# Financial Theory – Lecture 1

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Choice under certainty.
- Basics of financial securities and markets.

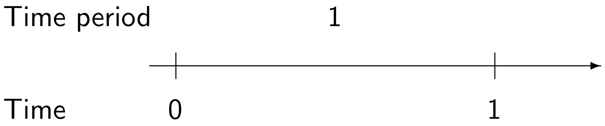
The second part of the lecture is based on

- Sections 1.1-1.7 in the course book.

# Choice under certainty

Consider the following situation:

- A consumer lives for one period.




  Note the difference between a time and a time period.

- The consumption at time $t$ is chosen at time $t$, and is denoted $c_t$. In this model $t = 0, 1$.

# Choice under certainty

- The individual has income $e_t$ at time $t = 0, 1$.
- There is a bank account with interest rate $r$ that the individual can use to lend and borrow money.

The income $e_0$ is diveded into consumption and savings $S$:

$$e_0 = c_0 + S.$$

# Choice under certainty

The individual's consumption at time 1 is given by

$$c_1 = e_1 + S \cdot (1 + r).$$

That is

$$\begin{aligned}\text{Time 1 consumption} &= \text{Time 1 income} \\ &\quad + (\text{Amount saved at time 0}) \cdot (1 + r).\end{aligned}$$

Note that $S$ can be both positive and negative, while $c_0, c_1 \geq 0$.

The consumer can choose any non-negative pair $(c_0, c_1)$ such that

$$c_0 + S = e_0 \quad \text{and} \quad c_1 = e_1 + S(1 + r).$$

# Choice under certainty

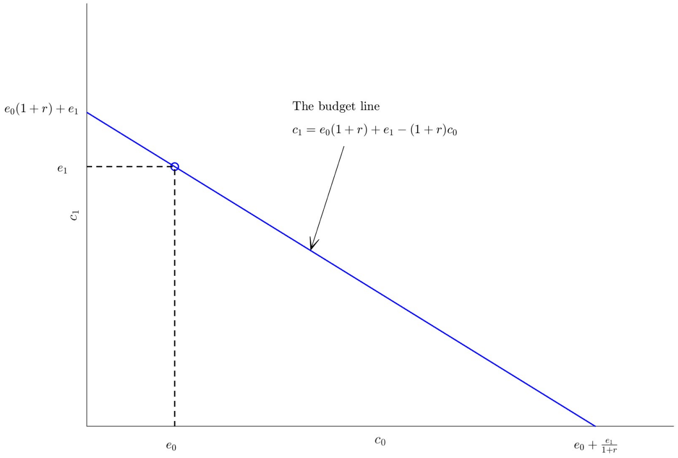
Now, since $c_0 + S = e_0$ we can write $S = e_0 - c_0$. This gives

$$\begin{aligned} c_1 &= e_1 + S(1+r) \\ &= e_1 + (e_0 - c_0) \cdot (1+r) \\ &= e_0(1+r) + e_1 - (1+r)c_0. \end{aligned}$$

This is the budget line.

$$c_1 = \underbrace{e_0(1+r) + e_1}_{\text{Intercept}} \underbrace{-(1+r)}_{\text{Slope}} c_0.$$

# Choice under certainty



# Choice under certainty

The budget line

$$c_1 = e_0(1+r) + e_1 - (1+r)c_0$$

can be written

$$c_0 + \frac{c_1}{1+r} = e_0 + \frac{e_1}{1+r}.$$

The interpretation of this is

Present value of consumption = Present value of income.

# Choice under certainty

Given the budget contraint, we need to choose the optimal consumption $(c_0, c_1)$.

The choice of consumption will depend on the individual's preferences, and will in general be different for different individuals.

The preferences are represented by a utility function, and we can found the optimal consumption pair by using indifference curves.

The individual wants to solve the optimisation problem

$$\max_{c_0, c_1} U(c_0, c_1) \quad \text{subject to} \quad c_0 + \frac{c_1}{1+r} = e_0 + \frac{e_1}{1+r},$$

where $U(c_0, c_1)$ is the utility of consuming the pair $(c_0, c_1)$.

To do this we form the Lagrangian:

$$L = U(c_0, c_1) + \lambda \left( e_0 + \frac{e_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial c_0} &= \frac{\partial U}{\partial c_0} - \lambda = 0 \quad \Leftrightarrow \quad \frac{\partial U}{\partial c_0} = \lambda \\
\frac{\partial L}{\partial c_1} &= \frac{\partial U}{\partial c_1} - \frac{\lambda}{(1+r)} = 0 \quad \Leftrightarrow \quad \frac{\partial U}{\partial c_1} = \frac{\lambda}{(1+r)} \\
\frac{\partial L}{\partial \lambda} &= e_0 + \frac{e_1}{1+r} - c_0 - \frac{c_1}{1+r} = 0.
\end{aligned}$$

# Choice under certainty

The two first conditions can be combined to

$$\frac{\frac{\partial U}{\partial c_0}}{\frac{\partial U}{\partial c_1}} = 1 + r.$$

Fixing a utility level $\overline{U}$,

$$\overline{U} = U(c_0, c_1),$$

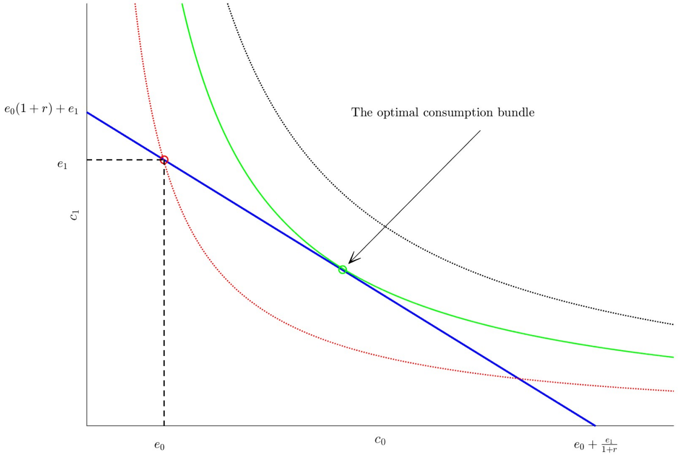
we get

$$0 = \frac{\partial U}{\partial c_0} dc_0 + \frac{\partial U}{\partial c_1} dc_1 \quad \Rightarrow \quad \frac{dc_1}{dc_0} = -\frac{\frac{\partial U}{\partial c_0}}{\frac{\partial U}{\partial c_1}}.$$

Hence, at optimum,

$$\frac{dc_1}{dc_0} = -(1 + r) \;\longleftarrow\; \text{Slope of the budget constraint.}$$

# Choice under certainty



# Choice under certainty

If there was no capital market (in this example represented by the bank account), then we have to consume

$$c_0 = e_0 \quad \text{and} \quad c_1 = e_1.$$

The introduction of a capital market results in the possibility to choose a consumption pair $(c_0, c_1)$ that can give the individual a higher utility than the original one.

# Choice under certainty

Compare this situation to the one you have seen in the courses in microeconomics:

$$\max_{x_1, x_2} U(x_1, x_2) \;\text{ subject to }\; p_1 x_1 + p_2 x_2 = m.$$

Recall our optimisation problem:

$$\max_{c_0, c_1} U(c_0, c_1) \;\text{ subject to }\; 1 \cdot c_0 + \frac{1}{1+r} \cdot c_1 = e_0 + \frac{e_1}{1+r}.$$

- The price of consumption at time 0 is 1.
- The price of consumption at time 1 is $\frac{1}{1+r}$.
- The income is $e_0 + \frac{e_1}{1+r}$.

# Financial markets

From Munk (p. 1):

> *A financial market is simply a market in which one or more financial assets are traded.*

A market can be physical or electronic. Nowadys they are mainly electronic.

- Primary markets: Issuance of new securities.
- Secondary markets: Trading of previously issued securities. Often on an exchange.
- Over-the-counter (OTC) markets: Trading issued securities that are not traded on an exchange.

# Financial markets

On a financial market suppliers of capital (who presently has an excess of capital) meet with users of capital (who presently need capital).

Capital is transferred both across time and across different countries and/or industries.

A financial market can also be used to transfer risk between different participants, pool resources and provide information.

Finally,

> *Financial markets are closely linked to the macroeconomy.*

(Munk, p. 17.)

# Securities

A security can be seen as a contract that gives the right to receive future benefits under some set of conditions.

There are many different types of financial assets – created in order to fulfill one or more demands of the buyer or seller.

Common types of securities are:

- Stocks.
- Bonds.
- Derivatives.

# Stocks

- Also referred to as "common stock".
- Represents a share of the ownership of a company.
- Cash flows generated by holding a stock are the dividends paid out.
- An owner of a stock, also known as a shareholder, has limited liability.
- On the other hand, the shareholders are residual claimants to the company's assets.

# Bonds and other debt-related securities

By issuing a bond, an entity (a company, a government, a municipality,...) can borrow money from investors.

The investor gets interest rate payments (called coupons) and at the maturity date the borrowed amount, known as the face value of the bond, is paid back.

There are other debt intruments – we will return to them when we discuss how bonds are valued.

- Money market: Debt instruments where all payments are within a year of issuance.
- Fixed-income market: Debt instruments where at least one payment is at a time over a year from the issuance.

# Derivative securities

Financial instruments whose value depend on one or more "underlying".

Examples of underlyings are stocks, interest rates and exchange rates.

Examples of derivates:

- Forward contracts.
- Futures contracts.
- Options.
- Swaps.

They are treated in Chapter 14-15 in Munk, and are not included in this course.

# Alternative asset classes

Other asset classes than the previous mentioned are usually referred to as alternative asset classes.

They include:

- Commodities.
- Real estate.
- Infrastructure.

# Indirect investing

Instead of buying financial assets directly, they can be bought indirectly.

- Mutual funds.
- Exchange-traded funds (ETFs).
- Real estate investment trusts (REITs).

# Main players

**Governments, municipalities and other public offices**
Mostly demand capital by issuing bonds to finance budget deficits. Some countries have sovereign wealth funds which supplies capital to the market

**Central banks**
Controls a country's supply of money. By allowing commerical banks to borrow money from the central bank the money supply increases, and by letting them lend money to central bank the money supply decreases.

Central banks can buy financial intruments in the market (quantitative easing).

# Main players

**Other players**

- Financial intermediaries.
    - Commercial banks.
    - Investment banks.
    - Pension funds.
    - Hedge funds.
- Foundations and endowments.

# Financial Theory – Lecture 2

Fredrik Armerin, Uppsala University, 2024

# Agenda

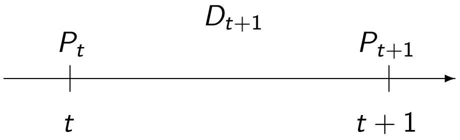
- Measuring the return on an investment.

The lecture is based on

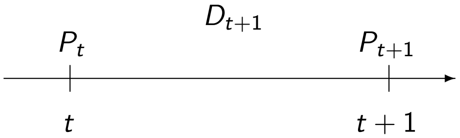
- Chapter 2 in the course book.

# Returns over a single period

- We let $P_t$ and $P_{t+1}$ denote the price of an asset at time $t$ and $t+1$ respectively.
- $D_{t+1}$ is the dividend (cash flow) paid out at time $t+1$ from the asset over the time period $(t, t+1]$.



# Returns over a single period




The rate of return of the asset over the time period $(t, t+1]$ is given by

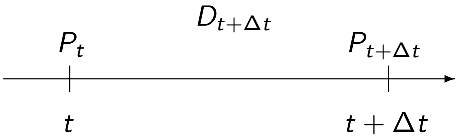
$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} = \frac{D_{t+1} + P_{t+1}}{P_t} - 1.$$

# Returns over a single period

More generally, if the times are $t$ and $t + \Delta t$, with $\Delta t$ being the time period over which we measure the return, we have

$$r_{t,t+\Delta t} = \frac{D_{t+\Delta t} + P_{t+\Delta t} - P_t}{P_t} = \frac{D_{t+\Delta t} + P_{t+\Delta t}}{P_t} - 1.$$

Here $D_{t,t+\Delta t}$ is the dividend over $(t, t + \Delta t]$ paid out at $t + \Delta t$.



# Returns over a single period

We can write

$$r_{t,t+1} = \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t},$$

where

$$\frac{D_{t+1}}{P_t} \text{ is the dividend yield}$$

and

$$\frac{P_{t+1} - P_t}{P_t} \text{ is the capital gain in percent.}$$

# Returns over a single period

**Example**

A stock has price $P_t = 125$ at time $t$ and price $P_{t+1} = 110$ at time $t + 1$.

During the time interval $(t, t+1]$ the stock pays a divident of 20.

In this case

$$r_{t,t+1} = \frac{20 + 110 - 125}{125} = \frac{5}{125} = 4\%,$$

$$\text{Dividend yield} = \frac{20}{125} = 16\%$$

and

$$\text{Capital gain} = \frac{110 - 125}{125} = -\frac{15}{125} = -12\%.$$

# Ex- and cum-dividend

In practise dividends are paid out at a given time $t$. What is the value of an asset at this time?

This depends on how you define the value.

- Ex-dividend: The dividend is not included in the price at the time of the dividend.
- Cum-dividend: The dividend is included in the price at the time of the dividend.

The book uses the ex-dividend principle, and so will we.

# Returns over a single period

There are alternative ways of measuring the return of an asset.

- The gross return:

$$R_{t,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = 1 + r_{t,t+1}$$

or

$$R_{t,t+\Delta t} = \frac{D_{t+\Delta t} + P_{t+\Delta}}{P_t} = 1 + r_{t,t+\Delta t}.$$

- The log-return:

$$r^{\log}_{t,t+1} = \ln\left(1 + r_{t,t+1}\right) = \ln R_{t,t+1},$$

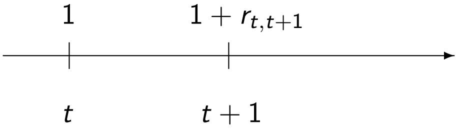
or

$$r^{\log}_{t,t+\Delta t} = \ln\left(1 + r_{t,t+\Delta t}\right) = \ln R_{t,t+\Delta t}.$$

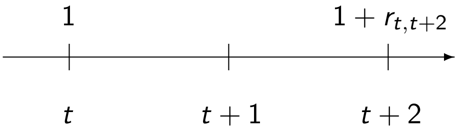
# Returns over multiple periods

When we study the return over multiple periods, we need to take the compounding of returns into account.

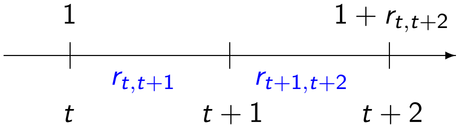
In one period:




In two periods:



# Returns over multiple periods




How is the two-period rate of return $r_{t,t+2}$ connetced to the two one-period returns $r_{t,t+1}$ and $r_{t+1,t+2}$?

- Start with 1 unit of currency at time $t$.
- At time $t+1$ this has grown to $1 \cdot (1 + r_{t,t+1}) = 1 + r_{t,t+1}$.
- This amount is now invested over the next time period.
- At time $t+2$ this has grown to $(1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) = 1 + r_{t,t+2}$.

# Returns over multiple periods

To summarize: It holds that

$$1 + r_{t,t+2} = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}),$$

or

$$r_{t,t+2} = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) - 1.$$

Since the gross return is 1 plus the rate of return we see that

$$R_{t,t+2} = R_{t,t+1} \cdot R_{t+1,t+2}.$$

Finally, for log-returns we have

$$\begin{aligned} r^{\log}_{t,t+2} &= \ln R_{t,t+2} \\ &= \ln \left( R_{t,t+1} \cdot R_{t+1,t+2} \right) \\ &= \ln R_{t,t+1} + \ln R_{t+1,t+2} \\ &= r^{\log}_{t,t+1} + r^{\log}_{t+1,t+2} \end{aligned}$$

# Returns over multiple periods

These results can be generalised to $n$ number of periods.

- For rates of return:

$$r_{t,t+n} = (1 + r_{t,t+1})(1 + r_{t+1,t+2}) \ldots (1 + r_{t+n-1,t+n}) - 1.$$

- For gross returns:

$$R_{t,t+n} = R_{t,t+1} R_{t+1,t+2} \ldots R_{t+n-1,t+n}.$$

- For log-returns:

$$r^{\log}_{t,t+n} = r^{\log}_{t,t+1} + r^{\log}_{t+1,t+2} + \ldots + r^{\log}_{t+n-1,t+n}.$$

Note that log-returns are additive.

# Returns over multiple periods

We have

$$\begin{aligned} r_{t,t+2} &= (1+r_{t,t+1}) \cdot (1+r_{t+1,t+2}) - 1 \\ &= r_{t,t+1} + r_{t+1,t+2} + r_{t,t+1} \cdot r_{t+1,t+2} \\ &\approx r_{t,t+1} + r_{t+1,t+2} \end{aligned}$$

if $r_{t,t+1}$ and $r_{t+1,t+2}$ are "small".

This can be generalised to

$$r_{t,t+n} \approx r_{t,t+1} + r_{t+1,t+2} \cdots + r_{t+n-1,t+n}.$$

Note that this approximation can be quite bad, and in general we should not use it.

# Annualising returns

In order to compare returns it is easier if they are given over the same time period.

This typically means that we transfer them into yearly returns.

These compounded annualised returns are also called effective annual returns.

If the calculated returns are monthly, then their annualised counterparties are

$$r_{\text{ann}} = (1 + r_{\text{mon}})^{12} - 1,$$

$$R_{\text{ann}} = \left(R_{\text{mon}}\right)^{12}$$

and

$$r^{\text{log}}_{\text{ann}} = 12 \cdot r^{\text{log}}_{\text{mon}}.$$

# The internal rate of return

Assume that we make an investment today at time $t = 0$ of $I$ and this investment will give us the cash flows $C_1, C_2, \ldots, C_T$ at times $t = 1, 2, \ldots, T$.

What is the return on this investment?

We say that $r$ is the internal rate of return (IRR) of the investment if $r$ satisfies

$$I = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t},$$

or

$$0 = -I + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}.$$

# The internal rate of return

How do we calculate the IRR?

Note that with $x = 1/(1+r)$ we can write

$$0 = -I + \sum_{t=1}^{T} C_t \left(\frac{1}{1+r}\right)^t = -I + \sum_{t=1}^{T} C_t x^t.$$

This is polynomial in $x$ of order $T$. For higher values of $T$ we need to use numerical methods.

# The internal rate of return

There are potential problems with the IRR:

- There might exist more than one IRR.
- There might exist no real valued IRR.

If the initial investment $I$ is positive and the future cash flows are greater than or equal to zero, then there exists a unique strictly positive IRR $r$.

This is the case for many bonds, and we will return to the IRR, known as the yield, when we study the pricing of bonds.

# Returns on short positions

To have a short position in an asset means that you have sold an asset you do not own.

This is called selling short or shorting the asset.

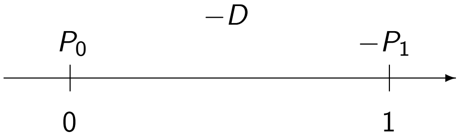
You do this if you believe that that the asset will decrease in value.

In practise this is done by borrowring the asset from someone and then sell it on the market. At some future time you buy it back on the market and return the asset to the lender.

# Returns on short positions

What is the return on a short position?

Let us look at the cash flows generated by the short selling:




$$r_{\text{short}} = \frac{P_0 - P_1 - D}{P_0}.$$

Hence,

$$r_{\text{short}} = -\frac{P_1 + D - P_0}{P_0} = -r_{\text{long}}.$$

# Excess returns

An excess return is the difference between two rates of return:

$$r^{\text{ex}}_{t,t+1} = r_{t,t+1} - r^{b}_{t,t+1}.$$

Here $r^{b}_{t,t+1}$ is the benchmark rate of return.

Examples of bechmark rate of returns:

- The risk-free rate of return.
- The rate of return of an index.
- The rate of return of an asset.

# Excess returns

Excess returns are also known as zero net investment portfolio returns.

- We want to invest 1 unit of currency in Asset 1 by selling Asset 2 short.
- The prices today at $t = 0$ are
$$P_{10} \quad \text{and} \quad P_{20}$$
respectively.[1]
- Today we can buy
$$\frac{1}{P_{10}} \text{ units of Asset 1,}$$
by short selling
$$\frac{1}{P_{20}} \text{ units of Asset 2.}$$

[1] For simplicity we assume that there are no dividends, but the formula will hold also when there are dividend payments.

# Excess returns

- Value today ($t = 0$):

$$-\frac{1}{P_{10}} \cdot P_{10} + \frac{1}{P_{20}} \cdot P_{20} = 0.$$

- Value of this investment when we sell it at $t = 1$:

$$\begin{aligned}\frac{1}{P_{10}} \cdot P_{11} - \frac{1}{P_{20}} \cdot P_{21} &= \frac{P_{11}}{P_{10}} - \frac{P_{21}}{P_{20}} \\ &= \frac{P_{11}}{P_{10}} - 1 - \left(\frac{P_{21}}{P_{20}} - 1\right) \\ &= \frac{P_{11} - P_{10}}{P_{10}} - \frac{P_{21} - P_{20}}{P_{20}} \\ &= r_1 - r_2.\end{aligned}$$

# Real and nominal returns

So far we have, implicitly, considered nominal returns. These returns are measured in monetary gains.

Real returns measure gains in purchasing power.

Real returns take the inflation into account when the returns are calculated.

# Returns on leveraged positions

Sometimes we take a loan to finance an investment. This is called levering up or gearing an investment.

Suppose that we have an amount $E_0$ and borrows the amount $L_0$ to get the value

$$V_0 = E_0 + L_0$$

to invest in a stock with.

The interest on the loan is $r_{\text{loan}}$, and the rate of return of the stock is

$$r_{\text{stock}} = \frac{P_1 - P_0 + D}{P_0},$$

where $D$ is the dividend payment.

# Returns on leveraged positions

We buy $V_0/P_0$ number of stocks.The value of the stocks at time 1 is

$$V_1 = \frac{V_0}{P_0}(P_1 + D) = V_0 \frac{P_1 + D}{P_0} = (E_0 + L_0)(1 + r_{\text{stock}}).$$

We need to pay back the loan with interest: $L_1 = L_0(1 + r_{\text{loan}})$. Hence, we have

$$\begin{aligned} E_1 &= V_1 - L_1 \\ &= (E_0 + L_0)(1 + r_{\text{stock}}) - L_0(1 + r_{\text{loan}}) \\ &= E_0(1 + r_{\text{stock}}) + L_0(r_{\text{stock}} - r_{\text{loan}}) \end{aligned}$$

left. The return on our own initial amount $E_0$ is

$$r = \frac{E_1 - E_0}{E_0}.$$

# Returns on leveraged positions

Using the previous expressions we get the return

$$
\begin{aligned}
r &= \frac{\overbrace{E_0(1+r_{\text{stock}}) + L_0(r_{\text{stock}} - r_{\text{loan}})}^{=E_1} - E_0}{E_0} \\
&= 1 + r_{\text{stock}} + \frac{L_0}{E_0}(r_{\text{stock}} - r_{\text{loan}}) - 1 \\
&= r_{\text{stock}} + \frac{L_0}{E_0}(r_{\text{stock}} - r_{\text{loan}})
\end{aligned}
$$

on the leveraged position.

Here $L_0/E_0$ is called the leverage ratio.

# Returns on portfolios

One of the most important concepts in this course is that of a portfolio.

A portfolio is a list of numbers marking how much of each asset an investor has.

Let there be $N$ number of assets to invest in. If we have $h_i$, $i = 1, 2, \ldots, N$ number of assets $i$ at time $t$, then the value of the portfolio at this time is

$$V_t = \sum_{i=1}^{N} h_i P_{it}.$$

The portfolio weight in asset $i$ at time $t$ is defined as

$$\pi_i = \frac{h_i P_{it}}{V_t}.$$

# Returns on portfolios

At time $t + 1$ the value has changed to

$$V_{t+1} = \sum_{i=1}^{N} h_i \big( D_{i,t+1} + P_{i,t+1} \big).$$

The rate of return on the portfolio is given by

$$\begin{aligned} r_p &= \frac{V_{t+1} - V_t}{V_t} \\ &= \frac{\sum_{i=1}^{N} h_i \big( D_{i,t+1} + P_{i,t+1} \big) - \sum_{i=1}^{N} h_i P_{it}}{V_t} \\ &= \frac{1}{V_t} \sum_{i=1}^{N} h_i \big( D_{i,t+1} + P_{i,t+1} - P_{it} \big) \end{aligned}$$

# Returns on portfolios

$$
\begin{aligned}
&= \sum_{i=1}^{N} \underbrace{\frac{h_i P_{it}}{V_t}}_{=\pi_i} \cdot \underbrace{\frac{D_{i,t+1} + P_{i,t+1} - P_{it}}{P_{it}}}_{=r_i} \\
&= \sum_{i=1}^{N} \pi_i r_i .
\end{aligned}
$$

Hence, the return of the portfolio is given by

$$
r_P = \pi_1 r_1 + \pi_2 r_2 + \cdots + \pi_N r_N = \sum_{i=1}^{N} \pi_i r_i .
$$

# Returns on portfolios

It can be shown that the gross return of a portfolio is given by

$$R_p = \pi_1 R_1 + \pi_2 R_2 + \cdots + \pi_N R_N = \sum_{i=1}^{N} \pi_i R_i.$$

However, in general we have

$$r_p^{\log} \neq \sum_{i=1}^{N} \pi_i r_i^{\log}.$$

# Financial Theory – Lecture 3

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# Agenda

- Brief recap of probability theory.
- Measuring risk.
- Vectors, matrices and multivariate random variables.

The lecture is based on

- Sections 3.1-3.5 and 4.2 in the course book.

# Risk in financial economics

From the course book:

> *Two key themes of this book are exactly how to measure the risk of an investment and by how much such risks are compensated in financial markets.*

(Munk, p. 16.)

# Random variables

There are discrete and continuous random variables.

For a discrete random variable we define the probability function

$$p_s = P(X = x_s)$$

of getting the outcome $x_s$, and for a continuous random variable we define the probability density function (pdf) $f_X(x)$ which has the property that for $a < b$

$$P(a < X \leq b) = \int_a^b f_X(x)dx.$$

**Remark.** There are random variables that has both a discrete and a continuous part. They are called mixed random variables.

# Random variables

For any random variable we define its cumulative probability function (cdf) as

$$F_X(x) = P(X \leq x).$$

It holds that

$$P(a < X \leq b) = F_X(b) - F_X(b).$$

The $k\%$ percentile for a contionuous random variable is the value $x$ that satisfies

$$P(X \leq x) = k\% \quad \Leftrightarrow \quad F_X(x) = k\% \quad \Leftrightarrow \quad x = F_X^{-1}(k\%).$$

# Expected values

The expected value of a discrete random variable is defined as

$$E[X] = \sum_s p_s x_s,$$

and for a continuous random variable as

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x) dx.$$

The expected value of a function of a random variable are given by

$$E[g(X)] = \sum_s p_s g(x_s)$$

and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x) dx$$

for a discrete and continuous random variable respectively.

# Variance and standard deviation

To measure the variability in the outcome of a random variable we use the variance

$$\mathrm{Var}[X] = E\left[\left(X - E\left[X\right]\right)^2\right]$$

or the standard deviation

$$\mathrm{Std}[X] = \sqrt{\mathrm{Var}[X]}.$$

The variance satisfies

$$\mathrm{Var}[X] = E\left[X^2\right] - \left(E\left[X\right]\right)^2.$$

# Calculation rules

Let $X$ be a random variable, and let $a$ and $b$ be two real numbers. Then

$$\begin{aligned} E[aX+b] &= aE[X]+b \\ \mathrm{Var}[aX+b] &= a^2\mathrm{Var}[X] \\ \mathrm{Std}[aX+b] &= |a|\mathrm{Std}[X]. \end{aligned}$$

# Higher order moments

Skew or skewness:

$$\text{Skew}[X] = \frac{E\left[\left(X - E[X]\right)^3\right]}{\text{Std}[X]^3}.$$

Kurtosis:

$$\text{Kurt}[X] = \frac{E\left[\left(X - E[X]\right)^4\right]}{\text{Std}[X]^4} - 3.$$

If $\text{Kurt}[X] > 0$, then we say that the distibution of $X$ is lepokurtic, or that it has heavy tails or fat tails.

# Higher order moments

If $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$, which we write as

$$X \sim N(\mu, \sigma^2),$$

then

$$\text{Skew}[X] = 0 \quad \text{and} \quad \text{Kurt}[X] = 0.$$

# The risk-return tradeoff

Let $r$ be a random rate of return over some time interval.

The expected value $E[r]$ is a measure of the reward we get from the investment.

Since we also can put our money in the bank and get the risk-free rate of return $r_f$, it is common to look at the excess return

$$E[r] - r_f.$$

This difference is called the risk premium.

To measure the risk in an investment, the standard deviation of the rate of return $\mathrm{Std}[r]$ is often used.

# The risk-return tradeoff

In order to measure the tradeoff between the risk and the reward in terms of the expected rate of return, the Sharpe ratio is often used:

$$\mathrm{SR} = \frac{E[r] - r_f}{\mathrm{Std}[r]}.$$

We will later in the course see that the Sharpe ratio arises in a natural way in finance, but at this point it is just one suggestion of how to measure the risk-return tradeoff.

In practical asset management other measures are also used, such as the Sortino ratio or the Maximum drawdown.

# Normally distributed log-returns

Assume that we want to model the rate of return $r$ as a normally distributed random variable.

Since $r$ can take both positive and negative values, this seems as an OK model.

One drawback, however, is that the rate of return cannot be lower than $-100\%$, i.e. $r \geq -1$.

The solution to this potential problem is to instead assume that the log-return is normally distributed.

# Normally distributed log-returns

A random variable $X$ is said to be lognormally distributed with parameters $m$ and $s^2$ if

$$\ln X \sim N(m, s^2).$$

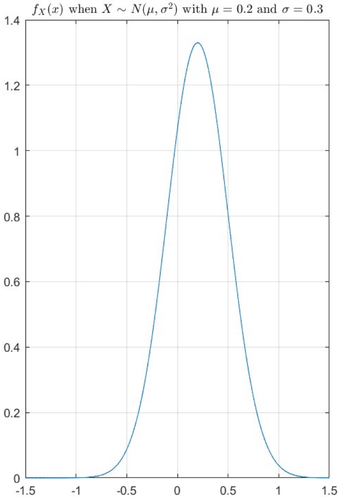
The first two moments of a lognormally distributed random variable $X$ are

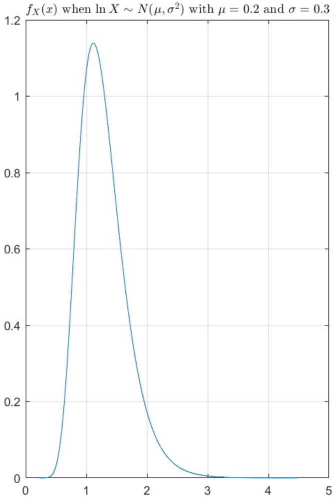
$$\begin{aligned} E\left[X\right] &= e^{m+\frac{s^2}{2}} \\ \mathrm{Var}[X] &= e^{2m+s^2}\left(e^{s^2}-1\right). \end{aligned}$$

Since $r^{\log} = \ln R$ we see that

$$r^{\log} \text{ is normally distributed } \Leftrightarrow R \text{ is lognormally distributed.}$$

# Normally distributed log-returns



# Covariance

The covariance between two random variables $X_1$ and $X_2$ is defined as

$$\mathrm{Cov}[X_1, X_2] = E\left[\left(X_1 - E\left[X_1\right]\right) \cdot \left(X_2 - E\left[X_2\right]\right)\right].$$

We see that

$$\mathrm{Cov}[X_1, X_1] = \mathrm{Var}[X_1].$$

It can further be shown that

$$\mathrm{Cov}[X_1, X_2] = E\left[X_1 X_2\right] - E\left[X_1\right] E\left[X_2\right],$$

which can be written

$$E\left[X_1 X_2\right] = E\left[X_1\right] E\left[X_2\right] + \mathrm{Cov}[X_1, X_2].$$

For random variables $X, Y$ and real numbers $a, b, c, d$

$$\mathrm{Cov}[aX + b, cY + d] = ac\mathrm{Cov}[X, Y].$$

# Correlation

The correlation between two random variables $X_1$ and $X_2$ is defined as

$$\mathrm{Corr}[X_1, X_2] = \frac{\mathrm{Cov}[X_1, X_2]}{\mathrm{Std}[X_1]\,\mathrm{Std}[X_2]}.$$

The correlation satisfies

$$-1 \leq \mathrm{Corr}[X_1, X_2] \leq 1 \quad \Leftrightarrow \quad \big|\mathrm{Corr}[X_1, X_2]\big| \leq 1.$$

Note that

$$\mathrm{Cov}[X_1, X_2] = \mathrm{Corr}[X_1, X_2]\,\mathrm{Std}[X_1]\,\mathrm{Std}[X_2].$$

# Alternative ways of measuring risk

Using the variance (standard deviation) to measure the risk goes back to Markowitz 1952.

There have been other suggestions.

- Mean absolute deviation (MAD): $E\left[\left|r - E\left[r\right]\right|\right]$.
- Semivariance.
- Value-at-Risk (VaR).
- Expected shortfall (ES) (sometimes called Conditional Value-at-Risk (CVaR) or Tail Value-at-Risk (TVaR)).
- Coherent risk measures.
- Convex risk measures.

# Value-at-Risk and Expected shortfall

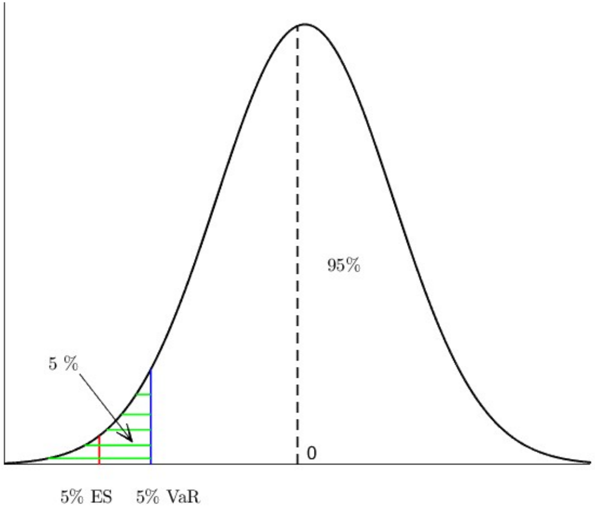
Say that we want to measure the risk of what can happen on the 5% worst days of trading for an investor.

**Value-at-Risk**
On 95% of the days, the maximum loss is equal to VaR.

**Expected shortfall**
On the 5% worst days, the mean loss is equal to ES.

VaR is about what can happen on a "good" day, while ES is about what can happen on a "bad" day.

# Value-at-Risk and Expected shortfall



# Value-at-Risk and Expected shortfall

From a computational point of view, the Value-at-Risk is a percentile of the profit distribution.

The profit can be defined either in units of currency or as the rate of return from the investment. See the book for details.

In some cases VaR and ES are defined in terms of the loss distribution. The only difference is that the sign of VaR and ES are changed.

# Risk measures

The systematic study of different types of risk measures starts with "Thinking coherently" from 1997 and "Coherent measures of risk" from 2001, both by Artzner, Delbaen, Embrechts and Heath.

- Value-at-Risk is not a coherent risk measure.
- Expected shortfall is a coherent risk measure.

Some argue that the definition of a coherent risk measure is too restrictive. This has lead to the more general concept of convex risk measures (and other alternatives as well).

# The Basel accords

The Basel accords regulates the supervision of banks from a risk perspective.

From Basel Committee on Banking Supervision, "MAR33 Internal models approach: capital requirements calculation":

**33.2** ES must be computed on a daily basis for the bank-wide internal models to determine market risk capital requirements. ES must also be computed on a daily basis for each trading desk that uses the internal models approach (IMA).

**33.3** In calculating ES, a bank must use a 97.5th percentile, one-tailed confidence level.

(This is equivalent to what the book calls 2.5% ES.)

Previously 1% VaR was used.

# Vectors and matrices

A vector is an orded collection of $N$ elements:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = (x_1, x_2, \ldots, x_N)^\top.$$

Here $\top$ denotes the transpose of a vector.

**Remark.** The notation $'$ is also used to denote transposition.

# Vectors and matrices

Recall that

$$\mathbf{x}+\mathbf{y}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}+\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}=\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_N+y_N \end{pmatrix}$$

and for a scalar $a$

$$a\mathbf{x}=a\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}=\begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}.$$

The inner product (or vector product or dot product) between two vectors is given by

$$\mathbf{x} \cdot \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N = \sum_{i=1}^{N} x_i y_i.$$

Note that

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}.$$

# Vectors and matrices

We let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

denote the vector of only 1's. For any vector $\mathbf{x}$

$$\mathbf{x} \cdot \mathbf{1} = x_1 + x_2 + \cdots + x_N = \sum_{i=1}^{N} x_i.$$

# Vectors and matrices

A matrix $A$ is a collection of elements in a rectangular array:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix}.$$

This vector has $M$ rows, $N$ columns and a total of $MN$ number of elements.

We say that $A$ is an $M \times N$ matrix.

In the book matrices are denoted using a double underline: $\underline{\underline{A}}$.

