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Counting strings in Dyck paths

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Abstract

This paper deals with the enumeration of Dyck paths according to the statistic "number of occurrences of τ", for an arbitrary string τ. In this direction, the statistic "number of occurrences of τ at height j" is considered. It is shown that the corresponding generating function can be evaluated with the aid of Chebyshev polynomials of the second kind. This is applied to every string of length 4. Further results are obtained for the statistic "number of occurrences of τ at even (or odd) height".

MSC: 05C05; 05C07; 05A15

Keywords: Dyck paths; Catalan numbers; Chebyshev polynomials; Strings

1. Introduction

A *Dyck path of semilength n* is a lattice path in the first quadrant, which begins at the origin $(0, 0)$, ends at $(2n, 0)$ and consists of steps $(1, 1)$ (called *rises*) and $(1, -1)$ (called *falls*). In a Dyck path a *peak* (resp. *valley*) is a point immediately preceded by a rise (resp. fall) and immediately followed by a fall (resp. rise). A *doublerise* (resp. *doublefall*) is a point immediately preceded and followed by a rise (resp. fall). The *height* of a point is its y-coordinate. A peak (resp. valley) is called *low* if it has height 1 (resp. 0). A peak (resp. valley) is called *small* if it is not immediately preceded by a doublerise (resp. doublefall) and immediately followed by a doublefall (resp. doublerise). Obviously, a low peak is a small peak. An *ascent* (resp. *descent*) of a Dyck path is a maximal sequence of consecutive rises (resp. falls).

It is clear that each Dyck path of semilength n is coded by a word $\alpha = a_1 a_2 \cdots a_{2n} \in \{u, d\}^*$, called *Dyck word*, so that every rise (resp. fall) corresponds to the letter u (resp. d). For example, the Dyck path of Fig. 1 is coded by the word $\alpha = uuudududdududduudd$.

Throughout this paper we denote with $\mathscr{D}$ the set of all Dyck paths (or equivalently Dyck words). Furthermore, the subset of $\mathscr{D}$ that contains all the paths α of semilength $l(\alpha) = n$ is denoted with $\mathscr{D}_n$. It is well known that $|\mathscr{D}_n| = C_n$, where $C_n = 1/n+1 \binom{2n}{n}$ is the nth Catalan number (A00108 of [17]), with generating function $C(z)$, which satisfies the relation

$$zC^2(z) - C(z) + 1 = 0.$$

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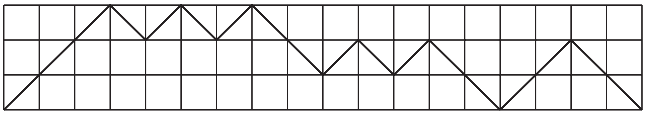

Fig. 1.

Furthermore, the powers of $C(z)$ are given in [7] by the relation

$$[z^n]C^s = \frac{s}{2n+s}\binom{2n+s}{n}, \tag{1}$$

for $(n, s) \neq (0, 0)$.

A word $\tau \in \{u, d\}^*$, called in this context *string*, *occurs* in a Dyck path α if $\alpha = \beta\tau\gamma$, where $\beta, \gamma \in \{u, d\}^*$. If a string τ does not occur in α we say that α *avoids* τ.

For example, the Dyck path of Fig. 1 has three occurrences of *udu*, whereas it avoids *ddd*.

A wide range of articles dealing with the occurrence of strings in Dyck paths appear frequently in the literature (e.g., see [2,4,5,7,10–13,16,18]).

For the study of the statistic "number of occurrences of τ" (or simply "number of τ's") we consider the generating function $F(t, z)$, where t counts the occurrences of τ, and z counts the semilength of the Dyck path. In other words, we have

$$F(t, z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k} t^k z^n,$$

where $a_{n,k}$ is the number of all $\alpha \in \mathscr{D}_n$ with k occurrences of τ. In particular, we will denote with $a_n = a_{n,0}$ the number of all $\alpha \in \mathscr{D}_n$ that avoid τ.

There is a general method to obtain the explicit formula for $a_{n,k}$, using some version of the Lagrange inversion formula [7, Appendix A], provided that the equation satisfied by the corresponding generating function has been produced. In this paper we apply this method several times, presenting it in detail only once (for the string *uuud*), since the other cases can be treated similarly.

The strings of length 2, namely uu, dd, ud and du, have been studied extensively; it has been proved (e.g., see [7]) that the corresponding statistics follow the Narayana distribution (A001263 of [17]). More generally, strings of length 2 have also been studied for k-colored Motzkin paths in [14].

The strings of length 3, namely *uuu*, *uud*, *udu*, *udd*, *duu*, *dud*, *ddu* and *ddd*, have also been studied extensively.

By symmetry with respect to a vertical axis, the statistics corresponding to each of the following pairs of strings: $\{duu, ddu\}$, $\{udu, dud\}$, $\{uuu, ddd\}$, $\{uud, udd\}$, are equidistributed.

The string $\tau = duu$ has been studied in [7] and it has been proved that the corresponding statistic follows the Touchard distribution, i.e.

$$tzF^2 - (1 - 2(1-t)z)F + 1 - (1-t)z = 0 \tag{2}$$

and that

$$a_{n,k} = \frac{2^{n-2k-1}}{n}\binom{n}{k}\binom{n-k}{k+1} = 2^{n-2k-1}C_k\binom{n-1}{2k}.$$

For a bijective proof of the above result, see [3].

The string $\tau = udu$ has been studied independently in [12] and [18], and it has been proved that the corresponding statistic follows the Donaghey distribution, i.e.

$$zF^2 - (1 + (1-t)z)F + 1 + (1-t)z = 0 \tag{3}$$

and that

$$a_{n,k} = \binom{n-1}{k} M_{n-1-k},$$

where $M_n=\sum_{k=0}^{[n/2]}\binom{n}{2k}C_k$ is the nth Motzkin number (A001006 of [17]). For a bijective proof of the above result, see [3] or [18].

