Lecture 2 (p43-49)

Constructing stochastic processes in continuous time

Motivation: Consider a stock with $S_0 = 100$.

How much do we expect it to fluctuate?

	1 day	1 month	1 year
Reasonable range for S	100 ± 1	100 ± 5	100 ± 15
ΔS	1	5	15
Δt	1	20	250
$\frac{\Delta S}{\sqrt{\Delta t}}$	1	$\frac{5}{\sqrt{20}} \approx 1$	$\frac{15}{\sqrt{250}} \approx 1$

Below we construct a stochastic process with variance t (st. dev. $\sqrt{t}$). Fix a time interval $[0,T]$.

Stage 1: Let $X_0^1 = 0$. At $t=0$, toss a coin.

If heads, let $X_T^1 = \sqrt{T}$.

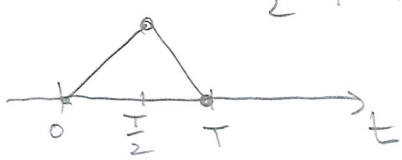
If tail, let $X_T^1 = -\sqrt{T}$.

Stage 2: Let $X_0^2 = 0$. Toss a coin at $t=0$.

Head $\Rightarrow X_{\frac{T}{2}}^2 = \sqrt{\frac{T}{2}}$

Tail $\Rightarrow X_{\frac{T}{2}}^2 = -\sqrt{\frac{T}{2}}$

Repeat at $t = \frac{T}{2}$, adding/subtracting $\sqrt{\frac{T}{2}}$.



<u>Stage n</u>: Let $X_0^n = 0$. At each time $t_k = \frac{k}{n}T$ toss a coin.

$$X^n_{t_{k+1}} = X^n_{t_k} + Y_k \quad , \text{ where } \quad Y_k = \begin{cases} \sqrt{\frac{T}{n}} & \text{prob. } \frac{1}{2} \\ -\sqrt{\frac{T}{n}} & \text{prob. } \frac{1}{2} \end{cases}$$

Clearly, $E[X^n_{t_k}] = E[Y_0 + Y_1 + \ldots + Y_{k-1}] = 0$.

Also, $\mathrm{Var}(X^n_{t_k}) \underset{\text{indep.}}{=} \mathrm{Var}(Y_0) + \ldots + \mathrm{Var}(Y_{k-1}) =$

$$= \frac{T}{n} \cdot k = t_k .$$

When $n \to \infty$, we obtain <u>Brownian motion</u> (or <u>Wiener process</u>).

More formally, we make the following definition.

<u>Def 4.1</u> A stochastic process W is a <u>Wiener process</u> (Brownian motion) if

i) $W_0 = 0$

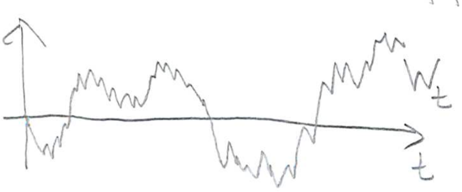
ii) it has <u>independent increments</u>, i.e

$W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent if $t_1 < t_2 \leq t_3 < t_4$

iii) $W_t - W_s$ is $N(0, \sqrt{t-s})$

iv) W_t is continuous.

<u>Thm 4.2</u> $t \mapsto W_t$ is of infinite variation and nowhere differentiable.



Our next goal: To define integrals

$\int_0^t g_s \, dW_s$, where g_t is a stochastic process

"determined" by $\{W_s : 0 \leq s \leq t\}$.

Def. 4.3 Let X_t be a stochastic process.

i) An event A is $\mathcal{F}_t^X$-measurable ($A \in \mathcal{F}_t^X$) if it is possible to determine whether A has happened or not based on observations of $\{X_s ; 0 \leq s \leq t\}$.

ii) If a random variable Z can be determined given observations of $\{X_s ; 0 \leq s \leq t\}$, then we also write $Z \in \mathcal{F}_t^X$.

iii) A stochastic process Y_t with $Y_t \in \mathcal{F}_t^X$ for all $t \geq 0$ is adapted to the filtration $\mathcal{F}_t^X$.

Ex: 1. $A = \{X_s \leq 7 \text{ for all } s \leq 9\} \in \mathcal{F}_9^X$

2. $Z = \int_0^5 X_s \, ds \in \mathcal{F}_5^X$

3. $Y_t = \sup_{0 \leq s \leq t} W_s$ is adapted to $\mathcal{F}_t^W$

4. $Y_t = \sup_{0 \leq s \leq t+1} W_s$ is not adapted to $\mathcal{F}_t^W$.

Def 4.4 A process g_t belongs to $\mathcal{L}^2$ if

i) g is adapted to $\mathcal{F}_t^W$

ii) $\int_0^t E[g_s^2] \, ds < \infty$

Ex: $\int_0^t E[W_s^2]\,ds = \int_0^t s\,ds = \frac{t^2}{2} < \infty$, so $W \in \mathcal{L}^2$.

