UPPSALA UNIVERSITET
Matematiska Institutionen
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Prov i matematik
1MA255 VT23
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Partial Differential Equations with Applications to Finance

Writing time: 08:00 - 13:00.
Instructions: There are 5 problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, 4 or 5, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no help materials are allowed.

GOOD LUCK!

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1. (8p) Let $u(t, x)$ be the solution of

$$\begin{cases} u_t - u_{xx} &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) &= 2x^2 + x, \quad x \in \mathbb{R} \end{cases}$$

i) (3p) Use the Feynman-Kac formula to solve this initial value problem.

ii) (5p) Use **i** to compute

$$\int_{\mathbb{R}} e^{-x^2}(2x^2 + x)dx.$$

Hint: use the fundamental solution to construct the solution of the initial value problem, and plug in specific values for (t, x).

2. (8p) Let $D = (-a, a)$, where $a > 0$. Let X_t be a Brownian motion with drift:

$$dX_t = \mu dt + dB_t, \quad X_0 = 0,$$

where $\mu \neq 0$. Let $\tau := \inf\{t > 0 : X_t \notin D\}$. Formulate suitable Dirichlet/Poisson problems and compute

i) (4p) $\mathbb{P}(X_\tau = a)$.

ii) (4p) $\mathbb{E}[X_\tau^+ + \tau]$, where $x^+ = \max(x, 0)$.
Hint: the general solution for the ODE $ay'' + by' + c = 0$ is

$$y(x) = C_1 e^{-\frac{b}{a}x} - \frac{c}{b}x + C_2$$

where C_1 and C_2 are arbitrary constants.

3. (8p) Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function and $D > 0$. Propose appropriate assumptions, and use the Fokker-Planck equation to show that the Gibbs distribution

$$p(x) = \frac{1}{Z} e^{-\frac{V(x)}{D}}$$

is the limiting density function for X_t that solves

$$dX_t = -DV'(X_t)dt + \sqrt{2D}dB_t,$$

and find the normalising factor Z.

4. (8p) Let X_t satisfy the SDE

$$dX_t = u_t dt + \sigma dB_t,$$

where $\sigma > 0$ and B_t is a standard Brownian motion, $u_t \in \mathbb{R}$ is a control process. Solve the following minimisation problem

$$V(x) = \inf_u \mathbb{E}_x[\int_0^\infty e^{-\rho t}(X_t^2 + \theta u_t^2)dt]$$

where ρ, θ are positive constants.

Hint: use the ansatz $V(x) = ax^2 + b$ *for some constants* a, b *where* $a > 0$.

5. (8p) Let the stochastic process X be a geometric Brownian motion, such that under the risk-neutral measure $\mathbb{Q}$,

$$dX_t = (r - \delta)X_t dt + \sigma X_t dB_t, \quad X_0 = x,$$

where $r > 0$ is the risk-free rate and $0 \leq \delta < r$ is the dividend.

i) (6p) When $\delta > 0$, find a price for the perpetual American call option, i.e., solve the following stopping problem:

$$V(x) = \sup_\tau \mathbb{E}_x^{\mathbb{Q}}[e^{-r\tau}(X_\tau - K)^+].$$

You **do not** need to prove the verification theorem for your solution.

ii) (2p) When $\delta = 0$, find the value function $V(x)$ and describe the structure of the continuation/stopping region.