19. (X_t) is a birth pr. with birth rates $\lambda_k = \lambda k$, $k \geq 1$

Basic ex. with $X_0 = 1$.

Forw. eq:

$$P'_{1k}(t) = \sum_j P_{1j}(t) \underbrace{q_{jk}}_{\substack{\lambda(k-1),\ j=k-1 \\ -\lambda k,\ j=k}}$$

$$\overset{(*)}{=} \lambda(k-1) P_{1,k-1}(t) - \lambda k P_{1k}(t)$$

So

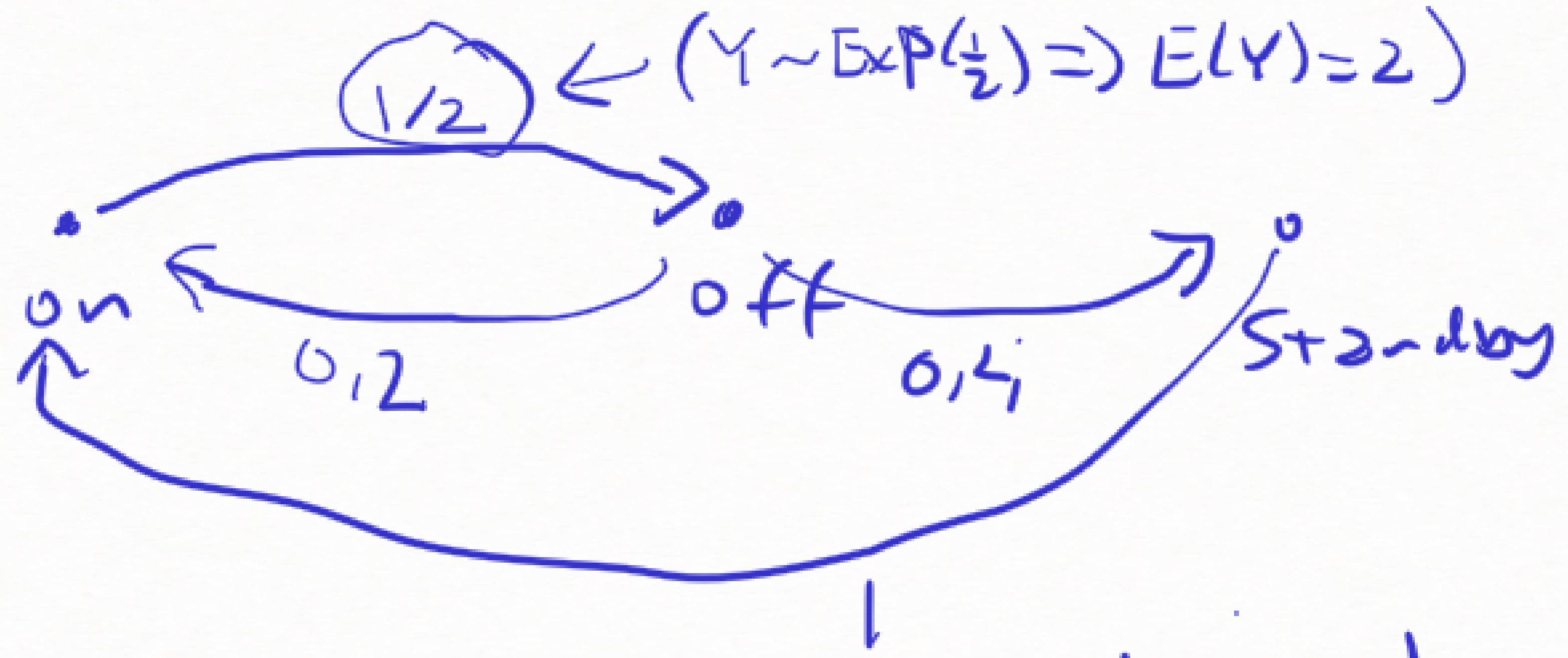
$$\frac{P_{1k}(t+h) - \cancel{P_{1k}(t)}}{\cancel{h}} = \underbrace{\left(\lambda(k-1)P_{1,k-1}(t) - \lambda k P_{1k}(t) + \frac{o(h)}{h}\right)h + P_{1k}(t)}_{RHS}$$

If $P_{1k}(t) = \underbrace{e^{-\lambda t}}_{(1-(1-e^{-\lambda t}))}(1-e^{-\lambda t})^{k-1}$ $= (1-e^{-\lambda t})^{k-1} - (1-e^{-\lambda t})^k$

then

By taking der. on both sides we get $(*)$

21



If X_t = state of switch at time t (in seconds) then (X_t) is has the above ~~tr.~~ diagram and thus (X_t) has generator

$$Q = \begin{array}{c|c} & \begin{matrix} on & off & sb. \end{matrix} \\ \begin{matrix} on \\ off \\ sb. \end{matrix} & \begin{pmatrix} -1/2 & 1/2 & 0 \\ 0,2 & -0.6 & 0,4 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

Note

(X_t) is irreducible.

