Part A

Question	Question title	Marks	Question type
i	Information		Information or resources
i	Part A		Information or resources
1	A1 (1 point)	1	Multiple Response
2	A2 (1 point)	1	Multiple Response
3	A3 (1 point)	1	Inline Gap Match
4	A4 (1 point)	1	Text area
5	A5 (1 point)	1	Inline Gap Match
6	A6 (1 point)	1	Multiple Response
7	A7 (1 point)	1	Inline Gap Match
8	A8 (1 point)	1	Inline Gap Match

Part B

Question	Question title	Marks	Question type
i	Part B		Information or resources
9	B1 (4 points)	4	Multiple Response
10	B2 (4 points)	4	Text area

Information

This exam consists of two parts: part A and part B.

Grade requirements

- Grade 3: At least 5/8 points on part A.
- Grade 4: At least 5/8 points on part A **and** at least 11/16 points **in total** (part A + part B).
- Grade 5: At least 5/8 points on part A **and** at least 14/16 points **in total** (part A + part B).

Note that if you score less than 5 points on part A, you fail the exam. We will not even grade part B.

You are allowed the following aids

- A formula sheet (pdf), which is available as a resource at the bottom of Inspera.
- Beta Mathematics Handbook
- Physics Handbook
- Calculator
- Pen and paper

ℹ Part A

Part A (mandatory)

1 A1 (1 point)

For each IBVP below, determine whether it is well posed.

To score 1 point on this problem, you need 3/3 correct answers.

IBVP 1

$$\begin{cases} u_t + u_x = 0, & 0 < x < 1, \quad t > 0 \\ u(1,t) = 1, & \qquad\qquad\quad\; t > 0 \\ u(x,0) = \sin(x), & 0 \le x \le 1 \end{cases}$$

Is IBVP 1 well posed?

- [ ] Well posed
- [ ] Not well posed ✔

IBVP 2

$$\begin{cases} u_t + u_x = \sin(x)\cos(t), & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, & \qquad\qquad\quad\; t > 0 \\ u(x,0) = \sin(x), & 0 < x < 1 \end{cases}$$

Is IBVP 2 well posed?

- [ ] Well posed ✔
- [ ] Not well posed

IBVP 3

$$\begin{cases} u_t + u_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, & \qquad\qquad\quad\; t > 0 \\ u(1,t) = 0, & \qquad\qquad\quad\; t > 0 \\ u(x,0) = \sin(\pi x), & 0 \le x \le 1 \end{cases}$$

Is IBVP 3 well posed?

- [ ] Well posed
- [ ] Not well posed ✔

Maximum marks: 1

2 A2 (1 point)

Consider the following PDE and boundary conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in (0, L), \quad t > 0 \\ u = 0, & x = 0, \quad t > 0 \\ c^2 u_x = -\alpha u_t, & x = L, \quad t > 0 \end{cases}$$

where $c > 0$ and $\alpha > 0$ are real constants. We assume that the solution is real.

The initial data are assumed to be smooth functions that are compatible with the boundary conditions, but are otherwise considered unknown.

Which of the following relations does the solution $u(x, t)$ satisfy for **any** smooth and compatible initial data? Select all correct alternatives. Zero, one, or more than one alternative may be correct!

To score 1 point on this problem, you need to select **all** correct alternatives and **no** incorrect alternatives.

Select zero or more alternatives:

- ☐ $||u||^2 = 0$
- ☐ $||u||^2 = -\alpha u_t(L, t)^2$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u||^2 \right) = 0$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u||^2 \right) = -\alpha u_t(L, t)^2$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u_t||^2 + c^2 ||u_x||^2 \right) = 0$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u_t||^2 + c^2 ||u_x||^2 \right) = -\alpha u_t(L, t)^2$ ✔
- ☐ $||u_t||^2 = 0$
- ☐ $||u_t||^2 = -\alpha u_t(L, t)^2$

