Lec 3

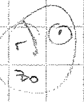
Topology of $\mathbb{C}$

The set

$$D_r(z_0) := \{ z \in \mathbb{C} : |z - z_0| < r \} \qquad (\text{or } D(z_0, r))$$

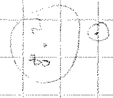
is called the open disk with center z_0 and radius r.

A subset M of $\mathbb{C}$ is called <u>open</u> if for every $z_0 \in M$ there exists an $r > 0$ s.t. $D_r(z_0) \subseteq M$.



<u>Ex.</u> $D_r(z_0)$ is open (hence the name open disk)

A subset M of $\mathbb{C}$ is called <u>closed</u> if its complement $M^c = \mathbb{C} \setminus M$ is open.



<u>Ex.</u> $\{ z \in \mathbb{C} : |z - z_0| \leq r \}$ is closed.

Let M be a subset of $\mathbb{C}$.

A point $z_0 \in M$ is called an <u>interior point</u> of M if there exists an $r > 0$ s.t. $D_r(z_0) \subseteq M$.

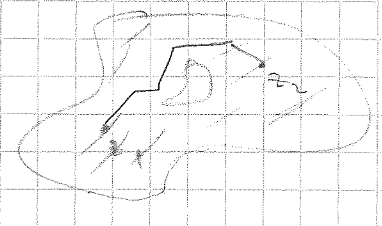
A point $z_0 \in \mathbb{C}$ is called a <u>boundary point</u> of M if $\forall r > 0$ it holds that $D_r(z_0) \cap M \neq \phi$ and $D_r(z_0) \cap M^c \neq \phi$.

The set of all interior points of M is denoted $\mathrm{int}(M)$ and the set of all boundary points of M is denoted ∂M.

It holds that

- M is closed $\Longleftrightarrow$ $\partial M \subseteq M$
- M is open $\Longleftrightarrow$ $\partial M \subseteq M^c$

An open set M is called (path-) connected if every pair of points $z_1, z_2 \in M$ can be connected by a polygonal path contained in M.



Remark: One can assume the polygonal paths to have segments parallel to the coordinate axes.

An open connected set is called a domain.

Thm Suppose that $u(x,y)$ is a real-valued fct defined in a domain $D \subseteq \mathbb{R}^2$. Suppose also that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

in all of D. Then u is constant in D.

A domain $D \subseteq \mathbb{C}$ is called simply connected if every closed curve in D can be, within D, continuously deformed to a point.

Limits and continuity

Def A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to have the limit z_0 or to converge to z_0, and we write

$$\lim_{n \to \infty} z_n = z_0$$

(or $z_n \to z_0$ as $n \to \infty$)

if for every given $\varepsilon > 0$ there exists an integer $N \geq 1$ s.t. $|z_n - z_0| < \varepsilon$ for all $n \geq N$.

Remark: $z_n \to z_0 \iff \operatorname{Re} z_n \to \operatorname{Re} z_0$ and $\operatorname{Im} z_n \to \operatorname{Im} z_0$

(Follows from $|x|, |y| \leq \sqrt{x^2+y^2} \leq |x|+|y|$)

Def Let f be a fcn defined in a punctured neighborhood of z_0.

We say that f has the limit w_0 as $z \to z_0$, and write

$$\lim_{z \to z_0} f(z) = w_0$$

if for every given $\varepsilon > 0$ there $\exists\, \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Remark: Limits are unique if they exist.

See figure

Thm For $z = x+iy$, let

$$u(x,y) = \operatorname{Re} f(z), \quad v(x,y) = \operatorname{Im} f(z).$$

Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

Then,

$$\lim_{z\to z_0} f(z) = w_0 \iff \begin{cases} \displaystyle\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \text{and} \\ \displaystyle\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Proof Exercise.

Def Let f be a fcn def in a neighborhood of z_0. Then f is said to be continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

A fcn f is said to be continuous on the (open) set M if it is continuous at each point of M.

The same proof as in the real case shows the following:

<u>Thm</u> If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

i) $\lim_{z \to z_0} \left(f(z) \pm g(z) \right) = A \pm B$

ii) $\lim_{z \to z_0} f(z) g(z) = AB$

iii) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ if $B \neq 0$

<u>Corollary</u>: If f and g are continuous at z_0, then so are $f \pm g$ and fg. The quotient f/g is cont. at z_0 if $g(z_0) \neq 0$.

<u>Ex</u> It is easy to show that $f(z) = $ const. and $f(z) = z$ are cont. in $\mathbb{C}$. It follows that polynomials are continuous in $\mathbb{C}$.

Rational fcns are cont. wherever the denominator is nonzero.

<u>Ex</u> $f(z) = e^z = e^x \cos y + i e^x \sin y$ is cont. in $\mathbb{C}$.

The complex derivative. Analytic functions.

In analogy with the real case we also make the following

Def Let f be a complex-valued fcn defined in a neighborhood of z_0. We say that f is differentiable at z_0 if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

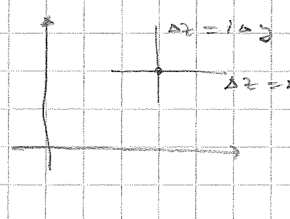
exists. The limit is then called the derivative of f at z_0, and is denoted $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

Remark: Δz is a complex number, so it can approach 0 "in many different ways". In order for the derivative to exist, the result must be independent of how $\Delta z \to 0$.

Ex $f(z) = \bar{z}$ is nowhere differentiable.

Proof: $$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{\overline{z_0+\Delta z} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

Now let $\Delta z \to 0$ in two different ways; see figure:



First let $\Delta z = \Delta x$. Then $\frac{\overline{\Delta z}}{\Delta z} = 1 \to 1$, $\Delta x \to 0$

If instead $\Delta z = i\Delta y$, then $\frac{\overline{\Delta z}}{\Delta z} = -1 \to -1$, $\Delta y \to 0$

□

Exercise Show that $f(z) = |z|^2$ is only diff

in the point $z = 0$, and that $f'(0) = 0$.

Ex. Let n be an integer ≥ 1.

Then, $\frac{d}{dz} z^n = n z^{n-1}$,

since by the binomial thm

$$\frac{(z+\Delta z)^n - z^n}{\Delta z} = \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} (\Delta z)^k - z^n}{\Delta z} =$$

$$= \frac{n z^{n-1} \Delta z + \binom{n}{2} z^{n-2} (\Delta z)^2 + \ldots}{\Delta z} \to n z^{n-1}, \Delta z \to 0$$

□

Also other rules hold (and are proven) in analogy

with the real case:

Thm If f and g are diff. at z, then

$$(f \pm g)'(z) = f'(z) \pm g'(z)$$

$$(cf)'(z) = c f'(z) \qquad (c \text{ constant})$$

$$(fg)'(z) = f'(z) g(z) + f(z) g'(z)$$

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z) g(z) - f(z) g'(z)}{(g(z))^2}, \quad g(z) \neq 0$$

Also the chain rule holds: If g is diff. at z

and f is diff at $g(z)$, then

$$(f \circ g)'(z) = f'(g(z))\, g'(z).$$

Def. A complex-valued fcn f is said to be analytic in a open set G if f is differentiable at every point of G. We say ~~that~~ f is analytic at z_0 if f is differentiable in a neighborhood of z_0.

If f is analytic in all of $\mathbb{C}$, then f is said to be entire.

The exercise shows that $f(z) = |z|^2$ is diff. at 0, but not analytic at 0.