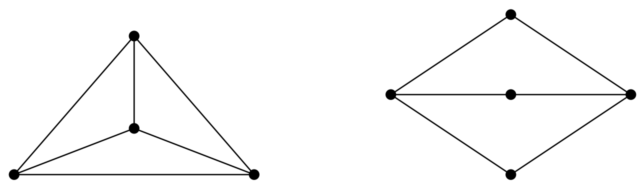
# 1 Basics of Planar Graphs

The following is a summary, hand-waving certain things which actually should be proven.

## *1.1 Plane Graphs*

A ***plane graph*** is a graph embedded in the plane such that no pair of lines intersect. The graph divides the plane up into a number of regions called ***faces***. Here are embeddings of $K_4$ and $K_{2,3}$.



**Theorem 1** *A plane graph has one face iff it is a forest.*

This uses Jordan's curve theorem, which is actually more difficult to prove than it looks. If one is to do this rigorously, we must descend into topology.

A graph is 2-connected if it does not have a cut-vertex. It is known that a graph is 2-connected iff every two vertices lie in a common cycle.

**Theorem 2** *If a graph is* 2*-connected, then every face is bounded by a cycle.*

Proof. If a vertex appears twice on a face, it must be a cut-vertex. $\diamond$

**Theorem 3** *Euler's formula:* $V - E + F = 2$ *for connected plane graph with* $V$ *vertices,* $E$ *edges and* $F$ *faces.*

Proof. By induction on $E$. If $E = V - 1$, then $G$ is a tree and we're done. Otherwise $E \geq V$ and there is a cycle containing some edge $e$. The removal of $e$ merges two faces. $\diamond$

It follows that:

**Theorem 4** *For plane graph* $E \leq 3V - 6$

Proof. Let $M$ be the number of edge–face pairs. Each edge appears twice. Each face appears at least three times. So we get $2E = M \geq 3F$. That is, $F \leq 2E/3$. Plug, into formula and algebra. $\diamond$

Consequence: $K_5$ is not planar.

**Theorem 5** *If $G$ is bipartite, then $E \leq 2V - 4$.*

Proof. Now, every face has length at least 4, and so we get $2E = M \geq 4F$. $\diamond$

Consequence: $K_{3,3}$ is not planar.

**Theorem 6** *A plane graph is maximally plane iff it is a triangulation. A triangulation has $3V - 6$ edges.*

Proof. If it is a triangulation, then cannot add an edge. If it is not a triangulation, then there is a face $f$ that is not a triangle. That is, it contains vertices $abcd \ldots$. Then we can add edge $ac$ in the face. But maybe edge $ac$ is already present. Then there are three internally disjoint $a$–$c$ paths. It follows that $bd$ cannot be already there. $\diamond$

### *1.2 Drawings*

A ***planar graph*** is one which has a plane embedding. Two drawings are ***topologically isomorphic*** if one can be continuously deformed into the other. If we wrap a drawing onto a sphere, and then off again, we can move any face to be the exterior face.

A block of a graph is a maximal subgraph without a cut-vertex.

**Theorem 7** *Graph $G$ is planar if and only if all its blocks are planar.*

Proof. Proof of planarity by induction. A graph that is not 2-connected has an end-block $B$: containing only one cut-vertex $v$. Take away $B - v$. Has planar embedding. So does $B$. Can merge by placing $v$ in the outer face of both, and identifying both $v$'s. $\diamond$

We omit the proofs of the following:

**Theorem 8** *If $G$ is 3-connected, then the face boundaries are its non-separating induced cycles.*

Indeed, Tutte showed that a 3-connected graph is planar if and only if every edge lies on exactly two non-separating induced cycles. For suitable definition of equivalent:

**Theorem 9 (Whitney)** *If $G$ is 3-connected, any two planar embeddings are equivalent.*

## 2 More on Planar Graphs

### *2.1 Special Embeddings*

Every planar graph has an embedding such that all edges are line segments.

### *2.2 Duality*

The dual of a plane graph is obtained by placing a vertex inside each face, and joining two such vertices if the faces are adjacent. Note that the dual has the same number of edges as the original. Indeed, each edge $e$ has a partner $e^*$ in the dual.

**Lemma 10** *A set of edges forms a cycle in the graph if and only if the set of dual edges form a minimal cut in the dual.*

This suggests the concept of a combinatorial duals and matroids and all that.

If the graph is suitably connected, then the dual of the dual is the original.

### *2.3 Chromatic Number*

Note that by Euler's formula, every planar graph has vertex of degree at most 5. It follows that every planar graph is 6-colorable.

Every planar graph is 4-colorable. Kempe thought he proved this. But proof wrong; his approach was salvaged to prove that planar graph is 5-colorable.

