Analysis of Time Series, L9

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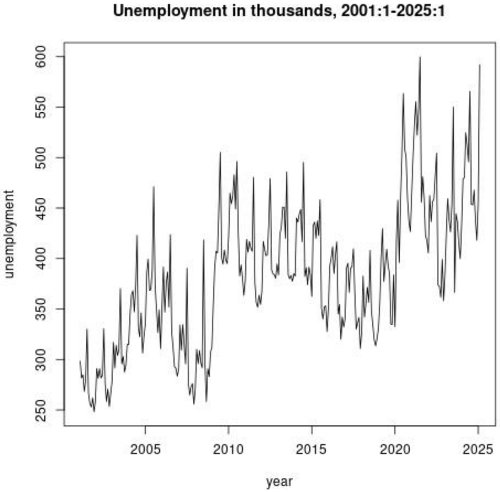
Today

3.9: Multiplicative Seasonal ARIMA (SARIMA) Models

- SARIMA models
- Empirical examples
- Menti

SARIMA models



SARIMA models

- Recall: ARMA(p, q)

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}$$

- or equivalently $\phi(B)x_t = \theta(B)w_t$ where

$$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p,$$
$$\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q.$$

- *Pure seasonal ARMA model*, ARMA$(P, Q)_s$

$$x_t = \Phi_1 x_{t-s} + \ldots + \Phi_P x_{t-Ps} + w_t + \Theta_1 w_{t-s} + \ldots + \Theta_Q w_{t-Qs}$$

- or equivalently $\Phi_P(B^s)x_t = \Theta_Q(B^s)w_t$ where

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \ldots - \Phi_p B^{Ps},$$
$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \ldots + \Theta_Q B^{Qs}.$$

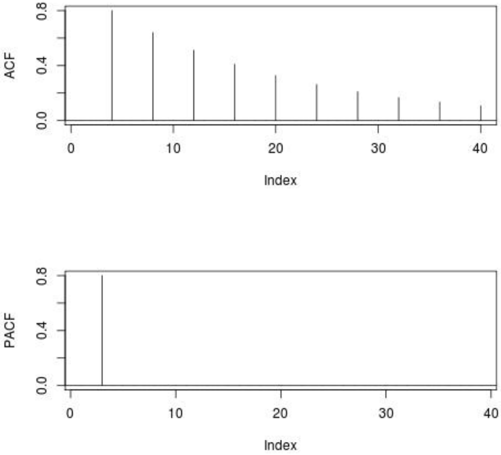
SARIMA models

Calculate the ACF of

1. $\text{ARMA}(0,1)_4$
2. $\text{ARMA}(1,0)_4$

SARIMA models

ACF and PACF for $x_t = 0.8x_{t-4} + w_t$



SARIMA models

- Pure seasonal ARMA$(P, Q)_s$ model

$$\Phi_P(B^s)x_t = \Theta_Q(B^s)w_t$$

where

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \dots - \Phi_p B^{Ps},$$
$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \dots + \Theta_Q B^{Qs}.$$

- Multiplicative (mixed) ARMA$(p, q) \times (P, Q)_s$ model

$$\Phi_P(B^s)\phi(B)x_t = \Theta_Q(B^s)\theta(B)w_t$$

where in addition

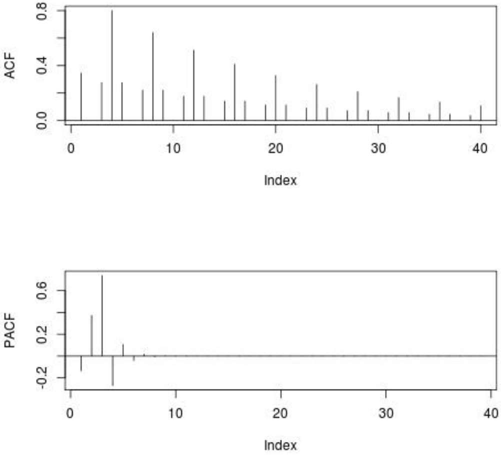
$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p,$$
$$\theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q.$$

