UPPSALA UNIVERSITET

Föreläsningsateckningar

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1. Options

Motivating Discussion:
Say a Swedish company has signed a contract to buy a machine from a US company for 100000USD to be paid at delivery 6 months from now. $T = \frac{1}{2}$ years.
Current exchange rate is 11SEK/USD. The buyer is suject to currency risk. There are 3 possible strategies to implement:

(1) Buy 100000USD today and deposit in the bank.
The risk is eliminated but money is tied up for a long time and the company may not have access to this money
(2) Buy a *forward contract* from a bank, i.e the bank delivers the sum you need at $T = \frac{1}{2} = t$, in return, the company payes some constant $K \cdot 100000USD$ at $T = t$, where K is chosen at $t = 0$ such that no transfer of money is needed at $t = 0$. Here, the bank takes all of the risk, but if the exchange rate drops below K then we would have preffered to do nothing.
(3) Buy a *European call option* on 100000USD, with strike price K and exercise date T. I.e, it gives the right but not the obligation to buy 100000USD at price $K \cdot 100000USD$ at time $T = t$. If exchange rate at T is $> K$, then we use the option. If its below at $t = T$ thne we do not use the option (right, not obligation)

The last one is a good choice, but not free. This leads to the 2 main problems in the course:

- How much is a fair price for an option?
- If you are the seller of an option, how to protect (hedge) from risk of exchange rate not going up?

Motivating Example in discrete time
At $t = 0$, we can trade in a market with 2 assets:

- *Bank account* (risk-free/non-risky asset)
  At $t = 0$ the value is 1 and at $t = 1$ the value is 1
- *Stock* (risky asset)
  At $t = 0$, $S_0 = 100$ then it either grows ($S_1 = 120$) or declines ($S_1 = 80$) with probability $p = 0.6$ and $p = 0.4$ respectively

Definition 1.1 Call option

A *call option* is a contract that gives its holder the right but not the obligation to buy one share of a stock at time T with predetermined price K. Thus, at time $t = 1$, the option is worth $S_1 - K$ if $S_1 > K$ and 0 else

What is a fair price of the option? The sensible thing to pay would be $p(S_1 - K)$. Assuming $K = 110$ in the above example, then $0.6(120 - 110) = 6$. But this is not the best price!

The idea is to replicate the option by finding a trading stategy using both the risk-free (B) and the risky asset (S) such that the value of the stock at $t = 1$ coincides with the value of hte option.
Is that possible? Yes. Let $x =$ amount in the bank at $t = 0$ and y be the number of shares of stock. We want to pick x, y such that regardles if stock goes up or down we have increase.
At $t = 1$

$$\left.\begin{aligned} &x + S_1 y = S_1 - K \\ &x + S_1 y = 0 \end{aligned}\right\}$$

If $K = 110$ and $S_1 = \{120, 80\}$, then $x = -20$ and $y = \frac{1}{4}$ since

$$\begin{cases} x + 120y = 10 \\ x + 80y = 0 \end{cases}$$

At $t = 0$. Our strategy is therefore to borrow 20 from the bank and buy $\frac{1}{4}$ of a share. The cost is $25 - 20 = 5$ which is less than 6.

At time $t=1$ our holdings are worth $\frac{1}{4}S_1 - 20 = \begin{cases} 10 & \text{if } S_1 = 120 \\ 0 & \text{if } S_1 = 80 \end{cases}$ which is exactly the same as the option.

Conclusion:
By the APT (Arbitrage pricing theory), the price of the call must be equal to the cost of setting up this portfolio.

Remark:
The probabilities do not influence the option value. They were never used in the calculation of the price.

Remark:
Let us change p into q such that $\mathbb{E}(S_1) = S_0 = 100$ in the example, which value of q satisfies this? It is symmetric in the example, so let $p = q = \frac{1}{2}$
Then $\mathbb{E}(\max\{S_1 - k, 0\}) = 10 \cdot \frac{1}{2} + 0 \cdot \frac{1}{5} = 5$
In general, the option price is $\mathbb{E}^Q\left(\frac{B_0}{B_1}\max\{S_1 - k, 0\}\right)$ where Q is chosen such that $\mathbb{E}^Q\left(\frac{B_0 S_1}{B_1}\right) = \frac{S_0}{B_0}$

Notation:
$a^+ = \max\{a, 0\}$. In particular,

$$(s-K)^+ = \begin{cases} s-K & \text{if } s \geq K \\ 0 & \text{if } s < K \end{cases}$$