It is known (A092107 of [17]) that for $\tau=uuu$ the generating function satisfies the equation

$$z(t+z-tz)F^2-(1-z+tz)F+1=0. \tag{4}$$

Furthermore, in this case it can be proved that

$$a_{n,k}=\frac{1}{n+1}\sum_{j=0}^{k}(-1)^{k-j}\binom{n+j}{n}\binom{n+1}{k-j}\sum_{i=j}^{[(n+j)/2]}\binom{n+j+1-k}{i+1}\binom{n-i}{i-j}.$$

In particular, for $k=0$, $a_n=1/n+1\sum_{i=0}^{[n/2]}\binom{n+1}{i}\binom{n+1-i}{i+1}$ and hence by a well-known formula for Motzkin numbers [1] we obtain that the number of all Dyck paths that avoid uuu is equal to M_n. For a bijective proof of the above result, see [3].

Finally, for the string $\tau=uud$ it has been proved in [15] that the corresponding generating function satisfies the equation

$$z((t-1)z+1)F^2-F+1=0 \tag{5}$$

and that

$$a_{n,k}=\frac{1}{n+1}\binom{n+1}{k}\sum_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n+1-k}{n-j}.$$

This number counts also the Dyck paths of semilength n with k long ascents (A091156 of [17]).

In Section 2, several new results in the same direction are presented, referring to the corresponding statistics for every string of length 4.

In Section 3, the statistic "number of τ's at height j" is studied and it is shown that for every string τ, the corresponding generating function F_j is expressed via F_0 and the Chebyshev polynomials of the second kind. Applying this result, the generating function F_j is evaluated for every string of length 4.

Finally, in Section 4, the statistics "number of τ's at even height" and "number of τ's at odd height" are studied. It is shown that for every string τ of length 3 there exist two strings τ_1, τ_2 of length 4 such that the statistics "number of τ's at even (resp. odd) height" and "number of τ_1's (resp. τ_2's)" are equidistributed.

Some of the results of this paper have been presented without proofs in [15].

2. Strings of length 4

There exist 16 strings τ of length 4, yielding 10 cases to be studied, since by symmetry with respect to a vertical axis, the statistic "number of τ's" for some of them (given here in pairs) are equidistributed: $\{uuud, uddd\}$, $\{uudd\}$, $\{uudu, dudd\}$, $\{uduu, ddud\}$, $\{udud\}$, $\{dduu\}$, $\{uuuu, dddd\}$, $\{uddu, duud\}$, $\{duuu, dddu\}$, $\{dudu\}$.

This statistic is known for every string of the first six sets.

Indeed, the statistic of the string $\tau=uuud$ is equidistributed with the statistic "number of branch nodes at odd height" in ordered trees ([8], A091958 of [17]). The corresponding generating function satisfies the equation

$$(t-1)z^3F^3+zF^2-F+1=0.$$

In order to evaluate the explicit formula of $a_{n,k}$ from the above equation, we write

$$A(z)=1+zH(A(z)),$$

where $H(\lambda)=(t-1)z^2\lambda^3+\lambda^2$ and $A(z)=F(t,z)$ considered as a function of z. By the Lagrange inversion formula [7] it follows that

$$[z^\sigma]A(z)=\frac{1}{\sigma}[\lambda^{\sigma-1}](H(1+\lambda))^\sigma.$$

Furthermore, by successive applications of the binomial formula, we obtain that

$$(H(1+\lambda))^{\sigma}=\sum_{v=0}^{3\sigma}\sum_{j=(v-2\sigma)^{+}}^{\sigma}\binom{\sigma}{j}\binom{2\sigma+j}{v}(t-1)^{j}z^{2j}\lambda^{v}.$$

For $v=\sigma-1$ it follows that

$$[z^{\sigma}]F(t,z)=\frac{1}{\sigma}\sum_{j=0}^{\sigma}\binom{\sigma}{j}\binom{2\sigma+j}{\sigma-1}(t-1)^{j}z^{2j}.$$

Then after some simple manipulations we have that

$$F(t,z)=1+\sum_{n=1}^{\infty}\sum_{k=0}^{[n/3]}\sum_{j=k}^{[n/3]}(-1)^{j-k}\frac{1}{n-2j}\binom{n-2j}{j}\binom{2n-3j}{n-2j-1}\binom{j}{k}t^{k}z^{n},$$

and hence, we finally have

$$a_{n,k}=\frac{1}{n+1}\binom{n+1}{k}\sum_{j=k}^{[n/3]}(-1)^{j-k}\binom{n+1-k}{j-k}\binom{2n-3j}{n}.$$

Moreover, we note that the number of Dyck paths that avoid *uuud* is counted by the Motzkin numbers.

For $\tau=uudd$ the corresponding generating function (A098978 of [17]) satisfies the equation

$$zF^{2}+(z^{2}(t-1)-1)F+1=0. \tag{6}$$

Furthermore, it can be proved that

$$a_{n,k}=\sum_{j=k}^{[n/2]}\frac{(-1)^{j-k}}{n-j}\binom{j}{k}\binom{n-j}{j}\binom{2n-3j}{n-j-1}.$$

For the avoiding sequence in this case, see A086581 and A025242 of [17].

For $\tau=uudu$ the corresponding generating function satisfies the equation

$$z(1-(1-t)z)F^{2}+((1-t)z^{2}-1)F+1=0. \tag{7}$$

More generally, the strings $u^{r}du$, for $r\geqslant 1$, have been studied recently in [11].

The occurrences of the string *uudu* are bijectively mapped to the occurrences of the string *uduu* by switching the middle two letters (that works because overlaps can only occur on the first or last *u*). Hence, for $\tau=uduu$ the corresponding generating function also satisfies Eq. (7).

Furthermore, using the Lagrange inversion formula, we obtain that the number of all Dyck paths of semilength n with k occurrences of *uudu* (or equivalently *uduu*) is given by

$$a_{n,k}=\sum_{j=k}^{[(n-1)/2]}\frac{(-1)^{j-k}}{n-j}\binom{j}{k}\binom{n-j}{j}\binom{2n-3j}{n-j+1}.$$

The first terms of this triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 22, 19, 1; 57, 66, 9; 154, 221, 53, 1.

It is also known (A094507 of [17]) that for $\tau=udud$ the corresponding generating function satisfies the equation

$$z(1+(1-t)z)F^{2}-(1+(1-t)z(z+1))F+1+(1-t)z=0, \tag{8}$$

while the avoiding sequence in this case counts also the irreducible stack sortable permutations (A078481 of [17]).

