Inference 2, 2023, lecture 7

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Today

Chap. 4. Estimation (continued):

- Unbiasedness and Mean Square Error (MSE)
- Best unbiased estimators

Unbiasedness and Mean Square Error (MSE)

- Let $\mathbf{X} = (X_1, \dots, X_n)$ be independent random variables distributed as X with parameter(s) θ.
- We want to estimate $\gamma = g(\theta)$. (Special case: $\gamma = \theta$.)
- The **bias** of T is given by

$$\mathrm{Bias}(T, \theta) = \mathrm{E}_\theta(T) - g(\theta).$$

Definition (4.7)

An estimator T for $\gamma = g(\theta)$ is called **unbiased** if

$$\mathrm{Bias}(T, \theta) = 0$$

for all $\theta \in \Theta$.

Unbiasedness and Mean Square Error (MSE)

Example 1:

- Suppose $\mathbf{X} = (X_1, ..., X_n)$ are independent random variables distributed as $X \sim \mathrm{Exp}(\beta)$ where β is the intensity.
- We have observations $\mathbf{x} = (x_1, ..., x_n)$.

1. Is $\hat{\beta}_{\mathrm{MLE}} = 1/\bar{x}$ unbiased for β?
2. Let $\mu = 1/\beta$. Is $\hat{\mu}_{\mathrm{MLE}} = \bar{x}$ unbiased for μ?

Unbiasedness and Mean Square Error (MSE)

Definition (4.6)

Let T be an estimator of $g(\theta)$. The **mean square error (MSE)** of T is given by

$$\mathrm{MSE}(T, \theta) = \mathrm{E}_\theta[\{T - g(\theta)\}^2].$$

Observe that, with $E(T) = \mu$, (why?)

$$\mathrm{MSE}(T, \theta) = \mathrm{Var}_\theta(T) + \{\mathrm{Bias}(T, \theta)\}^2.$$

Unbiasedness and Mean Square Error (MSE)

Example 2:

- Suppose we have n independent observations from $N(\mu, \sigma^2)$.
- Consider $\hat{\sigma}^2_{\mathrm{MLE}} = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ and $S^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

1. Is any of these unbiased?
2. Which has the smallest MSE?

Hint:
$V = \sigma^{-2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$, which means that $\mathrm{E}(V) = n - 1$ and $\mathrm{Var}(V) = 2(n-1)$.

Best unbiased estimators

Definition (4.8)

An unbiased estimator T^* for a parameter $\gamma = g(\theta)$ is called the **best unbiased estimator (BUE)**, if for any other unbiased estimator T,

$$\mathrm{Var}_\theta(T^*) \leq \mathrm{Var}_\theta(T)$$

for all $\theta \in \Theta$.

In a given (regular) situation

- is there a way to find a statistic which is BUE? (May be difficult!)
- for a given statistic, is it possible to know if it is BUE? (Possible!)

A **regular unbiased estimator** T fulfills

$$\int_{\mathcal{A}} T(\mathbf{x}) \frac{\partial}{\partial\theta} L(\theta; \mathbf{x}) d\mathbf{x} = \frac{\partial}{\partial\theta} \int_{\mathcal{A}} T(\mathbf{x}) L(\theta; \mathbf{x}) d\mathbf{x}.$$

Best unbiased estimators

Theorem (4.2)

The Cramér-Rao lower bound. *Suppose that regularity conditions 3 and 4 are satisfied and that the Fisher information satisfies* $0 < I_{\mathbf{X}}(\theta) < \infty$*.*
Let $\gamma = g(\theta)$ *where* g *is a continuously differentiable function with derivative* $g' \neq 0$*.*
If T *is a regular unbiased estimator for* γ*, then*

$$\mathrm{Var}_\theta(T) \geq \frac{\{g'(\theta)\}^2}{I_{\mathbf{X}}(\theta)}$$

for all $\theta \in \Theta$*. Equality holds if.f. for* $\mathbf{x} \in \mathcal{A}$ *and all* $\theta \in \Theta$*,*

$$T(\mathbf{x}) - g(\theta) = \frac{g'(\theta) V(\theta; \mathbf{x})}{I_{\mathbf{X}}(\theta)}$$

where V *is the score function.*

Best unbiased estimators

Cramér-Rao, special case: If $\gamma = \theta$,

$$\mathrm{Var}_\theta(T) \geq \frac{1}{I_{\mathbf{X}}(\theta)}$$

with equality if

$$T(\mathbf{x}) - \theta = \frac{V(\theta; \mathbf{x})}{I_{\mathbf{X}}(\theta)}.$$

Best unbiased estimators

In the general case, observe:

$$1 \geq \frac{\{g'(\theta)\}^2}{\mathrm{Var}_\theta(T) I_{\mathbf{X}}(\theta)}.$$

Definition (4.9)

The **efficiency** of an unbiased estimator T is

$$e(T, \theta) = \frac{\{g'(\theta)\}^2}{\mathrm{Var}_\theta(T) I_{\mathbf{X}}(\theta)}.$$

An estimator which attains the Cramér-Rao lower bound is said to be **efficient**.

It may be shown that, under regularity conditions, maximum likelihood estimators are asymptotically unbiased and efficient.

Best unbiased estimators

Theorem (4.3)

Suppose that the distribution of $\mathbf{X} = (X_1, ..., X_n)$ *belongs to a one-parameter exponential family in* ζ *and* T.
Then the sufficient statistic T *is an efficient estimator for the parameter* $\gamma = g(\theta) = \mathrm{E}_\theta(T)$.

Best unbiased estimators

Example 3: Check if the following estimators are efficient.

1. The sample mean when $X_i \sim N(\mu, \sigma^2)$ with known σ^2.
2. The MLE of the expectation parameter μ in the exponential distribution.
3. The sample mean in the $\mathrm{Po}(\lambda)$ distribution.

News of today

- Bias
- MSE (variance plus squared bias)
- Best unbiased estimator (has the smallest variance)
- Cramér-Rao inequality (gives smallest possible variance)
- Efficiency (lower bound divided by variance)
- Efficient estimator
- In the exponential family, the sufficient statistic is efficient.