Basic Topology (1MA179)

Johan Asplund

Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden

May 13, 2020

This version: June 15, 2022

Foreword

This document is originally a compilation of my lecture notes for the course Basic Topology (1MA179) given by me at Uppsala University during the spring semester 2020. The original lecture notes are based off of lecture notes by Georgios Dimitroglou Rizell (2012). The official course literature is

- J. Munkres, *Topology (2nd ed)*, Pearson Education (2014)

More or less everything in this compendium is contained in Munkres' book, and where appropriate we give indication of where the corresponding material is to be found in Munkres' book.

I am indebted to my students of the Basic Topology course at Uppsala University in the spring of 2020 for catching many errors in the original version. I am also grateful to Anders Israelsson and his students from the Basic Topology course at Uppsala University in the spring of 2022 for numerous corrections and suggestions leading to the current revised edition of these lecture notes.

Figure on front page: Code by Mark Wibrow.

Johan Asplund (June 2022)

Contents

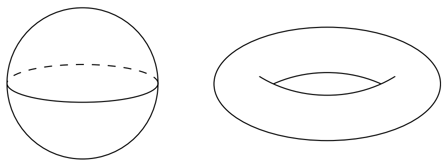
Part 1

Topological spaces, bases and first properties

Lecture 1

1.1. Introduction

Topology is the branch in mathematics that is about *shapes*, and in particular *shapes of spaces*.



Intuitively, a sphere (on the left) and a torus (on the right) are different shapes. But *why* are they different? It is easy to see in the figure above that the torus has a *hole* straight through the middle. But how would you express this fact in mathematical terms?

Another related question is about spaces we already know from previous courses, namely $\mathbb{R}^n$ for $n \geq 1$. $\mathbb{R}$ is a line and $\mathbb{R}^2$ is a plane. How would you show that these two "shapes" are in fact different mathematically? Often it is intuitively easy to understand why two different shapes or spaces are different, but difficult to express it mathematically. In this course we will start developing the necessary tools needed to express these intuitive feelings in mathematical language.

Another interesting part of the field of topology is called *knot theory*. A mathematical knot is a circle in three dimensional space that is "knotted". Imagine that you have a piece of string. Tie a knot on the string and finally glue the ends together. Then you have a closed loop which is knotted. This is also precisely what we would call a mathematical knot. One leading question in knot theory is the following: Given two knots, can you mathematically show that the are different?

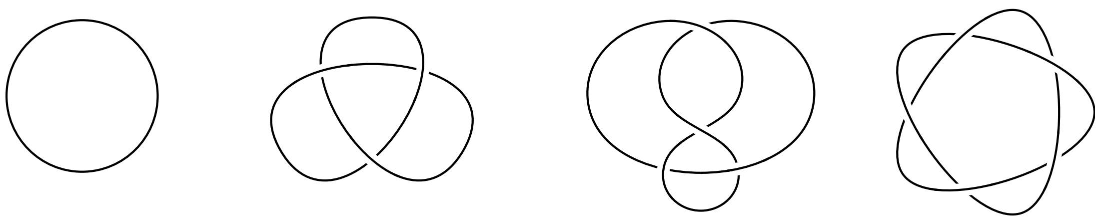


FIGURE 1. From left to right: The unknot, the trefoil knot, the figure eight knot and the cinquefoil knot.

These are just a small fraction of questions that is being studied in the field of topology. We will not encounter knot theory in this course. However, we will study the sphere, the torus and other surfaces later in this course, and by the end of the course we will be equipped with tools that enable us to mathematically prove that some of these spaces are in fact different.

As with all areas of mathematics, we need basic definitions and a framework to work with. We will start by studying *topological spaces*, and study many properties of them. We will later study maps (or *continuous functions*) between topological spaces, and eventually we will be able to apply this framework to study interesting spaces that we already have some intuitive knowledge about.

1.2. Topological spaces

- Munkres: §12

In previous courses you may have studied metric spaces. Topological spaces is a very important generalization of metric spaces, and it is the single most important object of study in this course.

NOTATION.
- The set I will always denote an index set of arbitrary cardinality
- If X is a set, we denote its power set by $P(X)$. That is, the set of all subsets of X.
- Throughout the course I will consequently use the notation $\subset$ to denote subset (some authors use $\subseteq$). Whenever I mean "proper subset" I will write $\subsetneq$.

Definition 1.2.1 (Topology)**.** *Let X be a set. A* topology *defined on X is a collection of subsets $\mathcal{T} \subset P(X)$ satisfying*

(T1) $\varnothing, X \in \mathcal{T}$
(T2) If $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$
(T3) If $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$

We call the tuple $(X, \mathcal{T})$ a topological space.

Remark 1.2.2. We can summarize the above definition by saying that $\mathcal{T}$ is a topology if it contains $\varnothing$ and X, is closed under arbitrary unions and closed under finite intersections.

Definition 1.2.3 (Open and closed subsets)**.** *Let $(X, \mathcal{T})$ be a topological space.*

- *We say that $U \subset X$ is* open *if and only if $U \in \mathcal{T}$. (We use the notation $U \overset{\text{open}}{\subset} X$.)*
- *We say that $F \subset X$ is* closed *if and only if $X \setminus F$ is open. (We use the notation $F \overset{\text{closed}}{\subset} X$.)*

Remark 1.2.4.
- A set can be both open and closed (in which case it is called *clopen*). For example, $X \subset X$ is always clopen.
- A set can also be neither open nor closed.

Example 1.2.5.
(1) Let X be any set, and consider $\mathcal{T} = P(X)$. This is called the *discrete topology. All sets* are open in this topology (and hence also closed).
(2) Let X be any set and consider $\mathcal{T} = \{\varnothing, X\}$. This is called the *indiscrete topology* (sometimes *trivial topology* is used). No set (except $\varnothing$ and X) is open.
(3) Let $X = \mathbb{R}$ and let $U \in \mathcal{T}$ if and only if every $x \in U$ is contained in some open interval (a, b) which is contained in U. This is called the standard topology on $\mathbb{R}$.
(4) Let $X = \mathbb{R}^n$. Similarly we let $U \in \mathcal{T}$ if and only if every $x \in U$ is contained in some open ball which is contained in U
$$B(x, \varepsilon) := \{y \in \mathbb{R}^n \mid |x - y| < \varepsilon\} \subset U\,.$$
This is again called the standard topology on $\mathbb{R}^n$.
(5) Let $X = \{a, b, c\}$ and define $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $\mathcal{T}$ is a topology for example. It is left as an exercise to check that $\mathcal{T}$ is closed under union and intersection (that is, to check that (T2) and (T3) holds).

Note that in the definition of a topology, there is only conditions on open sets. We can rewrite (T1)–(T3) in the definition as follows:

(T1) $\varnothing$ and X are open
(T2) If U_i is open for all $i \in I$, then $\bigcup_{i\in I} U_i$ is open
(T3) If $U_1, \dots, U_n$ are open then $U_1 \cap \dots \cap U_n$ is open

We can equivalently formulate the definition entirely in terms of closed sets as the following proposition says:

Proposition 1.2.6. *Let X be a set. Then $\mathcal{T}$ is a topology on X if and only if*

(1) $\varnothing$, X are closed
(2) If $F_i \overset{\text{closed}}{\subset} X$ for $i \in I$ then $\bigcap_{i\in I} F_i \overset{\text{closed}}{\subset} X$
(3) If $F_1, \dots, F_n \overset{\text{closed}}{\subset} X$, then $F_1 \cup \dots \cup F_n \overset{\text{closed}}{\subset} X$

Proof. Recall de Morgans's laws:

- $\bigcup_{i\in I} X \smallsetminus A_i = X \smallsetminus (\bigcap_{i\in I} A_i)$
- $\bigcap_{i\in I} X \smallsetminus A_i = X \smallsetminus (\bigcup_{i\in I} A_i)$

The rest is left as an exercise! □

Definition 1.2.7 (Neighborhood)**.** *Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. A* neighborhood *of x is any open set containing x.*

Remark 1.2.8. A small warning here! My definition aligns with Munkres' definition. Some authors choose to define the term "neighborhood" to mean a subset (not necessarily open) containing an open set containing the point.

We can characterize open subsets by the property that open sets should always contain other open sets.

Proposition 1.2.9. *Let $(X, \mathcal{T})$ be a topological space. We have that $U \overset{\text{open}}{\subset} X$ if and only if $\forall x \in U$ there is a neighborhood V_x of x such that $V_x \subset U$.*

Proof. **$\Leftarrow$:** Assume that for every point $x \in U$, there is a neighborhood V_x of x such that $V_x \subset U$. Then we consider the union $\bigcup_{x\in U} V_x$. Since each $V_x \overset{\text{open}}{\subset} X$, it is clear that $\bigcup_{x\in U} V_x \overset{\text{open}}{\subset} X$. It is also clear that $U = \bigcup_{x\in U} V_x$ (why?). Therefore $U \overset{\text{open}}{\subset} X$.

$\Rightarrow$: Assume now that $U \overset{\text{open}}{\subset} X$. Then for every $x \in U$, we have that U is a neighborhood of x, and $U \subset U$ is true.

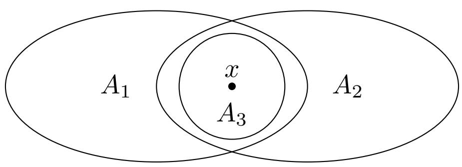
□

1.3. Basis of a topology

- Munkres: §13

Definition 1.3.10 (Basis)**.** *A* basis *on a set X is a collection $\mathcal{B} \subset P(X)$ satisfying the following:*

(B1) For every $x \in X$, there is some $A \in \mathcal{B}$ containing x
(B2) If $x \in A_1 \cap A_2$ where $A_1, A_2 \in \mathcal{B}$, then there is some A_3 such that $x \in A_3 \subset A_1 \cap A_2$



Definition 1.3.11 (Topology generated by basis)**.** *If X is a set, and B a basis. Then we define* the topology generated by $\mathcal{B}$, *denoted by $\mathcal{T}$ as*

$$U \in \mathcal{T} \overset{\text{def}}{\iff} U = \bigcup_{i \in I} A_i \text{ for } A_i \in \mathcal{B},\ \forall i \in I\,.$$

Proposition 1.3.12. *The topology $\mathcal{T}$ in the above definition is really a topology.*

Proof. We check (T1)–(T3) from the definition of a topology.

(T1):
- $\varnothing \in \mathcal{T}$ since $\varnothing = \bigcup_{i\in\varnothing} A_i$. This is the empty union!
- $X \in \mathcal{T}$ since by (B1), for every $x \in X$ there is some neighborhood $A_x \in \mathcal{B}$ of x. Then we have $X = \bigcup_{x\in X} A_x$.

(T2): Take a collection $\{U_i\}_{i\in I} \subset \mathcal{T}$. Then we have

$$U_i = \bigcup_{j\in J} A_{ij}\,,$$

for each $i \in I$, where $A_{ij} \in \mathcal{B}$. Then we have

$$U = \bigcup_{i\in I} U_i = \bigcup_{\substack{i\in I\\ j\in J}} A_{ij}\,,$$

which means $U \in \mathcal{T}$.

(T3): Pick $U_1, \ldots, U_n \in \mathcal{T}$. Then we first show that $U_1 \cap U_2 \in \mathcal{T}$. Namely, we write

$$U_1 = \bigcup_{i\in I} A_{1,i}$$
$$U_2 = \bigcup_{j\in J} A_{2,i}$$

since $U_1, U_2 \in \mathcal{T}$. Then we have

$$U_1 \cap U_2 = \left(\bigcup_{i\in I} A_{1,i}\right) \cap \left(\bigcup_{j\in J} A_{2,i}\right) = \bigcup_{\substack{i\in I\\ j\in J}} A_{1,i} \cap A_{2,j}\,.$$

Now for each $x \in U_1 \cap U_2$, there is some $i \in I$ and $j \in J$ such that $x \in A_{1,i} \cap A_{2,j}$. By (B2) this gives a set $A_x \in \mathcal{B}$ such that $A_x \subset A_{1,i} \cap A_{2,j}$, and then we write

$$U_1 \cap U_2 = \bigcup_{x\in U_1\cap U_2} A_x\,,$$

which shows that $U_1 \cap U_2 \in \mathcal{T}$. Now, we use induction. The above can be seen as the base case. The induction hypothesis is that we assume $V := U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}$. Then in the induction step we show $V \cap U_n \in \mathcal{T}$ by the exact same argument as above. By induction we then have that all finite intersections is contained in $\mathcal{T}$.

□

We now look at some examples of bases for topologies.

Example 1.3.13. (1) The open balls $\{B(x,r) \subset \mathbb{R}^n \mid x \in \mathbb{R}^n,\ r > 0\}$ form a basis. (B1) holds because we can take any ball centered at any point. (B2) holds, because if we pick a point in the intersection of two balls, we can just pick a sufficiently small ball and it will fit in the intersection! We can make the radius as small as we wish.

(2) Let X be a set. Then $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis for the discrete topology. It is a basis:
(B1): For any $x \in X$, we have $x \in \{x\}$.

(B2): For any $B_1, B_2 \in \mathcal{B}$ if we have $x \in B_1 \cap B_2$, then again we can pick $x \in \{x\} \subset B_1 \cap B_2$ because $\{x\}$ is a singleton.

Lastly, we argue that $\mathcal{B}$ is a basis of the discrete topology. This is the case, because *any* subset $U \subset X$ can be written as $U = \bigcup_{x\in U} \{x\}$ and each $\{x\} \in \mathcal{B}$.

We will now see a lemma which gives a sufficient condition for when a collection of open subsets forms basis.

Lemma 1.3.14. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C} \subset \mathcal{T}$ be a collection of open subsets. If for every $U \overset{\text{open}}{\subset} X$ and $x \in U$ there exists some $A \in \mathcal{C}$ such that $x \in A \subset U$, then $\mathcal{C}$ forms a basis for $\mathcal{T}$.*

We will not prove this, but it is Lemma 13.2 in the book.

We finish this lecture with some more examples.

Example 1.3.15. (1) Let $X = \mathbb{R}$. Then consider $\mathcal{B} = \{(a, b) \mid a < b\}$. This forms a basis (it is an exercise to show that $\mathcal{B}$ satisfies (B1) and (B2). The topology generated by $\mathcal{B}$ is called the *standard topology.*

(2) Let $X = \mathbb{R}$ and define

$$\mathcal{T} = \{U \subset \mathbb{R} \mid 1 \notin U \text{ or } \mathbb{R} \smallsetminus U \text{ finite}\} .$$

We will show that $\mathcal{T}$ is a topology by verifying (T1)–(T3).

(T1): It is clear that $\varnothing \in \mathcal{T}$ since $1 \notin \varnothing$. Also $\mathbb{R} \in \mathcal{T}$ since $\mathbb{R} \smallsetminus \mathbb{R} = \varnothing$ is a finite set.

(T2): Suppose $\{U_i\}_{i\in I} \subset \mathcal{T}$. We want to show that $\bigcup_{i\in I} U_i \in \mathcal{T}$. Note that by de Morgan's laws

$$\mathbb{R} \smallsetminus \left(\bigcup_{i\in I} U_i\right) = \bigcap_{i\in I} \mathbb{R} \smallsetminus U_i \, .$$

Now we have two cases:

(a) Any of the U_i is such that $\mathbb{R} \smallsetminus U_i$ is finite.

(b) No of the U_i is such that $\mathbb{R} \smallsetminus U_i$ is finite. That is, every U_i is so that $1 \notin U_i$.

Case (a): If any of the U_i is such that $\mathbb{R} \smallsetminus U_i$ is finite, then $\bigcap_{i\in I} \mathbb{R} \smallsetminus U_i$ is also finite, because for every $i \in I$ we have $\bigcap_{i\in I} \mathbb{R} \smallsetminus U_i \subset \mathbb{R} \smallsetminus U_i$. That means $\bigcup_{i\in I} U_i \in \mathcal{T}$.

Case (b): If $1 \notin U_i$ for every $i \in I$, then $1 \notin \bigcup_{i\in I} U_i$ by definition, and hence $\bigcup_{i\in I} U_i \in \mathcal{T}$ also in this case.

(T3): This is shown similarly. Pick $U_1, \ldots, U_n \in \mathcal{T}$. We then have have two more cases.

(a) Any of the U_i is such that $1 \notin U_i$

(b) All of the U_i is such that $\mathbb{R} \smallsetminus U_i$ is finite.

Case (a): If $1 \notin U_i$, then we also have $1 \notin U_1 \cap \cdots \cap U_n$ and hence $U_1 \cap \cdots \cap U_n \in \mathcal{T}$

Case (b): If all U_i are such that $\mathbb{R} \smallsetminus U_i$ is finite, then

$$\mathbb{R} \smallsetminus (U_1 \cap \cdots \cap U_n) = \bigcup_{i=1}^{n} \mathbb{R} \smallsetminus U_i \, ,$$

is a finite union of finite sets. Hence $\bigcup_{i=1}^{n} \mathbb{R} \smallsetminus U_i$ is finite, which means that $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Exercises

1.1. Let $X = \{a, b, c\}$, and define $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$. Show that $\mathcal{T}$ defines a topolopgy on X.

1.2. Let X be a set. Show that $\mathcal{T}$ is a topology on X if and only if

(a) $\varnothing, X$ are closed

(b) If $F_i \overset{\text{closed}}{\subset} X$ for $i \in I$, then $\bigcap_{i \in I} F_i \overset{\text{closed}}{\subset} X$

(c) If $F_1, \ldots, F_n \overset{\text{closed}}{\subset} X$, then $F_1 \cup \cdots \cup F_n \overset{\text{closed}}{\subset} X$

(Hint: Use de Morgan's laws!)

1.3. Let $X = \mathbb{R}$ and consider the set $\mathcal{B} = \{(a, b) \subset \mathbb{R} \mid a < b\}$. Show that $\mathcal{B}$ forms a basis.

