Le 11

Cauchy's integral formula & applications

Using Cauchy's integral thm one can prove the following important result:

Thm (Cauchy's integral formula)

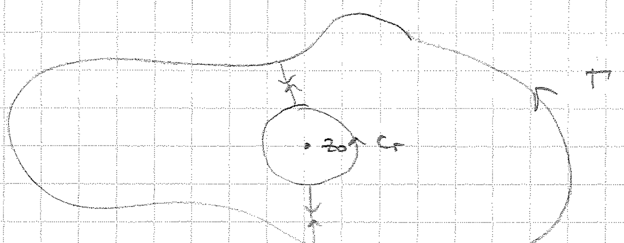
Suppose f analytic in a simply connected domain D. Let Γ be a simple closed positively oriented contour in D, and z_0 any point inside Γ.

Then,

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz$$

"Inside of Γ lies to the left on Γ as you traverse Γ according to its orientation"

Proof: $\frac{f(z)}{z-z_0}$ is analytic in $D \setminus \{z_0\}$.



As in the last example of Le 9, we see that

$$\int_\Gamma f(z)\, dz = \int_{C_r} \frac{f(z)}{z-z_0}\, dz =$$

$$= \int_{C_r} \frac{f(z_0)}{z-z_0}\, dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

Note that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz$ is indep. of r

Enough to show that $\lim_{r\to 0^+} \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$.

Let $M_r := \max_{z\in C_r} |f(z)-f(z_0)|$

f continuous $\Rightarrow$ $M_r \to 0$ as $r\to 0^+$.

So, by the ML-ineq.,

$$\left| \int_{C_r} \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\le \frac{M_r}{r}}\,dz \right| \le \frac{M_r}{\cancel{r}} \cdot 2\pi \cancel{r} \to 0,\ r\to 0^+$$

$\square$

<u>Remark</u>: $f(z_0)$ is determined by $f(z)$, $z\in\Gamma$!

So, Cauchy's integral formula says that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad (z \text{ inside } \Gamma)$$

By diff. under the integral sign it seems plausible that the following holds:

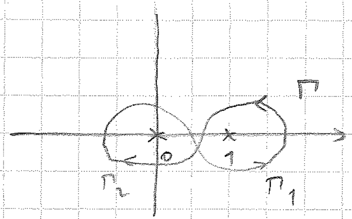
<u>Thm</u> (Cauchy's generalized integral formula)

If f is analytic inside and on a simple closed positively oriented contour, and z is a point inside Γ, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \quad (n=0,1,2,\dots)$$

The proof is by induction using the def. of derivative, I leave this out.

<u>Ex</u> Compute $\displaystyle\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz$, where Γ is as in the figure below:



<u>Sol:</u> Note that $\Gamma = \Gamma_1 + \Gamma_2$; Γ_1, Γ_2 simple closed

$$\Rightarrow \int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = \int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,dz + \int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z}\,dz$$

Γ_2 neg. oriented

$$= 2\pi i \frac{d}{dz}\Big(\underbrace{\frac{2z+1}{z}}_{=2+\frac{1}{z}}\Big)\Big|_{z=1} - 2\pi i \frac{2z+1}{(z-1)^2}\Big|_{z=0} = \underline{\underline{-4\pi i}}$$

$\square$

Note that Cauchy's generalized integral formula implies the following:

> <u>Thm</u> If f is analytic in a domain D, then all derivatives $f', f'', \dots$ exist and are analytic in D.

<u>Proof:</u> Apply Cauchy's gen. integral formula on a pos. oriented circle around an arbitrary $z_0 \in D$. $\square$

So, in particular the derivative of an analytic fcn is analytic.

