Proof of the existence part of the Riemann mapping theorem

Def. Let D be a domain in $\mathbb{C}$.

A family $\mathcal{F}$ of functions analytic on D is called a normal family if every sequence in $\mathcal{F}$ has a subsequence which converges normally on D.

Def. A family $\mathcal{F}$ of functions defined on a subset E of $\mathbb{C}$ is said to be uniformly bounded on E if there exists $B > 0$ s.t.

$$|f(z)| \leq B \quad \text{for all } z \in E \text{ and } f \in \mathcal{F}.$$

We shall need the following:

Thm (Montel's thm)

Suppose $\mathcal{F}$ is a family of analytic functions on a domain D such that $\mathcal{F}$ is uniformly bounded on every compact subset of D.

Then $\mathcal{F}$ is a normal family.

Proof: I leave this out. Idea: Cauchy's integral thm implies equicontinuity on compact subset of D; then Arzelà-Ascoli.

The existence proof has 3 steps

Step 1 Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$. I claim that there is an analytic fcn f mapping D bijectively onto a domain in $\mathbb{D}$ containing 0.

Indeed, choose $a \in \mathbb{C} \setminus D$. Since D is simply connected there is an analytic branch $g(z)$ of $\log(z-a)$ in D. Since $e^{g(z)} = z-a$, clearly g is one-to-one. Pick a $z_0 \in D$ and observe that

$$g(z) \neq g(z_0) + 2\pi i \qquad \forall z \in D,$$

for otherwise we exponentiate and find that $z = z_0$, hence $g(z) = g(z_0)$, a contradiction.

In fact, I claim that g stays strictly away from $g(z_0)+2\pi i$, in the sense that there exists a disk centered at $g(z_0)+2\pi i$ which contains no points of $g(D)$.

For otherwise there exists a sequence $\{z_n\}$ in D such that $g(z_n) \to g(z_0) + 2\pi i$.

We exponentiate this relation, and, using that the exponential fcn is continuous, find that $z_n \to z_0$. But this implies that $g(z_n) \to g(z_0)$, contradicting the above. Consider the map

$$h(z) = \frac{1}{g(z) - (g(z_0) + 2\pi i)}$$

Since g is one-to-one, so is h, so that h maps D bijectively onto $h(D)$. By the above $h(D)$ is bounded. By a translation and multiplication by a small positive number, we obtain an analytic fcn f which maps D bijectively onto a simply connected subset of $\mathbb{D}$ that contains the origin.
(We can assume $f(z_0) = 0$; and if we like also that $f'(z_0) > 0$)

<u>Step 2</u> By the first step, we may assume that D is a simply connected subset of $\mathbb{D}$ containing 0. Consider the family

$$\mathcal{F} = \{ f : D \to \mathbb{D} \mid f \text{ is analytic, one-to-one, and } f(0) = 0 \}$$

Clearly, $\mathcal{F}$ is non-empty, for it contains the identity map $f(z) = z$.

Also, $\mathcal{F}$ is uniformly bounded on (compact subsets) of D, by construction, since fns $f \in \mathcal{F}$ map D into $\mathbb{D}$.

We next show that there $\exists\, f \in \mathcal{F}$ which maximizes $|f'(0)|$. First note that the quantities $|f'(0)|$ are unif. bounded as f ranges over $\mathcal{F}$, by the Cauchy estimates or the Cauchy integral formula

$$f'(0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^2}\, dz$$

applied to a small circle γ in D centered at 0.

Next, let

$$s = \sup_{f \in \mathcal{F}} |f'(0)|$$

and choose a sequence $\{f_n\} \subset \mathcal{F}$ s.t. $|f_n'(0)| \to s$.

By Montel's thm this seq. has a subseq. converging normally to an analytic fcn f on D.

Since $s \geq 1$ (because $f(z) = z$ belongs to $\mathcal{F}$), f is not constant, hence one-to-one by Hurwitz's theorem. Also, by continuity we have that $|f(z)| \leq 1$ $\forall z \in D$, and by the (strict) maximum modulus thm $|f(z)| < 1$ $\forall z \in D$.

Since clearly $f(0) = 0$, we have that $f \in \mathcal{F}$ and that $|f'(0)| = s$.

<u>Step 3</u> We show next, that the fcn f from step 2 maps D <u>onto</u> $\mathbb{D}$.

We do this by showing that if f is <u>not</u> surjective we could construct a fcn $F \in \mathcal{F}$ with $|F'(0)| > s$.

So, suppose there $\exists\, a \in \mathbb{D}$ s.t. $f(z) \neq a \;\forall z \in D$

Let ψ_a be the conformal self-map of $\mathbb{D}$ given by

$$\psi_a(z) = \frac{a - z}{1 - \bar{a} z}, \quad z \in \mathbb{D}.$$

Note that ψ_a interchanges 0 and a.

Since D is simply connected, so is $U := (\psi_a \circ f)(D)$.

Moreover, U does not contain 0, so we may define an analytic branch $g(w)$ of $\sqrt{w}$ on U by

$$g(w) = e^{\frac{1}{2} \log w},$$

Next, consider the fcn

$$F = \psi_{g(a)} \circ g \circ \psi_a \circ f$$

I claim that $F \in \mathcal{F}$. Clearly, F is analytic, and $F(0) = 0$. Also, F maps $\mathbb{D}$ into $\mathbb{D}$. Finally, F is one-to-one, since this is true for all of the fcns $\psi_{g(a)}$, ψ_a, g and f.

Let h denote the square fcn $h(w) = w^2$ and put

$$\Phi = \psi_a^{-1} \circ h \circ \psi_{g(a)}^{-1}$$

Clearly then,

$$f = \Phi \circ F.$$

Now, Φ is an analytic fcn which maps $\mathbb{D}$ into $\mathbb{D}$, and $\Phi(0) = 0$. Clearly, Φ is not a rotation. [Φ is not even one-to-one, since h is not].

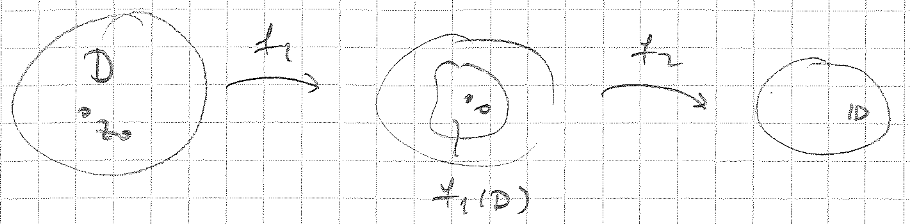
So by the infinitesimal version of Schwarz lemma, we conclude that $|\Phi'(0)| < 1$

Hence, $|f'(0)| = |\Phi'(\overset{F(0)}{\overset{\parallel}{0}}) F'(0)| < |F'(0)|$,

a contradiction which shows that f maps $\mathbb{D}$ onto $\mathbb{D}$.

Finally, we can multiply f by a complex constant of modulus 1 so that $f'(0) > 0$. $\square$

[More precisely, let f_1 be the map from step 1 taking $\mathbb{D}$ onto a subset of $\mathbb{D}$ and with $f_1(z_0) = 0$. See figure:



$f_2(0) = 0$.

Here f_2 is the map constructed in step 2, having maximal $|f_2'(0)|$.

Then $(f_2 \circ f_1)'(z_0) = f_2'(0)\, f_1'(z_0)$ so if we multiply $f_2 \circ f_1$ with a complex number of modulus 1 we can obtain a fcn f taking $\mathbb{D}$ onto $\mathbb{D}$ s.t.

$f(z_0) = 0$ and $f'(z_0) > 0$.]

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