Lecture 2

We start with a recap from lecture 1:

Recap.
- $(X, \mathcal{T})$ is a topological space if $\mathcal{T} \subset P(X)$ is a collection of subsets satisfying:
  (T1) $\varnothing, X \in \mathcal{T}$
  (T2) $\{U_i\}_{i\in I} \subset \mathcal{T}$ implies that $\bigcup_{i\in I} U_i \in \mathcal{T}$
  (T3) $U_1, \dots, U_n \in \mathcal{T}$ implies that $U_1 \cap \dots \cap U_n \in \mathcal{T}$
- If X is a set, then $\mathcal{B} \subset P(X)$ is a *basis* if
  (B1) $\forall x \in X$ there is some $A \in \mathcal{B}$ such that $x \in A$
  (B2) If $x \in A_1 \cap A_2$, where $A_1, A_2 \in \mathcal{B}$, then there is some $A_3 \in \mathcal{B}$ such that $x \in A_3 \subset A_1 \cap A_2$.

2.1. The poset- and order topologies

- Munkres: §14

Intuitively, an ordered set is a set with an *order relation*. Of course, the most natural example is $\mathbb{R}$ together with the relation $\leq$ which is used to order all numbers.

In the previous lecture we have already seen that $\mathcal{B} = \{(a, b) \mid a < b\}$ constitutes a basis for the standard topology on $\mathbb{R}$. In this context it is also known as the *order topology*.

2.1.1. Posets. As we tend to do, we will try and approach the concept of *order* from a mathematical point of view.

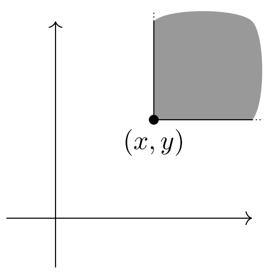
Definition 2.1.1 (Partially ordered set (poset))**.** *A tuple $(X, \preceq)$ of a set X together with a relation $\preceq$ is called a* partial order *if the relation $\preceq$ satisfies:*

Reflexivity: *$x \preceq x$ for every $x \in X$*
Transitivity: *$x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for every $x, y, z \in X$*
Anti-symmetry: *If $x \preceq y$ and $y \preceq x$, then $x = y$ for every $x, y \in X$*

Remark 2.1.2. Note that in this definition, we do *not* require that every pair is ordered. That is, our definition allows for points $x, y \in X$ such that neither $x \preceq y$ nor $y \preceq x$ is true.

Example 2.1.3.
- $(\mathbb{R}, \leq)$ is a poset. Make sure you understand this via the above definition.
- Consider $(\mathbb{R}^2, \preceq)$ where we define $\preceq$ via

$$(x, y) \preceq (z, w) \overset{\text{def}}{\iff} x \leq z \text{ and } y \leq w$$



The figure shows all pairs (z,w) such that $(x,y) \preceq (z,w)$ where (x,y) is fixed as in the picture.

For example, $(1,2)$ and $(2,1)$ are not comparable.

If we have a poset, we can define the *order topology* similar to what our intuition tells us as in the case with $(\mathbb{R}, \leq)$.

Proposition 2.1.4. *If $(X, \preceq)$ is a poset, then all the sets of the form*

$$B_a := \{x \in X \mid a \preceq x\} \subset X$$

constitutes a basis for a topology which we call the poset topology.

Proof. We need to show that B_a satisfies the axioms for a basis, (B1) and (B2).

(B1): For every $x \in X$ we have $x \in B_x$ since $x \preceq x$ is true by reflexivity.

(B2): If $x \in B_a \cap B_b$ then $a \preceq x$ and $b \preceq x$. We claim that B_x is such that $x \in B_x \subset B_a \cap B_b$.

(1) Note that for any $y \in B_x$ we have $x \preceq y$ by definition. We also have $a \preceq x$ since we assumed $x \in B_a \cap B_b$. Therefore by transitivity

$$a \preceq x \text{ and } x \preceq y \implies a \preceq y \;\Leftrightarrow\; y \in B_a$$

(2) Similarly, we have $b \preceq x$, so by transitivity again we have

$$b \preceq x \text{ and } x \preceq y \implies b \preceq y \;\Leftrightarrow\; y \in B_b$$

Hence we have $y \in B_a \cap B_b$ which means $B_x \subset B_a \cap B_b$ and so (B2) holds.

□

Example 2.1.5. Let $X = \{a, b, c, d\}$ and define $\preceq$ by

$$a \preceq b \preceq c, \qquad a \preceq d$$

and no other relations. Then the basis for the poset topology is

$$\mathcal{B} = \{B_a, B_b, B_c, B_d\} = \{\{a,b,c,d\}, \{b,c\}, \{c\}, \{d\}\} \,.$$

The open subsets in the poset topology on $(X, \preceq)$ are therefore unions of sets in $\mathcal{B}$. We have

$$\mathcal{T}_{\text{poset}} = \{\varnothing, X, \{c\}, \{d\}, \{d,c\}, \{b,c\}, \{b,c,d\}\} \,,$$

and we can easily check that this indeed defines a topology.

Definition 2.1.6. *Let $(X, \preceq)$ be a poset. If $\forall x, y \in X$ either $x \preceq y$ or $y \preceq x$ is true, then we say that $(X, \preceq)$ is a totally ordered set.*

Proposition 2.1.7. *Let $(X, \preceq)$ be a totally ordered set. Then all sets of the form*

$$\begin{cases} (-\infty, a) := \{x \in X \mid x \preceq a, \quad x \neq a\} \\ (a, \infty) := \{x \in X \mid a \preceq x, \quad x \neq a\} \\ (a, b) := \{x \in X \mid a \preceq x \preceq b, \quad x \neq a,\ x \neq b\} \end{cases}$$

form a basis of a topology on X called the order topology.

Example 2.1.8. $(\mathbb{R}, \leq)$ is a totally ordered set, and the order topology is equal to the standard topology.

2.1.2. Metric spaces.

- Munkres: §20

Metric spaces is a space where we can measure distances using a *metric*.

Definition 2.1.9 (Metric space)**.** *A metric space (X, d) is a set X together with a function*

$$d\colon\ X \times X \longrightarrow \mathbb{R}_{\geq 0}\,,$$

satisfying

(M1) $d(x, y) \geq 0$ with equality if and only if $x = y$
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$
(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (Triangle inequality)

Proposition 2.1.10. *Let (X, d) be a metric space. The collection of open balls*

$$B(x, r) := \{y \in X \mid d(x, r) < r\} \subset X\,,$$

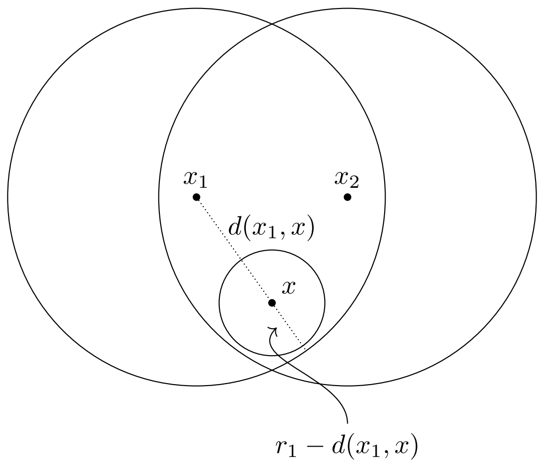
forms a basis of a topology on X we call the metric topology*.*

Proof. Again we prove (B1) and (B2).

(B1): For each $x \in X$ we have $x \in B(x, r)$ for any choice of radius by (M1).

(B2): Pick $x \in B(x_1, r_1) \cap B(x_2, r_2)$. Then our intuition says that we can pick a small enough ball around x so that it is completely contained in the intersection.

To prove it, we consider a ball $B(x, \varepsilon)$ of radius ε (which we determine below).



Set

$$\varepsilon = \min(r_1 - d(x, x_1), r_2 - d(x, x_2))\,.$$

Then we claim that $B(x, \varepsilon) \subset B(x_1, r_1) \cap B(x_2, r_2)$. Namely, pick any $z \in B(x, \varepsilon)$. Then we use the triangle inequality to get

$$d(x_1, z) \leq d(x_1, x) + d(x, z) < d(x_1, x) + \varepsilon \leq d(x_1, x) + (r_1 - d(x, x_1)) = r_1\,.$$

This implies $z \in B(x_1, r_1)$. Similarly, we show $d(x_2, z) < r_2$ which means $z \in B(x_2, r_2)$ and hence $B(x_1, r_1) \cap B(x_2, r_2)$ which means $x \in B(x, \varepsilon) \subset B(x_1, r_1) \cap B(x_2, r_2)$.

□

Example 2.1.11. (1) Let X be any set. Then define

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}\,.$$

Show that d satisfies (M1)–(M3) and therefore defines a metric. It is called the *discrete metric*. The metric topology generated by d is the one generated by balls

$$B(x,r) = \{y \in X \mid d(x,y) < r\} \ .$$

For every x, $B(x,r) = X$ if $r \geq 1$, and otherwise $B(x,r) = \{x\}$ if $r < 1$. Therefore we have

$$\mathcal{B} = \{\{x\} \mid x \in X\} \ ,$$

which is the same as the basis which generates the discrete topology. Therefore the metric topology with respect to d is equal to the discrete topology.

(2) Let $X = \mathbb{R}$, and define $d(x,y) = |x-y|$. This is the normal Euclidean metric on $\mathbb{R}$. The balls are open intervals

$$B(x,r) = \{y \in \mathbb{R} \mid |y-x| < r\} = \{y \in \mathbb{R} \mid -r+x < y < r+x\} = (x-r, x+r) \, .$$

Therefore the metric topology generated by d is equal to the standard topology on $\mathbb{R}$.

(3) Similarly, we consider $X = \mathbb{R}^n$, where $n \geq 1$, and again consider the Euclidean metric

$$d(x,y) = |x-y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \, .$$

It is generated by open balls and one can show that it is equal to the standard topology on $\mathbb{R}^n$.

Definition 2.1.12 (Metrizable)**.** *A topological space* $(X, \mathcal{T})$ *is called* metrizable *if there is a metric d defined on X such that the metric topology with respect to d is equal to $\mathcal{T}$.*

We will talk more about metrizable topological spaces later in the course.

Definition 2.1.13 (Equivalent metrics)**.** *Let X be a set, and consider two metrics d_1 and d_2 on X. We say that d_1 and d_2 are* equivalent *if and only if for each $x \in X$ there exist positive constants $\alpha, \beta > 0$ such that*

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y) \, ,$$

for any $y \in X$.

Proposition 2.1.14. *If X is a set equipped with two metrics d_1 and d_2. If d_1 and d_2 are equivalent, then the metric topology induced by d_1 and d_2 are equivalent.*

Example 2.1.15. Let $X = \mathbb{R}^n$ and $p \in [1, \infty]$. Then we can consider the metrics

$$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \, , & p < \infty \\ \max_{i=1,\dots,n} |x_i - y_i| \, , & p = \infty \end{cases} \, .$$

Then it is true that all d_p are equivalent metrics, and hence they all give rise to the same topology on $\mathbb{R}^n$. Try to show by hand that d_1, d_2 and d_∞ all are equivalent metrics.

Exercises

2.1. Let X be any set. Define $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$. Show that d is a metric.

2.2. Consider $X = \mathbb{R}^n$ and let $p \in [1, \infty]$. Define

$$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \, , & p < \infty \\ \max_{i=1,\dots,n} |x_i - y_i| \, , & p = \infty \end{cases} \, .$$

Show that d_1, d_2 and d_∞ are equivalent metrics.

Lecture 3

RECAP.

- $(X, \mathcal{T})$ is a topological space if $\mathcal{T} \subset P(X)$ is a collection of subsets satisfying:
  (T1) $\varnothing, X \in \mathcal{T}$
  (T2) $\{U_i\}_{i\in I} \subset \mathcal{T}$ implies that $\bigcup_{i\in I} U_i \in \mathcal{T}$
  (T3) $U_1, \ldots, U_n \in \mathcal{T}$ implies that $U_1 \cap \cdots \cap U_n \in \mathcal{T}$
- If X is a set, then $\mathcal{B} \subset P(X)$ is a *basis* if
  (B1) $\forall x \in X$ there is some $A \in \mathcal{B}$ such that $x \in A$
  (B2) If $x \in A_1 \cap A_2$, where $A_1, A_2 \in \mathcal{B}$, then there is some $A_3 \in \mathcal{B}$ such that $x \in A_3 \subset A_1 \cap A_2$.
- Poset topology: Let $(X, \preceq)$ be a poset. The basis of the poset topology consists of sets
$$B_a = \{x \in X \mid a \preceq x\} \ .$$
- Metric topology: Let (X, d) be a metric space. The basis of the metric topology consists of open balls
$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

3.2. Subspace topology

- Munkres: §16

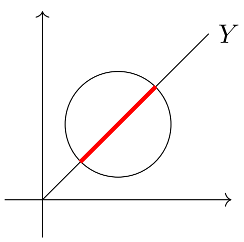
Let $(X, \mathcal{T})$ be a topological space. If $Y \subset X$ is a subset, then Y has a natural topology which is defined from $\mathcal{T}$. It is called the *subspace topology.*

Definition 3.2.1 (Subspace topology)**.** *Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$ a subset. The subspace topology on Y, denoted $\mathcal{T}_Y$ is defined as follows:*
$$U \in \mathcal{T}_Y \overset{\text{def}}{\iff} U = \widetilde{U} \cap Y \text{ for some } \widetilde{U} \in \mathcal{T} \,.$$

Proposition 3.2.2. *The subspace topology $\mathcal{T}_Y$ really defines a topology.*

PROOF. Verify the axioms for a topology (T1)–(T3). The proof is left as an exercise! □

Example 3.2.3. (1) Let $X = \mathbb{R}^2$ with the standard topology. Let $Y = \{(x, x) \mid x \in \mathbb{R}\}$ be the diagonal. Then the subspace topology on Y is what we would intuitively call the standard topology. It consists of open intervals on the diagonal line. But it is of course really intersections between open balls in $\mathbb{R}^2$ and Y.



(2) Let $X = \mathbb{R}$ with the standard topology. Let $Y = [0,1) \subset \mathbb{R}$ Then $[0, \frac{1}{2}) \subset Y$ is open, since $[0, \frac{1}{2}) = (-\frac{1}{2}, \frac{2}{1}) \cap Y$, and $(-\frac{1}{2}, \frac{1}{2}) \overset{\text{open}}{\subset} \mathbb{R}$. In fact $[0, \frac{1}{2})$ is *not* open in X because it can not be written as a union of open intervals, but it *is* open in the subspace topology in Y. It is therefore very important to pay attention to *where* a subset is open, whenever such a statement is made.

Lemma 3.2.4. *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then*

$$\mathcal{B}_Y := \{A \cap Y \mid A \in \mathcal{B}\} \ ,$$

is a basis for the subspace topology $\mathcal{T}_Y$.

Proof. We verify (B1) and (B2).

(B1): For $y \in Y$ we have $y \in X$. Therefore pick some $A_y \in \mathcal{B}$ such that $y \in A_y$. Then we have $y \in A_y \cap Y$ also, where $A_y \cap Y \in \mathcal{B}_Y$.

(B2): If $y \in (A_1 \cap Y) \cap (A_2 \cap Y) = (A_1 \cap A_2) \cap Y$ then $y \in A_1 \cap A_2$, where $A_1, A_2 \in \mathcal{B}$. Then pick $A_3 \in \mathcal{B}$ such that

$$y \in A_3 \subset A_1 \cap A_2 \, .$$

Since $y \in Y$ we therefore also have $y \in A_3 \cap Y \subset (A_1 \cap A_2) \cap Y$.

□

3.2.1. Comparing different topologies. Let X be a set. Then we can equip X with many different topologies as we have seen. It is natural to try and compare them

Definition 3.2.5. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we say that $\mathcal{T}_1 \subset \mathcal{T}_2$ if every open set in $\mathcal{T}_1$ is also open in $\mathcal{T}_2$.*

Not that not all topologies are comparable. (Therefore, $\subset$ is only a partial relation on the set of topologies on a set X!)

Example 3.2.6. Let $X = \{a, b, c\}$. Then consider

$$\begin{aligned} \mathcal{T}_1 &= \{\varnothing, X, \{a\}\} \\ \mathcal{T}_2 &= \{\varnothing, X, \{a, b\}, \{a\}\} \\ \mathcal{T}_3 &= \{\varnothing, X, \{b\}\} \ . \end{aligned}$$

Then we clearly see that $\mathcal{T}_1 \subset \mathcal{T}_2$. We also see that none of $\mathcal{T}_1$ and $\mathcal{T}_2$ are comparable with $\mathcal{T}_3$.

Proposition 3.2.7. *If $Y \overset{\text{open}}{\subset} X$, then $\mathcal{T}_Y \subset \mathcal{T}$.*

Proof. Exercise. □

3.3. Interior, closure, boundary

- Munkres: §17

Definition 3.3.8 (Interior and closure)**.** *Let $A \subset X$ be a subset. The* interior *of A is defined as*

$$\operatorname{int} A := \bigcup_{\substack{U \overset{\text{open}}{\subset} X \\ U \subset A}} U \, .$$

The closure *of A is defined as*

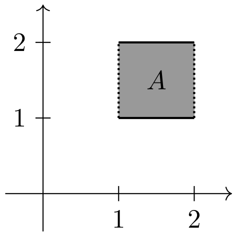
$$\operatorname{cl} A := \bigcap_{\substack{F \overset{\text{closed}}{\subset} X \\ A \subset F}} F \, .$$

Remark 3.3.9.
- Some authors use the notation $\mathring{A}$ to denote $\operatorname{int} A$, and $\overline{A}$ to denote $\operatorname{cl} A$. We will use this notation later on in the notes when it is convenient to do so.
- The slogan of the definition is that:

$$\begin{aligned} \operatorname{int} A = & \text{ largest open set contained in } A \\ \operatorname{cl} A = & \text{ smallest closed set containing } A \end{aligned}$$

- By definition we have $A \overset{\text{open}}{\subset} X \Leftrightarrow \operatorname{int} A = A$ and $A \overset{\text{closed}}{\subset} \Leftrightarrow \operatorname{cl} A = A$.
- It is furthermore clear from the definition that $\operatorname{int} A \subset A \subset \operatorname{cl} A$.

