UPPSALA UNIVERSITY
Department of Mathematics
Stephan Wagner

Probability and Martingales
1MS045
12 January 2023

Each problem counts 5 points. Grades are awarded according to the following scale: 0–17 grade U; 18–24 grade 3; 25–31 grade 4; 32–40 grade 5. Allowed tools: pen, paper, calculator. All solutions should be clearly explained.

Note: if not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

1. (a) State the first and the second *Borel–Cantelli lemma.* (2)

   (b) For every $n \geq 1$, let X_n, Y_n, Z_n be three integers in $\{1, 2, \ldots, n\}$, each taken uniformly at random. These random variables are all mutually independent. Prove: with probability 1, $X_n = Y_n$ occurs for infinitely many n, but $X_n = Y_n = Z_n$ only occurs for finitely many n. (3)

2. Let $X_1, X_2, \ldots$ be independent random variables with $P(X_i = -1) = P(X_i = 1) = \frac{1}{2}$ for all i. Use the Markov inequality to prove: if $c < 2$, then the probability that there are n identical consecutive values $X_i = X_{i+1} = \cdots = X_{i+n-1}$ somewhere among the first $N = \lfloor c^n \rfloor$ values $X_1, X_2, \ldots, X_N$ goes to 0 as $n \to \infty$. Show also that this probability tends to 1 if $c > 2$. (Note: $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.) (5)

3. State and prove *Kolmogorov's 0-1-law.* (5)

4. Let $X_1, X_2, \ldots$ be independent random variables with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$ for all i, and set $S_n = X_1 + X_2 + \cdots + X_n$ (with $S_0 = 0$).

   (a) Define *stopping time.* Which of the following is a stopping time with respect to the natural filtration? (2)

   - $\tau_1 = \inf\{n : S_n = 10\}$,
   - $\tau_2 = \sup\{n : S_n = 10\}$,
   - $\tau_3 = 1 + \sup\{n : S_n = n\}$.

   (b) State the *optional stopping theorem.* (1)

   (c) Show that $S_n^2 - n$ is a martingale. Use the optional stopping theorem to determine the expected value of $\tau = \inf\{n : |S_n| = 10\}$. (2)

5. A sequence of random variables $X_0, X_1, \ldots$ is defined recursively. We set $X_0 = 1$, and for $n > 0$, X_n is a uniformly random number in the interval $[X_{n-1}, 2X_{n-1}]$.

   (a) For which values of $c > 0$ is $c^n X_n$ a martingale, supermartingale or submartingale respectively (with respect to the natural filtration)? (2)

   (b) Prove that there is a constant c_0 such that the following holds: if $c > c_0$, then $c^n X_n \to \infty$ almost surely as $n \to \infty$, and if $0 < c < c_0$, then $c^n X_n \to 0$ almost surely as $n \to \infty$. Determine this constant. (3)

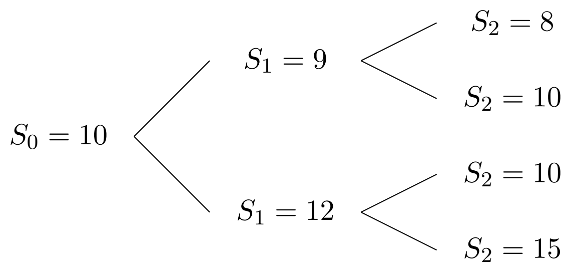
6. (a) Define *upcrossings*, and state *Doob's upcrossing lemma.* (2)

   (b) State and prove *Doob's convergence theorem.* You may use the upcrossing lemma without proof. (3)

7. (a) State the *first fundamental theorem of asset pricing* (for finite market models). (1)

   (b) State the *second fundamental theorem of asset pricing* (for finite market models). (1)

   (c) Consider a simple market model with a single asset S_t and two time steps as indicated in the following diagram: (3)



   The riskless bond is assumed to be $S_t^0 = 1$. Determine all equivalent martingale measures. Is this model viable? Is it complete?

8. Explain how the *Black–Scholes model* can be obtained as a limit of the *binomial model.* In particular, describe the steps leading to the following formula for the fair price of a European call option under the Black–Scholes model: (5)

$$\int_{-\infty}^{\infty} \left(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x} - e^{-rT}K\right)^+ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\, dx.$$

   Explain the meaning of all quantities involved in this formula (S_0, σ, T, r, K).