Math 314 Lecture #31
§16.4: Green's Theorem

Let C be a piecewise smooth, simple closed curve in $\mathbb{R}^2$, and D the region it encloses.

Parameterize C by $\vec{r}(t)$, $a \leq t \leq b$, that gives C the **positive orientation**, i.e., $\vec{r}(t)$ traverses C in the "counterclockwise" direction, or walking along C means your left hand is over D.

[A negative orientation is when $\vec{r}(t)$ traverses C in the "clockwise" direction.]

We introduce new notation for the line integral over a positively orientated, piecewise smooth, simple closed curve C; it is

$$\oint_C P dx + Q dy.$$

Green's Theorem. Let C be a positively oriented, piecewise smooth, simple closed curve. Let D be the region it encloses. If P and Q have continuous first-order partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Remarks. We recall notation for the boundary C of a region D; it is ∂D, i.e., $\partial D = C$. So Green's Theorem takes the form,

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Comparing Green's Theorem with the Fundamental Theorem of Calculus, $f(b) - f(a) = \int_a^b f'(x) dx$, we see that the left sides involve the boundary of a domain, and the right sides involves a "derivative" of some kind or another.

Proof of Green's Theorem when D is of type I and type II, i.e., D is a **simple region**.

[The proof of Green's Theorem over a more general region is accomplished by dividing that region into simple subregions.]

If we can show that

$$\oint_C P dx = -\iint_D \frac{\partial P}{\partial y} \, dA, \quad \oint_C Q dy = \iint_D \frac{\partial Q}{\partial x} \, dA,$$

then Green's Theorem for simple regions follows immediately.

A type I description of D is $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, where g_1, g_2 are continuous on $a \leq x \leq b$.

Then

$$\iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy dx = \int_a^b \big[P(x, g_2(x)) - P(x, g_1(x)) \big] \, dx.$$

The boundary curve C of D consists of four smooth pieces parameterized as

C_1: $x = t$, $y = g_1(t)$, $a \leq t \leq b$, (bottom curve)
C_2: $x = b$, $y = t$, $g_1(b) \leq t \leq g_2(b)$, (right curve)
C_3: $x = t$, $y = g_2(t)$, $a \leq t \leq b$, (top curve) and
C_4: $x = a$, $y = t$, $g_1(a) \leq t \leq g_2(a)$ (left curve).

The orientations of C_1 and C_2 are positive, but the orientations of C_3 and C_4 are negative, and so

$$C = C_1 + C_2 - C_3 - C_4.$$

Thus the line integral of P over C is

$$\begin{aligned}
\oint_C P dx &= \int_{C_1} P dx + \int_{C_2} P dx - \int_{C_3} P dx - \int_{C_4} P dx \\
&= \int_a^b P(t, g_1(t))\ dt + \int_{g_1(b)}^{g_2(b)} P(b,t)(0)\ dt \\
&\qquad - \int_a^b P(t, g_2(t))\ dt - \int_{g_2(a)}^{g_2(b)} P(a,t)(0)\ dt \\
&= \int_a^b P(t, g_1(t))\ dt - \int_a^b P(t, g_2(t))\ dt \\
&= \int_a^b \big[P(x, g_1(x)) - P(x, g_2(x))\big]\ dx \\
&= -\iint_D \frac{\partial P}{\partial y}\ dA.
\end{aligned}$$

Using a type II description of D gives the equality of the line integral of Q over C with the double integral of $\partial Q/\partial x$ over D. $\square$

<u>Outcome A</u>: Use Green's Theorem to compute a line integral over a positively oriented, piecewise smooth, simple closed curve in the plane.

Green's Theorem provides a *computational tool* for computing line integrals by converting it to a (hopefully easier) double integral.

Example. Let C be the curve $x^2 + y^2 = 4$, D the region enclosed by C, $P = xe^{-2x}$, $Q = x^4 + 2x^2y^2$.

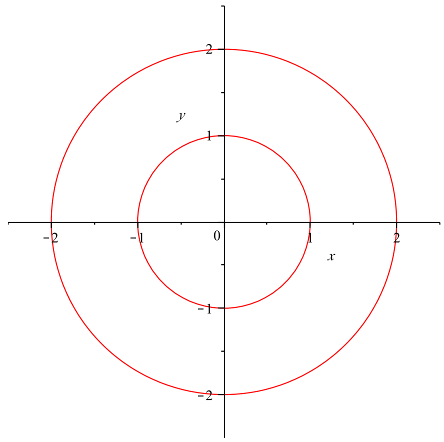
A positively oriented parameterization of C is $x(t) = 2\cos t$, $y(t) = 2\sin t$, $0 \leq t \leq 2\pi$.