SARIMA models

1. Write down the ARMA$(0,1) \times (1,0)_4$ model explicitly.
2. Derive its ACF.

SARIMA models

ACF and PACF for $x_t = 0.8x_{t-4} + w_t + 0.4w_{t-1}$



SARIMA models

Seasonal difference

$$\nabla_s x_t = (1 - B^s) x_t = x_t - x_{t-s}$$

Definition (3.12)

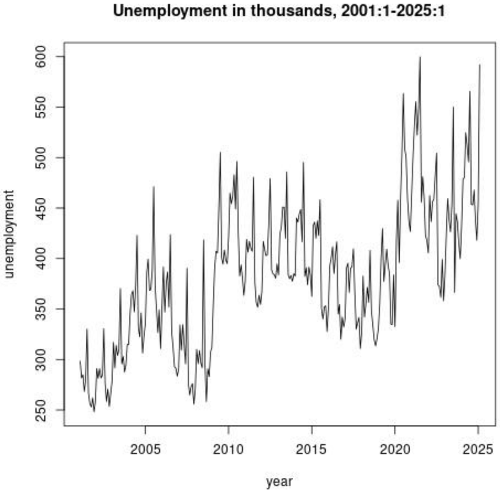
The multiplicative *seasonal autoregressive integrated moving average* (SARIMA) model is given by

$$\Phi_P(B^s)\phi(B)\nabla_s^D \nabla^d x_t = \delta + \Theta_Q(B^s)\theta(B) w_t,$$

where w_t is white noise.
It is denoted by ARIMA$(p, d, q) \times (P, D, Q)_s$.

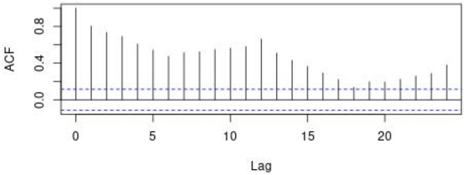
Write down the ARIMA$(0, 1, 1) \times (1, 1, 0)_4$ model without constant explicitly.

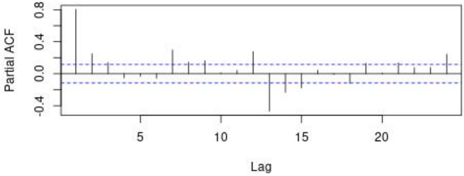
Empirical examples



Empirical examples

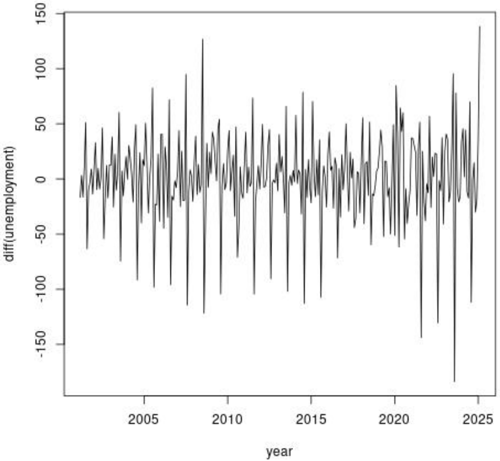
Unemployment, ACF and PACF. Slowly decaying ACF.
Try to take a difference!





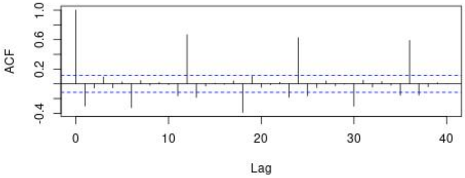
Empirical examples

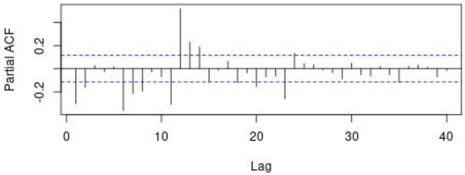
Plot of the differenced series:



Empirical examples

Differenced series, ACF and PACF. Slowly decaying ACF at lags 12, 24, 36,... (R: `acf(diff(u),40)` etc.) Try a seasonal difference.



