# Exercise 1

Consider the scalar initial-boundary value problem (IBVP)

$$\begin{array}{lll} cu_t = au_x + (bu_x)_x + du_{xxx}, & 0 \le x \le W, & t \ge 0, \\ \mathcal{L}_l u = g_l, & x = 0, & t \ge 0, \\ \mathcal{L}_r u = g_r, & x = W, & t \ge 0, \\ u = f, & 0 \le x \le W, & t = 0, \end{array}$$

where $c = c(x) > 0$ is a real-valued function; $b = b(x)$ is possibly complex-valued; and $a$ and $d$ are (possibly complex-valued) *constants*.

(a) Consider the case $d = 0$. What are the requirements on $a$, $b$, and $c$ for the PDE to be well-posed, disregarding the boundary conditions? That is, you may assume periodic boundary conditions.
*Hint:* Use the energy method.

(b) For $d = 0$, what are the requirements for the PDE to conserve some energy? Again consider periodic boundary conditions.

(c) For $d = 0$, derive at least two sets of well-posed boundary conditions. That is, find two different operators $\mathcal{L}_l$ (and $\mathcal{L}_r$) that yield a well-posed IBVP.

(d) Consider the case $a = c = 1$, $d = 0$, $b = 10$. Describe the expected behaviour of the solution.

(e) Now consider $d \neq 0$. What are the requirements for the PDE to be well-posed with periodic boundary conditions?
*Hint:* The term $du_{xxx}$ requires integrating by parts twice.

(f) For $d \neq 0$, derive one set of well-posed boundary conditions.
*Hint:* You will need 3 conditions in total due to the term $du_{xxx}$.

# Solutions

We start by applying the energy method for $d = 0$. The highest-order time-derivative is the term $u_t$, so we mulitply the PDE by $u^*$ and integrate in space:

$$(u, cu_t) = (u, au_x) + (u, (bu_x)_x). \; (*)$$

Integrating both terms in the RHS by parts leads to

$$(u, cu_t) = u^* au|_0^W + u^* bu_x|_0^W - (u_x, au) - (u_x, bu_x). \; (*)$$

Taking the complex conjugate of $(*)$ yields

$$(u_t, cu) = (au_x, u) + ((bu_x)_x, u) = (u_x, a^* u) + ((bu_x)_x, u),$$

where we used that $c \in \mathbb{R}$. Integrating the last term by parts yields

$$(u_t, cu) = (u_x, a^* u) + u_x^* b^* u|_0^W - (u_x, b^* u_x)). \ (*)$$

Adding $(*)$ and $(*)$ leads to the *energy rate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2 = [u^* a u + u^* b u_x + u_x^* b^* u]_0^W - (u_x, (a - a^*)u) - (u_x, (b + b^*)u_x),$$

where we used that

$$(u_t, cu) + (u, cu_t) = \frac{\mathrm{d}}{\mathrm{d}t}(u, cu) = \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2$$

and the norm notation is appropriate since $c \in \mathbb{R}$, $c > 0$.

(a) For the periodic problem, all boundary terms vanish. We then obtain the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2 = -(u_x, (a - a^*)u) - (u_x, (b + b^*)u_x).$$

Well-posedness requires a bound

$$\|u\| \leq K e^{\alpha t}\|f\|,$$

for some constants $\alpha, K$. We could derive such an estimate from an energy rate of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2 = \beta\|u\|_c^2,$$

but note that the terms in the RHS of our energy rate contain spatial derivatives and are not of this form. We have to ensure that these terms do not yield energy growth. We cannot control the sign of the term $-(u_x, (a - a^*)u)$, so we need it to vanish, which leads to the requirement

$$a - a^* = 0,$$

which is equivalent to $a \in \mathbb{R}$. If this condition is satisfied the term $au_x$ conserves energy.

The term $-(u_x, (b + b^*)u_x)$ is non-positive (and hence does not contribute to energy growth) if

$$b + b^* \geq 0,$$

which means that $\mathcal{R}(b)$ (the real part of $b$) must be non-negative. $\mathcal{R}(b) = 0$ yields energy conservation while $\mathcal{R}(b) > 0$ yields energy dissipation.

