Lecture 11

Volatility Mis-Specification (not in the book)

Assume that a trader believes in

$$dS_t = \mu(t,S_t)\,S_t\,dt + \sigma(t,S_t)\,S_t\,dW_t$$

whereas the stock actually follows

$$d\tilde{S}_t = \tilde{\mu}(t,\hat{S}_t)\,\hat{S}_t\,dt + \tilde{\sigma}(t,\hat{S}_t)\,\hat{S}_t\,d\hat{W}_t .$$

What happens if the trader tries to replicate a simple T-claim $\mathcal{X} = \Phi(\tilde{S}_T)$?

The trader solves
$$\begin{cases} F_t + \frac{\sigma^2}{2}s^2 F_{ss} + rsF_s - rF = 0 \\ F(T,s) = \Phi(s) \end{cases}$$

and constructs a portfolio $h = (h^B, h^S)$ with initial value $V_0^h = F(0,s)$, containing $F_s(t,\tilde{S}_t)$ shares of $\tilde{S}$ at each time (and $V_t^h - \tilde{S}_t F_s(t,S_t)$ in the bank account).

The <u>tracking error</u> $Y_t := V_t^h - F(t,\hat{S}_t)$ satisfies $Y_0 = 0$ and

$$dY_t = r(V_t^h - \tilde{S}_t F_s)\,dt + F_s\,d\tilde{S} - \left(F_t\,dt + F_s\,d\tilde{S}_t + \tfrac{1}{2}\tilde{\sigma}^2\tilde{S}_t^2 F_{ss}\,dt\right)$$

$$= rV_t^h\,dt - \underbrace{\left(F_t + \tfrac{1}{2}\sigma^2\tilde{S}^2 F_{ss} + r\tilde{S}_t F_s\right)}_{rF}dt + \frac{\sigma^2 - \tilde{\sigma}^2}{2}\hat{S}_t^2 F_{ss}\,dt$$

$$= rY_t\,dt + \frac{\sigma^2 - \tilde{\sigma}^2}{2}\tilde{S}_t^2 F_{ss}\,dt .$$

Thus, if $\sigma^2 \geq \tilde{\sigma}^2$ and $F_{ss} \geq 0$ then $Y(T) = V(T) - \Phi(\tilde{S}_T) \geq 0$

> A trader who overestimates volatility and who uses a model with a convex price will superreplicate the claim!

Asian options (In Ch. 8)

Asian options are options on the <u>average</u> of S.
An Asian call option pays $\mathcal{X} = \left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+$ at T.
<u>Not</u> a simple T-claim!

<u>Prop 8.6</u> Let $\mathcal{X} = \Phi(S_T, Z_T)$ where $Z_t = \int_0^t g(u, S_u)\,du$ for some function g. Let $F(t,s,z)$ solve

$$\begin{cases} F_t + \frac{\sigma^2 s^2}{2} F_{ss} + rsF_s + g(t,s)F_z - rF = 0 \\ F(T,s,z) = \Phi(s,z), \end{cases}$$

and let

$$\begin{cases} h_t^B = \dfrac{F(t,S_t,Z_t) - S_t F_s(t,S_t,Z_t)}{B_t} \\ h_t^S = F_s(t,S_t,Z_t). \end{cases}$$

Then h is self-financing and it replicates $\mathcal{X}$, with $\Pi_t(\mathcal{X}) = V_t^h = F(t,S_t,Z_t)$. Moreover,

$$F(t,s,z) = e^{-r(T-t)} E^Q_{t,s,z}\left[\Phi(S_T, Z_T)\right]$$

where the Q-dynamics are

$$\begin{cases} dS_u = rS_u\,du + \sigma(u,S_u)S_u\,dW_u^Q \\ S_t = s \\ dZ_u = g(u,S_u)\,du \\ Z_t = z \end{cases}$$

Pf: $V_t^h = h_t^B B_t + h_t^S S_t = F(t, S_t, Z_t)$, and in particular

$$V_T^h = F(T, S_T, Z_T) = \Phi(S_T, Z_T) = \mathcal{X}.$$

Moreover,

$$dV_t^h \overset{\text{Ito}}{=} F_t\,dt + F_s\,dS_t + F_z\,dZ_t + \tfrac{1}{2}F_{ss}(dS_t)^2$$

$$= \underbrace{\left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss} + g(t, S_t)F_z\right)}_{= r(F - S_t F_s) \text{ by BS PDE}} dt + F_s\,dS_t$$

$$= r(F - S_t F_s)\,dt + F_s\,dS_t = h_t^B\,dB_t + h_t^S\,dS_t,$$

so h is self-financing, and h replicates $\mathcal{X}$.

