Analysis of Time Series, L16

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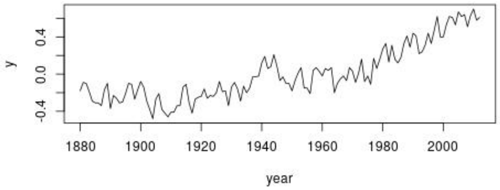
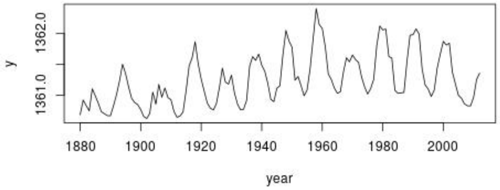
Today

- 5.5: Transfer functions
- 5.6: Multivariate ARMAX

Transfer functions

Global mean temperature (y_t) and solar irradiance (x_t), 1880-2012.





Transfer functions

- Input x_t (e.g. solar irradiance), output y_t (e.g. temperature), *both demeaned*.
- Lagged regression model

$$y_t = \sum_{j=0}^{\infty} \alpha_j x_{t-j} + \eta_t = \alpha(B) x_t + \eta_t,$$

where $\sum_j |\alpha_j| < \infty$, x_t and η_t stationary and independent.

- How do we estimate the α_j?

Transfer functions

(i) Prewhitening of input:

- Find an ARMA model for the input, $\phi(B)x_t = \theta(B)w_t$, where w_t is white noise.
- Hence,

$$y_t = \alpha(B)x_t + \eta_t = \alpha(B)\frac{\theta(B)}{\phi(B)}w_t + \eta_t.$$

(ii) Prewhitening of output:

$$\tilde{y}_t = \frac{\phi(B)}{\theta(B)}y_t = \alpha(B)w_t + \tilde{\eta}_t,$$

where

$$\tilde{\eta}_t = \frac{\phi(B)}{\theta(B)}\eta_t, \quad w_t = \frac{\phi(B)}{\theta(B)}x_t.$$

(iii) The α_j are given by the CCF (why?)

$$\gamma_{\tilde{y}w}(h) = E(\tilde{y}_{t+h}w_t) = \sigma_w^2\alpha_h.$$

Transfer functions

- $y_t = \alpha(B)x_t + \eta_t$

(i) $y_t = \alpha(B)\frac{\theta(B)}{\phi(B)}w_t + \eta_t$

(ii) $\tilde{y}_t = \alpha(B)w_t + \tilde{\eta}_t$

(iii) CCF $\gamma_{\tilde{y}w}(h) = \sigma_w^2 \alpha_h$.
Try the representation

$$\alpha(B) = \frac{\delta(B)}{\omega(B)}B^d,$$

where

$$\delta(B) = \delta_0 + \delta_1 B + ... + \delta_s B^s,$$
$$\omega(B) = 1 - \omega_1 B - \omega_2 B^2 - ... - \omega_r B^r.$$

Identify d, s and r from the CCF.

Transfer functions

- $y_t = \alpha(B)x_t + \eta_t$

(*i*) $y_t = \alpha(B)\frac{\theta(B)}{\phi(B)}w_t + \eta_t$

(*ii*) $\tilde{y}_t = \alpha(B)w_t + \tilde{\eta}_t$

(*iii*) CCF $\gamma_{\tilde{y}w}(h) = \sigma_w^2 \alpha_h$ gives $y_t = \frac{\delta(B)}{\omega(B)}B^d x_t + \eta_t$

(*iv*) Estimate the regression $\omega(B)y_t = \delta(B)B^d x_t + \omega(B)\eta_t$ i.e.

$$y_t = \omega_1 y_{t-1} + \ldots + \omega_r y_{t-r} + \delta_0 x_{t-d} + \delta_1 x_{t-d-1} + \ldots + \delta_s x_{t-d-s} + u_t,$$

where $u_t = \omega(B)\eta_t$.

