Laplace's Equation

3.1. Green's identities

For a smooth vector field $\vec{F} = (f^1, \dots, f^n)$, we define the **divergence** of $\vec{F}$ by

$$\operatorname{div} \vec{F} = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x_j} = \sum_{j=1}^{n} f^j_{x_j}.$$

Lemma 3.1 (**Divergence Theorem**). *Assume Ω is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $\vec{F} \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Then*

$$\int_\Omega \operatorname{div} \vec{F}(x)\, dx = \int_{\partial\Omega} \vec{F}(x) \cdot \nu(x)\, dS, \tag{3.1}$$

where $\nu(x)$ is the **outer unit normal** *to the boundary $\partial\Omega$ at x.*

For a smooth function $u(x)$, we denote the **gradient** of u by $Du = \nabla u = (u_{x_1}, \cdots, u_{x_n})$ and define the **Laplacian** of u by

$$\Delta u = \operatorname{div}(\nabla u) = \sum_{k=1}^{n} u_{x_k x_k}. \tag{3.2}$$

Lemma 3.2 (**Green's Identities**). *Assume Ω is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $u \in C^2(\bar{\Omega}), v \in C^1(\bar{\Omega})$. Then*

$$\int_\Omega v \Delta u\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS. \tag{3.3}$$

If $u, v \in C^2(\bar{\Omega})$, then

$$\int_\Omega (v \Delta u - u \Delta v)\, dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu})\, dS. \tag{3.4}$$

Equation (3.3) *is called* **Green's first identity**, *while Equation* (3.4) *is called* **Green's second identity**.

Proof. Assume $u \in C^2(\bar{\Omega}), v \in C^1(\bar{\Omega})$ and let $\vec{F} = v \nabla u$. Then $\vec{F} \in C^1(\bar{\Omega}; \mathbb{R}^n)$ and $\operatorname{div} \vec{F} = \nabla u \cdot \nabla v + v \Delta u$, and hence, by the Divergence Theorem,

$$\int_\Omega v \Delta u\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx = \int_{\partial\Omega} v \nabla u \cdot \nu\, dS = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS,$$

from which (3.3) follows. Now, assume $u, v \in C^2(\bar{\Omega})$. Exchanging u and v in (3.3), we obtain

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} dS. \tag{3.5}$$

Combining (3.3) and (3.5) yields (3.4). □

Definition 3.1. The equation $\Delta u = 0$ is called **Laplace's equation**. A C^2 function u satisfying $\Delta u = 0$ in an open set $\Omega \subseteq \mathbb{R}^n$ is called a **harmonic function** in Ω.

Dirichlet and Neumann (boundary) problems. The **Dirichlet (boundary) problem** for Laplace's equation is:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The **Neumann (boundary) problem** for Laplace's equation is:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Here f and g are given (boundary) functions.

Theorem 3.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$.*

(i) *Given any f, there can be at most one $C^2(\bar{\Omega})$-solution to the Dirichlet problem.*

(ii) *If the Neumann problem (3.7) has a $C^2(\bar{\Omega})$-solution u then $\int_{\partial\Omega} g\, dS = 0$; moreover, if Ω is connected, any two $C^2(\bar{\Omega})$-solutions to the Neumann problem must differ by a constant.*

Proof. In the Green's first identity (3.3), choose $u = v$, and we have

$$\int_\Omega u\Delta u\, dx = -\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu}\, dS. \tag{3.8}$$

(i) Let u_1, $u_2 \in C^2(\bar{\Omega})$ be any two solutions to the Dirichlet problem and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$; hence, by (3.8), $\nabla u = 0$; consequently $u \equiv$ constant on each connected component of Ω; however, since $u = 0$ on $\partial\Omega$, the constant must be zero. So $u \equiv 0$ and hence $u_1 = u_2$ in Ω.

(ii) Using (3.3) with $v \equiv 1$, we have

$$\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS.$$

Hence, if there is a $C^2(\bar{\Omega})$ solution u to (3.7) then $\int_{\partial\Omega} g(x) dS = 0$. Let u_1, $u_2 \in C^2(\bar{\Omega})$ be any two solutions to the Neumann problem (3.7) and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$; hence, by (3.8), $\nabla u = 0$ in Ω. If Ω is connected, then u must be a constant in Ω; therefore, any two C^2-solutions of the Neumann problem must differ by a constant. □

3.2. Fundamental solutions and Green's function

We try to seek a harmonic function $u(x)$ that depends only on the radius $r = |x|$, i.e., $u(x) = v(r)$, $r = |x|$ (**radial function**). Computing Δu for such a function leads to an ODE for v:

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

If $n = 1$, then $v(r) = r$ is a solution. For $n \geq 2$, let $s(r) = v'(r)$, and we have $s'(r) = -\frac{n-1}{r} s(r)$, which is a first-order linear ODE for $s(r)$; upon solving this ODE, we have that $s(r) = cr^{1-n}$. Consequently, we obtain a solution for $v(r)$ as follows:

$$v(r) = \begin{cases} Cr, & n = 1, \\ C \ln r, & n = 2, \\ Cr^{2-n}, & n \geq 3. \end{cases}$$

Note that $v(r)$ is well-defined for $r > 0$, but is singular at $r = 0$ when $n \geq 2$.

3.2.1. Fundamental solutions.

Definition 3.2. We call the function $\Phi(x) = \phi(|x|)$ the **fundamental solution** of Laplace's equation in $\mathbb{R}^n$, where

$$\phi(r) = \begin{cases} -\frac{1}{2} r, & n = 1, \\ -\frac{1}{2\pi} \ln r, & n = 2, \\ \frac{1}{n(n-2)\alpha_n} r^{2-n}, & n \geq 3. \end{cases} \tag{3.9}$$

Here, for $n \geq 3$, α_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 3.1. With the number α_n, it follows that a ball of radius ρ in $\mathbb{R}^n$ has the volume $\alpha_n \rho^n$ and the surface area $n\alpha_n \rho^{n-1}$. The constant appearing in the fundamental solution $\Phi(x)$ exactly assures the following theorem.

Theorem 3.4. *For any $f \in C_c^2(\mathbb{R}^n)$, define*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad (x \in \mathbb{R}^n).$$

Then $u \in C^2(\mathbb{R}^n)$ and solves the **Poisson's equation**

$$-\Delta u(x) = f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{3.10}$$

Proof. We only prove this for the case $n \geq 2$; the proof of the case $n = 1$ (where $\Phi(x) = -\frac{1}{2}|x|$) is left as an exercise.

1. Let $f \in C_c^2(\mathbb{R}^n)$. Fix any bounded open set $V \subset \mathbb{R}^n$ and take $x \in V$. Then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy = \int_{B(0,R)} \Phi(y) f(x-y)\, dy,$$

where $B(0,R)$ is a large ball in $\mathbb{R}^n$ such that $f(x-y) = 0$ for all $x \in V$ and $y \notin B(0, R/2)$. Since $\Phi(y)$ is integrable near $y = 0$ (see (3.11) below), by differentiation under the integral, we have

$$u_{x_i}(x) = \int_{B(0,R)} \Phi(y) f_{x_i}(x-y)\, dy, \quad u_{x_i x_j}(x) = \int_{B(0,R)} \Phi(y) f_{x_i x_j}(x-y)\, dy.$$

This proves that $u \in C^2(V)$. Since V is arbitrary, it follows that $u \in C^2(\mathbb{R}^n)$. Moreover

$$\Delta u(x) = \int_{B(0,R)} \Phi(y)\Delta_x f(x-y)\,dy = \int_{B(0,R)} \Phi(y)\Delta_y f(x-y)\,dy.$$

2. Fix $0 < \epsilon < R$. Write

$$\Delta u(x) = \int_{B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy + \int_{B(0,R)\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy =: I_\epsilon + J_\epsilon.$$

Now

$$|I_\epsilon| \le C\|D^2 f\|_{L^\infty} \int_{B(0,\epsilon)} |\Phi(y)|\,dy \le \begin{cases} C\int_0^\epsilon r|\ln r|\,dr & (n=2) \\ C\epsilon^2 & (n\ge 3). \end{cases} \tag{3.11}$$

Hence $I_\epsilon \to 0$ as $\epsilon \to 0^+$. For J_ϵ we apply the Green's second identity (3.4) with $\Omega = B(0,R)\setminus B(0,\epsilon)$ to have

$$\begin{aligned} J_\epsilon &= \int_\Omega \Phi(y)\Delta_y f(x-y)\,dy \\ &= \int_\Omega f(x-y)\Delta\Phi(y)\,dy + \int_{\partial\Omega}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right] dS_y \\ &= \int_{\partial\Omega}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right] dS_y \\ &= \int_{\partial B(0,\epsilon)}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right] dS_y, \end{aligned}$$

where $\nu_y = -\frac{y}{\epsilon}$ is the outer unit normal of $\partial\Omega$ on the sphere $\partial B(0,\epsilon)$. Now

$$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial\nu_y}\right| \le C|\phi(\epsilon)|\|Df\|_{L^\infty}\int_{\partial B(0,\epsilon)} dS \le C\epsilon^{n-1}|\phi(\epsilon)| \to 0$$

as $\epsilon \to 0^+$. Furthermore, $\nabla\Phi(y) = \phi'(|y|)\frac{y}{|y|}$; hence

$$\frac{\partial\Phi(y)}{\partial\nu_y} = \nabla\Phi(y)\cdot\nu_y = -\phi'(\epsilon) = \begin{cases} \frac{1}{2\pi}\epsilon^{-1} & (n=2) \\ \frac{1}{n\alpha_n}\epsilon^{1-n} & (n\ge 3), \end{cases} \qquad \text{for all } y\in\partial B(0,\epsilon).$$

That is, $\frac{\partial\Phi(y)}{\partial\nu_y} = \frac{1}{\int_{\partial B(0,\epsilon)} dS}$ and hence

$$\int_{\partial B(0,\epsilon)} f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}dS_y = \fint_{\partial B(0,\epsilon)} f(x-y)\,dS_y \to f(x),$$

as $\epsilon \to 0^+$. Combining all the above, we finally prove that

$$-\Delta u(x) = f(x) \quad \forall\, x\in\mathbb{R}^n.$$

□

Remark 3.2. The reason the function $\Phi(x)$ is called a fundamental solution of Laplace's equation is as follows. The function $\Phi(x)$ formally satisfies

$$-\Delta_x\Phi(x) = \delta_0 \quad \text{on } x\in\mathbb{R}^n,$$

where δ_0 is the **Dirac measure** concentrated at 0:

$$\langle\delta_0, f\rangle = f(0) \quad \forall\, f\in C_c^\infty(\mathbb{R}^n).$$

If $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$, we can formally compute that (in terms of distributions)

$$\begin{aligned} -\Delta u(x) &= -\int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\, dy = -\int_{\mathbb{R}^n} \Delta_y \Phi(x-y) f(y)\, dy \\ &= -\int_{\mathbb{R}^n} \Delta_y \Phi(y) f(x-y)\, dy = \langle \delta_0, f(x-\cdot)\rangle = f(x) \end{aligned}$$

3.2.2. Green's function. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $h \in C^2(\bar{\Omega})$ be any harmonic function in Ω.