Exercise:

- In the above example, find a replicating strategy for a put option (right but not obligated to sell one share) at price $K = 110$
- Find the vlaue of the option at $t = 0$

Answer:

$$\left.\begin{aligned} x &= 90 \\ y &= \frac{-3}{4} \end{aligned}\right\} \text{ option value of } 15$$

2. Continous time & Brownian Motion

2.1. Simple Random Walk.

Let X_i be i.i.d.r.v with $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}$
Let $S_n = \sum_{i=1}^{n} X_i$, then this is a stochastic process, still in discrete time. Do note that the expectation is 0 for the r.v. and that:

$$\mathbb{E}(S_n) = \sum_{k=1}^{n} \mathbb{E}(X_i) = 0$$

$$\operatorname{Var}(S_n) = \mathbb{E}(S_n^2) - \underbrace{(\mathbb{E}(S_n))^2}_{=0} = \sum_{k=1}^{n} \operatorname{Var}(X_i) = \sum_{k=1}^{n} 1 = n$$

Note that this was discrete time, how do we proceed to make this continuous?
We do this by scaling to finer time. Frist, fix a time interval:

Stage 1

Let $X_0^1 = 0$

At $t = 0$, toss a coin, $X_T^1 = \begin{cases} \sqrt{T} & \text{heads} \\ -\sqrt{T} & \text{tails} \end{cases}$.

Here $\mathbb{E}(X_T^1) = 0$ and $\operatorname{Var}\left(X_T^1\right) = T =$ elapsed time.

Stage 2

Add another time step. Let $X_0^2 = 0$, toss a coin, $X_{T/2}^2 = \begin{cases} \sqrt{\frac{T}{2}} & \text{heads} \\ -\sqrt{\frac{T}{2}} & \text{tails} \end{cases}$

Repeat at $t = \frac{T}{2}$, adding/subtracting $\sqrt{\frac{T}{2}}$

Stage n

Let $X_0^n = 0$, at each time $t_k = \frac{k}{n}T$, toss a coin.

Define $X_{t_{k+1}}^n = X_{t_k}^n + Y_k$ where $Y_k = \pm\sqrt{\frac{T}{2}}$ with prob. 1/2. Simulating our coin tosses.

Here

$$\mathbb{E}(X_{t_k}^n) = \mathbb{E}\left(\sum_{i=1}^{k-1} Y_i\right) = \sum_{i=1}^{k-1} \mathbb{E}(Y_i) = 0$$

$$\operatorname{Var}\left(X_{t_k}^n\right) = \operatorname{Var}\left(\sum_{i=1}^{n} Y_i\right) \overset{\text{indep}}{=} \sum_{i=1}^{k} = \frac{T}{n}k = t_k$$

Now the question becomes, what happens when $n \to \infty$? We obtain *Brownian Motion*, aka Weiner process.

Definition 2.2 Brownian Motion

Brownian Motion is a stochastic process W if:

- $W_0 = 0$
- Independent increments, i.e $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent (as long as they are not overlapping)
- $W_t - W_s \sim N(0, t - s)$
- $t \mapsto W_t$ is continuous

This is a nice definition and all, but does there even exists something which satsifies our definition?

Sats 2.1

$t \mapsto W_t$ is of infinite variation and nowhere differentiable
By infinite variation, it is meant

$$\lim_{n\to\infty}\sum_k \left|W_{t_{k+1}} - W_{t_k}\right| = \infty$$

A regular differentiable function has bounded variation. The next goal is to define the stochastic integral $\int_0^t g_s dW_s$, where g_t is a stochastic process determined by the Brownian motion W

Definition 2.3 Measurable w.r.t σ-algebra

Let X_t be a stochastic process. An event A is $\mathcal{F}_t^X$ measurable (denoted $A \in \mathcal{F}_t^X$) if it is possible to determine whether A has happened or not based on observations of $\{X_s : 0 \leq s \leq t\}$

Example:
$A = \{X_s \leq 7 : \forall s \leq 9\} \in \mathcal{F}_9^X$

Definition 2.4

If a random variable Z can be determined by observations of $\{X_s : 0 \leq s \leq t\}$, then $Z \in \mathcal{F}_t^X$

Example:

$$Z = \int_0^5 X_s d_s \in \mathcal{F}_5^X$$

If you only know X_5 up to 4, then you cannot determine Z

Definition 2.5

A stochastic process Y_t with $Y_t \in \mathcal{F}_t^X \quad \forall t$ is *adapted to the filtration* $\mathcal{F}_t^X$

