UPPSALA UNIVERSITET

Föreläsningsanteckningar

Variationskalkyl

Rami Abou Zahra

Inlämningsdatum
August 29, 2022

Contents

1. Introduction

One frequently encounters problems involving opitmisation (max, min). This is familiar from single/multi-variable calculus:

Let $f \in C'(I)$ (where $C'(I)$ means continous first derivative function space), find $\max(f)$ and $\min(f)$

For a stationary point we know that the derivative $f'(x) = 0$, and a stationary point is where the function is either at its peak (max) or lowest point

In several variables, we would have a gradient vector $\nabla f \equiv \bar{0}$

In Calculus of Variations (CoV) w ehave problems of the following from:

Find local minima for $J : C^2(\Omega) \to \mathbb{R}$: $C^2(\Omega) = \{y : \Omega \to \mathbb{R} : \text{y is twice continously differentiable}\}$

J is often called a **functional** and can be of the form:

$$J(y) = \int_\Omega f(x, y, \nabla y) dx$$

Where f is smooth in all three arguments

Example:

Let $p, q \in \mathbb{R}^2$ be 2 distinct points. Find the shortest curve connecting p, q. Because they are distinct, we know that either the x component or y component is different. Suppose that there is a function $y(x)$ connecting the 2 points, then the following:

$$p = (x_p, y(x_p)) \qquad q = (x_q, y(x_q))$$

Arc length is given by:

$$L(y) = \int_{x_p}^{x_q} \sqrt{1 + (y'(x))^2} dx$$

This would give us the length between p, q, however, since we want to find the shortest curve, we want to minimize (read: opitmize) this function L among the C' functions with end points fixed at p, q

Example: Minimal surfaces

Let C be a closed curve in $\mathbb{R}^3$ such that for $\Omega \subset \mathbb{R}^2$ bounded and $g : \partial\Omega \to \mathbb{R}$ so that $C : \{(x, y, g(x, y)) : (x, y) \in \Omega\}$

Suppose we want to minimize over surfaces S with $\partial S = C$ (boundary of S is C), sounds like an opitmisation problem!
We write $S = \text{graph}(u)$, where $u : \Omega \to \mathbb{R}$ and $u = g$ on $\partial\Omega$

The area of such a surface is:

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dxdy$$

Our opitmisation problem is now to find a minimizer u of A such that $u = g$ on $\partial\Omega$

Minimizers of A are called **minimal surfaces**. This problem is called Plateaus problem

Examples of minimal surfaces:

- 2D-plane in $\mathbb{R}^3$
- Helicoids
- Catenoid

Example: (Catenoid - Catenery)

Consider a thin cable hanging between 2 poles. What shape will such a cable attain?
Assuming the cable has uniform density ρ and fixed length L. The cable will arange itself in such a way such that it minimizes its potential energy.
In order to do so, suppose the line is given by the function $y(x)$ and that there is a force g acting on it:

$$W(y) = \int_{x_1}^{x_2} \rho\sqrt{1+(y')^2}gy(x)dx$$

Over functions:

$$y \in C'([x_1, x_2])$$

Such that $y(x_1) = y_1$ and $y(x_2) = y_2$
Operating with fixed length:

$$L = \int_{x_1}^{x_2} \sqrt{1+(y')^2}dx$$

Solutions to this problem are called *catenaries.* If you take a catenary and rotate it you get a catenoid, which is a minimal surface.

1.1. **The catenoid.**

Suppose I have a surface of revolution within an interval (x_1, x_2). In order to find the area, we can use our knowledge of infitesimals:

$$A(y) = \int_{x_1}^{x_2} 2\pi y(x)\sqrt{1+(y')^2}dx$$

Notice that this looks a lot like our function W, if we let $\rho \cdot g = 2\pi$ (they are constants, and we do not care of their values other than that they are independent)

1.2. **Hamiltons principle.**

Particles in gravitational fields will follow the trajectory that minimizes its energy. So, consider a moving particle with some mass m in some force field $f(x, y) = -\nabla V(x, y)$ (conservative field, gradient of some function). Denote its trajectory by $r : (-\varepsilon, \varepsilon)\mathbb{R}^2$ (with velocity $r^{\cdot}$).
Its kinetic energy is given by $T = \frac{1}{2}m\left|r^{\cdot}\right|^2$ and its potential energy is $V(x, y)$
We introduce the so called **lagrangian** which is $L = T - V$. Hamiltons principle states that the particle follows a path $r(t)$ between t_1 and t_2 that minimizes the **action functional**:

$$S(r) = \int_{t_1}^{t_2} L(t_1, r, r)dt = \int_{t_1}^{t_2} (T - V)dt$$