Financial Theory – Lecture 16

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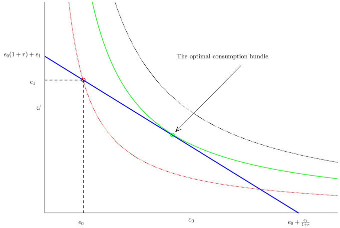
Agenda

- Summary of the course.

Choice under certainty



Financial markets

- Primary markets and secondary markets.
- Big asset classes: Stocks, bonds and derivatives.
- Alternative asset classes: Commodities, real estate infrastructure.

We also looked at the major players in financial markets.

Lecture 2

Returns

With prices P_t and P_{t+1}, and dividend payments D_{t+1}:

- The rate of return

$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}.$$

- The gross rate

$$R_{t,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = 1 + r_{t,t+1}.$$

- The log-return

$$r^{\log}_{t,t+1} = \ln R_{t,t+1} = \ln(1 + r_{t,t+1}).$$

Lecture 2

- Returns over multiple periods.
- Annualised returns.
- Ex dividend and cum dividend.
- Internal rate of return.
- Returns on short positions and excess returns.
- Real and nominal returns.
- Returns on leveraged positions.
- Returns of portfolios.

Lecture 3

Basic probability theory and formulas.

Measuring risk using:

- Standard deviation (or, equivalently, variance).
- Value-at-Risk (VaR).
- Expected shortfall (ES).

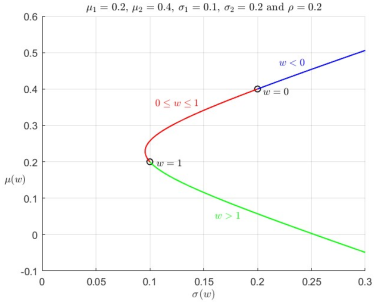
Measuring reward using the risk premium $E[r] - r_f$.

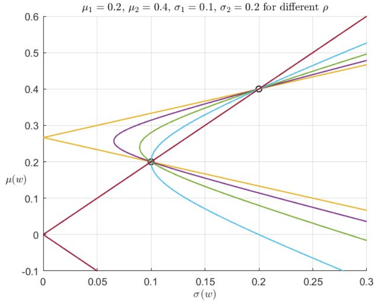
Measuring the risk-reward payoff by looking at the Sharpe ratio

$$\mathrm{SR} = \frac{E[r] - r_f}{\mathrm{Std}[r]}.$$

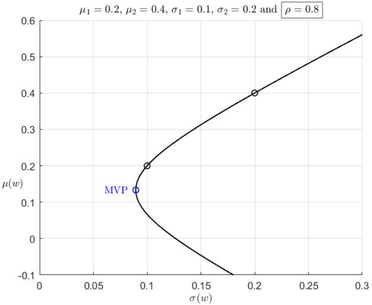
Vectors and matrices.

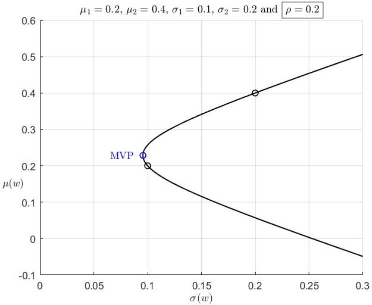
Two-asset portfolios



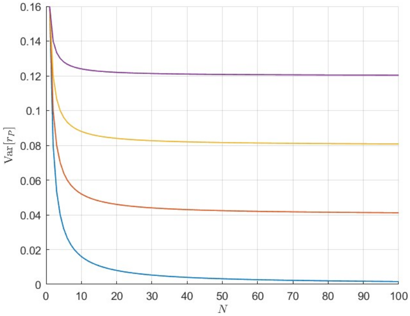


The minimum variance portfolio





- Diversification.



- Arbitrage portfolios. "Getting a lottery ticker for free."
- Replicating portfolios.
- Tracking portfolios.

