UPPSALA UNIVERSITET

INTRODUCTION TO PDE

Lecture Notes

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Contents

1. Introduction

Definition/Sats 1.1: Domain

An open connected set $\Omega \subset \mathbb{R}^n$ is called a *domain*

Definition/Sats 1.2: Bounded domain

A domain is *bounded* if its closure $\overline{\Omega}$ is compact

Definition/Sats 1.3: Smooth boundary

The boundary of a set (denoted $\partial\Omega$) is called *smooth* if it can be locally represented by a smooth function

1.1. **Examples of linear operators.**

- Gradient of $u \in C^1(\Omega)$, i.e $\nabla u = (u_{x_1}, \cdots, u_{x_n})$
- Laplacian of $u \in C^2(\Omega)$, i.e $\Delta u = u_{x_1x_1} + \cdots u_{x_nx_n}$
- Divergence of vector field

Definition/Sats 1.4: Classification of PDEs

A PDE is said to be **Quasilinear**: $\sum_{k,l=1}^n a^{kl}(x, u(x), Du(x))u_{x_kx_l} + b(x, u(x), Du(x)) = 0$

A PDE is said to be **Semilinear**: $\underbrace{\sum_{k,l=1}^n a^{kl}(x)u_{x_kx_l}}_{\text{principal term}} + b(x, u(x), Du(x)) = 0$

A PDE is said to be **linear**: $\sum_{k,l=1}^n a^{kl}(x)u_{x_kx_l} + \sum_{l=1}^n b^l(x)u_{x_l} + c(x)u(x) = f(x)$

Example: Heat equation

$$u_t - \Delta u = f \Leftrightarrow (\partial_t - \Delta)u$$

Take domain $\Omega \in \mathbb{R}^3$ and outward pointing unit $\overrightarrow{n}$
Let $u = u(x, y, z, t)$ be temperature at point $\overline{x} = (x, y, z)$ at time t, and let $q = (x, y, z, t, u)$ be the function describing the heat problem.
Heat production is dependent on temperature.
Let $\overline{Q} = \overline{Q}(x, y, z, t)$ be a vector field representing heat flux through $\partial\Omega$
The temperature change from t to $t + \Delta t$ corresponds to the flux in heat production as follows:

$$\int_\Omega (u(x, y, z, t + \Delta t) - u(x, y, z, t))dV = \int_t^{t+\Delta t} \int_\Omega q(x, y, z, t, u)dV dt - \int_t^{t+\Delta t} \int_{\partial\Omega} Q(x, y, z, t)\overrightarrow{n} dSdt$$

Divide both sides by Δt and take the limit as $\Delta t \to 0$ yields:

$$\int_\Omega u_t(x, y, z, t)dV = \int_\Omega q(x, y, z, t, u)dV - \int_{\partial\Omega} \overline{Q}(x, y, z, t)\overrightarrow{n} dS$$

We expect that in practice $\overline{Q} = -a\nabla u$ for $a > 0$
Note that the last term becomes

$$\int_{\partial\Omega} \overline{Q}(x, y, z, t)\overrightarrow{n} dS = \int_{\partial\Omega} -a\nabla u\overrightarrow{n} dS = -a \int_\Omega \nabla \bullet \nabla u dV$$

Move everything ot one side and integrating under same domain:

$$\int_\Omega (u_t(x, y, z, t) - q(x, y, z, t, y) - a\nabla \bullet \nabla u)dV = 0$$

This is precisely the heat equation.

The further study of the equation involves introducing/imposing further conditions:

- *Initial conditions*: At $t = 0$, $u(x, y, z, t) = \varphi(x, y, z)$
- *Dirichlet data*: Prescribes behaviour of boundary independent of time, i.e $u(x, y, z, t) = \psi(x, y, z) \quad \forall (x, y, z) \in \partial\Omega$
- *Neumann Conditions*: Prescribes heat production with boundary independent of time derivative: $u_{\overrightarrow{n}}(x, y, z, t) = \psi(x, y, z)$

There are other types, for example Robin conditions.
They can be mixed, and solutions can be determined if for example the combinations include stability etc.

1.2. **Other PDEs.**

- *Poisson equations*: $\Delta u = f$, if $f = 0$ then this is the Laplace equation (which has harmonic functions)
- *Wave equations*: $\Delta u - u_{tt}$
- *Minimal surface equation*: $\nabla \left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$
- Eikonal equation $|\nabla u| = 1$