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Matematical statistics
Inference II 1MS037
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Hand-in assignment 1, solutions

1. Consider the discrete random variable X with probability function

$$P(X=k)=\begin{cases} 9\theta_1^2, & \text{if } k=0,\\ 6\theta_1\theta_2, & \text{if } k=1,\\ \theta_2^2, & \text{if } k=2,\\ 0, & \text{otherwise,}\end{cases}$$

where $3\theta_1+\theta_2=1$.

Consider an independent sample $\mathbf{X}=(X_1,...,X_n)$ where all X_i, $i=1,2,...,n$, are distributed as X.

(a) Does the distribution belong to a strictly k-parametric exponential family? (3p)

Solution: For $m=0,1,2$, let $I_m(x_i)=1$ if $x_i=m$ and 0 otherwise. Moreover, let n_m be the number of observations that equal m. The probability function is

$$\begin{aligned} p(\mathbf{x};\theta_1,\theta_2) &= \prod_{i=1}^n (9\theta_1^2)^{I_0(x_i)}(6\theta_1\theta_2)^{I_1(x_i)}(\theta_2^2)^{I_2(x_i)}\\ &= 9^{n_0}6^{n_1}\theta_1^{2n_0+n_1}\theta_2^{n_1+2n_2}.\end{aligned}$$

Moreover, observe that given the total number of observations, n, we have $n=n_0+n_1+n_2$, i.e. $n_2=n-n_0-n_1$, which yields

$$p(\mathbf{x};\theta_1,\theta_2)=9^{n_0}6^{n_1}\theta_1^{2n_0+n_1}\theta_2^{2n-(2n_0+n_1)}.$$

At first sight, this appears to be a 2-parametric family, but introducing the restriction $\theta_2=1-3\theta_1$, we get

$$\begin{aligned} p(\mathbf{x};\theta_1) &= 9^{n_0}6^{n_1}\theta_1^{2n_0+n_1}(1-3\theta_1)^{2n-(2n_0+n_1)}\\ &= 9^{n_0}6^{n_1}(1-3\theta_1)^{2n}\left(\frac{\theta_1}{1-3\theta_1}\right)^{2n_0+n_1}\\ &= 9^{n_0}6^{n_1}(1-3\theta_1)^{2n}\exp\left\{(2n_0+n_1)\log\left(\frac{\theta_1}{1-3\theta_1}\right)\right\}\\ &= A(\theta_1)\exp\{\zeta_1(\theta_1)T_1(\mathbf{x})\}h(\mathbf{x}),\end{aligned}$$

where $A(\theta) = (1 - 3\theta_1)^{2n}$, $h(\mathbf{x}) = 9^{n_0} 6^{n_1}$, $\zeta_1(\theta) = \log\left(\frac{\theta_1}{1-3\theta_1}\right)$ and $T_1(\mathbf{x}) = 2n_0 + n_1$. (Observe that the n_j are functions of $\mathbf{x}$.) Hence, the probability function is on exponential form. We only have one natural parameter, so the distribution belongs to a strictly k-parametric exponential family with $k = 1$.

(b) Derive k and the corresponding sufficient statistic(s). (2p)

Solution: From (a), we have $k = 1$ and the sufficient statistic $T_1(\mathbf{x}) = 2n_0 + n_1$, where n_0 is the number of zeros and n_1 is the number of ones in the sample.

2. Consider a random sample $\mathbf{X} = (X_1, ..., X_n)$ where the X_i are independent continuous random variables with density function

$$f(x) = \frac{4}{\theta} x^3 \exp\left(-\frac{x^4}{\theta}\right),$$

for $x \geq 0$ and 0 otherwise.

(a) Calculate the score function. (2p)

Solution: Say that we have a sample $x_1, ..., x_n$. We may write down the likelihood function

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{4}{\theta} x_i^3 \exp\left(-\frac{x_i^4}{\theta}\right) = C\theta^{-n} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i^4\right),$$

where C is a constant in the sense that it does not depend on θ.
This gives us the log likelihood and its first two derivatives w.r.t θ as

$$\begin{aligned} l(\theta) &= \log C - n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i^4, \\ l'(\theta) &= -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^4, \\ l''(\theta) &= \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i^4, \end{aligned}$$

from which we see in particular that the score function is

$$V(\theta; \mathbf{X}) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} X_i^4.$$

(b) Calculate the Fisher information. (3p)

Hint: Without proof, you may use that $E(X^k) = \theta^{k/4}\Gamma\left(1 + \frac{k}{4}\right)$,
for $k = 1, 2, ...$, where $\Gamma(\cdot)$ is the Gamma function.