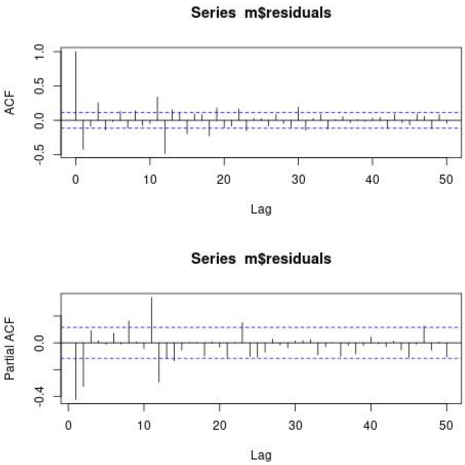


"Empty" seasonally differenced model:

```
> m=arima(u,order=c(0,1,0),seasonal=list(order=c(0,1,0),period=12));m

Call:
arima(x = u, order = c(0, 1, 0), seasonal = list(order = c(0, 1, 0), period = 12))


sigma^2 estimated as 1054:  log likelihood = -1352.2,  aic = 2706.39
```

ACF and PACF for residuals: Seasonal MA(1) behavior?

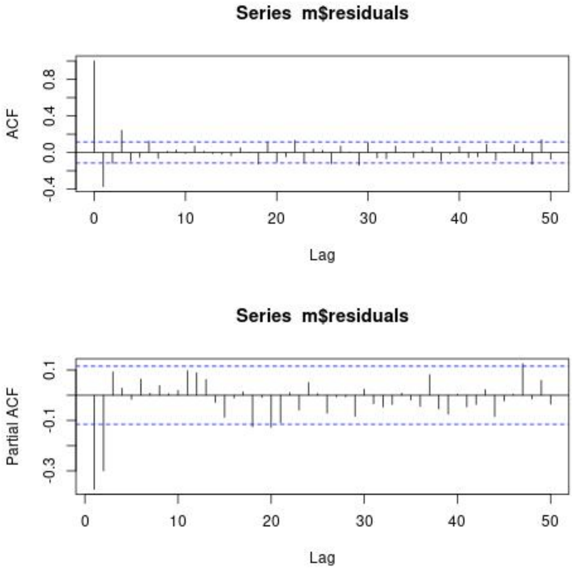
```
> acf(m$residuals,50)
> pacf(m$residuals,50)
```



Try ARIMA$(0,1,0) \times (0,1,1)_{12}$ (smallest AIC of purely seasonal models).

```
> m=arima(u,order=c(0,1,0),seasonal=list(order=c(0,1,1),period=12));m

Call:
arima(x = u, order = c(0, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
         sma1
      -0.8687
s.e.   0.0562

sigma^2 estimated as 626.1:  log likelihood = -1288.7,  aic = 2581.4
```

ACF and PACF for residuals: AR(2) behavior?

