1 (a) See Lecture 2 and Lecture 4

(b) Note that for every n, we have

$$P(X_n > n) = \int_n^\infty (a-1)x^{-a} I_{[1,\infty)}(x)\,dx$$

$$= (a-1)\int_n^\infty x^{-a}\,dx = (a-1)\left.\frac{x^{1-a}}{1-a}\right|_n^\infty$$

$$= n^{1-a}$$

For $a \le 2$, we have

$$\sum_{n=1}^\infty P(X_n > n) = \sum_{n=1}^\infty n^{1-a} = \infty,$$

so $X_n > n$ happens infinitely often with probability 1 by the second Borel-Cantelli lemma.

For $a > 2$,

$$\sum_{n=1}^\infty P(X_n > n) = \sum_{n=1}^\infty n^{1-a} < \infty,$$

so $X_n > n$ only happens finitely often with probability 1 by the first Borel-Cantelli lemma.

2 Suppose X is integrable. Then X_n is also integrable, since

$$\mathbb{E}(|X_n|) = \mathbb{E}(\min(X,n)) \le \mathbb{E}(X) < \infty$$

For every $\varepsilon > 0$, there exists K such that

$$\mathbb{E}(|X|;|X|>K) = \mathbb{E}(X;X>K) < \varepsilon$$

For this K, we also have

$$\underbrace{\mathbb{E}(|X_n|;|X_n|>K) \le \mathbb{E}(X;X>K)}_{<\varepsilon}$$

since $|X_n| = X_n \le X$ thus $|X_n| > K$ implies $X > K$

showing uniform integrability.

Conversely, assume that the family $\{X_n; n \ge 0\}$ is uniformly integrable. There exists $K > 0$ such that

$$\mathbb{E}(|X_n|;|X_n|>K) < 1 \quad \text{for all } n$$

$$\Rightarrow \mathbb{E}\left(\min(X,n);\ \min(X,n)>K\right) < 1 \quad \text{for all } n$$

Note that $\min(X,n)\cdot I_{\{\min(X,n)>K\}}$ is monotone in n; as $n\to\infty$, we have $\min(X,n)\to X$. Thus by monotone convergence,

$$\mathbb{E}\left(\min(X,n);\min(X,n)>K\right) = \mathbb{E}\left(\min(X,n)\cdot I_{\{\min(X,n)>K\}}\right)$$
$$\xrightarrow{n\to\infty} \mathbb{E}\left(X I_{\{X>K\}}\right) = \mathbb{E}(X; X>K)$$

It follows that

$$\mathbb{E}(|X|) = \mathbb{E}(X) = \mathbb{E}(X; X\leq K) + \mathbb{E}(X; X>K)$$
$$\leq K+1 < \infty,$$

so X is integrable.

③ See Lecture 5

④ (a) We have $\mathbb{E}(\theta^{X_n}|\mathcal{F}_{n-1}) = \mathbb{E}(\theta^{X_{n-1}+Y_n}|\mathcal{F}_{n-1})$

$$= \theta^{X_{n-1}}\,\mathbb{E}(\theta^{Y_n})$$
$$= \theta^{X_{n-1}}\left(\tfrac{1}{2}\theta + \tfrac{1}{4} + \tfrac{1}{4}\theta^{-1}\right)$$

For θ^{X_n} to be a martingale, we need

$$\tfrac{1}{2}\theta + \tfrac{1}{4} + \tfrac{1}{4}\theta^{-1} = 1 \iff \theta^2 - \tfrac{3}{2}\theta + \tfrac{1}{2} = 0$$
$$\iff \theta = 1 \text{ or } \theta = \tfrac{1}{2}$$

So $\left(\tfrac{1}{2}\right)^{X_n} = 2^{-X_n}$ is a martingale.

(b) Since $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = \mathbb{E}(X_{n-1}+Y_n|\mathcal{F}_{n-1})$

$$= X_{n-1} + \mathbb{E}(Y_n)$$
$$= X_{n-1} + \left(\tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot 0 + \tfrac{1}{4}\cdot(-1)\right)$$
$$= X_{n-1} + \tfrac{1}{4}$$

Taking $f(n) = \frac{n}{4}$, we obtain

$$\mathbb{E}\left(X_n - \tfrac{n}{4}\,\middle|\,\mathcal{F}_{n-1}\right) = X_{n-1} + \tfrac{1}{4} - \tfrac{n}{4} = X_{n-1} - \tfrac{(n-1)}{4}$$

showing that $X_n - \frac{n}{4}$ is a martingale.