Solution: The hint gives us

$$\mathrm{Var}(X_i^4) = E(X_i^8) - \{E(X_i^4)\}^2 = \theta^2\Gamma(3) - \{\theta\Gamma(2)\}^2 = 2\theta^2 - \theta^2 = \theta^2,$$

and we find the Fisher information as

$$I_{\mathbf{X}}(\theta) = \mathrm{Var}\{V(\theta; \mathbf{X})\} = \frac{1}{\theta^4}\sum_{i=1}^{n} \mathrm{Var}(X_i^4) = \frac{n}{\theta^2}.$$

Alternatively, we get from the results of (a) and the hint
(giving $E(X_i^4) = \theta$) that

$$I_{\mathbf{X}}(\theta) = -\mathrm{E}\left\{l''(\theta; \mathbf{X})\right\} = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} E(X_i^4) = \frac{n}{\theta^2}.$$

3. Consider a random sample $\mathbf{X} = (X_1, ..., X_n)$ where the X_i are i.i.d. and discrete with probability function

$$p(x;\theta) = \theta(1-\theta)^{x-1},\ x = 1, 2, ...,$$

where $0 \le \theta \le 1$.

(a) Show that the statistic $T = (X_1, ..., X_{n-1})$ is *not* sufficient for θ. (3p)

Solution: Sufficiency means that the probability of the sample conditional on the suggested T is no function of the parameter. We will check that it is not so. To this end, if the sample is $(x_1, ..., x_n)$, the probability in question is

$$\begin{aligned}
&P\{(X_1, ..., X_n) = (x_1, ..., x_n)|(X_1, ..., X_{n-1}) = (x_1, ..., x_{n-1})\} \\
&= \frac{P\{(X_1, ..., X_n) = (x_1, ..., x_n), (X_1, ..., X_{n-1}) = (x_1, ..., x_{n-1})\}}{P\{(X_1, ..., X_{n-1}) = (x_1, ..., x_{n-1})\}} \\
&= \frac{P\{(X_1, ..., X_n) = (x_1, ..., x_n)\}}{P\{(X_1, ..., X_{n-1}) = (x_1, ..., x_{n-1})\}} \\
&= \frac{P(X_1 = x_1, ..., X_n = x_n)\}}{P(X_1 = x_1, ..., X_{n-1} = x_{n-1})\}},
\end{aligned}$$

and by independence, this is

$$\begin{aligned}
&\frac{P(X_1 = x_1) \cdot ... \cdot P(X_{n-1} = x_{n-1})P(X_n = x_n)}{P(X_1 = x_1) \cdot ... \cdot P(X_{n-1} = x_{n-1})} \\
&= P(X_n = x_n) = \theta(1-\theta)^{x_n - 1},
\end{aligned}$$

which is a function of θ, as was to be proved.
Thus, T is not sufficient for θ.

(b) Find a sufficient statistic for θ. (2p)

Solution: If we have the sample $(x_1, ..., x_n)$, the likelihood is

$$L(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1} = \theta^n (1-\theta)^{-n}(1-\theta)^t,$$

where $t = \sum_{i=1}^n x_i$.
Hence, by the factorization theorem, $T = \sum_{i=1}^n X_i$ is sufficient.
(This is not the only possible sufficient statistic. For example, also $T = (X_1, ..., X_n)$ is sufficient.)

4. Consider a random sample $\mathbf{X} = (X_1, ..., X_n)$ where the X_i are i.i.d. and continuous random variables with density function

$$f(x) = \theta x^{-2} \exp(-\theta x^{-1}),$$

for $x \geq 0$, $\theta > 0$, and $f(x) = 0$ for $x < 0$.

Find a minimal sufficient statistic for θ. (5p)

Solution: Say that we have a sample $\mathbf{x} = (x_1, ..., x_n)$. This gives the likelihood as

$$\begin{aligned} L(\theta; \mathbf{x}) &= \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta x_i^{-2} \exp(-\theta x_i^{-1}) \\ &= \theta^n \left(\prod_{i=1}^{n} x_i \right)^{-2} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-1} \right). \end{aligned}$$

If $\mathbf{y} = (y_1, ..., y_n)$ is another sample, we have

$$\begin{aligned} \frac{L(\theta; \mathbf{x})}{L(\theta; \mathbf{y})} &= \frac{\theta^n \left(\prod_{i=1}^{n} x_i\right)^{-2} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-1}\right)}{\theta^n \left(\prod_{i=1}^{n} y_i\right)^{-2} \exp\left(-\theta \sum_{i=1}^{n} y_i^{-1}\right)} \\ &= \frac{\left(\prod_{i=1}^{n} y_i\right)^2}{\left(\prod_{i=1}^{n} x_i\right)^2} \exp\left\{ \theta \left(\sum_{i=1}^{n} y_i^{-1} - \sum_{i=1}^{n} x_i^{-1} \right) \right\}. \end{aligned}$$

This is no function of θ if $\sum_{i=1}^{n} y_i^{-1} = \sum_{i=1}^{n} x_i^{-1}$. Hence, by theorem 3.8, $T(\mathbf{X}) = \sum_{i=1}^{n} X_i^{-1}$ is minimal sufficient.

Alternatively, we can see directly from the likelihood that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i^{-1}$ is sufficient, and since the distribution is seen to belong to the exponential family, by corollary 3.4 $T(\mathbf{X})$ is also minimal sufficient.