4.4 Martingales

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration ("flow of information") (formally $\mathcal{F}_t$ is a σ-algebra of events, and $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$). If Y is a random variable, then $E[Y | \mathcal{F}_t]$ denotes the conditional expectation of Y given $\mathcal{F}_t$, i.e. the expected value of Y given all information up to t.

Ex: $E[W_t | \mathcal{F}_s] = E[W_s | \mathcal{F}_s] + E[W_t - W_s | \mathcal{F}_s] = W_s$

Def 4.7 X_t is an $\mathcal{F}_t$-martingale if

- X_t is $\mathcal{F}_t$-adapted
- $E[|X_t|] < \infty$ for all $t \geq 0$
- $E[X_t | \mathcal{F}_s] = X_s$ for $s \leq t$.

Ex: 1. Brownian motion W is a martingale.

2. $Y_t := W_t^2 - t$ is a martingale:

$$E[Y_t | \mathcal{F}_s] = E[W_t^2 - t | \mathcal{F}_s] = E[(W_t - W_s)^2 + 2W_sW_t - W_s^2 | \mathcal{F}_s] - t$$

$$= t - s + 2E[W_sW_t | \mathcal{F}_s] - E[W_s^2 | \mathcal{F}_s] - t$$

$$= 2W_s \underbrace{E[W_t | \mathcal{F}_s]}_{W_s} - W_s^2 - s$$

$$= W_s^2 - s = Y_s$$

3. $Y_t = \int_0^t g_u \, dW_u$ is a martingale:

$$E[Y_t \mid \mathcal{F}_s] = E\left[\int_0^s g_u \, dW_u \mid \mathcal{F}_s\right] + \underbrace{E\left[\int_s^t g_u \, dW_u \mid \mathcal{F}_s\right]}_{0}$$

$$= \int_0^s g_u \, dW_u = Y_s$$

4. W_t^3 is <u>not</u> a martingale:

$$E[W_t^3 \mid \mathcal{F}_s] = E\left[W_s^3 + (W_t - W_s)^3 - 3W_t W_s^2 + 3W_t^2 W_s \mid \mathcal{F}_s\right]$$

$$= W_s^3 + 0 - 3W_s^2 \underbrace{E[W_t \mid \mathcal{F}_s]}_{W_s} + 3W_s \underbrace{E[W_t^2 \mid \mathcal{F}_s]}_{t-s+W_s^2}$$

$$= W_s^3 + 3(t-s)W_s \neq W_s^3.$$

<u>Remark</u> A martingale is a "fair game".

<u>Ito's formula</u>

Assume $X_t = a + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s$

for some adapted processes μ_t and σ_t.

Short-hand notation: $\begin{cases} dX_t = \mu_t \, dt + \sigma_t \, dW_t \\ X_0 = a \end{cases}$

Let $f(t,x)$ be a $C^{1,2}$-function and define

$$Z_t := f(t, X_t).$$

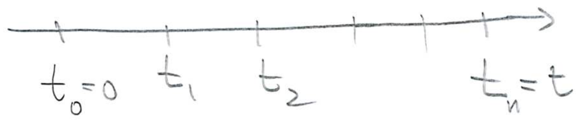
<u>Question</u>: What does dZ_t look like?

Recall: $\displaystyle\int_0^t W_s\, dW_s = \frac{W_t^2}{2} - \frac{t}{2}$

so $\displaystyle W_t^2 = t + 2\int_0^t W_s\, dW_s .$

Thus $d(W_t^2) = dt + 2W_t\, dW_t$

Fix n and let $t_k = \frac{k}{n} t$



Let $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$ and consider

$$S_n := \sum_{k=0}^{n-1} (\Delta W_{t_k})^2 .$$

We have

$$E[S_n] = \sum_{k=0}^{n-1} E\left[(\Delta W_{t_k})^2\right] = \sum_{k=0}^{n-1} \frac{t}{n} = t$$

and

$$Var(S_n) \underset{\substack{\uparrow \\ \text{indep.}}}{=} \sum_{k=0}^{n-1} Var\left((\Delta W_{t_k})^2\right) = n \cdot Var\left((\Delta W_{t_0})^2\right)$$

$$= n \cdot 2 \frac{t^2}{n^2} \longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

