UPPSALA UNIVERSITY PROBABILITY AND MARTINGALES
Department of Mathematics 1MS045
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Each problem counts 5 points. Grades are awarded according to the following scale: 0–17 grade U; 18–24 grade 3; 25–31 grade 4; 32–40 grade 5. Allowed tools: pen, paper, calculator. All solutions should be clearly explained.

Note: if not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

1. (a) State the first and the second *Borel–Cantelli lemma*. (2)

   (b) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with density $f(x) = (a-1)x^{-a}I_{[1,\infty)}(x)$ for some $a > 1$. Prove: if $a \leq 2$, then $X_n > n$ occurs infinitely often (a.s.). If $a > 2$, then $X_n > n$ only occurs finitely often (a.s). (3)

2. Let X be a non-negative random variable. Show that the family $\{X_n; n \geq 0\}$ of random variables defined by $X_n = \min(X, n)$ (for non-negative integers n) is uniformly integrable if and only if X is integrable. (5)

3. (a) State *Fatou's lemma* on sequences of non-negative measurable functions. (2)

   (b) Prove the *dominated convergence theorem*: given a measure space $(\Omega, \mathcal{F}, \mu)$, let $\{f_n; n \geq 0\}$ be a sequence of measurable functions with $f_n \to f$, and g an integrable function such that $|f_n(x)| \leq g(x)$ for all n and x. Then we have $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$. (3)

4. Let $Y_1, Y_2, \ldots$ be independent random variables with $P(Y_i = 1) = \frac{1}{2}$ and $P(Y_i = 0) = P(Y_i = -1) = \frac{1}{4}$ for all i, and consider the sum $X_n = \sum_{i=1}^n Y_i$.

   (a) Find a constant $\theta \neq 1$ such that θ^{X_n} is a martingale. (1)

   (b) Find a (deterministic) function $f(n)$ such that $X_n - f(n)$ is a martingale. (1)

   (c) Let a and b be positive integers. Determine the probability that X_n reaches the value a before the value $-b$. (3)

5. For each of the following statements, decide if it is true or false. If true, give a proof. If false, provide a counterexample.

   (a) If $\{X_n; n \geq 0\}$ is both a submartingale and a supermartingale (with respect to the same filtration), then it is a martingale. (1)

   (b) For every martingale $\{X_n; n \geq 0\}$ and every stopping time T with $P(T < \infty) = 1$, we have $\mathbb{E}(X_T) = \mathbb{E}(X_0)$. (2)

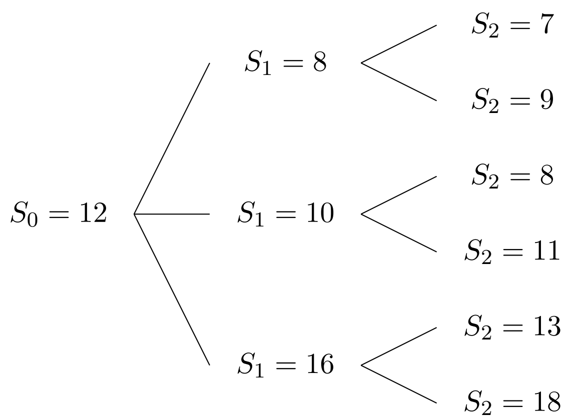
   (c) If $\{X_n; n \geq 0\}$ is a positive martingale (i.e., $X_n > 0$ for all n), then $\lim_{n\to\infty} X_n > 0$ holds almost surely. (2)

6. Prove *Doob's submartingale inequality*: for every non-negative submartingale $\{Z_n; n \geq 0\}$ and every $c > 0$, we have (5)

$$cP\big(\sup_{k \leq n} Z_k \geq c\big) \leq \mathbb{E}(Z_n).$$

7. (a) Define European call options and European put options, and derive the *Call-Put parity.* (2)

   (b) Consider a simple market model with a single asset S_t and two time steps as indicated in the following diagram: (3)



   The riskless bond is assumed to be $S_t^0 = 1$. Determine all equivalent martingale measures. Is this model viable? Is it complete?

8. Describe the *binomial model* (Cox–Ross–Rubinstein model) for option pricing. What conditions on the model parameters a, b, r are required for the model to be viable, and why? Explain how the Cox–Ross–Rubinstein formula for the price of a European call option is derived. (5)