Example 3.3.10. Consider $X = \mathbb{R}^2$ equipped with the standard topology, and consider the set $A = (1, 2) \times [1, 2] \subset \mathbb{R}^2$



Using some intuition we have that A is neither open nor closed because only *parts* of the boundary is included. We have

$$\begin{aligned} \operatorname{int} A &= (1, 2) \times (1, 2) \subset \mathbb{R}^2 \\ \operatorname{cl} A &= [1, 2] \times [1, 2] \subset \mathbb{R}^2. \end{aligned}$$

Proposition 3.3.11. *(1)* $\operatorname{int} A = X \setminus \operatorname{cl}(X \setminus A)$
(2) $\operatorname{cl} A = X \setminus \operatorname{int}(X \setminus A)$

Proof. (1) We have

$$X \setminus \operatorname{cl}(X \setminus A) = X \setminus \bigcap_{\substack{F \overset{\text{closed}}{\subset} X \\ (X \setminus A) \subset F}} F \overset{\text{de Morgan's law}}{=} \bigcup_{\substack{F \overset{\text{closed}}{\subset} X \\ (X \setminus A) \subset F}} X \setminus F\,.$$

Write $U := X \setminus F$, and by definition it is open. Then $X \setminus A \subset F \Leftrightarrow A \supset X \setminus F = U$. Therefore we can write this union as

$$\bigcup_{\substack{F \overset{\text{closed}}{\subset} X \\ (X \setminus A) \subset F}} X \setminus F = \bigcup_{\substack{U \overset{\text{open}}{\subset} X \\ U \subset A}} U = \operatorname{int} A\,.$$

(2) This is left as an exercise, but completely analogous.

□

Proposition 3.3.12. *(1)* $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$
(2) $\operatorname{cl}(A \cap B) \subset \operatorname{cl} A \cap \operatorname{cl} B$
(3) $\operatorname{int}(A \cup B) \supset \operatorname{int} A \cup \operatorname{int} B$
(4) $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$

Proof. Exercise! □

Remark 3.3.13 (Kuratowski closure-complement problem)**.** Let $A \subset X$ be a subset. How many subsets can be formed by taking successive closures and complements?

If we for example write $kA \coloneqq \mathrm{cl}\, A$ and $cA \coloneqq X \smallsetminus A$, then how many *different* sets can we get at most among the collection

$$kcA,\ kckcA,\ kA,\ kckA \text{ etc}$$

The answer is remarkably 14.

Definition 3.3.14 (Boundary)**.** *The* boundary *of a subset $A \subset X$ is defined as*

$$\partial A \coloneqq \mathrm{cl}\, A \cap \mathrm{cl}(X \smallsetminus A)\,.$$

Proposition 3.3.15. *(1) The boundary is*

$$\partial A = \{x \in X \mid \forall \text{ neighborhood } U \text{ of } x,\ :\ U \cap A \neq \varnothing \text{ and } U \cap (X \smallsetminus A) \neq \varnothing\}$$

(2) $A \overset{\text{open}}{\subset} X \iff \partial A = \mathrm{cl}\, A \smallsetminus A$
(3) $\mathrm{cl}\, A = \partial A \cup \mathrm{int}\, A$.
(4) $x \in \mathrm{cl}\, A$ *if and only if every neighborhood of* x *intersects* A.
(5) $\partial A = \mathrm{cl}\, A \smallsetminus \mathrm{int}\, A$.

PROOF. (1) We show that the complements agree. Namely that

$$X \smallsetminus \partial A = \{x \in X \mid \exists \text{ neighborhood } U \text{ of } x,\ :\ U \cap A = \varnothing \text{ or } U \cap (X \smallsetminus A) = \varnothing\}\ .$$

We first use Proposition 3.3.11 together with de Morgan's law to get

$$\begin{aligned} X \smallsetminus \partial A &= X \smallsetminus (\mathrm{cl}\, A \cap \mathrm{cl}(X \smallsetminus A)) \\ &= (X \smallsetminus \mathrm{cl}\, A) \cup (X \smallsetminus (\mathrm{cl}(X \smallsetminus A))) \\ &= \mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A \end{aligned}$$

To make notation easier we set

$$D \coloneqq \{x \in X \mid \exists \text{ neighborhood } U \text{ of } x,\ :\ U \cap A = \varnothing \text{ or } U \cap (X \smallsetminus A) = \varnothing\}\ .$$

We repeat now that we want to show

$$D = \mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A\,.$$

$\subset$**:** Then if $x \in D$ we have two cases. Either we have picked a neighborhood U such that $U \cap A = \varnothing$. Then we have $U \overset{\text{open}}{\subset} X \smallsetminus A$, which means $U \subset \mathrm{int}\, X \smallsetminus A$. In the other case we have $U \cap (X \smallsetminus A) = \varnothing$ which means $U \overset{\text{open}}{\subset} A$ and hence $U \subset \mathrm{int}\, A$. In either case we have $U \in \mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A$.

$\supset$**:** We again have two cases. If $x \in \mathrm{int}\, A$, then, because $\mathrm{int}\, A \overset{\text{open}}{\subset} A$, there is some neighborhood $U_x \overset{\text{open}}{\subset} X$ of x such that $U_x \subset A$ and hence $U_x \cap (X \smallsetminus A) = \varnothing$. If $x \in \mathrm{int}(X \smallsetminus A)$ a similar argument shows $U_x \cap A = \varnothing$, and hence $x \in D$.

(2) We already have from the above computation that

$$X \smallsetminus \partial A = \mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A \iff \partial A = X \smallsetminus (\mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A)\,.$$

We then have

$$\mathrm{cl}\, A \smallsetminus A = (X \smallsetminus \mathrm{int}(X \smallsetminus A)) \smallsetminus A = X \smallsetminus (\mathrm{int}(X \smallsetminus A) \cup A)\,.$$

Therefore

$$\begin{aligned} \partial A = \mathrm{cl}\, A \smallsetminus A &\iff X \smallsetminus (\mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A) = X \smallsetminus (\mathrm{int}(X \smallsetminus A) \cup A) \\ &\iff \mathrm{int}(X \smallsetminus A) \cup \mathrm{int}\, A = \mathrm{int}(X \smallsetminus A) \cup A \\ &\iff \mathrm{int}\, A = A \iff A \overset{\text{open}}{\subset} X\,. \end{aligned}$$

(3) This follows from the definition and some manipulation of unions and intersections.

$$\begin{aligned}\operatorname{int} A \cup \partial A &= \operatorname{int} A \cup (\operatorname{cl} A \cap \operatorname{cl}(X \setminus A)) \\ &= (\operatorname{int} A \cup \operatorname{cl} A) \cap (\operatorname{int} A \cup \operatorname{cl}(X \setminus A)) \\ &= \operatorname{cl} A \cap X = \operatorname{cl} A\,.\end{aligned}$$

(4) From (3) we have $\operatorname{cl} A = \partial A \cup \operatorname{int} A$. By (1) we have

$$\partial A = \{x \in X \mid \forall \text{ neighborhood } U \text{ of } x,\ :\ U \cap A \neq \varnothing \text{ and } U \cap (X \setminus A) \neq \varnothing\}\ ,$$

and since $\operatorname{int} A \subset A$ it is clear that every neighborhood U_x of x intersects A. This shows

$$x \in \operatorname{cl} A \implies \text{every neighborhood of } x \text{ intersects } A\,.$$

To show the converse, if we assume that $x \in X$ is such that every neighborhood U_x of x intersects A, then we either have $x \in \partial A \subset \operatorname{cl} A$ (by (3)), or we have $U_x \cap (X \setminus A) = \varnothing$ which means $U_x \subset A \subset \operatorname{cl} A$ and hence $x \in A \subset \operatorname{cl} A$. In either case we have $x \in \operatorname{cl} A$.

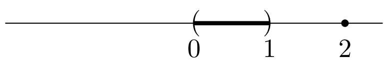
(5) Using (3) we obtain

$$\operatorname{cl} A \setminus \operatorname{int} A \overset{(3)}{=} (\partial A \cup \operatorname{int} A) \setminus \operatorname{int} A = \partial A\,.$$

□

Definition 3.3.16. *A subset* $A \subset X$ *is* dense *if* $\operatorname{cl} A = X$*.*

Example 3.3.17. (1) Consider $A := (0, 1) \cup \{2\} \subset \mathbb{R}$ where $\mathbb{R}$ is equipped with the standard topology.



Then we have

$$\begin{cases}\operatorname{int} A = (0, 1) \\ \operatorname{cl} A = [0, 1] \cup \{2\} \\ \partial A = \{0, 1, 2\}\end{cases}.$$

(2) Consider $\mathbb{Q} \subset \mathbb{R}$. The set of rationals does not contain any intervals (because otherwise it would be uncountable). Therefore $\operatorname{int} \mathbb{Q} = \varnothing$.

For every $x \in \mathbb{R}$, and for any interval around x, there is some rational number contained in the interval (try to prove this statement, or look it up!). Hence $\operatorname{cl} \mathbb{Q} = \mathbb{R}$ and so $\mathbb{Q}$ is dense in $\mathbb{R}$.

The boundary of $\mathbb{Q} \subset \mathbb{R}$ is

$$\partial \mathbb{Q} = \operatorname{cl} \mathbb{Q} \cap \operatorname{cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \cap \operatorname{cl}(\mathbb{R} \setminus \mathbb{Q}) = \operatorname{cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}\,.$$

Again, we have that the irrationals $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, because every interval in $\mathbb{R}$ contains an irrational number (if it did not, $\mathbb{Q}$ would not be countable!).

Exercises

3.1. Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset. Equip Y with the subspace topology $\mathcal{T}_Y$, which is defined as

$$U \in \mathcal{T}_Y \overset{\text{def}}{\iff} U = \widetilde{U} \cap Y \text{ for some } \widetilde{U} \in \mathcal{T}\,.$$

Show that $\mathcal{T}_Y$ really is a topology.

3.2. Let $(X, \mathcal{T})$ be a topological space. Show that if $Y \overset{\text{open}}{\subset} X$, then $\mathcal{T}_Y \subset \mathcal{T}$.

3.3. Let $A, B \subset X$ be subsets of a topological space. Show that

(a) $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$

(b) $\operatorname{cl}(A \cap B) \subset \operatorname{cl} A \cap \operatorname{cl} B$

(c) $\operatorname{int}(A \cup B) \supset \operatorname{int} A \cup \operatorname{int} B$

(d) $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$

Lecture 4

Recap. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset.

- The *interior* of A is
$$\operatorname{int} A = \bigcup_{\substack{U \overset{\text{open}}{\subset} X \\ U \subset A}} U\,.$$
- The *closure* of A is
$$\operatorname{cl} A = \bigcap_{\substack{F \overset{\text{closed}}{\subset} X \\ A \subset F}} F\,.$$
- The *boundary* of A is
$$\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \setminus A)$$

4.1. Limit points

- Munkres: §17

Definition 4.1.1 (Limit point of a subset)**.** *Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$ a subset. We say that $x \in X$ is a* limit point of A *if $x \in \operatorname{cl}(A \setminus \{x\})$.*

Define
$$A' \coloneqq \{x \in X \mid x \text{ limit point of } A\}$$

Remark 4.1.2.
- By a result from last time, we note that $x \in A'$ if and only if every neighborhood $U_x \overset{\text{open}}{\subset} X$ of x intersects A in a point different from x.
- Sometimes limit points are called "cluster points" or "accumulation points".

Example 4.1.3. Consider $A = (0, 1) \cup \{2\} \subset \mathbb{R}$ where $\mathbb{R}$ is equipped with the standard topology. Then
$$\begin{aligned} \operatorname{int} A &= (0,1) \\ \operatorname{cl} A &= [0,1] \cup \{2\} \\ \partial A &= \{0,1,2\} \\ A' &= [0,1] \end{aligned}$$
Note that $2 \notin A'$, since there exists a neighborhood of 2 (just pick a sufficiently small interval) which only intersects A in the point 2.

Proposition 4.1.4. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$ be a subset. Then*
$$\operatorname{cl} A = A \cup A'\,.$$

Proof. $\subset$**:** This is obvious since $A' \subset \operatorname{cl} A$ and $A \subset \operatorname{cl} A$.

$\supset$**:** Let $x \in \operatorname{cl} A$. If $x \in A$, then we are done. Else, if $x \notin A$, then every neighborhood of x intersects A. This means that every neighborhood intersects A in a point different from x (since x does not belong to A), and hence $x \in A'$.

□

Corollary 4.1.5. $A \overset{\text{closed}}{\subset} X$ *if and only if $A' \subset A$.*

Proof. $A \overset{\text{closed}}{\subset} X$ if and only if $A = \operatorname{cl} A = A \cup A'$ if and only if $A' \subset A$. $\square$

Example 4.1.6. Consider $X = \mathbb{R}$ with the standard topology. Consider the set

$$A = \left\{ \frac{1}{n} \;\middle|\; n = 1, 2, \dots \right\} .$$

We might write $A = \{a_n\}_{n=1}^{\infty}$ and define $a_n := \frac{1}{n}$ for each $n = 1, 2, \dots$. In standard calculus we have $\lim_{n\to\infty} \frac{1}{n} = 0$, and the intuition of a limit point in our context should be thought of in a similar way.

We will determine A'. We note that $0 \in \mathbb{R}$ is the unique point that satisfies $0 \in A'$. Every neighborhood of 0 intersects $\{a_n\}_{n=1}^{\infty}$. Therefore $0 \in A'$. Now, we have $A' = \{0\}$, because for any other point $y \in \mathbb{R} \setminus \{0\}$ we can pick a sufficiently small interval $I_y = (a, b)$ containing y such that $I_y \cap A = \varnothing$.

Example 4.1.7. Let (X, d) be a metric space. Recall that a basis for the metric topology is

$$\mathcal{B} = \{B(x, r) \mid x \in X,\ r > 0\} .$$

Let $A = \{a_n\}_{n=1}^{\infty} \subset X$ be a sequence. Write down what it means for $x \in A'$ in terms of the metric topology and open balls. Show that it is exactly the normal ε-δ definition of the limit that we are used to from real analysis.

Definition 4.1.8 (Limit of a sequence)**.** *Let $(X, \mathfrak{T})$ be a topological space and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points in X. We say that a_n* converges to a point $x \in X$ as $n \to \infty$ *if for any neighborhood $x \in U_x \overset{\text{open}}{\subset} X$ there is a positive integer $N > 0$ such that for all $n \geq N$ we have $a_n \in U_x$. If a_n converges to x we will write $a_n \to x$.*

Remark 4.1.9. Limits of sequences in general topological spaces do *not* have to be unique.

Remark 4.1.10. The reason we have the above definition exists is because in general a *sequence* is not a *set*. For instance the constant sequence $a_n = x$ for all $n \in \mathbb{Z}_+$ is a sequence with infinitely many terms while as a set it is equal to $\{x\}$.

It is clear that "the limit" of the constant sequence $a_n = x$ should be equal to x, but by the definition of limit points of a subset we have $\{x\}' = \varnothing$.

This is a very subtle distinction which is made clear in the next proposition.

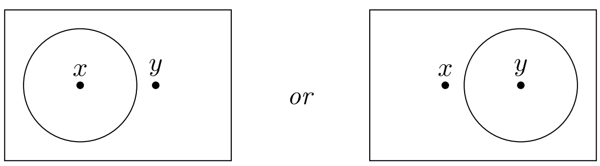
Proposition 4.1.11. *Suppose that $A := \{a_n\}_{n=1}^{\infty}$ is a sequence such that all the a_n's are distinct points. Show that $a_n \to x$ if and only if $x \in A'$.*

Proof. The proof of this is left as an exercise. It is similar to Example 4.1.7. $\square$

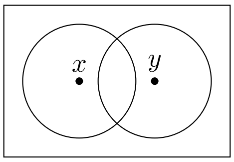
4.2. Separation axioms

- Munkres: §17

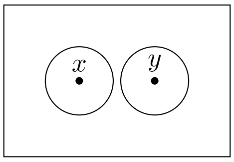
Definition 4.2.12 (T_0)**.** *A topological space is called T_0 if for all pairs of distinct points x, y we have either that there is a neighborhood U_x of x such that $y \notin U_x$ or that there is a neighborhood U_y of y such that $x \notin U_y$.*



Definition 4.2.13 (T_1)**.** *A topological space is called T_1 if for all pairs of distinct points x, y we have that there are neighborhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$.*



Definition 4.2.14 (Hausdorff)**.** *A topological space is called Hausdorff if for all pairs of distinct points x, y we have that there are neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \varnothing$.*



Remark 4.2.15.

- Sometimes Hausdorff is called T_2. In this course we will refer to this as Hausdorff however.
- It is clear from these definitions that Hausdorff $\implies T_1 \implies T_0$.

Example 4.2.16.

- Any metric space (X, d) is Hausdorff. For any two distinct points $x, y \in X$, we let $r < \frac{d(x,y)}{2}$. Then $B(x, r) \cap B(y, r) = \varnothing$.
- $\mathbb{R}$ equipped with the standard topology is a metric space, and hence it is Hausdorff. The picture is similar: For any two distinct points $x, y \in \mathbb{R}$, we can separate x and y by picking small enough intervals around x and y respectively.
- Let $X = \{a, b, c, d\}$ and consider the partial order $\preceq$ defined by
$$a \preceq b \preceq c, \quad a \preceq d\,.$$
Then the poset topology is
$$\mathcal{T} = \{\varnothing, X, \{c\}, \{d\}, \{b, c\}\}\ .$$
This topology is not T_1. Namely, the only neighborhood of a is the whole of X. For example, $\{b, c\}$ is a neighborhood of b not containing a, but X is the only neighborhood of a and it is not disjoint from b.

Proposition 4.2.17. *X is T_1 if and only if points are closed.*

Proof. **$\Rightarrow$:** Fix some arbitrary $x \in X$. For any other $y \in X \smallsetminus \{x\}$, there is some $U_y \overset{\text{open}}{\subset} X$ such that $x \notin U_y$. Then we have
$$X \smallsetminus \{x\} = \bigcup_{y \in X \smallsetminus \{x\}} U_y \overset{\text{open}}{\subset} X\,,$$
which means precisely that $\{x\} \overset{\text{closed}}{\subset} X$.