Finally, it is known (A114492 of [17]) that for $\tau = dduu$ the corresponding generating function satisfies the equation

$$z(t + (1-t)z)F^2 - (1 + (1-t)(z-2)z)F + 1 - (1-t)z = 0. \tag{9}$$

We note that the avoiding sequence in this case is equal to the avoiding sequence corresponding to the string $\tau = uudd$, shifted by one unit.

We now come to study the remaining cases.

2.1. The string uuuu

In this section we evaluate the generating function $F = F(t, z)$ for $\tau = uuuu$. For this, we consider the partition $\{\mathscr{A}_i\}$ of $\mathscr{D}$, where $\mathscr{A}_0$ contains only the empty path ε and $\mathscr{A}_i$ is the set of all Dyck paths with length of the first ascent equal to i, for every $i \geqslant 1$. Let $A_i = A_i(t, z)$ be the generating function of the set $\mathscr{A}_i$, where t counts the number of occurrences of *uuuu*. Clearly, since each element α of $\mathscr{A}_i$ can be uniquely written in the form $\alpha = u^i d\alpha_1 d\alpha_2 \cdots d\alpha_i$, where $\alpha_j \in \mathscr{D}$ for every $j \in [i]$, it follows that there exist (exactly $i-3$) new occurrences of *uuuu* in α, (in addition to those contributed by the α_i's) iff $i \geqslant 4$; thus, we have that

$$A_i = z^i F^i \quad \text{for } i \leqslant 3 \quad \text{and} \quad A_i = t^{i-3} z^i F^i \quad \text{for } i \geqslant 4.$$

It follows that

$$F = 1 + \sum_{i=1}^{3} z^i F^i + \sum_{i=4}^{\infty} t^{i-3} z^i F^i$$

and hence, after some simple manipulations we obtain that

$$F = 1 + tzF^2 + (1-t) \sum_{i=1}^{3} z^i F^i$$

which gives

$$(1-t)z^3 F^3 + z(t + z - tz)F^2 + ((1-t)z - 1)F + 1 = 0.$$

The first terms of the corresponding triangle formed by the coefficients of F, read by rows, are 1; 1; 2; 5; 13, 1; 36, 5, 1; 104, 21, 6, 1; 309, 84, 28, 7, 1.

The avoiding sequence is given (A036765 of [17]) by the formula

$$a_n = \frac{1}{n+1} \sum_{j=0}^{[n/2]} \binom{n+1}{n-2j} \binom{n+1}{j}.$$

A natural generalization of the above, is to consider the string $\tau = u^r$, $r \geqslant 2$. It can be proved that the generating function of the set of Dyck paths according to the number of occurrences of u^r and to the semilength satisfies the equation

$$F = 1 + tzF^2 + (1-t) \sum_{i=1}^{r-1} z^i F^i.$$

2.2. The string uddu

For the evaluation of the generating function $F = F(t, z)$ for $\tau = uddu$, we consider the set $\mathscr{A}$ of all Dyck paths with length of the last descent equal to 2, and its generating function $A = A(t, z)$, where t counts the number of occurrences of *uddu*. Clearly, every element α of $\mathscr{A}$ can be uniquely written in the form $\alpha = \alpha_1 u \alpha_2 udd$, where $\alpha_1, \alpha_2 \in \mathscr{D}$. Furthermore, exactly one (resp. two) new *uddu* occurs in α iff exactly one of (resp. both) α_1, α_2 belongs to $\mathscr{A}$; thus, we have that

$$A = z^2(F - A)^2 + 2z^2 t(F - A)A + z^2 t^2 A^2.$$

Moreover, since every non-empty $\alpha \in \mathscr{D}$ can be uniquely written in the form $\alpha = \beta u \gamma d$, where $\beta, \gamma \in \mathscr{D}$, we observe that a new *uddu* occurs in α iff $\beta \in \mathscr{A}$; thus we have that

$$F = 1 + ztAF + z(F - A)F.$$

From the above relations we obtain that the generating function F satisfies the equation

$$zF^3 - ((1-t)z + 1)F^2 + (1 + 2(1-t)z)F - (1-t)z = 0.$$

Using the Lagrange inversion formula we obtain that the number of all Dyck paths of semilength n with k occurrences of *uddu* is given by

$$a_{n,k} = \frac{1}{n}\binom{n}{k}\sum_{j=k}^{[(n-1)/2]}(-1)^{j-k}\binom{n-k}{j-k}\binom{2n-3j}{n-j+1}.$$

The first terms of the corresponding triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 23, 17, 2; 63, 54, 15; 178, 177, 69, 5; 514, 594, 273, 49.

2.3. The string duuu

For the evaluation of the generating function $F = F(t, z)$ for $\tau = duuu$, we consider the set $\mathscr{A}$ of all Dyck paths with length of the first ascent at least 3, and its generating function $A = A(t, z)$, where t counts the number of occurrences of *duuu*. Clearly, every element α of $\mathscr{A}$ can be uniquely written in the form $\alpha = uu\alpha_1 d\alpha_2 d\alpha_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{D}$ and $\alpha_1 \neq \varepsilon$. Furthermore, exactly one (resp. two) new *duuu* occurs in α iff exactly one of (resp. both) α_2, α_3 belongs to $\mathscr{A}$; thus, we have that

$$A = z^2(F-1)((F-A)^2 + 2t(F-A)A + t^2A^2).$$

Moreover, since every $\alpha \in \mathscr{D}\backslash\{\varepsilon\}$ can be uniquely written in the form $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathscr{D}$, we observe that a new *duuu* occurs in α iff $\gamma \in \mathscr{A}$; thus, we have that

$$F = 1 + ztAF + z(F - A)F.$$

From the above relations we obtain that the generating function F satisfies the equation

$$tzF^3 + (3(1-t)z - 1)F^2 - (3(1-t)z - 1)F + (1-t)z = 0.$$

Using the Lagrange inversion formula we obtain that the number of all Dyck paths of semilength n with k occurrences of *duuu* is given by

$$a_{n,k} = \frac{1}{n}\binom{n}{k}\sum_{j=3k+1}^{n}(-1)^{j-k+1}3^{n-j}\binom{n-k}{j-k}\binom{2j-2-3k}{j-1}.$$

The first terms of the corresponding triangle, read by rows, are 1; 1; 2; 5; 13, 1; 35, 7; 96, 36; 267, 159, 3.

