Financial Theory – Lecture 11

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Agenda

- Bonds.

The lecture is based on

- Sections 5.1-5.3 and 5.5 in the course book.

Bonds

A bond is a tradable loan contract.

When a bond is issued, a certain amount of money, the face value is borrowed.

During the life time of a bond are interest rate payments and repayments of the borrowed amount (amortisation).

The last time any payment is made to the owner of the bond is the maturity date of the bond.

Bonds

The bond payments are defined by

- the face value,
- the coupon rate
- and the amortisation principle of the bond.

We let $i = 1, 2, \ldots, n$ denote the payment dates.

Bonds

Notation:

$$\begin{aligned}
M_i &= \text{The total payment at time } i. \\
I_i &= \text{The interest payment at time } i. \\
X_i &= \text{The repayment of debt at time } i. \\
F_i &= \text{The outstanding debt at time } i \\
& \text{after the repayment of debt has been made.}
\end{aligned}$$

Then for $i = 1, 2, \ldots, n$

$$\begin{aligned}
M_i &= I_i + X_i \\
I_i &= qF_{i-1} \\
F_i &= F_{i-1} - X_i.
\end{aligned}$$

It also holds that

$$F_0 = F = \text{The face value, and } F_n = 0.$$

Bonds

By using $I_i = qF_{i-1}$ and

$$F_i = F_{i-1} - X_i \quad \Leftrightarrow \quad X_i = F_{i-1} - F_i$$

we can write

$$\begin{aligned} M_i &= I_i + X_i \\ &= qF_{i-1} + F_{i-1} - F_i \\ &= (1+q)F_{i-1} - F_i. \end{aligned}$$

Given M_i, $i = 1, 2, \ldots, n$ this is a recursion for F_i:

$$F_i = (1+q)F_{i-1} - M_i.$$

Bonds

- The face value is also known as the par value or the principal of the bond. I will use F to denote this amount (not F_0 as in the book).
- The outstanding debt is sometimes called the outstanding loan balance (OLB).
- Note that

$$\begin{aligned}\sum_{i=1}^{n} X_i &= \sum_{i=1}^{n} \left(F_{i-1} - F_i\right) \\ &= \underbrace{F_0}_{=F} - F_1 + F_1 - F_2 + \cdots + F_{n-2} - F_{n-1} + F_{n-1} - \underbrace{F_n}_{=0} \\ &= F,\end{aligned}$$

i.e. the total amount repaid is equal to the size of the loan.

Coupon bonds

Coupon bonds (or bullet bonds) have the following structure.

- $X_i = 0$ when $i = 1, 2, \ldots, n-1$ and $X_n = F$.
- $I_i = qF$, $i = 1, 2, \ldots, n$.

The constant q is referred to as the coupon rate.

When $q = 0$ (or $n = 1$), then the coupon is referred to as a zero coupon bond (ZCB).

A coupon bond has cash flows given by:

	qF	qF	$\cdots$	qF	$qF + F$
0	1	2	$\cdots$	$n-1$	n

Annuities

An annuity (or annuity bond) has the property that the payment M_i is equal for all times $i = 1, 2, \ldots, n$.

If $q = 0$, then

$$M_i = X_i = \frac{F}{n}$$

for an annuity.

The case $q > 0$ is more interesting (and harder to solve).

Annuities

Let $q > 0$. In this case we have the recusion

$$F_i = (1+q)F_{i-1} - M,$$

where M is a constant and independent of i (this is what characterises annuities).

