_{xy} \leq 1 \\
0 &\leq \frac{1}{n-1} \sum_{i=1}^n \left( \frac{x_i - \bar{x}}{s_x} + \frac{y_i - \bar{y}}{s_y} \right)^2 = 2 + 2r_{xy} \\
&\Rightarrow -1 \leq r_{xy}
\end{aligned}$$

□

### Bevis 2.2: Sample correlation satisfaction

There is yet another proof that may be more intuitive for the boundedness of the sample correlation coefficient satisfaction that goes as follows:

Recall that the covariance is an inner-product, and that the variance  $Var(x) = Cov(x, x)$

The "worst" that can happen is that  $x, y$  are independent, whereby the covariance  $Cov(x, y) = E(XY) - E(X)E(Y) = 0$  since  $E(XY) = E(X)E(Y)$

This gives the following:

$$\left| \frac{c_{x,y}}{s_x s_y} \right| = \left| \frac{Cov(x, y)}{\sqrt{Var(x)} \sqrt{Var(y)}} \right| \geq 0$$

Next, we will use the fact that the covariance is an inner product, and therefore the Cauchy-Schwartz inequality holds

Notice that  $Cov(x, x) = Var(x)$

$$\begin{aligned}
|Cov(x, y)|^2 &\leq |Var(x)Var(y)| \Leftrightarrow |Cov(x, y)| \leq \left| \sqrt{Var(x)} \sqrt{Var(y)} \right| \\
&= \left| \frac{Cov(x, y)}{\sqrt{Var(x)} \sqrt{Var(y)}} \right| \leq 1
\end{aligned}$$

From the properties of absolute values, we therefore get  $-1 \leq r_{x,y} \leq 1$

□

### 3. STATISTICAL INFERENCE

**Anmärkning:**

If  $X \sim Hyp$ , then for a large population  $X \sim Bin$  (because the chance of picking the same one in a large population is so small)

**Anmärkning:**

If  $X_1, \dots, X_n$  are independent and equally distributed  $N(\mu, \sigma^2)$  variables, then

$$\sum_{i=1}^n X_i \sim N(n\mu, n\sigma^2)$$

In the same way, the mean of these random variables is normally distributed with

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

**Example (\*):**

In an opinion poll, 1000 randomly selected voters are asked about their political sympathies. Let  $X$  be the number of these voters who sympathise with the party  $P$ .  $X \sim Hyp$ , but because the number of voters is so large, we may assume that  $X \sim Bin(1000, p)$

Suppose we know that 100 of the selected voters sympathise with  $P$ , what can  $p$  be?

Well, it makes sense that  $p = \frac{100}{1000}$ , the probability of choosing 100 from our 1000 people.

**Definition/Sats 3.1: Sample (stickprov)**

$x_1, x_2, \dots, x_n$  is a *sample* from the random variable  $X$  with distribution  $F_X$

If  $X = (X_1, \dots, X_n)$  are independent, we have a *random sample* from  $X$

We can purposely choose our random variable in such way that makes it easier for us to analyse. It also allows us to compare these observations.

**Example:**

Using the same environment as example (\*), we can either have a random variable  $X \sim Bin(1000, p)$ , or we can have let the 1000 people all have an associated Bernoulli distributed random variable.

In the first case, our observed sample has size  $n = 1$ , so our  $x_1 = 100$  is an observation of the random variable  $X_1$

In the second case, our observed sample has size  $n = 1000$ , with each  $X_1, \dots, X_{1000} \sim Be(p)$  and

$$\sum_{i=1}^{1000} x_i = 100$$

The pros in splitting up our situation into smaller random variables, is as touched upon previously, easier to analyse since it is easier to find independent random variables.

### 4. ESTIMATION

Suppose we already know what our distribution function  $F$  is. We know this  $F$  is associated to our random variable, that we have our measured observations on.

We also know that our distribution function normally takes on certain *parameters*, for example, the Poisson distribution

$$f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Is actually a function of both  $x$  and  $\lambda$ .

We call  $\lambda$  in our case, an *unknown parameter*. Given enough data, and knowing the distribution function, we should be able to estimate what the value of this parameter is.



