Superhedging

Given a claim $H = f(S_T)$ (function of an asset price at time N), an (x, H)-hedge is an admissible strategy with

$$V_0(\theta) = x \quad \text{and} \quad V_T(\theta) \geq H \quad (\text{a.s.})$$

The seller's price $\pi_S(H)$ can now be defined as:

$$\pi_S = \inf \{ z \geq 0 : \text{there exists a } (z, H)\text{-hedge} \}$$

→ guarantees the seller of H not to incur losses.

The buyer's price $\pi_B(H)$ is analogously

$$\pi_B = \sup \{ z \geq 0 : \text{there exists a } (-z, -H)\text{-hedge} \}$$

→ guarantees the buyer not to incur losses.

If there is a replicating strategy θ, then we actually have equality everywhere:

$$V_T(\theta) = H \Rightarrow \theta \text{ is a } (z, H)\text{-hedge}$$

$$-\theta \text{ is a } (-z, -H)\text{-hedge}$$

for $z = V_0(\theta)$. In this case, $\pi_B = \pi_S = \pi(H)$.

In general, we only have $\pi_B \le \pi_S$.

If we have an equivalent martingale measure Q, then for a seller's strategy θ we have

$$\mathbb{E}_Q(\bar{H}) \le \mathbb{E}_Q(\bar{V}_T(\theta)) = \mathbb{E}_Q(V_0(\theta)) = V_0(\theta)$$

$$\Rightarrow \quad \mathbb{E}_Q(\bar{H}) \le \pi_S \quad \text{upon taking inf.}$$

In the same way, $\mathbb{E}_Q(\bar{H}) \ge \pi_B$.

If the claim is attainable (i.e. $\exists$ a replicating strategy) we have equalities throughout.

Strategies involving contingent claims

We now expand our standard model, with asset prices $S_t^0, S_t^1, \dots, S_t^d$ by adding some attainable European claims $Z_t^1, Z_t^2, \dots, Z_t^m$.

A trading strategy is now a pair $\Phi = (\theta, \gamma)$ with initial value

"standard" strategy

trading with $Z_t^1, Z_t^2, \dots$

$$V_0(\Phi) = \theta_0 \cdot S_0 + \gamma_0 \cdot Z_0$$

It is self-financing if

$$\theta_t \cdot S_t + \gamma_t \cdot Z_t = \theta_{t+1} \cdot S_t + \gamma_{t+1} \cdot Z_t$$

<u>Theorem</u>: The model is arbitrage-free if and only if every attainable European claim with payoff Z has value process

$$Z_t = S_t^0 \, \mathbb{E}_Q\left(Z/S_T^0 \mid \tilde{\mathcal{F}}_t \right)$$

where Q is an equivalent martingale measure for the price process S.

<u>Proof</u>: By assumption there is a replicating strategy θ for Z. Its value is $V_t(\theta) = S_t^0 \, \mathbb{E}_Q\left(Z/S_T^0 \mid \mathcal{F}_t \right)$ by the martingale property. Now suppose $V_t(\theta) \neq Z_t$ on a set of positive measure.

Without loss of generality, we may assume $\mathbb{P}(Z_u > V_u(\theta)) > 0$ for some time u.

There now exists an arbitrage strategy:

→ do nothing until time u.

→ if the event $Z_u > V_u(\theta)$ does not occur, keep doing nothing

→ if $Z_u > V_u(\theta)$:

- sell Z for price Z_u
- invest in θ (at price $V_u(\theta)$)
- put positive difference in bank.

At time T, $V_T(\theta) = Z$, so θ and Z cancel. We are left with the difference. This happens with positive probability and is hence (weak) arbitrage, a contradiction in an arbitrage-free market.

Hence $Z_t = S_t^0 \mathbb{E}\left(\frac{Z}{S_T^0} \mid \mathcal{F}_t \right)$ a.s. for all t

This proves the first direction.

Assume now that all contingent claims follow this rule: $Z_t = S_t^0 \mathbb{E}_Q\left(\frac{Z}{S_T^0} \mid \mathcal{F}_t \right)$.

