Exam 12 January 2023

① (a) See Lecture 2 and Lecture 4

(b) Note that for every n, we have

$$P(X_n = Y_n) = \sum_{i=1}^{n} P(X_n = i \text{ and } Y_n = i)$$
$$= \sum_{i=1}^{n} P(X_n = i) P(Y_n = i)$$
$$= n \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n}$$

and similarly

$$P(X_n = Y_n = Z_n) = \sum_{i=1}^{n} P(X_n = i) P(Y_n = i) P(Z_n = i)$$
$$= n \cdot \left(\frac{1}{n}\right)^3 = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, but $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, it follows from the Borel-Cantelli lemmas that

$P(X_n = Y_n \text{ infinitely often}) = 1$, but

$P(X_n = Y_n = Z_n \text{ infinitely often}) = 0$

② Note that

$$P(X_i = X_{i+1} = \dots = X_{i+n-1}) = P(X_i = X_{i+1} = \dots = X_{i+n-1} = 1) + P(X_i = X_{i+1} = \dots = X_{i+n-1} = -1)$$
$$= \left(\tfrac{1}{2}\right)^n + \left(\tfrac{1}{2}\right)^n = 2^{1-n}$$

So if Z_N is the number of times that we have n consecutive identical values among the first N, we have

$$\mathbb{E}(Z_N) = \mathbb{E}\left(\sum_{i=1}^{N-n+1} I_{\{X_i = X_{i+1} = \dots = X_{i+n-1}\}}\right)$$
$$= \sum_{i=1}^{N-n+1} P(X_i = X_{i+1} = \dots = X_{i+n-1})$$
$$= (N-n+1) 2^{1-n}$$

If $N = \lfloor c^n \rfloor$, where $c < 2$, then

$$\mathbb{E}(Z_N) = (\lfloor c^n \rfloor - n + 1) 2^{1-n} \leq c^n \cdot 2^{1-n} = 2\left(\tfrac{c}{2}\right)^n \to 0 \quad \text{as } n \to \infty$$

Since $P(Z_N > 0) = P(Z_N \geq 1) \leq E(Z_N)$ by the Markov inequality, the first part follows.

For the second part, consider the $\lfloor \frac{N}{n} \rfloor$ disjoint n-tuples

$X_1, X_2, \dots, X_n$

$X_{n+1}, X_{n+2}, \dots, X_{2n}$

$X_{\lfloor \frac{N}{n} \rfloor n-n+1}, X_{\lfloor \frac{N}{n} \rfloor n-n+2}, \dots, X_{\lfloor \frac{N}{n} \rfloor n}$

Since they are disjoint, the events $X_1 = X_2 = \dots = X_n$, $X_{n+1} = X_{n+2} = \dots = X_{2n}$ etc are independent. It follows that

$$
\begin{aligned}
P(Z_N = 0) &\leq P(\{X_1, X_2, \dots, X_n \text{ not all equal}\} \cap \{X_{n+1}, X_{n+2}, \dots, X_{2n} \text{ not all equal}\} \cap \dots) \\
&= \prod_{i=1}^{\lfloor \frac{N}{n} \rfloor} \left(1 - P(X_{(i-1)n+1} = X_{(i-1)n+2} = \dots = X_{in})\right) \\
&= \left(1 - 2^{1-n}\right)^{\lfloor \frac{N}{n} \rfloor} \\
&= \left(1 - 2^{1-n}\right)^{2^{n-1} \left(\lfloor \frac{N}{n} \rfloor \cdot 2^{1-n}\right)}
\end{aligned}
$$

Since $\lfloor \frac{N}{n} \rfloor \cdot 2^{1-n} = \lfloor \frac{c^n}{n} \rfloor \cdot 2^{1-n} \to \infty$ for $c > 2$ and $\lim_{n \to \infty} \left(1 - 2^{1-n}\right)^{2^{n-1}} = \frac{1}{e}$, it follows that $P(Z_N = 0) \to 0$, which concludes the second part.

③ See Lecture 4

④ (a) A stopping time with respect to a filtration $\mathcal{F}_n$ is a random variable T with values in $\{0, 1, \dots, \infty\}$ such that

$$\{T = n\} \in \mathcal{F}_n$$

τ_1 is a stopping time:

$$\{\tau_1 = n\} = \{S_n = 10\} \cap \{S_1 \neq 10\} \cap \{S_2 \neq 10\} \dots \cap \{S_{n-1} \neq 10\}$$

can be determined from $S_1, S_2, \dots, S_n$ and is thus in $\mathcal{F}_n$.

τ_2 is not a stopping time:

$$\{\tau_2 = n\} = \{S_n = 10\} \cap \{S_{n+1} \neq 10\} \cap \{S_{n+2} \neq 10\} \dots$$

cannot be determined from $S_1, S_2, \dots, S_n$ alone.

