Analysis of Time Series, L11

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Today

- 4.3: Periodogram and Discrete Fourier Transform (DFT)
- 4.4: Nonparametric Spectral Estimation
- 4.5: Parametric Spectral Estimation
- Menti

Periodogram and DFT

Observations $x_1, \ldots, x_n$.

Definition (4.1)

The *discrete Fourier transform* (DFT) is defined as

$$d(\omega_j) = n^{-1/2} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t}, \quad j = 0, 1, \ldots, n-1,$$

where the frequencies $\omega_j = j/n$ are called the *Fourier* or *fundamental frequencies*.

Inverse DFT

$$x_t = n^{-1/2} \sum_{j=0}^{n-1} d(\omega_j) e^{2\pi i \omega_j t}.$$

Periodogram and DFT

Observations $x_1, \ldots, x_n$, $\omega_j = j/n$.

Definition (4.2)

The *periodogram* is defined as

$$I(\omega_j) = |d(\omega_j)|^2, \quad j = 0, 1, \ldots, n-1.$$

- Assume $j \neq 0$. It follows that (why?)

$$I(\omega_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) e^{-2\pi i \omega_j h}.$$

- Recall: for the spectral density, we have

$$f(\omega_j) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega_j h}.$$

Periodogram and DFT

The periodogram $I(\omega_j)$ is an estimate of the spectral density $f(\omega_j)$. Properties?

- Let $\omega_{j:n}$ be a sequence of fundamental frequencies such that $\omega_{j:n} \to \omega$ as $n \to \infty$.
- Asymptotic unbiasedness: as $n \to \infty$,

$$E\{I(\omega_{j:n})\} \to f(\omega).$$

- Asymptotic distribution: For a linear process, as $n \to \infty$,

$$I(\omega_{j:n}) \xrightarrow{d} \frac{\chi_2^2}{2} f(\omega).$$

- Approximate $1-\alpha$ confidence interval

$$\frac{2I(\omega_{j:n})}{\chi_2^2\left(1-\frac{\alpha}{2}\right)} \leq f(\omega) \leq \frac{2I(\omega_{j:n})}{\chi_2^2\left(\frac{\alpha}{2}\right)}$$

Periodogram and DFT

- Let $\omega_{j:n}$ be a sequence of fundamental frequencies such that $\omega_{j:n} \to \omega$ as $n \to \infty$.
- Asymptotic distribution: For a linear process, as $n \to \infty$,

$$I(\omega_{j:n}) \xrightarrow{d} \frac{\chi_2^2}{2} f(\omega).$$

- Hence,

$$\mathrm{var}\{I(\omega_{j:n})\} \to f(\omega)^2 \neq 0.$$

- The periodogram is *not* a consistent estimator of the spectral density!
- The solution is smoothing!

Nonparametric Spectral Estimation

- Smoothed periodogram

$$\bar{f}(\omega) = \frac{1}{L} \sum_{k=-m}^{m} I\left(\omega_j + \frac{k}{n}\right),$$

where $L = 2m + 1$ and $\omega_j = j/n$ is close to ω.

- For large n,

$$\bar{f}(\omega) \approx \frac{\chi^2_{2L}}{2L} f(\omega).$$

- Hence,

$$\mathrm{var}\{\bar{f}(\omega)\} \approx \frac{1}{L} f(\omega)^2 \to 0 \quad \text{as } L \to \infty.$$

- Dilemma:
  - L large gives small variance, large bias.
  - L small gives small bias, large variance.

Nonparametric Spectral Estimation

- For large n,

$$\bar{f}(\omega) \approx \frac{\chi^2_{2L}}{2L} f(\omega).$$

- Approximate $1-\alpha$ confidence interval

$$\frac{2L\bar{f}(\omega)}{\chi^2_{2L}\left(1-\frac{\alpha}{2}\right)} \leq f(\omega) \leq \frac{2L\bar{f}(\omega)}{\chi^2_{2L}\left(\frac{\alpha}{2}\right)}.$$

