UPPSALA UNIVERSITY
Dept of Mathematics
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Matematical statistics
Inference II 1MS037
2023–12–11

Hand-in assignment 3

There are three compulsory home assignments. You may hand in the assignments in groups of two students. To pass the assignment, 10p are needed. Solutions should be well motivated. Hand in your solutions via Studium *as a single pdf file.* If you have worked together with somebody else, it is enough to hand in for one of you as long as you write both names.

The deadline for this assignment is December 18.

1. Suppose $X_1, ..., X_5$ are independent Po(λ). The observations are $\mathbf{x} = (x_1, ..., x_5)$. Consider testing H_0: $\lambda = 1$ vs. H_1: $\lambda = 2$ at level α.

   (a) Let $t = \sum_{i=1}^{5} x_i$. Show that most powerful (randomized) test has test function of the type

   $$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } t > C, \\ \gamma & \text{if } t = C, \\ 0 & \text{if } t < C, \end{cases}$$

   where $0 \leq \gamma < 1$ and C is some constant. (2p)

   (b) Which β (probability of type II error) corresponds to $\alpha = 0.05$ (probability of type I error)? (2p)

   (c) Calculate one or two more (α, β) pairs as in (b) and sketch a plot of β (on the y axis) vs α. (1p)

2. Consider a random sample $\mathbf{X} = (X_1, ..., X_n)$ where the X_i are Exponentiallly distributed with intensity parameter β, i.e. with density function

   $$f(x) = \beta \exp(-\beta x), \quad x > 0,$$

   and 0 otherwise, with $\beta > 0$.

   Consider testing H_0: $\beta \geq 1$ vs H_1: $\beta < 1$ at level $\alpha = 0.05$.

   (a) Derive the uniformly most powerful (UMP) test.
   (Make sure to prove that it is really a UMP test. But you do not need to evaluate the test in form of a distribution here, just give the critical region in terms of a statistic.) (3p)

   (b) Assume that $n = 100$ and that $\sum_{i=1}^{100} x_i = 120$. Using the test in (a), do you reject H_0? (Asymptotic approximations are permitted.) (2p)

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3. Assume that we have one observation x of $X \sim \text{Bin}(5, p)$, where $0 \leq p \leq 1$.

   (a) Test H_0: $p = 1/4$ vs H_1: $p \neq 1/4$ with critical region $C = \{0, 4, 5\}$, i.e. the test function $\varphi(x) = 1$ if $x \in C$ and 0 otherwise.

   Is this test unbiased? Motivate you answer. (2p)

   (b) Test H_0: $p = p_0$ vs H_1: $p \neq p_0$ with critical region $C = \{0, 4, 5\}$.
   Is there any p_0 such that this test is unbiased? (3p)

4. Let X_1, X_2, X_3, X_4 be i.i.d. $N(\mu_1, \sigma^2)$, and let Y_1, Y_2 be i.i.d. $N(\mu_2, 4\sigma^2)$ independent of the X_i. All parameters are unknown.
   Consider $\mathbf{Z} = (X_1, X_2, X_3, X_4, Y_1, Y_2)$.

   Derive the UMP α-similar test for H_0: $\sigma^2 \leq \Delta_0$ vs H_1: $\sigma^2 > \Delta_0$. (5p)

   Hint: Compare to one of the problems in the course book.