LINEAR ALGEBRA III EXAM

Course: 1MA026 **Time:** 2013-03-15 14:00-19:00

Only writing tools are allowed. Solutions may be written in Swedish or English. Motivate your answers carefully. Each problem is worth 5 points. For grade 3/4/5 you will need 18/25/32 (including bonus points from the assignments) of the total score. Good luck!

1. We define an inner product on the vector space $\mathcal{C}[0,1]$ of continuous functions $[0,1] \to \mathbb{R}$ by $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx$.
a) Find an orthonormal basis of the subspace $S = span\{1, x^2\}$.
b) Find the function in S closest to $x^3 + 5$ with respect to our chosen inner product.
SUGGESTED SOLUTION:
We apply Gram-Schmidt to the basis $\{1, x^2\}$.
a) $\langle 1, 1 \rangle = 1$ so we take $e_1 := 1$. $\langle 1, x^2 \rangle = \frac{1}{3}$ so take $e_2' = x^2 - \frac{1}{3} \cdot 1$. Since $\langle e_2', e_2' \rangle = \frac{4}{45}$, we can take $e_2 := \frac{e_2'}{\sqrt{\frac{4}{45}}} = \frac{3\sqrt{5}}{2}(x^2 - \frac{1}{3})$. Thus the set $\{1, \frac{3\sqrt{5}}{2}(x^2 - \frac{1}{3})\}$ is an orthonormal basis for S.
b) We have $\langle x^3 + 5, e_1 \rangle = \frac{21}{4}$ and $\langle x^3 + 5, e_2 \rangle = \frac{\sqrt{5}}{8}$, so the point in S closest to $x^3 + 5$ is the projection

$$\langle x^3 + 5, e_1 \rangle e_1 + \langle x^3 + 5, e_2 \rangle e_2 = \frac{21}{4} + \frac{\sqrt{5}}{8}\frac{3\sqrt{5}}{2}(x^2 - \frac{1}{3}) = \frac{1}{16}(79 + 15x^2).$$

Picture!

2. Find a matrix T which satisfies $\mu_T(t) = (t-1)^2(t-2)$ and $p_T(t) = (\mu_T(t))^2$.
SUGGESTED SOLUTION:
We may assume T is in Jordan form with diagonal $(1, 1, 1, 1, 2, 2)$. The largest Jordan block for the eigenvalue 1 should have size 2 and the largest Jordan block for the eigenvalue 2 should have size 1. Thus we can for example take the matrix

$$T := \begin{pmatrix} 1 & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}$$

3. Consider the real vector space $Mat_n(\mathbb{R})$ of $n \times n$-matrices with real entries. Let

$$\mathcal{S} := \{A \in Mat_n(\mathbb{R}) | A^t = -A\}$$

be the set of skew-symmetric matrices. Prove that
a) The set $\mathcal{S}$ is a subspace of $Mat_n(\mathbb{R})$.
b) We have $Mat_n(\mathbb{R}) = \mathcal{S} \oplus \mathcal{T}$ where $\mathcal{T}$ is the subspace of symmetric matrices.
SUGGESTED SOLUTION:
a) Let $A, B \in \mathcal{S}; \lambda \in \mathbb{R}$. Then $(A+B)^t = A^t + B^t = -A - B = -(A+B)$ so $A + B \in \mathcal{S}$. Similarly, $(\lambda A)^t = \lambda A^t = \lambda(-A) = -(\lambda A)$ so $(\lambda A) \in \mathcal{S}$. Thus $\mathcal{S}$ is a subspace.
b) $A \in \mathcal{S} \cap \mathcal{T} \implies A = A^t = -A \implies A = 0$, so $\mathcal{S} \cap \mathcal{T} = \{0\}$. For any $A \in Mat_n(\mathbb{R})$ we can write $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$ where the first term is symmetric and the second is skew-symemtric. Hence $Mat_n(\mathbb{R}) = \mathcal{S} + \mathcal{T}$. This shows that $Mat_n(\mathbb{R}) = \mathcal{S} \oplus \mathcal{T}$.

4. Let
$$A := \begin{bmatrix} 2 & -2 & 2 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & 3 & -2 & 3 \end{bmatrix}.$$
Then $p_A(t) = (t-2)^4$. Find an invertible matrix S and a matrix J in Jordan form such that $S^{-1}AS = J$.
SUGGESTED SOLUTION:
The only eigenvalue is 2. Let $T = A - 2I$. We know that $T^4 = 0$, and by computing some powers we find that already $T^3 = 0$ but $T^2 \neq 0$. Thus the minimal polynomial of A is $(t-2)^3$, so the largest Jordan block has size 3 and so the remaining block has to have size 1. Thus we take
$$J := \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$
A first vector of a Jordan-chain of lenght 3 is any element in $(Im T^2 \setminus Im T) \cap Ker T$. Computing the corresponding spaces, we pick $v_1 = (0, -1, -1, 1)$. Solving $T^2 v_3 = v_1$ we take $v_3 = (0, 1, 0, 0)$ and then $v_2 = Tv_3 = (-2, 0, 0, 3)$. Finally we take some $w_1 \in Ker\, T$ but not in the span of v_1. We pick $w_1 = (-1, 0, 0, 1)$. We put these vectors as columns in a matrix S in order corresponding to the Jordan form:
$$S := [v_1 \;\; v_2 \;\; v_3 \;\; w_1] = \begin{bmatrix} 0 & -2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}.$$
Then with the matrices J and S as above, we have $S^{-1}AS = J$.