We can write the following for our data:

$$x = (x_1, x_2, \dots, x_n)$$

$$X = (X_1, X_2, \dots, X_n)$$

#### Definition/Sats 4.1: Estimate (skattning)

An *estimate*  $\theta^* = \theta^*(x)$  is a function of the sample  $x$

The estimate is an observation of the estimator  $\theta^*(X)$

This attempts to put the previous paragraphs into "functions". Given a sample from our random variable (given in  $x = x_1, \dots$ ), we want to find the unknown parameter  $\theta$ . We can then construct a general formula for *any* data given that it comes from the random variable we have agreed upon beforehand. This is the *estimate* function, as defined above, of a given sample  $x$

#### Example:

Let  $x = (x_1, x_2, x_3, x_4, x_5) = (200, 185, 210, 190, 190)$  be a random sample from  $X \sim N(\mu, 100)$  (every  $X_i \sim N(\mu, 100)$ )

In order to estimate  $\mu$  by the sample mean, we induce the function  $\mu^*$  on our sample set and calculate the mean:

$$\mu^*(x) = \bar{x} = 195$$

Since the estimate is an observation of the *estimator*, let's look at the estimator:

$$\mu^*(X) = \bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i \sim N(\mu, 100) \Rightarrow \bar{X} \sim N(\mu, 100/5) = N(\mu, 20)$$

#### Example:

Using the poll example from above, what we really did when we said that  $p$  reasonably has to be  $\frac{100}{1000}$  is determine an estimate  $p^*$

The greater the sample size the better the estimate (because less and less variance)

#### Anmärkning:

The estimator is not distributed with the same distribution, since the estimator is not always an integer. We saw this in the above example, where the estimator was  $N(\mu, 20)$  distributed but our estimate was  $N(\mu, 100)$  distributed.

If we have different estimates, we need to make a reasonable choice such that our error is as little as possible (this is why we introduce estimators)

#### 4.1. Properties of estimates.

The purpose of our estimates is to estimate  $\theta$ . When we calculate using our function  $\theta^*$  on our sample data we get a value that may or may not deviate from the actual value  $\theta$

We can take  $\theta^* - \theta = E(\theta^*(X)) - \theta + (\theta^* - E(\theta^*(X)))$

It turns out, this is equal to the systematic error + random error

#### Definition/Sats 4.2: Unbiased (väntevärdesriktigt)

An estimate  $\theta^*$  is said to be *unbiased* if it satisfies  $E(\theta^*(X)) - \theta = 0$

This is the same as saying it has no systematic error (therefore, we only have the random error left)

**Example:**

We show this by:

$$E(\mu^*(X)) = E(\bar{X}) = \mu$$

Let

$$p^*(X) = \frac{X}{1000} \quad X \sim \text{Bin}(1000, p)$$

Is  $p^*(x)$  an unbiased estimate of  $p$ ?

Take the expected value function on both sides:

$$\begin{aligned} E\left(\frac{X}{1000}\right) &= E(p^*(X)) \quad X \sim \text{Bin}(1000, p) \\ \frac{1}{1000}E(X) &= \frac{1}{1000} \cdot 1000 \cdot p = p \\ \Rightarrow E(p^*(X)) &= p \Rightarrow p - p = 0 \end{aligned}$$

If we have more than one unbiased estimate, which is the best one? Well, in that case we need to start looking at the random error. We can study this by looking at the variance

**Definition/Sats 4.3: Efficiency comparison of estimates**

If  $\theta_1^*$  and  $\theta_2^*$  are unbiased estimates of  $\theta$  and

$$V(\theta_1^*(X)) \leq V(\theta_2^*(X))$$

For all  $\theta$  with strict inequality for some. We say that  $\theta_1^*$  is more *efficient* than  $\theta_2^*$  (less random error)

**Example:**

See slide 6 & 7

**Example:** (stratification)

We are interested in the proportion  $p$  of Swedish citizens that last year have traveled by plane in connection with work. Take a sample consisting of  $n = 1000$  people

It is safe to assume that the number of men that take the plane will be greater than the number of woman, we can denote this using  $p \pm a$ . Since we are looking at the combinations of ways we can choose men and women from our set, a reasonable distribution would be a binomial distribution for men  $\text{Bin}(500, p + a)$  while for women  $\text{Bin}(500, p - a)$ . One can also look at Bernoulli distributions and get the same results.

$$\begin{aligned} p_1^*(X) &= \frac{1}{1000}X \Rightarrow E(p_1^*) = p \\ p_2^*(y, z) &= \frac{1}{1000}y + \frac{1}{1000}z \\ &\Rightarrow \frac{1}{1000}E(y) + \frac{1}{1000}E(z) \\ \frac{1}{1000}500(p + a) + \frac{1}{1000}500(p - a) &= p \\ V(p_1^*) &= \frac{p(1-p)}{1000} \geq V(p_2^*) = \frac{p(1-p)}{1000} - \frac{a^2}{1000} \end{aligned}$$

**Definition/Sats 4.4: Standard error (medelfelet)**

We want to assign a numerical value to the dispersion of an estimate.

