

UPPSALA UNIVERSITET

FÖRELÄSNINGSANTECKNINGAR

Inferensteori

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1. TODO

- Experiment in r (QQ-plot of exp vs $n(0,1)$ data)
- Understand .dat files
- Add proof from book of theorem 4.9
- Problems 7.2.2 in the book
- Stora talens lag
- MLE better than methods of moments
- Derivatan av binomial
- Statistical significant change
- Pivotal storhet

2. DATA ANALYSIS (K6)

Vi kommer undersöka statistisk säkerställd skillnad (Opinion polls example), hypotestestning (räknar sannolikheten att hypotesen är sann).

Anmärkning:

Vanligtvis antar vi att datan är normalfördelad, men inte i alla fall (såsom stickprov av lön)

2.1. Location Measures.

A data set is given by x_1, \dots, x_n

Definition/Sats 2.1: Sample mean

Sample mean is given by:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Definition/Sats 2.2: Median

The "middle value" of the sorted data. Different from the mean.

If n is even, the median is defined as the mean of the two middle values

Definition/Sats 2.3: Mode

This doesn't work if it's continuous data but it can be made discrete (such as age/time)

Mode is the most common data value

Example:

Let our data points be:

32 34 41 44 45 50 50 54 55 57 58 60 63

Find mean, median mode:

Mean: 13 data sets $\Rightarrow n = 13$:

$$\frac{1}{13}(32 + 34 + 41 + 44 + 45 + 50 + 50 + 54 + 55 + 57 + 58 + 60 + 63) \approx 49.46$$

Median: The middle value is 50

Mode: 50 is the only data value appearing more than once.

Anmärkning:

In this example, the median = mode. This is not always the case!

2.2. Dispersion measures.

Describes the "spread" of the data, such as the variance. We have the following:

Definition/Sats 2.4: Sample variance

The sample variance is given by:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Definition/Sats 2.5: Sample standard variance

Is given by:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

Also called *sample standard deviation*

Definition/Sats 2.6: Range

Variationsbredden. The difference between the largest and the smallest values of the data

Definition/Sats 2.7: Inter quartile range

Kvartilavståndet is the difference between the upper and lower quartiles.
If we have an odd amount of data it is including the median!

Definition/Sats 2.8: Mid-range

The mean between the biggest and smallest value in the sample

Definition/Sats 2.9: Lower/Upper quartile

The *lower quartile* is the median of the lower half of the data material including the median if n is odd

The *upper quartile* is the median of the upper half of the data material including the median if n is odd

Example:

0 0 1 1 2 2

Here, the mean is given by $\frac{(1+1+2+2)}{6} = 1$.

Therefore, the sample variance is given by $\frac{4}{5}$ and the sample standard deviation $\sqrt{\frac{4}{5}}$

We can find the inter quartile range by looking at the half, like this:

$[0 \underbrace{0}_{\Delta} 1] [1 \underbrace{2}_{\Delta} 2]$

Therefore, the inter quartile range here is $2 - 0 = 2$

2.3. Graphical illustration.**Stem av leafplots:**

```
u = c(32,34,...)
stem(u)
```

Boxplots:

Uses quartiles, max min, and median. Useful if you want a quick look at the dispersion of data.

Bar chart:

Good for illustrating the frequency of each data point, but for large data points the data is hard to read

Histogram:

Attempts to fix the readability issues with the bar chart and is easier to compare with probability density functions.

Easier to manipulate data for readability (use bigger/smaller intervals) (one can ask what the optimal width for a histogram would be)

Very often you can ask if the data follow a normal distribution, which can be hard by just looking at the histogram (because the width varies)

Thoughts:

Dynamically widths on histograms, the more sparse data the greater the width and the more dense, the less the width

QQ-plot:

Is the data normally distributed? You order your data and construct a table with your data and compare it with if it was normally distributed:

$$\Phi(z) = \frac{i - 0.5}{n}$$

If data was perfectly normal on both axis, x_i would be a linear function of z , ie. normally distributed $N(0, 1)$

We plot z on the x -axis and x_i on the y -axis

The name comes from quantile-quantile-plot (QQ-plot). It is a graphical way of comparing two probability distributions (sannolikhetsfördelning)

2.4. Data materials in several dimensions.

We can calculate correlation through sample covariance:

Definition/Sats 2.10: Sample covariance

$$c_{xy} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

Anmärkning:

Not scale invariant (if you measure x in meters and go to cm then it is not the same). Therefore we need to norm it with something, which is where the correlation comes in:

Definition/Sats 2.11: Sample correlation coefficient

$$r_{xy} = \frac{c_{xy}}{s_x s_y}$$

Where s_x and s_y are the sample standard deviations for x and y

Definition/Sats 2.12: Sample correlation satisfies

The sample correlation coefficient satisfies

$$-1 \leq r_{xy} \leq 1$$

If it is 1, then there is a strong positive correlation (the linear regression has a line with positive derivative), similarly for negative.

When it is 0 there is no *linear* relation. There might be other, for example quadratic relation.

