UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Inference Theory II, 1MS037
2019–01–04

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Suppose $X_1, ..., X_n$ are independent negative binomial with parameters (r, p), where r is a positive integer and $0 < p < 1$, i.e. the probability mass function of X_i is
$$p(x) = \binom{r+x-1}{x} p^r (1-p)^x$$
for $x = 0, 1, 2, ...$ Suppose that r is known and that p is unknown.
   (a) Find a sufficient statistic for p. (3p)
   (b) Find a minimal sufficient statistic for p. (It could be the same one as in (a).) (2p)

2. Suppose that the discrete random variable X can take the values 1, 2, 3 according to
$$P(X=1) = \theta_1^2,\ P(X=2) = \theta_2^2, P(X=3) = 2\theta_1\theta_2,$$
where $\theta_1 + \theta_2 = 1$. Consider an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X.
   (a) Does the distribution belong to a strictly k-parametric family? In that case, determine k, the natural parameter(s) and the sufficient statistic(s). (2p)
   (b) Show that the Fisher information for θ_1 is $\frac{2n}{\theta_1(1-\theta_1)}$. (2p)
   (c) Is there any unbiased estimator of θ_1 with variance strictly less than $\frac{\theta_1(1-\theta_1)}{2n}$? Motivate your answer. (2p)

3. Suppose $X_1, ..., X_n$ are independent, distributed according to a continuous uniform distribution on $[-\theta, \theta]$. We have observations $x_1, ..., x_n$.
   (a) Show that the maximum likelihood estimate (MLE) is $\hat{\theta}_{\text{MLE}} = \max_i |x_i|$, where $|a|$ is the absolute value of a. (3p)
   (b) Assume that the observations are $2.5, -3.2, -0.5, 2.0$. Consider testing H_0: $\theta = 4$ vs H_1: $\theta > 4$, using the MLE as test statistic.
   Calculate the p value. (3p)

   Hint: If X is uniform on $[-\theta, \theta]$, then $|X|$ is uniform on $[0, \theta]$. This fact may be used without proof.

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4. Suppose $X_1, ..., X_n$ are independent, distributed as $X \sim N(\mu, 1)$. We have observations $x_1, ..., x_n$.

   (a) Show that $\bar{X}$ is a sufficent statistic for μ. (1p)

   (b) Let $\theta = P(X \leq a)$ for some real number a. In the rest of this problem, the aim is to estimate θ. Show that the estimator

   $$U = \begin{cases} 1, & \text{if} \quad X_1 \leq a, \\ 0, & \text{if} \quad X_1 > a \end{cases}$$

   is unbiased for θ. (2p)

   (c) Find a unbiased estimator of θ with smaller variance than U. (3p)

   Hint: It may be used without proof that $(X_1 | \bar{X} = t)$ is normal with expectation t and variance $1 - n^{-1}$.

5. Consider testing that the observation x comes from a discrete distribution with probability function $p_0(x)$ vs the alternative that it comes from a discrete distribution with probability function $p_1(x)$, where these two probability functions are given in the following table:

x	1	2	3	4	5	6
$p_0(x)$	0.10	0.02	0.05	0.20	0.30	0.33
$p_1(x)$	0.05	0.10	0.20	0.25	0.20	0.20

   (a) Which is the most powerful (MP) test at level $\alpha = 0.05$? (2p)

   (b) Calculate the size of the type II error and the power for the MP test. (2p)

   (c) Calculate sizes of the errors of type I and II as well as the power for the test with critical region $\{x = 3\}$. Compare to the power for the MP test. (2p)

6. Suppose $X_1, ..., X_n$ are independent, distributed as X which is exponential with intensity $\beta > 0$, i.e. with density function

   $$f(x) = \beta \exp(-\beta x), \quad x > 0,$$

   and 0 otherwise. Let $x_1, ..., x_n$ be the observations.

   (a) Show that this distribution belongs to a one-parameter exponential family. (1p)

   (b) Give the natural parameter and the sufficient statistic. (1p)

   (c) Consider testing H_0: $\beta \geq \beta_0$ vs H_1: $\beta < \beta_0$. Show that the uniformly most powerful (UMP) test has critical region $\bar{x} > C$ where $\bar{x}$ is the mean of the observations and C is some constant. (3p)

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7. Suppose we have one observation of X_1, which is Poisson with parameter θ, and one of X_2, which is Poisson with parameter $\theta + \delta$, where X_1 and X_2 are independent. The parameter space consists of all $\theta > 0$ and all $\delta \geq 0$.

   (a) If $\delta = 0$, is $X_1 + X_2$ complete and sufficient for θ? Why or why not?(2p)

   (b) Consider testing H_0: $\delta = 0$ vs H_1: $\delta > 0$.
   Derive the UMP α-similar test. Do we reject at level 0.05 if $x_1 = 2$ and $x_2 = 5$? (4p)
   Hint: It may be used without proof that if $\delta = 0$, then $(X_2|X_1 + X_2 = t)$ is $\text{Bin}(t, 1/2)$.

GOOD LUCK!