By Green's Theorem we have

$$\begin{aligned}
\oint_C P dx + Q dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA \\
&= \iint_D (4x^3 + 4xy^2 - 0)\ dA = \iint_D 4x(x^2 + y^2)\ dA \\
&= \int_0^{2\pi} \int_0^2 (r\cos\theta)(r^2) r\ dr d\theta = \int_0^{2\pi} \int_0^2 4r^4 \cos\theta\ dr d\theta \\
&= 0.
\end{aligned}$$

Example. Let $D = \{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le 2\pi\}$, an annulus.

The boundary of D consists of two smooth simple closed curves $C_1 : x^2 + y^2 = 1$ and $C_2 : x^2 + y^2 = 2^2$, and so $\partial D = C_1 \cup C_2$. Here is a rendering of D and its boundary curves.



Does Green's Theorem apply to $\int_{C_1 \cup C_2} P dx + Q dy$?

Yes, it does, *after* we cut D along, say the curve C_3 that is positive x-axis between 1 and 2, so that D is now simply connected and the boundary of D is C_1 counterclockwise, followed by C_3 from right to left, followed by C_2 *clockwise*, followed by C_3 from left to right:

$$C_1 \cup C_2 = C_1 - C_3 + C_2 + C_3.$$

Then

$$\int_{C_1 \cup C_2} P dx + Q dy = \oint_{C_1 - C_3 + C_2 + C_3} P dx + Q dy = \iint_D (Q_y - P_x)\, dA,$$

where the line integrals over $-C_3$ and C_3 cancel.

<u>Outcome B</u>: Use Green's Theorem to compute the area of a simply connected region.

Recall that $\iint_D dA$ gives the area of A.

Are there choices of P and Q such that $Q_x - P_y = 1$ on D? Yes, here are some

$$P = 0, Q = x, \quad P = -y, Q = 0, \quad P = -y/2, Q = x/2.$$

For these choices of P and Q, Green's Theorem gives

$$\oint_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA = \iint_D dA.$$

This is the basis for the theory of certain **planimeters** (instruments for measuring area): one can "walk" along the shoreline of England to measure the area of its landmass.

Example. The area of the region D bounded by the cardioid $C : r = 1 - \cos\theta$, $0 \le \theta \le 2\pi$, is

$$\iint_D dA = \oint_C x dy.$$

We parameterize C by θ and polar coordinates:

$$\begin{aligned} x &= r\cos\theta = (1-\cos\theta)\cos\theta = \cos\theta - \cos^2\theta, \\ y &= r\sin\theta = (1-\cos\theta)\sin\theta = \sin\theta - \cos\theta\sin\theta. \end{aligned}$$

Thus the line integral is

$$\begin{aligned} \oint_C x dy &= \int_0^{2\pi} (\cos\theta - \cos^2\theta)(\cos\theta + \sin^2\theta - \cos^2\theta)\ d\theta \\ &= \int_0^{2\pi} \big(\cos^2\theta + \cos\theta\sin^2\theta - \cos^3\theta - \cos^3\theta - \cos^2\theta\sin^2\theta + \cos^4\theta\big)\ d\theta \\ &= \int_0^{2\pi} \cos^2\theta\big(1 - \sin^2\theta + \cos^2\theta\big)\ d\theta \\ &= \int_0^{2\pi} \cos^2\theta\big(\cos^2\theta + \cos^2\theta\big)\ d\theta \\ &= 2\int_0^{2\pi} \cos^4\theta\ d\theta \\ &= 2\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2}. \end{aligned}$$

In the middle of this calculation, the integrals of odd functions $\cos\theta\sin^2\theta$ and $\cos^3\theta$ over $[0, 2\pi]$ are zero.

Outcome C: Use Green's Theorem to compute a line integral of a vector field.

For the line integral of a vector field $\vec{F} = \langle P, Q\rangle$ over a positively oriented piecewise smooth simple closed curve $C : \vec{r}(t)$, $a \leq t \leq b$, we apply Green's Theorem to get

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\ dA$$

where D is the region enclosed by C.

What happens when $\vec{F}$ is conservative? The integrand $Q_x - P_y$ of the double integral is 0, and so the line integral of $\vec{F}$ over C is 0 too.

When $\vec{F}$ is not conservative, i.e., when $Q_x - P_y \neq 0$ on D, but D is a type I or type II region, the double integral sometimes gives a simpler means to compute the line integral.