# Vectors and matrices

Let $A$ be an $N \times N$ matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be two column vectors of length $N$. Then

$$A\mathbf{x} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \vdots \\ \square \end{pmatrix}$$

is a column vector of length $N$.

We can now multiply this vector with $\mathbf{y}$:

$$\mathbf{y} \cdot A\mathbf{x} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j A_{ij},$$

and the result is a scalar.

# Vectors and matrices

A square matrix $A$ is symmetric if

$$A = A^\top,$$

which is the same as requiring that

$$A_{ij} = A_{ji}, \text{ for every } i, j = 1, 2, \ldots, N \text{ when } i \neq j.$$

An import observation is that if $A$ is symmetric, then

$$\mathbf{y} \cdot A\mathbf{x} = \mathbf{x} \cdot A\mathbf{y}.$$

The main example of a symmetric matrix in this course is the variance-covariance matrix.

# Vectors and matrices

We let $I$ denote the identity matrix:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

i.e. it is a square matrix with 1's on the diagonal an 0's off the diagonal. This matrix has the property that

$$A\,I = I\,A = A$$

for any matrix $A$ such that the multiplication makes sense (i.e. if $A$ is $N \times N$, then $I$ must be $N \times N$).

In the book the notation $\underline{\underline{1}}$ is used for the identity matrix.

The inverse of a square matrix $A$ (if it exist!) is a matrix denoted $A^{-1}$ which satisfies

$$AA^{-1} = A^{-1}A = I.$$

**Example** If the matrix $A$ is $1 \times 1$ then it is equal to a scalar $a$. If $a \neq 0$, then

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \leftarrow \text{The } 1 \times 1 \text{ identity matrix.}$$

As for numbers, not every matrix is invertible.

# Vectors and matrices

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a $2 \times 2$ matrix.

Then it is invertible if and only if $ad - bc \neq 0$ and in this case

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Exercise:** Check this!

# Multivariate random variables

Let $X_1, X_2, \ldots, X_N$ be random variables and let $a_1, a_2, \ldots, a_N$ be real numbers. Then

$$
\begin{aligned}
E\left[\sum_{i=1}^{N} a_i X_i\right] &= \sum_{i=1}^{N} a_i E\left[X_i\right] \\
\mathrm{Var}\left[\sum_{i=1}^{N} a_i X_i\right] &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \mathrm{Cov}[X_i, X_j]
\end{aligned}
$$

The last formula is sometimes written

$$
\mathrm{Var}\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i^2 \mathrm{Var}[X_i] + \sum_{i \neq j}^{N} a_i a_j \mathrm{Cov}[X_i, X_j]
$$

(and see Theorem 3.2 in Munk for a third way).

# Multivariate random variables

An $N$-dimensional random variable $\boldsymbol{X}$ is a random vector:

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix} = (X_1, X_2, \ldots, X_N)^\top.$$

The mean vector is given by

$$\boldsymbol{\mu} = E[\boldsymbol{X}] = \begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{pmatrix}.$$

Using this we can write

$$E\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i E[X_i] = \mathbf{a} \cdot \boldsymbol{\mu}.$$

# Multivariate random variables

The variance-covariance (or covariance) matrix is given by

$$\Sigma = \mathrm{Var}(\boldsymbol{X}) = \begin{pmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \cdots & \mathrm{Cov}[X_1, X_N] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2, X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_N, X_1] & \mathrm{Cov}[X_N, X_2] & \cdots & \mathrm{Var}[X_N] \end{pmatrix},$$

that is

$$\Sigma_{ij} = \mathrm{Cov}[X_i, X_j], \ \ i, j = 1, \ldots, N.$$

Since $\mathrm{Cov}[X_i, X_j] = \mathrm{Cov}[X_j, X_i]$, the matrix $\Sigma$ is symmetric:

$$\Sigma = \Sigma^\top.$$

Futhermore

$$\mathrm{Var}\left[\sum_{i=1}^N a_i X_i\right] = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \mathrm{Cov}[X_i, X_j] = \mathbf{a} \cdot \Sigma \mathbf{a}.$$

# Financial Theory – Lecture 4

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Two-asset portfolios.
- Diversification.
- Some special types of portfolios.
- Portfolio mathematics.

The lecture is based on

- Chapter 4.1-4.2 and 4.4-4.5 in the course book.

# Basic definitions in portfolio models

We let $\boldsymbol{r} = (r_1, r_2, \ldots, r_N)^\top$ denote a vector of rates of return of $N$ assets.

A vector of portfolio weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ is a vector satisfying

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 1.$$

The rate of return of a portfolio is denoted $r_p$, or $r(\boldsymbol{\pi})$ if we want to emphasise the specific portfolio weight vector.

We have previously shown that

$$r_p = r(\boldsymbol{\pi}) = \sum_{i=1}^{N} \pi_i r_i = \boldsymbol{\pi} \cdot \mathbf{r}.$$

# Two-asset portfolios

Now assume that there exists only two assets: $N = 2$.

Let $w$ be the portfolio weight in asset 1, i.e. the portfolio weights are

$$(\pi_1, \pi_2)^T = (w, 1 - w)^T.$$

Note that $w$ can be any number. The return on this portfolios is

$$r(w) = wr_1 + (1 - w)r_2.$$

We let for $i = 1, 2$

$$E[r_i] = \mu_i, \; \text{Std}[r_i] = \sigma_i \text{ and } \text{Corr}[r_1, r_2] = \rho,$$

and assume that $\mu_1 \neq \mu_2$.

# Two-asset portfolios

The mean return of the portfolio is

$$
\begin{aligned}
\mu(w) &= E\left[r(w)\right] \\
&= E\left[wr_1 + (1-w)r_2\right] \\
&= w\mu_1 + (1-w)\mu_2,
\end{aligned}
$$

and the variance of the rate of return of the portfolio is

$$
\begin{aligned}
\sigma^2(w) &= \text{Var}[r(w)] \\
&= \text{Var}[wr_1 + (1-w)r_2] \\
&= \text{Var}[wr_1] + 2\text{Cov}[wr_1, (1-w)r_2] + \text{Var}[(1-w)r_2] \\
&= w^2\sigma_1^2 + 2w(1-w)\text{Cov}[r_1, r_2] + (1-w)^2\sigma_2^2 \\
&= w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2.
\end{aligned}
$$

# Two-asset portfolios

Using the expression for $\mu(w)$ we can solve for the weight as long as $\mu_1 \neq \mu_2$:

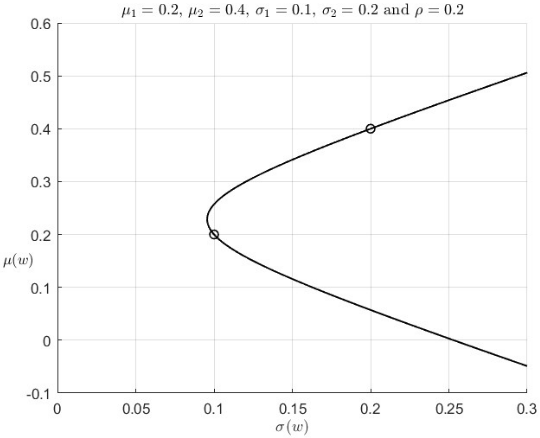
$$\mu(w) = w\mu_1 + (1-w)\mu_2 = \mu_2 + w(\mu_1 - \mu_2) \quad \Leftrightarrow \quad w = \frac{\mu(w) - \mu_2}{\mu_1 - \mu_2}.$$

Now use this expression in the formula for $\sigma(w) = \sqrt{\sigma^2(w)}$:

$$\begin{aligned} \sigma(w) &= \sqrt{w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2} \\ &= \dots \\ &= \sqrt{K_0 + K_1\mu(w) + K_2\mu(w)^2} \end{aligned}$$

for some constants $K_0, K_1$ and $K_2$ depending on $\mu_1$, $\mu_2$, $\sigma_1$, $\sigma_2$ and $\rho$ (see p. 107 in Munk for details).

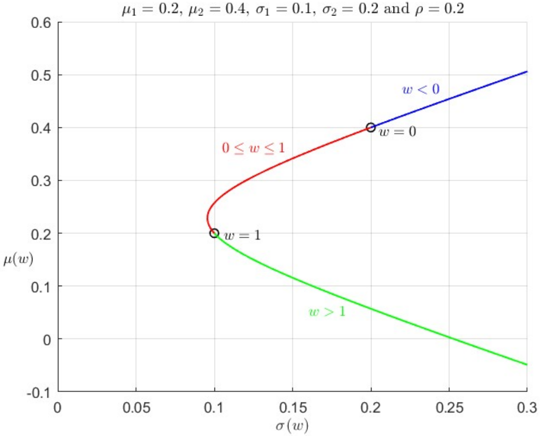
# Two-asset portfolios




By letting $w$ vary we can get any expected return we want – given that we accept the standard deviation of that portfolio.

# Two-asset portfolios

The weight $w$ can be any real number.




When $w \in [0, 1]$ then also $1 - w \in [0, 1]$, and there is no short-selling in any of the assets (a "long-only portfolio").

# Two-asset portfolios

Which portfolios has the lowest possible variance and how large is this variance?

We use

$$\sigma^2(w) = w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2$$

and look for a portfolio with

$$\frac{\partial\sigma^2(w)}{\partial w} = 2w\sigma_1^2 + 2\big(1-w-w\big)\rho\sigma_1\sigma_2 - 2(1-w)\sigma_2^2 = 0.$$

Solving this equation yields the portfolio weights

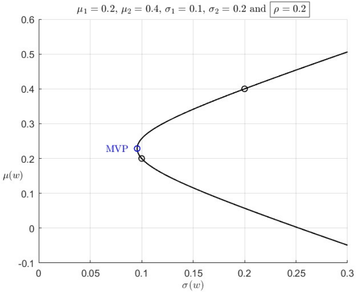
$$w_{\min} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} \quad \text{and} \quad 1 - w_{\min} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}.$$

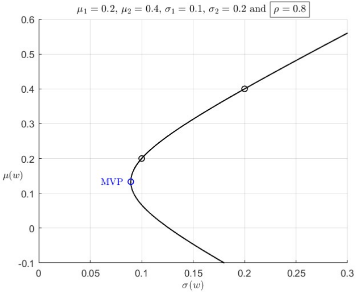
# Two-asset portfolios

The minimum variance is given by

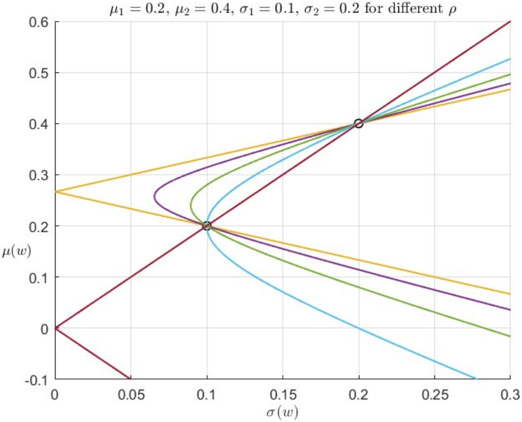
$$\sigma^2(w_{\min}) = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}.$$

Check this by doing Exercise 4.1 in the course book, and there you can also find an expression for $\mu(w_{\min})$.



# Two-asset portfolios



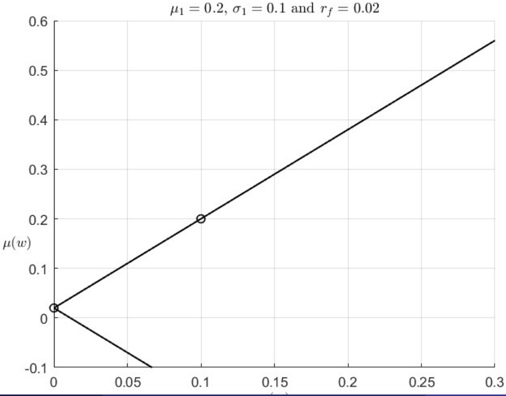

| Colour | Yellow | Purple | Green | Blue | Red |
|---|---|---|---|---|---|
| $\rho$ | $-1$ | $-0.5$ | 0 | 0.5 | 1 |

# Two-asset portfolios

Now assume that asset 2 is a risk-free asset with rate of return $r_f$.

In this case

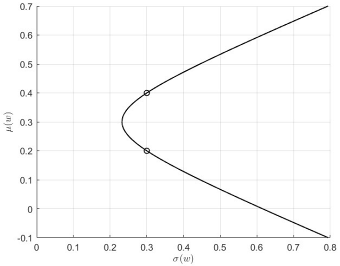
$$\mu(w) = w\mu_1 + (1 - w)r_f \quad \text{and} \quad \sigma(w) = \sqrt{w^2\sigma_2^2} = |w|\sigma_2.$$



# Two special cases

$\underline{\sigma_1 = \sigma_2}$

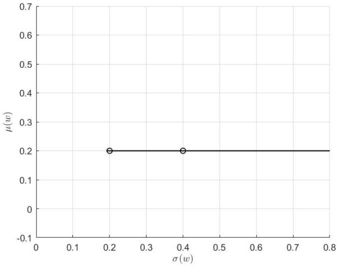
In this case a typical situation is as follows:




This is a situation that is OK.

$\underline{\mu_1 = \mu_2}$

In this case a typical situation is as follows:




This is not a realistic situation from an economic point of view.

# Risk reduction through diversification

Now consider a market with $N$ assets.

On this market we form a portfolio with equal weights in each asset:

$$\pi_1 = \pi_2 = \ldots = \pi_N.$$

Since they need to sum to one, we have

$$\pi_i = \frac{1}{N},\ i = 1, 2, \ldots, N.$$

# Risk reduction through diversification

How large is the variance of the rate return $r_p$ of this equally weighted portfolio?

$$
\begin{aligned}
\mathrm{Var}[r_p] &= \mathrm{Var}\left[\sum_{i=1}^{N} \pi_i r_i\right] \\
&= \mathrm{Var}\left[\sum_{i=1}^{N} \frac{1}{N} r_i\right] \\
&= \frac{1}{N^2}\mathrm{Var}\left[\sum_{i=1}^{N} r_i\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{Cov}[r_i, r_j] \\
&= \frac{1}{N^2}\left(\sum_{i=1}^{N} \mathrm{Var}[r_i] + \sum_{i \neq j}^{N} \mathrm{Cov}[r_i, r_j]\right)
\end{aligned}
$$

# Risk reduction through diversification

There are $N$ variance and $N(N-1)$ covariance terms. Let

$$\overline{\mathrm{Var}}_N = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Var}[r_i]$$

and

$$\overline{\mathrm{Cov}}_N = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} \mathrm{Cov}[r_i, r_j]$$

be their respective averages.

We assume that as the number of assets grow, i.e. when $N \to \infty$, they converge to $\overline{\mathrm{Var}}$ and $\overline{\mathrm{Cov}}$ respectively.

# Risk reduction through diversification

Now,

$$
\begin{aligned}
\mathrm{Var}[r_p] \quad &= \quad \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}[r_i] + \frac{1}{N^2}\sum_{i\neq j}^{N}\mathrm{Cov}[r_i, r_j] \\
&= \quad \frac{1}{N}\cdot\frac{1}{N}\sum_{i=1}^{N}\mathrm{Var}[r_i] + \frac{N-1}{N}\cdot\frac{1}{N(N-1)}\sum_{i\neq j}^{N}\mathrm{Cov}[r_i, r_j] \\
&= \quad \frac{1}{N}\overline{\mathrm{Var}}_N + \frac{N-1}{N}\overline{\mathrm{Cov}}_N \\
&\xrightarrow{N\to\infty} \quad 0\cdot\overline{\mathrm{Var}} + 1\cdot\overline{\mathrm{Cov}} \\
&= \quad \overline{\mathrm{Cov}}.
\end{aligned}
$$

**Conlusion:** By investing in more and more asset we can diminish the risk, but the lower limit is given by $\overline{\mathrm{Cov}}$.

# Risk reduction through diversification

What is the intuition behind the previous result?

Let $r_i$ be the rate of return of asset $i$ and let $r_m$ be the return of a market index.

Then we can always write

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

for some $\alpha_i$, $\beta_i = \text{Cov}[r_i, r_m]/\text{Var}[r_m]$ and $\text{Cov}[r_m, \varepsilon_i] = 0$ (cf. with OLS).

Hence,

$$\text{Var}[r_i] = \text{Var}[\alpha_i + \beta_i r_m + \varepsilon_i] = \beta_i^2 \sigma_m^2 + \sigma_i^2,$$

where

$$\sigma_m^2 = \text{Var}[r_m] \text{ and } \sigma_i^2 = \text{Var}[\varepsilon_i].$$

# Risk reduction through diversification

If we make the assumption that $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0$ if $i \neq j$, then we can interpret $\varepsilon_i$ as the firm specific variation in the return $r_i$.

Under this assumption

$$\mathrm{Cov}[r_i, r_j] = \mathrm{Cov}[\alpha_i + \beta_i r_m + \varepsilon_i, \alpha_j + \beta_j r_m + \varepsilon_j] = \beta_i \beta_j \sigma_m^2.$$

It follows that

$$\overline{\mathrm{Cov}}_N = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} \mathrm{Cov}[r_i, r_j] = \frac{\sigma_m^2}{N(N-1)} \sum_{i \neq j}^{N} \beta_i \beta_j.$$

The limit of this as $N \to \infty$ does not depend on the variances $\sigma_i^2$.

We can diversify away the firm specific risks, but not the market risk.

# Risk reduction through diversification

Now let all the assets have the same standard deviation $\sigma$ and the same pairwise correlation $\rho \geq 0$.

**Remark.** One can show that if $N$ random variables have the same pairwise correlation $\rho$, then this correlation has to satisfy $\rho \geq -1/(N-1)$.

In this case

$$\overline{\mathrm{Var}}_N = \frac{1}{N}\sum_{i=1}^{N}\sigma^2 = \frac{1}{N}\cdot N\sigma^2 = \sigma^2$$

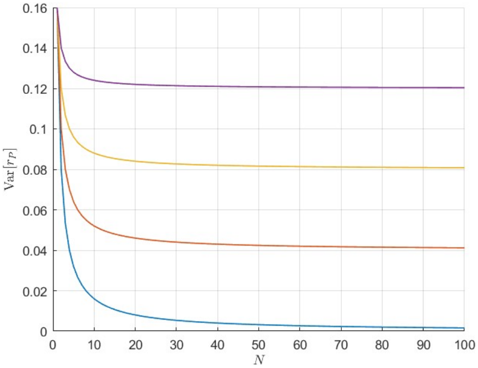
and

$$\overline{\mathrm{Cov}}_N = \frac{1}{N(N-1)}\sum_{i\neq j}^{N}\rho\sigma^2 = \frac{1}{N(N-1)}\cdot N(N-1)\rho\sigma^2 = \rho\sigma^2.$$

# Risk reduction through diversification

Now we get

$$\mathrm{Var}[r_p] = \frac{1}{N}\overline{\mathrm{Var}}_N + \frac{N-1}{N}\overline{\mathrm{Cov}}_N = \frac{\sigma^2}{N} + \left(1 - \frac{1}{N}\right)\rho\sigma^2.$$




Corrrelation $\rho = 0, 0.25, 0.5, 0.75$ from bottom and up.

# Arbitrage portfolios

An arbitrage is a portfolio with the following properties:

1) It costs zero to buy.
2) It has a payoff that is non-negative, and with strictly positive probability the payoff is strictly positive.

One can think of this as getting a free lottery ticket.

How many units of this portfolio would you like?

Infinitely many!

# Arbitrage portfolios

The principle of no arbitrage states that on a market in equilibrium, there can be no arbitrage opportunities, i.e. it is not possible to construct an arbitrage on a market in equilibrium.

This approach is very powerful when determining the price of derivatives such as options and futures.

A model of a market that rules out arbitrage opportunities is said to be free of arbitrage or arbitrage free.

# Replicating portfolios

A replicating portfolio for a given asset is a portfolio that exactly replicates the cash flows of the given asset.

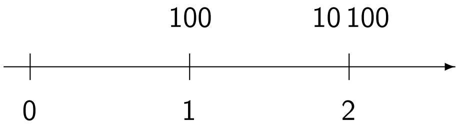
The law of one price states that if two assets have the same cash flows, then they must have the same price.

The law of one price is a weak requirement, and in fact if a market model is free of arbitrage, then the law of one price holds (but the converse is not true).

By using the law of one price, we see that the price of the replicating portfolio and the asset it replicates must be the same.

# Replicating portfolios

**Example.** Asset A is paying out 100 euros one year from now and 10 100 euros in two years from now.




The price of asset B that is paying out 100 euros in one years time is 98.53 euros, and the price of asset C that is paying out 100 euros in two years time is 97.85.

We see that asset A is replicated by the portfolio consisting of 1 unit of asset B and 101 units of asset C. Hence, this is the replicating portfolio.

Using the law of one price we get

$$\text{Price of A} = 1 \cdot 98.53 + 101 \cdot 97.85 = 9\,981.38 \text{ euros.}$$

# Tracking portfolios

A tracking portfolio has as its goal to be close to the value of an asset. It is not unusual that the asset being tracked is an index.

The tracking error (TE) measures how much the tracking portfolio deviates from the asset it is tracking and is defined as

$$r_p - r_b,$$

where $r_p$ is the tracking portfolio and $r_b$ the return on the given asset.

One way to quantitatively measure the TE is to calculate $\mathrm{Std}[r_p - r_b]$.

# Portfolio mathematics

Given is the vector of rates of return

$$\boldsymbol{r} = (r_1, r_2, \dots, r_N)^\top.$$

From now on we let

$$\boldsymbol{\mu} = E[\boldsymbol{r}]$$

and

$$\Sigma = \mathrm{Var}[\boldsymbol{r}]$$

denote the mean vector and the variance-covariance matrix of the rate of return vector respectively.

Recall

$$r_p = r(\boldsymbol{\pi}) = \sum_{i=1}^{N} \pi_i r_i = \boldsymbol{\pi} \cdot \boldsymbol{r}.$$

# Portfolio mathematics

Then the mean rate of return of a portfolio is

$$
\begin{aligned}
E\left[r(\boldsymbol{\pi})\right] &= E\left[\sum_{i=1}^{N} \pi_i r_i\right] \\
&= \sum_{i=1}^{N} \pi_i E\left[r_i\right] \\
&= \sum_{i=1}^{N} \pi_i \mu_i \\
&= \boldsymbol{\pi} \cdot \boldsymbol{\mu}.
\end{aligned}
$$

# Portfolio mathematics

The variance of the portfolio rate of return is given by

$$\begin{aligned}
\mathrm{Var}[r(\boldsymbol{\pi})] &= \mathrm{Var}\left[\sum_{i=1}^{N} \pi_i r_i\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \mathrm{Cov}[r_i, r_j] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij} \\
&= \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.
\end{aligned}$$

# Portfolio optimisation

Using the $N$ assets we can form portfolios with special characteristics.

Typically we want to minimise or maximise some property of a portfolio.

Since we must have

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 1$$

this leads to optimisation with constraints.

**Example**

1) The portfolio with the smallest variance:

$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{N} \pi_i = 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \end{array}$$

# Portfolio optimisation

2) The long-only portfolio with the smallest variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & \pi_i \geq 0, i = 1, 2, \ldots, N \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \geq \mathbf{0} \end{aligned}$$

3) The portfolio with expected rate of return $\bar{\mu}$ that has the smallest variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \overline{\mu}. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \overline{\mu}. \end{aligned}$$

# Financial Theory – Lecture 5

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Mean-variance analysis with only risky assets.
- Choice under uncertainty.

The lecture is based on

- Chapter 7 in the course book.

# Mean-variance analysis with risky assets

We consider a market with $N$ risky assets.

By a "risky asset" we mean an asset whose rate of return has a strictly positive standard deviation.

The rate of return of these assets are collected in the vector **r**.

Recall the notation

$$\boldsymbol{\mu} = E\left[\boldsymbol{r}\right]$$

and

$$\Sigma = \mathrm{Var}[\boldsymbol{r}].$$

# Mean-variance analysis with risky assets

The first goal for us is to find the portfolio with mean $\bar{\mu}$ that has the smallest variance.

Mathematically we formulate this problem as

$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \overline{\mu}. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \overline{\mu}. \end{array}$$

The Lagrangian of this problem is

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda_1 (1 - \boldsymbol{\pi} \cdot \mathbf{1}) + \lambda_2 (\overline{\mu} - \boldsymbol{\pi} \cdot \boldsymbol{\mu}).$$

Here $\lambda_1$ and $\lambda_2$ are the (Lagrangian) multipliers.

# Mean-variance analysis with risky assets

We need to be able to take the derivative of functions like

$$f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{a} \quad \text{and} \quad g(\boldsymbol{x}) = \boldsymbol{x} \cdot A\boldsymbol{x}$$

where $\boldsymbol{a}$ is a vector and $A$ is a matrix.

Since

$$f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{a} = \sum_{i=1}^{N} x_i a_i$$

we have

$$\frac{\partial f}{\partial x_i} = a_i, \; i = 1, \ldots, N.$$

We write this as

$$\frac{\partial f}{\partial \boldsymbol{x}} = \boldsymbol{a}.$$

We also have

$$g(\boldsymbol{x}) = \boldsymbol{x} \cdot A\boldsymbol{x} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j A_{ij}.$$

If the matrix $A$ is symmetric then

$$\frac{\partial g}{\partial x_i} = 2(A\boldsymbol{x})_i,$$

where $(A\boldsymbol{x})_i$ is the element of row $i$ of $A\boldsymbol{x}$.

We write this as

$$\frac{\partial g}{\partial \boldsymbol{x}} = 2A\boldsymbol{x}.$$

# Mean-variance analysis with risky assets

Let us return to the Lagrangian:

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda_1 (1 - \boldsymbol{\pi} \cdot \mathbf{1}) + \lambda_2 (\overline{\mu} - \boldsymbol{\pi} \cdot \boldsymbol{\mu}).$$

The first-order condition with respect to $\boldsymbol{\pi}$ is

$$\frac{\partial L}{\partial \boldsymbol{\pi}} = 2\Sigma \boldsymbol{\pi} - \lambda_1 \mathbf{1} - \lambda_2 \boldsymbol{\mu} = 0.$$

This can be written

$$2\Sigma \boldsymbol{\pi} = \lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu} \quad \Leftrightarrow \quad \Sigma \boldsymbol{\pi} = \frac{\lambda_1}{2} \mathbf{1} + \frac{\lambda_2}{2} \boldsymbol{\mu}.$$

How do we solve for $\boldsymbol{\pi}$? Multiply both sides with $\Sigma^{-1}$!

# Mean-variance analysis with risky assets

$$\underbrace{\Sigma^{-1}\Sigma}_{=I}\boldsymbol{\pi} = \Sigma^{-1}\left(\frac{\lambda_1}{2}\mathbf{1} + \frac{\lambda_2}{2}\boldsymbol{\mu}\right)$$

Since $I\boldsymbol{\pi} = \boldsymbol{\pi}$ we get

$$\boldsymbol{\pi} = \frac{\lambda_1}{2}\Sigma^{-1}\mathbf{1} + \frac{\lambda_2}{2}\Sigma^{-1}\boldsymbol{\mu}.$$

We need to find the Lagrange multipliers $\lambda_1$ and $\lambda_2$ $\longrightarrow$ use the constraints.

# Mean-variance analysis with risky assets

Portfolio weights sum to 1:

$$\begin{aligned}
1 &= \boldsymbol{\pi} \cdot \mathbf{1} = \mathbf{1} \cdot \boldsymbol{\pi} \\
&= \mathbf{1} \cdot \left( \frac{\lambda_1}{2} \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \Sigma^{-1} \boldsymbol{\mu} \right) \\
&= \frac{\lambda_1}{2} \mathbf{1} \cdot \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu}
\end{aligned}$$

Expected rate of return equal to $\bar{\mu}$:

$$\begin{aligned}
\bar{\mu} &= \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \boldsymbol{\pi} \\
&= \boldsymbol{\mu} \cdot \left( \frac{\lambda_1}{2} \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \Sigma^{-1} \boldsymbol{\mu} \right) \\
&= \frac{\lambda_1}{2} \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu}
\end{aligned}$$

# Mean-variance analysis with risky assets

Now introduce the parameters

$$\begin{aligned} A &= \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu} \\ B &= \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} = \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu} \\ C &= \mathbf{1} \cdot \Sigma^{-1} \mathbf{1} \\ D &= AC - B^2. \end{aligned}$$

Using these, the two equations above can be written

$$\begin{cases} 1 = \frac{\lambda_1}{2} C + \frac{\lambda_2}{2} B \\ \bar{\mu} = \frac{\lambda_1}{2} B + \frac{\lambda_2}{2} A \end{cases}$$

We want to solve for $\lambda_1$ and $\lambda_2$.

# Mean-variance analysis with risky assets

The solution is given by

$$\begin{cases} \lambda_1 &=& 2\dfrac{A - B\bar{\mu}}{D} \\ \\ \lambda_2 &=& 2\dfrac{C\bar{\mu} - B}{D}. \end{cases}$$

Inserting them in the expression for $\boldsymbol{\pi}$ results in

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D}\Sigma^{-1}\mathbf{1} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

These are the portfolio weights in the portfolio with mean rate of return $\bar{\mu}$ whose rate of return has the smallest variance.

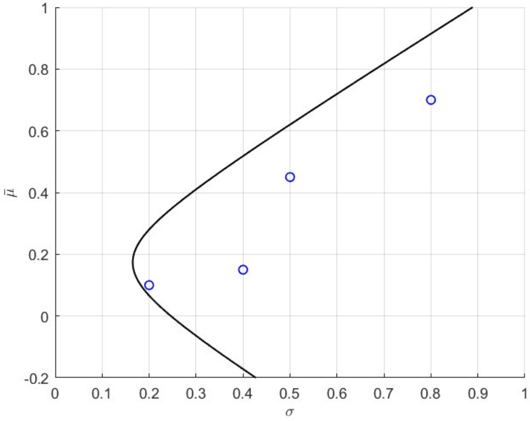
# Mean-variance analysis with risky assets

Using the optimal weights from the previous slide we can calculate the standard deviation of this portfolio:

$$\begin{aligned}
\sigma(\bar{\mu}) &= \mathrm{Std}[r(\boldsymbol{\pi}(\bar{\mu}))] \\
&= \sqrt{\boldsymbol{\pi}(\bar{\mu}) \cdot \Sigma \boldsymbol{\pi}(\bar{\mu})} \\
&= \dots \\
&= \sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}.
\end{aligned}$$

This is called the mean-variance frontier or the portfolio frontier.

# Mean-variance analysis with risky assets




An example with four assets ($N = 4$).

# Mean-variance analysis with risky assets

A portfolio whose mean and standard devition is on the mean-variance frontier is called a frontier portfolio.

We say that a portfolio is on the portfolio frontier if its standard deviation and mean is on the frontier.

The variance $\sigma^2$ for a given $\bar{\mu}$ is

$$\sigma^2 = \frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}.$$

This can be written

$$\frac{\sigma^2}{1/C} - \frac{\left(\bar{\mu} - B/C\right)^2}{D/C^2} = 1.$$

Mathematically this is an equation of a hyperbola in the $(\sigma, \bar{\mu})$-plane.

# Mean-variance analysis with risky assets

Let us now turm to the problem of finding the portfolio with the smallest variance, not taking its expected rate of return into account.

The minimum-variance portfolio (MVP) $\boldsymbol{\pi}_{\text{min}}$ solves the problem

$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \text{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{N} \pi_i = 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1. \end{array}$$

To solve this problem we again set up the Lagrangian

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda(1 - \boldsymbol{\pi} \cdot \mathbf{1})$$

and proceed as above

# Mean-variance analysis with risky assets

In this case we get (check this!)

$$\boldsymbol{\pi}_{\text{min}} = \frac{1}{C}\Sigma^{-1}\mathbf{1} = \frac{1}{\mathbf{1} \cdot \Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}.$$

The standard deviation and mean of this portfolio is

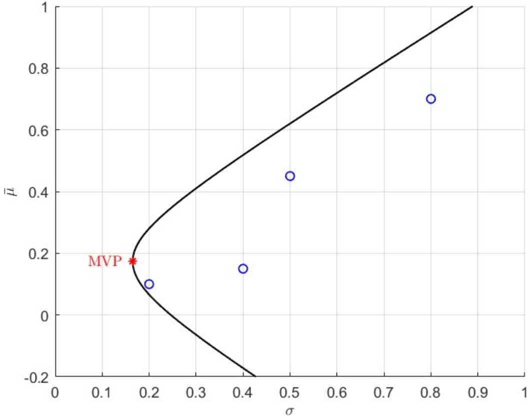
$$\sigma_{\text{min}} = \frac{1}{\sqrt{C}}$$

and

$$\mu_{\text{min}} = \frac{B}{C}$$

respectively.

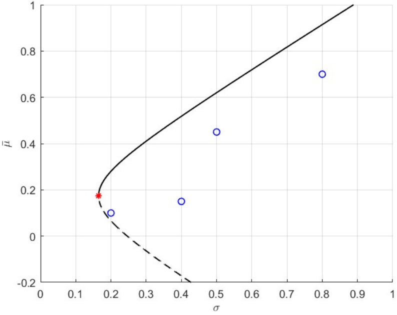
# Mean-variance analysis with risky assets



# Mean-variance analysis with risky assets

Looking at the mean-variance frontier, we see that for standard deviations larger than $\sigma_{\text{min}} = 1/\sqrt{C}$ there are two portfolios having this standard deviation.

Since an investor wants to maximise the expected return and minimise the standard deviation, no rational investor will hold a portfolio on the half of the mean-variance frontier that is below the MVP.



# Mean-variance analysis with risky assets

Recall the general formula

$$\boldsymbol{\pi}(\bar{\mu}) = \underbrace{\frac{A - B\bar{\mu}}{D}}_{\text{Scalar}} \underbrace{\boldsymbol{\Sigma^{-1}\mathbf{1}}}_{\text{Vector}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

Note that

$$\boldsymbol{\pi}_{\text{min}} = \frac{1}{C}\boldsymbol{\Sigma^{-1}\mathbf{1}} \quad \Leftrightarrow \quad \boldsymbol{\Sigma^{-1}\mathbf{1}} = C\boldsymbol{\pi}_{\text{min}}.$$

**Conclusion:** The vector $\Sigma^{-1}\mathbf{1}$ is up to a scaling equal to the MVP.

We can write the general optimal weight vector as

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

# Mean-variance analysis with risky assets

We have seen that $\boldsymbol{\pi}(\bar{\mu})$ can be written as a constant, depending on the chosen level $\bar{\mu}$, times the MVP plus another constant, also depending on $\bar{\mu}$, times the vector $\Sigma^{-1}\boldsymbol{\mu}$.

Define the portfolio

$$\boldsymbol{\pi}_{\text{slope}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{\mu} = \frac{1}{\mathbf{1}\cdot\Sigma^{-1}\boldsymbol{\mu}}\Sigma^{-1}\boldsymbol{\mu}.$$

**Note:** 1) We want a portfolio that is proportional to $\Sigma^{-1}\boldsymbol{\mu}$.

2) We multiply with $1/\mathbf{1}\cdot\Sigma^{-1}\boldsymbol{\mu}$ so that $\boldsymbol{\pi}_{\text{slope}}$ is a portfolio, i.e. its elements sum to 1.

# Mean-variance analysis with risky assets

Using that

$$\boldsymbol{\pi}_{\text{slope}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{\mu} \quad \Leftrightarrow \quad \Sigma^{-1}\boldsymbol{\mu} = B\boldsymbol{\pi}_{\text{slope}}$$

we get

$$\begin{aligned}
\boldsymbol{\pi}(\bar{\mu}) &= \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu} \\
&= \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{(C\bar{\mu} - B)B}{D}\boldsymbol{\pi}_{\text{slope}} \\
&= a_{\text{min}}(\bar{\mu})\boldsymbol{\pi}_{\text{min}} + a_{\text{slope}}(\bar{\mu})\boldsymbol{\pi}_{\text{slope}}.
\end{aligned}$$

# Mean-variance analysis with risky assets

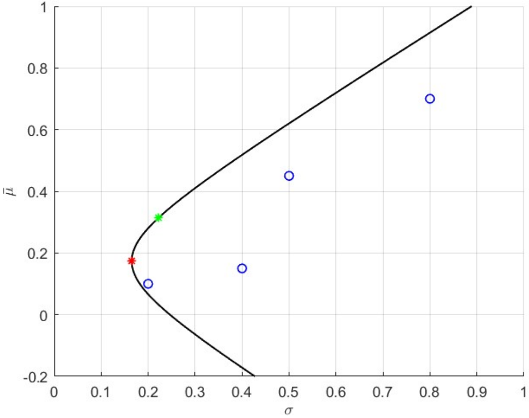
We have

$$
\begin{aligned}
a_{\min}(\bar{\mu}) + a_{\text{slope}}(\bar{\mu}) &= \frac{(A - B\bar{\mu})C}{D} + \frac{(C\bar{\mu} - B)B}{D} \\
&= \frac{AC - BC\bar{\mu} + BC\bar{\mu} - B^2}{D} \\
&= \frac{AC - B^2}{D} = 1.
\end{aligned}
$$

The interpretation is that for each $\bar{\mu}$, the optimal portfolio $\boldsymbol{\pi}(\bar{\mu})$ can be written as a combination of the two portfolios $\boldsymbol{\pi}_{\min}$ and $\boldsymbol{\pi}_{\text{slope}}$.