Then along came Appel and Haken and Koch. Recently, Robertson, Sanders, Seymour, Thomas redid.

Exercise: show that if every vertex has even degree then a planar graph is 3-colorable.

### *2.4 Hamiltonicity*

Tutte's theorem: a 4-connected planar graph is hamiltonian.

For a long time it was thought that 3-connected cubic planar implies hamiltonian. But Tutte found counterexample.

Every now and again someone suggests that every maximal planar graph is hamiltonian. This can be disproved by repeatedly doing the following: triangulate each region. Chung showed that 3-connected planar graph has $\Omega(\sqrt{n})$ cycle.

### *2.5 Crossing Number*

The crossing number of a graph is the minimum number of crossings in a drawing of the graph in the plain. For example, crossing numbers of $K_5$ and $K_{3,3}$ are both 1.

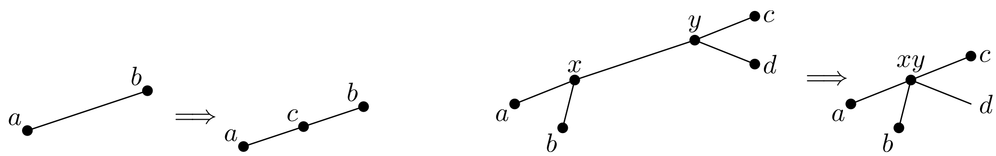
The crossing number of $K_n$ is unsolved, as is the crossing number of $K_{m,m}$. There are conjectures; that is, there are natural attempts for the drawing that are believed to be best possible.

## 3 Kuratowski's Theorem

### *3.1 Subdivisions and Minors*

A ***subdivision*** of a graph is obtained by adding vertices of degree two into edges.

The ***contraction*** of edge $e$ is to delete the ends of $e$ and build a new vertex adjacent to all vertices originally adjacent to one or both the ends of $e$. A graph $G$ is a ***minor*** of $H$ if $G$ can be obtained from $H$ by deleting vertices and contracting edges.




It is straightforward that:

**Lemma 11** *If $G$ is a subdivision of $H$, then $H$ is a minor of $G$.*

There is a converse if $H$ has small degree. For example: $G$ contains a subdivision of $K_3$ if and only if $G$ contains $K_3$ as a minor.

**Lemma 12** *If $H$ has maximum degree at most 3, then graph $G$ contains a subdivision of $H$ if and only if it contains $H$ as a minor.*

### *3.2 Kuratowski's Theorem*

Clearly, taking a subdivision of a graph preserves whether it is planar or not.

**Theorem 13** *A graph is planar if and only if it does not contain a subdivision of either $K_5$ or $K_{3,3}$.*

Proof. We saw earlier that $K_5$ and $K_{3,3}$ are not planar. So neither is any of their subdivisions.

Suppose $G$ is not planar. We may assume $G$ is minimal nonplanar; that is, removal of any edge or vertex yields a planar graph. In particular it follows that $G$ is 2-connected. Further, we may assume that $G$ is not the subdivision of another graph.

Now, claim $G$ has minimum degree at least 3. Suppose vertex $v$ has degree 2 with neighbors $x$ and $y$. If $x$ and $y$ not already adjacent, then splice $v$ out—delete $v$ and add edge $xy$. Then resultant graph is such that $G$ is subdivision thereof. And if $x$ and $y$ are already adjacent, then one can delete $v$, embed $G - v$ and add back in $v$ in either region where $xy$ is a boundary. Thus the claim is proven.

Now, it follows that there is an edge edge $e = uv$ such that $G-e$ is 2-connected. (Proof omitted.)

By the assumption of $G$, the graph $G - e$ is planar.

Consider a cycle $C$ of $G - e$ containing $u$ and $v$. We would like to add edge $e$. So we may assume that there are barriers to doing this inside and outside $C$. If we make the interior of $C$ as large as possible, we can assume properties outside $C$. Then argue that no matter how place inside $C$, get a $K_5$ or $K_{3,3}$ subdivision. (Details omitted.)

### *3.3 Wagner's Theorem*

Clearly, contractions preserve planarity. That is, if $G$ is planar then so is any contraction of $G$.

**Theorem 14** *A graph is planar if and only if it contains either $K_5$ or $K_{3,3}$ as a minor.*

One direction is a consequence of Kuratowski's theorem: if $G$ is not planar it contains a subdivsion of one of the graphs and hence a minor.