The avoiding sequence a_n counts also the directed animals of size n, as well as the Grand–Dyck paths starting with u and avoiding *udu* (A005773 of [17]).

2.4. The string dudu

For the evaluation of the generating function $F = F(t, z)$ for $\tau = dudu$, we consider the set $\mathscr{A}$ of all Dyck paths of semilength greater than 2, that start with a low peak, and its generating function $A = A(t, z)$, where t counts the number of occurrences of *dudu*. Clearly, every element α of $\mathscr{A}$ can be uniquely written in the form $\alpha = ud\beta$, where $\beta \in \mathscr{D}\backslash\{\varepsilon\}$. Furthermore, exactly one new *dudu* occurs in α iff $\beta \in \mathscr{A}$; thus, we have that

$$A = ztA + z(F - 1 - A).$$

Moreover, since every $\alpha \in \mathscr{D}\backslash\{\varepsilon\}$ can be uniquely written in the form $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathscr{D}$, we observe that a new *dudu* occurs in α iff $\gamma \in \mathscr{A}$; thus, we have that

$$F = 1 + ztFA + zF(F - A).$$

From the above relations we obtain that the generating function F satisfies the equation

$$zF^2 + ((1-t)(z-1)z - 1)F + (1-t)z + 1 = 0. \tag{10}$$

Notice that F coincides with the generating function of the set of Dyck paths according to the number of ascents of length 1 that start at an odd level and to the semilength (A102405 of [17]). Hence the statistics "number of *dudu*'s" and "number of ascents of length 1 that start at an odd level" are equidistributed.

Finally, using the Lagrange inversion formula, we obtain that the number of all Dyck paths of semilength n that avoid *dudu*, is given by the formula

$$a_n = \sum_{j=0}^{[n/2]} \frac{1}{n-j}\binom{n-j}{j}\sum_{i=0}^{n-2j}\binom{n-2j}{i}\binom{j+i}{n-2j-i+1}.$$

3. Strings at height j

We say that a string τ *occurs at height* j in a Dyck path, where $j \in \mathbb{N}$, if the minimum height of the points of τ in this occurrence is equal to j. For example, the Dyck path of Fig. 1 has two occurrences of the string *udu* at height 2 and one at height 1, whereas it has one occurrence of the string *dudd* at height 1 and one at height 0. It is clear that the string *ud* occurs at height j in a Dyck path iff the corresponding peak lies at level $j+1$.

An occurrence of a string τ at height 0 is usually referred to as a *low occurrence* of τ. The statistic "number of low occurrences of τ" (or simply "number of low τ's") has been studied for $\tau = ud$ and $\tau = du$ in [7], for $\tau = udu$ in [18], as well as for $\tau = u^r du$ and $\tau = (ud)^r u$, with $r \in \mathbb{N}^*$, in [11]. In this section we study the statistic "number of τ's at height j" with corresponding generating function $F_j(t,z)$, where t counts the occurrences of τ at height j. In particular, we write $L(t,z) = F_0(t,z)$ for the generating function that counts the low occurrences of τ and $l_{n,k} = [t^k z^n]L$ for the number of all $\alpha \in \mathscr{D}_n$ with k low occurrences of τ.

For $\tau = ud$ or $\tau = du$, the generating function F_j is expressed in [10] via Chebyshev polynomials of the second kind and the Catalan generating function. The connection between Dyck paths and the Chebyshev polynomials of the second kind has been already exhibited in [9].

In the following, it is shown that an analogous procedure can be applied for an arbitrary string τ.

We recall that the *Chebyshev polynomials of the second kind* (see A053117 of [17]) are defined by

$$U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta},$$

where $r \in Z$. It is clear that $U_{-1}(z) = 0$, $U_0(z) = 1$, $U_1(z) = 2z$ and

$$U_r(z) = 2zU_{r-1}(z) - U_{r-2}(z).$$

We consider the sequence of functions $(R_j(z))_{j\in\mathbb{N}}$, such that

$$R_j(z) = \frac{U_{j-1}(1/2\sqrt{z})}{\sqrt{z}U_j(1/2\sqrt{z})}.$$

It is easy to check that $R_0(z) = 0$ and $R_j(z)$ is a rational function of z, satisfying the following relations:

$$R_j(z) = \frac{1}{1 - zR_{j-1}(z)} \tag{11}$$

and

$$zR_j^2(z) - R_j(z) + 1 = \frac{1}{U_j^2(1/2\sqrt{z})}. \tag{12}$$

Furthermore, we will show that every sequence $(f_j(z))_{j\in\mathbb{N}}$ of functions that satisfy relation (11), i.e.

$$f_j(z)=\frac{1}{1-zf_{j-1}(z)} \quad \text{for every } j\geqslant 1, \tag{13}$$

is given by the formula

$$f_j(z)=R_j(z)+\frac{1}{U_j^2(1/2\sqrt{z})(1/f_0(z)-zR_j(z))}, \tag{14}$$

or equivalently, using (12),

$$f_j(z)=\frac{R_j(z)-f_0(z)R_j(z)+f_0(z)}{1-zf_0(z)R_j(z)}.$$

Indeed, using relation (11) we can verify that the sequence on the right-hand side of the above equality satisfies the recursion of Eq. (13) (the initial value can be easily checked).

We now come to evaluate the generating function $F_j(t,z)$, where $j\in\mathbb{N}$, for an arbitrary string τ. Using the first return decomposition of a non-empty Dyck path $\alpha=u\beta d\gamma$, where $\beta,\gamma\in\mathscr{D}$, we can easily deduce that

$$F_j(t,z)=1+zF_{j-1}(t,z)F_j(t,z)$$

for every $j\in\mathbb{N}^*=\mathbb{N}\backslash\{0\}$, so that the sequence $(F_j(t,z))_{j\in\mathbb{N}}$ satisfies Eq. (13). It follows that $F_j(t,z)$ is given by relation (14).

From the above discussion we obtain the main result of this section.

Proposition 1. *For every string τ, the generating function $F_j(t,z)$ is given by*

$$F_j(t,z)=R_j(z)+\frac{1}{U_j^2(1/2\sqrt{z})(1/L(t,z)-zR_j(z))}$$

for every $j\in\mathbb{N}^*$.