- Equivalent to

$$\log\{\bar{f}(\omega)\} - \log\left\{\frac{\chi^2_{2L}\left(1-\frac{\alpha}{2}\right)}{2L}\right\} \leq \log\{f(\omega)\}$$

$$\leq \log\{\bar{f}(\omega)\} + \log\left\{\frac{2L}{\chi^2_{2L}\left(\frac{\alpha}{2}\right)}\right\}.$$

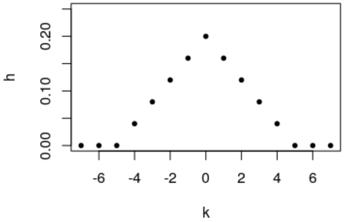
Nonparametric Spectral Estimation

More general:

- Smoothed periodogram

$$\hat{f}(\omega) = \sum_{k=-m}^{m} h_k I\left(\omega_j + \frac{k}{n}\right),$$

  where $\sum_{k=-m}^{m} h_k = 1$.
- Example of $\{h_k\}$:



Nonparametric Spectral Estimation

- Smoothed periodogram, where $\sum_{k=-m}^{m} h_k = 1$,

$$\hat{f}(\omega) = \sum_{k=-m}^{m} h_k I\left(\omega_j + \frac{k}{n}\right),$$

- Assume that if $n, m \to \infty$ such that $\frac{m}{n} \to 0$, then $\sum_{k=-m}^{m} h_k^2 \to 0$.
- If so, then as $n \to \infty$,

$$E\left\{\hat{f}(\omega)\right\} \to f(\omega)$$

and

$$\hat{f}(\omega) \approx \frac{\chi^2_{2L_h}}{2L_h} f(\omega), \quad L_h = \left(\sum_{k=-m}^{m} h_k^2\right)^{-1}.$$

- With $h_k = 1/L$, we have $\sum_{k=-m}^{m} h_k^2 = 1/L$ and $L_h = L$.

Nonparametric Spectral Estimation

- Smoothed periodogram

$$\hat{f}(\omega) = \sum_{|k| \leq m} h_k I(\omega_j + k/n)$$

- Inserting $I(\omega) = \sum_{|h|<n} \hat{\gamma}(h) e^{-2\pi i \omega h}$ with $\omega = \omega_j + k/n$ yields (why?)

$$\hat{f}(\omega) = \sum_{h} g(h/n) \hat{\gamma}(h) e^{-2\pi i \omega_j h},$$

where $g(h/n) = \sum_{|k| \leq m} h_k e^{-2\pi i k h/n}$.

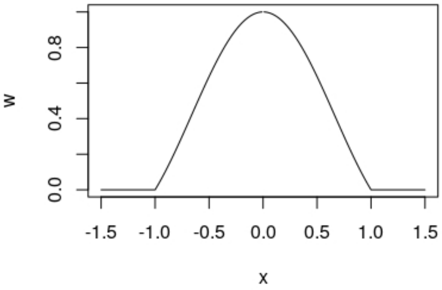
- Suggests estimators of the form

$$\tilde{f}(\omega) = \sum_{|h| \leq r} w(h/r) \hat{\gamma}(h) e^{-2\pi i \omega h}.$$

- The function $w(x)$ is called the *lag window*.

Nonparametric Spectral Estimation

- The lag window $w(\cdot)$ has to satisfy
  - (*i*) $w(0) = 1$,
  - (*ii*) $|w(x)| \leq 1$ and $w(x) = 0$ for $|x| > 1$,
  - (*iii*) $w(x) = w(-x)$.
- Example of $w(x)$:



Nonparametric Spectral Estimation

- The *smoothing window*

$$W_r(\omega) = \sum_{|h| \leq r} w(h/r) e^{-2\pi i \omega h}.$$

- Inversion formula (not in the book)

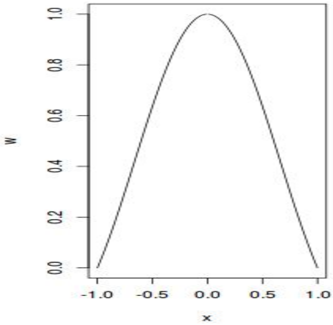
$$w(h/r) = \int_{-1/2}^{1/2} W_r(\omega) e^{2\pi i \omega h} d\omega.$$

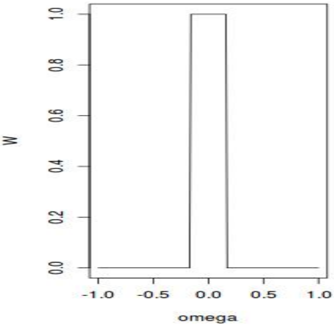
- Corresponds to $g(h/n) = \sum_{|k| \leq m} h_k e^{-2\pi i k h/n}$, i.e $W_r(\omega)$ is the "continuous counterpart" of h_k.
- $W_r(\omega)$ "smooths over frequencies" and $w(x)$ "smooths over lags".