Let us pause and check that our conclusions seem reasonable. For $a \in \mathbb{R}$, we recognize $au_x$ as an *advection* term, which conserves energy. For $b \in \mathbb{R}, b > 0$, we recognize $(bu_x)_x$ as a diffusion term, which dissipates energy. Note that with $b < 0$, this term corresponds to the *backwards* heat/diffusion equation, which we know is ill-posed! For $b$ purely imaginary (and constant), the term is instead of Schrödinger type, which conserves energy. All of these observations match the requirements that we derived.

*Answer:* The requirements for well-posedness are

$$a - a^* = 0, \quad b + b^* \geq 0.$$

(b) See the discussion under a).

*Answer:* The requirements for energy conservation are

$$a - a^* = 0, \quad b + b^* = 0.$$

(c) Now we assume that the requirements for a well-posed periodic problem are met and turn our attention to the boundary terms ($BT$). Since the volume terms do not yield energy growth, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2 \leq [u^* a u + u^* b u_x + u_x^* b^* u]_0^W =: BT$$

The PDE is a linear, scalar equation of second order in space, so we know that we will need two boundary conditions (BC) in total. We assume that $b \neq 0$ so that the PDE is indeed of second order. If the homogeneous version of a BC (i.e., $\mathcal{L}_l u = 0$) is well-posed, then so is any inhomogeneous version ($\mathcal{L}_l u = g_l$), so we focus on homogeneous BC for simplicity. We seek conditions that ensure $BT \leq 0$. One option is

$$\mathcal{L}_l u = u, \quad \mathcal{L}_r u = u,$$

which yields $BT = 0$. To find other well-posed BC, we will make an educated guess. Since $BT$ contains $u_x$, we anticipate that well-posed BC will include $u_x$ and possibly $u$. For the right boundary, we make the ansatz

$$\mathcal{L}_r u = b u_x + \alpha u,$$

and investigate which values of $\alpha$ (if any) yield well-posedness. We obtain

$$\begin{aligned} u^* a u + u^* b u_x + u_x^* b^* u &= a|u|^2 - u^* \alpha u - \alpha^* u^* u \\ &= a|u|^2 - \alpha|u|^2 - \alpha^*|u|^2 = (a - 2\mathcal{R}(\alpha))|u|^2 \end{aligned}$$

This corresponds to a well-posed boundary condition that does not contribute to energy growth if

$$a - 2\mathcal{R}(\alpha) \leq 0.$$

This makes sense because we have already concluded that $a$ must be real. At the left boundary, the sign is reversed, so we would instead need

$$a - 2\mathcal{R}(\alpha) \geq 0.$$

We have found the following well-posed BC for the left boundary:

$$\mathcal{L}_l u = u$$

or

$$\mathcal{L}_l u = b u_x + \alpha u, \quad a - 2\mathcal{R}(\alpha) \geq 0.$$

For the right boundary, we have found

$$\mathcal{L}_r u = u$$

or

$$\mathcal{L}_r u = b u_x + \alpha u, \quad a - 2\mathcal{R}(\alpha) \leq 0.$$

Any of the four combinations of one BC for the left boundary and one for the right boundary yields a well-posed IBVP.

(d) The term $a u_x$ is an advection term, which for positive $a$ transports the solution to the *left*, with speed $a/c$. For real and positive $b$, the term $(b u_x)_x$ is a diffusion term, which damps all nonzero wavenumbers (high wavenumbers dissipate the fastest) and tends to flatten out the solution. The larger $b$ is, the faster the diffusion. This equation is known as the *advection-diffusion* equation.

