<u>Recall</u>: • A stopping time $T$ is a random variable such that $\{T \le n\} \in \tilde{\mathcal{F}}_n$

• The stopped process $X_{T\wedge n} = \begin{cases} X_n & n \le T \\ X_T & T > n \end{cases}$,

$X^T = \lim_n X_{T\wedge n}$, if it exists.

• Doob's Optional stopping: $\mathbb{E}(X^T) = \mathbb{E}(X_0)$ for "nice" $X_n$, $T$.

The following lemma is useful to show that $\mathbb{E}(T) < \infty$ for specific stopping times.

<u>Lemma</u> Suppose there exists $\varepsilon > 0$ & a positive integer $N$ such that $\underbrace{\mathbb{P}(T \le n + N \mid \tilde{\mathcal{F}}_n) \ge \varepsilon}$ for all $n$.

Then $\mathbb{E}(T) < \infty$.

"The probability of stopping at any point within the next $N$ steps is at least $\varepsilon > 0$".

Proof: We have

$$\mathbb{P}(T > N) \le 1 - \varepsilon \qquad \text{(first } N \text{ steps)}$$

$$\mathbb{P}(T > 2N \mid T > N) \le 1 - \varepsilon \qquad \text{(steps } N+1, \ldots, 2N)$$

$$\mathbb{P}(T > 3N \mid T > 2N) \le 1 - \varepsilon \qquad \ldots$$

So,
$$\mathbb{E}(T) \le N \cdot \varepsilon + 2N\varepsilon(1-\varepsilon) + 3N\varepsilon(1-\varepsilon)^2 + \ldots$$
$$= N\varepsilon\left(1 + 2(1-\varepsilon) + 3(1-\varepsilon)^2 + \ldots\right)$$
$$= N\varepsilon \frac{1}{(1-(1-\varepsilon))^2} = \frac{N}{\varepsilon} < \infty \qquad \square$$

Example: Consider the simple random walk

$$X_n = \begin{cases} X_{n-1} + 1 & \text{with prob } \frac{1}{2} \\ X_{n-1} - 1 & \text{---"---} \ \frac{1}{2} \end{cases}, \quad X_0 = 0.$$

Take $T = \min\{n : |X_n| = a\}$, then $\mathbb{E}(T) < \infty$.

It follows by taking $N = a$, $\varepsilon = \frac{1}{2^a}$.

More generally, we can consider

$T = \min\{n : X_n \geq a \text{ or } X_n \leq -b\}$.

Since $|X_k - X_{k-1}| = 1$, the third (or second item) of Doob's optional stopping theorem applies.

This allows us to answer questions such as:

- What is the probability that we reach $a$ before $-b$?
- What is the expected time for one of the two to happen.

We get $\mathbb{E}(X_T) = \mathbb{E}(X_0) = 0$ (from DOST)

$$\Leftrightarrow \quad a \cdot \mathbb{P}(X_T = a) + (-b)\,\mathbb{P}(X_T = -b) = 0$$

Further $\mathbb{P}(X_T = a) + \mathbb{P}(X_T = -b) = 1$

$$\Rightarrow \quad \mathbb{P}(X_T = a) = \frac{b}{a+b}, \quad \mathbb{P}(X_T = -b) = \frac{a}{a+b}$$

Now look at $X_n^2$:

$$\mathbb{E}(X_n^2 \mid \tilde{F}_{n-1}) = \frac{1}{2}(X_{n-1}+1)^2 + \frac{1}{2}(X_{n-1}-1)^2$$

$$= X_{n-1}^2 + 1$$

It follows that

$$\mathbb{E}(X_n^2 - n \mid \tilde{F}_{n-1}) = X_{n-1}^2 + 1 - n = X_{n-1}^2 - (n-1).$$

Hence $Y_n = X_n^2 - n$ is a martingale !!

The $2^{nd}$ & $3^{rd}$ item of DOST apply and

$$\mathbb{E}(Y_T) = \mathbb{E}(Y_0) = 0$$

$Y_T = X_T^2 - T$ = either $a^2 - T$ or $b^2 - T$.