Nonparametric Spectral Estimation

Default in R: The sinc lag window, Daniell smoothing window (here $r = 3$)

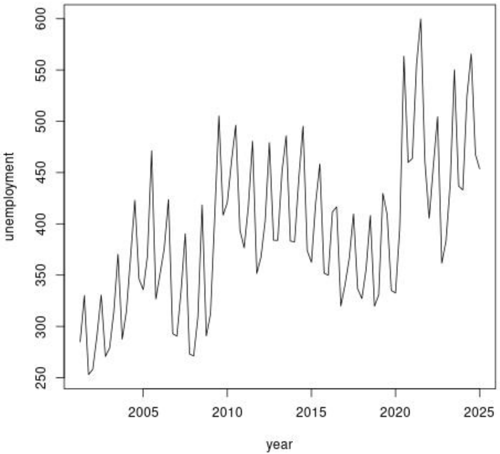
$$w(x) = \frac{\sin(\pi x)}{\pi x}, \qquad W_r(\omega) = \begin{cases} r, & |\omega| \leq 1/(2r), \\ 0, & \text{otherwise.} \end{cases}$$



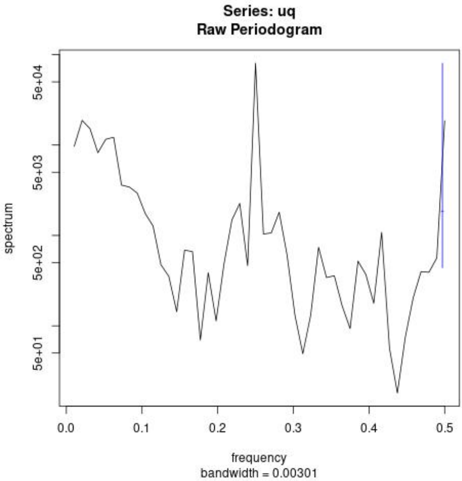


Nonparametric Spectral Estimation

Unemployment, quarterly data (period length 4, and maybe 40).



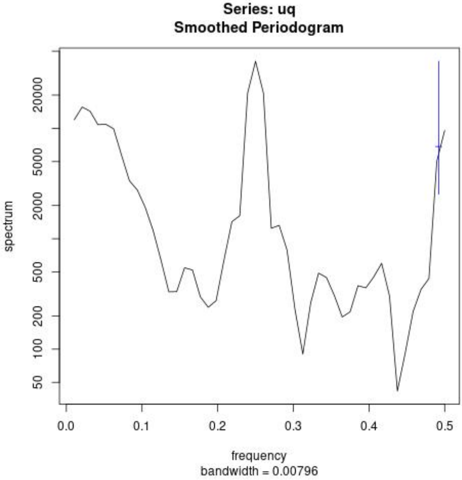
```
> spec.pgram(uq)
```



Ragged!

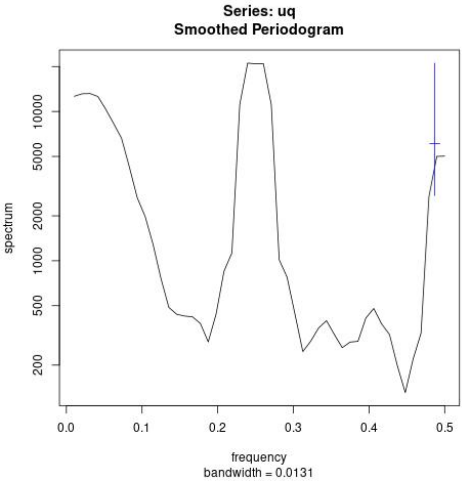
(Observe the log scale on the y axis and the 95% c.i. lenght in blue.)

```
> spec.pgram(uq,spans=2)
```



