Exam - Fourier analysis

Department of Mathematics	Exam in Fourier Analysis, 5 credits
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Writing time: 08:00–13:00. Allowed equipments: writing materials, table of formulæ. There are 8 problems in this exam. You have to motivate every step in your solution to get the full score from a question.

To pass the exam you need at least one point on exercise 1b, 2 and 3 (or similar exercises). You can obtain the grades 3, 4 and 5 on the exam by the requirements given in the table below.

Grade	Requirements				
3	3 A	7 B	2 C		18 total
4	4 A	10 B	4 C		25 total
5	4 A	10 B	4 C	4*	32 total
Max	8 A	24 B	8 C	10*	40 total

Learning Outcomes:

- Basic concepts and theorems (A points)
- Basic numeracy skill (B points)
- Ordinary or Partial differential equations (C points)

1. (a) State the uniqueness theorem for the Laplace transform. 2 A

   (b) Solve the ODE

$$\begin{cases} y'(t) + y(t) = 3 \\ y(0) = 2 \end{cases}$$

   using some method that has been taught during the course. 3 C

Solution: (a) If $F(s) = G(s)$, then $f(t) = g(t)$ a.e. for $t > 0$.

(b) The Laplace transform yields

$$\begin{aligned} sY(s) - y(0) + Y(s) &= \frac{3}{s} \\ Y(s)(s+1) &= \frac{3}{s} + 2 \\ Y(s) = \frac{2s+3}{s(s+1)} &= \frac{A}{s} + \frac{B}{s+1} \end{aligned}$$

This gives $A = 3, B = -1$, so

$$Y(s) = \frac{3}{s} - \frac{1}{s+1}$$

The inverse transform now gives

$$y(t) = 3 - e^{-t}$$

2. Let f be an even, 1 periodic function with $f(x) = x^2$ for $0 \leq x < 1/2$.

   (a) Find the Fourier series of f. 2 B

   (b) Calculate the series

   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

   1 B

   (c) Calculate the series

   $$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

   1 A, 1 B

Solution: (a) Even $\Rightarrow b_n = 0$. If $n > 0$,

$$\begin{aligned}
a_n &= 2\int_{-1/2}^{1/2} x^2 \cos(2\pi n x)\,\mathrm{d}x = 4\int_0^{1/2} x^2 \cos(2\pi n x)\,\mathrm{d}x \\
&= \cancel{\left[2x^2 \frac{\sin(2\pi n x)}{\pi n}\right]_0^{1/2}} - \int_0^{1/2} 4x \frac{\sin(2\pi n x)}{\pi n}\,\mathrm{d}x \\
&= \left[2x \frac{\cos(2\pi n x)}{\pi^2 n^2}\right]_0^{1/2} - \cancel{\int_0^{1/2} 2 \frac{\cos(2\pi n x)}{\pi^2 n^2}\,\mathrm{d}x} = \frac{\cos(\pi n)}{\pi^2 n^2} = \frac{(-1)^n}{\pi^2 n^2}
\end{aligned}$$

For $n = 0$:

$$a_0 = 4\int_0^{1/2} x^2\,\mathrm{d}x = 4\left[\frac{x^3}{3}\right]_0^{1/2} = \frac{1}{6}$$

So

$$f(x) \sim \frac{1}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2\pi n x)}{\pi^2 n^2}$$

(b) Take $x = 0$. Since f is continuous in that point we have equality with the Fourier series there. Hence

$$0 = f(0) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2\pi n 0)}{\pi^2 n^2} = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Solving for the series yields

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

(c) We use Parseval's formula:

$$\int_{-1/2}^{1/2} |f(x)|^2 \, \mathrm{d}x = 2\int_0^{1/2} x^4 \, \mathrm{d}x = \frac{2}{5}\left[x^5\right]_0^{1/2} = \frac{1}{5 \cdot 2^4}$$
$$\frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 = \frac{1}{2^4 \cdot 9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\pi^4 n^4}$$

So

$$\frac{1}{5 \cdot 2^4} = \frac{1}{2^4 \cdot 9} + \frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2 \cdot \pi^4 \cdot (9-5)}{2^4 \cdot 5 \cdot 9} = \frac{\pi^4}{90}$$

3. Use the Fourier transform to calculate

$$\int_{\mathbb{R}} \frac{e^{ix}}{(x+1)^2+1} \, \mathrm{d}x$$

5 B

Solution: Define $f(x) = \frac{1}{(x+1)^2+1}$ and $g(x) = \frac{1}{x^2+1}$, so that $f(x) = g(x+1)$. Now $\hat{g}(\xi) = \pi e^{-|\xi|}$ and thus $\hat{f}(\xi) = \pi e^{i\xi - |\xi|}$. We write

$$\int_{\mathbb{R}} \frac{e^{ix}}{(x+1)^2+1} \, \mathrm{d}x = \hat{f}(-1) = \pi e^{-(1+i)}$$

