Le 6

More about Möbius transformations

<u>Recall</u> A Möbius transformation is a mapping of the form

$$T(z) = \frac{az+b}{cz+d} \qquad (a, b, c, d \in \mathbb{C})$$

where $ad - bc \neq 0$.

Particular cases are:

1) $T(z) = z + b$; translation
2) $T(z) = az = |a| e^{i \arg a} z$; rotation & magnification
3) $T(z) = \frac{1}{z}$; inversion.

Note now that (if $c \neq 0$)

$$T(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} =$$

$$= \frac{a}{c} - \frac{ad-bc}{c^2} \frac{1}{z + \frac{d}{c}}$$

$\Rightarrow$ Every Möbius transformation is a composition of Möbius transformations of the type 1), 2), 3).

<u>Thm</u> Every Möbius transformation maps "circles" onto "circles".

<u>Remark</u>: Recall that a "circle" in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a circle or line in $\mathbb{C}$. A line in $\mathbb{C}$ is a "circle" through ∞ in $\hat{\mathbb{C}}$.

Proof. It is easy to see that mappings of the form 1) and 2) map circles onto circles and lines onto lines. ⇒ Enough to prove that inversion

$$T(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2} = u+iv$$

maps "circles" onto "circles". [Note: $u^2+v^2 = \frac{1}{x^2+y^2}$]

A "circle" in $\hat{\mathbb{C}}$ has eq.

$$A(x^2+y^2) + Cx + Dy + E = 0$$

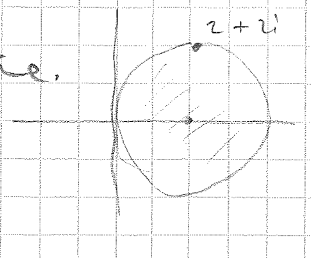
$$\Leftrightarrow \quad A + C\,\frac{x}{x^2+y^2} + D\,\frac{y}{x^2+y^2} + E\,\frac{1}{x^2+y^2} = 0$$

$$\Leftrightarrow \quad E(u^2+v^2) + Cu - Dv + A = 0 \ ; \quad \text{"circle"}$$ ⊠

Ex. Determine the image of the disk $|z-2|<2$ under the Möbius transformation $T(z) = \dfrac{z}{2z-8}$.

Sol. First determine the image of $C: |z-2| = 2$.

Since $T(4) = \infty$, $T(C)$ must be a line.



Since $T(0) = 0$ and

$$T(2+2i) = \frac{2+2i}{2(2+2i)-8} = \frac{2+2i}{-4+4i} = -\frac{1}{2}\,\frac{1+i}{1-i} = -\frac{i}{2}$$

⇒ $T(C)$ is a line through $w=0$ and $w = -\frac{i}{2}$.

⇒ $T(C)$ is the imaginary axis. Since $T(2) = -\frac{1}{2}$, clearly $|z-2|<2$ maps onto the left half-plane $\operatorname{Re} w < 0$ ⊠

Given a "circle" C_z in the z-plane and a "circle" C_w in the w-plane, can one find a Möbius transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $T(C_z) = C_w$? Yes!

<u>The cross-ratio</u>

<u>Def.</u> Let $z_1, z_2, z_3 \in \hat{\mathbb{C}}$; all different.

Put

$$(z, z_1, z_2, z_3) := \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \in \hat{\mathbb{C}}$$

If some of the z_i is ∞, the right-hand side should be interpreted as

$$(z, z_1, z_2, z_3) = \begin{cases} \dfrac{z_2 - z_3}{z - z_3}, & \text{if } z_1 = \infty \\[2ex] \dfrac{z - z_1}{z - z_3}, & \text{if } z_2 = \infty, \\[2ex] \dfrac{z - z_1}{z_2 - z_1}, & \text{if } z_3 = \infty \end{cases}$$

(z, z_1, z_2, z_3) is called the <u>cross-ratio</u> of the four points.

Note: $S(z) = (z, z_1, z_2, z_3)$ is a Möbius transf. s.t.

$$S(z_1) = 0, \quad S(z_2) = 1, \quad S(z_3) = \infty.$$

By an earlier proposition it is the unique Möbius transformation mapping z_1, z_2, z_3 to $0, 1, \infty$.

<u>Thm</u> Given a tripple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ of distinct points, and another tripple $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ of distinct points, there is a unique Mobius transformation T s.t. $T(z_i) = w_i$, $i = 1,2,3$.

The mapping $w = T(z)$ is found by solving

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

<u>Proof</u> By our earlier proposition there is at most one such mapping. We now prove that there is exactly one by constructing it.

$$\begin{tikzcd} z_1 \arrow[d, "S"'] & z_2 \arrow[d] & z_3 \arrow[d] \\ 0 & 1 & \infty \\ w_1 \arrow[u, "U"] & w_2 \arrow[u] & w_3 \arrow[u] \end{tikzcd}$$

Put $S(z) = (z, z_1, z_2, z_3)$, $U(w) = (w, w_1, w_2, w_3)$.

$\Rightarrow$ $T(z) := (U^{-1} \circ S)(z) = U^{-1}(S(z))$ is a Mobius transformation s.t.

$$T(z_1) = U^{-1}(S(z_1)) = U^{-1}(0) = w_1 \quad \text{etc.}$$

Clearly,

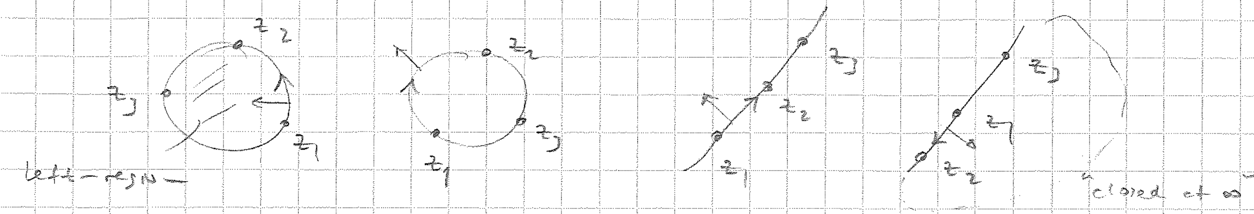
$$w = T(z) \Leftrightarrow w = U^{-1}(S(z)) \Leftrightarrow U(w) = S(z)$$

$$\Leftrightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3).$$

$\square$

Of course, this theorem can be used to construct a T as above, mapping C_z to C_w.

