UPPSALA UNIVERSITET
Matematiska institutionen — Financial Derivatives
Erik Ekström — 2024-08-30

Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points. No means of assistance other than pencil and paper are allowed. Motivate your answers carefully!

1. Determine $\mathbb{E}[\cos(y+2W_t)]$, where W is a Brownian motion.

2. Solve the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x,y) + 2\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(t,x,y) - \frac{\partial^2 u}{\partial x \partial y}(t,x,y) = 0 \\ u(T,x,y) = x + (x-y)^2. \end{cases}$$

3. Consider the standard Black-Scholes model with volatility $\sigma = 0.2$, interest rate $r = 0.05$ and with initial stock price $S_0 = 20$. Determine the arbitrage-free price at time 0 of a contract which at time $T = 1$ pays the holder the amount

$$\mathcal{X} = \int_{1/2}^{1} S_t \, dt.$$

4.

(i) Let W^1 and W^2 be Brownian motions with instantaneous correlation ρ (i.e. $dW_t^1 dW_t^2 = \rho\, dt$). Determine a constant $b > 0$ such that the process $W(t) := b(2W_t^1 + 3W_t^2)$ is a Brownian motion.

(ii) Explain briefly the following notions:
   (a) replication of a T-claim;
   (b) gamma of an option;
   (c) inversion of the yield curve.

5. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time 0 of a contract which at time T pays the holder the amount S_T provided $S_T \geq a$, and 0 otherwise.

6. In a market consisting of a bank account with a constant interest rate r and a non-dividend paying stock S, consider a T-claim that pays

$$\mathcal{X} = \frac{S_{T_0} + S_T}{2}$$

at time T, where $0 < T_0 < T$.

a) Find a replicating strategy for $\mathcal{X}$.
b) What is the arbitrage-free price of $\mathcal{X}$ at time 0?
c) What is the arbitrage-free price of $\mathcal{X}$ at time $t \in (T_0, T)$?

7. A certain T-claim on an underlying asset S pays its holder the amount

$$\mathcal{X} = \begin{cases} 50 - S_T & \text{if } S_T \leq 25 \\ S_T & \text{if } S_T > 25 \end{cases}$$

at time T. The claim $\mathcal{X}$ trades at price 30, a zero-coupon T-bond with face value 50 trades at 48, and a call option on S with maturity T and strike price 25 trades at 5. Show how to construct an arbitrage if the current stock price is 26.

8. Consider the Ho-Lee model

$$\begin{cases} dr_t = \Theta(t)\,dt + \sigma\,dW_t, \quad t \geq 0 \\ r_0 = r \end{cases}$$

for the short rate under the pricing measure, where $\Theta(\cdot)$ is a function of time and σ is a constant. Determine the term structure, i.e. calculate bond prices at time $t = 0$ in this model for all possible maturities T. Explain briefly the advantage of allowing Θ to be time-dependent.

GOOD LUCK!