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Matematiska institutionen
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Financial Derivatives
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Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points.
You may use the textbook and notes from the course. No communication with other people is allowed. Your solutions should be uploaded in Studium by 1.20pm as a single pdf-file.
If you have any questions during the exam, please contact Erik Ekström (tel 0739 98 60 81) or Yuqiong Wang (tel 072 564 61 94).
Motivate your answers carefully!

1. Let $X(t)$ be the solution of the stochastic differential equation

$$\begin{cases} dX(t) = 2\,dt + X(t)\,dW(t) \\ X(0) = x, \end{cases}$$

where $x > 0$. Determine $\mathbb{E}[X(t)]$ and $Var(X(t))$.

2. Find a continuous function $u : [0,T] \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$ that satisfies

$$\frac{\partial u}{\partial t}(t,x,y) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{y^2}{2}\frac{\partial^2 u}{\partial y^2}(t,x,y) = 0$$

for $(t,x,y) \in [0,T) \times \mathbb{R} \times (0,\infty)$ and with $u(T,x,y) = (x+y)^2$.

3. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time 0 of a contract which at time T pays the holder the amount $2S(T)$ provided $S(T) < b$, and 0 otherwise.

4. Answer the following short questions. Provide a short motivation in each case.

(i) If W_1 and W_2 are independent Brownian motions, determine a constant $a > 0$ such that the process $W(t) := a(2W_1(t) + 5W_2(t))$ is a Brownian motion.

(ii) Let X be a geometric Brownian motion with drift μ and volatility σ. Determine a constant c so that $e^{-ct}X^3(t)$ is a martingale.

(iii) Is it true that the arbitrage-free price C of a call option with strike K on a stock with price s has to satisfy $C \geq s - K$? (The stock does not pay dividends, and the short rate is non-negative.)

5. In a market consisting of a bank account with a constant interest rate r and a non-dividend paying stock S, consider a T-claim that pays

$$\mathcal{X} = \frac{S(T_0) + S(T)}{2}$$

at time T, where $0 < T_0 < T$.

a) Find a replicating strategy for $\mathcal{X}$.
b) What is the arbitrage-free price of $\mathcal{X}$ at time 0?
c) What is the arbitrage-free price of $\mathcal{X}$ at time $t \in (T_0, T)$?

6. Consider a European call option and a European put option written on the same underlying stock, which pays no dividends. Both options mature three months from now and have strike price 150. Moreover, assume that a three-month zero-coupon bond with face value 50 trades at 49. Your broker quotes the prices 6 and 12 for the call and the put, respectively. Show how to construct an arbitrage if the current stock price is 140.

7. Consider a model

$$dr(t) = \left(\frac{\sigma^2}{4} - a\sqrt{r(t)}\right) dt + \sigma\sqrt{r(t)}\, dW(t)$$

for the short rate under the pricing measure, where a and σ are constants. Show that bond prices $p(t,T)$ at time t with maturity T can be determined on the form

$$p(t,T) = \exp\{A(t,T)r(t) + B(t,T)\sqrt{r(t)} - C(t,T)\}$$

for some deterministic functions $A(t,T)$, $B(t,T)$ and $C(t,T)$.

Comment: You do not need to determine the functions A, B and C, but it suffices that you show how to find them.

8. Consider a model consisting of a stock S with volatility σ that pays a continuous dividend yield $\delta > 0$, and a bank account with interest rate $r \geq 0$. In this model, consider a simple T-claim $\mathcal{X} = g(S(T))$, where g is a increasing function. Show that the price of $\mathcal{X}$ is decreasing in the dividend yield δ.

GOOD LUCK!