UPPSALA UNIVERSITET
Matematiska institutionen — Financial Derivatives
Erik Ekström — 2021-08-18

Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points.
You may use the textbook and notes from the course. No communication with other people is allowed. Your solutions should be uploaded in Studium by 1.20pm as a single pdf-file. If you have any questions during the exam, please contact Erik Ekström (tel 0739986081).
Motivate your answers carefully!

1. Let $X(t)$ be the solution of the stochastic differential equation

$$\begin{cases} dX(t) = (1 - \frac{X(t)}{2})\,dt + X(t)\,dW(t) \\ X(0) = x, \end{cases}$$

where $x > 0$. Determine $\mathbb{E}[X(1)]$ and $\mathbb{E}[X^2(1)]$.

2. Use the Feynman-Kac theorem to solve the terminal-value problem

$$2\frac{\partial u}{\partial t}(t,x) + 8x^2\frac{\partial^2 u}{\partial x^2}(t,x) + x\frac{\partial u}{\partial x}(t,x) = u(t,x)$$
$$u(T,x) = \frac{x^{1/2} + x^{-1/2}}{2}.$$

in the strip $[0,T] \times (0,\infty)$.

3. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time 0 of a contract which at time T pays the holder the amount

$$\mathcal{X} = \begin{cases} S(T) - K & \text{if } S(T) \geq K \\ K & \text{if } S(T) < K. \end{cases}$$

4. Answer the following short questions. Provide a short motivation in each case.

(i) If W_1 and W_2 are Brownian motions with instantaneous correlation ρ (i.e. $dW_1(t)dW_2(t) = \rho\,dt$), can one find a constant $b > 0$ such that the process $W(t) := b(2W_1(t) + 3W_2(t))$ is a Brownian motion?

(ii) Let X be a geometric Brownian motion with drift 0 and volatility σ. For each n, determine a constant c (depending on n) so that $e^{-ct}X^n(t)$ is a martingale.

(iii) Is it true that the arbitrage-free price P of a put option with strike K on a stock with price s has to satisfy $P \geq K - s$? (The stock does not pay dividends, and the short rate is non-negative.)

5. In a market consisting of a bank account with a constant interest rate r and a stock S which pays a proportional discrete dividend $\delta S(T_0-)$ at T_0, consider a T-claim that pays

$$\mathcal{X} = S(T_1)$$

at time T, where $0 < T_0 < T_1 < T$.

a) Find a replicating strategy for $\mathcal{X}$.
b) What is the arbitrage-free price of $\mathcal{X}$ at time 0?

6. Consider a European call option and a European put option written on the same underlying stock, which pays no dividends. Both options mature six months from now and have strike price 200. Moreover, assume that a six-month zero-coupon bond with face value 50 trades at 49. Your broker quotes the prices 12 and 10 for the call and the put, respectively. Show how to construct a model-independent arbitrage if the current stock price is 200.

7. Consider a model

$$dr(t) = \left(\sigma^2 - b\sqrt{r(t)}\right)\,dt + 2\sigma\sqrt{r(t)}\,dW(t)$$

for the short rate under the pricing measure, where b and σ are constants. Show that bond prices $p(t,T)$ at time t with maturity T can be determined on the form

$$p(t,T) = \exp\{A(t,T)r(t) + B(t,T)\sqrt{r(t)} - C(t,T)\}$$

for some deterministic functions $A(t,T)$, $B(t,T)$ and $C(t,T)$.

Comment: You do not need to determine the functions A, B and C, but it suffices that you show how to find them.

8. Consider a currency derivative that gives the right (but not the obligation) to buy 100 USD at a given price K SEK at a given future date T. Assume that the US interest rate r_f, the Swedish (domestic) interest rate r_d and the volatility σ of the exchange rate are positive constants. Which of the following statements are correct? Motivate your answers.

The value of the currency derivative is

(i) increasing in r_d;
(ii) decreasing in r_f;
(iii) neither increasing nor decreasing in σ.

GOOD LUCK!