Bevis 2.1: Sample correlation satisfaction

$$\begin{aligned}
0 &\leq \frac{1}{n-1} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{s_x} - \frac{y_i - \bar{y}}{s_y} \right)^2 \\
&= \underbrace{\frac{1}{s_x^2} \frac{1}{n-1} \sum_i (x_i - \bar{x})^2}_{s_x^2} + \underbrace{\frac{1}{s_y^2} \frac{1}{n-1} \sum_i (y_i - \bar{y})^2}_{s_y^2} - 2 \underbrace{\frac{1}{s_x s_y} \frac{1}{n-1} \sum_i (x_i - \bar{x})(y_i - \bar{y})}_{c_{xy}} \\
&= 2 - 2r_{xy} \Rightarrow r_{xy} \leq 1 \\
0 &\leq \frac{1}{n-1} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{s_x} + \frac{y_i - \bar{y}}{s_y} \right)^2 = 2 + 2r_{xy} \\
&\Rightarrow -1 \leq r_{xy}
\end{aligned}$$

□

Bevis 2.2: Sample correlation satisfaction

There is yet another proof that may be more intuitive for the boundedness of the sample correlation coefficient satisfaction that goes as follows:

Recall that the covariance is an inner-product, and that the variance $Var(x) = Cov(x, x)$

The "worst" that can happen is that x, y are independent, whereby the covariance $Cov(x, y) = E(XY) - E(X)E(Y) = 0$ since $E(XY) = E(X)E(Y)$

This gives the following:

$$\left| \frac{c_{x,y}}{s_x s_y} \right| = \left| \frac{Cov(x, y)}{\sqrt{Var(x)} \sqrt{Var(y)}} \right| \geq 0$$

Next, we will use the fact that the covariance is an inner product, and therefore the Cauchy-Schwartz inequality holds

Notice that $Cov(x, x) = Var(x)$

$$\begin{aligned}
|Cov(x, y)|^2 &\leq |Var(x)Var(y)| \Leftrightarrow |Cov(x, y)| \leq \left| \sqrt{Var(x)} \sqrt{Var(y)} \right| \\
&= \left| \frac{Cov(x, y)}{\sqrt{Var(x)} \sqrt{Var(y)}} \right| \leq 1
\end{aligned}$$

From the properties of absolute values, we therefore get $-1 \leq r_{x,y} \leq 1$

□

3. STATISTICAL INFERENCE

Anmärkning:

If $X \sim Hyp$, then for a large population $X \sim Bin$ (because the chance of picking the same one in a large population is so small)

Anmärkning:

If X_1, \dots, X_n are independent and equally distributed $N(\mu, \sigma^2)$ variables, then

$$\sum_{i=1}^n X_i \sim N(n\mu, n\sigma^2)$$

In the same way, the mean of these random variables is normally distributed with

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

Example (*):

In an opinion poll, 1000 randomly selected voters are asked about their political sympathies. Let X be the number of these voters who sympathise with the party P . $X \sim Hyp$, but because the number of voters is so large, we may assume that $X \sim Bin(1000, p)$

Suppose we know that 100 of the selected voters sympathise with P , what can p be?

Well, it makes sense that $p = \frac{100}{1000}$, the probability of choosing 100 from our 1000 people.

Definition/Sats 3.1: Sample (stickprov)

x_1, x_2, \dots, x_n is a *sample* from the random variable X with distribution F_X

If $X = (X_1, \dots, X_n)$ are independent, we have a *random sample* from X

We can purposely choose our random variable in such way that makes it easier for us to analyse. It also allows us to compare these observations.

Example:

Using the same environment as example (*), we can either have a random variable $X \sim Bin(1000, p)$, or we can have let the 1000 people all have an associated Bernoulli distributed random variable.

In the first case, our observed sample has size $n = 1$, so our $x_1 = 100$ is an observation of the random variable X_1

In the second case, our observed sample has size $n = 1000$, with each $X_1, \dots, X_{1000} \sim Be(p)$ and

$$\sum_{i=1}^{1000} x_i = 100$$

The pros in splitting up our situation into smaller random variables, is as touched upon previously, easier to analyse since it is easier to find independent random variables.

4. ESTIMATION

Suppose we already know what our distribution function F is. We know this F is associated to our random variable, that we have our measured observations on.

We also know that our distribution function normally takes on certain *parameters*, for example, the Poisson distribution

$$f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Is actually a function of both x and λ .

We call λ in our case, an *unknown parameter*. Given enough data, and knowing the distribution function, we should be able to estimate what the value of this parameter is.

We can write the following for our data:

$$x = (x_1, x_2, \dots, x_n)$$

$$X = (X_1, X_2, \dots, X_n)$$

Definition/Sats 4.1: Estimate (skattning)

An *estimate* $\theta^* = \theta^*(x)$ is a function of the sample x

The estimate is an observation of the estimator $\theta^*(X)$

This attempts to put the previous paragraphs into "functions". Given a sample from our random variable (given in $x = x_1, \dots$), we want to find the unknown parameter θ . We can then construct a general formula for *any* data given that it comes from the random variable we have agreed upon beforehand. This is the *estimate* function, as defined above, of a given sample x

Example:

Let $x = (x_1, x_2, x_3, x_4, x_5) = (200, 185, 210, 190, 190)$ be a random sample from $X \sim N(\mu, 100)$ (every $X_i \sim N(\mu, 100)$)

