Financial Theory – Lecture 7

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Agenda

- More on the CAPM.
- Optimal investments.
- The stochastic discount factor.

The lecture is based on

- Chapters 7 and 10 in the course book.

The Capital Asset Pricing Theory

We ended the last lecture with the CAPM equation:

$$E[r_i] - r_f = \beta_i (E[r_m] - r_f).$$

The beta can be written

$$\begin{aligned}
\beta_i &= \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]} \\
&= \frac{\mathrm{Corr}[r_i, r_m]\mathrm{Std}[r_i]\mathrm{Std}[r_m]}{\mathrm{Std}[r_m]^2} \\
&= \mathrm{Corr}[r_i, r_m]\frac{\mathrm{Std}[r_i]}{\mathrm{Std}[r_m]} \\
&= \rho_{im}\frac{\sigma_i}{\sigma_m}.
\end{aligned}$$

The Capital Asset Pricing Theory

If $r_p = \boldsymbol{\pi} \cdot \boldsymbol{r}$ is the return of a portfolio, then the beta of the portfolio is

$$\beta_p = \sum_{i=1}^{N} \pi_i \beta_i.$$

Any rate of return r is generated by a portfolio of the basic N assets.

For such a rate of return

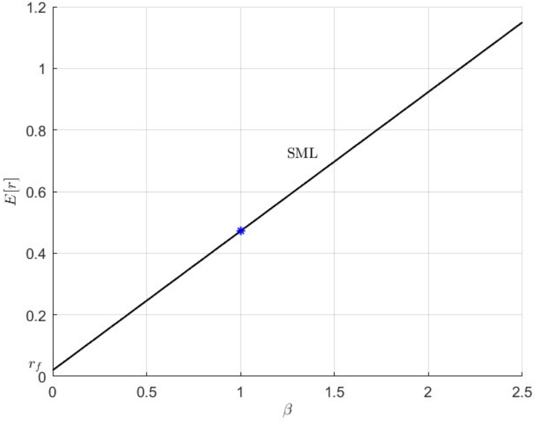
$$E[r] = r_f + \beta(E[r_m] - r_f),$$

where

$$\beta = \frac{\text{Cov}[r, r_m]}{\text{Var}[r_m]}.$$

The Capital Asset Pricing Theory

The mean of a rate of return as a function of its beta-value is the security market line (SML).



The Capital Asset Pricing Theory

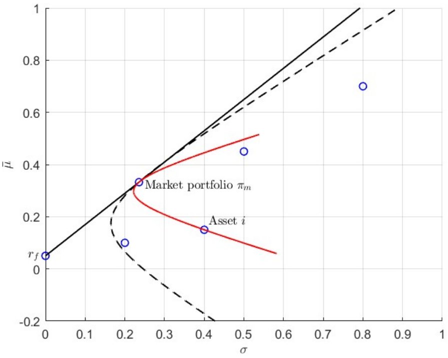
We can write the CAPM equation as

$$
\begin{aligned}
E\left[r_i\right] - r_f &= \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}\left(E\left[r_m\right] - r_f\right) \\
&= \frac{E\left[r_m\right] - r_f}{\mathrm{Var}[r_m]}\mathrm{Cov}[r_i, r_m]
\end{aligned}
$$

Important observation: The risk premium of an asset is proportional to the covariance between the rate of return of the asset and the rate of return on the market portfolio.

The Capital Asset Pricing Theory

Here is an alternative proof of the CAPM equation.



1) Form a two-asset portfolio of asset i and the market portfolio:

$$r_p(w) = wr_i + (1-w)r_m.$$

2) Observe that

$$\left.\frac{d\mu_p}{d\sigma_p}\right|_{w=0} = \frac{E\left[r_m\right] - r_f}{\sigma_m}.$$

3) Remember that

$$\mu_p(w) = w\mu_i + (1-w)E\left[r_m\right]$$

and

$$\sigma_p(w) = \sqrt{w^2\sigma_i^2 + 2\mathrm{Cov}[r_i, r_m]w(1-w) + (1-w)^2\sigma_m^2}.$$

