Financial Theory – Lecture 2

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Agenda

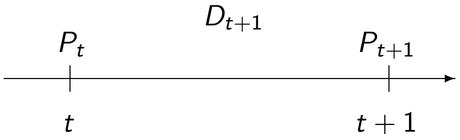
- Measuring the return on an investment.

The lecture is based on

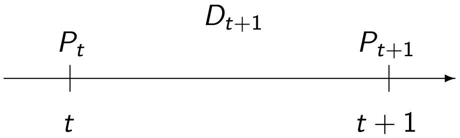
- Chapter 2 in the course book.

Returns over a single period

- We let P_t and P_{t+1} denote the price of an asset at time t and $t+1$ respectively.
- D_{t+1} is the dividend (cash flow) paid out at time $t+1$ from the asset over the time period $(t, t+1]$.



Returns over a single period



The rate of return of the asset over the time period $(t, t+1]$ is given by

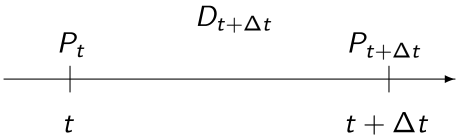
$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} = \frac{D_{t+1} + P_{t+1}}{P_t} - 1.$$

Returns over a single period

More generally, if the times are t and $t + \Delta t$, with Δt being the time period over which we measure the return, we have

$$r_{t,t+\Delta t} = \frac{D_{t+\Delta t} + P_{t+\Delta t} - P_t}{P_t} = \frac{D_{t+\Delta t} + P_{t+\Delta t}}{P_t} - 1.$$

Here $D_{t,t+\Delta t}$ is the dividend over $(t, t + \Delta t]$ paid out at $t + \Delta t$.



Returns over a single period

We can write

$$r_{t,t+1} = \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t},$$

where

$$\frac{D_{t+1}}{P_t} \text{ is the dividend yield}$$

and

$$\frac{P_{t+1} - P_t}{P_t} \text{ is the capital gain in percent.}$$

Returns over a single period

Example

A stock has price $P_t = 125$ at time t and price $P_{t+1} = 110$ at time $t+1$.

During the time interval $(t, t+1]$ the stock pays a divident of 20.

In this case

$$r_{t,t+1} = \frac{20 + 110 - 125}{125} = \frac{5}{125} = 4\%,$$

$$\text{Dividend yield} = \frac{20}{125} = 16\%$$

and

$$\text{Capital gain} = \frac{110 - 125}{125} = -\frac{15}{125} = -12\%.$$

Ex- and cum-dividend

In practise dividends are paid out at a given time t. What is the value of an asset at this time?

This depends on how you define the value.

- Ex-dividend: The dividend is not included in the price at the time of the dividend.
- Cum-dividend: The dividend is included in the price at the time of the dividend.

The book uses the ex-dividend principle, and so will we.

Returns over a single period

There are alternative ways of measuring the return of an asset.

- The gross return:

$$R_{t,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = 1 + r_{t,t+1}$$

or

$$R_{t,t+\Delta t} = \frac{D_{t+\Delta t} + P_{t+\Delta}}{P_t} = 1 + r_{t,t+\Delta t}.$$

- The log-return:

$$r^{\log}_{t,t+1} = \ln\left(1 + r_{t,t+1}\right) = \ln R_{t,t+1},$$

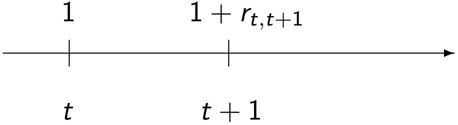
or

$$r^{\log}_{t,t+\Delta t} = \ln\left(1 + r_{t,t+\Delta t}\right) = \ln R_{t,t+\Delta t}.$$

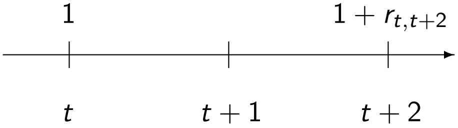
Returns over multiple periods

When we study the return over multiple periods, we need to take the compounding of returns into account.

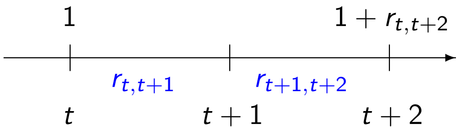
In one period:



In two periods:



Returns over multiple periods



How is the two-period rate of return $r_{t,t+2}$ connetced to the two one-period returns $r_{t,t+1}$ and $r_{t+1,t+2}$?

- Start with 1 unit of currency at time t.
- At time $t+1$ this has grown to $1 \cdot (1 + r_{t,t+1}) = 1 + r_{t,t+1}$.
- This amount is now invested over the next time period.
- At time $t+2$ this has grown to
  $(1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) = 1 + r_{t,t+2}$.