Portfolio mathematics with vectors and matrices
The portfolio weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ satisfies

$$1 = \sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1}.$$

Mean and variance of the portfolio rate of return:

$$E\left[r(\boldsymbol{\pi})\right] = \sum_{i=1}^{N} \pi_i \mu_i = \boldsymbol{\pi} \cdot \boldsymbol{\mu}.$$

$$\mathrm{Var}[r(\boldsymbol{\pi})] = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \mathrm{Cov}[r_i, r_j] = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij} = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

Mean-variance analysis with only risky assets

We found the portfolio that for a given expected rate of return $\bar{\mu}$ has the least variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \overline{\mu}. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \overline{\mu}. \end{aligned}$$

By using the Lagrange multiplier method we found that the optimal weights are

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D} \Sigma^{-1} \mathbf{1} + \frac{C\bar{\mu} - B}{D} \Sigma^{-1} \boldsymbol{\mu}$$

with

$$A = \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu}, \quad B = \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} = \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu}, \quad C = \mathbf{1} \cdot \Sigma^{-1} \mathbf{1}$$

and $D = AC - B^2$.

The mean-variance frontier, or the portfolio frontier, is given by

$$\sigma(\bar{\mu}) = \sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}.$$

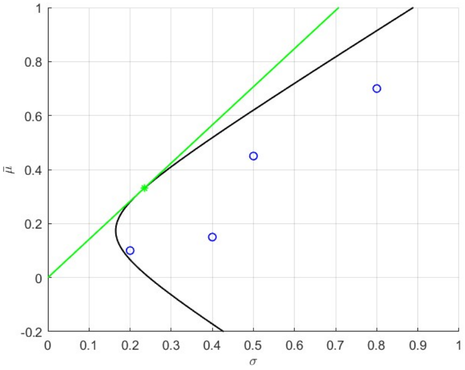
We have two-fund separation:

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D} C\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D} B\boldsymbol{\pi}_{\text{slope}},$$

where $\boldsymbol{\pi}_{\text{min}}$ solves

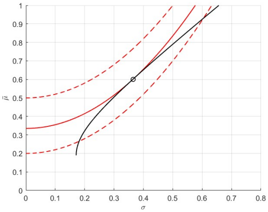
$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \text{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \sum\limits_{i=1}^{N} \pi_i = 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1. \end{array}$$

The portfolio weights $\boldsymbol{\pi}_{\text{slope}}$ are of the frontier portfolio with the largest slope of the line $\bar{\mu} = k\sigma$ in the $(\bar{\mu}, \sigma)$-plane for some k.



Portfolio choice

- Choosing the optimal portfolio.



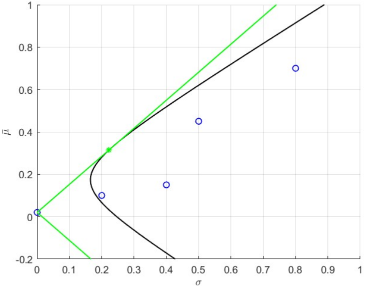
- Measuring the attitude towards risk using the absolute risk aversion,

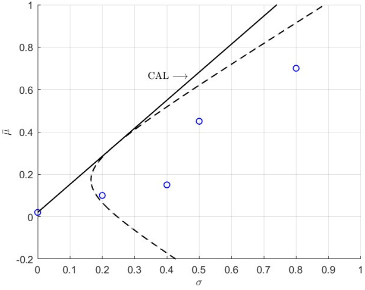
$$ARA(x) = -\frac{u''(x)}{u'(x)},$$

or the relative risk aversion,

$$RRA(x) = -\frac{xu''(x)}{u'(x)}.$$

Mean-variance analysis with risky assets and a risk-free asset
In this case we get the mean-variance frontier and the capital allocation line (CAL).

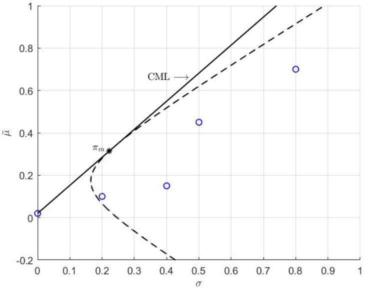




When the market is in equilibrium, the market portfolio is equal to the tangent portfolio:

$$\pi_{\text{mkt}} = \pi_{\text{tan}}.$$

This results in the capital market line.