In order to estimate μ by the sample mean, we induce the function μ^* on our sample set and calculate the mean:

$$\mu^*(x) = \bar{x} = 195$$

Since the estimate is an observation of the *estimator*, let's look at the estimator:

$$\mu^*(X) = \bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i \sim N(\mu, 100) \Rightarrow \bar{X} \sim N(\mu, 100/5) = N(\mu, 20)$$

Example:

Using the poll example from above, what we really did when we said that p reasonably has to be $\frac{100}{1000}$ is determine an estimate p^*

The greater the sample size the better the estimate (because less and less variance)

Anmärkning:

The estimator is not distributed with the same distribution, since the estimator is not always an integer. We saw this in the above example, where the estimator was $N(\mu, 20)$ distributed but our estimate was $N(\mu, 100)$ distributed.

If we have different estimates, we need to make a reasonable choice such that our error is as little as possible (this is why we introduce estimators)

4.1. Properties of estimates.

The purpose of our estimates is to estimate θ . When we calculate using our function θ^* on our sample data we get a value that may or may not deviate from the actual value θ

We can take $\theta^* - \theta = E(\theta^*(X)) - \theta + (\theta^* - E(\theta^*(X)))$

It turns out, this is equal to the systematic error + random error

Definition/Sats 4.2: Unbiased (väntevärdesriktigt)

An estimate θ^* is said to be *unbiased* if it satisfies $E(\theta^*(X)) - \theta = 0$

This is the same as saying it has no systematic error (therefore, we only have the random error left)

Example:

We show this by:

$$E(\mu^*(X)) = E(\bar{X}) = \mu$$

Let

$$p^*(X) = \frac{X}{1000} \quad X \sim \text{Bin}(1000, p)$$

Is $p^*(x)$ an unbiased estimate of p ?

Take the expected value function on both sides:

$$\begin{aligned} E\left(\frac{X}{1000}\right) &= E(p^*(X)) \quad X \sim \text{Bin}(1000, p) \\ \frac{1}{1000}E(X) &= \frac{1}{1000} \cdot 1000 \cdot p = p \\ \Rightarrow E(p^*(X)) &= p \Rightarrow p - p = 0 \end{aligned}$$

If we have more than one unbiased estimate, which is the best one? Well, in that case we need to start looking at the random error. We can study this by looking at the variance

Definition/Sats 4.3: Efficiency comparison of estimates

If θ_1^* and θ_2^* are unbiased estimates of θ and

$$V(\theta_1^*(X)) \leq V(\theta_2^*(X))$$

For all θ with strict inequality for some. We say that θ_1^* is more *efficient* than θ_2^* (less random error)

Example:

See slide 6 & 7

Example: (stratification)

We are interested in the proportion p of Swedish citizens that last year have traveled by plane in connection with work. Take a sample consisting of $n = 1000$ people

It is safe to assume that the number of men that take the plane will be greater than the number of woman, we can denote this using $p \pm a$. Since we are looking at the combinations of ways we can choose men and women from our set, a reasonable distribution would be a binomial distribution for men $\text{Bin}(500, p + a)$ while for women $\text{Bin}(500, p - a)$. One can also look at Bernoulli distributions and get the same results.

$$\begin{aligned} p_1^*(X) &= \frac{1}{1000}X \Rightarrow E(p_1^*) = p \\ p_2^*(y, z) &= \frac{1}{1000}y + \frac{1}{1000}z \\ &\Rightarrow \frac{1}{1000}E(y) + \frac{1}{1000}E(z) \\ \frac{1}{1000}500(p + a) + \frac{1}{1000}500(p - a) &= p \\ V(p_1^*) &= \frac{p(1-p)}{1000} \geq V(p_2^*) = \frac{p(1-p)}{1000} - \frac{a^2}{1000} \end{aligned}$$

Definition/Sats 4.4: Standard error (medelfelet)

We want to assign a numerical value to the dispersion of an estimate.

Therefore, we define the *standard error* of the estimate θ^* is an estimate of the standard deviation $D(\theta^*)$

Denoted by $d(\theta^*(x)) = d(\theta^*)$

Recall that the standard deviation is given by \sqrt{Var}

Example:

Given $x = 100$ (one observation) of the random variable $X \sim \text{Bin}(1000, p)$, estimate $p^* = \frac{x}{1000} = 0.1$ and calculate the standard error of this estimate.

4.2. Asymptotic properties.

The accuracy of an estimate should improve as the sample size increases, seems reasonable

Definition/Sats 4.5: Bias

The *bias* (väntevärdesfelet) for the estimate θ^* is defined as

$$B(\theta^*) = E(\theta^*) - \theta$$

Anmärkning:

An unbiased estimate has bias 0

Definition/Sats 4.6: Asymptotically unbiased

If the bias $B(\theta_n^*)$ tends to zero as $n \rightarrow \infty$ for all θ , the estimate θ_n^* is said to be *asymptotically unbiased*

Think of it like, the more data you gather, the less the bias.

