Exam June 2023

Problem 5

Given the utility function:

$$U(c_0, c_1) = u(c_0) + \delta u(c_1)$$

where $0 < \delta < 1$, and $u(\cdot)$ is increasing and concave.

Part a

Write down the stochastic discount factor (SDF): Maximize:

$$\max_{x_t} u(c_t(x_t)) + \delta E_t[u(c_{t+1}(x_{t+1}))]$$

where x_t is the portfolio choice. First-order condition with respect to x_t:

$$u'(c_t)\frac{\partial c_t}{\partial x_t} + \delta E_t u'(c_{t+1})\frac{\partial c_{t+1}}{\partial x_t} = 0$$

$$\frac{\partial c_t}{\partial x_t} = P_{it}, \quad \frac{\partial c_{t+1}}{\partial x_t} = D_{it+1} + P_{it+1}$$

Thus:

$$u'(c_t)(-P_{it}) + \delta E_t u'(c_{t+1})(D_{it+1} + P_{it+1}) = 0$$

$$P_{it} = \delta E_t \left[u'(c_{t+1})(D_{it+1} + P_{it+1})\right] / u'(c_t)$$

From this, we derive:

$$1 = \delta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \left(\frac{D_{it+1} + P_{it+1}}{P_{it}}\right)\right] \Rightarrow 1 = \delta E_t m_{t+1} R_{t+1}$$

The stochastic discount factor (SDF) is:

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$$

Part b

Derive an expression for the risk-free rate R_f and how it depends on u and δ.
We know:

$$1 = E_t[m_{t+1}R_f] \Rightarrow 1 = R_f E_t[m_{t+1}]$$

which leads to:

$$R_f = \frac{1}{E_t[m_{t+1}]}$$

Substitute the expression for m_{t+1}:

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$$

Thus:

$$R_f = \frac{u'(c_t)}{\delta E_t[u'(c_{t+1})]}$$

and $R_{t+1} = 1 + r_f$.

Part c

Find the risk-free rate R_f if $u(x) = \ln(x)$ and consumption growth is log-normally distributed with mean μ and standard deviation σ.
Since $u'(c) = \frac{1}{c}$, then:

$$u'(c_t) = \frac{1}{c_t}, \quad u'(c_{t+1}) = \frac{1}{c_{t+1}}$$

and:

$$m_{t+1} = \delta \frac{c_t}{c_{t+1}}$$

Let $c_{t+1}/c_t = \exp(g_{t+1})$, where g_{t+1} is the consumption growth rate.
Therefore:

$$m_{t+1} = \delta \exp(-g_{t+1})$$

Using the properties of log-normal distribution:

$$E_t[\exp(-g_{t+1})] = \exp\left(-\mu + \frac{\sigma^2}{2}\right)$$

Thus:

$$R_f = \frac{1}{\delta \exp\left(-\mu + \frac{\sigma^2}{2}\right)} = \exp\left(\mu - \frac{\sigma^2}{2}\right)$$

Finally, the risk-free rate is:

$$R_f = 1 + r_f = \delta^{-1} \exp(\mu - \frac{\sigma^2}{2})$$

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Problem 3

Part b

$$f_{n,m} = \left[\frac{(1+y_n)^n}{(1+y_m)^m}\right]^{1/n-m} \iff (1+y_n)^n = (1+y_m)^m(1+f_{n,m})^{n-m} \quad f_{1,4}\ ?$$

$$(1+y_4)^4 = (1+y_1)^1 \quad (1+f_{1,4})^{4-1}$$

$$1.08^4 = (1.05) \times (1+f_{1,4})^3$$

$$\frac{1.3604}{1.05} \quad \text{for} \quad f_{1,4} \approx 1.09$$

$$f_{1,4} \approx 0.09 \quad > \quad 0.08$$

Part c

Lecture 12 page 23 , zero coupon bond maturity at time t is $D_t = n - t$ where n is time to maturity. Form the portfolio for total duration to be 3

$$D_1 = 1 \quad \text{and} \quad D_3 = 4$$

Define x as the fraction of the investment in Bond 1, so $1 - x$ is the fraction in Bond 3. Then, for the portfolio duration:

$$D = x \cdot D_1 + (1 - x) \cdot D_3$$

Substitute $D = 3$, $D_1 = 1$, and $D_3 = 4$:

$$3 = x \cdot 1 + (1 - x) \cdot 4$$

$$3 = x + 4 - 4x$$

$$-1 = -3x$$

$$x = \frac{1}{3}$$

Part d

$F = 100$ for all bonds. P_1, P_2, P_3. Is $P_3 < P_1 < P_2$?

