GRAPH THEORY – EXAM

25 AUGUST 2022

Please note: This exam consists of 5 problems, worth 8p each. In order to pass this part of the course examination, you will need to obtain at least 18 points out of 40.

For solving this exam, no aids are allowed. When using colours, please abstain from using the colour red.

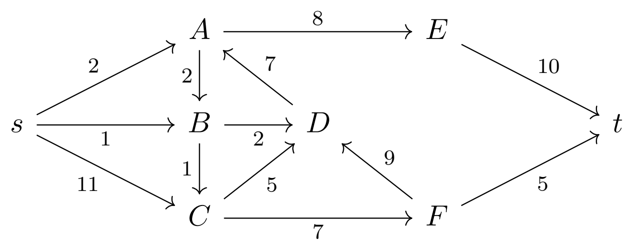
Unless stated otherwise, all graphs in this exam are assumed to be finite simple graphs.

– Good luck!

Problem 1. For each of the following statements, decide whether they are right or wrong, and prove (or disprove) them. (2p each)

(a) If a tree on $n \geq 2$ vertices has no vertices of degree 2, then it contains at least $n/2$ leaves.
(b) A k-regular graph must have an even number of vertices for $k = 27$.
(c) Any 3-regular graph has chromatic number at least 3.
(d) For every $k \geq 2$ there exists a k-regular graph that is Hamiltonian.

Problem 2. Consider the following flow network:



(a) Employ the Ford-Fulkerson algorithm to find a maximum flow. Your solution should contain all relevant steps of the algorithm. (3p)
(b) Find a minimum s-t-cut in the above network. (1p)
(c) Let G be a graph, and let $\prec$ be an ordering of its vertices. Describe the greedy colouring algorithm. (2p)
(d) Show that the ordering in (c) can always be chosen in a such a way that the greedy colouring algorithm requires exactly $\chi(G)$ colours. (2p)

Problem 3. (a) State and prove Hall's marriage theorem about matchings in bipartite graphs. (3p)
(b) Show that a tree has at most one perfect matching. (2p)
(c) Give an example of a graph that is *not* a tree and has a unique perfect matching. (You need to show that the perfect matching is unique!) (3p)

Problem 4. (a) State Euler's formula, relating the number of vertices, edges, and faces of a planar connected graph $G = (V, E)$. (1p)

(b) Use Euler's formula to show that K_5 and $K_{3,3}$ are non-planar. *Hint:* Show first that Euler's formula implies $|E| \leq 3|V| - 6$ for planar graphs on at least 3 vertices, and $|E| \leq 2|V| - 4$ for planar bipartite graphs on at least 3 vertices. (3p)

(c) State the Wagner-Kuratowski theorem(s). (2p)

(d) Show that the Petersen graph is non-planar. (2p)

Problem 5. (a) How is the random graph $G(n, p)$ constructed? (2p)

(b) We discussed in the lectures that $p(n) = n^{-2/(k-1)}$ is the critical function for the appearance of K_k as a subgraph in $G(n, p)$. Explain what this statement means. (2p)

(c) Recall the first-moment-method, stating that for a random variable X taking only values in the non-negative integers, we have $\mathbf{P}[X > 0] \leq \mathbf{E}[X]$. Apply this method to show that for $p(n) = n^{-1+\varepsilon}$ (where $0 < \varepsilon < 1$ is an arbitrary constant), the probability of having an isolated vertex in $G(n, p)$ goes to zero as $n \to \infty$. *Hint:* You may use without proof that $n(1 - n^{-1+\varepsilon})^{n-1} \to 0$ as $n \to \infty$. (2p)

(d) For any $p = p(n)$, what can be said about the probability that $G(n, p)$ contains both an isolated vertex and a K_4, as $n \to \infty$? (2p)