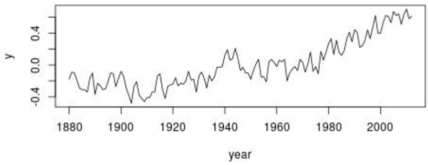
(*v*) Construct $\eta_t = \omega^{-1}(B)u_t$ and fit an ARMA model $\phi_\eta(B)\eta_t = \theta_\eta(B)z_t$ where z_t is white noise.

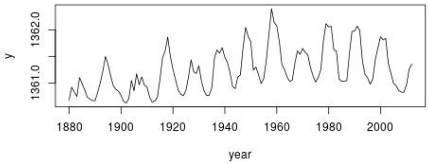
- Final model (why?):

$$\phi_\eta(B)\omega(B)y_t = \phi_\eta(B)\delta(B)B^d x_t + \omega(B)\theta_\eta(B)z_t.$$

Transfer functions

Global mean temperature (y_t) and solar irradiance (x_t), 1880-2012.
Fit a transfer function model for the years 1880-1969.

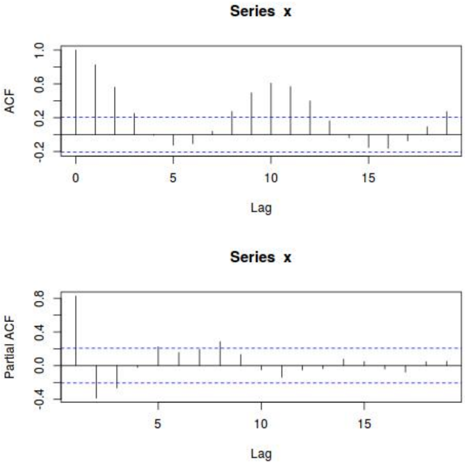




```
> x1=x[seq(1,90)]
> x=x1-mean(x1)
> y1=y[seq(1,90)]
> y=y1-mean(y1)
> par(mfrow=c(2,1))
> acf(x)
> pacf(x)
```

Transfer functions

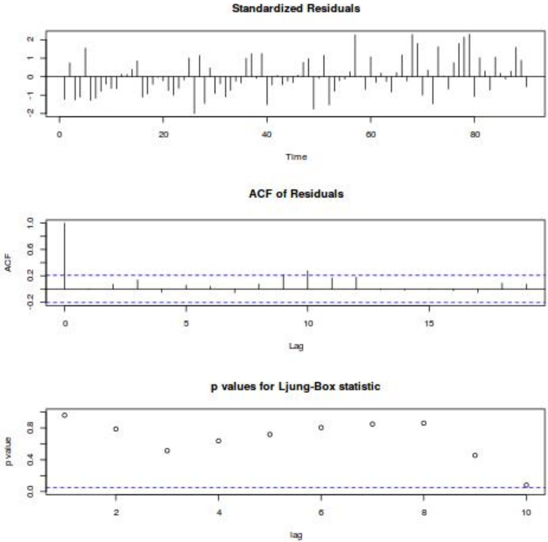
ACF (tails off?) and PACF (cuts off after lag 3?) for demeaned solar irradiance (x_t):



Try AR(3) without constant:

```
> m=arima(x,order=c(3,0,0),include.mean=FALSE)
> m

Call:
arima(x = x, order = c(3, 0, 0), include.mean = FALSE)

Coefficients:
         ar1      ar2      ar3
      1.0785  -0.1091  -0.2584
s.e.  0.1024   0.1553   0.1031

sigma^2 estimated as 0.03541:  log likelihood = 21.73,
aic = -35.45
```

```
> tsdiag(m)
```


Standardized Residuals
Time
ACF of Residuals
ACF
Lag
p values for Ljung-Box statistic
p value
lag

Transfer functions

Input: demeaned solar irradiance.

