Statistical Risk Analysis
Chapter 5: Conditional Distributions with Applications

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Dependence

We often assume the measurements are independent. However, if we have several measurements on the same individual, the measurements are often dependent.

- Suppose that we have two random variables X and Y. The distribution function is

$$F_{X,Y}(x,y) = \mathrm{P}(X \leq x, Y \leq y).$$

  It is often called the joint distribution.

- The marginal distributions are

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \leq x) = F_{X,Y}(x,\infty), \\ F_Y(y) &= \mathrm{P}(Y \leq y) = F_{X,Y}(\infty,y). \end{aligned}$$

Probability Mass Function

If X and Y take a finite or countable number of values (e.g., 0, 1, 2, ...), then the distribution

$$F_{X,Y}(x, y) = \sum_{j \leq x} \sum_{k \leq y} p_{jk},$$

where $p_{jk} = P(X = j, Y = k)$ is the joint probability mass function (pmf).

The marginal probability mass function can be computed by

$$P(X = j) = \sum_{k=0}^{\infty} \mathrm{P}(X = j, Y = k),$$

$$P(Y = k) = \sum_{j=0}^{\infty} \mathrm{P}(X = j, Y = k).$$

Example: A Multinomial Distribution

Consider a multinomial distribution of three outcomes. Let X be the number of the first outcome and Y be the number of the second outcome after n trials. Its pmf is

$$\mathrm{P}\left(X=j,Y=k\right) = \frac{n!}{j!k!\left(n-j-k\right)!}p_A^j p_B^k\left(1-p_A-p_B\right)^{n-j-k}$$

where $0\leq p_A\leq 1$ and $0\leq p_B\leq 1$ are the probabilities of the first and second outcome respectively.
The marginal is

$$\begin{aligned}\mathrm{P}\left(X=j\right) &= \frac{n!}{j!\left(n-j\right)!}p_A^j\left(1-p_A\right)^{n-j},\\ \mathrm{P}\left(Y=k\right) &= \frac{n!}{k!\left(n-k\right)!}p_B^k\left(1-p_B\right)^{n-k}.\end{aligned}$$

That is, $X\in\mathrm{Bin}\left(n,p_A\right)$ and $Y\in\mathrm{Bin}\left(n,p_B\right)$.

Probability Density Function

If $F_{X,Y}(x,y)$ is differentiable with respect to x and y, the derivative

$$f(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

is called the joint probability density function (pdf). Then

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,dtds, \\ P(X \in A, Y \in B) &= \int_{x \in A}\int_{y \in B} f(s,t)\,dtds. \end{aligned}$$

The marginal densities of X and Y are obtained by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,t)\,dt \qquad f_Y(y) = \int_{-\infty}^{\infty} f(s,y)\,ds.$$

Example: Bivariate Normal Distribution

For $\rho \in (-1, 1)$, the joint density of a bivariate normal distribution is

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{u}{2(1-\rho^2)}\right\},$$

where

$$u = \frac{(x - m_X)^2}{\sigma_X^2} - \frac{2\rho(x - m_X)(y - m_y)}{\sigma_X\sigma_Y} + \frac{(y - m_y)^2}{\sigma_Y^2}.$$

We denote it by $(X, Y) \in N\left(m_X, m_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$.

The marginal is $X \in N\left(m_X, \sigma_X^2\right)$ and $Y \in N\left(m_Y, \sigma_Y^2\right)$.

Expectation

Let $h(X,Y)$ be a function of X and Y. Then,

$$E[h(X,Y)] = \begin{cases} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(j,k)\, p_{jk}, & (X,Y) \text{ is discrete,} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx dy, & (X,Y) \text{ is continuous.} \end{cases}$$

In the special case where $h(X,Y) = aX + bY$ for constants a and b, we have

$$E[aX + bY] = aE[X] + bE[Y].$$

Covariance

The covariance between X and Y is defined by

$$\operatorname{Cov}[X,Y] = E[XY] - E[X]E[Y].$$

1. Covariance is symmetric:

$$\operatorname{Cov}[Y,X] = E[YX] - E[Y]E[X] = \operatorname{Cov}[X,Y].$$

2. The covariance between X and X is the variance:

$$\operatorname{Cov}[X,X] = E\left[X^2\right] - E[X]E[X] = V(X).$$

3. The covariance matrix of (X,Y) is

$$\begin{bmatrix} V[X] & \operatorname{Cov}[X,Y] \\ \operatorname{Cov}[X,Y] & V[Y] \end{bmatrix}.$$

Covariance In Variance

Variance of a linear combination is

$$V\left[aX+bY+c\right] = a^2V\left[X\right]+2ab\mathrm{Cov}\left[X,Y\right]+b^2V\left[Y\right].$$

In general,

$$\begin{aligned} V\left[\sum_{i=1}^{n}a_iX_i+c\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}\left[X_i,X_j\right] \\ &= \sum_{i=1}^{n}a_i^2V\left[X_i\right]+\sum_{i=1}^{n}\sum_{j\neq i}\mathrm{Cov}\left[X_i,X_j\right] \\ &= \sum_{i=1}^{n}a_i^2V\left[X_i\right]+2\sum_{i=2}^{n}\sum_{j<i}\mathrm{Cov}\left[X_i,X_j\right]. \end{aligned}$$

Pearson Correlation

The correlation between X and Y is

$$\rho_{XY} = \frac{\operatorname{Cov}[X, Y]}{D[X]\, D[Y]} \in [-1, 1],$$

where $D[X] = \sqrt{V[X]}$ and $D[Y] = \sqrt{V[Y]}$.

