Financial Theory – Lecture 13

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Agenda

- Asset allocation.
- Household finance.

The lecture is based on

- Sections 1.8, 9.1-9.2, 10.1.3, and 13.1-13.2.

Investments in practise

Usually we think of the investment process in the following steps.

1) The strategic asset allocation. This is the choice of in which proportion each of the major asset classes should have in the portfolio. Typically done by senior management/investors.
2) The security selection. Given the strategic asset allocation, this is the choice which securities should be bought in each of the classes.
3) The tactical asset allocation. This is the timing of when the securities are bought.

It has been shown (see the references on p. 18 in the course book), that the strategic asset allocation decision explains more than 90% of the time series variation in quarterly returns.

The alpha

We know that if the assumptions of the CAPM holds then

$$E[r_i] - r_f = \beta_i(E[r_m] - r_f),$$

where r_m is the return on the market portfolio, and

$$\beta_i = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}.$$

But if we regress the excess return $r_i - r_f$ on the excess market return $r_m - r_f$,

$$r_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i,$$

then we will probably get an intersection α_i – the alpha – that is different from zero.

The alpha

Recall

$$r_{t,t+1} = r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t},$$

which we can write

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}},$$

and also

$$P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}}\right].$$

We have shown that if $E_t[r_{t+1}] = r$, a constant, then

$$P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1 + r}\right] = \frac{E_t[D_{t+1} + P_{t+1}]}{1 + r}.$$

The alpha

The price of the asset given by the CAPM is

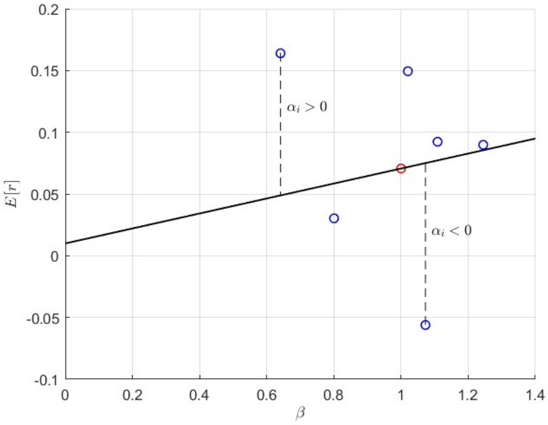
$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r_f + \beta\big(E\left[r_m\right] - r_f\big)}.$$

Now assume that we instead use $r = r_f + \alpha + \beta\big(E\left[r_m\right] - r_f\big)$ as discount factor:

$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r_f + \alpha + \beta\big(E\left[r_m\right] - r_f\big)}.$$

- $\alpha = 0$: The price is equal to the CAPM price.
- $\alpha > 0$: The price is lower than the CAPM price $\rightarrow$ the asset is underpriced.
- $\alpha < 0$: The price is higher than the CAPM price $\rightarrow$ the asset is overpriced.

The alpha



The SML with the market portfolio (red circle) and six assets (blue circles).

The Treynor-Black model

A systematic way of using the information that assets are under/over-priced is given by the Treynor-Black model.

- There are N number of risky assets and a risk-free asset.
- The Single-Index Model holds.
- We have found that J of the risky assets have an alpha $\neq 0$:

$$r_i = r_f + \alpha_i + \beta_i \big(r_m - r_f \big) + \varepsilon_i, \ \ i = 1, 2, \ldots, J.$$

- We form a portfolio with weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_J)^\top$ using these J assets – this is called the active portfolio, and its rate of return is denoted r_A.
- Finally we form a portfolio with weight w_A in the active portfolio and weight $1 - w_A$ in the market portfolio:

