OU-process

A solution of the SDE

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

is a stochastic process $\{X_t,\ t \in [0,T]\}$ that satisfies

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s, \quad 0 \le t \le T.$$

Some examples

1) (Exercise) Assignment 4.15

2) Assignment 4.6 $\longleftrightarrow$ Example 5.1.1

$$dX_t = rX_t\,dt + \alpha X_t\,dB_t$$

- geometric BM
- linear population growth
- multiplicative returns

Front cover figure

3) Ornstein-Uhlenbeck process, Exercise 5.5

$$dX_t = \mu X_t\,dt + \sigma\,dB_t, \quad \sigma > 0, \quad -\infty < \mu < \infty.$$

$X_0 = x_0$ fixed

or $X_0 \in \mathcal{F}_0$ random variable

Soln a) Consider $f(t, X_t) = e^{-\mu t} X_t$

$$\begin{cases} f(t,x) = e^{-\mu t} x \\ f_t'(t,x) = -\mu e^{-\mu t} x \\ f_x'(t,x) = e^{-\mu t} \\ f_{xx}''(t,x) = 0 \end{cases}$$

By the Itô formula

$$df(t, X_t) = \Big(f_t'(t, X_t) + \mu X_t f_x'(t, X_t) + \frac{\sigma^2}{2}\underbrace{f_{xx}''(t, X_t)}_{=0}\Big)dt + \sigma f_x'(t, X_t)\,dB_t$$

$$= \underbrace{\left(-\mu X_t e^{-\mu t} + \mu X_t e^{-\mu t}\right)dt}_{=0} + \sigma e^{-\mu t}\,dB_t$$

Hence $e^{-\mu t} X_t = X_0 + \sigma \int_0^t e^{-\mu s}\,dB_s$

and so

$$X_t = e^{\mu t} X_0 + \sigma e^{\mu t} \int_0^t e^{-\mu s}\, dB_s \quad , 0 \le t \le T$$

b) $E(X_t) = e^{\mu t} E(X_0) + 0$

Assume X_0 and $\{B_t\}$ are independent, then

$$Var(X_t) = e^{2\mu t} Var(X_0) + \sigma^2 e^{2\mu t} \underbrace{Var\left(\int_0^t e^{-\mu s} dB_s\right)}_{\int_0^t e^{-2\mu s} ds = \frac{1}{2\mu}(1 - e^{-2\mu t})}$$

$$= e^{2\mu t} Var(X_0) + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$$

~~Rk~~ Remark: Take $\mu = -\beta$, $\beta > 0$

$E(X_0) = 0$ and $Var(X_0) = \frac{\sigma^2}{2\beta}$

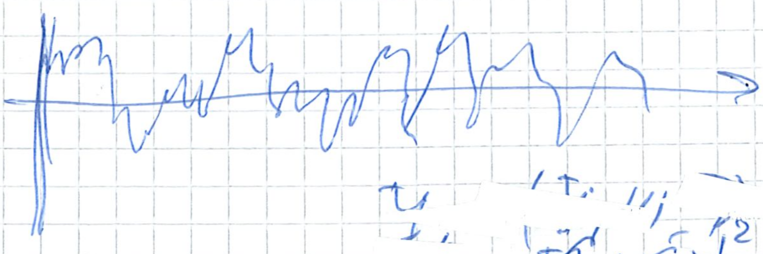
Then $Var(X_t) = \frac{\sigma^2}{2\beta} e^{-2\beta t} - \frac{\sigma^2}{2\beta}(e^{-2\beta t} - 1) = \frac{\sigma^2}{2\beta} = Var(X_0)$!

Moreover, take $X_0 \in N(0, \frac{\sigma^2}{2\beta})$. Then $X_t \in N(0, \frac{\sigma^2}{2\beta})$

is the stationary OU-process:

$$dX_t = \underbrace{-\beta X_t\, dt}_{} + \sigma dB_t \qquad \text{Friction term}$$

i.e. $X_t = e^{-\beta t} X_0 + \sigma e^{-\beta t} \int_0^t e^{\beta s} dB_s$.



[illegible]

4) [illegible]

[illegible]

A random process $X : (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \to (\mathbb{R}, \mathcal{B})$

is a solution of the SDE

$$(*) \begin{cases} dX_t = f(t, X_t)\,dt + g(t, X_t)\,dB_t, & 0 \le t \le T \\ X_0 = x_0 \end{cases}$$

if
- X is adapted
- $\int_0^T f(s, X_s)\,ds + \int_0^T g(s, X_s)\,dB_s < \infty$ $\mathbb{P}$-a.s.
- $X_t = x_0 + \int_0^t f(s, X_s)\,ds + \int_0^t g(s, X_s)\,dB_s$, $0 \le t \le T$ $\mathbb{P}$-as

Thm 5.1.5 Suppose we have

Lipschitz condition: $|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le C \cdot |x-y|$

Growth — " — : $|f(t,x)|^2 + |g(t,x)|^2 \le C(1+|x|^2)$.