An estimate is

$$\rho^*_{XY} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}}.$$

Examples

1. Consider

$$\mathrm{P}(X=j, Y=k) = \frac{n!}{j!k!(n-j-k)!} p_A^j p_B^k (1-p_A-p_B)^{n-j-k}.$$

The covariance between X and Y is

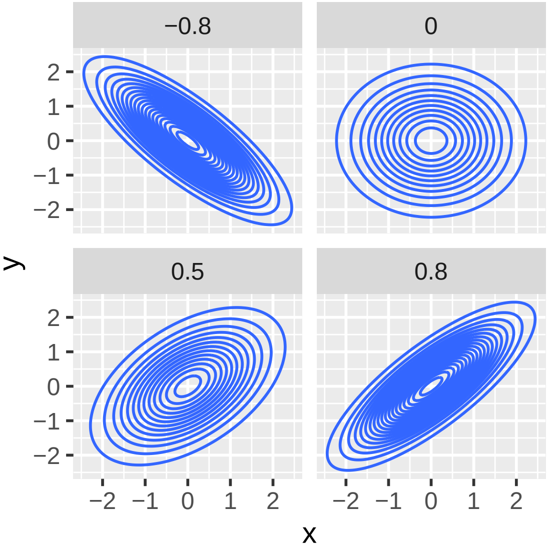
$$\mathrm{Cov}[X, Y] = -np_A p_B.$$

2. Consider $(X, Y) \in N\left(m_X, m_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$.

$$\mathrm{Cov}[X, Y] = \rho\sigma_X\sigma_Y,$$

and ρ is the correlation between X and Y.

Contour of Bivariate Normal Density



Application of Bivariate Normal

Let ℓ be the log-likelihood function.

1. Suppose that we want to estimate one parameter θ. The distribution of $\left[\ddot{\ell}\left(\theta^*\right)\right]^{-1/2}\left(\Theta^* - \theta\right)$ can often be approximated by $N\left(0,1\right)$.
2. Suppose that we want to estimate two parameters θ_1 and θ_2. The distribution of can often be approximated by

$$\left[\ddot{\ell}\left(\theta_1^*, \theta_2^*\right)\right]^{-1/2}\left(\begin{bmatrix}\Theta_1^* \\ \Theta_2^*\end{bmatrix} - \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}\right) \in AsN\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right).$$

The covariance matrix of Θ_1^* and Θ_2^* can be approximated by

$$-\begin{bmatrix}\frac{\partial^2 \ell\left(\theta_1^*, \theta_2^*\right)}{\partial \theta_1^2} & \frac{\partial^2 \ell\left(\theta_1^*, \theta_2^*\right)}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \ell\left(\theta_1^*, \theta_2^*\right)}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ell\left(\theta_1^*, \theta_2^*\right)}{\partial \theta_2^2}\end{bmatrix}^{-1}.$$

Example: MLE of Multinomial Distribution

Suppose that we have data from a multinomial distribution with pmf

$$\mathrm{P}\left(X=j, Y=k\right) = \frac{n!}{j!k!\left(n-j-k\right)!}p_A^j p_B^k \left(1-p_A-p_B\right)^{n-j-k}.$$

Approximate the distribution of p_A^* and p_B^*.

Conditioning

The conditional probability is

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}.$$

1. The conditional probability mass function is

$$\mathrm{P}(X = j \mid Y = k) = \frac{\mathrm{P}(X = j, Y = k)}{\mathrm{P}(Y = k)}.$$

   We have $\sum_{j=-\infty}^{\infty} \mathrm{P}(X = j \mid Y = k) = 1$.

2. The conditional probability density function is

$$f(x \mid y) = \frac{f(x, y)}{f(y)}, \quad \text{if } f(y) > 0 \text{ and zero otherwise.}$$

   We have $F(x \mid y) = \int_{-\infty}^{x} f(t \mid y)\, dt$.