$$r_p = w_A r_A + (1 - w_A) r_m.$$

The Treynor-Black model

The return of the active portfolio is

$$
\begin{aligned}
r_A &= \boldsymbol{\pi} \cdot \mathbf{r} = \sum_{i=1}^{J} \pi_i r_i = \sum_{i=1}^{J} \pi_i \Big(r_f + \alpha_i + \beta_i \big(r_m - r_f \big) + \varepsilon_i \Big) \\
&= \sum_{i=1}^{J} \pi_i r_f + \sum_{i=1}^{J} \pi_i \alpha_i + \sum_{i=1}^{J} \pi_i \beta_i \big(r_m - r_f \big) + \sum_{i=1}^{J} \pi_i \varepsilon_i \\
&= r_f \underbrace{\sum_{i=1}^{J} \pi_i}_{=1} + \underbrace{\sum_{i=1}^{J} \pi_i \alpha_i}_{=\alpha_A} + \big(r_m - r_f \big) \underbrace{\sum_{i=1}^{J} \pi_i \beta_i}_{=\beta_A} + \underbrace{\sum_{i=1}^{J} \pi_i \varepsilon_i}_{=\varepsilon_A} \\
&= r_f + \alpha_A + \beta_A \big(r_m - r_f \big) + \varepsilon_A .
\end{aligned}
$$

The Treynor-Black model

The model uses a two-step procedure.

Step 1
Find, given $\boldsymbol{\pi}$, the optimal weight w_A^* that maximises the Sharpe ratio

$$\begin{aligned} \mathrm{SR}(r_p) &= \frac{E\left[r_p\right] - r_f}{\mathrm{Std}[r_p]} \\ &= \frac{E\left[w_A r_A + (1 - w_A) r_m\right] - r_f}{\mathrm{Std}\left[w_A r_A + (1 - w_A) r_m\right]}. \end{aligned}$$

The Treynor-Black model

One can show (Theorem 13.1 in the course book) that

$$w_A^* = \frac{\dfrac{\alpha_A}{\mathrm{Var}[\varepsilon_A]}}{\dfrac{E\left[r_m\right] - r_f}{\mathrm{Var}[r_m]} + (1-\beta_A)\dfrac{\alpha_A}{\mathrm{Var}[\varepsilon_A]}}$$

and

$$\begin{aligned} \mathrm{SR}(r_p)^2 &= \left(\frac{E\left[r_m\right] - r_f}{\mathrm{Std}[r_m]}\right)^2 + \left(\frac{\alpha_A}{\mathrm{Std}[\varepsilon_A]}\right)^2 \\ &= \mathrm{SR}_m^2 + IR_A^2. \end{aligned}$$

The contribution to the overall Sharpe ratio from the active portfolio is given by the size of the information ratio IR_A.

The Treynor-Black model

Step 2

Find the best (optimal) choice of $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_J)^\top$.

We have $\mathrm{SR}(r_p)^2 = \mathrm{SR}_m^2 + IR_A^2$ and only IR_A^2 dependes on the choice of $\boldsymbol{\pi}$.

Hence, we maximise

$$IR_A^2 = \left(\frac{\alpha_A}{\mathrm{Std}[\varepsilon_A]}\right)^2 = \left(\frac{\sum_{i=1}^J \pi_i \alpha_i}{\mathrm{Std}\left[\sum_{i=1}^J \pi_i \varepsilon_i\right]}\right)^2$$

under the constraint $\boldsymbol{\pi} \cdot \mathbf{1} = 1$. The result is

$$\pi_i = \frac{\dfrac{\alpha_i}{\mathrm{Var}[\varepsilon_i]}}{\displaystyle\sum_{j=1}^J \frac{\alpha_j}{\mathrm{Var}[\varepsilon_j]}}, \; i = 1, 2, \ldots, J.$$

(See Theorem 13.2 in the course book.)

The Treynor-Black model

From p. 487 in the course book:

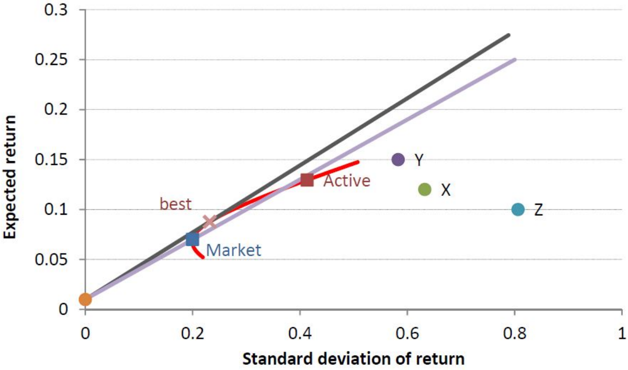


Figure 13.1: The mean-variance diagram with Treynor-Black.
The diagram is constructed using the inputs explained in Example 13.1.

