Financial Theory – Lecture 12

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Agenda

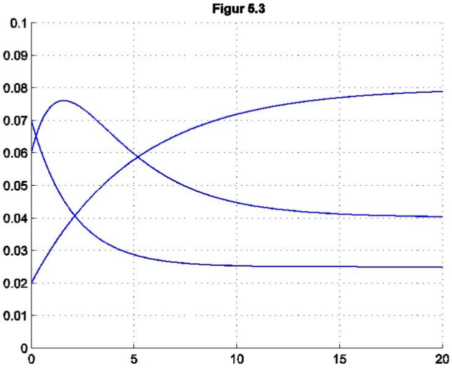
- Yield curves.
- Duration and convexity.

The lecture is based on

- Chapter 5 and Section 6.4 in the course book.

Yield curves

For a set of bonds with the same characteristics (e.g. they are equally risky and have the same amortisation principle, etc.), the yield as a function of time to maturity is known as the yield curve or the term structure of interest rates.



The function y_t for $t \in [0, 20]$. (Armerin & Song p. 131.)

Yield curves

The form of the yield curve contains information about the economy in which the bonds are traded.

Often an inverted yield curve tends to indicate that the economy is in a recession, or is moving into a recession.

But what can be said about the form of a general yield curve?

Explanations for the form of yield curve

The level of the yield curve is affected by the general time preference of investors.

If they are impatient, then they want to consume now, which results in

$$\text{a high supply of bonds and a low demand} \longrightarrow$$

$$\text{the price on bonds} \downarrow \longleftrightarrow \text{the yields} \uparrow.$$

Explanations for the form of yield curve

If the economy is expected to grow, then, since consumers want to even out consumption over their lives, they want to consume more today. Thus, they want to

$$\text{borrow today} \longrightarrow \text{an increase in the supply of bonds today} \longrightarrow$$

$$\text{the price of bonds} \downarrow \longleftrightarrow \text{the yield} \uparrow .$$

The conclusion is that when there is an expected expansion of the economy, the yield curve is upward sloping.

Explanations for the form of yield curve

Another factor influencing the form of the yield curve is uncertainty.

When there is increased uncertainty, risk-averse investors prefer govenment bonds, which results in

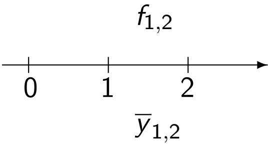
$$\text{increased demand of default-free bonds } \longrightarrow$$

$$\text{the price of default-free bonds} \uparrow \;\longleftrightarrow\; \text{yields on default-free bonds } \downarrow .$$

Older explanations for the form of yield curves

Expectation hypothesis

The expected future yields are equal to their respective forward rate.



In formulas:

$$\overline{y}_{1,2} = f_{1,2},$$

or

$$1 + \overline{y}_{1,2} = \frac{(1 + y_2)^2}{1 + y_1}.$$

And in the same way for other time spans $(m, n]$.

Older explanations for the form of yield curves

Liquidity preference hypothesis
Investors requires a premium from holding bonds with longer maturities.

Reasons for this:

- No or very illiquid secondary market.
- The market can be dominated by short-term investors, who need a premium when buying long-term bonds.

Older explanations for the form of yield curves

Market segmentation hypothesis
Buyers and issuers of bonds have different time spans for their investments.

They are only interested in their respective part of the yield curve.

This means that the yield on short-term bonds and the yield for long-term bonds are defined by supply and demand for these two segments of the market – the market is segmented rather than connected.

Older explanations for the form of yield curves

Preferred habitat hypothesis
Investors have their preferred length of their investment (as in the Market segmentation hypothesis), but if the extra return is sufficiently high, then they can adjust their positions.

This implies that there will be a premium for maturities where there is lower demand.

Using the yireld curve

Let

$$(M_1, M_2, \ldots, M_n)$$

be a stream of deterministic cash flows.

What is the value of this?

We can think of this as a portfolio of ZCB's.

This means that the value is

$$\sum_{i=1}^{n} M_i Z_{0,i} = \sum_{t=1}^{T} \frac{M_i}{(1+y_i)^i}.$$

Note: Here I follow the book and write $y_i = y_{0,i}$.