Since $F_0 = F$ we get

$$\begin{array}{rcl}
F_1 & = & (1+q)F - M \\
F_2 & = & (1+q)\big[(1+q)F - M\big] - M \\
 & = & (1+q)^2 F - M[1 + (1+q)] \\
F_3 & = & (1+q)\big[(1+q)^2 F - M[1+(1+q)]\big] - M \\
 & = & (1+q)^3 F - M[1 + (1+q) + (1+q)^2] \\
\vdots & \vdots & \vdots
\end{array}$$

Finally,

$$\underbrace{F_n}_{=0} = (1+q)^n F - M\sum_{j=0}^{n-1}(1+q)^j \quad \Leftrightarrow \quad M\sum_{j=0}^{n-1}(1+q)^j = (1+q)^n F.$$

Now we use that for $\alpha \neq 1$

$$\sum_{j=0}^{n-1}\alpha^j = \frac{1-\alpha^n}{1-\alpha}.$$

With $\alpha = 1+q$:

$$\sum_{j=0}^{n-1}(1+q)^j = \frac{1-(1+q)^n}{-q} = \frac{(1+q)^n - 1}{q} \quad \Rightarrow$$

Annuities

$$M\frac{(1+q)^n-1}{q}=(1+q)^nF.$$

Divide with $(1+q)^n$:

$$M\underbrace{\frac{1-(1+q)^{-n}}{q}}_{=A(q,n)}=F.$$

The constant $A(q,n)$ is called the annuity factor.

Using this we can write

$$M=\frac{F}{A(q,n)}.$$

One can show that the outstanding payment at time i is

$$F_i=MA(q,n-i)=M\frac{1-(1+q)^{-(n-i)}}{q}.$$

Serial bonds

A serial bond pays back the face value in equal amounts:

$$X_i = \frac{F}{n}, \;\; i = 1, 2, \ldots, n.$$

If follows that the outstanding debt is

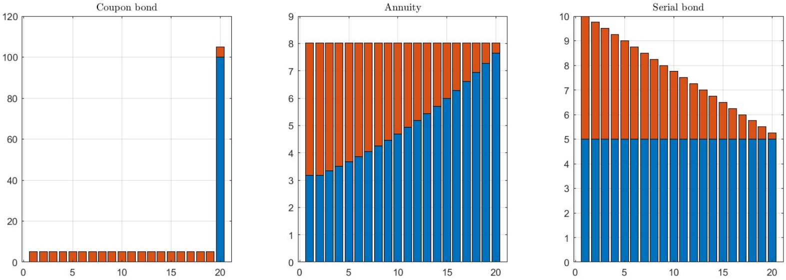
$$F_i = \frac{F}{n} \cdot (n - i) = F \left(1 - \frac{i}{n}\right),$$

and that the interest rate payment is

$$I_i = qF_{i-1} = qF \left(1 - \frac{i-1}{n}\right).$$

Bond cash flow examples

Let $n = 20$, $F = 100$ and $q = 0.05$.

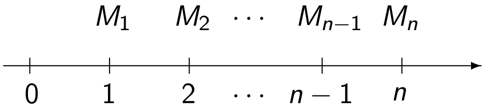


Blue = Amortisation. Red = Interest rate payments.

Bond prices

So far we have considered the cash flows of different bonds. What about the price of a bond?

Consider being at time 0 and let $M_i > 0$, $i = 1, 2, \ldots, n$, be the (deterministic) payments of the bond.



What is the value today ($i = 0$) of getting the cash flow M_i at time $i = 1, \ldots, n$?

Bond prices

Assume a constant interest rate r.

If I have have the amount $M_i/(1+r)^i$ at time zero, then this has grown to

$$\frac{M_t}{(1+r)^i} \cdot (1+r)^i = M_i$$

at time i.

Hence,

Having $\dfrac{M_i}{(1+r)^i}$ at time zero is the same has having M_i at time i.

Bond prices

With a constant discount rate r, the price B_0 at time 0 of the bond is

$$B_0 = \sum_{i=1}^{n} \frac{M_i}{(1+r)^i} = \sum_{i=1}^{n} M_i(1+r)^{-i}.$$

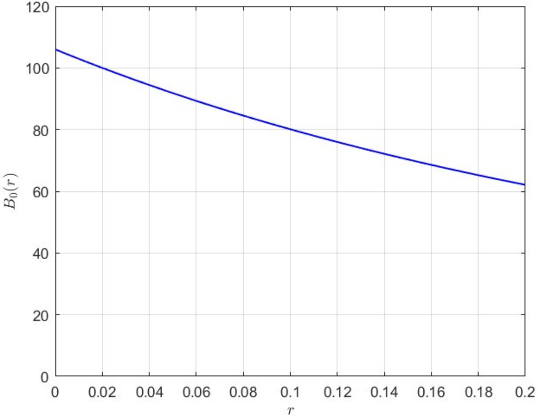
It follows that

$$\frac{\partial B_0}{\partial r} = -\sum_{i=1}^{n} iM_i(1+r)^{-i-1} < 0$$

$$\frac{\partial^2 B_0}{\partial r^2} = \sum_{i=1}^{n} i(i+1)M_i(1+r)^{-i-2} > 0$$