4) Calculate

$$\frac{d\mu_p}{d\sigma_p} = \frac{\frac{d\mu_p}{dw}}{\frac{d\sigma_p}{dw}} = \frac{\mu_i - E\left[r_m\right]}{\frac{1}{\sigma_p(w)}\left(w\sigma_i^2 + \mathrm{Cov}[r_i, r_m](1-2w) - (1-w)\sigma_m^2\right)}$$

5) Use 2):

$$\frac{E\left[r_m\right] - r_f}{\sigma_m} = \left.\frac{d\mu_p}{d\sigma_p}\right|_{w=0} = \frac{\mu_i - E\left[r_m\right]}{\frac{1}{\sigma_m}\left(\mathrm{Cov}[r_i, r_m] - \sigma_m^2\right)}.$$

6) Simplify $\longrightarrow$

$$\mu_i = r_f + \frac{\mathrm{Cov}[r_i, r_m]}{\sigma_m^2}(E\left[r_m\right] - r_f) = r_f + \beta_i(E\left[r_m\right] - r_f).$$

The Capital Asset Pricing Theory

CAPM is a pricing model. Where are the prices?

Recall that

$$r_i = \frac{D_i + P_{i1} - P_{i0}}{P_{i0}}.$$

From CAPM:

$$E\left[\frac{D_i + P_{i1} - P_{i0}}{P_{i0}}\right] = E\left[r_i\right] = r_f + \beta_i(E\left[r_m\right] - r_f)$$

$$\frac{1}{P_{i0}} E\left[D_i + P_{i1}\right] = 1 + r_f + \beta_i(E\left[r_m\right] - r_f) \qquad (*)$$

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right]}{1 + r_f + \beta_i(E\left[r_m\right] - r_f)}.$$

Given expectations of the future dividend payment and price, we use CAPM to calculate the discount rate to get today's price.

The Capital Asset Pricing Theory

Write Equation $(*)$ from the previous slide as

$$E\left[D_i + P_{i1}\right] = P_{i0}(1 + r_f) + P_{i0}\beta_i(E\left[r_m\right] - r_f).$$

Now

$$P_{i0}\beta_i = P_{i0}\frac{\mathrm{Cov}[r_i, r_m]}{\sigma_m^2} = \frac{\mathrm{Cov}\big[P_{i0}(1 + r_i), r_m\big]}{\sigma_m^2} = \frac{\mathrm{Cov}[P_{i1}, r_m]}{\sigma_m^2},$$

so

$$E\left[D_i + P_{i1}\right] = P_{i0}(1 + r_f) + \frac{E\left[r_m\right] - r_f}{\sigma_m^2}\mathrm{Cov}[P_{i1}, r_m].$$

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right] - \frac{E[r_m] - r_f}{\sigma_m^2}\mathrm{Cov}[P_{i1}, r_m]}{1 + r_f}.$$

This is sometimes called the certainty equivalent pricing version of CAPM: It tells us which cash flow we should use if we want to discount using the risk-free rate.

The Capital Asset Pricing Theory

What if there is no risk-free rate in the economy?

There is a version of CAPM that is valid when there is no risk-free rate.

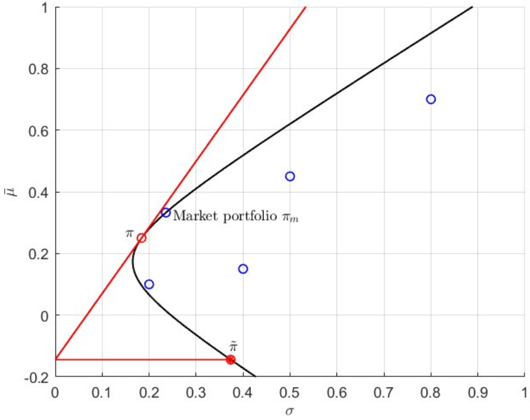
We need the following result: For every portfolio $\boldsymbol{\pi}$ on the upper part of the efficient frontier, there is a portfolio $\tilde{\boldsymbol{\pi}}$ on the lower part of the frontier that is uncorrelated with the first portfolio.

The uncorrelated portfolio $\tilde{\boldsymbol{\pi}}$ has expected rate of return

$$E\left[r(\tilde{\boldsymbol{\pi}})\right] = \frac{A - BE\left[r(\boldsymbol{\pi})\right]}{B - CE\left[r(\boldsymbol{\pi})\right]}.$$

This is Theorem 7.5 in the course book. In Exercise 7.6 you are asked to prove this.