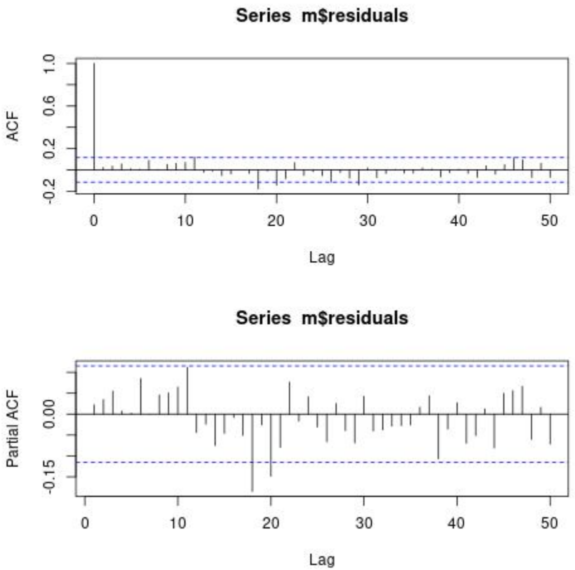


Try ARIMA$(2,1,0) \times (1,1,1)_{12}$. (Smaller AIC than the 'closest' models.)

```
> m=arima(u,order=c(2,1,0),seasonal=list(order=c(1,1,1),period=12));m

Call:
arima(x = u, order = c(2, 1, 0), seasonal = list(order = c(1, 1, 1), period = 12))

Coefficients:
          ar1      ar2    sar1     sma1
      -0.5384  -0.3427  0.1075  -0.8991
s.e.   0.0592   0.0587  0.0748   0.0574

sigma^2 estimated as 471.3:  log likelihood = -1250.03,  aic = 2510.07
```

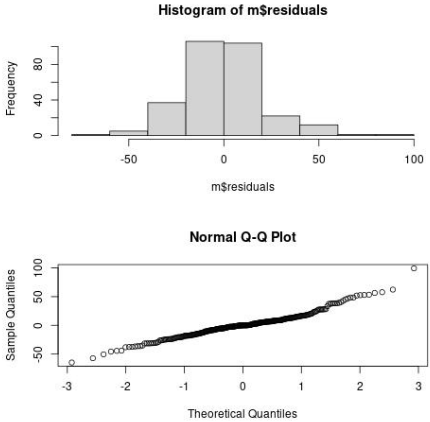
Observe: the `sar1` coefficient is not significant.

ACF and PACF for residuals: White noise behavior?



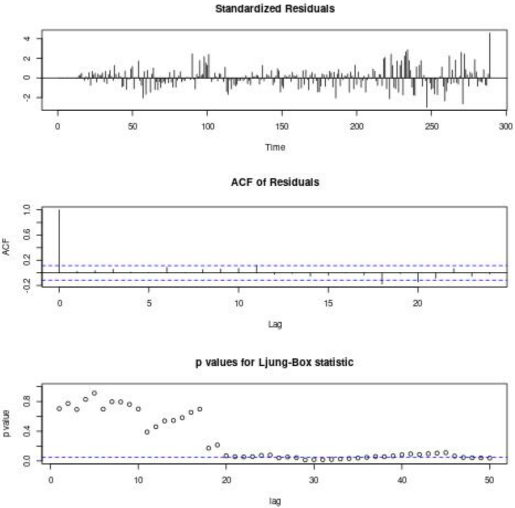
Histogram and qqplot for residuals:

```
> par(mfrow=c(2,1))
> hist(m$residuals)
> qqnorm(m$residuals)
```



Diagnostics for residuals: Too small P values!

```
> tsdiag(m,50)
```

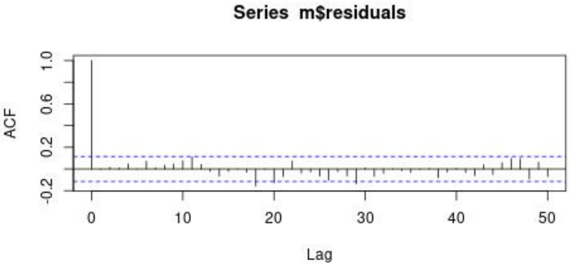


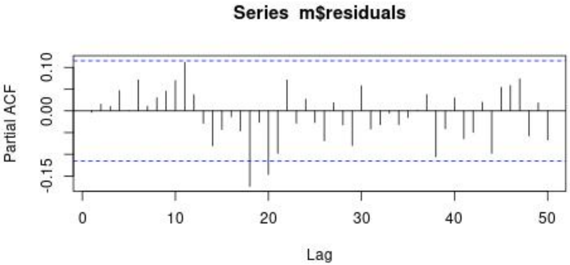
Instead, try ARIMA$(3, 1, 0) \times (0, 1, 1)_{12}$. (A little bit higher AIC, but maybe better diagnostics?)

```
> m=arima(u,order=c(3,1,0),seasonal=list(order=c(0,1,1),period=12));m

Call:
arima(x = u, order = c(3, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
          ar1      ar2     ar3     sma1
      -0.5065  -0.3016  0.0704  -0.8534
s.e.   0.0625   0.0679  0.0624   0.0532

sigma^2 estimated as 475.8:  log likelihood = -1250.42,  aic = 2510.85
```

