Lecture 5:

Integration

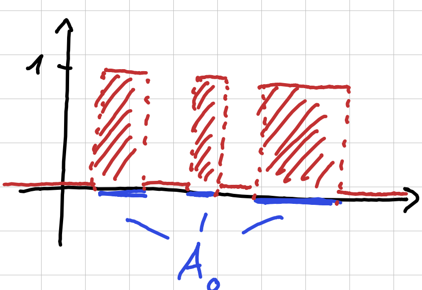
Let (S, Σ, μ) be a measure space and let f be (Σ) measurable.

We define $\int f \, d\mu$ in three steps:

1) We define $\int$ for indicator functions.

Let $I_{A_0}(s) = \begin{cases} 1 & s \in A_0 \\ 0 & s \notin A_0 \end{cases}$ for $A_0 \in \Sigma$.

We set $\int I_{A_0} \, d\mu = \mu(A_0)$



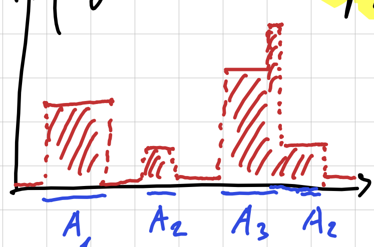
2) We define $\int$ for finite linear combinations of indicator functions:

$g(s) = \sum_{k=1}^{n} a_k I_{A_k}(s)$, $a_k \in \mathbb{R}_0^+$, $A_k \in \Sigma$; by

$$\int g \, d\mu = \sum_{k=1}^{n} a_k \int I_{A_k} \, d\mu$$

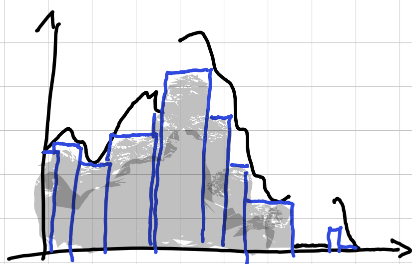
$$= \sum_{k=1}^{n} a_k \, \mu(A_k)$$

(g is called a step function)



3) For arbitrary measurable and non-negative $f \in m\Sigma^+$ we define

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \text{ is a non-negative step function}, \ g(s) \leq f(s) \right\}$$



Note: If f is already a step function, 3) and 2) coincide.

We may also write $\mu(f)$ for $\int f \, d\mu$.

3*) We extend 3) to all measurable functions $f \in m\Sigma$ by defining

$$f^+(s) = \begin{cases} f(s) & \text{if } f(s) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad f^-(s) = \begin{cases} -f(s) & \text{if } f(s) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Both f^+, f^- are non-negative and $f = f^+ - f^-$.

We define $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$

Properties of $\int \cdot \, d\mu$:

1) Linearity: $\int a \cdot f + b \cdot g \, d\mu = a \cdot \int f \, d\mu + b \cdot \int g \, d\mu$

for $a, b \in \mathbb{R}$

2) Monotonicity:

If $f \leq g$ for all $s \in S$ (or even for almost all w.r.t. μ), then $\int f \, d\mu \leq \int g \, d\mu$

3) "Triangle Inequality":

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| = \int |f| \, d\mu$$

Example: If μ is the Lebesgue measure on $\mathbb{R}$ then this is the familiar "Lebesgue Integral". If both Lebesgue and Riemann integral exist for f, then they agree.

Example: Let μ be the counting measure on the positive integers.

Then $\int f \, d\mu = \sum_{n=1}^{\infty} f(n)$.

We will later restrict ourselves to probability spaces and re-interpret $\int f \, d\mu$ as the expectation of f w.r.t. μ.

We can also restrict the domain:

$$\int_A f \, d\mu = \int f \cdot I_A \, d\mu, \quad A \in \Sigma.$$

indicator of A

We say that f is (μ) integrable if $\int f^+ d\mu$ and $\int f^- d\mu$ are finite.

If this is not the case the integral is undefined. (We could e.g. have "$\infty - \infty$" in the definition)

We write $\mathcal{L}^1(S, \Sigma, \mu)$ for the space of integrable functions, i.e. all $f \in \mathcal{L}^1(S, \Sigma, \mu)$ are integrable.

Note that if $f(s) = \pm\infty$ for some $s \in S$, then f can only be integrable if $\mu(\{s : f(s) = \pm\infty\}) = 0$.

<u>Lemma</u> If f is a non-negative measurable function with $\int f \, d\mu = 0$, then $\mu(\{f > 0\}) = 0$, i.e. $f(s) = 0$ μ-almost-everywhere.

Proof: Note that $\{s : f(s) > 0\}$ $= \bigcup_{n\in\mathbb{N}} \{s : f(s) > \frac{1}{n}\}$. So, either

$\mu(\{s : f(s) > \frac{1}{n}\}) = 0$ for all n, which gives

$\mu(\{f > 0\}) = 0$, or $\mu(\{f > \frac{1}{n}\}) > 0$ some n. Let $A = \{f > \frac{1}{n}\}$. Then, $f(s) \geq \frac{1}{n} I_A(s)$ and by monotonicity

$$\int f \, d\mu \geq \int \frac{1}{n} I_A \, d\mu = \frac{1}{n}\mu(A) > 0,$$

a contradiction. Hence our claim follows. $\square$

Question: When is it true that

$$\int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu \quad ?$$

Not always: Let $f_n = I_{[n, n+1)}$.
Then $\int_{\mathbb{R}} f_n \, dx = 1$ for all $n \in \mathbb{N}$

but $f_n(x) \to 0$ for all x. So $\int \lim f_n(x) \, dx = 0$.

