LINEAR ALGEBRA III   Period 3, 2023

Lecture 1 (2023-01-20)

See Studium page for:
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Goal: Go beyond Linear Algebra 2:
- more generality: discuss vector spaces (possibly of infinite dimension) over arbitrary fields (not just $\mathbb{R}$ or $\mathbb{C}$)
- Recall: spectral theorem: given a real symmetric $n\times n$ matrix S, can find an ON basis $v_1,\dots,v_n$ of $\mathbb{R}^n$, so that in this basis S is given by a diagonal matrix D:

$$D = C^{-1}.S.C = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \leftarrow \text{has eigenvalues (w/ suitable multiplicities) in the diagonal.}$$

where $C = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ is the change of basis matrix.

We will see a more general version, where S doesn't have to be real or symmetric.
- Jordan normal form: given any $n\times n$ matrix A w/ $\mathbb{C}$-entries, can find a basis $v_1,\dots,v_n$ of $\mathbb{C}^n$ st

$$J = C^{-1}.A.C, \quad C \text{ is change of basis matrix}$$

$$= \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_m \end{bmatrix}, \quad \underbrace{J_k}_{\text{Jordan block}} = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix} \leftarrow \text{eigenvalue } \lambda_k \text{ on diagonal}$$

Def: An *abelian group* is a set G with an element 0 and an operation $+: G\times G \to G$ st

i) $a+(b+c) = (a+b)+c$ (associativity) } G is a semigroup

ii) $0+a = a = a+0$ (0 is identity element for $+$) } monoid

iii) For all $a\in G$ there is $-a$ st $a+(-a) = 0 = (-a)+a$ (existence of inverses) } group

iv) $a+b = b+a$ (commutativity)

Exs: • $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

• $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ $\left(\cong \mathbb{Z}/(a \sim a+m)\right)$

To define $+$, may need to subtract m (arithmetic mod m)

Ex: in $\mathbb{Z}_4 = \{0,1,2,3\}$, we have

$$2+3 = \underbrace{5-4}_{\text{because } 5 \text{ is not in } \{0,1,2,3\}} = 1$$

Exercise: Prove that $\mathbb{Z}_m$ is an abelian gp for every integer $m \geq 0$.

Def: A field is a set F with different elements 0 and 1, and operations $+ : F \times F \to F$, $\cdot : F \times F \to F$ st:

$$\left.\begin{array}{ll} \text{i)} & a+(b+c) = (a+b)+c \\ \text{ii)} & 0+a = a \\ \text{iii)} & \forall_{a\in F}\ \exists_{-a \in F} \text{ st } a+(-a) = 0 \\ \text{iv)} & a+b = b+a \end{array}\right\} \text{F with } 0 \text{ \& } + \text{ is an abelian group}$$

$$\left.\begin{array}{ll} \text{v)} & a\cdot(b\cdot c) = (a\cdot b)\cdot c \\ \text{vi)} & 1\cdot a = a \\ \text{vii)} & \forall_{a \in F\setminus\{0\}}\ \exists_{a^{-1} \in F\setminus\{0\}} \text{ st } a\cdot(a^{-1}) = 1 \\ \text{viii)} & a\cdot b = b \cdot a \end{array}\right\} F\setminus\{0\} \text{ w/ } 1 \text{ \& } \cdot \text{ is an abelian group}$$

ix) $a\cdot(b+c) = a\cdot b + a \cdot c$ (distributivity of $\cdot$ wrt $+$)

Note: A field is a ring with $0 \neq 1$ & (vii) & (viii).

Exs: • $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (not $\mathbb{Z}$: $2^{-1} \notin \mathbb{Z}$)

• $\mathbb{Z}_p$, if p is prime

Exercise: • Show that $(-1)^{-1} = -1$ (ie that $(-1)\cdot(-1) = 1$)

• Write the addition and multiplication tables for $\mathbb{Z}_3$

• Write the addition and multipl. tables for the only field that has 4 elements (denote it by $F(4)$).

<u>Def</u>: The <u>characteristic</u> of F is the smallest $n \in \mathbb{Z}_{>0}$ st $\overbrace{1+\dots+1}^{n} = 0$. If there is no such n, say it's 0. Write $\operatorname{char}(F)$.

<u>Exercise</u>:
- $\operatorname{char}(F)$ is always either 0 or a prime p.
- if $\operatorname{char}(F) = p$, then $\forall_{a \in F}$ $\overbrace{a+\dots+a}^{p} = 0$.
- $\operatorname{char}(\mathbb{Z}_p) = p$; $\operatorname{char}(F(4)) = 2$.
- $\operatorname{char}(\mathbb{Q}) = 0$ (hence also $\operatorname{char}(\mathbb{R}) = \operatorname{char}(\mathbb{C}) = 0$)

<u>Def</u>: A <u>vector space</u> V over a field F is a set with an element 0 and operations $+ : V \times V \to V$ (<u>addition</u>) and $\cdot : F \times V \to V$ (<u>scalar multiplication</u>) s.t.

i) $u + (v + w) = (u + v) + w$
ii) $0 + v = v$
iii) $\forall_{v \in V}\ \exists_{-v \in V}$ st $v + (-v) = 0$
iv) $u + v = v + u$

} V w/ $+$ & 0 is an abelian group

v) $1 \cdot v = v$ (multiplicative identity)
vi) $(a+b) \cdot v = a \cdot v + b \cdot v$
vii) $a \cdot (v + w) = a \cdot v + a \cdot w$

} distributivity

viii) $(a \cdot b) \cdot v = a \cdot (b \cdot v)$ (associativity)

<u>Note</u>: When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we often have more structure, like an inner product. That's not the case for a general $\mathbb{F}$.

<u>Exercise</u>: Show that
- $0 \cdot v = 0$ $\quad \forall_{v \in V}$
- $(-1) \cdot v = -v$ $\quad \forall_{v \in V}$

<u>Ex</u>:
- F is a vector space over F (compare def of vector space with def. of field).
- $F^n = \overbrace{F \times \dots \times F}^{n \text{ times}} = \{ (a_1, \dots, a_n) : \text{all } a_k \in F \}$.