This is called two-fund separation: Any portfolio on the mean-variance frontier is the combination of the two portfolios $\boldsymbol{\pi}_{\min}$ and $\boldsymbol{\pi}_{\text{slope}}$.

# Mean-variance analysis with risky assets



More generally:

If the constant $B \neq 0$, then any two frontier portfolios $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ can be used to span the frontier, i.e. there exists a number $w$ such that

$$\boldsymbol{\pi}(\bar{\mu}) = w\boldsymbol{\pi}_1 + (1 - w)\boldsymbol{\pi}_2.$$

# Mean-variance analysis with risky assets

Is there any economic interpretation of $\boldsymbol{\pi}_{\text{slope}}$?

Yes!

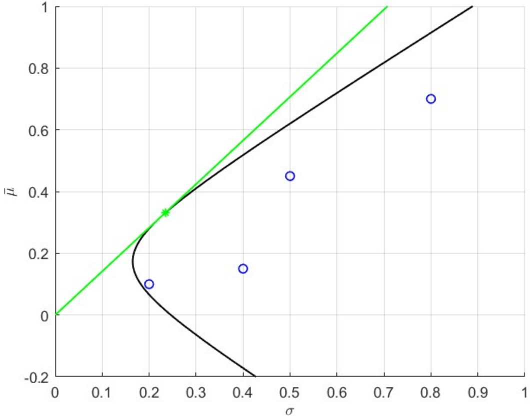
Consider a straight line

$$\bar{\mu} = k\sigma$$

in the $(\bar{\mu}, \sigma)$-plane for some $k$.

The portfolio $\boldsymbol{\pi}_{\text{slope}}$ represents the frontier portfolio with the largest slope of this line.

# Mean-variance analysis with risky assets



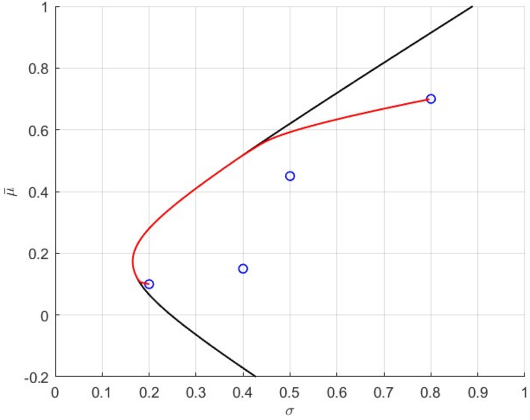
# Mean-variance analysis with portfolio constraints

So far we have allowed short-selling. Let us look at the problem where short-selling is not allowed.

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & \sum_{i=1}^{N} \pi_i \mu_i = \bar{\mu} \\ & \pi_i \geq 0, i = 1, 2, \ldots, N \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \bar{\mu} \\ & \boldsymbol{\pi} \geq \mathbf{0}. \end{aligned}$$

These type of problems are in general much harder to solve than when there are only equility constraints.

# Mean-variance analysis with portfolio constraints



# Mean-variance analysis with portfolio constraints

Different types of constraints:

- No short-selling in one or several assets.
- Maximum fraction invested in one or several assets.
- Maximum fraction invested in a sector or country.

# Choice under uncertainty

We know that a rational investor will choose a portfolio on the upper efficient frontier. But which portfolio will be chosen?

The choice depends on the investor's attitude towards risk.

An investor has initial wealth $W_0$ and invests in an asset with rate of return $r$. Then the future wealth is

$$W = W_0(1 + r).$$

Given axioms of investor behaviour, one can show the existence of a utility function representing the investor's risk preferences such that

$$E\left[u(W)\right]$$

is the utility of getting the cash flow $W$.

# Choice under uncertainty

Let $u$ be such a utility function.

In general it is

1) increasing, $u'(x) > 0$, and
2) concave, $u''(x) \leq 0$.

**Examples of utility functions**

- $u(x) = \ln x$.
- $u(x) = -e^{-x}$.
- $u(x) = \sqrt{x}$.
- $u(x) = x$.

How can we measure the level of attitude towards risk of an investor?

It is possible to show that the coefficient of absolute risk aversion

$$\mathrm{ARA}(x) = -\frac{u''(x)}{u'(x)}$$

is a good measure of this.

As an alternative the coefficient of relative risk aversion

$$\mathrm{RRA}(x) = -\frac{x\,u''(x)}{u'(x)}$$

can be used.

# CARA utility functions

If

$$u(x) = -e^{-ax}$$

then

$$u'(x) = ae^{-ax} \quad \text{and} \quad u''(x) = -a^2 e^{-ax},$$

so

$$\text{ARA}(x) = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a.$$

These are known as CARA utility functions for constant absolute risk aversion.

# CRRA utility functions

If

$$u(x) = \begin{cases} \dfrac{x^{1-\gamma} - 1}{1-\gamma} & \text{if } \gamma > 0, \gamma \neq 1 \\ \\ \ln x & \text{if } \gamma = 1 \end{cases}$$

then

$$u'(x) = x^{-\gamma} \quad \text{and} \quad u''(x) = -\gamma x^{-\gamma-1},$$

so

$$\mathrm{RRA}(x) = -\frac{x \cdot \left(-\gamma x^{-\gamma-1}\right)}{x^{-\gamma}} = \gamma.$$

These are known as CRRA utility functions for constant relative risk aversion.

# CARA utility and normally distributed rates of returns

Consider a market with $N$ risky assets such that:

1) The rates of return vector $\boldsymbol{r}$ has a multivariate normal distribution.
2) The investor has a CARA utility function with parameter $a > 0$.

This means that

$$W = W_0\big((1 + r(\boldsymbol{\pi})\big) = W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}$$

is normally distributed with mean

$$E\left[W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}\right] = W_0 + W_0 E\left[\boldsymbol{\pi} \cdot \boldsymbol{r}\right] = W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{\mu}$$

and variance

$$\mathrm{Var}[W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}] = W_0^2\mathrm{Var}[\boldsymbol{\pi} \cdot \boldsymbol{r}] = W_0^2\boldsymbol{\pi} \cdot \Sigma\boldsymbol{\pi}.$$

# CARA utility and normally distributed rates of returns

One can show (see the book!) that in this model an investor is indifferent between portfolios that have the same value of

$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} - \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

Hence, indifference curves are given by

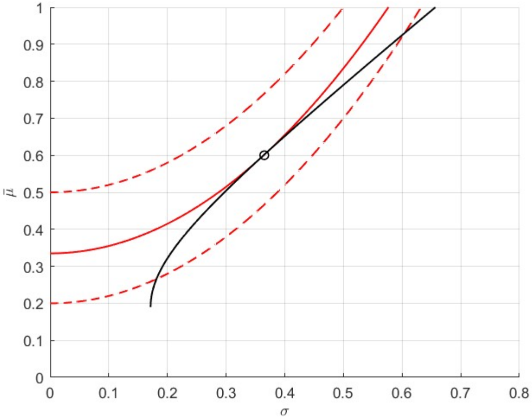
$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} - \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} = K$$

or

$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} = K + \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi},$$

where we let the level of utility $K$ vary.

# CARA utility and normally distributed rates of returns




Utility increases in the north-west direction.

# Financial Theory – Lecture 6

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Markets with a risk-free asset.
- The Capital Asset Pricing Model.

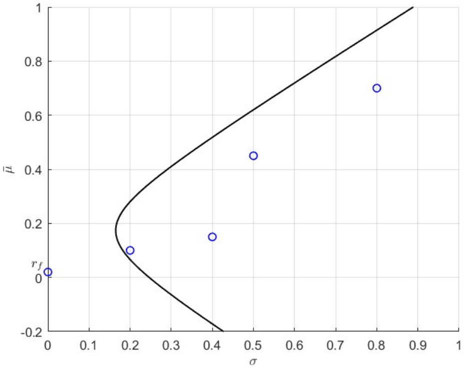
The lecture is based on

- Chapters 7 and 10 in the course book.

# Mean-variance analysis with a risk-free asset

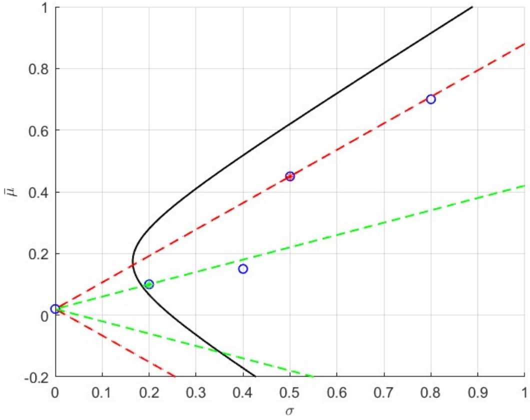
We now assume that there are $N$ risky assets represented by the vector $\mathbf{r}$ and additionally a risk-free asset with rate of return $r_f$.

A risk-free asset has standard deviation equal to 0, so it is represented by a point on the $\bar{\mu}$-axis.



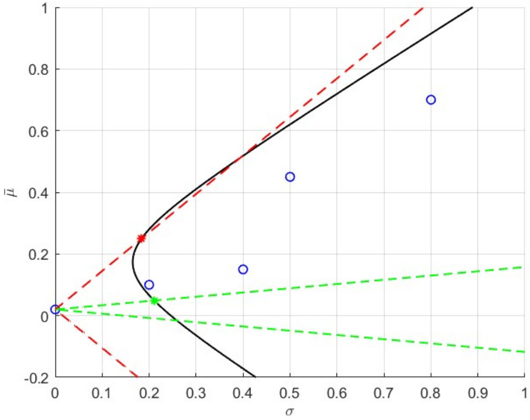
# Mean-variance analysis with a risk-free asset

The combination of an investment in the risk-free asset and any risky asset will result in straight lines in the $(\sigma, \bar{\mu})$-plane.



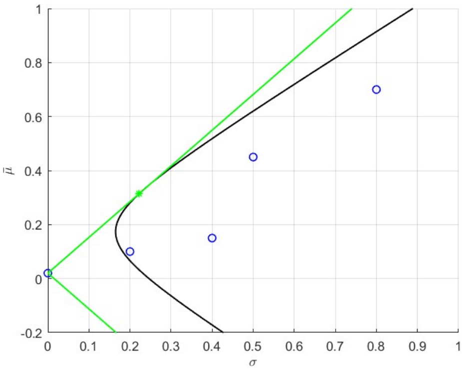
# Mean-variance analysis with a risk-free asset

But we can also combine the risk-free asset with a portfolio on the mean-variance frontier.



# Mean-variance analysis with a risk-free asset

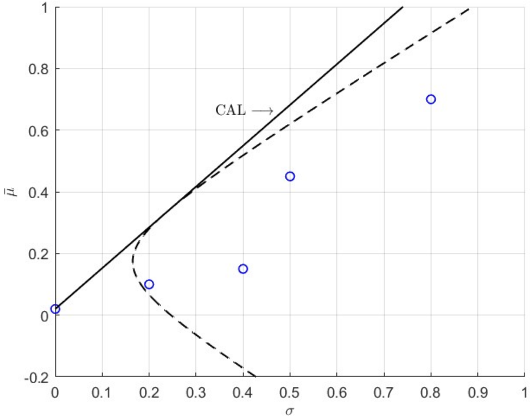
The best frontier when there is a risk-free is achieved if we combine the risk-free asset with the portfolio on the portfolio frontier that makes the straight line tangent to the mean-variance frontier.




This is tangent portfolio with weights $\boldsymbol{\pi}_{\tan}$.

# Mean-variance analysis with a risk-free asset

An investor will only choose a portfolio on the upper straight line. This is called the capital allocation line (CAL).



# Mean-variance analysis with a risk-free asset

Again we have two-fund separation: Every portfolio on the mean-variance frontier can be written as a combined portfolio of the risk-free asset and the tangent portfolio.

Actually, this requires that $B - Cr_f \neq 0$ as we will soon see.

# Mean-variance analysis with a risk-free asset

How do we find the tangent portfolio?

Recall the Sharpe ratio:

$$SR = \frac{\bar{\mu} - r_f}{\sigma}.$$

The Sharpe ratio for a portfolio with weights $\boldsymbol{\pi}$ is

$$SR = \frac{\boldsymbol{\pi} \cdot \boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}}}.$$

Since we know that the tangent portfolio is on the mean-variance frontier, we can write

$$SR = \frac{\bar{\mu} - r_f}{\sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}}$$

and maximise this over $\bar{\mu}$.

This is how it is done in the book. I'll take a different route.

# Mean-variance analysis with a risk-free asset

Again we want to find the portfolio with mean $\bar{\mu}$ that has the smallest variance.

Let $\boldsymbol{\pi}$ be the portfolio weights in the risky assets and let $\pi_0$ be the portfolio weight in the risk-free asset. Thus

$$\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1.$$

The return on this portfolio can be written

$$r_p = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}.$$

Then

$$E\left[r_p\right] = E\left[\pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}\right] = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu}$$

and

$$\mathrm{Var}[r_p] = \mathrm{Var}[\pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{r}] = \mathrm{Var}[\boldsymbol{\pi} \cdot \mathbf{r}] = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

# Mean-variance analysis with a risk-free asset

Now

$$\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1 \quad \Leftrightarrow \quad \pi_0 = 1 - \boldsymbol{\pi} \cdot \mathbf{1}$$

and we can write

$$E\left[r_p\right] = \pi_0 r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu} = (1 - \boldsymbol{\pi} \cdot \mathbf{1}) r_f + \boldsymbol{\pi} \cdot \boldsymbol{\mu} = r_f + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1}).$$

This leads to the problem

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & r_f + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1}) = \overline{\mu}. \end{aligned}$$

**Note:** We have replaced $\pi_0$ with $1 - \boldsymbol{\pi} \cdot \mathbf{1}$, so the condition $\pi_0 + \boldsymbol{\pi} \cdot \mathbf{1} = 1$ is already in place.

# Mean-variance analysis with a risk-free asset

Lagrangian:

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda\big(\bar{\mu} - r_f - \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1})\big).$$

FOC:

$$\frac{\partial L}{\partial \boldsymbol{\pi}} = 2\Sigma\boldsymbol{\pi} - \lambda(\boldsymbol{\mu} - r_f\mathbf{1}) = 0,$$

or

$$\Sigma\boldsymbol{\pi} = \frac{\lambda}{2}(\boldsymbol{\mu} - r_f\mathbf{1}) \quad \Rightarrow \quad \boldsymbol{\pi} = \frac{\lambda}{2}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}).$$

At this point we are only interested in the tangent portfolio.

The tangent portfolio $\boldsymbol{\pi}_{\text{tan}}$ satisfies $\mathbf{1} \cdot \boldsymbol{\pi}_{\text{tan}} = 1 \longrightarrow$

$$1 = \mathbf{1} \cdot \boldsymbol{\pi}_{\text{tan}} = \frac{\lambda_{\text{tan}}}{2}\mathbf{1} \cdot \Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) \quad \Rightarrow \quad \frac{\lambda_{\text{tan}}}{2} = \frac{1}{\mathbf{1} \cdot \Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}.$$

# Mean-variance analysis with a risk-free asset

Using this we can write

$$\begin{aligned}
\boldsymbol{\pi}_{\text{tan}} &= \frac{1}{\mathbf{1}\cdot\Sigma^{-1}(\boldsymbol{\mu}-r_f\mathbf{1})}\Sigma^{-1}(\boldsymbol{\mu}-r_f\mathbf{1}) \\
&= \frac{1}{\mathbf{1}\cdot\Sigma^{-1}\boldsymbol{\mu}-r_f\mathbf{1}\cdot\Sigma^{-1}\mathbf{1}}\Sigma^{-1}(\boldsymbol{\mu}-r_f\mathbf{1}) \\
&= \frac{1}{B-Cr_f}\Sigma^{-1}(\boldsymbol{\mu}-r_f\mathbf{1}).
\end{aligned}$$

We see that if $B - Cr_f = 0$, then there is division by zero.

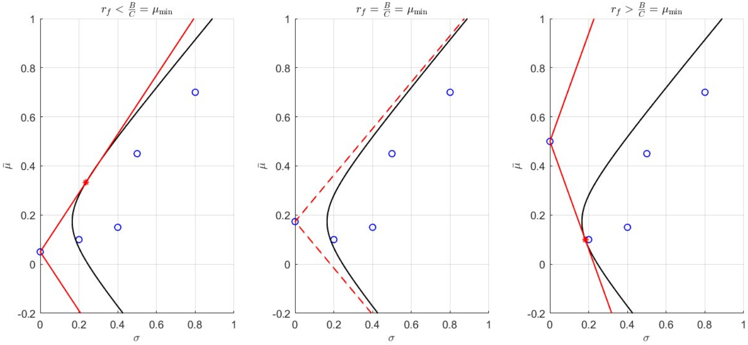
What is the problem?

$$B - Cr_f = 0 \quad\Leftrightarrow\quad B = Cr_f \quad\Leftrightarrow\quad r_f = \frac{B}{C} = \mu_{\text{min}}.$$

# Mean-variance analysis with a risk-free asset

When $r_f = B/C$ there is no tangent to the mean-variance frontier.

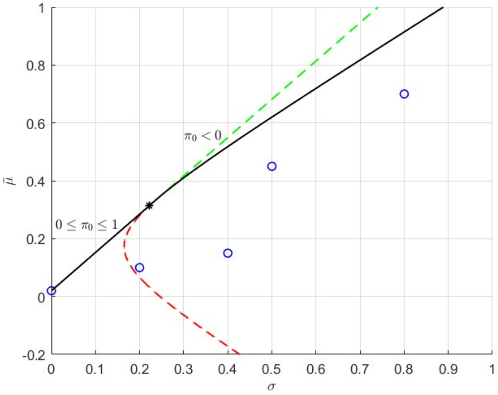
There are three cases:




The first one is the most reasonable from an economic point of view.

# Mean-variance analysis with borrowing constraints

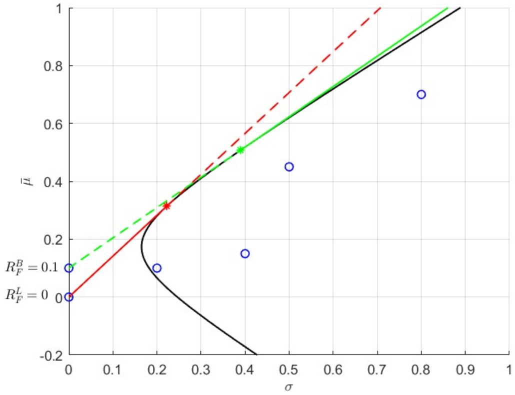
Sometimes an investor is credit constrained in the sense that he or she cannot borrow any money.



# Mean-variance analysis with different borrowing and lending rates

In reality, the lending rate is lower than the borrowing rate.

In this case the mean-variance frontier is given by the following.



# Mean-variance analysis with a risk-free asset

Let $\boldsymbol{\pi}$ be any portfolio weights.

The covariance between the rate of return $r_i$ and the portfolio rate of return $\boldsymbol{\pi}^T \boldsymbol{r}$ is given by

$$
\begin{aligned}
\mathrm{Cov}[r_i, \boldsymbol{\pi}^T \boldsymbol{r}] &= \mathrm{Cov}\left[r_i, \sum_{j=1}^{N} \pi_j r_j\right] \\
&= \sum_{j=1}^{N} \mathrm{Cov}[r_i, r_j]\pi_j \\
&= \sum_{j=1}^{N} \Sigma_{ij}\pi_j \\
&= \left(\Sigma\boldsymbol{\pi}\right)_i \longleftarrow \text{The element of row } i \text{ in } \Sigma\boldsymbol{\pi}
\end{aligned}
$$

# Mean-variance analysis with a risk-free asset

We will now prove that

$$\mathrm{Cov}[r_i, r_{\tan}] = \left(E\left[r_i\right] - r_f\right) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

Start with

$$\mathrm{Cov}[r_i, r_{\tan}] = \mathrm{Cov}[r_i, \boldsymbol{\pi}_{\tan} \cdot \boldsymbol{r}] = \left(\Sigma \boldsymbol{\pi}_{\tan}\right)_i.$$

Now

$$\Sigma \boldsymbol{\pi}_{\tan} = \frac{1}{B - Cr_f} \Sigma \Sigma^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}) = \frac{1}{B - Cr_f} (\boldsymbol{\mu} - r_f \mathbf{1}),$$

so

$$\mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \cdot \left(E\left[r_i\right] - r_f\right).$$

Multiply this with $\pi_{\tan,i}$:

$$\pi_{\tan,i} \mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \cdot \pi_{\tan,i} \left(E\left[r_i\right] - r_f\right).$$

# Mean-variance analysis with a risk-free asset

Sum this equation over $i$:

$$\sum_{i=1}^{N} \pi_{\tan,i} \mathrm{Cov}[r_i, r_{\tan}] = \frac{1}{B - Cr_f} \sum_{i=1}^{N} \pi_{\tan,i} (\mu_i - r_f)$$

$$\Leftrightarrow$$

$$\mathrm{Cov}\Bigg[\underbrace{\sum_{i=1}^{N} \pi_{i,\tan} r_i}_{r_{\tan}}, r_{\tan}\Bigg] = \frac{1}{B - Cr_f} \Bigg( \underbrace{\sum_{i=1}^{N} \pi_{\tan,i} \mu_i}_{=E[r_{\tan}]} - r_f \underbrace{\sum_{i=1}^{N} \pi_{\tan,i}}_{=1} \Bigg)$$

$$\Leftrightarrow$$

$$\mathrm{Cov}[r_{\tan}, r_{\tan}] = \mathrm{Var}[r_{\tan}] = \frac{1}{B - Cr_f} \cdot \left( E\left[r_{\tan}\right] - r_f \right).$$

# Mean-variance analysis with a risk-free asset

It follows that

$$\frac{1}{B - Cr_f} = \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

Finally,

$$\begin{aligned}\mathrm{Cov}[r_i, r_{\tan}] &= \left(E\left[r_i\right] - r_f\right) \cdot \frac{1}{B - Cr_f} \\ &= \left(E\left[r_i\right] - r_f\right) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.\end{aligned}$$

We will use this result later today!

# The Capital Asset Pricing Theory

Consider an economy with a number of investors.

The investors all agree that the expected rate of return vector is $\mu$ and that the variance-covariance matrix of the returns is $\Sigma$.

There also exists a risk-free asset with rate of return $r_f$.

The investors are mean-variance optimisers, which means that each invester prefers a high mean to a lower one and a low variance to a higher one.

# The Capital Asset Pricing Theory

The market portfolio is the portfolio of all risky assets in the economy.

In the market portfolio every asset that is traded is present.

It consists of all stocks, bonds, real estate, funds and derivatives as well as anything that investors invest in.

# The Capital Asset Pricing Theory

**Example** Consider an economy with the following assets:

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 25 | 25 000 | 6.25% |
| B | 1 750 | 100 | 175 000 | 43.75% |
| C | 1 250 | 80 | 100 000 | 25% |
| D | 500 | 200 | 100 000 | 25% |
| | | | 400 000 | 100% |

$$\text{Market cap} = \text{Units} \cdot \text{Price per unit.}$$

$$\text{Market weight} = \frac{\text{Market cap}}{\text{Total market cap}}.$$

# The Capital Asset Pricing Theory

Every investor will invest in a portfolio on the upper efficient frontier (which is a stright line since there is a risk-free asset in the economy).

From two-fund separation we know that every investor will only invest in a combination of the risk-free asset and the tangent portfolio.

Hence, the only risky assets the investors hold is a fraction of the tangent portfolio.

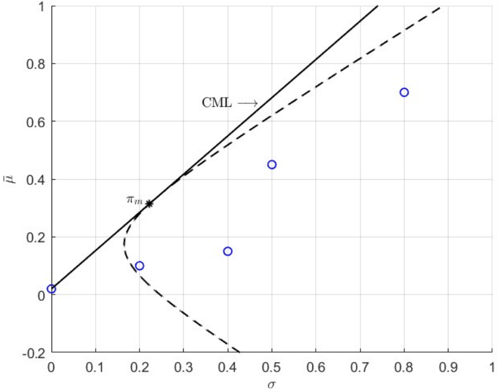
This implies that if the market is in equilibrium, then the market portfolio is equal to the tangent portfolio:

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\text{tan}}$$

From now on, the subindex $m$ stands for quantities connected to the market portfolio.

# The Capital Asset Pricing Theory

In this case the possible portfolios that rational investors choose from is called the capital market line (CML).



# The Capital Asset Pricing Theory

The CML intersects the $\bar{\mu}$-axis at $r_f$ and has slope

$$\frac{E[r_m] - r_f}{\sigma_m} = \text{The SR of the market portfolio.}$$

The equation is of the CML is

$$\bar{\mu} = r_f + \frac{E[r_m] - r_f}{\sigma_m}\sigma.$$

This can be written

$$\frac{\bar{\mu} - r_f}{\sigma} = \frac{E[r_m] - r_f}{\sigma_m};$$

all portfolios on the CML has the same SR.

# The Capital Asset Pricing Theory

Recall that

$$\mathrm{Cov}[r_i, r_{\tan}] = \big(E\left[r_i\right] - r_f\big) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

With $r_{\tan} = r_m$:

$$\big(E\left[r_m\right] - r_f\big)\mathrm{Cov}[r_i, r_m] = \big(E\left[r_i\right] - r_f\big)\mathrm{Var}[r_m]$$

This can be written

$$E\left[r_i\right] - r_f = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}\big(E\left[r_m\right] - r_f\big).$$

This is the Capital Asset Pricing Model (CAPM) equation.

# The Capital Asset Pricing Theory

The CAPM equation can be written:

$$E\left[r_i\right] - r_f = \beta_i(E\left[r_m\right] - r_f),$$

where

- $r_m$ is the rate of return of the market portfolio, and
- $\beta_i = \dfrac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}$ is the beta-value.

We used:

1) The mathematical formula

$$\mathrm{Cov}[r_i, r_{\tan}] = (E\left[r_i\right] - r_f) \cdot \frac{\mathrm{Var}[r_{\tan}]}{E\left[r_{\tan}\right] - r_f}.$$

2) The economics result

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\tan}$$

about a market in equilibrium.

Consider an economy with the following assets:

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 25 | 25 000 | 0.25 |
| B | 3 000 | 25 | 75 000 | 0.75 |
| | | | 100 000 | 1 |

Now assume that the tangent portfolio is $\boldsymbol{\pi}_{\text{tan}} = (0.5, 0.5)^\top$.

The tangent portfolio when all investors use the same $\boldsymbol{\mu}$ and $\Sigma$ is the portfolio of risky assets every investor wants to hold.

Since the price of both assets is the same, but the supply of asset A is bigger than for asset B, this can not be a market in equilibrium.

What will happen is that the price of asset A will increase to 30, and the price of asset B will decrease to 10 resulting in

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{\tan}.$$

| Asset | Units | Price per unit | Market cap | Market weight |
|---|---|---|---|---|
| A | 1 000 | 30 | 30 000 | 0.5 |
| B | 3 000 | 10 | 30 000 | 0.5 |
| | | | 60 000 | 1 |

# Financial Theory – Lecture 7

Fredrik Armerin, Uppsala University, 2024

# Agenda

- More on the CAPM.
- Optimal investments.
- The stochastic discount factor.

The lecture is based on

- Chapters 7 and 10 in the course book.

# The Capital Asset Pricing Theory

We ended the last lecture with the CAPM equation:

$$E[r_i] - r_f = \beta_i(E[r_m] - r_f).$$

The beta can be written

$$\begin{aligned}
\beta_i &= \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]} \\
&= \frac{\mathrm{Corr}[r_i, r_m]\mathrm{Std}[r_i]\mathrm{Std}[r_m]}{\mathrm{Std}[r_m]^2} \\
&= \mathrm{Corr}[r_i, r_m]\frac{\mathrm{Std}[r_i]}{\mathrm{Std}[r_m]} \\
&= \rho_{im}\frac{\sigma_i}{\sigma_m}.
\end{aligned}$$

# The Capital Asset Pricing Theory

If $r_p = \boldsymbol{\pi} \cdot \boldsymbol{r}$ is the return of a portfolio, then the beta of the portfolio is

$$\beta_p = \sum_{i=1}^{N} \pi_i \beta_i.$$

Any rate of return $r$ is generated by a portfolio of the basic $N$ assets.

For such a rate of return

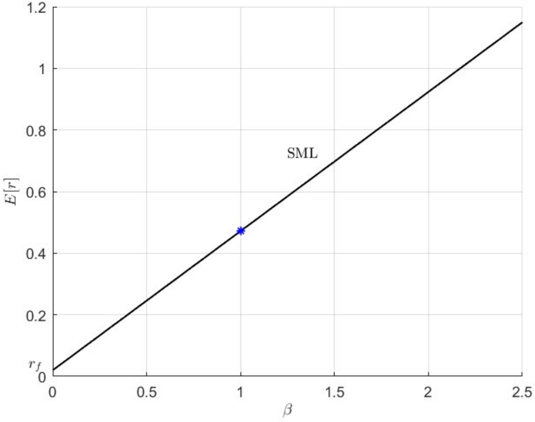
$$E[r] = r_f + \beta(E[r_m] - r_f),$$

where

$$\beta = \frac{\mathrm{Cov}[r, r_m]}{\mathrm{Var}[r_m]}.$$

# The Capital Asset Pricing Theory

The mean of a rate of return as a function of its beta-value is the security market line (SML).



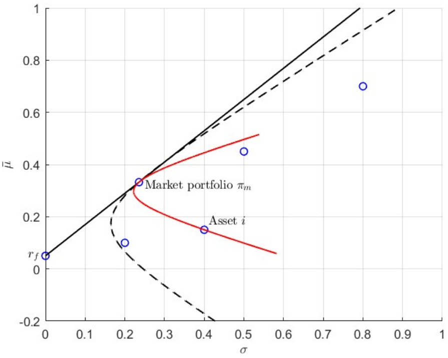
# The Capital Asset Pricing Theory

We can write the CAPM equation as

$$\begin{aligned} E[r_i] - r_f &= \frac{\text{Cov}[r_i, r_m]}{\text{Var}[r_m]}(E[r_m] - r_f) \\ &= \frac{E[r_m] - r_f}{\text{Var}[r_m]}\text{Cov}[r_i, r_m] \end{aligned}$$

**Important observation:** The risk premium of an asset is proportional to the covariance between the rate of return of the asset and the rate of return on the market portfolio.

Here is an alternative proof of the CAPM equation.




1) Form a two-asset portfolio of asset $i$ and the market portfolio:

$$r_p(w) = wr_i + (1-w)r_m.$$

2) Observe that

$$\left.\frac{d\mu_p}{d\sigma_p}\right|_{w=0} = \frac{E\left[r_m\right] - r_f}{\sigma_m}.$$

3) Remember that

$$\mu_p(w) = w\mu_i + (1-w)E\left[r_m\right]$$

and

$$\sigma_p(w) = \sqrt{w^2\sigma_i^2 + 2\mathrm{Cov}[r_i, r_m]w(1-w) + (1-w)^2\sigma_m^2}.$$

# The Capital Asset Pricing Theory

4) Calculate

$$\frac{d\mu_p}{d\sigma_p} = \frac{\frac{d\mu_p}{dw}}{\frac{d\sigma_p}{dw}} = \frac{\mu_i - E\left[r_m\right]}{\frac{1}{\sigma_p(w)}\left(w\sigma_i^2 + \mathrm{Cov}[r_i, r_m](1-2w) - (1-w)\sigma_m^2\right)}$$

5) Use 2):

$$\frac{E\left[r_m\right] - r_f}{\sigma_m} = \left.\frac{d\mu_p}{d\sigma_p}\right|_{w=0} = \frac{\mu_i - E\left[r_m\right]}{\frac{1}{\sigma_m}\left(\mathrm{Cov}[r_i, r_m] - \sigma_m^2\right)}.$$

6) Simplify $\longrightarrow$

$$\mu_i = r_f + \frac{\mathrm{Cov}[r_i, r_m]}{\sigma_m^2}(E\left[r_m\right] - r_f) = r_f + \beta_i(E\left[r_m\right] - r_f).$$

# The Capital Asset Pricing Theory

CAPM is a pricing model. Where are the prices?

Recall that

$$r_i = \frac{D_i + P_{i1} - P_{i0}}{P_{i0}}.$$

From CAPM:

$$E\left[\frac{D_i + P_{i1} - P_{i0}}{P_{i0}}\right] = E\left[r_i\right] = r_f + \beta_i(E\left[r_m\right] - r_f)$$

$$\frac{1}{P_{i0}}E\left[D_i + P_{i1}\right] = 1 + r_f + \beta_i(E\left[r_m\right] - r_f) \qquad (*)$$

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right]}{1 + r_f + \beta_i(E\left[r_m\right] - r_f)}.$$

Given expectations of the future dividend payment and price, we use CAPM to calculate the discount rate to get today's price.

# The Capital Asset Pricing Theory

Write Equation ($*$) from the previous slide as

$$E\left[D_i + P_{i1}\right] = P_{i0}(1 + r_f) + P_{i0}\beta_i(E\left[r_m\right] - r_f).$$

Now

$$P_{i0}\beta_i = P_{i0}\frac{\mathrm{Cov}[r_i, r_m]}{\sigma_m^2} = \frac{\mathrm{Cov}\left[P_{i0}(1 + r_i), r_m\right]}{\sigma_m^2} = \frac{\mathrm{Cov}[P_{i1}, r_m]}{\sigma_m^2},$$

so

$$E\left[D_i + P_{i1}\right] = P_{i0}(1 + r_f) + \frac{E\left[r_m\right] - r_f}{\sigma_m^2}\mathrm{Cov}[P_{i1}, r_m].$$

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right] - \frac{E[r_m] - r_f}{\sigma_m^2}\mathrm{Cov}[P_{i1}, r_m]}{1 + r_f}.$$

This is sometimes called the certainty equivalent pricing version of CAPM: It tells us which cash flow we should use if we want to discount using the risk-free rate.

# The Capital Asset Pricing Theory

What if there is no risk-free rate in the economy?

There is a version of CAPM that is valid when there is no risk-free rate.

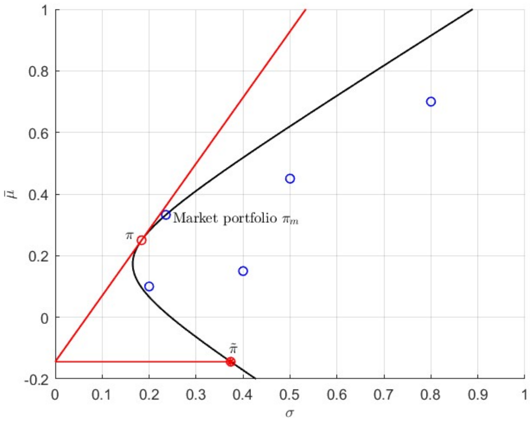
We need the following result: For every portfolio $\boldsymbol{\pi}$ on the upper part of the efficient frontier, there is a portfolio $\tilde{\boldsymbol{\pi}}$ on the lower part of the frontier that is uncorrelated with the first portfolio.

The uncorrelated portfolio $\tilde{\boldsymbol{\pi}}$ has expected rate of return

$$E\left[r(\tilde{\boldsymbol{\pi}})\right] = \frac{A - BE\left[r(\boldsymbol{\pi})\right]}{B - CE\left[r(\boldsymbol{\pi})\right]}.$$

This is Theorem 7.5 in the course book. In Exercise 7.6 you are asked to prove this.

# The Capital Asset Pricing Theory




To find the uncorrelated portfolio, draw the tangent line at the point $(\mathrm{Std}[r(\pi)], E\,[r(\pi)])$.

The portfolio $\tilde{\pi}$ is the mean-variance efficient portfolio that has the mean where the tangent line hits the $\bar{\mu}$-axis.

Let $\boldsymbol{\pi}_z$ denote the portfolio that is uncorrelated with the market portfolio.

Then one can show that

$$E[r_i] = E[r(\boldsymbol{\pi}_z)] + \beta_i \big( E[r_m] - E[r(\boldsymbol{\pi}_z)] \big).$$

This is the zero-beta CAPM or Black's CAPM.

# Optimal portfolios

Let us return to the optimal choice of one investor.

The investor wants to maximise

$$E[r_p] - \frac{\gamma}{2}\mathrm{Var}[r_p].$$

We assume that there exists a risk-free asset and $N$ risky assets.