(*i*)
- Estimated model for input:

$$x_t = 1.0785x_{t-1} - 0.1091x_{t-2} - 0.2584x_{t-3} + w_t,$$

  i.e. $w_t = (1 - 1.0785B + 0.1091B^2 + 0.2584B^3)x_t$.
- Hence,

$$y_t = \alpha(B)\frac{1}{1 - 1.0785B + 0.1091B^2 + 0.2584B^3}w_t + \eta_t.$$

(*ii*) Prewhitening of output:

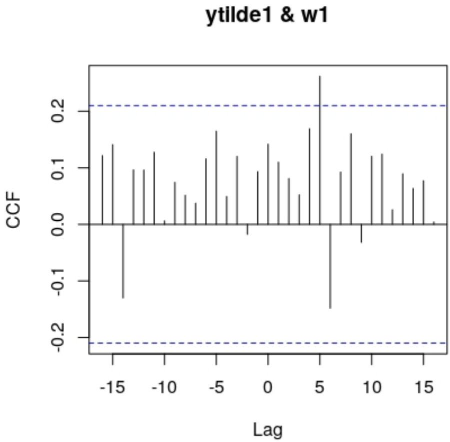
$$\tilde{y}_t = (1 - 1.0785B + 0.1091B^2 + 0.2584B^3)y_t = \alpha(B)w_t + \tilde{\eta}_t$$

(*iii*) The α_j are given by the CCF $\gamma_{\tilde{y}w}(h) = \sigma_w^2\alpha_h$, for $h \geq 0$.

```
> phi1=as.numeric(m$coef[1])
> phi2=as.numeric(m$coef[2])
> phi3=as.numeric(m$coef[3])
> ytilde=filter(y,filter=c(1,-phi1,-phi2,-phi3),method="convolution",sides=1)
> w=filter(x,filter=c(1,-phi1,-phi2,-phi3),method="convolution",sides=1)
> n=length(y)
> ytilde1=ytilde[seq(4,n)]
> w1=w[seq(4,n)]
> ccf(ytilde1,w1,ylab="CCF")
```

Transfer functions

CCF:



$d = 5$? Slowly decaying to the right means AR form? Trial and error!

Transfer functions

- $$\alpha(B) = \frac{\delta(B)}{\omega(B)} B^d,$$

  where

  $$\delta(B) = \delta_0 + \delta_1 B + ... + \delta_s B^s,$$
  $$\omega(B) = 1 - \omega_1 B - \omega_2 B^2 - ... - \omega_r B^r.$$

- Trial and error (regressions...):
  Try $d = 1$, $s = 0$, $r = 4$ with zero AR coefficients for lags 2,3., i.e.
  $y_t = \frac{\delta_0}{1-\omega_1 B - \omega_4 B^4} B x_t + \eta_t$.

(iv) Estimate the regression $(1 - \omega_1 B - \omega_4 B^4) y_t = \delta_0 B x_t + u_t$ i.e.

$$y_t = \delta_0 x_{t-1} + \omega_1 y_{t-1} + \omega_4 y_{t-4} + u_t,$$

where $u_t = (1 - \omega_1 B - \omega_4 B^4)\eta_t$.

```
> y0=y[seq(5,n)]
> y1=y[seq(4,n-1)]
> y4=y[seq(1,n-4)]
> x1=x[seq(4,n-1)]
> r=lm(y0~x1+y1+y4-1);summary(r)

Call:
lm(formula = y0 ~ x1 + y1 + y4 - 1)

Residuals:
      Min        1Q    Median        3Q       Max
-0.211752 -0.074580  0.000089  0.074073  0.206602

Coefficients:
   Estimate Std. Error t value Pr(>|t|)
x1  0.05830    0.02931   1.989   0.0500 *
y1  0.60071    0.08371   7.177 2.76e-10 ***
y4  0.19932    0.08188   2.434   0.0171 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.09714 on 83 degrees of freedom
Multiple R-squared:  0.6243,Adjusted R-squared:  0.6107
F-statistic: 45.98 on 3 and 83 DF,  p-value: < 2.2e-16
```

