Problem Session 3 Solutions.

1) $\mathbb{E}(|X|^p)^{1/p} = \|X\|_p$ and we have seen that $\|X\|_p \leq \|X\|_q$ for $q \geq p$.

So the limit on LHS exists (but may be infinite) by monotonicity.

Let $c = \inf\{K \geq 0 : \mathbb{P}(|X| > K) = 0\}$

Choosing $0 < \varepsilon < c$ arbitrarily, we get (assuming $0 < c < \infty$),

$$\mathbb{E}(|X|^p) = \int_{\{0 \leq |X| < c-\varepsilon\}} |X|^p \, d\mathbb{P} + \int_{\{c-\varepsilon \leq |X| < c+\varepsilon\}} |X|^p \, d\mathbb{P}$$

$$\leq (1-q_\varepsilon)(c-\varepsilon)^p + q_\varepsilon (c+\varepsilon)^p \leq (c+\varepsilon)^p,$$

where $q_\varepsilon = \mathbb{P}(\{c-\varepsilon \leq |X| < c+\varepsilon\}) > 0$. So,

$$\lim_{p\to\infty} \mathbb{E}(|X|^p)^{1/p} \leq \lim_{p\to\infty} c+\varepsilon = c+\varepsilon.$$

Further, $\mathbb{E}(|X|^p)^{1/p} \geq ((1-q_\varepsilon)\cdot 0 + q_\varepsilon (c-\varepsilon)^p)^{1/p} = q_\varepsilon^{1/p}(c-\varepsilon)$

and $\lim_{p\to\infty} \|X\|_p \geq c-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get the desired conclusion. (for $0 < c < \infty$)

It remains to check $c = 0$ and $c = \infty$. In the former $X = 0$ a.s. and the conclusion follows trivially. In the latter case, consider $A_n = \{|X| \geq n\}$. Then $q_n = \mathbb{P}(A_n) > 0$ for all n.

Hence $\mathbb{E}(|X|^p)^{\frac{1}{p}} \geq (q_n \cdot n^p)^{\frac{1}{p}} = q_n^{\frac{1}{p}} \cdot n$

and $\lim_{p\to\infty} \|X\|_p \geq n$. But n is arbitrary

and so $\lim_{p\to\infty} \|X\|_p = \infty$.

2) Note that $\mathbb{E}(\mathrm{Var}(X|\mathcal{G})) = \mathbb{E}(\mathbb{E}((X - \mathbb{E}(X|\mathcal{G}))^2|\mathcal{G}))$

$= \mathbb{E}((X - \mathbb{E}(X|\mathcal{G}))^2)$

$= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - 2\mathbb{E}(X\,\mathbb{E}(X|\mathcal{G}))$

and $\mathrm{Var}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - \mathbb{E}(\mathbb{E}(X|\mathcal{G}))^2$

$= \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - \mathbb{E}(X)^2$

So $\mathbb{E}(\mathrm{Var}(X|\mathcal{G})) + \mathrm{Var}(\mathbb{E}(X|\mathcal{G}))$

$= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - 2\mathbb{E}(X\,\mathbb{E}(X|\mathcal{G})) + \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - \mathbb{E}(X)^2$

$= \mathrm{Var}(X) + 2(\mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) - \mathbb{E}(X\,\mathbb{E}(X|\mathcal{G})))$

It remains to show

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})^2) = \mathbb{E}(X\,\mathbb{E}(X|\mathcal{G}))$$

But this follows since $\mathbb{E}(X|\mathcal{G})$ is a $\mathcal{G}$-measurable random variable and

$$\mathbb{E}(X\,\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(\mathbb{E}(X\,\mathbb{E}(X|\mathcal{G})\,|\mathcal{G}))$$

$$= \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})^2).$$

3)

a) We have

$$\mathbb{E}(X_n\,|\,\tilde{\mathcal{F}}_{n-1}) = \mathbb{E}(\theta^{\sum^n Y_k}\,|\,\tilde{\mathcal{F}}_{n-1})$$

$$= \theta^{\sum_{k=1}^{n-1} Y_k}\,\mathbb{E}(\theta^{Y_n}\,|\,\tilde{\mathcal{F}}_{n-1}) = X_{n-1}\,\mathbb{E}(\theta^{Y_n})$$

$$= X_{n-1}\,(p\theta + (1-p)\theta^{-1}).$$

So, X_n is a martingale if $p\theta + (1-p)\theta^{-1} = 1$

$$\Rightarrow \quad p\theta^2 - \theta + 1 - p = 0$$

$$\Rightarrow \quad \theta = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm \sqrt{(2p-1)^2}}{2p}$$

$$= 1 \quad \text{or} \quad \frac{2-2p}{2p} = \frac{1-p}{p}.$$

b)