Let z_1, z_2, z_3 be three distinct points on a "circle" C_z in $\hat{\mathbb{C}}$. Note that C_z is oriented by the order of these points, that is, C_z acquires an orientation by proceeding through z_1, z_2, z_3 in succession.

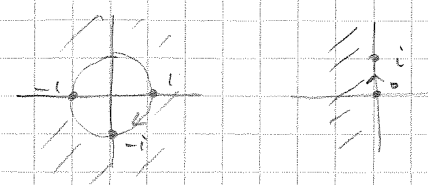


Since a Möbius transformation is conformal, it maps the region to the left of C_z, oriented by z_1, z_2, z_3, to the region to the left of $C_w = T(C_z)$, oriented by w_1, w_2, w_3.

__Ex__ Find a Möbius transformation that maps the region $|z| > 1$ onto $\operatorname{Re} w < 0$.

__Sol.__ Let $z_1 = 1$, $z_2 = -i$, $z_3 = -1$ and

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$



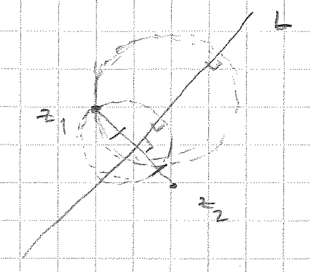
$$\frac{w-0}{w-\cancel{\infty}} \cdot \frac{i-\cancel{\infty}}{i-0} = \frac{z-1}{z+1} \cdot \frac{-i+1}{-i-1} \iff$$

$$\iff w = -i \cdot \frac{1-i}{1+i} \, \frac{z-1}{z+1} = -\frac{z-1}{z+1} = \frac{1-z}{1+z}.$$

Answer: $T(z) = \dfrac{1-z}{1+z}.$

Symmetry-preserving property

Two points z_1 and z_2 are said to be symmetric w.r.t. a line L if L is the perpendicular bisector of the line-segment joining z_1 and z_2; see figure

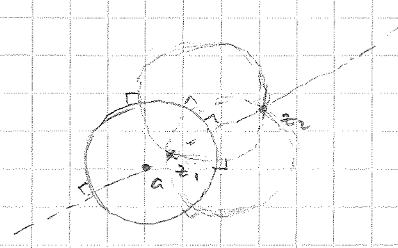


This means that every circle or line through z_1 and z_2 intersects L orthogonally.

This motivates:

<u>Def</u> Two points z_1 and z_2 are said to be symmetric w.r.t. a <u>circle C</u> if every circle or line through z_1 and z_2 intersects C orthogonally.

See figure:



In particular, the center a of C and ∞ are symmetric w.r.t. C.

<u>Thm</u> (Symmetry principle)

Let C_z be a circle or line in the z-plane, and $w = T(z)$ be any Möbius transformation. Then two points z_1 and z_2 are symmetric w.r.t. C_z if and only if their images $w_1 = T(z_1)$ and $w_2 = T(z_2)$ are symmetric w.r.t. the image $C_w = T(C_z)$ under T.

<u>Proof</u>: [Two points are symmetric w.r.t. a given "circle" iff every "circle" containing the points intersects the given "circle" orthogonally.]

Möbius transformations preserve the class of "circles", and they also preserve orthogonality; hence they preserve the symmetric condition. ∎

<u>Ex</u> Let C be a (proper) circle with center a and radius R. Given a point $\alpha \in \hat{\mathbb{C}}$ there is a unique point α^* s.t. α and α^* are symmetric w.r.t. C.

First observe that

$$T(z) = (z, a-R, a+iR, a+R) =$$

$$= \frac{z-(a-R)}{z-(a+R)} \cdot \frac{\alpha+iR-(\alpha+R)}{\alpha+iR-(\alpha-R)} =$$

$$= \frac{z-(a-R)}{z-(a+R)} \cdot \frac{(-1)(1-i)}{1+i} = i\,\frac{z-(a-R)}{z-(a+R)}$$

maps C onto the real line. Thus,

α^* is symmetric to α w.r.t. C iff

$T(\alpha^*)$ is symmetric to $T(\alpha)$ w.r.t. $\mathbb{R}$.

But this holds iff

$$T(\alpha^*) = \overline{T(\alpha)}$$

$\Longleftrightarrow$

$$i\,\frac{\alpha^*-(a-R)}{\alpha^*-(a+R)} = -i\,\frac{\bar{\alpha}-(\bar{a}-R)}{\bar{\alpha}-(\bar{a}+R)}$$

Solving this for α^* gives

$$(\alpha^*-a)\overline{(\alpha-a)} = R^2$$

or

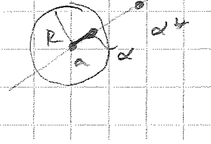
$$\alpha^* = a + \frac{R^2}{\overline{\alpha-a}} = a + \frac{R^2}{|\alpha-a|^2}(\alpha-a).$$

Thus,

$$\arg(\alpha^*-a) = \arg(\alpha-a)$$

and

$$|\alpha^*-a|\,|\alpha-a| = R^2$$



$\square$