Linear Algebra III

Problem sessions

Lecture 5: Problem session 1

1. Prove that if $a \in F$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.
2. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, but U is not subspace of $\mathbb{R}^2$.
3. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.
4. Prove that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}, \mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin. Prove that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all the planes in $\mathbb{R}^3$ through the origin.
5. Prove that if $\dim(V) < \infty$ and $U \subset V$ is a subspace, then $\dim(U) < \infty$.
6. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.
7. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.
8. Let U be a subspace of $\mathbb{R}^5$ defined by
$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2, x_3 = 7x_4\}.$$
Find a basis of U.
9. Show that if $\varphi : V \to W$ is an isomorphism, then it takes a basis of V to a basis of W.
10. Suppose U and W are subspaces of $\mathbb{R}^8$ such that $\dim(U) = 3$, $\dim(W) = 5$, and $U + W = \mathbb{R}^8$. Prove that $U \cap W = \{0\}$.
11. Let $U = \{(x, x, y, y) \in F^4 \mid x, y \in F\}$. Find a subspace W of F^4 such that $F^4 = U \oplus W$.
12. For subspaces U_1, U_2, U_3 of a finite-dimensional vector space, prove or give counterexample to the following:
$$\begin{aligned}\dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3) - \\ - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$
13. What is the dimension of $\mathbb{R}$ as a vector space over $\mathbb{Q}$?
14. Prove that if T is a linear map from F^4 to F^2 such that $\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in F^4 \mid x_1 = 5x_2, x_3 = 7x_4\}$, then T is surjective.
15. Suppose that V and W are finite-dimensional vector spaces, that B is an ordered basis of V and B' is an ordered basis of W. Prove that if T is an invertible linear map from V to W, then the rows of $\mathcal{M}(T, B, B')$ linearly independent. Show that the same is true about the columns of $\mathcal{M}(T, B, B')$.

16. Suppose that V and W are finite-dimensional vector spaces. Let B_1, B_1' be ordered bases of V and B_2, B_2' be ordered bases of W. Let $T : V \to W$ be a linear map. What is the relation between the matrices $\mathcal{M}(T, B_1, B_2)$ and $\mathcal{M}(T, B_1', B_2')$?

17. (a) Define the vector space of formal power series in F as

$$\mathcal{PS}(F) = \left\{ \sum_{k=0}^{\infty} a_k x^k \mid a_k \in F \right\},$$

where we do not make any requirements on the convergence of these series. Write an isomorphism from $\mathcal{PS}(F)$ to F^∞.

(b) The space of all polynomials

$$\mathcal{P}(F) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(F)$$

is a vector subspace of $\mathcal{PS}(F)$. Find the images of $\mathcal{P}_n(F)$ and of $\mathcal{P}(F)$ under the isomorphism in the previous part.

(c) Prove that $\mathcal{P}(F)$ and $\mathcal{PS}(F)$ are infinite dimensional.

18. Let $U = \{f \in \mathcal{P}(\mathbb{R}) \mid f(3) = 0\}$. Then prove that U is a subspace of $\mathcal{P}(\mathbb{R})$ and find $\mathcal{P}(\mathbb{R})/U$.

19. Prove that $F^n \otimes_F F^m \cong F^{nm}$.

20. Write $(4, 3) \otimes (1, 2) \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ as a linear combination of $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$.

21. Let $F \subset G$ be an inclusion of fields, let V be a vector space over F and let W be a vector space over G. Observe that W is also a vector space over F (thinking of W as a vector space over F is called *restriction of scalars*).

(a) Show that

$$G \otimes_F V$$

is a vector space over G.

Note: This process is called *extension of scalars*. The special case of $F = \mathbb{R}$ and $G = \mathbb{C}$ is called *complexification*.

(b) How is the dimension of V as an F-vector space (denoted $\dim_F(V)$) related to the dimension of $G \otimes_F V$ as a G-vector space ($dim_G(G \otimes_F V)$)?

(c) How is the dimension of W as an F-vector space ($\dim_F W$) related to the dimension of W as a G-vector space ($\dim_G W$)?