Maximum marks: 1

3 A3 (1 point)

Consider the time-independent advection-diffusion equation,

$$cu_x - \epsilon u_{xx} = F,$$

where $c > 0$ and $\epsilon > 0$ are real constants and $F = F(x)$ is a forcing function. Discretizing using the finite element method with piecewise linear basis functions leads to

$$B\text{\boldsymbol}\xi = \mathbf{F},$$

which is a linear system of equations for $\text{\boldsymbol}\xi$. The right-hand side $\mathbf{F}$ is an $n \times 1$ vector that depends on F. The matrix B is $n \times n$ and satisfies

$$B = cD + \epsilon A,$$

where

$$A = 1/h \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

We are interested in solving the advection-diffusion equation for many different forcing functions F. This will lead to a sequence of linear systems to solve, where B remains the same and $\mathbf{F}$ varies:

$$B\text{\boldsymbol}\xi_i = \mathbf{F}_i, \quad i = 1, 2, \ldots.$$

For this sequence of systems, which of the following three solution methods is the most suitable?

1. Gaussian elimination,
2. LU factorization,
3. The conjugate gradient method.

Rank the three methods. Assume that n is of order 10^5.

To score 1 point, you need to rank all three methods correctly.

In the next problem, you will be asked to motivate your ranking.

Help

Gaussian elimination | LU factorization | The conjugate gradient method

Most suitable: LU factorization ✔

Second-most suitable: Gaussian elimination ✔

Least suitable: The conjugate gradient meth ✔

Maximum marks: 1

4 A4 (1 point)

Motivate your ranking in the previous problem. For each of the three methods (Gaussian elimination, LU factorization, conjugate gradient), explain why you ranked it the way you did.

Fill in your answer here

Maximum marks: 1

5 A5 (1 point)

Error displaying question "A5 (1 point)".
The question seems to be corrupt.

Maximum marks: 1

6 A6 (1 point)

For each PDE below, select the spatial discretization method that would be the **most suitable**. Consider both how easy/difficult it would be to implement the method and what execution time would be required to provide a numerical solution with an error below the prescribed error tolerance.

To score 1 point, you must answer all three questions correctly.

PDE 1
$u_t + u_x = 0, \quad 0 < x < 1, \quad t > 0$
Relative error tolerance: 10^{-6}.
Which method is best for PDE 1?

- A fourth order finite difference method ✔
- The finite element method with piecewise linear basis functions

PDE 2
$\nabla \cdot a\nabla u = F, \quad (x, y) \in \Omega$
where $a = a(x, y)$ and $F = F(x, y)$ are known functions.
Relative error tolerance: 10^{-2}.
Which method is best for PDE 2?

- A sixth order finite difference method
- The finite element method with piecewise linear basis functions ✔