Example:
$Y_t = \sup_{0\leq s\leq t} W_s$ is adapted to $\mathcal{F}_t^W$

Definition 2.6

The process $g_t \in \mathcal{L}^2$ if

- g is adapted to $\mathcal{F}_t^W$
- $\int_0^t \mathbb{E}(g_s^2)ds < \infty$

Example:

Brownian motion $\in \mathcal{L}^2$, its adapted to $\mathcal{F}_t^W$ and $\int_0^t \mathbb{E}(\overbrace{W_s^2}^{\sim N(0,\sqrt{s})})ds = \int_0^t sds = \dfrac{t^2}{2} < \infty$

2.2. Stochastic integration.
Assume $g \in \mathcal{L}^2$. If g is simple (i.e $g_s = g_{t_k}$ for $s \in [t_k, t_{k+1}]$), then we define

$$\int_0^t g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})$$

For egeneral $g \in \mathcal{L}^2$, we can approximate g using step functions which are simple such that

$$\int_0^t \mathbb{E}((g_s - g_s^n)^2) ds \to 0 \quad \text{as } n \to \infty$$

Then, one defines the stochastic integral as

$$\int_0^t g_s dW_s = \lim_{n\to\infty} g_s^n dW_s$$

Remark
One can show that the limit indeed exists and does not depend on the sequence used for approximation.

Remark:
Forward increments are used! The integrand is fixed at t_k, and we look at forward movements of the Brownian motion.

Remark:
Steiltjes integration si not possible since paths are not of unbounded variation.

Proposition:
Assume $g \in \mathcal{L}^2$ and adapted to a filtration, then:
(1) $\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0$
(2) $\mathbb{E}\left(\left(\int_0^t g_s dW_s\right)^2\right) = 0 = \int_0^t \mathbb{E}(g_s^2) dW_s$ (Ito isometry)
(3) $X_t = \int_0^t g_s dW_s$, then X_t is $\mathcal{F}^W$-adapted

Bevis 2.1

Assume g is simple (if it was not, then approximate using step functions).

(1)

$$\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0 = \mathbb{E}\left(\sum_{k=1}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right) = \sum_{k=0}^{n-1} \mathbb{E}\left(\underbrace{g_{t_k}}_{\text{indep.}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep.}}\right)$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}) \mathbb{E}\underbrace{\left(W_{t_{k+1}} - W_{t_k}\right)}_{\sim N(0,\sigma^2)} = 0$$

(2) This is the variance of a stochastic integral:

$$\mathbb{E}\left(\left(\sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}}-W_{t_k})\right)^2\right) = \mathbb{E}\left(\sum_{k=0}^{n-1} g_{t_k}^2(W_{t_{k+1}}-W_{t_j})\right)^2 + 2\sum_{j<k} \underbrace{g_{t_k}g_{t_j}}_{\in\mathcal{F}_{t_k}} \underbrace{(W_{t_{k+1}}-W_{t_k})}_{\text{indep. of } \mathcal{F}_{t_k}} \underbrace{(W_{t_{j+1}}W_{t_j})}_{\in\mathcal{F}_{t_k}}$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left(g_{t_k}^2(W_{t_{k+1}}-W_{t_k})^2\right) + 2\sum_{j<k}\mathbb{E}\left(g_{t_k}g_{t_j}(W_{t_{k+1}}-W_{t_k})(W_{t_{j+1}}-W_{t_j})\right)$$

$$= \sum_{k=0}^{n-1}\mathbb{E}(g_{t_k}^2)\mathbb{E}\left(\underbrace{(W_{t_{k+1}}-W_{t_k})^2}_{t_{k+1}-t_k}\right) + 2\sum_{j<k}\mathbb{E}(\cdots)\underbrace{\mathbb{E}(W_{t_k+1}-W_{t_k})}_{=0}$$

$$= \int_0^t \mathbb{E}(g_{t_k}^2)dW_s$$

□

2.3. **Properties of the stochastic integarl.**

Examples:

$\int_0^t 1dW_s = W_t - W_0 = W_t$, but that is $\int_0^t W_s dW_s$? W_s is not piecewise constant, but we may approximate it by letting $g_t^n = W_{t_k}$ for $t \in [t_k, t_{k+1})$. What happens here is essentially discretisation but for finer and finer time.