22. From Axler's *Linear algebra done right*, 3rd edition: Section 1.C: 1, 2, 3, 22; 2.A: 15, 17; 2.B: 4; 2.C: 6; 3.A: 2; 3.B: 4, 6, 15; 3.E: 1, 3, 4, 13, 14.

Lecture 10: Problem session 2

1. Show that the space of power series $\mathcal{PS}(F)$ is isomorphic to the dual to the space of polynomials, $\mathcal{P}(F)'$.

   Hint: Use a basis for $\mathcal{P}(F)$.

   Remark: Notions from set theory that we will not go into in this course allow one to show that if V is an infinite dimensional vector space, then V is not isomorphic to V'. Therefore, $\mathcal{P}(F)$ is not isomorphic to $\mathcal{PS}(F)$.

2. Let $F = \mathbb{Z}/2$ (the field with 2 elements) and consider the vector space $V = (\mathbb{Z}/2)^2$.

   (a) Show that the quadratic form $q_1 : V \to F$ given by $q_1(x_1, x_2) = x_1^2 + x_2^2$ satisfies $q_1(v) = B(v, v)$ for every $v \in V$, for some symmetric bilinear form $B : V \times V \to F$.

   (b) Consider now the quadratic form $q_2 : V \to F$ given by $q_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$. Show that there is no symmetric bilinear form $B : V \times V \to F$ such that $q_2(v) = B(v, v)$.

3. Let V be a vector space over F and let $U \subset V$ be a subspace. Define the annihilator of U to be

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \text{ for every } u \in U\}.$$

   (a) Show that U^0 is a subspace of V'.

   (b) Assuming that if $\dim V < \infty$, write an isomorphism $V/U \to U^0$. This implies that $U \oplus U^0 \cong V$. Note: In exercise 37 of Axler 3.F, you are asked to prove that there is a canonical isomorphism $U^0 \cong (V/U)'$, with no finite dimensionality assumption. Observe that this implies part (b).

4. Let V be a finite dimensional vector space with a bilinear form $B : V \times V \to F$ and let $v_1, \ldots, v_n$ be an ordered bases of V. Recall that the associated matrix of B is $A = (a_{ij})_{i,j=1}^n$, with $a_{ij} = B(v_i, v_j)$. Given a vector $v \in V$, denote by $[v]$ the column vector in F^n associated to v in this basis (the entries of $[v]$ are the coefficients of the expression of v in the chosen basis).

   (a) Show that $B(v, w) = [v]^t A[w]$ (where the subscript t denotes transposition).

   (b) Let $v_1', \ldots, v_n'$ be another basis of V, with respect to which the matrix associated to B is A'. Explain how A and A' are related.

5. Equip $\mathbb{R}^3$ with the standard inner product. Find orthonormal bases of the subspaces of $\mathbb{R}^3$ spanned by the following vectors:

   (a) $(1, 1, -1)$ and $(1, 0, 1)$.

   (b) $(2, 1, 1)$ and $(1, 3, -1)$.

6. Let P be the point $(2, 1, 3) \in \mathbb{R}^3$.

   (a) Find the point Q on the plane $-x + 2y - 2z = 0$ that is the closest to P.

   (b) Find the point R on the plane $-x + 2y - 2z = 5$ that is the closest to P.

   (c) Show that P, Q and R are colinear (meaning: there is a line in $\mathbb{R}^3$ that contains the three points).

7. On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

   (a) Apply the Gram–Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

   (b) Find the matrix of $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, defined by the differentiation operator i.e. $T(p) = p'$, with respect to the basis $1, x, x^2$.

   (c) Find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, such that the matrix of T with respect to this basis is upper-triangular.

8. Let $V = C([-1, 1], \mathbb{R})$ be the real vector space of continuous functions $f : [-1, 1] \to \mathbb{R}$, with $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)dx$. Show that all functions of the form $\sin(k\pi x)$ and $\cos(l\pi x)$, where $k, l > 0$ are integers, are pairwise orthonormal.

9. Let V be a vector space over a field F equal to $\mathbb{R}$ or $\mathbb{C}$. A map $\langle .,. \rangle : V \times V \to F$ is a *semi-inner product* if $\langle a_1v_1 + a_2v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$, $\langle v, w\rangle = \overline{\langle w, v\rangle}$ and $\langle v, v\rangle \geq 0$ (it may not be an inner product only because we do not require it to be positive definite). Prove the Cauchy–Schwarz inequality: $|\langle v, w\rangle| \leq \|v\|.\|w\|$.