There are circum-stances that allow interchanging limit & integral:

<u>Monotone Convergence Theorem</u>

Let f_n be a sequence of non-negative measurable functions s.t. $f_n \uparrow f$, i.e. $f_n(x)$ is non-decreasing in n and $\lim_n f_n(x) = f(x)$ for all x. Then, $\mu(f_n) \uparrow \mu(f)$, i.e.

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu .$$

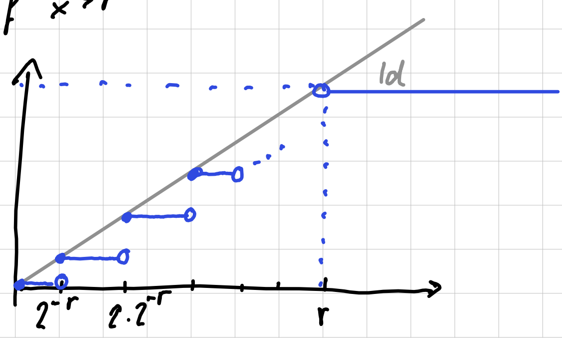
We can always approximate measurable functions by monotone sequences of step functions. Set

$$\alpha^{(r)}(x) = \begin{cases} 0 & \text{if } x = 0 \\ (i-1)2^{-r} & \text{if } (i-1)2^{-r} \le x \le i\,2^{-r} \le r \\ r & \text{if } x > r \end{cases}$$

Note that $\alpha^{(r)}(x) \le x$,

$\lim_{r\to\infty} \alpha^{(r)}(x) = x$ and

$\alpha^{r}(x)$ nondecr. in r.



Setting $f^{(r)}(x) = \alpha^{(r)}(f(x))$, $f \in m\Sigma^+$;
we get a function which:

- $f^{(r)}$ is a step function
- $f^{(r)} \uparrow f$.

This gives a general proof strategy:

→ prove something for indicators
→ extend to step functions by linearity
→ extend to arbitrary $f \in m\Sigma^+$ by approx with step functions and monotonicity
→ extend to $f \in m\Sigma$ by splitting into positive and negative part.

Lemma: Suppose f, g are integrable and $f = g$ almost everywhere. Then, $\int f \, d\mu = \int g \, d\mu$.

Proof (Sketch): Follow the above recipe: Consider $\int f - g \, d\mu$. This is $= 0$ trivially for indicator functions. Following steps above shows $\int f - g \, d\mu = 0$ generally. $\square$

Corollary If $f_n \uparrow f$ almost everywhere, then $\mu(f_n) \uparrow \mu(f)$ is still true.

Fatou's Lemma: Suppose f_n is a sequence of non-neg. measurable functions. Then,

$$\mu\left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} \mu(f_n)$$

Proof: Consider the sequence $g_k = \inf_{n \geq k} f_n$. Then, $\lim_{k\to\infty} g_k = \liminf_{k\to\infty} f_k$.

Since g_k is monotone ($g_k \uparrow \liminf f_k$) we can apply monotone convergence and $\mu\left(\liminf_{k\to\infty} f_k\right) = \lim_k \mu(g_k)$. But

$g_k \leq f_n$ for all $n \geq k$, so $\mu(g_k) \leq \mu(f_n)$ for all $n \geq k$. In particular, $\mu(g_k) \leq \inf_{n\geq k} \mu(f_n)$. Hence $\mu\left(\liminf_{k\to\infty} f_k\right) = \lim \mu(g_k) \leq \lim_k \inf_{n \geq k} \mu(f_n)$

$= \liminf_{n\to\infty} \mu(f_n)$ as required. $\square$

Corollary (Reversed Fatou Lemma)

Suppose $f_n \le g$ for some nonnegative integrable functions. Then,

$$\mu\left(\limsup_{n\to\infty} f_n\right) \ge \limsup_{n\to\infty} \mu(f_n).$$

Proof: Follows from Fatou's lemma applied to $g - f_n$. $\square$

Dominated Convergence Theorem

Let f_n be a seq of measurable functions and assume $|f_n| \le g$ for some integrable g. If $f_n \to f$ pointwise, then

- $\mu(|f_n - f|) = \int |f_n - f| \, d\mu \to 0$
- $\mu(f_n) = \int f_n \, d\mu \to \int f \, d\mu = \mu(f)$.

Proof: $|f_n - f| \le |f_n| + |f| \le 2g$.

By reverse Fatou lemma, we have

$$\limsup_{n\to\infty} \mu(|f_n - f|) \le \mu\left(\limsup |f_n - f|\right) = 0$$

$$\Rightarrow \quad 0 \leq \liminf \mu(|f_n - f|) \leq \limsup \mu(|f_n - f|) = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \mu(|f_n - f|) = 0.$$

It follows that $|\mu(f_n) - \mu(f)| = |\mu(f_n - f)|$

$\leq \mu(|f_n - f|) \to 0$. So $\mu(f_n) \to \mu(f)$. $\square$

↑ triangle inequality

<u>Scheffé's Lemma</u>:

Suppose f_n, f are non-negative functions s.t. $f_n \to f$ (almost) everywhere. Then,

$$\mu(f_n) \to \mu(f) \text{ if and only if } \mu(|f_n - f|) = 0.$$