Plan for F1–F3

- Intro
- Some foundations
- Basic examples

Two views; a random process is

I) a collection $X = \{X_t\}_{t \in T}$ of random variables

fix t, $\omega \to X_t(\omega)$

II) a random function

ω fixed, $t \to X_t(\omega)$

Def

Sample paths: $X : (t,\omega) \longrightarrow X_t(\omega)$

is jointly measurable map

from $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}, m \otimes P)$ to $(\mathbb{R}, \mathcal{B})$

$\longleftarrow$ c.f.

if $\{(t,\omega) : X_t(\omega) \in B\} \in \mathcal{B}(T) \otimes \mathcal{F}$ Fubini

for all $B \in \mathcal{B}$

We may also consider this as a map

$X : (\Omega', \mathcal{F}', P') \longrightarrow (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$

where $\mathcal{B}(\mathbb{R}^T)$ = smallest σ-algebra s.t.

all projections $\pi_t : x \to x_t$

are measurable

or $\mathcal{B}(\mathbb{R}^T) = \sigma(\text{cylinder sets})$

smallest σ-algebra generated by

But suppose, for example, that we want the sample paths to be continuous

However, it turns out

$$\{\text{cont functions } t \to x_t\} \notin \mathcal{B}(\mathbb{R}^T),$$

since $\mathcal{B}(\mathbb{R}^T)$ is generated by ~~sets~~ countable sets.

~~So~~ Now recall the distribution measure, $\mu_t(B) = \mathbb{P}(X_t \in B)$, $t \in T$

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), \mu)$$

?

Def The finite-dimensional distribution of X is the family of measures $\mu_{t_1,\dots,t_n}(B_1,\dots,B_n)$.

$$= \mathbb{P}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) \quad n \geq 1$$

It is convenient to identify the elements in $\mathbb{R}^T$ with

canonical sample paths $\omega = (\omega_t, t \in T)$

and ~~put~~ let $X_t(\omega) = \omega_t$ be

a canonical representation.

Kolmogorov's extension theorem

(Daniell-Kolmogorov)
(Ionescu Tulcea)

Start, conversely, with a family of f.d.d's, which is

consistent:

$$- \mu_{t_1 t_2}(B_1, B_2) = \mu_{t_2 t_1}(B_2, B_1) \quad \text{etc}$$

$$- \mu_{t_1 t_2}(B, \mathbb{R}) = \mu_{t_1}(B) \quad \text{etc}$$

same for all $t_1, \ldots, t_n$, $n \geq 1$
and all $B_1, \ldots, B_n \in \mathcal{B}$

F2:4

Then there exists a unique prob. measure $\mathbb{P}$, ~~and~~ a canonical prob. space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$ and a random process $\{X_t, \mathcal{F}_t\}_{t \in T}$,

s.t. the f.d.d of $(X_{t_1}, \dots, X_{t_n}) = (\omega_{t_1}, \dots, \omega_{t_n})$ is given by $\mu_{t_1 \dots t_n}$, all $t_1, \dots, t_n$ and $n \geq 1$.

Note however, ~~[illegible]~~ $(\Omega, \mathcal{F})$ may not be suitable, practical, ...

Important example

Let $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$, $t > 0$, $x, y \in \mathbb{R}$

Then $\mu_t(B) = \int_B p_t(0,y)\,dy$, $t > 0$, $B \in \mathcal{B}$

$$\mu_{t_1, \dots, t_n}(B_1, \dots, B_n) = \int_{B_1} \int \dots \int_{B_n} p_{t_1}(0, x_1) p_{t_2 - t_1}(x_1, x_2) \dots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \dots dx_n$$

is a family of consistent f.d.d.'s.

F2:5

By Kolmogorov extension there exists a stochastic process $\{B_t, t \geq 0\}$ on a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$, s.t.

1) $B_0 = 0$

2) $B_t - B_s \sim N(0, t-s)$, $0 \leq s \leq t$

3) the increments $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent

Let's call this <u>pre-Brownian motion</u>

Proof 1) clear

2)

$$\mathbb{P}(B_t - B_s \in A) = \iint_{\{(x,y):\, y-x \in A\}} p_s(0,x)\, p_{t-s}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} p_s(0,x) \int_{\{y:\, y-x \in A\}} p_{t-s}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} p_s(0,x) \int_A \underbrace{p_{t-s}(x, x+u)}_{p_{t-s}(0,u)\,!}\, du\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} p_s(0,x)\, dx}_{=1} \int_A p_{t-s}(0,u)\, du$$

$$= \int_A p_{t-s}(0,u)\, du = \mathbb{P}(B_{t-s} \in A)$$

so $B_t - B_s \overset{d}{=} B_{t-s} \sim N(0, t-s)$

F2:6

To obtain Brownian motion we need the predicates in Einstein's ~~Ged~~ Gedanken experiment, 1905, namely

1) $B_0 = 0$
2) $B_t - B_s \sim N(0, t-s)$, $0 \leq s \leq t$
Stationary Gaussian increments
3) Independent increments
4) Continuous paths

But "$P(\{B_t\}_{t \in T}$ continuous)"
is not defined, since
$\{t \to x_t \text{ cont.}\} \notin \mathcal{B}(\mathbb{R}^T)$

So, we take a different approach, Kolmogorov-Čhentsov Thm