Suppose f analytic. Recall that

$$f'(z) = \overset{1)}{\overbrace{u_x}^{\tilde{u}}} + i\,\overbrace{v_x}^{\tilde{v}} = \overset{2)}{\overbrace{v_y}^{\tilde{u}}} - i\,\overbrace{u_y}^{\tilde{v}}$$

Since f' is analytic, and in partic. continuous, it follows that $u, v \in C^1$. Since f'' exists and is continuous, and

$$f''(z) = u_{xx} + i v_{xx} = v_{xy} - i u_{xy}$$

$$f''(z) = v_{yx} - i u_{yx} = -u_{yy} - i v_{yy}$$

it follows that also all second derivatives are cont., etc.

Thm If $f = u + iv$ is analytic in a domain D, then $u, v \in C^\infty(D)$.

Remark: This completes the proof that u, v are harmonic.

$=$

Suppose f continuous in a domain D, and that

$\int_\Gamma f(z)\,dz = 0 \quad \forall$ closed contours in D.

$\overset{\text{Thm on path indep.}}{\Longrightarrow}$ f has an antiderivative F in D,

i.e. $F' = f$ in D.

Since F is analytic, so is $F' = f$ according to the theorem above. We have proven the following:

<u>Thm</u> (Morera)

If f is continuous in a domain D, and $\int_\Gamma f(z)\,dz = 0$ for all closed contours Γ in D, then f is analytic in D.

<u>Consequences of Cauchy's (gen.) integral formula</u>

<u>Thm</u> (Cauchy estimate)

Let f be analytic inside and on a circle C_R of radius R centered at z_0.

Suppose $|f(z)| \leq M$ for all $z \in C_R$.

Then, it holds that

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{R^n} \quad (n = 0, 1, 2, \ldots)$$

Proof: Give C_R a positive orientation.

Then, by Cauchy's gen. integral formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta .$$

For $\zeta \in C_R$ it holds that

$$\left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| \leq \frac{M}{R^{n+1}}$$

The length of C_R, $L(C_R)$, is $2\pi R$

So by the ML-ineq,

$$\Rightarrow \left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n! \, M}{R^n}$$

$\boxtimes$

Suppose f entire, $|f(z)| \leq M \quad \forall z \in \mathbb{C}$

Then the Cauchy estimate implies that

$$|f'(z_0)| \leq \frac{M}{R}$$

This is true $\forall R > 0 \Rightarrow |f'(z_0)| = 0$, i.e. $f'(z_0) = 0$

This is true $\forall z_0 \in \mathbb{C} \Rightarrow f'(z) = 0 \Rightarrow f(z) = \text{const.}$

We have proven the following:

Thm (Liouville)
The only bounded entire functions are the constant fcns

Liouville's thm can be used to prove the following well-known result:

Thm (Fundamental thm of algebra)
Every non-constant polynomial with complex coefficients has at least one zero.

Proof: Let $P(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$.

Suppose that $P(z)$ has no zeros.

Put $f(z) = \frac{1}{P(z)}$. Then f is entire.

We next show that $f(z)$ is bounded.

1) $P(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right)$, so

$$\frac{P(z)}{z^n} \longrightarrow a_n \quad \text{as} \quad z \to \infty$$

$$\Rightarrow \exists \rho \text{ s.a. } |z| \geq \rho \Rightarrow \left| \frac{P(z)}{z^n} \right| \geq \frac{|a_n|}{2}$$

$$\Rightarrow |f(z)| = \frac{1}{|P(z)|} \leq \frac{2}{|z|^n |a_n|} \leq \frac{2}{\rho^n |a_n|}, \quad |z| \geq \rho.$$

2) For $|z| \leq \rho$ the fcn $|f(z)|$ is a cont. fcn on a compact set $\Rightarrow$ $|f(z)|$ has a maximum, and in partic. is bounded.

Thus: $\frac{1}{P(z)}$ is a bounded entire fcn, and must therefore be constant according to Liouville's theorem. But then $P(z)$ must be constant (so $n=0$).

In other words: the only polynomials without zeros are the constant ones.

$\boxtimes$