Back to measurability

(Riedle Def 1.1.5?) + Def. 3.1.1.

Often a random process is defined on a filtered prob. space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Here the <u>filtration</u> $(\mathcal{F}_t)_{t\geq 0}$ is a family of σ-algebras, s.t.

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s \leq t$$

<u>Def</u> We say that $\{X_t\}_{t\geq 0}$ is <u>adapted</u> to $(\mathcal{F}_t)$, if $X_t \in \mathcal{F}_t$, $t \geq 0$ | $X_t^{-1}(B) \in \mathcal{F}_t$ all $B \in \mathcal{B}(\mathbb{R})$

<u>Ex</u> $(N_t)_{t\geq 0}$ Poisson process

$$\{T_n \leq t\} = \{N_t \geq n\} \in \mathcal{F}_t \quad \forall t$$

<u>Interpretation</u>: $\mathcal{F}_t$ contains all information about N_s, $0 \leq s \leq t$, i.e. the history of (N_t) up to time t.

F4-F5: Conditional expectation and martingales

$X, Y, \ldots$ r.v's on $(\Omega, \mathcal{F}, P)$

Def The conditional expectation of X, given $B \in \mathcal{F}$, is

$$E[X \mid B] = \frac{1}{P(B)} \int_B X \, dP .$$

Special case: $X = 1_A$, $A \in \mathcal{F}$.

$$E[1_A \mid B] = \frac{1}{P(B)} \int_{A \cap B} dP$$

$$= \frac{P(A \cap B)}{P(B)} =: P(A \mid B)$$

Example $E[X \mid \Omega] = \frac{1}{P(\Omega)} \int_\Omega X \, dP = E[X]$

"given that something happened"

Def The cond. expectation of X given $Y = k$, where Y discrete r.v.

$P(Y = k) > 0$, $0 \le k \le k_0$ or $0 \le k < \infty$,

is $E[X \mid Y = k] = \frac{1}{P(Y=k)} \int_{\{Y=k\}} X \, dP$

F4:2

Def $\mathbb{E}(X \mid Y)$ is the random variable taking the value $\mathbb{E}(X \mid Y=k)$, if $Y=k$

Important property:

$$\mathbb{E}[\mathbb{E}(X \mid Y)] = \mathbb{E}[X]$$

Proof

We have $\boxed{\int_B \mathbb{E}[X \mid B]\, dP} = \int_B \frac{1}{P(B)} \int_B X\, dP\, dP$

$$= \frac{P(B)}{P(B)} \int_B X\, dP = \boxed{\int_B X\, dP} \quad (*)$$

all $B \in \mathcal{F}$

Thus

$$\mathbb{E}[\mathbb{E}(X \mid Y)] = \int_\Omega \mathbb{E}[X \mid Y]\, dP$$

$$= \sum_k \int_{\{Y=k\}} \mathbb{E}[X \mid Y]\, dP$$

$$= \sum_k \int_{\{Y=k\}} \mathbb{E}[X \mid \underbrace{Y=k}_{B \text{ in } (*)}]\, dP$$

$$\overset{(*)}{=} \sum_k \int_{\{Y=k\}} X\, dP = \int_\Omega X\, dP = \mathbb{E}[X]$$

$\square$

<u>Def</u> For X such that $\mathbb{E}|X|<\infty$ and Y arbitrary, the cond. expectation of X given Y is a ~~the~~ random variable, such that

1, $\mathbb{E}(X|Y) \in \sigma(Y)$

2, For any $A \in \sigma(Y)$,

$$\int_A \mathbb{E}(X|Y)\,d\mathbb{P} = \int_A X\,d\mathbb{P}$$

<u>Proposition</u> If $\sigma(Y) = \sigma(Z)$ then $\mathbb{E}(X|Y) = \mathbb{E}(X|Z)$ a.s.

<u>Final definition</u> X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\mathbb{E}|X|<\infty$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub σ-alg. Then $\mathbb{E}(X|\mathcal{G})$ is a r.v. such that

1, $\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable

2, $$\int_A \mathbb{E}(X|\mathcal{G})\,d\mathbb{P} = \int_A X\,d\mathbb{P}$$

for any $A \in \mathcal{G}$.

But can we find such random var.'s?

Interpretation

$(\Omega, \mathcal{F}, \mathbb{P})$ $X \in \mathcal{F}$; $\mathcal{F}$ contains all information about X

A sub σ-algebra $\mathcal{G} \subset \mathcal{F}$ carries partial information; perhaps $X \notin \mathcal{G}$

Can we find $\hat{X} \in \mathcal{G}$ such that $\hat{X}$ represents/approximates X based only on $\mathcal{G}$?

