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PROV I MATEMATISK STATISTIK
Markov Processes
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Permitted aids: Pocket calculator

1. For a homogeneous Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{0,1\}$ it is known that $P(X_{n+1} = 1 \,|X_n = 1) = 0.5$, and $\lim_{n\to\infty} P(X_n = 1) = 0.25$.

   Calculate $P(X_2 = 0 \,|X_0 = 0)$. (5p)

   Solution: The transition matrix is of the form

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} x & 1-x \\ 0.5 & 0.5 \end{pmatrix},$$

where $0 < x < 1$ since $\lim_{n\to\infty} P(X_n = 1) = 0.25$. Hence $(X_n)_{n\geq 0}$ is irreducible and aperiodic and has a unique stationary distribution. Since the stationary distribution $\pi = (\pi_0, \pi_1) = (0.75, 0.25)$ satisfies $\pi\mathbf{P} = \pi$, it follows that $0.75x + 0.25 \cdot 0.5 = 0.75$, i.e. $x = 5/6$, and therefore

$$\mathbf{P} = \begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix}.$$

It follows that

$$\mathbf{P}^2 = \begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 7/9 & 2/9 \\ 2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{pmatrix}$$

Therefore $P(X_2 = 0|X_0 = 0) = p_{00}^{(2)} = 7/9$.

2. Let $(X_t)_{t\geq 0}$ be a Markov process on $S = \{0,1,2\}$ with intensity matrix

$$\mathbf{Q} = \begin{pmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

(a) Find the limit

$$\lim_{t\to\infty} P(X_t = 0 | X_0 = 0).$$

(2p)

(b) Find the expected value of T, where $T = \inf\{t \geq 0 : X_t = 2 | X_0 = 1\}$ is the time it takes to reach state 2 from state 1. (2p)

(c) Let $(Y_t)_{t\geq 0}$ be a Markov process with generator $\tilde{\mathbf{Q}} = 2\mathbf{Q}$. Explain how the transition probabilities of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are related. (2p)

Solution:

(a) The Markovprocess (X_t) is irreducible (since there is a path between all states in the model graph of $(X_t)_{t\geq 0}$) and has a finite state space. It follows from the convergence theorem that

$$\lim_{t\to\infty} P(X_t = 0 | X_0 = 0) = \pi_0,$$

where $\pi = (\pi_0, \pi_1, \pi_2)$ is the unique probability vector solving the equation

$$\pi\mathbf{Q} = \mathbf{0}.$$

This system has solution $\pi = (1/3, 1/3, 1/3)$.
Thus

$$\lim_{t\to\infty} P(X_t = 0 | X_0 = 0) = 1/3.$$

(b) Let T_{i2} be the time it takes to reach state 2 from state i, $i = 0, 1$. We want to find $E(T) = ET_{12}$. We can write $T_{i2} = L_i + N_i$, where L_i denotes the time it takes to leave state i and N_i denotes the time it takes to reach state 2 from the new state reached after the first jump from i. From the intensity matrix we see that $L_0 \sim Exp(3)$ and $L_1 \sim Exp(2)$ and thus $E(L_0) = 1$ and $E(L_1) = 1/2$ and that the jump chain has transition matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}.$$

We thus get

$$\begin{aligned} E(T_{12}) &= E(L_1) + E(N_1) = E(L_1) + P(\text{first jump } 1 \to 0) E(N_1 | \text{first jump } 1 \to 0) \\ &= 1/2 + (1/2) \cdot E(T_{02}), \end{aligned}$$

and

$$\begin{aligned} E(T_{02}) &= E(L_0) + E(N_0) = E(L_0) + P(\text{first jump } 0 \to 1) E(N_0 | \text{first jump } 0 \to 1) \\ &= 1/3 + (2/3) \cdot E(T_{12}). \end{aligned}$$

Thus $E(T_{12}) = 1/2 + (1/2) \cdot E(T_{02}) = 1/2 + (1/2) \cdot (1/3 + (2/3) \cdot E(T_{12})) = 2/3 + (1/3) \cdot E(T_{12})$, i.e. $E(T_{12}) = 1$.

(c) Let $\tilde{p}_{ij}(t) = P(Y_t = j|Y_0 = i)$, and $p_{ij}(t) = P(X_t = j|X_0 = i)$. Since (X_t) and (Y_t) are irreducible Markov processes with finite state spaces it follows that their matrices of transition probabilities satisfies $\mathbf{P}(t) = e^{t\mathbf{Q}}$ and $\tilde{\mathbf{P}}(t) = e^{t\tilde{\mathbf{Q}}} = e^{2t\mathbf{Q}}$ respectively, i.e. $\tilde{p}_{ij}(t) = p_{ij}(2t)$ for any $i, j \in S$.

3. N black balls and N white balls are placed in two urns so that each urn contains N balls. After each unit of time one ball is selected at random from each urn, and the two balls thus selected are interchanged. Let the number of black balls in the first urn denote the state of the system.