Observe that the `ar3` coefficient is not significant.
(But removing it gave worse Ljung-Box P values than for this model.)

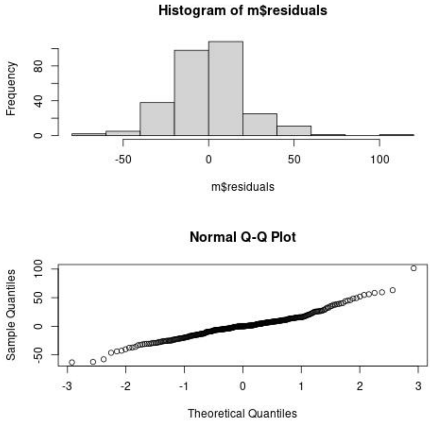
ACF and PACF for residuals: White noise behavior?





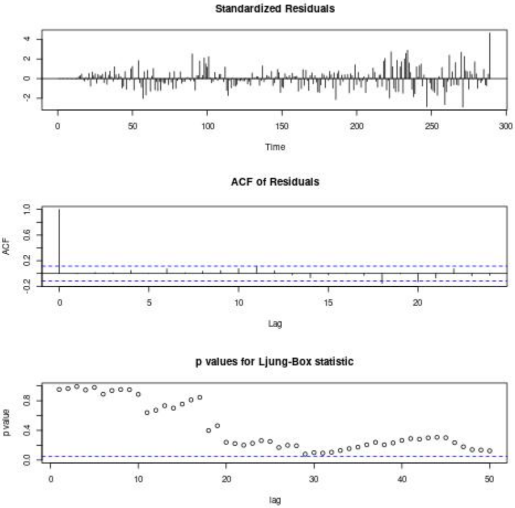
Histogram and qqplot for residuals:

```
> par(mfrow=c(2,1))
> hist(m$residuals)
> qqnorm(m$residuals)
```

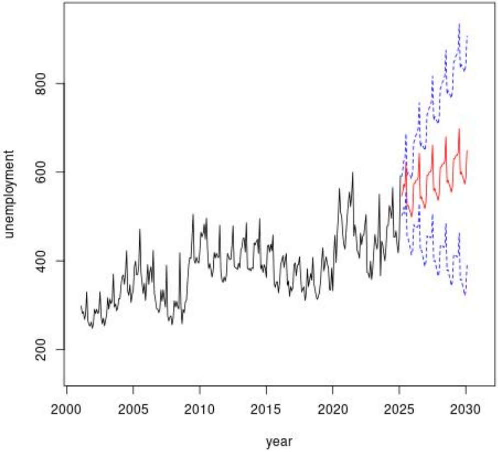


Diagnostics for residuals: Better P values!

```
> tsdiag(m,50)
```



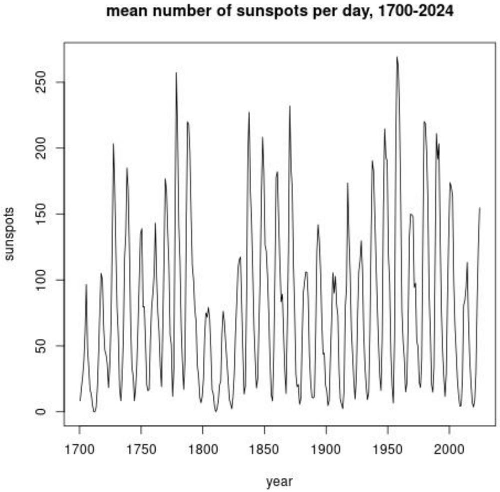
Plot of forecasts (5 years ahead):



