# Exercise 1

Consider the scalar initial-boundary value problem (IBVP)

$$\begin{aligned}
&cu_t = au_x + (bu_x)_x + du_{xxx}, && 0 \le x \le W, && t \ge 0,\\
&\mathcal{L}_l u = g_l, && x = 0, && t \ge 0,\\
&\mathcal{L}_r u = g_r, && x = W, && t \ge 0,\\
&u = f, && 0 \le x \le W, && t = 0,
\end{aligned}$$

where $c = c(x) > 0$ is a real-valued function; $b = b(x)$ is possibly complex-valued; and $a$ and $d$ are (possibly complex-valued) *constants.*

(a) Consider the case $d = 0$. What are the requirements on $a$, $b$, and $c$ for the PDE to be well-posed, disregarding the boundary conditions? That is, you may assume periodic boundary conditions.
*Hint:* Use the energy method.

(b) For $d = 0$, what are the requirements for the PDE to conserve some energy? Again consider periodic boundary conditions.

(c) For $d = 0$, derive at least two sets of well-posed boundary conditions. That is, find two different operators $\mathcal{L}_l$ (and $\mathcal{L}_r$) that yield a well-posed IBVP.

(d) Consider the case $a = c = 1$, $d = 0$, $b = 10$. Describe the expected behaviour of the solution.

(e) Now consider $d \neq 0$. What are the requirements for the PDE to be well-posed with periodic boundary conditions?
*Hint:* The term $du_{xxx}$ requires integrating by parts twice.

(f) For $d \neq 0$, derive one set of well-posed boundary conditions.
*Hint:* You will need 3 conditions in total due to the term $du_{xxx}$.

# Exercise 2

Consider the IBVP

$$\begin{aligned}
&\mathbf{C}\mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u} + \mathbf{F}, && 0 \leq x \leq W, && t \geq 0,\\
&\mathcal{L}_l \mathbf{u} = g_l, && x = 0, && t \geq 0,\\
&\mathcal{L}_r \mathbf{u} = g_r, && x = W, && t \geq 0,\\
&\mathbf{u} = \mathbf{f}, && 0 \leq x \leq W, && t = 0,
\end{aligned}$$

where $\mathbf{F} = \mathbf{F}(x,t)$ is the forcing function, $\mathbf{f}$ is the initial data, $\mathbf{A} = \mathbf{A}^*$ is a *constant* matrix, $\mathbf{B}$ and $\mathbf{C}$ are variable-coefficient matrices, and $\mathbf{C} = \mathbf{C}^* > 0$. $\mathbf{A}$ and $\mathbf{C}$ have the structure

$$\mathbf{C} = \begin{bmatrix} c_1(x) & 0 & 0 \\ 0 & c_2(x) & 0 \\ 0 & 0 & c_3(x) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & \alpha \end{bmatrix},$$

where $c_i \in \mathbb{R}$ and $\alpha$ is a real constant.

(a) Use the energy method to derive an energy rate for the IBVP with $\mathbf{F} = 0$. Under which conditions can you show that the PDE with periodic boundary conditions is well-posed?

(b) How many boundary conditions should be prescribed at each boundary in the cases $\alpha = 0$, $\alpha = -8$ and $\alpha = -10$?

(c) Consider the case $\alpha = 0$. Derive at least one set of well-posed boundary conditions. You may assume that the solution is real-valued.