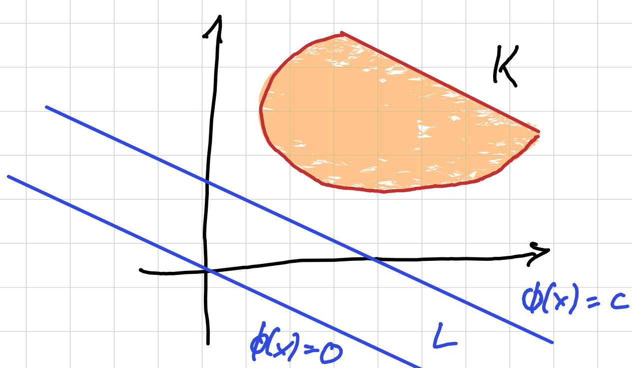
The separating hyperplane theorem

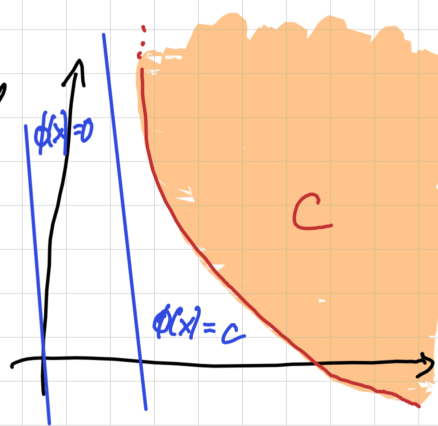
Theorem (separating hyperplane theorem)
Let L be a linear subspace of $\mathbb{R}^n$ and K a convex compact subset of $\mathbb{R}^n$, disjoint from L. Then there exists a linear functional $\phi: \mathbb{R}^n \to \mathbb{R}$ s.t.

- $\phi(x) = 0$ for $x \in L$
- $\phi(x) \geq c$ for all $x \in K$, where $c > 0$.

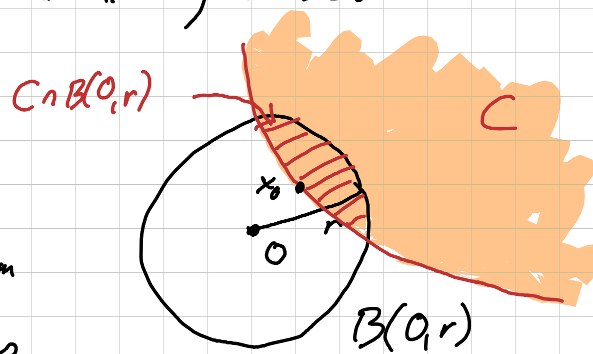


For the proof we will first prove the following lemma:

Lemma: Let $C \subseteq \mathbb{R}^d$ be closed and convex, s.t. $0 \notin C$. Then there exists linear $\phi: \mathbb{R}^d \to \mathbb{R}$ and $c > 0$ s.t. $\phi(x) \geq c$ for all $x \in C$.



Proof: If C is empty the statement is trivial, so we assume C is non-empty. Take $r > 0$ large s.t. the closed ball $B(0,r) = \{x \in \mathbb{R}^d : \|x\| \leq r\}$ intersects C.



The intersection is non-empty and closed, hence compact. Since the norm $\|.\|$ is continuous it has a minimum in $C \cap B(0,r)$. That is, there exists $x_0 \in C \cap B(0,r)$ s.t. $\|z\| \geq \|x_0\|$ for all $z \in C \cap B(0,r)$. Further, $\|x_0\| \leq r < \|z\|$ for all $z \in C$. By convexity, if $z \in C$, we have $\lambda x_0 + (1-\lambda) z \in C$ for all $\lambda \in [0,1]$.