The Treynor-Black model

We need to estimate the parameters of the model. We have seen that especially for the expected return, where we find the alpha's, it is hard to get a good precision in the estimates.

One way to take care of this is to shrink the estimates of the alpha's towards zero.

One example is to shrink by using the R_i^2 in the regression from which α_i is estimated:

$$\alpha_i \;\rightarrow\; R_i^2 \cdot \alpha_i.$$

Another way is to only consider the alpha's that are larger than or equal to some threshold.

Bayesian statistics

Assume that data suggests that

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \Sigma).$$

We now take a Bayesian approach.

This means that we consider some or all parameters in the model as being random variables.

Bayesian statistics

The idea is to then use Bayes' formula in the following formal way

$$\begin{aligned} P(\text{Parameters}|\text{Data}) &= \frac{P(\text{Data}|\text{Parameters}) \cdot P(\text{Parameters})}{P(\text{Data})} \\ &\propto P(\text{Data}|\text{Parameters}) \cdot P(\text{Parameters}). \end{aligned}$$

Here

- $P(\text{Parameters})$ is our initial assumption of the distribution of the parameter(s) – the prior distribution or just the prior.
- $P(\text{Data}|\text{Parameters})$ is the likelihood function.
- $P(\text{Parameters}|\text{Data})$ is the resulting distribution of the parameter(s) after we have observed the data – the posterior distribution or just the posterior.

In summary:

$$\text{Posterior} \propto \text{Likelihood} \cdot \text{Prior}$$

Bayesian statistics

We now consider $\boldsymbol{\mu}$ as a random vector, not a vector of parameters, while we still consider the elements of Σ as being parameters.

As $\boldsymbol{\mu}$ is now a random vector, we need to make some assumptions regarding its distribution.

We take the prior distribution to be a multivariate normal distribution:

$$\boldsymbol{\mu} \sim N(\mathbf{m}, V).$$

You as an analyst, based on intensive research, has views on some of the expected values of the returns, or on portfolios of returns.

These views are usually on one of the following two forms.

- An absolute view: The expected return on asset no 3 should be 3%:

$$\mu_3 = 0.03.$$

- A relative view: The excess return of asset 4 with respect to asset 2 should be 2%:

$$\mu_4 - \mu_2 = 0.02.$$

The Black-Litterman model

We collect these views in a matrix P and a vector $\mathbf{Q}$.

If we assume that there are $N = 4$ risky assets, then

$$P = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right]$$

and

$$\mathbf{Q} = \left[\begin{array}{c} 0.03 \\ 0.02 \end{array} \right]$$

with the examples from the previous slide.

The Black-Litterman model

In order to introduce the uncertainty we have about our views, we add noise:

$$P\boldsymbol{\mu} = \mathbf{Q} + \varepsilon_v,$$

where

$$\varepsilon_v \sim N(\mathbf{0}, \Omega) \quad \text{and} \quad \Omega = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_K \end{pmatrix}.$$

How certain we are of our views is reflected in the size of the ω's: The more certain we are, the smaller is the ω of that view.

It follows that

$$P\boldsymbol{\mu} \sim N(\mathbf{Q}, \Omega).$$

The Black-Litterman model

To summarise:

- The data tells us that

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \Sigma) \text{ with } \boldsymbol{\mu} \sim N(\mathbf{m}, V).$$

- Our views on some portfolio(s) of the assets are that

$$P\boldsymbol{\mu} \sim N(\mathbf{Q}, \Omega).$$

How do we combine the views with the data?

We calculate the posterior distribution.

The Black-Litterman model

Using Bayesian statistics we get the following posterior distribution for the mean $\boldsymbol{\mu}$ when we use our personal views:

$$\boldsymbol{\mu} \sim N(\hat{\mathbf{m}}, \hat{V})$$

with

$$\hat{\mathbf{m}} = \mathbf{m} + VP^T(PVP^T + \Omega)^{-1}\big(\mathbf{Q} - P\mathbf{m}\big)$$

and

$$\hat{V} = \big(V^{-1} + P^T\Omega^{-1}P\big)^{-1}.$$

The Black-Litterman model

The Black-Litterman model is mainly considered as a way of taking your personal views into account when finding the optimal portfolio.