   Hint: Write $\langle v, w\rangle = e^{i\theta}|\langle v, w\rangle|$, for some $\theta \in \mathbb{R}$ (if $F = \mathbb{R}$, take $\theta = 0$). Think about the quadratic function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \|v + te^{i\theta}w\|^2 \geq 0$.

10. Let $V = C([-1, 1], \mathbb{R})$ and let $U \subset V$ be the subspace consisting of functions such that $f(0) = 0$. Clearly, $U \neq V$. Show that $U^\perp = 0$.

    Note: This example shows that the formula $U \oplus U^\perp = V$, which holds when $\dim(U) < \infty$, may fail if $\dim(U) = \infty$.

11. Let $(V, \langle ., .\rangle)$ be a finite dimensional inner product space, defined over $\mathbb{R}$. Show that the following map is an isomorphism: $f : V \to V'$ given by $f(v) = f_v \in V'$, such that $f_v(w) = \langle v, w\rangle$ for all $w \in V$. Think about the analogue for inner product spaces over $\mathbb{C}$.

12. Let $(V, \langle ., .\rangle)$ be a finite dimensional inner product space, defined over $\mathbb{R}$. Let $U \subset V$ be a subspace. Recall the definition of the annihilator $U^0 \subset V'$ in Exercise 3. Write an explicit isomorphism $U^\perp \to U^0$. Think about the analogue for inner product spaces over $\mathbb{C}$.

13. Let V be a vector space over F and let $f : V \times \ldots \times V \to F$, be a k-multilinear form on V. Recall that f is alternating if we always have $f(\ldots, v, \ldots, v, \ldots) = 0$. Show that f is alternating iff $f(v_1, \ldots, v_k) = 0$ whenever the vectors $v_1, \ldots, v_k$ are linearly dependent.

14. The following is a matrix in $\mathrm{Mat}(3 \times 3, \mathbb{Z}/7)$, defined over the field with 7 elements:

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 5 \\ 3 & 0 & 6 \end{pmatrix}.$$

    Compute $\det(A) \in \mathbb{Z}/7$. If A is invertible, then find its inverse.

15. Let $F = \mathbb{Z}/2$ and $V = F^2$. Construct a bilinear form $B : V \times V \to F$ such that for every $v, w \in V$

$$B(v, w) = B(w, v)$$

    and for every $v \in V$ we have $B(v, v) \neq 0$.

    Note: since $\mathrm{char}(\mathbb{Z}/2) = 2$, we have $x = -x$ for every $x \in \mathbb{Z}/2$. Therefore, we can write the first formula above as $B(v, w) = -B(w, v)$. We saw that that would imply that B is alternating if we had $\mathrm{char}(F) \neq 2$. This exercise shows that that the same implication does not hold if $\mathrm{char}(F) = 2$.

16. A *group* is a set G with a map $m : G \times G \to G$ (called *multiplication*, and sometimes denoted $m(x, y) = xy$) and an element $e \in G$ (called *identity*) such that:

    - $(xy)z = x(yz)$ for all $x, y, z \in G$ (multiplication is associative);
    - $ex = x = xe$ for all $x \in G$ (e is the identity);
    - for every $x \in G$ there is $y \in G$ such that $xy = e = yx$. Write $x^{-1} = y$ (existence of inverses).

    Recall that, given a positive integer n, $\mathrm{Mat}(n \times n, F)$ is the space of $n \times n$-matrices over a field F.

    (a) Show that $\mathrm{Mat}(n \times n, F)$ is not a group.
    (b) Let $GL(n, F) = \{A \in \mathrm{Mat}(n \times n, F) \mid A \text{ is invertible}\}$. Show that $GL(n, F)$ is a group (it is called the *general linear group*).
    (c) Let $SL(n, F) = \{A \in \mathrm{Mat}(n \times n, F) \mid \det(A) = 1\}$. Show that $SL(n, F)$ is a group (it is called the *special linear group*).