In view of the previous result, for the evaluation of $F_j(t,z)$ it is enough to evaluate $L(t,z)$.

For example, if $\tau=udu$ then $L(t,z)=1+zC(z)/(1+z(1-t-C(z)))$ (see [18]), so that for every $j\in\mathbb{N}$ we obtain that

$$F_j(t,z)=R_j(z)+\frac{1}{U_j^2(1/2\sqrt{z})(1-zC(z)/(1+z(1-t)-zR_j(z)))}$$

giving for $F_j(t,z)$ an expression equivalent to that of Theorem 4.1 in [18].

The generating functions $L(t,z)$ for the remaining strings of length 3 are evaluated in [15].

In the sequel we evaluate the generating function $L(t,z)$ (and so, also $F_j(t,z)$) for every string of length 4.

3.1. The string uuuu

For the evaluation of the generating function $L=L(t,z)$ for the case $\tau=uuuu$, we consider the partition $\{A_i\}$ used in Section 2.1. Clearly, the low occurrences of *uuuu* in every $\alpha=u^i d\alpha_1 d\alpha_2\cdots d\alpha_i$, for $i\leqslant 3$, are the same as those in α_i, whereas for $i\geqslant 4$ we have an additional low occurrence; thus we have that

$$\begin{aligned}L&=1+\sum_{i=1}^{3} z^iC^{i-1}L+t\sum_{i=4}^{\infty} z^iC^{i-1}L\\&=1+\left(zC-(1-t)\sum_{i=4}^{\infty} z^iC^{i-1}\right)L.\end{aligned}$$

It follows easily that

$$L(t,z)=\frac{C(z)}{1+(1-t)z^4C^5(z)}.$$

If we expand the above generating function into a geometric series and use formula (1), we obtain that the number of all $\alpha\in\mathscr{D}_n$ with k low *uuuu*'s is equal to

$$l_{n,k}=\sum_{j=k}^{[n/4]}(-1)^{j-k}\frac{5j+1}{n+j+1}\binom{j}{k}\binom{2n-3j}{n+j}.$$

The first terms of this triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 105, 27; 319, 110; 1002, 427, 1; 3235, 1616, 11; 10685, 6034, 77.

The same procedure can be applied in order to show that for $\tau=u^r$, where $r\in\mathbb{N}^*$, we have

$$L(t,z)=\frac{C(z)}{1+(1-t)z^rC^{r+1}(z)}$$

with coefficients

$$l_{n,k}=\sum_{j=k}^{[n/r]}(-1)^{j-k}\frac{(r+1)(j-k+1)+1}{n+j+1}\binom{j}{k}\binom{2n-(r-1)j}{n+j}.$$

3.2. The strings uudd, uddu and dduu

The statistic "number of low *uudd*'s" is known (see A1144086 in [17]), with corresponding generating function

$$L(t,z)=\frac{C(z)}{1+(1-t)z^2C(z)}.$$

Furthermore, the number of all $\alpha\in\mathscr{D}_n$ with k low *uudd*'s is equal to

$$l_{n,k}=\sum_{j=k}^{[n/2]}(-1)^{j-k}\frac{j+1}{n-j+1}\binom{j-k+1}{k}\binom{2n-3j}{n-j}.$$

The first terms of this triangle, read by rows, are 1; 1; 1, 1; 3, 2; 10, 3, 1; 31, 8, 3; 98, 27, 6, 1; 321, 88, 16, 4; 1078, 287, 54, 10, 1; 3686, 960, 183, 28, 5.

In the sequel we will show that the statistics "number of *uddu*'s at height *j*" and "number of *uudd*'s at height $j+1$" are equidistributed, for every $j\in\mathbb{N}$.

For this, we define a mapping $\phi_j:\mathscr{D}\to\mathscr{D}$ as follows. Given a path $\alpha\in\mathscr{D}$ we construct $\phi_j(\alpha)$ using the following three rules:

i. Subtract 2 from the *y*-coordinates of each pair of consecutive points P, Q of α, where P is a peak at level $j+3$ immediately preceded by a doublerise, and Q is a doublefall.
ii. Add 2 to the *y*-coordinates of each pair of consecutive points P', Q' of α, where P' is a doublefall at level $j+1$ immediately preceded by a peak, and Q' is a valley.
iii. Retain the other points unchanged.

For example, if

$$\alpha=uuduuuddudduuuddududdd$$

then

$$\phi_1(\alpha)=uuduudduuuddudduududdd;$$

(see Fig. 2).

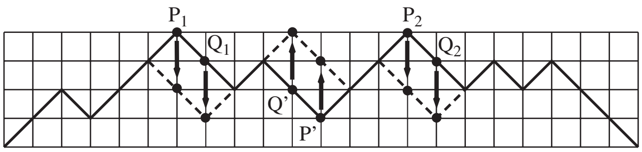

Fig. 2.

It is easy to check that ϕ_j is an involution of $\mathscr{D}$ such that $l(\phi_j(\alpha)) = l(\alpha)$, and that the number of *uddu*'s at height j in α is equal to the number of *uudd*'s at height $j+1$ in $\phi_j(\alpha)$.

By constructing a similar involution of $\mathscr{D}$, we can show that the statistics "number of *dduu*'s at height j" and "number of *uddu*'s at height $j+1$" are equidistributed.

Thus, applying Proposition 1 for $\tau = uudd$ and $j = 1, 2$ we obtain $L(t, z)$ for $\tau = uddu$, $\tau = dduu$, respectively.

In fact, for $\tau = uddu$ we obtain that

$$L(t, z) = 1 + \frac{zC^2(z)}{1 + (1-t)z^2C^2(z)}$$

with coefficients

$$l_{n,k} = \frac{2(k+1)}{n+1} \sum_{j=k}^{[(n-1)/2]} (-1)^{j-k} \binom{j+1}{k+1} \binom{2n-2j-1}{n}.$$

The first terms of this triangle, read by rows, are 1; 1; 2; 4, 1; 10, 4; 29, 12, 1; 90, 36, 6; 290, 114, 24, 1; 960, 376, 86, 8; 3246, 1272, 303, 40, 1.

For $\tau = dduu$ we obtain that

$$L(t, z) = \frac{C(z)(1 + (1-t)z^2C^2(z))}{1 + (1-t)(1-z)z^2C^3(z)}.$$

The first terms of the corresponding triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 106, 25, 1; 327, 94, 8; 1045, 342, 42, 1; 3433, 1230, 189, 10.