$\Rightarrow \mathbb{E}(X_T^2) = \mathbb{E}(T)$ and

$a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = \mathbb{E}(T)$, so we find that

$$\mathbb{E}(T) = \frac{a^2 b + b^2 a}{a+b} = \frac{ab(a+b)}{a+b} = ab$$

# The Convergence Theorem

Are there conditions under which a martingale converges to a limit $X_\infty$? (Limit may still be random)

<u>Example</u>: $X_0 = 0$, $X_n = X_{n-1} \begin{cases} + \frac{1}{2^n} & \text{with prob. } \frac{1}{2} \\ - \frac{1}{2^n} & \text{—"—} \ \frac{1}{2} \end{cases}$

We can express $X_n$ as

$$X_n = \sum_{k=1}^{n} \frac{1}{2^k} Y_k \qquad \text{with } Y_k = \pm 1$$

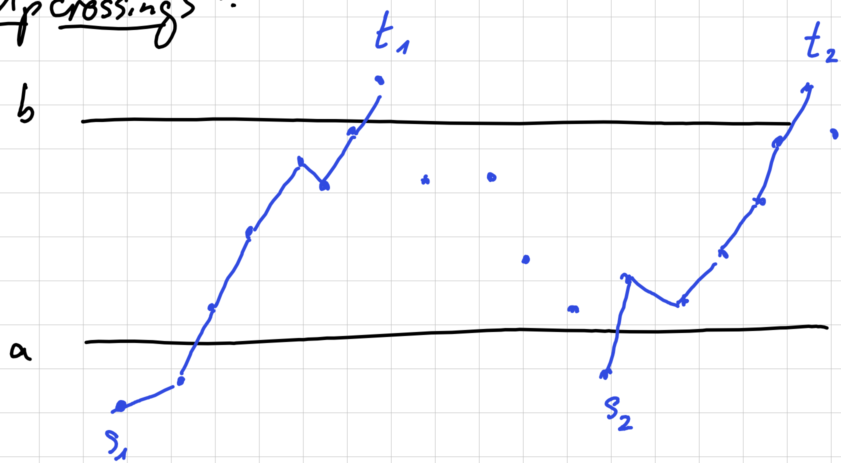
$$X_\infty = \sum_{k=1}^{\infty} \frac{1}{2^k} Y_k$$ always exists because sum is absolutely convergent.

In fact, $X_\infty$ is uniformly distributed on $[-1, 1]$.

[Bernoulli Convolutions Project]

We want to establish conditions under which martingales converge almost surely.

## Upcrossings:




Fix $a<b$; an upcrossing starts from a value below a and ends with a value above b.

Formally, let $X_n$ be an adapted process, and let $U_N[a,b](\omega)$ be the largest $k$ such that there exist times

$$0 \leq s_1 < t_1 < s_2 < \dots < s_k < t_k \leq N$$

with $X_{s_i}(\omega) < a$ and $X_{t_i}(\omega) > b$ for all $i$.

Consider the previsible process that is equal to 1 within an upcrossing and 0 otherwise.

$$C_1 = I_{\{X_0 < a\}}$$

$$C_n = I_{\{C_{n-1}=1\}} \cdot I_{\{X_{n-1} \leq b\}} + I_{\{C_{n-1}=0\}} \, I_{\{X_{n-1} < a\}}$$

