Last time we finished by introducing the complex logarithm as a multivalued fcn.

For $z \neq 0$ we defined $\log z$ by

$$\log z := \ln|z| + i \arg z = \ln|z| + i(\operatorname{Arg} z + k2\pi),\ k \in \mathbb{Z}$$

Recall that $\operatorname{Arg} z \in (-\pi, \pi]$ (Principal branch of $\arg z$)

Branches of the logarithm

If in the defn above we replace $\arg z$ by $\operatorname{Arg} z$, we obtain a singlevalued fcn; a so-called branch of $\log z$.

$$\boxed{\operatorname{Log} z := \ln|z| + i \operatorname{Arg} z,\quad z \in \mathbb{C}\setminus\{0\}}$$

This is called the principal logarithm.

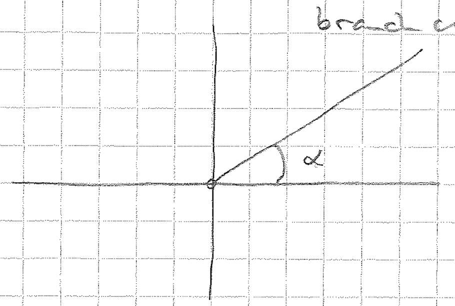
Note that $\operatorname{Log} z$ extends "the usual logarithm" (defined on $(0,\infty)$) to $\mathbb{C}\setminus\{0\}$.

Note also that $\operatorname{Log} z$ is "discontinuous" along the negative real line; its so-called branch cut.

We shall see that $\operatorname{Log} z$ is differentiable in $\mathbb{C}\setminus(-\infty, 0]$.

There are other branches of $\log z$; e.g.

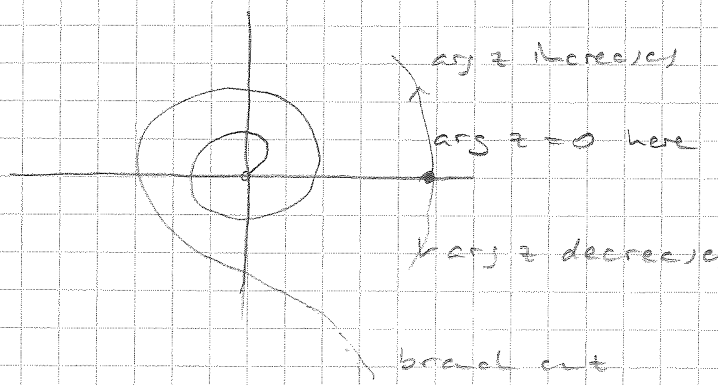
1)



$\log z := \ln|z| + i\arg_\alpha z,$

$\arg_\alpha z \in (\alpha, \alpha + 2\pi]$

2)



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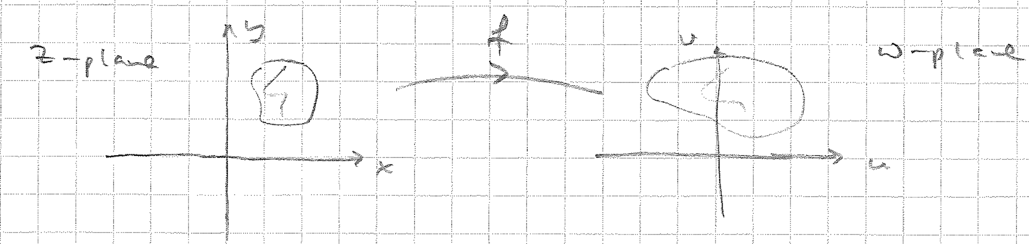
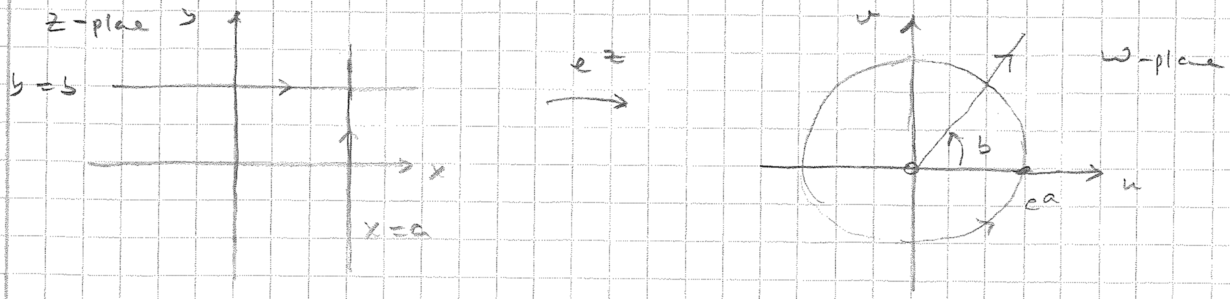
To visualize and understand a fcn $f: \mathbb{C} \to \mathbb{C}$ one can check how it maps various region.