$(a_1,\dots,a_m)+(b_1,\dots,b_m)=(a_1+b_1,\dots,a_m+b_m)$
$a\cdot(b_1,\dots,b_m)=(a\cdot b_1,\dots,a\cdot b_m)$

- $F^\infty=\{(a_1,a_2,\dots) : \text{all } a_k\in F\}$
  This is the space of sequences of elements in F
- $\mathcal{P}_m(F)=\{a_0+a_1x+\dots+a_mx^m : \text{all } a_k\in F\}$
  space of polynomial of deg $\leq m$, with coefficients in F.
- Given a set S, $F^S=\{\text{functions from } S \text{ to } F\}$
  Note: $\mathcal{P}_m(F)\subset F^F$

Exercise: Let $C^k(\mathbb{R})$ denote the space of functions $f:\mathbb{R}\to\mathbb{R}$ that are k-times diff, with $f^{(k)}$ (k-th order derivative) continuous. Show that $C^k(\mathbb{R})$ is an $\mathbb{R}$-vector space.

Exercise: Let $K_1\subset K_2$ be an inclusion of fields, preserving $0,1,+,\cdot$ (say that K_1 is a subfield of K_2, or that K_2 is a field extension of K_1). Show that K_2 is a v.s. over K_1.
In particular, $\mathbb{C}$ is a v.s. over $\mathbb{R}$ and over $\mathbb{Q}$, and $\mathbb{R}$ is a v.s. over $\mathbb{Q}$.

Review: From now on, let V be a vector space over a field F.

Def: A subset $U\subset V$ is a vector subspace if it contains $0\in V$ as is a vector space for the operations $+$ and $\cdot$ in V.

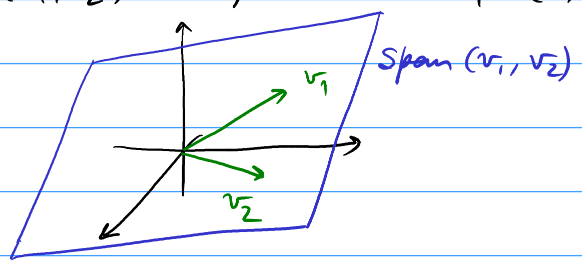
Def: A linear combination is a vector of the form
$v=a_1v_1+\dots+a_kv_k$, where the $a_i\in F$ and the $v_i\in V$.
Important: The sum is finite.

Def: Given a collection of vectors $v_1, v_2, \ldots \in V$, their <u>span</u> is the vector subspace

$$\mathrm{Span}(v_1, v_2, \ldots) = \left\{ \begin{array}{l} \text{linear combinations of vectors} \\ \text{from the collection} \end{array} \right\} \subset V$$

(Frequently, we will take the span of a finite collection $\mathrm{span}(v_1, \ldots, v_m)$.)

Exercise: Write a single equation defining the subspace $\mathrm{Span}(v_1, v_2) \subset \mathbb{R}^3$, where $v_1 = (5, -2, 1)$ and $v_2 = (1, -1, 3)$.



Def: A collection of vectors $v_1, v_2, \ldots \in V$ is called <u>linearly independent</u> if, for every linear combination

$$a_1 v_1 + \ldots + a_m v_m = 0,$$

we have $a_1 = a_2 = \ldots = a_m = 0$.

(Frequently we will talk about linear independence of <u>finite</u> collections of vectors.)

Prop: A subset $U \subset V$ is a vector subspace if $0 \in U$ and

i) $v_1, v_2 \in U \Rightarrow v_1 + v_2 \in U$ (U is <u>closed under addition</u>)

ii) $a \in F,\ v \in U \Rightarrow a \cdot v \in U$ (U is <u>closed under scalar multipl.</u>)

Ex: • if $m \le n$, then can think of F^m as a subspace of F^n, by identifying $(x_1, \ldots, x_m) \in F^m$ with $(x_1, \ldots, x_m, 0, \ldots, 0) \in F^n$.

• given a subset $S \subset V$, we have the subspace

$$\mathrm{Span}(S) = \{ \underbrace{a_1 v_1 + \ldots + a_k v_k}_{\text{linear combinations}} : a_i \in F \text{ and } v_i \in V \} \subset V$$

Def: A subset $B \subset V$ is a **basis** of V if
i) $\mathrm{Span}(B) = V$ ("B generates all of V")
ii) the subset is linearly independent:
if $a_1 v_1 + \ldots + a_k v_k = 0$ with $a_i \in F$ and $v_i \in B$,
then all the $a_i = 0$ ("there is no redundancy")

End of review.

Exercise:
- Find a basis for $\mathcal{P}_m(F)$ (polynomials of degree up to m).
- Find a basis for $\mathcal{P}(F) = \bigcup_{m=1}^{\infty} \mathcal{P}_m(F) = F[x]$ (polynomials of any degree)
- Think about finding a basis for the vector space of power series
$\mathcal{PS}(F) = \{ \sum_{k=1}^{\infty} a_k x^k \mid a_k \in F \} = F[[x]]$ (no convergence requirements)
- Think about trying to find a basis for F^∞.

Lecture 2 01-23

Last time: Fields and vector spaces.

Thm/Def: a) Every vector space V has a basis.
b) The number of elements in a basis for V is the same for every basis of V. Call it the **dimension** of V.

Remark:
- if $\dim(V)$ is not finite, write $\dim(V) = \infty$.
- The proof of (a) for a general vector space (possibly of ∞ dimension) needs the Axiom of Choice (actually, it is in some sense equivalent to the AC).
(The proof of (b) in general also needs the Axiom of Choice, or at least the weaker Ultrafilter Lemma.)

Even more is true:

Prop: Every linearly independent subset $S \subset V$ can be extended to a basis B of V (meaning: there is a basis B s.t. $S \subset B$).

