UPPSALA UNIVERSITY
Department of Mathematics
Rolf Larsson

Exam in Mathematical Statistics
Inference Theory II, 1MS037
2021–06–08

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Suppose $X_1, ..., X_n$ are independent, distributed as X which has density function
$$f(x;\alpha) = \frac{2\alpha x}{(1+x^2)^{\alpha+1}},$$
for $x > 0$, and 0 otherwise.
   (a) Does this distribution belong to an exponential family?
   If so, which is the natural parameter? (2p)
   (b) Find a sufficient statistic for α. (2p)
   (c) Which is the smallest variance that an unbiased estimator of α can attain? (2p)

2. Suppose $X_1, ..., X_n$ are independent and normally distributed with expectation μ and variance σ^2. Let $\theta = (\mu, \sigma^2)$.
   Which of the following statistics are sufficient for θ, and which are also minimal sufficient for θ? Motivate your answer. (6p)
   (a) $\sum_{i=1}^n X_i$
   (b) $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$
   (c) $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^3)$
   (d) $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i^3)$

3. Suppose $X_1, ..., X_n$ are independent, distributed as X which is discrete with probability function
$$p(x;\theta) = \begin{cases} \frac{1}{4+\theta} & \text{if } x = 0, \\ \frac{3}{4+\theta} & \text{if } x = 1, \\ \frac{\theta}{4+\theta} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Calculate the maximum likelihood estimate (MLE) of θ. (2p)
   (b) Assume that the observations are $2, 0, 2, 2$. Consider testing H_0: $\theta = 4$ vs H_1: $\theta > 4$, using the MLE as test statistic.
   Calculate the p value. (3p)

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4. Suppose X_1 and X_2 are independent and Poisson with parameter λ.

   (a) Show that $S = X_1 + X_2$ is a sufficient statistic for λ. (2p)

   (b) Show that the statistic $T = X_1$ is unbiased for λ. (1p)

   (c) Use the Rao-Blackwell theorem to find an unbiased estimator of λ with smaller variance than the variance of T. (3p)

5. Consider testing that the observation x comes from a discrete distribution with probability function $p_0(x)$ vs the alternative that it comes from a discrete distribution with probability function $p_1(x)$, where these two probability functions are given in the following table:

x	1	2	3	4	5
$p_0(x)$	0.3	0.2	0.2	0.2	0.1
$p_1(x)$	0.1	0.1	0.2	0.3	0.3

   (a) Which is the most powerful (MP) test at level $\alpha = 0.2$? (2p)

   (b) Calculate the size of the type II error and the power for the MP test. (2p)

   (c) Calculate sizes of the errors of type I and II as well as the power for the test with critical region $\{x = 4\}$. Compare to the power for the MP test. (2p)

6. Suppose $X_1, ..., X_n$ are independent, distributed as X which has density function

$$f(x; \beta) = \frac{2x}{\beta} \exp\left(-\frac{x^2}{\beta}\right), \quad x > 0,$$

   and 0 otherwise. Let $x_1, ..., x_n$ be the observations.

   (a) Show that this distribution belongs to a one-parameter exponential family. (1p)

   (b) Give the natural parameter and the sufficient statistic. (1p)

   (c) Consider testing H_0: $\beta \leq \beta_0$ vs H_1: $\beta > \beta_0$. Show that the uniformly most powerful (UMP) test has critical region $\sum_{i=1}^{n} x_i^2/n > C$ where C is some constant. (3p)

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7. Suppose $X_1, ..., X_n$ are independent and normally distributed with expectation μ and variance 1. We want to test H_0: $\mu = 0$ vs H_1: $\mu \neq 0$ at level α.

Define z_α through $P(Z < z_\alpha) = \alpha$ where Z is standard normal.
Let $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$, and let T_{obs} be the observed value of $T = \sqrt{n}\bar{X}$.

(a) Which of the following tests (if any) are unbiased size α tests, and why? (3p)

i. The test that rejects if.f. $T_{obs} > z_{1-\alpha}$.
ii. The test that rejects if.f. $|T_{obs}| > z_{1-\frac{\alpha}{2}}$.
iii. The test that rejects if.f. $T_{obs} < z_{\frac{\alpha}{4}}$ or $T_{obs} > z_{1-\frac{3\alpha}{4}}$.

(b) Is any of the tests in (a) uniformly most powerful unbiased (UMPU), and in that case, which one? Explain why! (3p)

GOOD LUCK!