Therefore, we define the *standard error* of the estimate  $\theta^*$  is an estimate of the standard deviation  $D(\theta^*)$

Denoted by  $d(\theta^*(x)) = d(\theta^*)$

Recall that the standard deviation is given by  $\sqrt{Var}$

**Example:**

Given  $x = 100$  (one observation) of the random variable  $X \sim Bin(1000, p)$ , estimate  $p^* = \frac{x}{1000} = 0.1$  and calculate the standard error of this estimate.

**4.2. Asymptotic properties.**

The accuracy of an estimate should improve as the sample size increases, seems reasonable

**Definition/Sats 4.5: Bias**

The *bias* (väntevärdesfelet) for the estimate  $\theta^*$  is defined as

$$B(\theta^*) = E(\theta^*) - \theta$$

**Anmärkning:**

An unbiased estimate has bias 0

**Definition/Sats 4.6: Asymptotically unbiased**

If the bias  $B(\theta_n^*)$  tends to zero as  $n \rightarrow \infty$  for all  $\theta$ , the estimate  $\theta_n^*$  is said to be *asymptotically unbiased*

Think of it like, the more data you gather, the less the bias.

**Example:**

Let  $x_1, \dots, x_n$  be a random sample from  $N(\mu, \sigma^2)$  where  $\mu$  is unknown. We want to estimate  $\sigma^2$

The estimate  $\sigma_n^{2*} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$  is biased, but it is asymptotically unbiased

The estimate  $s_n^2$  is unbiased for  $\sigma^2$ , why?

$$\begin{aligned}
s^2 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \sum_{i=1}^n x_i^2 - 2\bar{x} \underbrace{\sum_{i=1}^n x_i}_{nx} - n\bar{x}^2 \\
&= \frac{1}{n-1} \left( \sum_{i=1}^n x_i^2 - n\bar{x}^2 \right) \\
E(s^2) &= \frac{1}{n-1} \left( \sum_{i=1}^n E(x_i^2) - nE(\bar{x}^2) \right) \\
E(x_i^2) &= V(x_i) + (E(x_i))^2 = \sigma^2 + \mu^2 \\
E(\bar{x}^2) &= V(\bar{x}) + (E(\bar{x}))^2 = \frac{\sigma^2}{n} + \mu^2 \\
E(s^2) &= \frac{1}{n-1} \left( \sum_{i=1}^n (\sigma^2 + \mu^2) - n \left( \frac{\sigma^2}{n} + \mu^2 \right) \right) \\
&= \frac{1}{n-1} (n(\sigma^2 + \mu^2) - \sigma^2 - n\mu^2) = \frac{1}{n-1} (n-1)\sigma^2 = \sigma^2
\end{aligned}$$

#### Definition/Sats 4.7: Consistent estimate

The estimate  $\theta_n^*$  is said to be *consistent* for  $\theta$  if the corresponding estimator converges to  $\theta$  in probability for all  $\theta$

#### Definition/Sats 4.8: Convergence in probability

The estimator  $\theta_n^*$  converges to  $\theta$  in probability if  $\forall \varepsilon > 0$ :

$$\lim_{n \rightarrow \infty} P(|\theta_n^* - \theta| > \varepsilon) = 0$$

#### Definition/Sats 4.9

If the estimate  $\theta_n^*$  is asymptotically unbiased and

$$\lim_{n \rightarrow \infty} V(\theta_n^*) = 0$$

Then it is consistent

#### Example:

Let  $x_1, \dots, x_n$  be a random sample from  $N(\mu, \sigma^2)$  where  $\sigma^2$  is known.  
Estimate  $\mu$  by

$$\mu_n^* = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Show that the estimate is unbiased:  $\mu_n^* - \mu = 0$

Calculate the variance for the corresponding estimator:  $\frac{\sigma^2}{n}$

Show that the estimate is consistent:  $\lim_{n \rightarrow \infty} \frac{\sigma^2}{n} = 0$

If an estimate is not consistent, then it does not matter if the sample size is increased, it wont yield better results.