Proof: If $\dim(V) = \infty$, need the AC.

So, consider only the case $\dim(V) = n < \infty$.

$S = \{v_1, \ldots, v_m\}$, with $m \le n$. If $m = n$, then S is a basis and we are done. Otherwise, take any $v_{m+1} \in V$ that is not in $\mathrm{Span}(S)$. Clearly, $\{v_1, \ldots, v_{m+1}\}$ is lin. indep: if $a_1 v_1 + \ldots + a_{m+1} v_{m+1} = 0$, then there are 2 possible cases:

- if $a_{m+1} \neq 0$, then $v_{m+1} = -\frac{1}{a_{m+1}}(a_1 v_1 + \ldots + a_m v_m) \in \mathrm{Span}(S)$, which contradicts our assumption on v_{m+1};
- if $a_{m+1} = 0$, then $a_1 v_1 + \ldots + a_m v_m = 0 \Rightarrow$ all $a_i = 0$ because S is assumed linearly independent.

We conclude that $\{v_1, \ldots, v_{m+1}\}$ is linearly independent, as wanted. Continue this process until you have a lin. indep. subset of V with n elements. That is the basis we are after. $\blacksquare$

Exercise: Find a basis for $\mathcal{P}_3(\mathbb{C})$ that contains $\{x+1, x^2+1\}$.

Let $\{U_i\}_{i \in I}$ be a family of subspaces of V (meaning: there is a set I and for every $i \in I$ there is a subspace $U_i \subset V$).

Goal: Build more subspaces of V out of the U_i.

Prop: The intersection of the subspaces is a subspace

$$\bigcap_{i \in I} U_i = \{v \in V : v \in U_i \text{ for all } i \in I\} \subset V.$$

Proof: 0 is contained in all vector subspaces, hence it is contained in the intersection.

Closure under addition: if $u, v \in \bigcap_{i \in I} U_i$, then $u, v \in U_i$ for all $i \in I$. Since the U_i are subspaces, they all contain $u+v$. Hence, $u+v \in \bigcap_{i \in I} U_i$.

Closure under scalar multipl.: similar. $\blacksquare$

Note: Frequently, we'll consider finite intersections $U_1 \cap \ldots \cap U_m$.

<u>Exercise</u>: Find two subspaces $U_1, U_2 \subset \mathbb{R}^2$ such that the <u>union</u> $U_1 \cup U_2 = \{ v \in V : v \in U_1 \text{ or } v \in U_2 \}$ is <u>not</u> a subspace of $\mathbb{R}^2$.

<u>Q</u>: What is the "smallest" subspace of V containing $U_1 \cup U_2$? Answer:

<u>Def</u>: Let $\{U_i\}_{i\in I}$ be a family of subspaces of V.
The <u>sum</u> of these subspaces is

$$\sum_{i\in I} U_i = \Big\{ \sum_{i\in I} u_i \in V : \text{the } u_i \in U_i \text{ and only } \underline{\text{finitely many}}\ u_i \text{ can be } \neq 0 \Big\} \subset V.$$

$$= \operatorname{Span}\Big(\bigcup_{i\in I} U_i \Big)$$

<u>Note</u>: Frequently, we will consider <u>finite</u> sums

$$U_1 + \dots + U_m = \{ u_1 + \dots + u_m \in V : \text{the } u_i \in U_i \}.$$

<u>Prop</u>: $\sum_{i\in I} U_i$ is a subspace of V.

<u>Proof</u>: The sum contains 0 and is closed under addition and scalar multiplication, as can be easily checked. ∎

<u>Exercise</u>: Let $U_1 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 3x_1 + x_2 - x_4 = 0 \text{ and } x_1 + 2x_2 + 5x_3 = 0 \}$
and $U_2 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 - x_3 = 0 \text{ and } 4x_2 + x_4 = 0 \}$.
Find $U_1 + U_2$.

<u>Def</u>: Let $\{U_i\}_{i\in I}$ be a family of subspaces of V.
If every element in $\sum_{i\in I} U_i$ can be written <u>uniquely</u> as a sum $\sum_{i\in I} u_i$, with $u_i \in U_i$ and only <u>finitely</u> many $u_i \neq 0$, then say that $\sum_{i\in I} U_i$ is the <u>internal direct sum</u> of the subspaces $U_i \subset V$. Write $\bigoplus_{i\in I} U_i$. ← because it is a subspace of a <u>specified</u> V

<u>Note</u>: If there are only <u>finitely</u> many U_i, write $U_1 \oplus \dots \oplus U_m$.

<u>Prop</u>: $\sum_{i\in I} U_i = \bigoplus_{i\in I} U_i$ iff $\Big\{ 0 = \sum_{i\in I} u_i$ with $u_i \in U_i$ and only fin. many $u_i \neq 0 \Rightarrow$ all $u_i = 0$.

(So: to show that the sum is an internal $\underline{\text{direct}}$ sum, it is enough to check that $\underline{\underline{0}}$ can only be written as a sum in one way.)

<u>Proof</u>: see Axler.

<u>Prop</u>: $U_1 + U_2$ is an internal direct sum iff $U_1 \cap U_2 = \{0\}$.

<u>Proof</u>: $(\Rightarrow)$: Let $v \in U_1 \cap U_2$. Then, $v \in U_1$ and $v \in U_2$, so we can write

$$v = \underset{U_1}{\underset{\cap}{v}} + \underset{U_2}{\underset{\cap}{0}} = \underset{U_1}{\underset{\cap}{0}} + \underset{U_2}{\underset{\cap}{v}}.$$

But since $U_1 + U_2 = U_1 \oplus U_2$, we can only do this one way. Hence, $v = 0$, as wanted.

$(\Leftarrow)$ Assume that $U_1 \cap U_2 = \{0\}$.