5. Prove that an operator on a finite dimensional vector space is nilpotent if and only if it only has zero as an eigenvalue.
SUGGESTED SOLUTION:
If $T : V \to V$ has some nonzero eigenvalue λ with corresponding eigenvector v, then $T^n v = \lambda^n v \neq 0$ for all $n \geq 0$, so $T^n \neq 0$ and T is not nilpotent. Conversely, assuming all the eigenvalues of T are zero, we have $p_T(t) = t^n$ where $n = \dim V$, and by Cayley-Hamilton, $p_T(T) = T^n = 0$ so T is nilpotent.
Remark: Alternatively one could look at the Jordan form of T and note that SJS^{-1} is nilpotent if and only if J is.

6. Let φ_t be an operator on a complex inner product space V with matrix
$$[\varphi_t] = \begin{pmatrix} 1 & t & 0 \\ 0 & t^2 & t \\ t & 0 & 1 \end{pmatrix}$$
with respect to some orthonormal basis. For which $t \in \mathbb{R}$ does there exist an orthonormal basis for V consisting of eigenvectors of φ_t?
SUGGESTED SOLUTION:
By the complex spectral theorem, an orthonormal basis for V consisting of eigenvectors of φ_t exists if and only if φ_t is normal, that is, if and only if $\varphi_t \varphi_t^* = \varphi_t^* \varphi_t$. Since

the basis is orthonormal this is true if and only if $[\varphi_t][\varphi_t]^* = [\varphi_t]^*[\varphi_t]$. Multiplying the corresponding matrices we see that this is equivalent to

$$\begin{pmatrix} 1+t^2 & t^3 & t \\ t^3 & t^4+t^2 & t \\ t & t & 1+t^2 \end{pmatrix} = \begin{pmatrix} 1+t^2 & t & t \\ t & t^4+t^2 & t^3 \\ t & t^3 & 1+t^2 \end{pmatrix}$$

which holds if and only if $t^3 = t$. Conclusion: An orthonormal basis for V consisting of eigenvectors of φ_t exists precisely for $t \in \{-1, 0, 1\}$.

7. Compute e^B where

$$B = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hint: First write B in Jordan form.

SUGGESTED SOLUTION:

Jordanizing a 2×2 matrix is easy, we find that $B = SJS^{-1}$ with

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Now $J = -I + N$ where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent. Thus $e^A = S^{-1}e^J S = S^{-1}e^{-I+N}S = S^{-1}e^{-I}e^N S = S^{-1}e^{-1}(I+N)S = e^{-1}S^{-1}(I+N)S$. Computing the last matrix product we get $e^A = e^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

8. Let X be a finite set and let $P(X) := \{Y | Y \subset X\}$ be the set of subsets of X. We define addition on $P(X)$ by $A + B := (A \cup B) \backslash (A \cap B)$, and scalar multiplication of $\mathbb{Z}_2$ on $P(X)$ in the only possible way. Then $P(X)$ becomes a vector space over $\mathbb{Z}_2$.

a) What is the additive identity of the vector space?

b) What is the additive inverse of an element S of the vector space?

c) Prove that the vector space is isomorphic to $\mathbb{Z}_2^{|X|}$.

SUGGESTED SOLUTION:

a) The additive identity is the empty set $\varnothing$, since $A + \varnothing = (A \cup \varnothing) \backslash (A \cap \varnothing) = A \backslash \varnothing = A$ for all $A \in P(X)$.

b) The additive inverse of each A is A itself, since $A + A = (A \cup A) \backslash (A \cap A) = A \backslash A = \varnothing = 0$.

c) Let B be the subset of $P(X)$ consisting of singleton sets: $B = \{\{x\} | x \in X\}$. Then each $A \in P(X)$ has a unique expression $A = \sum_{\{x\} \in B} \lambda_{\{x\}} \{x\}$, where $\lambda_{\{x\}}$ is the indicator function $\lambda_{\{x\}} = I(x \in A)$: it is one if x is in A and 0 otherwise. Thus B is a basis of $P(X)$ and $|B| = |X| = \dim P(X)$. This gives an explicit isomorphism $P(X) \to \mathbb{Z}_2^{|X|}$ if we map elements $\{x\}$ bijectivly to the standard basis of $\mathbb{Z}_2^{|X|}$.