Returns over multiple periods

To summarize: It holds that

$$1 + r_{t,t+2} = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}),$$

or

$$r_{t,t+2} = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) - 1.$$

Since the gross return is 1 plus the rate of return we see that

$$R_{t,t+2} = R_{t,t+1} \cdot R_{t+1,t+2}.$$

Finally, for log-returns we have

$$\begin{aligned} r^{\log}_{t,t+2} &= \ln R_{t,t+2} \\ &= \ln \left(R_{t,t+1} \cdot R_{t+1,t+2} \right) \\ &= \ln R_{t,t+1} + \ln R_{t+1,t+2} \\ &= r^{\log}_{t,t+1} + r^{\log}_{t+1,t+2} \end{aligned}$$

Returns over multiple periods

These results can be generalised to n number of periods.

- For rates of return:

$$r_{t,t+n} = (1 + r_{t,t+1})(1 + r_{t+1,t+2}) \dots (1 + r_{t+n-1,t+n}) - 1.$$

- For gross returns:

$$R_{t,t+n} = R_{t,t+1} R_{t+1,t+2} \dots R_{t+n-1,t+n}.$$

- For log-returns:

$$r^{\log}_{t,t+n} = r^{\log}_{t,t+1} + r^{\log}_{t+1,t+2} + \dots + r^{\log}_{t+n-1,t+n}.$$

Note that log-returns are additive.

Returns over multiple periods

We have

$$\begin{aligned} r_{t,t+2} &= (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) - 1 \\ &= r_{t,t+1} + r_{t+1,t+2} + r_{t,t+1} \cdot r_{t+1,t+2} \\ &\approx r_{t,t+1} + r_{t+1,t+2} \end{aligned}$$

if $r_{t,t+1}$ and $r_{t+1,t+2}$ are "small".

This can be generalised to

$$r_{t,t+n} \approx r_{t,t+1} + r_{t+1,t+2} \ldots + r_{t+n-1,t+n}.$$

Note that this approximation can be quite bad, and in general we should not use it.

Annualising returns

In order to compare returns it is easier if they are given over the same time period.

This typically means that we transfer them into yearly returns.

These compounded annualised returns are also called effective annual returns.

If the calculated returns are monthly, then their annualised counterparties are

$$r_{\text{ann}} = (1 + r_{\text{mon}})^{12} - 1,$$

$$R_{\text{ann}} = \left(R_{\text{mon}}\right)^{12}$$

and

$$r_{\text{ann}}^{\log} = 12 \cdot r_{\text{mon}}^{\log}.$$

The internal rate of return

Assume that we make an investment today at time $t = 0$ of I and this investment will give us the cash flows $C_1, C_2, \ldots, C_T$ at times $t = 1, 2, \ldots, T$.

What is the return on this investment?

We say that r is the internal rate of return (IRR) of the investment if r satisfies

$$I = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t},$$

or

$$0 = -I + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}.$$

The internal rate of return

How do we calculate the IRR?

Note that with $x = 1/(1+r)$ we can write

$$0 = -I + \sum_{t=1}^{T} C_t \left(\frac{1}{1+r}\right)^t = -I + \sum_{t=1}^{T} C_t x^t.$$

This is polynomial in x of order T. For higher values of T we need to use numerical methods.

The internal rate of return

There are potential problems with the IRR:

- There might exist more than one IRR.
- There might exist no real valued IRR.

If the initial investment I is positive and the future cash flows are greater than or equal to zero, then there exists a unique strictly positive IRR r.

This is the case for many bonds, and we will return to the IRR, known as the yield, when we study the pricing of bonds.

Returns on short positions

To have a short position in an asset means that you have sold an asset you do not own.

This is called selling short or shorting the asset.

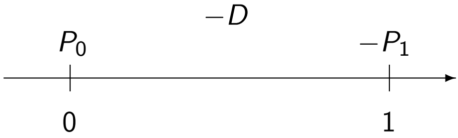
You do this if you believe that that the asset will decrease in value.

In practise this is done by borrowring the asset from someone and then sell it on the market. At some future time you buy it back on the market and return the asset to the lender.

Returns on short positions

What is the return on a short position?

Let us look at the cash flows generated by the short selling:



$$r_{\text{short}} = \frac{P_0 - P_1 - D}{P_0}.$$

Hence,

$$r_{\text{short}} = -\frac{P_1 + D - P_0}{P_0} = -r_{\text{long}}.$$

Excess returns

An excess return is the difference between two rates of return:

$$r^{\text{ex}}_{t,t+1} = r_{t,t+1} - r^{b}_{t,t+1}.$$

Here $r^{b}_{t,t+1}$ is the benchmark rate of return.

Examples of bechmark rate of returns:

- The risk-free rate of return.
- The rate of return of an index.
- The rate of return of an asset.

Excess returns

Excess returns are also known as zero net investment portfolio returns.

- We want to invest 1 unit of currency in Asset 1 by selling Asset 2 short.
- The prices today at $t = 0$ are

$$P_{10} \quad \text{and} \quad P_{20}$$

  respectively.[1]
- Today we can buy

$$\frac{1}{P_{10}} \text{ units of Asset 1,}$$

  by short selling

$$\frac{1}{P_{20}} \text{ units of Asset 2.}$$

[1] For simplicity we assume that there are no dividends, but the formula will hold also when there are dividend payments.

