THE CENTRAL LIMIT THEOREM VIA FOURIER TRANSFORMS

For $f \in L^1(\mathbb{R})$, we define $\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}dt$. so that for $f(t) = e^{-t^2/2}$, we have $\widehat{f}(x) = \sqrt{2\pi}e^{-x^2/2}$.

Theorem: Let $X_1 X_2, ...$ be independent and identically distributed random variables with $E(X_i) = 0$ and $\mathrm{var}(X_i) = 1$ Let $S_n = X_1 + X_2 + \cdots + X_n$. Then,

$$\lim_{n\to\infty} P(\alpha\sqrt{n} \le S_n \le \beta\sqrt{n}) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2}dt.$$

Proof: We want to determine the density function of $S_n/\sqrt{n}$. Recall that if X has density f_X and Y has density f_Y, then $X+Y$ has density $f_X * f_Y$, provided X and Y are independent. Since all the X_i's are identically distributed with density function f (say) and the X_i's are independent, S_n has density function given by

$$f^{*n} := f * \cdots * f.$$

Now if X has density $f(t)$, λX has density $\lambda^{-1}f(t/\lambda)$ (Exercise). Therefore $S_n/\sqrt{n}$ has density $g^{*n}(t)$ where $g(t) = \sqrt{n}f(\sqrt{n}t)$. We want to show

$$\lim_{n\to\infty} g^{*n}(t) \to \frac{1}{\sqrt{2\pi}}e^{-t^2/2}.$$

Taking Fourier transforms of both sides, this is equivalent to

$$\widehat{g}(t)^n \to e^{-t^2/2}.$$

But $\widehat{g}(t) = \widehat{f}(t/\sqrt{n})$. Taking the Taylor expansion, we have

$$f(t/\sqrt{n}) = \widehat{f}(0) + \widehat{f}'(0)t/\sqrt{n} + \widehat{f}''(0)t^2/2n + O(1/n^{3/2}).$$

Now

$$\widehat{f}(0) = \int_{-\infty}^{\infty} f(t)dt = 1,$$

since f is a probability density function. Also,

$$\widehat{f}'(0) = -i\int_{-\infty}^{\infty} tf(t)e^{-ixt}dt\Big|_{x=0} = \int_{-\infty}^{\infty} tf(t)dt = E(X) = 0$$

with $X = X_i$ by our hypothesis. Moreover, for $X = X_i$,

$$\widehat{f}''(0) = -\int_{-\infty}^{\infty} t^2 f(t)e^{-t^2/2}dt\Big|_{x=0} = -\int_{-\infty}^{\infty} t^2 f(t)dt = -\mathrm{var}(X) = -1.$$

Thus, our Taylor expansion simplifies to

$$1 - t^2/2n + O(1/n^{3/2}).$$

Using basic calculus, we immediately see that

$$\lim_{n\to\infty} \left(1 - \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n = e^{-t^2/2}.$$

This completes the proof. $\square$