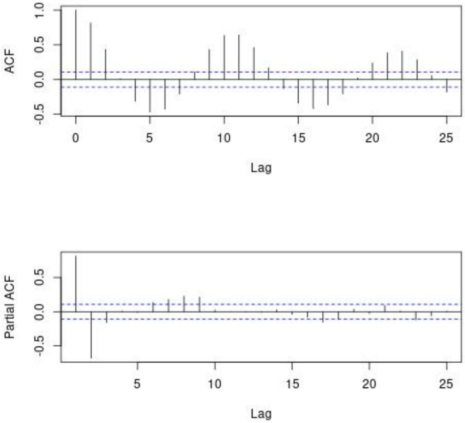
R code for this plot:

```
> fore=predict(m,n.ahead=60)
> plot(seq(2001+1/12,2025+1/12,1/12),u,type='l',xlab='year',
xlim=c(2001,2031),ylim=c(150,950),ylab='unemployment')
> lines(seq(2025+2/12,2030+1/12,1/12),fore$pred,col='red')
> lines(seq(2025+2/12,2030+1/12,1/12),fore$pred+2*fore$se,col='blue',
lty='dashed')
> lines(seq(2025+2/12,2030+1/12,1/12),fore$pred-2*fore$se,col='blue',
lty='dashed')
```

Empirical examples



Empirical examples

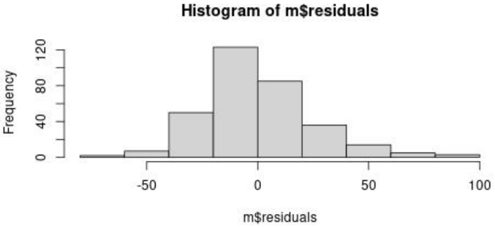


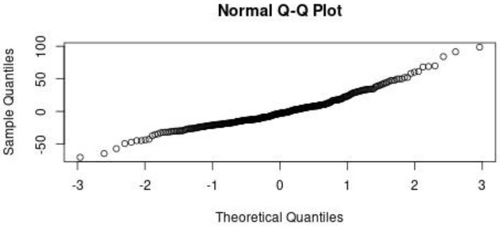
10-11 year season. AR(3) structure of PACF?

Try ARIMA$(3,0,0) \times (0,0,1)_{10}$ (smallest AIC among similar models):

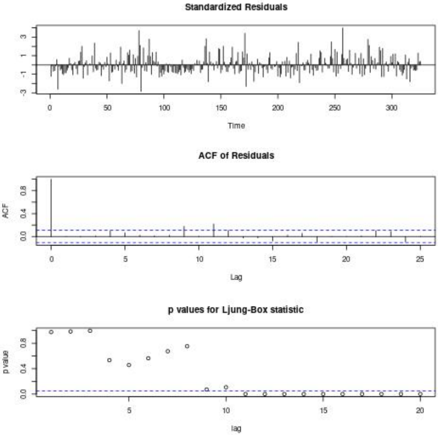
```
> m=arima(s,order=c(3,0,0),seasonal=list(order=c(0,0,1),period=10));m

Call:
arima(x = s, order = c(3, 0, 0), seasonal = list(order = c(0, 0, 1), period = 10))

Coefficients:
         ar1      ar2      ar3    sma1  intercept
      1.2052  -0.4028  -0.1670  0.2309    78.5312
s.e.  0.0587   0.0866   0.0548  0.0581     4.6094

sigma^2 estimated as 611.7:  log likelihood = -1505.21,  aic = 3022.41
```

Empirical examples





Empirical examples



Small P values for Ljung-Box! Problem: the season is not exactly 10 years.

News of today

- Seasonal differencing
- SARIMA models