This yields the approximation

$$\int_0^t \mathbb{E}\left((g_s^n - W_s)^2\right)ds = \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \underbrace{\mathbb{E}\left((W_s - W_{t_k})^2\right)}_{s-t_k} \leftarrow \quad \text{variance of increment of BM}$$

$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(s-t_k)ds = \sum_{k=0}^{n-1}\frac{1}{2}(t_{k+1}-t_k)^2 = \sum_{k=0}^{n-1}\frac{1}{2}\Delta t$$

$$\Delta t = \frac{t}{n} \Rightarrow \frac{1}{2}(\Delta t)^2\frac{t}{\Delta t} = \frac{\Delta t}{2}t \to 0 \quad \text{as } n\to\infty$$

$$\Rightarrow \sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}}-W_{t_k}) = \frac{1}{2}\sum_{k=0}^{n-1}\left(W_{t_{k+1}}^2 - W_{t_k}^2(W_{t_{k+1}}-W_{t_k})^2\right) = \frac{1}{2}W_{t_n} - \underbrace{\frac{1}{2}\sum_{k=0}^{n-1}(W_{t_{k+1}}-W_{t_k})^2}_{I_n}$$

We claim $I_n \to t$ as $n \to \infty$:

$$\mathbb{E}(I_n) = \underbrace{\mathbb{E}\left(\sum_{k=0}^{n-1}(W_{t_{k+1}}-W_{t_k})^2\right)}_{\text{2nd moment}} = \sum_{k=0}^{n-1}(t_{k+1}-t_k) = t_n = t$$

Need to check $\mathbb{E}((I_n - t)^2) = 0$:

$$\mathbb{E}\left((\sum_{k=0}^{n-1}(W_{t_{k+1}}-W_{t_k})^2 - \overbrace{(t_{k+1}-t_k)}^{\Delta t})\right)^2$$

$$= \sum_{k=0}^{n-1}\mathbb{E}\left(((W_{t_{k+1}}-W_{t_k})^2-\Delta t)^2\right) + \sum_{j\neq k}\mathbb{E}\left(((W_{t_{k+1}}-W_{t_k})^2-\Delta t)((W_{t_{j+1}}-W_{t_j})-\Delta t)\right)$$

$$= \sum_{j\neq k}\mathbb{E}\left((W_{t_{k+1}}W_{t_k})^4\right) - (\Delta t)^2 = \sum_{k=0}^{n-1} 2(\Delta t)^2 \sim \Delta t \to 0$$

hus, $I_n \to t$ as $n \to \infty$, so

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{t}{2}$$

Remark:

Lets prove if $X \sim N(0,\sigma)$, then $\mathbb{E}(X^4) = 3\sigma^2$

$$\mathbb{E}(X^4) = \int z^4 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-z^2}{2\sigma^2}\right\} \overset{\text{parts}}{\Rightarrow} - \left[z^3 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}\right]_{-\infty}^{\infty} - \int 3z^2 \frac{\sigma^2}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\pi\sigma^3\right\} dz$$

$$= 3\sigma^2 \cdot \underbrace{\int z^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}}_{\sigma^2} = 3\sigma^4$$

3. Martingales

Let $\mathcal{F}_t$ be a filtration, "information generated by B; up to a time t".
If Y is a random variable, then $\mathbb{E}(Y \mid \mathcal{F}_t)$ is the conditional expectation given all information up to time t

Example:

$$\mathbb{E}(W_s \mid \mathcal{F}_t) = W_t$$

Definition 3.7 Martingale

A process X is a martingale if X is $\mathcal{F}_t$-adapted. X_t integrable, i.e

- $\mathbb{E}(|X_t|) < \infty \quad \forall t$
- $\mathbb{E}(X_s \mid \mathcal{F}_t) = X_t$ for $s > t$

Example:
W_t is a martingale, $W_t^2 - t$ is a martingale since

$$Y_t := W_t^2 - t \qquad \mathbb{E}(Y_t \mid \mathcal{F}_s) = \mathbb{E}(W_t^2 - t \mid \mathcal{F}_s)$$
$$= \mathbb{E}((W_t - W_s)^2 + 2W_sW_t - W_s^2 \mid \mathcal{F}_s) - t$$
$$= t - s + 2\mathbb{E}(W_sW_t \mid \mathcal{F}_s) - \mathbb{E}(W_s^2 \mid \mathcal{F}_s) - t = 2W_s \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} W_s^2 - s$$
$$= W_s^2 - s = Y_s$$