The Capital Asset Pricing Theory



To find the uncorrelated portfolio, draw the tangent line at the point $(\mathrm{Std}[r(\pi)], E[r(\pi)])$.

The portfolio $\tilde{\pi}$ is the mean-variance efficient portfolio that has the mean where the tangent line hits the $\bar{\mu}$-axis.

Let $\boldsymbol{\pi}_z$ denote the portfolio that is uncorrelated with the market portfolio.

Then one can show that

$$E\left[r_i\right] = E\left[r(\boldsymbol{\pi}_z)\right] + \beta_i \big(E\left[r_m\right] - E\left[r(\boldsymbol{\pi}_z)\right] \big).$$

This is the zero-beta CAPM or Black's CAPM.

Optimal portfolios

Let us return to the optimal choice of one investor.

The investor wants to maximise

$$E[r_p] - \frac{\gamma}{2}\mathrm{Var}[r_p].$$

We assume that there exists a risk-free asset and N risky assets.

The investor will invest a fraction w in the tangent portfolio and the fraction $1 - w$ in the risk-free asset:

$$r_p = r(w) = w r_{\text{tan}} + (1 - w) r_f.$$

We also let

$$\mu_{\text{tan}} = E[r_{\text{tan}}] \quad \text{and} \quad \sigma^2_{\text{tan}} = \mathrm{Var}[r_{\text{tan}}].$$

Optimal portfolios

Let

$$\begin{aligned} f(w) &= E[r(w)] - \frac{\gamma}{2}\mathrm{Var}[r(w)] \\ &= E[wr_{\tan} + (1-w)r_f] - \frac{\gamma}{2}\mathrm{Var}[wr_{\tan} + (1-w)r_f] \\ &= w\mu_{\tan} + (1-w)r_f - \frac{\gamma}{2}w^2\sigma_{\tan}^2. \end{aligned}$$

$$\text{FOC: } f'(w) = \mu_{\tan} - r_f - \gamma w \sigma_{\tan}^2 = 0 \quad \Rightarrow \quad w^* = \frac{\mu_{\tan} - r_f}{\gamma\sigma_{\tan}^2}.$$

Which is the optimal portfolio $\boldsymbol{\pi}^*$ in the risky assets?

$$\begin{aligned} \boldsymbol{\pi}^* &= w^*\boldsymbol{\pi}_{\tan} \\ &= \frac{\mu_{\tan} - r_f}{\gamma\sigma_{\tan}^2}\boldsymbol{\pi}_{\tan} \end{aligned}$$

Optimal portfolios

We know that

$$\frac{\sigma_{\tan}^2}{\mu_{\tan} - r_f} = \frac{1}{B - Cr_f}.$$

Use this:

$$\boldsymbol{\pi}_{\tan} = \frac{1}{B - Cr_f}\Sigma^{-1}\big(\boldsymbol{\mu} - r_f\mathbf{1}\big) = \frac{\sigma_{\tan}^2}{\mu_{\tan} - r_f}\Sigma^{-1}\big(\boldsymbol{\mu} - r_f\mathbf{1}\big).$$

We get

$$\boldsymbol{\pi}^* = \frac{\mu_{\tan} - r_f}{\gamma\sigma_{\tan}^2}\frac{\sigma_{\tan}^2}{\mu_{\tan} - r_f}\Sigma^{-1}\big(\boldsymbol{\mu} - r_f\mathbf{1}\big) = \frac{1}{\gamma}\Sigma^{-1}\big(\boldsymbol{\mu} - r_f\mathbf{1}\big).$$

Optimal portfolios

Let us now introduce n number of investors, each with his or her own constant $ARA_j = \gamma_j$, $j = 1, \ldots, n$.