Bond prices

With a constant discount rate, the price $B_0(r)$ as a funtion of the rate r is a decreasing and convex function.



Bond prices

- For a ZCB with face value F maturing at n the price at $t < n$ is given by
$$Z_{t,n} = \frac{F}{(1+r)^{n-t}} = F(1+r)^{-(n-t)}.$$
- For a coupon bond with face value F, coupon rate q and maturity date n:
$$\begin{aligned} B_t &= \sum_{i=t+1}^{n} \frac{qF}{(1+r)^{i-t}} + \frac{F}{(1+r)^{n-t}} \\ &= \frac{qF}{r}\left(1 - \frac{1}{(1+r)^{n-t}}\right) + \frac{F}{(1+r)^{n-t}}. \end{aligned}$$

For the pricing of annuities and serial bonds, see Theorem 5.1 and its proof in the course book.

Bond prices

Auction date :	2023-09-27
Auction type :	Nominal bond
Loan :	1065
ISINcode :	SE0017830730
Coupon % :	1.750
Maturity :	2033-11-11
Offered/tendered :	1,500
Tendered :	4,020
Allocated institutional:	1,500
Tender ratio :	2.68
Number of bids :	26
Number of accepted bids :	9
Yield avg :	2.9397(89.715)
Low :	2.9340(89.762)
Cutoff :	2.9430(89.689)
% of Eq Price Lvl :	66.67

Perpetuities

A perpetuity (or a perpetual bond, or a consol bond) is a bond which only has interest payments and in which the face value is never paid back.

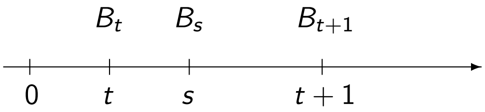
We can think of this as a coupon bond with $n = \infty$.

The price of a perpetuity is given by

$$\begin{aligned} B_0 &= \sum_{i=1}^{\infty} \frac{qF}{(1+r)^i} = qF\left(\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i - 1\right) = qF\left(\frac{1}{1-\frac{1}{1+r}} - 1\right) \\ &= qF\left(\frac{1+r}{1+r-1} - 1\right) = qF\left(\frac{1+r}{r} - \frac{r}{r}\right) \\ &= \frac{qF}{r}. \end{aligned}$$

More on bond prices

What if the time at which we want to price the bond is not an integer?



In this case, the cash flows from time $t+1$ and onwards should be included in the value. It follows that

$$B_s = (1+r)^{s-t} \cdot B_t.$$

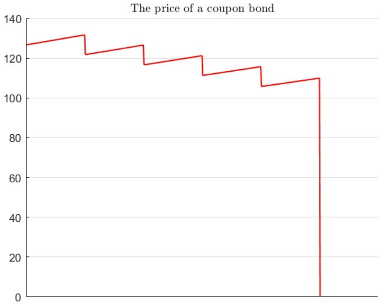
Note that in general

$$B_s \neq \frac{1}{(1+r)^{t+1-s}} \cdot B_{t+1}$$

since the value at time $t+1$ does not include the cash flow at time $t+1$.

More on bond prices

The price of a bond over time looks like this:



The accrued interest for a coupon bond is given by

$$Q_t^{\text{acc}} = tqF,$$

where t is the fraction of time that has evolved since the latest dividend payment.

