Final Exam

Fourier Analysis, 1MA211

Uppsala university

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Examinator: Fredrik Viklund. Every problem gives at most 5 *points. There are* 8 *problems. Tentative grade limits (including any extra points from the homework assignments): at least* 18 *points implies* 3*, at least* 25 *points implies* 4*, and at least* 32 *points implies* 5*. You are allowed to use writing utensils and the attached formula sheet. Good luck!*

1. Compute the Fourier series of the even 2π-periodic function defined by t^2 on $[0, \pi]$ and use the result to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Suppose that $f \in C^2(\mathbb{T})$. Show that the Fourier series of f converges at every point of the unit circle $\mathbb{T}$. You don't have to discuss what the series converges to. Hint: Show that $|c_n(f'')|$ is bounded and this to bound $|c_n(f)|$.

3. Consider the inner product space $L^2([-1, 1])$.

   (a) What is the definition of the inner product for this space?

   (b) What is the definition of the norm derived from the inner product?

(c) Compute the norms of $\{1, x, e^{2ix}\}$.

(d) Which (if any) of these functions are pairwise orthogonal in $L^2([-1,1])$?

4. Solve the following PDE:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \cos x, \quad x \in (0,\pi), \quad t > 0; \\ u(0,t) &= 0,\ u(\pi,t) = 0, \quad t > 0; \\ u(x,0) &= \cos x, \quad x \in (0,\pi), \end{aligned}$$

and determine the steady state solution, $\lim_{t\to\infty} u(x,t)$

5. Show that:

(a) $\widehat{f'(t)}(\omega) = i\omega \hat{f}(\omega)$.

(b) $\widehat{(f \star g)(t)}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$.

(c) Let $f(t) = \sin x$ for $-1 \le t \le 1$ and $f(t) = 0$ otherwise. Compute $\hat{f}(\omega)$ using the definition of the Fourier transform.

You may assume that all involved integrals are absolutely convergent.

6. Compute the integral

$$\int_{-\infty}^{\infty} \frac{t^2}{(1+t^2)^4} dt.$$

7. Find $f \in L^1(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} f(s) e^{-(t-s)^2/2} ds = e^{-t^2/4}.$$

8. Solve following ODE:

$$y''(t) - 3y'(t) + 2y(t) = 4e^{2t}, \quad t > 0;$$

with initial conditions

$$y(0) = -3, \quad y'(0) = 5.$$