Thus $S_n \to t$ as $n \to \infty$ (in L^2). This motivates to write $\displaystyle\int_0^t (dW_s)^2 = t$, or

$$\boxed{dW_t^2 = dt} .$$

Taylor expansion:

$$dZ_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{\partial^2 f}{\partial t^2}(dt)^2$$

$$+ \frac{\partial^2 f}{\partial t \partial x} dt\, dX_t + \text{higher order terms}$$

$$= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW + \text{higher order terms}$$

Thm 4.11 (Ito's formula) If $dX_t = \mu_t dt + \sigma_t dW_t$, and $Z_t := f(t, X_t)$, then

$$dZ_t = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t.$$

(here $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(t, X_t)$ and similarly for other derivatives of f).

Alternative formulation:

$$dZ_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2,$$

where $(dX_t)^2$ is calculated using
$(dt)^2 = 0$
$dt\, dW_t = 0$
$(dW_t)^2 = dt$.

Ex 4.15: Compute $\int_0^t W_s\, dW_s$ (again).

Solution: Let $Z_t = W_t^2$. By Ito's formula,

$$dZ_t = 2W_t\, dW_t + \frac{1}{2}\cdot 2(dW_t)^2$$

$$= dt + 2W_t\, dW_t.$$

Thus $W_t^2 = Z_t = t + 2\int_0^t W_s\, dW_s$ so $\boxed{\int_0^t W_s\, dW_s = \frac{W_t^2}{2} - \frac{t}{2}}$.

Ex 4.16 Compute $E[W_t^4]$.

Solution: Let $Z_t = W_t^4$. By Ito's formula,

$$dZ_t = 4W_t^3\,dW_t + \tfrac{1}{2}12W_t^2(dW_t)^2$$
$$= 6W_t^2\,dt + 4W_t^3\,dW_t$$

Thus

$$W_t^4 = Z_t = 6\int_0^t W_s^2\,ds + 4\int_0^t W_s^3\,dW_s.$$

Taking expectations gives

$$E[W_t^4] = 6\int_0^t \underbrace{E[W_s^2]}_{s}\,ds + 4\underbrace{E\left[\int_0^t W_s^3\,dW_s\right]}_{0}$$
$$= 6\int_0^t s\,ds = 3t^2.$$

Alternatively, without using Ito's formula,

$$E[W_t^4] = \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx = \left\{\begin{matrix}\text{integration}\\ \text{by parts}\end{matrix}\right\}$$
$$= \underbrace{\left[x^3 \frac{t}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\right]_{-\infty}^{\infty}}_{0} + \int_{\mathbb{R}} 3x^2 \frac{t}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx$$
$$= 3t\,\mathrm{Var}(W_t) = 3t^2.$$

Ex 4.17 Compute $E[e^{\alpha W_t}]$.

Solution: Let $Z_t = e^{\alpha W_t}$. Ito's formula gives

$$dZ_t = \alpha e^{\alpha W_t} dW_t + \frac{1}{2}\alpha^2 e^{\alpha W_t}(dW_t)^2$$
$$= \frac{\alpha^2}{2} e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t$$
$$= \frac{\alpha^2}{2} Z_t\, dt + \alpha Z_t\, dW_t$$

Integrating gives

$$Z_t = 1 + \frac{\alpha^2}{2}\int_0^t Z_s\, ds + \alpha \int_0^t Z_s\, dW_s$$

so

$$E[Z_t] = 1 + E\left[\frac{\alpha^2}{2}\int_0^t Z_s\, ds\right] + \underbrace{E\left[\alpha \int_0^t Z_s\, dW_s\right]}_{0}$$
$$= 1 + \frac{\alpha^2}{2}\int_0^t E[Z_s]\, ds$$

Denote $m(t) := E[Z_t]$. Then

$$\begin{cases} \dot{m}(t) = \frac{\alpha^2}{2} m(t) \\ m(0) = 1 \end{cases}$$

which has the solution $m(t) = e^{\frac{\alpha^2}{2}t}$

Answer: $E[e^{\alpha W_t}] = e^{\frac{\alpha^2}{2}t}$