Transfer functions

(*iv*) Estimated regression

$$y_t = 0.058x_{t-1} + 0.60y_{t-1} + 0.20y_{t-4} + u_t,$$

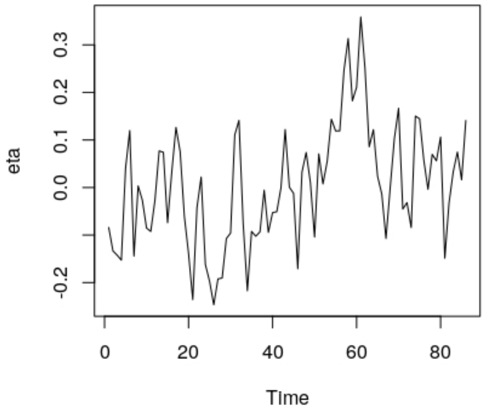
where $u_t = (1 - 0.60B - 0.20B^4)\eta_t$.

(*v*) Construct $\eta_t = (1 - 0.60B - 0.20B^4)^{-1}u_t$ and fit an ARMA model $\phi_\eta(B)\eta_t = \theta_\eta(B)z_t$ where z_t is white noise.

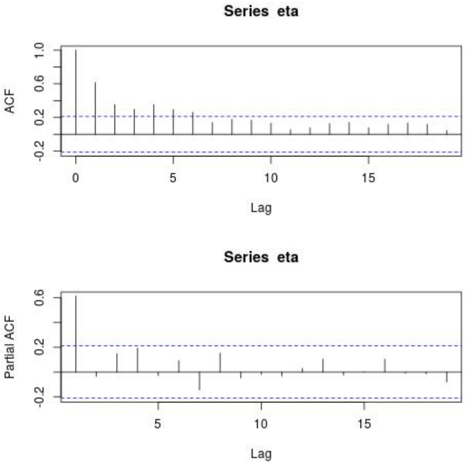
```
> u=r$res
> omega1=as.numeric(r$coef[2])
> omega2=as.numeric(r$coef[3])
> eta=filter(u,filter=c(omega1,0,0,omega2),method="recursive")
> plot(eta,type='l')
> par(mfrow=c(2,1))
> acf(eta)
> pacf(eta)
```

Transfer functions

Plot of η_t



Transfer functions



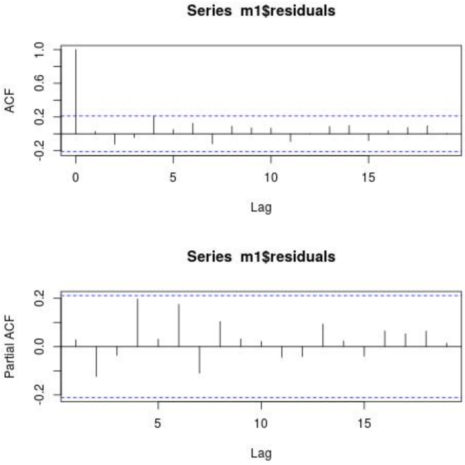
The ACF tails off and the PACF cuts off after lag 1.

Try AR(1). Model estimation for η_t:

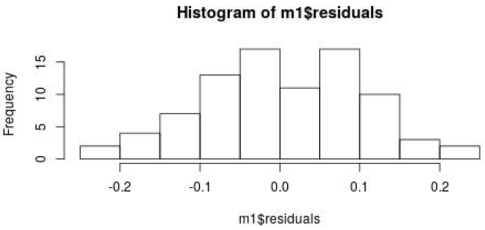
```
> m1=arima(eta,order=c(1,0,0),include.mean=FALSE);m1

Call:
arima(x = eta, order = c(1, 0, 0), include.mean = FALSE)

Coefficients:
         ar1
      0.6193
s.e.  0.0843

sigma^2 estimated as 0.00956:  log likelihood = 77.68,
aic = -151.37
```