## 5. METHODS OF ESTIMATION

## 5.1. Methods of moments.

Often, this method works quite well but it also tends to fail (**CHECK**)

**Definition/Sats 5.1: Method of moments**

Let  $x_1, \dots, x_n$  be a random sample from the random variable  $X$  with  $E(X) = m(\theta)$  where  $\theta$  is the parameter in the distribution for  $X$

If  $\theta$  is one dimensional, the moment estimate  $\theta = \theta^*$  solves the equation  $m(\theta) = \bar{x}$

**Example:**

Slide 1

$$m(\beta) = E(X) = \frac{1}{\beta}$$

$$\text{Solve } \bar{x} = m(\beta) = \frac{1}{\beta} \Rightarrow \beta = \frac{1}{\bar{x}}$$

Therefore, moment estimate is  $\beta^* = \frac{1}{\bar{x}}$

**Example:**

Slide 2

$$m(p) = E(X) = np$$

$$\text{Solve } x = \bar{x} = m(p) = np \Rightarrow p = \frac{x}{n} \text{ (recall political party example)}$$

Moment estimation is  $p^* = \frac{x}{n}$

**Example:**

Slide 3 (a third of those 15 birds have rings, so we can get 30 from there)

The number of birds captured with ring =  $X \sim \text{Hyp}(N, n, m)$ , where  $N$  = number of birds,  $n$  = how many were captured the second day (15) and  $m$  = how many with a ring in total the second day = 10:

$$p_X(x) = \frac{\binom{m}{x} \binom{N-m}{n-x}}{\binom{N}{n}}$$

We want to estimate  $N$  using the method of moments:

$$E(X) = n \frac{m}{N} = 15 \frac{10}{N} = \frac{150}{N}$$

Solve for  $f = x = \bar{x} = \frac{150}{N} \Rightarrow N = \frac{150}{5} = 30$ . Moment estimation is  $N^* = 30$

**Definition/Sats 5.2: Method of moments with multiple parameters**

If the parameter  $\theta = (\theta_1, \theta_2)$ , then the moment estimates solves the system:

$$E(X) = m_1(\theta_1, \theta_2) = \bar{x}$$

$$E(X^2) = m_2(\theta_1, \theta_2) = \frac{1}{n} \sum_{i=1}^n x_i^2$$

**Example:**  
Slide 4

$$m_1(\mu, \sigma^2) = E(X) = \mu$$

$$m_2(\mu, \sigma^2) = E(X^2) = V(X) + (E(X))^2 = \sigma^2 + \mu^2$$

Solve

$$\begin{cases} \mu = \bar{x} \\ \sigma^2 + \mu^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 \end{cases}$$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Almost looks like  $s^2$ , but here in the denominator we have  $n$  instead of  $n - 1$

Moment estimates are:

$$\begin{cases} \mu^* = \bar{x} \\ \sigma^{2*} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \end{cases}$$

### Definition/Sats 5.3

Let  $x_1, \dots, x_n$  be a random sample from the random variable  $X$  where  $V(X) = \sigma^2$

Then the sample variance is given by:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

is an *unbiased estimate* of  $\sigma^2$

## 5.2. Maximum likelihood.

**Example:**  
Slide 5

### Definition/Sats 5.4: Maximum likelihood

Let  $x_1, \dots, x_n$  be a random sample from  $X$  which has distribution  $F(X; \theta)$

The likelihood function  $L(\theta)$  is defined by:

$$L(\theta) = \begin{cases} \prod_{i=1}^n p(x_i; \theta) & X \text{ discrete} \\ \prod_{i=1}^n n f(x_i; \theta) & X \text{ continuous} \end{cases}$$

The *maximum likelihood* estimate (MLE, ML-skattning) of  $\theta$  is the  $\theta$  that maximizes the likelihood function

**Example:**  
Slide 6

**Example:**  
Slide 7

**Example:**  
Slide 8

**Example:**  
Let  $x_1, \dots, x_n$  be a random sample from  $X \sim N(\mu, \sigma^2)$  where  $\mu$  and  $\sigma^2$  are both unknown.

Estimate  $\mu$  and  $\sigma^2$  by MLE:

$$\begin{aligned}
 f_X(x; \mu, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \\
 L(\mu, \sigma^2) &= \prod_{i=1}^n f_X(x_i; \mu, \sigma^2) \\
 &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \\
 &= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \\
 l(\mu, \sigma^2) &= \ln \left( (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \right) \\
 &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\
 \frac{\partial l}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = \frac{1}{\sigma^2} \left( \sum_{i=1}^n x_i - n\mu \right) = \frac{n}{\sigma^2} (\bar{x} - \mu) \\
 \frac{\partial l}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 \\
 \frac{\partial l}{\partial \mu} &= 0 \Rightarrow \mu = \bar{x} \\
 \frac{\partial l}{\partial \sigma^2} &= 0 \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (\mu = \bar{x})
 \end{aligned}$$

MLE is therefore:

$$\begin{aligned}
 \mu^* &= \bar{x} \\
 \sigma^{2*} &= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2
 \end{aligned}$$

**Example:**

Slide 9 (not supposed to follow, research level) (ibland för att få fram skattning måste man ta till med numeriska metoder)