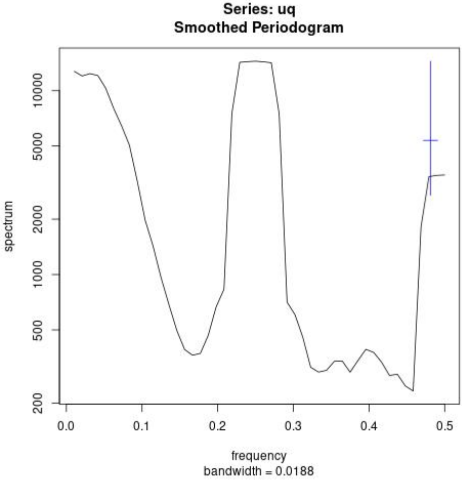
More smooth! (And shorter 95% c.i. lenght.)

```
> spec.pgram(uq,spans=4)
```



Even more smooth!

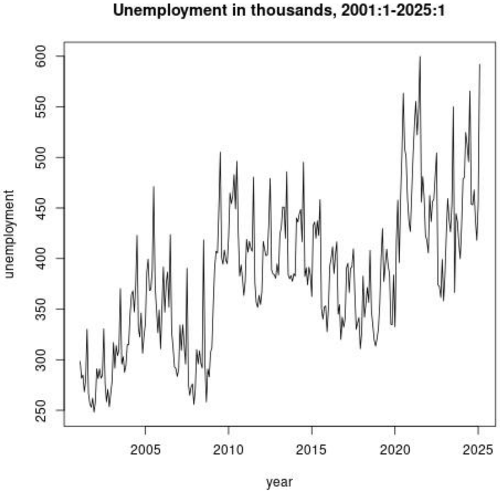
```
> spec.pgram(uq,spans=6)
```



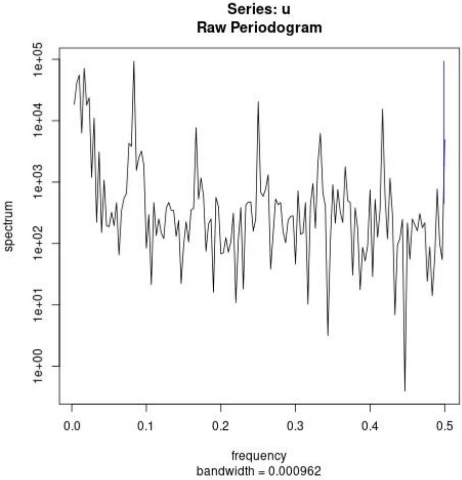
Too much smoothing?

Nonparametric Spectral Estimation

Unemployment, monthly data (period length 12, and maybe 120).

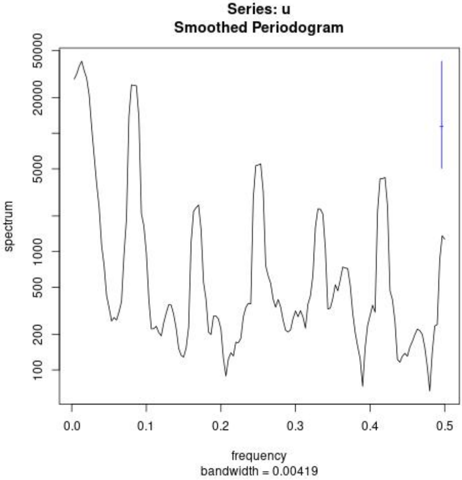


```
> spec.pgram(u)
```



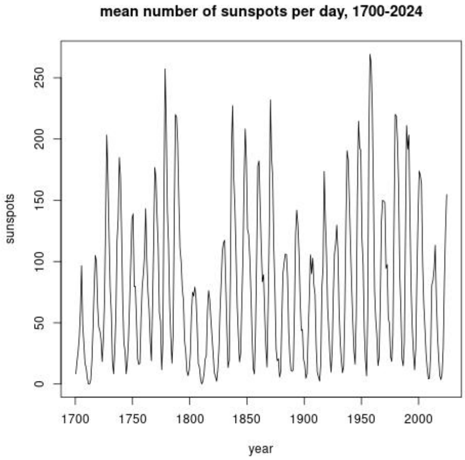
Ragged!

```
> spec.pgram(u,spans=4)
```