Given any function $u \in C^2(\bar{\Omega})$, fix $x \in \Omega$ and $0 < \epsilon < \operatorname{dist}(x, \partial\Omega)$. Let $\Omega_\epsilon = \Omega \setminus B(x,\epsilon)$. Apply Green's second identity

$$\int_{\Omega_\epsilon} (u(y)\Delta v(y) - v(y)\Delta u(y))\, dy = \int_{\partial\Omega_\epsilon} \left(u(y)\frac{\partial v}{\partial \nu}(y) - v(y)\frac{\partial u}{\partial \nu}(y) \right) dS$$

to functions $u(y)$ and $v(y) = \Gamma(x,y) = \Phi(y-x) - h(y)$ on Ω_ϵ, where $\Phi(y) = \phi(|y|)$ is the fundamental solution above, and since $\Delta v(y) = 0$ on Ω_ϵ, we have

$$\tag{3.12} \begin{aligned} -\int_{\Omega_\epsilon} \Gamma(x,y)\Delta u(y)\, dy &= \int_{\partial\Omega_\epsilon} u(y)\frac{\partial \Gamma}{\partial \nu_y}(x,y) dS - \int_{\partial\Omega_\epsilon} \Gamma(x,y)\frac{\partial u}{\partial \nu_y}(y) dS \\ &= \int_{\partial\Omega} u(y)\frac{\partial \Gamma}{\partial \nu_y}(x,y) dS - \int_{\partial\Omega} \Gamma(x,y)\frac{\partial u}{\partial \nu_y}(y) dS \\ &\quad + \int_{\partial B(x,\epsilon)} u(y) \left(\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial h}{\partial \nu}(y) \right) dS \\ &\quad - \int_{\partial B(x,\epsilon)} (\Phi(y-x) - h(y))\frac{\partial u}{\partial \nu}(y) dS, \end{aligned}$$

where $\nu = \nu_y$ is the outer unit normal at $y \in \partial\Omega_\epsilon = \partial\Omega \cup \partial B(x,\epsilon)$. Note that $\nu_y = -\frac{y-x}{\epsilon}$ at $y \in \partial B(x,\epsilon)$. Hence $\frac{\partial \Phi}{\partial \nu_y}(y-x) = -\phi'(\epsilon) = \frac{1}{\int_{\partial B(x,\epsilon)} dS}$ for $y \in \partial B(x,\epsilon)$. So, in (3.12), letting $\epsilon \to 0^+$ and noting that

$$\begin{aligned} &\int_{\partial B(x,\epsilon)} u(y) \left(\frac{\partial \Phi}{\partial \nu_y}(y-x) - \frac{\partial h}{\partial \nu_y}(y) \right) dS_y \\ &= \fint_{\partial B(x,\epsilon)} u(y) dS_y - \int_{\partial B(x,\epsilon)} u(y)\frac{\partial h}{\partial \nu_y}(y) dS_y \to u(x) \end{aligned}$$

and

$$\int_{\partial B(x,\epsilon)} (\Phi(y-x) - h(y))\frac{\partial u}{\partial \nu_y}(y) dS \to 0,$$

we deduce

Theorem 3.5 (Representation formula)**.** *Let $\Gamma(x,y) = \Phi(y-x) - h(y)$, where $h \in C^2(\bar{\Omega})$ is harmonic in Ω. Then, for all $u \in C^2(\bar{\Omega})$,*

$$\tag{3.13} u(x) = \int_{\partial\Omega} \left[\Gamma(x,y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(x,y) \right] dS - \int_{\Omega} \Gamma(x,y)\Delta u(y)\, dy \quad (x \in \Omega).$$

This formula permits us to solve for u *if* we know the values of Δu in Ω and *both* u and $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. However, for Poisson's equation with Dirichlet boundary condition, $\partial u/\partial \nu$ is not known (and cannot be prescribed arbitrarily). We must modify this formula to remove the boundary integral term involving $\partial u/\partial \nu$.

Given $x \in \Omega$, we assume that there exists a **corrector function** $h = h^x \in C^2(\bar{\Omega})$ solving the special Dirichlet problem:

$$\begin{cases} \Delta_y h^x(y) = 0 & (y \in \Omega), \\ h^x(y) = \Phi(y-x) & (y \in \partial\Omega). \end{cases} \tag{3.14}$$

Definition 3.3. We define **Green's function** for domain Ω to be the function

$$G(x,y) = \Phi(y,x) - h^x(y) \quad (x \in \Omega,\ y \in \bar{\Omega},\ x \neq y).$$

Then $G(x,y) = 0$ for $y \in \partial\Omega$ and $x \in \Omega$; hence, by (3.13),

$$u(x) = -\int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu_y}(x,y) dS - \int_\Omega G(x,y)\Delta u(y)\, dy. \tag{3.15}$$

The function

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) \quad (x \in \Omega,\ y \in \partial\Omega)$$

is called **Poisson's kernel** for domain Ω. Given a function g on $\partial\Omega$, the function

$$K[g](x) = \int_{\partial\Omega} K(x,y) g(y) dS_y \quad (x \in \Omega)$$

is called the **Poisson integral of g with kernel K**.

Remark 3.3. A corrector function h^x, if exists for bounded domain Ω, must be unique. Here we require the corrector function h^x exist in $C^2(\bar{\Omega})$, which may not be possible for general bounded domains Ω. However, for bounded domains Ω with smooth boundary, existence of h^x in $C^2(\bar{\Omega})$ is guaranteed by the general existence and regularity theory and consequently for such domains the Green's function always exists and is unique; we do not discuss these issues in this course.

Theorem 3.6 (Representation by Green's function)**.** *If $u \in C^2(\bar{\Omega})$ solves the Dirichlet problem*

$$\begin{cases} -\Delta u(x) = f(x) & (x \in \Omega), \\ u(x) = g(x) & (x \in \partial\Omega), \end{cases}$$

then

$$u(x) = \int_{\partial\Omega} K(x,y) g(y) dS + \int_\Omega G(x,y) f(y)\, dy \quad (x \in \Omega). \tag{3.16}$$

Theorem 3.7 (Symmetry of Green's function)**.** *$G(x,y) = G(y,x)$ for all $x, y \in \Omega,\ x \neq y$.*

Proof. Fix $x, y \in \Omega$, $x \neq y$. Let

$$v(z) = G(x,z), \quad w(z) = G(y,z) \qquad (z \in U).$$

Then $\Delta v(z) = 0$ for $z \neq x$ and $\Delta w(z) = 0$ for $z \neq y$ and $v|_{\partial\Omega} = w|_{\partial\Omega} = 0$. For sufficiently small $\epsilon > 0$, we apply Green's second identity on $\Omega_\epsilon = \Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$ for functions $v(z)$ and $w(z)$ to obtain

$$\int_{\partial\Omega_\epsilon} \left(v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z) \right) dS = 0.$$

This implies

$$\int_{\partial B(x,\epsilon)} \left(v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z) \right) dS = \int_{\partial B(y,\epsilon)} \left(w(z) \frac{\partial v}{\partial \nu}(z) - v(z) \frac{\partial w}{\partial \nu}(z) \right) dS, \tag{3.17}$$

where ν denotes the *inward* unit normal vector on $\partial B(x,\epsilon) \cup \partial B(x,\epsilon)$.

We compute the limits of two terms on both sides of (3.17) as $\epsilon \to 0^+$. For the term on LHS, since $w(z)$ is smooth near $z = x$,

$$\left| \int_{\partial B(x,\epsilon)} v(z) \frac{\partial w}{\partial \nu}(z) dS \right| \leq C\epsilon^{n-1} \sup_{z \in \partial B(x,\epsilon)} |v(z)| = o(1).$$

Also, $v(z) = \Phi(x - z) - h^x(z) = \Phi(z - x) - h^x(z)$, where the corrector h^x is smooth in Ω. Hence

$$\lim_{\epsilon \to 0^+} \int_{\partial B(x,\epsilon)} w(z) \frac{\partial v}{\partial \nu}(z) dS = \lim_{\epsilon \to 0^+} \int_{\partial B(x,\epsilon)} w(z) \frac{\partial \Phi}{\partial \nu}(z - x) dS = w(x).$$

So

$$\lim_{\epsilon \to 0^+} \text{LHS of (3.17)} = -w(x).$$

Similarly,

$$\lim_{\epsilon \to 0^+} \text{RHS of (3.17)} = -v(y),$$

proving $w(x) = v(y)$, which exactly shows that $G(y, x) = G(x, y)$. $\square$

Remark 3.4. (1) Strong Maximum Principle below implies that $G(x, y) > 0$ for all $x, y \in \Omega$, $x \neq y$. (**Homework!**) Since $G(x, y) = 0$ for $y \in \partial\Omega$, it follows that $\frac{\partial G}{\partial \nu_y}(x, y) \leq 0$, where ν_y is outer unit normal of Ω at $y \in \partial\Omega$. (In fact, we have $\frac{\partial G}{\partial \nu_y}(x, y) < 0$ for all $x \in \Omega$ and $y \in \partial\Omega$.)