**Example:**

Slide 1, täthetsfunk given by  $f_X(x) = \frac{1}{\theta}$  if  $0 \leq x \leq \theta$  and 0 otherwise, this can be shown using the indicator function  $I\{0 \leq x \leq \theta\}$

Likelihood is given by

$$\prod_{i=1}^n f_X(x_i) = \prod_{i=1}^n \frac{1}{\theta} I\{0 \leq x_i \leq \theta\}$$

This product is only  $\frac{1}{\theta^n} \neq 0$  if all  $x_i \in [0, \theta]$ , otherwise it is 0

This can be expressed using an indicationr function as follows:

$$\frac{1}{\theta^n} I\{0 \leq \min x_i \leq \max x_i \leq \theta\}$$

**Anmärkning:**

The max-point is when  $\theta = \max x_i$ , which is therefore  $\theta^* = 3.0$  (from slide 1)

The method of moment is given by  $\bar{x} = E(X) = m(\theta) = \frac{\theta}{2}$

Then  $\theta = 2\bar{x}$  and  $\bar{x} = 1.25 \Rightarrow \theta^* = 2.5$

But 2.5 is an unreasonable number since  $3 > 2.5$

**Anmärkning:**

MLE is always an underestimate, so we can scale it up a little

**5.3. Least Squares.**

**Definition/Sats 5.5: Least Squares**

Let  $x_1, \dots, x_n$  be a random sample from the random variable  $X$  with  $E(X) = m(\theta)$   
Moreover, let

$$Q(\theta) = \sum_{i=1}^n \{x_i - m(\theta)\}^2$$

The value of  $\theta$  that minimizes  $Q(\theta)$  is called the *least squares* estimates of  $\theta$

**Example:**

Slide 2.

$$\begin{aligned} m(\beta) &= E(X) = \frac{1}{\beta} \\ Q(\beta) &= \sum_{i=1}^n (x_i - \frac{1}{\beta})^2 \\ Q'(\beta) &= \frac{2}{\beta^2} \sum_{i=1}^n (x_i - \frac{1}{\beta}) = \frac{2}{\beta^2} \sum_{i=1}^n x_i - \frac{n}{\beta} \\ &= \frac{2n}{\beta^2} (\bar{x} - \frac{1}{\beta}) = \frac{2n\bar{x}}{\beta^2} - \frac{2n}{\beta^3} \\ Q''(\beta) &= \frac{-4n\bar{x}}{\beta^3} + \frac{6n}{\beta^4} \\ 0 &= Q'(\beta) \Rightarrow \bar{x} = \frac{1}{\beta} \Rightarrow \beta = \frac{1}{\bar{x}} \\ Q''\left(\frac{1}{\bar{x}}\right) &= -\frac{4n\bar{x}}{(1/\bar{x})^3} + \frac{6n}{(1/\bar{x})^4} = 2n\bar{x}^4 > 0 \text{ is a minimum-point} \end{aligned}$$

Therefore, the LSE (least square estimate) is  $\beta^* = \frac{1}{\bar{x}}$ , same as moment estimator.

The fact that it was equal to the moment estimator was no coincidence, this is actually a special case.

**Example:**

Slide 3

$$\begin{aligned} E(X) &= m(\theta) & m'(\theta) &\neq 0 \\ Q(\theta) &= \sum_{i=1}^n \{x_i - m(\theta)\}^2 \\ Q'(\theta) &= -2m'(\theta) \sum_{i=1}^n \{x_i - m(\theta)\} \\ &= -2n \underbrace{m'(\theta)}_{\neq 0} \{\bar{x} - m(\theta)\} \end{aligned}$$



If the sign of the second derivative is positive, then we will see that this method of estimation yields the same answer as the moments of estimation, so let us verify the second derivative:

$$\begin{aligned}
 Q'(\theta) &= 0 \Leftrightarrow \bar{x} = m(\theta) \\
 Q'(\theta) &= -2n\bar{x}m'(\theta) + 2nm'(\theta)m(\theta) \\
 Q''(\theta) &= -2nm''(\theta) + 2nm''(\theta)m(\theta) + 2nm'(\theta)m'(\theta) \\
 &= -2nm''(\theta) \underbrace{\{\bar{x} - m(\theta)\}}_{=0} + 2n \underbrace{\{m'(\theta)\}^2}_{>0} > 0
 \end{aligned}$$

Ok! LSE estimation solves the equation  $\bar{x} = m(\theta)$   $\square$

Therefore, if the question is to calculate with LSE, you can do it using the moment of estimation which is easier

### Definition/Sats 5.6

Let  $x_1, \dots, x_n$  be a random sample where the corresponding random variables  $X_1, \dots, X_n$  have  $E(X_i) = m_i(\theta)$