$$P_i = \sum \frac{M_i}{(1+y)^i}$$

$$P_1 = \frac{100}{1 + 0.05} \approx 95.23$$

$$P_3 = \frac{100}{(1 + 0.08)^4} = \frac{100}{1.3604} \approx 73.50$$

$$P_2 = \frac{6}{1.06} + \frac{6}{(1.06)^2} + \frac{100}{(1.06)^2} = 5.66 + 94.37 \approx 100$$

$$P_3 < P_1 < P_2 \quad \text{yes!}$$

Problem 4

Check the proof in the book page 270 chapter 7-One period portfolio choice

Part a

$$M_1 = 0.2, \quad M_2 = 0.4, \quad \sigma_1 = 0.1, \quad \sigma_2 = 0.2$$

Minimum variance portfolio:

$$\min_{\pi} \quad \boldsymbol{\pi' \Sigma \pi}$$

Subject to:

$$\boldsymbol{1'\pi} = 1$$

$$L = \boldsymbol{\pi'\Sigma\pi} - 2\lambda(\boldsymbol{1'\pi} - 1)$$

$$\frac{\partial L}{\partial \pi} = 2\boldsymbol{\Sigma\pi} - 2\lambda\boldsymbol{1} = 0$$

$$2\boldsymbol{\Sigma\pi} = 2\lambda\boldsymbol{1}$$

$$\boldsymbol{\Sigma\pi} = \lambda\boldsymbol{1}$$

$$\pi = \frac{\lambda}{2}\boldsymbol{\Sigma^{-1}1}$$

Substitute this into the constraint:

$$\boldsymbol{1'\pi} = \boldsymbol{\pi'1} = 1 \quad \Rightarrow \boldsymbol{1'}\left(\boldsymbol{\frac{\lambda}{2}}\boldsymbol{\Sigma^{-1}1}\right) \quad \frac{\lambda}{2}\boldsymbol{1'\Sigma^{-1}1} = 1 \quad \frac{\lambda}{2} = \frac{1}{C} \quad \text{where} \quad C = \boldsymbol{1'\Sigma^{-1}1}$$

Thus:

$$\pi_{min} = \frac{1}{C}\boldsymbol{\Sigma^{-1}1}$$

$$\mu_{min} = \boldsymbol{\mu\pi} = \boldsymbol{\mu'}\left(\frac{1}{C}\boldsymbol{\Sigma^{-1}1}\right) = \frac{1}{C}\boldsymbol{\mu'\Sigma^{-1}1} = \frac{B}{C}$$

For a square matrix we know:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Given:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{\Sigma^{-1}} = \begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix}$$

$$\boldsymbol{\Sigma^{-1}1} = \begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 25 \end{bmatrix}$$

$$\boldsymbol{1'\Sigma^{-1}1} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 100 \\ 25 \end{bmatrix} = 125 \quad \Rightarrow \frac{1}{\boldsymbol{1'\Sigma^{-1}1}} = \frac{1}{C} = \frac{1}{125}$$

$$\boldsymbol{\mu'\Sigma^{-1}1} = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 100 \\ 25 \end{bmatrix} = 30$$

$$\boldsymbol{\mu\pi} = \frac{1}{C}\boldsymbol{\mu'\Sigma^{-1}1} = \frac{30}{125} = \frac{6}{25} = 0.24$$

Part b

Call x be the $\mathrm{Cov}(r_1, r_2)$

$$\mathbf{\Sigma} = \begin{bmatrix} 0.01 & x \\ x & 0.04 \end{bmatrix} \quad \mathbf{\Sigma^{-1}} = \frac{1}{0.04 \cdot 0.01 - x^2} \begin{bmatrix} 0.04 & -x \\ -x & 0.01 \end{bmatrix} = \frac{1}{0.0004 - x^2} \begin{bmatrix} 0.04 & -x \\ -x & 0.01 \end{bmatrix}$$

Since $\dfrac{1}{0.0004 - x^2}$ is a constant, and we can take this out from the matrix multiplication and then add it at the end.