So, $\|x_0\|^2 \leq \|\lambda x_0 + (1-\lambda) z\|^2$

$$x_0 \cdot x_0 \leq (\lambda x_0 + (1-\lambda) z) \cdot (\lambda x_0 + (1-\lambda) z)$$
$$= \lambda^2 x_0 \cdot x_0 + (1-\lambda)^2 z \cdot z + 2\lambda(1-\lambda)\, x_0 \cdot z$$
$$(1-\lambda^2)\, x_0 \cdot x_0 \leq (1-\lambda)^2 z \cdot z + 2\lambda(1-\lambda)\, x_0 \cdot z$$

$$(1-\lambda)(1+\lambda)\, x_0 \cdot x_0 \le (1-\lambda)^2 z . z + 2\lambda (1-\lambda)\, x_0 \cdot z$$

$$(1+\lambda)\, x_0 \cdot x_0 \le (1-\lambda)\, z . z + 2\lambda\, x_0 \cdot z$$

and taking $\lambda \to 1$,

$$2\, x_0 \cdot x_0 \le 2\, x_0 \cdot z \quad \Rightarrow \quad x_0 \cdot x_0 \le x_0 \cdot z .$$

Hence the linear functional $\phi : z \mapsto x_0 \cdot z$ satisfies $\phi(z) = x_0 \cdot z \ge c := x_0 \cdot x_0 = \|x_0\|^2 > 0$. □

<u>Proof</u> (of the separating hyperplane theorem):

Let $C = K - L = \{ k - l : k \in K,\ l \in L \}$.

This set is convex: for $x_1 = k_1 - l_1 \in C$ and $x_2 = k_2 - l_2 \in C$ we have

$$\lambda x_1 + (1-\lambda) x_2 = \lambda k_1 - \lambda l_1 + (1-\lambda) k_2 - (1-\lambda) l_2$$

$$= \underbrace{\lambda k_1 + (1-\lambda) k_2}_{\in K} - \underbrace{(\lambda l_1 + (1-\lambda) l_2)}_{\in L} \in K - L = C .$$

C is also closed. Let $c_n = k_n - l_n$ be a sequence that converges (in $\mathbb{R}^d$) as $n \to \infty$.

By compactness of K, there exists a subsequence n_r s.t. $k_{n_r} \to k_\infty$ for some $k_\infty \in K$.

Since $c_n \to c_\infty$ we have that

$l_{n_r} = k_{n_r} - c_{n_r}$ converges to $l_\infty = k_\infty - c_\infty$.

Since L is closed, $l_\infty \in L$ and

$c_\infty = k_\infty - l_\infty \in K - L$ and as c_n

arbitrary, C is closed.

We now apply the lemma above and conclude that

there exists a functional ϕ s.t. $\phi(x) \geq c > 0$

for all $x \in C$. Hence, for all $k \in K, l \in L$,

$$\phi(k-l) = \phi(k) - \phi(l) \geq c.$$

If $\phi(l) \neq 0$ for some $l \in L$ we get, for all $a \in \mathbb{R}$,

$\phi(k - a \cdot l) = \phi(k) - a\,\phi(l)$ and choosing $|a|$ large

and of the same sign as $\phi(l)$ gives

$$0 < c \leq \phi(k - a\,l) = \phi(k) - a\,\phi(l) < 0,$$

a contradiction. Hence $\phi(l) = 0$ and

$\phi(k) \geq c > 0$ for all $k \in K$. $\square$

Construction of Martingale Measures

A probability space is called finitely generated if every measurable function takes at most finitely many distinct values.
Ω can be partitioned into n disjoint sets $\Omega = \Omega_1 \cup \Omega_2 \cup .. \cup \Omega_n$ s.t. every measurable function is constant on Ω_i.
Without loss of generality we can assume $\Omega_i = \{\omega_i\}$ are singletons and the probability measure is determined by $p_i = \mathbb{P}(\Omega_i) = \mathbb{P}(\{\omega_i\})$.

Recall A model is viable if there is no arbitrage strategy. An attainable strategy θ is uniquely determined under the self-financing property (linear equation). The gains are

$$\overline{G}_t(\theta) = \sum_{u=1}^{t} \theta_u \cdot \Delta \overline{S}_u .$$

We now assume our model is finitely generated. Then, the condition of being viable / arbitrage free is equivalent to gains $G_t(\theta)$ not belonging to $\mathcal{C} = \{ X : X(\omega_i) \geq 0$ for all i, $X(\omega_i) > 0$ for at least one $i \}$

Th$^{\underline{m}}$ (First Fundamental Theorem of Asset Pricing)
Assume that the model is finitely generated.
The following are equivalent:

i) The model is viable

ii) There exists an equivalent martingale measure Q

Proof: ii) => i) Recall:

If $\bar{S}$ is a martingale under Q, then so is $\bar{G}_t(\theta)$ for every attainable θ, as it is a martingal transform.