If $0 = u_1 + u_2$, with $u_1 \in U_1$ & $u_2 \in U_2$, then

Prop. above

$u_2 = -u_1 \in U_1 \Rightarrow u_2 \in U_1 \cap U_2 \Rightarrow u_2 = u_1 = 0 \Rightarrow U_1 + U_2 = U_1 \oplus U_2$. ∎

<u>Prop</u>: Let $U \subset V$ be a vector subspace. Then, there is another vector subspace $W \subset V$ st $U \oplus W = V$.

<u>Proof</u>: Let $B \subset U$ be a basis for U (we saw above that it always exists).

Extend B to a basis $\tilde{B}$ of V, st $B \subset \tilde{B}$ (we saw above that the extension always exists).

Let $W = \text{Span}(\underbrace{\tilde{B} \setminus B}_{\text{the elements in } \tilde{B} \text{ that are not in } B})$.

<u>Claim</u>: $U \oplus W = V$.

- $U + W$ is a direct sum: $U \cap W = \text{Span}(B) \cap \text{Span}(\tilde{B} \setminus B) = \{0\}$.
  Now, use the previous result.
- $U + W = V$: because $U + W \subset V$ is a vector subspace containing the basis $\tilde{B}$. ∎

<u>Note</u>: We will see later that, if V has an inner product, then can take $W = U^{\perp}$ (orthogonal complement).

Thm (Dimension formula): Given $U_1, U_2 \subset V$ subspaces, with $\dim V < \infty$,

$$\boxed{\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)}.$$

Proof: Pick a basis $u_1, \dots, u_k$ for $U_1 \cap U_2$.

By a result from last time, can extend it to:

- a basis $u_1, \dots, u_k, v_1, \dots, v_\ell$ for U_1 and
- a basis $u_1, \dots, u_k, w_1, \dots, w_m$ for U_2.

Claim: $u_1, \dots, u_k, v_1, \dots, v_\ell, w_1, \dots, w_m$ is a basis for $U_1 + U_2$.

Exercise: Prove the claim.

Claim $\Rightarrow$ Thm:

$$\dim(U_1+U_2) = k + \ell + m = \underbrace{(k+\ell)}_{\dim U_1} + \underbrace{(k+m)}_{\dim U_2} - \underbrace{k}_{\dim(U_1 \cap U_2)} = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

∎

Cor: $\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2$.

Proof: If $U_1 + U_2 = U_1 \oplus U_2$, then $U_1 \cap U_2 = \{0\}$. ∎

Lecture 3 01-24

Linear maps aka **linear transformations** aka **homomorphisms of vector spaces**

Def: Given V, W vector spaces over a field F, a map $\varphi : V \to W$ is **linear** if

$$\varphi(a_1 v_1 + a_2 v_2) = a_1 \varphi(v_1) + a_2 \varphi(v_2) \quad \text{for every } a_1, a_2 \in F,\ v_1, v_2 \in V.$$

Prop: The set of linear maps

$$\mathcal{L}(V, W) = \{\varphi : V \to W : \varphi \text{ is linear}\}$$

is a vector space.

Prop: The composition of linear maps

$$\begin{tikzcd} U \arrow[r, "\varphi"'] \arrow[rr, bend left, "\psi \circ \varphi"] & V \arrow[r, "\psi"'] & W \end{tikzcd}$$

is a linear map.

<u>Def</u>: $\varphi \in \mathcal{L}(V,W)$ is an <u>isomorphism</u> if it is a bijection.
the inverse function $\varphi^{-1}: W \to V$ is also linear.
In this case, say that V and W are <u>isomorphic</u>. Write $V \cong W$.

<u>Thm</u>: If V and W are finite dimensional, then
$V \cong W$ iff $\dim(V) = \dim(W)$.

<u>Cor</u>: $\dim(V) < \infty \Rightarrow V \cong F^{\dim(V)}$.

<u>Exercise</u>: Show that $\mathcal{P}_m(F) \cong F^{m+1}$.
Show that $\mathcal{PS}(F) \cong F^{\infty}$.
{power series with coefficients in F

<u>Def</u>: Given $\varphi \in \mathcal{L}(V,W)$, there are the following subspaces:

- The <u>kernel</u>: $\ker(\varphi) = \{ v \in V : \varphi(v) = 0 \} \subset V$.
  Also called <u>null space</u>, $\operatorname{null}(\varphi)$.
- The <u>image</u>: $\operatorname{im}(\varphi) = \{ \varphi(v) \in W : v \in V \} \subset W$.
  Also called <u>range</u>, $\operatorname{range}(\varphi)$.

<u>Rank-nullity Thm</u>: given $\varphi \in \mathcal{L}(V,W)$, $\dim V = \overbrace{\dim \operatorname{null}(\varphi)}^{\text{"nullity"}} + \overbrace{\dim \operatorname{im}(\varphi)}^{\text{"rank"}}$

<u>Proof</u>: Recall that there is a subspace $U \subset V$ st $\operatorname{null}(\varphi) \oplus U = V$.

<u>Claim</u>: The restriction of φ to U, $\varphi|_U : U \to \operatorname{im}(\varphi)$, is an iso.

Claim $\Rightarrow$ thm: $\dim V = \dim \operatorname{null}(\varphi) + \underbrace{\dim U}_{= \dim \operatorname{im}(\varphi)}$

dim of direct sum

∎

Matrices

Let $M(m \times n, F) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in F \right\}$

denote the space of $m \times n$ matrices with entries in F.

Prop: $M(m\times n, F)$ is an F-vector space of dimension $m \cdot n$.

Prop: Let $\dim(V) = n$ and $\dim(W) = m$ and pick <u>ordered</u> bases $(v_1, \dots, v_n)$ for V and $(w_1, \dots, w_m)$ for W. The map

$$\mathcal{M} : \mathcal{L}(V, W) \longrightarrow M(m\times n, F)$$

such that $\mathcal{M}(\varphi) = \begin{bmatrix} | & & | \\ \varphi(v_1) & \cdots & \varphi(v_n) \\ | & & | \end{bmatrix}$

is an isomorphism of vector spaces.

