Lc 10

Goursat's argument (Sketch)

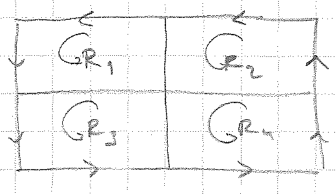
Thm (Goursat)

Let R be a rectangle, and let f be analytic on R.

Then,

$$\int_{\partial R} f(z)\,dz = 0$$

Proof: Decompose R into four sub-rectangles by bisecting the sides:



Then, $\displaystyle\int_{\partial R} f\,dz = \sum_{j=1}^{4} \int_{\partial R_j} f\,dz$

$$\Rightarrow \left|\int_{\partial R} f\,dz\right| \le \sum_{j=1}^{4} \left|\int_{\partial R_j} f\,dz\right|$$

So there is some rectangle $R^{(1)}$ among R_1, R_2, R_3, R_4 s.t.

$$\left|\int_{\partial R^{(1)}} f\,dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f\,dz\right|$$

Next decompose $R^{(1)}$ into four sub-rectangles by bisecting the sides. Similarly one of these, say $R^{(2)}$, satisfies

$$\left|\int_{\partial R^{(2)}} f\,dz\right| \geq \frac{1}{4}\left|\int_{R^{(1)}} f\,dz\right| \geq \frac{1}{4^2}\left|\int_{\partial R} f\,dz\right|$$

We continue to obtain a seq. of rectangles

$$R^{(1)} \supseteq R^{(2)} \supseteq R^{(3)} \supseteq \cdots$$

Clearly, $\displaystyle\left|\int_{\partial R^{(n)}} f\,dz\right| \geq \frac{1}{4^n}\left|\int_{\partial R} f\,dz\right|.$

Let L be the length of ∂R and L_n the length of $\partial R^{(n)}$. Clearly then

$$L_n = \frac{1}{2^n} L$$

It is not difficult to show that $\bigcap_{n=1}^{\infty} R^{(n)}$ consists of a single point z_0. Since f is differentiable at z_0,

$$\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \longrightarrow 0 \quad , \quad z \to z_0$$

Let $\varepsilon > 0$ be given. Then $\exists\ \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| < \varepsilon ,$$

i.e.

$$\left| f(z) - f(z_0) - f'(z_0)(z-z_0) \right| \leq \varepsilon |z - z_0| , \quad |z-z_0| < \delta .$$

Choose n so large that $R^{(n)}$ belongs to the disk $|z - z_0| < \delta$. Then

$$\left| \int_{\partial R} f(z)\, dz \right| \leq 4^n \left| \int_{\partial R^{(n)}} f(z)\, dz \right| =$$

$$= 4^n \left| \int_{\partial R^{(n)}} \underbrace{\Big(f(z) - \overset{\to 0}{f(z_0)} - f'(z_0)\overset{\to 0}{(z-z_0)} \Big)}_{|\ \ | \leq \varepsilon |z-z_0| \leq \varepsilon \operatorname{diam} R^{(n)} = \varepsilon \cdot \frac{1}{2^n} \operatorname{diam} R} dz \right| \leq$$

ML-ineq

$$\leq 4^n \cdot \varepsilon \frac{1}{2^n} \operatorname{diam} R \cdot \frac{1}{2^n} L = L \cdot \operatorname{diam} R \cdot \varepsilon ,$$

where $\operatorname{diam} R$ is the length of the diagonal of R.

True for any $\varepsilon > 0 \Rightarrow \int_{\partial R} f\, dz = 0$ ▢

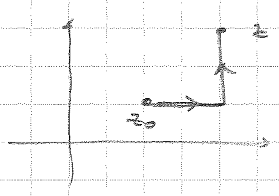
Thm Let D be an open disk centered at z_0. Let f be continuous in D, and assume that for each rectangle R contained in D we have

$$\int_{\partial R} f(z)\,dz = 0.$$

For any point $z \in D$, define

$$F(z) = \int_{\Gamma_z} f(s)\,ds$$

where Γ_z is the contour below:

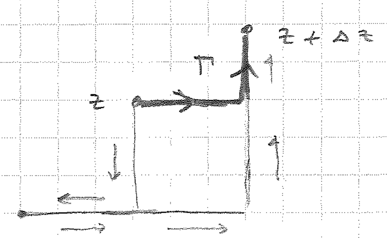


Then, $F'(z) = f(z)$.