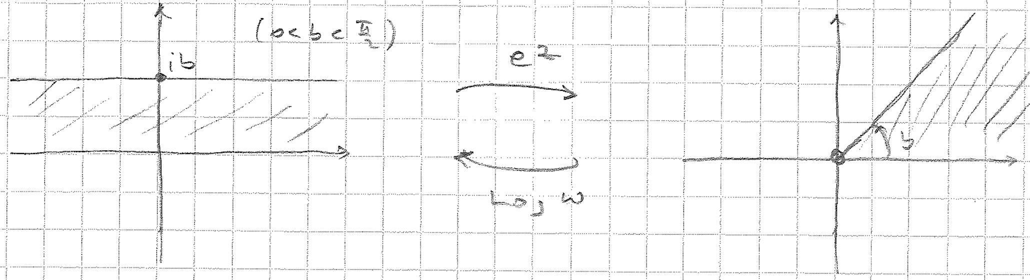
One usually writes $f(z) = f(x+iy) = w = u+iv$ and draw two copies of $\mathbb{C}$; a z-plane for the domain and a w-plane for the range.

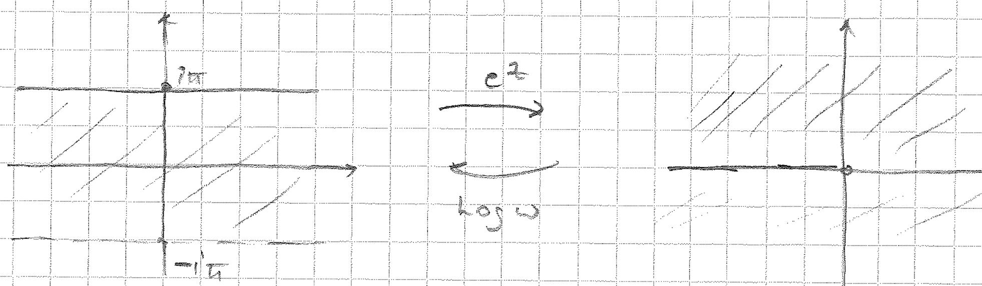


Mapping properties of e^z and $\log z$

$f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z = e^x(\cos y + i \sin y)$

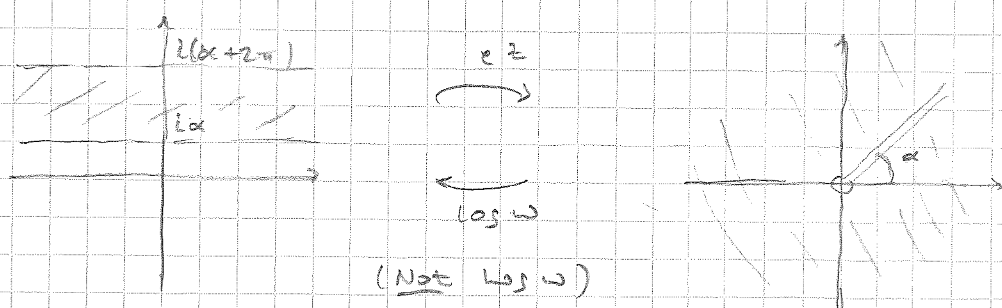






$-\pi < \operatorname{Im} z \le \pi \longrightarrow \mathbb{C} \setminus \{0\}$

$-\pi < \operatorname{Im} z < \pi \longrightarrow \mathbb{C} \setminus (-\infty, 0]$



(Not Log w)

Complex powers

Given $z \in \mathbb{C}$ consider the equation

$$w^n = z \qquad (*)$$

The set of all solutions w of $(*)$ is denoted $z^{\frac{1}{n}}$, and called the nth root of z.

If $z=0$, clearly $w=0$. Suppose now $z \neq 0$.

Write $w = |w| e^{i\alpha}$, $z = |z| e^{i\theta}$.

By deMoivre's formula $(*)$ becomes

$$|w|^n e^{in\alpha} = |z| e^{i\theta}$$

Clearly then

$$\begin{cases} |w| = \sqrt[n]{|z|} \\ n\alpha = \theta + k 2\pi, \ k \in \mathbb{Z} \end{cases} \iff \begin{cases} |w| = \sqrt[n]{|z|} \\ \alpha = \frac{\theta}{n} + k \frac{2\pi}{n}, \ k \in \mathbb{Z} \end{cases}$$

Every $k \in \mathbb{Z}$ gives a solution of $(*)$.