$\begin{matrix} | \\ \varphi(v_i) \\ | \end{matrix}$ is the column i $\begin{matrix} a_{1i} \\ \vdots \\ a_{mi} \end{matrix}$, st

$\varphi(v_i) = a_{1i} w_1 + \dots + a_{mi} w_m$

Cor: $\dim \mathcal{L}(V,W) = \dim(V) \cdot \dim(W)$.

Note: $\mathcal{M}(\varphi)$ depends on the <u>choices</u> of ordered bases.

Under such isomorphisms, composition

$$\mathcal{L}(V,W) \times \mathcal{L}(U,V) \to \mathcal{L}(U,W) \quad \text{corresponds to}$$

matrix multiplication

$$M(\dim(W) \times \dim(V), F) \times M(\dim(V) \times \dim(U), F) \to M(\dim(W), \dim(U), F).$$

More concretely, given $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ <u>and</u> choices of <u>ordered</u> bases for U, V, W

$$\mathcal{M}(\psi \circ \varphi) = \mathcal{M}(\psi) \cdot \mathcal{M}(\varphi)$$

composition of linear maps

product of matrices

Note: If you are familiar with the language of <u>ring theory</u>, and if $\dim(V) = n < \infty$, then if you pick a basis $v_1, \dots, v_n$ for V as above, the map

$$\mathcal{M} : \mathcal{L}(V, V) \longrightarrow M(n\times n, F)$$

is a <u>ring isomorphism</u>.

<u>Important</u>: Review how the matrix $\mathcal{M}(\varphi)$ changes if one changes the bases of V and W.

<u>Def</u>: Given a collection of vector spaces $\{U_i\}_{i\in I}$, their

- <u>direct product</u> is the vector space (union of sets)

$$\prod_{i\in I} U_i = \{\text{functions } f : I \to \bigcup_{i\in I} U_i \;:\; \text{for all } i\in I,\ f(i)\in U_i\}$$

  * addition: $(f_1+f_2)(i) = f_1(i)+f_2(i) \in U_i$
  * scalar mult: $(af)(i) = a\cdot f(i) \in U_i$

- <u>external direct sum</u> is the vector subspace

$$\bigoplus_{i\in I} U_i = \{ f \in \prod_{i\in I} U_i \;:\; \text{only finitely many } f(i)\neq 0\}.$$

<u>Note</u>: If the collection of vector spaces is finite, then their direct product and external direct sum are the same.

Write $U_1 \oplus \dots \oplus U_m$ for the <u>external direct sum</u> of finitely many vector spaces $U_1,\dots,U_m$. This can be identified with the set

$U_1\times\dots\times U_m = \{(u_1,\dots,u_m) : \text{all } u_i\in U_i\}$ ("<u>Cartesian product</u>"), with

* addition: $(u_1,\dots,u_m)+(v_1,\dots,v_m) = (u_1+v_1,\dots,u_m+v_m)$
* scalar mult: $a\cdot(u_1,\dots,u_m) = (au_1,\dots,au_m)$.

<u>Note</u>: We'll see that the external and internal direct sums of subspaces are isomorphic. To minimize confusion, we sometimes write $U_1\times\dots\times U_m$ for the external direct sum.

<u>Prop</u>: If $\dim(U_1)<\infty,\dots,\dim(U_m)<\infty$, then

$$\dim(U_1\oplus\dots\oplus U_m) = \dim(U_1)+\dots+\dim(U_m).$$

<u>Ex</u>: $\overbrace{F\oplus\dots\oplus F}^{n \text{ times}} = F^n$.

<u>Note</u>: Since $\dim(V)=\dim(W)<\infty \Rightarrow V\cong W$, the Prop. above already implies the following one, for finite direct sums of finite dimensional vector subspaces.

Prop: $\boxed{\text{Internal direct sum} \cong \text{external direct sum.}}$ More explicitly: Given a collection of subspaces $\{U_i\}_{i\in I} \subset V$, the internal and external direct sums of the U_i are isomorphic.

Proof: To ease notation, suppose we have finitely many subspaces (but the same argument applies in the general case). The linear map

$$U_1 \times \dots \times U_m \longrightarrow U_1 + \dots + U_m$$

$$(u_1, \dots, u_m) \longmapsto u_1 + \dots + u_m$$

is an isomorphism: surjective by def of $\sum U_i$; injective because $0 = \sum u_i$ uniquely. $\boxtimes$

Note: Since the internal and external direct sum are isomorphic, we'll just say "direct sum" and write $\oplus$.

Quotient space

Let $U \subset V$ be a subspace. Define a relation on V:

$$v \sim w \iff v - w \in U.$$

Exercise: Show that $\sim$ is an equivalence relation:

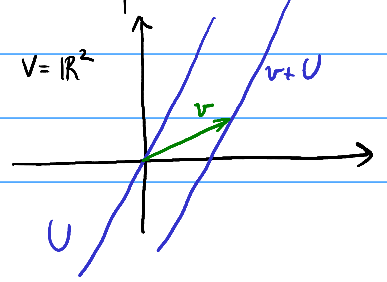
- reflexive: $v \sim v$ ($v - v = 0 \in U$ since U is a subspace)
- symmetric: $v \sim w \Rightarrow w \sim v$
  ($v \sim w \iff v - w \in U \iff -(v-w) = w - v \in U \iff w \sim v$) because U is a subspace
- transitive: $u \sim v$ and $v \sim w \Rightarrow u \sim w$
  ($u \sim v$ & $v \sim w \Rightarrow u - v \in U$ & $v - w \in U \Rightarrow (u-v)+(v-w) = u - w \in U \iff u \sim w$)
  $\leftarrow$ U subspace

Consider the set of equivalence classes, or cosets:

$$V/U = \{ v + U : v \in V \}.$$

Can think of $v + U$ as an affine subspace of V (doesn't have to contain 0)

Ex:



V is the union of all the cosets.
Can write $[v]$ instead of $v + U$.

Prop: V/U is a vector space with

- addition: $[v]+[w]=[v+w]$.
- scalar multiplication: $a\cdot[v]=[av]$.