Proof: As in the proof of the path indep. theorem we have that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z}\int_{\Gamma} \big(f(s) - f(z)\big)\,ds,$$

where Γ is the contour shown below:



By the ML-ineq.

$$\left|\frac{1}{\Delta z}\int_{\Gamma}\big(f(s)-f(z)\big)\,ds\right| \le \frac{1}{|\Delta z|}\max_{s\in\Gamma}|f(s)-f(z)|\cdot\underbrace{(|\Delta x|+|\Delta y|)}_{\le 2|\Delta z|}$$

$$\le 2\max_{s\in\Gamma}|f(s)-f(z)| \to 0 \text{ by contn}$$

So, $F'(z) = f(z)$.

□

Combining the two theorems above with the indep. of path theorem, we get the following local result:

<u>Thm</u> Let D be an open disk and suppose that f is analytic in D. Then f has an antiderivative in D, contour integrals are indep. of path, and integrals over closed contours are 0.

<u>Homotopy</u> Let D be a domain, $I = [0,1]$.

<u>Def 1</u> Suppose that $\gamma_0, \gamma_1 : I \to D$ are continuous and that $\gamma_0(0) = \gamma_1(0) = z_0$, $\gamma_0(1) = \gamma_1(1) = z_1$.

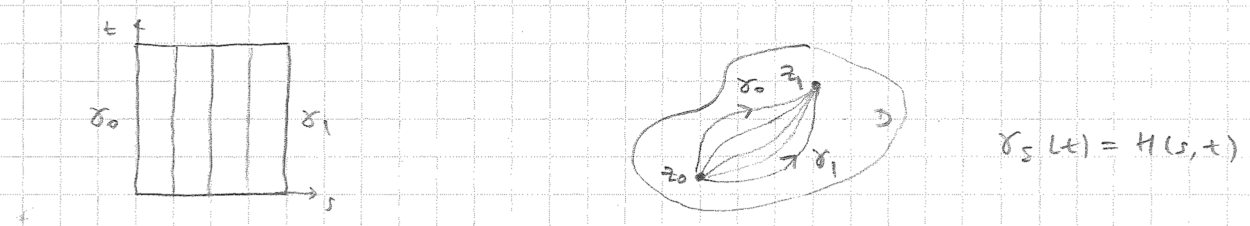
We say that <u>γ_0 is homotopic to γ_1 with endpoints fixed in D</u> if there is a continuous mapping $H : I \times I \to D$ s.t.

(i) $H(0,t) = \gamma_0(t) \quad \forall t \in I$

(ii) $H(1,t) = \gamma_1(t) \quad \forall t \in I$

(iii) $H(s,0) = z_0$, $H(s,1) = z_1 \quad \forall s \in I$

See figure:



<u>Def 2</u> Suppose that $\gamma_0, \gamma_1 : I \to D$ are continuous and that $\gamma_0(0) = \gamma_0(1) = z_0$, $\gamma_1(0) = \gamma_1(1) = z_1$ (closed curves)

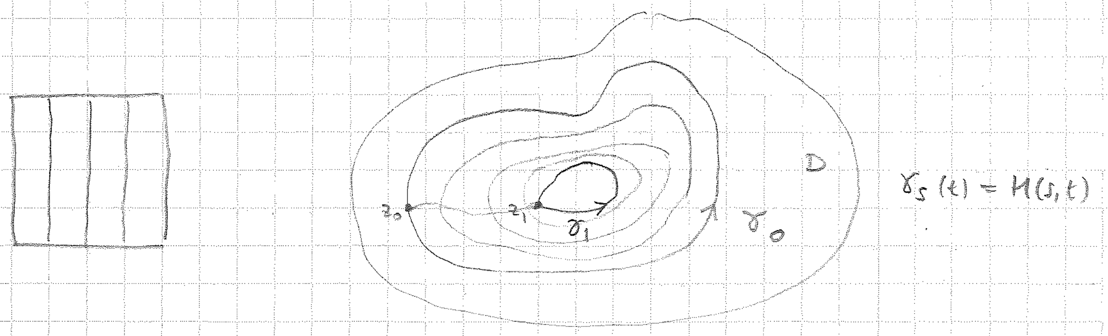
We say that <u>γ_0 and γ_1 are homotopic as closed curves in D</u> if there is a continuous mapping $H : I \times I \to D$ s.t.