The investor will invest a fraction $w$ in the tangent portfolio and the fraction $1 - w$ in the risk-free asset:

$$r_p = r(w) = wr_{\tan} + (1 - w)r_f.$$

We also let

$$\mu_{\tan} = E[r_{\tan}] \quad \text{and} \quad \sigma^2_{\tan} = \mathrm{Var}[r_{\tan}].$$

# Optimal portfolios

Let

$$\begin{aligned} f(w) &= E\left[r(w)\right] - \frac{\gamma}{2}\mathrm{Var}[r(w)] \\ &= E\left[wr_{\tan} + (1-w)r_f\right] - \frac{\gamma}{2}\mathrm{Var}[wr_{\tan} + (1-w)r_f] \\ &= w\mu_{\tan} + (1-w)r_f - \frac{\gamma}{2}w^2\sigma_{\tan}^2. \end{aligned}$$

$$\text{FOC: } f'(w) = \mu_{\tan} - r_f - \gamma w \sigma_{\tan}^2 = 0 \quad \Rightarrow \quad w^* = \frac{\mu_{\tan} - r_f}{\gamma\sigma_{\tan}^2}.$$

Which is the optimal portfolio $\boldsymbol{\pi}^*$ in the risky assets?

$$\begin{aligned} \boldsymbol{\pi}^* &= w^*\boldsymbol{\pi}_{\tan} \\ &= \frac{\mu_{\tan} - r_f}{\gamma\sigma_{\tan}^2}\boldsymbol{\pi}_{\tan} \end{aligned}$$

# Optimal portfolios

We know that

$$\frac{\sigma_{\mathrm{tan}}^2}{\mu_{\mathrm{tan}} - r_f} = \frac{1}{B - Cr_f}.$$

Use this:

$$\boldsymbol{\pi}_{\mathrm{tan}} = \frac{1}{B - Cr_f}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) = \frac{\sigma_{\mathrm{tan}}^2}{\mu_{\mathrm{tan}} - r_f}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}).$$

We get

$$\boldsymbol{\pi}^* = \frac{\mu_{\mathrm{tan}} - r_f}{\gamma\sigma_{\mathrm{tan}}^2}\frac{\sigma_{\mathrm{tan}}^2}{\mu_{\mathrm{tan}} - r_f}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) = \frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}).$$

# Optimal portfolios

Let us now introduce $n$ number of investors, each with his or her own constant $ARA_j = \gamma_j$, $j = 1, \ldots, n$.

In equilibrium, the tangent portfolio is equal to the market portfolio, so investor $j$ will hold the portfolio

$$w_j = \frac{E\left[r_m\right] - r_f}{\gamma_j \mathrm{Var}[r_m]}.$$

Since we are in equilibrium it must hold that supply equals demand:

$$\sum_{j=1}^{n} w_j = 1.$$

# Optimal portfolios

$$1=\sum_{j=1}^{n} w_j=\sum_{j=1}^{n}\frac{E\left[r_m\right]-r_f}{\gamma_j \mathrm{Var}[r_m]}=\frac{E\left[r_m\right]-r_f}{\mathrm{Var}[r_m]}\sum_{j=1}^{n}\frac{1}{\gamma_j}=\frac{E\left[r_m\right]-r_f}{\mathrm{Var}[r_m]}\cdot\frac{1}{\bar{\gamma}}.$$

Here

$$\bar{\gamma}=\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}$$

is a measure of the market's average risk aversion.

We can use

$$\bar{\gamma}=\frac{E\left[r_m\right]-r_f}{\mathrm{Var}[r_m]}$$

to write

$$w_j=\frac{\bar{\gamma}}{\gamma_j}.$$

# Optimal portfolios

We can also write

$$E[r_m] - r_f = \bar{\gamma}\mathrm{Var}[r_m].$$

The market risk premium is increasing in

1) the amount of risk as measured by $\mathrm{Var}[r_m]$, and in
2) the average risk averision as measured by $\bar{\gamma}$.

# Optimal portfolios

We have seen in Lecture 5 that

$$\text{CARA with parameter } a \ + \ \text{Multivariate normal returns}$$

$$\Rightarrow$$

$$\text{maximise } E\left[r_p\right] - \frac{a}{2}\text{Var}[r_p].$$

However, for more general random variables it might happen that

$$X \geq Y \quad \not\Rightarrow \quad E\left[X\right] - \frac{a}{2}\text{Var}[X] \geq E\left[Y\right] - \frac{a}{2}\text{Var}[Y].$$

The function

$$f(X) = E\left[X\right] - \frac{a}{2}\text{Var}[X]$$

is not always monotone.

It is easy to construct an example.

| $X$ | $Y$ | Prob |
|---|---|---|
| 1 | 1 | $1/3$ |
| 2 | 2 | $1/3$ |
| 5 | 3 | $1/3$ |

Since $X \geq Y$ a rational investor should prefer $X$ to $Y$.

But if we let $a = 2$, then

$$E[X] - \text{Var}[X] = -\frac{2}{9} < \frac{4}{3} = E[Y] - \text{Var}[Y].$$

# Time-dependent CAPM

CAPM is a one-period model. If we want to let the model meet data we should take into account that the returns can vary over time.

This results in CAPM equations like

$$E_t\left[r_{i,t+1} - r_{f,t+1}\right] = \beta_{i,t} E_t\left[r_{m,t+1} - r_{f,t+1}\right],$$

where

$$\beta_{i,t} = \frac{\mathrm{Cov}_t\left[r_{i,t+1}, r_{m,t+1}\right]}{\mathrm{Var}_t\left[r_{m,t+1}\right]}.$$

Here the $t$-subscript in $E_t$, $\mathrm{Cov}_t$ and $\mathrm{Var}_t$ means that we should take into account all information up to and including time $t$.

For now, we only consider one-period models, but by considering the intervals $(t, t+1]$ for $t = 0, 1, 2, \ldots$ we can move to time-dependent models.

# Consumption-based CAPM

> *Investments move consumption opportunities across time and across states of the world. Individual investors ultimately care about the consumption they get out of their investments.*

(Munk p. 400.)

**Question:** When would an investor prefer an extra amount of money?
**Answer:** When there are bad times.

**Conclusion:** An asset that delivers cash flows when there are bad times will be much in demand $\longrightarrow$ It's price will be high $\longrightarrow$ It's return will be low.

# Consumption-based CAPM

We build a model with investors and see how it can be used to price assets.

- At time $t = 0$: Endowment $e_0$, consumption $c_0$ and an investment portfolio $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ is bought.
- At time $t = 1$: Endowment $e_1$, consumption $c_1$ and the investment portfolio $\mathbf{x}$ is sold.
- The portfolio costs $\sum_{i=1}^{N} x_i P_{i0}$ at time $t = 0$ and results in the cash flow $\sum_{i=1}^{N} x_i (D_i + P_{i1})$ at time $t = 1$.
- An investor's utility is given by

$$u(c_0) + e^{-\delta} E\left[u(c_1)\right],$$

where $u$ is an increasing and concave utility function and $\delta$ measures the investor's time preference rate.

# Consumption-based CAPM

The investor wants to solve

$$\max_{\mathbf{x}} \left\{ u(c_0) + e^{-\delta} E\left[u(c_1)\right] \right\}$$

$$\text{s.t.}\ \ c_0 + \sum_{i=1}^{N} x_i P_{i0} = e_0$$

$$c_1 = e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1}).$$

Note that $e_0, e_1$ are given exogenously and that $c_0, c_1$ are given endogenously when the portfolio $\mathbf{x}$ is determined.

We solve this problem by replacing $c_0, c_1$ from the constraints into the objective function.

# Consumption-based CAPM

We get the function

$$f(\mathbf{x}) = u\left(e_0 - \sum_{i=1}^{N} x_i P_{i0}\right) + e^{-\delta} E\left[u\left(e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1})\right)\right]$$

to maximise.

$$\text{FOC:} \quad \frac{\partial f}{\partial x_i} = -P_{i0} u'(c_0) + e^{-\delta} E\left[(D_i + P_{i1}) u'(c_1)\right] = 0$$

$$\Leftrightarrow$$

$$E\left[e^{-\delta} \frac{D_i + P_{i1}}{P_{i0}} \cdot \frac{u'(c_1)}{u'(c_0)}\right] = 1 \quad \Leftrightarrow \quad E\left[(1 + r_i) \underbrace{e^{-\delta} \frac{u'(c_1)}{u'(c_0)}}_{=m}\right] = 1.$$

# Consumption-based CAPM

Now,

$$\begin{aligned} 1 &= E\left[(1+r_i)m\right] \\ &= E\left[1+r_i\right]E\left[m\right] + \mathrm{Cov}[1+r_i, m] \\ &= \big(1+E\left[r_i\right]\big)E\left[m\right] + \mathrm{Cov}[r_i, m]. \end{aligned}$$

This implies

$$1+E\left[r_i\right] = \frac{1}{E\left[m\right]} - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_i, m],$$

or

$$E\left[r_i\right] = \frac{1}{E\left[m\right]} - 1 - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_i, m].$$

This hold for any return. Assume that there exists a risk-free asset:

$$E\left[r_f\right] = r_f = \frac{1}{E\left[m\right]} - 1 - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_f, m] = \frac{1}{E\left[m\right]} - 1.$$

# The stochastic discount factor

Making the substitution $1/E[m] - 1 = r_f$ results in

$$E[r_i] = r_f - (1 + r_f)\mathrm{Cov}[r_i, m],$$

or

$$E[r_i] = r_f + \mathrm{Cov}\big[r_i, -(1 + r_f)m\big].$$

This equation looks like the CAPM equation.

Instead of the rate of return of the market portfolio, $-(1 + r_f)m$ is used. This is called the consumption-based CAPM.

# The stochastic discount factor

The random variable

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)}$$

is called a stochastic discount factor (SDF). We use it to discount random cash flows to get its price today:

$$\begin{aligned} P_{i0} &= E\left[m\big(D_i + P_{i1}\big)\right] \\ &= E[m]\, E[D_i + P_{i1}] + \mathrm{Cov}[m, D_i + P_{i1}] \\ &= \frac{E[D_i + P_{i1}]}{1 + r_f} + \mathrm{Cov}[m, D_i + P_{i1}] \end{aligned}$$

# The stochastic discount factor

Recall

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right]}{1 + r_f} + \mathrm{Cov}[m, D_i + P_{i1}].$$

We have a high price $P_{i0}$ today if $\mathrm{Cov}[m, D_i + P_{i1}]$ is high.

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)} = \text{positive constant} \cdot u'(c_1).$$

$\boxed{\text{High value of } m} = \boxed{\text{High value of } u'(c_1)} = \boxed{\text{Bad future state of the world}}.$

# The CRRA-lognormal model

How the explicit expression that gives the risk premium looks like depends on the assumptions we make on the utility functions of the investors and the distributional properties of the rates of return.

**The CRRA-lognormal model**

- Consumption growth is lognormally distributed,
- Each consumer has a CRRA utility function: $u'(x) = x^{-\gamma}$.

In this case

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)} = e^{-\delta} \left( \frac{c_1}{c_0} \right)^{-\gamma} = e^{-\delta} e^{-\gamma \ln(c_1/c_0)} = e^{-\delta - \gamma \ln(c_1/c_0)}.$$

We will come back to this model when we look at models in macro-finance.

# Financial Theory – Lecture 8

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Factor models.
- The arbitrage pricing theory (APT).

The lecture is based on

- Chapter 11 in the course book.

# Factor models

Again, we have a market of $N$ risky assets with returns $r_1, \ldots, r_N$.

A factor model is a model that explains the risk premium on every asset by it's exposure to a (low) number of factors.

In a factor model the two major questions are

1) Which factors should be used?
2) How large is the risk premium with respect to each of the factors?

# Factor models

Examples of potential priced factors are:

- Macroeconomic variables such as GDP growth rate.
- Prices on raw materials such as oil and metals.
- Rates of return of indexes, specific portfolios or individual assets.

# One-factor models

A one-factor model is defined as follows.

There exists a random variable $F$ such that

$$r_i = E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i,\ \ i = 1, 2, \ldots, N,$$

and where for every $i, j = 1, 2, \ldots, N$

(i) $\text{Cov}[F, \varepsilon_i] = 0$.

(ii) $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0$ when $i \neq j$.

Here $F$ is the factor affecting all returns and $\varepsilon_i$ represents asset-specific return uncertainty.

Note that

$$E\left[\varepsilon_i\right] = 0.$$

# One-factor models

In this model

$$\begin{aligned}
\text{Cov}[r_i, F] &= \text{Cov}\Big[E\,[r_i] + \beta_i\big(F - E\,[F]\,\big) + \varepsilon_i, F\Big] \\
&= \beta_i \text{Var}[F] \quad \Leftrightarrow \quad \beta_i = \frac{\text{Cov}[r_i, F]}{\text{Var}[F]}
\end{aligned}$$

$$\begin{aligned}
\text{Var}[r_i] &= \text{Var}\Big[E\,[r_i] + \beta_i\big(F - E\,[F]\,\big) + \varepsilon_i\Big] \\
&= \beta_i^2 \text{Var}[F] + \text{Var}[\varepsilon_i]
\end{aligned}$$

$$\begin{aligned}
\text{Cov}[r_i, r_j] &= \text{Cov}\Big[E\,[r_i] + \beta_i\big(F - E\,[F]\,\big) + \varepsilon_i, \\
&\qquad\qquad E\,[r_j] + \beta_j\big(F - E\,[F]\,\big) + \varepsilon_j\Big] \\
&= \beta_i \beta_j \text{Var}[F]
\end{aligned}$$

# One-factor models

The variance

$$\text{Var}[r_i] = \beta_i^2 \text{Var}[F] + \text{Var}[\varepsilon_i]$$

consists of two terms.

- $\beta_i^2 \text{Var}[F]$ is called the systematic risk.
- $\text{Var}[\varepsilon_i]$ is called the non-systematic risk, the idiosyncratic risk or the asset(firm)-specific risk.

**Remark.** Sometimes $\beta_i \text{Std}[F]$ and $\text{Std}[\varepsilon_i]$ are used in these definitions.

# One-factor models

The equation

$$r_i = E[r_i] + \beta_i \big(F - E[F]\big) + \varepsilon_i$$

can be written

$$r_i = a_i + \beta_i F + \varepsilon_i,$$

where

$$a_i = E[r_i] - \beta_i E[F].$$

Alternatively, it can be written

$$r_i - r_f = a_i' + \beta_i F + \varepsilon_i,$$

where $a_i' = a_i - r_f$.

When estimating and/or testing these type of models, we need to run regressions. Depending on what we are interested in, we can use any of the model specifications above.

# The Single-Index model

In the Single-Index model, also called the Market Model, the only factor is equal to the return on the market portfolio:

$$F = r_m.$$

It is usually expressed using excess returns:

$$r_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i.$$

The condition $E[\varepsilon_i] = 0$ can be written

$$0 = E[r_i - r_f - \alpha_i - \beta_i(r_m - r_f)] \quad \text{or} \quad \alpha_i = E[r_i - r_f - \beta_i(r_m - r_f)].$$

If all $\alpha_i = 0$, then

$$E[r_i] - r_f = \beta_i\big(E[r_m] - r_f\big),$$

which is the same equation as in CAPM.

# The Single-Index model

In the Single-Index Model with $\alpha_i = 0$ for all $i$ it is not possible to find any portfolio that has a higher Sharpe ratio than the market portfolio.

To see this, consider the portfolio with weights $\boldsymbol{\pi}$ and return $r_p = \sum_{i=1}^{N} \pi_i r_i$:

$$\begin{aligned}
E[r_p] - r_f &= \sum_{i=1}^{N} \pi_i E[r_i] - r_f \\
&= \sum_{i=1}^{N} \pi_i \big( E[r_i] - r_f \big) \\
&= \big\{ \text{The Single-Index Model with all } \alpha_i = 0. \big\} \\
&= \sum_{i=1}^{N} \pi_i \beta_i \big( E[r_m] - r_f \big) \\
&= \big( E[r_m] - r_f \big) \sum_{i=1}^{N} \pi_i \beta_i .
\end{aligned}$$

# The Single-Index model

We now use that the beta for a portfolio is

$$\beta_p = \frac{\mathrm{Cov}[r_p, r_m]}{\mathrm{Var}[r_m]} = \frac{\mathrm{Corr}[r_p, r_m]\mathrm{Std}[r_p]}{\mathrm{Std}[r_m]} =: \frac{\rho_{pm}\sigma_p}{\sigma_m}$$

and

$$SR_m = \frac{E\left[r_m\right] - r_f}{\sigma_m} \quad \Leftrightarrow \quad E\left[r_m\right] - r_f = SR_m \sigma_m.$$

We get

$$\begin{aligned} E\left[r_p\right] - r_f &= \left(E\left[r_m\right] - r_f\right) \underbrace{\sum_{i=1}^{N} \pi_i \beta_i}_{=\beta_p} \\ &= SR_m \sigma_m \beta_p \\ &= SR_m \sigma_m \frac{\rho_{pm}\sigma_p}{\sigma_m} \\ &= SR_m \rho_{pm} \sigma_p. \end{aligned}$$

# The Single-Index model

This last relationship can be written

$$\frac{E\left[r_p\right] - r_f}{\sigma_p} = \rho_{pm} SR_m,$$

or

$$SR_p = \rho_{pm} SR_m.$$

Since $\rho_{pm} \leq 1$ the highest Sharpe ratio is achieved when $\rho_{pm} = 1$.

One way of getting this is to choose $p = m$.

# The Single-Index model

Assume that there is an asset $i$ with $\alpha_i > 0$.

How can we benefit from this?

If we invest 1 unit of currency in asset $i$, then we get the payoff

$$1 + r_i = 1 + r_f + \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i.$$

But we can do better.

Short $\beta_i$ units of currency of the market portfolio and invest the money in the risk-free asset. This results in the payoff

$$\beta_i(1 + r_f) - \beta_i(1 + r_m) = \beta_i(r_f - r_m).$$

# The Single-Index model

By adding the zero net investment portfolio $\beta_i(r_f - r_m)$ we get the total payoff

$$\begin{aligned} 1 + r_i + \beta_i(r_f - r_m) &= 1 + r_f + \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i + \beta_i(r_f - r_m) \\ &= 1 + r_f + \alpha_i + \varepsilon_i. \end{aligned}$$

Now we are only exposed to the asset-specific $\varepsilon_i$ – the cash flow does not depend on the movement of the market.

The attractiveness of the position can be measured using the information ratio

$$IR = \frac{\alpha_i}{\mathrm{Std}[\varepsilon_i]}.$$

# Multi-factor models

One factor is probably not enough in order to get a reasonable model.

When we have a multi-factor model with $K$ factors, we say that we have a $K$-factor model.

For $i = 1, 2, \ldots, N$:

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big) + \varepsilon_i,$$

where for every $i$ and $k$

$$\mathrm{Cov}[F_k, \varepsilon_i] = 0 \;\text{ and }\; \mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0 \text{ when } i \neq j.$$

Note that we again have

$$E[\varepsilon_i] = 0.$$

# Multi-factor models

With the vectors

$$\boldsymbol{\beta}_i = \begin{pmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{iK} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_K \end{pmatrix} \text{ and } E\left[\mathbf{F}\right] = \begin{pmatrix} E\left[F_1\right] \\ E\left[F_2\right] \\ \vdots \\ E\left[F_K\right] \end{pmatrix}$$

we can write the $K$-factor model as

$$r_i = E\left[r_i\right] + \boldsymbol{\beta}_i \cdot \Big(\mathbf{F} - E\left[\mathbf{F}\right]\Big) + \varepsilon_i.$$

The variance of $r_i$ now becomes

$$\mathrm{Var}[r_i] = \boldsymbol{\beta}_i \cdot \Sigma_F \boldsymbol{\beta}_i + \mathrm{Var}[\varepsilon_i],$$

where

$$\Sigma_F = \mathrm{Var}[\mathbf{F}].$$

# Multi-factor models

One can further show that

$$\mathrm{Cov}[r_i, \mathbf{F}] = \Sigma_F \boldsymbol{\beta}_i$$

and

$$\mathrm{Cov}[r_i, r_j] = \boldsymbol{\beta}_i \cdot \Sigma_F \boldsymbol{\beta}_j.$$

From the first equation we see that

$$\Sigma_F^{-1}\mathrm{Cov}[r_i, \mathbf{F}] = \Sigma_F^{-1}\Sigma_F \boldsymbol{\beta}_i = \boldsymbol{\beta}_i.$$

Compare this with multivariate OLS.

Note that the moments $\mathrm{Cov}[r_i, r_j]$ and $\mathrm{Var}[r_i]$ are determined by $\Sigma_F$ and the beta vectors.

# Multi-factor models

One reason multi-factors are popular, is that it reduces the number of parameters that has to be estimated.

With $N$ assets there are in general $N$ means and $\frac{(N+1)N}{2}$ elements in the variance-covariance to estimate.

With a $K$-factor model there are $N$ means and

$$\underbrace{\frac{(K+1)K}{2}}_{\text{Elements in } \Sigma_F} + \underbrace{(K+1)N}_{\beta\text{'s and } \alpha\text{'s}}$$

factor model parameters.

# Multi-factor models

| Number of factors $K$ | $N = 10$ | $N = 100$ | $N = 500$ |
|---|---|---|---|
| 1 | 21 | 201 | 1 001 |
| 2 | 33 | 606 | 3 006 |
| 5 | 75 | 3 651 | 18 051 |
| Without factors | 55 | 5 050 | 125 250 |

Number of parameters in the variance-covariance matrix for different $K$-factor models.

See also Table 11.2 in Munk.

# Arbitrage pricing theory

Consider the $K$-factor model

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big) + \varepsilon_i.$$

What can be said of $E[r_i]$?

It turns out that if we rule out arbitrage opportunities then there exists risk premia $\mathrm{RP}_k$, $k = 1, 2, \ldots, K$ such that

$$E[r_i] = r_f + \beta_{i1}\mathrm{RP}_1 + \beta_{i2}\mathrm{RP}_2 + \ldots + \beta_{iK}\mathrm{RP}_K.$$

This is known as the Arbitrage pricing theory (APT).

Note that the risk premium $\mathrm{RP}_k$ is connected to factor $F_k$ and that the $\beta$'s are the same as the ones defining the model.

# Arbitrage pricing theory

The general proof of the APT is surprisingly hard and technical from a mathematical point of view.

There is, however, a special case which is more straightforward to show.

Assume that

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big).$$

These models are known as exact factor models since there is no $\varepsilon_i$.

# Arbitrage pricing theory

In this model choose a portfolio $\boldsymbol{\pi}$ such that

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 0 \quad \text{and} \quad \sum_{i=1}^{N} \pi_i \beta_{ik} = \boldsymbol{\pi} \cdot \boldsymbol{\beta}^k = 0$$

for every $k = 1, 2, \ldots, K$.

For such a portfolio

$$\begin{aligned} \boldsymbol{\pi} \cdot \mathbf{r} = \sum_{i=1}^{N} \pi_i r_i &= \sum_{i=1}^{N} \pi_i \left( E\left[r_i\right] + \sum_{k=1}^{K} \beta_{ik} (F_k - E\left[F_k\right]) \right) \\ &= \sum_{i=1}^{N} \pi_i E\left[r_i\right] + \sum_{k=1}^{K} \left(F_k - E\left[F_k\right]\right) \sum_{i=1}^{N} \pi_i \beta_{ik} \\ &= \sum_{i=1}^{N} \pi_i E\left[r_i\right] = \boldsymbol{\pi} \cdot E\left[\mathbf{r}\right]. \end{aligned}$$

# Arbitrage pricing theory

A portfolio with

$$\sum_{i=1}^{N} \pi_i = 0$$

costs zero to buy.

A portfolio with

$$\sum_{i=1}^{N} \pi_i \beta_{ik} = 0 \text{ for every } k$$

has no risk.

A risk-free portfolio that costs zero must have zero payoff – otherwise we would have an arbitrage. Hence we have

$$\left.\begin{array}{l} \boldsymbol{\pi} \cdot \mathbf{1} = 0 \\ \boldsymbol{\pi} \cdot \boldsymbol{\beta}^k = 0 \text{ for } k = 1, \ldots, K \end{array}\right\} \Rightarrow \boldsymbol{\pi} \cdot E[\mathbf{r}] = 0.$$

# Arbitrage pricing theory

Using basic linear algebra (I will spare you the details) it follows that there exists $\mathrm{RP}_0$, $\mathrm{RP}_1, \ldots, \mathrm{RP}_K$ such that

$$E[r_i] = \mathrm{RP}_0 + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

This is the exact APT.

Although the conditions are strong, it shows the major principle of the APT.

If there exists a risk-free asset, then $\mathrm{RP}_0 = r_f$ and we get

$$E[r_i] = r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

# Arbitrage pricing theory

For a general $K$-factor model the APT says that when the number $N$ of assets is large, then

$$E\left[r_i\right] \approx r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

The exact mathematical result is that with $N = \infty$

$$\sum_{i=1}^{\infty} \left( E\left[r_i\right] - r_f - \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k \right)^2 < \infty.$$

From now on we follow the book and say that if APT holds, then

$$E\left[r_i\right] = r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

# Arbitrage pricing theory

In the Single-Index Model

$$r_i = E[r_i] + \beta_i\big(r_m - E[r_m]\big) + \varepsilon_i$$

we get

$$E[r_i] = r_f + \beta_i \mathrm{RP}_m$$

from APT.

Let the rate of return be $r_m$ in this equation:

$$E[r_m] = r_f + \underbrace{\beta_m}_{=1} \mathrm{RP}_m = r_f + \mathrm{RP}_m \quad \Rightarrow \quad \mathrm{RP}_m = E[r_m] - r_f .$$

We recover the CAPM equation again:

$$E[r_i] = r_f + \beta_i\big(E[r_m] - r_f\big).$$

# Arbitrage pricing theory

How should we choose the factors?

In order to be able to estimate the parameters in the model we need to have factors that are observable.

One way is to choose factors that are returns – such as the return $r_m$ of the market portfolio.

If we want to use, say, inflation as a factor but want factors that we can observe more often than at the times the inflation is updated by then we can use a factor mimicking portfolio.

A factor mimicking portfolio is a portfolio of traded assets that closely follows a factor.

It is also common to choose factors that are excess returns.

# Arbitrage pricing theory

Which factors should we use in order to get a good model to price assets?

Consider the $K$-factor model

$$r_i = \alpha_i + \beta_{i1}\big(F_1 - E\left[F_1\right]\big) + \ldots + \beta_{iK}\big(F_K - E\left[F_K\right]\big) + \varepsilon_i.$$

The pricing model implied by this factor model is

$$E\left[r_i\right] = r_f + \beta_{i1}\mathrm{RP}_1 + \beta_{i2}\mathrm{RP}_2 + \ldots + \beta_{iK}\mathrm{RP}_K.$$

When looking at data, the question is:

Is the risk premium $\mathrm{RP}_k$ different from zero?

# The Fama-French model

A model that is good at pricing assets is the Fama-French model, also known as the Fama-French three factor model.

It is a model formulated in excess returns.

- The first factor is the market excess return: $r_m - r_f$.
- The second factor is the difference between the return on small stocks minus the return on large stocks: SMB
- The third factor is the return on stocks with a high book-to-market ratio minus the return on stocks of low book-to-market ratios: HML

The Fama-French model is

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \beta_{i,\text{SMB}}\text{SMB} + \beta_{i,\text{HML}}\text{HML} + \varepsilon_i.$$

# The Fama-French model

The definition of a small/large firm is defined in terms of the firm's market cap ($=$ Number of stocks $\cdot$ Price of the stock).

The definition of a high/low book-to-market ratio firm is defined as the book value of the firm divided by it's market cap.

A firm with a low book-to-market value is known as a growth firm and a firm with a high book-to-market value is known as a value firm.

# The Fama-French-Carhart model

The Fama-French model turned out to be succesful in explaining the expected rate of return of assets.

But more factors that explain the expected rate of return have been introduced after the Fama-French model was suggested in 1992.

One extension is the four factor model that was suggested by Carhart in 1997 and that added a momentum factor.

Momentum is the fact that a stock that has increased in value will continue to increase in the future.

# The Fama-French-Carhart model

It turns out that this is in general true in the short-run, but that there is a reversal (the opposite of a momentum effect) in the long-run.

The four factor Fama-French-Carhart model is

$$\begin{aligned} r_i - r_f &= \alpha_i + \beta_{i,m}(r_m - r_f) + \beta_{i,\mathrm{SMB}}\mathrm{SMB} + \beta_{i,\mathrm{HML}}\mathrm{HML} \\ &\quad + \beta_{i,\mathrm{WML}}\mathrm{WML} + \varepsilon_i. \end{aligned}$$

Here WML is the excess return of winners-minus-losers.

Sometimes WML is called UMD for "up-minus-down".

# The factor zoo

Which factors should be used?

In theory, every factor whose risk premium is different from zero is a "priced factor".

But we only have estimated values on the risk premia, so we are interested in the values that are significantly different from zero.

There has been a lot of research in trying to find different priced factors.

Over one hundred factors have been identified in the academic literature. This has lead to the term the factor zoo.

# The factor zoo

Is it enough that a factor has a risk premium that is significantly different from zero?

Theoretically yes, but it would be nice to have an economic interpretation of the factor.

The rule of thumb is that if the "$t$-stat" if larger than 2, then the parameter is significantly different from zero.

But recently there have been a demand of an even higher $t$-stat in order for a factor to be accepted as a pricing factor.

In order to use factors in practise there are usually transaction costs involved. These should be included when testing if a factor is pricing assets.

# The factor zoo

These are "the largest animals in the zoo".

- Value: Usually measured by book-to-market ratio, but there are other suggestions on how to measure value.
- Momentum: A portfolio of winners-minus-losers when looked at a given time period (e.g. measured over the last 12 months).
- Quality: Measures how well managed a company is (several measures exists). The excess return of quality-minus-junk (QMJ) can be used as a factor.
- Defensive: This is also known as a low-risk factor. Create an excess return of low risk stocks minus high risk stocks and use it as a factor.
- Size: The difference between the return on small companies minus the return on large companies.

See Section 11.6.2 in the course book for a more detailed description.

# Financial Theory – Lecture 9

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Data from financial markets.
- Some financial econometrics.
- Efficient markets.
- Behavioural finance.

The lecture is based on

- Chapter 12 in the course book.
- Some additional literature.

# Prices on financial markets

We use price data to create a return time series.

I would recommend that you use prices as your raw data and then calculate the rate of return or the log-return.

The data source I use is Refinitiv Eikon. This is available through the library at Ekonomikum.

There are introductory courses via Zoom and you can also ask the library for a one-on-one introduction to Refinitiv Eikon at the computer in the library.

There is also data available from Swedish House of Finance.

An exchange is usually using a limit order book.

There is a bid and an ask price.

The bid price is the highest price a buyer want to buy for, and the ask price is the lowest price a seller wants to sell for.

**Orderdjup**

| Antal | Köp | Sälj | Antal |
|---|---|---|---|
| 34 | **76,52** | **76,53** | 1 125 |
| 1 635 | 76,51 | 76,54 | 3 596 |
| 1 510 | 76,50 | 76,55 | 1 849 |
| 4 489 | 76,49 | 76,56 | 7 469 |
| 4 726 | 76,48 | 76,57 | 8 170 |

$$\text{Bid price} = 76.52. \quad \text{Ask price} = 76.53.$$
$$\text{Bid-ask spread} = 76.53 - 76.52 = 0.01.$$

(Market depth for Ericsson B, 29 August 2022 from Avanza.)

# Prices on financial markets

On and after a trading day the following prices are collected:

- Closing: The last price during the day at which a trade was made.
- Opening: The first price during the day at which a trade was made.
- High: The highest price during the day at which a trade was made.
- Low: The lowest price during the day at which a trade was made.

Both academics and practitioners are also interested in how large the volume, i.e. the number of shares that was traded, is in a transaction.

The VWAP is the volume-weighted average price.

# Estimating the risk premia

Start with a $K$-factor model:

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big) + \varepsilon_i$$

and $\mathrm{Cov}[F_k, \varepsilon_i] = 0$ for every $k$ and $i$, and $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0$ when $i \neq j$.

We then add the condition of no arbitrage to the model $\longrightarrow$ the APT equation

$$E[r_i] = \mathrm{RP}_0 + \beta_{i1}\mathrm{RP}_1 + \ldots + \beta_{iK}\mathrm{RP}_K,$$

with $\mathrm{RP}_0 = r_f$ is there is a risk-free asset.

There are two standard ways of estimating the risk premia:

1) The two-stage or cross-sectional approach.
2) The Fama-MacBeth approach.

# Estimating the risk premia

Assume that data $r_{it}$ are given for $i = 1, 2, \ldots, N$ assets and for $t = 1, 2, \ldots, T$ times.

For ease of exposition I assume that the only factor is the rate of return on the market portfolio, i.e. we want to estimate the risk premium in the Single-Index Model.

In both approaches we start by using the data and for each asset $i$ estimate the $\beta_{im}$'s by running the regressions

$$r_{it} = a_i + \beta_{im} r_{mt} + \varepsilon_{it}.$$

This leads to estimated values on the beta's: $\hat{\beta}_{im}$, $i = 1, \ldots, N$.

# Estimating the risk premia

In the two-step approach we treat the estimated betas are the true betas and run the regression

$$\bar{r}_i = \mathrm{RP}_0 + \mathrm{RP}_m \hat{\beta}_{im} + \eta_i,$$

where

$$\bar{r}_i = \frac{1}{T} \sum_{t=1}^{T} r_{it}$$

and $\eta_i$ is an error term.

This gives the estimates $\widehat{\mathrm{RP}}_0$ and $\widehat{\mathrm{RP}}_m$.

# Estimating the risk premia

In the Fama-MacBeth approach we again treat the estimated betas as the true betas, but now we run the regressions

$$r_{it} = \mathrm{RP}_{0t} + \mathrm{RP}_{mt}\hat{\beta}_{im} + \eta_{it}$$

for $t = 1, 2, \ldots, T$.

This gives the estimates $\widehat{\mathrm{RP}}_{0t}$ and $\widehat{\mathrm{RP}}_{mt}$.

We then get the estimates of the risk premia from

$$\widehat{\mathrm{RP}}_0 = \frac{1}{T}\sum_{t=0}^{T}\widehat{\mathrm{RP}}_{0t} \quad \text{and} \quad \widehat{\mathrm{RP}}_m = \frac{1}{T}\sum_{t=0}^{T}\widehat{\mathrm{RP}}_{mt}.$$

# Estimating the risk premia

If there is a risk-free rate, then we estimate the betas using the time series, but regressions in the second step are

$$\bar{r}_i - r_f = \mathrm{RP}_m \hat{\beta}_{im} + \eta_i$$

and

$$r_{it} - r_f = \mathrm{RP}_{mt} \hat{\beta}_{im} + \eta_{it}$$

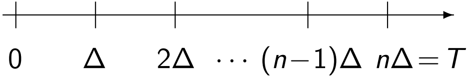
for $t = 1, 2, \ldots, T$ respectively.

Note that there is no constant in these regressions.

# Estimating the mean and the variance of returns

The difference between estimation of the mean and the variance of a time series was shown in the following example in Merton (1980).

The total time length $T$ is divided into $n$ parts each length $\Delta$:




One example is $T = 10$ years and $\Delta = 1$ month.

We model the return $r_t$ over period $t$ for $t = 1, 2, \ldots, n$ as

$$r_t = \mu\Delta + \sigma\sqrt{\Delta}\,\varepsilon_t,$$

where each $\varepsilon_t \sim N(0, 1)$ and independent of each other.

# Estimating the mean and the variance of returns

In this model each $r_t$ is independent of the others.

Furthermore
$$r_t \sim N(\mu\Delta, \sigma^2\Delta).$$

This is a discrete time version of a Brownian motion (or Wiener process) with drift.

# Estimating the mean and the variance of returns

To estimate $\mu$ we use the estimator

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{n} r_t = \frac{1}{\Delta}\cdot\frac{1}{n}\sum_{t=1}^{n} r_t.$$

This is an unbiased estimator,

$$E\left[\hat{\mu}\right] = \frac{1}{T}\sum_{t=1}^{n} E\left[r_t\right] = \frac{1}{n\Delta}\cdot n\cdot\mu\Delta = \mu,$$

with variance

$$\mathrm{Var}[\hat{\mu}] = \mathrm{Var}\left[\frac{1}{T}\sum_{t=1}^{n} r_t\right] = \frac{1}{T^2}\sum_{t=1}^{n}\mathrm{Var}[r_t] = \frac{1}{T^2}\cdot n\cdot\sigma^2\Delta = \frac{\sigma^2}{T}.$$

# Estimating the mean and the variance of returns

For $\sigma^2$ we use the estimator

$$\widehat{\sigma^2} = \frac{1}{T} \sum_{t=1}^{n} r_t^2.$$

This estimator is not unbiased:

$$\begin{aligned}
E\left[\widehat{\sigma^2}\right] &= \frac{1}{T} \sum_{t=1}^{n} E\left[r_t^2\right] \\
&= \frac{1}{T} \sum_{t=1}^{n} \left(\mathrm{Var}[r_t] + E\left[r_t\right]^2\right) \\
&= \frac{1}{T} \cdot n\left(\sigma^2 \Delta + \mu^2 \Delta^2\right) \\
&= \sigma^2 + \mu^2 \Delta = \sigma^2 + \mu^2 \frac{T}{n}.
\end{aligned}$$

# Estimating the mean and the variance of returns

Merton prefers this estimator to

$$\frac{1}{T}\sum_{t=1}^{n}\left(r_t-\bar{r}\right)^2$$

because you do not need to estimate the mean $\bar{r}$ in order to find its value..

Also note that

$$E\left[\widehat{\sigma^2}\right]=\sigma^2+\mu^2\frac{T}{n}\to\sigma^2$$

as $n\to\infty$.

One can show that

$$\mathrm{Var}\left[\widehat{\sigma^2}\right] = \frac{2\sigma^2}{n} + \frac{4\mu^2 T}{n^2}.$$

Recall that

$$\mathrm{Var}\left[\hat{\mu}\right] = \frac{\sigma^2}{T}.$$

If we want a better precision (=lower variance) in the estimtion of $\sigma$ we can inrease the sampling frequence (increase $n$).

