UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Kurs: 1MA022
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Complex Analysis

Writing time: 08:00–13:00.
Other than writing utensils and paper, no help materials are allowed.

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1. Suppose that $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain D, such that $u(x, y) = -v^2(x, y)$ for all $z = x + iy \in D$. Show that $f(z) = u(x, y) + iv(x, y)$ is analytic in D only if f is a constant function.

2. Find a conformal mapping that transforms the domain

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \cup \{z \in \mathbb{C} : |z| < 1\}$$

onto the left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z-2)^3} - \frac{1}{(z+3)}$$

in the annulus $A = \{z \in \mathbb{C} : 2 < |z| < 3\}$.

4. Let γ be a piecewise smooth, simple closed curve in a domain D. Assume that $f : D \longrightarrow \mathbb{C}$ is analytic and at each point z belonging to the trace of γ the following inequality is satisfied:

$$|f(z) - 1| < |f(z)| + 1.$$

Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

Hint: Characterize geometrically the domain $\{w \in \mathbb{C} : |w - 1| < |w| + 1\}$ and observe that $\operatorname{Log} w$ is analytic in this domain.

5. Use the residue theorem to calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+2)^2}\, dx.$$

6. Show that all zeros the polynomial $p(z) = z^5 - z + 16$ are contained in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. How many of the zeros have positive real part?

7. Find a formula for the analytic function $f : \mathbb{C} \setminus \{1, -1\} \longrightarrow \mathbb{C}$ which has the following properties:

- f has simple zeros at $\pm i$ and a double zero at 0;
- f has double poles at ± 1 with residues ± 1 respectively;
- f has a removable singularity at ∞.

8. Suppose that $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : \bar{D} \longrightarrow \mathbb{C}$ is a continuous function, which is analytic in D. Assume that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \partial D$. Show that $|f(z)| \leq |z|$ for all $z \in D$. Show also that if $|f(a)| = |a|$ at some point $a \in D$, then in fact $f(z) = cz$ for some constant c such that $|c| = 1$.
Hint: The function $f(z)/z$ has a removable singularity at 0.

GOOD LUCK!