Using the yield curve

Conclusion: When valuing a deterministic stream of cash flows, we use the yields of ZCB's.

In order for this to work, we need the term structure of ZCB's.

In general, the non-defaultable bonds given have coupons.

Bootstrapping

Assume that we have the following bonds:

Bond No	Time to maturity	Coupon	Face value	Price
1	1	0	100	95
2	2	5	100	98
3	4	2	100	97

How does the term structure of ZCB's look like?

First of all, we have (Bond 1)

$$95 = \frac{100}{1 + y_1} \;\Rightarrow\; y_1 = \frac{100}{95} - 1 \approx 0.0526.$$

To get y_2 we can not use Bond 2 directly, since this is not a ZCB.

Bootstrapping

But we can create a ZCB with time to maturity equal to 2.

Consider the portfolio with

$$1 \text{ of Bond 2 and } -0.05 \text{ of Bond 1.}$$

The result is:

Time	Cash flow
0	$1 \cdot 98 - 0.05 \cdot 95 = 93.25$
1	$1 \cdot 5 - 0.05 \cdot 100 = 0$
2	$1 \cdot (5 + 100) = 105$

This portfolio is like a ZCB with time to maturity 2, face value 105 and price 93.25.

Bootstrapping

Using this fictitious ZCB, we can get y_2:

$$93.25 = \frac{105}{(1+y_2)^2} \quad \Leftrightarrow \quad y_2 = \left(\frac{105}{93.25}\right)^{1/2} - 1 \approx 0.0611.$$

This technique is known as bootstrapping.

What about the spot rates y_3 and y_4?

Bootstrapping

Bond 3 has coupon payments at $t = 1, 2, 3$ that we need to "remove".

This is possible for the coupon payments at $t = 1$ and $t = 2$.

For the coupon payment at $t = 3$, there is no unique way of removing this to create a ZCB.

We need to find the two rates y_3 and y_4 that satisfies

$$97 = \frac{2}{1 + 0.0526} + \frac{2}{(1 + 0.0611)^2} + \frac{2}{(1 + y_3)^3} + \frac{102}{(1 + y_4)^4}.$$

To proceed we have to make some assumptions about the form on the term structure.

Duration

Assume that we have a bond with cash flows $M_1, M_2, \dots, M_n$.

We know that the price of this bond is

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i} = \sum_{i=1}^{n} M_i (1+y)^{-i}.$$

What happens if y changes?

$$\begin{aligned} \frac{\partial B_0}{\partial y} &= -\sum_{i=1}^{n} i\, M_i (1+y)^{-i-1} \\ &= -\frac{1}{1+y} \sum_{i=1}^{n} i\, M_i (1+y)^{-i}. \end{aligned}$$

Duration

This can be written

$$\begin{aligned}\frac{\partial B_0}{\partial y} &= -\frac{B_0}{1+y} \cdot \frac{\sum_{i=1}^{n} i\, M_i (1+y)^{-i}}{B_0} \\ &= -\frac{B_0}{1+y} \cdot \sum_{i=1}^{n} i\, w_i,\end{aligned}$$

where

$$w_i = \frac{M_i (1+y)^{-i}}{B_0}.$$

Duration

The duration at time 0 is defined as

$$D_0 = \sum_{i=1}^{n} i \cdot \frac{\frac{M_i}{(1+y)^i}}{B_0} = \sum_{i=1}^{n} i\, w_i.$$

Note that $0 \leq w_i \leq 1$ and

$$\sum_{i=1}^{n} w_i = 1.$$

The duration is a weighted average of the times at which the cash flows are paid.

The higher the cash flow M_i is, the higher is the weight w_i.

