UPPSALA UNIVERSITET
Matematiska Institutionen
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PROV I MATEMATIK
Graph Theory
8 January 2016

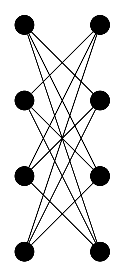
Time: 8.00-13.00.
Each question is worth 5 marks. You need 9 marks to pass the exam, which is one of the requirements for passing the course. You may write your answers in English or Swedish. Good luck!

1. (a) Let G be a graph. Explain what is meant by a proper colouring of the vertices of G and define the chromatic number $\chi(G)$ of G.

   (b) Outline the greedy colouring algorithm.

   (c) For any graph G, show that $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph. For which graphs is $\chi(G) = \Delta(G) + 1$?

   (d) Let G be the graph below. Suppose you are given some ordering on the vertices. What are the possible numbers of colours used by the greedy algorithm? For each possible number, give an ordering on the vertices for which the greedy algorithm uses this number of colours.



2. We call a graph property $\mathcal{P}$ stable if adding *a single edge* to a graph that has $\mathcal{P}$ cannot destroy that property. In other words, for any graph G that has $\mathcal{P}$, we have that if H is any graph obtained by adding a single edge to G (between two vertices not already joined by an edge), then H also has $\mathcal{P}$.

   Determine which of the following properties are stable.

   (a) The diameter of the graph is at most 3.

   (b) The graph is bipartite.

   (c) The largest independent set has size exactly 3.

   (d) There is no perfect matching.

   (e) The graph is planar.

   If your answer is that a property is stable, explain briefly why. If your answer is that a property is not stable, construct a graph with that property, such that the addition of a single edge makes the graph not have that property anymore.

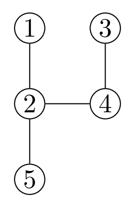
   [An example of a non–stable property is the property of being disconnected. For instance, the graph E_2 is disconnected, but adding an edge between the two vertices makes the graph connected.]

3. (a) Let G be a graph. What is meant by a spanning tree of G?

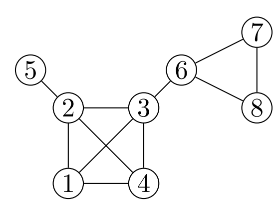
   (b) State Cayley's formula for the number of spanning trees of K_n.

   (c) Outline the bijection between trees with vertex set $\{1, 2, \ldots, n\}$ and Prüfer sequences of length $n - 2$. (You need not prove that this is a bijection, only explain how to construct a tree from a given Prüfer sequence, and vice versa.) Given that you have defined a bijection, explain why Cayley's formula follows.

   (d) Illustrate your bijection in (c) by finding the tree corresponding to the Prüfer sequence $(4, 4, 4, 5)$, and by determining the Prüfer sequence corresponding to the following tree.



   (e) Determine the number of spanning trees of the following graph. Explain your answer.



4. Let G be a directed graph, in which each edge is given a capacity (some non–negative real number), with two special vertices s (the source) and t (the sink). Assume s has no incoming edges and t has no outgoing edges.

   (a) Explain what is meant by an $s-t$ flow and its *value*.

   (b) Explain what is meant by an $s-t$ cut and its *capacity*.

   (c) State the max flow – min cut theorem.

   (d) Use the Ford–Fulkerson algorithm (which you need not explain, but you should in each step draw the current flow and the corresponding residual graph) to find an $s-t$ flow of maximum value in the flow network below.

   (e) Using your answer in (d), or otherwise, find an $s-t$ cut of minimum capacity in the flow network below.