More on bond prices

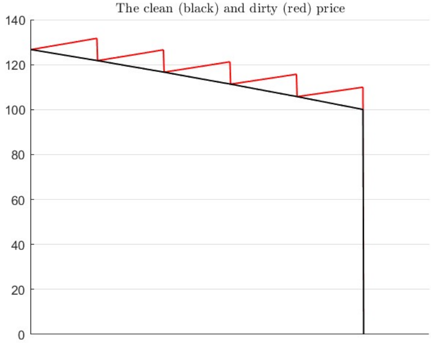
The price

$$B_t^{\text{list}} = B_t - Q_t^{\text{acc}}$$

is the listed bond price (i.e. the price shown for buyers and sellers).

But since B_t is the true value of the bond, you also need to pay the accrued interest if you buy the bond (and you get the accrued interest if you sell the bond).

B_t is called the dirty price, and B_t^{list} the clean price.



More on bond prices

For the bond types we have considered:

- $r = q \quad \Rightarrow \quad B_0 = F$. The bond is trading at par.
- $r < q \quad \Rightarrow \quad B_0 > F$. The bond is trading at a premium.
- $r > q \quad \Rightarrow \quad B_0 < F$. The bond is trading at a discount.

More on bond prices

How the price and the rate at which we discount a bond are connected is depending on the instruments.

There is also a day count convention that defines how we should calculate the fraction between two dividend dates.

Examples

- $\boxed{30/360}$ Each month of the year has 30 days, and the year has 360 days.
- $\boxed{\text{Actual}/360}$ Actual number of days, and the year has 360 days.
- $\boxed{\text{Actual}/\text{Actual}}$ Actual number of days, and the year has it actual number of days.

Yield-to-maturity

The yield-to-maturity, or just yield, y is the internal rate of return (IRR) of a bond:

$$B_0^{\text{mkt}} = \sum_{i=1}^{n} \frac{M_i}{(1+y)^i}.$$

Here B_0^{mkt} is the observed market value of the bond.

Two basic, but very important, observations are:

$$y \uparrow \;\Leftrightarrow\; B_0(y) \downarrow$$

and

$$y \downarrow \;\Leftrightarrow\; B_0(y) \uparrow.$$

Yield-to-maturity

The yield is the average rate of return we get if we hold the bond until maturity.

In general, as for the IRR, numerical methods are needed to calculate the yield.

For ZCB's, on the other hand, it is easy:

$$Z_{0,n}^{\mathrm{mkt}} = \frac{F}{(1+y_n)^n} \quad \Leftrightarrow \quad y_n = \left(\frac{F}{Z_{0,n}^{\mathrm{mkt}}} \right)^{1/n} - 1.$$

Here y_n is the yield at time 0 for a ZCB maturing at n.

Returns and yields

In general, the yield is not equal to the rate of return of a bond. With yields y_t and y_{t+1} of the bond we have

$$B_t = \sum_{i=t+1}^{n} \frac{M_i}{(1+y_t)^{i-t}} \quad \text{and} \quad B_{t+1} = \sum_{i=t+2}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}}.$$

Note that

$$\begin{aligned} M_{t+1} + B_{t+1} &= M_{t+1} + \sum_{i=t+2}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}} \\ &= \sum_{i=t+1}^{n} \frac{M_i}{(1+y_{t+1})^{i-(t+1)}} \\ &= (1+y_{t+1}) \sum_{i=t+1}^{n} \frac{M_i}{(1+y_{t+1})^{i-t}}. \end{aligned}$$

Returns and yields

It follows that the one period rate of return for this bond is

$$\begin{aligned} r_{t,t+1} &= \frac{B_{t+1} + M_{t+1} - B_t}{B_t} = \frac{B_{t+1} + M_{t+1}}{B_t} - 1 \\ &= (1 + y_{t+1}) \frac{\sum_{i=t+1}^{n} M_i (1 + y_{t+1})^{-(i-t)}}{\sum_{i=t+1}^{n} M_i (1 + y_t)^{-(i-t)}} - 1. \end{aligned}$$

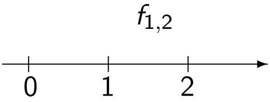
We see that if $y_{t+1} = y_t$, then

$$r_{t,t+1} = y_t = \text{The yield at time } t.$$

But in general the return is not equal to the yield.