(2) Since $G(x, y)$ is harmonic in $y \in \Omega \setminus \{x\}$, by the symmetry property, we know that $G(x, y)$ is also harmonic in $x \in \Omega \setminus \{y\}$. In particular, $G(x, y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$; hence Poisson's kernel $K(x, y) = -\frac{\partial G}{\partial \nu_y}(x, y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$.

(3) We always have that $K(x, y) \geq 0$ for all $x \in \Omega$ and $y \in \partial\Omega$ and that, by Green's representation theorem,

$$\int_{\partial\Omega} K(x, y)\, dS_y = 1 \quad (x \in \Omega).$$

The following general theorem implies that the Poisson integral gives a solution to the Dirichlet problem to Laplace's equation.

Theorem 3.8. *Let Ω be an open set in $\mathbb{R}^n$. Assume a function $K \colon \Omega \times \partial\Omega \to \mathbb{R}$ satisfies*

(i) $K(x, y) \geq 0$ *for all* $x \in \Omega$ *and* $y \in \partial\Omega$;

(ii) $K(\cdot, y)$ *is harmonic in* Ω *for each* $y \in \partial\Omega$;

(iii) $D_x^\alpha K(x, \cdot) \in L^1(\partial\Omega)$ *for all* $x \in \Omega$ *and multi-indexes* α *with* $|\alpha| \leq 2$;

(iv) $\int_{\partial\Omega} K(x, y)\, dS_y = 1$ *for all* $x \in \Omega$;

(v) *for each* $x^0 \in \partial\Omega$ *and* $\delta > 0$,

$$\lim_{x \to x^0,\, x \in \Omega} \int_{\partial\Omega \setminus B(x^0,\delta)} K(x, y)\, dS_y = 0.$$

Let $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ *and define*

$$u(x) = \int_{\partial\Omega} K(x, y) g(y)\, dS_y \quad (x \in \Omega).$$

Then u *is harmonic in* Ω *and satisfies*

$$\lim_{x \to x^0,\, x \in \Omega} u(x) = g(x^0) \quad (x^0 \in \partial\Omega). \tag{3.18}$$

Proof. That u is harmonic in Ω follows easily from (ii) and (iii). Let $M = \|g\|_{L^\infty}$. For $\varepsilon > 0$, let $\delta > 0$ be such that

$$|g(y) - g(x^0)| < \varepsilon \quad \forall\, y \in \partial\Omega, \;\; |y - x^0| < \delta.$$

Then, by (i) and (iv),

$$\begin{aligned}
|u(x) - g(x^0)| &= \left| \int_{\partial\Omega} K(x,y)(g(y) - g(x^0))\, dS_y \right| \le \int_{\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y \\
&\le \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y + \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\, dS_y \\
&\le \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y \\
&\le \varepsilon + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y.
\end{aligned}$$

Hence, by (v),

$$\limsup_{x\to x^0,\, x\in\Omega} |u(x) - g(x^0)| \le \varepsilon + 2M \limsup_{x\to x^0,\, x\in\Omega} \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y = \varepsilon.$$

This proves (3.18) and completes the proof. □

3.2.3. Green's functions for half spaces and balls. Although Green's function is defined above for a bounded domain with smooth boundary, it can be similarly defined for unbounded domains or domains with nonsmooth boundaries; however, the representation formula may not be valid for such domains. Green's functions for certain special domains Ω can be explicitly found from the fundamental solution $\Phi(x)$.

Case 1. Green's function for a half-space. Let

$$\Omega = \mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then its **reflection** with respect to the hyper-plane $x_n = 0$ is defined to be the point

$$\hat{x} = (x_1, x_2, \cdots, x_{n-1}, -x_n).$$

Clearly $\hat{\hat{x}} = x\, (x \in \mathbb{R}^n)$, $\hat{x} = x\, (x \in \partial\mathbb{R}^n_+)$ and $\Phi(x) = \Phi(\hat{x})\, (x \in \mathbb{R}^n)$. In this case, we can easily see that the corrector can be chosen as $h^x(y) = \Phi(y - \hat{x})$.

Definition 3.4. Green's function for half-space $\mathbb{R}^n_+$ is defined by

$$G(x,y) = \Phi(y - x) - \Phi(y - \hat{x}) \quad (x \in \mathbb{R}^n_+, \; y \in \bar{\mathbb{R}}^n_+, \; x \neq y).$$

Note that

$$\begin{aligned}
\frac{\partial G}{\partial y_n}(x,y) &= \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \hat{x}) \\
&= \frac{-1}{n\alpha_n}\left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \hat{x}|^n} \right].
\end{aligned}$$

So, the corresponding **Poisson's kernel of half-space** $\mathbb{R}^n_+$ is given by

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{\partial G}{\partial y_n}(x,y) = \frac{2x_n}{n\alpha_n |x - y|^n} \quad (x \in \mathbb{R}^n_+, \; y \in \partial\mathbb{R}^n_+).$$

If, for $y \in \partial\mathbb{R}^n_+$, we write $y = (y', 0)$ with $y' \in \mathbb{R}^{n-1}$, then

$$K(x,y) = \frac{2x_n}{n\alpha_n (|x' - y'|^2 + x_n^2)^{n/2}} := H(x, y')$$

and the Poisson integral $u = K[g]$ of a function $g \in C(\partial\mathbb{R}^n_+)$ can be written as

$$u(x) = \int_{\mathbb{R}^{n-1}} H(x,y')g(y')dy' = \frac{2x_n}{n\alpha_n}\int_{\mathbb{R}^{n-1}} \frac{g(y')\,dy'}{(|x'-y'|^2+x_n^2)^{n/2}} \quad (x\in\mathbb{R}^n_+). \tag{3.19}$$

Theorem 3.9 (Poisson's formula for half-space)**.** *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and $u = K[g]$ is defined by* (3.19)*. Then,*

(i) *$u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ is harmonic in $\mathbb{R}^n_+$,*

(ii) *for all $x^0 \in \partial\mathbb{R}^n_+$,*

$$\lim_{x\to x^0,\, x\in\mathbb{R}^n_+} u(x) = g(x^0).$$

Proof. It is easily verified that $D^\alpha_x H(x,y') \in L^1(\mathbb{R}^{n-1}_{y'})$ for all $x\in\mathbb{R}^n_+$ and multi-indexes α and that $H(x,y')$ is harmonic in $x \in \mathbb{R}^n_+$ for each $y'\in\mathbb{R}^{n-1}$; also, a complicated computation shows that

$$\int_{\mathbb{R}^{n-1}} H(x,y')\,dy' = 1 \quad (x\in\mathbb{R}^n_+).$$

Hence the conclusion (i) follows easily. Conclusion (ii) will follow from the general theorem Theorem 3.8 if we verify the condition (v) there. So let $x^0 \in \partial\mathbb{R}^n_+$ and $\delta > 0$. Then , if $|x-x^0| < \delta/2$ and $|y'-x^0| \geq \delta$, then we have

$$|y'-x^0| \leq |y'-x| + \delta/2 \leq |y'-x| + \frac{1}{2}|y'-x^0|;$$

so $|y'-x| \geq \frac{1}{2}|y'-x^0|$ and hence $H(x,y') \leq \frac{2^{n+1}x_n}{n\alpha_n}|y'-x^0|^{-n}$. Therefore

$$\int_{\mathbb{R}^{n-1}\setminus B(x^0,\delta)} H(x,y')\,dy' \leq \frac{2^{n+1}x_n}{n\alpha_n}\int_{\mathbb{R}^{n-1}\setminus B(x^0,\delta)} |y'-x^0|^{-n}\,dy'$$
$$= \frac{2^{n+1}x_n}{n\alpha_n}\left((n-1)\alpha_{n-1}\int_\delta^\infty r^{-n}r^{n-2}\,dr\right) = 2^{n+1}\frac{(n-1)\alpha_{n-1}}{n\alpha_n\delta}\,x_n \to 0,$$

as $x_n \to 0^+$ if $x \to x^0$ in $\mathbb{R}^n_+$, which proves (v) of Theorem 3.8. ☐

Case 2. Green's function for a ball. Let

$$\Omega = B(0,1) = \{x\in\mathbb{R}^n \mid |x| < 1\}$$

be the unit ball in $\mathbb{R}^n$. If $x \in \mathbb{R}^n \setminus \{0\}$, the point

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the **inversion point** of x with respect to unit sphere $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is called the **inversion with respect to unit sphere**.

Given $x \in B(0,1)$, $x \neq 0$, we try to find the corrector $h^x(y)$ in the form of

$$h^x(y) = \Phi(b(x)(y-\tilde{x})).$$

For this we need to have

$$|b(x)||y-\tilde{x}| = |y-x| \quad (y \in \partial B(0,1)).$$

Note that, if $y \in \partial B(0,1)$ then $|y| = 1$ and

$$|y-\tilde{x}|^2 = 1 - 2y\cdot\tilde{x} + |\tilde{x}|^2 = 1 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2} = \frac{|y-x|^2}{|x|^2}.$$

So we can choose $b(x) = |x|$. Consequently, for $x \neq 0$, the corrector h^x is given by

$$h^x(y) = \Phi(|x|(y - \tilde{x})) \quad (y \in \bar{B}(0,1)).$$

Definition 3.5. Green's function for unit ball $B(0,1)$ is given by

$$G(x,y) = \begin{cases} \Phi(y-x) - \Phi(|x|(y-\tilde{x})) & (x \neq 0, \ \ x \neq y), \\ G(y,0) = \Phi(y) - \Phi(|y|\tilde{y}) = \Phi(y) - \phi(1) & (x = 0, \ \ y \neq 0). \end{cases}$$

(Note that $G(0,y)$ cannot be given by the first formula since $\tilde{0}$ is undefined, but it is found from the symmetry of G: $G(0,y) = G(y,0)$ for $y \neq 0$.)