In equilibrium, the tangent portfolio is equal to the market portfolio, so investor j will hold the portfolio

$$w_j = \frac{E\left[r_m\right] - r_f}{\gamma_j \mathrm{Var}[r_m]}.$$

Since we are in equilibrium it must hold that supply equals demand:

$$\sum_{j=1}^{n} w_j = 1.$$

Optimal portfolios

$$1 = \sum_{j=1}^{n} w_j = \sum_{j=1}^{n} \frac{E\left[r_m\right] - r_f}{\gamma_j \text{Var}[r_m]} = \frac{E\left[r_m\right] - r_f}{\text{Var}[r_m]} \sum_{j=1}^{n} \frac{1}{\gamma_j} = \frac{E\left[r_m\right] - r_f}{\text{Var}[r_m]} \cdot \frac{1}{\bar{\gamma}}.$$

Here

$$\bar{\gamma} = \frac{1}{\sum_{j=1}^{n} \frac{1}{\gamma_j}}$$

is a measure of the market's average risk aversion.

We can use

$$\bar{\gamma} = \frac{E\left[r_m\right] - r_f}{\text{Var}[r_m]}$$

to write

$$w_j = \frac{\bar{\gamma}}{\gamma_j}.$$

Optimal portfolios

We can also write

$$E[r_m] - r_f = \bar{\gamma}\mathrm{Var}[r_m].$$

The market risk premium is increasing in

1) the amount of risk as measured by $\mathrm{Var}[r_m]$, and in
2) the average risk averision as measured by $\bar{\gamma}$.

Optimal portfolios

We have seen in Lecture 5 that

$$\text{CARA with parameter } a \; + \; \text{Multivariate normal returns}$$

$$\Rightarrow$$

$$\text{maximise } E[r_p] - \frac{a}{2}\text{Var}[r_p].$$

However, for more general random variables it might happen that

$$X \geq Y \quad \not\Rightarrow \quad E[X] - \frac{a}{2}\text{Var}[X] \geq E[Y] - \frac{a}{2}\text{Var}[Y].$$

The function

$$f(X) = E[X] - \frac{a}{2}\text{Var}[X]$$

is not always monotone.

Optimal portfolios

It is easy to construct an example.

X	Y	Prob
1	1	$1/3$
2	2	$1/3$
5	3	$1/3$

Since $X \geq Y$ a rational investor should prefer X to Y.

But if we let $a = 2$, then

$$E[X] - \text{Var}[X] = -\frac{2}{9} < \frac{4}{3} = E[Y] - \text{Var}[Y].$$

Time-dependent CAPM

CAPM is a one-period model. If we want to let the model meet data we should take into account that the returns can vary over time.

This results in CAPM equations like

$$E_t\left[r_{i,t+1} - r_{f,t+1}\right] = \beta_{i,t} E_t\left[r_{m,t+1} - r_{f,t+1}\right],$$

where

$$\beta_{i,t} = \frac{\mathrm{Cov}_t\left[r_{i,t+1}, r_{m,t+1}\right]}{\mathrm{Var}_t\left[r_{m,t+1}\right]}.$$

Here the t-subscript in E_t, Cov_t and Var_t means that we should take into account all information up to and including time t.

For now, we only consider one-period models, but by considering the intervals $(t, t+1]$ for $t = 0, 1, 2, \ldots$ we can move to time-dependent models.

Consumption-based CAPM

> *Investments move consumption opportunities across time and across states of the world. Individual investors ultimately care about the consumption they get out of their investments.*

(Munk p. 400.)

Question: When would an investor prefer an extra amount of money?
Answer: When there are bad times.

Conclusion: An asset that delivers cash flows when there are bad times will be much in demand $\longrightarrow$ It's price will be high $\longrightarrow$ It's return will be low.

Consumption-based CAPM

We build a model with investors and see how it can be used to price assets.

- At time $t = 0$: Endowment e_0, consumption c_0 and an investment portfolio $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ is bought.
- At time $t = 1$: Endowment e_1, consumption c_1 and the investment portfolio $\mathbf{x}$ is sold.
- The portfolio costs $\sum_{i=1}^{N} x_i P_{i0}$ at time $t = 0$ and results in the cash flow $\sum_{i=1}^{N} x_i (D_i + P_{i1})$ at time $t = 1$.
- An investor's utility is given by

$$u(c_0) + e^{-\delta} E\left[u(c_1)\right],$$

where u is an increasing and concave utility function and δ measures the investor's time preference rate.