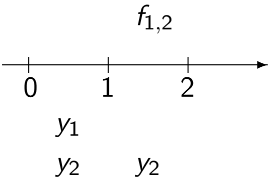
Forward rates

Today, at $t = 0$, someone if offering us the interest rate $f_{1,2}$ for the time period $(1, 2]$:



Question: How large should this interest be?

Forward rates



Consider the following two strategies.

- **Strategy 1:** Invest 1 today for 1 year, and also enter into the contract of getting the interest rate $f_{1,2}$ over $(1, 2]$.

$$\text{Payoff at } t = 2: \;\; (1 + y_1) \cdot (1 + f_{1,2}).$$

- **Strategy 2:** Invest 1 today for 2 years.

$$\text{Payoff at } t = 2: \;\; (1 + y_2)^2.$$

Forward rates

Since both investments are risk-free, they must have the same payoff:

$$(1 + y_1)(1 + f_{1,2}) = (1 + y_2)^2,$$

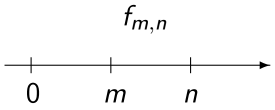
or

$$f_{1,2} = \frac{(1 + y_2)^2}{1 + y_1} - 1.$$

That is, given the ZCB yields y_1 and y_2, there is only one interest rate (the $f_{1,2}$ given by the equation above) that makes it impossible to create an arbitrage.

Forward rates

We can generalise this.



The forward rate for the period from m to n, denoted $f_{m,n}$, satisfies

$$(1 + y_n)^n = (1 + y_m)^m (1 + f_{m,n})^{n-m},$$

or

$$f_{m,n} = \left(\frac{(1 + y_n)^n}{(1 + y_m)^m} \right)^{1/(n-m)} - 1.$$

Note that the value of the forward rate $f_{m,n}$ is known at time 0.

Forward rates

With $m = n - 1$ we get

$$(1 + y_n)^n = (1 + y_{n-1})^{n-1}(1 + f_{n-1,n}).$$

By iterating this we see that

$$(1 + y_n)^n = (1 + y_1)(1 + f_{1,2})(1 + f_{2,3}) \cdots (1 + f_{n-1,n}).$$

Since y_1 is known at time 0, it holds that $f_{0,1} = y_0$.

Hence, $1 + y_n$ is the geometric average of

$$(1 + f_{0,1}), (1 + f_{1,2}), \cdots, (1 + f_{n-1,n}).$$

Defaultable bonds

A defaultable bond is a bond with risky payments.

Typically the risk lies in the fact that an issuer of a bond can not (or will not) make coupon and/or amortisation payments.

Bonds issued by stable states are generally considered non-defaultable, i.e. there is no risk of their bonds to default.

Bonds issued by firms are considered more risky, and there are rating institutes rating the quality of a firm based on the probability of not being able to pay the cash flows of issued bonds.

Defaultable bonds

TABLE 1 Bond Ratings by Moody's, Standard and Poor's, and Fitch

	Rating Agency		
Moody's	**S&P**	**Fitch**	**Definitions**
Aaa	AAA	AAA	Prime Maximum Safety
Aa1	AA+	AA+	High Grade High Quality
Aa2	AA	AA	
Aa3	AA−	AA−	
A1	A+	A+	Upper Medium Grade
A2	A	A	
A3	A−	A−	
Baa1	BBB+	BBB+	Lower Medium Grade
Baa2	BBB	BBB	
Baa3	BBB−	BBB−	
Ba1	BB+	BB+	Noninvestment Grade
Ba2	BB	BB	Speculative
Ba3	BB−	BB−	
B1	B−	B−	Highly Speculative
B2	B	B	
B3	B−	B−	
Caa1	CCC+	CCC	Substantial Risk
Caa2	CCC	—	In Poor Standing
Caa3	CCC−	—	
Ca	—	—	Extremely Speculative
C	—	—	May Be in Default
—	—	DDD	Default
—	—	D	
—	D	D	

Table 1 on p. 165 in Mishkin, F. S. (2016), "The Economics of Money, Banking, and Financial Markets" (11th Ed.), *Pearson*.