Since $\Phi_{y_i}(y) = \phi'(|y|)\frac{y_i}{|y|} = \frac{-y_i}{n\alpha_n |y|^n}$ ($y \neq 0$ and $n \geq 2$), we deduce that, if $x \neq 0$, $x \neq y$, then

$$\begin{aligned} G_{y_i}(x,y) &= \Phi_{y_i}(y-x) - \Phi_{y_i}(|x|(y-\tilde{x}))|x| \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y-x|^n} - \frac{|x|^2((\tilde{x})_i - y_i)}{(|x||y-\tilde{x}|)^n}\right] \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y-x|^n} - \frac{x_i - |x|^2 y_i}{(|x||y-\tilde{x}|)^n}\right] \end{aligned}$$

So, if $y \in \partial B(0,1)$, since $|x||y-\tilde{x}| = |y-x|$ and $\nu_y = y$, we have

$$\begin{aligned} \frac{\partial G}{\partial \nu_y}(x,y) &= \sum_{i=1}^n G_{y_i}(x,y) y_i = \frac{1}{n\alpha_n}\sum_{i=1}^n \left[\frac{x_i y_i - y_i^2}{|y-x|^n} - \frac{x_i y_i - |x|^2 y_i^2}{(|x||y-\tilde{x}|)^n}\right] \\ &= \frac{1}{n\alpha_n}\frac{|x|^2 - 1}{|y-x|^n} \quad (x \in B(0,1) \setminus \{0\}). \end{aligned}$$

The same formula also holds for $x = 0$ and $y \in \partial B(0,1)$.

Therefore, the **Poisson's kernel for unit ball** $B(0,1)$ is given by

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{1 - |x|^2}{n\alpha_n |y-x|^n} \quad (x \in B(0,1), \ \ y \in \partial B(0,1)).$$

Given $g \in C(\partial B(0,1))$, its Poisson integral $u = K[g]$ is given by

$$u(x) = \int_{\partial B(0,1)} K(x,y) g(y)\, dS_y = \frac{1 - |x|^2}{n\alpha_n}\int_{\partial B(0,1)} \frac{g(y)\, dS_y}{|y-x|^n} \quad (x \in B(0,1)). \tag{3.20}$$

By Green's representation formula, the $C^2(\bar{B}(0,1))$-solution u of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{on } \partial B(0,1), \end{cases}$$

is given by the formula (3.20).

Suppose u is the C^2-solution to the Dirichlet problem on a closed ball $\bar{B}(a,R)$, of center a and radius R:

$$\begin{cases} \Delta u = 0 & \text{in } B(a,R), \\ u = g & \text{on } \partial B(a,R). \end{cases}$$

Let $\tilde{u}(x) = u(a + Rx)$ and $\tilde{g}(x) = g(a + Rx)$ for $x \in \bar{B}(0,1)$. Then $\tilde{u}$ solves the Dirichlet problem on unit ball $B(0,1)$ with boundary data $\tilde{g}$. By formula (3.20) with $\tilde{g}$ we have, for all $x \in B(a,R)$,

$$u(x) = \tilde{u}\left(\frac{x-a}{R}\right) = \frac{1 - |\frac{x-a}{R}|^2}{n\alpha_n}\int_{\partial B(0,1)} \frac{g(a+Ry)\, dS_y}{|y - \frac{x-a}{R}|^n}$$

$$= \frac{R^2 - |x-a|^2}{n\alpha_n R^2} \int_{\partial B(a,R)} \frac{g(z) R^{1-n}\, dS_y}{R^{-n}|z-x|^n} \quad (z = a + Ry).$$

Hence, changing z back to y,

$$u(x) = \frac{R^2 - |x-a|^2}{n\alpha_n R} \int_{\partial B(a,R)} \frac{g(y)\, dS_y}{|y-x|^n} = \int_{\partial B(a,R)} K(x,y;a,R) g(y)\, dS_y, \tag{3.21}$$

where

$$K(x,y;a,R) = \frac{R^2 - |x-a|^2}{n\alpha_n R |y-x|^n} \quad (x \in B(a,R), \;\; y \in \partial B(a,R))$$

is the **Poisson's kernel for general ball** $B(a,R)$.

The formula (3.21) is called the **Poisson's formula on ball** $B(a,R)$. This formula has a special consequence if we take $x = a$, which gives

$$u(a) = \frac{R}{n\alpha_n} \int_{\partial B(a,R)} \frac{g(y)}{|y-a|^n}\, dS_y = \fint_{\partial B(a,R)} g(y)\, dS_y.$$

(Note that $|\partial B(a,R)| = n\alpha_n R^{n-1}$.) Therefore, if u is harmonic in a domain Ω and $B(a,r) \subset\subset \Omega$ (this means $\bar{B}(a,r) \subset \Omega$), then

$$u(a) = \fint_{\partial B(a,r)} u(y)\, dS_y. \tag{3.22}$$

This is the **mean-value property** for harmonic functions; we will give another proof in the next section.

Theorem 3.10 (Poisson's formula for a ball)**.** *Assume $g \in C(\partial B(a,R))$ and $u = K[g]$ is defined by* (3.21)*. Then,*

(i) *$u \in C^\infty(B(a,R))$ is harmonic in $B(a,R)$;*

(ii) *for each $x^0 \in \partial B(a,R)$,*

$$\lim_{x \to x^0,\, x \in B(a,R)} u(x) = g(x^0).$$

Proof. The result (i) follows since the Poisson kernel $K(x,y;a,R)$ is harmonic and C^∞ on x in $B(a,R)$ for all $y \in \partial B(a,R)$, while the result (ii) follows from Theorem 3.8 because, for each $x^0 \in \partial B(a,R)$ and $\delta > 0$,

$$\lim_{x \to x^0,\, x \in B(a,R)} \int_{\partial B(a,R) \setminus B(x^0,\delta)} K(x,y;a,R)\, dS_y = \int_{\partial B(a,R) \setminus B(x^0,\delta)} K(x^0,y;a,R)\, dS_y = 0.$$

□

3.3. Mean-value property

For two sets U and V in $\mathbb{R}^n$ we write $V \subset\subset U$ if $\bar{V}$ is a *compact subset* of U.

Theorem 3.11 (Mean-value property for harmonic functions)**.** *Let $u \in C^2(\Omega)$ be harmonic. Then*

$$u(x) = \fint_{\partial B(x,r)} u(y) dS = \fint_{B(x,r)} u(y)\, dy$$

for each ball $B(x,r) \subset\subset \Omega$.

Proof. The first equality (called the **spherical mean-value property**) has been proved above. We give a different proof. Let $B(x,r) \subset\subset \Omega$. For each $\rho \in (0,r]$, let

$$h(\rho) = \fint_{\partial B(x,\rho)} u(y)\, dS_y = \fint_{\partial B(0,1)} u(x+\rho z)\, dS_z.$$

Then, by Green's first identity,

$$\begin{aligned}
h'(\rho) &= \fint_{\partial B(0,1)} \nabla u(x+\rho z)\cdot z\, dS_z = \fint_{\partial B(x,\rho)} \nabla u(y)\cdot \frac{y-x}{\rho}\, dS_y \\
&= \fint_{\partial B(x,\rho)} \nabla u(y)\cdot \nu_y\, dS_y = \fint_{\partial B(x,\rho)} \frac{\partial u(y)}{\partial \nu_y}\, dS_y \\
&= \frac{1}{n\alpha_n \rho^{n-1}} \int_{\partial B(x,\rho)} \frac{\partial u(y)}{\partial \nu_y}\, dS_y = \frac{1}{n\alpha_n\rho^{n-1}} \int_{B(x,\rho)} \Delta u(y)\, dy \\
&= \frac{\rho}{n} \fint_{B(x,\rho)} \Delta u(y)\, dy.
\end{aligned}$$

Hence, since $\Delta u = 0$ in Ω, it follows that $h'(\rho) = 0$ on $\rho \in (0,r]$ and so h is constant on $(0,r]$; hence,

$$h(r) = h(0^+) = \lim_{\rho\to 0^+} \fint_{\partial B(x,\rho)} u(y)\, dS_y = u(x),$$

which proves the spherical mean-value property. From this, we have

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,\rho)} u(y)\, dS_y \right) d\rho = u(x) \int_0^r n\alpha_n \rho^{n-1}\, d\rho = u(x)\alpha_n r^n,$$

which proves the **ball mean-value property**: $u(x) = \fint_{B(x,r)} u(y)\, dy$. □

Theorem 3.12 (Converse to mean-value property)**.** *Let $u \in C^2(\Omega)$ satisfy*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS_y$$

for all $x \in \Omega$ and $0 < r < r_x \le \operatorname{dist}(x, \partial\Omega)$, where $r_x > 0$ is a number depending on x. Then u is harmonic in Ω.

Proof. Suppose $\Delta u(x_0) \neq 0$ for some $x_0 \in \Omega$. WLOG, assume $\Delta u(x_0) > 0$. Then there exists a ball $B(x_0, r)$ with $0 < r < r_{x_0}$ such that $\Delta u(y) > 0$ on $\bar{B}(x_0, r)$. Consider the function

$$h(\rho) = \fint_{\partial B(x_0,\rho)} u(y)\, dS_y \quad (0 < \rho < r_{x_0}).$$

The assumption says that h is constant on $(0, r_{x_0}]$. However, by the computation as above, $h'(r) = \frac{r}{n}\fint_{B(x_0,r)} \Delta u(y)\, dy > 0$, giving a desired contradiction. □

This result actually holds under a much weaker assumption that u is only continuous.

Theorem 3.13. *Let $u \in C(\Omega)$ satisfy*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS_y$$

for all $x \in \Omega$ and $0 < r < r_x \le \operatorname{dist}(x, \partial\Omega)$, where $r_x > 0$ is a number depending on x. Then u is harmonic in Ω.