3.3. The strings uudu, uuud, uduu and duuu

The statistic "number of low *uudu*'s" is known, with corresponding generating function

$$L(t, z) = \frac{C(z)}{1 + (1-t)z^3C^3(z)}.$$

This result is given in [11] in a more general setup.

It follows easily that the corresponding coefficients are equal to

$$l_{n,k} = \frac{1}{n+1} \sum_{j=k}^{[n/3]} (-1)^{j-k}(3j+1) \binom{j}{k} \binom{2n-3j}{n}.$$

The first terms of this triangle, read by rows, are 1; 1; 2; 4, 1; 10, 4; 28, 14; 85, 46, 1; 271, 151, 7; 893, 502, 35; 3013, 1697, 151, 1; 10351, 5828, 607, 10.

In the sequel we will show that the statistics "number of *uudu*'s at height j", "number of *uuud*'s at height j" and "number of *uduu*'s at height j" are equidistributed, for every $j \in \mathbb{N}$.

For this, for every $j \in \mathbb{N}$, we consider the involution ψ_j of $\mathscr{D}$ according to which every peak at level $j+2$ in a Dyck path is turned into a valley at level j, and vice versa (see Theorem 2.1 in [10]).

It is easy to check that the number of *uudu*'s at height j in α is equal to the number of *uuud*'s at height j in $\psi_{j+1}(\alpha)$, for every $\alpha \in \mathscr{D}$. This shows that the statistics "number of *uudu*'s at height j" and "number of *uuud*'s at height j" are equidistributed.

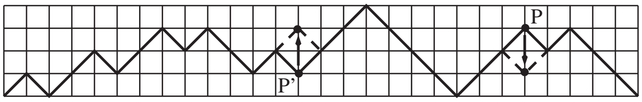

Fig. 3.

We now define a mapping $\chi_j : \mathscr{D} \to \mathscr{D}$ as follows: given a path $\alpha \in \mathscr{D}$ we construct $\chi_j(\alpha)$ using the following three rules:

i. Subtract 2 from the y-coordinate of each peak P at level $j+2$, which is immediately preceded by a doublerise and immediately followed by a valley, provided that the first valley (if it exists) on the left of this peak is not a small valley at level j.
ii. Add 2 to the y-coordinate of each small valley P' at level j, which is immediately followed by a doublerise, provided that the first peak on the right of this valley is not a small peak at level $j+2$.
iii. Retain the other points unchanged.

For example, if

$$\alpha = uduuduududdduduuudddduuududddd$$

then

$$\chi_1(\alpha) = uduuduududdduuduuddddduuduuddd;$$

(see Fig. 3).

It is easy to check that this mapping is an involution of $\mathscr{D}$ such that $l(\chi_j(\alpha)) = l(\alpha)$ and the number of *uudu*'s at height j in α is equal to the number of *uduu*'s at height j in $\chi_j(\alpha)$.

Thus, the statistics "number of *uudu*'s at height j" and "number of *uduu*'s at height j" are equidistributed.

Finally, using the involution ψ_j, we can show that the statistics "number of *duuu*'s at height j" and "number of *uduu*'s at height $j+1$" are equidistributed.

Thus, applying Proposition 1 for $\tau = uduu$ and $j=1$, we obtain the generating function $L(t,z)$ for $\tau = duuu$:

$$L(t,z) = 1 + \frac{zC^2(z)}{1+(1-t)z^3C^4(z)}$$

with coefficients

$$l_{n,k} = \sum_{j=k}^{[(n-1)/3]} (-1)^{j-k} \frac{2j+1}{n-j+1} \binom{j}{k} \binom{2n-2j-1}{n-j}.$$

The first terms of this triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 105, 27; 320, 108, 1; 1011, 409, 10; 3289, 1508, 65; 10957, 5491, 347, 1.

3.4. The strings udud and dudu

For the evaluation of the generating function $L(t,z)$ for the case $\tau = udud$, we consider the set $\mathscr{A}$ of all Dyck paths that start with a low peak, and its generating function $A' = A'(t,z)$, where t counts the number of low *udud*'s.

Clearly, every $\alpha \in \mathscr{A}$ can be uniquely written in the form $\alpha = ud\beta$, where $\beta \in D$. Furthermore, an additional low *udud* occurs in α iff $\beta \in \mathscr{A}$. Thus, we have that

$$A' = ztA' + z(L - A').$$

Moreover, using the first return decomposition of a non-empty Dyck path $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathscr{D}$, we observe that a new low *udud* occurs in α iff $\beta = \varepsilon$ and $\gamma \in \mathscr{A}$. Thus, we have that

$$L = 1 + ztA' + z(L - A') + z(C-1)L,$$

which gives

$$L(t,z)=\frac{(1+(1-t)z)C(z)}{1+(1-t)z(1+zC(z))}.$$

The first terms of the corresponding triangle, read by rows, are 1; 1; 1, 1; 4, 1; 11, 2, 1; 33, 6, 2, 1; 105, 17, 7, 2, 1; 343, 56, 19, 8, 2, 1; 1148, 185, 64, 21, 9, 2, 1.

Finally, using the involution ψ_j, we can show that the statistics "number of *dudu*'s at height *j*" and "number of *udud*'s at height $j+1$" are equidistributed.

Thus, applying Proposition 1 for $\tau = udud$ and $j=1$, we obtain the generating function $L(t,z)$ for $\tau = dudu$:

$$L(t,z)=1+zC(z)+\frac{z^2C^3(z)}{1+(1-t)zC(z)}$$

with coefficients

$$l_{n,k}=\delta_{0k}c_{n-1}+\frac{1}{n+1}\sum_{j=k}^{n-2}(-1)^{j-k}(j+3)\binom{j}{k}\binom{2n-j-2}{n},$$

where δ_{0k} is the Kronecker symbol.

The first terms of this triangle, read by rows, are 1; 1; 2; 4, 1; 11, 2, 1; 32, 7, 2, 1; 99, 22, 8, 2, 1; 318, 73, 26, 9, 2, 1; 1051, 246, 90, 30, 10, 2, 1.