Consumption-based CAPM

The investor wants to solve

$$\max_{\mathbf{x}} \left\{ u(c_0) + e^{-\delta} E\left[u(c_1)\right] \right\}$$

$$\text{s.t. } c_0 + \sum_{i=1}^{N} x_i P_{i0} = e_0$$

$$c_1 = e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1}).$$

Note that e_0, e_1 are given exogenously and that c_0, c_1 are given endogenously when the portfolio $\mathbf{x}$ is determined.

We solve this problem by replacing c_0, c_1 from the constraints into the objective function.

Consumption-based CAPM

We get the function

$$f(\mathbf{x}) = u\left(e_0 - \sum_{i=1}^{N} x_i P_{i0}\right) + e^{-\delta} E\left[u\left(e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1})\right)\right]$$

to maximise.

$$\text{FOC:} \quad \frac{\partial f}{\partial x_i} = -P_{i0} u'(c_0) + e^{-\delta} E\left[(D_i + P_{i1}) u'(c_1)\right] = 0$$

$$\Leftrightarrow$$

$$E\left[e^{-\delta} \frac{D_i + P_{i1}}{P_{i0}} \cdot \frac{u'(c_1)}{u'(c_0)}\right] = 1 \quad \Leftrightarrow \quad E\left[(1 + r_i)\underbrace{e^{-\delta} \frac{u'(c_1)}{u'(c_0)}}_{=m}\right] = 1.$$

Consumption-based CAPM

Now,

$$\begin{aligned} 1 &= E\left[(1+r_i)m\right] \\ &= E\left[1+r_i\right]E\left[m\right] + \mathrm{Cov}[1+r_i, m] \\ &= \big(1+E\left[r_i\right]\big)E\left[m\right] + \mathrm{Cov}[r_i, m]. \end{aligned}$$

This implies

$$1 + E\left[r_i\right] = \frac{1}{E\left[m\right]} - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_i, m],$$

or

$$E\left[r_i\right] = \frac{1}{E\left[m\right]} - 1 - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_i, m].$$

This hold for any return. Assume that there exists a risk-free asset:

$$E\left[r_f\right] = r_f = \frac{1}{E\left[m\right]} - 1 - \frac{1}{E\left[m\right]}\mathrm{Cov}[r_f, m] = \frac{1}{E\left[m\right]} - 1.$$

The stochastic discount factor

Making the substitution $1/E[m] - 1 = r_f$ results in

$$E[r_i] = r_f - (1 + r_f)\mathrm{Cov}[r_i, m],$$

or

$$E[r_i] = r_f + \mathrm{Cov}\big[r_i, -(1 + r_f)m\big].$$

This equation looks like the CAPM equation.

Instead of the rate of return of the market portfolio, $-(1 + r_f)m$ is used. This is called the consumption-based CAPM.

The stochastic discount factor

The random variable

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)}$$

is called a stochastic discount factor (SDF). We use it to discount random cash flows to get its price today:

$$\begin{aligned} P_{i0} &= E\left[m\big(D_i + P_{i1}\big)\right] \\ &= E[m]\, E[D_i + P_{i1}] + \mathrm{Cov}[m, D_i + P_{i1}] \\ &= \frac{E[D_i + P_{i1}]}{1 + r_f} + \mathrm{Cov}[m, D_i + P_{i1}] \end{aligned}$$

The stochastic discount factor

Recall

$$P_{i0} = \frac{E\left[D_i + P_{i1}\right]}{1 + r_f} + \text{Cov}[m, D_i + P_{i1}].$$

We have a high price P_{i0} today if $\text{Cov}[m, D_i + P_{i1}]$ is high.

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)} = \text{positive constant} \cdot u'(c_1).$$

$\boxed{\text{High value of } m} = \boxed{\text{High value of } u'(c_1)} = \boxed{\text{Bad future state of the world}}$.

The CRRA-lognormal model

How the explicit expression that gives the risk premium looks like depends on the assumptions we make on the utility functions of the investors and the distributional properties of the rates of return.

The CRRA-lognormal model

- Consumption growth is lognormally distributed,
- Each consumer has a CRRA utility function: $u'(x) = x^{-\gamma}$.

In this case

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)} = e^{-\delta} \left(\frac{c_1}{c_0} \right)^{-\gamma} = e^{-\delta} e^{-\gamma \ln(c_1/c_0)} = e^{-\delta - \gamma \ln(c_1/c_0)}.$$

We will come back to this model when we look at models in macro-finance.