Since $\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1}) = X_{n-1} + \mathbb{E}(Y_n)$

$= X_{n-1} + p + (1-p)(-1)$

$= X_{n-1} + 2p - 1.$

Hence for $f(n) = -(2p-1)n$ we get

$$\mathbb{E}(X_n + f(n) \mid \tilde{\mathcal{F}}_{n-1}) = X_{n-1} - (2p-1)n + 2p-1$$

$$= X_{n-1} - (2p-1)(n-1)$$

$$= X_{n-1} - f(n-1).$$

c) Let $T = \min\{n : X_n \leq -b \text{ or } X_n \geq a\}$.

Since $\mathbb{P}(T \leq \max\{a,b\} + n \mid \mathcal{F}_n)$

$\geq \min\{p^{a+b}, (1-p)^{a+b}\} > 0,$

we have $\mathbb{E}(T) < \infty$. Consider the martingale

$Y_n = \theta^{X_{n \wedge T}}$ for $\theta = \frac{1-p}{p}$.

Since $|Y_n| \leq \max\{\theta^a, \theta^{-b}\}$ and $T < \infty$ a.s.

(ii) in DOST applies and $\mathbb{E}(Y^T) = \mathbb{E}(Y_0) = \theta^0 = 1$

This gives $\mathbb{P}(X^T = a)\,\theta^a + \mathbb{P}(X^T = -b)\,\theta^{-b} = 1$

and $\mathbb{P}(X^T = a) + \mathbb{P}(X^T = -b) = 1$

So $\mathbb{P}(X^T = a)(\theta^a - \theta^{-b}) = 1 - \theta^{-b}$

and $\mathbb{P}(X^T = a) = \dfrac{1-\theta^{-b}}{\theta^a - \theta^{-b}} = \dfrac{\theta^b - 1}{\theta^{a+b} - 1} = \dfrac{1-\theta^b}{1-\theta^{a+b}}$

d) Let $Z_n = X_n - f(n)$. This is a martingale and

$$|Z_n - Z_{n-1}| \leq |X_n - X_{n-1}| + |f(n) - f(n+1)|$$
$$= 1 + |2p-1| < \infty.$$

Since $\mathbb{E}(T) < \infty$, (iii) of DOST applies and

$\mathbb{E}(Z^T) = \mathbb{E}(Z_0) = \mathbb{E}(X_0 + f(0)) = 0$. Now,

$$\mathbb{E}(Z^T) = \mathbb{E}(X^T + f(T)) = \mathbb{E}(X^T) + \mathbb{E}(f(T)) = 0$$
$$= \mathbb{P}(X^T = a)a + \mathbb{P}(X^T = -b)(-b) + (2p-1)\mathbb{E}(T)$$
$$= \frac{1-\theta^b}{1-\theta^{a+b}} a - \frac{\theta^b - \theta^{a+b}}{1-\theta^{a+b}} b + (2p-1)\mathbb{E}(T)$$

and $\mathbb{E}(T) = -\dfrac{(1-\theta^b)a - (\theta^b - \theta^{a+b})b}{(2p-1)(1-\theta^{a+b})}$.

4) If X_n is previsible, X_n is $\mathcal{F}_{n-1}$ measureable.

Hence, $\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = X_n \mathbb{E}(1 \mid \mathcal{F}_{n-1}) = X_n$.

But since X_n is a martingale, $X_n = \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$ almost surely. Hence, $X_n = X_{n-1} = \ldots = X_0$ a.s. for all n.