$\Leftarrow$: Since $\{x\}, \{y\} \overset{\text{closed}}{\subset} X$ we have $X \smallsetminus \{x\}, X \smallsetminus \{y\} \overset{\text{open}}{\subset} X$. Then define $U_y := X \smallsetminus \{x\}$ and $U_x := X \smallsetminus \{y\}$. These are two neighborhoods of x and y respectively such that $x \notin U_y$ and $y \notin U_x$. Hence X is T_1.

$\square$

Proposition 4.2.18. *Let $(X, \mathcal{T})$ be a Hausdorff topological space and consider a sequence $\{a_n\}_{n=1}^{\infty}$. If $a_n \to x$ and $a_n \to y$, then $x = y$. That is, limits in Hausdorff spaces are unique.*

PROOF. The proof is left as an exercise. $\square$

Example 4.2.19. Consider $\mathbb{R}$ and equip it with the *open ray topology*. The open ray topology is defined as

$$\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\} \ .$$

The open ray topology is T_0: For any pair of distinct points $x, y \in \mathbb{R}$ we can without loss of generality assume $x < y$. Then $y \in (y - \varepsilon, \infty)$ but $x \notin (y - \varepsilon, \infty)$ given that $\varepsilon > 0$ is small enough.

The open ray topology is not T_1: Let x and y be as above. Any neighborhood of x contains y since $x < y$. Therefore it can not be T_1. Another way of showing this is to show that singletons are not closed. Consider $\{x\}$ and compute $\operatorname{cl}\{x\}$ and draw the conclusion that singletons are not closed and hence $\mathbb{R}$ equipped with the is not T_1.

Example 4.2.20. We consider $\mathbb{R}$ equipped with the open ray topology again. Let $A = \{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{1}{n}$. Find A'.

We see that $A' = (-\infty, 1)$ since for any $y \in (-\infty, 1)$, any neighborhood of y contains all points greater than y, and hence contains at least the point $1 \in A$. So in this case there are uncountably many limits of the sequence $\{a_n\}_{n=1}^{\infty}$.

Exercises

4.1. Let (X, d) be a metric space, and let $A = \{a_n\}_{n=1}^{\infty} \subset X$ be a sequence. Show that $x \in A'$ is equivalent to the normal ε-δ definition of limits.

4.2. Consider $\mathbb{R}$ equipped with the open ray topology $\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$. For any $x \in \mathbb{R}$, compute $\operatorname{cl}\{x\}$ and draw the conclusion that the open ray topology is not T_1.

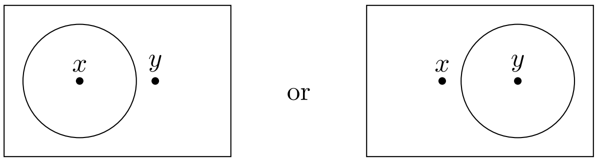
4.3. Let $(X, \mathcal{T})$ be a Hausdorff topological space and consider a sequence $\{a_n\}_{n=1}^{\infty}$. If $a_n \to x$ and $a_n \to y$, show that $x = y$.

(*Hint: Pick two disjoint neighborhoods of x and y and write down what $a_n \to x$ and $a_n \to y$ means. You may use Theorem 5.1.2 from Lecture 5!*)

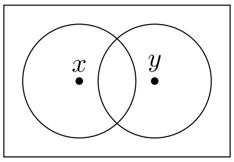
Lecture 5

Recap. Let $(X, \mathcal{T})$ be a topological space. Let x and y be two distinct points. Then X is called

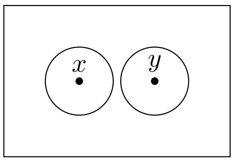
- T_0 if there is a neighborhood U_x of x or a neighborhood U_y of y not containing the other point.



- T_1 if there are neighborhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$



- Hausdorff if there are disjoint neighborhoods U_x of x and U_y of y.



5.1. More separation axioms

- Munkres: §17

Before we move on, we will discuss the separation axioms a little bit more. The Hausdorff axiom is a particularly nice one, which any *reasonable* space satisfies. Sequences having unique limits can be seen as a property that is convenient to have. Especially in applications (to analysis and other areas).

Theorem 5.1.1. *If X is a Hausdorff topological space, then any $Y \subset X$ equipped with the subspace topology is also Hausdorff.*

Proof. Exercise. $\square$

Theorem 5.1.2. *Let X be an infinite T_1 topological space and $A \subset X$ a subspace. Then $x \in A'$ if and only if every neighborhood of x contains infinitely many points of A.*

Proof. **$\Leftarrow$:** If every neighborhood intersects A in infinitely many points, it means in particular that it intersects A in a point different from x, which means exactly that $x \in A'$.

⇒: Assume by contradiction that there is some $U \overset{\text{open}}{\subset} X$ neighborhood of x such that

$$U \cap (A \smallsetminus \{x\}) = \{a_1, \ldots, a_n\} \ .$$

Recall that $\{a_1, \ldots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}$ is closed since points are closed in a T_1 space. Then

$$\widetilde{U} := X \smallsetminus \{a_1, \ldots, a_n\} \ ,$$

is open since $\{a_1, \ldots, a_n\}$ is closed. Then we have $x \in \widetilde{U} \cap U$ and also $\widetilde{U} \cap (A \smallsetminus \{x\}) = \varnothing$, which contradicts the assumption that $x \in A'$.ence we need to have that the intersection $U \cap A$ is infinite.

□

5.2. First countability

- Munkres: §21, §30

Definition 5.2.3 (Countable basis)**.** *A topological space X has a* countable basis at $x \in X$ *if there exists a collection $\{B_n\}_{n=1}^{\infty}$ of neighborhoods of x such that for any neighborhood $U \overset{\text{open}}{\subset} X$ of x, there is some $N > 0$ such that $B_N \subset U$.*

Remark 5.2.4. Compare with lemma 13.2 in the book to see why it is reasonable to call this a *local basis* at x.

Definition 5.2.5 (First countable)**.** *A topological space $(X, \mathfrak{T})$ is called* first countable *if every point has a countable basis.*

Proposition 5.2.6. *Metric spaces are first countable.*

Proof. Pick any $x \in X$. Then the collection $\{B\left(x, \frac{1}{n}\right)\}_{n=1}^{\infty}$ of balls of radius $\frac{1}{n}$ centered at x is a countable basis. □

One of the main reasons we discuss first countability here is because of the following lemma. It characterizes the closure of a set as all the points that can be seen as a limit of some sequence.

Lemma 5.2.7 (Sequence lemma)**.** *Let $(X, \mathfrak{T})$ be a first countable topological space and let $A \subset X$ be a subset. Then $x \in \operatorname{cl} A$ if and only if there is a sequence $\{a_n\}_{n=1}^{\infty} \subset A$ such that $a_n \to x$.*

Proof. **⇐:** Assume that we have a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to x$. By definition this means that for any neighborhood $U \overset{\text{open}}{\subset} X$ of x, there is some $N > 0$ such that for every $n \geq N$ we have $a_n \in U$. Therefore $U \cap A \neq \varnothing$ which means precisely that every neighborhood of x intersects A, so $x \in \operatorname{cl} A$.

⇒: Let $x \in \operatorname{cl} A$. The idea is to pick a countable basis $\{B_i\}_{i=1}^{\infty}$ at x and define

$$U_n := \bigcap_{i=1}^{n} B_i \, .$$

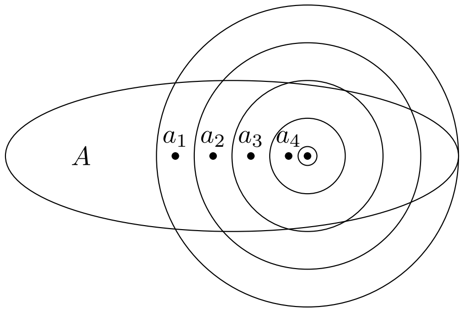
Then we have

$$U_1 \supset U_2 \supset U_3 \supset \cdots \ ,$$

and the idea is to define a sequence $\{a_n\}_{n=1}^{\infty}$ by picking

$$a_k \in A \cap U_k \, ,$$

arbitrarily, for each $k \geq 1$.



With this choice, we need to show that $a_n \to x$. Therefore, pick any neighborhood $U \overset{\text{open}}{\subset} X$ of x. Since $\{B_i\}_{i=1}^{\infty}$ is a countable basis, it means that there exists some $N > 0$ such that $B_N \subset U$. Then it also implies $\bigcap_{i=1}^{N} B_i \subset U$ and in fact it implies

$$\bigcap_{i=1}^{n} B_i \subset U,\ \forall n \geq N\,.$$

Therefore there is some $N > 0$ such that for every $n \geq N$ we have $a_n \in U$, so $a_n \to x$ by definition.

□

Corollary 5.2.8. *Let X be a first countable topological space. Then $F \overset{\text{closed}}{\subset} X$ if and only if any convergent sequence $\{a_n\}_{n=1}^{\infty}$ in F has its limit contained in F.*

Proof. This follows from the sequence lemma, and the fact that F is closed if and only if $F = \operatorname{cl} F$. □

We will now finish this part about separation axioms and first countability by a list of examples.

Example 5.2.9. (1) Let X be any set, and let $\mathcal{T} = \{\varnothing, X\}$ be the indiscrete topology.

- It is not T_0 since $X \overset{\text{open}}{\subset} X$ is the only neighborhood of any point. This means that it is not T_1 and not Hausdorff either.
- It is first countable. Since for every $x \in X$, the collection $\{X\}_{i=1}^{\infty}$ is a countable basis, since again, X is the only neighborhood of any point. In fact, every point has a *finite* basis!

(2) Let X be any set and let $\mathcal{T}$ be the discrete topology. That is, every set is open in X.

- It is Hausdorff since for any two distinct points, $\{x\}$ and $\{y\}$ are two disjoint neighborhoods of x and y respectively. Therefore it is also T_1 and T_0.
- It is also first countable, since $\{\{x\}\}_{i=1}^{\infty}$ is a countable (again, its even finite!) basis. This is because $\{x\} \subset U$ for any neighborhood U of x.

(3) Let (X, d) be a metric space.

- It is Hausdorff as we have already seen before. To get disjoint neighborhoods of two different points, we pick sufficiently small balls (of radius $r < d(x, y)$).
- It is also first countable by the previous proposition.

(4) Consider the totally ordered set $(\mathbb{R}, <)$, equipped with the poset topology. This is what we called the *open ray topology* in the last lecture. Recall that a basis consists of all half-infinite rays

$$\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}\ .$$

- It is T_0 like we saw in the previous lecture, but not T_1 and hence not Hausdorff.

- It is first countable as well. It is left as an exercise to show it!

(5) Consider $\mathbb{R}$ equipped with the standard topology. Since the standard topology is the same as the metric topology, we have that it is Hausdorff and first countable.

5.3. Product topology

- Munkres: §15

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, there is a natural topology on their product $X \times Y$.

Definition 5.3.10 (Product topology)**.** *Let X and Y be two topological spaces as above. Then the* product topology *is the one with basis given by*

$$\mathcal{B} = \left\{ U \times V \subset X \times Y \,\middle|\, U \overset{\text{open}}{\subset} X,\ V \overset{\text{open}}{\subset} Y \right\} .$$

Theorem 5.3.11. *If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for the topologies on X and Y respectively, then*

$$\mathcal{B}_{X \times Y} := \{ B \times C \mid B \in \mathcal{B}_X,\ C \in \mathcal{B}_Y \} ,$$

is a basis for the product topology on $X \times Y$.

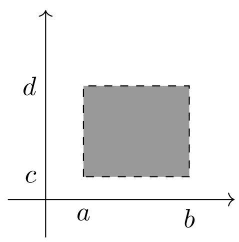
Proof. Remember Lemma 13.2 from the book: It suffices to check that for any $W \overset{\text{open}}{\subset} X \times Y$ and $(x, y) \in W$, that there is some $B \times C \in \mathcal{B}_{X \times Y}$ such that $(x, y) \in B \times C \subset W$.

First we pick any $W \overset{\text{open}}{\subset} X \times Y$ and $(x, y) \in W$. Then by definition there is some open $U \times V \subset X \times Y$ such that $(x, y) \in U \times V \subset W$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases of X and Y respectively, we pick $B \in \mathcal{B}_X$ and $C \in \mathcal{B}_Y$ such that

$$x \in B \subset U$$
$$y \in C \subset V$$

Then we have $(x, y) \in B \times C \subset U \times V \subset W$. □

Example 5.3.12. We have $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The product topology is generated by products of intervals $(a, b) \times (c, d)$



Proposition 5.3.13. *If X and Y are Hausdorff topological spaces, then $X \times Y$ equipped with the product topology is Hausdorff.*

Proof. Exercise. □

We will discuss the product topology more later.

Exercises

5.1. Let X be a Hausdorff topological space and let $Y \subset X$ be any subset equipped with the subspace topology. Show that Y is Hausdorff.

5.2. Consider $\mathbb{R}$ equipped with the open ray topology $\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$. Show that the open ray topology is first countable.

5.3. Let X and Y be Hausdorff topological spaces. Show that $X \times Y$ equipped with the product topology is Hausdorff.

Lecture 6 (Problem Session 1)

Problem §13.4.

(a) Let $\{\mathcal{T}_\alpha\}_{\alpha\in I}$ be a family of topologies on X. Show that $\bigcap_{\alpha\in I}\mathcal{T}_\alpha$ is a topology on X. Is $\bigcup_{\alpha\in I}\mathcal{T}_\alpha$ a topology on X?

(b) Let $\{\mathcal{T}_\alpha\}_{\alpha\in I}$ be a family of topologies on X. Show that there exists a unique smallest topology on X containing all the topologies $\mathcal{T}_\alpha$ and a unique largest topology on X which is contained in all the topologies $\mathcal{T}_\alpha$.

(c) Let $X = \{a, b, c\}$, and define

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\}, \qquad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}\ .$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Problem §13.7. Consider the following topologies on $\mathbb{R}$.

- $\mathcal{T}_1 =$ the standard topology
- $\mathcal{T}_2 =$ the topology with basis $\mathcal{B} = \{(a, b), (a, b) \setminus K \mid a < b\}$ where $K := \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$. (This topology is called $\mathbb{R}_K$ in the book.)
- $\mathcal{T}_3 =$ the finite complement topology on $\mathbb{R}$. That is $U \in \mathcal{T}_3$ if and only if $\mathbb{R}\setminus U = \mathbb{R}$ or $\mathbb{R} \setminus U$ is a finite subset.
- $\mathcal{T}_4 =$ upper limit topology. It is generated by the basis $\mathcal{B} = \{(a, b] \mid a < b\}$.
- $\mathcal{T}_5 =$ the topology generated by $\mathcal{B} = \{(-\infty, a) \mid a \in \mathbb{R}\}$.

Determine, for each of these topologies, which of the others it contains.

Problem §16.2. Let X be a set. Equip X with two topologies $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T} \subsetneq \mathcal{T}'$. If $Y \subset X$, what can be said about the corresponding subspace topologies $\mathcal{T}_Y$ and $\mathcal{T}'_Y$?

Problem §16.3. Let $Y := [-1, 1] \subset \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology. Consider the following sets:

$$A := \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{2} < |x| < 1\right\}$$
$$B := \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{2} < |x| \le 1\right\}$$
$$C := \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{2} \le |x| < 1\right\}$$
$$D := \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{2} \le |x| \le 1\right\}$$
$$E := \left\{x \in \mathbb{R} \;\middle|\; 0 < |x| < 1, \quad \frac{1}{x} \notin \mathbb{Z}_+\right\}$$

Which of these sets are open in Y? Which are open in $\mathbb{R}$?

Problem §17.5. Let $(X, \preceq)$ be a totally ordered set equipped with the order topology. That is the topology generated by the following sets:

- $(a, b) = \{x \mid a \preceq x \preceq b, \quad x \neq a,\ x \neq b\}$
- $[a_0, b) = \{x \mid a_0 \preceq x \preceq b, \quad x \neq b\}$ if $a_0 := \min X$ exists, and
- $(a, b_0] = \{x \mid a \preceq x \preceq b_0, \quad x \neq a\}$ if $b_0 := \max X$ exits.

Show that $\overline{(a, b)} \subset [a, b]$. When does equality hold?

Problem §17.12. Let $(X, \mathcal{T})$ be a Hausdorff topological space and let $Y \subset X$ be equipped with the subspace topology $\mathcal{T}_Y$. Show that $\mathcal{T}_Y$ is Hausdorff.

Problem §17.19.

(a) Show that $\operatorname{int} A \cap \partial A = \varnothing$ and $\operatorname{cl} A = \operatorname{int} A \cup \partial A$
(b) Show that $\partial A = \varnothing \ \Leftrightarrow\ A$ is clopen (both open and closed)
(c) Show that U open $\ \Leftrightarrow\ \partial U = \operatorname{cl} U \setminus U$
(d) Let U be open. Is it true that $U = \operatorname{int}(\operatorname{cl}(U))$? Motivate.

Problem §17.6.

(a) Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$. Compute $\operatorname{cl} K$ with respect to the topologies in problem §13.7.
(b) Which of the topologies in problem §13.7. are Hausdorff? Which ones are T_1?

Bonus problem. Show that the finite complement topology on $\mathbb{R}$ (it is $\mathcal{T}_3$ in problem §13.7) is not first countable.

(*Hint: For a countable local basis* $\{B_n\}_{n=1}^{\infty}$ *at* x *in this topology we have* $\{x\} = \bigcap_{n=1}^{\infty} B_n$.)

Part 2

Quotients, connectedness and compactness

Lecture 7

RECAP. Recall that if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are two topological spaces, we can endow $X \times Y$ with a natural topology with basis

$$\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y\} .$$

It is called the *product topology.*

7.1. Infinite products

- Munkres: §19

Before we move on, we will take a brief moment and discuss infinitely many products. Let $\{X_i\}_{i\in I}$ be a family of topological spaces indexed by some set I. We then consider the (possibly infinite) product $\prod_{i\in I} X_i$.