Example:

Let x_1, \dots, x_n be a random sample from $N(\mu, \sigma^2)$ where μ is unknown. We want to estimate σ^2

The estimate $\sigma_n^{2*} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$ is biased, but it is asymptotically unbiased

The estimate s_n^2 is unbiased for σ^2 , why?

$$\begin{aligned}
s^2 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \sum_{i=1}^n x_i^2 - 2\bar{x} \underbrace{\sum_{i=1}^n x_i}_{nx} - n\bar{x}^2 \\
&= \frac{1}{n-1} \left(\sum_{i=1}^n x_i^2 - n\bar{x}^2 \right) \\
E(s^2) &= \frac{1}{n-1} \left(\sum_{i=1}^n E(x_i^2) - nE(\bar{x}^2) \right) \\
E(x_i^2) &= V(x_i) + (E(x_i))^2 = \sigma^2 + \mu^2 \\
E(\bar{x}^2) &= V(\bar{x}) + (E(\bar{x}))^2 = \frac{\sigma^2}{n} + \mu^2 \\
E(s^2) &= \frac{1}{n-1} \left(\sum_{i=1}^n (\sigma^2 + \mu^2) - n \left(\frac{\sigma^2}{n} + \mu^2 \right) \right) \\
&= \frac{1}{n-1} (n(\sigma^2 + \mu^2) - \sigma^2 - n\mu^2) = \frac{1}{n-1} (n-1)\sigma^2 = \sigma^2
\end{aligned}$$

Definition/Sats 4.7: Consistent estimate

The estimate θ_n^* is said to be *consistent* for θ if the corresponding estimator converges to θ in probability for all θ

Definition/Sats 4.8: Convergence in probability

The estimator θ_n^* converges to θ in probability if $\forall \varepsilon > 0$:

$$\lim_{n \rightarrow \infty} P(|\theta_n^* - \theta| > \varepsilon) = 0$$

Definition/Sats 4.9

If the estimate θ_n^* is asymptotically unbiased and

$$\lim_{n \rightarrow \infty} V(\theta_n^*) = 0$$

Then it is consistent

Example:

Let x_1, \dots, x_n be a random sample from $N(\mu, \sigma^2)$ where σ^2 is known.
Estimate μ by

$$\mu_n^* = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Show that the estimate is unbiased: $\mu - \mu = 0$

Calculate the variance for the corresponding estimator: $\frac{\sigma^2}{n}$

Show that the estimate is consistent: $\lim_{n \rightarrow \infty} \frac{\sigma^2}{n} = 0$

If an estimate is not consistent, then it does not matter if the sample size is increased, it wont yield better results.

5. METHODS OF ESTIMATION

5.1. Methods of moments.

Often, this method works quite well but it also tends to fail (**CHECK**)

Definition/Sats 5.1: Method of moments

Let x_1, \dots, x_n be a random sample from the random variable X with $E(X) = m(\theta)$ where θ is the parameter in the distribution for X

If θ is one dimensional, the moment estimate $\theta = \theta^*$ solves the equation $m(\theta) = \bar{x}$

Example:

Slide 1

$$m(\beta) = E(X) = \frac{1}{\beta}$$

$$\text{Solve } \bar{x} = m(\beta) = \frac{1}{\beta} \Rightarrow \beta = \frac{1}{\bar{x}}$$

Therefore, moment estimate is $\beta^* = \frac{1}{\bar{x}}$

Example:

Slide 2

$$m(p) = E(X) = np$$

$$\text{Solve } x = \bar{x} = m(p) = np \Rightarrow p = \frac{x}{n} \text{ (recall political party example)}$$

Moment estimation is $p^* = \frac{x}{n}$

Example:

Slide 3 (a third of those 15 birds have rings, so we can get 30 from there)

The number of birds captured with ring = $X \sim \text{Hyp}(N, n, m)$, where N = number of birds, n = how many were captured the second day (15) and m = how many with a ring in total the second day = 10:

$$p_X(x) = \frac{\binom{m}{x} \binom{N-m}{n-x}}{\binom{N}{n}}$$

We want to estimate N using the method of moments:

$$E(X) = n \frac{m}{N} = 15 \frac{10}{N} = \frac{150}{N}$$

Solve for $f = x = \bar{x} = \frac{150}{N} \Rightarrow N = \frac{150}{5} = 30$. Moment estimation is $N^* = 30$

Definition/Sats 5.2: Method of moments with multiple parameters

If the parameter $\theta = (\theta_1, \theta_2)$, then the moment estimates solves the system:

$$E(X) = m_1(\theta_1, \theta_2) = \bar{x}$$

$$E(X^2) = m_2(\theta_1, \theta_2) = \frac{1}{n} \sum_{i=1}^n x_i^2$$

Example:
Slide 4

$$m_1(\mu, \sigma^2) = E(X) = \mu$$

$$m_2(\mu, \sigma^2) = E(X^2) = V(X) + (E(X))^2 = \sigma^2 + \mu^2$$

Solve

$$\begin{cases} \mu = \bar{x} \\ \sigma^2 + \mu^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 \end{cases}$$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Almost looks like s^2 , but here in the denominator we have n instead of $n - 1$

Moment estimates are:

$$\begin{cases} \mu^* = \bar{x} \\ \sigma^{2*} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \end{cases}$$

Definition/Sats 5.3

Let x_1, \dots, x_n be a random sample from the random variable X where $V(X) = \sigma^2$