Stochastic integration

Assume $g \in \mathcal{L}^2$.

1. If g is simple, i.e. $g_s = g_{t_k}$ for $s \in [t_k, t_{k+1})$ where $0 = t_0 < t_1 < \dots < t_n = t$, define

$$\int_0^t g_s\,dW_s := \sum_{k=0}^{n-1} g_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

2. For a general $g \in \mathcal{L}^2$, approximate g with simple g^n such that $\int_0^t E[(g_s^n - g_s)^2]\,ds \to 0$ as $n \to \infty$.

Define $\int_0^t g_s\,dW_s := \lim_{n\to\infty} \int_0^t g_s^n\,dW_s$ (limit in $\mathcal{L}^2$)

Remarks 1. One can show that the limit

- exists
- does not depend on the approximating sequence.

2. Forward increments are used!

3. Riemann-Stieltje integration not possible since W_t has paths of infinite variation.

Prop. 4.5 Assume $g \in \mathcal{L}^2$. Then

(i) $E\left[\int_0^t g_s\,dW_s\right] = 0$

(ii) $E\left[\left(\int_0^t g_s\,dW_s\right)^2\right] = \int_0^t E[g_s^2]\,ds$ (Ito isometry)

(iii) $X_t = \int_0^t g_s\,dW_s$ is $\mathcal{F}_t^W$-adapted.

Pf: Assume g is simple (general case follows by approximation).

(i)
$$E\left[\int_0^t g_s\,dW_s\right] = E\left[\sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right]$$
$$= \sum_{k=0}^{n-1} E\left[g_{t_k}(W_{t_{k+1}} - W_{t_k})\right]$$
indep.
$$= \sum_{k=0}^{n-1} E[g_{t_k}]\underbrace{E[W_{t_{k+1}} - W_{t_k}]}_{0} = 0$$

(ii)
$$E\left[\left(\int_0^t g_s\,dW_s\right)^2\right] = E\left[\left(\sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right)^2\right]$$
$$= \sum_{k=0}^{n-1} E\left[g_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2\right] + 2\sum_{j<k} E\left[\underbrace{g_{t_j} g_{t_k}(W_{t_{j+1}} - W_{t_j})}\underbrace{(W_{t_{k+1}} - W_{t_k})}\right]$$
indep. indep.
$$= \sum_{k=0}^{n-1} E[g_{t_k}^2]\underbrace{E[(W_{t_{k+1}} - W_{t_k})^2]}_{t_{k+1}-t_k} + 2\sum_{j<k} E[g_{t_j} g_{t_k}(W_{t_{j+1}} - W_{t_j})]\underbrace{E[W_{t_{k+1}} - W_{t_k}]}_{0}$$
$$= \int_0^t E[g_s^2]\,ds$$

Ex: Calculate $\int_0^t W_s\,dW_s$.

Solution: Let $g_t^n = W_{t_k}$ for $t \in [t_k, t_{k+1})$.

Then
$$\int_0^t E[(g_s^n - W_s)^2]\,ds = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \underbrace{E[(W_s - W_{t_k})^2]}_{s-t_k}\,ds$$
$$= \sum_{k=0}^{n-1} \frac{(t_{k+1} - t_k)^2}{2} \longrightarrow 0 \text{ as } \Delta t \to 0.$$

Thus g^n approximates the integrand.

$$\sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}} - W_{t_k}) = \frac{1}{2}\sum_{k=0}^{n-1}\left(W_{t_{k+1}}^2 - W_{t_k}^2 - (W_{t_{k+1}} - W_{t_k})^2\right)$$ 6

$$= \frac{1}{2} W_t^2 - \frac{1}{2}\sum_{k=0}^{n-1}(W_{t_{k+1}} - W_{t_k})^2$$

We have

$$E\left[\left(\sum_k (W_{t_{k+1}} - W_{t_k})^2 - t\right)^2\right] = E\left[\left(\sum_k\left\{(W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\right\}\right)^2\right]$$

$$= \sum_k E\left[\left((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\right)^2\right] +$$

$$+ \sum_{j\neq k} \underbrace{E\left[(W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\right]}_{0} \underbrace{E\left[(W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)\right]}_{0}$$

$$= \sum_k \mathrm{Var}\left((W_{t_{k+1}} - W_{t_k})^2\right)$$

$$= \sum_k \left\{ \underbrace{E\left[(W_{t_{k+1}} - W_{t_k})^4\right]}_{3(t_{k+1} - t_k)^2} - \underbrace{\left(E\left[(W_{t_{k+1}} - W_{t_k})^2\right]\right)^2}_{(t_{k+1} - t_k)^2} \right\}$$

$$= 2\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \longrightarrow 0 \quad \text{as } \Delta t \to 0.$$

$$\boxed{\int_0^t W_s\, dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}}$$