UPPSALA UNIVERSITET

Föreläsningsanteckningar

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1. σ-algebras & Measure spaces

1.1. σ-algebras.

Definition 1.1 σ-algebra

A collection of subsets Σ of a set S is called a *σ-algebra* if:
- $\varnothing \in \Sigma$
- Is an algebra:
  - Closed under complements such that for $A \in \Sigma \Rightarrow A^c = S \backslash A \in \Sigma$
  - Closed under unions such that $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$
- Closed under countably infinite unions $A_i \in \Sigma$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

Example:
$\Sigma = \{\varnothing, S\}$ is a σ-algebra on any set S.
Another example is $\mathcal{P}(S)$, which denotes the powerset.
Another example is $S = \mathbb{N}$, then $\Sigma = \{\varnothing, \mathbb{N}, \{2k : k \in \mathbb{N}\}, \{2k+1 : k \in \mathbb{N}\}\}$

Remark:
There exists many equivalent definitions of a σ-algebra. For example, instead of the first axiom of $\varnothing \in \Sigma$, an equivalent definition could be "Σ is non-empty", since then $\exists A \in \Sigma \Rightarrow A^c \in \Sigma \Rightarrow A \cup A^c = S \in \Sigma \Rightarrow (A \cup A^c)^c = \varnothing \in \Sigma$

Remark:
Closed under unions $\Rightarrow$ closed under finite unions since $A_1, \cdots, A_n \in \Sigma \Rightarrow A_1 \cup A_2 \in \Sigma, A_1 \cup A_2 \cup A_3 = \underbrace{(A_1 \cup A_2)}_{\in \Sigma} \cup A_3$, thus by induction $A_1 \cup \cdots \cup A_n \in \Sigma$

This does *not* imply Σ is closed under countable unions.

Counter-example:
Consider $S = [0, 1) \subseteq \mathbb{R}$. Let Σ be all finite unions of disjoint sets on the form $[a, b)$ such that $0 \leq a \leq b < 1$ (if $a = b \Rightarrow \varnothing$).
First and all algebra axioms are fulfilled, but the last one is not since we cvan consider $A_n = \left[\frac{1}{n}, 1\right)$.
Then $\bigcup_{i=2}^{\infty} = (0, 1) \notin \Sigma$

An algebra Σ is an algebra in an algebraic sense.
The symmetric difference $A \triangle B = (A \backslash B) \cup (B \backslash A)$. This behaves like "+" on Σ and intersections behave like multiplication.
Just like one would expect from an algebra, the multiplication is distributive over addition, eg. $C \cap (A \triangle B) = (C \cap A) \triangle (C \cap B)$

1.2. Measures.

Let Σ be a σ-algebra on S, and let μ_0 be a function from Σ_0 to $[0, \infty] = [0, \infty) \cup \{\infty\}$, essentially a function that assigns some value to subsets of Σ.
Intuitively, a measure should increase if we measure something bigger.

Definition 1.2 Additive and σ-additive measures

A measure μ_0 is called *additive* if $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$ where A, B are disjoint sets.

A measure μ_0 is called *σ-additive* if this holds for ocuntable unions, i.e if A_n are pairwise disjoint, then $\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n)$

Remark:
We say that μ_0 is a measure if μ_0 is σ-additive and $\mu_0(\varnothing) = 0$

Example:
$S = \{1, 2, \cdots, 6\}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A) = \dfrac{1}{6}|A|$. Note here that $\mu_0(S) = 1$

Definition 1.3 Probability measures

All measures that sum up to 1 are called *probability measures*

Example:
$S = \mathbb{N}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A \in \Sigma) = |A|$. Here $\mu_0(S) = \infty$

Example:
$S = \mathbb{N}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A \in \Sigma) = \begin{cases} 0 & \text{if} \quad |A| < \infty \\ \infty & \text{if} \quad |A| = \infty \end{cases}$
This is an example of an additive but not σ-additive measure, since if $A_n = \{n\}$, then $\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$, but $\sum_{n=1}^{\infty} \mu_0(A_n) = -1$

1.3. **Measure spaces.**

Definition 1.4 Measure space triplet

A *measure space* is a triplet (S, Σ, μ) where S is some set, Σ is a σ-algebra over S, and μ is a σ-additive function $\mu : \Sigma \to [0, \infty]$ such that $\mu(\varnothing) = 0$

Definition 1.5 Probability space

If $\mu(S) = 1$, then the triplet is called a *probability space.*

Example: (finite measure space)
Let $S = \{s_1, \cdots, s_k\}$ where $k \in \mathbb{N}$ be a set of outcomes. We also associate probabilities $p_1, \cdots, p_k$ to each $s_1, \cdots, s_k$ such that $\sum_i p_i = 1$. Let $\mu(A) = \sum_{s_i \in A} p_i \ \forall A \subseteq S$. If we let $\Sigma = \mathcal{P}(S)$, then (S, Σ, μ) is a measure and a probability space.

Example: (Lebesgue measure)
Let $S = \mathbb{R}$, $\Sigma = \mathcal{B}(\mathbb{R})$ be the Borel σ-algebra (smallest σ-algebra that makes open sets measureable, note that $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$) and let μ be something measuring length on finite unions of disjoint open intervals $A = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ such that $\mu(A) = |b_1 - a_1| + \cdots + |b_n - a_n|$

This μ is called the Lebesgue measure ($\mathcal{L}$)
Restricting S to $[0, 1]$, then we have a probability measure

$$\mu = \mathcal{L}\mid_{[0,1]}(A) = \mathcal{L}(A \cap [0,1]) \Rightarrow ([0,1], \mathcal{B}([0,1], \mathcal{L}\mid_{[0,1]})) \quad \text{is a probability measure}$$

This is a formulation of uniform random numbers in $[0, 1]$

1.4. **Properties of measures.**

For a measure space, we have the following properties:

(1) $\mu(A \cup B) \leq \mu(A) + \mu(B)$
(2) $\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$
(3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) - \mu(A_1 \cap A_2) - \cdots - \mu(A_{n-1} \cap A_n) + \mu(A_1 \cap A_2 \cap A_3) \cdots + (-1)^{n+1} \mu(A_1 \cap A_2 \cdots \cap A_n)$

Note that for the first two points, we have previously assumed that A, B were disjoint. This would be the case for "joint" sets.

Bevis 1.1

Consider $\mu(A) = \mu(A \backslash B \cup (A \cap B)) = \mu(A \backslash B) + \mu(A \cap B)$ and proceed. $\square$

Remark:
For point 4, check Math Stackexchange