Therefore, by no-arbitrage, $\Pi_t(\mathcal{X}) = V_t^h = F(t, S_t, Z_t)$.

Finally, the stochastic representation follows from Feynman-Kac.

Exercise 8.3 $\mathcal{X} = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_u\,du$, paid at T_2.

What is the value of the T_2-claim $\mathcal{X}$ at $t < T_1$?

$$E_{t,s}^Q\left[e^{-r(T_2-t)}\frac{1}{T_2-T_1}\int_{T_1}^{T_2} S_u\,du\right] = \frac{e^{-r(T_2-t)}}{T_2-T_1}\int_{T_1}^{T_2}\underbrace{E_{t,s}[S_u]}_{s e^{r(u-t)}}\,du$$

$$= \frac{e^{-r(T_2-t)}}{T_2-T_1}\cdot\frac{s}{r}\left(e^{r(T_2-t)} - e^{r(T_1-t)}\right)$$

$$= \frac{s}{r(T_2-T_1)}\left(1 - e^{-r(T_2-T_1)}\right)$$

Answer: Price is $\dfrac{S_t}{r(T_2-T_1)}\left(1 - e^{-r(T_2-T_1)}\right)$

Remark: What is the value of X in Exercise 8.3 at $t \in [T_1, T_2]$?

$$X = \frac{1}{T_2-T_1}\int_{T_1}^{T_2} S_u\,du = \underbrace{\frac{1}{T_2-T_1}\int_{T_1}^{t} S_u\,du}_{\text{known at } t} + \underbrace{\frac{1}{T_2-T_1}\int_{t}^{T_2} S_u\,du}_{Y}$$

Price of Y: $E^Q_{t,s}\left[e^{-r(T_2-t)}\frac{1}{T_2-T_1}\int_t^{T_2} S_u\,du\right] =$

$$= \frac{e^{-r(T_2-t)}}{T_2-T_1}\int_t^{T_2} s e^{r(u-t)}\,du$$

$$= \frac{s}{r(T_2-T_1)}\left(1-e^{-r(T_2-t)}\right)$$

Answer: $\frac{1}{T_2-T_1}\left(e^{-r(T_2-t)}\int_{T_1}^{t} S_u\,du + \frac{S_t}{r}\left(1-e^{-r(T_2-t)}\right)\right)$

8.3 Completeness vs Absence of Arbitrage

Ex: (i) The BS-model $\begin{cases} dB_t = rB_t\,dt \\ dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \end{cases}$

is arbitrage-free and complete.

(ii) The model

$$dB_t = rB_t\,dt$$
$$dS^1_t = \mu_1 S^1_t\,dt + \sigma_1 S^1_t\,dW_t$$
$$dS^2_t = \mu_2 S^1_t\,dt + \sigma_2 S^2_t\,dW_t$$

same BM

is complete, but (typically) not arbitrage-free (construct a portfolio in S^1, S^2 with no dW-term with local mean rate of return $\neq r$).

(iii) The model $dB_t = rB_t\,dt$

$$dS_t = \mu S_t\,dt + \sigma_1 S_t\,dW^1_t + \sigma_2 S_t\,dW^2_t$$ is

arbitrage-free but not complete ($X = W^1_T$ cannot be replicated).

Meta-theorem 8.3.1

Let M = # traded assets excluding B

R = # random sources (BM's, Poisson processes, ...)

Then

Absence of arbitrage $\iff M \leq R$

Completeness $\iff M \geq R$

Absence of arbitrage and completeness $\iff M = R$