Problems and solutions in part A

1 Concept

	Stochastic method	Deterministic model	Stochastic model	Deterministic method
$\frac{dF}{dt} = \beta F R - \gamma F$ $\frac{dR}{dt} = \alpha R - \beta F R$		✓		
SSA algorithm	✓			
Monte Carlo simulation	✓			
R,F integer-valued $R \xrightarrow{\alpha} 2R$ $R + F \xrightarrow{\beta} 2F$ $F \xrightarrow{\gamma} \varnothing$			✓	
QR iteration				✓

Make the following statements as correct as possible. The same term may (but does not have to) be used multiple times.

Help

1. The QR iteration computes all eigenvalues of a matrix A .
2. Brownian motion $B(t)$ is normally distributed with variance t.
3. If an $n \times n$ matrix is orthogonal , then it is non-singular.
4. The number of singular values of a matrix depends on its dimensions
5. Using Gram-Schmidt, one can compute the QR decomposition of a matrix.

singular value decomposition	power method	QR iteration
variance	QR decomposition	inverse power method
orthogonal	condition number	dimensions
SSA algorithm	rank	standard deviation
symmetric	diagonal	

2 Algorithm

2.1 Algo-LS

You are given the data set

x	-5	-1	0	2	4
y	1	-5	6	-4	2
z	-5	2	12.6	-1	0.5

Use the ansatz $p(x, y) = c_0 + c_1x + c_2y$ to fit the data in z, formulate the normal equations, and solve them. Write down the polynomial p that you get.

Proposed solution: This leads to $Ac = b$ with

$$A = \begin{pmatrix} 1 & -5 & 1 \\ 1 & -1 & -5 \\ 1 & 0 & 6 \\ 1 & 2 & -4 \\ 1 & 4 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -5 \\ 2 \\ 12.6 \\ -1 \\ 0.5 \end{pmatrix}$$

The normal equations are given by $A^TAc = A^Tb$. There holds

$$A^TA = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 46 & 0 \\ 0 & 0 & 82 \end{pmatrix}, \quad A^Tb = \begin{pmatrix} 9.1 \\ 23 \\ 65.6 \end{pmatrix},$$

which results in the solution

$$c = \begin{pmatrix} 1.82 \\ 0.5 \\ 0.8 \end{pmatrix}.$$

Therefore, $p(x, y) = 1.82 + 0.5x + 0.8y$.

2.2 Algo-MC

Felicity buys several things to eat each day. In the morning she buys a croissant at the local supermarket for 10 sek. For lunch she goes to a buffet place where the price depends on the weight. We assume that the price she pays each day follows the uniform distribution $U(90, 100)$ (in sek). In the afternoon she gets some kind of cake. Her choice varies. We assume that the price can be modeled by a given probability density function (pdf) $f(x)$. Assume there exists a function sample_f(), which returns one random number that is distributed according to the pdf $f(x)$ every time that it is called. Using Monte-Carlo simulation, estimate how much money Felicity spends on food outside her home on average each day. Write pseudo code for this problem.

Proposed solution: Input N
for i in range(N):
    A = 10
    B = random sample drawn from $U(90, 110)$
    # or: B = np.random.uniform(90,110)
    C = sample_f()
    result[i] = A+B+C
avg_costs = mean(result)

3 Analysis

3.1 Ana 1

We use Monte-Carlo simulation to estimate $G = \int_0^1 g(x)\, dx$ with $g(x) = \frac{6}{1+x^2}$. We sample N samples $X_i, i = 1, \ldots, N$, from the uniform distribution $U(0, 1)$ and estimate the true value as

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} g(X_i).$$

We also know that

$$\frac{1}{99} \sum_{i=1}^{100} \left(g(X_i) - \hat{\mu}_{100}\right)^2 = 1.21$$

and that $\hat{\mu}_{100} = 4.75$

Use that information to determine the interval (with 2 decimals), within which the true integral value G lies in with 95% probability using $N = 100$. Which theorem is behind the approach you are using?

Proposed solution: To estimate the error, we use the formula

$$[\hat{\mu}_{100} - \frac{1.96 * \sqrt{Var}}{\sqrt{N}}, \hat{\mu}_{100} + \frac{1.96 * \sqrt{Var}}{\sqrt{N}}] = [4.75 - \frac{1.96 * 1.1}{10}, 4.75 + \frac{1.96 * 1.1}{10}]$$

With

$$\frac{1.96 * 1.1}{10} \approx 0.22$$

this gives the interval [4.53, 4.97].

The formula for the error comes from the central limit theorem.

3.2 Ana 2

The SVD of a matrix A is given by

$$U = \begin{pmatrix} \frac{\sqrt{6}}{3} & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \Sigma = \begin{pmatrix} 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) What is rank(A)?

(b) What is cond$_2(A)$ (the condition number measured in the 2-norm)?

(c) What is the rank-1-matrix that best approximates A in the 2-norm?

Proposed solution:

(a) There are 2 non-zero singular values: $\sigma_1 = 2\sqrt{3}$ and $\sigma_2 = 2 \Rightarrow$ rank=2

(b) $\text{cond}_2(A) = \frac{\sigma_1}{\sigma_4}$ with $\sigma_1 = 2\sqrt{3}$ and $\sigma_2 = 2$ and $\sigma_3 = \sigma_4 = 0 \Rightarrow \text{cond}_2(A) = \frac{\sigma_1}{\sigma_4} = \frac{2\sqrt{3}}{0} = \infty$.

correct answer: cond(A) = (maximum sigma)/(minimum positive sigma) = sqrt(3)

(c) We compute

$$\sigma_1 u_1 v_1^T = 2\sqrt{3} \begin{pmatrix} \frac{\sqrt{6}}{3} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

3.3 Ana 3

You are given the following measurement data

x	500	1000	1500	2000	2500
y	2	8	10	13	20

Your goal is to find a quadratic polynomial that fits the given data best in a least-squares sense (representing y as a function of x). We want to set up the resulting overdetermined system $Ac = b$ in such a way that the condition number of A becomes as small as we can make it. Describe how you achieve that and write down the resulting matrix A with all its entries.

Note: For any computations you might do, use 1 decimal (1 digit after the dot).

Proposed solution: Instead of fitting

$$p(x) = c_0 + c_1 x + c_2 x^2,$$

we fit

$$p(x) = c_0 + c_1 \frac{(x - \bar{x})}{\sigma} + c_2 \frac{(x - \bar{x})^2}{\sigma^2},$$

i.e., we center and scale our data. We center around the mean $\bar{x} = 1500$. For computing the standard deviation σ, we first compute

$$\sum (x_i - \bar{x})^2 = 1000^2 + 500^2 + 500^2 + 1000^2 = 2,500,000$$

Then, using the unbiased estimator we get

$$s_1^2 = \frac{1}{4} 2,500,000 = 625,000 \quad \Rightarrow \quad s_1 \approx 790.6$$

Using the biased estimator we get

$$s_2^2 = \frac{1}{5} 2,500,000 = 500,000 \quad \Rightarrow \quad s_2 \approx 707.1$$

Both versions are fine. The matrix A has then entries

$$A_1 = \begin{pmatrix} 1 & -1.3 & 1.6 \\ 1 & -0.6 & 0.4 \\ 1 & 0 & 0 \\ 1 & 0.6 & 0.4 \\ 1 & 1.3 & 1.6 \end{pmatrix}.$$

And A_2 is given by

$$A_2 = \begin{pmatrix} 1 & -1.4 & 2.0 \\ 1 & -0.7 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0.7 & 0.5 \\ 1 & 1.4 & 2.0 \end{pmatrix}.$$