Proof idea: addition is well-defined: if $v_1\sim v_2$ and $w_1 \sim w_2$,

$$[v_1]+[w_1]=[v_1+w_1]=[v_1+\underbrace{v_2-v_1}_{\in U}+w_1+\underbrace{w_2-w_1}_{\in U}]=[v_2+w_2]=[v_2]+[w_2]$$

Similar for scalar mult.

Need to check vector space axioms. Tedious but easy. ∎

Prop: The map $\pi: V \to V/U$, $v \mapsto [v]$ is well-defined and linear. Call it projection.

First isomorphism thm: For every $\varphi \in \mathcal{L}(V,W)$,

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi"'] & W \\ V/\mathrm{null}\,\varphi \arrow[ur, dashed, "\exists!\,\tilde\varphi"'] \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

This means that there is a unique linear map $\tilde\varphi: V/\mathrm{null}\,\varphi \to W$ st $\varphi=\tilde\varphi\circ\pi$.

$\tilde\varphi$ is injective and defines an isomorphism $\tilde\varphi: V/\mathrm{null}\,\varphi \to \mathrm{im}\,\varphi$.

Also, $V \cong \mathrm{null}\,\varphi \oplus \mathrm{im}\,\varphi \cong \mathrm{null}\,\varphi \oplus V/\mathrm{null}\,\varphi$

external direct sums

Proof: Given $[v]\in V/\mathrm{null}(\varphi)$, $\tilde\varphi([v])$ is uniquely specified by $\tilde\varphi([v])=\tilde\varphi(\pi(v))=\varphi(v)$. Need to check:

- $\tilde\varphi$ is well-defined: $[v]=[w] \Rightarrow v-w\in\mathrm{null}(\varphi) \Rightarrow \tilde\varphi([v])-\tilde\varphi([w])=\varphi(v)-\varphi(w)=\varphi(v-w)=0$.
- $\tilde\varphi$ is linear: $\tilde\varphi(a[v]+b[w]) \overset{\pi\text{ linear}}{=} \tilde\varphi([av+bw]) \overset{\varphi\text{ linear}}{=} a\varphi(v)+b\varphi(w) = a\tilde\varphi([v])+b\tilde\varphi([w])$.
- $\tilde\varphi$ is injective: $\tilde\varphi([v])=\varphi(v)=0 \Rightarrow v\in\mathrm{null}(\varphi) \Rightarrow [v]=0$.
- $\tilde\varphi$ is surjective onto $\mathrm{im}\,\varphi$: if $w=\varphi(v)\in\mathrm{im}(\varphi)$, then $w=\tilde\varphi([v])\in\mathrm{im}(\tilde\varphi)$.
- $V\cong\mathrm{null}\,\varphi\oplus\mathrm{im}\,\varphi$: follows from the Claim in proof of Rank-Nullity Thm. ∎

Cor: For every subspace $U\subset V$, have $V\cong U\oplus V/U$ — external $\oplus$

Proof: Find $W\subset V$ st $V=U\oplus W$. — internal $\oplus$

Apply 1st iso thm to $\varphi: V\to W$, $u+w\mapsto w$, where $u\in U$ and $w\in W$.

Note that $U=\mathrm{null}\,\varphi$ and $W\cong V/U$. ∎

Cor: $\dim V/U = \dim V - \dim U$.

Universal property of the quotient: Let $U \subset V$ be a subspace and $\varphi \in \mathcal{L}(V,W)$. If $U \subset \ker\varphi$, then there is a unique linear map $\tilde{\varphi}: V/U \to W$ making the diagram commute:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi"'] & W \\ V/U \arrow[ur, dashed, "\exists!\,\tilde{\varphi}"'] \arrow[u, phantom, "="] & \end{tikzcd}.$$

Exercise: How are U and $\ker\varphi$ related if $\tilde{\varphi}$ is injective?

Lecture 4 02-27

Tensor product

Idea: Let V, W be vector spaces over a field F.

Want: a vector space $V \otimes W$, consisting of linear combos. of terms of the form $v \otimes w$, with $v \in V$ and $w \in W$.

this should be bilinear (meaning: linear in V and in W):

(A) $(a_1 v_1 + a_2 v_2) \otimes w - a_1(v_1 \otimes w) - a_2(v_2 \otimes w) = 0$ and

(B) $v \otimes (a_1 w_1 + a_2 w_2) - a_1(v \otimes w_1) - a_2(v \otimes w_2) = 0$.

Note: $V \oplus W$ doesn't work like that:

$(a_1 v_1 + a_2 v_2, a_1 w_1 + a_2 w_2) - a_1(v_1, w_1) - a_2(v_2, w_2) = 0$

Let's construct $V \otimes W$ "by hand" to have these properties:

Step 1: Given any set S, let $F(S)$ be a vector space for which the set S forms a basis. An element in $F(S)$ is a linear combination

$$a_1 \cdot s_1 + \dots + a_m \cdot s_m, \quad \text{all } a_i \in F \text{ and all } s_i \in S.$$

Say that $F(S)$ is the free vector space on the set S.

Note: One can take $F(S) = \bigoplus_{i \in S} F$

Ex: $\mathcal{P}_m(F) = F(\{1, x, \dots, x^m\})$. Cartesian product of the sets V, W

Step 2: Take the vector space $F(V \times W)$ - this is step 1 with $S = V \times W$.

Elements in $F(V\times W)$ are lin. combos

$$a_1 \cdot (v_1, w_1) + \dots + a_m \cdot (v_m, w_m).$$

"$F(V \times W)$ is huge".

Step 3: Let $U \subset F(V\times W)$ be the subspace spanned by the set of all elements of the forms (inspired by (A) & (B) above)

(A') $(a_1 v_1 + a_2 v_2, w) - a_1 (v_1, w) - a_2 (v_2, w)$ or

(B') $(v, a_1 w_1 + a_2 w_2) - a_1 (v, w_1) - a_2 (v, w_2)$

Step 4: Define $V\otimes W = F(V\times W)/U$.