Proof. See Lemma 3.26 below. □

3.4. Maximum principles

Theorem 3.14 (Maximum principle for harmonic functions). *Let Ω be bounded open in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω.*

(i) (**Weak maximum principle**) *We have that $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.*

(ii) (**Strong maximum principle**) *If, in addition, Ω is connected and there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then $u(x) \equiv u(x_0)$ for all $x \in \Omega$.*

Proof. Note that (ii) implies (i) (**Explain why?**) To prove (2), let

$$S = \{x \in \Omega \mid u(x) = u(x_0)\}.$$

This set is nonempty since $x_0 \in S$. It is relatively closed in Ω since u is continuous. We show that S is open; hence $S = \Omega$ since Ω is connected. Let $x \in S$; so $u(x) = u(x_0) = \max_{\bar{\Omega}} u$. Assume $B(x,r) \subset\subset \Omega$. By the ball mean-value property,

$$u(x) = \fint_{B(x,r)} u(y)dy \le \fint_{B(x,r)} u(x)\,dy = u(x).$$

So the equality holds, which implies $u(y) = u(x)$ for all $y \in B(x,r)$; hence $B(x,r) \subset S$ and thus S is open. □

If u is harmonic then $-u$ is also harmonic; hence the **minimum principles** also hold. In particular, we have the following **positivity property** for harmonic functions:

Corollary 3.15. *Let Ω be connected, bounded and open in $\mathbb{R}^n$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω and $u|_{\partial\Omega} \ge 0$ but $\not\equiv 0$, then $u(x) > 0$ for all $x \in \Omega$.*

From the maximum principle, we easily have the uniqueness of Dirichlet problem.

Theorem 3.16 (Uniqueness for Dirichlet problem). *Let Ω be bounded open in $\mathbb{R}^n$. Then, given f, g, the Dirichlet problem for Poisson's equation*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

can have at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Theorem 3.17 (C^∞**-regularity of harmonic functions**). *If $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$.*

Proof. Let $a \in \Omega$ and $B(a,R) \subset\subset \Omega$. Then $g(y) = u(y)$ is continuous on $y \in \partial B(a,R)$. Let $\tilde{u}(x) = \int_{\partial B(a,R)} K(x,y;a,R) g(y)\,dS_y$ for $x \in B(a,R)$ be the Poisson integral of g on $B(a,R)$, and extend $\tilde{u}$ to $\partial B(a,R)$ by defining $\tilde{u}(y) = g(y) = u(y)$ for $y \in \partial B(a,R)$. Then u and $\tilde{u}$ are both solutions to Laplace's equation with the same boundary boundary data $g(y) = u(y)$ in $C^2(B(a,R)) \cap C(\bar{B}(a,R))$. Hence, by the uniqueness theorem above, $u \equiv \tilde{u}$ in $B(a,R)$. However, by Theorem 3.10, $\tilde{u} \in C^\infty(B(a,R))$; this proves that $u \in C^\infty(\Omega)$. □

3.5. Estimates of higher-order derivatives and Liouville's theorem

Theorem 3.18 (Local estimates on derivatives). *Assume u is harmonic in Ω. Then*

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x,r))} \tag{3.23}$$

for each ball $B(x,r) \subset\subset \Omega$, each $k = 1, 2, \cdots$, and each multi-index α of order $|\alpha| = k$.

Proof. We prove by induction on k.

1. If $k = 1$, say $\alpha = (1, 0, \cdots, 0)$ and so $D^\alpha u = u_{x_1} = D_1 u$. Since $u \in C^\infty$ and is harmonic, $D_1 u$ is also harmonic; hence, if $B(x,r) \subset\subset \Omega$, then

$$D_1 u(x) = ⨍_{B(x,r/2)} D_1 u(y)\, dy = \frac{2^n}{\alpha_n r^n} \int_{B(x,r/2)} D_1 u(y)\, dy$$
$$= \frac{2^n}{\alpha_n r^n} \int_{\partial B(x,r/2)} u(y) \nu_1(y)\, dS_y.$$

So

$$|D_1 u(x)| \le \frac{2^n}{\alpha_n r^n} \int_{\partial B(x,r/2)} |u(y)|\, dS_y \le \frac{2n}{r} \max_{\partial B(x,r/2)} |u|.$$

However, for each $y \in \partial B(x, r/2)$, one has $B(y, r/2) \subset B(x,r) \subset\subset \Omega$, and hence

$$|u(y)| = \left| ⨍_{B(y,r/2)} u(z) dz \right| \le \frac{2^n}{\alpha_n r^n} \|u\|_{L^1(B(x,r))}.$$

Combining these estimates, we have

$$|D_1 u(x)| \le \frac{2^{n+1} n}{\alpha_n r^{n+1}} \|u\|_{L^1(B(x,r))}.$$

This proves (3.23) when $k = 1$ with constant $C_1 = \frac{2^{n+1} n}{\alpha_n}$.

2. Assume now $k \ge 2$ and let $|\alpha| = k$. Then for some i we have $\alpha = \beta + (0, \cdots, 1, 0, \cdots, 0)$, where $|\beta| = k - 1$ and 1 is in the i-th place. So $D^\alpha u = (D^\beta u)_{x_i}$ and hence, as in Step 1,

$$|D^\alpha u(x)| \le \frac{nk}{r} \|D^\beta u\|_{L^\infty(B(x,r/k))}.$$

If $y \in B(x, r/k)$ then $B(y, \frac{k-1}{k} r) \subset B(x,r)$; hence, by the induction assumption for $D^\beta u$ at y, we have

$$|D^\beta u(y)| \le \frac{C_{k-1}}{(\frac{k-1}{k} r)^{n+k-1}} \|u\|_{L^1(B(y, \frac{k-1}{k} r))} \le \frac{C_{k-1} (\frac{k}{k-1})^{n+k-1}}{r^{n+k-1}} \|u\|_{L^1(B(x,r))}.$$

Combining the previous estimates we derive that

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x,r))},$$

where $C_k \ge C_{k-1} nk (\frac{k}{k-1})^{n+k-1}$. For example, we can choose

$$C_k = \frac{(2^{n+1} nk)^k}{\alpha_n} \quad \forall\, k = 1, 2, \cdots.$$

□

Theorem 3.19 (Liouville's Theorem)**.** *Suppose u is a bounded harmonic function on whole $\mathbb{R}^n$. Then u is constant.*

Proof. Let $|u(y)| \le M$ on $y \in \mathbb{R}^n$. By (3.23) with $k = 1$, for each $i = 1, 2, \cdots, n$,

$$|u_{x_i}(x)| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x,r))} \le \frac{C_1}{r^{n+1}} M \alpha_n r^n = \frac{M C_1 \alpha_n}{r}.$$

This inequality holds for all $r > 0$ since $B(x,r) \subset\subset \mathbb{R}^n$. Taking $r \to \infty$ gives $u_{x_i}(x) = 0$ for all $x \in \mathbb{R}^n$ and all $i = 1, 2, \cdots, n$. Hence $\nabla u \equiv 0$ and so u is a constant on $\mathbb{R}^n$. □

Theorem 3.20 (Representation formula)**.** *Let $n \geq 3$ and $f \in C_c^\infty(\mathbb{R}^n)$. Then any bounded solution of $-\Delta u = f$ on $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C$$

for some constant C.

Proof. Since $n \geq 3$ and thus $\Phi(y) \to 0$ as $|y| \to \infty$, it follows that the **Newton potential**

$$\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad (x \in \mathbb{R}^n)$$

is bounded on $\mathbb{R}^n$; indeed, if $\operatorname{supp} f \subset B(0,R)$ then, for all $|x| \geq R+1$ and $|y| \leq R$, we have $|x-y| \geq |x| - |y| > 1$ and hence $\Phi(x-y) \leq \frac{1}{n(n-2)\alpha_n}$; so

$$|\tilde{u}(x)| \leq \int_{B(0,R)} \Phi(x-y)|f(y)|\, dy \leq \frac{R^n}{n(n-2)} \|f\|_{L^\infty},$$

and thus $\tilde{u}$ is bounded on $\mathbb{R}^n$. This $\tilde{u}$ solves the Poisson equation $-\Delta \tilde{u} = f$ and hence $u - \tilde{u}$ is bounded harmonic on $\mathbb{R}^n$. By Liouville's theorem, $u = \tilde{u} + C$ for a constant C. ☐

Remark 3.5. If $n = 2$, the representation formula may not hold; for example, if $\int_{\mathbb{R}^n} f(y)\, dy \neq 0$ then, as $|x| \to \infty$,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \sim \Phi(x) \int_{\mathbb{R}^n} f(y)\, dy$$

is not bounded.

Theorem 3.21 (Compactness of sequence of harmonic functions)**.** *Suppose $\{u^j\}$ is a sequence of harmonic functions in Ω and*

$$|u^j(x)| \leq M \quad (x \in \Omega, \ \ j = 1, 2, \cdots).$$

Let $V \subset\subset \Omega$. Then there exists a subsequence $\{u^{j_k}\}$ and a harmonic function $\bar{u}$ in V such that

$$\lim_{k \to \infty} \|u^{j_k} - \bar{u}\|_{L^\infty(V)} = 0.$$

Proof. Let $0 < r < \operatorname{dist}(V, \partial\Omega)$. Then $B(x,r) \subset\subset \Omega$ for all $x \in \bar{V}$. Applying the derivative estimates,

$$|Du^j(x)| \leq \frac{C_1 M}{r} \quad \forall\, x \in \bar{V}, \ \ j = 1, 2, \cdots.$$

Consequently, by **Arzela-Ascoli's theorem**, the family $\{u^j\}$ is uniformly bounded and equi-continuous on $\bar{V}$, and hence there exists a subsequence $\{u^{j_k}\}$ converging uniformly in $\bar{V}$ to a continuous function $\bar{u}$ on $\bar{V}$. This uniform limit $\bar{u}$ certainly satisfies the mean-value property in V and hence must be harmonic in V. ☐

Theorem 3.22 (Harnack's Inequality)**.** *For each subdomain $V \subset\subset \Omega$, there exists a constant $C = C(V, \Omega)$ such that the inequality*

$$\sup_V u \leq C \inf_V u$$

holds for all nonnegative harmonic *functions u in Ω.*

Proof. Let $r = \frac{1}{4}\operatorname{dist}(V, \partial\Omega)$. Let $x, y \in V$ with $|x-y| \le r$. Then $B(y,r) \subset B(x,2r) \subset\subset \Omega$; hence, for all nonnegative harmonic functions u in Ω,

$$u(x) = \frac{1}{\alpha_n (2r)^n} \int_{B(x,2r)} u(z)\,dz \ge \frac{1}{\alpha_n 2^n r^n} \int_{B(y,r)} u(z)\,dz = \frac{1}{2^n} ⨍_{B(y,r)} u(z) dz = \frac{1}{2^n} u(y).$$

Therefore, $\frac{1}{2^n} u(y) \le u(x) \le 2^n u(y)$ for all $x, y \in V$ with $|x-y| \le r$.