$$\mathbf{1'\Sigma^{-1}1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0.04 & -x \\ -x & 0.01 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.04 - x \\ -x + 0.01 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{0.05 - 2x}{0.0004 - x^2}$$

$$\frac{1}{C} = \frac{0.0004 - x^2}{0.05 - 2x}$$

$$\pi = \frac{1}{C}\mathbf{\Sigma^{-1}1} = \frac{0.0004 - x^2}{0.05 - 2x} \begin{bmatrix} 0.04 - x \\ 0.01 - x \end{bmatrix} \frac{1}{0.0004 - x^2}$$

$$\boldsymbol{\pi} = \begin{bmatrix} \dfrac{0.04 - x}{0.05 - 2x} \\ \dfrac{0.01 - x}{0.05 - 2x} \end{bmatrix}$$

Since we are given $(\pi_1, \pi_2) = (1, 0)$

$$\mathrm{Cov}(r_1, r_2) = 0.01 \quad \rho = \frac{\mathrm{Cov}(r_1, r_2)}{\sigma_1 \sigma_2} \quad \rho = 1/2$$

Part c

$$\mu_p = \pi_1 \mu_1 + \pi_2 \mu_2$$

$$\sigma_p^2 = \pi_1^2 \theta_1^2 + \pi_2^2 \sigma_2^2 \rightarrow \text{ no covariance because they are uncorrelated}$$

$$\pi_1 + \pi_2 = 1$$

$$\mu_1 = 0.2 \quad \mu_2 = 0.4 \quad \sigma_1 = 0.1 \quad \sigma_2 = 0.2$$

$$\mu_p = \pi_1 \cdot 0.2 + (1 - \pi_1) \cdot 0.4 \quad = 0.4 - 0.2\pi_1 \quad \rightarrow \frac{\bar{\mu} - 0.4}{-0.2} = \pi_1 \quad \rightarrow \pi_1 = 2 - 5\mu$$

$$\sigma_p^2 = {\pi_1}^2 0.01 + \underbrace{(1 - \pi_1)^2}_{1 - 2\pi_1 + \pi_1^2} 0.04$$

$$\sigma_p^2 = \pi_1^2 0.01 + 0.04 - 0.08\pi + 0.04\pi_1^2$$
$$= 0.05{\pi_1}^2 - 0.08\pi_1 + 0.04$$

$$\sigma_p^2 = 0.05 \left[2 - 5\bar{\mu}\right]^2 - 0.08 \left[2 - 5\bar{\mu}\right] + 0.04$$

$$\sigma_p^2 = 4 - 20\bar{\mu} + 25\bar{\mu}^2$$

$$\sigma_p^2 = 0.2 - \bar{\mu} + 1.25\bar{\mu}^2 - 0.16 + 0.4\bar{\mu} + 0.04$$

$$\sigma_p^2 = 1.25\bar{\mu}^2 - 0.6\bar{\mu} + 0.08$$

$$\sigma(\bar{\mu}) = \sqrt{1.25\bar{\mu}^2 - 0.6\bar{\mu} + 0.08}$$

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Problem 1

Part a

Write down Gordon's formula for the stock price P as a function of the dividends D_t and define the different parts of it.
$P_t = \dfrac{(1+g)D_t}{r-g}$ where g as the expected dividend growth rate.

Part b

A zero coupon bond has 4 years to maturity and is trading at a yield of $y = 0.043$. Determine the duration of the bond.

$D_t = n - t$ for a zero capon bond maturing at n, the duration at time $t < n$ is $n - t$. Thus $D_0 = 4 - 0 = 4$ because the bond only has one cash flow at maturity.

Part d

d) An investor has utility function $u(x) = -5e^{-3x}$. Derive the investors' coefficient of absolute risk aversion

$$\begin{aligned} \text{ARA} &= \frac{-u''(x)}{u'(x)} \quad \Rightarrow u''(x) = -45e^{-3x} \quad \text{and} \quad u'(x) \quad = 15e^{-3x} \\ &= \frac{-(-45)e^{-3x}}{15e^{-3x}} = 3 \end{aligned}$$

Problem 2

Part a

The weights must satisfy

$$\pi_1 + \pi_2 + \pi_3 = 1$$

since investor has invested the same amount of 300k on each asset

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

$$
\begin{aligned}
\beta_p &= \pi_1\beta_1 + \pi_2\beta_2 + \pi_3\beta_3 \\
&= \frac{1}{3}\cdot 0.9 + \frac{1}{3}\times 1.2 + \frac{1}{3}\times 1.8 \\
&= 0.3 + 0.4 + 0.6 \\
&= 1.3
\end{aligned}
$$