In the case of infinite products, there are actually more than one "natural" choice of topology.

Definition 7.1.1 (Product topology)**.** *A basis for the* product topology *on the product* $\prod_{i\in I} X_i$ *is given by*

$$\mathcal{B}_{prod} := \left\{ \prod_{i\in I} U_i \;\middle|\; U_i \overset{\text{open}}{\subset} X_i,\ U_i = X_i \text{ except for finitely many } i \in I \right\} .$$

Definition 7.1.2 (Box topology)**.** *A basis for the* box topology *on* $\prod_{i\in I} X_i$ *is given by*

$$\mathcal{B}_{box} := \left\{ \prod_{i\in I} U_i \;\middle|\; U_i \overset{\text{open}}{\subset} X_i \right\} .$$

Remark 7.1.3. When $|I| < \infty$, the product topology is of course equal to the box topology. Since the distinction only occurs for infinite index sets, it is not very easy to visualize.

The box topology contains strictly more open sets than the product topology.

Unless otherwise stated, we will by default equip $\prod_{i\in I} X_i$ with the *product topology.* We will discuss later on why we want the product topology, and not the box topology by default.

7.2. Continuous functions

- Munkres: §18

Thus far in the course we have been interested in topological spaces and some of its basic properties (T_0, T_1, Hausdorff, first countability). We will now talk about *relationships between different topological spaces.*

Definition 7.2.4 (Continuous function)**.** *Let X and Y be two topological spaces. Let $f\colon X \longrightarrow Y$ be a function.*

(1) We say that f is continuous *if*

$$U \overset{\text{open}}{\subset} Y \implies f^{-1}(U) \overset{\text{open}}{\subset} X .$$

(2) We say that f is continuous at $x \in X$ *if*

$$\forall U \text{ neighborhood of } f(x)\, \exists V \text{ neighborhood of } x \,:\, f(V) \subset U\,.$$

Example 7.2.5. Consider $X = \{a,b,c\}$ equipped with the topology $\mathcal{T}_X = \{\varnothing, X, \{a\}, \{a,b\}\}$ and $Y = \{a,b,c,d\}$ equipped with the topology $\mathcal{T}_Y = \{\varnothing, Y, \{b\}, \{b,c\}, \{b,d\}, \{b,c,d\}\}$. Consider the functions

$$\begin{aligned} f\colon X &\longrightarrow Y \\ a &\longmapsto b \\ b &\longmapsto c \\ c &\longmapsto a \end{aligned} \qquad \begin{aligned} g\colon X &\longrightarrow Y \\ a &\longmapsto b \\ b &\longmapsto c \\ c &\longmapsto c \end{aligned} \qquad \begin{aligned} h\colon X &\longrightarrow Y \\ a &\longmapsto a \\ b &\longmapsto a \\ c &\longmapsto a \end{aligned}$$

Then f is continuous. This is because

$$\begin{aligned} f^{-1}(\{b\}) &= \{a\} \in \mathcal{T}_X \\ f^{-1}(\{b,c\}) &= \{a,b\} \in \mathcal{T}_X \\ f^{-1}(\{b,d\}) &= \{a\} \in \mathcal{T}_X \\ f^{-1}(\{b,c,d\}) &= \{a,b\} \in \mathcal{T}_X \end{aligned}$$

It is an exercise to check whether g and h are continuous or not.

Proposition 7.2.6. *Let X and Y be topological spaces with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively, and consider the function $f\colon X \longrightarrow Y$. Then f is continuous if and only if $\forall B \in \mathcal{B}_Y$ we have $f^{-1}(B) \overset{\text{open}}{\subset} X$.*

Proof. By definition we have $U \overset{\text{open}}{\subset} Y \;\Leftrightarrow\; U = \bigcup_{i\in I} B_i$ for some $B_i \in \mathcal{T}_Y$. We also have that preimages commute with unions:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)\,.$$

Then we see that

$$f^{-1}(B_i) \overset{\text{open}}{\subset} X \;\Leftrightarrow\; f^{-1}(U) \overset{\text{open}}{\subset} X\,.$$

$\square$

Proposition 7.2.7. *Let X, Y and Z be topological spaces.*

(a) If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two continuous functions, then $g \circ f\colon X \longrightarrow Z$ is continuous.

(b) The function $f\colon X \longrightarrow Y$ is continuous if and only if

$$F \overset{\text{closed}}{\subset} Y \implies f^{-1}(F) \overset{\text{closed}}{\subset} X\,.$$

(c) The function $f\colon X \longrightarrow Y$ is continuous if and only if f is continuous at every point $x \in X$.

Proof. (a) Exercise!

(b) We have $F \overset{\text{closed}}{\subset} Y \;\Leftrightarrow\; Y \smallsetminus F \overset{\text{open}}{\subset} Y$ and also $f^{-1}(Y \smallsetminus F) = X \smallsetminus f^{-1}(F)$. It is an exercise to work out the rest of the details.

(c) $\Rightarrow$**:** Pick any $x \in X$ and any neighborhood U of $f(x)$. Then we pick $V = f^{-1}(U)$. By continuity of f, we have $V \overset{\text{open}}{\subset} X$ (and we have that $x \in V$ which follows from $f(x) \in U$). Therefore

$$f(V) = f\left(f^{-1}(U)\right) \subset U\,,$$

which means that f is continuous at any point $x \in X$.

$\Leftarrow$: Pick any $U \overset{\text{open}}{\subset} Y$. Then $\forall x \in f^{-1}(U)$ there is a neighborhood V_x of x such that $f(V_x) \subset U$. Therefore

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x\,,$$

is open, since $V_x \overset{\text{open}}{\subset} X$.

$\square$

Remark 7.2.8. Note that our definition of continuity at a point generalizes the ε-δ definition from analysis! From analysis we know that $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ if

$$\forall \varepsilon > 0\, \exists \delta > 0 \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon\,.$$

We can quite directly compare this to our definition of continuity:

$$\underbrace{\forall \varepsilon > 0}_{\text{“}\forall U \text{ nghd of } f(x_0)\text{”}} \quad \underbrace{\exists \delta > 0}_{\text{“}\exists V \text{ nghd of } x_0\text{”}} \quad \underbrace{|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon}_{\text{“}f(V) \subset U\text{”}}\,.$$

Some of the topological constructions we have seen so far in the course can be expressed in terms of continuity of functions.

Proposition 7.2.9. *Let X be a topological space and $Y \subset X$ a subset equipped with the subspace topology.*

(1) The inclusion $i\colon Y \longrightarrow X$, $y \longmapsto y$ is continuous.

(2) The subspace topology is the smallest topology such that the inclusion $i\colon Y \longrightarrow X$ is continuous.

Proof. (1) Exercise.

(2) Denote by $\mathcal{T}_Y$ the subspace topology on Y coming from X. Now, the statement of the proposition can be rewritten to read: for any topology $\mathcal{T}'$ on Y, we have that

$$i\colon (Y, \mathcal{T}') \longrightarrow X \text{ continuous} \implies \mathcal{T}_Y \subset \mathcal{T}'\,.$$

Let $U \overset{\text{open}}{\subset} X$. Then we always have (regardless of whether i is continuous or not) $i^{-1}(U) = U \cap Y \in \mathcal{T}_Y$ by definition of the subspace topology. But if i is continuous it means that $i^{-1}(U) = U \cap Y \in \mathcal{T}'$, which means $\mathcal{T}_Y \subset \mathcal{T}'$.

$\square$

Proposition 7.2.10. *Let X and Y be topological spaces and equip $X \times Y$ with the product topology.*

(1) The projections

$$\begin{aligned} \pi_X\colon X \times Y &\longrightarrow X \\ (x, y) &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi_Y\colon X \times Y &\longrightarrow Y \\ (x, y) &\longmapsto y \end{aligned}$$

are continuous.

(2) For any fixed $x_0 \in X$ and $y_0 \in Y$, the inclusions

$$\begin{aligned} i_{X,y_0}\colon X &\longrightarrow X \times Y \\ x &\longmapsto (x, y_0) \end{aligned} \qquad\qquad \begin{aligned} i_{Y,x_0}\colon Y &\longrightarrow X \times Y \\ y &\longmapsto (x_0, y) \end{aligned}$$

are continuous.

Proof. (1) Let $U \overset{\text{open}}{\subset} X$ and $V \overset{\text{open}}{\subset} Y$. Then we have

$$\begin{aligned} \pi_X^{-1}(U) &= U \times Y \overset{\text{open}}{\subset} X \times Y \\ \pi_Y^{-1}(V) &= X \times V \overset{\text{open}}{\subset} X \times Y\,, \end{aligned}$$

by the definition of the product topology.

(2) We only show that i_{X,y_0} is continuous. The other inclusion is shown to be continuous in a similar way. If $W = U \times V \overset{\text{open}}{\subset} X \times Y$ such that $y_0 \notin V$ then $i_{X,y_0}^{-1}(W) = \varnothing \overset{\text{open}}{\subset} X$. Else if $y_0 \in V$ we have $i_{X,y_0}^{-1}(W) = U \overset{\text{open}}{\subset} X$ by definition of the product topology. Therefore, by Proposition 7.2.6 we are done.

$\square$

7.3. Homeomorphisms

- Munkres: §18

We will now define what it means for two topological spaces to be "equivalent".

Definition 7.3.11 (Homeomorphism)**.** *Let X and Y be topological spaces. A continuous function $f\colon X \longrightarrow Y$ is called a* homeomorphism *if it is bijective and has a continuous inverse $f^{-1}\colon Y \longrightarrow X$.*

If there is a homeomorphism between two topological spaces X and Y we say that X and Y are homeomorphic*, and we denote it by $X \cong Y$.*

Remark 7.3.12. Suppose that $f\colon X \longrightarrow Y$ is a continuous function such that $f^{-1}\colon Y \longrightarrow X$ exists. Then what does it mean for f^{-1} to be continuous? By definition it means

$$V \overset{\text{open}}{\subset} X \implies (f^{-1})^{-1}(V) \overset{\text{open}}{\subset} Y\,,$$

and $(f^{-1})^{-1}(V) = f(V)$, since f is bijective. This means precisely that f is an open map, and therefore we can summarize: f is a homeomorphism if and only if it is a bijective map such that

$$U \overset{\text{open}}{\subset} X \;\Leftrightarrow\; f(U) \overset{\text{open}}{\subset} Y\,.$$

Homeomorphisms are continuous maps that themselves send open sets to open sets while being bijective. There is not only a one-to-one correspondence between the underlying sets of the topological spaces involved, but there is also a one-to-one-correspondence between the open sets! In this sense, homeomorphisms is an equivalence of topological spaces.

As such, homeomorphisms will preserve "topological properties" of topological spaces. We will see more on this later!

Example 7.3.13. (1) We have $(-1,1) \cong \mathbb{R}$ by

$$\begin{aligned} f\colon (-1,1) &\longrightarrow \mathbb{R} \\ x &\longmapsto \tan\left(\frac{x\pi}{2}\right)\,. \end{aligned}$$

We can write down an explicit inverse

$$\begin{aligned} f^{-1}\colon \mathbb{R} &\longrightarrow (-1,1) \\ x &\longmapsto \frac{2\arctan(x)}{\pi}\,. \end{aligned}$$

(a) Fix some $a > 0$. Then the map

$$\begin{aligned} f_a\colon (-1,1) &\longrightarrow (-a,a) \\ x &\longmapsto ax \end{aligned}$$

is a homeomorphism (it is an exercise to show this).

(b) Fix some $t > 0$. Then the map

$$\begin{aligned} g_t \colon\ (a,b) &\longrightarrow (a+t, b+t) \\ x &\longmapsto x+t\,, \end{aligned}$$

is a homeomorphism (it is an exercise to show this).
in fact, the maps f, f_a and g_t together show that $(a,b) \cong \mathbb{R}$ for any open interval (a,b)!

(c) Any open ball $B(0,1) \subset \mathbb{R}^n$ in the standard metric on $\mathbb{R}^n$ is homeomorphic to the entire $\mathbb{R}^n$. We can write down an explicit map

$$\begin{aligned} f \colon\ B(0,1) &\longrightarrow \mathbb{R}^n \\ \boldsymbol{x} = (x_1, \dots, x_n) &\longmapsto \left(\frac{x_1}{1-|\boldsymbol{x}|}, \dots, \frac{x_n}{1-|\boldsymbol{x}|}\right)\,, \end{aligned}$$

where

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\,,$$

is the Euclidean metric. In fact, we can write down an explicit inverse to f!

$$\begin{aligned} f^{-1} \colon\ \mathbb{R}^n &\longrightarrow B(0,1) \\ \boldsymbol{x} = (x_1, \dots, x_n) &\longmapsto \left(\frac{x_1}{1+|\boldsymbol{x}|}, \dots, \frac{x_n}{1+|\boldsymbol{x}|}\right)\,. \end{aligned}$$

We finish with the following proposition.

Proposition 7.3.14. *Let X_1, X_2, Y_1, Y_2 be topological spaces.*

(1) If $f \colon X_1 \longrightarrow X_2$ and $g \colon X_2 \longrightarrow Y_1$ are homeomorphisms, then $g \circ f \colon X_1 \longrightarrow Y_1$ is a homeomorphism.

(2) If $X_1 \cong Y_1$ and $X_2 \cong Y_2$, then $X_1 \times X_2 \cong Y_1 \times Y_2$.

Proof. (1) If f and g are bijective, then so is $g \circ f$. It follows from part (a) of Proposition 7.2.7 that $g \circ f$ is continuous. Moreover, given that the inverses f^{-1} and g^{-1} are continuous it follows that $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ is continuous as well. Therefore $g \circ f$ is a homeomorphism.

(2) Let $f_1 \colon X_1 \longrightarrow Y_1$ and $f_2 \colon X_2 \longrightarrow Y_2$ be homeomorphisms. Then

$$\begin{aligned} f_1 \times f_2 \colon\ X_1 \times X_2 &\longrightarrow Y_1 \times Y_2 \\ (x_1, x_2) &\longmapsto (f_1(x_1), f_2(x_2))\,, \end{aligned}$$

is a homeomorphism. It is clear that $f_1 \times f_2$ is bijective since each of f_1 and f_2 is bijective. By definition of the product topology and because preimages and unions commute, it suffices to show continuity for open sets of the form $W = U \times V \subset Y_1 \times Y_2$ where $U \overset{\text{open}}{\subset} Y_1$ and $V \overset{\text{open}}{\subset} Y_2$. We have

$$(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V) \overset{\text{open}}{\subset} X_1 \times X_2\,,$$

since both f_1 and f_2 are continuous. This also shows that the inverse

$$\begin{aligned} (f_1 \times f_2)^{-1} = f_1^{-1} \times f_2^{-1} \colon\ Y_1 \times Y_2 &\longrightarrow X_1 \times X_2 \\ (y_1, y_2) &\longmapsto (f_1^{-1}(y_1), f_2^{-1}(y_2))\,. \end{aligned}$$

is continuous. Namely, if we let $U \overset{\text{open}}{\subset} X_1$ and $V \overset{\text{open}}{\subset} X_2$ we get

$$(f_1^{-1} \times f_2^{-1})^{-1}(U \times V) = (f_1^{-1})^{-1}(U) \times (f_2^{-1})^{-1}(V) \overset{\text{open}}{\subset} Y_1 \times Y_2\,,$$

since f_1^{-1} and f_2^{-1} both are continuous. Hence $f_1 \times f_2$ is a homeomorphism.

□

Remark 7.3.15. In fact, item (2) in Proposition 7.3.14 is true in great generality. If I is any index set and $X_i \cong Y_i$ for every $i \in I$, then $\prod_{i\in I} X_i \cong \prod_{i\in I} Y_i$ where the products are equipped with either the product topology or the box topology.

Exercises

7.1. Let X, Y and Z be topological spaces, and suppose that $W \subset X$ is a subset equipped with the subspace topology.

(a) If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two continuous functions, show that the composition

$$g \circ f\colon X \longrightarrow Z\,,$$

is continuous.

(b) Show that the inclusion $i\colon W \longrightarrow X$ is continuous.

7.2. Consider $X = \{a, b, c\}$ equipped with the topology $\mathcal{T}_X = \{\varnothing, X, \{a\}, \{a, b\}\}$ and $Y = \{a, b, c, d\}$ equipped with the topology $\mathcal{T}_Y = \{\varnothing, Y, \{b\}, \{b, c\}, \{b, d\}, \{b, c, d\}\}$. Consider the following functions:

$$\begin{aligned} g\colon X &\longrightarrow Y \\ a &\longmapsto b \\ b &\longmapsto c \\ c &\longmapsto c \end{aligned} \qquad\qquad \begin{aligned} h\colon X &\longrightarrow Y \\ a &\longmapsto a \\ b &\longmapsto b \\ c &\longmapsto a \end{aligned}$$

Is any of the functions g and h continuous?

7.3. The goal of this exercise is to show that $(-1, 1) \cong (a, b)$ for any numbers $a, b \in \mathbb{R}$ with $a < b$.

(a) Fix some $a > 0$. Show that the function

$$\begin{aligned} f_a\colon (-1, 1) &\longrightarrow (-a, a) \\ x &\longmapsto ax \end{aligned}$$

is a homeomorphism.

(b) Fix some $t > 0$ and let $a < b$ be two arbitrary reals. Show that the function

$$\begin{aligned} g_t\colon (a, b) &\longrightarrow (a + t, b + t) \\ x &\longmapsto x + t \end{aligned}$$

is a homeomorphism.

(c) Using part (a) and (b), write down an explicit homeomorphism $(-1, 1) \cong (1, 5)$.

Lecture 8

Recap. Let X and Y be topological spaces.

- $f\colon X \longrightarrow Y$ is said to be *continuous* if and only if
$$U \overset{\text{open}}{\subset} Y \implies f^{-1}(U) \overset{\text{open}}{\subset} X\,.$$
- $f\colon X \longrightarrow Y$ is said to be a *homeomorphism* (we denote it by $X \cong Y$) if and only if f is a continuous bijection such that there exists an inverse $f^{-1}\colon Y \longrightarrow X$ that is continuous as well.