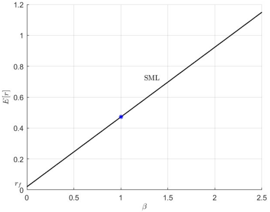
The Capital Asset Pricing Model (CAPM)
We can also derive the CAPM equation

$$E[r_i] = r_f + \beta_i \big(E[r_M] - r_f \big),$$

where the beta value is

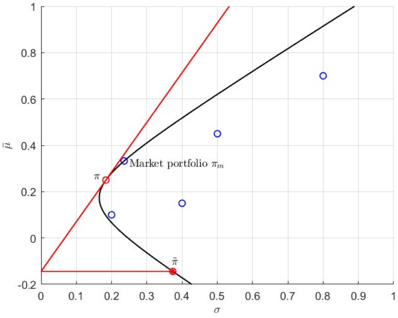
$$\beta_i = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}.$$

The security market line (SML).



Lecture 7

The CAPM without a risk-free asset – use the market portfolio's "zero-beta portfolio".



We also looked at optimal portfolios with CARA utility and normally distributed returns.

Consumption-based CAPM

The problem

$$\max_{\mathbf{x}} \left\{ u(c_0) + e^{-\delta} E\left[u(c_1)\right] \right)$$

$$\text{s.t. } c_0 + \sum_{i=1}^{N} x_i P_{i0} = e_0$$

$$c_1 = e_1 + \sum_{i=1}^{N} x_i (D_i + P_{i1})$$

leads to the consumption-based CAPM equation

$$E\left[r_i\right] = r_f + \mathrm{Cov}[r_i, -(1 + r_f)m],$$

where

$$m = e^{-\delta} \frac{u'(c_1)}{u'(c_0)}$$

is the stochastic discount factor (SDF).

Factor models

Models with one factor:

$$r_i = E\left[r_i\right] + \beta_i \big(F - E\left[F\right] \big) + \varepsilon_i, \; i = 1, 2, \ldots, N,$$

and where for every $i, j = 1, 2, \ldots, N$

(i) $\text{Cov}[F, \varepsilon_i] = 0$.

(ii) $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0$ when $i \neq j$.

One special case is when $F = r_m$. This is the Single-Index Model or the Market Model.

Models with K factors: For $i = 1, 2, \ldots, N$:

$$r_i = E\left[r_i\right] + \beta_{i1}\big(F_1 - E\left[F_1\right]\big) + \ldots + \beta_{iK}\big(F_K - E\left[F_K\right]\big) + \varepsilon_i,$$

where for every i and k

$$\mathrm{Cov}[F_k, \varepsilon_i] = 0 \quad \text{and} \quad \mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0 \text{ when } i \neq j.$$

Arbitrage pricing theory (APT)
If there are no arbitrage opportunities, then there exists factor risk premia RP_k such that

$$E\left[r_i\right] = \mathrm{RP}_0 + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

If there is a risk-free rate, then $\mathrm{RP}_0 = r_f$.

- The Fama-French model.
- The Fama-French-Carhart model.
- The Factor zoo.

Lecture 9

Looking at data:

- Bid and ask price.
- Closing, opening, high and low.
- How to estimate the risk premia.
- Why it is harder to correctly estimate the mean than the standard deviation of asset returns.

Efficiency

- Weak-form.
- Semistrong-form.
- Strong-form.

Markets are often "efficiently inefficient".

Behavioural finance and economics

- Prospect theory.
- Behavioural biases:
    - Anchoring.
    - Loss aversion or Disposition effect.
    - Overconfidence.
    - Mental accounting.
    - Framing.
- Behavioural game theory.

