EXAM - SOLUTIONS

13-14-2023 March 15, 2023

1. Concept 1. Correct answers on Test.

2. Concept 2. Correct answers on Test.

3. Algorithm 1 - A. This is an exercise from the non-mandatory exercise of problem session 2.

$$\min \left\| \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \\ 2 & 5 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} m_N \\ m_O \end{bmatrix} - \begin{bmatrix} 30.006 \\ 76.012 \\ 44.013 \\ 108.010 \\ 46.006 \\ 92.001 \end{bmatrix} \right\| \tag{3.1}$$

(1point)

Solve for instance with normal equations to get $m_N = 14.0072$ and $m_O = 15.9985$. (1 point)

4. Algorithm 2 - A. Here we want to know the variance and the mean of standard framing timber. We use a Monte Carlo method since we can only use the electricity price as a black box function.

In pseudo Code:

```
N = large number
for i = range(N)
p[i] = 3 + normal_distr(mu_T, sigma_T) + timber_el_pris()
p_mean = mean(p)
p_variance = variance(p)
```

We obtain the mean price and the variance of the price.

5. Analysis 1 - A. We want to have a uncertainty interval on our price and we are given the mean and variance of the random variable p_i the price of a piece of timber.

We can invoke the central limit theorem, since the price for a single piece of framing timber is i.i.d). The CLT states that the average price

$$\bar{p}_i = \frac{1}{n}\sum_{i=1}^{n} p_i$$

converges (in distribution) to a normal distribution in the limit $N \to \infty$ with mean p_{mean} and variance $p_{variance}$ after scaling with $\sqrt{N}$.

$$\sqrt{N}(\bar{p}_i - p_{mean}) \mapsto \mathcal{N}(0, p_{variance}) \tag{5.1}$$

(1 points for correctly stating the central limit theorem and how it applies to the problem. Note: The CLT does not say that p_i becomes normally distributed for large N.) The uncertainty interval decreases proportionally to $\sim \frac{1}{\sqrt{N}}$, and for large N and we get and 95% uncertainty interval of

$$|e| \approx 1.96\sqrt{\frac{p_{variance}}{N}} \quad (1)$$

A common mistake was to write a Monte Carlo simulation with two loops, M and N and trying to approximate the variance and mean. However, the questions asks specifically for the uncertainty after producting N pieces and you are given the true mean and variance, so you can directly invoke the theorem.

6. Analysis 2 -A. The eigenvectors are orthonormal, so A is a symmetric matrix. Further, all eigenvalues are positive so the singular valued decomposition and eigenvalue decomposition are the same (up to sorting).

7. Algorithm B.

7.1. a).

1. A system that is too poorly conditioned to use the normal equations. In case there are no measurement error involved in constructing the matrix and right-hand side $\mathrm{cond}(A) < 10^{15}$
2. Matrix is non singular (Included in point 1)

7.2. b). See our worksheet. Gramm Schmidt orthogonalization:

$$\mathbf{a}_2 = \mathbf{a}_0 - \mathbf{a}_1 = \mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}/|\mathbf{b}|^2$$

with $\mathbf{a}_2 \perp \mathbf{b}$ easily verified by computing the inner product.

7.3. c). $\mathbf{q}_2$ has the direction of $\mathbf{a}_2$ and length 1. $\mathbf{q}_2$ is orthogonal to $\mathbf{q}_1$ by construction. We now read the QR decomposition along the columns of $\mathbf{R}$ to obtain

$$\begin{aligned}
r_{11} &= |\mathbf{b}| && (7.1)\\
\mathbf{q}_1 &= \mathbf{b}/r_{11} = \mathbf{a}_1/|\mathbf{a}_1| && (7.2)\\
\mathbf{q}_2 &= \mathbf{a}_2/|\mathbf{a}_2| && (7.3)\\
r_{12} &= \mathbf{a}\cdot\mathbf{b}/r_{11} && (7.4)\\
r_{22} &= |\mathbf{a}_2| && (7.5)\\
& && (7.6)
\end{aligned}$$

8. Analysis 2 - B.

8.1. a). With $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$ we get

$$\mathbf{B} = \mathbf{A}^4 = \mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{Q}\Lambda\mathbf{Q}^{-1} = \mathbf{Q}\Lambda^4\mathbf{Q}^{-1} \tag{8.1}$$

So $\mathbf{B}$ has the same eigenvectors as $\mathbf{A}$ but the eigenvalues are the quartics of the eigenvalues of $\mathbf{A}$.

8.2. b). From the previous exercise it is easy to establish that

$$\mathbf{A}^n = \mathbf{Q}\Lambda^n\mathbf{Q}^{-1}$$

such that

$$f(\mathbf{A}) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\mathbf{A}^k = \mathbf{Q}(\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\Lambda^k)\mathbf{Q}^{-1} = \mathbf{Q}f(\Lambda)\mathbf{Q}^{-1} \tag{8.2}$$

8.3. c). For symmetric real matriciees the eigenvectors are orthonormal and $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$. Further for diagonal matricees, it holds that

$$\Lambda^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

Therefore,

$$\begin{aligned} f(\mathbf{A}) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \mathbf{A}^k \\ &= \mathbf{Q} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \Lambda^k \mathbf{Q}^T \\ &= \sum_{i=1}^{n} \mathbf{q}_i \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \lambda_i^k \mathbf{q}_i^T \\ &= \sum_{i=1}^{n} \mathbf{q}_i f(\lambda_i) \mathbf{q}_i^T \end{aligned}$$