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2. Functions and partial derivatives

We solve the problems together in the exercise sessions. Note that these problems are optional and for learning purposes: solving these does not provide extra points. Actual home assignments (giving you extra points) are given separately.

It is advised to take a look of the problems beforehand. Note that some of the problems might be very challenging, so do not feel bad if you are unable to solve them independently: we will go through the solutions together!

Problems

2.1 Describe (e.g. by drawing) the following curves:

- $r(t) = (t, t^2), \quad -2 \leq t \leq 2,$
- $r(t) = (3\cos t, 3\sin t), \quad 0 \leq t \leq 2\pi,$
- $r(t) = (2\cos t, 3\sin t), \quad 0 \leq t \leq 2\pi,$
- $r(t) = (1 + \cos t, -1 + \sin t), \quad \pi \leq t \leq 2\pi,$
- $r(t) = (-1 + t, 1 - 2t), \quad t \in \mathbb{R},$
- $r(t) = (-1 + t^3, 1 - 2t^3), \quad t \in \mathbb{R},$
- $r(t) = (t, t, 1 - t), \quad 0 \leq t \leq 1,$
- $r(t) = (t, 2\cos t, 2\sin t), \quad 0 \leq t \leq 2\pi,$

2.2 Study the following limits:

- $\lim_{(x,y)\to(1,1)} \frac{xy-1}{x-1}$,
- $\lim_{(x,y)\to(1,1)} \frac{x-y}{x-1}$,
- $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\log(x^2 + y^2)$,
- $\lim_{(x,y)\to(1,0)} \frac{y^2 \log x}{(x-1)^2+y^2}$,
- $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2}$,
- $\lim_{x^2+y^2\to\infty} \frac{\log(x^2+y^2)}{x^2+y^2+xy}$,
- $\lim_{x^2+y^2\to\infty} xye^{-x^2-y^2}$.

2.3 Compute the gradient of the following function.

- $f(x, y) = (xy^2 + 1)^5$,

- $f(x,y) = \frac{x+y}{x-y}$,
- $f(x,y,z) = \log|xy+xz+yz|$,
- $f(x,y,z) = x^{y^z}$.

2.4 Find the tangent plane of the paraboloid $z = x^2 + 4y^2$ at point $(1,1,5)$.

2.5 Let $Q(p,v) = g(pv) - f(pv)\log p$, where g and f are differentiable one variable functions. Show that Q satisfy

$$v\frac{\partial Q}{\partial v} - p\frac{\partial Q}{\partial p} = f(pv), \quad p, v > 0.$$

This equation arises from thermodynamics.

2.6 Suppose that $f(x,y)$ is differentiable on $\mathbb{R}^2$ and partial derivatives satisfy

$$f'_x - 3f'_y = 0.$$

Show that a line $3x + y = 1$ is a contour for f.

2.7 Let f be a differentiable one variable function. Show that

- $u(x,y) = f(2x+3y)$ satisfy $3u'_x - 2u'_y = 0$.
- $u(x,y) = f(xy)$ satisfy $xu'_x - yu'_y = 0$.

2.8 Let $F(x,y)$ be differentiable and set $f(t) = F(t,-t)$ and $g(t) = F(t,2t)$. If $f'(0) = 2$ and $g'(0) = 0$, what are the values $F'_x(0,0)$ and $F'_y(0,0)$?

2.9 A function $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ given by $F = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is called potential field with a potential function $U : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla U = F$. Show that

$$U(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a potential function to field $F = -\frac{1}{|(x,y,z)|^3}(x,y,z)$.

2.10 Compute the directional derivative of $f(x,y,z) = \frac{xy^2z^3}{2+x}$ at a point $(2,2,1)$ into the direction $(-4,2,-4)$.

2.11 Let $f(x,y) = y^2/x$ and and a point $(x_0,y_0) = (2,1)$. Compare $\Delta f = f(2+\Delta x, 1+\Delta y) - f(2,1)$ and $df = f'_x(2,1)\Delta x + f'_y(2,1)\Delta y$ for

- $\Delta x = 0.1, \Delta y = 0.3$,
- $\Delta x = 0.01, \Delta y = 0.03$.

2.12 Show that a good approximation for

$$\frac{1}{\sqrt{3+x+\sqrt{1-y}}}$$

with small x and y is given by

$$\frac{1}{\sqrt{3+x+\sqrt{1-y}}} \approx \frac{1}{2} - \frac{x}{16} + \frac{y}{32}.$$

2.13 Prove the following (Lemma 4.1 in the book): f is differentiable at (a,b) if and only if

$$f(a+h,b+k) = f(a,b) + Ah + Bk + h\rho_1(h,k) + k\rho_2(h,k)$$

for some ρ_1, ρ_2 satisfying $\rho_i(h,k) \to 0$ as $(h,k) \to (0,0)$.

2.14 (continuation) Prove the following (Theorem 4.3 in the book): Suppose f has partial derivatives f'_x and f'_y that are continuous at (a,b). Then f is differentiable at (a,b). Justify yourself that analogous result is valid in $\mathbb{R}^n$.

2.15 Prove the following (Theorem 4.5 in the book): If f is differentiable on a connected set D and $\nabla f = 0$ on D, then f is a constant.

2.16 Prove the following (Theorem 4.6 in the book): If f is differentiable at (a,b) and $v = (v_1,v_2)$ is a vector with $|v| = 1$. Then $f'_v(a,b) = \nabla f(a,b) \cdot v$. Conclude that we have

$$-|\nabla f(a,b)| \leq f'_v(a,b) \leq |\nabla f(a,b)|$$

and the direction of highest increase (decrease) is the direction of the gradient (negative gradient).

2.17 Let $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that partial derivatives f''_{xy}, f''_{yx} exists at $(0,0)$ and we have $f''_{xy}(0,0) = -1$ and $f''_{yx}(0,0) = 1$.

2.18 Let $h(x,y) = f(g_1(x,y), g_2(x,y))$, where $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $g_i : \mathbb{R}^2 \mapsto \mathbb{R}$ for $i = 1,2$. Show that we have

$$h'_x(x,y) = f'_x(g_1(x,y), g_2(x,y))g'_{1,x}(x,y) + f'_y(g_1(x,y), g_2(x,y))g'_{2,x}(x,y).$$