(e) Now we consider $d \neq 0$. We repeat the energy analysis for the term $d u_{xxx}$ only. We have

$$\begin{aligned}(u, d u_{xxx}) &= u^* d u_{xx}|_0^W - (u_x, d u_{xx}) \\ &= [u^* d u_{xx} - u_x^* d u_x]_0^W + (u_{xx}, d u_x)\end{aligned}$$

Taking the complex conjugate yields

$$(d u_{xxx}, u) = u_{xx}^* d^* u|_0^W - (u_{xx}, d^* u_x).$$

The contribution to the energy rate is

$$(u, d u_{xxx}) + (d u_{xxx}, u) = [u^* d u_{xx} + u_{xx}^* d^* u - u_x^* d u_x]_0^W + (u_{xx}, d u_x) - (u_{xx}, d^* u_x).$$

For the periodic problem, the boundary terms vanish. We cannot say anything about the sign of the volume terms so we need them to vanish, which leads to the condition

$$d - d^* = 0,$$

or, equivalently, $d$ must be real.

(f) Now we consider the boundary terms only. Adding boundary terms from c) and e) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_c^2 &\leq [u^* a u + u^* b u_x + u_x^* b^* u]_0^W + [u^* d u_{xx} + u_{xx}^* d^* u - u_x^* d u_x]_0^W \\ &= [u^*(a u + b u_x + d u_{xx}) + (u_x^* b^* - u_{xx}^* d^*) u - d|u_x|^2]_0^W\end{aligned}$$

Because we are dealing with a scalar, third-order PDE, we know that we will need three BC in total, probably two at one boundary and one at the other. Most terms vanish if $u = 0$, so it is reasonable to require $u = 0$ at both boundaries. Assuming $d > 0$, the remaining term $[-d|u_x|^2]_0^W$ provides dissipation at the right boundary and growth at the left boundary (and the other way around if $d < 0$). We need to impose $u_x = 0$ on the boundary that corresponds to growth.

*Answer:* We can impose $u = 0$ at both boundaries and $u_x = 0$ at either the left boundary, if $d > 0$, or the right boundary, if $d < 0$.

## Exercise 2

Consider the IBVP

$$\begin{array}{lll} \mathbf{C}\mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u} + \mathbf{F}, & 0 \le x \le W, & t \ge 0, \\ \mathcal{L}_l \mathbf{u} = g_l, & x = 0, & t \ge 0, \\ \mathcal{L}_r \mathbf{u} = g_r, & x = W, & t \ge 0, \\ \mathbf{u} = \mathbf{f}, & 0 \le x \le W, & t = 0, \end{array}$$

where $\mathbf{F} = \mathbf{F}(x,t)$ is the forcing function, $\mathbf{f}$ is the initial data, $\mathbf{A} = \mathbf{A}^*$ is a *constant* matrix, $\mathbf{B}$ and $\mathbf{C}$ are variable-coefficient matrices, and $\mathbf{C} = \mathbf{C}^* > 0$. $\mathbf{A}$ and $\mathbf{C}$ have the structure

$$\mathbf{C} = \begin{bmatrix} c_1(x) & 0 & 0 \\ 0 & c_2(x) & 0 \\ 0 & 0 & c_3(x) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & \alpha \end{bmatrix},$$

where $c_i \in \mathbb{R}$ and $\alpha$ is a real constant.

(a) Use the energy method to derive an energy rate for the IBVP with $\mathbf{F} = 0$. Under which conditions can you show that the PDE with periodic boundary conditions is well-posed?

(b) How many boundary conditions should be prescribed at each boundary in the cases $\alpha = 0$, $\alpha = -8$ and $\alpha = -10$?

(c) Consider the case $\alpha = 0$. Derive at least one set of well-posed boundary conditions. You may assume that the solution is real-valued.

## Solutions

(a) Taking the inner product of the PDE and $\mathbf{u}$ yields (with $\mathbf{F} = 0$)

$$(\mathbf{u}, \mathbf{u}_t)_C = (\mathbf{u}, \mathbf{A}\mathbf{u}_x) + (\mathbf{u}, \mathbf{B}\mathbf{u}) = \mathbf{u}^*\mathbf{A}\mathbf{u}|_0^W - (\mathbf{u}_x, \mathbf{A}\mathbf{u}) + (\mathbf{u}, \mathbf{B}\mathbf{u}).$$