Moreover, let

$$Q(\theta) = \sum_{i=1}^n \{x_i - m_i(\theta)\}^2$$

### Anmärkning:

This situation may not be handled by the method of moments nor by maximum likelihood, unless the distribution of  $X_i$  is specified (**is not the distribution of  $X_i$  given by definition of estimation?**)

Used in regression

### Example:

slide 4

$$\begin{aligned}
 m_i(v) &= E(Y_i) = \text{time} = \frac{x_i}{v} \\
 Q(v) &= \sum_{i=1}^n \left(y_i - \frac{x_i}{v}\right)^2 \\
 Q'(v) &= \frac{2}{v^2} \sum_{i=1}^n x_i \left(y_i - \frac{x_i}{v}\right) = \frac{2}{v^2} \left(\sum_{i=1}^n x_i y_i - \frac{1}{v} \sum_{i=1}^n x_i^2\right) \\
 0 = Q'(v) &\Rightarrow v = \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i y_i} = \frac{S_{xx}}{S_{xy}} \\
 Q'(v) &= \frac{2}{v^2} S_{xy} - \frac{2}{v^3} S_{xx} \\
 Q''(v) &= -\frac{4}{v^3} S_{xy} + \frac{6}{v^4} S_{xx} \\
 Q''\left(\frac{S_{xx}}{S_{xy}}\right) &= -\frac{4}{(S_{xx}/S_{xy})^3} S_{xy} + \frac{6}{(S_{xx}/S_{xy})^4} S_{xx} = \frac{2S_{xy}^4}{S_{xx}^3} > 0 \\
 v^* &= \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i y_i} = \frac{60^2 + \dots + 400^2}{60 \cdot 6.39 + \dots + 400 \cdot 43.03} \approx 9.54
 \end{aligned}$$

#### 5.4. Parameter estimation in standard distributions.

**Example:**

Two normally distributed random samples with  $v = \sigma^2$  for both of them

$$\begin{aligned}
 L(\mu_1, \mu_2, \sigma^2) &= \prod_{i=1}^{n_1} f_X(x_i) \prod_{j=1}^{n_2} f_Y(y_j) \\
 &= \prod_{i=1}^{n_1} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(x_i - \mu_1)^2} \cdot \prod_{j=1}^{n_2} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(y_j - \mu_2)^2} \\
 &= C v^{-(n_1+n_2)/2} e^{-\frac{1}{2v}A} \\
 A &= \sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \sum_{j=1}^{n_2} (y_j - \mu_2)^2 \\
 l = \log(L(\mu_1, \mu_2, v)) &= -\log(C) - \frac{n_1 + n_2}{2} \log(v) - \frac{1}{2v}A \\
 \frac{\partial l}{\partial \mu_1} &= -\frac{1}{2v} \frac{\partial}{\partial \mu_1} A = -\frac{1}{v} \sum_{i=1}^{n_1} (x_i - \mu_1) = -\frac{n_1}{v} (\bar{x} - \mu_1) = 0 \Rightarrow \mu_1 = \bar{x} \\
 \frac{\partial l}{\partial \mu_2} &= 0 \Rightarrow \mu_2 = \bar{y} \\
 \frac{\partial l}{\partial v} &= -\frac{n_1 + n_2}{2v} + \frac{1}{2v^2} A = 0 \\
 &\Leftrightarrow v = \frac{A}{n_1 + n_2}
 \end{aligned}$$

$$\begin{aligned}
 \mu_1^* = \bar{x} \quad \mu_2^* = \bar{y} \quad v^* &= \frac{1}{n_1 + n_2} \left( \underbrace{\sum_{i=1}^n (x_i - \bar{x})^2}_{(n-1)S_x^2} + \underbrace{\sum_{j=1}^n (y_j - \bar{y})^2}_{(n-1)S_y^2} \right) \\
 &= \frac{(n_1 - 1)S_x^2 + (n_2 - 1)S_y^2}{n_1 + n_2}
 \end{aligned}$$

$$\begin{aligned}
 E(S_x^2) &= \sigma^2 = E(S_y^2) \\
 E(V^*) &= \frac{(n_1 - 1)E(S_x^2) + (n_2 - 1)E(S_y^2)}{n_1 + n_2} = \frac{n_1 - 1 + n_2 - 1}{n_1 + n_2} \sigma^2 \neq \sigma^2 \\
 \text{Men } S_p^2 &= \frac{(n_1 - 1)S_x^2 + (n_2 - 1)S_y^2}{n_1 + n_2 - 2} \text{ är väntevärdesriktigt}
 \end{aligned}$$

