Lecture 5

5. Stochastic Differential Equations

Let
- a d-dimensional Brownian motion W
- $\mu : [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- $\sigma : [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$
- $x_0 \in \mathbb{R}^n$

be given. A stochastic differential equation is an equation of the form

$$(*) \qquad \begin{cases} dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t \\ X_0 = x_0 \end{cases}$$

or, equivalently, $X_t = x_0 + \int_0^t \mu(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s$.

Prop 5.1 Assume $\|\mu(t,x)-\mu(t,y)\| + \|\sigma(t,x)-\sigma(t,y)\| \leq K\|x-y\|$ and $\|\mu(t,x)\| + \|\sigma(t,x)\| \leq K\|x\|$ for some K.

Then there exists a unique solution X_t to the SDE $(*)$. Moreover,

i) X is $\mathcal{F}^W$-adapted

ii) X_t has continuous trajectories

iii) X is a Markov process.

Geometric Brownian motion $(n=1)$

Consider $\begin{cases} dX_t = \alpha X_t\,dt + \sigma X_t\,dW_t \\ X_0 = x \end{cases}$. constants

<u>Note</u>: If $\sigma = 0$, then $dX_t = \alpha X_t dt$ so $X_t = x_0 e^{\alpha t}$.

Let $Z_t = \ln X_t$. Then

$$dZ_t \underset{\text{Ito}}{=} \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma W_t$$

so

$$Z_t = \ln x_0 + \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

and $X_t = e^{Z_t} = x_0\, e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t}$.

Moreover, $E[X_t] = x_0 + E\left[\int_0^t \alpha X_s\, ds\right] + \underbrace{E\left[\int_0^t \sigma X_s\, dW_s\right]}_{0}$

$$= x_0 + \alpha \int_0^t E[X_s]\,ds$$

so if $m(t) := E[X_t]$ we find that $\begin{cases} \dot m(t) = \alpha m(t) \\ m(0) = x_0 \end{cases}$.

Thus $m(t) = x_0 e^{\alpha t}$.

<u>Result</u>: The solution of $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ X_0 = x_0 \end{cases}$

is $X_t = x_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$.

Moreover, $E[X_t] = x_0 e^{\alpha t}$.

Ex: Consider the SDE $\begin{cases} dX_t = -X_t\,dt + dW_t \\ X_0 = x \end{cases}$

(this is a mean-reverting Ornstein-Uhlenbeck process).

Trick: Let $Y_t := e^t X_t$. Then

$$dY_t = e^t X_t\,dt + e^t\,dX_t$$
$$= e^t\,dW_t$$

so $Y_t = x + \int_0^t e^s\,dW_s$.

Thus $X_t = e^{-t} Y_t = x e^{-t} + e^{-t}\int_0^t e^s\,dW_s$.

Moreover, $E[X_t] = x e^{-t}$.

Terminology: The solution X of an SDE

$$\begin{cases} dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW \\ X_0 = x_0 \end{cases}$$

is called a <u>diffusion process</u>.

μ is the <u>drift</u> and σ is the <u>diffusion coefficient</u>.

5.5 Partial Differential Equations

Consider the following terminal value problem:

Given functions σ, μ, ϕ, find a function $F(t,x)$ such that

$$(*) \quad \begin{cases} \dfrac{\partial F}{\partial t}(t,x) + \dfrac{\sigma^2(t,x)}{2}\dfrac{\partial^2 F}{\partial x^2}(t,x) + \mu(t,x)\dfrac{\partial F}{\partial t}(t,x) = 0 \\ \qquad\qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{R} \\ F(T,x) = \phi(x). \end{cases}$$

If $F(t,x)$ satisfies $(*)$, define X_s by

$$\begin{cases} dX_s = \mu(s,X_s)\,ds + \sigma(s,X_s)\,dW_s \\ X_t = x \end{cases}$$

and let $Z_s = F(s,X_s)$. Then

$$dZ_s \underset{\text{Ito}}{=} \frac{\partial F}{\partial s}\,ds + \frac{\partial F}{\partial x}\,dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX_s)^2$$

$$= \underbrace{\left(\frac{\partial F}{\partial s} + \mu\frac{\partial F}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}\right)}_{0} ds + \sigma\frac{\partial F}{\partial x}\,dW_s$$

$$= \sigma\frac{\partial F}{\partial x}\,dW_s.$$

Integrate: $$Z_T = Z_t + \int_t^T \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)\,dW_s$$

Take expectation: $$E[Z_T] = Z_t = F(t,x)$$

$$\| $$

$$E[F(T,X_T)] \underset{(*)}{=} E[\phi(X_T)]$$

We write $F(t,x) = E_{t,x}[\phi(X_T)]$ to indicate that $X_t = x$

We have thus proved the following:

Prop. 5.5 (Feynman-Kac)

If $F(t,x)$ satisfies
$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 F}{\partial x^2}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} = 0 & (t<T) \\ F(T,x) = \Phi(x) \end{cases}$$

then $F(t,x) = E_{t,x}\left[\Phi(X_T)\right]$ where $\begin{cases} dX_s = \mu(s,X_s)ds + \sigma(s,X_s)\,dW_s \\ X_t = x \end{cases}$

Ex 5.7 Solve the PDE
$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} = 0 \\ F(T,x) = x^2 \end{cases}$$

(σ constant).

Let X_s be the solution of $\begin{cases} dX_s = \sigma\,dW_s \\ X_t = x \end{cases}$

i.e. $X_s = x + \sigma(W_s - W_t)$.

By Feynman-Kac,

$$F(t,x) = E_{t,x}\left[X_T^2\right] = E\left[\left(x + \sigma(W_T - W_t)\right)^2\right] =$$

$$= x^2 + 2x\sigma\, E\left[W_T - W_t\right] + \sigma^2 E\left[(W_T - W_t)^2\right]$$

$$= x^2 + \sigma^2(T-t).$$

Answer: $F(t,x) = x^2 + \sigma^2(T-t)$