Duration

We can write

$$\frac{\partial B_0}{\partial y} \cdot \frac{1}{B_0} = -\frac{\sum_{i=1}^{n} i\, w_i}{1+y} = -\frac{D_0}{1+y}.$$

Approximating the derivate, we get

$$\frac{\Delta B_0}{\Delta y} \cdot \frac{1}{B_0} \approx \frac{\partial B_0}{\partial y} \cdot \frac{1}{B_0} = -\frac{D_0}{1+y},$$

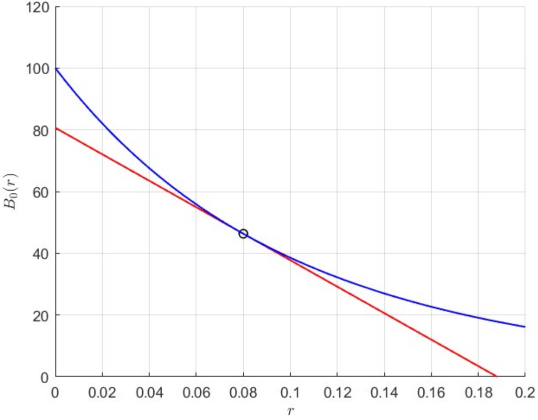
or

$$\frac{\Delta B_0}{B_0} \approx -\frac{D_0}{1+y} \Delta y$$

The duration meaures how sensitive the price of a deterministic stream of cash flows is to a change in the yield.

This duration is also called the Macauly duration.

Duration



Duration

The modified duration is defined by

$$D_0^* = \frac{1}{1+y} D_0.$$

Using this, we can write

$$\frac{\partial B_0}{\partial y} = -D_0^* B_0$$

and

$$\frac{\Delta B_0}{B_0} \approx -D_0^* \Delta y.$$

At time t for a ZCB with time to maturity n, the duration is given by

$$D_t = n - t.$$

Convexity

The convexity of a bond at time 0 is defined as

$$C_0 = \sum_{i=1}^{n} i(i+1)\frac{M_i(1+y)^{-i-2}}{B_0} = \sum_{i=1}^{n} i(i+1)w_i.$$

The convexity satisfies

$$C_0 = \frac{(1+y)^2}{B_0} \cdot \frac{\partial^2 B_0}{\partial y^2} \quad \Leftrightarrow \quad \frac{\partial^2 B_0}{\partial y^2} = \frac{C_0 B_0}{(1+y)^2}.$$

The modified convexity at time zero is defined as

$$C_0^* = \frac{C_0}{(1+y)^2}.$$

Convexity

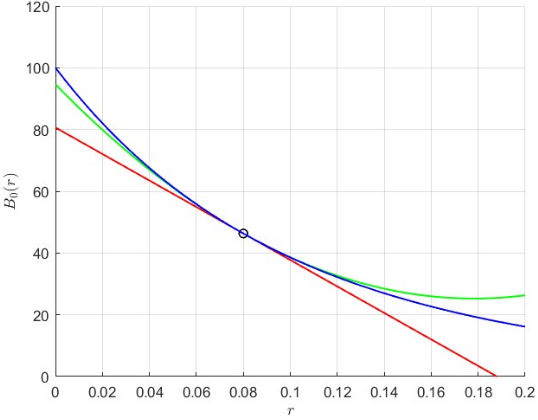
By using the second order Taylor expansion

$$\Delta B_0 \approx \frac{\partial B_0}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 B_0}{\partial y^2}\left(\Delta y\right)^2$$

we can write

$$\begin{aligned} \frac{\Delta B_0}{B_0} &\approx -\frac{D_0}{1+y}\Delta y + \frac{1}{2}\frac{C_0}{(1+y)^2}\left(\Delta y\right)^2 \\ &= -D_0^*\Delta y + \frac{1}{2}C_0^*\left(\Delta y\right)^2. \end{aligned}$$

Convexity



Fisher-Weil duration

In general, we can use a ZCB yield curve to price other bonds:

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+y_i)^i}.$$

In order to define a duration in this case we consider the sensitivity of the price if the yield curve is changed according to

$$1+y_i \;\rightarrow\; (1+y_i)\cdot(1+\delta),\;\; i=1,2,\ldots,n,$$

for some constant δ.

We get

$$B_0(\delta) = \sum_{i=1}^{n} \frac{M_i}{\big((1+y_i)(1+\delta)\big)^i} = \sum_{i=1}^{n} M_i(1+y_i)^{-i}(1+\delta)^{-i}$$

Fisher-Weil duration

and from this

$$B_0'(\delta) = -\sum_{i=1}^{n} i\, M_i (1+y_i)^{-i} (1+\delta)^{-i-1}.$$

The quantity

$$D_0^{\mathrm{FW}} = \sum_{i=1}^{n} i \cdot \frac{M_i (1+y_i)^{-i}}{B_0}$$

is the Fisher-Weil duration. We see that

$$B_0(\delta) \approx B_0(0) + B_0'(0)\delta = B_0 - B_0 D_0^{\mathrm{FW}} \delta,$$

or

$$\frac{\Delta B_0}{B_0} \approx -\delta D_0^{\mathrm{FW}}.$$

Immunisation

Now consider the case when we have liabilities in the form of a stream of cash flows whose value is influenced by the yield curve.