The model also, however, adresses the problem of estimating the means.

The idea is to use "reverse engineering". To simplify we assume that there is no uncertainty in $\boldsymbol{\mu}$, i.e. $V = 0$.

Recall from Lecture 7 that

$$\boldsymbol{\pi} = \frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})$$

is the optimal portfolio when $\mathbf{r}$ is normally distributed, the investor has a CARA utility function with parameter γ and there is a risk-free asset.

The Black-Litterman model

The market portfolio was then given by

$$\boldsymbol{\pi}_{\text{mkt}} = \frac{1}{\bar{\gamma}} \Sigma^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}),$$

where $\bar{\gamma}$ is determined by the investors' γ's.

Using this equation, we can write

$$\boldsymbol{\mu} = r_f \mathbf{1} + \bar{\gamma} \Sigma \boldsymbol{\pi}_{\text{mkt}}.$$

We have also seen that

$$\bar{\gamma} = \frac{E\left[r_m\right] - r_f}{\text{Var}[r_m]} \longrightarrow \boldsymbol{\mu} = r_f \mathbf{1} + \frac{E\left[r_m\right] - r_f}{\text{Var}[r_m]} \Sigma \boldsymbol{\pi}_{\text{mkt}}.$$

Hence, we can estimate the vector of expected rate of returns $\boldsymbol{\mu}$ by using the market portfolio, Σ and the mean and variance of the market portfolio.

Household finance

Chapter 9 in the course book contains several examples of household finance.

We will look at the following topics from that chapter:

- Labour income.
- Housing.

Labour income

For an individual considering his or her portfolio choice over time, the human capital is important.

$$\text{Human capital} = \text{PV}\big(\text{Future labour income}\big).$$

Let F_t and L_t denote the financial and human capital at time t respectively. The total wealth is

$$W_t = F_t + L_t,$$

the human capital share of the total wealth is

$$h_t = \frac{L_t}{F_t + L_t} = \frac{L_t}{W_t},$$

and

$$\ell_t = \frac{L_t}{F_t}$$

is the human-to-financial wealth ratio.

Labour income

The return on the financial capital is (here we use that $\boldsymbol{\pi}_t \cdot \mathbf{1} + \pi_{t,f} = 1$):

$$\frac{F_{t+1} - F_t}{F_t} = \boldsymbol{\pi}_t \cdot \mathbf{r}_{t+1} + \pi_{t,f} r_f = \boldsymbol{\pi}_t \cdot \mathbf{r}_{t+1} + \left(1 - \boldsymbol{\pi}_t \cdot \mathbf{1}\right) r_f.$$

Let

$$r_{t+1}^L = \frac{L_{t+1} - L_t}{L_t}$$

denote the rate of return on human capital and assume that

$$E_t\left[r_{t+1}^L\right] = \mu_L \quad \text{and} \quad \mathrm{Var}_t\big[r_{t+1}^L\big] = \sigma_L^2.$$

Labour income

The individual at time t wants to solve

$$\max_{\boldsymbol{\pi}_t} \left\{ E_t\left[\frac{W_{t+1}}{W_t}\right] - \frac{1}{\gamma}\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right] \right\},$$

i.e. it has mean-variance preferences. We get

$$\begin{aligned}
\frac{W_{t+1}}{W_t} &= \frac{F_{t+1} + L_{t+1}}{F_t + L_t} \\
&= \frac{F_t}{F_t + L_t}\big(1 + \boldsymbol{\pi}_t \cdot \mathbf{r}_{t+1} + (1 - \boldsymbol{\pi}_t \cdot \mathbf{1})r_f\big) + \frac{L_t}{F_t + L_t}(1 + r^L_{t+1}) \\
&= (1 - h_t)\big(1 + r_f + \boldsymbol{\pi}_t \cdot (\mathbf{r}_{t+1} - r_f\mathbf{1})\big) + h_t(1 + r^L_{t+1}).
\end{aligned}$$