Note however, that since cosine and sine are 2π-period, only $k = 0, 1, \dots, n-1$ gives <u>different</u> solutions of $(*)$.

That is

$$z^{\frac{1}{n}} = \sqrt[n]{|z|}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)}, \quad k = 0, 1, \dots, n-1.$$

Suppose $z \neq 0$. For $n \in \mathbb{Z}$ it holds that

$$z^n = e^{n \log z}$$

for every value that $\log z$ attains.

It is also true, that for $n = 1, 2, 3, \ldots$,

$$z^{\frac{1}{n}} = e^{\frac{1}{n} \log z},$$

i.e. the right-hand side attains n values.

We therefore make the following:

Def: For $\alpha \in \mathbb{C}$ we let

$$z^{\alpha} := e^{\alpha \log z}, \quad z \neq 0.$$

This makes z^{α} (in general) a multivalued fcn.

Ex. Compute i^{-2i}

Sol. $i^{-2i} = e^{-2i \log i}$

Now, $\log i = \text{Log}\,|i| + i \arg i =$

$$= \text{Ln}\, 1 + i(\text{Arg}\, i + k2\pi) = i\left(\frac{\pi}{2} + k2\pi\right), \; k \in \mathbb{Z}$$

$$\Rightarrow \; i^{-2i} = e^{-2i \cdot i\left(\frac{\pi}{2} + k2\pi\right)} =$$

$$= e^{(4k+1)\pi}, \; k \in \mathbb{Z}$$

$\square$

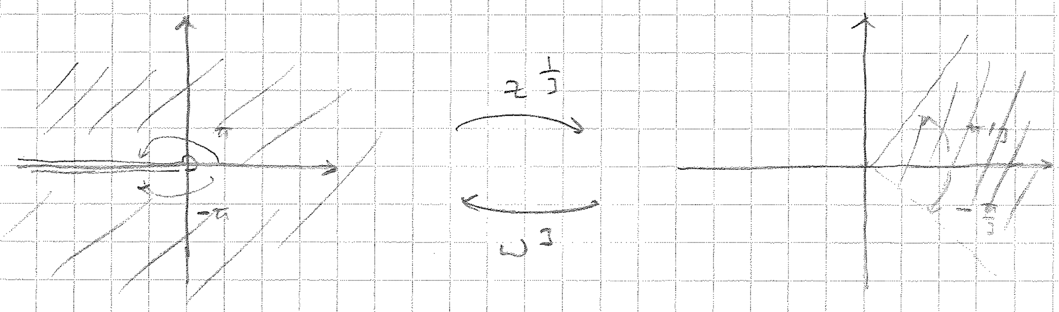
By choosing a branch of $\log z$ (i.e. of $\arg z$) in the def. of z^α one obtains a branch of z^α. For example, the principal branch of z^α is defined by

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

<u>Ex.</u> The principal branch of $z^{\frac{1}{3}}$ is given by

$$z^{\frac{1}{3}} = e^{\frac{1}{3}\operatorname{Log} z} = e^{\frac{1}{3}(\ln|z| + i \operatorname{Arg} z)} =$$

$$= |z|^{\frac{1}{3}} e^{i \frac{\operatorname{Arg} z}{3}}.$$



<u>Trigonometric and hyperbolic fcns</u>

We have that $\begin{cases} e^{iy} = \cos y + i \sin y \\ e^{-iy} = \cos y - i \sin y \end{cases}$ $(y \in \mathbb{R})$

$$\Rightarrow \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \qquad (y \in \mathbb{R})$$

<u>Def</u> For $z \in \mathbb{C}$ we define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

We also define $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$ etc.

Since $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$ $(x \in \mathbb{R})$

we furthermore make the following.

Def For $z \in \mathbb{C}$ we define

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Ex. Solve $\sin z = 2$

Sol. $\sin z = 2 \iff \frac{e^{iz} - e^{-iz}}{2i} = 2$

$\iff e^{iz} - e^{-iz} - 4i = 0 \iff (e^{iz})^2 - 4ie^{iz} - 1 = 0$

$\iff e^{iz} = 2i \pm \sqrt{-4+1} = 2i \pm i\sqrt{3}$

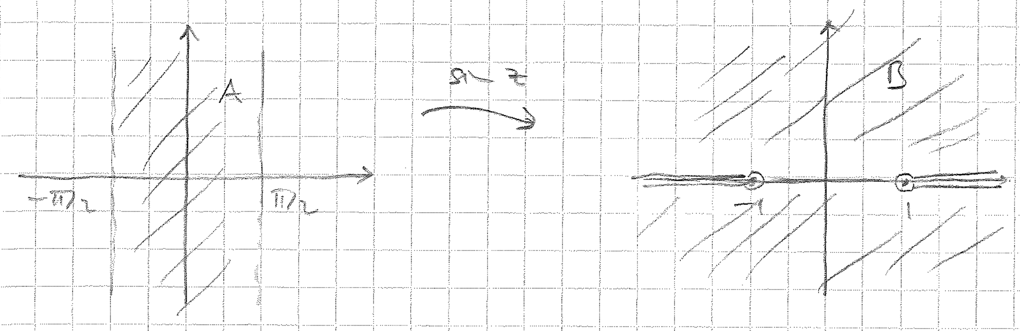
$\iff e^{iz} = i\underbrace{(2 \pm \sqrt{3})}_{>0}$