By MP conv. thm it follows that

$P(X_t = \text{"on"}) \to \pi_0$
$P(X_t = \text{"off"}) \to \pi_1$ as $t \to \infty$ where
$P(X_t = \text{"s.b"}) \to \pi_2$

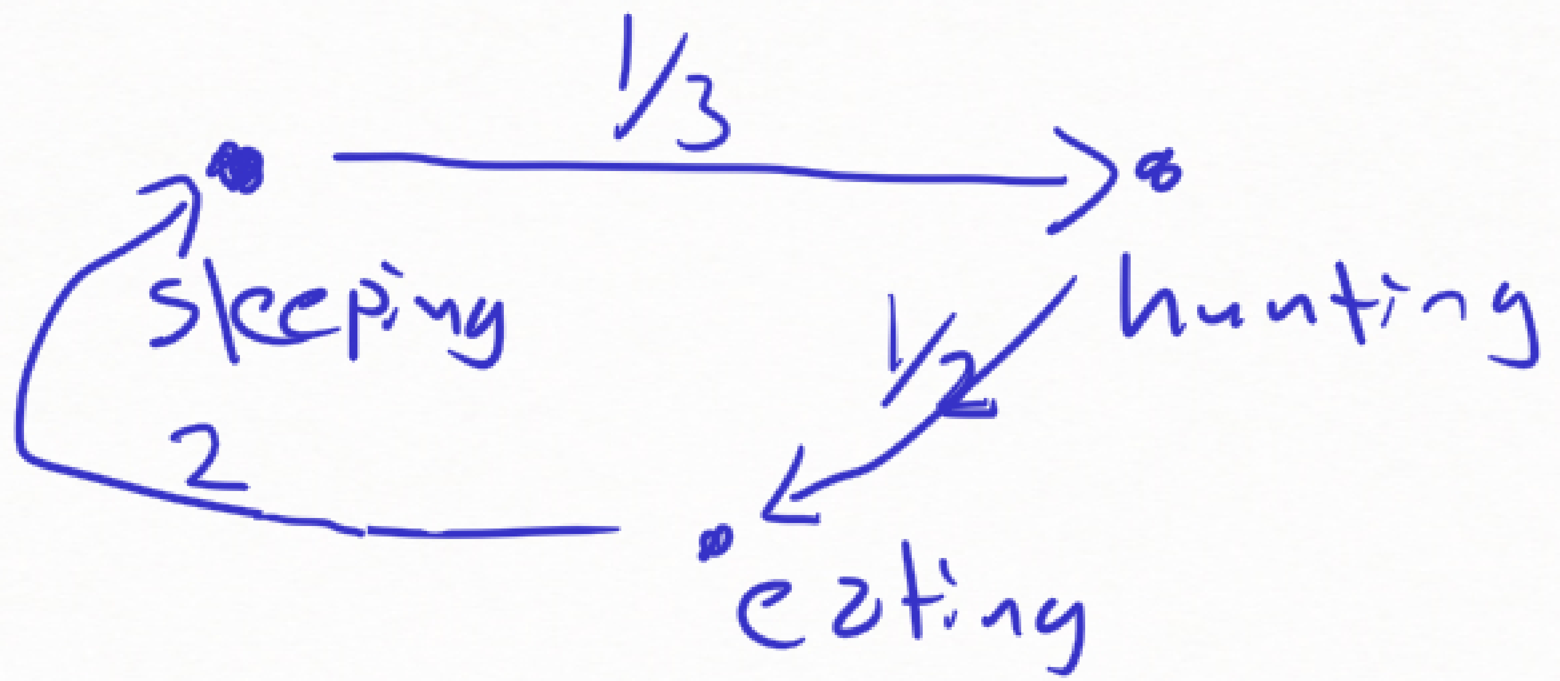
$\pi = (\pi_0, \pi_1, \pi_2)$ solves $\pi Q = 0$
$\uparrow$
Prob vector