4. Strings at even and odd height

In this section we deal with the statistics "number of occurrences of τ at even height" and "number of occurrences of τ at odd height" (or simply "number of τ's at even height" and "number of τ's at odd height"), with corresponding generating functions $E=E(t,z)$ and $O=O(t,z)$.

Using the first return decomposition of non-empty Dyck paths we can easily show that

$$O(t,z)=\frac{1}{1-zE(t,z)}. \tag{15}$$

These statistics have been studied for strings of length 2.

For $\tau = uu$ it is known (A091894 of [17]) that the statistics "number of *uu*'s at odd height" and "number of *duu*'s" are equidistributed.

Thus, using (2) and (15) we deduce that in this case $E(t,z)$ satisfies Eq. (5), so that the statistics "number of *uu*'s at even height" and "number of *uud*'s" are equidistributed.

For $\tau = ud$ it is known (A091867 of [17]) that the generating function $E(t,z)$ satisfies the equation

$$(1+(1-t)z)E^2-(1+(1-t)z)E+1=0.$$

Furthermore, it is known (A091869 of [17]) that the statistics "number of *ud*'s at odd height" and "number of *udu*'s" are equidistributed.

Moreover, we show bijectively that the statistic "number of *du*'s at even height" is equidistributed with the above two statistics.

For this, we construct a bijection $\psi : \mathscr{D} \to \mathscr{D}$ such that the number of *ud*'s at odd height in α is equal to the number of *du*'s at even height in $\psi(\alpha)$, for every $\alpha \in \mathscr{D}$.

We define $\psi(\varepsilon)=\varepsilon$ and for $\alpha \neq \varepsilon$, $\psi(\alpha)=(\psi_{2\rho}\circ\psi_{2(\rho-1)}\circ\cdots\circ\psi_2\circ\psi_0)(\alpha)$, where ψ_j is the involution used in Section 3.3 and $\rho=[(h-1)/2]$; here h denotes the height of the path α.

For example, if

$$\alpha = uuddudu uuuddudddduududddud,$$

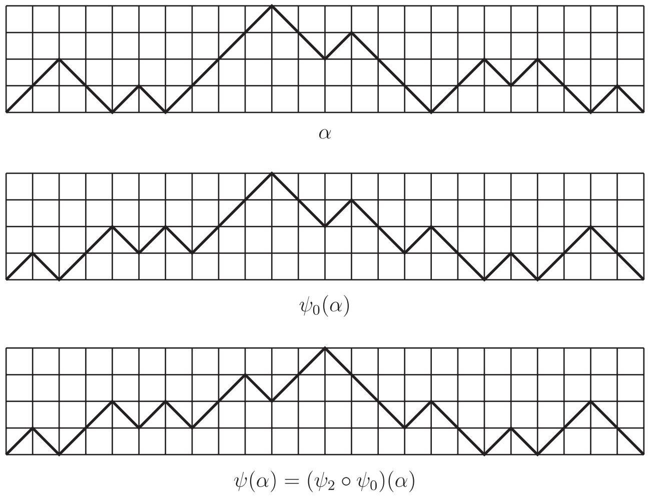

Fig. 4.

then

$$\psi(\alpha) = uduududuuduudddudduduudd;$$

(see Fig. 4).

It is easy to check that ψ satisfies the required conditions.

Finally, using Eqs. (3) and (15) we easily deduce that for $\tau = du$ the generating function $O(t, z)$ satisfies Eq. (4), so that the statistics "number of *du*'s at odd height" and "number of *uuu*'s" are equidistributed.

In the sequel we study these statistics for every string τ of length 3. We will show that in each case there exist strings τ_1, τ_2 of length 4 such that the statistics "number of τ's at even height" (resp. "number of τ's at odd height") and "number of τ_1's" (resp. "number of τ_2's") are equidistributed.

4.1. The strings *uuu* and *duu*

For the evaluation of the generating function $E(t, z)$ for the case $\tau = uuu$ we consider the partition $\{\mathscr{A}_i\}$ used in Sections 2.1 and 3.1. Clearly, the occurrences of *uuu* at even height in every $\alpha = u^i d\alpha_1 d\alpha_2 \cdots d\alpha_i \in \mathscr{A}_i$, where $i \geqslant 1$, consist of the ones at even height in each α_j for $i - j$ even, as well as the ones at odd height in each α_j for $i - j$ odd, where $j \in [i]$, together with $[(i-1)/2]$ occurrences in the first ascent u^i. It follows that

$$\begin{aligned} E &= 1 + \sum_{i=1}^{\infty} z^i t^{[(i-1)/2]} E^{[(i+1)/2]} O^{[i/2]} \\ &= 1 + \sum_{i=1}^{\infty} z^{2i} t^{i-1} E^i O^i + \sum_{i=1}^{\infty} z^{2i-1} t^{i-1} E^i O^{i-1} \\ &= 1 + \frac{zE(zO+1)}{1 - z^2 tEO}. \end{aligned}$$

Furthermore, using Eq. (15) we obtain that

$$z(1-(1-t)z)E^2 + ((1-t)z^2 - 1)E + 1 = 0$$

and

$$z(t + (1-t)z)O^2 - (1 + (1-t)(z-2)z)O + 1 - (1-t)z = 0.$$

Thus, $E(t,z)$ and $O(t,z)$ satisfy Eqs. (7) and (9), respectively. It follows that the statistics "number of *uuu*'s at even height" (resp. "number of *uuu*'s at odd height") and "number of *uudu*'s" (resp. "number of *dduu*'s") are equidistributed.

Using the above technique, we can show that the generating functions $E(t,z)$ and $O(t,z)$ for $\tau = duu$ also satisfy Eqs. (7) and (9), respectively, so that the statistics "number of *duu*'s at even height" (resp. "number of *duu*'s at odd height") and "number of *uudu*'s" (resp. "number of *dduu*'s") are equidistributed.

Instead of using the above algebraic method, we will proceed bijectively, by showing that the statistics "number of *uuu*'s at even (resp. odd) height" and "number of *duu*'s at even (resp. odd) height" are equidistributed. For this, we define inductively two involutions ϕ, ψ of $\mathscr{D}$ (which could be of a more general interest) such that

$$v_e(\phi(\alpha)) = \mu_e(\alpha) \quad \text{and} \quad v_o(\psi(\alpha)) = \mu_o(\alpha) \tag{16}$$

for every $\alpha \in \mathscr{D}$, where $\mu_e(\alpha)$ (resp. $v_e(\alpha)$) is the number of *uuu*'s (resp. *duu*'s) at even height in α and $\mu_o(\alpha)$ (resp. $v_o(\alpha)$) is the number of *uuu*'s (resp. *duu*'s) at odd height in α.