$\Rightarrow \mathbb{E}(\bar{G}_t(\theta)) = \mathbb{E}(\bar{G}_0(\theta)) = 0$

$\Rightarrow$ if $\bar{G}_t(\theta) \geq 0$ a.s. then $\bar{G}_t(\theta) = 0$. So there is no arbitrage, and the model is viable.

i) $\Rightarrow$ ii)

First, $L = \{ \overline{G}_T(\theta) : \theta \text{ attainable strategy}\}$ is a linear subspace of the space of all random variables:

- $0 \in L$
- θ_1, θ_2 attainable strategies $\Rightarrow$ $\theta_1 + \theta_2$ attainable strategy
- θ_1 attainable strategy, $c \in \mathbb{R}$ $\Rightarrow$ $c\,\theta_1$ attainable strategy.

Since $\overline{G}_T(\theta)$ is linear in θ, linearity of L follows.

Further, $K = \{ X \text{ non-neg. rand. var. with } \mathbb{E}_P(X) = 1 \}$ is compact w.r.t. $\|\cdot\|_1$ since any open cover (U_i) of K must contain a U_i which contains an element $X \in K$.

But then $B(X, \varepsilon) \supseteq \{ Y : \|Y\|_1 = 1 \} \supseteq \{ Y \geq 1 : \mathbb{E}(Y) = 1 \} = K$

Hence U_i is a cover of K and K is compact.

K is also convex, since

$X_1, X_2 \in K$, $\lambda \in [0,1] \Rightarrow \mathbb{E}(\lambda X_1 + (1-\lambda) X_2)$

$= \lambda \mathbb{E}(X_1) + (1-\lambda)\mathbb{E}(X_2) = \lambda + (1-\lambda) = 1$ and

$\lambda X_1 + (1-\lambda) X_2 \in K$.

Note that $K \cap L = \emptyset$ since any attainable strategy with $\mathbb{E}(\bar{G}_T(\theta)) = 1$ would be arbitrage. We can apply the separating hyperplane theorem and there exists a linear functional φ that is zero on L and greater than $c > 0$ on K. By finiteness, we can express φ as

$$\varphi(X) = \sum_{i=1}^{n} q_i X(\omega_i) \quad \text{for some constants } q_i$$

In particular consider the rand. var. $\xi_i = \frac{1}{p_i} I_{\omega_i}$ then ξ_i is non-negative and $\mathbb{E}(\xi_i) = \frac{1}{p_i} \mathbb{E}(I_{\omega_i}) = \frac{\mathbb{P}(\omega_i)}{p_i} = 1$.

So $\xi_i \in K$ and $\varphi(\xi_i) = q_i \cdot \frac{1}{p_i} \geq c > 0$.

Hence $q_i > 0$ for all i. Define Q by

$$Q(\{\omega_i\}) = \frac{q_i}{\sum_j q_j} > 0$$

This is a probability measure as $\sum_i \frac{q_i}{\sum_j q_j} = \frac{\sum_i q_i}{\sum_j q_j} = 1$

We have $\mathbb{E}_Q(\bar{G}_T(\theta)) = \frac{1}{\sum_j q_j} \underbrace{\sum_i q_i \bar{G}_T(\theta)(\omega_i)}_{= \varphi(\bar{G}_T(\theta)) = 0} = 0$ for all attainable θ.

as $\bar{G}_T(\theta) \in L$

Now Q is an equivalent martingale measure

$\rightsquigarrow$ $\rightarrow$ equivalent to $\mathbb{P}$ since $Q(\{\omega_i\}) > 0$

$\rightarrow$ $\mathbb{E}_Q(\bar{G}_T(\theta)) = 0$ for all predictable processes, so $\mathbb{E}_Q(\Delta \bar{S}_t^i \mid \tilde{\mathcal{F}}_{t-1}) = 0$ for all t.

This completes the proof. $\square$

<u>Remark:</u> This holds in greater generality for non-finite models.

Completeness of Market Models

Recall that a market model is complete if every contingent claim X has a replicating (generating) strategy θ: a strategy with $V_T(\theta) = X$.

Recall that we are only considering finite models.