(c) Consider the stopping time $\tau = \inf\{n : X_n = a \text{ or } X_n = -b\}$.
By the optional stopping theorem, we have

$$\mathbb{E}(2^{-X_\tau}) = \mathbb{E}(2^{-X_0}) = 1$$

On the other hand, $X_\tau \in \{a, -b\}$ by definition.

$\Rightarrow \mathbb{E}(2^{X_\tau}) = 2^{-a} P(X_\tau = a) + 2^{b} P(X_\tau = -b)$

$P(X_\tau = a)$ is exactly the probability to reach a before $-b$.

We obtain

$$2^{-a} P(X_\tau = a) + 2^{b} \left(1 - P(X_\tau = a)\right) = 1$$

$$\Rightarrow P(X_\tau = a) = \frac{2^{b} - 1}{2^{b} - 2^{-a}}.$$

5 (a) True: If both $\mathbb{E}(X_n | \mathcal{F}_{n-1}) \geq X_{n-1}$ (submartingale)
and $\mathbb{E}(X_n | \mathcal{F}_{n-1}) \leq X_{n-1}$ (supermartingale)
then $\mathbb{E}(X_n | \mathcal{F}_{n-1}) = X_{n-1}$, so X_n is a martingale.

(b) False: Consider for example the martingale given by $X_0 = 1$ and

$$X_n = \begin{cases} 2X_{n-1} & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

Now $T = \inf\{n : X_n = 0\}$ satisfies $P(T < \infty) = 1$,
but $\mathbb{E}(X_T) = 0 \neq 1 = \mathbb{E}(X_0)$

(c) False: Consider for example the martingale given by $X_0 = 1$ and

$$X_n = \begin{cases} 2X_{n-1} & \text{with probability } \frac{1}{3} \\ \frac{1}{2}X_{n-1} & \text{with probability } \frac{2}{3} \end{cases}$$

We can write $X_n = X_{n-1} Y_n$, where

$$Y_n = \begin{cases} 2 & \text{with probability } \frac{1}{3} \\ \frac{1}{2} & \text{with probability } \frac{2}{3} \end{cases}$$

Note that

$$\ln X_n = \ln Y_1 + \ln Y_2 + \dots + \ln Y_n$$

and $\mathbb{E}(\ln Y_n) = \frac{1}{3} \cdot \ln 2 + \frac{2}{3} \cdot \ln \frac{1}{2} = -\frac{1}{3} \ln 2 < 0$

By the strong law of large numbers,

$$\frac{\ln X_n}{n} \to -\frac{1}{3}\ln 2 \quad \text{almost surely}$$

Thus also

$$\lim_{n\to\infty} \ln X_n = -\infty \qquad \text{almost surely}$$

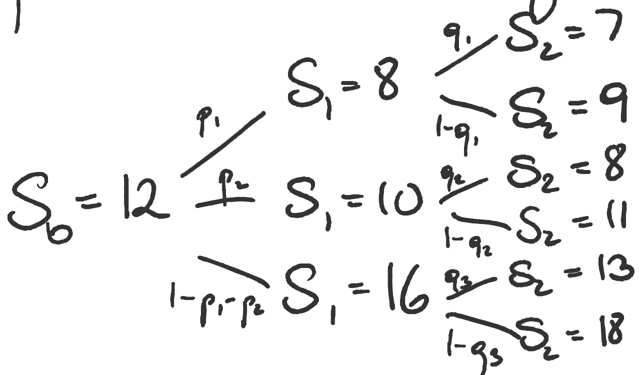
$$\lim_{n\to\infty} X_n = 0$$

even though clearly $X_n > 0$ for all n.

⑥ See Lecture 14

⑦ (a) See Lecture 15

(b) The probabilities in the diagram



should be such that $\bar{S}_t = S_t$ becomes a martingale.
We need

$$12 = 8p_1 + 10p_2 + 16(1-p_1-p_2) \iff 8p_1 + 6p_2 = 4$$
$$\iff 4p_1 + 3p_2 = 2$$
$$\iff p_2 = \tfrac{1}{3}(2-4p_1)$$

It follows that $1-p_1-p_2 = \frac{1}{3} + \frac{p_1}{3}$; p_1 can be arbitrary in the interval $(0, \frac{1}{2})$.

Moreover,

$$8 = 7q_1 + 9(1-q_1) \qquad 10 = 8q_2 + 11(1-q_2) \qquad 16 = 13q_3 + 18(1-q_3)$$
$$\iff q_1 = \tfrac{1}{2} \qquad \iff q_2 = \tfrac{1}{3} \qquad \iff q_3 = \tfrac{2}{5}$$

There exists an equivalent martingale measure, but it is not unique. Thus the model is viable, but not complete.

8 See Lectures 15 and 16