Write $\boxed{V \otimes_F W}$ if it is useful to specify the field.

Write denote $[(v,w)] \in V\otimes W$ by $v\otimes w$. Call this a simple tensor.

Prop: (A) and (B) hold.

Proof: By Steps 3 and 4 above.

Ex: In $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$:

- $(1,0)\otimes(0,1) + (1,0)\otimes(1,1) \overset{(B)}{=} (1,0)\otimes\big((0,1)+(1,1)\big) = (1,0)\otimes(1,2)$ is a simple tensor
- $(1,0)\otimes(1,0) + (0,1)\otimes(0,1)$ is not a simple tensor

Special cases:
- $a(v\otimes w) \overset{\text{by (A)}}{=} (av)\otimes w \overset{\text{by (B)}}{=} v\otimes (aw)$ for all $a\in F$, $v\in V$, $w\in W$.
- $0\otimes w = 0 = v\otimes 0$
- $F\otimes_F V \cong V \cong V\otimes_F F$

Thm: Given bases B for V and $\tilde{B}$ for W, the set $\{ v\otimes w \in V\otimes W : v\in B \text{ and } w\in\tilde{B}\}$ is a basis for $V\otimes W$.

Cor: $\dim(V\otimes W) = \dim(V)\cdot\dim(W)$

(compare with $\dim(V\oplus W) = \dim(V)+\dim(W)$ and $\dim \mathcal{L}(V,W) = \dim(V)\cdot\dim(W)$).

Observe that there is a bilinear map $\begin{array}{ccc} V\times W & \xrightarrow{\tau} & V\otimes W \\ (v,w) & \longmapsto & v\otimes w \end{array}$ (linear on V & on W). "tensor"

Meaning: $\tau(a_1v_1+a_2v_2, w) = (a_1v_1+a_2v_2)\otimes w = a_1(v_1\otimes w) + a_2(v_2\otimes w) = a_1\tau(v_1,w)+a_2\tau(v_2,w)$.

$\tau(v, a_1w_1+a_2w_2) = \dots = a_1\cdot\tau(v,w_1) + a_2\cdot\tau(v,w_2)$.

The next result is useful to construct linear maps out of $V \otimes W$:

<u>Universal property of $\otimes$</u>: Given a bilinear map $\varphi : V \times W \to U$,

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[d, "\tau"'] & U \\ V \otimes W \arrow[ur, dashed, "\exists! \tilde{\varphi}"'] & \end{tikzcd}$$

This means that there is a unique linear map $\tilde{\varphi} \in \mathcal{L}(V \otimes W, U)$, s.t. $\tilde{\varphi} \circ \tau = \varphi$.

<u>Proof</u>: We must define $\tilde{\varphi}$ on simple tensors as

$\tilde{\varphi}(v \otimes w) = \tilde{\varphi}(\tau(v,w)) = \varphi(v,w)$. Extend $\tilde{\varphi}$ by linearity to lin. combs.

Need to show that $\tilde{\varphi}$ is <u>linear</u> (although φ and τ are <u>bilinear</u>). $\square$

<u>Slogan</u>: "$\otimes$ converts bilinearity into linearity".

<u>Properties of $\otimes$</u>:
- $V \otimes W \cong W \otimes V$ (commutativity)
- $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ (associativity)
- $U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$ (distributivity)
- $\mathcal{L}(U \otimes V, W) \cong \mathcal{L}(U, \mathcal{L}(V, W))$

Lecture 6 02-03

<u>Dual spaces</u>

Let V be a vector space over F.

<u>Def</u>: The <u>dual space</u> of V is the vector space

$$V' = \mathcal{L}(V, F) \quad \text{(sometimes also denoted } V^*\text{)}.$$

An element $\varphi \in V'$ is called a <u>linear functional</u> on V (or a <u>covector</u>).

<u>Exs</u>: • Given a set S and a field F, the set

$$F^S = \{ \text{functions } f : S \to F \}$$

is a vector space over F ($(f+g)(x) = f(x) + g(x)$, $(a \cdot f)(x) = a \cdot f(x)$).

Given any $x_0 \in S$, there is a linear functional $\varphi_{x_0} \in (F^S)'$:

$$\varphi_{x_0}(f) = f(x_0).$$

• More concrete version: $\varphi_4 \in (\mathcal{P}_m(\mathbb{R}))'$ s.t. $\varphi_4(p) = p(4)$ for $p \in \mathcal{P}_m(\mathbb{R})$.

- Let $V = C([2,5], \mathbb{R})$ be the vector space of continuous functions $f : [2,5] \to \mathbb{R}$.
  The map $\varphi : V \to \mathbb{R}$ st $\varphi(f) = \int_2^5 f(x)\,dx$ is in V'.

In these examples, a linear functional is a "function of functions".

Suppose that $\dim(V) = m < \infty$ and <u>fix a basis</u> $v_1, \dots, v_m$.
Define $\varphi_1, \dots, \varphi_m \in V'$ by
$$\varphi_i(a_1 v_1 + \dots + a_m v_m) = a_i .$$

<u>Prop</u>: $\varphi_1, \dots, \varphi_m$ is a basis for V'. In particular, $\dim V' = m$ and $V' \cong V$.
Call $\varphi_1, \dots, \varphi_m$ the <u>dual basis</u> of $v_1, \dots, v_m$.