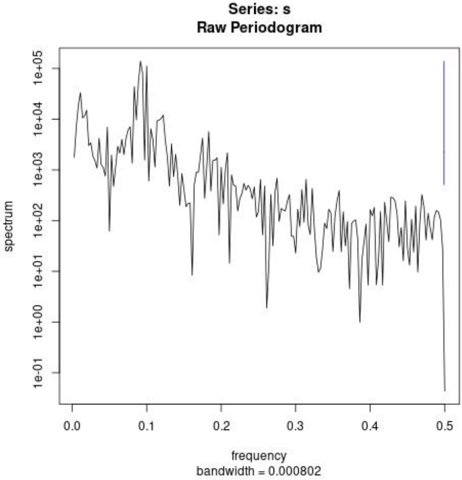
Peaks at multiples of $1/12 \approx 0.08$ and one at about $1/120 \approx 0.008$.

Nonparametric Spectral Estimation



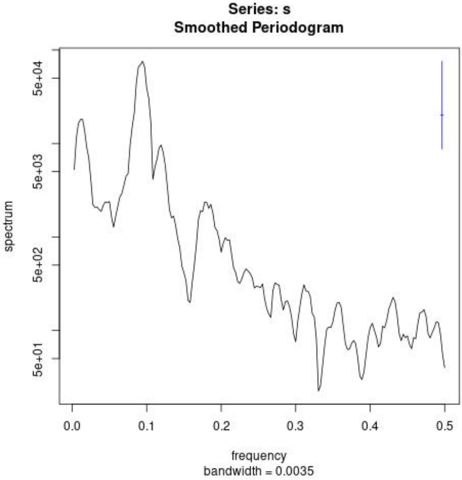
Period length 10 or 11.

```
> spec.pgram(s)
```



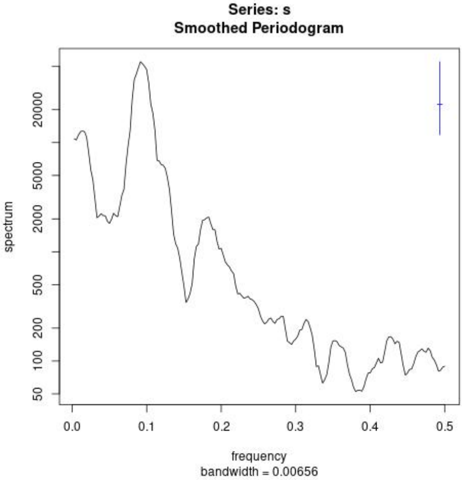
Ragged!

```
> spec.pgram(s,spans=4)
```



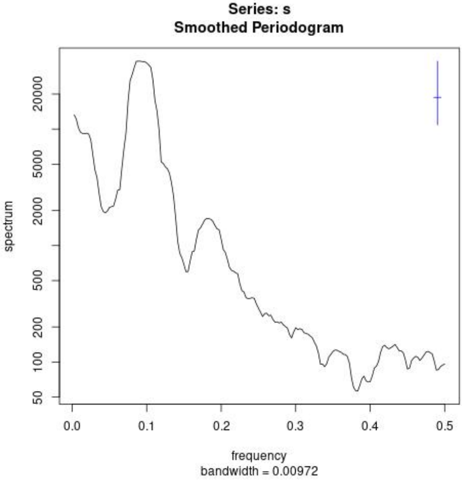
Still too ragged?

```
> spec.pgram(s,spans=8)
```



Maybe the smoothing is good enough. Main peaks at 0.1 and 0.2. Peak at around 0.01 maybe because of a 100 year period?

```
> spec.pgram(s,spans=12)
```



Too much smoothing?

Parametric Spectral Estimation

- AR(p): $\phi(B)x_t = w_t$, i.e.