4. Calculate, using separation of variables, the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0,1), t \in (0,\infty) \\ u(0,t) = 1, u(1,t) = 2 & t \in (0,\infty) \\ u(x,0) = 1 & x \in (0,1) \end{cases}$$

5 C

Solution: First we need to homogenise the problem, since we don't have 0 in the BC. Make the ansatz $u(x,t) = v(x,t) + \varphi(x)$. Then

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \varphi''(x), & 0 < x < 1, t > 0, \\ v(0,t) + \varphi(0) = 1, v(1,t) + \varphi(1) = 2, & t > 0. \end{cases}$$

We want

$$\begin{cases} \varphi''(x) = 0 \\ \varphi(0) = 1, \varphi(1) = 2 \end{cases}$$

This has solution $\varphi(x) = Ax + B$. $\varphi(0) = 1 \Rightarrow B = 1$, $\varphi(1) = 2, \Rightarrow A = 1$

Now the original problem has turned into

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} & x \in (0,1), t \in (0,\infty) \\ v(0,t) = v(1,t) = 0 & t \in (0,\infty) \\ v(x,0) = -x & x \in (0,1) \end{cases}$$

Make the assumption $v(x,t) = X(x)T(t)$. Then $X(x)T'(t) = X''(x)T(t)$ or

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

We consider the X-equation. Divide into three cases.

Case 1 - $\lambda < 0$
The solution is given by

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

Now BC yields $X(0) = X(1) = 0$ so

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{\sqrt{-\lambda}} + C_2 e^{-\sqrt{-\lambda}} = 0 \end{cases}$$

The system of equation corresponds to matrix with determinant $\neq 0$ so the unique solution is $C_1 = C_2 = 0$.

Case 2 - $\lambda = 0$
We have $X(x) = C_1 x + C_2$. $X(0) = X(1) = 0 \Rightarrow C_1 = C_2 = 0$.

Case 3 - $\lambda > 0$
The solution is given by

$$X(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right)$$

$X(0) = 0 \Rightarrow C_1 = 0$. $X(1) = 0$ yields

$$C_2 \sin\left(\sqrt{\lambda}\right) = 0$$

which has the solution

$$\sqrt{\lambda_n} = n\pi, \quad n \in \mathbb{Z}_+.$$

or

$$\lambda_n = \pi^2 n^2$$

Case 3 yields

$$X_n(x) = C_n \sin(n\pi x), \quad n \in \mathbb{Z}_+.$$

Turning to T we want to find the solution to the equation

$$T_n'(t) = -\pi^2 n^2 T_n(t)$$

which is

$$T_n(t) = D_n e^{-n^2\pi^2 t}$$

Hence the general solution will be given

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

By the IC (of v)

$$-x = v(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

This is a Fourier series of an odd function and hence we have to extend it in an odd sense, which is $f(x) = -x$ for $-1 \leq x < 0$ and $C_n = b_n$ are the Fourier coefficients. Hence

$$\begin{aligned} C_n &= \int_{-1}^{1} f(x) \sin(nx)\,\mathrm{d}x = -2\int_0^1 x \sin(n\pi x)\,\mathrm{d}x \\ &= 2\left[x \frac{\cos(n\pi x)}{n\pi}\right]_0^1 - 2\cancel{\int_0^1 \frac{\cos(n\pi x)}{n\pi}}\,\mathrm{d}x = 2\frac{\cos(n\pi)}{n\pi} = \frac{2(-1)^n}{n\pi}. \end{aligned}$$

Let us put the pieces together! The formula $u(x,t) = v(x,t) + x + 1$ now gives

$$u(x,t) = x + 1 + \sum_{k=0}^{\infty} \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x)$$

5. (a) Assume that f is a function of at most polynomial growth. Write down the expression of how $f \in \mathscr{S}'(\mathbb{R})$ acts on a test function $\varphi \in \mathscr{S}(\mathbb{R})$. 2 A

   (b) Show that $xg = \mathrm{p.v.}\,\frac{1}{x}$ in $\mathscr{S}'(\mathbb{R})$, where

$$g(\varphi) := \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2}\,\mathrm{d}x \text{ and } \mathrm{p.v.}\,\frac{1}{x}(\varphi) := \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