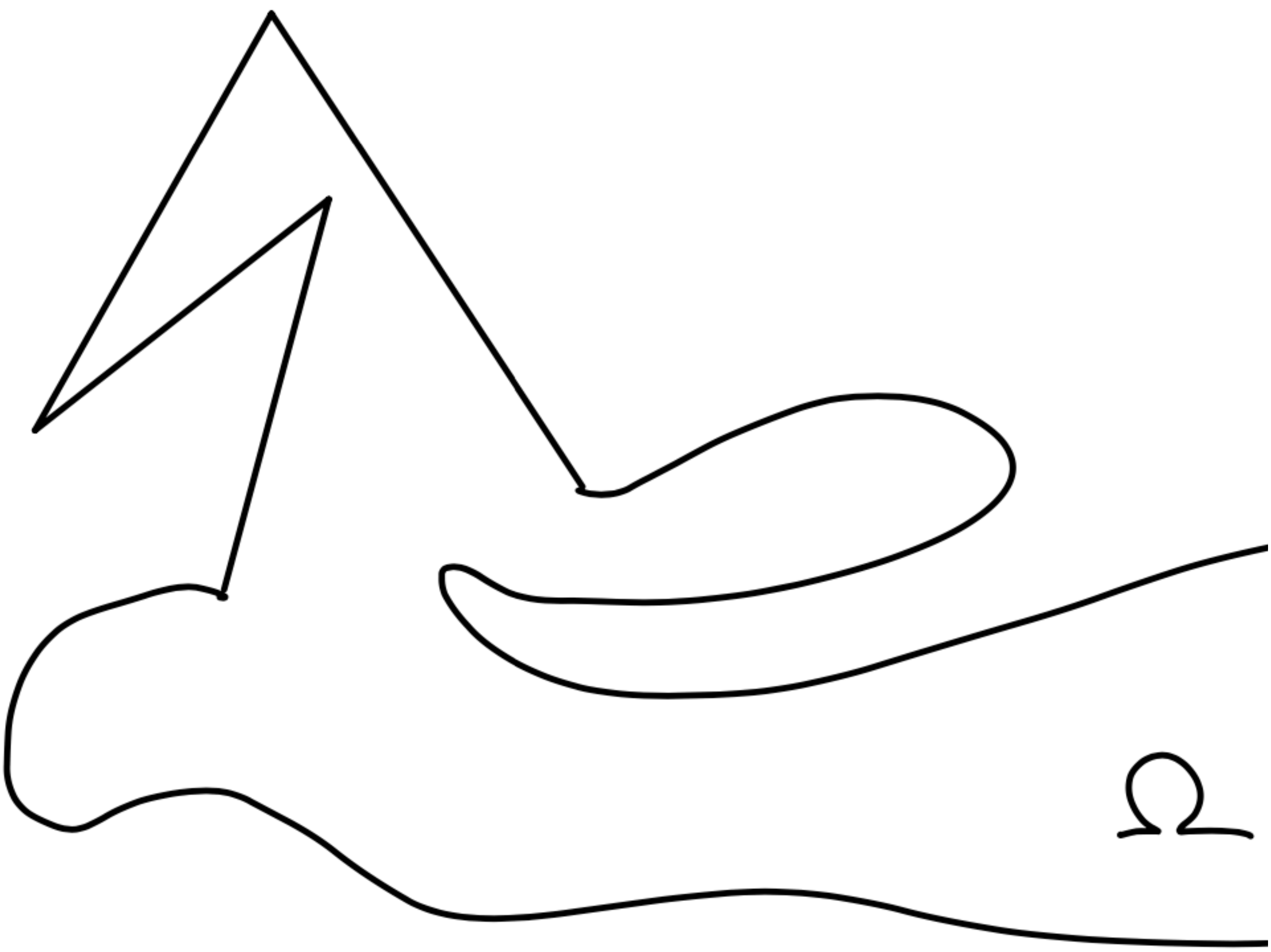

Figure: The domain Ω for PDE 2.

PDE 3
$u_t + i(u_{xx} + u_{yy}) = 0, \quad 0 < x < 1, \ 0 < y < 1, \ t > 0$

Relative error tolerance: 10^{-6}

Which method is best for PDE 3?

- ☐ A sixth order finite difference method ✔
- ☐ The finite element method with piecewise linear basis functions

Maximum marks: 1

7 A7 (1 point)

You are trying to discretize the heat equation,

$$u_t = u_{xx}, \quad x \in (0, L)$$

with the SBP-SAT method. There are three different sets of well-posed boundary conditions (BC) to consider. The BC at the left boundary ($x = 0$) has already been imposed correctly. Your task is to select **consistent and stable** SATs for the right boundary ($x = L$).

The SBP operator D_2 is defined as in the formula sheet that is available as a resource.

BC 1

$u(0,t) = 0$
$u_x(L,t) = 0$

Discretization of BC 1

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT1$

BC 2

$u(0,t) = 0$
$u(L,t) = 0$

Discretization of BC 2

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT2$

BC 3

$u(0,t) = 0$
$u_x(L,t) + \alpha u(L,t) = 0$
where $\alpha > 0$ is a real scalar.

Discretization of BC 3

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT3$

What should SAT1, SAT2 and SAT3 be for the semi-discrete approximations to be **consistent and stable**? Drag and drop boxes below.

To score 1 point, you need to get all three answers right.

