Lecture 5: Connections with Partial Differential Equations and The Feynman-Kac Theorem

1 Stochastic Differential Equations

1.1 SDEs

- A *stochastic differential equation (SDE)* is an equation of the form

$$dX_t = \beta(t, X_t)dt + \gamma(t, X_t)dW_t. \tag{1}$$

Here $\beta(t, x)$ and $\gamma(t, x)$ are given functions, called the *drift* and *diffusion*, respectively.

Example 1 (GBM)

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

Example 2 (Hull-White)

$$dR_t = (a - bR_t)dt + \sigma dW_t$$

Example 3 (Cox-Ingersoll-Ross)

$$dR_t = (a - bR_t)dt + \sigma\sqrt{R_t}dW_t$$

1.2 Markov Property

- **Theorem 1** *Let $X_t, t \geq 0$ be a solution to the SDE (1) with initial condition $X_0 = x$. Then, for $0 \leq t \leq T$ and function $h(\cdot)$, there exists a function $g(\cdot, \cdot)$ such that*

$$E[h(X_T)|\mathcal{F}_t] = g(t, X_t).$$

In other words, $X_t, t \geq 0$ is a Markovian process.

2 Partial Differential Equations

2.1 Feynman-Kac Theorem

- **Theorem 2** *Consider the stochastic differential equation*

$$dX_t = \beta(t, X_t)du + \gamma(t, X_t)dW_t.$$

Let $h(y)$ be a function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = E[h(X_T)|X_t = x].$$

Then g satisfies the following partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x)$$

for all x.

Theorem 3 (Feynman-Kac Theorem with Discounting) *Consider the stochastic differential equation*

$$dX_t = \beta(u, X_u)du + \gamma(u, X_u)dW_u$$

again. Let $h(y)$ be a function and let r be constant. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$f(t, x) = E\left[e^{-r(T-t)}h(X_T)\middle| X_t = x\right].$$

Then f satisfies the following partial differential equation

$$f_t(t, x) + \beta(t, x)f_x(t, x) + \frac{1}{2}\gamma^2(t, x)f_{xx}(t, x) = rf(t, x)$$

and the terminal condition

$$f(T, x) = h(x)$$

for all x.

Theorem 4 (Multivariate Feynman-Kac Theorem) *Let W_t^1 and W_t^2 be two independent Brownian motions. Consider the following two-dimensional stochastic differential equation*

$$\begin{aligned} dX_t &= \beta_1(u, X_u, Y_u)du + \gamma_{11}(u, X_u, Y_u)dW_u^1 + \gamma_{12}(u, X_u, Y_u)dW_u^2 \\ dY_t &= \beta_2(u, X_u, Y_u)du + \gamma_{21}(u, X_u, Y_u)dW_u^1 + \gamma_{22}(u, X_u, Y_u)dW_u^2 \end{aligned}$$

again. Let $h(x, y)$ be a function and let r be constant. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x, y) = E\left[e^{-r(T-t)}h(X_T, Y_T)\middle| X_t = x, Y_t = y\right].$$

Then g satisfies the following partial differential equation

$$\begin{aligned} g_t &+ \beta_1 g_x + \beta_2 g_y \\ &+\frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2)g_{xx} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2)g_{yy} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22})g_{xy} = rg \end{aligned}$$

and the terminal condition

$$g(T, x, y) = h(x, y)$$

for all x and y.

2.2 Interest Rate Models

- A *Short-rate model* under a risk-neutral probability measure is given by

$$dR_t = \beta(t, R_t)dt + \gamma(t, R_t)dW_t.$$

The *discount process* is then

$$D_t = e^{-\int_0^t R_s ds}.$$

This gives us the *zero-coupon bond pricing formula*

$$B(t,T) = \tilde{E}[e^{-\int_0^t R_s ds}|\mathcal{F}_t].$$

By the Markovian property of R_t, we know that $B(t,T) = f(t,R_t)$ for some function f. Furthermore, f should satisfy

$$f_t(t,r) + \beta(t,r)f_r(t,r) + \frac{1}{2}\gamma^2(t,r)f_{rr}(t,r) = rf(t,x)$$

and $f(T,r) = 1$.

Example 4 (Hull-White and CIR revisited)

Homework Set 5 (Due on Nov 17)

1. Exercise 6.1 in Page 282 of Shreve's book.
2. Exercise 6.4 in Page 285 of Shreve's book.
3. Exercise 6.5 in Page 286 of Shreve's book.
4. Exercise 6.6 in Page 286 of Shreve's book.
5. Exercise 6.7 in Page 288 of Shreve's book.