If we want a better precision (=lower variance) in the estimtion of $\mu$ we must inrease the sampling period (increase $T$), i.e. collect more data.

# Efficiency

We now turn to the question of how informationally efficient financial markets are.

The efficient market hypothesis (EMH) states that

> *The prices in a financial market fully reflect all available information.*

In this definition, what is meant by all avilable information?

To be more precise, there are three different versions of efficiency.

# Efficiency

Some inital comments.

- Getting information is costly. It takes time to collect information and it can be costly to obtain it.
- We expect that information that is cheap to acquire is probably known to more investors than information that is costly to acquire.
- In an efficient market prices move only when new information arrives.
- When testing for efficiency you have the "joint hypothesis problem": You have to also have a model for expected returns.

# Weak-form efficiency

**Weak-form efficiency**

Definition:

- All information in historical trading such as prices and volumes is fully reflected in today's prices.

One concequence of this is that technical analysis of asset prices is useless.

We test this type of EMH by looking at if it is possible to predict returns by using historical trading data.

This type of data can typically be obtained at a low cost.

# Semistrong-form efficiency

**Semistrong-form efficiency**

Definition:

- All historical publically availabe information is fully reflected in today's prices.

This means that we should also include quarterly and yearly reports from companies, announcements (such as a profit warning and the sacking of a CEO) and other relevant public news.

To test this type of efficiency event studies can be used. There are also alternative ways of testing for the semistrong form.

# Strong-form efficiency

**Strong-form efficiency**

Definition:

- All historical informaton, public and private, is fully reflected in today's prices.

One concequence of this is that there is no extra gain for traders with inside information.

This is indeed a strong assumption.

# Efficiency today

Nowadays most economicst (but not all!) agree that markets are efficient to some degree, but that it is not prefectly true.

Since there are many asset managers making money by collecting and processing information it is reasonable to take the stand that markets are enough inefficient for them to be compensated for the cost they have colleting and processing information.

This has led Pedersen to use the phrase "efficiently inefficient".

# Empirical results

There is a difference in what the researchers knew in the 1970's, and what they knew in the 2000's.

The following two slides are based on p. 389-391 in Cochrane (2005).

# Empirical results

**In the 1970's**

- CAPM explains why some assets have higher average returns than others.
- Stock returns are close to unpredictable.
- Professional asset managers do not outperform passive portfolios when corrected for risk.

# Empirical results

**In the 2000's**

- There are assets whose expected return cannot be explained by their beta values with respect to the market return.
- Stock returns are predictable over longer time horizons, but close to unpredictable measured daily, weekly or monthly.
- Some fund managers seem to outperform simple indexes, even after corrected for risk.

# Normative and positive results

So far, we have mainly taken a normative view.

One example is the mean-variance approach to investments: Given the vector of mean returns and the covariance matrix, we calculate the optimal weights.

But is this how investors behave?

That is, what are the positive results of financial economics?

# Behavioural finance and economics

There are a number of empirical facts about how individuals and investors behave that seems to be at odds with assumptions and theory.

They are known as biases, and are studied in behavioural finance and behavioural economics.

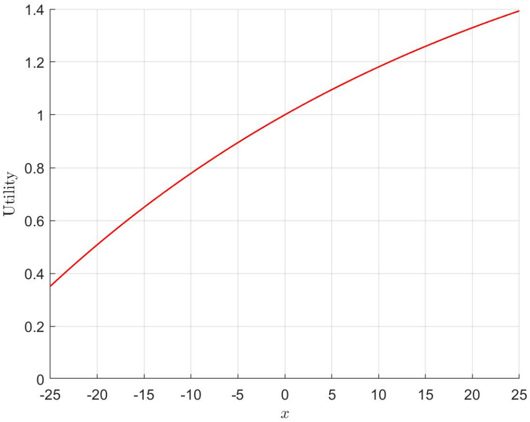
# The Allais paradox

Already in 1953, Maurice Allais in "Le comportement de l'homme rationnel devant le risque, critique des postulats et axiomes de l'école Américaine" raised a critique against the axioms of expected utility theory.

Allais argued that individuals do not behave as predicted by expected utility theory.

# Prospect theory

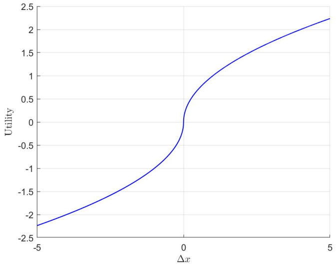
Recall the notion of utility over monetary outcomes.



# Prospect theory

A refinement of expected utility theory is prospect theory.

- Utility is measured relative to a reference point.




- The utility function over changes in monetary values is concave in gains and convex in losses.
- Probabilities are weighted using a weighting function.

The extension cumulative prospect theory has better theoretical properties.

# Behavioural biases

*Anchoring*
Individuals and investors anchor their views.

**Example**
When estimating the expected return of a new asset, expected returns of existing assets might influence the estimation fo this new asset.

*Loss aversion or Disposition effect*
There is an aversion in realising losses.

**Example**
Investments that has decreased in value are hard to sell for an investor.

# Behavioural biases

*Overconfidence*
Individuals tend to too confident in their views.

**Example**
Investors are too sure about their subjective views on estimates, and their own ability in general.

*Mental accounting*
Different attitudes to risk for different parts of an investor's portfolio. The portfolio is not considered as a whole

**Example**
Gains made can be considered as less problematic to lose than the money that made up the original investment.

# Behavioural biases

*Framing*

The answer to a question is dependent of how it is framed.

**Example**

In finance, an example is if an investment opportunity is described using possible gains or possible losses.

# Behavioral game theory

In some cases the experimental outcomes of games seems to contradict rational behavior.

A standard example is the ultimatum game.

Another interesting example is the dictator game.

An example from auctions is the winner's curse: In a common value auction (where the item being auctioned has the same value for all bidders) the winning bid exceeds the value of the item.

# Answers to the critique from behavioural finance

Financial economists such as Stephen A. Ross and John H. Cochrane have contested the claims of behavioural finance.

> *Note that to obtain a significant effect on equilibrium prices, such behavioural biases have to be systematic across individuals.*

From Munk (2013).

# References

[1] Cochrane, J. H. (2005), "Asset Pricing", revised edition, *Princeton University Press*.

[2] Merton, R. C. (1980), "On Estimating the Expected Return on the Market: An Explanatory investigation", *Journal of Financial Economics*, pp. 323-361.

[3] Munk, C. (2013), "Financial Asset Pricing Theory", *Oxford University Press*.

# Financial Theory – Lecture 10

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Valuation of stocks.

The lecture is based on

- Chapter 6 in the course book.

# Prices and returns over time

Recall:

$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}$$

is the rate of return for an asset over $(t, t+1]$.

To simplify notation, we write this equation as

$$r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}.$$

This is our "master equation".

It is an accounting identity, i.e. it holds by definition.

# Prices and returns over time

$$r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} \quad \Leftrightarrow \quad 1 + r_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t},$$

or

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}}.$$

Now use

$$P_{t+1} = \frac{D_{t+2} + P_{t+2}}{1 + r_{t+2}}.$$

Then

$$\begin{aligned} P_t &= \frac{D_{t+1} + \frac{D_{t+2} + P_{t+2}}{1 + r_{t+2}}}{1 + r_{t+1}} \\ &= \frac{D_{t+1}}{1 + r_{t+1}} + \frac{D_{t+2}}{(1 + r_{t+1})(1 + r_{t+2})} + \frac{P_{t+2}}{(1 + r_{t+1})(1 + r_{t+2})}. \end{aligned}$$

# Prices and returns over time

We iterate this in total $T - t$ times to arrive at

$$\begin{aligned} P_t \;=\;& \frac{D_{t+1}}{1+r_{t+1}} + \frac{D_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \cdots \\ &+ \frac{D_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)} \\ &+ \frac{P_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)}. \end{aligned}$$

This equation is written ex post, i.e. after the randomness has been resolved.

To be useful, we need the ex ante version, i.e. before the randomness is resolved.

# Prices and returns over time

We achieve this by taking the expectation with respect to the information up to and including time $t$, i.e. we take $E_t$ on both sides of the equation on the previous slide.

Since $P_t$ is known at time $t$, we have $P_t = E_t[P_t]$:

$$\begin{aligned} P_t = E_t\left[P_t\right] &= E_t\left[\frac{D_{t+1}}{1+r_{t+1}} + \frac{D_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \dots \right. \\ &\quad + \frac{D_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)} \\ &\quad \left. + \frac{P_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)}\right]. \end{aligned}$$

Now we want to let $T \to \infty$. If the last term goes to zero, then the price is given by an infinite sum of expected discounted dividends.

We'll soon get back to this equation.

# Prices and returns over time

Now we divide the equation

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}}$$

with $D_t$:

$$\frac{P_t}{D_t} = \frac{\frac{D_{t+1}}{D_t} + \frac{D_{t+1}}{D_t} \cdot \frac{P_{t+1}}{D_{t+1}}}{1 + r_{t+1}}.$$

This is a recursion for the price-dividend ratio $P_t/D_t$.

It depends on the return and the dividend growth $D_{t+1}/D_t$.

It turns out that the price-dividend ratio has better statistical properties than the price. Furthermore, it has been shown that the price-dividend ratio can predict returns on several assets for longer horizons (1+ years).

# Constant discount rate

We now make the following assumption:

$$E_t[r_{t+1}] = r = \text{a constant.}$$

What does this mean? Write

$$\begin{aligned} r_{t+1} &= E_t[r_{t+1}] + \big(r_{t+1} - E_t[r_{t+1}]\big) \\ &= r + \varepsilon_{t+1}. \end{aligned}$$

We see that

$$E_t\left[\varepsilon_{t+1}\right] = E_t\left[r_{t+1} - E_t\left[r_{t+1}\right]\right] = E_t\left[r_{t+1}\right] - E_t\left[r_{t+1}\right] = 0.$$

The rate of return $r_{t+1}$ is the constant $r$ plus zero-mean noise.

We can use the CAPM or APT equation to determine the discount rate $r$.

# Constant discount rate

Recall that the "master equation" can be written

$$1 + r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}.$$

Take $E_t$ on both sides:

$$\underbrace{E_t\left[1 + r_{t+1}\right]}_{=1+E_t[r_{t+1}]=1+r} = E_t\left[\frac{P_{t+1} + D_{t+1}}{P_t}\right] = \frac{1}{P_t} E_t\left[P_{t+1} + D_{t+1}\right]$$

$$\Leftrightarrow$$

$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r} = E_t\left[\frac{P_{t+1} + D_{t+1}}{1 + r}\right].$$

# Constant discount rate

Using the same iteration technique as previously we get

$$
\begin{aligned}
P_t &= E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} + \ldots + \frac{D_T}{(1+r)^{T-t}} + \frac{P_T}{(1+r)^{T-t}}\right] \\
&= \frac{E_t\left[D_{t+1}\right]}{1+r} + \frac{E_t\left[D_{t+2}\right]}{(1+r)^2} + \ldots + \frac{E_t\left[D_T\right]}{(1+r)^{T-t}} + \frac{E_t\left[P_T\right]}{(1+r)^{T-t}} \\
&= \sum_{j=1}^{T-t} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j} + \frac{E_t\left[P_T\right]}{(1+r)^{T-t}}.
\end{aligned}
$$

Assuming that

$$\frac{E_t\left[P_T\right]}{(1+r)^{T-t}} \to 0 \quad \text{as} \quad T \to \infty,$$

we get

$$P_t = \sum_{j=1}^{\infty} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j}.$$

# Constant growth models

Now assume that the expected growth rate in dividends is a constant $g$:

$$E_t\left[\frac{D_{t+1}-D_t}{D_t}\right]=g \quad \Leftrightarrow \quad E_t\left[D_{t+1}\right]=(1+g)D_t.$$

Iterating this we get

$$\begin{aligned} E_t\left[D_{t+2}\right] &= E_t\left[(1+g)D_{t+1}\right] \\ &= (1+g)\cdot E_t\left[D_{t+1}\right] \\ &= (1+g)\cdot(1+g)D_t \\ &= (1+g)^2 D_t. \end{aligned}$$

More generally,

$$E_t\left[D_{t+j}\right]=(1+g)^j D_t \text{ for } j=1,2,\ldots$$

# Constant growth models

Now,

$$
\begin{aligned}
P_t &= \sum_{j=1}^{\infty} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j} \\
&= \sum_{j=1}^{\infty} \frac{(1+g)^j D_t}{(1+r)^j} \\
&= D_t \cdot \underbrace{\sum_{j=1}^{\infty} \left(\frac{1+g}{1+r}\right)^j}_{=S}
\end{aligned}
$$

# Constant growth models

Recall: If $|\alpha| < 1$, then

$$\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}.$$

This implies

$$\sum_{j=1}^{\infty} \alpha^j = \sum_{j=0}^{\infty} \alpha^j - 1 = \frac{1}{1-\alpha} - 1 = \frac{\alpha}{1-\alpha}.$$

We get

$$S = \sum_{j=1}^{\infty} \left(\frac{1+g}{1+r}\right)^j = \frac{\frac{1+g}{1+r}}{1-\frac{1+g}{1+r}} = \frac{1+g}{1+r-(1+g)} = \frac{1+g}{r-g}.$$

For this to be well defined, we need

$$\frac{1+g}{1+r} < 1 \quad \Leftrightarrow \quad 1+g < 1+r \quad \Leftrightarrow \quad g < r.$$

# Constant growth models

Using

$$P_t = S \cdot D_t \quad \text{and} \quad S = \frac{1+g}{r-g}$$

we arrive at

$$\begin{aligned} P_t &= \frac{(1+g)D_t}{r-g} \\ &= \frac{E_t\left[D_{t+1}\right]}{r-g}. \end{aligned}$$

This is known as Gordon's formula.

# Constant growth models

Using Gordon's formula we get the following in our model.

- The price-dividend ratio is

$$\frac{P_t}{D_t} = \frac{1+g}{r-g}.$$

- The discount rate can be written

$$\begin{aligned} r &= g + \frac{(1+g)D_t}{P_t} \\ &= g + \frac{E_t\left[D_{t+1}\right]}{P_t} \\ &= g + E_t\left[\frac{D_{t+1}}{P_t}\right] \\ &= \text{Growth rate of dividends} + \text{Expected dividend yield.} \end{aligned}$$

# Rational bubbles

Let us return to the equation

$$P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right]. \qquad (*)$$

If

$$E_t\left[\frac{P_T}{(1+r)^{T-t}}\right] \to 0 \quad \text{as} \quad T \to \infty,$$

then we have seen that we get the solution

$$P_t = \frac{(1+g)D_t}{r-g}.$$

But without this condition there are other solutions.

The solution $P_t$ above is known as the bubble-free solution.

Now let $\hat{P}_t$ be any solution to Equation $(*)$.

# Rational bubbles

We have

$$\hat{P}_t = E_t\left[\frac{\hat{P}_{t+1} + D_{t+1}}{1+r}\right] \quad \text{and} \quad P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right].$$

Subtract these two equations:

$$\begin{aligned}
\hat{P}_t - P_t &= E_t\left[\frac{\hat{P}_{t+1} + D_{t+1}}{1+r}\right] - E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right] \\
&= E_t\left[\frac{\hat{P}_{t+1}}{1+r}\right] - E_t\left[\frac{P_{t+1}}{1+r}\right] \\
&= E_t\left[\frac{\hat{P}_{t+1} - P_{t+1}}{1+r}\right].
\end{aligned}$$

# Rational bubbles

Now let

$$M_t = \frac{\hat{P}_t - P_t}{(1+r)^t}$$

$$\Leftrightarrow$$

$$\hat{P}_t - P_t = (1+r)^t M_t.$$

One can show that $E_t[M_{t+1}] = M_t$.

# Rational bubbles

We can thus write

$$\hat{P}_t = P_t + (1+r)^t M_t,$$

or

Any price satisfying Equation $(*)$ = Bubble-free solution + Bubble.

We call

$$(1+r)^t M_t$$

a rational bubble since it satisfies Equation $(*)$.

# A firm's dividend policy

Let $e_t$ denote the earnings of a firm during year $t$. The earnings are used to reinvest the amount $I_t$ in the firm, and to pay the amount $D_t$ as dividends to the share holders:

$$e_t = I_t + D_t.$$

The earnings grow due to the return $r_e$ on the investments or on the equity:

$$e_{t+1} = e_t + r_e I_t.$$

It is common to refer to $r_e$ as the return on equity (ROE).

We let $b$ denote the plowback ratio, i.e. the fraction of the earnings that are invested:

$$I_t = b e_t.$$

Throughout, we assume that both $r_e$ and $b$ are constants.

# A firm's dividend policy

**Remark**

We can talk about earnings, dividends and price per share, in which case we call earnings earnings per share (EPS).

Or, we can talk about the total amount of these three quantities.

They only differ by a multiple of the number of shares.

# A firm's dividend policy

The choice of how much of the earnings that should be paid out to the share holders is the firm's dividend policy.

Now,

$$e_{t+1} = e_t + r_e \underbrace{I_t}_{=be_t} = (1 + br_e)e_t \quad \Rightarrow \quad \frac{e_{t+1} - e_t}{e_t} = br_e.$$

The growth rate in earnings is equal to $br_e$. For the dividends:

$$D_t = e_t - I_t = e_t - be_t = (1 - b)e_t.$$

# A firm's dividend policy

It follows that

$$\frac{D_{t+1} - D_t}{D_t} = \frac{(1-b)e_{t+1} - (1-b)e_t}{(1-b)e_t} = \frac{e_{t+1} - e_t}{e_t} = br_e.$$

**Conclusion:** If the plowback ratio and return on new investements are constant, then the growth rate of dividends is constant and equal to

$$g = br_e.$$

**Exercise:** Show that also

$$\frac{P_{t+1} - P_t}{P_t} = r_e b$$

in this model.

# A firm's dividend policy

An important "multiple" is the price-earnings ratio (P/E).

In this model

$$P_t = \frac{(1+g)D_t}{r-g} = \frac{(1+br_e)(1-b)e_t}{r-br_e} \quad \Rightarrow \quad \frac{P_t}{e_t} = \frac{(1+br_e)(1-b)}{r-br_e}.$$

We also have

$$e_{t+1} = (1+br_e)e_t \;\; \text{(ex post)} \quad \Rightarrow \quad E_t\left[e_{t+1}\right] = (1+br_e)e_t \;\; \text{(ex ante)}.$$

The forward price-earnings ratio is in this model given by

$$\frac{P_t}{E_t\left[e_{t+1}\right]} = \frac{1-b}{r-br_e}.$$

# A firm's dividend policy

Why do firm's pay dividends?

- It is a way of share holders to get cash flows without having to sell stocks.
- Some funds are only allowed to use dividends and other cash payments in their distribution of funds; they are not allowed to sell anything of their capital.

# Investment opportunities

Consider a firm whose dividend policy is

$$e_t = D_t \quad \Leftrightarrow \quad b = 0,$$

i.e. all earnings are paid out as dividends.

The value of this firm is

$$\frac{e_t}{r}.$$

To see this we use Gordon's formula:

$$\begin{aligned} \frac{(1+g)D_t}{r-g} &= \left\{ g = br_e = 0 \text{ and } D_t = e_t \right\} \\ &= \frac{e_t}{r}. \end{aligned}$$

# Investment opportunities

Now let

$$O_t = P_t - \frac{e_t}{r},$$

i.e. we can write

$$P_t = \frac{e_t}{r} + O_t.$$

Here $O$ is the present value of growth opportunities (PVGO).

$$\text{Price today} = \text{Value of assets in place} + \text{PVGO}.$$

We can write the P/E ratio as

$$\frac{P_t}{e_t} = \frac{1}{r} + \frac{O_t}{e_t} = \frac{1}{r}\left(1 + \frac{O_t}{e_t/r}\right).$$

# Two-period growth models

Let us consider the price $P_t$ decomposed into two sums:

$$\begin{aligned} P_t &= E_t\left[\sum_{j=1}^{T-t}\frac{D_{t+j}}{(1+r)^j} + \sum_{j=T-t+1}^{\infty}\frac{D_{t+j}}{(1+r)^j}\right] \\ &= \underbrace{E_t\left[\sum_{j=1}^{T-t}\frac{D_{t+j}}{(1+r)^j}\right]}_{=S_1} + \underbrace{E_t\left[\sum_{j=T-t+1}^{\infty}\frac{D_{t+j}}{(1+r)^j}\right]}_{=S_2} \end{aligned}$$

Now assume:

- The dividends grow with constant rate $G \neq r$ from $t+1$ to $T$ (we assume that $t < T$).
- The dividends grow with constant rate $g < r$ from $T+1$ onwards.

# Two-period growth models

With this model

$$\begin{aligned} S_1 &= E_t\left[\sum_{j=1}^{T-t}\frac{D_{t+j}}{(1+r)^j}\right] = \sum_{j=1}^{T-t}\frac{E_t\left[D_{t+j}\right]}{(1+r)^j} \\ &= \sum_{j=1}^{T-t}\frac{D_t(1+G)^j}{(1+r)^j} = D_t\sum_{j=1}^{T-t}\left(\frac{1+G}{1+r}\right)^j. \end{aligned}$$

Now we use that when $\alpha \neq 1$

$$\sum_{j=0}^{N}\alpha^s = \frac{1-\alpha^{N+1}}{1-\alpha},$$

with $N = T - t$:

$$\sum_{j=1}^{T-t}\left(\frac{1+G}{1+r}\right)^j = \frac{1-\left(\frac{1+G}{1+r}\right)^{T-t+1}}{1-\frac{1+G}{1+r}} - 1 = \cdots$$

# Two-period growth models

$$\cdots = (1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r-G}.$$

Hence,

$$S_1 = D_t(1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r-G}.$$

One can show that

$$S_2 = D_t \frac{1+g}{r-g} \left(\frac{1+G}{1+r}\right)^{T-t}$$

# Two-period growth models

To summarise:

$$
\begin{aligned}
P_t &= S_1 + S_2 \\
&= D_t \left[ (1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r - G} + \frac{1+g}{r-g} \cdot \left(\frac{1+G}{1+r}\right)^{T-t} \right].
\end{aligned}
$$

# Two-period growth models

For a finite number of cash flows one can have

$$\text{Growth rate} \geq \text{Discount rate}.$$

But this can not be true for an infinite number of cash flows, since this would imply

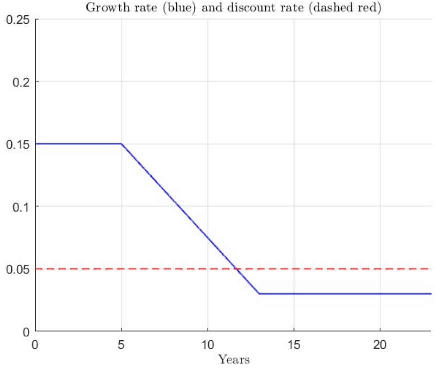
$$P_t = \infty.$$

This is one reason for having more than one time period when modelling dividend growth rate: You can have exceptional growth in the dividends for the first period (which includes a finite number of times).

The final growth rate is the long run steady state growth rate.

# Three-period growth models

We can extend to three (or, of course, any number of periods). One way of using a three-period model is the following.




- Assume an initial high constant growth rate.
- Assume final (lower) steady state growth rate.
- Assume that the growth rate diminishes linearly from the initial high to the steady state one.

# Three-period growth models

Another version is the following:

- Estimate explicitly the dividends in the first period.
- Assume a high growth rate in the second period.
- Assume (lower) steady state growth in the last premium.

This model could e.g. be used to value growth stocks.

# Free cash flows

The "discounted dividend" approach to valuing firms is the typical way we use it in (financial) economics.

A practitioner valuing a firm may find it hard to estimate dividends, and there are firms who do not (yet) have paid any dividends.

An alternative is to take a more "business" or "accounting" approach.

The free cash flow (FCF) of a firm is defined as

> *... the after-tax cash flow generated by the firms operations which is available for distribution among shareholders and creditors.*

(Munk p. 221.)

# Free cash flows

The value of the firm is then calculated as

$$V_t = \sum_{j=1}^{\infty} \frac{E_t \left[ \mathrm{FCF}_{t+j} \right]}{\left(1 + r_{\text{firm}}\right)^j}$$

Here the discount rate $r_{\text{firm}}$ is the weighted average cost of capital (WACC).

Since we are valuing the whole form, not just its equity, we need to discount using the WACC.

In order to get the value of the equity, we need to subtract the value of the firm's debt.

# Financial Theory – Lecture 11

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Bonds.

The lecture is based on

- Sections 5.1-5.3 and 5.5 in the course book.

# Bonds

A bond is a tradable loan contract.

When a bond is issued, a certain amount of money, the face value is borrowed.

During the life time of a bond are interest rate payments and repayments of the borrowed amount (amortisation).

The last time any payment is made to the owner of the bond is the maturity date of the bond.

# Bonds

The bond payments are defined by

- the face value,
- the coupon rate
- and the amortisation principle of the bond.

We let $i = 1, 2, \ldots, n$ denote the payment dates.

# Bonds

Notation:

$$
\begin{aligned}
M_i &= \text{The total payment at time } i. \\
I_i &= \text{The interest payment at time } i. \\
X_i &= \text{The repayment of debt at time } i. \\
F_i &= \text{The outstanding debt at time } i \\
&\phantom{=} \text{after the repayment of debt has been made.}
\end{aligned}
$$

Then for $i = 1, 2, \ldots, n$

$$
\begin{aligned}
M_i &= I_i + X_i \\
I_i &= qF_{i-1} \\
F_i &= F_{i-1} - X_i.
\end{aligned}
$$

It also holds that

$$F_0 = F = \text{The face value, and } F_n = 0.$$

# Bonds

By using $I_i = qF_{i-1}$ and

$$F_i = F_{i-1} - X_i \quad \Leftrightarrow \quad X_i = F_{i-1} - F_i$$

we can write

$$\begin{aligned} M_i &= I_i + X_i \\ &= qF_{i-1} + F_{i-1} - F_i \\ &= (1+q)F_{i-1} - F_i. \end{aligned}$$

Given $M_i$, $i = 1, 2, \ldots, n$ this is a recursion for $F_i$:

$$F_i = (1+q)F_{i-1} - M_i.$$

# Bonds

- The face value is also known as the par value or the principal of the bond. I will use $F$ to denote this amount (not $F_0$ as in the book).
- The outstanding debt is sometimes called the outstanding loan balance (OLB).
- Note that

$$\begin{aligned}\sum_{i=1}^{n} X_i &= \sum_{i=1}^{n} \left(F_{i-1} - F_i\right) \\ &= \underbrace{F_0}_{=F} - F_1 + F_1 - F_2 + \cdots + F_{n-2} - F_{n-1} + F_{n-1} - \underbrace{F_n}_{=0} \\ &= F,\end{aligned}$$

i.e. the total amount repaid is equal to the size of the loan.

# Coupon bonds

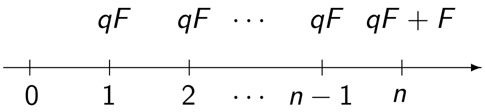
Coupon bonds (or bullet bonds) have the following structure.

- $X_i = 0$ when $i = 1, 2, \ldots, n-1$ and $X_n = F$.
- $I_i = qF$, $i = 1, 2, \ldots, n$.

The constant $q$ is referred to as the coupon rate.

When $q = 0$ (or $n = 1$), then the coupon is referred to as a zero coupon bond (ZCB).

A coupon bond has cash flows given by:



# Annuities

An annuity (or annuity bond) has the property that the payment $M_i$ is equal for all times $i = 1, 2, \ldots, n$.

If $q = 0$, then

$$M_i = X_i = \frac{F}{n}$$

for an annuity.

The case $q > 0$ is more interesting (and harder to solve).

# Annuities

Let $q > 0$. In this case we have the recusion

$$F_i = (1+q)F_{i-1} - M,$$

where $M$ is a constant and independent of $i$ (this is what characterises annuities).

Since $F_0 = F$ we get

$$\begin{array}{rcl}
F_1 & = & (1+q)F - M \\
F_2 & = & (1+q)\big[(1+q)F - M\big] - M \\
 & = & (1+q)^2 F - M[1 + (1+q)] \\
F_3 & = & (1+q)\big[(1+q)^2 F - M[1+(1+q)]\big] - M \\
 & = & (1+q)^3 F - M[1 + (1+q) + (1+q)^2] \\
\vdots & \vdots & \vdots
\end{array}$$

# Annuities

Finally,

$$\underbrace{F_n}_{=0} = (1+q)^n F - M \sum_{j=0}^{n-1} (1+q)^j \quad \Leftrightarrow \quad M \sum_{j=0}^{n-1} (1+q)^j = (1+q)^n F.$$

Now we use that for $\alpha \neq 1$

$$\sum_{j=0}^{n-1} \alpha^j = \frac{1-\alpha^n}{1-\alpha}.$$

With $\alpha = 1+q$:

$$\sum_{j=0}^{n-1} (1+q)^j = \frac{1-(1+q)^n}{-q} = \frac{(1+q)^n - 1}{q} \quad \Rightarrow$$

# Annuities

$$M\frac{(1+q)^n-1}{q}=(1+q)^nF.$$

Divide with $(1+q)^n$:

$$M\underbrace{\frac{1-(1+q)^{-n}}{q}}_{=A(q,n)}=F.$$

The constant $A(q,n)$ is called the annuity factor.

Using this we can write

$$M=\frac{F}{A(q,n)}.$$

One can show that the outstanding payment at time $i$ is

$$F_i=MA(q,n-i)=M\frac{1-(1+q)^{-(n-i)}}{q}.$$

# Serial bonds

A serial bond pays back the face value in equal amounts:

$$X_i = \frac{F}{n}, \;\; i = 1, 2, \ldots, n.$$

If follows that the outstanding debt is

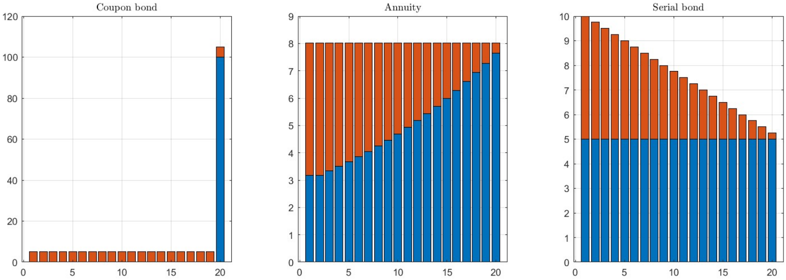
$$F_i = \frac{F}{n} \cdot (n - i) = F \left(1 - \frac{i}{n}\right),$$

and that the interest rate payment is

$$I_i = qF_{i-1} = qF \left(1 - \frac{i-1}{n}\right).$$

# Bond cash flow examples

Let $n = 20$, $F = 100$ and $q = 0.05$.

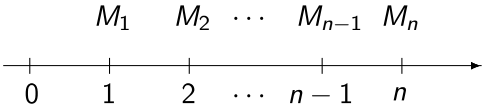



Blue = Amortisation. Red = Interest rate payments.

# Bond prices

So far we have considered the cash flows of different bonds. What about the price of a bond?

Consider being at time 0 and let $M_i > 0$, $i = 1, 2, \ldots, n$, be the (deterministic) payments of the bond.




What is the value today ($i = 0$) of getting the cash flow $M_i$ at time $i = 1, \ldots, n$?

# Bond prices

Assume a constant interest rate $r$.

If I have have the amount $M_i/(1+r)^i$ at time zero, then this has grown to

$$\frac{M_t}{(1+r)^i} \cdot (1+r)^i = M_i$$

at time $i$.

Hence,

Having $\dfrac{M_i}{(1+r)^i}$ at time zero is the same has having $M_i$ at time $i$.

# Bond prices

With a constant discount rate $r$, the price $B_0$ at time 0 of the bond is

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+r)^i} = \sum_{i=1}^{n} M_i (1+r)^{-i}.$$

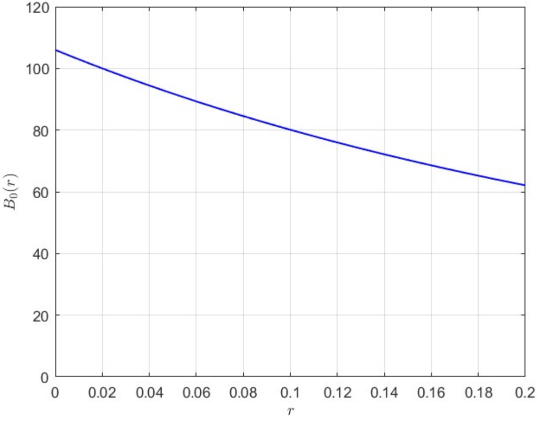
It follows that

$$\frac{\partial B_0}{\partial r} = -\sum_{i=1}^{n} i M_i (1+r)^{-i-1} < 0$$

$$\frac{\partial^2 B_0}{\partial r^2} = \sum_{i=1}^{n} i(i+1) M_i (1+r)^{-i-2} > 0$$

# Bond prices

With a constant discount rate, the price $B_0(r)$ as a funtion of the rate $r$ is a decreasing and convex function.



# Bond prices

- For a ZCB with face value $F$ maturing at $n$ the price at $t < n$ is given by
$$Z_{t,n} = \frac{F}{(1+r)^{n-t}} = F(1+r)^{-(n-t)}.$$
- For a coupon bond with face value $F$, coupon rate $q$ and maturity date $n$:
$$\begin{aligned} B_t &= \sum_{i=t+1}^{n} \frac{qF}{(1+r)^{i-t}} + \frac{F}{(1+r)^{n-t}} \\ &= \frac{qF}{r}\left(1 - \frac{1}{(1+r)^{n-t}}\right) + \frac{F}{(1+r)^{n-t}}. \end{aligned}$$

For the pricing of annuities and serial bonds, see Theorem 5.1 and its proof in the course book.

# Bond prices

| | |
|---|---|
| **Auction date :** | 2023-09-27 |
| **Auction type :** | Nominal bond |
| **Loan :** | 1065 |
| **ISINcode :** | SE0017830730 |
| **Coupon % :** | 1.750 |
| **Maturity :** | 2033-11-11 |
| **Offered/tendered :** | 1,500 |
| **Tendered :** | 4,020 |
| **Allocated institutional:** | 1,500 |
| **Tender ratio :** | 2.68 |
| **Number of bids :** | 26 |
| **Number of accepted bids :** | 9 |
| **Yield avg :** | 2.9397(89.715) |
| **Low :** | 2.9340(89.762) |
| **Cutoff :** | 2.9430(89.689) |
| **% of Eq Price Lvl :** | 66.67 |

# Perpetuities

A perpetuity (or a perpetual bond, or a consol bond) is a bond which only has interest payments and in which the face value is never paid back.

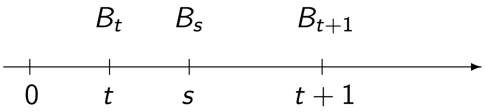
We can think of this as a coupon bond with $n = \infty$.

The price of a perpetuity is given by

$$
\begin{aligned}
B_0 &= \sum_{i=1}^{\infty} \frac{qF}{(1+r)^i} = qF\left(\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i - 1\right) = qF\left(\frac{1}{1-\frac{1}{1+r}} - 1\right) \\
&= qF\left(\frac{1+r}{1+r-1} - 1\right) = qF\left(\frac{1+r}{r} - \frac{r}{r}\right) \\
&= \frac{qF}{r}.
\end{aligned}
$$

# More on bond prices

What if the time at which we want to price the bond is not an integer?




In this case, the cash flows from time $t+1$ and onwards should be included in the value. It follows that

$$B_s = (1+r)^{s-t} \cdot B_t.$$

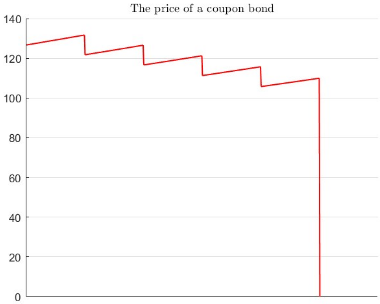
Note that in general

$$B_s \neq \frac{1}{(1+r)^{t+1-s}} \cdot B_{t+1}$$

since the value at time $t+1$ does not include the cash flow at time $t+1$.

# More on bond prices

The price of a bond over time looks like this:




The accrued interest for a coupon bond is given by

$$Q_t^{\mathrm{acc}} = tqF,$$

where $t$ is the fraction of time that has evolved since the latest dividend payment.

# More on bond prices

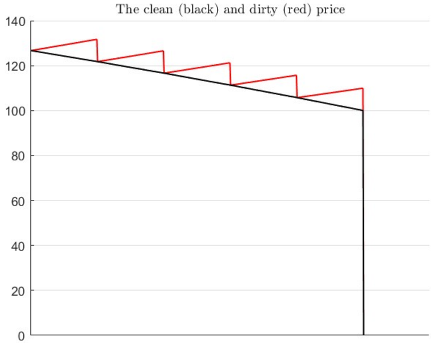
The price

$$B_t^{\text{list}} = B_t - Q_t^{\text{acc}}$$

is the listed bond price (i.e. the price shown for buyers and sellers).