τ_3 is a stopping time:

$$\sup\{n : S_n = n\} = \sup\{n : X_1 = X_2 = \dots = X_n = 1\}$$

$$\Rightarrow \{\tau_3 = n\} = \{X_1 = X_2 = \dots = X_{n-1} = 1\} \cap \{X_n = -1\} \in \mathcal{F}_n$$

(b) See Lecture 11

(c)
$$\begin{aligned}
\mathbb{E}(S_n^2 - n \mid \mathcal{F}_{n-1}) &= \mathbb{E}\left(\left((S_{n-1}+X_n)^2 - n\right) \mid \mathcal{F}_{n-1}\right) \\
&= \mathbb{E}(S_{n-1}^2 + 2X_n S_{n-1} + X_n^2 - n \mid \mathcal{F}_{n-1}) \\
&= S_{n-1}^2 + 2S_{n-1}\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) + \mathbb{E}(\underbrace{X_n^2}_{\text{always} = 1} - n \mid \mathcal{F}_{n-1}) \\
&= S_{n-1}^2 + 2S_{n-1}\underbrace{\mathbb{E}(X_n)}_{=0} + 1 - n \\
&= S_{n-1}^2 - (n-1)
\end{aligned}$$

showing that $S_n^2 - n$ is a martingale.

By the optional stopping theorem, we have

$$\mathbb{E}(\underbrace{S_\tau^2}_{=\,100 \text{ by definition}} - \tau) = \mathbb{E}(S_0 - 0) = 0$$

$$\Rightarrow 100 - \mathbb{E}(\tau) = 0 \Rightarrow \mathbb{E}(\tau) = 100$$

5 (a) We can express X_n as $X_n = Y_n \cdot X_{n-1}$, where Y_n is uniform on $[1,2]$.

Note that

$$\begin{aligned}
\mathbb{E}(c^n X_n \mid \mathcal{F}_{n-1}) &= c^n \,\mathbb{E}(Y_n X_{n-1} \mid \mathcal{F}_{n-1}) \\
&= c^n \,\mathbb{E}(Y_n) X_{n-1} \\
&= c \cdot \frac{3}{2} \cdot c^{n-1} X_{n-1}
\end{aligned}$$

So $c^n X_n$ is a martingale if $c = \frac{2}{3}$, a supermartingale if $c < \frac{2}{3}$ and a submartingale if $c > \frac{2}{3}$.

(b) We have

$$\begin{aligned}
\ln(c^n X_n) &= n \ln c + \ln(Y_1 + \dots + Y_n) \\
&= \ln(cY_1) + \ln(cY_2) + \dots + \ln(cY_n)
\end{aligned}$$

Now
$$\begin{aligned}
\mathbb{E}(\ln(cY_i)) &= \int_1^2 \ln(cx)\,dx = \ln c + x(\ln x - 1)\Big|_1^2 \\
&= \ln c + 2\ln 2 - 1 = \ln\left(\frac{4c}{e}\right)
\end{aligned}$$

If $c > \frac{e}{4}$, it follows from the strong law of large numbers that

$$\frac{\ln(c^n X_n)}{n} \to \ln\left(\frac{4c}{e}\right) > 0 \quad \Rightarrow \ln(c^n X_n) \to \infty$$

and thus $c^n X_n \to \infty$. Likewise, for $c < \frac{e}{4}$, we have

$$\frac{\ln(c^n X_n)}{n} \to \ln\left(\frac{4c}{e}\right) < 0 \quad \Rightarrow \ln(c^n X_n) \to -\infty$$

and thus $c^n X_n \to 0$. This proves the statement with $c_0 = \frac{e}{4}$.

⑥ See Lecture 12

⑦ (a), (b): See Lectures 19 and 20

(c) A martingale measure is determined by the probabilities of the different transitions:

$$\begin{tikzcd}[column sep=small]
 & & S_2 = 8 \\
 & S_1 = 9 \arrow[ur, "q_2"] \arrow[r, "1-q_2"'] & S_2 = 10 \\
S_0 = 10 \arrow[ur, "q_1"] \arrow[dr, "1-q_1"'] & & \\
 & S_1 = 12 \arrow[r, "q_3"] \arrow[dr, "1-q_3"'] & S_2 = 10 \\
 & & S_2 = 15
\end{tikzcd}$$

For $\bar{S}_t = S_t$ to be a martingale, we need

- $9q_1 + 12(1-q_1) = 10 \Leftrightarrow 2 = 3q_1 \Leftrightarrow q_1 = \frac{2}{3}$
- $8q_2 + 10(1-q_2) = 9 \Leftrightarrow 1 = 2q_2 \Leftrightarrow q_2 = \frac{1}{2}$
- $10q_3 + 15(1-q_3) = 12 \Leftrightarrow 3 = 5q_3 \Leftrightarrow q_3 = \frac{3}{5}$

Since there is a martingale measure, the model is viable. Since it is also unique, the model is complete.

⑧ See Lecture 18