Problem session 2

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1. (2020-08-19)A metric space (X, d) is called discrete if $d(x, y) = 1$ for all points $x \neq y$ in X. Prove that

a) a discrete metric space (X, d) is compact if and only if the set X is finite, and

b) a discrete metric space (X, d) is complete.

2. (2021-06-08)On the set $\mathbb{Z}^2$ of integer points in the plane, denote $L = (0, 0)$ and define the distance function $d : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ by

$$d(P, Q) := \begin{cases} 0, & P = Q \\ |P - L| + |Q - L|, & P \neq Q \end{cases}$$

where $P, Q \in \mathbb{Z}^2$ and $|(x, y)| := \sqrt{x^2 + y^2}$ is the Euclidean metric on $\mathbb{R}^2$.

a) Show that $\left(\mathbb{Z}^2, d\right)$ defines a metric space. (Amusingly, if L is London, d might be called the British Railway metric).

b) Does this metric space have the Heine-Borel property? Explain!

3.(2020-03-16) Let X be a metric space and let A be a subset of X. Assume that A is not closed. Prove that there exists a Cauchy sequence (x_n) in A which does not converge to any point in A.

4.(2020-08-19) Give an example of a sequence of nonempty compact subsets $C_1, C_2, \ldots$ of $\mathbb{R}^2$ (equipped with its standard metric) such that the union $\cup_{n=1}^{\infty} C_n$ is an open set. Prove your claim.

5. (2021-03-15) Give examples or claim non-existence (with brief motivations) of:

a) A bounded subset of $\mathbb{R}^2$ with the same cardinality as $\mathbb{R}$.

b) A bounded metric space which is complete but not compact.

6. (2019-06-15) Give an example of an open cover of the interval $(0, 1]$ which has no finite subcover.(Note: You must prove that your open cover indeed does not have any finite subcover.)

7. (Rudin 4.2) If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

8. (Rudin 4.4) Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Also, one must look at the following exercises 4.1, 4.3, 4.4, 4.5,4.6, 4.7, and 4.8 in Rudin's book.