But since $B_t$ is the true value of the bond, you also need to pay the accrued interest if you buy the bond (and you get the accrued interest if you sell the bond).

$B_t$ is called the dirty price, and $B_t^{\text{list}}$ the clean price.



# More on bond prices

For the bond types we have considered:

- $r = q \quad \Rightarrow \quad B_0 = F$. The bond is trading at par.
- $r < q \quad \Rightarrow \quad B_0 > F$. The bond is trading at a premium.
- $r > q \quad \Rightarrow \quad B_0 < F$. The bond is trading at a discount.

# More on bond prices

How the price and the rate at which we discount a bond are connected is depending on the instruments.

There is also a day count convention that defines how we should calculate the fraction between two dividend dates.

**Examples**

- $\boxed{30/360}$ Each month of the year has 30 days, and the year has 360 days.
- $\boxed{\text{Actual}/360}$ Actual number of days, and the year has 360 days.
- $\boxed{\text{Actual}/\text{Actual}}$ Actual number of days, and the year has it actual number of days.

# Yield-to-maturity

The yield-to-maturity, or just yield, $y$ is the internal rate of return (IRR) of a bond:

$$B_0^{\text{mkt}} = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i}.$$

Here $B_0^{\text{mkt}}$ is the observed market value of the bond.

Two basic, but very important, observations are:

$$y \uparrow \;\Leftrightarrow\; B_0(y) \downarrow$$

and

$$y \downarrow \;\Leftrightarrow\; B_0(y) \uparrow.$$

# Yield-to-maturity

The yield is the average rate of return we get if we hold the bond until maturity.

In general, as for the IRR, numerical methods are needed to calculate the yield.

For ZCB's, on the other hand, it is easy:

$$Z_{0,n}^{\mathrm{mkt}} = \frac{F}{(1+y_n)^n} \quad \Leftrightarrow \quad y_n = \left(\frac{F}{Z_{0,n}^{\mathrm{mkt}}}\right)^{1/n} - 1.$$

Here $y_n$ is the yield at time 0 for a ZCB maturing at $n$.

# Returns and yields

In general, the yield is not equal to the rate of return of a bond. With yields $y_t$ and $y_{t+1}$ of the bond we have

$$B_t = \sum_{i=t+1}^{n} \frac{M_i}{(1+y_t)^{i-t}} \quad \text{and} \quad B_{t+1} = \sum_{i=t+2}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}}.$$

Note that

$$\begin{aligned} M_{t+1} + B_{t+1} &= M_{t+1} + \sum_{i=t+2}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}} \\ &= \sum_{i=t+1}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}} \\ &= (1+y_{t+1}) \sum_{i=t+1}^{n} \frac{M_i}{(1+y_{t+1})^{i-t}}. \end{aligned}$$

# Returns and yields

It follows that the one period rate of return for this bond is

$$\begin{aligned} r_{t,t+1} &= \frac{B_{t+1} + M_{t+1} - B_t}{B_t} = \frac{B_{t+1} + M_{t+1}}{B_t} - 1 \\ &= (1 + y_{t+1}) \frac{\sum_{i=t+1}^{n} M_i (1 + y_{t+1})^{-(i-t)}}{\sum_{i=t+1}^{n} M_i (1 + y_t)^{-(i-t)}} - 1. \end{aligned}$$

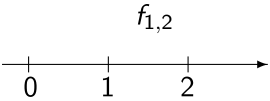
We see that if $y_{t+1} = y_t$, then

$$r_{t,t+1} = y_t = \text{The yield at time } t.$$

But in general the return is not equal to the yield.

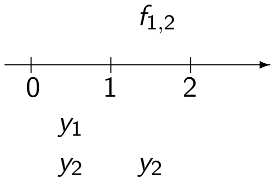
# Forward rates

Today, at $t = 0$, someone if offering us the interest rate $f_{1,2}$ for the time period $(1, 2]$:




**Question:** How large should this interest be?

# Forward rates




Consider the following two strategies.

- **Strategy 1:** Invest 1 today for 1 year, and also enter into the contract of getting the interest rate $f_{1,2}$ over $(1, 2]$.

$$\text{Payoff at } t = 2: \ (1 + y_1) \cdot (1 + f_{1,2}).$$

- **Strategy 2:** Invest 1 today for 2 years.

$$\text{Payoff at } t = 2: \ (1 + y_2)^2.$$

# Forward rates

Since both investments are risk-free, they must have the same payoff:

$$(1 + y_1)(1 + f_{1,2}) = (1 + y_2)^2,$$

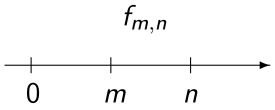
or

$$f_{1,2} = \frac{(1 + y_2)^2}{1 + y_1} - 1.$$

That is, given the ZCB yields $y_1$ and $y_2$, there is only one interest rate (the $f_{1,2}$ given by the equation above) that makes it impossible to create an arbitrage.

# Forward rates

We can generalise this.




The forward rate for the period from $m$ to $n$, denoted $f_{m,n}$, satisfies

$$(1 + y_n)^n = (1 + y_m)^m (1 + f_{m,n})^{n-m},$$

or

$$f_{m,n} = \left( \frac{(1 + y_n)^n}{(1 + y_m)^m} \right)^{1/(n-m)} - 1.$$

Note that the value of the forward rate $f_{m,n}$ is known at time 0.

# Forward rates

With $m = n - 1$ we get

$$(1 + y_n)^n = (1 + y_{n-1})^{n-1}(1 + f_{n-1,n}).$$

By iterating this we see that

$$(1 + y_n)^n = (1 + y_1)(1 + f_{1,2})(1 + f_{2,3}) \cdots (1 + f_{n-1,n}).$$

Since $y_1$ is known at time 0, it holds that $f_{0,1} = y_0$.

Hence, $1 + y_n$ is the geometric average of

$$(1 + f_{0,1}), (1 + f_{1,2}), \cdots, (1 + f_{n-1,n}).$$

# Defaultable bonds

A defaultable bond is a bond with risky payments.

Typically the risk lies in the fact that an issuer of a bond can not (or will not) make coupon and/or amortisation payments.

Bonds issued by stable states are generally considered non-defaultable, i.e. there is no risk of their bonds to default.

Bonds issued by firms are considered more risky, and there are rating institutes rating the quality of a firm based on the probability of not being able to pay the cash flows of issued bonds.

# Defaultable bonds

**TABLE 1** Bond Ratings by Moody's, Standard and Poor's, and Fitch

| | Rating Agency | | |
|---|---|---|---|
| **Moody's** | **S&P** | **Fitch** | **Definitions** |
| Aaa | AAA | AAA | Prime Maximum Safety |
| Aa1 | AA+ | AA+ | High Grade High Quality |
| Aa2 | AA | AA | |
| Aa3 | AA− | AA− | |
| A1 | A+ | A+ | Upper Medium Grade |
| A2 | A | A | |
| A3 | A− | A− | |
| Baa1 | BBB+ | BBB+ | Lower Medium Grade |
| Baa2 | BBB | BBB | |
| Baa3 | BBB− | BBB− | |
| Ba1 | BB+ | BB+ | Noninvestment Grade |
| Ba2 | BB | BB | Speculative |
| Ba3 | BB− | BB− | |
| B1 | B− | B− | Highly Speculative |
| B2 | B | B | |
| B3 | B− | B− | |
| Caa1 | CCC+ | CCC | Substantial Risk |
| Caa2 | CCC | — | In Poor Standing |
| Caa3 | CCC− | — | |
| Ca | — | — | Extremely Speculative |
| C | — | — | May Be in Default |
| — | — | DDD | Default |
| — | — | D | |
| — | D | D | |

Table 1 on p. 165 in Mishkin, F. S. (2016), "The Economics of Money, Banking, and Financial Markets" (11th Ed.), *Pearson*.

# Financial Theory – Lecture 12

Fredrik Armerin, Uppsala University, 2024

# Agenda

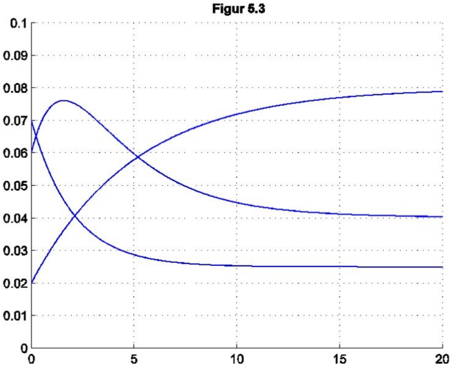
- Yield curves.
- Duration and convexity.

The lecture is based on

- Chapter 5 and Section 6.4 in the course book.

# Yield curves

For a set of bonds with the same characteristics (e.g. they are equally risky and have the same amortisation principle, etc.), the yield as a function of time to maturity is known as the yield curve or the term structure of interest rates.




The function $y_t$ for $t \in [0, 20]$. (Armerin & Song p. 131.)

# Yield curves

The form of the yield curve contains information about the economy in which the bonds are traded.

Often an inverted yield curve tends to indicate that the economy is in a recession, or is moving into a recession.

But what can be said about the form of a general yield curve?

# Explanations for the form of yield curve

The level of the yield curve is affected by the general time preference of investors.

If they are impatient, then they want to consume now, which results in

$$\text{a high supply of bonds and a low demand } \longrightarrow$$

$$\text{the price on bonds } \downarrow \longleftrightarrow \text{ the yields } \uparrow .$$

# Explanations for the form of yield curve

If the economy is expected to grow, then, since consumers want to even out consumption over their lives, they want to consume more today. Thus, they want to

$$\text{borrow today} \longrightarrow \text{an increase in the supply of bonds today} \longrightarrow$$
$$\text{the price of bonds} \downarrow \longleftrightarrow \text{the yield} \uparrow .$$

The conclusion is that when there is an expected expansion of the economy, the yield curve is upward sloping.

# Explanations for the form of yield curve

Another factor influencing the form of the yield curve is uncertainty.

When there is increased uncertainty, risk-averse investors prefer govenment bonds, which results in

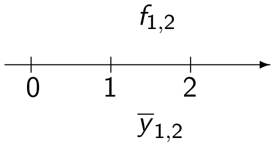
$$\text{increased demand of default-free bonds} \longrightarrow$$

$$\text{the price of default-free bonds} \uparrow \;\longleftrightarrow\; \text{yields on default-free bonds} \downarrow .$$

# Older explanations for the form of yield curves

**Expectation hypothesis**

The expected future yields are equal to their respective forward rate.




In formulas:

$$\overline{y}_{1,2} = f_{1,2},$$

or

$$1 + \overline{y}_{1,2} = \frac{(1 + y_2)^2}{1 + y_1}.$$

And in the same way for other time spans $(m, n]$.

# Older explanations for the form of yield curves

**Liquidity preference hypothesis**
Investors requires a premium from holding bonds with longer maturities.

Reasons for this:

- No or very illiquid secondary market.
- The market can be dominated by short-term investors, who need a premium when buying long-term bonds.

# Older explanations for the form of yield curves

**Market segmentation hypothesis**
Buyers and issuers of bonds have different time spans for their investments.

They are only interested in their respective part of the yield curve.

This means that the yield on short-term bonds and the yield for long-term bonds are defined by supply and demand for these two segments of the market – the market is segmented rather than connected.

# Older explanations for the form of yield curves

**Preferred habitat hypothesis**
Investors have their preferred length of their investment (as in the Market segmentation hypothesis), but if the extra return is sufficiently high, then they can adjust their positions.

This implies that there will be a premium for maturities where there is lower demand.

# Using the yireld curve

Let

$$(M_1, M_2, \ldots, M_n)$$

be a stream of deterministic cash flows.

What is the value of this?

We can think of this as a portfolio of ZCB's.

This means that the value is

$$\sum_{i=1}^{n} M_i Z_{0,i} = \sum_{t=1}^{T} \frac{M_i}{(1+y_i)^i}.$$

**Note:** Here I follow the book and write $y_i = y_{0,i}$.

# Using the yield curve

**Conclusion:** When valuing a deterministic stream of cash flows, we use the yields of ZCB's.

In order for this to work, we need the term structure of ZCB's.

In general, the non-defaultable bonds given have coupons.

# Bootstrapping

Assume that we have the following bonds:

| Bond No | Time to maturity | Coupon | Face value | Price |
|---|---|---|---|---|
| 1 | 1 | 0 | 100 | 95 |
| 2 | 2 | 5 | 100 | 98 |
| 3 | 4 | 2 | 100 | 97 |

How does the term structure of ZCB's look like?

First of all, we have (Bond 1)

$$95 = \frac{100}{1 + y_1} \;\Rightarrow\; y_1 = \frac{100}{95} - 1 \approx 0.0526.$$

To get $y_2$ we can not use Bond 2 directly, since this is not a ZCB.

# Bootstrapping

But we can create a ZCB with time to maturity equal to 2.

Consider the portfolio with

$$1 \text{ of Bond 2 and } -0.05 \text{ of Bond 1.}$$

The result is:

| Time | Cash flow |
|---|---|
| 0 | $1 \cdot 98 - 0.05 \cdot 95 = 93.25$ |
| 1 | $1 \cdot 5 - 0.05 \cdot 100 = 0$ |
| 2 | $1 \cdot (5 + 100) = 105$ |

This portfolio is like a ZCB with time to maturity 2, face value 105 and price 93.25.

# Bootstrapping

Using this fictitious ZCB, we can get $y_2$:

$$93.25 = \frac{105}{(1+y_2)^2} \quad \Leftrightarrow \quad y_2 = \left(\frac{105}{93.25}\right)^{1/2} - 1 \approx 0.0611.$$

This technique is known as bootstrapping.

What about the spot rates $y_3$ and $y_4$?

# Bootstrapping

Bond 3 has coupon payments at $t = 1, 2, 3$ that we need to "remove".

This is possible for the coupon payments at $t = 1$ and $t = 2$.

For the coupon payment at $t = 3$, there is no unique way of removing this to create a ZCB.

We need to find the two rates $y_3$ and $y_4$ that satisfies

$$97 = \frac{2}{1 + 0.0526} + \frac{2}{(1 + 0.0611)^2} + \frac{2}{(1 + y_3)^3} + \frac{102}{(1 + y_4)^4}.$$

To proceed we have to make some assumptions about the form on the term structure.

# Duration

Assume that we have a bond with cash flows $M_1, M_2, \ldots, M_n$.

We know that the price of this bond is

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i} = \sum_{i=1}^{n} M_i(1+y)^{-i}.$$

What happens if $y$ changes?

$$\begin{aligned}\frac{\partial B_0}{\partial y} &= -\sum_{i=1}^{n} i\, M_i(1+y)^{-i-1} \\ &= -\frac{1}{1+y}\sum_{i=1}^{n} i\, M_i(1+y)^{-i}.\end{aligned}$$

# Duration

This can be written

$$\begin{aligned}\frac{\partial B_0}{\partial y} &= -\frac{B_0}{1+y} \cdot \frac{\sum_{i=1}^{n} i\, M_i (1+y)^{-i}}{B_0} \\ &= -\frac{B_0}{1+y} \cdot \sum_{i=1}^{n} i\, w_i,\end{aligned}$$

where

$$w_i = \frac{M_i (1+y)^{-i}}{B_0}.$$

# Duration

The duration at time 0 is defined as

$$D_0 = \sum_{i=1}^{n} i \cdot \frac{\frac{M_i}{(1+y)^i}}{B_0} = \sum_{i=1}^{n} i\, w_i.$$

Note that $0 \leq w_i \leq 1$ and

$$\sum_{i=1}^{n} w_i = 1.$$

The duration is a weighted average of the times at which the cash flows are paid.

The higher the cash flow $M_i$ is, the higher is the weight $w_i$.

# Duration

We can write

$$\frac{\partial B_0}{\partial y} \cdot \frac{1}{B_0} = -\frac{\sum_{i=1}^{n} i\, w_i}{1+y} = -\frac{D_0}{1+y}.$$

Approximating the derivate, we get

$$\frac{\Delta B_0}{\Delta y} \cdot \frac{1}{B_0} \approx \frac{\partial B_0}{\partial y} \cdot \frac{1}{B_0} = -\frac{D_0}{1+y},$$

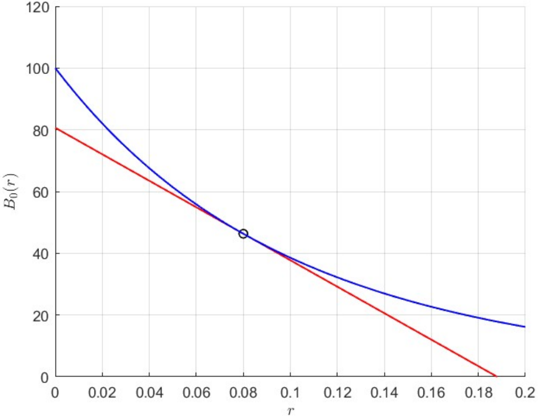
or

$$\frac{\Delta B_0}{B_0} \approx -\frac{D_0}{1+y} \Delta y$$

The duration meaures how sensitive the price of a deterministic stream of cash flows is to a change in the yield.

This duration is also called the Macauly duration.

# Duration



# Duration

The modified duration is defined by

$$D_0^* = \frac{1}{1+y} D_0.$$

Using this, we can write

$$\frac{\partial B_0}{\partial y} = -D_0^* B_0$$

and

$$\frac{\Delta B_0}{B_0} \approx -D_0^* \Delta y.$$

At time $t$ for a ZCB with time to maturity $n$, the duration is given by

$$D_t = n - t.$$

# Convexity

The convexity of a bond at time 0 is defined as

$$C_0 = \sum_{i=1}^{n} i(i+1)\frac{M_i(1+y)^{-i-2}}{B_0} = \sum_{i=1}^{n} i(i+1)w_i.$$

The convexity satisfies

$$C_0 = \frac{(1+y)^2}{B_0} \cdot \frac{\partial^2 B_0}{\partial y^2} \quad \Leftrightarrow \quad \frac{\partial^2 B_0}{\partial y^2} = \frac{C_0 B_0}{(1+y)^2}.$$

The modified convexity at time zero is defined as

$$C_0^* = \frac{C_0}{(1+y)^2}.$$

# Convexity

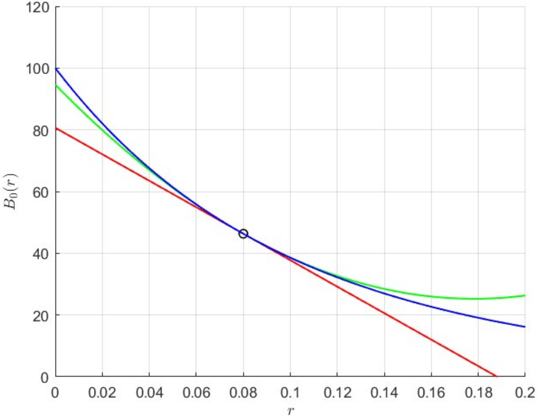
By using the second order Taylor expansion

$$\Delta B_0 \approx \frac{\partial B_0}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 B_0}{\partial y^2}\left(\Delta y\right)^2$$

we can write

$$\begin{aligned}\frac{\Delta B_0}{B_0} &\approx -\frac{D_0}{1+y}\Delta y + \frac{1}{2}\frac{C_0}{(1+y)^2}\left(\Delta y\right)^2 \\ &= -D_0^*\Delta y + \frac{1}{2}C_0^*\left(\Delta y\right)^2.\end{aligned}$$

# Convexity



# Fisher-Weil duration

In general, we can use a ZCB yield curve to price other bonds:

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+y_i)^i}.$$

In order to define a duration in this case we consider the sensitivity of the price if the yield curve is changed according to

$$1 + y_i \;\to\; (1+y_i)\cdot(1+\delta),\; i = 1, 2, \ldots, n,$$

for some constant $\delta$.

We get

$$B_0(\delta) = \sum_{i=1}^{n} \frac{M_i}{\big((1+y_i)(1+\delta)\big)^i} = \sum_{i=1}^{n} M_i (1+y_i)^{-i}(1+\delta)^{-i}$$

# Fisher-Weil duration

and from this

$$B_0'(\delta) = -\sum_{i=1}^{n} i\, M_i (1+y_i)^{-i}(1+\delta)^{-i-1}.$$

The quantity

$$D_0^{\mathrm{FW}} = \sum_{i=1}^{n} i \cdot \frac{M_i(1+y_i)^{-i}}{B_0}$$

is the Fisher-Weil duration. We see that

$$B_0(\delta) \approx B_0(0) + B_0'(0)\delta = B_0 - B_0 D_0^{\mathrm{FW}}\delta,$$

or

$$\frac{\Delta B_0}{B_0} \approx -\delta D_0^{\mathrm{FW}}.$$

# Immunisation

Now consider the case when we have liabilities in the form of a stream of cash flows whose value is influenced by the yield curve.

In order to decrease the interest rate risk we want to invest in a portfolio that resembles, in some sense, the cash flows.

The best would be to buy a portfolio of ZCB's exactly matching the cash flows in the liability, but this is usually not possible, and if it is possible it might be costly.

# Immunisation

Instead we buy a portfolio whose value and duration matches that of our liability – this is knows as immunisation.

The idea is that if there is a small change in the yield curve, then the value of our portfolio should change approximately as much as our liabilities (cf. the figure above).

# Immunisation

Let $\bar{B}$ and $\bar{D}$ denote the value and duration of the liabilities.

The value is invested in two bonds with price $B_1$ and $B_2$ respectively, and durations $D_1$ and $D_2$ respectively, where we assume that $D_1 \neq D_2$.

Let $N_1$ and $N_2$ be the number of bond 1 and 2 we buy.

Then the value of the bond portfolio $B_p$ must satisfy the budget constraint

$$B_p = N_1 B_1 + N_2 B_2 = \bar{B}.$$

# Immunisation

One can show that the Fisher-Weil duration $D$ of a portfolio with value $B$ and consisting of $N_1, N_2, \ldots, N_m$ number of bonds with prices $B_1, B_2, \ldots, B_m$ and durations $D_1, D_2, \ldots, D_m$ satisfies

$$B = \sum_{j=1}^{m} N_j B_j$$

and

$$BD = \sum_{j=1}^{m} N_j B_j D_j.$$

The relation holds approximately for the Macaulay duration.

# Immunisation

If we want the portfolio to have the same duration as the liabilities, then we should choose $(N_1, N_2)$ such that

$$\begin{cases} N_1 B_1 + N_2 B_2 &= \bar{B} \\ N_1 B_1 D_1 + N_2 B_2 D_2 &= \bar{B}\bar{D}. \end{cases}$$

The solution is given by

$$N_1 = \frac{\bar{B}}{B_1} \cdot \frac{\bar{D} - D_2}{D_1 - D_2} \quad \text{and} \quad N_2 = \frac{\bar{B}}{B_2} \cdot \frac{\bar{D} - D_1}{D_2 - D_1}.$$

# Equity duration

Recall that, under the assumptions from Lecture 10, the price of a stock is given by

$$P_t = \sum_{i=1}^{\infty} \frac{E_t\left[D_{t+i}\right]}{(1+r)^i} = \sum_{i=1}^{\infty} E_t\left[D_{t+i}\right](1+r)^{-i}.$$

Taking the derivative of this with respect to $r$ gives

$$\frac{\partial P_t}{\partial r} = -\sum_{i=1}^{\infty} i\, E_t\left[D_{t+j}\right](1+r)^{-i-1}.$$

Now define

$$w_{i,t} = \frac{E_t\left[D_{t+i}\right](1+r)^{-i}}{P_t}$$

and

$$\mathrm{DUR}_t = \sum_{i=1}^{\infty} i\, w_{i,t}.$$

# Equity duration

Then we can write

$$\frac{\partial P_t}{\partial r} = -\frac{1}{1+r}\mathrm{DUR}_t P_t.$$

In the Gordon growth model, we have

$$P_t = \frac{(1+g)D_t}{r-g},$$

so

$$\frac{\partial P_t}{\partial r} = -\frac{(1+g)D_t}{(r-g)^2} = -\frac{P_t}{r-g}.$$

It follow that

$$\mathrm{DUR}_t = -\frac{1+r}{P_t} \cdot \left(-\frac{P_t}{r-g}\right) = \frac{1+r}{r-g}.$$

# Equity duration

It has been found empirically that stocks with low equity duration tends to deliver higher returns than stocks that has high equity duration.

This has led to the introduction of a, usually downward sloping, equity term structure (how the return is connected to the equity duration).

It has also been found that the slope of the equity term structure is upward sloping in bad times.

# Financial Theory – Lecture 13

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Asset allocation.
- Household finance.

The lecture is based on

- Sections 1.8, 9.1-9.2, 10.1.3, and 13.1-13.2.

# Investments in practise

Usually we think of the investment process in the following steps.

1) The strategic asset allocation. This is the choice of in which proportion each of the major asset classes should have in the portfolio. Typically done by senior management/investors.
2) The security selection. Given the strategic asset allocation, this is the choice which securities should be bought in each of the classes.
3) The tactical asset allocation. This is the timing of when the securities are bought.

It has been shown (see the references on p. 18 in the course book), that the strategic asset allocation decision explains more than 90% of the time series variation in quarterly returns.

# The alpha

We know that if the assumptions of the CAPM holds then

$$E[r_i] - r_f = \beta_i(E[r_m] - r_f),$$

where $r_m$ is the return on the market portfolio, and

$$\beta_i = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}.$$

But if we regress the excess return $r_i - r_f$ on the excess market return $r_m - r_f$,

$$r_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i,$$

then we will probably get an intersection $\alpha_i$ – the alpha – that is different from zero.

# The alpha

Recall

$$r_{t,t+1} = r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t},$$

which we can write

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}},$$

and also

$$P_t = E_t \left[ \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}} \right].$$

We have shown that if $E_t [r_{t+1}] = r$, a constant, then

$$P_t = E_t \left[ \frac{D_{t+1} + P_{t+1}}{1 + r} \right] = \frac{E_t [D_{t+1} + P_{t+1}]}{1 + r}.$$

# The alpha

The price of the asset given by the CAPM is

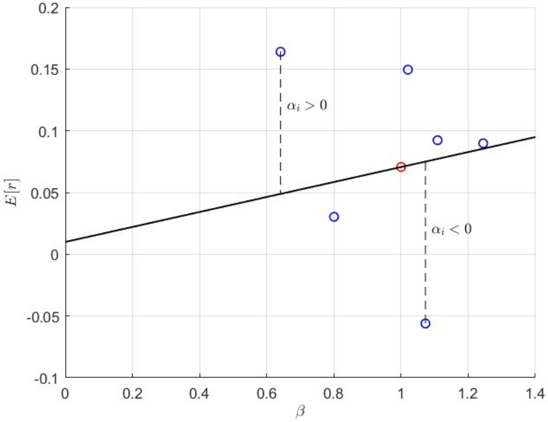
$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r_f + \beta\big(E\left[r_m\right] - r_f\big)}.$$

Now assume that we instead use $r = r_f + \alpha + \beta\big(E\left[r_m\right] - r_f\big)$ as discount factor:

$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r_f + \alpha + \beta\big(E\left[r_m\right] - r_f\big)}.$$

- $\alpha = 0$: The price is equal to the CAPM price.
- $\alpha > 0$: The price is lower than the CAPM price $\rightarrow$ the asset is underpriced.
- $\alpha < 0$: The price is higher than the CAPM price $\rightarrow$ the asset is overpriced.

# The alpha




The SML with the market portfolio (red circle) and six assets (blue circles).

# The Treynor-Black model

A systematic way of using the information that assets are under/over-priced is given by the Treynor-Black model.

- There are $N$ number of risky assets and a risk-free asset.
- The Single-Index Model holds.
- We have found that $J$ of the risky assets have an alpha $\neq 0$:

$$r_i = r_f + \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i, \ i = 1, 2, \ldots, J.$$

- We form a portfolio with weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_J)^\top$ using these $J$ assets – this is called the active portfolio, and its rate of return is denoted $r_A$.
- Finally we form a portfolio with weight $w_A$ in the active portfolio and weight $1 - w_A$ in the market portfolio:

$$r_p = w_A r_A + (1 - w_A) r_m.$$

# The Treynor-Black model

The return of the active portfolio is

$$
\begin{aligned}
r_A &= \boldsymbol{\pi}\cdot\mathbf{r} = \sum_{i=1}^{J}\pi_i r_i = \sum_{i=1}^{J}\pi_i\Big(r_f+\alpha_i+\beta_i\big(r_m-r_f\big)+\varepsilon_i\Big)\\
&= \sum_{i=1}^{J}\pi_i r_f+\sum_{i=1}^{J}\pi_i\alpha_i+\sum_{i=1}^{J}\pi_i\beta_i\big(r_m-r_f\big)+\sum_{i=1}^{J}\pi_i\varepsilon_i\\
&= r_f\underbrace{\sum_{i=1}^{J}\pi_i}_{=1}+\underbrace{\sum_{i=1}^{J}\pi_i\alpha_i}_{=\alpha_A}+\big(r_m-r_f\big)\underbrace{\sum_{i=1}^{J}\pi_i\beta_i}_{=\beta_A}+\underbrace{\sum_{i=1}^{J}\pi_i\varepsilon_i}_{=\varepsilon_A}\\
&= r_f+\alpha_A+\beta_A\big(r_m-r_f\big)+\varepsilon_A.
\end{aligned}
$$

# The Treynor-Black model

The model uses a two-step procedure.

**Step 1**
Find, given $\boldsymbol{\pi}$, the optimal weight $w_A^*$ that maximises the Sharpe ratio

$$\begin{aligned}\mathrm{SR}(r_p) &= \frac{E\left[r_p\right] - r_f}{\mathrm{Std}[r_p]} \\ &= \frac{E\left[w_A r_A + (1-w_A) r_m\right] - r_f}{\mathrm{Std}\left[w_A r_A + (1-w_A) r_m\right]}.\end{aligned}$$

# The Treynor-Black model

One can show (Theorem 13.1 in the course book) that

$$w_A^* = \frac{\dfrac{\alpha_A}{\text{Var}[\varepsilon_A]}}{\dfrac{E\left[r_m\right] - r_f}{\text{Var}[r_m]} + (1-\beta_A)\dfrac{\alpha_A}{\text{Var}[\varepsilon_A]}}$$

and

$$\begin{aligned} \text{SR}(r_p)^2 &= \left(\frac{E\left[r_m\right] - r_f}{\text{Std}[r_m]}\right)^2 + \left(\frac{\alpha_A}{\text{Std}[\varepsilon_A]}\right)^2 \\ &= \text{SR}_m^2 + IR_A^2. \end{aligned}$$

The contribution to the overall Sharpe ratio from the active portfolio is given by the size of the information ratio $IR_A$.

**Step 2**

Find the best (optimal) choice of $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_J)^\top$.

We have $\mathrm{SR}(r_p)^2 = \mathrm{SR}_m^2 + IR_A^2$ and only $IR_A^2$ dependes on the choice of $\boldsymbol{\pi}$.

Hence, we maximise

$$IR_A^2 = \left(\frac{\alpha_A}{\mathrm{Std}[\varepsilon_A]}\right)^2 = \left(\frac{\sum_{i=1}^J \pi_i \alpha_i}{\mathrm{Std}\left[\sum_{i=1}^J \pi_i \varepsilon_i\right]}\right)^2$$

under the constraint $\boldsymbol{\pi} \cdot \mathbf{1} = 1$. The result is

$$\pi_i = \frac{\dfrac{\alpha_i}{\mathrm{Var}[\varepsilon_i]}}{\displaystyle\sum_{j=1}^J \frac{\alpha_j}{\mathrm{Var}[\varepsilon_j]}}, \; i = 1, 2, \ldots, J.$$

(See Theorem 13.2 in the course book.)

# The Treynor-Black model

From p. 487 in the course book:

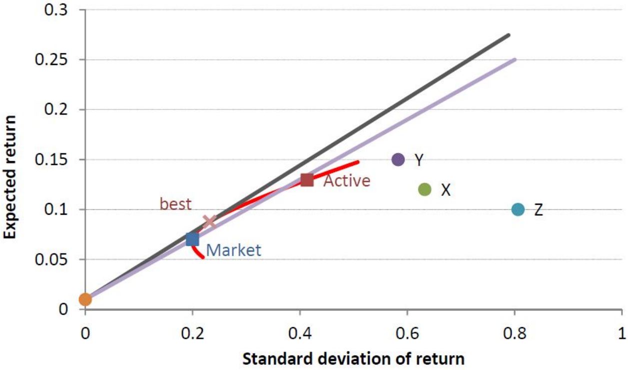



**Figure 13.1: The mean-variance diagram with Treynor-Black.**
The diagram is constructed using the inputs explained in Example 13.1.

# The Treynor-Black model

We need to estimate the parameters of the model. We have seen that especially for the expected return, where we find the alpha's, it is hard to get a good precision in the estimates.

One way to take care of this is to shrink the estimates of the alpha's towards zero.

One example is to shrink by using the $R_i^2$ in the regression from which $\alpha_i$ is estimated:

$$\alpha_i \;\rightarrow\; R_i^2 \cdot \alpha_i .$$

Another way is to only consider the alpha's that are larger than or equal to some threshold.

# Bayesian statistics

Assume that data suggests that

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \Sigma).$$

We now take a Bayesian approach.

This means that we consider some or all parameters in the model as being random variables.

# Bayesian statistics

The idea is to then use Bayes' formula in the following formal way

$$\begin{aligned} P(\text{Parameters}|\text{Data}) &= \frac{P(\text{Data}|\text{Parameters}) \cdot P(\text{Parameters})}{P(\text{Data})} \\ &\propto P(\text{Data}|\text{Parameters}) \cdot P(\text{Parameters}). \end{aligned}$$

Here

- $P(\text{Parameters})$ is our initial assumption of the distribution of the parameter(s) – the prior distribution or just the prior.
- $P(\text{Data}|\text{Parameters})$ is the likelihood function.
- $P(\text{Parameters}|\text{Data})$ is the resulting distribution of the parameter(s) after we have observed the data – the posterior distribution or just the posterior.

In summary:

$$\text{Posterior} \propto \text{Likelihood} \cdot \text{Prior}$$

# Bayesian statistics

We now consider $\boldsymbol{\mu}$ as a random vector, not a vector of parameters, while we still consider the elements of $\Sigma$ as being parameters.

As $\boldsymbol{\mu}$ is now a random vector, we need to make some assumptions regarding its distribution.

We take the prior distribution to be a multivariate normal distribution:

$$\boldsymbol{\mu} \sim N(\mathbf{m}, V).$$

# The Black-Litterman model

You as an analyst, based on intensive research, has views on some of the expected values of the returns, or on portfolios of returns.

These views are usually on one of the following two forms.

- An absolute view: The expected return on asset no 3 should be 3%:

$$\mu_3 = 0.03.$$

- A relative view: The excess return of asset 4 with respect to asset 2 should be 2%:

$$\mu_4 - \mu_2 = 0.02.$$

We collect these views in a matrix $P$ and a vector $\mathbf{Q}$.

If we assume that there are $N = 4$ risky assets, then

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{Q} = \begin{bmatrix} 0.03 \\ 0.02 \end{bmatrix}$$

with the examples from the previous slide.

# The Black-Litterman model

In order to introduce the uncertainty we have about our views, we add noise:

$$P\boldsymbol{\mu} = \mathbf{Q} + \varepsilon_v,$$

where

$$\varepsilon_v \sim N(\mathbf{0}, \Omega) \ \text{ and } \ \Omega = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_K \end{pmatrix}.$$

How certain we are of our views is reflected in the size of the $\omega$'s: The more certain we are, the smaller is the $\omega$ of that view.

It follows that

$$P\boldsymbol{\mu} \sim N(\mathbf{Q}, \Omega).$$

# The Black-Litterman model

To summarise:

- The data tells us that

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \Sigma) \text{ with } \boldsymbol{\mu} \sim N(\mathbf{m}, V).$$

- Our views on some portfolio(s) of the assets are that

$$P\boldsymbol{\mu} \sim N(\mathbf{Q}, \Omega).$$

How do we combine the views with the data?

We calculate the posterior distribution.

# The Black-Litterman model

Using Bayesian statistics we get the following posterior distribution for the mean $\boldsymbol{\mu}$ when we use our personal views:

$$\boldsymbol{\mu} \sim N(\hat{\mathbf{m}}, \hat{V})$$

with

$$\hat{\mathbf{m}} = \mathbf{m} + VP^T(PVP^T + \Omega)^{-1}\big(\mathbf{Q} - P\mathbf{m}\big)$$

and

$$\hat{V} = \big(V^{-1} + P^T\Omega^{-1}P\big)^{-1}.$$

# The Black-Litterman model

The Black-Litterman model is mainly considered as a way of taking your personal views into account when finding the optimal portfolio.

The model also, however, adresses the problem of estimating the means.

The idea is to use "reverse engineering". To simplify we assume that there is no uncertainty in $\boldsymbol{\mu}$, i.e. $V = 0$.

Recall from Lecture 7 that

$$\boldsymbol{\pi} = \frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})$$

is the optimal portfolio when $\mathbf{r}$ is normally distributed, the investor has a CARA utility function with parameter $\gamma$ and there is a risk-free asset.

# The Black-Litterman model

The market portfolio was then given by

$$\boldsymbol{\pi}_{\text{mkt}} = \frac{1}{\bar{\gamma}} \Sigma^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}),$$

where $\bar{\gamma}$ is determined by the investors' $\gamma$'s.