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 1, 2, \cdots, N$. Then it follows that

$$u(x) \ge \frac{1}{2^{n(N+1)}} u(y) \quad \forall\, x,\ y \in V.$$

This completes the proof. □

3.6. Perron's method for Dirichlet problem of Laplace's equation

(*This material is not covered in Evans's book, but can be found in other textbooks, e.g., John's book mentioned in the syllabus.*)

Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C(\partial\Omega)$. We now discuss **Perron's method** of **subharmonic functions** to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Definition 3.6. We say a function $u \in C(\Omega)$ is **subharmonic** in Ω if for every $\xi \in \Omega$ the inequality

$$u(\xi) \le ⨍_{\partial B(\xi,\rho)} u(x)\,dS := M_u(\xi, \rho)$$

holds for all sufficiently small $\rho > 0$.

We denote by $\sigma(\Omega)$ the set of all subharmonic functions in Ω.

Lemma 3.23. *For $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$,*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Homework. □

Definition 3.7. For any $u \in C(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we define the **harmonic lifting** of u on $B(\xi,\rho)$ to be the function

$$u_{\xi,\rho}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus B(\xi,\rho), \\ \frac{\rho^2 - |x-\xi|^2}{n\alpha_n \rho} \int_{\partial B(\xi,\rho)} \frac{u(y)}{|y-x|^n}\,dS_y & \text{if } x \in B(\xi,\rho). \end{cases} \tag{3.24}$$

Note that in $B(\xi,\rho)$ the function $u_{\xi,\rho}$ is simply the Poisson integral of $u|_{\partial B(\xi,\rho)}$ and hence is harmonic in $B(\xi,\rho)$ and takes the boundary value $u|_{\partial B(\xi,\rho)}$ on $\partial B(\xi,\rho)$; therefore, $u_{\xi,\rho}$ is in $C(\Omega)$.

Lemma 3.24. *For $u \in \sigma(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we have $u_{\xi,\rho} \in \sigma(\Omega)$ and*

$$u(x) \le u_{\xi,\rho}(x) \quad \forall\, x \in \Omega. \tag{3.25}$$

Proof. We first prove (3.25). If $x \notin B(\xi,\rho)$ then $u(x) = u_{\xi,\rho}(x)$. Note that $u - u_{\xi,\rho}$ is subharmonic in $B(\xi,\rho)$ and equals zero on $\partial B(\xi,\rho)$; hence, by Lemma 3.23, $u - u_{\xi,\rho} \le 0$ in $B(\xi,\rho)$; this proves (3.25). We now prove that $u_{\xi,\rho}$ is subharmonic in Ω. We must show that for any $x \in \Omega$

$$u_{\xi,\rho}(x) \le ⨍_{\partial B(x,r)} u_{\xi,\rho}(y)\, dS_y = M_{u_{\xi,\rho}}(x,r) \tag{3.26}$$

for all sufficiently small $r > 0$. We first assume $x \notin \partial B(\xi,\rho)$; then there exists a ball $B(x,r') \subset\subset \Omega$ such that either $B(x,r') \subset \Omega \setminus \bar{B}(\xi,\rho)$ or $B(x,r') \subset B(\xi,\rho)$; hence, either $u_{\xi,\rho}(y) = u(y)$ for all $y \in B(x,r')$ or $u_{\xi,\rho}(y)$ is harmonic in $y \in B(x,r')$. In the first case, (3.26) holds if $0 < r < r'$ is sufficiently small since u is subharmonic, while in the second case, (3.26) holds if $0 < r < r'$ is sufficiently small since $u_{\xi,\rho}$ is harmonic. We now prove (3.26) if $x \in \partial B(\xi,\rho)$. In this case, by (3.25),

$$u_{\xi,\rho}(x) = u(x) \le M_u(x,r) \le M_{u_{\xi,\rho}}(x,r)$$

for all sufficiently small $r > 0$. ☐

Lemma 3.25. *For $u \in \sigma(\Omega)$, we have $u(\xi) \le M_u(\xi,\rho)$ whenever $B(\xi,\rho) \subset\subset \Omega$.*

Proof. Let $B(\xi,\rho) \subset\subset \Omega$. Then

$$u(\xi) \le u_{\xi,\rho}(\xi) = M_{u_{\xi,\rho}}(\xi,\rho) = M_u(\xi,\rho).$$

☐

Lemma 3.26. *Let $u \in C(\Omega)$. Then u is harmonic in Ω if and only if $\pm u \in \sigma(\Omega)$.*

Proof. Suppose $u, -u \in \sigma(\Omega)$. Then for all balls $B(\xi,\rho) \subset\subset \Omega$, by Lemma 3.24,

$$u \le u_{\xi,\rho}, \quad -u \le (-u)_{\xi,\rho} = -u_{\xi,\rho}.$$

This implies $u \equiv u_{\xi,\rho}$; hence u is harmonic in $B(\xi,\rho)$. Consequently u is harmonic in Ω. ☐

Let $g \in C(\partial\Omega)$. Define $\sigma_g(\Omega) = \{u \in C(\bar{\Omega}) \cap \sigma(\Omega) \mid u \le g \text{ on } \partial\Omega\}$, and

$$w_g(x) = \sup_{u \in \sigma_g(\Omega)} u(x) \quad (x \in \Omega). \tag{3.27}$$

Suppose $m = \min_{\partial\Omega} g$ and $M = \max_{\partial\Omega} g$. Then m, M are finite numbers and $m \in \sigma_g(\Omega)$; so $\sigma_g(\Omega)$ is nonempty. Also, by Lemma 3.23,

$$u(x) \le M \quad \forall\, u \in \sigma_g(\Omega), \;\; x \in \Omega.$$

Hence the function w_g is well-defined in Ω.

Lemma 3.27. *Let $v_1, v_2, \cdots, v_k \in \sigma_g(\Omega)$ and $v = \max\{v_1, v_2, \cdots, v_k\}$. Then $v \in \sigma_g(\Omega)$.*

Proof. Homework. ☐

Theorem 3.28. *The function w_g defined by (3.27) is harmonic in Ω.*

Proof. Let $\xi \in \Omega$, $B(\xi,\rho) \subset\subset \Omega$ and $0 < \rho' < \rho$ be given.

1. Assume $\{x^k\}_{k=1}^\infty$ is any sequence of points in $B(\xi,\rho')$. For each x^k, let $\{u_k^j\}_{j=1}^\infty$ be a sequence in $\sigma_g(\Omega)$ such that

$$w_g(x^k) = \lim_{j\to\infty} u_k^j(x^k) \quad (k = 1, 2, \cdots).$$

Define $u^j(x) = \max\{m, u_1^j(x), u_2^j(x), \cdots, u_j^j(x)\}$ for $x \in \bar{\Omega}$. Then $u^j \in \sigma_g(\Omega),\ m \le u^j(x) \le w_g(x)$ and

$$\lim_{j\to\infty} u^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Let $v^j = u^j_{\xi,\rho}$ be the harmonic lifting of u^j on $B(\xi, \rho)$. Then $v^j \in \sigma_g(\Omega)$ and $u^j(x) \le v^j(x) \le w_g(x)$ in Ω, and so

$$\lim_{j\to\infty} v^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Since $\{v^j\}$ is a bounded sequence of harmonic functions in $B(\xi, \rho)$, by the **compactness theorem** (Theorem 3.21), there exists a subsequence $\{v^{j_m}\}$ uniformly converging to a harmonic function W on $B(\xi, \rho')$. Hence

$$W(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots). \tag{3.28}$$

Note that the harmonic function W depends on the choice of points $\{x^k\}$ and the subsequence $\{v^{j_m}\}$. However the function w_g is independent of all these choices.

2. We first show that w_g is continuous in $B(\xi, \rho')$. Let $y^0 \in B(\xi, \rho')$, and let $\{y^k\}$ be any sequence in $B(\xi, \rho')$ converging to y^0. Define $x^1 = y^0$ and $x^k = y^k$ for all $k = 2, 3, \cdots$. With this sequence $\{x^k\}$ in $B(\xi, \rho')$ as in Step 1, we have, by (3.28) and the continuity of W, that

$$w_g(y^0) = W(y^0) = \lim_{k\to\infty} W(x^k) = \lim_{k\to\infty} w_g(x^k) = \lim_{k\to\infty} w_g(y^k).$$

This proves the continuity of w_g at $y^0 \in B(\xi, \rho')$.

3. We now prove that w_g is harmonic in $B(\xi, \rho')$. To show this, let $\{x^k\}$ be a **dense sequence** in $B(\xi, \rho')$. Then, by (3.28) and the continuity of w_g, it follows that $w_g \equiv W$ in $B(\xi, \rho')$. Since W is harmonic in $B(\xi, \rho')$, so is w_g in $B(\xi, \rho')$.

Finally, since $B(\xi, \rho')$ can be arbitrary, it follows that w_g is harmonic in whole Ω. □

The harmonic function w_g constructed is the candidate of a solution to our Dirichlet problem. To guarantee this, we need to study the behavior of w_g near the boundary under some specific property of the boundary $\partial\Omega$.