First we set $\phi(\varepsilon) = \psi(\varepsilon) = \varepsilon$. For a non-empty Dyck path α we use the decomposition $\alpha = \beta u \gamma d$, where $\beta, \gamma \in \mathscr{D}$; then we define $\psi(\alpha) = \psi(\beta) u \phi(\gamma) d$. The mapping ϕ is defined according to the form of the last prime component of the Dyck path:

If $\alpha = \beta ud$, where $\beta \in \mathscr{D}$, then

$$\phi(\alpha) = \phi(\beta)ud.$$

If on the other hand $\alpha = \beta uu\gamma d\delta d$, where $\beta, \gamma, \delta \in \mathscr{D}$, then

$$\phi(\alpha) = \begin{cases} \phi(\beta)uu\phi(\gamma)d\psi(\delta)d & \text{if } \beta, \gamma \neq \varepsilon, \\ \phi(\gamma)uu\phi(\beta)d\psi(\delta)d & \text{otherwise.} \end{cases}$$

For example, if

$$\alpha = uuududdduuduuuudddd$$

then

$$\phi(\alpha) = uuududuudddduuduudddd$$

and

$$\psi(\alpha) = uuududdduuduuuudddd.$$

It is easy to check that the mappings ϕ, ψ are involutions. Furthermore, in order to show equalities (16) we restrict ourselves to the non-trivial case where $\alpha = \beta uu\gamma d\delta d$. Then, we have

$$\mu_e(\alpha) = \mu_e(\beta) + \mu_e(\gamma) + \mu_0(\delta) + [\gamma \neq \varepsilon]$$

and

$$v_e(\phi(\alpha)) = v_e(\phi(\beta)) + v_e(\phi(\gamma)) + v_o(\psi(\delta)) + [\phi(\gamma) \neq \varepsilon].$$

Here $[P]$ is the Iverson notation: $[P] = 1$ if P is true and $[P] = 0$ if P is false.

Now, making use of the induction hypothesis, the first of equalities (16) follows at once; the proof of the second is immediate and it is omitted.

4.2. The string uud

For the evaluation of the generating function $E = E(t,z)$ for the case $\tau = uud$, we consider the first return decomposition of a non-empty Dyck path $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathscr{D}$. Clearly, the occurrences of *uud* at even height in α consist of the ones at odd height in β, as well as the ones at even height in γ, together with a new occurrence of *uud* that appears iff $\beta = ud\delta$, where $\delta \in \mathscr{D}$.

It follows that

$$E = 1 + z(tzO + O - zO)E.$$

Furthermore, using Eq. (15) we obtain that

$$zE^2 + (z^2(t-1) - 1)E + 1 = 0$$

and

$$z(1-(1-t)z)O^2 + ((1-t)z^2 - 1)O + 1 = 0.$$

Thus, $E(t, z)$ and $O(t, z)$ satisfy Eqs. (6) and (7), respectively. It follows that the statistics "number of *uud*'s at even height" (resp. "number of *uud*'s at odd height") and "number of *uudd*'s"(resp. "number of *uudu*'s") are equidistributed.

4.3. The string udu

The case $\tau = udu$ is treated similarly. Here, for $\alpha = u\beta d\gamma$ a new occurrence of *udu* at even height appears in α iff $\beta = \varepsilon$ and $\gamma \neq \varepsilon$.

It follows that

$$E = 1 + z(t(E-1) + (O-1)E + 1).$$

Finally, using Eq. (15) we obtain that

$$z(1+(1-t)z)E^2 - (1 + (1-t)z(z+1))E + 1 + (1-t)z = 0$$

and

$$zO^2 + ((1-t)(z-1)z - 1)O + (1-t)z + 1 = 0.$$

Thus, $E(t, z)$ and $O(t, z)$ satisfy Eqs. (8) and (10), respectively. It follows that the statistics "number of *udu*'s at even height" (resp. "number of *udu*'s at odd height") and "number of *udud*'s"(resp. "number of *dudu*'s") are equidistributed.

In the rest of this section we deal with the statistic "number of high occurrences of τ" (an occurrence of a string τ is considered *high* if it is not low).

For $\tau = ud$, it is known (see [6]) that this statistic is equidistributed with the statistic "number of *du*'s". More generally, a bijection on *k*-colored Motzkin paths is constructed in [14] in order to show that these statistics remain equidistributed even for *k*-colored Motzkin paths. This bijection can be used in order to show the equidistribution of the statistics "number of high $(ud)^r$'s" and "number of $(du)^r$'s", for every $r \in \mathbb{N}^*$.

Using the first return decomposition of non-empty Dyck paths we can easily check that the generating function $H = H(t, z)$ of the statistic "number of high τ's" is given by the relation

$$H(t, z) = \frac{1}{1 - zF(t, z)}, \tag{17}$$

where $F(t, z)$ denotes the generating function corresponding to the statistic "number of τ's".

Given three strings τ, τ_1 and τ_2 such that

$$E_\tau(t, z) = F_{\tau_1}(t, z) \quad \text{and} \quad O_\tau(t, z) = F_{\tau_2}(t, z),$$

from relations (15) and (17) we obtain that

$$H_{\tau_1}(t, z) = F_{\tau_2}(t, z).$$

(Here the subscripts denote the strings to which the generating functions refer.)

Thus, from the previous discussion it follows that the statistics "number of high τ_1's" and "number of τ_2's" are equidistributed for each of the following pairs (τ_1, τ_2) : (uud, duu), (udu, uuu), (udu, uuu), $(uduu, dduu)$, $(uudd, uduu)$, $(udud, dudu)$.

It would be interesting to find for which strings τ there exist strings τ_1 and τ_2 of length one more than the length of τ, such that the statistics “number of τ’s at even height” and “number of τ’s at odd height” are equidistributed with the statistics “number of τ_1’s” and “number of τ_2’s”, respectively.

As we have seen, this is true for every string τ of length 3, whereas it is not true for $\tau = ud$.

Acknowledgments

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