Using this equation, we can write

$$\boldsymbol{\mu} = r_f \mathbf{1} + \bar{\gamma} \Sigma \boldsymbol{\pi}_{\text{mkt}}.$$

We have also seen that

$$\bar{\gamma} = \frac{E[r_m] - r_f}{\text{Var}[r_m]} \longrightarrow \boldsymbol{\mu} = r_f \mathbf{1} + \frac{E[r_m] - r_f}{\text{Var}[r_m]} \Sigma \boldsymbol{\pi}_{\text{mkt}}.$$

Hence, we can estimate the vector of expected rate of returns $\boldsymbol{\mu}$ by using the market portfolio, $\Sigma$ and the mean and variance of the market portfolio.

# Household finance

Chapter 9 in the course book contains several examples of household finance.

We will look at the following topics from that chapter:

- Labour income.
- Housing.

# Labour income

For an individual considering his or her portfolio choice over time, the human capital is important.

$$\text{Human capital} = \text{PV}\big(\text{Future labour income}\big).$$

Let $F_t$ and $L_t$ denote the financial and human capital at time $t$ respectively. The total wealth is

$$W_t = F_t + L_t,$$

the human capital share of the total wealth is

$$h_t = \frac{L_t}{F_t + L_t} = \frac{L_t}{W_t},$$

and

$$\ell_t = \frac{L_t}{F_t}$$

is the human-to-financial wealth ratio.

# Labour income

The return on the financial capital is (here we use that $\boldsymbol{\pi}_t \cdot \mathbf{1} + \pi_{t,f} = 1$):

$$\frac{F_{t+1} - F_t}{F_t} = \boldsymbol{\pi}_t \cdot \mathbf{r}_{t+1} + \pi_{t,f} r_f = \boldsymbol{\pi}_t \cdot \mathbf{r}_{t+1} + \big(1 - \boldsymbol{\pi}_t \cdot \mathbf{1}\big) r_f.$$

Let

$$r^L_{t+1} = \frac{L_{t+1} - L_t}{L_t}$$

denote the rate of return on human capital and assume that

$$E_t\left[r^L_{t+1}\right] = \mu_L \quad \text{and} \quad \mathrm{Var}_t\big[r^L_{t+1}\big] = \sigma^2_L.$$

# Labour income

The individual at time $t$ wants to solve

$$\max_{\boldsymbol{\pi}_t} \left\{ E_t\left[\frac{W_{t+1}}{W_t}\right] - \frac{1}{\gamma}\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right]\right\},$$

i.e. it has mean-variance preferences. We get

$$\begin{aligned}
\frac{W_{t+1}}{W_t} &= \frac{F_{t+1}+L_{t+1}}{F_t+L_t} \\
&= \frac{F_t}{F_t+L_t}\big(1+\boldsymbol{\pi}_t\cdot\mathbf{r}_{t+1}+(1-\boldsymbol{\pi}_t\cdot\mathbf{1})r_f\big)+\frac{L_t}{F_t+L_t}(1+r^L_{t+1}) \\
&= (1-h_t)\big(1+r_f+\boldsymbol{\pi}_t\cdot(\mathbf{r}_{t+1}-r_f\mathbf{1})\big)+h_t(1+r^L_{t+1}).
\end{aligned}$$

# Labour income

Now

$$
\begin{aligned}
E_t\left[\frac{W_{t+1}}{W_t}\right] &= (1-h_t)\big(1+r_f+\boldsymbol{\pi}_t\cdot\big(E_t\left[\mathbf{r}_{t+1}\right]-r_f\mathbf{1}\big)\big) \\
&\quad +h_t\left(1+E_t\left[r^L_{t+1}\right]\right) \\
&= (1-h_t)\big(1+r_f+\boldsymbol{\pi}_t\cdot(\boldsymbol{\mu}-r_f\mathbf{1})\big)+h_t(1+\mu_L), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right] &= \mathrm{Var}_t\big[(1-h_t)\boldsymbol{\pi}_t\cdot\mathbf{r}_{t+1}+h_t r^L_{t+1}\big] \\
&= (1-h_t)^2\boldsymbol{\pi}_t\cdot\Sigma\boldsymbol{\pi}_t+2(1-h_t)h_t\boldsymbol{\pi}_t\cdot\mathrm{Cov}_t[\mathbf{r}_{t+1},r^L_{t+1}] \\
&\quad +h_t^2\sigma_L^2.
\end{aligned}
$$

# Labour income

The solution to the maximisation problem is (see the course book for a derivation):

$$\begin{aligned}\boldsymbol{\pi}_t &= \frac{1}{\gamma}(1+\ell_t)\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) - \ell_t\Sigma^{-1}\mathrm{Cov}_t[\mathbf{r}_{t+1}, r^L_{t+1}] \\ &= \underbrace{\frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}_{=(1)} + \underbrace{\ell_t\Sigma^{-1}\left(\frac{1}{\gamma}(\boldsymbol{\mu} - r_f\mathbf{1}) - \mathrm{Cov}_t[\mathbf{r}_{t+1}, r^L_{t+1}]\right)}_{=(2)}.\end{aligned}$$

(1) Optimal weights without labour income.
(2) Hedge against labour income risk.

When there is one risky asset:

$$\pi_t = \frac{\mu - r_f}{\gamma\sigma^2} + \ell_t\left(\frac{\mu - r_f}{\gamma\sigma^2} - \frac{\mathrm{Cov}_t[r_{t+1}, r^L_{t+1}]}{\sigma^2}\right).$$

# Labour income

The older we are, the smaller is the fraction $\ell_t = L_t / F_t$.

In the book the following approximate table is derived (see footnote 3 on p. 351 and Section 9.1.2):

| $\ell_t$ | Age |
|---|---|
| 1 | 97 |
| 2 | 84 |
| 5 | 55 |
| 10 | 44 |
| 20 | 35 |
| 50 | 26 |

# Labour income

Consider the case with one risky asset and a constant

$$\text{Cov}_t[r_{t+1}, r^L_{t+1}] = \rho_{SL}\sigma_S\sigma_L.$$

From p. 351 in the course book:

| | $\gamma = 1$ | | | | $\gamma = 5$ | | | | $\gamma = 10$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\ell_t$ | stock | rf | exp | std | stock | rf | exp | std | stock | rf | exp | std |
| 0 | 125 | -25 | 7.3 | 25 | 25 | 75 | 2.3 | 5 | 13 | 87 | 1.6 | 3 |
| 1 | 245 | -145 | 13.3 | 49 | 45 | 55 | 3.3 | 9 | 20 | 80 | 2.0 | 4 |
| 2 | 365 | -265 | 19.3 | 73 | 65 | 35 | 4.3 | 13 | 28 | 72 | 2.4 | 6 |
| 5 | 725 | -625 | 37.3 | 145 | 125 | -25 | 7.3 | 25 | 50 | 50 | 3.5 | 10 |
| 10 | 1325 | -1225 | 67.3 | 265 | 225 | -125 | 12.3 | 45 | 88 | 12 | 5.4 | 18 |
| 20 | 2525 | -2425 | 127.3 | 505 | 425 | -325 | 22.3 | 85 | 163 | -63 | 9.1 | 33 |
| 50 | 6125 | -6025 | 307.3 | 1225 | 1025 | -925 | 52.3 | 205 | 388 | -288 | 20.4 | 78 |

**Table 9.2: Optimal portfolios with human capital.**
The table shows the percentages of financial wealth optimally invested in the stock and the riskfree asset, as well as the expectation and standard deviation of the financial return in percent. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$.

# Labour income

From p. 352 in the course book:

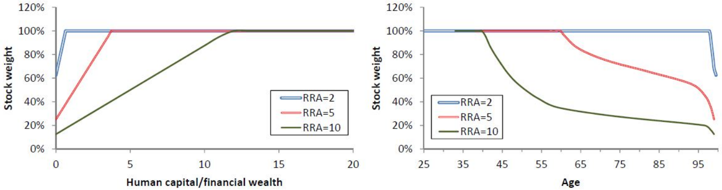



**Figure 9.2: Optimal stock weight with human capital.**
The figure shows the constrained optimal stock weight as a function of the human capital to financial wealth ratio (left panel) and age (right panel) for three different values of the relative risk aversion coefficient $\gamma$. The stock weight is restricted to the interval from 0% to 100%. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$.

# Housing

Housing (i.e. the financial position in owning a house or apartment) is a large part of many households investments.

Consider the previous labour income model where we let the risky "investment universe" consist of a stock and investment in housing:

$$\mathbf{r} = \begin{pmatrix} r_S \\ r_H \end{pmatrix}.$$

Throughout we let "S", "L" and "H" denote parameters connected to the stock, human capital and housing respectively.

# Housing

The amount of our investment portfolio we invest in the risk-free asset is

$$1 - \pi_S - \pi_H$$

(the human capital is not part of the investment decision; there is no $\pi_L$).

We assume that if you borrow money, then you have to take out a mortgage on your real estate investment – short selling stocks is not allowed

Furthermore, you are only allowed to mortgage $1 - \kappa$ of the value of the real estate. This means that

$$\pi_S \geq 0 \text{ and } 1 - \pi_S - \pi_H \geq -(1 - \kappa)\pi_H = -\pi_H + \kappa\pi_H,$$

or

$$\pi_S \geq 0 \text{ and } \pi_S + \kappa\pi_H \leq 1.$$

# Housing

This leads to the maximisation problem

$$\max_{(\pi_S,\pi_H)} \quad \left\{ E_t\left[\frac{W_{t+1}}{W_t}\right] - \frac{1}{\gamma}\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right]\right\}$$

$$\text{s.t.} \qquad \pi_S \geq 0$$
$$\pi_S + \kappa\pi_H \leq 1.$$

As in the labour income problem, the objective function is quadratic in the weights, so it is straightforward to numerically derive them

# Housing

From p. 361 in the course book:

| | $\gamma = 1$ | | | $\gamma = 5$ | | | $\gamma = 10$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $\ell$ | stock | house | rf | stock | house | rf | stock | house | rf |
| Panel A: Baseline case with max 80% LTV, $\kappa = 0.2$ | | | | | | | | | |
| 0 | 52 | 240 | -192 | 20 | 52 | 28 | 10 | 26 | 64 |
| 1 | 13 | 434 | -347 | 35 | 96 | -31 | 16 | 44 | 41 |
| 2 | 0 | 500 | -400 | 51 | 140 | -91 | 21 | 61 | 17 |
| 5 | 0 | 500 | -400 | 50 | 250 | -200 | 39 | 115 | -53 |
| 10 | 0 | 500 | -400 | 16 | 420 | -336 | 60 | 200 | -160 |
| 20 | 0 | 500 | -400 | 0 | 500 | -400 | 32 | 340 | -272 |
| 50 | 0 | 500 | -400 | 0 | 500 | -400 | 0 | 500 | -400 |
| Panel B: max 60% LTV, $\kappa = 0.4$ | | | | | | | | | |
| 0 | 33 | 167 | -100 | 20 | 52 | 28 | 10 | 26 | 64 |
| 1 | 4 | 241 | -145 | 35 | 96 | -31 | 16 | 44 | 41 |
| 2 | 0 | 250 | -150 | 47 | 133 | -80 | 21 | 61 | 17 |
| 5 | 0 | 250 | -150 | 30 | 174 | -105 | 39 | 115 | -53 |
| 10 | 0 | 250 | -150 | 3 | 242 | -145 | 36 | 159 | -95 |
| 20 | 0 | 250 | -150 | 0 | 250 | -150 | 12 | 220 | -132 |
| 50 | 0 | 250 | -150 | 0 | 250 | -150 | 0 | 250 | -150 |
| Panel C: no borrowing, $\kappa = 1$ | | | | | | | | | |
| 0 | 62 | 38 | 0 | 20 | 52 | 28 | 10 | 26 | 64 |
| 1 | 100 | 0 | 0 | 31 | 69 | 0 | 16 | 44 | 41 |
| 2 | 100 | 0 | 0 | 38 | 62 | 0 | 21 | 61 | 17 |
| 5 | 100 | 0 | 0 | 60 | 40 | 0 | 31 | 69 | 0 |
| 10 | 100 | 0 | 0 | 95 | 5 | 0 | 43 | 57 | 0 |
| 20 | 100 | 0 | 0 | 100 | 0 | 0 | 67 | 33 | 0 |
| 50 | 100 | 0 | 0 | 100 | 0 | 0 | 100 | 0 | 0 |
| Panel D: Higher borrowing than lending rate, $r_{\text{bor}} = 2\%, r_{\text{len}} = 1\%$ | | | | | | | | | |
| 0 | 68 | 160 | -128 | 20 | 52 | 28 | 10 | 26 | 64 |
| 1 | 45 | 274 | -219 | 30 | 70 | 0 | 16 | 44 | 41 |
| 2 | 22 | 388 | -310 | 42 | 83 | -25 | 21 | 61 | 17 |
| 5 | 0 | 500 | -400 | 69 | 154 | -123 | 31 | 69 | 0 |
| 10 | 0 | 500 | -400 | 51 | 244 | -195 | 50 | 100 | -50 |
| 20 | 0 | 500 | -400 | 15 | 424 | -339 | 66 | 172 | -138 |
| 50 | 0 | 500 | -400 | 0 | 500 | -400 | 30 | 352 | -282 |

**Table 9.6: Optimal portfolios with borrowing constraints.** The table shows percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 9.4 are assumed. In Panels B, C, and D the numbers in blue are larger than in the baseline case of Panel A, numbers in red are smaller, whereas the remaining numbers are unchanged.

# Financial Theory – Lecture 14

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Empirical aspects.

The lecture is based on

- Sections 5.7 and 6.5 in the course book.

# Stock market returns

We'll start looking at data for the stocks in the S&P 500 index.

When looking at the stocks from an index, we can construct a value weighted (VW) or equally weighted (EW) portfolio.

- VW:
$$\pi_i = \frac{\text{Asset market cap}}{\text{Total market cap}}, \; i = 1, 2, \dots, n.$$
- EW:
$$\pi_i = \frac{1}{n}, \; i = 1, 2, \dots, n.$$

# Stocks market returns

From p. 229 in the course book:

| | Period | Mean | Std dev | Skew | Kurt | Min | Max |
|---|---|---|---|---|---|---|---|
| | | | Annual returns | | | | |
| VW nominal | 1927-2019 | 11.9% | 19.9% | -0.446 | 0.098 | -45.5% | 53.3% |
| VW nominal | 1946-2019 | 12.4% | 17.0% | -0.351 | 0.109 | -36.6% | 52.8% |
| VW nominal | 1990-2019 | 11.5% | 17.5% | -0.767 | 0.623 | -36.6% | 37.7% |
| EW nominal | 1927-2019 | 14.3% | 24.1% | 0.042 | 1.081 | -53.0% | 95.7% |
| EW nominal | 1946-2019 | 13.9% | 19.0% | -0.176 | 0.193 | -40.1% | 58.0% |
| EW nominal | 1990-2019 | 13.1% | 18.5% | -0.637 | 1.236 | -40.1% | 47.4% |
| Inflation | 1946-2019 | 3.69% | 3.36% | 1.942 | 4.960 | -2.07% | 18.1% |
| VW real | 1946-2019 | 8.60% | 17.4% | -0.301 | 0.144 | -36.7% | 53.9% |
| EW real | 1946-2019 | 10.1% | 19.1% | -0.204 | 0.146 | -40.2% | 55.2% |
| 1Y riskfree | 1946-2019 | 3.99% | 3.12% | 0.919 | 0.886 | 0.02% | 14.7% |
| VW excess | 1946-2019 | 8.41% | 17.4% | -0.320 | 0.126 | -38.2% | 51.9% |
| EW excess | 1946-2019 | 9.96% | 19.3% | -0.125 | 0.267 | -41.7% | 56.4% |
| | | | Monthly returns | | | | |
| VW nominal | 1927-2019 | 0.94% | 5.41% | 0.356 | 9.794 | -28.7% | 41.4% |
| VW nominal | 1946-2019 | 0.96% | 4.13% | -0.425 | 1.654 | -21.6% | 16.8% |
| VW nominal | 1990-2019 | 0.88% | 4.09% | -0.612 | 1.260 | -16.7% | 11.4% |
| EW nominal | 1927-2019 | 1.14% | 6.74% | 1.472 | 18.574 | -31.0% | 68.0% |
| EW nominal | 1946-2019 | 1.08% | 4.69% | -0.325 | 2.508 | -25.6% | 23.1% |
| EW nominal | 1990-2019 | 1.01% | 4.65% | -0.492 | 2.125 | -20.9% | 18.5% |
| Inflation | 1946-2019 | 0.30% | 0.45% | 2.577 | 29.203 | -1.92% | 5.88% |
| VW real | 1946-2019 | 0.66% | 4.17% | -0.421 | 1.458 | -21.8% | 15.6% |
| EW real | 1946-2019 | 0.78% | 4.73% | -0.320 | 2.310 | -25.8% | 22.7% |
| 1M riskfree | 1946-2019 | 0.32% | 0.25% | 0.976 | 1.174 | 0.00% | 1.35% |
| VW excess | 1946-2019 | 0.64% | 4.15% | -0.446 | 1.627 | -22.2% | 16.3% |
| EW excess | 1946-2019 | 0.76% | 4.70% | -0.345 | 2.485 | -26.2% | 22.5% |

**Table 6.2: Summary statistics for the S&P 500 index.**
Data on the value-weighted (VW) and equally-weighted (EW) returns on the stocks in the S&P 500 index as well as the inflation rate were downloaded from CRSP through WRDS on June 22, 2020. Data on the 1 month and the 1 year riskfree rate were downloaded from the homepage of Kenneth French on June 22, 2020. The mean shown in the table is the arithmetic average.

# Stocks market returns

From p. 235 in the course book:

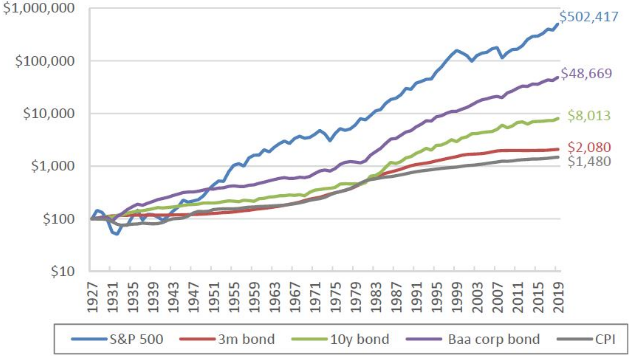



**Figure 6.5: Stocks look great in the long run.**
The graph shows cumulative returns on U.S. asset classes from 1927 to 2019. The data were downloaded on June 24, 2020 from the homepage of Professor Aswath Damodaran at New York University, see http://pages.stern.nyu.edu/~adamodar/.

# Stocks market returns

From p. 236 in the course book:

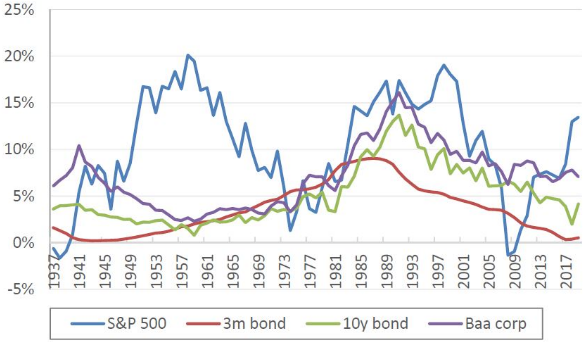



**Figure 6.6: Trailing 10-year average returns.**
The graphs show the trailing 10-year geometric average rate of return for the S&P 500 stock market index, 3-month Treasury bills, 10-year Treasury bonds, and Baa-rated corporate bonds over the period 1937-2019. The data were downloaded on June 24, 2020 from the homepage of Professor Aswath Damodaran at New York University, see http://pages.stern.nyu.edu/~adamodar/.

# Stocks market returns

Are returns predictable?

On a larger scale, there are empirical studies indicating that average stock market returns are counter-cyclical: they are higher in bad times than in good times.

Again, we see this from

$$r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} \quad \Leftrightarrow \quad P_t = \frac{P_{t+1} + D_{t+1}}{1 + r_{t+1}}.$$

# Stocks market returns

Several studies also show momentum in the short run and reversal in the long run. This observation lead to the introduction of the momentum factor.

Can some other variables predict stock returns?

The price-dividend ratio and the dividend yield has been suggested.

# Stocks market returns

We can write

$$\begin{aligned}
\ln\left(1+r_{t+1}\right) &= \ln\left(P_{t+1}+D_{t+1}\right)-\ln P_t \\
&= \ln\left(P_{t+1}+D_{t+1}\right)-\ln P_{t+1}+\ln P_{t+1}-\ln P_t \\
&= \ln\left(1+\frac{D_{t+1}}{P_{t+1}}\right)+\ln P_{t+1}-\ln P_t \\
&= \ln\left(1+e^{d_{t+1}-p_{t+1}}\right)+p_{t+1}-p_t,
\end{aligned}$$

where $p=\ln P$ and $d=\ln D$. Using an approximation here (the Campbell-Shiller approximation) and iterating we get decompositions that can be used in testing for prediction.

# Stocks market returns

There have been several other suggestions, but there seems not to be any strong indication of specific variables being able to, in general, predict stock returns.

# Stocks market returns

The cross-sectional investigation of returns focus on if we can predict the return of an asset by looking at the return(s) of other assets.

Typically this includes factor models and testing CAPM or some APT model by using the two-stage regression or the Fama-MacBeth method discussed in Lecture 9.

# Asset returns

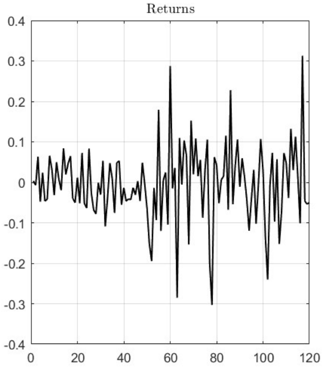
Moving from the general to the specific, we often make assumptions such as:

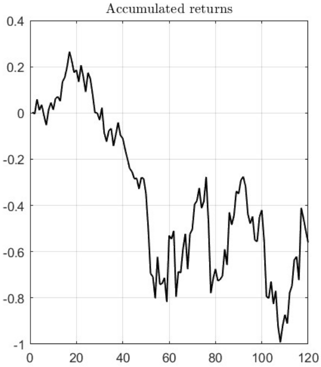
- The rate of returns $r_1, r_2, \ldots, r_n$ are IID.
- Each $r_i$, $i = 1, 2, \ldots, n$ has a distribution depending on some parameters.

We want to estimate parameters and test hypotheses.

We will see an example of 10 years monthly data for a stock.

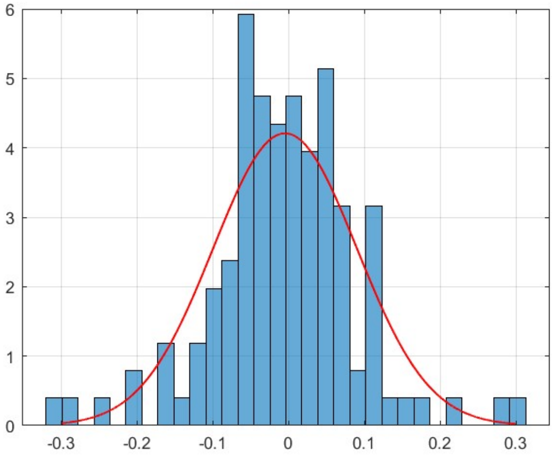
A time series description of the data.



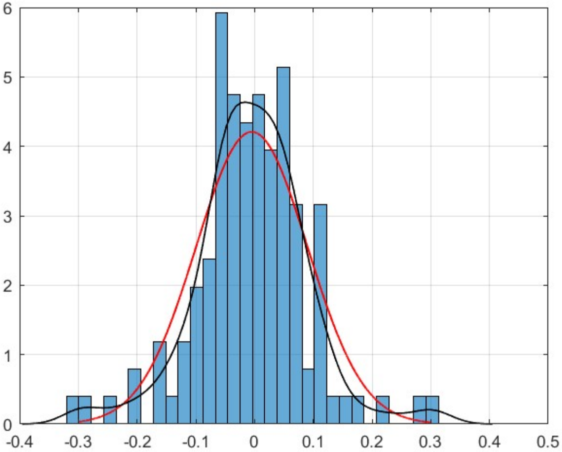
# Asset returns

A histogram with an estimated normal density $\hat{\mu} = -0.0047$ and $\hat{\sigma} = 0.0948$.



# Asset returns

A histrogram with an estimated normal density together with a kernal estimator.



# Asset returns

Are asset returns correlated over time?

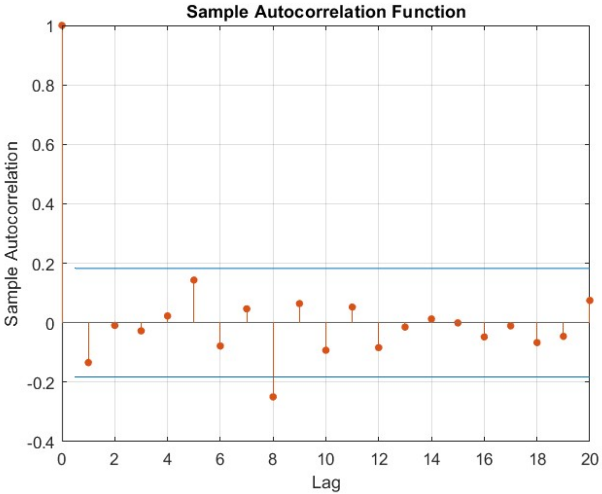
The autocorrelation function is given by

$$\rho_k = \mathrm{Corr}[r_t, r_{t+k}].$$

If the returns are IID, then they also are uncorrelated, and we should have $\rho_k = 0,\ k = 1, 2, \ldots$.

# Asset returns

The autocorrelation function for $k = 1, 2, \ldots, 20$.



# Asset returns

How do we estimate parameters?

"Old method": Use maximum likelihood (ML).

"Modern method": Use generalised method of moments (GMM).

# Method of moments

Assume that $x_1, x_2 \ldots, x_n$ are observations from a random variable $X$ with mean $\mu$.

Then

$$E[X - \mu] = 0.$$

Now replace $E$ with $\frac{1}{n}\sum_{i=1}^{n}$ to get an estimator:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_n) = 0 \quad \Leftrightarrow \quad \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The (weak) law of large numbers implies that

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu.$$

# GMM

Let $r_1, r_2, \ldots, r_T$ be a time series of rate of returns (random variables).

We need some technical assumptions on the time series of returns for the following method to work (stationary and ergodic).

Assume that there exists a function $g$ such that for each $t$

$$E\left[g(r_t, b)\right] = 0.$$

We want to estimate the parameter $b$.

Again, replace $E$ with $\frac{1}{T}\sum_{i=1}^{T}$:

$$\frac{1}{T}\sum_{i=1}^{T} g(r_i^{\text{obs}}, \hat{b}_T) = 0.$$

# GMM

If the parameter is one-dimensional, then we have 1 equation and 1 unkown $\longrightarrow$ we can (in theory) calculate $\hat{b}_T$.

In general, we have $N$ conditions:

$$E\left[g_i(r_t, b)\right] = 0, \ \ i = 1, 2, \ldots, N,$$

and a parameter vector with $K < N$ number of elements.

We then choose $K$ linear combinations (this can be done in several ways) to get estimates, and use the rest $N - K$ equations to get goodness-of-fit tests.

# GMM

In general, the estimator is unbiased:

$$E\left[\hat{b}_n\right] = b$$

and consistent:

$$\hat{b}_n \xrightarrow{P} b \text{ as } n \to \infty.$$

There are also typically some asymptotic distribution result(s) which we can use to test hypothesis.

# GMM

If the parameter vector $b$ is $K$-dimensional, then (from a version of the CLT)

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} g(r_t, b) \xrightarrow{d} N(\mathbf{0}, S),$$

where

$$S = \sum_{s=-\infty}^{\infty} E\left[ g(r_t, b) g(r_{t-s}, b)^\top \right]$$

is a $K \times K$-dimensional asymptotic variance-coraviance matrix.

How do we estimate $S$?

Assume that the the $r_t$'s are uncorrelated (easy, but not realistic) or use a few time lags á la Newey-West.

# Bond returns and interest rates

*Average returns and volatilities increase with maturity, whereas Sharpe ratios decrease with maturity.*

(Munk, p. 174.)

From p. 175 in the course book:

| | Inflation | 1M | 3M | 1Y | 2Y | 5Y | 7Y | 10Y | 20Y | 30Y |
|---|---|---|---|---|---|---|---|---|---|---|
| Avg return | 3.67% | 3.84% | 4.25% | 4.72% | 4.95% | 5.50% | 5.82% | 5.59% | 6.18% | 6.16% |
| Standard dev | 1.59% | 0.92% | 1.05% | 1.79% | 2.75% | 4.99% | 6.14% | 7.36% | 9.86% | 11.35% |
| Sharpe ratio | | | 0.398 | 0.492 | 0.406 | 0.334 | 0.323 | 0.238 | 0.238 | 0.204 |

Table 5.2: **Return statistics for U.S. Treasury bonds.**
The statistics are based on monthly observations over the period from January 1946 to December 2021 downloaded from CRSP U.S. Treasury and Inflation Indexes on July 4, 2022. The returns are nominal. The statistics shown are annualized from monthly statistics. For the average return and standard deviation for both bond returns and the inflation rate, the annualization follows Eqs. (3.83) and (3.84). The annualized Sharpe ratio for a given maturity is calculated as the difference of the annualized average return for that maturity minus the annualized average return on 1-month bills, divided by the annualized standard deviation for the given maturity.

# Bond returns and interest rates

We can see the same feature in risky bond (p. 176 in the course book):

| | Intermediate maturity ($\approx$ 5 years) | | | | Long maturity ($\approx$ 10 years) | | | |
|---|---|---|---|---|---|---|---|---|
| | AAA | AA | A | BAA | AAA | AA | A | BAA |
| Average excess return | 2.38% | 2.53% | 2.76% | 3.44% | 3.12% | 3.80% | 3.75% | 4.60% |
| Standard deviation | 5.02% | 4.99% | 5.28% | 5.48% | 10.45% | 9.74% | 9.67% | 9.82% |
| Sharpe ratio | 0.47 | 0.51 | 0.52 | 0.63 | 0.30 | 0.39 | 0.39 | 0.47 |

Table 5.3: **Return statistics for U.S. corporate bonds.**
The statistics are annualized and based on data from Barclays corporate bond indexes over the period from January 1973 to August 2014. Intermediate maturity corresponds to a duration of about 5 years, and long maturity to a duration of about 10 years. Source: Table 5 in van Binsbergen and Koijen (2017).

# Bond returns and interest rates

Bonds with close maturities are highly correlated (p. 177 in the course book):

| | 1M | 3M | 6M | 1Y | 2Y | 3Y | 5Y | 7Y | 10Y | 20Y | 30Y |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1M | 1.00 | 0.97 | 0.93 | 0.88 | 0.72 | 0.60 | 0.46 | 0.37 | 0.36 | 0.30 | 0.30 |
| 3M | 0.97 | 1.00 | 0.98 | 0.93 | 0.78 | 0.67 | 0.53 | 0.45 | 0.43 | 0.37 | 0.36 |
| 6M | 0.93 | 0.98 | 1.00 | 0.97 | 0.83 | 0.73 | 0.58 | 0.49 | 0.46 | 0.39 | 0.38 |
| 1Y | 0.88 | 0.93 | 0.97 | 1.00 | 0.91 | 0.82 | 0.67 | 0.58 | 0.54 | 0.46 | 0.44 |
| 2Y | 0.72 | 0.78 | 0.83 | 0.91 | 1.00 | 0.97 | 0.86 | 0.77 | 0.72 | 0.63 | 0.59 |
| 3Y | 0.60 | 0.67 | 0.73 | 0.82 | 0.97 | 1.00 | 0.95 | 0.87 | 0.82 | 0.73 | 0.68 |
| 5Y | 0.46 | 0.53 | 0.58 | 0.67 | 0.86 | 0.95 | 1.00 | 0.98 | 0.94 | 0.87 | 0.83 |
| 7Y | 0.37 | 0.45 | 0.49 | 0.58 | 0.77 | 0.87 | 0.98 | 1.00 | 0.99 | 0.94 | 0.90 |
| 10Y | 0.36 | 0.43 | 0.46 | 0.54 | 0.72 | 0.82 | 0.94 | 0.99 | 1.00 | 0.97 | 0.95 |
| 20Y | 0.30 | 0.37 | 0.39 | 0.46 | 0.63 | 0.73 | 0.87 | 0.94 | 0.97 | 1.00 | 0.99 |
| 30Y | 0.30 | 0.36 | 0.38 | 0.44 | 0.59 | 0.68 | 0.83 | 0.90 | 0.95 | 0.99 | 1.00 |

Table 5.4: **Bond correlations.**
The table shows correlations between monthly changes in yields of U.S. Treasury bonds of different maturities in the period January 2012 to December 2021. Source: `http://www.federalreserve.gov/releases/h15/data.htm`, data retrieved on July 4, 2022.

Implication: There is little diversification effect in investing in bonds with close maturities.

# Bond returns and interest rates

Other features include:

- High autocorrelation for short-term interest rates.
- Interest rates, especially short-term, tend to be mean reverting.

# Yield curves

Take as given the yields

$$(y_1, y_2, \dots, y_n)^\top.$$

We can think of this as an $n$-dimensional random vector. One way of analysing the yields is to use principal component analysis (PCA).

This is a way to get $k < n$ number of vectors – principal components – that "explains" the major part of the movement of the yields.

# Yield curves

For PCA to be a successful method, we want two conditions to be satisfied.

- The number $k$ of principal components should be low.
- We should be able to give an interpretation to the principal components.

# Yield curves

If we do a PCA on the yields, we typically get three components that explains a large part of the yield curve.

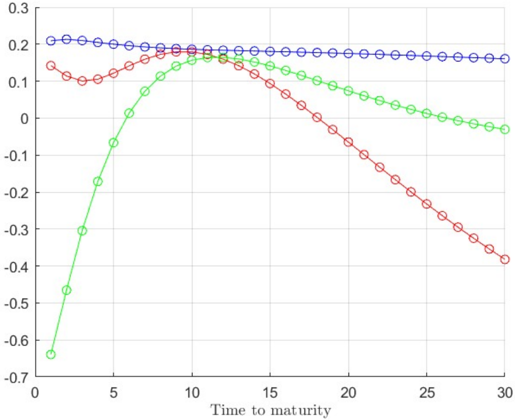
We can also give them an interpretation:

- Level.
- Slope.
- Curvature.

# Yield curves

A PCA performed on monthly US nominal bonds for about 35 years.

By using the three first principal components, we take account of 99.52% of the variation.



# Yield curves

| Number of principal components | Cumulative sum of variation |
|---|---|
| 1 | 0.938827060103672 |
| 2 | 0.970281483447598 |
| 3 | 0.995221012006218 |
| 4 | 0.999255414190555 |
| 5 | 0.999924874885835 |
| 6 | 0.999990576439362 |
| 7 | 0.999998952231860 |
| 8 | 0.999999923342092 |
| 9 | 0.999999994210718 |
| 10 | 0.999999999522157 |

# Financial Theory – Lecture 15

Fredrik Armerin, Uppsala University, 2024

# Agenda

- Macro-finance.
- ESG investing

The ESG investing part is based on

- Section 13.4 in the course book.

# Introduction

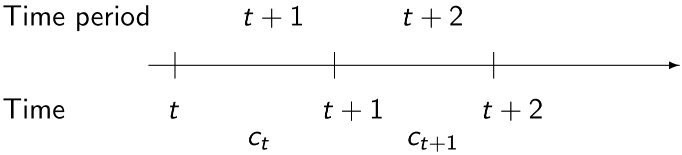
What is macro-finance?

> *Macro-finance studies the relationship between asset prices and economic fluctuations.*

John H. Cochrane (2017), "Macro-Finance", *Review of Finance*, p. 945-985.

# The utility of a consumer over time

A consumer has at time $t$ to choose his/her consumption $c_t$ over the next period.




But a consumer also has to choose consumption over all future time periods.

# The utility of a consumer over time

What is the utility of a consumption stream

$$(c_t, c_{t+1}, c_{t+2}, \ldots)?$$

We use expected utility, and let the utility of this stream at time $t$ be given by

$$E_t\left[U(t, c_t, c_{t+1}, c_{t+2}, \ldots)\right].$$

In order to get more explicit results, we assume some more structure on the utility function.

# The utility of a consumer

- The utility function is additive:

$$
\begin{aligned}
U(t, c_t, c_{t+1}, c_{t+2}, \ldots) &= U_t(c_t) + U_t(c_{t+1}) \\
&\quad U_t(c_{t+2}) + \cdots \\
&= \sum_{s=0}^{\infty} U_t(c_{t+s}).
\end{aligned}
$$

- We have time-separability:

$$U_t(c_{t+s}) = \delta^s u(c_{t+s}).$$

This leads to the following utility at time $t$

$$E_t\left[\sum_{s=0}^{\infty} \delta^s u(c_{t+s})\right].$$

# The utility of a consumer

This is a generalisation of the utility function

$$u(c_0) + \delta E\left[u(c_1)\right]$$

we used when deriving the consumption-based CAPM equation.

