The Poincaré-Lindstedt Method: the van der Pol oscillator

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The purpose of this document is to give a detailed overview of how the Poincaré-Lindstedt method can be used to approximate the limit cycle in the van der Pol system

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

where $\epsilon > 0$ is small.

1. Introduce a new time scale $\tau = \omega t$ so that the new period becomes 2π. The van der Pol equation becomes
$$\omega^2 x'' + \epsilon\omega(x^2 - 1)x' + x = 0,$$
where $x' = \omega^{-1}\dot{x}$ represents the derivative of x with respect to the new parameter τ, and similar for x''.

2. Substitute series expansions for
$$x_\epsilon(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \cdots \quad \text{and} \quad \omega_\epsilon = \omega_0 + \epsilon\omega_1 + \cdots$$
into the equation. Note that $\omega_0 = 1$ since the solution has period 2π when $\epsilon = 0$. Substitute the same expansions into the initial conditions and find the resulting initial conditions for $x_i(t)$.

   For the van der Pol equation we have hence
$$\begin{aligned}(1 + \epsilon\omega_1 + \cdots)^2(x_0''(\tau) + \epsilon x_1''(\tau) + \cdots) &+ \\ \epsilon(1 + \epsilon\omega_1 + \cdots)[(x_0(\tau) + \epsilon x_1(\tau) + \cdots)^2 - 1) &+ \\ x_0(\tau) + \epsilon x_1(\tau) + \cdots &= 0.\end{aligned}$$

3. Collect terms of the same order in ϵ. We get the following equations (up to second order):
$$\begin{aligned}&x_0'' + x_0 = 0\\ &x_1'' + x_1 = -2\omega_1 x_0'' - (x_0^2 - 1)x_0'\\ &x_2'' + x_2 = -(\omega_1^2 + 2\omega_2)x_0'' - 2\omega_1 x_1'' - (x_0^2 - 1)(x_1' + \omega_1 x_0') - 2x_0 x_1 x_0'.\end{aligned}$$

   Now suppose the initial conditions are $x(0) = a$ and $\dot{x}(0) = 0$, with a a constant which is not yet determined. The initial conditions for the order-by-order equations then become
$$x_0(0) = a, \quad x_1(0) = x_2(0) = 0 \quad \text{and} \quad x_0'(0) = x_1'(0) = x_2'(0) = 0.$$

4. Solve the resulting equations for $x_i(\tau)$, $i = 0, 1, \ldots$. Use the freedom in choosing the coefficients ω_i to eliminate resonant forces. *Adapt the initial conditions to correspond to a periodic orbit.* As to the latter, this is a special feature of the Poincaré-Lindstedt method, which is designed to determine *periodic solutions*: any attempt to choose the initial conditions to correspond to a non-periodic orbit will lead to resonant forces which cannot be eliminated through a specific choice of ω_1.

   For the zeroth-order equation, we have $x_0(\tau) = a\cos\tau$. Substituting this into the equation for x_1 we obtain

$$x_1'' + x_1 = 2a\omega_1 \cos\tau - a\left(1 - \frac{a^2}{4}\right)\sin\tau + \frac{a^3}{4}\sin 3\tau.$$

   There are two distinct resonant forces here: the term proportional to $\cos\tau$ can be eliminated by choosing $\omega_1 = 0$, while in order to suppress the term proportional to $\sin\tau$ we have to choose the initial conditions so that $a = \pm 2$. Let us pick $a = 2$. The resulting equation becomes $x_1'' + x_1 = 2\sin 3\tau$, with solution $x_1(\tau) = \sin^3\tau$.

   Using this scheme, one can now solve the equations up to any desired order, but the calculations become progressively more difficult, often making it necessary to use a computer algebra package. As a last step, let us determine ω_2, which will be the first non-trivial correction to the frequency. For this, we need to consider the equation for x_2, which becomes

$$x_2'' + x_2 = (4\omega_2 + 11)\cos\tau - 31\cos^3\tau + 20\cos^5\tau.$$

   The right-hand side can be rewritten as

$$\left(4\omega_2 + \frac{1}{4}\right)\cos\tau + A\cos 3\tau + B\cos 5\tau,$$

   where we don't need the coefficients A, B of the higher harmonics. In order to eliminate the resonant force, we need to choose

$$\omega_2 = -\frac{1}{16}.$$

   The resulting solution is

$$x(t) = 2\cos\omega t + \epsilon\sin^3\omega t + \cdots, \quad \text{with} \quad \omega = 1 - \frac{1}{16}\epsilon^2 + \cdots.$$

   We hence recover the fact that, for weak nonlinearities, the limit cycle of the van der Pol oscillator is approximately circular with radius 2.