Yes; $\hat{X} = \mathbb{E}[X \mid \mathcal{G}]$!

To find $\hat{X}$, suppose first $X \geq 0$.

Consider the measure space $(\Omega, \mathcal{G}, \mathbb{P})$

On $(\Omega, \mathcal{G})$ define $\mathbb{Q}$ by

$$\mathbb{Q}(A) = \int_A X \, d\mathbb{P}, \quad A \in \mathcal{G}.$$

This is a measure with $\mathbb{Q}(\Omega) = \ldots = \mathbb{E}[X] < \infty$

[Check this!]

Then $\boxed{\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0, \quad A \in \mathcal{G}}$

that is, $\mathbb{Q}$ is absolutely cont. wrt $\mathbb{P}$

notation $\boxed{\mathbb{Q} \ll \mathbb{P}}$

F4: ~~B~~
extra

- More generally, if $Y \in \mathcal{G}$

then $\mathbb{E}[X \cdot Y \mid \mathcal{G}] \overset{a.s}{=} Y \cdot \mathbb{E}[X \mid \mathcal{G}]$

- $\mathbb{E}[X + Y \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{G}] + \mathbb{E}[Y \mid \mathcal{G}]$

<u>Proof</u>: $\mathbb{E}[\mathbb{E}(X+Y \mid \mathcal{G}) \cdot 1_G] \overset{\text{def of cond exp.}}{=} \mathbb{E}[(X+Y) 1_G]$

$\overset{\text{prop of Leb. int}}{=} \mathbb{E}[X 1_G] + \mathbb{E}[Y 1_G]$

$\overset{\text{def}}{=} \mathbb{E}[\mathbb{E}(X \mid \mathcal{G}) 1_G] + \mathbb{E}[\mathbb{E}(Y \mid \mathcal{G}) 1_G]$

$\overset{\text{Leb int}}{=} \mathbb{E}[(\mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})) 1_G]$, all $G \in \mathcal{G}$.

$\square$

In general, for real-valued X,

$X = X^+ - X^-$, if $\mathbb{E}[|X|] < \infty$, ~~define~~

define $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}(X^+ \mid \mathcal{G}) + \mathbb{E}[X^- \mid \mathcal{G}]$

In part. $X = 1_A$, $A \in \mathcal{F}$; define <u>cond.</u> <u>prob</u>: $\mathbb{P}(A \mid \mathcal{G}) = \mathbb{E}[1_A \mid \mathcal{G}]$.

Thus, by the Radon-Nikodym theorem there exists a random variable $\hat{X}$, i.e. $\hat{X} \in \mathcal{G}$, such that

$$Q(A) = \int_A \hat{X}\, dP \quad , \text{ all } A \in \mathcal{G}$$

This means

$$Q(A) = \int_A X\, dP = \int_A \hat{X}\, dP \quad , \quad A \in \mathcal{G},$$

so we have found $E[X|\mathcal{G}]$, namely $\hat{X} = E[X|\mathcal{G}]$ a.s.

Proposition (General properties)

1, $E(aX+bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G})$

2, $E[E(X|\mathcal{G})] = E[X]$

3, If $Y \in \mathcal{G}$ then $E[XY|\mathcal{G}] = Y \cdot E[X|\mathcal{G}]$
"pull out what is known"

4, If X is independent of $\mathcal{G}$ then $E[X|\mathcal{G}] = E[X]$

5, If $X \geq 0$ then $E[X|\mathcal{G}] \geq 0$

6, $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$ if $\mathcal{H} \subset \mathcal{G}$

"tower property"

Proof

1) By linearity of the Lebesgue integral

2) As $\Omega \in \mathcal{G}$,

$$\int_\Omega E(X|\mathcal{G})\,dP = \int_\Omega X\,dP = E(X)$$

3) First, take $Y = I_A$, $A \in \mathcal{G}$. In this case

$$\int_B I_A\, E[X|\mathcal{G}]\,dP = \int_{A\cap B} E[X|\mathcal{G}]\,dP$$

$$= \int_{A\cap B} X\,dP = \int_B I_A \cdot X\,dP, \quad B \in \mathcal{G},$$

which implies $I_A E[X|\mathcal{G}] = E[I_A \cdot X|\mathcal{G}]$ a.s.

Similarly, the result holds for $Y = \sum_{k=1}^{m} a_k I_{A_k}$. In general, approximate Y by such simple functions.

4) ~~Since~~ X independent of $\mathcal{G}$ means that X and I_B are independent for all $B \in \mathcal{G}$.

Thus, $\int_B E[X]\,dP = E(X)E[I_B] \overset{\text{ind}}{=} E[X \cdot I_B]$

$$= \int_B X\,dP, \text{ all } B \in \mathcal{G}.$$

$\square$