Taking the conjugate transpose yields

$$(\mathbf{u}_t, \mathbf{u})_C = (\mathbf{u}_x, \mathbf{A}^*\mathbf{u}_x) + (\mathbf{u}, \mathbf{B}^*\mathbf{u}),$$

where we used that $\mathbf{C} = \mathbf{C}^*$. Adding the two equations above, using that $\mathbf{C} = \mathbf{C}^* > 0$, leads to the energy rate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{\mathbf{C}}^2 = \mathbf{u}^*\mathbf{A}\mathbf{u}|_0^W - (\mathbf{u}_x, (\mathbf{A} - \mathbf{A}^*)\mathbf{u}) + (\mathbf{u}, (\mathbf{B} + \mathbf{B}^*)\mathbf{u}).$$

For periodic BC, all boundary terms vanish. One term in the RHS vanishes because $\mathbf{A}^* = \mathbf{A}$. Note that the matrix $\mathbf{B}$ may cause energy growth, but as long as $\mathbf{B}$ is bounded we have

$$(\mathbf{u}, (\mathbf{B} + \mathbf{B}^*)\mathbf{u}) \le \beta\|\mathbf{u}\|_{\mathbf{C}}^2,$$

for some constant $\beta$. This kind of growth does not destroy well-posedness, because we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{\mathbf{C}}^2 \leq \beta\|\mathbf{u}\|_{\mathbf{C}}^2.$$

The last estimate shows that the growth is not faster than exponential.

(b) Now we address the boundary terms, which are

$$BT = \mathbf{u}^*\mathbf{A}\mathbf{u}|_0^W.$$

Since $\mathbf{A}$ is Hermitian, it can be diagonalized by a unitary matrix $\mathbf{T}$. That is, we have

$$\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^*,$$

where $\mathbf{\Lambda}$ is diagonal and has the eigenvalues of $\mathbf{A}$ on the diagonal. It follows that

$$BT = \mathbf{u}^*\mathbf{A}\mathbf{u}|_0^W = (\mathbf{T}^*\mathbf{u})^*\mathbf{\Lambda}(\mathbf{T}^*\mathbf{u})|_0^W.$$

Let $\mathbf{w} = \mathbf{T}^*\mathbf{u}$. We have

$$BT = \mathbf{w}^*\mathbf{\Lambda}\mathbf{w}|_0^W = \sum_i \lambda_i|w_i|^2|_0^W,$$

where $w_i$ denotes a component of $\mathbf{w}$ and the $\lambda_i$ are the eigenvalues of $\mathbf{A}$. For every positive eigenvalue there is a term that causes growth at the right boundary and for every negative eigenvalue there is a term that causes growth at the left boundary. We have to control that growth by imposing BC. For every positive eigenvalue, we impose a BC at the right boundary and for every negative eigenvalue we impose a BC at the left boundary.

It is useful to remember that the number of BC at each boundary, for hyperbolic first-order systems (in 1D), depends on the eigenvalues of the matrix in front of $\mathbf{u}_x$.

For $\alpha > -8$ there are two positive eigenvalues and one negative. For $\alpha < -8$ there are two negative and one positive. For $\alpha = -8$, there is one positive, one zero, and one negative. In the last case, we only need one BC at each boundary.

(c) Let $\mathbf{u} = [u_1, u_2, u_3]^T$. Assuming that $\mathbf{u}$ is real, the boundary terms are

$$BT = 2u_1^2 + 2u_1u_2 + 4u_2u_3 + \alpha u_3^2|_0^W.$$

We seek boundary conditions such that, with zero boundary data, we obtain $BT \leq 0$.

Let us first consider $\alpha = 0$, in which case the last term vanishes. It is tempting to set $u_2 = 0$ at both boundaries. The term with $u_1^2$ causes growth at the right boundary, but damping at the left. Imposing $u_1 = 0$ at the right boundary ensures $BT \leq 0$. This agrees with our analysis in b), which tells us that we need two BC at the right boundary and one BC at the left boundary for this case.

*Answer:* $u_2 = 0$ at $x = 0$ and $u_1 = 0$, $u_2 = 0$ at $x = 1$.

*Remark:* There are many other sets of well-posed BC for this problem. One can use the eigenvectors of $\mathbf{A}$ (which form the columns of $\mathbf{T}$) to find other sets.