Final Exam
Graph Theory, 1MA170, Period 2 2017

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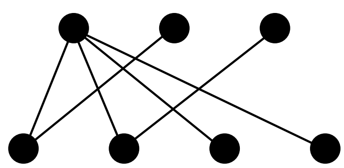
The exam consists of 5 questions worth 10 points each. Choose any 4 questions to answer. If you attempt all 5, the best 4 will count for your final grade. Answer each question carefully and with attention to details, citing any results from the lecture notes that you use. You may write your solutions in Swedish or English. Calculators are not allowed.

Good luck!

1. (Matchings)
   (a) Define *perfect matching.* (1)
   (b) State the Augmenting path algorithm. Use it to find a maximum matching in the graph below, clearly indicating each step. (3)



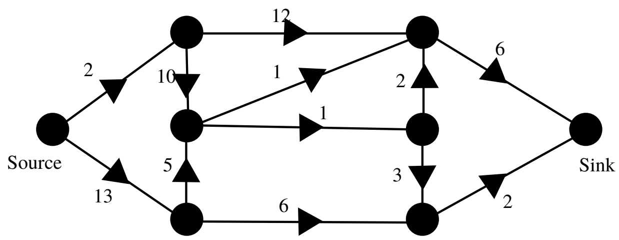
   (c) Use Hall's theorem to prove that if $k > 0$, every k-regular bipartite graph has a perfect matching. (3)
   (d) Let G be the Petersen graph. You may assume that if M is a perfect matching in G, then $G \setminus M$ forms two 5-cycles. Determine the number of perfect matchings in G by proving that every edge of G lies in 4 5-cycles and counting the number of 5-cycles, or otherwise. (3)

2. (Planarity)
   (a) Define *planar graph.* (1)
   (b) Prove that a planar graph has a vertex of degree at most 5. (3)
   (c) Recall Wagner's theorem: a graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor. Prove that Wagner's theorem is equivalent to Kuratowski's theorem. (4)
   (d) Show that the Petersen graph is nonplanar by finding both K_5 and $K_{3,3}$ as minors. (2)

3. (Flows in networks)

   (a) Define *flow* and *source/sink cut*. (2)

   (b) State the Ford-Fulkerson algorithm. Use it to determine the maximum flow in the network below, clearly indicating each step in the algorithm. (4)



   (c) State the max-flow min-cut theorem, and explain how the Ford-Fulkerson algorithm can be used to prove it for rational edge weights. (A detailed proof is not necessary, only the main ideas of the proof). (4)

4. (Probabilistic methods)

   (a) Explain briefly in your own words how the probabilistic method works. (2)

   (b) Show that some n-vertex tournament has at least $n!/2^{n-1}$ Hamilton paths. (3)

   (c) Let M be a matching of size m. Choose a set X uniformly at random from the vertices saturated by the matching. What is the expected number of edges in M with both endpoints in X? (3)

   (d) What is the probability of $G_{n,p}$ taking the value of a given labelled graph G with m edges? (2)

5. (Bipartite graphs)

   (a) Define *subgraph*. (1)

   (b) Prove that every graph has a bipartite subgraph with at least half the edges using each of the following techniques:

      i. Induction. (2)

      ii. Probabilistic methods. (2)

      iii. Any other method. (2)

   (c) Show that the hypercube is bipartite. (3)