### 5.5. Estimation of distributions.

**Example:**

slide 6

We have a random sample from a random variable  $X$  with an unknown distribution

More generally, estimate the distribution function  $F_X(x) = P(X \leq x) \quad \forall x$

**Definition/Sats 5.7: Empirical distribution function**

Let  $x_1, \dots, x_n$  be a random sample from a random variable  $X$ . The *empirical distribution function* for  $X$  is defined as

$$F_n(x) = \frac{A_n(x)}{n}$$

where  $A_n(x)$  is the number of observations in the sample that are smaller than or equal to  $x$

**Anmärkning:**

For every given  $x$ ,  $F_n(x)$  is unbiased and consistent estimate of  $F_X(x)$

## 6. LESSON 1

## 6.1. 727.

Kalle lägger patiens, en gång per kväll, tills den går ut för första gången.  
Under en vecka får han observationerna

3 7 10 5 12 8 4

Bestäm ML-skattningen av  $p = P(\text{patiensen går ut})$

**Lösning:**

Här är slumpvariabeln ffg fördelad.

Låt  $X$  vara antalet gånger tills patiensen går ut, då är fördelningsfunktionen:

$$p_X(k) = (1-p)^{k-1}$$

Vi räknar med ML-skattning, vilket är:

$$\begin{aligned} L(p) &= \prod_{i=1}^n p_X(x_i) = \prod_{i=1}^n (1-p)^{x_i-1} p \\ &= (1-p)^{\sum_{i=1}^n x_i - n} p^n \end{aligned}$$

Vi logariterar:

$$\begin{aligned} l(p) &= \ln \{L(p)\} = \left( \sum_{i=1}^n x_i - n \right) \ln(1-p) + n \ln(p) \\ l'(p) &= - \left( \sum_{i=1}^n x_i - n \right) \frac{1}{1-p} + \frac{n}{p} \\ l''(p) &= - \left( \sum_{i=1}^n x_i - n \right) \left( \frac{1}{1-p} \right)^2 - \frac{n}{p^2} < 0 \Rightarrow \max \\ 0 = l'(p) &\Rightarrow p = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}} = \frac{1}{7} \end{aligned}$$

ML-skattningen är  $p^* = \frac{1}{7}$  vilket är samma som momentskattningen (så blir det ofta men inte alltid)

## 6.2. 7.210.

En enarmad bandit (spel) med vinstchans  $p$   
Albert has spelat 10 ggr och fått 2 vinster  
Beata spelade tills första vinst, gång 7  
ML-skatta  $p$

**Lösning:**

Låt  $X_1$  = antal vinster under 10 spel. Denna slumpvariabel är Binomialfördelad med  $10, p$

Låt  $X_2$  = antalet spel till första vinst. Denna slumpvariabel är ffg fördelad med parameter  $p$

$$\begin{aligned}
L(p) &= p_{X_1}(x_1; p) p_{X_2}(x_2; p) \quad x_1 = 2 \quad x_2 = 7 \\
&= \binom{10}{x_1} p^{x_1} (1-p)^{10-x_1} \cdot (1-p)^{x_2-1} p \\
&= \binom{10}{x_1} p^{x_1+1} (1-p)^{9-x_1+x_2} \\
l(p) &= \ln \{L(p)\} = \ln \binom{10}{x_1} + (x_1+1) \ln(p) + (9-x_1+x_2) \ln(1-p) \\
l'(p) &= (x_1+1) \frac{1}{p} - (9-x_1+x_2) \frac{1}{1-p} \\
l''(p) &= -(x_1+1) \frac{1}{p^2} - (9-x_1+x_2) \frac{1}{(1-p)^2} < 0 \quad \text{om } 9-x_1+x_2 > 0 \quad \text{vilket vi har eftersom } 9-2+7 > 0 \\
0 = l'(p) &\Rightarrow p = \frac{x_1+1}{x_2+10} = \frac{2+1}{7+10} = \frac{3}{17} \approx 0.176
\end{aligned}$$

ML-skattningen är då  $p^* = \frac{3}{17}$

### 6.3. 7.2.12.

Taxi problemet. 7 taxibilar observeras. De är numrerade  $1, \dots, N$   
 Obs, numren 070, 234, 166, 7, 65, 17, 4

ML-skatta  $N$

$X$  = numret på en taxibil, diskret likformigt fördelad på  $(1, 2, \dots, N)$

Sannolikhetsfunktionen är då:

$$p_X(k) = \begin{cases} \frac{1}{N}, & 1 \leq k \leq N \\ 0, & \text{annars} \end{cases}$$

$N$  kan vara hur stort som helst,  $N \in \mathbb{N}^+ =$  rummet av alla positiva heltal