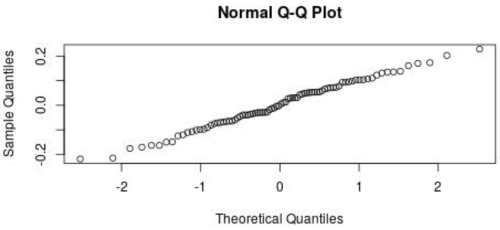
Analyse residuals of η_t model (white noise?):

```
> par(mfrow=c(2,1))
> acf(m1$res)
> pacf(m1$res)
```

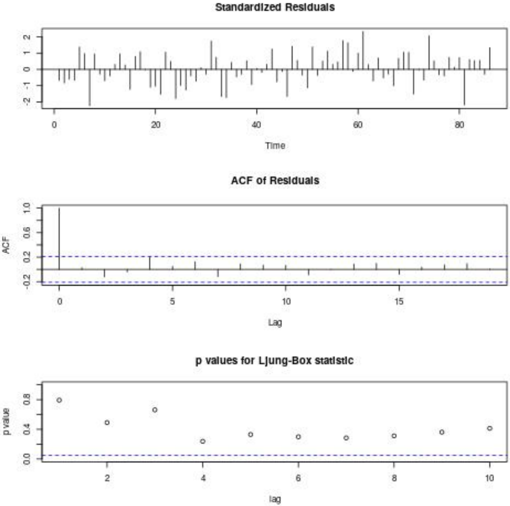


```
> hist(m1$res)
> qqnorm(m1$res)
```





```
> tsdiag(m1)
```


Standardized Residuals
2
1
0
-1
-2
0
20
40
60
80
Time
ACF of Residuals
ACF
1.0
0.6
0.2
-0.2
0
5
10
15
Lag
p values for Ljung-Box statistic
p value
0.8
0.4
0.0
2
4
6
8
10
lag

Transfer functions

- The final model

$$\phi_\eta(B)\omega(B)y_t = \phi_\eta(B)\delta(B)B^d x_t + \omega(B)\theta_\eta(B)z_t$$

is estimated as

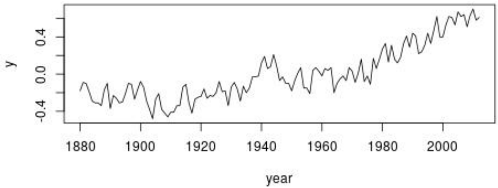
$$\begin{aligned}&(1 - 0.62B)(1 - 0.60B - 0.20B^4)y_t \\ &= (1 - 0.62B)0.058Bx_t + (1 - 0.60B - 0.20B^4)z_t,\end{aligned}$$

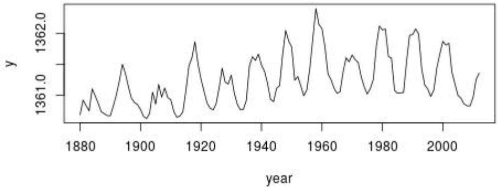
where $z_t = (1 - 0.62B)\eta_t$ is white noise.

- White noise properties are already checked, since the z_t are the residuals of the η_t model!

Transfer functions

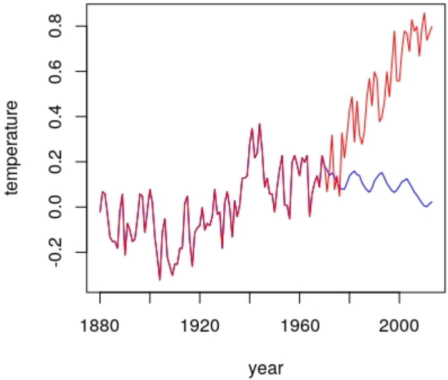
Global mean temperature (y_t) and solar irradiance (x_t), 1880-2012.