Stock valuation

The price ex post

$$P_t = \frac{P_{t+1} + D_{t+1}}{1 + r_{t+1}},$$

and the price ex ante

$$P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1 + r_{t+1}}\right].$$

If $E_t\left[D_{t+1}\right] = (1+g)D_t$ and $E_t\left[r_{t+1}\right] = r$, then (under the assumption of no rational bubble)

$$P_t = \frac{E_t\left[D_{t+1}\right]}{r-g} = \frac{(1+g)D_t}{r-g}$$

as long as $r > g$. This is Gordon's formula.

A firm's dividend policy

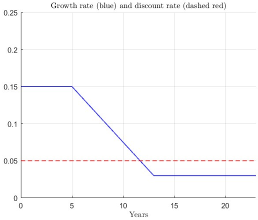
- The firm's earnings e_t are invested or paid out as dividends: $e_t = I_t + D_t$.
- The return on equity, or return on investments, r_e: $e_{t+1} = e_t + r_e I_t$.
- The plowback ratio b: $I_t = be_t$.
- Growth rate in dividends: $g = br_e$.
- Price-earnings (P/E) ratio:

$$\frac{P_t}{e_t} = \frac{(1 + br_e)(1 - b)}{r - br_e}.$$

- Present value of growth opportunities (PVGO) O_t: $O_t = P_t - e_t/r$.

More on stock valuation

- Two- and three period growth models.



- Valuation using free cash flows:

$$V_t = \sum_{j=1}^{\infty} \frac{E_t\left[\mathrm{FCF}_{t+j}\right]}{\left(1 + r_{\mathrm{firm}}\right)^j},$$

where r_{firm} is the weighted average cost of capital (WACC).

Bonds

Tradable loan contracts.

$$\begin{aligned} M_i &= \text{The total payment at time } i. \\ I_i &= \text{The interest payment at time } i. \\ X_i &= \text{The repayment of debt at time } i. \\ F_i &= \text{The outstanding debt at time } i \\ &\quad \text{after the repayment of debt has been made.} \end{aligned}$$

Then for $i = 1, 2, \ldots, n$

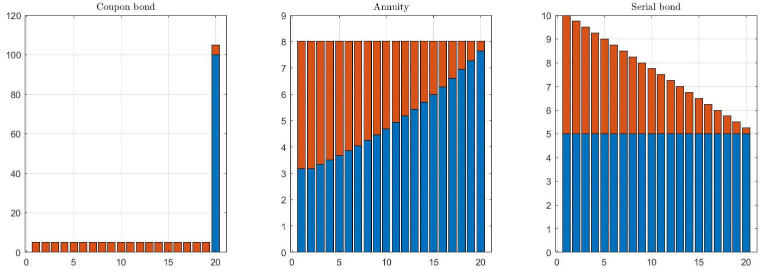
$$\begin{aligned} M_i &= I_i + X_i \\ I_i &= qF_{i-1} \\ F_i &= F_{i-1} - X_i. \end{aligned}$$

It also holds that

$$F_0 = F = \text{The face value, and } F_n = 0.$$

Lecture 11

- Coupon bonds.
- Annuities.
- Serial bonds.



Coupon bonds price

$$\begin{aligned} B_t &= \sum_{i=t+1}^{n} \frac{qF}{(1+r)^{i-t}} + \frac{F}{(1+r)^{n-t}} \\ &= \frac{qF}{r}\left(1 - \frac{1}{(1+r)^{n-t}}\right) + \frac{F}{(1+r)^{n-t}}. \end{aligned}$$

Special case: Zero coupon bonds (ZCB's) with price

$$Z_{t,n} = \frac{F}{(1+r)^{n-t}} = F(1+r)^{-(n-t)}.$$

- Accrued interest, clean price and dirty price.
- Trading at par, premium or discount.

Yield-to-maturity (YTM) or just yield

The internal rate of return (IRR) of a bond:

$$B_0^{\text{mkt}} = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i}.$$

$$y \uparrow \;\Leftrightarrow\; B_0(y) \downarrow \quad \text{and} \quad y \downarrow \;\Leftrightarrow\; B_0(y) \uparrow .$$

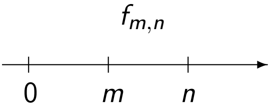
For ZCB's:

$$Z_{0,n}^{\text{mkt}} = \frac{F}{(1+y_n)^n} \;\Leftrightarrow\; y_n = \left(\frac{F}{Z_{0,n}^{\text{mkt}}}\right)^{1/n} - 1.$$

The difference between the yield and the actual return of a bond.