(d) Let F be $\mathbb{R}$ or $\mathbb{C}$ and denote by $\langle .,. \rangle_F$ the standard inner product on F^n. Say that $A \in \mathrm{Mat}(n \times n, F)$ preserves the inner product if $\langle Av, Aw \rangle_F = \langle v, w \rangle_F$ for all vectors $v, w \in F^n$. For $F = \mathbb{R}$, let $O(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, A \text{ preserves } \langle .,. \rangle_{\mathbb{R}}\}$. Show that $O(n, \mathbb{R})$ is a group (called the *orthogonal group*). Find and example of $A \in O(2, \mathbb{R})$ that is not the identity matrix.

(e) Let now $F = \mathbb{C}$. Show that $U(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \,|\, A \text{ preserves } \langle .,. \rangle_{\mathbb{C}}\}$ is a group (called the *unitary group*). Find an example of $A \in U(2, \mathbb{C})$ that is not the identity matrix.

17. Using the terminology from the previous question, show that if $A \in O(n, \mathbb{R})$ then $\det(A) = \pm 1$. Show that if $A \in U(n, \mathbb{C})$ then $|\det(A)| = 1$.

    Hint: Write $\langle v, w \rangle = v^t \overline{w}$. On the right, v^t is thought of as a row vector (the transpose of the column vector v) and $\overline{w}$ is thought of as a column vector (the complex conjugate of the column vector w).

18. Let $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ be the permutation such that

$$\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 4,\ \sigma(4) = 2.$$

    Write σ as a product of transpositions and compute $(-1)^\sigma$.

19. From Axler's *Linear algebra done right*, 3rd edition: Section 3.F: 1, 2, 7, 30, 31; 6.A: 8, 10, 31; 6.B: 2, 4, 5, 9, 17; 6.C: 2, 3, 4, 5, 11, 12, 14.

Lecture 15: Problem session 3

1. Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all the eigenvalues and the corresponding eigenspaces of T.

2. (a) What are the possible Jordan forms of $T \in \mathcal{L}(V)$ with characteristic polynomial $(z+2)^2(z-5)^3$?

   (b) What are the possible Jordan forms of $T \in \mathcal{L}(V)$ with $(z+2)^2(z-5)^3$ as characteristic polynomial and such that the eigenspace corresponding to -2 is 1-dimensional and the eigenspace corresponding to 5 is 2-dimensional?

3. For each of the following matrices, determine the Jordan for, and find a Jordan basis:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

4. Find the characteristic polynomial and the minimal polynomial of the following matrices:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

5. Let $T \in \mathcal{L}(V)$, $v \in V$ and $m > 0$ integer such that $T^{m-1}(v) \neq 0$ but $T^m(v) = 0$. Prove that the vectors $v, T(v), T^2(v), \ldots, T^{m-1}(v)$ are linearly independent.

6. Let $T \in \mathcal{L}(V)$ and let $U, W \subset V$ be subspaces invariant under T (meaning: $\mathrm{im}(T|_U) \subset U$ and $\mathrm{im}(T|_W) \subset W$). Assume that $V = U \oplus W$. Suppose the matrix of $T_{|U} : U \to U$ is A with respect to the ordered basis $u_1, \ldots, u_p$ and the matrix of $T_{|V} : V \to V$ is B with respect to the ordered basis $w_1, \ldots, w_q$. Show that the matrix of T with respect to the ordered basis $u_1, \ldots, u_p, v_1, \ldots, w_q$ is the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

7. Recall that if one picks an ordered basis $v_1, \ldots, v_n$ for a vector space V over F, there are induced isomorphisms $V \overset{\cong}{\to} F^n$ and $f : \mathcal{L}(V) \overset{\cong}{\to} \mathrm{Mat}(n \times n, F)$. Given $T \in V$, define its trace as

$$\mathrm{tr}(T) = \mathrm{tr}(f(T))$$

where the trace of a square matrix is the sum of the elements in the pricipal diagonal. Show that this is well-defined, in the sense that different choices of ordered bases for V yield the same definition of $\mathrm{tr}(T)$.

8. Let $P_A(x) = \det(xI - A) = a_0 + \ldots + a_n x^n$ be the characteristic polynomial of $A \in \mathrm{Mat}(n \times n, F)$. Show that $a_0 = (-1)^n \det(A)$ and $a_{n-1} = -\mathrm{tr}(A)$.