   (a) Write down the transition matrix of this Markov chain. (3p)

   (b) Find the unique reversible (and hence stationary) distribution.
   (You are allowed to "guess" what this distribution is, and then verify that it is reversible. A correct solution in the special case $N = 2$ gives 1p.) (3p)

Solution:

(a) Let X_n be the number of black balls in the first urn. If for some $0 \le i \le N$, and $n \ge 1$, $X_{n-1} = i$, then $X_n = i+1$, if a white ball is chosen from the first urn and a black ball is chosen from the second urn at time n. This happens with probability $((N-i)/N) \cdot ((N-i)/N)$.
Similarly, if $X_{n-1} = i$, then $X_n = i-1$, if a black ball is chosen from the first urn and a white ball is chosen from the second urn at time n. This happens with probability $(i/N) \cdot (i/N)$. In the remaining case, if $X_n = i$ and both chosen balls have the same colour, then $X_{n+1} = i$.
The sequence (X_n) is thus a Markov chain on $S = \{0, 1, ..., N\}$ with $p_{i,i+1} = P(X_{n+1} = i+1|X_n = i) = (N-i)^2/N^2$, $p_{i,i-1} = P(X_{n+1} = i-1|X_n = i) = i^2/N^2$, and $p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1} = 2i(N-i)/N^2$, $0 \le i \le N$. The transition matrix is thus

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & & \cdot & 0 \\ 1/N^2 & 2(N-1)/N^2 & (N-1)^2/N^2 & 0 & \cdot & 0 \\ 0 & 4/N^2 & 4(N-2)/N^2 & (N-2)^2/N^2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & (N-1)^2/N^2 & 2(N-1)/N^2 & 1/N^2 \\ \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix}.$$

(b) In the long run the probability of having k black (and thus $N-k$ white balls) in the first urn should be $\pi_k = \frac{\binom{N}{k}\binom{N}{N-k}}{\binom{2N}{N}} = \frac{\binom{N}{k}^2}{\binom{2N}{N}}$, that is "the number of ways of choosing k black balls and $N-k$ white balls to the first urn" divided by "the number of ways of choosing N balls to the first urn".
Since

$$\begin{aligned} \pi_k p_{k,k+1} &= \frac{\binom{N}{k}^2}{\binom{2N}{N}} \frac{(N-k)^2}{N^2} = \frac{(N!/(k!(N-k-1)!))^2}{\binom{2N}{N}} \frac{1}{N^2} \\ &= \frac{\binom{N}{k+1}^2}{\binom{2N}{N}} \frac{(k+1)^2}{N^2} = \pi_{k+1} p_{k+1,k} \end{aligned}$$

it follows that this distribution is reversible.

4. Customers arrive to a service system according to a Poisson process with intensity parameter λ. The system has one single service person and service times are independent and exponentially distributed random variables with expectation $1/\mu$, $(0 < \lambda < \mu)$. Customers arriving when the service person is busy will wait for service in a queue. Every time when the queue becomes empty, the service person goes for a coffee break which last an exponentially distributed time with expected value $1/\gamma$. Customers arriving during such periods will wait in the queue.

   (a) Model this system with a Markov process with states (i, j), where i denotes the number of customers in the system (including, if existing, the customer at service), and $j = 0$ if the service person has a coffee break, and $j = 1$ otherwise.

   Draw a model graph of the states and transition intensities between the states. (3p)

   (b) Calculate the probability that the service person will manage to get back from a given coffee break before a customer arrives. (3p)

Solution:

(a) The state of the system at time t can be modelled by a Markov process X_t with phase diagram

$$
\begin{tikzcd}
(0,1) \arrow[r, "\lambda"] & (1,1) \arrow[r, shift left, "\lambda"] \arrow[dl, "\mu"'] & (2,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] & (3,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] & (4,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] & \cdot \arrow[l, shift left, "\mu"] & \cdot & \cdot \\
(0,0) \arrow[u, "\gamma"] \arrow[r, "\lambda"] & (1,0) \arrow[u, "\gamma"] \arrow[r, "\lambda"] & (2,0) \arrow[u, "\gamma"] \arrow[r, "\lambda"] & (3,0) \arrow[u, "\gamma"] \arrow[r, "\lambda"] & (4,0) \arrow[u, "\gamma"] \arrow[r, "\lambda"] & \cdot & \cdot & \cdot
\end{tikzcd}
$$

(b) The probability that the service person will manage to get back from a given coffee break before a customer arrives is the probability that the jump chain makes the transition from $(0, 0)$ to $(0, 1)$. This probability is given by $\gamma/(\lambda + \gamma)$.

5. Consider the Markov chain $(X_n)_{n=0}^{\infty}$ on $S = \{..., -2, -1, 0, 1, 2, ...\}$, obtained by random (independent) iterations with the functions

$$f_1(x) = x + 7, \qquad f_2(x) = x - 3, \quad \text{and} \qquad f_3(x) = \begin{cases} -2 & x \le -2 \\ x & -2 < x < 2 \\ 2 & x \ge 2 \end{cases}.$$

In each iteration step we choose function to iterate with equal probabilities. The Markov chain considered can thus be represented as $X_{n+1} = f_{I_{n+1}}(X_n)$, where $(I_n)_{n=1}^{\infty}$ is a sequence of independent random variables with $P(I_n = 1) = P(I_n = 2) = P(I_n = 3) = 1/3$ for each fixed n.