Then the sample variance is given by:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

is an *unbiased estimate* of σ^2

5.2. Maximum likelihood.

Example:
Slide 5

Definition/Sats 5.4: Maximum likelihood

Let x_1, \dots, x_n be a random sample from X which has distribution $F(X; \theta)$

The likelihood function $L(\theta)$ is defined by:

$$L(\theta) = \begin{cases} \prod_{i=1}^n p(x_i; \theta) & X \text{ discrete} \\ \prod_{i=1}^n n f(x_i; \theta) & X \text{ continuous} \end{cases}$$

The *maximum likelihood* estimate (MLE, ML-skattning) of θ is the θ that maximizes the likelihood function

Example:
Slide 6

Example:
Slide 7

Example:
Slide 8

Example:
Let x_1, \dots, x_n be a random sample from $X \sim N(\mu, \sigma^2)$ where μ and σ^2 are both unknown.

Estimate μ and σ^2 by MLE:

$$\begin{aligned}
 f_X(x; \mu, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \\
 L(\mu, \sigma^2) &= \prod_{i=1}^n f_X(x_i; \mu, \sigma^2) \\
 &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \\
 &= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \\
 l(\mu, \sigma^2) &= \ln \left((2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \right) \\
 &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\
 \frac{\partial l}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = \frac{1}{\sigma^2} \left(\sum_{i=1}^n x_i - n\mu \right) = \frac{n}{\sigma^2} (\bar{x} - \mu) \\
 \frac{\partial l}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 \\
 \frac{\partial l}{\partial \mu} &= 0 \Rightarrow \mu = \bar{x} \\
 \frac{\partial l}{\partial \sigma^2} &= 0 \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (\mu = \bar{x})
 \end{aligned}$$

MLE is therefore:

$$\begin{aligned}
 \mu^* &= \bar{x} \\
 \sigma^{2*} &= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2
 \end{aligned}$$

Example:

Slide 9 (not supposed to follow, research level) (ibland för att få fram skattning måste man ta till med numeriska metoder)

Example:

Slide 1, täthetsfunkt given by $f_X(x) = \frac{1}{\theta}$ if $0 \leq x \leq \theta$ and 0 otherwise, this can be shown using the indicator function $I\{0 \leq x \leq \theta\}$

Likelihood is given by

$$\prod_{i=1}^n f_X(x_i) = \prod_{i=1}^n \frac{1}{\theta} I\{0 \leq x_i \leq \theta\}$$

This product is only $\frac{1}{\theta^n} \neq 0$ if all $x_i \in [0, \theta]$, otherwise it is 0

This can be expressed using an indicationr function as follows:

$$\frac{1}{\theta^n} I\{0 \leq \min x_i \leq \max x_i \leq \theta\}$$

Anmärkning:

The max-point is when $\theta = \max x_i$, which is therefore $\theta^* = 3.0$ (from slide 1)

The method of moment is given by $\bar{x} = E(X) = m(\theta) = \frac{\theta}{2}$

Then $\theta = 2\bar{x}$ and $\bar{x} = 1.25 \Rightarrow \theta^* = 2.5$

But 2.5 is an unreasonable number since $3 > 2.5$

Anmärkning:

MLE is always an underestimate, so we can scale it up a little

5.3. Least Squares.

Definition/Sats 5.5: Least Squares

Let x_1, \dots, x_n be a random sample from the random variable X with $E(X) = m(\theta)$
Moreover, let

$$Q(\theta) = \sum_{i=1}^n \{x_i - m(\theta)\}^2$$

The value of θ that minimizes $Q(\theta)$ is called the *least squares* estimates of θ

Example:

Slide 2.

$$\begin{aligned} m(\beta) &= E(X) = \frac{1}{\beta} \\ Q(\beta) &= \sum_{i=1}^n (x_i - \frac{1}{\beta})^2 \\ Q'(\beta) &= \frac{2}{\beta^2} \sum_{i=1}^n (x_i - \frac{1}{\beta}) = \frac{2}{\beta^2} \sum_{i=1}^n x_i - \frac{n}{\beta} \\ &= \frac{2n}{\beta^2} (\bar{x} - \frac{1}{\beta}) = \frac{2n\bar{x}}{\beta^2} - \frac{2n}{\beta^3} \\ Q''(\beta) &= \frac{-4n\bar{x}}{\beta^3} + \frac{6n}{\beta^4} \\ 0 &= Q'(\beta) \Rightarrow \bar{x} = \frac{1}{\beta} \Rightarrow \beta = \frac{1}{\bar{x}} \\ Q''\left(\frac{1}{\bar{x}}\right) &= -\frac{4n\bar{x}}{(1/\bar{x})^3} + \frac{6n}{(1/\bar{x})^4} = 2n\bar{x}^4 > 0 \text{ is a minimum-point} \end{aligned}$$

Therefore, the LSE (least square estimate) is $\beta^* = \frac{1}{\bar{x}}$, same as moment estimator.

The fact that it was equal to the moment estimator was no coincidence, this is actually a special case.

