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Financial Derivatives
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Assignment 1

Please hand in your solutions by November 28. To obtain two (one) bonus points at the exam (only the exam in January 2025), a total of 80% (60%) of the total score in Assignments 1-2 are required. Hand in your solutions during the lecture or in my mail box (House 1, Floor 4). Solutions should be hand-written or in pdf-format.

1. Define $\alpha_k(t) = \mathbb{E}[W_t^k]$ for $k = 0, 1, 2, ...$

(a) Use Ito's formula to show that

$$\alpha_k(t) = \frac{1}{2}k(k-1)\int_0^t \alpha_{k-2}(s)\, ds.$$

(b) Deduce that $E[W_t^4] = 3t^2$.

(c) Find $E[W_t^6]$.

2. Consider the SDE

$$\begin{cases} dX_t = (b - aX_t)\, dt + \sigma\, dW_t \\ X_0 = x_0, \end{cases}$$

where a, b, σ and x_0 are constants (X is then an Ornstein-Uhlenbeck process).

(a) Solve the equation.
(Hint: Let $Y_t = e^{at}(X_t - b/a)$.)

(b) Determine the expected value $\mathbb{E}[X_t]$.

(c) Determine $Var(X_t)$.

3. Solve the terminal value problem

$$\begin{cases} \frac{\partial F(t,x,y)}{\partial t} + \frac{9}{2}\frac{\partial^2 F(t,x,y)}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F(t,x,y)}{\partial y^2} + \frac{\partial^2 F(t,x,y)}{\partial x \partial y} + \frac{\partial F(t,x,y)}{\partial x} = 0 \quad (t,x,y) \in [0,T) \times \mathbb{R}^2 \\ F(T,x,y) = xy^2. \end{cases}$$

4. Solve the terminal value problem

$$\begin{cases} \frac{\partial F(t,x)}{\partial t} + 2x^2\frac{\partial^2 F(t,x)}{\partial x^2} - x\frac{\partial F(t,x)}{\partial x} + 2x = 0 \quad (t,x) \in [0,T) \times \mathbb{R} \\ F(T,x) = x^3. \end{cases}$$

Hint: You may use Exercise 5.10 from the book.