UPPSALA UNIVERSITET
Matematiska Institutionen
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Prov i matematik
1MA255, VT22
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Partial Differential Equations with Applications to Finance

Writing time: 14:00 - 19:00.
Instructions: There are 5 problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, 4 or 5, the score has to be at least 25 or 32 points, respectively. Bonus points do not apply. Other than writing utensils and paper, no help materials are allowed.

GOOD LUCK!

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1. (8p)

i) (3p) Use the Feynman-Kac formula to solve the following initial value problem:

$$\begin{cases} u_t - u_{xx} &= 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(0,x) &= x^2 + 3x, \quad x \in \mathbb{R} \end{cases}$$

ii) (5p) Use the fundamental solution of the heat equation and combine it with the result in i) to compute the value of

$$\int_{\mathbb{R}} (x^2 + 3x) e^{-x^2} dx.$$

2. (8p) Suppose that $r > 0$, $a_j > 0, b_j \in \mathbb{R}$ for $j = 1, \dots, n$. Consider the set

$$D = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} (a_j x_j^2 + b_j x_j) < r\}.$$

Let $X_t = kB_t, k \in \mathbb{R}$, where B_t is an n-dimensional Brownian motion starting at $(0, \dots, 0) \in \mathbb{R}^n$. Determine the expected exit time of X_t from D by constructing a suitable Poisson problem.

3. (8p) Consider a geometric Brownian motion X_t

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

starting at $X_0 = x_0$ a.s..

i) (2p) Write down the Kolmogorov forward (Fokker-Planck) equation satisfied by $\rho(t,x)$, the probability density function of X_t.

ii) (5p) Denote the n-th moment of X_t by $M_n(t)$, show that

$$\frac{d}{dt}M_1(t) = \mu M_1(t),$$

and

$$\frac{d}{dt}M_n(t) = \left(\mu n + \frac{\sigma^2}{2}n(n-1)\right) M_n(t), \quad n \geq 2.$$

You may assume that the density function decays sufficiently fast at infinity.

iii) (1p) Use ii) to calculate $\mathbb{E}[X_t^2]$.

4. (8p) Let X_t^α satisfy the SDE

$$dX_t^\alpha = \alpha_t dt + \sigma dB_t,$$

where B_t is a standard Brownian motion, $\alpha_t \in \mathbb{R}$ is the control process. Solve the following minimisation problem

$$V(x) = \inf_{\alpha\in\mathbb{R}} \mathbb{E}_x[\int_0^\infty e^{-\rho t}((X_t^\alpha)^2 + \theta\alpha_t^2)dt]$$

where ρ, θ are positive constants.You **do not** need to prove the verification theorem for your solution.

Hint: *use the ansatz* $V(x) = ax^2 + b$ *for some constants* a, b.

5. (8p) Consider the stock price process X_t with the following dynamics:

$$dX_t = (r-\delta)X_t dt + \sigma X_t dB_t,$$

where $r > 0$ and $\delta \geq 0$. Consider the price of an American call option:

$$V(x) = \sup_\tau \mathbb{E}_{0,x}[e^{-r\tau}(X_\tau - K)^+]$$

for some strike $K > 0$.

i) When $\delta > 0$, solve the optimal stopping problem above. You **do not** need to prove the verification theorem for your solution.

ii) When $\delta = 0$, argue that it is never optimal to exercise.