8.1. Quotient topology

- Munkres: §22

Definition 8.1.1 (Open and closed maps). *Let X and Y be two topological spaces and consider a function $f\colon X \longrightarrow Y$. We say that*

(1) f is open *if $U \overset{\text{open}}{\subset} X \implies f(U) \overset{\text{open}}{\subset} Y$*

(2) f is closed *if $U \overset{\text{closed}}{\subset} X \implies f(U) \overset{\text{closed}}{\subset} Y$*

Proposition 8.1.2. *Let X and Y be two topological spaces and equip $X \times Y$ with the product topology. Then the projection maps*

$$\begin{aligned} \pi_X\colon X \times Y &\longrightarrow X \\ (x,y) &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi_Y\colon X \times Y &\longrightarrow Y \\ (x,y) &\longmapsto y \end{aligned}$$

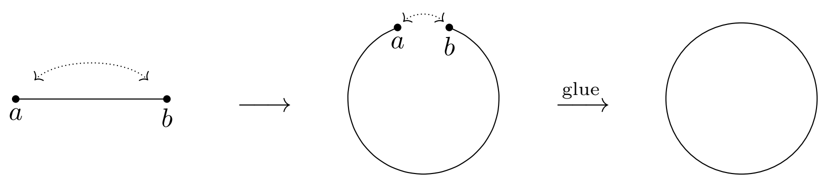
are open maps.

Proof. This basically follows from the definition of the product topology. If $W \overset{\text{open}}{\subset} X \times Y$, then by definition we have $W = \bigcup_{i\in I} U_i \times V_i$, where $U_i \overset{\text{open}}{\subset} X$ and $V_i \overset{\text{open}}{\subset} Y$ for every $i \in I$. Therefore

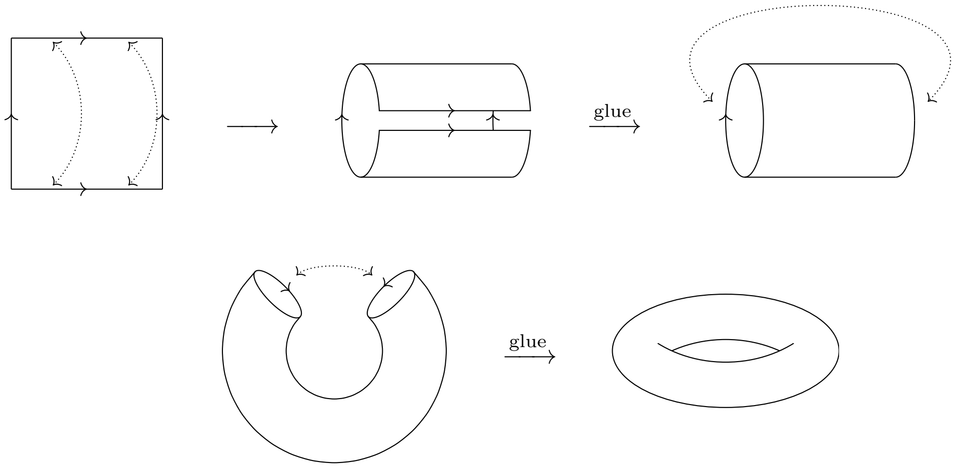
$$\pi_X(W) = \pi_X\left(\bigcup_{i\in I} U_i \times V_i\right) = \bigcup_{i\in I} \pi_X(U_i \times V_i) = \bigcup_{i\in I} U_i \overset{\text{open}}{\subset} X\,.$$

Similarly we have $\pi_Y(W) = \bigcup_{i\in I} V_i \overset{\text{open}}{\subset} Y$. $\square$

The intuition and motivation for the quotient topology is we want to "gluing together topological spaces". For instance, consider the open interval $(a,b) \subset \mathbb{R}$. Then we can "glue together its ends" to obtain a circle



This can be made precise using the quotient topology. Using a similar technique we can glue together the edges of a square to obtain a torus.



Both of these constructions and more can be done via *quotient maps* and the quotient topology.

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then we denote by $X/\sim$ the set of equivalence classes. Recall that $X/\sim$ partitions X into disjoint subsets.

Example 8.1.3. (1) Let X be any set. Define $\sim$ by $x \sim y$ for every $\forall x, y \in X$. Then

$$X/\sim = \{X\} .$$

Namely, every element in X is related to every other element, so X is the only equivalence class.

(2) Consider instead the relation $\sim$ defined by

$$x \sim y \overset{\text{def}}{\Leftrightarrow} x = y .$$

Then

$$X/\sim = \{\{x\} \mid x \in X\} .$$

It is bijective to X, since every equivalence class is a singleton.

(3) For a non-trivial example, consider $X = \mathbb{R}$. Then define

$$x \sim y \overset{\text{def}}{\Leftrightarrow} x - y \in \mathbb{Z} .$$

Alternatively $x \sim y \Leftrightarrow \exists k \in \mathbb{Z} : x = y + k$. The equivalence classes are

$$\mathbb{R}/\sim = \{x + \mathbb{Z} \mid x \in X\} ,$$

where $x + \mathbb{Z}$ is shorthand for the set

$$x + \mathbb{Z} = \{x + k \mid k \in \mathbb{Z}\} .$$

(4) Let X and Y be any sets. If $f\colon X \longrightarrow Y$ is any surjective function, then

$$x \sim y \overset{\text{def}}{\Leftrightarrow} f(x) = f(y) ,$$

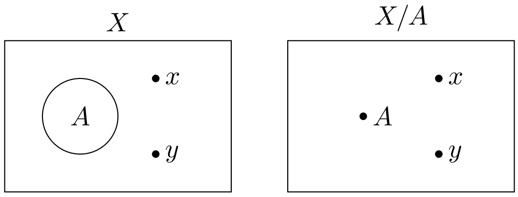
is an equivalence relation, and

$$X/\sim = \{f^{-1}(y) \mid y \in Y\} .$$

From this, we have an obvious bijection between $X/\sim$ and Y.

(5) If X is any set, and $A \subset X$ a subset. Define
$$x \sim y \overset{\text{def}}{\Leftrightarrow} x = y \text{ or } x, y \in A\,.$$
We should interpret this as shrinking the subset A to a single point. We identify all points inside of A. We have
$$X/{\sim} = \{A\} \cup \{\{x\} \mid x \in X \setminus A\}\;.$$
Everything inside of A is identified to a single equivalence class, and everything not in A is not identified with anything.

This particular equivalence relation is so important that we give it its own notation. We write X/A.



We now define the quotient topology.

Definition 8.1.4 (Quotient topology)**.** *Let X be a topological space and $\sim$ an equivalence relation on X. Consider*
$$\begin{aligned} p\colon\; X &\longrightarrow X/{\sim} \\ x &\longmapsto [x]\,. \end{aligned}$$
The quotient topology *on $X/{\sim}$ is defined by*
$$U \overset{\text{open}}{\subset} X/{\sim} \overset{\text{def}}{\Leftrightarrow} p^{-1}(U) \overset{\text{open}}{\subset} X\,.$$

Proposition 8.1.5. *The quotient topology above really defines a topology.*

Proof. We check the axioms.

(T1): We have $p^{-1}(\varnothing) = \varnothing \overset{\text{open}}{\subset} X$ and by surjectivity of p we have
$$p^{-1}(X/{\sim}) = X \overset{\text{open}}{\subset} X\,.$$

(T2), (T3): Both (T2) and (T3) follows immediately from the equalities
$$p^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} p^{-1}(U_i)$$
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$
together with the definition of the quotient topology.

$\square$

Remark 8.1.6. The quotient topology is the largest topology for which p is continuous.

Example 8.1.7. We take a closer look at example (3) in Example 8.1.3 above and use it to motivate why the quotient topology should be thought of as some kind of gluing. It might be a little hard to imagine, but this space $\mathbb{R}/{\sim}$, where
$$x \sim y \overset{\text{def}}{\Leftrightarrow} x - y \in \mathbb{Z}\,,$$
is in fact a circle. The circle is defined as
$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2\,,$$

equipped with the subspace topology coming from the standard metric topology on $\mathbb{R}^2$. As in the definition of the quotient topology we have the quotient projection

$$\begin{aligned} p\colon \mathbb{R} &\longrightarrow \mathbb{R}/\sim \\ r &\longmapsto r+\mathbb{Z} = \{r+k \mid k \in \mathbb{Z}\}\ . \end{aligned}$$

Consider the function

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow S^1 \subset \mathbb{R}^2 \\ r &\longmapsto (\cos(2\pi r), \sin(2\pi r))\,. \end{aligned}$$

It has period 1 meaning by definition that $f(r) = f(r+1)$ for any $r \in \mathbb{R}$. Note that $r \sim r+1$, so that r and $r+1$ are identified in $\mathbb{R}/\sim$. It is a fact that f induces a bijection of sets

$$\begin{aligned} \tilde{f}\colon \mathbb{R}/\sim &\longrightarrow S^1 \\ [r] &\longmapsto (\cos(2\pi r), \sin(2\pi r))\,. \end{aligned}$$

We can (later in this lecture) show that $\tilde{f}$ is a homeomorphism.

Definition 8.1.8 (Quotient map)**.** *Let X and Y be topological spaces. Let $p\colon X \longrightarrow Y$ be a surjective map. Then p is a* quotient map *if*

$$U \overset{\text{open}}{\subset} Y \ \Leftrightarrow\ p^{-1}(U) \overset{\text{open}}{\subset} X\,.$$

Remark 8.1.9. Observe the equivalence above! Equivalently a map is a quotient map if it is continuous, surjective and

$$p^{-1}(U) \overset{\text{open}}{\subset} X \implies U \overset{\text{open}}{\subset} Y\,.$$

Proposition 8.1.10. *Suppose $p\colon X \longrightarrow Y$ is a continuous surjective map.*

(1) If p is an open map, then p is a quotient map.
(2) If p is a closed map, then p is a quotient map.

Proof. (1) Since continuity is equivalent (by definition) with

$$U \overset{\text{open}}{\subset} X \implies p^{-1}(U) \overset{\text{open}}{\subset} X\,,$$

it suffices to show the other direction given that p is open. Namely, let $p^{-1}(U) \overset{\text{open}}{\subset} X$. By definition of p being open we then have

$$p(p^{-1}(U)) \overset{\text{open}}{\subset} Y\,,$$

but $p(p^{-1}(U)) = U$ by surjectivity of p, so we are done.

(2) Exercise.

□

Proposition 8.1.11. *(1) If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two quotient maps, then $g \circ f\colon X \longrightarrow Z$ is a quotient map.*
(2) Let $f\colon X \longrightarrow Z$ be an injective quotient map. Then f is a homeomorphism.

Example 8.1.12. (1) Consider $X = [0,1] \cup [2,3] \subset \mathbb{R}$ and $Y = [0,2] \subset \mathbb{R}$. Define a map $p\colon X \longrightarrow Y$ via

$$p(x) = \begin{cases} x, & x \in [0,1] \\ x-1, & x \in [2,3] \end{cases}\,.$$

Then p is surjective, continuous and closed. Surjectivity is clear. To prove continuity it suffices to show it for basis elements. Let $U \overset{\text{open}}{\subset} [0,2]$ be an interval.

If $1 \notin U$, then $p^{-1}(U)$ is completely contained in either $[0,1]$ or $[2,3]$, which is still open. If $1 \in U$, let us say $U = (1-a, 1+b)$ for some $0 < a, b < 1$. Then

$$p^{-1}((1-a,1+b)) = (1-a,1] \cup [2, 2+b),$$

which is still open in X, because of the definition of the subspace topology:

$$(1-a,1] \cup [2,2+b) = (1-a,2+b) \cap X,$$

where $(1-a, 2+b) \overset{\text{open}}{\subset} \mathbb{R}$. Therefore p is continuous. We also see that p is a closed map by the following argument. Consider first the two restrictions of p to $[0,1]$ and $[2,3]$ respectively.

$$\begin{aligned} p_1 := p|_{[0,1]} \colon\ [0,1] &\longrightarrow [0,1] \xrightarrow{\text{inclusion}} [0,2] \\ x &\longmapsto x \end{aligned}$$

$$\begin{aligned} p_2 := p|_{[2,3]} \colon\ [2,3] &\longrightarrow [1,2] \xrightarrow{\text{inclusion}} [0,2] \\ x &\longmapsto x-1 \end{aligned}$$

First of all, the two inclusion maps are closed, because if $Y \overset{\text{closed}}{\subset} X$, and if Y is equipped with the subspace topology, then the inclusion $Y \longrightarrow X$ is closed. Next, the two maps $x \longmapsto x$ and $x \longmapsto x-1$ are homeomorphisms, so they are closed. Because p_1 and p_2 are compositions of closed maps, they are closed. Let $U \overset{\text{closed}}{\subset} X$ and write

$$U = U_1 \cap U_2,$$

where $U_1 = U \cap [0,1]$ and $U_2 = U \cap [0,2]$. Since U is closed, both U_1 and U_2 are closed as well. We now apply the function p, and we get

$$p(U) = p_1(U) \cup p_2(U).$$

Using that p_1 and p_2 are closed yields that $p(U) \overset{\text{closed}}{\subset} Y$.

By Proposition 8.1.10 it means that p is a quotient map.

(2) Now consider $X = \{a,b,c,d\}$ with the topology $\mathcal{T}_X = \{\varnothing, \{a,b,c,d\}, \{b\}, \{c\}, \{b,c\}\}$, and $Y = \{a,b,c\}$ with the topology $\mathcal{T}_Y = \{\varnothing, \{a,b,c\}, \{a\}\}$. Then define the map

$$\begin{aligned} f\colon\ X &\longrightarrow Y \\ a &\longmapsto b \\ b &\longmapsto a \\ c &\longmapsto c \\ d &\longmapsto c \end{aligned}$$

Then f is a quotient map that is neither open nor closed. It is an exercise to check this!

Proof. (1) By definition of quotient map we have

$$U \overset{\text{open}}{\subset} Z \;\Leftrightarrow\; g^{-1}(U) \overset{\text{open}}{\subset} Y \;\Leftrightarrow\; f^{-1}(g^{-1}(U)) \overset{\text{open}}{\subset} X,$$

and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$, which shows that $g \circ f$ is a quotient map:

(2) Exercise. (Hint: Show that f is an open map.)

□

We will now see the most important theorem in this lecture.

THEOREM 8.1.13. *Let $g\colon X \longrightarrow Z$ be a surjective continuous map. Let $\sim$ be an equivalence relation on X defined by*

$$x \sim y \overset{def}{\Leftrightarrow} g(x) = g(y)\,,$$

and equip $X/{\sim}$ with the quotient topology.

(1) g induces a bijective continuous map $f\colon X/{\sim} \longrightarrow Z$ which satisfies $f \circ p = g$. Moreover, f is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X/{\sim} \arrow[r, "f"'] & Z \end{tikzcd}$$

(2) If Z is Hausdorff, then so is $X/{\sim}$.

PROOF. The induced map f is defined as

$$\begin{aligned} f\colon X/{\sim} &\longrightarrow Z \\ [x] &\longmapsto g(x)\,. \end{aligned}$$

(1) Since f is defined on *sets*, we need to check that the definition of f does not matter which element in the set we pick. That is, if $[x_1] = [x_2]$, then we need to check that $g(x_1) = g(x_2)$. This is called that f is *well-defined*.

f is well-defined: If $[x_1] = [x_2]$ we have by definition that $x_1 \sim x_2$ so $g(x_1) = g(x_2)$. This means $f([x_1]) = g(x_1) = g(x_2) = f([x_2])$, and so f is well-defined.

f is injective: If $g(x_1) = g(x_2)$, then $x_1 \sim x_2$, which means $[x_1] = [x_2]$ by definition, so f is injective.

f is surjective: Since g is surjective we have that $\forall z \in \mathbb{Z}\ \exists x \in X$ such that $g(x) = z$. Hence there is some $[x] \in X/{\sim}$ such that $f([x]) = g(x) = z$, so f is surjective.

f is continuous: By continuity of g we have

$$U \overset{\text{open}}{\subset} Z \implies g^{-1}(U) \overset{\text{open}}{\subset} X\,.$$

Because $f \circ p = g$, we have

$$g^{-1}(U) = p^{-1}(f^{-1}(U)) \overset{\text{open}}{\subset} X \;\Leftrightarrow\; f^{-1}(U) \overset{\text{open}}{\subset} X/{\sim}\,.$$

The equivalence is due to the definition of the quotient topology. Therefore f is continuous.

f homeomorphism $\implies$ g quotient map: By Proposition 8.1.10 f is a quotient map since a homeomorphism is continuous, surjective and open. Therefore, since $g = f \circ p$, g is a composition of quotient maps so it is a quotient map itself.

g quotient map $\implies$ f homeomorphism: If g is a quotient map we have

$$U \overset{\text{open}}{\subset} Z \;\Leftrightarrow\; g^{-1}(U) \overset{\text{open}}{\subset} Z\,.$$

Then by the exact same proof as above where we showed that f is continuous, we show that f is a quotient map. Then f is an injective quotient map which means that it is a homeomorphism by Proposition 8.1.11.

(2) Let Z be Hausdorff. Let $[x], [y] \in X/{\sim}$ be two distinct points. Therefore $x \nsim y \Leftrightarrow g(x) \neq g(y)$. Pick disjoint neighborhoods $U_x, U_y \overset{\text{open}}{\subset} Z$ of $g(x)$ and $g(y)$ respectively. Then $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are disjoint neighborhoods of $[x]$ and $[y]$ respectively, and so $X/{\sim}$ is Hausdorff.

□

Example 8.1.14. We continue Example 8.1.7 and can now prove that

$$\begin{aligned} \tilde{f} \colon \mathbb{R}/{\sim} &\longrightarrow S^1 \\ [r] &\longmapsto (\cos(2\pi r), \sin(2\pi r))\,, \end{aligned}$$

is in fact a homeomorphism. It is easy to verify that the following diagram *commutes*:

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"] \arrow[dr, "f"] & \\ \mathbb{R}/{\sim} \arrow[r, "\tilde{f}"'] & S^1 \end{tikzcd},$$

meaning that $f = \tilde{f} \circ p$ where f and p are both defined as in Example 8.1.7. Furthermore we have that f is a quotient map (check this carefully!). Therefore, by part (1) of Theorem 8.1.13 it follows that $\tilde{f}$ is a homeomorphism.

