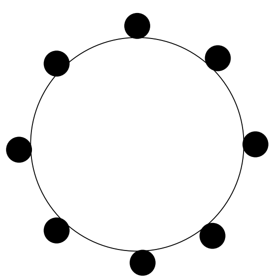
2-Connected Graphs

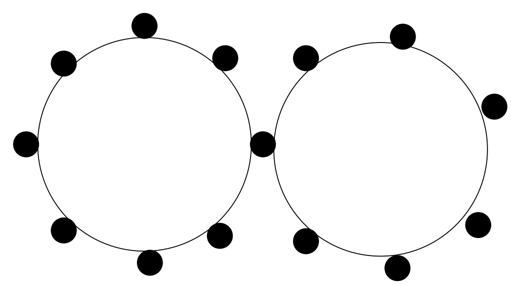
Definition 1

A graph is **connected** if for any two vertices $x, y \in V(G)$, there is a path whose endpoints are x and y.

A connected graph G is called **2-connected**, if for every vertex $x \in V(G)$, $G - x$ is connected.

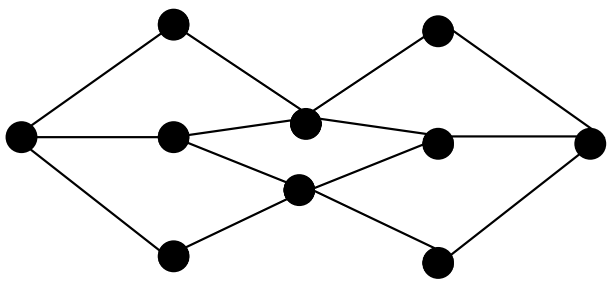


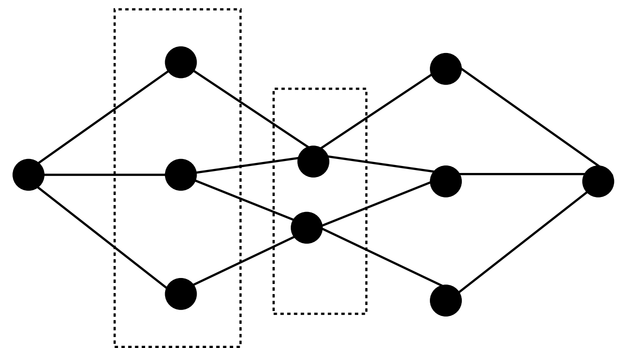
2−connected graph



1−connected graph

A **separating set** or **vertex cut** of a connected graph G is a set $S \subset V(G)$ such that $G - S$ is disconnected.

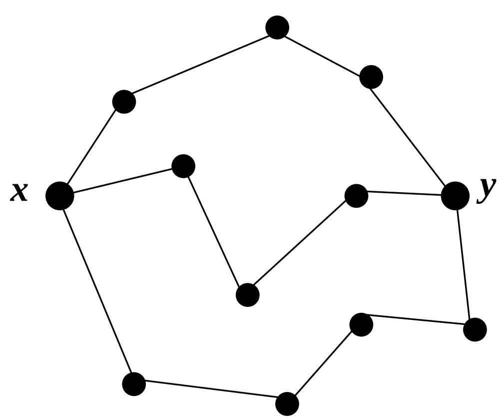




The **connectivity** of G, denoted $\kappa(G)$ is the smallest size of a vertex set S such that $G - S$ is disconnected or has only one vertex.

Two paths connecting two given vertices x and y are called **internally disjoint** if x and y are the only common vertices for the paths.

A **disconnecting set** of edges F is such that $G - F$ has more connected components than G does.



A connected graph G is called k-**edge-connected** if every disconnecting edge set has at least k edges.

The **edge-connectivity** of a connected graph G, written $\kappa'(G)$, is the minimum size of a disconnecting set.

An **edge cut** is a set of edges of the form $[S, \overline{S}]$ for some $S \subset V(G)$. Here $[S, \overline{S}]$ denotes the set of edges xy, where $x \in S$ and $y \in \overline{S}$.

Theorem 1 *(Whitney, 1927) A connected graph G with at least three vertices is 2-connected iff for every two vertices $x, y \in V(G)$, there is a cycle containing both.*

Proving $\Leftarrow$ (*sufficient condition*): If every two vertices belong to a cycle, no removal of one vertex can disconnect the graph.

Proving $\Rightarrow$ (*necessary condition*): If G is 2-connected, every two vertices belong to a cycle.

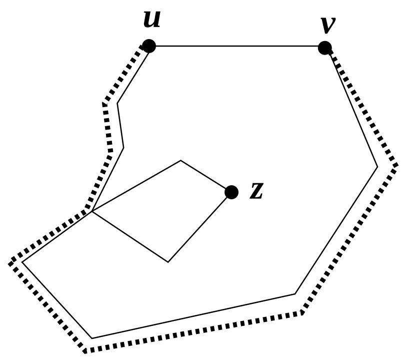
We will prove it by induction on the distance $dist(u, v)$ between two vertices in the graph.

BASE.

Since the vertices are distinct, the smallest distance is 1. This means that the vertices u and v are adjacent.

Let z be any vertex in G different from u and v. Because of the removal of u (resp. v) does not disconnect G, there is a path P_1 (resp. P_2) which connects u (resp. v) with z and does not contain v (resp. u).

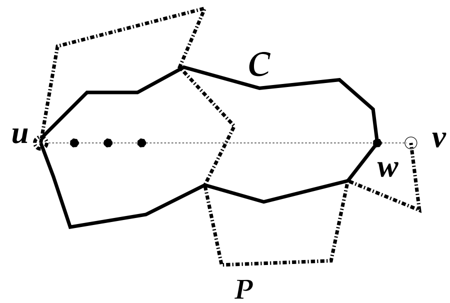
The cycle containing u and v consists of the edge (u, v) and a path from u to v obtained from the walk from v to z, using P_2 followed by the reverse of the path P_1 from z to u (see figure below.

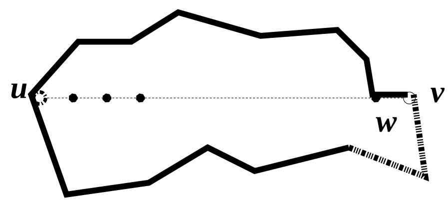


INDUCTIVE STEP.

Now, let the proposition be true for all pairs of vertices on the distance $\leq k$, and let $dist(u, v) = k + 1$, where $k \geq 0$. Consider the shortest path from u to v and let w be the vertex on the path which is adjacent to v. Since $dist(u, w) = k$, there is a cycle C containing u and w.

Furthermore, since the removal of w does not disconnect u from v, let here is a path P which connects u with v but does not contain w. A cycle containing u and v can be constructed from C and P the way it is illustrated in the figure below (*give a rigorous description !*).





Problem 1 *Let G be a connected graph, and let H be obtained from G by adding edges xy iff $dist_G(x, y) = 2$. Prove that H is 2-connected.*

Problem 2 *Let graph G satisfy the following condition: for every edge xy there are two cycles C_1 and C_2 such that their intersection is xy. Prove that G is 3-edge-connected.*

Problem 3 *Prove that Petersen graph is 3-edge-connected*