Labour income

Now

$$
\begin{aligned}
E_t\left[\frac{W_{t+1}}{W_t}\right] &= (1-h_t)\big(1+r_f+\boldsymbol{\pi}_t\cdot\left(E_t\left[\mathbf{r}_{t+1}\right]-r_f\mathbf{1}\right)\big) \\
&\quad +h_t\left(1+E_t\left[r^L_{t+1}\right]\right) \\
&= (1-h_t)\big(1+r_f+\boldsymbol{\pi}_t\cdot(\boldsymbol{\mu}-r_f\mathbf{1})\big)+h_t(1+\mu_L), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right] &= \mathrm{Var}_t\big[(1-h_t)\boldsymbol{\pi}_t\cdot\mathbf{r}_{t+1}+h_t r^L_{t+1}\big] \\
&= (1-h_t)^2\boldsymbol{\pi}_t\cdot\Sigma\boldsymbol{\pi}_t+2(1-h_t)h_t\boldsymbol{\pi}_t\cdot\mathrm{Cov}_t[\mathbf{r}_{t+1},r^L_{t+1}] \\
&\quad +h_t^2\sigma_L^2.
\end{aligned}
$$

Labour income

The solution to the maximisation problem is (see the course book for a derivation):

$$\begin{aligned}\boldsymbol{\pi}_t &= \frac{1}{\gamma}(1+\ell_t)\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) - \ell_t\Sigma^{-1}\mathrm{Cov}_t[\mathbf{r}_{t+1}, r^L_{t+1}] \\ &= \underbrace{\frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}_{=(1)} + \underbrace{\ell_t\Sigma^{-1}\left(\frac{1}{\gamma}(\boldsymbol{\mu} - r_f\mathbf{1}) - \mathrm{Cov}_t[\mathbf{r}_{t+1}, r^L_{t+1}]\right)}_{=(2)}.\end{aligned}$$

(1) Optimal weights without labour income.
(2) Hedge against labour income risk.

When there is one risky asset:

$$\pi_t = \frac{\mu - r_f}{\gamma\sigma^2} + \ell_t\left(\frac{\mu - r_f}{\gamma\sigma^2} - \frac{\mathrm{Cov}_t[r_{t+1}, r^L_{t+1}]}{\sigma^2}\right).$$

Labour income

The older we are, the smaller is the fraction $\ell_t = L_t/F_t$.

In the book the following approximate table is derived (see footnote 3 on p. 351 and Section 9.1.2):

ℓ_t	Age
1	97
2	84
5	55
10	44
20	35
50	26

Labour income

Consider the case with one risky asset and a constant

$$\mathrm{Cov}_t[r_{t+1}, r^L_{t+1}] = \rho_{SL}\sigma_S\sigma_L.$$

From p. 351 in the course book:

	$\gamma = 1$				$\gamma = 5$				$\gamma = 10$			
ℓ_t	stock	rf	exp	std	stock	rf	exp	std	stock	rf	exp	std
0	125	-25	7.3	25	25	75	2.3	5	13	87	1.6	3
1	245	-145	13.3	49	45	55	3.3	9	20	80	2.0	4
2	365	-265	19.3	73	65	35	4.3	13	28	72	2.4	6
5	725	-625	37.3	145	125	-25	7.3	25	50	50	3.5	10
10	1325	-1225	67.3	265	225	-125	12.3	45	88	12	5.4	18
20	2525	-2425	127.3	505	425	-325	22.3	85	163	-63	9.1	33
50	6125	-6025	307.3	1225	1025	-925	52.3	205	388	-288	20.4	78

Table 9.2: Optimal portfolios with human capital.
The table shows the percentages of financial wealth optimally invested in the stock and the riskfree asset, as well as the expectation and standard deviation of the financial return in percent. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$.

Labour income

From p. 352 in the course book:

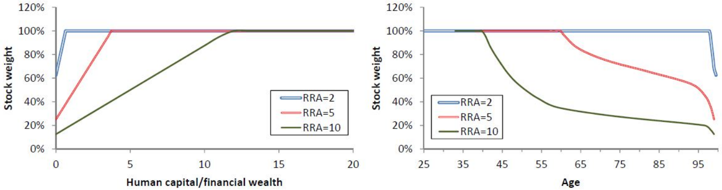


Figure 9.2: Optimal stock weight with human capital.
The figure shows the constrained optimal stock weight as a function of the human capital to financial wealth ratio (left panel) and age (right panel) for three different values of the relative risk aversion coefficient γ. The stock weight is restricted to the interval from 0% to 100%. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$.