<u>Proof</u>:
- <u>lin indep</u>: if $a_1\varphi_1 + \dots + a_m \varphi_m = 0$, then for every $i = 1, \dots, m$
  $(a_1\varphi_1 + \dots + a_m\varphi_m)(v_i) = a_i = 0 \Rightarrow \varphi_1, \dots, \varphi_m$ are lin. indep.
- <u>generate</u>: let $\varphi \in V'$. Claim: $\varphi = \varphi(v_1)\varphi_1 + \dots + \varphi(v_m)\varphi_m$.
  Proof of Claim: for every $v = a_1 v_1 + \dots + a_m v_m \in V$,
  $\varphi(v) = \varphi(a_1 v_1 + \dots + a_m v_m) = a_1 \varphi(v_1) + \dots + a_m \varphi(v_m)$ and
  $(\varphi(v_1)\varphi_1 + \dots + \varphi(v_m)\varphi_m)(v) = \varphi(v_1)\varphi_1(a_1v_1 + \dots + a_m v_m) + \dots + \varphi(v_m)\varphi_m(a_1 v_1 + \dots + a_m v_m)$
  $= \varphi(v_1)\cdot a_1 + \dots + \varphi(v_m)\cdot a_m$, as we wanted to show. $\blacksquare$

<u>Notes</u>:
- If $\dim(V) = m < \infty$, we already knew that
  $V' = \mathcal{L}(V, F) \cong \mathrm{Mat}(1 \times m, F) \cong F^m$, hence $\dim(V') = m$
  need choice of ordered basis for V
- If $\dim(V) = \infty$, then V' is <u>not</u> isomorphic to V.
  <u>Reason</u> (beyond the scope of this course): the cardinality of V' is bigger than the cardinality of V, <u>if</u> $\dim(V) = \infty$.
  <u>Ex</u>: $(\mathcal{P}(F))' \cong \mathcal{PS}(F)$ and $\mathcal{PS}(F) \not\cong \mathcal{P}(F)$.

<u>Important</u>: We saw above that if $\dim V = m < \infty$, then $V' \cong V$.
Fixing a basis $v_1, \dots, v_m$, we have the isomorphism
$$f : V \longrightarrow V' \quad \text{st} \quad f(a_1 v_1 + \dots + a_m v_m) = a_1\varphi_1 + \dots + a_m \varphi_m .$$
The isomorphism <u>depends on the choice of basis</u>. This is sometimes referred to as a <u>non-canonical isomorphism</u>.

<u>The double dual</u>:

Let $V'' = (V')'$ and define $f \in \mathcal{L}(V, V'')$ as follows: given $v \in V$, $f(v) = f_v \in V'' = \mathcal{L}(V', F)$ s.t., for every $\varphi \in V'$, we have

$$\boxed{f_v(\varphi) = \varphi(v)}$$

(f is linear because φ is linear)

<u>Prop 1</u>: $f \in \mathcal{L}(V, V'')$ is injective.

<u>Proof</u>: Take $v \in V$ s.t. $f(v) = f_v = 0$. For every $\varphi \in V'$, $f_v(\varphi) = \varphi(v) = 0$. The next claim implies that $v = 0$, as wanted:

<u>Claim</u>: if $v \neq 0$, then there is some $\varphi \in V'$ st $\varphi(v) \neq 0$.

Proof of Claim: Take $U \subset V$ st $\text{span}(v) \oplus U = V$ and define $\varphi \in V'$ by $\varphi(av + u) = a$ for all $a \in F$ and all $u \in U$. ∎

<u>Prop 2</u>: If $\dim V = n < \infty$, then $f \in \mathcal{L}(V, V'')$ is surjective.

<u>Proof</u>: We already know that f is injective and that $n = \dim V = \dim V' = \dim V''$. By the rank-nullity thm,

$$n = \underbrace{\dim \text{null} f}_{=0} + \dim \text{im} f \Rightarrow \dim \text{im} f = n = \dim V'$$

$$\Rightarrow f \text{ is surjective.}$$ ∎

Prop 1 & Prop 2 imply:

<u>Thm</u>: If $\dim V < \infty$, then $f \in \mathcal{L}(V, V'')$ is an isomorphism.

<u>Note</u> We say that such f is a <u>canonical</u> isomorphism.

Meaning: f is an isomorphism that <u>does not depend</u> on a choice of basis for V.

<u>Homework</u>: If $\dim(V) < \infty$ and $\dim(W) < \infty$, then there is a <u>canonical isomorphism</u> $\mathcal{L}(V, W) \cong V' \otimes W$.

↑ ie, independent of choices of bases for V, W

The elements of both vector spaces "take vectors in V and return vectors in W".

__Notation__: given $v \in V$ and $\varphi \in V'$, write
$$\underbrace{\langle v, \varphi \rangle_V}_{} = \varphi(v).$$
This is not an inner product. φ and v live in different spaces

__Def__: Let $T \in \mathcal{L}(V,W)$. Its dual map is $T' \in \mathcal{L}(W',V')$, s.t. for every $\varphi \in W'$ and every $v \in V$, $(T'(\varphi))(v) = \varphi(T(v))$.
Equivalently, can write $\langle v, T'(\varphi) \rangle_V = \langle T(v), \varphi \rangle_W$.

__Prop__: Let V, W be fin. dim. vector spaces and fix ordered bases $v_1, \dots, v_n$ of V and $w_1, \dots, w_m$ of W.
Let $T \in \mathcal{L}(V,W)$. Then,
$$\underbrace{\mathcal{M}(T')}_{\text{matrix for } T' \text{ with respect to the dual bases of } W', V'} = \Big(\underbrace{\mathcal{M}(T)}_{\text{matrix for } T \text{ with respect to the chosen bases}}\Big)^{t} \leftarrow \text{transpose matrix}$$

__Recall__: $\begin{bmatrix} 3 & -1 & 0 \\ 5 & 2 & -7 \end{bmatrix}^{t} = \begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 0 & 7 \end{bmatrix}$.

__Exercise__: Define $D : \mathcal{P}_3(F) \to \mathcal{P}_2(F)$ by
$$D(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2 \quad (\text{"derivative"})$$

a) Show that $D \in \mathcal{L}(\mathcal{P}_3(F), \mathcal{P}_2(F))$

b) Fix the standard bases $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3(F)$ and $\{1, x, x^2\}$ of $\mathcal{P}_2(F)$.
Write the matrix representative of the dual map D', with respect to the dual bases of $(\mathcal{P}_2(F))'$ and $(\mathcal{P}_3(F))'$.