In order to decrease the interest rate risk we want to invest in a portfolio that resembles, in some sense, the cash flows.

The best would be to buy a portfolio of ZCB's exactly matching the cash flows in the liability, but this is usually not possible, and if it is possible it might be costly.

Immunisation

Instead we buy a portfolio whose value and duration matches that of our liability – this is knows as immunisation.

The idea is that if there is a small change in the yield curve, then the value of our portfolio should change approximately as much as our liabilities (cf. the figure above).

Immunisation

Let $\bar{B}$ and $\bar{D}$ denote the value and duration of the liabilities.

The value is invested in two bonds with price B_1 and B_2 respectively, and durations D_1 and D_2 respectively, where we assume that $D_1 \neq D_2$.

Let N_1 and N_2 be the number of bond 1 and 2 we buy.

Then the value of the bond portfolio B_p must satisfy the budget constraint

$$B_p = N_1 B_1 + N_2 B_2 = \bar{B}.$$

Immunisation

One can show that the Fisher-Weil duration D of a portfolio with value B and consisting of $N_1, N_2, \dots, N_m$ number of bonds with prices $B_1, B_2, \dots, B_m$ and durations $D_1, D_2, \dots, D_m$ satisfies

$$B = \sum_{j=1}^{m} N_j B_j$$

and

$$BD = \sum_{j=1}^{m} N_j B_j D_j.$$

The relation holds approximately for the Macaulay duration.

Immunisation

If we want the portfolio to have the same duration as the liabilities, then we should choose (N_1, N_2) such that

$$\begin{cases} N_1 B_1 + N_2 B_2 = \bar{B} \\ N_1 B_1 D_1 + N_2 B_2 D_2 = \bar{B}\bar{D}. \end{cases}$$

The solution is given by

$$N_1 = \frac{\bar{B}}{B_1} \cdot \frac{\bar{D} - D_2}{D_1 - D_2} \quad \text{and} \quad N_2 = \frac{\bar{B}}{B_2} \cdot \frac{\bar{D} - D_1}{D_2 - D_1}.$$

Equity duration

Recall that, under the assumptions from Lecture 10, the price of a stock is given by

$$P_t = \sum_{i=1}^{\infty} \frac{E_t\left[D_{t+i}\right]}{(1+r)^i} = \sum_{i=1}^{\infty} E_t\left[D_{t+i}\right](1+r)^{-i}.$$

Taking the derivative of this with respect to r gives

$$\frac{\partial P_t}{\partial r} = -\sum_{i=1}^{\infty} i\, E_t\left[D_{t+j}\right](1+r)^{-i-1}.$$

Now define

$$w_{i,t} = \frac{E_t\left[D_{t+i}\right](1+r)^{-i}}{P_t}$$

and

$$\mathrm{DUR}_t = \sum_{i=1}^{\infty} i\, w_{i,t}.$$

Equity duration

Then we can write

$$\frac{\partial P_t}{\partial r} = -\frac{1}{1+r}\mathrm{DUR}_t P_t.$$

In the Gordon growth model, we have

$$P_t = \frac{(1+g)D_t}{r-g},$$

so

$$\frac{\partial P_t}{\partial r} = -\frac{(1+g)D_t}{(r-g)^2} = -\frac{P_t}{r-g}.$$

It follow that

$$\mathrm{DUR}_t = -\frac{1+r}{P_t} \cdot \left(-\frac{P_t}{r-g}\right) = \frac{1+r}{r-g}.$$

Equity duration

It has been found empirically that stocks with low equity duration tends to deliver higher returns than stocks that has high equity duration.

This has led to the introduction of a, usually downward sloping, equity term structure (how the return is connected to the equity duration).

It has also been found that the slope of the equity term structure is upward sloping in bad times.