Example:

Slide 3

$$\begin{aligned} E(X) &= m(\theta) & m'(\theta) &\neq 0 \\ Q(\theta) &= \sum_{i=1}^n \{x_i - m(\theta)\}^2 \\ Q'(\theta) &= -2m'(\theta) \sum_{i=1}^n \{x_i - m(\theta)\} \\ &= -2n \underbrace{m'(\theta)}_{\neq 0} \{\bar{x} - m(\theta)\} \end{aligned}$$

If the sign of the second derivative is positive, then we will see that this method of estimation yields the same answer as the moments of estimation, so let us verify the second derivative:

$$\begin{aligned}
 Q'(\theta) &= 0 \Leftrightarrow \bar{x} = m(\theta) \\
 Q'(\theta) &= -2n\bar{x}m'(\theta) + 2nm'(\theta)m(\theta) \\
 Q''(\theta) &= -2nm''(\theta) + 2nm''(\theta)m(\theta) + 2nm'(\theta)m'(\theta) \\
 &= -2nm''(\theta) \underbrace{\{\bar{x} - m(\theta)\}}_{=0} + 2n \underbrace{\{m'(\theta)\}^2}_{>0} > 0
 \end{aligned}$$

Ok! LSE estimation solves the equation $\bar{x} = m(\theta)$ \square

Therefore, if the question is to calculate with LSE, you can do it using the moment of estimation which is easier

Definition/Sats 5.6

Let x_1, \dots, x_n be a random sample where the corresponding random variables X_1, \dots, X_n have $E(X_i) = m_i(\theta)$

Moreover, let

$$Q(\theta) = \sum_{i=1}^n \{x_i - m_i(\theta)\}^2$$

Anmärkning:

This situation may not be handled by the method of moments nor by maximum likelihood, unless the distribution of X_i is specified (**is not the distribution of X_i given by definition of estimation?**)

Used in regression

Example:

slide 4

$$\begin{aligned}
 m_i(v) &= E(Y_i) = \text{time} = \frac{x_i}{v} \\
 Q(v) &= \sum_{i=1}^n \left(y_i - \frac{x_i}{v}\right)^2 \\
 Q'(v) &= \frac{2}{v^2} \sum_{i=1}^n x_i \left(y_i - \frac{x_i}{v}\right) = \frac{2}{v^2} \left(\sum_{i=1}^n x_i y_i - \frac{1}{v} \sum_{i=1}^n x_i^2\right) \\
 0 = Q'(v) &\Rightarrow v = \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i y_i} = \frac{S_{xx}}{S_{xy}} \\
 Q'(v) &= \frac{2}{v^2} S_{xy} - \frac{2}{v^3} S_{xx} \\
 Q''(v) &= -\frac{4}{v^3} S_{xy} + \frac{6}{v^4} S_{xx} \\
 Q''\left(\frac{S_{xx}}{S_{xy}}\right) &= -\frac{4}{(S_{xx}/S_{xy})^3} S_{xy} + \frac{6}{(S_{xx}/S_{xy})^4} S_{xx} = \frac{2S_{xy}^4}{S_{xx}^3} > 0 \\
 v^* &= \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i y_i} = \frac{60^2 + \dots + 400^2}{60 \cdot 6.39 + \dots + 400 \cdot 43.03} \approx 9.54
 \end{aligned}$$

5.4. Parameter estimation in standard distributions.

Example:

Two normally distributed random samples with $v = \sigma^2$ for both of them

$$\begin{aligned}
 L(\mu_1, \mu_2, \sigma^2) &= \prod_{i=1}^{n_1} f_X(x_i) \prod_{j=1}^{n_2} f_Y(y_j) \\
 &= \prod_{i=1}^{n_1} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(x_i - \mu_1)^2} \cdot \prod_{j=1}^{n_2} \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(y_j - \mu_2)^2} \\
 &= C v^{-(n_1+n_2)/2} e^{-\frac{1}{2v}A} \\
 A &= \sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \sum_{j=1}^{n_2} (y_j - \mu_2)^2 \\
 l = \log(L(\mu_1, \mu_2, v)) &= -\log(C) - \frac{n_1 + n_2}{2} \log(v) - \frac{1}{2v}A \\
 \frac{\partial l}{\partial \mu_1} &= -\frac{1}{2v} \frac{\partial}{\partial \mu_1} A = -\frac{1}{v} \sum_{i=1}^{n_1} (x_i - \mu_1) = -\frac{n_1}{v} (\bar{x} - \mu_1) = 0 \Rightarrow \mu_1 = \bar{x} \\
 \frac{\partial l}{\partial \mu_2} &= 0 \Rightarrow \mu_2 = \bar{y} \\
 \frac{\partial l}{\partial v} &= -\frac{n_1 + n_2}{2v} + \frac{1}{2v^2} A = 0 \\
 &\Leftrightarrow v = \frac{A}{n_1 + n_2}
 \end{aligned}$$

$$\begin{aligned}
 \mu_1^* = \bar{x} \quad \mu_2^* = \bar{y} \quad v^* &= \frac{1}{n_1 + n_2} \left(\underbrace{\sum_{i=1}^n (x_i - \bar{x})^2}_{(n-1)S_x^2} + \underbrace{\sum_{j=1}^n (y_j - \bar{y})^2}_{(n-1)S_y^2} \right) \\
 &= \frac{(n_1 - 1)S_x^2 + (n_2 - 1)S_y^2}{n_1 + n_2}
 \end{aligned}$$

$$\begin{aligned}
 E(S_x^2) &= \sigma^2 = E(S_y^2) \\
 E(V^*) &= \frac{(n_1 - 1)E(S_x^2) + (n_2 - 1)E(S_y^2)}{n_1 + n_2} = \frac{n_1 - 1 + n_2 - 1}{n_1 + n_2} \sigma^2 \neq \sigma^2 \\
 \text{Men } S_p^2 &= \frac{(n_1 - 1)S_x^2 + (n_2 - 1)S_y^2}{n_1 + n_2 - 2} \text{ är väntevärdesriktigt}
 \end{aligned}$$