$\iff iz = \log(i(2 \pm \sqrt{3})) = \ln(2 \pm \sqrt{3}) + i(\frac{\pi}{2} + k2\pi)$ $(k \in \mathbb{Z})$

$\iff z = \frac{\pi}{2} + k2\pi - i\underbrace{\ln(2 \pm \sqrt{3})}_{\pm \ln(2+\sqrt{3})}$, $k \in \mathbb{Z}$. $\square$

Mapping properties of $\sin z$

Let $f(z) = \sin z$ in $-\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}$.



Claim: $f : A \to B$ is a bijective mapping (one-to-one and onto)

$z = x + iy$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(z) = \sin(x+iy) \overset{!}{=} \sin x \cosh y + i \cos x \sinh y$

$f(z) \in \mathbb{R} \Leftrightarrow \cos x \sinh y = 0 \underset{\cos x > 0}{\Longleftrightarrow} \sinh y = 0 \Leftrightarrow y = 0$

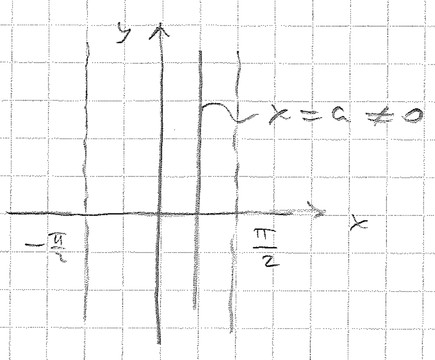
If $y = 0$ (and $-\frac{\pi}{2} < x < \frac{\pi}{2}$)

$f(z) = \sin x \cosh y = \sin x \in (-1, 1)$.

So, if $z \in A \Rightarrow f(z) \in B$.

Put $u = \sin x \cosh y$, $v = \cos x \sinh y$

Next consider images of vertical/horizontal lines.



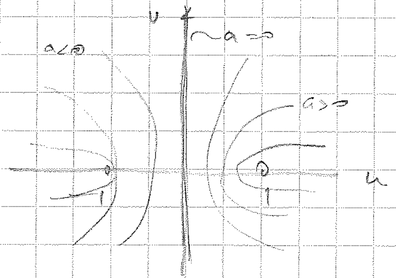
If $x = a \neq 0$, then

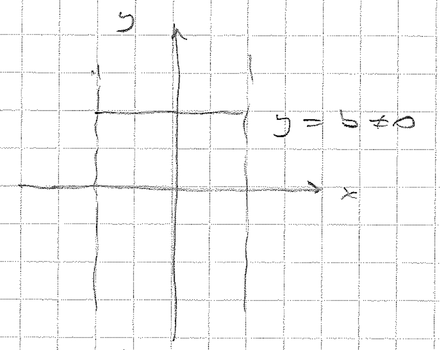
$$\cosh y = \frac{u}{\sin a}, \quad \sinh y = \frac{v}{\cos a}$$

Since $(\cosh y)^2 - (\sinh y)^2 = 1$

$$\Rightarrow \left(\frac{u}{\sin a}\right)^2 - \left(\frac{v}{\cos a}\right)^2 = 1$$

This is a hyperbola in the w-plane.





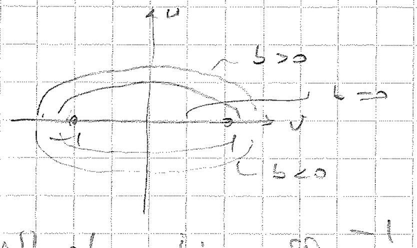
If $y = b \neq 0$, then $\sin x = \frac{u}{\cosh b}$, $\cos x = \frac{v}{\sinh b}$

Since $\cos^2 x + \sin^2 x = 1$.

$$\Rightarrow \left(\frac{u}{\cosh b}\right)^2 + \left(\frac{v}{\sinh b}\right)^2 = 1$$

This is a half-ellipse. Note that

$$v > 0 \iff \sinh b > 0 \iff b > 0$$



Question: What is $\sin^{-1} w$ for $w \in \mathbb{C}$?