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- Finish Le 20

Mapping properties of analytic fcns

Thm (Mapping thm)

Let f be analytic and not constant on a domain D, and let $z_0 \in D$. Suppose $f(z) - w_0$ has a zero of order $m \geq 1$ at z_0. ($f(z_0) = w_0$)

Then there $\exists$ $\varepsilon_0 > 0$ s.t. for every ε, $0 < \varepsilon < \varepsilon_0$ there $\exists$ $\delta > 0$ s.t. for all w with $0 < |w - w_0| < \delta$ the equation

$$f(z) = w$$

has exactly m distinct roots in the disk $0 < |z - z_0| < \varepsilon$.

Proof. Since f is not constant, the zeros of $f(z) - w_0$ are isolated. Thus there exists an $\tilde{\varepsilon}_0$ s.t. $f(z) - w_0$ has no zeros in the punctured disk $0 < |z - z_0| \leq \tilde{\varepsilon}_0$.

Since $f(z) - w_0$ is continuous and nonzero on the cpt set $|z - z_0| = \varepsilon$ ($0 < \varepsilon < \tilde{\varepsilon}_0$) there $\exists$ $\delta > 0$ s.t.

$$|f(z) - w_0| \geq \delta > 0 \quad \text{for} \quad |z - z_0| = \varepsilon$$

Now,

$$f(z) - w = f(z) - w_0 + w_0 - w = F(z) + H(z)$$

where $F(z) = f(z) - w_0$, $H(z) = w_0 - w$.

Thus, if $|w - w_0| < \delta$, then

$$|H(z)| < \delta \leq |F(z)|.$$

By Rouché's thm $F(z) = f(z) - w_0$ and

$F(z) + H(z) = f(z) - w$ has equally many

zeros in the disc $|z - z_0| < \varepsilon$ if $|w - w_0| < \delta$;

that is m zeros counting multiplicity.

However, since f' is not identically zero on D,

the zeros of f' are isolated. Thus,

$f'(z)$ must be nonzero in $0 < |z - z_0| < \varepsilon_0$

for some $\varepsilon_0 < \tilde{\varepsilon}_0$. Hence the m zeros

of $f(z) - w$, $0 < |w - w_0| < \delta$, in $|z - z_0| < \varepsilon < \varepsilon_0$

are simple and hence distinct.

[Note: If $w \neq w_0$, clearly $f(z_0) = w_0 \neq w$] $\boxtimes$

Alt. If $f(z) - w_0$ has a zero of order m at z_0

then $f(z) - w_0 = (z - z_0)^m h(z)$, where h anal. & $h(z_0) \neq 0$

$\Rightarrow f(z) - w_0 = \left((z - z_0) h^{\frac{1}{m}}(z)\right)^m = (g(z))^m =: (\zeta(z))^m$

where $\zeta(z) = g(z) = (z - z_0) h^{\frac{1}{m}}(z)$ has simple zero at z_0.

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<u>Corollary</u> (Open mapping thm)

If f is analytic on a domain D, and f is not constant, then f maps open sets onto open sets, i.e. $f(U)$ is open for every open subset U of D.

<u>Proof</u>: Suppose $f(z_0) = w_0$, $z_0 \in U$.

We want to show that there is an open disk around w_0 contained in $f(U)$.

But by the mapping thm $\exists$ $\varepsilon > 0$, $\delta > 0$ s.t. $D_\varepsilon(z_0) \subseteq U$ and $f(D_\varepsilon(z_0)) \supseteq D_\delta(w_0)$.

Hence $D_\delta(w_0) \subseteq f(U)$ $\boxtimes$

<u>Corollary</u>

If f is analytic and one-to-one on a domain D, then $f'(z) \neq 0$ for all $z \in D$.

In particular, the inverse f^{-1} of f is is analytic on the range $f(D)$ of D.

<u>Proof</u> If $f'(z_0) = 0$ for some $z_0 \in D$ then $f(z) - f(z_0)$ would have a zero of order $m \geq 2$ at z_0. By the mapping thm it cannot be locally one-to-one near z_0.

Analyticity of f^{-1} follows by the inverse fcn thm. $\boxtimes$

Thm (Schwarz lemma)

Let $f(z)$ be analytic for $|z| < 1$.

Suppose $|f(z)| \leq 1$ for all $|z| < 1$, and $f(0) = 0$.

Then,

$$|f(z)| \leq |z| \quad , \quad |z| < 1. \qquad (*)$$

Further, if equality holds in $(*)$ at some point $z_0 \neq 0$, then $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

Proof: We factor $f(z)$ as $f(z) = z g(z)$, where g is analytic in $|z| < 1$, and apply the maximum modulus principle to g.

