Summary of complex number formulae, Part 3, MATH103

Note on principal values of arguments

The principal value of the argument of a complex number is an angle θ satisfying $-\pi < \theta \leq \pi$. Thus $z = re^{i\theta}$ will have θ in this range when the principal value is used. However,

$$e^{2\pi i} = 1, \text{ so that } z = re^{i(\theta+2k\pi)} = re^{i\theta}(e^{2\pi i})^k = re^{i\theta}$$

for any integer k. That is, the angle can be changed by any multiple of 2π without changing z. You are always allowed to alter the argument of a complex number by a multiple of 2π (**note** 2π **not** π !) without changing the complex number. The principal value is simply what we get when we adjust the argument, if necessary, to lie between $-\pi$ and π. For example, $z = 2e^{5\pi i/4} = 2e^{-3\pi i/4}$, subtracting 2π from the argument $5\pi/4$, and the principal value of the argument of z is $-3\pi/4$. Unless specifically asked you need not check that you have given the principal value of the argument.

Method of solving equations of the form $z^n = a$ where a is complex

(i) Find the modulus $|a|$ and argument α of a, and write $a = |a|e^{i\alpha}$.
(ii) Write $z = re^{i\theta}$ where r and θ are unknowns to be solved for.
(iii) Therefore $z^n = r^n e^{in\theta}$.
(iv) Now write the original equation as $r^n e^{in\theta} = |a|e^{i\alpha}$.

Using the result at the bottom of summary page 2 we deduce that

- $r^n = |a|$, giving $r = |a|^{1/n}$ (unique solution remembering $r > 0$);
- $n\theta = \alpha + 2k\pi$ for some integer k, that is
$$\theta = \frac{\alpha}{n} + \frac{2k\pi}{n}.$$

In fact we can always restrict attention to the values $k = 0, 1, \ldots n-1$. Thus the solutions are $z = re^{i\theta}$ for the above values of r and θ (n values in all). Note that the angles θ might not lie between $-\pi$ and π, that is might not be the principal values of the argument of z.

Note that the solutions of $z^n = a$ are always equally spaced round the circle of radius $r = |a|^{1/n}$ centred at the origin. The angle between successive solutions is always $2\pi/n$.

If a question asks for it, you can now turn the values of z into $x + iy$ form using $x = r\cos\theta,\ y = r\sin\theta$.

Note that in addition to the sines and cosines on summary sheet 2, you should also know that $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$. For example $\cos(5\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2$ and $\sin(2\pi/3) = \sin(\pi/3) = \sqrt{3}/2$.

The quadratic formula also works when the coefficients are complex numbers, that is

$$az^2 + bz + c = 0 \text{ has solutions } z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

when a, b, c are complex numbers. The interpretation of the square root here is that we find the two complex numbers whose squares are $b^2 - 4ac$. In practice, you will only be given exercises to do where you are first told to verify that some complex number w, when squared, gives $b^2 - 4ac$, so it follows that the square roots of $b^2 - 4ac$ are exactly $\pm w$.

If z is a solution of $z^n = a$ then $\overline{z}$ is a solution of $z^n = \overline{a}$.