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Prov i matematik
Kurs: 1MA022
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Complex Analysis

Writing time: 08:00–13:00.
Other than writing utensils and paper, no help materials are allowed.

1. Suppose that $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain D, such that $u(x, y) = -v^2(x, y)$ for all $z = x + iy \in D$. Show that $f(z) = u(x, y) + iv(x, y)$ is analytic in D only if f is a constant function.

2. Find a conformal mapping that transforms the domain

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \cup \{z \in \mathbb{C} : |z| < 1\}$$

onto the left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z-2)^3} - \frac{1}{(z+3)}$$

in the annulus $A = \{z \in \mathbb{C} : 2 < |z| < 3\}$.

4. Let γ be a piecewise smooth, simple closed curve in a domain D. Assume that $f : D \longrightarrow \mathbb{C}$ is analytic and at each point z belonging to the trace of γ the following inequality is satisfied:

$$|f(z) - 1| < |f(z)| + 1.$$

Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

Hint: Characterize geometrically the domain $\{w \in \mathbb{C} : |w - 1| < |w| + 1\}$ and observe that $\operatorname{Log} w$ is analytic in this domain.

5. Use the residue theorem to calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+2)^2}\, dx.$$

6. Show that all zeros the polynomial $p(z) = z^5 - z + 16$ are contained in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. How many of the zeros have positive real part?

7. Find a formula for the analytic function $f : \mathbb{C} \setminus \{1, -1\} \longrightarrow \mathbb{C}$ which has the following properties:

- f has simple zeros at $\pm i$ and a double zero at 0;
- f has double poles at ± 1 with residues ± 1 respectively;
- f has a removable singularity at ∞.

8. Suppose that $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : \bar{D} \longrightarrow \mathbb{C}$ is a continuous function, which is analytic in D. Assume that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \partial D$. Show that $|f(z)| \leq |z|$ for all $z \in D$. Show also that if $|f(a)| = |a|$ at some point $a \in D$, then in fact $f(z) = cz$ for some constant c such that $|c| = 1$.
Hint: The function $f(z)/z$ has a removable singularity at 0.

GOOD LUCK!

SOLUTIONS

Solution 1: If f is analytic, then the Cauchy-Riemann equations hold: $u_x = v_y$ and $u_y = -v_x$. But since $u = -v^2$, we also have $u_x = -2vv_x$ and $u_y = -2vv_y$. Therefore $u_x = 2vu_y = -4v^2v_y = -4v^2u_x$. So $u_x(1+4v^2) = 0$ identically in D. Consequently $u_x \equiv 0$, and so $v_y \equiv 0$ because of the Cauchy-Riemann equations. Thus u depends only on y and v depends only on x. Since $u = -v^2$, the required conclusion follows.

Solution 2: Let $Q_I, Q_{II}, Q_{III}, Q_{IV}$ denote the 1st, 2nd, 3rd and 4th quadrant in the plane. We want to map $Q_I \cup Q_{II} \cup D(0,1)$ onto $Q_{II} \cup Q_{III}$. The composition of the following mappings will do:

- $z \mapsto z+1$ maps $Q_I \cup Q_{II} \cup D(0,1)$ onto $Q_I \cup Q_{II} \cup D(1,1)$;
- $z \mapsto 1/z$ maps $Q_I \cup Q_{II} \cup D(1,1)$ onto $Q_{III} \cup Q_{IV} \cup \{z \in \mathbb{C} : \operatorname{Re} z > 1/2\}$;
- $z \mapsto z - 1/2$ maps $Q_{III} \cup Q_{IV} \cup \{z \in \mathbb{C} : \operatorname{Re} z > 1/2\}$ onto $Q_{III} \cup Q_{IV} \cup Q_I$;
- $z \mapsto \frac{\sqrt{3}+i}{2} \cdot z^{2/3}$ maps $Q_{III} \cup Q_{IV} \cup Q_I$ onto $Q_I \cup Q_{II}$ (because $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$);
- $z \mapsto iz$ maps $Q_I \cup Q_{II}$ onto $Q_{II} \cup Q_{III}$.