Let $r < 1$. If $|z| = r$, then

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

By the maximum modulus principle,

$$|g(z)| \leq \frac{1}{r} \qquad \forall z \text{ s.t } |z| \leq r$$

Letting $r \to 1$, we obtain

$$|g(z)| \leq 1 \quad , \quad |z| < 1.$$

This clearly implies $(*)$.

Now, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$ and by the (strict) maximum modulus principle $g(z)$ is constant, say $g(z) = \lambda$. Clearly $|\lambda| = 1$, and $f(z) = \lambda z$. $\square$

The following infinitesimal version of Schwarz lemma holds.

<u>Thm</u> Let $f(z)$ be analytic in $|z| < 1$. If $|f(z)| \leq 1$ for $|z| < 1$, and $f(0) = 0$, then

$$|f'(0)| \leq 1,$$

with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

<u>Proof</u>: The inequality follows by letting $z \to 0$ in Schwarz lemma.

Use the factorization $f(z) = z\, g(z)$, as before. Note that $f'(0) = g(0)$, so if we have equality then $|g(0)| = 1$, and we conclude as above that $g(z)$ is constant in view of the (strict) maximum modulus principle. Writing $g(z) = \lambda$, then $|\lambda| = 1$ and $f(z) = \lambda z$. $\square$

Conformal self-maps of the unit disk

Let $\mathbb{D} = \{ |z| < 1 \}$. A conformal self-map of the unit disk $\mathbb{D}$ is an analytic fcn from $\mathbb{D}$ to itself that is one-to-one and onto.

The conformal self-maps form a group under composition. Note: the inverse of a conformal self-map is analytic, by the corollary above.

Lemma If $g(z)$ is a conformal self-map of the unit disk $\mathbb{D}$ such that $g(0)=0$, then $g(z)$ is a rotation, that is, $g(z) = e^{i\varphi} z$ for some fixed φ, $0 \le \varphi < 2\pi$.

Proof: Apply Schwarz lemma to g and g^{-1}. Since $g(0)=0$ and $|g(z)| < 1$, $|g(z)| \le |z|$. Similarly, $g^{-1}(0)=0$ and $|g^{-1}(w)| < 1$, so $|g^{-1}(w)| \le |w|$, which for $w = g(z)$ becomes $|z| \le |g(z)|$. Thus $|g(z)| = |z|$.

Since $g(z)/z$ has constant modulus, it is constant. Hence $g(z) = \lambda z$ for a λ with $|\lambda| = 1$.

Thm The conformal self-maps of the unit disk $\mathbb{D}$ are precisely the Möbius transformations of the form

$$f(z) = e^{i\varphi} \frac{z-a}{1-\bar{a}z}, \quad |z|<1,$$

where a is complex, $|a|<1$, and $0 \le \varphi \le 2\pi$.

Proof: Let $g(z) = \dfrac{z-a}{1-\bar{a}z}$, $|a|<1$

Note that

$$|e^{i\theta} - a| = |e^{-i\theta} - \bar{a}| = |1 - \bar{a}e^{i\theta}|, \quad 0 \le \theta \le 2\pi$$

Hence, g maps $\partial\mathbb{D}$ onto $\partial\mathbb{D}$.

Since g is a Möbius transformation, and $g(a)=0$, it follows that g maps $\mathbb{D}$ onto $\mathbb{D}$.

Consequently, g is a conformal self-map, and so is f above.

Let h be an arbitrary conformal self-map of $\mathbb{D}$, and let $a = h^{-1}(0) \in \mathbb{D}$

Then $h \circ g^{-1}$ is a conformal self-map of $\mathbb{D}$, and $(h \circ g^{-1})(0) = h(a) = 0$. By the lemma

$(h \circ g^{-1})(w) = e^{i\varphi} w$. Writing $w = g(z)$, $h(z) = e^{i\varphi} g(z)$ ☒

Note that the parameters a and φ are uniquely defined by f. Indeed, $a = f^{-1}(0)$ and since

$$f'(z) = e^{i\varphi} \cdot \frac{1 \cdot (1-\bar{a}z) - (z-a)\cdot(-\bar{a})}{(1-\bar{a}z)^2} =$$

$$= e^{i\varphi} \frac{1-|a|^2}{(1-\bar{a}z)^2}, \quad |z|<1,$$

the parameter φ is uniquely defined (mod 2π) as the argument of $f'(0)$.

Thus, there is a one-to-one correspondence between conformal self-maps of $\mathbb{D}$ and points of the parameter-space $\mathbb{D} \times \partial\mathbb{D}$.