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Prov i matematik
Reell Analys, 1MA226
2014-12-17

Skrivtid: 8-13. Inga hjälpmedel. Lösningar skall åtföljas av förklarande text/figurer. För betyget 3 *krävs minst* 18*p, för* 4 *- minst* 25*p och för* 5 *- minst* 32*p.*

1. (5 points)

a) Use the root test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$ converges.

b) Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$ converges absolutely, or diverges.

2. (6 points) Consider the "Gauss map"

$$g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = frac\left(\frac{1}{x}\right)$$

defined on the unit interval $(0, 1]$. Here, $\lfloor x \rfloor$ is the largest integer not greater than x, and $frac(x)$ is the fractional part of x.

Define $g(0) = 0$.

a) Find all points in $(0, 1]$ where the function is discontinuous. What kind of discontinuities are these?

b) Sketch the function.

c) What kind of discontinuity is the point 0? Prove it.

3. (4 points) Suppose that f is a differentiable function on $\mathbb{R}$ with a bounded derivative. Prove that f is uniformly continuous.

4. (5 points) Suppose f is a real valued function on $[0, 1]$ and $f \in \mathcal{R}([\epsilon, 1])$ for every $0 < \epsilon < 1$. Define,

$$\int_0^1 f(x)dx = \lim_{\epsilon \to 0_+} \int_\epsilon^1 f(x)dx, \tag{1}$$

whenever this limit exists.

a) If $f \in \mathcal{R}([0, 1]$ show that this definition coincides with the old one, i.e. show that the upper and lower Riemann sums on $[\epsilon, 1]$ converge to those on $[0, 1]$ *as* $\epsilon \to 0$.

b) Construct a function for which the limit (1) exists but it fails to exist for $|f|$ (*Hint:* try oscillatory functions in a neighborhood of 0.)

5. (5 points) Let $\{f_n\}$ be a sequence of non-negative monotone functions which are pointwise bounded from above by $x^{-\alpha}$, $\alpha > 1$.

a) Prove that f_n are integrable over $[0,1]$ in the sense of definition (1) (that is, that the limit in (1) exists).

b) Put

$$F_n(x) = \int_0^x f_n(t)dt.$$

Prove that for any $\epsilon > 0$, there is a subsequence F_{n_k} which converges uniformly in $[\epsilon, 1]$.

6. (5 points) Consider the sequence $\{f_n\}$ of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

a) On which subset of $\mathbb{R}$ does this sequence converge uniformly? To what function $f(x)$?

b) On which subset of $\mathbb{R}$ does the sequence of derivatives converge uniformly? To what function? Make a conclusion about when

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

7. (5 points) Consider the system $F(x,y,u,v) = 0$ given by

$$\begin{aligned} 0 &= x^2 - y^2 - u^3 + v^2 + 4, \\ 0 &= 2xy + y^2 - 2u^2 + 3v^4 + 8. \end{aligned}$$

It has a solution $(x,y,u,v) = (2,-1,2,1)$. Prove that there is a unique function $f : U \mapsto W$, where U and W are some neighborhoods of $(2,-1)$ and $(2,1)$, respectively, such that $F(x,y,f_1(x,y),f_2(x,y)) = 0$ for every $(x,y) \in U$.

8. (5 points) Let $f : U \subset \mathbb{R} \mapsto \mathbb{R}$ be *continuously differentiable* with a hyperbolic derivative at points of U(that is, $|f'(x)| \neq 1$). Fix $x_0 \in \text{int}(U)$, define the following

$$M = (1 - f'(x_0))^{-1}$$

(note, since $|f'(x_0)| \neq 1$, $1 - f'(x_0)$ is invertible.

Consider the following *Newton map* associated to f:

$$\phi(x) = x + f(x_0 + M(x - x_0)) - (x_0 + M(x - x_0))$$

a) Compute the derivative of the map ϕ in terms of M, $f'(x_0)$ and $f'(x_0 + M(x - x_0))$.

b) Using the fact that f is *continuously* differentiable, show that

$$\lim_{x\to x_0} \phi'(x) = 0$$

c) Suppose that there exists an open set $V \ni x_0$ such that $\phi(V) \subset V$, and $|\phi'(x)| < \kappa < 1$ for all $x \in V$. Use the Contraction Mapping Principle to prove that V contains the unique fixed point of ϕ. Conclude that f also has the unique fixed point in V.

Remark: In practice, one can often simply iterate ϕ many times to find the fixed point of f.