Help

$-H^{-1}\mathbf{d}_r(\mathbf{e}_r^T\mathbf{u} - 0)$

$H^{-1}\mathbf{e}_r(\mathbf{e}_r^T\mathbf{u} + \alpha\mathbf{d}_r^T\mathbf{u} - 0)$

$-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} + \alpha\mathbf{e}_r^T\mathbf{u} - 0)$

$-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} - 0)$

$H^{-1}\mathbf{e}_r(D_1\mathbf{u} - 0)$

SAT 1: $-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} - 0)$ ✔

SAT 2: $-H^{-1}\mathbf{d}_r(\mathbf{e}_r^T\mathbf{u} - 0)$ ✔

SAT 3: $-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} + \alpha\mathbf{e}_r^T\mathbf{u} - 0)$ ✔ .

Maximum marks: 1

8 A8 (1 point)

Consider the finite difference approximations below, where h denotes the grid spacing. Drag and drop boxes to pair the approximations with the correct description.

To score 1 point, you have to get all three boxes right.

Help

First order approximation of $u'(x)$	Fourth order approximation of $u'(x)$
First order approximation of $u''(x)$	Second order approximation of $u''(x)$
First order approximation of $u''''(x)$	Not a consistent approximation of any derivative
Fourth order approximation of $u''(x)$	Second order approximation of $u'(x)$

$\dfrac{u(x+h)-u(x)}{h}$: First order approximation of $u'(x)$

$\dfrac{u(x+h)-u(x-h)}{2h}$: Second order approximation of $u'(x)$

$\dfrac{u(x+h)-2u(x)+u(x-h)}{h^2}$: Second order approximation of $u''(x)$.

Maximum marks: 1

i Part B

Part B (for grades 4 and 5).

The questions in part B will only be graded if you score at least 5/8 on part A.

9 B1 (4 points)

Consider the following PDE and initial condition:

$$\begin{aligned} &u_t = u_{xx} - u_x, \quad 0 \le x \le 1, \quad t \ge 0\ , \\ &u = f, \quad\quad\quad\quad 0 \le x \le 1, \quad t = 0\ . \end{aligned}$$

We assume that the solution is real.

Which of the following choices of boundary conditions lead to a well-posed initial-boundary value problem?

Select one or more alternatives:

- $\begin{cases} 2u_x - u = 0, & x = 0\ , \\ u_x = 0, & x = 1\ . \end{cases}$ ✔
- $\begin{cases} u = 0, & x = 0\ , \\ u_x - u = 0, & x = 1\ . \end{cases}$
- $\begin{cases} u_x = 0, & x = 0\ , \\ u_x = 0, & x = 1\ . \end{cases}$
- $\begin{cases} u = 0, & x = 0\ , \\ u = 0, & x = 1\ . \end{cases}$ ✔
- $\begin{cases} u_x - u = 0, & x = 0\ , \\ u_x + u = 0, & x = 1\ . \end{cases}$ ✔

Maximum marks: 4

10 B2 (4 points)

Consider the advection equation with wave speed $c > 0$,

$$u_t + cu_x = 0, \quad 0 < x < L,$$

with periodic boundary conditions. Introduce a grid

$$x_j = jh, \quad j = 0, 1, \ldots N-1,$$

where $h = \frac{L}{N}$ is the grid spacing. Consider the following two finite difference approximations in space:

Semi-discrete approximation 1:

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + c\frac{u_{j+1} - u_{j-1}}{2h} = 0$$

Semi-discrete approximation 2:

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + c\frac{u_{j+1} - u_{j-1}}{2h} = \varepsilon h \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

where ε is a scalar parameter that we may select the value of. Here, u_j denotes the numerical solution at grid point j such that $u_j(t) \approx u(x_j, t)$.

Explain the differences between approximation 1 and approximation 2 for $\varepsilon > 0$ and $\varepsilon < 0$. Compare the methods in terms of stability and accuracy. For approximation 2, describe how you expect the numerical solution to behave for $\varepsilon > 0$ and for $\varepsilon < 0$.

Fill in your answer here:

Maximum marks: 4