3 B

Solution: (a)

$$f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$$

(b)

$$\begin{aligned} xg(\varphi) = g(x\varphi) &= \lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{x\varphi(x) - 0\varphi(0)}{x^2}\,\mathrm{d}x \\ &= \lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \mathrm{p.\,v.}\,\frac{1}{x}(\varphi). \end{aligned}$$

6. (a) State Riemann-Lebesgue Lemma for the Fourier transform. 2 A

   (b) Prove that the Fourier transform of $f \in L^1(\mathbb{R})$ is continuous. Motivate every step in your proof! Hint: It is enough to show that $\lim\limits_{\xi\to\xi_0} \hat{f}(\xi) = \hat{f}(\xi_0)$ using Lebesgue Dominated Convergence Theorem. 3 B

Solution: (a) For $f \in L^1(\mathbb{R})$,

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0$$

(b) Observe that $\left|f(x)e^{-ix\xi}\right| = |f(x)| \in L^1(\mathbb{R})$, so LDCT applies.

$$\begin{aligned} \lim_{\xi\to\xi_0} \hat{f}(\xi) &= \lim_{\xi\to\xi_0} \int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x = \int_{\mathbb{R}} f(x) \lim_{\xi\to\xi_0} e^{-ix\xi}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)e^{-ix\xi_0}\,\mathrm{d}x = \hat{f}(\xi_0) \end{aligned}$$

7.* Let $g \in \mathcal{C}^2(\mathbb{T})$ and define T_g as the mapping

$$T_g f(x) := \sum_{n=-\infty}^{\infty} \hat{g}(n)\hat{f}(n)e^{inx}$$

for $f \in L^2(\mathbb{T})$. **Show that** $\|T_g f\|_{L^2(\mathbb{T})} \leq C_g \|f\|_{L^2(\mathbb{T})}$, where C_g only depends on the function g.

1 A, 4 B

Solution: By the uniqueness theorem we have

$$\widehat{T_g f}(n) = \hat{g}(n)\hat{f}(n),$$

so using that $\hat{g}(n)$ is uniformly bouneded, i.e. $|\hat{g}(n)| \leq C_g$ we have

$$\|T_g f\|_{L^2(\mathbb{T})}^2 = 2\pi \sum_{n=-\infty}^{\infty} \left|\hat{g}(n)\hat{f}(n)\right|^2 \leq 2\pi {C_g}^2 \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 = {C_g}^2 \|f\|_{L^2(\mathbb{T})}^2$$

8.* Assume that the kernel $K_n(x)$ satisfies the following properties:

(i) $K_n(x)$ is even and positive for all $n \in \mathbb{N}$.

(ii) $\int_{\mathbb{T}} K_n(x)\,\mathrm{d}x = 1, \quad \forall n \in \mathbb{N}.$

(iii) For every $\delta > 0$, $\lim_{n\to\infty} \int_\delta^\pi K_n(y)\,\mathrm{d}y = 0.$

Prove that $\lim_{n\to\infty} \|K_n * f - f\|_{L^1(\mathbb{T})} = 0$

Hint: for all $f \in L^1(\mathbb{T})$, $\lim_{y\to 0} \|f(x-y) - f(x)\|_{L^1_x(\mathbb{T})} = 0.$

5 B

Solution: We have, using Minkowski integral inequality,

$$\begin{aligned}
\|K_n * f(x) - f(x)\|_{L^1(\mathbb{T})} &\leq \int_{\mathbb{T}} \|f(x-y) - f(x)\|_{L^1_x(\mathbb{T})} K_n(y)\,\mathrm{d}y \\
&= \int_{|y|\leq\delta} \|f(x-y) - f(x)\|_{L^1_x(\mathbb{T})} K_n(y)\,\mathrm{d}y \\
&+ \int_{|y|>\delta} \|f(x-y) - f(x)\|_{L^1_x(\mathbb{T})} K_n(y)\,\mathrm{d}y =: I + II
\end{aligned}$$

We see that $\lim_{\delta\to 0} I = 0$ by the hint.

We analyse II.

$$\begin{aligned}
II &\leq \int_{|y|>\delta} \left(\|f(x-y)\|_{L^1_x(\mathbb{T})} + \|f(x)\|_{L^1_x(\mathbb{T})}\right) K_n(y)\,\mathrm{d}y \\
&= \int_{|y|>\delta} 2\|f(x)\|_{L^1_x(\mathbb{T})} K_n(y)\,\mathrm{d}y = 2\|f(x)\|_{L^1_x(\mathbb{T})} \int_{|y|>\delta} K_n(y)\,\mathrm{d}y
\end{aligned}$$

This shows that $\lim_{n\to\infty} II = 0$.