   (a) Show that this Markov chain is irreducible. (1p)

   (b) Show that this Markov chain is positive recurrent and hence that there exists a unique stationary distribution π. (3p)

   (c) Explain how we can obtain non-biased samples from π. (2p)

Solution:

(a) If i and j are arbitrary states states and $k = j - i$, then there exists integers n_1 and n_2 such that $7n_1 - 3n_2 = k$. Therefore it is possible to go from state i to state j by applying function f_1 n_1 times and function f_2 n_2 times. The Markov chain is therefore irreducible.

(b) Note first that $f_3 \circ f_1 \circ f_3(x) = 2$, for any x. Let $T = \inf\{n \geq 1 : X_n = 2 | X_0 = 2\}$, and $N = \inf\{k \geq 1 : \ I_{3k-2} = 3, \ I_{3k-1} = 1, \ I_{3k} = 3\}$. The random variable N is geometrically distributed with parameter $(1/3)^3$, and thus $E(N) < \infty$. Since $T \leq 3N$ it follows that $E(T) \leq 3E(N) < \infty$. Thus state 2 is positive recurrent, and since the Markov chain is irreducible it follows that all states are positive recurrent, i.e. the Markov chain is positive recurrent and thus there exists a stationary distribution π.

(c) Let $\tilde{T} = \inf\{n : f_{I_1} \circ f_{I_2} \circ ... \circ f_{I_n}(x)$, does not depend on $x\}$. Then $f_{I_1} \circ f_{I_2} \circ ... \circ f_{I_{\tilde{T}}}(x)$ is a π-distributed random variable.
(Note that $\tilde{T} \leq 3N$, and since $E\tilde{T} < \infty$ it follows that $P(\tilde{T} < \infty) = 1$.)
We can thus apply the Propp-Wilson perfect sampling algorithm and simulate independent realisations, i_j of the uniformly distributed random variables I_j, $j = 1, 2, ..., \tilde{t}$, where $\tilde{t}$ is an integer such that $y := f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_{\tilde{t}}}(x)$, does not depend on x. Then y is an observation from π.

6. Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be independent standard Brownian motions.

(a) Show that $Z_t = X_t - Y_t$ is a Brownian motion. What is the variance parameter for (Z_t)? (3p)

(b) Express $P(Y_t < X_t + 5, \text{ for all } 0 \leq t \leq 10)$ in terms of the distribution function of a standard normal random variable. (3p)

Solution:

(a) We need to check the defining properties of a Brownian motion:
1. $Z_0 = X_0 - Y_0 = 0 - 0 = 0$.
2. Independent increments:
If $0 = t_0 < t_1 < t_2 < ... < t_n$, then the increments $Z_{t_j} - Z_{t_{j-1}} = (X_{t_j} - Y_{t_j}) - (X_{t_{j-1}} - Y_{t_{j-1}}) = (X_{t_j} - X_{t_{j-1}}) - (Y_{t_j} - Y_{t_{j-1}})$, are independent, since the increments $(X_{t_k} - X_{t_{k-1}})$, and $(Y_{t_j} - Y_{t_{j-1}})$, $1 \leq j, k \leq n$ are independent.
Stationary, normally distributed, increments:
$Z_{t_j} - Z_{t_{j-1}} = (X_{t_j} - X_{t_{j-1}}) - (Y_{t_j} - Y_{t_{j-1}})$, and since $(X_{t_j} - X_{t_{j-1}}) \sim N(0, t_j - t_{j-1})$, and $(Y_{t_j} - Y_{t_{j-1}}) \sim N(0, t_j - t_{j-1})$ are independent, and sums of independent normally distributed random variables are normal, it follows that $Z_{t_j} - Z_{t_{j-1}} \sim N(0, 2(t_j - t_{j-1}))$. In particular $Z_t \sim N(0, 2t)$.
3. $Z_t = X_t - Y_t$ has continuous trajectories, since both trajectories of X_t and Y_t are continuous, and differences between continuous functions are continuous.
Thus (Z_t) is a Brownian motion with variance parameter 2.

(b) Let $B_t = -Z_t/\sqrt{2}$. Then (B_t) is a standard Brownian motion. By the maximum principle it is known that $\max_{0 \leq s \leq t} B_s$ has the same distribution as $|B_t|$. It follows that $P(Y_t < X_t + 5, \text{ for all } 0 \leq t \leq 10) = P(\max_{0 \leq t \leq 10} B_t < 5/\sqrt{2}) = P(|B_{10}| < 5/\sqrt{2}) = P(-5/\sqrt{20} < B_{10}/\sqrt{10} < 5/\sqrt{20}) = 2\Phi(5/\sqrt{20}) - 1$, where Φ denotes the distribution function of a standard normal random variable.