Exercises

8.1. Suppose that $p \colon X \longrightarrow Y$ is a surjective continuous map. Show that if p is a closed map, then p is a quotient map.

8.2. Consider $X = \{a, b, c, d\}$ with the topology $\mathcal{T}_X = \{\varnothing, \{a, b, c, d\}, \{b\}, \{c\}, \{b, c\}\}$, and $Y = \{a, b, c\}$ with the topology $\mathcal{T}_Y = \{\varnothing, \{a, b, c\}, \{a\}\}$. Then define the map

$$\begin{aligned} f \colon X &\longrightarrow Y \\ a &\longmapsto b \\ b &\longmapsto a \\ c &\longmapsto c \\ d &\longmapsto c \end{aligned}$$

Show that f is a quotient map that is neither open nor closed.

8.3. Let $f \colon X \longrightarrow Y$ be an injective quotient map. Show that f is a homeomorphism. *(Hint: Show that f is an open map.)*

Lecture 9

9.1. Connectedness

- Munkres: §23

We will now discuss one main topological property, called *connectedness*.

Definition 9.1.1 (Separation)**.** *A* separation *of a topological space X is a pair of disjoint, non-empty subsets $U, V \overset{\text{open}}{\subset} X$ such that*

$$X = U \cup V .$$

Definition 9.1.2 (Connected)**.** *Let X be a topological space.*

- *X is* disconnected *if there is a separation of X.*
- *X is* connected *if there is no separation. Equivalently, X is connected if*

$$X = U \cup V \implies U = \varnothing \text{ or } V = \varnothing .$$

Proposition 9.1.3. *A topological space X is connected if and only if X and $\varnothing$ are the only closed and open (clopen) subsets.*

Proof. We will instead show the contrapositive statement. That is X is disconnected if and only if $\exists$ non-empty proper clopen subset $U \subset X$.

⇒: We assume that X is disconnected. We pick a separation $X = U \cup V$, $U, V \overset{\text{open}}{\subset} X$, $U, V \neq \varnothing$. Then since $U \overset{\text{open}}{\subset} X$ we have $X \smallsetminus U \overset{\text{closed}}{\subset} X$. But we also have

$$X \smallsetminus U = V \overset{\text{open}}{\subset} X ,$$

which means that $X \smallsetminus U$ (and hence also U) is clopen.

⇐: Now assume $U \subset X$ is clopen. Then we can write down a separation of X:

$$X = U \cup (X \smallsetminus U) ,$$

where $X \smallsetminus U \overset{\text{open}}{\subset} X$ since $U \overset{\text{closed}}{\subset} X$. Hence X is disconnected.

□

Example 9.1.4. (1) Consider $X = \{a, b, c\}$. Equip X with four different topologies:

$$\begin{aligned} \mathcal{T} &= \{\varnothing, X, \{a\}, \{b, c\}\} \\ \mathcal{T}' &= \{\varnothing, X, \{a\}, \{a, b\}\} \\ \mathcal{T}'' &= \{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\} \\ \mathcal{T}''' &= \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} \end{aligned}$$

Then $(X, \mathcal{T})$ is disconnected, because there is a separation: $X = \{a\} \cup \{b, c\}$.

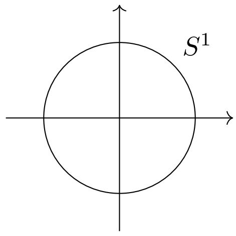
However, $(X, \mathcal{T}')$ is connected, because there is no separation. In this situation it is easy to check all the possible unions among the open sets.

It is an exercise to check whether $(X, \mathcal{T}'')$ and $(X, \mathcal{T}''')$ are connected or not.

(2) $X = (0,1) \cup (2,3) \subset \mathbb{R}$ equipped with the subspace topology is disconnected. This is almost too trivial, because we have already defined X as a separation.

$$X = \underbrace{(0,1)}_{\substack{\text{open} \\ \subset X}} \cup \underbrace{(2,3)}_{\substack{\text{open} \\ \subset X}} .$$

(3) The circle $S^1 \subset \mathbb{R}^2$ is connected.



It is easy to "see" in the picture that it is connected, but it is not trivial to show.

(4) $\mathbb{Q} \subset \mathbb{R}$ equipped with the subspace topology is disconnected. It is an exercise to write down a separation!

(5) $\mathbb{R}^n$ for $n \geq 1$ is connected. This is not trivial either! We will in fact see a proof of this at the end of this lecture.

(6) If X is any set with at least two elements equipped with the discrete topology (that is, every subset is open), then it is disconnected. Every set in the discrete topology is clopen and so X is disconnected.

(7) If X is any set equipped with the indiscrete topology (that is, $\varnothing$ and X are the only open sets), then X is connected. Indeed, there are no proper non-empty open sets, so there is no separation!

Connectedness is a property which is formulated in terms of open sets. This is what we call a *topological property.* It is something that has to do with open sets. Therefore it must also be preserved under homeomorphisms, because homeomorphisms give a bijective correspondence between open sets of two topological spaces.

Theorem 9.1.5. *Suppose X and Y are two homeomorphic topological spaces. Then X is connected if and only if Y is connected.*

Proof. Like before we show the contrapositive statement. That is X is disconnected if and only if Y is disconnected.

$\Leftarrow$: Suppose Y is disconnected. Pick a separation $Y = U \cup V$. Then since $X = f^{-1}(Y)$ (since f is bijective), we have

$$X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) .$$

By bijectivity and continuity of f, both $f^{-1}(U)$ and $f^{-1}(V)$ are open, non-empty and disjoint. Hence X is disconnected.

$\Rightarrow$: If X is disconnected with a separation $X = U \cup V$, then we have similarly that $f(X) = Y$ by bijectivity of f, and

$$Y = f(U \cup V) = f(U) \cup f(V) ,$$

where $f(U)$ and $f(V)$ are open, non-empty and disjoint since f is bijective and an open map.

$\square$

Lemma 9.1.6. *If $X = U \cup V$ is a separation of X, and $Y \subset X$ (equipped with the subspace topology) is connected, then either $Y \subset U$ or $Y \subset V$.*

Proof. Since $X = U \cup V$ is a separation we have in particular that $U \cap V = \varnothing$. Therefore $U \cap Y$ and $V \cap Y$ are open (by definition of the subspace topology) and disjoint as well. We then have

$$(U \cap Y) \cup (V \cap Y) = (U \cup V) \cap Y = X \cap Y = Y\,.$$

Because of the assumption that Y is connected, we have $U \cap Y = \varnothing$ or $V \cap Y = \varnothing$, which means precisely that $Y \subset U$ or $Y \subset V$. $\square$

Theorem 9.1.7. *Let $Y_i \subset X$ be a family of connected subspaces (equipped with the subspace topology) indexed by $i \in I$, which have a common point $x \in Y_i \ \forall i \in I$. Then*

$$\bigcup_{i\in I} Y_i \subset X\,,$$

is connected.

Proof. Assume that there is a separation $\bigcup_{i\in I} Y_i = U \cup V$. Without loss of generality we can assume $x \in U$ and $x \notin V$. Since each Y_i is connected, and $Y_i \subset \bigcup_{i\in I} Y_i = U \cup V$, we have that $Y_i \subset U$ or $Y_i \subset V$ by Lemma 9.1.6. But since $x \in Y_i$ we need to have $Y_i \subset U$ for every $i \in I$. This means $\bigcup_{i\in I} Y_i \subset U$ and hence $V = \varnothing$. Therefore $\bigcup_{i\in I} Y_i$ is connected. $\square$

Theorem 9.1.8. *Let $Z \subset X$ be a connected subspace. If $Z \subset Y \subset \overline{Z}$, then Y is connected.*

Proof. Assume $Y = U \cup V$ is a separation again. By Lemma 9.1.6, we have $Z \subset U$ or $Z \subset V$. Without loss of generality, assume $Z \subset U$. Then we have $\overline{Z} \subset \overline{U}$, so $Y \subset \overline{Z} \subset \overline{U}$, but $\overline{U} \cap V = \varnothing$, which must mean $V = \varnothing$. $\square$

Remark 9.1.9. Note that in this theorem we used the following two basic facts:

(1) $Z \subset U \implies \overline{Z} \subset \overline{U}$
(2) If U and V are two disjoint open sets, then $\overline{U} \cap V = \varnothing$.

Theorem 9.1.10. *Suppose X is a connected topological space. If $f\colon X \longrightarrow Y$ is a continuous function, then $f(X) \subset Y$ is connected.*

Proof. It is true that f is surjective onto its image. Assume for a contradiction that $f(X)$ is disconnected and that

$$f(X) = U \cup V\,,$$

is a separation. Then

$$X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)\,.$$

We have $f^{-1}(U), f^{-1}(V) \overset{\text{open}}{\subset} X$ by continuity of f, and they are non-empty because f is surjective onto its image. They are also disjoint. This is a separation of X, which contradicts the assumption that X was connected. Therefore $f(X)$ can not have a separation. $\square$

Theorem 9.1.11. *The product $\prod_{i\in I} X_i$ equipped with the product topology is connected if and only if X_i is connected $\forall i \in I$.*

Proof. **$\Rightarrow$:** We will prove the contrapositive statement. Namely that $\exists j \in I$ such that X_j is disconnected implies that $\prod_{i\in I} X_i$ is disconnected.

Assume that there is some $j \in I$ such that X_j is not connected, and pick a separation $X_j = U \cup V$. Then define for each $i \in I$ $A_i, B_i \subset X_i$ by

$$A_i = \begin{cases} U, & i = j \\ X_i, & i \neq j \end{cases}, \qquad B_i = \begin{cases} V, & i = j \\ X_i, & i \neq j \end{cases}.$$

Then we have that

$$\prod_{i\in I} X_i = \left(\prod_{i\in I} A_i\right) \cup \left(\prod_{i\in I} B_i\right),$$

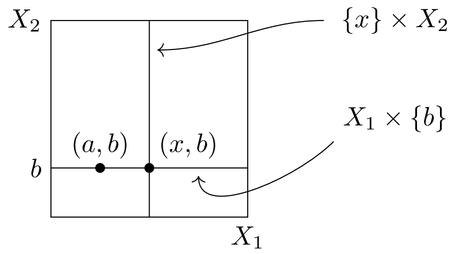
is a separation. They are clearly non-empty and open by definition of the product topology. They are disjoint because $U \cap V = \varnothing$. Therefore $\prod_{i\in I} X_i$ is disconnected.

$\Leftarrow$: We will only prove this for $|I| < \infty$. (The statement is however true for every index set I, see exercise §23.10 in Munkres.) By induction it suffices to show it for $|I| = 2$. Suppose therefore that X_1 and X_2 connected, and fix some $(a, b) \in X_1 \times X_2$. Then note

$$\begin{cases} X_2 \cong \{p\} \times X_2 \\ X_1 \cong X_1 \times \{q\} \end{cases},$$

for any $p \in X_1$, $q \in X_2$. For every $x \in X_1$, define

$$T_x := (X_1 \times \{b\}) \cup \{\{x\} \times X_2\} \subset X_1 \times X_2 .$$



Then T_x is connected, because $\{x\} \times X_2$ and $X_1 \times \{b\}$ have the point (x, b) in common, and by Theorem 9.1.7. Here we also use that X_1 and X_2 are connected.

Finally we have

$$X_1 \times X_2 = \bigcup_{x\in X_1} T_x ,$$

is connected since all the T_x have the point (a, b) in common.

□

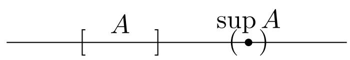
We will now finally show that $\mathbb{R}$ is connected and hence that $\mathbb{R}^n$ is connected. In order to do that, we will need the following lemma. The lemma might be known from a course in analysis already, but we include it for sake of completion.

Lemma 9.1.12. *If $A \subset \mathbb{R}$ is closed and bounded from above, then A has a maximal element.*

If $B \subset \mathbb{R}$ is closed and bounded from below, then B has a minimal element.

Proof. We only show the first statement as the second one is completely analogous. By the supremum property of $\mathbb{R}$, we have that $\sup A \in \mathbb{R}$ exists. The goal is to show that $\sup A \in A$.

Going for a contradiction, assume that $\sup A \notin A$, or equivalently $\sup A \in \mathbb{R} \setminus A$ Then since A is closed, its complement is open. Therefore we can find a small open set in $\mathbb{R} \setminus A$ containing $\sup A$ which is disjoint from A.



That is, there is some $\varepsilon > 0$ such that

$$(\sup A - \varepsilon, \sup A + \varepsilon) \cap A = \varnothing .$$

By definition $a \leq \sup A$ we have $\forall a \in A$. Since $\sup A$ is the *least* upper bound, we can pick a such that

$$\sup A - \frac{\varepsilon}{2} \leq a \leq \sup A\,.$$

But this gives $a \in (\sup A - \varepsilon, \sup A + \varepsilon)$ which is a contradiction. Therefore we must have $\sup A \in A$, and instead we might denote it $\sup A = \max A$ and call it the maximal element of A. □

THEOREM 9.1.13. *$\mathbb{R}$ is connected.*

PROOF. Suppose for a contradiction that there is a separation $\mathbb{R} = U \cup V$. Without loss of generality assume that we can pick points $x \in U$ and $y \in V$ such that $x < y$, and consider the closed interval $[x, y] \subset \mathbb{R}$ equipped with the subspace topology. Then define

$$U_0 := U \cap [x, y], \qquad V_0 := V \cap [x, y]\,.$$

Both of these sets are open, non-empty and disjoint in $[x, y]$. So we get a separation

$$[x, y] = U_0 \cup V_0\,.$$

Now, because $U \overset{\text{closed}}{\subset} \mathbb{R}$ and $[x, y] \overset{\text{closed}}{\subset} \mathbb{R}$ we have $U_0 \overset{\text{closed}}{\subset} \mathbb{R}$. It also has an upper bound, so there exists a maximal element $c = \max U_0 \in U_0$ by Lemma 9.1.12. Then we have

$$x \leq c < y\,,$$

for $y \in V_0$. Note that y needs to be strictly greater than c, because $y \notin U_0$. We also have that U_0 is open, so we can take a sufficiently small interval around c. That is, there is some small $\varepsilon > 0$ such that

$$(c - \varepsilon, c + \varepsilon) \cap [x, y] \subset U_0\,.$$

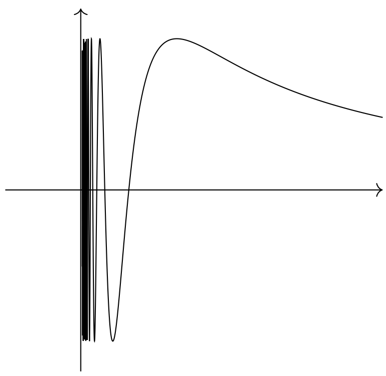
But this is a contradiction, because then we have in particular $c + \frac{\varepsilon}{2} \in U_0$ which contradicts the maximality of c. Hence $\mathbb{R}$ is connected. □

Remark 9.1.14. Note that this same proof shows that all connected subspaces of $\mathbb{R}$ are of the form:

$$\mathbb{R},\ (-\infty, a),\ (-\infty, a],\ (a, \infty),\ [a, \infty),\ (a, b),\ [a, b],\ (a, b],\ [a, b),\ \varnothing \subset \mathbb{R}\,.$$

This theorem together with Theorem 9.1.11 shows that $\mathbb{R}^n$ for $n \geq 1$ is connected.

Example 9.1.15. Consider the *topologists sine curve.* It is the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \in (0, 1]$.



It is connected since the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous, and the function $x \longmapsto (x, \sin\left(\frac{1}{x}\right))$ is therefore continuous. Since the graph is the image of the latter function by the use of Theorem 9.1.10.

9.1.1. Distinguishing spaces by connectedness properties. Using connectedness, we can in many situations actually show that spaces are not homeomorphic. The main tool is the following theorem.

THEOREM 9.1.16. *Let* $X \cong Y$. *Then* $\forall x \in X\ \exists y \in Y$ *such that* $X \smallsetminus \{x\} \cong Y \smallsetminus \{y\}$.

PROOF. This follows from the fact that if $f\colon X \longrightarrow Y$ is a homeomorphism, then the restriction of f to any subset $U \subset X$ is a homeomorphism onto its image, that is

$$f|_U : U \longrightarrow f(U)\,,$$

is a homeomorphism. We then apply this to $U = X \smallsetminus \{x\}$, for any $x \in X$. ☐

We illustrate the power of connectedness and this theorem by solving exercise §24.1 in Munkres.

Example 9.1.17. Show that no two of the spaces $(0,1)$, $(0,1]$ and $[0,1]$ are homeomorphic.

Let $X := (0,1]$ and $Y := (0,1)$. The key point is to note that if we remove the point $1 \in X$ we have $X \smallsetminus \{1\} = (0,1)$ which is connected. If we remove any point $y \in Y$ from Y we have

$$Y \smallsetminus \{y\} = (0,y) \cup (y,1)\,,$$

which is disconnected (we have just written down a separation!). Therefore $X \smallsetminus \{1\}$ is connected, while $Y \smallsetminus \{y\}$ is not. By Theorem 9.1.5 we therefore have

$$X \smallsetminus \{1\} \ncong Y \smallsetminus \{y\}\ .$$

Therefore by Theorem 9.1.16 we have $X \ncong Y$. This shows that $(0,1)$ and $(0,1]$ can not be homeomorphic.

It is now an exercise to show that $(0,1]$ and $[0,1]$ are not homeomorphic.

Exercises

9.1. Equip $X = \{a,b,c\}$ with the following topologies:

$$\mathcal{T}'' = \{\varnothing, X, \{a\}, \{b\}, \{a,c\}, \{a,b\}\}$$
$$\mathcal{T}''' = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}\ .$$

Decide whether $(X, \mathcal{T}'')$ and $(X, \mathcal{T}''')$ are connected or disconnected.