Transfer functions

Demeaned observations 1880-2013 (in red) and prediction 1970-2013 (in blue) using known x_t values and updated (predicted) y_t values.



Multivariate ARMAX

- Multivariate regression

$$\begin{cases} y_{t1} & = & \beta_{11} z_{t1} + \ldots + \beta_{1r} z_{tr} + w_{t1}, \\ & \vdots & \\ y_{tk} & = & \beta_{k1} z_{t1} + \ldots + \beta_{kr} z_{tr} + w_{tk}. \end{cases}$$

- i.e.

$$\begin{pmatrix} y_{t1} \\ y_{t2} \\ \vdots \\ y_{tk} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1r} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k1} & \beta_{k2} & \ldots & \beta_{kr} \end{pmatrix} \begin{pmatrix} z_{t1} \\ z_{t2} \\ \vdots \\ z_{tr} \end{pmatrix} + \begin{pmatrix} w_{t1} \\ w_{t2} \\ \vdots \\ w_{tk} \end{pmatrix}$$

  i.e. $\mathbf{y}_t = B\mathbf{z}_t + \mathbf{w}_t$.

- For example, with $r = 1$ and $k = 2$,

$$\begin{cases} y_{t1} & = & \beta_{11} z_{t1} + w_{t1}, \\ y_{t2} & = & \beta_{21} z_{t1} + w_{t2}, \end{cases} \qquad \begin{pmatrix} y_{t1} \\ y_{t2} \end{pmatrix} = \begin{pmatrix} \beta_{11} \\ \beta_{12} \end{pmatrix} z_{t1} + \begin{pmatrix} w_{t1} \\ w_{t2} \end{pmatrix}$$

Multivariate ARMAX

VAR(1):

$$\begin{pmatrix} x_{t1} \\ x_{t2} \\ \vdots \\ x_{tk} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1k} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1} & \phi_{k2} & \cdots & \phi_{kk} \end{pmatrix} \begin{pmatrix} x_{t-1,1} \\ x_{t-1,2} \\ \vdots \\ x_{t-1,k} \end{pmatrix} + \begin{pmatrix} w_{t1} \\ w_{t2} \\ \vdots \\ w_{tk} \end{pmatrix}$$

i.e. $\mathbf{x}_t = \mathbf{\Phi} \mathbf{x}_{t-1} + \mathbf{w}_t$.

Multivariate ARMAX

Extensions:

- VAR(p)

$$\mathbf{x}_t = \sum_{j=1}^{p} \boldsymbol{\Phi}_j \mathbf{x}_{t-j} + \mathbf{w}_t.$$

- VARX(p)

$$\mathbf{x}_t = \Gamma \mathbf{u}_t + \sum_{j=1}^{p} \boldsymbol{\Phi}_j \mathbf{x}_{t-j} + \mathbf{w}_t,$$

  where Γ is $k \times r$, $\mathbf{u}_t$ is $r \times 1$.

- Example with $r = 2$:

$$\Gamma \mathbf{u}_t = \begin{pmatrix} \gamma_0 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \gamma_0 + \gamma_1 t.$$

Multivariate ARMAX

VARMA models:

- VARMA(1,1)

$$\mathbf{x}_t = \mathbf{\Phi x}_{t-1} + \mathbf{w}_t + \mathbf{\Theta w}_{t-1}.$$

- Unicity problem: VARMA(1,1) with

$$\mathbf{\Phi} = \begin{pmatrix} 0 & \phi + \theta \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\Theta} = \begin{pmatrix} 0 & -\theta \\ 0 & 0 \end{pmatrix}$$

  is equivalent to VARMA(1,0), $\mathbf{x}_t = \mathbf{\Phi}_1 \mathbf{x}_{t-1} + \mathbf{w}_t$, with

$$\mathbf{\Phi}_1 = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix}.$$

  Why?

