UPPSALA UNIVERSITET Tentamen i Reell Analys 10p
Matematiska institutionen **1MA226**
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Duration: 08:00–13:00. The exam consists of 8 problems, each worth 5 points. All solutions should be provided with details and appropriate justifications. No calculators are allowed.

1. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:
   (a) $x_n = \frac{n-1}{n+1}\cos\frac{n\pi}{3}$.
   (b) $x_n = \frac{\log n-(1+\cos n\pi)n}{\log 2n}$. Here log is the natural logarithm in the base e i.e. $\log = \ln$.

2. The integral $\int_0^\infty \sin x^2\,dx$ is called a *Fresnel integral* and arises in wave optics. Show that this integral converges, by proving that the sequence $a_n := \int_0^n \sin x^2\,dx$ converges in $\mathbb{R}$. Hint: Use the fact that $\sin x^2 = \frac{-1}{2x}\frac{d}{dx}(\cos x^2)$.

3. Show that the function $F(x) = \sum_{n=1}^\infty \frac{1}{(n+x)^2}$ is continuous in the interval $[0,\infty)$. Thereafter calculate the exact numerical value of the integral $\int_0^1 F(x)\,dx$.

4. Show that a bounded continuous function $f : [1,\infty) \mapsto \mathbb{R}$ is identically equal to zero if and only if $\int_1^\infty f(x)x^{-n}\,dx = 0$ for $n = 8, 9, 10, \ldots$. In order to obtain full credit, you need to fully justify all the steps of your solution.

5. (a) Let $f \in C^1(\mathbb{R})$ and $f(0) = 0$. Show that $\frac{f(x)}{x}$ is in $f \in C(\mathbb{R})$.
   (b) Let $f \in C^\infty$ in a neighbourhood of the point x_0. Assume that there exist positive numbers δ and M such that for any $x \in (x_0-\delta, x_0+\delta)$ one has the estimate $|\frac{d^k f(x)}{dx^k}| \le M\frac{k!}{\delta^k}$. Show that under these assumptions
$$f(x) = \sum_{k=0}^\infty \frac{1}{k!}\frac{d^k f(x_0)}{dx^k}(x-x_0)^k.$$
   Note that this means that the estimate above implies that $f(x)$ is *analytic* at x_0.

6. Let $K(x,y) \in C([0,1]\times[0,1])$ and assume that $|K(x,y)| < 1$ for all $(x,y) \in [0,1]\times[0,1]$. Let $g(x) \in C[0,1]$. Show that there exists a unique solution $f(x)$ to the following *Fredhom's integral equation*,
$$f(x) = g(x) + \int_0^1 K(x,y)u(y)\,dy.$$

7. If one identifies $\mathbb{R}^2$ and $\mathbb{C}$ via $z = x + iy$, $i = \sqrt{-1}$, then the multiplication by i on $\mathbb{C}$ corresponds to applying the matrix (the so called *complex structure matrix*)

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let $E \subset \mathbb{R}^2$ be open and $f : E \to \mathbb{R}^2$ be a map in $C^1(E)$. Set $f(x,y) = (u(x,y), v(x,y))$ and let $f'(x,y)$ denote the Jacobian matrix of f. Show that the condition $f'(x,y)J = Jf'(x,y)$ is equivalent to certain explicit relationships which hold among the partial derivatives of u and v.

8. The system

$$\begin{cases} \sin(x+y) + \sin(y+z) + z = 0 \\ \cos(x+y) + \cos(y+z) + y - 2 = 0 \end{cases}$$

is satisfied at the point $(0,0,0)$. Show that (x,y) can be solved in a neighbourhood of $(0,0)$ as a function of z (for z near 0). Calling that function $f(z)$, calculate explicitly $f'(0)$.

Happy Yuletide!