Alt: $|f(t,x)| + |g(t,x)| \le C'(1+|x|)$

Then there exists a unique solution $\{X_t\}_{0 \le t \le T}$ of $(*)$. The solution is continuous and

$$\mathbb{E}\left[\sup_{t \le T} |X_t|^2\right] \le \alpha(1 + |X_0|^2), \quad \text{for some } \alpha > 0$$

The multi-dimensional version is in Thm 5.5.2

Example $X_t = \begin{pmatrix} \cos B_t \\ \sin B_t \end{pmatrix}$, $\begin{cases} dX_t = -\frac{1}{2} X_t\,dt + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X_t\,dB_t \\ X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

$d=1$, $n=2$

Note $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$

<u>Proof</u>: ~~{Uniqueness}~~ is based on the Gronwall lemma, Exercise 5.17; v continuous

Consider two solutions: X_t, $X_0 = Z$ $\quad dX_t = b\,dt + \sigma\,dB_t$

$\hat{X}_t$, $\hat{X}_0 = \hat{Z}$ $\quad d\hat{X}_t = \hat{b}\,dt + \hat{\sigma}\,dB_t$

$$E|X_t - \hat{X}_t|^2 = E\left[\left(Z - \hat{Z} + \int_0^t (b - \hat{b})\,ds + \int_0^t (\sigma - \hat{\sigma})\,dB_s\right)^2\right], \ldots$$

Use $(x+y+z)^2 = x^2+y^2+z^2 + \underbrace{2xy}_{\le x^2+y^2} + \underbrace{2xz}_{< x^2+z^2} + \underbrace{2yz}_{< y^2+z^2}$

$\le 3(x^2+y^2+z^2)$ $\qquad 2xy = x^2+y^2-(x-y)^2 \le x^2+y^2$

$$E(|X_t - \hat{X}_t|^2) \le 3\left(E|Z-\hat{Z}|^2 + tE\int_0^t (b-\hat{b})^2\,ds + E\int_0^t (\sigma-\hat{\sigma})^2\,ds\right)$$

$$\overset{\text{Lip}}{\le} 3\left(E|Z-\hat{Z}|^2 + (t+1)D^2\int_0^t E|X_s - \hat{X}_s|^2\,ds\right)$$

Put $v(t) = E[(X_t - \hat{X}_t)^2]$. The v is cont. and

$$v(t) \le C + A\int_0^t v(s)\,ds, \qquad C = 3E|Z-\hat{Z}|^2, \quad A = 3D^2(1+T)$$

Gronwall $\Rightarrow$ $v(t) \le Ce^{At}$, $0 \le t \le T$.

With $Z = \hat{Z}$ we obtain $v(t) = 0$ $\forall t$

Now, $E[(X_t - \hat{X}_t)^2] = 0$ implies $P(X_t = \hat{X}_t) = 1$, $\forall t$

so X and $\hat{X}$ are <u>versions</u> of each other.

~~Hence~~ ~~It~~ Since they are continuous, they are even indistinguishable, i.e

$$P(X_t = \hat{X}_t \ \forall t) = 1$$

F11:5

Existence (Picard iteration, as for ODE)

Put $Y_t^0 = X_0$, and for $k \geq 0$, recursively

$$Y_t^{k+1} = X_0 + \int_0^t b(s, Y_s^k)\,ds + \int_0^t \sigma(s, Y_s^k)\,dB_s$$

Then, as above

$$\sup_{t \leq T} E\left[|Y_t^{k+1} - Y_t^k|^2\right] \leq 3(1+T)D^2 \int_0^T \underbrace{E\left[(Y_s^k - Y_s^{k-1})^2\right]}_{\sup_{r \leq s}} ds$$

and, by induction

$$\sup_{t \leq T} E(\;) \leq \frac{A_2^{k+1} t^{k+1}}{(k+1)!} \to 0, \; k \to \infty$$

Similarly

$$P\left(\sup_t |Y_t^{k+1} - Y_t^k| > \tfrac{1}{2^k}\right) \leq 2^{2(k+1)} D^2 (1+T) \underbrace{\int_0^T E\left(Y_t^k - Y_t^{k-1}\right)^2 dt}_{\leq \int_0^T \frac{A_2^k t^k}{k!} dt}$$

$\therefore \sum_k P(\;) < \infty$

By Borel-Cantelli

$$P\left(\sup_t |Y_t^{k+1} - Y_t^k| > \tfrac{1}{2^k}, \text{ i.o.}\right) = 0$$

Hence $\{Y_t^k\}$ is a Cauchy sequence in $L^2(P)$, uniformly convergent in time.

Hence $X_t = \lim_{k \to \infty} Y_t^{k+1}$ exists and is continuous in time.