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Retake Exam in Mathematics/Omtenta
1MA255, VT24
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Partial Differential Equations with Applications to Finance

Instructions: There are five problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no other materials are allowed. In the problems 4 and 5, you do not need to provide a proof to the respective verification theorems. **Good luck!**

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1. (8p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x) : t > 0, \quad x > 0\}$ with $u(0,x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t,0) = 0$ for $t > 0$.

(a) (3p) Construct a suitable extension of the initial condition to the whole space.

(b) (5p) Show that

$$u(t,x) = \int_0^\infty u_0(y) h(t,x,y)\,\mathrm{d}y$$

for some function $h(t,x,y)$. Find the function h. *Note: h is not simply the fundamental solution!*

2. (8p) Let D denote a bounded interval $(-a,b) \in \mathbb{R}$ with $a,b > 0$ and let W_t be the standard Brownian motion. Propose a suitable boundary value problem so that u solves

$$u(x) := \mathbb{E}_x\left[W^p_{\tau_D} + \tau_D\right],$$

where $p > 0$ is a constant and $\tau_D = \inf\{t > 0 | W_t \notin D\}$ is the so-called first exit time from D, and calculate $u(0)$.

3. (8p) Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function and $D > 0$. Under suitable assumptions, use the Fokker-Planck equation (Kolmogov Forward) to show that the Gibbs distribution

$$p(x) = \frac{1}{Z} e^{-\frac{V(x)}{D}}$$

is the limiting density function for X_t, which follows the dynamics

$$\mathrm{d}X_t = -DV'(X_t)\,\mathrm{d}t + \sqrt{2}D\,\mathrm{d}W_t, \quad X_0 = 0,$$

and find the normalizing constant Z. Be careful in noting down possible assumptions on p as you go.

4. (8p) Consider the Merton's asset allocation problem with one risky asset

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$$

and a risk-free rate $r = 0$. Here μ, σ are constants and W_t the standard Brownian motion. Let further X^α_t denote the wealth process where α_t is the share invested in the risky asset at time t. Write down the dynamics of X_t^α and find the value function and the optimal control α^* in the Merton's problem,

$$V(t,x) = \sup_\alpha \mathbb{E}_{t,x}\left[\Phi(X_T^\alpha)\right]$$

for some termination time T, where the utility function satisfies the so-called Kelly criterion: $\Phi(x) = \log x$. *Note: Use an ansatz $\hat{V}(t,x) = \Phi(x) + \lambda(t)$ for some suitable function λ with $\lambda(T) = 0$, which you have to solve from the corresponding HJB equation.*

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_x\left[e^{-r\tau}\left(e^{\frac{X_\tau}{4}} - 1\right)\right],$$

where $\mathrm{d}X_t = r\,\mathrm{d}t + 2\sqrt{r}\,\mathrm{d}W_t$, W_t is the standard Brownian motion, and $r > 0$.