Lecture 11

Forward rates



The forward rate for the period from m to n, denoted $f_{m,n}$, satisfies

$$(1+y_n)^n = (1+y_m)^m (1+f_{m,n})^{n-m},$$

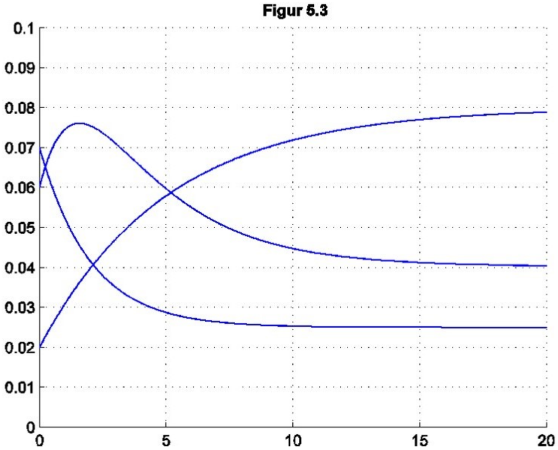
or

$$f_{m,n} = \left(\frac{(1+y_n)^n}{(1+y_m)^m}\right)^{1/(n-m)} - 1.$$

Defaultble bonds

- Bonds for which the issuer of the bond can not (or will not) make coupon and/or amortisation payments.
- Bonds issued by stable states are generally considered non-defaultable, i.e. there is no risk of their bonds to default.
- There are rating institutes rating the quality of a firm based on the probability

Yield curves



Explanations for the form of the yield curve (new and old).

- Use yield curves from ZCB's to discount.
- Bootstrapping.
- The duration of a bond:

$$D_0 = \sum_{i=1}^{n} i \cdot \frac{\frac{M_i}{(1+y)^i}}{B_0} = \sum_{t=1}^{n} i\, w_i \quad \text{and} \quad \frac{\Delta B_0}{B_0} \approx -\frac{D_0}{1+y}\Delta y.$$

- The modified duration:

$$D_0^* = \frac{1}{1+y} D_0.$$

- Convexity and modified convexity.
- Fisher-Weil duration.
- Equity duration.

Immunisation
Buy a portfolio whose value and duration matches that of the liability we have.

With two bonds having prices B_1 and B_2, and durations D_1 and D_2 this leads to the system of equations

$$\begin{cases} N_1 B_1 + N_2 B_2 & = \bar{B} \\ N_1 B_1 D_1 + N_2 B_2 D_2 & = \bar{B}\bar{D} \end{cases}$$

for N_1 and N_2 (the number of bonds 1 and 2 respectively to buy).

Lecture 13

Asset allocation

- The investment process.
- The Treynor-Black model: Taking advantage of mispriced assets.
- The Black-Litterman model: Incorporating your personal views and using CAPM to estimate the mean vector of rates of return.

Lecture 13

Household finance

Using the portfolio investment framework to also include other sources of cash flows.

- Labour income: Total wealth = Financial wealth + Human capital, where
$$\text{Human capital} = \text{PV(Future labour income)}.$$
- Housing: As an additional source of investment we can consider housing. Natural constraints are that households do not shortsell financial assets and they have mortgage limitations.

Empirical aspects
Mainly given you an orientation about what types of econometric questions you might want to ask, and what techniques that are available.

Lecture 15

Macro-finance

The standard model:

- Additive and time-separable individuals.
- CRRA utility funtion: $u'(x) = x^{-\gamma}$, $\gamma \geq 0$.
- Growth rate in consumption is normally distributed.

Extensions:

- Habit formation: Internal and external.
- Recursive utility: Epstein-Zin.
- Heterogeneous preferences: Different investors have different risk aversion.
- Ambiguity aversion: Uncertainty about which are the correct probabilites to use.

We also looked at ESG investing.