$Y_t = \int_0^t g_u dW_u$ is a martingale since:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = \mathbb{E}\left(\int_0^s g_u dW_u \mid \mathcal{F}_s\right) + \mathbb{E}\left(\int_s^t g_u dW_u \mid \mathcal{F}_s\right) = \int_0^s g_u dW_u = Y_s$$

However, W_t^3 is *not* a martingale:

$$\mathbb{E}(W_t^3 \mid \mathcal{F}_s) = \mathbb{E}(W_s^3 + (W_t - W_s)^3 - 3W_tW_s^2 + 3W_t^2W_s \mid \mathcal{F}_s)$$
$$= W_s^3 + 0 - 3W_s^2 \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} + 3W_s \underbrace{\mathbb{E}(W_t^2 \mid \mathcal{F}_s)}_{t-s+W_s^2}$$
$$= W_s^3 + 3(t-s)W_s \neq W_s^3$$

Remark: A martingale is a "fair game"

4. Itos formula

Assume

$$X_t = a + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

for some adapted process μ_t and σ_t. Short-hand notation $\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t \\ X_0 = a \end{cases}$

Let $f(t, x)$ be a $C^{1,2}$-function and define $Z_t = f(t, X_t)$, what does dZ_t look like?

Recall:

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$$

so $W_t^2 = t + 2\int_0^t W_s dW_s$, thus

$$d(W_t^2) = dt + 2W_t dW_t$$

Fix n and let $t_k = \frac{k}{n}t$
Let $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$ and consider

$$S_n = \sum_{k=0}^{n-1} (\Delta W_{t_k})^2$$

We have

$$\mathbb{E}(S_n) = \sum_{k=0}^{n-1} \mathbb{E}\left((\Delta W_{t_k})^2\right) = \sum_{k=0}^{n-1} \frac{t}{n} = t$$

and

$$\mathrm{Var}\,(S_n) \overset{\text{indep.}}{=} \sum_{k=0}^{n-1} \mathrm{Var}\left((\Delta W_{t_k})^2\right) = n\mathrm{Var}\left((\Delta W_{t_0})^2\right) = n \cdot 2\frac{t^2}{n^2} \to 0 \quad \text{as } n \to \infty$$

Thus $S_n \to t$ as $n \to \infty$ (in $\mathcal{L}^2$). This motivates to write

$$\int_0^t (dW_s^2) = t$$
$$\Leftrightarrow dW_t^2 = dt$$

4.1. **Taylor Expansion.**

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2} + \frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t + \text{ higher order terms} \\ &= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW + \text{ higher order terms} \end{aligned}$$

Sats 4.2: Itos formula

If $dX_t = \mu_t dt + \sigma_t dW_t$ and $Z_t = f(t, X_t)$, then

$$dZ_t = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

Here $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(t, X_t)$ and similarly for other derivatives of f

Alternative formulation:

$$dZ_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

Where $(dX_t)^2$ is calculated using

- $(dt)^2 = 0$

- $dtdW_t = 0$
- $(dW_t)^2 = dt$

Example:
Compute $\int_0^t W_s dW_s$. Let $Z_t = W_t^2$, then by Itos formula

$$\begin{aligned} dZ_t &= 2W_t dW_t + \frac{1}{2} \cdot 2(dW_t)^2 \\ &= dt + 2W_t dW_t \end{aligned}$$

Thus $W_t^2 = Z_t = t + 2\int_0^t W_s dW_s$, so $\int_0^t W_s dW_s = \dfrac{W_t^2}{2} - \dfrac{t}{2}$

Example:
Compute $\mathbb{E}(W_t^4)$
Let $Z_t = W_t^4$, then by Itos formula

$$\begin{aligned} dZ_t &= 4W_t^3 dW_t + \frac{1}{2} \cdot 12 W_t^2 (dW_t)^2 \\ &= 6W_t^2 dt + 4W_t^3 dW_t \end{aligned}$$

Thus

$$W_t^4 = Z_t = 6\int_0^t W_s^2 ds + 4\int_0^t W_s^3 dW_s$$

Taking expectation yields

$$\begin{aligned} \mathbb{E}(W_t^4) &= 6\int_0^t \underbrace{\mathbb{E}(W_s^2)}_{s} ds + 4\underbrace{\mathbb{E}\left(\int_0^t W_s^3 dW_s\right)}_{=0} \\ &= 6\int_0^t s ds = 3t^2 \end{aligned}$$