Multivariate ARMAX

VAR(1)

- $$\mathbf{x}_t = \mathbf{\Phi}\mathbf{x}_{t-1} + \mathbf{w}_t.$$
- Equivalent: Error correction form
$$\nabla \mathbf{x}_t = \mathbf{\Phi}\mathbf{x}_{t-1} + \mathbf{w}_t - \mathbf{x}_{t-1} = \mathbf{\Phi}_1 \mathbf{x}_{t-1} + \mathbf{w}_t$$
where $\mathbf{\Phi}_1 = \mathbf{\Phi} - I$, with I as the identity matrix.
- If $\mathbf{\Phi}$ has (at least) one eigenvalue that is equal to one, then $\mathbf{\Phi}_1$ has (at least) one eigenvalue that is equal to zero. Then, $\mathbf{\Phi}_1$ is singular and $\mathbf{x}_t$ is non stationary.

Multivariate ARMAX

VAR(1), dimension 2:

- Error correction form

$$\begin{pmatrix} \nabla x_{t1} \\ \nabla x_{t2} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} x_{t-1,1} \\ x_{t-1,2} \end{pmatrix} + \begin{pmatrix} w_{t1} \\ w_{t2} \end{pmatrix}$$

- Reduced rank (cointegration)

$$\begin{pmatrix} \nabla x_{t1} \\ \nabla x_{t2} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} x_{t-1,1} \\ x_{t-1,2} \end{pmatrix} + \begin{pmatrix} w_{t1} \\ w_{t2} \end{pmatrix}$$

- i.e.

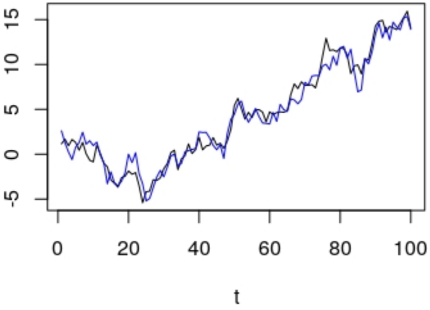
$$\begin{aligned} \nabla x_{t1} &= \alpha_1(\beta_1 x_{t-1,1} + \beta_2 x_{t-1,2}) + w_{t1}, \\ \nabla x_{t2} &= \alpha_2(\beta_1 x_{t-1,1} + \beta_2 x_{t-1,2}) + w_{t2}. \end{aligned}$$

- *Cointegrating relation* $\beta_1 x_{t-1,1} + \beta_2 x_{t-1,2}$.

Multivariate ARMAX

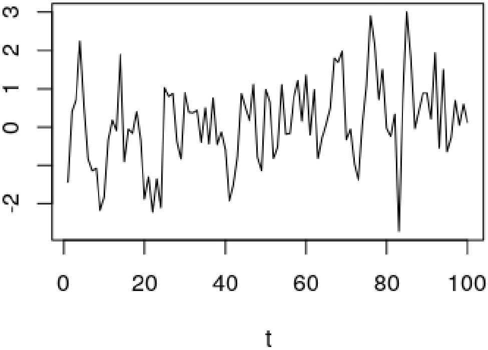
Simulation example (x_{t1} in black, x_{t2} in blue):

$$\nabla x_{t1} = w_{t1},$$
$$\nabla x_{t2} = 0.5(x_{t-1,1} - x_{t-1,2}) + w_{t2}.$$



Multivariate ARMAX

Plot of the cointegrating relation, $x_{t1} - x_{t2}$ (observe the different scale on the y axis):



Multivariate ARMAX

Clive Granger, Nobel prize 2003.

https://www.nobelprize.org/nobelprizes/economic-sciences/laureates/2003/granger-photo.html

Multivariate ARMAX

Sören Johansen

http://www.economics.ku.dk/staff/vip/?pure=en/persons/34220

News of today

- Transfer function modeling
- VAR
- VARX
- VARMA
- Cointegration