5.5. Estimation of distributions.

Example:

slide 6

We have a random sample from a random variable X with an unknown distribution

More generally, estimate the distribution function $F_X(x) = P(X \leq x) \quad \forall x$

Definition/Sats 5.7: Empirical distribution function

Let x_1, \dots, x_n be a random sample from a random variable X . The *empirical distribution function* for X is defined as

$$F_n(x) = \frac{A_n(x)}{n}$$

where $A_n(x)$ is the number of observations in the sample that are smaller than or equal to x

Anmärkning:

For every given x , $F_n(x)$ is unbiased and consistent estimate of $F_X(x)$

6. LESSON 1

6.1. 727.

Kalle lägger patiens, en gång per kväll, tills den går ut för första gången.
Under en vecka får han observationerna

3 7 10 5 12 8 4

Bestäm ML-skattningen av $p = P(\text{patiensen går ut})$

Lösning:

Här är slumpvariabeln ffg fördelad.

Låt X vara antalet gånger tills patiensen går ut, då är fördelningsfunktionen:

$$p_X(k) = (1-p)^{k-1}$$

Vi räknar med ML-skattning, vilket är:

$$\begin{aligned} L(p) &= \prod_{i=1}^n p_X(x_i) = \prod_{i=1}^n (1-p)^{x_i-1} p \\ &= (1-p)^{\sum_{i=1}^n x_i - n} p^n \end{aligned}$$

Vi logariterar:

$$\begin{aligned} l(p) &= \ln \{L(p)\} = \left(\sum_{i=1}^n x_i - n \right) \ln(1-p) + n \ln(p) \\ l'(p) &= - \left(\sum_{i=1}^n x_i - n \right) \frac{1}{1-p} + \frac{n}{p} \\ l''(p) &= - \left(\sum_{i=1}^n x_i - n \right) \left(\frac{1}{1-p} \right)^2 - \frac{n}{p^2} < 0 \Rightarrow \max \\ 0 = l'(p) &\Rightarrow p = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}} = \frac{1}{7} \end{aligned}$$

ML-skattningen är $p^* = \frac{1}{7}$ vilket är samma som momentskattningen (så blir det ofta men inte alltid)

6.2. 7.210.

En enarmad bandit (spel) med vinstchans p
Albert has spelat 10 ggr och fått 2 vinster
Beata spelade tills första vinst, gång 7
ML-skatta p

Lösning:

Låt X_1 = antal vinster under 10 spel. Denna slumpvariabel är Binomialfördelad med $10, p$

Låt X_2 = antalet spel till första vinst. Denna slumpvariabel är ffg fördelad med parameter p

$$\begin{aligned}
L(p) &= p_{X_1}(x_1; p) p_{X_2}(x_2; p) \quad x_1 = 2 \quad x_2 = 7 \\
&= \binom{10}{x_1} p^{x_1} (1-p)^{10-x_1} \cdot (1-p)^{x_2-1} p \\
&= \binom{10}{x_1} p^{x_1+1} (1-p)^{9-x_1+x_2} \\
l(p) &= \ln \{L(p)\} = \ln \binom{10}{x_1} + (x_1+1) \ln(p) + (9-x_1+x_2) \ln(1-p) \\
l'(p) &= (x_1+1) \frac{1}{p} - (9-x_1+x_2) \frac{1}{1-p} \\
l''(p) &= -(x_1+1) \frac{1}{p^2} - (9-x_1+x_2) \frac{1}{(1-p)^2} < 0 \quad \text{om } 9-x_1+x_2 > 0 \quad \text{vilket vi har eftersom } 9-2+7 > 0 \\
0 = l'(p) &\Rightarrow p = \frac{x_1+1}{x_2+10} = \frac{2+1}{7+10} = \frac{3}{17} \approx 0.176
\end{aligned}$$

ML-skattningen är då $p^* = \frac{3}{17}$

6.3. 7.2.12.