$\Leftrightarrow$

$$\begin{cases} -\frac{\pi_0}{2} + 0,2\pi_1 + \pi_2 = 0 \\ \frac{\pi_0}{2} - 0,6\pi_1 = 0 \\ 0,4\pi_1 - \pi_2 = 0 \end{cases} \qquad \sum \pi_i = 1, \quad \pi_i \geq 0$$

so $\pi = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right)$

22



If X_t denotes state of tiger at time t (measured in hours), then (X_t) is a MP with stationary distr

$$\pi = (\pi_{sleep.}, \pi_{hunt}, \pi_{eat})$$

$$= \frac{1}{5.5}\left(3, 2, \tfrac{1}{2}\right) = \left(\tfrac{6}{11}, \tfrac{4}{11}, \tfrac{1}{11}\right)$$

and model graph as above

23) (X_t) BD-process with birthrates $\lambda_i = \frac{\lambda}{\sqrt{i+1}}$

& deathrates $\mu_i = \mu\sqrt{i}$

($\lambda > 0$ $\mu > 0$)

By the conv. thm for irr. BD-process we know that

$P_{ij}(t) \to \pi_j$ as $t \to \infty$ where

$$\pi_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} \pi_0 \overset{\text{here}}{=} \frac{\frac{\lambda}{\sqrt{1}} \frac{\lambda}{\sqrt{2}} \cdots \frac{\lambda}{\sqrt{j}}}{\mu\sqrt{1}\,\mu\sqrt{2} \cdots \mu\sqrt{j}} \pi_0 = \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!} \pi_0$$

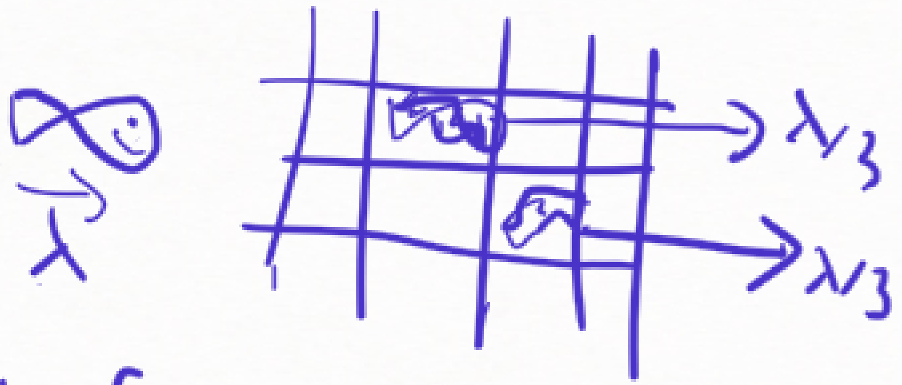
*

and $\pi_0 = \Big(\underbrace{1 + \frac{\left(\frac{\lambda}{\mu}\right)^1}{1!} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} + \cdots}_{e^{\lambda/\mu}}\Big)^{-1} = e^{-\lambda/\mu}$

so $\pi_j = \frac{(\lambda/\mu)^j}{j!} e^{-\lambda/\mu}$ $j \geq 0$ so $\pi = (\pi_j)$ is the $Po\left(\frac{\lambda}{\mu}\right)$ distribution

24) BD process with $\lambda_i = \lambda$, $\mu_i = \mu i$ (M/M/∞ model) $\Rightarrow \pi_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} \pi_0 = * \cdot \pi_0$ so π is $Po\left(\frac{\lambda}{\mu}\right)$

28



Let $X_t = \#$ fishes in net at time t
(measured in hours)

Then (X_t) is a BD-process with (M/M/∞) model

birth rates $\lambda_i = \lambda$ and

death rates $\mu_i = \frac{\lambda i}{3}$

ex. 24

with stationary distr. $\pi \sim Po\left(\underbrace{\frac{\lambda}{\lambda/3}}_{3}\right)$

By the conv. thm we have that

$$P(X_t \geq 4) \to \sum_{k=4}^{\infty} \pi_k = 1 - \underbrace{\pi_0}_{e^{-3}\frac{3^0}{0!}} - \underbrace{\pi_1}_{e^{-3}\frac{3^1}{1!}} - \underbrace{\pi_2}_{e^{-3}\frac{3^2}{2!}} - \underbrace{\pi_3}_{e^{-3}\frac{3^3}{3!}} \quad \text{as } t \to \infty$$

$$= 1 - e^{-3}\underbrace{\left(1 + 3 + \frac{9}{2} + \frac{9}{2}\right)}_{13} = 1 - 13e^{-3} \approx 0{,}35$$

27

a) A fails with int. 2/day

B —||—

repair int $= \frac{24}{4} = 6/\text{day}$

States	{no machine works,	only A works,	only B works,	both machines work}
Repair man works on	A	B	A	—

If X_t = state of mach after t days

(X_t) is a MP with tr. diagram