Definition 3.8. Given a boundary point $\eta \in \partial\Omega$, a function Q_η is said to be a **Barrier function** at η if $Q_\eta \in C(\bar{\Omega}) \cap \sigma(\Omega)$ such that

$$Q_\eta(\eta) = 0, \quad Q_\eta(x) < 0 \quad (x \in \bar{\Omega} \setminus \{\eta\}).$$

In this case, we say that the point $\eta \in \partial\Omega$ is **regular** or η is a **regular boundary point** of Ω.

Theorem 3.29. *If $\eta \in \partial\Omega$ is regular, then*

$$\lim_{x\to\eta,\, x\in\Omega} w_g(x) = g(\eta).$$

Proof. 1. We first prove

$$\liminf_{x\to\eta,\, x\in\Omega} w_g(x) \ge g(\eta).$$

Let $\varepsilon > 0$, $K > 0$ be given constants and define $u(x) = g(\eta) - \varepsilon + KQ_\eta(x)$ on $\bar{\Omega}$. Then $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$, $u(\eta) = g(\eta) - \varepsilon$, and $u(x) \le g(\eta) - \varepsilon$ on $\partial\Omega$. Since g is continuous, there exists a $\delta > 0$ such that $g(x) > g(\eta) - \varepsilon$ for all $x \in B(\eta, \delta) \cap \partial\Omega$. Hence $u(x) \le g(x)$ on $B(\eta, \delta) \cap \partial\Omega$. Since $Q_\eta < 0$ on the compact set $\partial\Omega \setminus B(\eta, \delta)$, it follows that $Q_\eta(x) \le -\gamma$ on $\partial\Omega \setminus B(\eta, \delta)$, where $\gamma > 0$ is a number. We now let $K = \frac{M-m}{\gamma} \ge 0$. Then

$$u(x) = g(\eta) - \varepsilon + KQ_\eta(x) \le M - K\gamma = m \le g(x) \quad (x \in \partial\Omega \setminus B(\eta, \delta)).$$

Therefore $u \le g$ on whole $\partial\Omega$. So $u \in \sigma_g(\Omega)$. Consequently, $u(x) \le w_g(x)$ for all $x \in \Omega$; so,

$$g(\eta) - \varepsilon = \lim_{x\to\eta,\, x\in\Omega} u(x) \le \liminf_{x\to\eta,\, x\in\Omega} w_g(x),$$

which completes the first step.

2. We now prove

$$\limsup_{x\to\eta,\, x\in\Omega} w_g(x) \le g(\eta).$$

We consider the function w_{-g} defined similarly with $-g$; namely,

$$w_{-g}(x) = \sup_{v\in\sigma_{-g}(\Omega)} v(x) \quad (x \in \Omega).$$

For each pair of $u \in \sigma_g(\Omega)$ and $v \in \sigma_{-g}(\Omega)$, it follows that $u + v \in \sigma(\Omega) \cap C(\bar\Omega)$ and $u+v \le g+(-g) = 0$ on $\partial\Omega$. Hence, by Lemma 3.23, $u+v \le 0$ on $\bar\Omega$; therefore, $u(x) \le -v(x)$ $(x \in \Omega)$ for all such pairs. Cosequently,

$$w_g(x) = \sup_{u\in\sigma_g(\Omega)} u(x) \le \inf_{v\in\sigma_{-g}(\Omega)} (-v(x)) = -\sup_{v\in\sigma_{-g}(\Omega)} v(x) = -w_{-g}(x);$$

that is, $w_g \le -w_{-g}$ in Ω, which is valid without the regularity of $\partial\Omega$. From this, by applying Step 1 to $-g$, we have

$$\limsup_{x\to\eta,\, x\in\Omega} w_g(x) \le \limsup_{x\to\eta,\, x\in\Omega} (-w_{-g}(x)) = -\liminf_{x\to\eta,\, x\in\Omega} w_{-g}(x) \le -(-g(\eta)) = g(\eta),$$

completing the proof. ☐

Theorem 3.30 (Solvability of Dirichlet problems). *Let $\Omega \subset \mathbb{R}^n$ be bounded open. Then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in C^2(\Omega) \cap C(\bar\Omega)$ for every continuous boundary function $g \in C(\partial\Omega)$ if and only if every boundary point $\eta \in \partial\Omega$ is regular.

Proof. 1. Suppose every boundary point $\eta \in \partial\Omega$ is regular. Let w_g be the function defined by (3.27) and let

$$u(x) = \begin{cases} w_g(x) & (x \in \Omega), \\ g(x) & (x \in \partial\Omega). \end{cases}$$

Then $u \in C^2(\Omega) \cap C(\bar\Omega)$ is a solution to the Dirichlet problem.

2. Assume the Dirichlet problem is solvable for all continuous boundary data g. Given any $\eta \in \partial\Omega$, let $g(x) = -|x-\eta|$. Let $u = Q_\eta$ be the $C^2(\Omega) \cap C(\bar\Omega)$-solution to the Dirichlet problem with this g. Then Q_η is a barrier function at η. (Explain why?) Therefore, $\eta \in \partial\Omega$ is regular. ☐

Remark 3.6. A domain Ω is said to satisfy the **exterior ball property** at a boundary point $\eta \in \partial\Omega$ if there exists a closed ball $B = \bar B(x_0, \rho)$ in the exterior domain $\mathbb{R}^n \setminus \Omega$ such that $B \cap \partial\Omega = \{\eta\}$; in this case, η is regular because we can choose the barrier function Q_η to be

$$Q_\eta(x) = \Phi(x - x_0) - \phi(\rho),$$

where $\Phi(x) = \phi(|x|)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. Domain Ω is said to satisfy the **exterior ball property** if it satisfies this property at every boundary point. For such domains the Dirichlet problem is uniquely solvable for any continuous boundary data. Note that every strictly convex domain satisfies the exterior ball property.

3.7. Maximum principles for second-order linear elliptic equations

(*This material is from Section 6.4 of the textbook.*)

3.7.1. Second-order linear elliptic PDEs. Consider the second-order linear differential operator

$$Lu(x) = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b^i(x)D_iu(x) + c(x)u(x),$$

where $D_iu = u_{x_i}$, $D_{ij}u = u_{x_ix_j}$ and $a^{ij}(x)$, $b^i(x)$, $c(x)$ are given functions in an open set Ω in $\mathbb{R}^n$ for all $i, j = 1, 2, \cdots, n$. With loss of generality, we assume $a^{ij}(x) = a^{ji}(x)$ for all i, j.

Definition 3.9. The operator L is called **elliptic** in Ω if there exists $\lambda(x) > 0$ $(x \in \Omega)$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda(x)\sum_{i=1}^{n} \xi_i^2 \quad \forall\, x \in \Omega, \quad \xi \in \mathbb{R}^n. \tag{3.29}$$

If $\lambda(x) \geq \lambda_0 > 0$ in Ω, we say that L is **uniformly elliptic** in Ω.

So, if L is elliptic in Ω, then for each $x \in \Omega$ the symmetry matrix $(a^{ij}(x))$ is *positive definite*, with all eigenvalues $\geq \lambda(x)$.

Lemma 3.31. *If $A = (a_{ij})$ is an $n \times n$ symmetric nonnegative definite matrix then there exists an $n \times n$ matrix $B = (b_{ij})$ such that $A = B^TB$, i.e.,*

$$a_{ij} = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n).$$

Proof. Exercise. □

3.7.2. Weak maximum principle.

Lemma 3.32. *Let L be as given above and satisfy* (3.29) *with $\lambda(x) \geq 0$ in Ω, and let $u \in C^2(\Omega)$ satisfy $Lu < 0$ in Ω. If $c(x) \geq 0$, then u cannot attain a nonnegative maximum in Ω. If $c(x) \equiv 0$, then u cannot attain a maximum in Ω.*

Proof. Let $Lu < 0$ in Ω. Suppose $u(x_0)$ is maximum for some $x_0 \in \Omega$. Then, by derivative test, $D_ju(x_0) = 0$ for each $j = 1, 2, \cdots, n$ and

$$\frac{d^2u(x_0 + t\xi)}{dt^2}\Big|_{t=0} = \sum_{i,j=1}^{n} D_{ij}u(x_0)\xi_i\xi_j \leq 0$$

for all $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. By the lemma above, we write

$$a^{ij}(x_0) = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n),$$

where $B = (b_{ij})$ is an $n \times n$ matrix. Hence

$$\sum_{i,j=1}^{n} a^{ij}(x_0)D_{ij}u(x_0) = \sum_{k=1}^{n}\sum_{i,j=1}^{n} D_{ij}u(x_0)b_{ki}b_{kj} \leq 0,$$

which implies that $Lu(x_0) \geq c(x_0)u(x_0) \geq 0$ either when $c \geq 0$ and $u(x_0) \geq 0$ or when $c \equiv 0$. This is a contradiction. □

Theorem 3.33 (Weak maximum principle with $c = 0$). *Let Ω be bounded open in $\mathbb{R}^n$ with $c \equiv 0$ and let L be elliptic in Ω and*

$$|b^i(x)|/\lambda(x) \le M \quad (x \in \Omega,\ i = 1, 2, \cdots, n) \tag{3.30}$$

for some constant $M > 0$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \le 0$ in Ω; that is, u is a **sub-solution** of L. *Then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Let $\alpha > 0$ and $v(x) = e^{\alpha x_1}$. Then

$$Lv(x) = (-a^{11}(x)\alpha^2 + b^1(x)\alpha)e^{\alpha x_1} = \alpha a^{11}(x)\left[-\alpha + \frac{b^1(x)}{a^{11}(x)}\right]e^{\alpha x_1} < 0$$

if $\alpha > M + 1$ because $\frac{|b^1(x)|}{a^{11}(x)} \le \frac{|b^1(x)|}{\lambda(x)} \le M$. Then consider the function $w(x) = u(x) + \varepsilon v(x)$ for $\varepsilon > 0$. Then $Lw = Lu + \varepsilon Lv < 0$ in Ω. So by Lemma 3.32, for all $x \in \bar{\Omega}$,

$$u(x) + \varepsilon v(x) \le \max_{\partial\Omega}(u + \varepsilon v) \le \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v.$$

Letting $\varepsilon \to 0^+$ proves the theorem. □

Remark 3.7. (a) The weak maximum principle still holds if $(a^{ij}(x))$ is nonnegative definite, i.e., $\lambda(x) \ge 0$ in Ω, but satisfies $\frac{|b^k(x)|}{a^{kk}(x)} \le M$ for some $k = 1, 2, \cdots, n$. (In this case use $v = e^{\alpha x_k}$.)