currently in an upcrossing

not completed yet

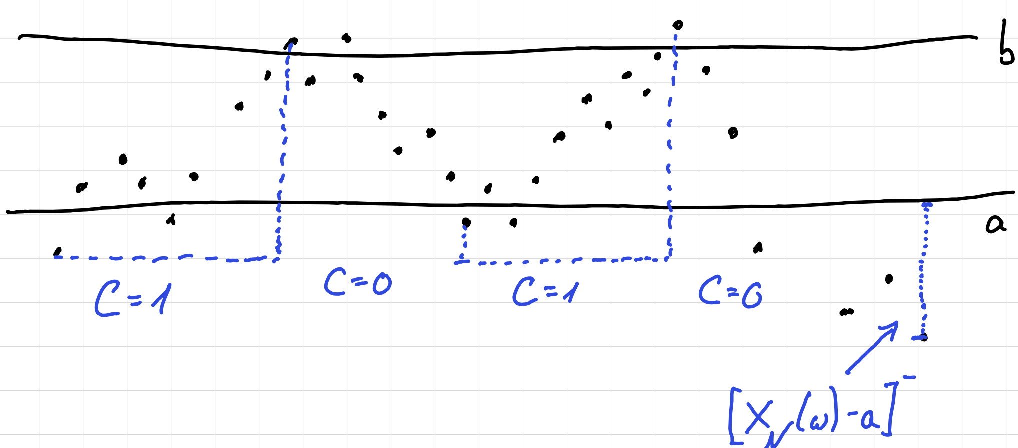
currently not in an upcrossing

starting a new upcrossing

The transformed sequence $Y = C \bullet X$ satisfies

$$Y_n(\omega) \geq \underbrace{(b-a)\; U_N[a,b](\omega)}_{\text{within each upcrossing}} - \underbrace{[X_N(\omega) - a]^-}_{\text{correction for last incomplete upcrossing.}}$$

$$\sum (X_i - X_{i-1}) \geq (b-a)$$



Apply expectation to both sides to get:

Doob's upcrossing lemma: If $X$ is a supermartingale, then

$$(b-a)\,\mathbb{E}(U_N[a,b]) \leq \mathbb{E}((X_n-a)^-)$$

This follows since the transform of a supermartingale by a non-negative pre-visible process is still a supermartingale:

So $Y$ is a supermartingale, thus

$$\mathbb{E}(Y_N) \leq \mathbb{E}(Y_0)=0.$$

Corollary: If $X$ is a supermartingale with $\sup_n \mathbb{E}(|X_n|)<\infty$, then we have

$$(b-a)\,\mathbb{E}(U_\infty[a,b]) \leq |a| + \sup_n \mathbb{E}(|X_n|) < \infty,$$

where $U_\infty[a,b] = \lim_{N\to\infty} U_N[a,b]$.

In particular, $U_\infty[a,b]$ is a.s. finite.

Proof: We have

$$(b-a)\,\mathbb{E}(U_N[a,b]) \leq \mathbb{E}((X_N-a)^-)$$

$$\leq \mathbb{E}(|X_N - a|) \leq \mathbb{E}(|X_N|) + |a|$$

$$\leq \sup_n \mathbb{E}(|X_n - a|) + |a|.$$

We take $N \to \infty$ and apply MCT. $\square$

## Doob's Convergence Theorem

Let $X_n$ be a supermartingale with $\sup_n \mathbb{E}(|X_n|) < \infty$. Then, $X_\infty = \lim_{n\to\infty} X_n$ exists a.s. and is finite.

( to make $X_\infty$ well-defined where limit does not exist one can define it as $X_\infty = \limsup_{n\to\infty} X_n$.
The statement above becomes $\lim_{n\to\infty} X_n = X_\infty$ a.s. and $X_\infty \neq \pm\infty$ a.s.)

Proof: Suppose that for some $\omega \in \Omega$, the limit does not exist (even as $\pm\infty$). Then there are $a, b \in \mathbb{Q}$ such that

$$\liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)$$

This means that $X_n(\omega)$ drops below $a$ and rises above $b$ infinitely many times.

So $U_\infty[a,b] = \infty$.

We conclude

$$E = \{\omega \in \Omega : \liminf_n X_n(\omega) \neq \limsup_n X_n(\omega)\}$$

$$\subseteq \bigcup_{\substack{a,b\in\mathbb{Q} \\ a<b}} \{\omega \in \Omega : U_\infty[a,b](\omega) = \infty\}$$

which is a countable union of null sets and $\mathbb{P}(E) = 0$. Hence limits exist almost surely.

It remains to show that limit is finite a.s.

By Fatou's lemma, $\mathbb{E}(|X_\infty|) = \mathbb{E}(\liminf_n X_n)$

$$\leq \liminf_n \mathbb{E}|X_n| \leq \sup_n \mathbb{E}|X_n| < \infty$$

by assumption. This completes the proof. $\square$

Remark: In particular, the theorem holds if $|X_n| \leq K \quad \forall n \quad$ (a.s.).

Remark: If $X_n$ is a non-negative supermartingale then $\mathbb{E}(|X_n|) = \mathbb{E}(X_n) \leq \mathbb{E}(X_0)$ for all $n$, and the condition holds, provided $\mathbb{E}(X_0) < \infty$.

[ Non-negative martingale convergence theorem:

non-negative martingales converge ]