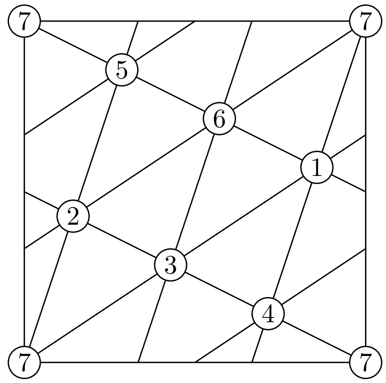
On the other hand, if $G$ is planar, then all its minors are planar; so it cannot contain either $K_5$ or $K_{3,3}$ as a minor. $\diamond$

## 4 Graphs Embedded on Surfaces

### *4.1 The Torus*

The plane and the sphere are equivalent for embedding purposes. The simplest next surface is the ***torus*** (or donut). This can be thought of as a sphere with one hole in it, or a sphere with one handle added.

One can also flatten the handle to form a bridge: a rectangle where the left-to-right and top-to-bottom edges happen simultaneously. If we invert this picture, or if we cut a hollow torus, we see that: A torus can be obtained from a ***rectangle*** where the top and bottom are glued together, while the left and right are glued together. We say that the opposites sides are identified. Here is $K_7$ embedded on the torus, using this picture: the vertex at all four corners is the same vertex.




Now, it is important that there are two types of cycles. A ***contractible*** cycle can be continuously deformed or contracted to a single point. In the plane, all cycles are contractible, but on other surfaces they are not. In the above example, cycle 136 is contractible, but cycle 123 is noncontractible. If you slice through the torus on a noncontractible cycle, this will break the handle; if you slice through the torus on a contractible cycle, you will get two pieces, a sphere and a torus.

A region is called a ***2-cell*** if its boundary is a contractible cycle. An embedding is a ***2-cell*** embedding if every face is a 2-cell.

## *4.2 The Orientable Surface of Genus $h$*

In general we can add $h$ handles. The resultant surface is called $S_h$ and $h$ is the genus of the surface. There is also a way to depict this is as a $4h$-sided polygon with opposite sides suitably identified. There is an Euler's formula:

**Theorem 15** *For a 2-cell embedding of a graph $G$ on $S_h$, it holds that $V - E + F = 2 - 2h$.*

## *4.3 The Genus of a Graph*

The ***genus*** of a graph is the minimum number of handles for a surface needed to embed the graph.

Given each complete graph, one can apply Euler formula to get a lower bound on the genus. It is possibly surprising, but showing that the complete graph embeds on the surface it is supposed to takes a lot of proving! Finished by Ringel–Youngs.

In contrast, proving that the complete bipartite graph embeds on the lowest genus that Euler allows is not too hard. It is probably not too surprising that the genus of a graph is the sum of the genera of its blocks. (Battle, Harary, Kodama and Youngs).

## 5 Minors and Well-quasi-orderings

### *5.1 The Trees are Well-quasi-ordered by Topological Minor*

A quasi-ordering is a reflexive transitive relation. A set $X$ is ***well-quasi-ordered*** by $\leq$ if given any infinite subsequence $\{x_k\}$ of $X$ there are two elements $x_i$ and $x_j$ with $i < j$ and $x_i \leq x_j$.

**Lemma 16** *A set is WQO iff it contains neither an infinite antichain (no two elements comparable) nor an infinite strictly decreasing sequence.*

For two subsets $A$ and $B$ of $X$, we define $A \leq B$ if there is a 1–1 mapping $f$ from $A$ to $B$ such that $a \leq f(a)$ for all $a \in A$.

**Lemma 17** *If $X$ is WQO by $\leq$, then so is the set of finite subsets of $X$.*

We say that graph $G$ is a ***topological minor*** of $H$ is $H$ contains a subgraph that is a subdivision of $G$. People also say that $G$ is a homeomorphic subgraph of $H$.

**Theorem 18 (Kruskal)** *The finite trees are WQO by the topological minor relation.*

That is, given any infinite sequence of trees, there is $i < j$ such that tree $T_j$ contains a subdivision of tree $T_i$ as a subgraph.

### *5.2 The Minor Theorems*

**Theorem 19 (Robertson & Seymour)** *The finite graphs are WQO by the minor relation.*

As a consequence we get:

**Theorem 20** *Every graph property closed under minors has a characterization by a finite list of forbidden minors.*

In particular, for every surface there is a finite list of forbidden minors. Actually, they proved this consequence as a stepping stone to the overall theorem.

There is an $O(n^3)$ algorithm to test for existence of fixed minor.

For example, a ***linkless embedding*** of a graph is an embedding of the graph in 3 dimensions such that there are no two interlocking cycles. Having a linkless embedding is clearly closed under taking minors. So before the problem was not known to be decidable; now it is known to be in P.

But note that the algorithms have huge hidden constants, and are nonconstructive in parts.

## 6 References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