5). Since $Y = \mathbb{E}(X \mid \mathcal{G})$ is, by definition, $\mathcal{G}$-measurable we have,

$$\mathbb{E}((X-Y)^2 \mid \mathcal{G}) = \mathbb{E}(X^2 \mid \mathcal{G}) + \mathbb{E}(Y^2 \mid \mathcal{G}) - 2\mathbb{E}(XY \mid \mathcal{G})$$

$$= Y^2 + Y^2 \mathbb{E}(1 \mid \mathcal{G}) - 2Y\mathbb{E}(X \mid \mathcal{G})$$

$$= 2Y^2 - 2Y^2 = 0.$$

Now $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(Y)$

and $\mathbb{E}(X-Y) = 0$. Since

$$Var(X-Y) = \mathbb{E}(|X-Y|^2) - \mathbb{E}(X-Y)^2$$

$$= \mathbb{E}(\mathbb{E}(|X-Y|^2 \mid \mathcal{G})) - 0 = 0$$

$X - Y = 0$ a.s. $\square$

6) We get the joint density

$$f_{X,Y}(x,y) = \frac{1}{\pi} I_{\{x^2+y^2 \le 1\}}$$ and so

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \frac{1}{\pi}\int I_{\{x^2 \le 1-y^2\}}\,dx$$

$$= \frac{1}{\pi}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx = \frac{2}{\pi}\sqrt{1-y^2}$$

so $$f_{X|Y}(x \mid y) = \frac{I_{\{x^2+y^2 \le 1\}}}{2\sqrt{1-y^2}}$$ for $-1 < y < 1$.

Hence $\mathbb{E}(X \mid Y) = \mathbb{E}(X \mid Y=y)$

$$= \int x \, f_{X|Y}(x|y) \, dx = \int \frac{x \, I_{\{x^2+y^2 \leq 1\}}}{2\sqrt{1-y^2}} \, dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \, dx = \frac{1}{2\sqrt{1-y^2}} \left[\frac{x^2}{2} \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 0.$$

and $\mathbb{E}(|X| \mid Y) = \mathbb{E}(|X| \mid Y=y)$

$$= \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |x| \, dx = \frac{1}{2\sqrt{1-y^2}} \, 2 \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} = \frac{1-y^2}{2\sqrt{1-y^2}}$$

Now let $Z = |Y|$. The joint density is

$$f_{X,Z}(x,z) = \frac{2}{\pi} I_{\{x^2+z^2 \leq 1\}} I_{\{z \geq 0\}}$$

Then $$f_Z(z) = \int \frac{2}{\pi} I_{\{x^2+z^2 \leq 1\}} I_{\{z \geq 0\}} \, dx$$

$$= \frac{2}{\pi} \sqrt{1-z^2} \qquad z \in [0,1)$$

and $$f_{X|Z}(x|z) = \frac{\frac{2}{\pi} I_{\{x^2+z^2 \leq 1\}} I_{\{z \geq 0\}}}{2/\pi \, \sqrt{1-z^2}}$$

$$= \frac{I_{\{x^2+z^2 \leq 1\}} I_{\{z \geq 0\}}}{\sqrt{1-z^2}} = 2 f_{X|Y}(x|y) \text{ for } y \geq 0$$

Hence $\mathbb{E}(X \mid Z) = 0$

and $\mathbb{E}(|X| \mid Z) = \dfrac{1-Z^2}{\sqrt{1-Z^2}} = \sqrt{1-Z^2}$

7) a) We compute

$$\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1}) = \mathbb{E}(e^{S_n - n/2} \mid \tilde{\mathcal{F}}_{n-1})$$

$$= e^{S_{n-1}} e^{-n/2}\, \mathbb{E}(e^{Y_n} \mid \tilde{\mathcal{F}}_{n-1}) = e^{S_{n-1}} e^{-n/2}\, \mathbb{E}(e^{Y_n}) = (\dagger)$$

Now $\mathbb{E}(e^{Y_n}) = \displaystyle\int_{-\infty}^{\infty} e^{y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - 2y)}\, dy. \qquad (*)$$

But $y^2 - 2y = (y-1)^2 - 1$ and

$$(*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-1)^2} e^{1/2}\, dy = e^{1/2}$$

Since $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2}$ is the density of $\mathcal{N}(1,1)$.