ML-skatta  $N$  (observationer  $x_1, \dots, x_n$ ):

$$L(N) = \prod_{i=1}^n p_X(x_i) = \begin{cases} \left(\frac{1}{N}\right)^n & \text{om } \forall x_i \leq N \\ 0 & \text{annars} \end{cases}$$

(Logaritmera/derivera funkar ej här, man måste tyvärr tänka)

ML-skattningen  $N^*$  inträffar i  $\max x_i = 234$

Momentskattning:  $m(n) = E(X) = \frac{N+1}{2}$

Lös  $\bar{x} = \frac{N+1}{2} \Rightarrow N = 2\bar{x} - 1$

Vi har  $\bar{x} = 84.3$ , momentskattningen blir 167.6, vilket blir en orimlig skattning eftersom vi har en observation som är större.

## 6.4. 7.2.14.

$x_1, x_2$  mätningar av en storhet med värdet  $\mu$

$x_3$  mätning av en storhet med värdet  $2\mu$

Mätningar saknar systematiska fel, men har en slumpfel standardavvikelse  $\sigma$

Bestäm MK-skattningen av  $\mu$  och visa att den är väntevärdesriktigt

Minimera  $Q(\mu) = (x_1 - \mu)^2 + (x_2 - \mu)^2 + (x_3 - 2\mu)^2$ , detta kan vi lösa med derivering:

$$\begin{aligned} Q'(\mu) &= -2(x_1 - \mu) - 2(x_2 - \mu) - 4(x_3 - 2\mu) \\ &= -2x_1 - 2x_2 - 4x_3 + 12\mu \\ Q''(\mu) &= 12 > 0 \quad \text{ger min} \\ 0 = Q'(\mu) &\Leftrightarrow \mu = \frac{1}{12}(2x_1 + 2x_2 + 4x_3) = \frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{1}{3}x_3 \end{aligned}$$

MK-skattningen  $\mu^*$

Estimatorn  $\mu^*(X_1 + X_2 + X_3) = \frac{1}{6}X_1 + \frac{1}{6}X_2 + \frac{1}{3}X_3$

Då blir:

$$\begin{aligned} E\{\mu^*(X_1; X_2; X_3)\} &= \frac{1}{6}E(X_1) + \frac{1}{6}E(X_2) + \frac{1}{3}E(X_3) \\ &= \frac{1}{6}\mu + \frac{1}{6}\mu + \frac{1}{3}2\mu = \mu \end{aligned}$$

Då är  $\mu^*$  väntevärdesriktigt

En annan skattning är:

$$\mu' = \frac{2x_1 + 2x_2 + x_3}{6}$$

Är den väntevärdesriktigt? Vi tittar på motsvarande estimator:

$$\begin{aligned} \mu'(X_1, X_2, X_3) &= \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{6}X_3 \\ E\{\mu'(X_1, X_2, X_3)\} &= \frac{1}{3}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{6}E(X_3) \\ &= \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{6}2\mu = \mu \end{aligned}$$

Ok!

Vilken skattning är effektivast?

Då jämför vi varianserna:

$$\begin{aligned} V(\mu) &= \frac{6}{36}\sigma^2 \\ V(\mu') &= \frac{9}{36}\sigma^2 \end{aligned}$$

## 6.5. 702.

Observationer: 4.0, 1.1, 0.2, 1.2, 2.5, 2.0, 0.7, 1.0 är ett stickprov från en Raylerghfördelning med täthetsfunktion:

$$F_X(x) = axe^{-\frac{ax^2}{2}} \quad x \leq 0$$

ML-skatta  $a$ :

$$\begin{aligned} L(a) &= \prod_{i=1}^n f_X(x_i) = \prod_{i=1}^n a x_i e^{-\frac{a x_i^2}{2}} \\ &= a^n \left( \prod_{i=1}^n x_i \right) e^{-\frac{a}{2} \sum_{i=1}^n x_i^2} \end{aligned}$$

$$l(a) = \ln \{L(a)\} = n \ln(a) + \sum_{i=1}^n \ln x_i - \frac{a}{2} \sum_{i=1}^n x_i^2$$

$$l'(a) = \frac{n}{a} - \frac{1}{2} \sum_{i=1}^n x_i^2$$

$$l''(a) = -\frac{n}{a^2} < 0 \Rightarrow \max$$

$$0 = l'(a) \Rightarrow a = \frac{2n}{\sum_{i=1}^n x_i^2}$$

$$\text{ML skattning } a^* = \frac{2n}{\sum_{i=1}^n x_i^2} = \frac{2 \cdot 8}{30.43} \approx 0.526$$