Alternatively, without using Itos formula

$$\begin{aligned} \mathbb{E}(W_t^4) = \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx &\overset{\text{parts.}}{=} \left[x^3 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)}\right]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \\ &= 3t \operatorname{Var}(W_t) = 3t^2 \end{aligned}$$

Example:
Compute $\mathbb{E}(e^{\alpha W_t})$
Let $Z_t = e^{\alpha W_t}$. Itos formula yields

$$\begin{aligned} dZ_t &= \alpha e^{\alpha W_t} dW_t + \frac{1}{2}\alpha^2 e^{\alpha W_t} (dW_t)^2 \\ &= \frac{\alpha^2}{2} e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t \\ &= \frac{\alpha^2}{2} Z_t dt + \alpha Z_t dW_t \end{aligned}$$

Integration yields

$$Z_t = 1 + \frac{\alpha^2}{2}\int_0^t Z_s ds + \alpha\int_0^t Z_s dW_s$$

So

$$\begin{aligned} \mathbb{E}(Z_t) &= 1 + \mathbb{E}\left(\frac{\alpha^2}{2}\int_0^t Z_s ds\right) + \underbrace{\mathbb{E}\left(\alpha\int_0^t Z_s dW_s\right)}_{=0} \\ &= 1 + \frac{\alpha^2}{2}\int_0^t \mathbb{E}(Z_s) ds \end{aligned}$$

Let $m(t) = \mathbb{E}(Z_t)$, then

$$\begin{cases} \dfrac{dm}{dt} = \dfrac{\alpha^2}{2} m(t) \\ m(0) = 1 \end{cases}$$

Which has the solution $m(t) = e^{\frac{\alpha^2}{2}t}$

4.2. **Multi-dimensional Ito formula.**
Assume $dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$ where W^i are d independent Brownian motions.
On a matrix form:

$$\underbrace{dX_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1} dt + \underbrace{\sigma_t}_{n\times d} \underbrace{dW_t}_{d\times 1}$$

Let $Z_t = f(t, X_t)$ where $f : [0,\infty] \times \mathbb{R}^2 \to \mathbb{R}$ is $C^{1,2}$

Sats 4.3: Itos multi-dimensional formula

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

Where

- $dW_t^i dW_t^j = 0$ if $i \neq j$
- $\left(dW_t^i\right) = dt$
- $(dt)^2 = dtdW_t = 0$

Alternatively

$$dZ_t = \left(\frac{\partial f}{\partial t} + \sum_{i=1}^n \mu_t^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_t^{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \sigma_t^i dW_t$$

Where $C = \sigma\sigma^*$ and σ^i is the i:th row of σ
Indeend,

$$\begin{aligned} dX_t^i dX_t^j &= \left(\sum_{j)1}^d \sigma^{ik} dW^k \right) \left(\sum_{l=1}^d \sigma^{jl} dWl \right) \\ &= \left(\sum_{k=1}^d \sigma^{ik} \sigma^{jl} \right) dt \\ &= (\sigma\sigma^*)^{ij} dt \end{aligned}$$

Example:
If $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ dY_t = \gamma Y_t dt + \delta Y_t dV_t \end{cases}$ and $Z_t = X_t Y_t$; find dZ_t

Itos formula yields

$$\begin{aligned} dZ_t &= Y_t dX_t + X_t dY_t + \frac{1}{2} \cdot 2 dX_t dY_t \\ &= (\alpha + \gamma) Z_t dt + Z_t(\sigma dW_t + \delta dV_t) \end{aligned}$$

Setting $\overline{W}_t = \dfrac{1}{\sqrt{\sigma^2 + \delta^2}} (\sigma W_t + \delta V_t)$, then $\overline{W}$ is a Brownian Motion and

$$dZ_t = (\alpha + \gamma) Z_t dt + \sqrt{\sigma^2 + \delta^2} Z_t d\overline{W}_t$$

5. Correlated Brownian Motions

Let $\overline{W} = \begin{bmatrix} \overline{W}^1 \\ \vdots \\ \overline{W}^d \end{bmatrix}$ where $\overline{W}^1, \cdots, \overline{W}^d$ are independent

Consider $W = \delta \overline{W}$ where

$$\delta = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1d} \\ \vdots & \vdots & \vdots \\ \delta_{d1} & \cdots & \delta_{dd} \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_d \end{bmatrix}$$