Excess returns

- Value today ($t = 0$):

$$-\frac{1}{P_{10}} \cdot P_{10} + \frac{1}{P_{20}} \cdot P_{20} = 0.$$

- Value of this investment when we sell it at $t = 1$:

$$\begin{aligned}
\frac{1}{P_{10}} \cdot P_{11} - \frac{1}{P_{20}} \cdot P_{21} &= \frac{P_{11}}{P_{10}} - \frac{P_{21}}{P_{20}} \\
&= \frac{P_{11}}{P_{10}} - 1 - \left(\frac{P_{21}}{P_{20}} - 1\right) \\
&= \frac{P_{11} - P_{10}}{P_{10}} - \frac{P_{21} - P_{20}}{P_{20}} \\
&= r_1 - r_2.
\end{aligned}$$

Real and nominal returns

So far we have, implicitly, considered nominal returns. These returns are measured in monetary gains.

Real returns measure gains in purchasing power.

Real returns take the inflation into account when the returns are calculated.

Returns on leveraged positions

Sometimes we take a loan to finance an investment. This is called levering up or gearing an investment.

Suppose that we have an amount E_0 and borrows the amount L_0 to get the value

$$V_0 = E_0 + L_0$$

to invest in a stock with.

The interest on the loan is r_{loan}, and the rate of return of the stock is

$$r_{\text{stock}} = \frac{P_1 - P_0 + D}{P_0},$$

where D is the dividend payment.

Returns on leveraged positions

We buy V_0/P_0 number of stocks.The value of the stocks at time 1 is

$$V_1 = \frac{V_0}{P_0}(P_1 + D) = V_0 \frac{P_1 + D}{P_0} = (E_0 + L_0)(1 + r_{\text{stock}}).$$

We need to pay back the loan with interest: $L_1 = L_0(1 + r_{\text{loan}})$. Hence, we have

$$\begin{aligned} E_1 &= V_1 - L_1 \\ &= (E_0 + L_0)(1 + r_{\text{stock}}) - L_0(1 + r_{\text{loan}}) \\ &= E_0(1 + r_{\text{stock}}) + L_0(r_{\text{stock}} - r_{\text{loan}}) \end{aligned}$$

left. The return on our own initial amount E_0 is

$$r = \frac{E_1 - E_0}{E_0}.$$

Returns on leveraged positions

Using the previous expressions we get the return

$$
\begin{aligned}
r &= \frac{\overbrace{E_0(1+r_{\text{stock}})+L_0(r_{\text{stock}}-r_{\text{loan}})}^{=E_1}-E_0}{E_0} \\
&= 1+r_{\text{stock}}+\frac{L_0}{E_0}(r_{\text{stock}}-r_{\text{loan}})-1 \\
&= r_{\text{stock}}+\frac{L_0}{E_0}(r_{\text{stock}}-r_{\text{loan}})
\end{aligned}
$$

on the leveraged position.

Here L_0/E_0 is called the leverage ratio.

Returns on portfolios

One of the most important concepts in this course is that of a portfolio.

A portfolio is a list of numbers marking how much of each asset an investor has.

Let there be N number of assets to invest in. If we have h_i, $i = 1, 2, \ldots, N$ number of assets i at time t, then the value of the portfolio at this time is

$$V_t = \sum_{i=1}^{N} h_i P_{it}.$$

The portfolio weight in asset i at time t is defined as

$$\pi_i = \frac{h_i P_{it}}{V_t}.$$

Returns on portfolios

At time $t+1$ the value has changed to

$$V_{t+1} = \sum_{i=1}^{N} h_i \big(D_{i,t+1} + P_{i,t+1} \big).$$

The rate of return on the portfolio is given by

$$\begin{aligned} r_P &= \frac{V_{t+1} - V_t}{V_t} \\ &= \frac{\sum_{i=1}^{N} h_i \big(D_{i,t+1} + P_{i,t+1} \big) - \sum_{i=1}^{N} h_i P_{it}}{V_t} \\ &= \frac{1}{V_t} \sum_{i=1}^{N} h_i \big(D_{i,t+1} + P_{i,t+1} - P_{it} \big) \end{aligned}$$

Returns on portfolios

$$
\begin{aligned}
&= \sum_{i=1}^{N} \underbrace{\frac{h_i P_{it}}{V_t}}_{=\pi_i} \cdot \underbrace{\frac{D_{i,t+1} + P_{i,t+1} - P_{it}}{P_{it}}}_{=r_i} \\
&= \sum_{i=1}^{N} \pi_i r_i.
\end{aligned}
$$

Hence, the return of the portfolio is given by

$$
r_P = \pi_1 r_1 + \pi_2 r_2 + \cdots + \pi_N r_N = \sum_{i=1}^{N} \pi_i r_i.
$$

Returns on portfolios

It can be shown that the gross return of a portfolio is given by

$$R_p = \pi_1 R_1 + \pi_2 R_2 + \cdots + \pi_N R_N = \sum_{i=1}^{N} \pi_i R_i.$$

However, in general we have

$$r_p^{\log} \neq \sum_{i=1}^{N} \pi_i r_i^{\log}.$$