Consider a self-financing strategy $\Phi = (\theta, \gamma)$ with $V_0(\Phi) = 0$, $V_T(\Phi) \geq 0$ (a.s.).

$$\mathbb{E}_Q\left(\bar{V}_t(\Phi) \mid \tilde{\mathcal{F}}_{t-1}\right)$$

$$= \mathbb{E}_Q\Big(\underbrace{\sum_i \theta_t^i \bar{S}_t^i}_{\substack{\text{martingale} \\ \text{under } Q}} + \sum_j \underset{\text{pre-visible}}{\gamma_t^j}\, \bar{Z}_t^j \mid \tilde{\mathcal{F}}_{t-1}\Big)$$

$$\mathbb{E}\left(\bar{Z}_t^j \mid \tilde{\mathcal{F}}_{t-1}\right)$$

$$= \mathbb{E}\left(\frac{Z_t^j}{S_t^0} \mid \tilde{\mathcal{F}}_{t-1}\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left(\frac{Z^j}{S_T^0} \mid \tilde{\mathcal{F}}_t\right) \mid \mathcal{F}_{t-1}\right)$$

$$= \mathbb{E}\left(\frac{Z^j}{S_T^0} \mid \tilde{\mathcal{F}}_{t-1}\right) = \bar{Z}_{t-1}^j$$

Combining gives

$= \bar{V}_{t-1}(\Phi)$ and $\bar{V}_t(\Phi)$ is a martingale.

$$\Rightarrow \mathbb{E}(\bar{V}_T(\theta)) = \mathbb{E}(\bar{V}_0(\theta)) = 0$$

$\Rightarrow \bar{V}_T(\theta) = 0$ a.s. since $V_T(\theta) \geq 0$ a.s. by assumption.

It follows that there is no arbitrage. $\square$

From the Binomial Model to Black-Scholes

In the binomial model, the price changes by either $1+a$ or $1+b$ in each step $(a<b)$. The risk free rate is r (factor of $1+r$ per time step).

The equivalent martingale measure is determined by a single probability q:

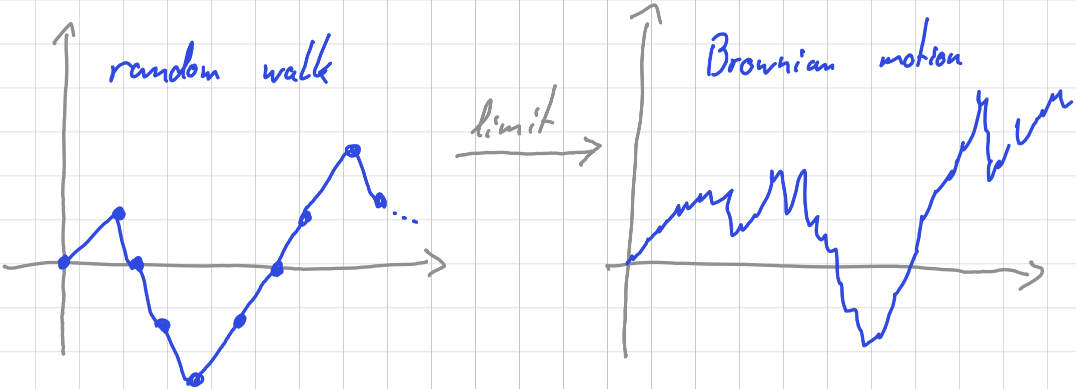
$$S_t \begin{cases} \nearrow (1+b)\,S_t & \text{with prob. } 1-q = \dfrac{r-a}{b-a} \\ \searrow (1+a)\,S_t & \text{with prob. } q = \dfrac{b-r}{b-a} \end{cases}$$

where $a<r<b$.

