Proof of some fundamental relationships in survival analysis

Philip Fowler

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1 Definitions

Let X be a continuous random variable measuring the time to some prespecified event. Then define the density function $f(x)$, the cumulative distribution function $F(x)$, the survival function $S(x)$, the hazard function $h(x)$ and the cumulative hazard function as the following:

$$f(x) = \lim_{\Delta x \to 0} \frac{Pr\left(x \leq X < x + \Delta x\right)}{\Delta x}$$

$$F(x) = Pr(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

$$S(x) = 1 - F(x) = Pr(X > x) = \int_{x}^{\infty} f(t)dt$$

$$h(x) = \lim_{\Delta x \to 0} \frac{Pr\left(x \leq X < x + \Delta x | X \geq x\right)}{\Delta x}$$

$$H(x) = \int_{-0}^{x} h(t)dt$$

2 Relationships between the entities

We will now prove some well-know relationships between $f(x), F(x), S(x), h(x)$ and $H(x)$.

2.1 $f(x) = -\frac{dS(x)}{dx}$

This follows directly from noting that $S(x) = 1 - F(x)$ and that $f(x) = dF(x)/dx$. That is, by definition

$$f(x) = \frac{dF(x)}{dx} = \frac{d\left[1 - S(x)\right]}{dx} = -\frac{dS(x)}{dx}$$

Q.E.D.

2.2 $h(x) = f(x)/S(x)$

Remembering that $Pr(A, B) = Pr(A|B)Pr(B)$ and that thus $Pr(A|B) = Pr(A, B)/Pr(B)$ it follows that:

$$\lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x | X \ge x\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x, X \ge x\right)}{\Delta x Pr(X \ge x)}$$

Noting that $x \le X$ is the same statement as $X \ge x$, it follows that

$$\lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x, X \ge x\right)}{\Delta x Pr(X \ge x)} = \lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x\right)}{\Delta x Pr(X \ge x)}$$

We now rewrite the right hand side for clarity:

$$\lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x\right)}{\Delta x Pr(X \ge x)} = \lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x\right)}{\Delta x} \frac{1}{Pr(X \ge x)}$$

From here we just use the definition of $f(x)$ and $S(x)$ and see that

$$\lim_{\Delta x \to 0} \frac{Pr\left(x \le X < x + \Delta x\right)}{\Delta x} \frac{1}{Pr(X \ge x)} = f(x)\frac{1}{S(x)} = \frac{f(x)}{S(x)}$$

Q.E.D.

2.3 $h(x) = -\frac{d\ln[S(x)]}{dx}$

Remembering that $S(x) = 1 - F(x)$ it follows that

$$-\frac{d\ln\left[S(x)\right]}{dx} = -\frac{d\ln\left[1 - F(x)\right]}{dx}$$

Next we recall the chain rule of differentiation, which states that for two functions f and g, it follows that $\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$. We also recall that the derivative of $\ln x$ is $1/x$. Thus

$$-\frac{d\ln[1-F(x)]}{dx} = -\frac{1}{1-F(x)}\frac{d[1-F(x)]}{dx}$$

The derivative of a sum is the sum of the derivatives. Thus

$$-\frac{1}{1-F(x)}\frac{d[1-F(x)]}{dx} = -\frac{1}{1-F(x)}\left[\frac{d1}{dx} - \frac{dF(x)}{dx}\right]$$

Recalling that $f(x) = dF(x)/dx$ and that the derivative of a constant is 0, it follows that

$$-\frac{1}{1-F(x)}\left[\frac{d1}{dx} - \frac{dF(x)}{dx}\right] = -\frac{1}{1-F(x)}[0 - f(x)]$$

Simplifying the expression by canceling out the minus signs and removing the 0, we get

$$-\frac{1}{1-F(x)}[0 - f(x)] = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}$$

As we saw in Section 2.2, $\frac{f(x)}{S(x)} = h(x)$. Thus it follows that $h(x) = -\frac{d\ln[S(x)]}{dx}$.
Q.E.D.

2.4 $H(x) = -\ln S(x)$

Since $H(x) = \int_{-0}^{x} h(t)dt$, it follows that

$$H(x) = -\ln S(x) \Leftrightarrow \int_{-0}^{x} h(t)dt = -\ln S(x)$$

Since $\frac{dH(x)}{dx} = h(x)$, it follows that

$$H(x) = -\ln S(x) \quad \Leftrightarrow \quad \frac{dH(x)}{x} = -\frac{d\ln S(x)}{dx}$$

Thus it follows that $H(x) = -\ln S(x)$ is true if $\frac{dH(x)}{x} = -\frac{d\ln S(x)}{dx}$ is true, which was proven in Section 2.3.
Q.E.D.