Part b

$\sigma_1 = 0.2$ and $\sigma_2 = 0.4$
Determine the variance of the minimum variance portfolio if the rate of returns ore uncorrelated and short-selling is allowed (weights can be negative).

$$r_p = \pi r_1 + (1-\pi) r_2$$

$$\operatorname{Var}(r_p) = \pi^2 \operatorname{Var}(r_1) + (1-\pi)^2 \operatorname{Var}(r_2) + 2\pi(1-\pi)\rho \operatorname{Srd}(r_1) \operatorname{Std}(r_2)$$

$$\rho = 0$$

$$\operatorname{Var}(r_p) = \pi^2\sigma_1^2 + (1-\pi)^2\sigma_2^2$$

$$\min_\pi \operatorname{Var}(r_p) \quad \rightarrow \frac{\partial \operatorname{Var}(r_p)}{\partial \pi} = 0$$

$$2\pi\sigma_1^2 + 2(\pi-1)\sigma_2^2 = 0 \quad \rightarrow \pi(\sigma_1^2+\sigma_2^2) - \sigma_2^2 = 0$$

$$\pi_{\min} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \frac{0.16}{0.20} = 0.8$$

$$
\begin{aligned}
&\pi_{\min} = 0.8 \\
&\operatorname{var}(r_p) = (0.8)^2 \cdot 0.04 + (0.2)^2 0.16 \\
&= 0.64\times 0.04 + 0.04\times 0.16 \\
&= 0.04(0.64+0.16) \\
&= 0.04\times 0.8 \\
&= 0.032
\end{aligned}
$$

Part c

Determine the $\alpha\%$ one manth value at risk of r satisfies $\ln(1+r) \sim N\left(\mu, \sigma^2\right)$

$$P(r \leq \mathrm{VaR}) = \alpha\%$$

$$P(1+r \leq 1+\mathrm{VaR}) = \alpha\%$$

$$P(\ln(1+r) \leq \ln(1+\mathrm{VaR})) \leq \alpha\%$$

$$\ln(1+r) \sim N\left(\mu,\sigma^2\right) \text{ then}$$

$$N(\frac{\ln(1+\mathrm{VaR})) - \mu}{\sigma}) = \alpha\%$$

$$N^{-1}(\alpha\%) = \frac{\ln(1+\text{VaR}) - \mu}{\sigma}$$

$$\ln(1+\text{VaR}) - \mu = \sigma N^{-1}(\alpha\%)$$
$$\ln(1+\text{VaR}) = \mu + \sigma N^{-1}(\alpha\%)$$
$$1 + \text{VaR} = e^{\left[\mu + \sigma N^{-1}(\alpha\%)\right]}$$

$$\text{VaR} = e^{\left[\mu + \sigma N^{-1}(\alpha y_n)\right]} - 1$$

Part d

Consider the father model

$$r_i = E[r_i] + \beta_{11}(F_1 - E[F_1]) + \beta_{i2}(F_2 - E[F_2]) + \varepsilon_i$$

Write down the equation for the expected return $E[r_i]$ if there exists a risk free asset and the conditions of the APT holds

$$E[r_i] = r_f + \beta_{i1} RP_1 + \beta_{i2} RP_2$$

Part e

Show that if the trust n years of yields to maturity are all equal $y_1 = y_2 = \cdots y_n = y_1$, then every forward rate defined by this yield curve is equal to y.

$$(1+y_n)^n = (1+y_m)^m (1+f_{m,n})^{n-m}$$
$$(1+y)^n = (1+y)(1+f_{1,n})^{n-1}$$
$$(1+y)^{n-1} = (1+f_{1,n})^{n-1} \rightarrow f_{1,n} = y$$
$$(1+y)^n = (1+y)^2 (1+f_{2,n})^{n-2}$$
$$(1+y)^{n-2} = (1+f_{2,n})^{n-2} \rightarrow f_{2,n} = y$$

all forward rate defined by this yield curve is y.

Problem 3

Check the proof in the book page 270-271 chapter 7-One period portfolio choice

Problem 4

Same as the exercise 6.1 and the last part of the question is same as the exercise 2.2 (Chapter 2 page 34-35)