Taxi problemet. 7 taxibilar observeras. De är numrerade $1, \dots, N$
 Obs, numren 070, 234, 166, 7, 65, 17, 4

ML-skatta N

X = numret på en taxibil, diskret likformigt fördelad på $(1, 2, \dots, N)$

Sannolikhetsfunktionen är då:

$$p_X(k) = \begin{cases} \frac{1}{N}, & 1 \leq k \leq N \\ 0, & \text{annars} \end{cases}$$

N kan vara hur stort som helst, $N \in \mathbb{N}^+ =$ rummet av alla positiva heltal

ML-skatta N (observationer x_1, \dots, x_n):

$$L(N) = \prod_{i=1}^n p_X(x_i) = \begin{cases} \left(\frac{1}{N}\right)^n & \text{om } \forall x_i \leq N \\ 0 & \text{annars} \end{cases}$$

(Logaritmera/derivera funkar ej här, man måste tyvärr tänka)

ML-skattningen N^* inträffar i $\max x_i = 234$

Momentskattning: $m(n) = E(X) = \frac{N+1}{2}$

Lös $\bar{x} = \frac{N+1}{2} \Rightarrow N = 2\bar{x} - 1$

Vi har $\bar{x} = 84.3$, momentskattningen blir 167.6, vilket blir en orimlig skattning eftersom vi har en observation som är större.

6.4. 7.2.14.

x_1, x_2 mätningar av en storhet med värdet μ

x_3 mätning av en storhet med värdet 2μ

Mätningar saknar systematiska fel, men har en slumpfel standardavvikelse σ

Bestäm MK-skattningen av μ och visa att den är väntevärdesriktigt

Minimera $Q(\mu) = (x_1 - \mu)^2 + (x_2 - \mu)^2 + (x_3 - 2\mu)^2$, detta kan vi lösa med derivering:

$$\begin{aligned} Q'(\mu) &= -2(x_1 - \mu) - 2(x_2 - \mu) - 4(x_3 - 2\mu) \\ &= -2x_1 - 2x_2 - 4x_3 + 12\mu \\ Q''(\mu) &= 12 > 0 \quad \text{ger min} \\ 0 = Q'(\mu) &\Leftrightarrow \mu = \frac{1}{12}(2x_1 + 2x_2 + 4x_3) = \frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{1}{3}x_3 \end{aligned}$$

MK-skattningen μ^*

Estimatorn $\mu^*(X_1 + X_2 + X_3) = \frac{1}{6}X_1 + \frac{1}{6}X_2 + \frac{1}{3}X_3$

Då blir:

$$\begin{aligned} E\{\mu^*(X_1; X_2; X_3)\} &= \frac{1}{6}E(X_1) + \frac{1}{6}E(X_2) + \frac{1}{3}E(X_3) \\ &= \frac{1}{6}\mu + \frac{1}{6}\mu + \frac{1}{3}2\mu = \mu \end{aligned}$$

Då är μ^* väntevärdesriktigt

En annan skattning är:

$$\mu' = \frac{2x_1 + 2x_2 + x_3}{6}$$

Är den väntevärdesriktigt? Vi tittar på motsvarande estimator:

$$\begin{aligned} \mu'(X_1, X_2, X_3) &= \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{6}X_3 \\ E\{\mu'(X_1, X_2, X_3)\} &= \frac{1}{3}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{6}E(X_3) \\ &= \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{6}2\mu = \mu \end{aligned}$$

Ok!

Vilken skattning är effektivast?

Då jämför vi varianserna:

$$\begin{aligned} V(\mu) &= \frac{6}{36}\sigma^2 \\ V(\mu') &= \frac{9}{36}\sigma^2 \end{aligned}$$

6.5. 702.

Observationer: 4.0, 1.1, 0.2, 1.2, 2.5, 2.0, 0.7, 1.0 är ett stickprov från en Raylerghfördelning med täthetsfunktion:

$$F_X(x) = axe^{-\frac{ax^2}{2}} \quad x \leq 0$$

ML-skatta a :

$$\begin{aligned} L(a) &= \prod_{i=1}^n f_X(x_i) = \prod_{i=1}^n a x_i e^{-\frac{a x_i^2}{2}} \\ &= a^n \left(\prod_{i=1}^n x_i \right) e^{-\frac{a}{2} \sum_{i=1}^n x_i^2} \end{aligned}$$

$$l(a) = \ln \{L(a)\} = n \ln(a) + \sum_{i=1}^n \ln x_i - \frac{a}{2} \sum_{i=1}^n x_i^2$$

$$l'(a) = \frac{n}{a} - \frac{1}{2} \sum_{i=1}^n x_i^2$$

$$l''(a) = -\frac{n}{a^2} < 0 \Rightarrow \max$$

$$0 = l'(a) \Rightarrow a = \frac{2n}{\sum_{i=1}^n x_i^2}$$

$$\text{ML skattning } a^* = \frac{2n}{\sum_{i=1}^n x_i^2} = \frac{2 \cdot 8}{30.43} \approx 0.526$$

7. KONFIDENSINTERVALL/INTERVALLSKATTNING

Instead of estimating the parameter, we estimate an interval in which the parameter is most likely to "be" in.

Attempts to answer the question of "was it random?"

Recall:

Let $Z \sim N(0, 1)$. The α quantile λ_α is defined through $\alpha = P(Z > \lambda_\alpha)$

By symmetry, this gives $P(-\lambda_{\alpha/2} < Z < \lambda_{\alpha/2}) = 1 - \alpha$

If $1 - \alpha = 0.95$ then we have $\lambda_{\alpha/2} = \lambda_{0.025} = 1.96$ and:

$$P(-1.96 < Z < 1.96) = 0.95$$

Example:

Slide 7 example 2

We always start with an estimator:

$$\mu^*(X) = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5} \sim N(\mu, 20)$$

Then standardize it