1. Vectors and shapes

We solve the problems together in the exercise sessions. Note that these problems are optional and for learning purposes: solving these does not provide extra points. Actual home assignments (giving you extra points) are given separately.

It is advised to take a look of the problems beforehand. Note that some of the problems might be very challenging, so do not feel bad if you are unable to solve them independently: we will go through the solutions together!

Problems

1.1 Describe (e.g. by drawing) the following sets in $\mathbb{R}^2$:

- $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$,
- $\{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\}$,
- $\{(x,y) \in \mathbb{R}^2 : \max(|x|,|y|) \leq 1\}$,
- $\{(x,y) \in \mathbb{R}^2 : |xy| < \frac{1}{4}\}$,
- $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2x - 4y < 11\}$,
- $\{(x,y) \in \mathbb{R}^2 : |x + 2y| \leq 2\}$,
- $\{(x,y) \in \mathbb{R}^2 : y > x^2, 0 < x < 1, y \leq 1\}$,
- $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2 \leq -4 - 4x - 4y - x^2 - y^2\}$.

1.2 (continuation) Characterise the inner points of the sets in the previous problem.

1.3 Describe (e.g. by drawing) the following sets in $\mathbb{R}^2$:

- $\{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + 4y^2 \leq 4\}$,
- $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \geq 0\}$,
- $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \leq 2\}$,
- $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$,
- $\{(x,y) \in \mathbb{R}^2 : 2^x 3^y \leq 1\}$.

1.4 Verify that the plane $x + y + z - 5 = 0$ is a tangent plane to the ball (why it is a ball?)

$$x^2 - 2x + y^2 - 4y + z^2 + 2z + 3 = 0.$$

Find the common point.

1.5 Describe (e.g. by drawing) the following sets given in the polar coordinates (r, ϕ):

- $0 \leq r \leq 2, \quad 0 \leq \phi \leq 2\pi,$
- $0 \leq r \leq 1, \quad \frac{3\pi}{4} \leq \phi \leq \frac{5\pi}{4},$
- $1 \leq r \leq 3, \quad 0 \leq \phi \leq 2\pi,$
- $2 \leq r \leq 3, \quad \frac{3\pi}{2} \leq \phi \leq 2\pi,$
- $r = 1, \quad 0 \leq \phi \leq 2\pi,$
- $0 \leq r \leq 2, \quad \phi = \frac{\pi}{4}.$

1.6 Describe (e.g. by drawing) the following sets given in the spherical coordinates (r, θ, ϕ):

- $0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$
- $r = 2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$
- $0 \leq r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2},$
- $1 \leq r \leq 2, \quad \frac{\pi}{2} \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$
- $0 \leq r \leq 1, \quad \theta = \frac{\pi}{4}, \quad 0 \leq \phi \leq \pi.$

1.7 Prove triangle inequality in $\mathbb{R}^n$. That is, for vectors $\overline{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{y} = (y_1, y_2, \ldots, y_n)$ we have

$$|\overline{x} + \overline{y}| \leq |\overline{x}| + |\overline{y}|.$$

1.8 Let $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$ be vectors in $\mathbb{R}^2$. Prove that we have the identity

$$\overline{x} \cdot \overline{y} = |\overline{x}||\overline{y}| \cos\theta = x_1 y_1 + x_2 y_2$$

for the scalar product, where θ is the angle between vectors $\overline{x}$ and $\overline{y}$. Elaborate whether the identity remains valid in $\mathbb{R}^3$. What about $\mathbb{R}^n$?

1.9 Prove Cauchy-Schwarz inequality in $\mathbb{R}^n$. That is, we have $\overline{x} \cdot \overline{y} \leq |\overline{x}||\overline{y}|$.