Housing

Housing (i.e. the financial position in owning a house or apartment) is a large part of many households investments.

Consider the previous labour income model where we let the risky "investment universe" consist of a stock and investment in housing:

$$\mathbf{r} = \begin{pmatrix} r_S \\ r_H \end{pmatrix}.$$

Throughout we let "S", "L" and "H" denote parameters connected to the stock, human capital and housing respectively.

Housing

The amount of our investment portfolio we invest in the risk-free asset is

$$1 - \pi_S - \pi_H$$

(the human capital is not part of the investment decision; there is no π_L).

We assume that if you borrow money, then you have to take out a mortgage on your real estate investment – short selling stocks is not allowed

Furthermore, you are only allowed to mortgage $1 - \kappa$ of the value of the real estate. This means that

$$\pi_S \geq 0 \quad \text{and} \quad 1 - \pi_S - \pi_H \geq -(1-\kappa)\pi_H = -\pi_H + \kappa\pi_H,$$

or

$$\pi_S \geq 0 \quad \text{and} \quad \pi_S + \kappa\pi_H \leq 1.$$

Housing

This leads to the maximisation problem

$$\max_{(\pi_S,\pi_H)} \quad \left\{ E_t\left[\frac{W_{t+1}}{W_t}\right] - \frac{1}{\gamma}\mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right]\right\}$$

$$\begin{aligned} \text{s.t.} \quad & \pi_S \geq 0 \\ & \pi_S + \kappa\pi_H \leq 1. \end{aligned}$$

As in the labour income problem, the objective function is quadratic in the weights, so it is straightforward to numerically derive them

Housing

From p. 361 in the course book:

	$\gamma = 1$			$\gamma = 5$			$\gamma = 10$		
ℓ	stock	house	rf	stock	house	rf	stock	house	rf
Panel A: Baseline case with max 80% LTV, $\kappa = 0.2$									
0	52	240	-192	20	52	28	10	26	64
1	13	434	-347	35	96	-31	16	44	41
2	0	500	-400	51	140	-91	21	61	17
5	0	500	-400	50	250	-200	39	115	-53
10	0	500	-400	16	420	-336	60	200	-160
20	0	500	-400	0	500	-400	32	340	-272
50	0	500	-400	0	500	-400	0	500	-400
Panel B: max 60% LTV, $\kappa = 0.4$									
0	33	167	-100	20	52	28	10	26	64
1	4	241	-145	35	96	-31	16	44	41
2	0	250	-150	47	133	-80	21	61	17
5	0	250	-150	30	174	-105	39	115	-53
10	0	250	-150	3	242	-145	36	159	-95
20	0	250	-150	0	250	-150	12	220	-132
50	0	250	-150	0	250	-150	0	250	-150
Panel C: no borrowing, $\kappa = 1$									
0	62	38	0	20	52	28	10	26	64
1	100	0	0	31	69	0	16	44	41
2	100	0	0	38	62	0	21	61	17
5	100	0	0	60	40	0	31	69	0
10	100	0	0	95	5	0	43	57	0
20	100	0	0	100	0	0	67	33	0
50	100	0	0	100	0	0	100	0	0
Panel D: Higher borrowing than lending rate, $r_{\text{bor}} = 2\%, r_{\text{len}} = 1\%$									
0	68	160	-128	20	52	28	10	26	64
1	45	274	-219	30	70	0	16	44	41
2	22	388	-310	42	83	-25	21	61	17
5	0	500	-400	69	154	-123	31	69	0
10	0	500	-400	51	244	-195	50	100	-50
20	0	500	-400	15	424	-339	66	172	-138
50	0	500	-400	0	500	-400	30	352	-282

Table 9.6: Optimal portfolios with borrowing constraints. The table shows percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 9.4 are assumed. In Panels B, C, and D the numbers in blue are larger than in the baseline case of Panel A, numbers in red are smaller, whereas the remaining numbers are unchanged.