(b) If Ω is unbounded but contained in a slab $|x_1| < N$, then the proof is still valid if the maximum is changed supremum.

Theorem 3.34 (Weak maximum principle with $c \ge 0$). *Let Ω be bounded open in $\mathbb{R}^n$ and L be elliptic in Ω satisfying* (3.30). *Let $c(x) \ge 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then*

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+ \quad \text{if } Lu \le 0 \text{ in } \Omega;$$

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| \quad \text{if } Lu = 0 \text{ in } \Omega,$$

where $u^+(x) = \max\{u(x), 0\}$.

Proof. 1. Let $Lu \le 0$ in Ω. Let $\Omega^+ = \{x \in \Omega \mid u(x) > 0\}$. If Ω^+ is empty then the result is trivial. Assume $\Omega^+ \ne \emptyset$; then $L_0 u \equiv Lu - c(x)u(x) \le 0$ in Ω^+. Note that $\partial(\Omega^+) = [\Omega \cap \partial\Omega^+] \cup [\partial\Omega^+ \cap \partial\Omega]$, from which we easily see that $\max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+$; hence, by Theorem 3.33,

$$\max_{\bar{\Omega}} u = \max_{\overline{\Omega^+}} u = \max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+.$$

2. Let $Lu = 0$. We apply Step 1 to u and $-u$ to complete the proof. □

Remark 3.8. The weak maximum principle for a sub-solution u can not be replaced by $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. In fact, for any $u \in C^2(\bar{\Omega})$ satisfying

$$0 > \max_{\bar{\Omega}} u > \max_{\partial\Omega} u,$$

if we choose a constant $\theta > -\|Lu\|_{L^\infty(\Omega)}/\max_{\bar{\Omega}} u > 0$, then $\tilde{L}u = Lu + \theta u \le 0$ in Ω. But the zero-th order coefficient of $\tilde{L}$ is $c(x,t) + \theta > 0$.

The weak maximum principle easily implies the following uniqueness result for Dirichlet problems.

Theorem 3.35 (**Uniqueness of solutions**). *Let Ω be bounded open in $\mathbb{R}^n$ and let the linear operator L with $c(x) \geq 0$ be elliptic in Ω and satisfy* (3.30). *Then, given any functions f and g, the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Remark 3.9. The uniqueness result fails if $c(x) < 0$ in Ω. For example, if $n = 1$, then function $u(x) = \sin x$ solves the elliptic problem $Lu \equiv -u'' - u = 0$ in $\Omega = (0, \pi)$ with $u(0) = u(\pi) = 0$; but $u \not\equiv 0$.

3.7.3. Strong maximum principle. The following important result, known as **Hopf's Lemma**, describes some behavior of sub-solutions of the elliptic equations at boundary maximum points under certain circumstances.

Theorem 3.36 (**Hopf's Lemma**). *Let L be uniformly elliptic with bounded coefficients in a ball B and let $u \in C^2(B) \cap C^1(\bar{B})$ satisfy $Lu \leq 0$ in B. Assume $x^0 \in \partial B$ and $u(x) < u(x^0)$ for all $x \in B$.*

(a) *If $c \equiv 0$ in B, then $\frac{\partial u}{\partial \nu}(x^0) > 0$, where ν is outer unit normal to ∂B.*

(b) *If $c(x) \geq 0$ in B, then the same conclusion holds provided $u(x^0) \geq 0$.*

(c) *If $u(x^0) = 0$, then the same conclusion holds no matter what sign of $c(x)$ is.*

Proof. Note that the condition that x^0 is a boundary maximum point of u readily implies that $\frac{\partial u}{\partial \nu}(x^0) \geq 0$; the importance of **Hopf's Lemma** is that it asserts that under certain conditions the strict inequality $\frac{\partial u}{\partial \nu}(x^0) > 0$ holds.

1. Without loss of generality, assume $B = B(0, R)$. Consider function

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

Let $\tilde{L}u \equiv Lu - c(x)u + c^+(x)u$, where $c^+(x) = \max\{c(x), 0\}$. The operator $\tilde{L}$ has the coefficient of the zero-th order term $c^+ \geq 0$ and hence the weak maximum principle applies to $\tilde{L}$. We compute

$$\begin{aligned} \tilde{L}v(x) &= \left[-4 \sum_{i,j=1}^n a^{ij}(x)\alpha^2 x_i x_j + 2\alpha \sum_{i=1}^n (a^{ii}(x) - b^i(x)x_i) \right] e^{-\alpha|x|^2} + c^+(x)v(x) \\ &\leq \left[-4\lambda_0 \alpha^2 |x|^2 + 2\alpha \operatorname{tr}(a^{ij}(x)) + 2\alpha |b(x)||x| + c^+(x) \right] e^{-\alpha|x|^2} < 0 \end{aligned}$$

on $\frac{R}{2} \leq |x| \leq R$ if $\alpha > 0$ is sufficiently large; we fix such an α below.

2. For each $\varepsilon > 0$, consider function $w_\varepsilon(x) = u(x) - u(x^0) + \varepsilon v(x)$. Then

$$\tilde{L}w_\varepsilon(x) = \varepsilon \tilde{L}v(x) + Lu(x) + (c^+(x) - c(x))u(x) - c^+(x)u(x^0) \leq 0$$

on $\frac{R}{2} \leq |x| \leq R$ in all the cases of (a), (b) and (c) of the theorem.

3. By assumption, $u(x) < u(x^0)$ on $|x| = \frac{R}{2}$; hence there exists an $\varepsilon > 0$ such that $w_\varepsilon(x) < 0$ on $|x| = \frac{R}{2}$. In addition, since $v|_{\partial B} = 0$, we have $w_\varepsilon(x) = u(x) - u(x^0) \leq 0$ on $|x| = R$. Hence the weak maximum principle implies that $w_\varepsilon(x) \leq 0$ for all $\frac{R}{2} \leq |x| \leq R$. But $w_\varepsilon(x^0) = 0$; this implies

$$0 \leq \frac{\partial w_\varepsilon}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \varepsilon \frac{\partial v}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - 2\varepsilon R \alpha e^{-\alpha R^2}.$$

Therefore
$$\frac{\partial u}{\partial \nu}(x^0) \geq 2\varepsilon R\alpha e^{-\alpha R^2} > 0,$$
as desired. □

Theorem 3.37 (**Strong maximum principle**). *Let Ω be bounded, open and connected in $\mathbb{R}^n$ and L be uniformly elliptic with bounded coefficients in Ω and let $u \in C^2(\Omega)$ satisfy $Lu \leq 0$ in Ω.*

(a) *If $c(x) \geq 0$, then u cannot attain a nonnegative maximum in Ω unless u is constant.*

(b) *If $c \equiv 0$, then u cannot attain a maximum in Ω unless u is constant.*

Proof. Assume $c(x) \geq 0$ in Ω and u attains the maximum M at some point in Ω; also assume $M \geq 0$ if $c(x) \geq 0$. Suppose, for the contrary, that u is not constant in Ω. Then, the following two sets,
$$\Omega^- = \{x \in \Omega \mid u(x) < M\}; \quad \Omega_0 = \{x \in \Omega \mid u(x) = M\},$$
are both nonempty, with Ω^- open and $\Omega_0 \neq \Omega$ relatively closed in Ω. Since Ω is connected, the set Ω_0 is not open. Assume $a \in \Omega_0$ is not an interior point of Ω_0; so, there exists a sequence $\{x^k\}$ not in Ω_0 but converging to a. Hence, for some ball $B(a, r) \subset\subset \Omega$ and an integer $N \in \mathbb{N}$, we have that $x^k \in B(a, r/2)$ for all $k \geq N$. Fix $k = N$ and let
$$S = \{\rho > 0 \mid B(x^N, \rho) \subset \Omega^-\}.$$
Then $S \subset \mathbb{R}$ is nonempty and bounded above by $r/2$. Let $\bar{\rho} = \sup S$; then $0 < \bar{\rho} \leq r/2$ and thus $B(x^N, \bar{\rho}) \subset B(a, r) \subset\subset \Omega$. Hence it follows that $B(x^N, \bar{\rho}) \subset \Omega^-$ and $\Omega_0 \cap \partial B(x^N, \bar{\rho}) \neq \emptyset$. So let $x^0 \in \Omega_0 \cap \partial B(x^N, \bar{\rho})$ and then $u(x) < u(x^0)$ for all $x \in B(x^N, \bar{\rho})$. We apply the Hopf's lemma above to the ball $B = B(x^N, \bar{\rho})$ at the point $x^0 \in \partial B$ to obtain that $\frac{\partial u}{\partial \nu}(x^0) > 0$, where ν is the outer normal of ∂B at x^0. However, as u has a maximum at the interior point $x^0 \in \Omega_0$ of Ω, we have that $Du(x^0) = 0$, which gives the desired contradiction. □

Finally we state without proof the following **Harnack's inequality** for nonnegative solutions of second-order elliptic PDEs, which extends the result for harmonic functions. For smooth coefficients, this result follows as a special case of **Harnack's inequality** for parabolic equations proved later.

Theorem 3.38 (**Harnack's Inequality**). *Let $V \subset\subset \Omega$ be connected and L be uniformly elliptic in Ω with bounded coefficients. Then there exists a constant $C = C(V, \Omega, L) > 0$ such that*
$$\sup_V u \leq C \inf_V u$$
for all nonnegative solutions u of $Lu = 0$ in Ω.