We derived the fair price for a European call using this model with

$$\mathbb{E}_Q\big(\underbrace{\beta_T}_{\text{discounting factor}}\underbrace{(S_T-K)}_{\text{payoff}}{}^{+} \mid \mathcal{F}_0\big)$$

We now let the number of time steps N go to ∞, while decreasing the time between steps to 0, to obtain a continuous model:



We define a, b, r in such a way that the process converges:

Let $h_N = \frac{T}{N}$ (length of one time step), and

total time

total time steps

$\varrho_N = r h_N$ (risk-free rate). Observe that

$$(1+\varrho_N)^N = \left(1 + r\frac{T}{N}\right)^N \xrightarrow[N\to\infty]{} e^{rT}.$$

Let a_N, b_N satisfy

$$\log\left(\frac{1+b_N}{1+\varrho_N}\right) = \sigma\sqrt{h_N} = \sigma\sqrt{\frac{T}{N}}$$

$$\log\left(\frac{1+a_N}{1+\varrho_N}\right) = -\sigma\sqrt{h_N} = \sigma\sqrt{\frac{T}{N}},$$

chosen s.t. the process converges

turns multiplication into addition

where σ is a constant that measures the volatility of an asset. We get

$$1+a_N = (1+\varrho_N)e^{-\sigma\sqrt{h_N}}, \quad 1+b_N = (1+\varrho_N)e^{\sigma\sqrt{h_N}}$$

and we find

$$q_N = \frac{b_N - r_N}{b_N - a_N} = \frac{(1+r_N)\, e^{\sigma\sqrt{T/N}} - (1+r_N)}{(1+r_N)\, e^{\sigma\sqrt{T/N}} - (1+r_N)\, e^{-\sigma\sqrt{T/N}}} \xrightarrow[N\to\infty]{} \frac{1}{2}$$

and $1 - q_N \to \frac{1}{2}$.

The discounted price process becomes

$$\bar{S}_N^{(N)} = S_0 \, (1+r_N)^{-N} \prod_{k=1}^{N} R_k^{(N)}$$

either $1+a_N$ or $1+b_N$

dependence on N.

final time

$$= S_0 \prod_{k=1}^{N} \left(\frac{R_k^{(N)}}{1+r_N} \right)$$

$\begin{cases} \text{either } \dfrac{1+a_N}{1+r_N} = e^{-\sigma\sqrt{T/N}} \\ \text{or } \dfrac{1+b_N}{1+r_N} = e^{\sigma\sqrt{T/N}} \end{cases}$

$$= S_0 \exp\left(\sum_{k=1}^{N} Y_k^{(N)} \right)$$

where $Y_k^{(N)}$ is $\pm \sigma\sqrt{T/N}$. The sum has <u>mean</u>

$$\mathbb{E}_Q\left(\sum_{k=1}^{N} Y_k^{N} \right) = N\left((1-q_N)\,\sigma\sqrt{\tfrac{T}{N}} + q_N\left(-\sigma\sqrt{\tfrac{T}{N}}\right) \right)$$

$$= \sigma\sqrt{T\cdot N}\,(1-2q_N) \xrightarrow[N\to\infty]{} -\frac{\sigma^2}{2} T$$

and variance

$$\mathrm{Var}_Q\left(\sum_{k=1}^{N} Y_k^{(N)} \right) = N\, \mathrm{Var}_Q\left(Y_k^{(N)}\right) \xrightarrow[N\to\infty]{} \sigma^2 T$$

The central limit theorem gives us that $\sum_{k=1}^{N} Y_k^{(N)}$ converges in distribution to a normal distribution:

$$\sum_{k=1}^{N} Y_k^{(N)} \to \mathcal{N}\left(-\frac{\sigma^2}{2}T, \sigma^2 T\right).$$

This justifies our choice of $\pm\sigma\sqrt{\frac{T}{N}}$. If the factors were too large there would be no convergence, if too small there would be a trivial deterministic limit.

The discounted final price $\bar{S}_N^{(N)}$ under the martingale measure is distributed as

$$\exp\left(\mathcal{N}\left(-\frac{\sigma^2}{2}T, \sigma^2 T\right)\right) \sim \exp\left(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}\cdot\mathcal{N}(0,1)\right).$$

We can now plug this into the general formula

$$\mathbb{E}_Q\left(\beta_T \cdot (\text{payoff at time } T)\right)$$

and we obtain the Black-Scholes formula:

European call:

density of $\mathcal{N}(0,1)$

$$\int_{-\infty}^{\infty} \underbrace{\left(S_0 e^{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}x} - e^{-rT}K\right)}_{\text{discounted payoff with underlying asset price distributed as above.}}^{+} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \, dx$$

European put:

$$\int_{-\infty}^{\infty} \left(e^{-rT}K - S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x}\right)^{+} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \, dx.$$