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t.$$

- Spectral density

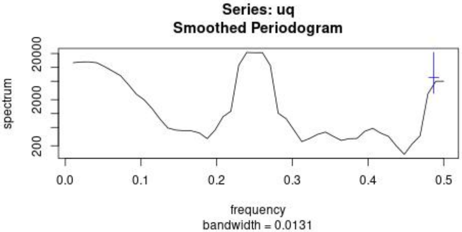
$$f(\omega) = \frac{\sigma_w^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

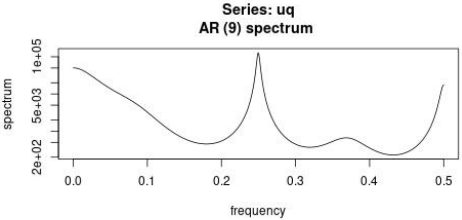
- Idea: Fit an AR(p) model to data and estimate $f(\omega)$ by inserting the parameter estimates.
- The order p may be found by minimizing AIC.
- In R: `spec.ar(x)`
- Approximate $1-\alpha$ confidence interval (as $n, p \to \infty$, $p^3/n \to 0$)

$$\frac{\hat{f}(\omega)}{1+\sqrt{\frac{2p}{n}}z_{\alpha/2}} \leq f(\omega) \leq \frac{\hat{f}(\omega)}{1-\sqrt{\frac{2p}{n}}z_{\alpha/2}}.$$

Parametric Spectral Estimation

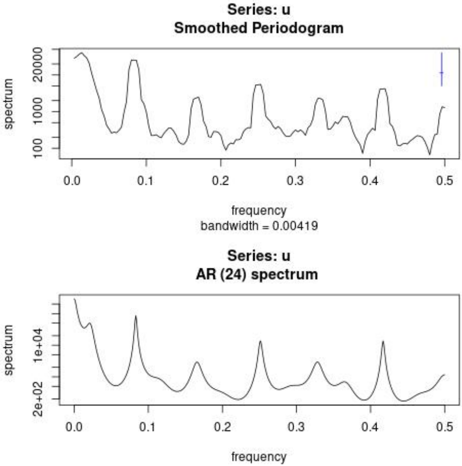
For the quarterly unemployment series
(the upper figure is the non parametric estimate with spans=4):





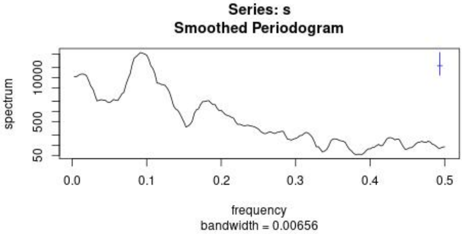
Parametric Spectral Estimation

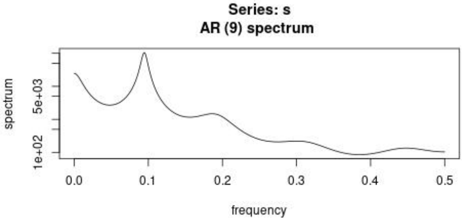
For the monthly unemployment series
(the upper figure is the non parametric estimate with spans=4):



Parametric Spectral Estimation

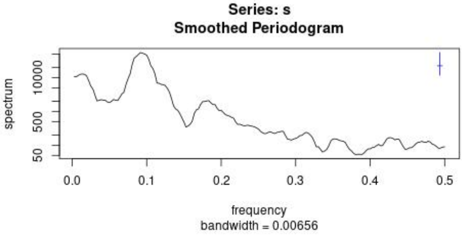
For the sunspot series
(the upper figure is the non parametric estimate with spans=8):

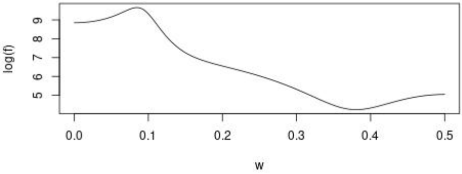




Parametric Spectral Estimation

Comparing to the spectral density for the estimated ARIMA$(3,0,0)\times(0,0,1)_{10}$ model (too smooth?):





News of today

- The periodogram
  - definition
  - asymptotic distribution
  - non consistency
- Non parametric spectral estimation
  - smoothing the periodogram
  - confidence interval
  - the lag window
  - the smoothing window
- Parametric spectral estimation
  - Fitting an AR model
  - confidence interval