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Eigenvalue Decomposition

I wrote about eigenvalues and eigenvectors a while back, here. In this post, I'll show how determining the eigenvalues and eigenvectors of a matrix (2 by 2 in this case) is pretty much all of the work of what's called eigenvalue decomposition. We'll start with this matrix, which represents a linear transformation:

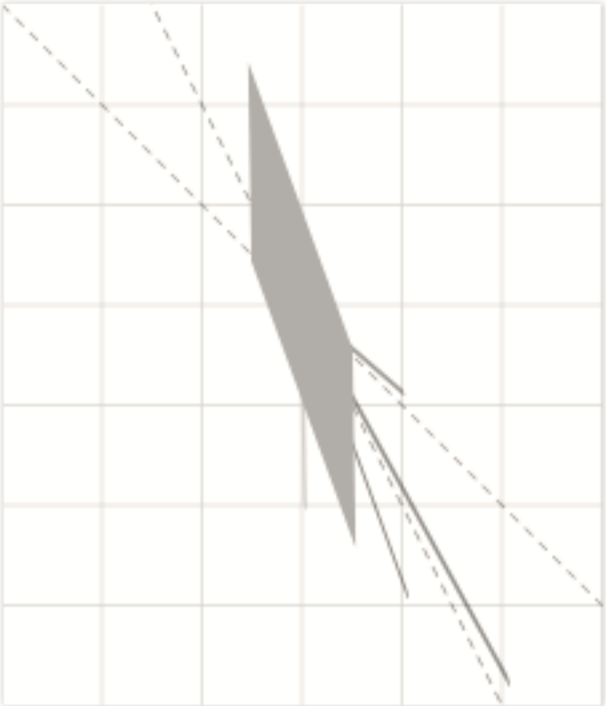
$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

You can see the action of this matrix at the right (sort of). It sends the (1, 0) vector to (0, −2) and the (0, 1) vector to (1, −3).

The eigenvectors of this transformation are any nonzero vectors that do not change their direction during this transformation, but only scale up or down (or stay the same) by a factor of λ as a result of the transformation. So,

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \end{bmatrix} = \lambda \begin{bmatrix} r_1 \end{bmatrix}$$

eigenvalues to be $\lambda_1 = -2$ and $\lambda_2 = -1$. And the corresponding eigenvectors are of the form $(\mathbf{r}, -2\mathbf{r})$ and $(-\mathbf{r}, \mathbf{r})$, respectively.



The red vector (representing the eigenvector $(-\mathbf{r}, \mathbf{r})$) at right starts at $(-1, 1)$. It is scaled by the eigenvalue of -1 during the transformation—meaning it simply turns in the opposite direction and its magnitude doesn't change. Any vector of the form $(-\mathbf{r}, \mathbf{r})$ will behave this way during this tranformation.

The purple vector (representing the eigenvector $(\mathbf{r}, -2\mathbf{r})$) starts at $(-1, 2)$. It is scaled by the eigenvalue of -2 during the transformation—meaning it turns in the opposite direction and is scaled by a factor of 2. Any vector of the form $(\mathbf{r}, -2\mathbf{r})$ will behave this way during this tranformation.

And Now for the Decomposition

We can now use the equation above and plug in each eigenvalue and its corresponding eigenvector to create two matrix equations.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We can combine the items on the left side of each equation and the items on the right side of each equation into one matrix equation.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

This leaves us with [original matrix][eigenvector matrix] = [eigenvector matrix][eigenvalue matrix]. Finally, we multiply both sides by the inverse of the eigenvector matrix, in order to remove it from the left side of the equation. We can't remove it from the right side, because matrix multiplication is not commutative. That leaves us with the final decomposition (hat tip to Math the Beautiful for some of the ideas in this post):

Original Matrix | Eigenvectors Matrix | Eigenvalues Matrix

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Inverse of Eigenvectors Matrix

Multiplying these three matrices together, or combining the transformations represented by the matrices as we showed here, will result in the original matrix.

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