Law of Total Probability

Theorem (Theorem 5.2 and 5.3, Law of Total Probability)

Consider an event B. Then,

$$P(B) = \begin{cases} \sum_{i=0}^{\infty} P(B \mid Y = i) P(Y = i) & Y \text{ is discrete,} \\ \int_{-\infty}^{\infty} P(B \mid Y = y) f_Y(y)\, dy & Y \text{ is continuous.} \end{cases}.$$

If X and Y have joint density $f(x, y)$ and B is a statement about X, then

$$P(B \mid Y = y) = \int_{x \in B} f(x \mid y)\, dx.$$

Example: Number of Cracks

1. Suppose that number of cracks of a tanker follows a Poisson distribution with mean m. A devices can detect a crack with probability 0.999. Let $B = \{\text{All creacks have been detected}\}$. Find $\mathrm{P}(B)$.
2. Suppose that the strength of concrete M follows a normal distribution $N\left(\mu, \sigma^2\right)$. Suppose that, given the strength m, the number of cracks of a product made by such concrete follows a Poisson distribution with mean $m(1-m)$. Find the probability that the number of cracks is 2.
3. Suppose that we have two fuses. The times that the fuses last are X and Y respectively. We assume that they follow independent exponential distribution with mean θ. Find the probability that $X \leq Y$.

Independence

Suppose that two random variables X and Y are independent. Then,

1. $\text{Cov}\left[X, Y\right] = 0$ and $E\left[XY\right] = E\left[X\right]E\left[Y\right]$.
2. If X and Y are discrete random variables, then

$$
\begin{aligned}
p_{jk} &= \mathrm{P}\left(X = j\right)\mathrm{P}\left(Y = k\right), \\
\mathrm{P}\left(X = j \mid Y = k\right) &= \mathrm{P}\left(X = j\right).
\end{aligned}
$$

3. If X and Y are discrete continuous variables, then

$$
\begin{aligned}
f\left(x, y\right) &= f_X\left(x\right)f_Y\left(y\right), \\
f\left(x \mid Y = y\right) &= f\left(x\right).
\end{aligned}
$$

Bayes' Formula

- For events, we have seen the Bayes' formula

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(B \mid A)\,\mathrm{P}(A)}{\mathrm{P}(B)}.$$

- The conditional cdf of Y given event B is $P(Y \leq y \mid B)$.
- The pdf or pmf of this conditional distribution is

$$f_{Y|B}(y) = \frac{\mathrm{P}(B \mid Y = y)\, f_Y(y)}{\mathrm{P}(B)},$$

where

$$\mathrm{P}(B) = \begin{cases} \sum_{y} \mathrm{P}(B \mid Y = y)\,\mathrm{P}(Y = y), & Y \text{ is discrete,} \\ \int_{-\infty}^{\infty} \mathrm{P}(B \mid Y = y)\, f_Y(y)\, dy, & Y \text{ is continuous.} \end{cases}$$

Example: Bayes' Formula

1. Let X be the maximal weight that will be carried by the wire for a period of 1 year. Assume that

$$\mathrm{P}\left(X \leq x\right) = \exp\left\{-e^{-(x-b)/a}\right\}, \quad x \in \mathbb{R},$$

   where $a = 156$ and $b = 910$.

2. Let Y be the strength of the wire (capacity to carry a load). We assume that Y follows a Weibull distribution with the density

$$f\left(y\right) = \frac{c}{\alpha}\left(\frac{y}{\alpha}\right)^{c-1}\exp\left\{-\left(\frac{y}{\alpha}\right)^{c}\right\}, \quad y \geq 0,$$

   where $c = 5.79$ and $\alpha = 1080$.

3. We also assume that X and Y are independent.

Let $B = \{\text{Safe operation during 1 year}\}$. Find $\mathrm{P}\left(B\right)$.

Example: Continue

Suppose that the wire has been used for one year and event B is true. We can update our knowledge about the wire to

$$
\begin{aligned}
f_{Y|B}(y) &= \frac{P(B \mid Y = y) f_Y(y)}{P(B)} \\
&= \frac{\exp\left\{-e^{-(x-910)/156}\right\} \cdot \frac{5.79}{1080}\left(\frac{y}{1080}\right)^{4.79} \exp\left\{-\left(\frac{y}{1080}\right)^{5.79}\right\}}{0.533}
\end{aligned}
$$

Let $C = \{\text{Safe operation during the next year}\}$. Hence,

$$
P(C) = \int_0^\infty P(C \mid Y = y) f_{Y|B}(y)\, dy \approx 0.705.
$$

Conditional independence

Denote by X_1 and X_2 the maximal load during the first and second year, respectively. We assume have that X_1 and X_2 are independent. Let $B_1 = \{X_1 < Y\}$ and $B_2 = \{X_2 > Y\}$.

1. If B_1 and B_2 were independent, then

$$P\left(\text{the wire survives two years}\right) = P\left(B_1\right) P\left(B_2\right) = (0.533)^2 .$$

2. If B_1 and B_2 were not independent, we may only have

$$P\left(B_1 \cap B_2 \mid Y = y\right) = P\left(B_1 \mid Y = y\right) P\left(B_2 \mid Y = y\right)$$

Then,

$$\begin{aligned} P\left(\text{the wire survives two years}\right) &= P\left(B_2 \mid B_1\right) P\left(B_1\right) \\ &= 0.705 \cdot 0.533 \end{aligned}$$