Le 22

The Riemann mapping theorem

Question: Given two open sets U and V in $\mathbb{C}$, does there exist a bijective analytic map

$$f : U \to V \quad ?$$

Recall that then f^{-1} is also analytic.

If so, we say that U and V are conformally equivalent. Let us take $V = \mathbb{D}$.

The problem is then:

Given an open subset U of $\mathbb{C}$, what conditions on U guarantee that there is an analytic bijection $f : U \to \mathbb{D}$?

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g^{-1}\circ f"'] & \mathbb{D} & V \arrow[l, "g"'] \end{tikzcd}$$

Certain conditions are obviously necessary.

First, if $U = \mathbb{C}$ no such map exists, since by Liouville's thm f would have to be constant. Since $\mathbb{D}$ is connected, in fact simply connected, the same must be true for U. It is remarkable that these necessary conditions are indeed sufficient.

<u>Thm</u> (Riemann mapping thm)

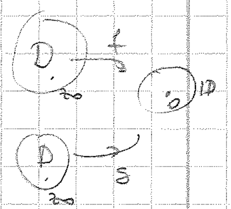
Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$.

If $z_0 \in D$, then there exists a unique analytic bijection $f : D \to \mathbb{D}$ such that

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0.$$

The uniqueness statement is straightforward, since if f and g are two maps that satisfy these conditions, then

$$h = f \circ g^{-1}$$



is a conformal self-map of $\mathbb{D}$ that fixes the origin. Therefore, $h(w) = e^{i\varphi} w$.

However,

$$h'(0) = (f \circ g^{-1})'(0) = f'(g^{-1}(0))(g^{-1})'(0) =$$

$$= f'(g^{-1}(0)) \cdot \frac{1}{g'(g^{-1}(0))} =$$

$$= f'(z_0) \cdot \frac{1}{g'(z_0)} > 0$$

so that it must be that $e^{i\varphi} = 1$.

Thus $(f \circ g^{-1})(w) = w$. Writing $w = g(z)$, we have that $f(z) = g(z)$.

We shall now show the existence of the conformal map f. The idea of proof is as follows. We consider all one-to-one analytic maps f taking D into $\mathbb{D}$ and z_0 to 0. From these we wish to choose an f whose image fills out all of $\mathbb{D}$, and this can be achieved by making $|f'(z_0)|$ as large as possible.