At every $t$ there is a non-negative endowment $e_t$ given to the investor.

At time $t$, the investor chooses the portfolio

$$\mathbf{x}_t = \left(x_{1t}, x_{2t}, \ldots, x_{Nt}\right)^T$$

and holds it during $(t, t+1]$.

# The utility of a consumer

The budget constraint when portfolio $\mathbf{x}_t$ is bought at time $t$ is

$$\begin{aligned} c_t + \sum_{i=1}^{N} x_{it} P_{it} &= e_t + \sum_{i=1}^{N} x_{i,t-1} \big( P_{it} + D_{it} \big) \\ \text{Consumption} + \text{Cost of new portfolio} &= \text{Endowment} + \text{Value of old portfolio} \end{aligned}$$

Using vector notation:

$$c_t + \mathbf{x}_t \cdot \mathbf{P}_t = e_t + \mathbf{x}_{t-1} \cdot (\mathbf{P}_t + \mathbf{D}_t).$$

In the same way, we get the following budget constraint when portfolio $\mathbf{x}_t$ is sold:

$$c_{t+1} + \mathbf{x}_{t+1} \cdot \mathbf{P}_{t+1} = e_{t+1} + \mathbf{x}_t \cdot (\mathbf{D}_{t+1} + \mathbf{P}_{t+1}).$$

# The utility of a consumer

At time $t$ we want to maximise

$$E_t\left[\sum_{s=0}^{\infty}\delta^s u(c_{t+s})\right] = u(c_t) + E_t\left[\delta u(c_{t+1}) + \delta^2 u(c_{t+1})\ldots\right]$$

over $\mathbf{x}_t$ subject to the two constraints from the previous slide.

Only the first two terms contain $\mathbf{x}_t$ and replacing $c_t$ and $c_{t+1}$ from the previous slide results in

$$\max_{\mathbf{x}_t} u\big(c_t(\mathbf{x}_t)\big) + E_t\left[\delta u\big(c_{t+1}(\mathbf{x}_t)\big)\right].$$

The FOC with respect to $x_{it}$ is

$$u'(c_t)\frac{\partial c_t}{\partial x_{it}} + \delta E_t\left[u'(c_{t+1})\frac{\partial c_{t+1}}{\partial x_{it}}\right] = 0.$$

# The utility of a consumer

Using the constraints we get

$$u'(c_t) \cdot (-P_{it}) + \delta E_t \left[ u'(c_{t+1}) \cdot (D_{i,t+1} + P_{i,t+1}) \right] = 0$$

$$\Leftrightarrow$$

$$P_{it} = E_t \left[ \delta \frac{u'(c_{t+1})}{u'(c_t)} \cdot (D_{i,t+1} + P_{i,t+1}) \right].$$

$$\Leftrightarrow$$

$$1 = E_t \left[ \delta \frac{u'(c_{t+1})}{u'(c_t)} \cdot \frac{D_{i,t+1} + P_{i,t+1}}{P_{it}} \right] = E_t \left[ \delta \frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{i,t+1} \right].$$

# The stochastic discount factor

The stochastic discount factor (SDF) for discounting from $t+1$ to $t$ is

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}.$$

Looking at $m_{t+1}$, we see again that the value of the SDF is high in the future states where $u'(c_{t+1})$ is high, and $u'(c_{t+1})$ is high when $c_{t+1}$ is low.

The SDF from $t$ to 0 is given by

$$\begin{aligned} m_t \cdot m_{t-1} \cdot \ldots \cdot m_1 &= \delta \frac{u'(c_t)}{u'(c_{t-1})} \cdot \delta \frac{u'(c_{t-1})}{u'(c_{t-2})} \cdot \ldots \cdot \delta \frac{u'(c_1)}{u'(c_0)} \\ &= \delta^t \frac{u'(c_t)}{u'(c_0)}. \end{aligned}$$

# Taking a positive view

We will now continue by taking a positive rather than normative view.

Instead of taking the returns $R_{it}$ as given and finding the optimal consumption, we instead take the consumption as given and wants to know what this says about the returns.

In order to do this, we need to use aggregate consumption. The same FOC will hold if we e.g. assume the existence of a representative consumer.

# The risk premium

If there exists a risk-free asset with gross return $R_{f,t}$, then

$$1 = E_t\left[m_{t+1} R_{f,t}\right] \quad \Leftrightarrow \quad 1 = R_{f,t} E_t\left[m_{t+1}\right] \quad \Leftrightarrow \quad R_{f,t} = \frac{1}{E_t\left[m_{t+1}\right]}.$$

Now write the pricing equation

$$1 = E_t\left[m_{t+1} R_{i,t+1}\right]$$

as

$$1 = \underbrace{E_t\left[m_{t+1}\right]}_{=1/R_{f,t}} E_t\left[R_{i,t+1}\right] + \mathrm{Cov}_t\left[m_{t+1}, R_{i,t+1}\right],$$

or

$$R_{f,t} = E_t\left[R_{i,t+1}\right] + R_{f,t}\mathrm{Cov}_t\left[m_{t+1}, R_{i,t+1}\right].$$

# The risk premium

This can be written

$$E_t\left[R_{i,t+1}\right]-R_{f,t}=R_{f,t}\mathrm{Cov}_t\Big[-m_{t+1},R_{i,t+1}\Big],$$

or

$$E_t\left[r_{i,t+1}\right]-r_{f,t}=R_{f,t}\mathrm{Cov}_t\Big[-m_{t+1},r_{i,t+1}\Big].$$

We recognise the left-hand side as the risk premium.

> *Explaining why different assets have different risk premia is the main goal of asset pricing theory.*

Kerry Back (2017), "Asset Pricing and Portfolio Choice Theory", 2nd Ed., *Oxford University Press*.

# Bounding the Sharpe ratio

Now,

$$\begin{aligned}
\left|E_t\left[r_{i,t+1}\right] - r_{f,t}\right| &= \left|R_{f,t}\mathrm{Cov}_t\left[-m_{t+1}, r_{i,t+1}\right]\right| \\
&= R_{f,t}\left|\mathrm{Std}_t[m_{t+1}]\mathrm{Std}[r_{i,t+1}]\mathrm{Corr}_t[m_{t+1}, r_{i,t+1}]\right| \\
&\leq R_{f,t}\mathrm{Std}_t[m_{t+1}]\mathrm{Std}[r_{i,t+1}].
\end{aligned}$$

This implies that

$$|\mathrm{SR}_{i,t}| \leq R_{f,t}\mathrm{Std}_t[m_{t+1}].$$

Hence, the standard deviation of the SDF bounds the Sharpe ratio.

# CRRA utility functions

When

$$u'(x) = x^{-\gamma},\ \gamma \geq 0,$$

we have the CRRA (Constant Relative Risk Aversion) class of utility functions.

For such a utility function

$$\begin{aligned} m_{t+1} &= \delta \frac{u'(c_{t+1})}{u'(c_t)} \\ &= \delta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \\ &= \delta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}. \end{aligned}$$

# CRRA utility functions

With

$$c_t^{\log} = \ln c_t \quad \Leftrightarrow \quad c_t = e^{c_t^{\log}}$$

we can write

$$\begin{aligned} m_{t+1} &= \delta \left( \frac{e^{c_{t+1}^{\log}}}{e^{c_t^{\log}}} \right)^{-\gamma} = \delta \left( e^{c_{t+1}^{\log} - c_t^{\log}} \right)^{-\gamma} \\ &= \delta e^{-\gamma (c_{t+1}^{\log} - c_t^{\log})} \\ &= \delta e^{-\gamma \Delta c_{t+1}^{\log}}. \end{aligned}$$

Here, $\Delta c_{t+1}^{\log}$ is the consumption growth.

# CRRA utility functions

When

$$m_{t+1} = \delta e^{-\gamma \Delta c_{t+1}^{\log}}$$

we get

$$E_t\left[r_{i,t+1}\right] - r_f = -R_{f,t}\delta \mathrm{Cov}_t\left[e^{-\gamma \Delta c_{t+1}^{\log}}, r_{i,t+1}\right].$$

Hence, the risk premium is dependent on the growth in consumption.

We now use the Taylor approximation

$$e^{-\gamma \Delta c_{t+1}^{\log}} \approx 1 - \gamma \Delta c_{t+1}^{\log}.$$

This implies

$$\begin{aligned} E_t\left[r_{i,t+1}\right] - r_f &\approx -R_{f,t}\delta \mathrm{Cov}_t\left[1 - \gamma \Delta c_{t+1}^{\log}, r_{i,t+1}\right] \\ &= R_{f,t}\delta\gamma \mathrm{Cov}_t\left[\Delta c_{t+1}^{\log}, r_{i,t+1}\right]. \end{aligned}$$

# The CRRA-lognormal model

**Assumptions**

- The utility function is CRRA:

$$u'(x) = x^{-\gamma}$$

  for some $\gamma \geq 0$.

- The growth rate in consumption is IID and normally distributed:

$$\Delta c_{t+1}^{\log} \sim N(g, \sigma_g).$$

# The CRRA-lognormal model

Recall the following result: If $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$, i.e. $X \sim N(\mu, \sigma^2)$, then

$$E\left[e^X\right] = e^{\mu + \frac{\sigma^2}{2}}.$$

It follows from this that

$$E\left[e^{aX}\right] = e^{a\mu + \frac{a^2\sigma^2}{2}}.$$

# The CRRA-lognormal model

The pricing equation for the risk-free return:

$$E_t\left[\delta e^{-\gamma \Delta c_{t+1}^{\log}} R_{f,t}\right] = 1.$$

$$\Leftrightarrow$$

$$\delta E_t\left[e^{-\gamma \Delta c_{t+1}^{\log}}\right] = \frac{1}{R_{f,t}}.$$

Under our assumptions (use the formula above with $a = -\gamma$):

$$\delta e^{-\gamma g + \frac{\gamma^2 \sigma_g^2}{2}} = \frac{1}{R_{f,t}},$$

or

$$1 + r_{f,t} = R_{f,t} = e^{-\ln \delta + \gamma g - \frac{\gamma^2 \sigma_g^2}{2}}.$$

Given the parameters of the model, this should be the risk-free rate.

# The CRRA-lognormal model

Now,

$$\begin{aligned}\mathrm{Cov}_t\left[\Delta c_{t+1}^{\log}, r_{i,t+1}\right] &= \mathrm{Corr}_t\left[\Delta c_{t+1}^{\log}, r_{i,t+1}\right] \cdot \mathrm{Std}_t\left[\Delta c_{t+1}^{\log}\right] \cdot \mathrm{Std}_t\left[r_{i,t+1}\right] \\ &= \rho_t \cdot \sigma_g \cdot \sigma_{it}.\end{aligned}$$

It follows that the risk premium is

$$\begin{aligned}E_t\left[r_{i,t+1}\right] - r_{f,t} &\approx R_{f,t}\delta\gamma\mathrm{Cov}_t\left[\Delta c_{t+1}^{\log}, r_{i,t+1}\right] \\ &= R_{f,t}\delta\gamma\rho_t\sigma_g\sigma_{it} \\ &\approx \left\{R_{f,t}\delta = e^{\gamma g - \frac{\gamma^2\sigma_g^2}{2}} \approx 1\right\} \\ &\approx \gamma\rho_t\sigma_g\sigma_{it},\end{aligned}$$

and the Sharpe ratio

$$\frac{E_t\left[r_{i,t+1}\right] - r_{f,t}}{\sigma_{it}} \approx \gamma\rho_t\sigma_g.$$

# The equity premium puzzle

Now let the theory meet data. Using

$$E_t\left[r_{i,t+1}\right] - r_f \approx \gamma \rho_t \sigma_g \sigma_{it},$$

we can estimate the market's risk premium $\gamma$.

This analysis was done by Mehra & Prescott in their paper "The Equity premium: A puzzle" from 1985.

This resulted in a very high value of $\gamma$.

Using reasonable values on the parameters, it can be shown that $\gamma$ is around 65, while a value of $\gamma$ between 2 and 5 seems more realistic (p. 295 in "Financial Asset Pricing Theory" by Claus Munk).

This is know as the equity premium puzzle.

How can this puzzle be resolved?

# Extended models

To explain among other things the equity premium puzzle, the following are examples of what have been suggested.

- Habit formation.
- Recursive utility.
- Heterogeneous preferences.
- Ambiguity aversion.

For a fuller list, see Cochrane's paper on macro-finance.

# Habit formation

There are two types of habit formation.

**Internal habit formation**
The level of habit $h_t$ is formed from previous consumption:

$$h_t = f(c_{t-1}, c_{t-2}, \ldots)$$

**External habit formation**
"Keeping up with the Joneses." Consumption is compared to an external habit benchmark $X_t$:

$$\text{Utility} = u(c_t - X_t).$$

# Recursive utility

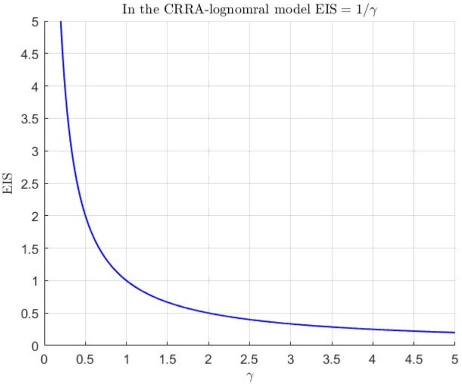
One problem with the standard model is that $\gamma$ both measures risk-aversion and intertemporal effects.

The elasticity of intertemporal substitution in consumption (EIS) is defined as

$$\text{EIS} = -\frac{d\ln(c_{t+1}/c_t)}{d\ln\left(u'(c_{t+1})/u'(c_t)\right)}.$$

With $u'(x) = x^{-\gamma}$:

$$\text{EIS} = \frac{1}{\gamma}.$$



# Recursive utility

One way to disentangle these two effects is to introduce recursive utility.

The general formula is

$$U_t = f(c_t, \mathrm{CE}_t(U_{t+1})),$$

where $f$ is a function and $\mathrm{CE}_t(U_{t+1})$ is the certainty equivalent at time $t$ of the utility at time $t+1$:

$$u\big(\mathrm{CE}_t(U_{t+1})\big) = E_t\left[u\big(U_{t+1}\big)\right]$$

for some utility function $u$.

One example is the Epstein-Zin recursive utility, which uses $u'(x) = x^{-\gamma}$.

# Heterogeneous preferences

Different investors have different risk aversion.

Groups of investors with low risk aversion will buy more stocks than groups with higher risk aversion.

This means that when the stock market goes down, the wealth of the less risk averse will decrease and the average (wealth weighted) risk aversion decreases.

# Ambiguity aversion

If the probabilities are not known, then we need to understand how the model behaves.

The case when probabilites are not known is referred to as Knightian uncertainty, or ambiguity.

There are several ways of handle this type of uncertainty.

One way is to use some type of max-min preferences.

# A general approach

It turns out that in the cases we have considered above, the SDF can be written

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)} \cdot Y_{t+1}$$

for a stochastic process $Y_t$.

# ESG investing

The recent years, there has been a hightend interest in ESG investing.

# ESG investing

Regarding environmental aspects, it is common to look at the differences between "Green" and "Brown" investments.

Firms and countries issue green bonds, which are bonds that is only allowed to finance green investments.

One potential problem is greenwashing.

# ESG investing

Not all investments are purely Green or Brown. Instead investors take ESG profiles into account when they choose their investments.

There are several ways of doings this:

- Active ownership in the firms invested in.
- Systematic and explicit inclusion of ESG in the investment analysis or in the portfolio construction.
- In the security selection, choose the firms with the best ESG profile.
- Include/exclude firms with good/bad ESG profile.

# ESG investing

Introduction of ESG rating in portfolio optimisation is an ongoing area of research.

This leads to "ESG portfolio frontiers".

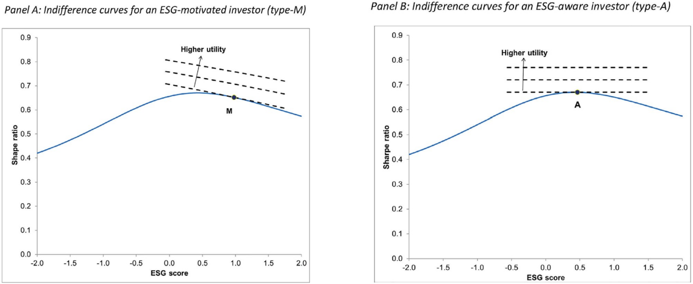



**Figure 13.8: The ESG-efficient frontier and investor indifference curves.** This is Figure 3 from Pedersen, Fitzgibbons, and Pomorski (2021).

# ESG investing

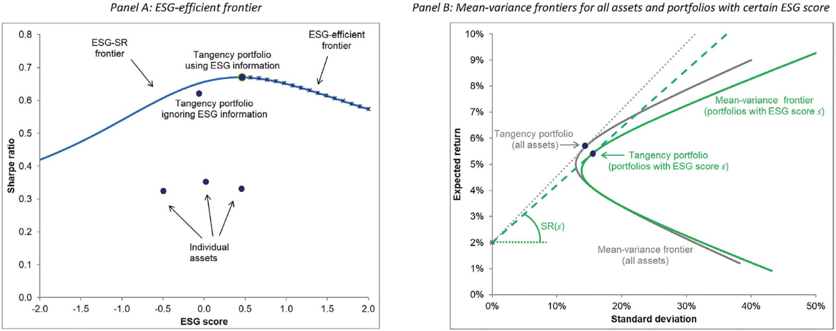



**Figure 13.9: The ESG-efficient frontier and mean-variance frontiers.**
This is Figure 1 from Pedersen, Fitzgibbons, and Pomorski (2021).

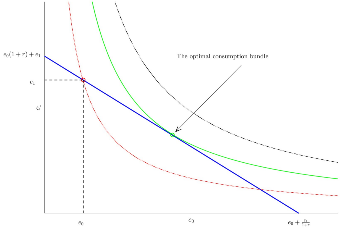
# Financial Theory – Lecture 16

Fredrik Armerin, Uppsala university, 2024

# Agenda

- Summary of the course.

**Choice under certainty**




**Financial markets**

- Primary markets and secondary markets.
- Big asset classes: Stocks, bonds and derivatives.
- Alternative asset classes: Commodities, real estate infrastructure.

We also looked at the major players in financial markets.

# Lecture 2

**Returns**

With prices $P_t$ and $P_{t+1}$, and dividend payments $D_{t+1}$:

- The rate of return

$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}.$$

- The gross rate

$$R_{t,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = 1 + r_{t,t+1}.$$

- The log-return

$$r_{t,t+1}^{\log} = \ln R_{t,t+1} = \ln(1 + r_{t,t+1}).$$

# Lecture 2

- Returns over multiple periods.
- Annualised returns.
- Ex dividend and cum dividend.
- Internal rate of return.
- Returns on short positions and excess returns.
- Real and nominal returns.
- Returns on leveraged positions.
- Returns of portfolios.

# Lecture 3

Basic probability theory and formulas.

Measuring risk using:

- Standard deviation (or, equivalently, variance).
- Value-at-Risk (VaR).
- Expected shortfall (ES).

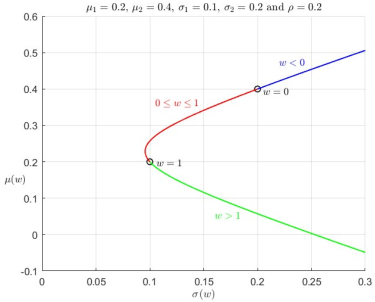
Measuring reward using the risk premium $E[r] - r_f$.

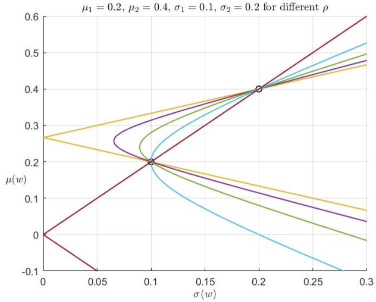
Measuring the risk-reward payoff by looking at the Sharpe ratio

$$\text{SR} = \frac{E[r] - r_f}{\text{Std}[r]}.$$

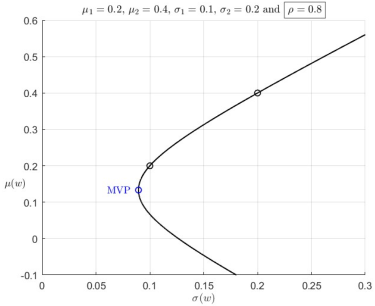
Vectors and matrices.

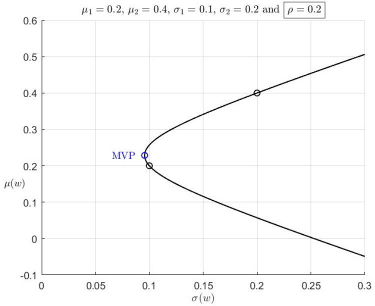
## Two-asset portfolios



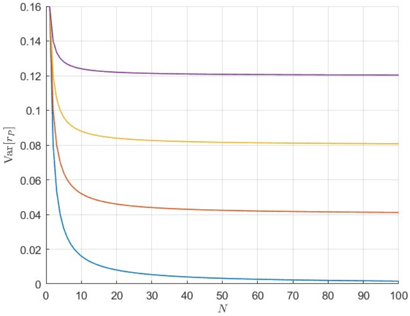



## The minimum variance portfolio



- Diversification.




- Arbitrage portfolios. "Getting a lottery ticker for free."
- Replicating portfolios.
- Tracking portfolios.

**Portfolio mathematics with vectors and matrices**

The portfolio weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ satisfies

$$1 = \sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1}.$$

Mean and variance of the portfolio rate of return:

$$E\left[r(\boldsymbol{\pi})\right] = \sum_{i=1}^{N} \pi_i \mu_i = \boldsymbol{\pi} \cdot \boldsymbol{\mu}.$$

$$\mathrm{Var}[r(\boldsymbol{\pi})] = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \mathrm{Cov}[r_i, r_j] = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij} = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

**Mean-variance analysis with only risky assets**

We found the portfolio that for a given expected rate of return $\bar{\mu}$ has the least variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \bar{\mu}. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \bar{\mu}. \end{aligned}$$

By using the Lagrange multiplier method we found that the optimal weights are

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D} \Sigma^{-1} \mathbf{1} + \frac{C\bar{\mu} - B}{D} \Sigma^{-1} \boldsymbol{\mu}$$

with

$$A = \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu}, \quad B = \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} = \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu}, \quad C = \mathbf{1} \cdot \Sigma^{-1} \mathbf{1}$$

and $D = AC - B^2$.

The mean-variance frontier, or the portfolio frontier, is given by

$$\sigma(\bar{\mu}) = \sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}.$$

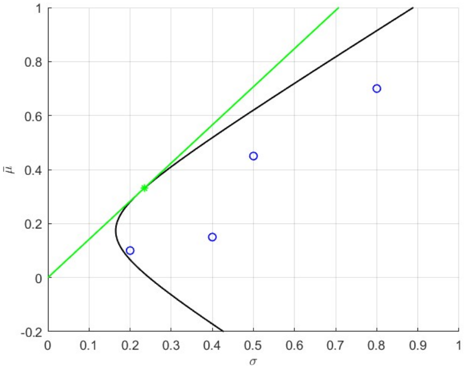
We have two-fund separation:

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D} C\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D} B\boldsymbol{\pi}_{\text{slope}},$$

where $\boldsymbol{\pi}_{\text{min}}$ solves

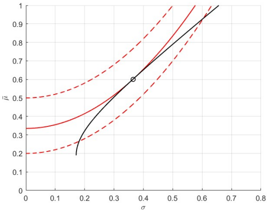
$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \text{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \sum\limits_{i=1}^{N} \pi_i = 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma\boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1. \end{array}$$

The portfolio weights $\boldsymbol{\pi}_{\text{slope}}$ are of the frontier portfolio with the largest slope of the line $\bar{\mu} = k\sigma$ in the $(\bar{\mu}, \sigma)$-plane for some $k$.



## Portfolio choice

- Choosing the optimal portfolio.




- Measuring the attitude towards risk using the absolute risk aversion,

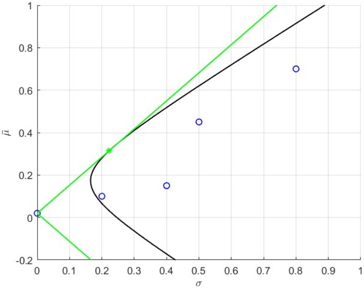
$$ARA(x) = -\frac{u''(x)}{u'(x)},$$

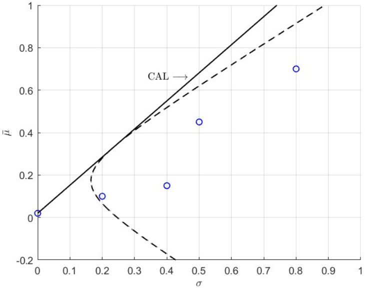
or the relative risk aversion,

$$RRA(x) = -\frac{xu''(x)}{u'(x)}.$$

# Lecture 6

**Mean-variance analysis with risky assets and a risk-free asset**
In this case we get the mean-variance frontier and the capital allocation line (CAL).

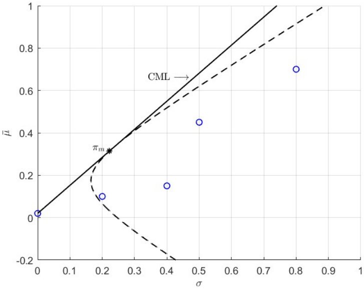


When the market is in equilibrium, the market portfolio is equal to the tangent portfolio:

$$\boldsymbol{\pi}_{\mathsf{mkt}} = \boldsymbol{\pi}_{\mathsf{tan}}.$$

This results in the capital market line.



**The Capital Asset Pricing Model (CAPM)**

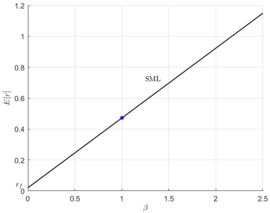
We can also derive the CAPM equation

$$E[r_i] = r_f + \beta_i \big( E[r_M] - r_f \big),$$

where the beta value is

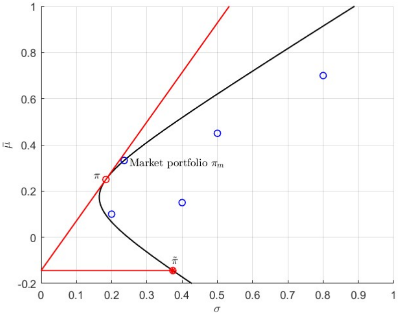
$$\beta_i = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}.$$

The security market line (SML).



# Lecture 7

The CAPM without a risk-free asset – use the market portfolio's "zero-beta portfolio".




We also looked at optimal portfolios with CARA utility and normally distributed returns.

**Consumption-based CAPM**

The problem

$$\max_{\mathbf{x}} \left\{ u(c_0) + e^{-\delta} E\left[u(c_1)\right] \right)$$

$$\text{s.t. } c_0 + \sum_{i=1}^{N} x_i P_{i0} = e_0$$

$$c_1 = e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1})$$

leads to the consumption-based CAPM equation

$$E\left[r_i\right] = r_f + \text{Cov}[r_i, -(1 + r_f)m],$$

where

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)}$$

is the stochastic discount factor (SDF).

**Factor models**

Models with one factor:

$$r_i = E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i,\ i = 1, 2, \ldots, N,$$

and where for every $i, j = 1, 2, \ldots, N$

(i) $\mathrm{Cov}[F, \varepsilon_i] = 0$.

(ii) $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0$ when $i \neq j$.

One special case is when $F = r_m$. This is the Single-Index Model or the Market Model.

Models with $K$ factors: For $i = 1, 2, \ldots, N$:

$$r_i = E\left[r_i\right] + \beta_{i1}\big(F_1 - E\left[F_1\right]\big) + \ldots + \beta_{iK}\big(F_K - E\left[F_K\right]\big) + \varepsilon_i,$$

where for every $i$ and $k$

$$\mathrm{Cov}[F_k, \varepsilon_i] = 0 \;\; \text{and} \;\; \mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0 \text{ when } i \neq j.$$

**Arbitrage pricing theory (APT)**
If there are no arbitrage opportunities, then there exists factor risk premia $\mathrm{RP}_k$ such that

$$E[r_i] = \mathrm{RP}_0 + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

If there is a risk-free rate, then $\mathrm{RP}_0 = r_f$.

- The Fama-French model.
- The Fama-French-Carhart model.
- The Factor zoo.

# Lecture 9

Looking at data:

- Bid and ask price.
- Closing, opening, high and low.
- How to estimate the risk premia.
- Why it is harder to correctly estimate the mean than the standard deviation of asset returns.

**Efficiency**

- Weak-form.
- Semistrong-form.
- Strong-form.

Markets are often "efficiently inefficient".

# Lecture 9

**Behavioural finance and economics**

- Prospect theory.
- Behavioural biases:
  - Anchoring.
  - Loss aversion or Disposition effect.
  - Overconfidence.
  - Mental accounting.
  - Framing.
- Behavioural game theory.

**Stock valuation**

The price ex post

$$P_t = \frac{P_{t+1} + D_{t+1}}{1 + r_{t+1}},$$

and the price ex ante

$$P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1 + r_{t+1}}\right].$$

If $E_t[D_{t+1}] = (1+g)D_t$ and $E_t[r_{t+1}] = r$, then (under the assumption of no rational bubble)

$$P_t = \frac{E_t[D_{t+1}]}{r-g} = \frac{(1+g)D_t}{r-g}$$

as long as $r > g$. This is Gordon's formula.

### A firm's dividend policy

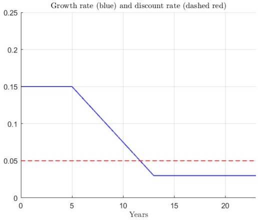
- The firm's earnings $e_t$ are invested or paid out as dividends: $e_t = I_t + D_t$.
- The return on equity, or return on investments, $r_e$: $e_{t+1} = e_t + r_e I_t$.
- The plowback ratio $b$: $I_t = be_t$.
- Growth rate in dividends: $g = br_e$.
- Price-earnings (P/E) ratio:

$$\frac{P_t}{e_t} = \frac{(1 + br_e)(1 - b)}{r - br_e}.$$

- Present value of growth opportunities (PVGO) $O_t$: $O_t = P_t - e_t/r$.

## More on stock valuation

- Two- and three period growth models.




- Valuation using free cash flows:

$$V_t = \sum_{j=1}^{\infty} \frac{E_t\left[\mathrm{FCF}_{t+j}\right]}{\left(1 + r_{\mathrm{firm}}\right)^j},$$

where $r_{\mathrm{firm}}$ is the weighted average cost of capital (WACC).

**Bonds**

Tradable loan contracts.

$$\begin{aligned} M_i &= \text{The total payment at time } i. \\ I_i &= \text{The interest payment at time } i. \\ X_i &= \text{The repayment of debt at time } i. \\ F_i &= \text{The outstanding debt at time } i \\ & \quad \text{after the repayment of debt has been made.} \end{aligned}$$

Then for $i = 1, 2, \ldots, n$

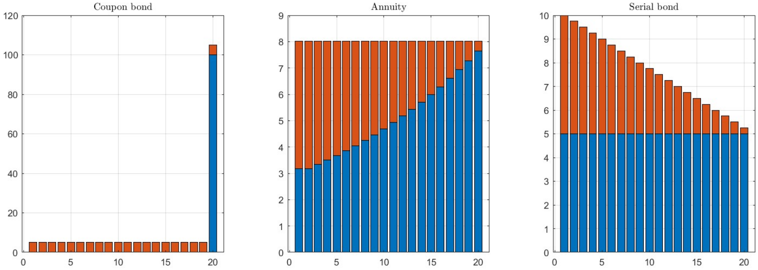
$$\begin{aligned} M_i &= I_i + X_i \\ I_i &= qF_{i-1} \\ F_i &= F_{i-1} - X_i. \end{aligned}$$

It also holds that

$$F_0 = F = \text{The face value, and } F_n = 0.$$

# Lecture 11

- Coupon bonds.
- Annuities.
- Serial bonds.



# Lecture 11

Coupon bonds price

$$\begin{aligned} B_t &= \sum_{i=t+1}^{n} \frac{qF}{(1+r)^{i-t}} + \frac{F}{(1+r)^{n-t}} \\ &= \frac{qF}{r}\left(1 - \frac{1}{(1+r)^{n-t}}\right) + \frac{F}{(1+r)^{n-t}}. \end{aligned}$$

Special case: Zero coupon bonds (ZCB's) with price

$$Z_{t,n} = \frac{F}{(1+r)^{n-t}} = F(1+r)^{-(n-t)}.$$

- Accrued interest, clean price and dirty price.
- Trading at par, premium or discount.

# Lecture 11

**Yield-to-maturity (YTM) or just yield**
The internal rate of return (IRR) of a bond:

$$B_0^{\text{mkt}} = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i}.$$

$$y \uparrow \ \Leftrightarrow \ B_0(y) \downarrow \quad \text{and} \quad y \downarrow \ \Leftrightarrow \ B_0(y) \uparrow .$$

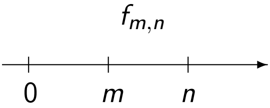
For ZCB's:

$$Z_{0,n}^{\text{mkt}} = \frac{F}{(1+y_n)^n} \ \Leftrightarrow \ y_n = \left( \frac{F}{Z_{0,n}^{\text{mkt}}} \right)^{1/n} - 1.$$

The difference between the yield and the actual return of a bond.

# Lecture 11

**Forward rates**




The forward rate for the period from $m$ to $n$, denoted $f_{m,n}$, satisfies

$$(1+y_n)^n = (1+y_m)^m(1+f_{m,n})^{n-m},$$

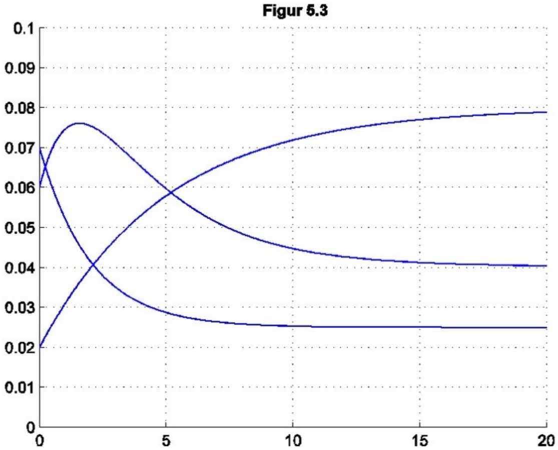
or

$$f_{m,n} = \left(\frac{(1+y_n)^n}{(1+y_m)^m}\right)^{1/(n-m)} - 1.$$

**Defaultble bonds**

- Bonds for which the issuer of the bond can not (or will not) make coupon and/or amortisation payments.
- Bonds issued by stable states are generally considered non-defaultable, i.e. there is no risk of their bonds to default.
- There are rating institutes rating the quality of a firm based on the probability

## Yield curves




Explanations for the form of the yield curve (new and old).

- Use yield curves from ZCB's to discount.
- Bootstrapping.
- The duration of a bond:

$$D_0 = \sum_{i=1}^{n} i \cdot \frac{\frac{M_i}{(1+y)^i}}{B_0} = \sum_{t=1}^{n} i\, w_i \quad \text{and} \quad \frac{\Delta B_0}{B_0} \approx -\frac{D_0}{1+y}\Delta y.$$

- The modified duration:

$$D_0^* = \frac{1}{1+y} D_0.$$

- Convexity and modified convexity.
- Fisher-Weil duration.
- Equity duration.

**Immunisation**

Buy a portfolio whose value and duration matches that of the liability we have.

With two bonds having prices $B_1$ and $B_2$, and durations $D_1$ and $D_2$ this leads to the system of equations

$$\begin{cases} N_1 B_1 + N_2 B_2 & = & \bar{B} \\ N_1 B_1 D_1 + N_2 B_2 D_2 & = & \bar{B}\bar{D} \end{cases}$$

for $N_1$ and $N_2$ (the number of bonds 1 and 2 respectively to buy).

# Lecture 13

**Asset allocation**

- The investment process.
- The Treynor-Black model: Taking advantage of mispriced assets.
- The Black-Litterman model: Incorporating your personal views and using CAPM to estimate the mean vector of rates of return.

**Household finance**
Using the portfolio investment framework to also include other sources of cash flows.

- Labour income: Total wealth = Financial wealth + Human capital, where
$$\text{Human capital} = \text{PV(Future labour income)}.$$
- Housing: As an additional source of investment we can consider housing. Natural constraints are that households do not shortsell financial assets and they have mortgage limitations.

# Lecture 14

**Empirical aspects**
Mainly given you an orientation about what types of econometric questions you might want to ask, and what techniques that are available.

# Lecture 15

**Macro-finance**

The standard model:

- Additive and time-separable individuals.
- CRRA utility funtion: $u'(x) = x^{-\gamma}$, $\gamma \geq 0$.
- Growth rate in consumption is normally distributed.

Extensions:

- Habit formation: Internal and external.
- Recursive utility: Epstein-Zin.
- Heterogeneous preferences: Different investors have different risk aversion.
- Ambiguity aversion: Uncertainty about which are the correct probabilites to use.

We also looked at ESG investing.