(i) $H(0,t) = \gamma_0(t) \quad \forall t \in I$

(ii) $H(1,t) = \gamma_1(t) \quad \forall t \in I$

(iii) $H(s,0) = H(s,1) \quad \forall s \in I$

See figure:



<u>Def</u> A domain D is called <u>simply connected</u> if every closed curve in D is homotopic to a point (= constant closed curve) in D.

<u>Thm</u> (Deformation thm)

Suppose that f is analytic in a domain D.

(i) If Γ_0 and Γ_1 are contours from z_0 to z_1 which are homotopic with endpoints fixed in D, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz$$

(ii) If Γ_0 and Γ_1 are closed contours which are homotopic as closed curves in D, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz$$

Proof: $H: I \times I \to D$ homotopy from Γ_0 to Γ_1

$H(I \times I)$ is a compact subset of D, open.

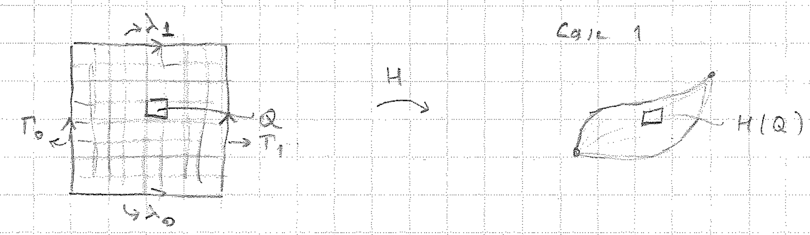
$\Rightarrow \exists\ \varepsilon > 0$ s.t. $\forall\ z \in H(I \times I)$ the disk $D_\varepsilon(z) \subset D$.

H is uniformly continuous $\Rightarrow \exists\ \delta > 0$ s.t.

$$|(s,t) - (\tilde{s},\tilde{t})| < \delta \Rightarrow |H(s,t) - H(\tilde{s},\tilde{t})| < \varepsilon.$$

Subdivide $I \times I$ into subsquares Q with diam $Q < \delta$.

See figure:



For every subsquare Q, $H(Q)$ belongs to a disk

contained in D. Orient each ∂Q counterclockwise,

and $H(\partial Q)$ accordingly. By Cauchy's integral thm for a disk

$$\Rightarrow \int_{H(\partial Q)} f(z)\,dz = 0 \quad \forall\ Q.$$

$$\Rightarrow 0 = \sum_Q \int_{H(\partial Q)} f(z)\,dz = \int_{H(\partial(I \times I))} f(z)\,dz$$

$$= \int_{\lambda_0} + \int_{\Gamma_1} - \int_{\lambda_1} - \int_{\Gamma_0}$$

(i) λ_0, λ_1 constant $\Rightarrow \int_{\lambda_0} = \int_{\lambda_1} = 0 \Rightarrow \int_{\Gamma_0} = \int_{\Gamma_1}$

(ii) $\lambda_0 = \lambda_1 \Rightarrow \int_{\lambda_0} = \int_{\lambda_1} \Rightarrow \int_{\Gamma_0} = \int_{\Gamma_1}$

$\square$

<u>Remark:</u> One can argue that the above proof only works for a smooth homotopy. Now, if $\gamma : I \to D$ is a contour, then we can find a partition $P = \{t_0, \ldots, t_n\}$ of I,

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

s.t. $\gamma([t_i, t_{i+1}])$ belongs to a disk D_i contained in D. If F_i is an antiderivative for f in D_i then letting γ_i denote the restriction of γ to $[t_i, t_{i+1}]$, it follows that

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n-1} \int_{\gamma_i} f(z)\,dz =$$

$$= \sum_{i=1}^{n-1} \Big(F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i)) \Big) \quad (*)$$

For any continuous curve $\gamma : [0,1] \to D$ one can define $\int_\gamma f(z)\,dz$, if f is analytic in D, by $(*)$ for any partition P of $[0,1]$ s.t. $\gamma([t_i, t_{i+1}])$ belongs to a disk D_i, $D_i \subset D$, and F_i is an antiderivative of f in D_i. Independent of the choice of P, D_i and F_i!

Corollary

Suppose that f is analytic in a simply connected domain, and let Γ be any closed contour in D. Then,

$$\int_\Gamma f(z)\,dz = 0 .$$