The outcome is

$$f(z) = \frac{i\sqrt{3}-1}{2^{5/3}} \left(\frac{1-z}{1+z}\right)^{2/3}.$$

Solution 3: For $|z| > 2$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}.$$

Since

$$\left(\frac{1}{z-2}\right)'' = \left(-\frac{1}{(z-2)^2}\right)' = \frac{2}{(z-2)^3},$$

we have

$$\frac{1}{(z-2)^3} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}\right)'' = \sum_{n=1}^{\infty}\frac{n(n+1)2^{n-2}}{z^{n+2}}.$$

For $|z| < 3$ we have

$$\frac{1}{z+3} = \frac{1}{3}\frac{1}{\frac{z}{3}+1} = \frac{1}{3}\frac{1}{1-\left(-\frac{z}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{(-3)^n} = -\sum_{n=0}^{\infty}\frac{z^n}{(-3)^{n+1}}.$$

Hence

$$f(z) = \sum_{n=3}^{\infty} \frac{(n-2)(n-1)2^{n-4}}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{(-3)^{n+1}}.$$

Solution 4: Note that $\{w \in \mathbb{C} : |w-1| < |w|+1\} = \mathbb{C} \setminus (-\infty, 0]$. Indeed, if $w \in \mathbb{C} \setminus \mathbb{R}$, then $|w-1| < |w|+1$ because the points $0, 1, w$ form a triangle. If $w > 0$, then obviously $\max\{w-1, 1-w\} < w+1$. If $w \leq 0$, then $|w-1| = -w+1 = |w|+1$. Since Log is analytic in $\mathbb{C} \setminus (-\infty, 0]$, the function Log$f(z)$ is analytic in D with derivative f'/f. Since γ is closed the result follows.

Solution 5: If $R > 2$, the function

$$f(z) = \frac{e^{iz}}{(z^2+2)^2}$$

has only one singularity within the upper half of $\bar{D}(0, R)$, a pole of order 2 at $i\sqrt{2}$. Also if Γ_R denotes the upper semicircle of radius R and centre at 0 oriented counterclockwise, then in view of Jordan's lemma

$$\left| \int_{\Gamma_R} f(z)\, dz \right| \leq \frac{1}{(R^2-1)^2} \int_{\Gamma_R} |e^{iz}||dz| < \frac{\pi}{(R^2-1)^2} \to 0 \text{ as } R \to \infty.$$

Since

$$\text{Res}[f, i\sqrt{2}] = \left(\frac{e^{iz}}{(z+i\sqrt{2})^2} \right)' \Big|_{z=i\sqrt{2}} = -\frac{ie^{-\sqrt{2}}}{16}(2+\sqrt{2}),$$

the integral is equal to

$$\frac{\pi e^{-\sqrt{2}}}{4} \left(1 + \frac{1}{\sqrt{2}} \right).$$

Solution 6: If $|z| = 1$, then obviously $|16 - z| > |z^5|$, so the polynomial has no zeros in the unit disc by Rouche's theorem. If $|z| = 2$, then $|z^5| > |16 - z|$, so – again by Rouche's theorem – the polynomial has 5 roots in the disk $D(0, 2)$. Combining these two statements, we get the first assertion. Let $R > 2$. If $z = ir$, for $r \in [-R, R]$, then $p(z) = 16 + ir(r^4 - 1)$. So the change of argument as z travels from iR to $-iR$ along the imaginary axis is approximately $-\pi$ for large R. When z travels along the right-half of the circle $\partial D(0, R)$ counterclockwise from $-iR$ to iR, the change of argument of $p(z)$ is for lage R dictated by the dominant term z^5, and therefore is approximately 5π. Hence, in view of the argument principle, the number of roots with positive real part is 2.

Solution 7: For some analytic function $g : \mathbb{C} \longrightarrow \mathbb{C}$ we have

$$f(z) = \frac{z^2(z^2+1)}{(z-1)^2(z+1)^2} g(z), \qquad z \in \mathbb{C}.$$

Since f has a removable singularity at ∞, so does g and so by Liouville's theorem g is constant $g \equiv A$. Then

$$\pm 1 = \operatorname{Res}[f, \pm 1] = \frac{d}{dz} \big((z \mp 1)^2 f(z)\big) \Big|_{z=\pm 1} = \pm A.$$

So $A = 1$.

Solution 8: Since $g(z) = f(z)/z$ has a removable singularity at zero and its modulus is bounded by 1 on ∂D, we get the first inequality from the maximum modulus principle. If $|f(a)| = |a|$ at some point $a \in D$, then g has a global maximum 1 there and so $g(z) \equiv c$ for some constant $c \in \partial D$.