9. Let $T \in \mathcal{P}_2(\mathbb{R})$ be given by

$$T(p(x)) = p(0) + p'(1)x + p''(2)x^2.$$

Compute $\det(T)$ and $\mathrm{tr}(T)$ ($\det(T)$ was defined in the lectures and $\mathrm{tr}(T)$ is defined in the previous exercise).

10. Let V be a complex vector space of dimension n and let $T \in \mathcal{L}(V)$. Recall that if $\lambda \in \mathbb{C}$ is an eigenvalue of T, then its generalized eigenspace is

$$G(\lambda, T) = \{v \in V \mid (T - \lambda I)^k(v) = 0 \text{ for some } k > 0\}.$$

(a) Show that $G(\lambda, T) = \mathrm{null}\,((T - \lambda I)^n)$.

(b) Show that $G(\lambda, T) \cap \mathrm{im}\,((T - \lambda I)^n) = \{0\}$ and conclude that $G(\lambda, T) \oplus \mathrm{im}\,((T - \lambda I)^n) = V$.

11. Give an example of a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ that cannot be diagonalized.

12. Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of the operator on $\mathbb{C}^2$ whose matrix with respect to the standard basis is $\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$.

13. Let $(V, \langle ., . \rangle)$ be a finite dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Show that $T^{**} = T$.

14. Let $(V, \langle ., . \rangle)$ be a finite dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Show that $T \in \mathcal{L}(V)$ is normal iff $\|T^*v\| = \|Tv\|$ for every $v \in V$.

Hint: You may find useful the fact that

$$\langle v, w \rangle = \begin{cases} \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + \|v+iw\|^2 i - \|v-iw\|^2 i\right) & \text{if } F = \mathbb{C} \\ \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right) & \text{if } F = \mathbb{R} \end{cases}$$

15. (a) Find $A \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ which is not diagonalizable as a real matrix, but which is diagonalizable as a complex matrix (diagonalizable when thought of as an element of $A \in \mathrm{Mat}(2 \times 2, \mathbb{C})$).

(b) Find a finite dimensional real inner product space $(V, \langle ., . \rangle)$ with a normal operator $T \in \mathcal{L}(V)$ that is not self-adjoint.

16. Let $v_1, \ldots, v_n$ and $v_1', \ldots, v_n'$ be two orthonormal bases of an inner product space $(V, \langle ., . \rangle)$, and let C be the corresponding change of basis matrix. Show that $C^{-1} = C^t$.

17. Let V be a finite dimensional complex vector space, and let $R, S \in \mathcal{L}(V)$ be diagonalizable operators. This mean that there is a basis of V in which R is given by a diagonal matrix, and there is a basis of V in which S is given by a diagonal matrix. Say that R and S are *simultaneously diagonalizable* if there is a basis of V that diagonalizes R and S.

Show that if R and S are simultaneously diagonalizable, then $RS = SR$ (the two operators commute).

You don't need to show it, but the converse is also true: if $RS = SR$, then R and S are simultaneously diagonalizable.

18. Let $(V, \langle ., \rangle)$ be a finite dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $T \in \mathcal{L}(V)$ be a self-adjoint operator. Show that T is positive iff all the eigenvalues of T are non-negative.

19. Let $(V, \langle ., \rangle)$ be a finite dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $T \in \mathcal{L}(V)$.

    (a) Relate the eigenvalues of T^*T with the eigenvalues of T.

    (b) Relate the eigenvalues of $\sqrt{T^*T}$ (also called the singular values of T) with the eigenvalues of T.

20. From Axler's *Linear algebra done right*, 3rd edition: Section 5.A: 1, 10, 12, 15, 18, 21, 32, 35; 5.B: 1; 5.C: 3, 15, 16; 7.A: 8, 9, 10, 15, 17, 21; 7.B: 2, 3, 6, 9, 11; 7.C: 5, 6, 13; 7.D: 7; 8.A: 3, 5, 6, 7, 8, 12, 16, 17, 18, 19; 8.B: 3, 6, 7, 10; 8.C: 1, 4, 6, 8, 20; 8.D: 1, 2, 4, 7.