So $(\dagger) = e^{S_{n-1}} e^{\frac{1}{2} - n/2} = e^{S_{n-1} - \frac{(n-1)}{2}} = X_{n-1}$.

Hence X_n is a martingale.

b) Note that X_n is also non-negative.

Hence, by Doob's convergence theorem, X_n converges a.s. to some r.v. $X_\infty \geq 0$.

Since $Y_n \sim \mathcal{N}(0,1)$ the sum satisfies $S_n \sim \mathcal{N}(0,n)$

We can compute $\mathbb{E}|S_n| = 2\int_0^\infty \frac{1}{\sqrt{2\pi n}} e^{-x^2/2n} \cdot x \, dx$

$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$. Let $\frac{1}{2} < t < 1$. Then

$$\mathbb{P}(S_n \geq n^t) \leq \mathbb{P}(|S_n| \geq n^t) \leq n^{-t}\mathbb{E}|S_n| = \sqrt{\frac{2}{\pi}}\, n^{\frac{1}{2}-t}$$

$$\mathbb{P}\left(e^{S_n - n/2} \geq e^{n^t - n/2}\right) \leq \sqrt{\frac{2}{\pi}}\, n^{\frac{1}{2}-t}$$

$$\Leftrightarrow \quad \mathbb{P}\Big(X_n \geq \underbrace{e^{n^t - n/2}}_{\to 0 \text{ as } n\to\infty}\Big) \leq \sqrt{\frac{2}{\pi}}\, n^{\frac{1}{2}-t} \to 0 \quad \text{as } n \to \infty$$

Thus $\mathbb{P}(X_n < e^{n^t - n/2}) \geq 1 - \sqrt{\frac{2}{\pi}}\, n^{\frac{1}{2}-t}$

and $X_n \to 0$ in probability.

To see that this implies $X_\infty = 0$ a.s.

ie let $\varepsilon > 0$ be arbitrary and let N be large enough such that $\mathbb{P}(\underbrace{X_N \leq \tfrac{\varepsilon}{2}}_{A}) \geq 1 - \varepsilon/2$

and $\mathbb{P}(\underbrace{\{|X_N - X_\infty| \leq \varepsilon/2\}}_{B}) \geq 1 - \varepsilon/2$

Then
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$$
$$\geq 1 - \varepsilon/2 + 1 - \varepsilon/2 - 1 = 1 - \varepsilon.$$

Further $A \cap B = \{ \omega : \ |X_\infty| \leq \varepsilon/2 + |X_N| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \}$

And hence $\mathbb{P}(|X_\infty| \leq \varepsilon) = \mathbb{P}(X_\infty \leq \varepsilon) \geq 1 - \varepsilon$.

As $\varepsilon > 0$ was arbitrary we must have $X_\infty = 0$ almost surely.

c) $\mathbb{E}(X_n^r \mid \tilde{\mathcal{F}}_{n-1}) = \mathbb{E}(e^{rS_n - \frac{r}{2}n} \mid \tilde{\mathcal{F}}_{n-1})$

$= e^{rS_{n-1} - \frac{r}{2}n}\, \mathbb{E}(e^{rY_n})$. Again,

$$\mathbb{E}(e^{rY_n}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{rx}\, dx = e^{r^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-r)^2}{2}} dx$$

$= e^{r^2/2}$ and

$$\mathbb{E}(X_n^r \mid \tilde{\mathcal{F}}_{n-1}) = e^{rS_{n-1} - \frac{r}{2}(n-1)} \; e^{r^2/2 - r/2}$$

$$= X_{n-1}^r \; e^{r^2/2 - r/2} \qquad (\ddagger)$$

Since $r^2 \leq r$ for $0 < r \leq 1$

and $r^2 \geq r$ for $r \geq 1$,

$r^2/2 - r/2$ is non-positive in the former and non-negative in the latter case. $(\ddagger)$ becomes

$$\mathbb{E}(X_n^r \mid \tilde{\mathcal{F}}_{n-1}) \begin{cases} \geq X_{n-1}^r & r \geq 1 \\ \leq X_{n-1}^r & 0 < r \leq 1 \end{cases}$$