9.2. Show that $\mathbb{Q} \subset \mathbb{R}$ equipped with the subspace topology is disconnected.

9.3. Show that $(0,1] \ncong [0,1]$.

Lecture 10

10.1. Path-connectedness

- Munkres: §24

We will now see a stronger form of path-connectedness, which might feel more intuitive than the existence of separations.

Definition 10.1.1 (Path)**.** *Let X be a topological space and let $x, y \in X$ be any two points. A* path *in X from x to y is a continuous function*

$$\gamma\colon\ [a,b] \longrightarrow X\,,$$

such that $\gamma(a) = x$ and $\gamma(b) = y$.

Remark 10.1.2. Because $[a,b] \cong [0,1]$ it is usually the case that we pick the closed unit interval $[0,1]$ as the domain of the path.

Definition 10.1.3. *(1) The* inverse *(or reverse) of a path $\gamma\colon\ [0,1] \longrightarrow X$ from x to y is defined by*

$$\begin{aligned} \gamma^{-1}\colon\ [0,1] &\longrightarrow X \\ t &\longmapsto \gamma^{-1}(t) := \gamma(1-t)\,. \end{aligned}$$

(2) Let $\gamma_1, \gamma_2\colon\ [0,1] \longrightarrow X$ be two paths from x to y and y to z respectively. Define their concatenation *$\gamma_2\gamma_1\colon\ [0,1] \longrightarrow X$ by*

$$(\gamma_2\gamma_1)(t) := \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2} \\ \gamma_2\left(2t-1\right), & \frac{1}{2} \le t \le 1 \end{cases}$$

Remark 10.1.4. Sometimes the notation $\gamma_2 \star \gamma_1$ or $\gamma_2 \cdot \gamma_1$ is used for concatenation. In Munkres, the notation $\gamma_1 * \gamma_2$ is used (where the concatenation is read left to right!).

We should read it from right to left. First follow the path γ_1 and then follow the path γ_2. It is the same as how we read function compositions $g \circ f$: First apply f, then apply g.

Proposition 10.1.5. *(1) If γ is a path from x to y, then γ^{-1} is a path from y to x.*
(2) If γ_1 and γ_2 are paths from x to y and y to z respectively, then $\gamma_2\gamma_1$ is a path from x to z.

Proof. (1) By definition γ^{-1} is the composition of the following maps:

$$\begin{aligned} [0,1] \longrightarrow\ & [0,1] \xrightarrow{\ \gamma\ } X \\ t \longmapsto\ & 1-t \longmapsto \gamma(1-t) \end{aligned},$$

and since both of these maps are continuous, so is the composition. If $\gamma(0) = x$ and $\gamma(1) = y$, then we have $\gamma^{-1}(0) = \gamma(1) = y$ and $\gamma^{-1}(1) = \gamma(0) = x$, so γ^{-1} is indeed a path from y to x.

(2) We will prove that if $U \overset{\text{closed}}{\subset} X$, then $(\gamma_2\gamma_1)^{-1}(U) \overset{\text{closed}}{\subset} [0,1]$. We look at the two restrictions of $\gamma_2\gamma_1$ to $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ respectively:

$$\begin{aligned}\gamma_2\gamma_1|_{[0,\frac{1}{2}]} &: \left[0,\frac{1}{2}\right] \longrightarrow X\\ \gamma_2\gamma_1|_{[\frac{1}{2},1]} &: \left[\frac{1}{2},1\right] \longrightarrow X\end{aligned}$$

They are given by

$$\begin{aligned}\gamma_2\gamma_1|_{[0,\frac{1}{2}]}(t) &= \gamma_1(2t)\\ \gamma_2\gamma_1|_{[0,\frac{1}{2}]}(t) &= \gamma_2(2t-1)\ ,\end{aligned}$$

both of which are continuous, since each of the maps are compositions of two continuous functions. Therefore we have

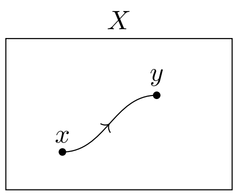
$$\begin{aligned}\left(\gamma_2\gamma_1|_{[0,\frac{1}{2}]}\right)^{-1}(U) &= (\gamma_2\gamma_1)^{-1}(U) \cap \left[0,\frac{1}{2}\right] \overset{\text{closed}}{\subset} [0,1]\\ \left(\gamma_2\gamma_1|_{[\frac{1}{2},1]}\right)^{-1}(U) &= (\gamma_2\gamma_1)^{-1}(U) \cap \left[\frac{1}{2},1\right] \overset{\text{closed}}{\subset} [0,1]\,.\end{aligned}$$

Therefore, since $[0,\frac{1}{2}] \cup [\frac{1}{2},1] = [0,1]$ we have

$$\begin{aligned}(\gamma_2\gamma_1)^{-1}(U) = &\left((\gamma_2\gamma_1)^{-1}(U) \cap \left[0,\frac{1}{2}\right]\right) \cup \left((\gamma_2\gamma_1)^{-1}(U) \cap \left[\frac{1}{2},1\right]\right)\\ &\overset{\text{closed}}{\subset} [0,1]\,,\end{aligned}$$

which means that $\gamma_2\gamma_1$ is continuous. We also have $(\gamma_2\gamma_1)(0) = \gamma_1(0) = x$ and $(\gamma_2\gamma_1)(1) = \gamma_2(2-1) = \gamma_2(1) = z$. This means that $\gamma_2\gamma_1$ is a path from x to z. □

Definition 10.1.6 (Path-connected)**.** *Let X be a topological space. We say that X is* path-connected *if for any two points $x, y \in X$ there is a path from x to y.*



It is easy to see directly from the definition that path-connectedness is *stronger* than connectedness.

THEOREM 10.1.7. *Suppose that X is a path-connected topological space. Then X is connected.*

PROOF. We prove the contrapositive. Assume X is disconnected with a separation $X = U \cup V$, where $U, V \overset{\text{open}}{\subset} X$ are non-empty and $U \cap V = \varnothing$. Pick two points $x \in U$ and $y \in V$. Suppose we have a path

$$\gamma\colon\ [a,b] \longrightarrow X\,,$$

such that $\gamma(a) = x$. Since $[a,b] \subset \mathbb{R}$ is connected and γ is continuous, we have that $\gamma([a,b]) \subset X$ is connected. Therefore we have $\gamma([a,b]) \subset U$ because of Lemma 9.1.6 and the assumption $\gamma(a) = x$. But then it means that $\gamma(b) \in U$, so we can not have $\gamma(b) = y$, which means that there is no path from x to y. Hence X is not path-connected. □

Remark 10.1.8. Note that the proof of Theorem 10.1.7 relies on the fact that $[a, b] \subset \mathbb{R}$ is connected. Our proof that $\mathbb{R}$ is connected (Theorem 9.1.13) can be repeated almost word by word to give that $[a, b] \subset \mathbb{R}$ is connected.

Example 10.1.9. (1) Let X be any set and equip it with the trivial topology $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is path-connected, since any function

$$\gamma\colon [0,1] \longrightarrow X\,,$$

is continuous.

(2) Let $n > 1$ and consider $\mathbb{R}^n$. It is path-connected, since if we pick any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then the straight line between $\boldsymbol{x}$ and $\boldsymbol{y}$ is a continuous function:

$$\begin{aligned}\gamma\colon [0,1] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto (1-t)\boldsymbol{x} + t\boldsymbol{y}\,.\end{aligned}$$

This is clearly continuous (since each component of the map is a continuous function), and $\gamma(0) = \boldsymbol{x}$, $\gamma(1) = \boldsymbol{y}$.

By Theorem 10.1.7, this gives a new proof that $\mathbb{R}^n$ for $n > 1$ is connected.

(3) For the same reason as above (we can always choose a path which is a straight line), any ball $B(\boldsymbol{x}, r) \subset \mathbb{R}^n$ is path-connected and hence connected.

(4) Consider $X = \{a, b, c\}$ and equip it with the two topologies

$$\begin{aligned}\mathcal{T}_1 &= \{\varnothing, X, \{a\}, \{a,b\}\} \\ \mathcal{T}_2 &= \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}\end{aligned}$$

We already saw in Example 9.1.4 that $(X, \mathcal{T}_1)$ is connected. We also see that $(X, \mathcal{T}_2)$ is connected.

Now we have also that $(X, \mathcal{T}_1)$ is path-connected. By Proposition 10.1.5 it suffices to construct paths from a to b and from b to c, since then we get a path from a to c by concatenation. We define two maps $\gamma_{a\to b}, \gamma_{b\to c}\colon [0,1] \longrightarrow X$ as

$$\gamma_{a\to b}(t) := \begin{cases} a, & 0 \le t < \frac{1}{2} \\ b, & \frac{1}{2} \le t \le 1 \end{cases}, \qquad \gamma_{b\to c}(t) := \begin{cases} b, & 0 \le t < \frac{1}{2} \\ c, & \frac{1}{2} \le t \le 1 \end{cases}.$$

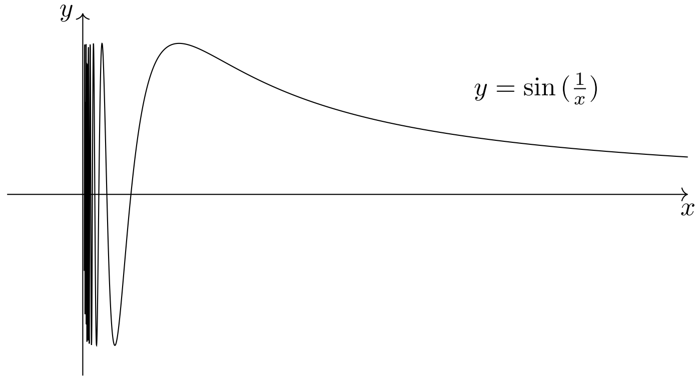
We then check explicitly that they are continuous. The only (non-trivial) open subsets in $\mathcal{T}_1$ are $\{a\}$ and $\{a, b\}$. We have

$$\begin{aligned}(\gamma_{a\to b})^{-1}(\{a\}) &= \left[0, \frac{1}{2}\right) \overset{\text{open}}{\subset} [0,1] & (\gamma_{b\to c})^{-1}(\{a\}) &= \varnothing \overset{\text{open}}{\subset} [0,1] \\ (\gamma_{a\to b})^{-1}(\{a,b\}) &= [0,1] \overset{\text{open}}{\subset} [0,1] & (\gamma_{b\to c})^{-1}(\{a,b\}) &= \left[0, \frac{1}{2}\right) \overset{\text{open}}{\subset} [0,1]\,.\end{aligned}$$

which means that both $\gamma_{a\to b}$ and $\gamma_{b\to c}$ are continuous paths from a to b and b to c respectively.

It is an exercise to show that $(X, \mathcal{T}_2)$ also is path-connected.

10.1.1. The topologists sine curve. We will now take a look at a topological space which is connected but not path-connected. The most famous of such spaces is the topologists sine curve, which we already have seen in Lecture 9. It is the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \in (0, \infty)$.



More precisely, it is the following subset of $\mathbb{R}^2$

$$S := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \;\middle|\; x \in (0,\infty) \right\} \subset \mathbb{R}^2 ,$$

equipped with the subspace topology. The function $x \longmapsto (x, \sin(\frac{1}{x}))$ is continuous, and S is the continuous image of a connected set $(0,\infty) \subset \mathbb{R}$, which means that S is connected. This we have already seen in Lecture 9.

Then we consider its closure $\overline{S}$. It is an exercise to "prove by inspection" using the graph of the function $f(x) = \sin(\frac{1}{x})$ that the closure is

$$\overline{S} = S \cup (\{0\} \times [-1,1]) .$$

That is, S together with the part of the y-axis for which $-1 \leq y \leq 1$. Since S is connected, so is $\overline{S}$. However, we will now show that $\overline{S}$ is *not* path-connected!

Before we prove that $\overline{S}$ is not path-connected, we state an important theorem about continuous functions and sequences.

Theorem 10.1.10. *Let $f\colon X \longrightarrow Y$ be a function. Then if f is continuous, then*

$$x_n \to x \implies f(x_n) \to f(x) .$$

If we assume that X is first countable, then the converse is true. Namely, if $x_n \to x \implies f(x_n) \to f(x)$, then f is continuous.

Proof. **⇒:** Assume f is continuous. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V) \overset{\text{open}}{\subset} X$ is a neighborhood of x by continuity. Hence $\exists N > 0$ such that $x_n \in f^{-1}(V)$ for $n \geq N$ by the assumption that $x_n \to x$. This implies precisely that $f(x_n) \in V$ for $n \geq N$. Hence $f(x_n) \to f(x)$.

⇐: Now assume X is first countable. Assume $x_n \to x \implies f(x_n) \to f(x)$. We will first prove that if $A \subset X$ is a subset, then $f(\overline{A}) \subset \overline{f(A)}$. Namely, if $x \in \overline{A}$ (that is $f(x) \in f(\overline{A})$), then by the sequence lemma (Lemma 5.2.7) there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ in X. Therefore, by assumption we have $f(x_n) \to f(x)$. Therefore, since $f(x_n) \in f(A)$, the sequence lemma gives us that $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subset \overline{f(A)}$.

The rest of the proof is left as an exercise: If $f(\overline{A}) \subset \overline{f(A)}$, then f is continuous.

□

Proposition 10.1.11. *The closure of the topologists sine curve $\overline{S} \subset \mathbb{R}^2$ is connected but not path-connected.*

Proof. Suppose that there exists a path $\gamma\colon [0,1] \longrightarrow \overline{S}$ from $(0,0)$ to some $(x,y) \in S$. The set

$$C := \gamma^{-1}(\{0\} \times [-1,1]) \subset [0,1] ,$$

is closed, since $\{0\} \times [-1,1] \overset{\text{closed}}{\subset} \overline{S}$. We then by assumption have $0 \in C$ and $1 \notin C$. By the maximum property (Lemma 9.1.12), C has a maximum,

$$b := \max C \in C\,,$$

which satisfies $0 \leq b < 1$. Then consider the following restriction of γ

$$\gamma|_{[b,1]} : [b,1] \longrightarrow \overline{S}\,,$$

and write it as

$$\gamma|_{[b,1]}(t) = (x(t), y(t))\,.$$

Then $x(b) = 0$ while $x(t) > 0$ for $t \in (b,1]$. Moreover, by definition we have $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t \in (b,1]$.

We will now construct a sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to b$, such that $y(t_n) = (-1)^n$. Then $y(t_n)$ does not converge, which means that y can not be continuous by Theorem 10.1.10, so $\gamma|_{[b,1]}$ can not be continuous, so finally γ can not be continuous.

To find this sequence $\{t_n\}_{n=1}^{\infty}$ we do the following: Given n, choose some u_n with $x(b) = 0 < u_n < x\left(b + \frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u_n}\right) = (-1)^n$. If we use the continuity of x, we apply the intermediate value theorem from calculus to get a number

$$b < t_n < b + \frac{1}{n}\,,$$

such that $x(t_n) = u_n$, and $y(t_n) = \sin\left(\frac{1}{u_n}\right) = (-1)^n$. This gives our desired sequence and we are done. $\square$

We will now see a condition which guarantees that connectedness implies path-connectedness.

Theorem 10.1.12. *A topological space X is path-connected if and only if it is connected and every point has a path-connected neighborhood.*

Proof. **⇒:** Path-connectedness implies connectedness by Theorem 10.1.7, and the entire space X is a neighborhood of any point, which is path-connected by assumption.

⇐: Now fix some point $x \in X$. Define

$$U_x = \{y \in X \mid \text{there exists a path } \gamma\colon [0,1] \longrightarrow X \text{ from } y \text{ to } x\}\,.$$

Our goal is to prove that $U_x = X$, because then it means precisely that X is path-connected, because we can take inverses of paths and concatenate paths. Since X is assumed to be connected and by Proposition 9.1.3, it suffices to show that U_x is clopen.

U_x **is open:** Let $y \in U_x$, and pick a path $\gamma_{y\to x}$ from y to x. By assumption, there is a path-connected neighborhood V_y of y. Then, for any $z \in V_y$, there is a path $\eta_{z\to y}$ from z to y, which by concatenation gives a path from z to x, which means $V_y \subset U_x$. Therefore we have

$$U_x = \bigcup_{y\in U_x} V_y\,,$$

which then is open, since V_y are open subsets of X.

U_x **is closed:** We show that its complement $X \setminus U_x$ is open in a similar way. We pick some $y \in X \setminus U_x$, then there is no path from y to x. Let V_y be a path-connected neighborhood of y and let $z \in V_y$. Then we claim that there is no path from z to x. Because if such a path $\eta_{z\to x}$ from z to x existed, we could pick a path $\gamma_{y\to z}$ from y to z whose concatenation would yield a path from y to x. Hence there is no path from z to x and therefore $z \in X \setminus U_x$. Consequently $V_y \subset X \setminus U_x$, and we can write

$$X \setminus U_x = \bigcup_{y \in X \setminus U_x} V_y \,,$$

which then is open, since V_y are open subsets of $X \setminus U_x$.

□

Exercise 10.1.13. Use Theorem 10.1.12 to show that any open and connected subset of $\mathbb{R}^n$ in fact is path-connected.

10.2. Local connectedness

- Munkres: §25

We can now break up a topological space X into components that are connected (or path-connected) using an equivalence relation.

Definition 10.2.14 (Connected components)**.** *Let X be a topological space. Define an equivalence relation defined by*

$$x \sim y \overset{def}{\Leftrightarrow} \exists A \subset X \ : \ x, y \in A \text{ where } A \text{ is connected}.$$

The equivalence classes of $\sim$ are called connected components.

Proposition 10.2.15. *Let X be a topological space, and denote its connected components by C_i, $i \in I$.*

(1) $X = \bigcup_{i \in I} C_i$
(2) If $A \subset X$ is connected, then $A \subset C_i$ for some $i \in I$.
(3) C_i is connected for each $i \in I$.

Proof. (1) This is true, because C_i is by definition equivalence classes.
(2) By definition, $\forall x, y \in A$ we have $x \sim y$, which means precisely $A \subset C_i$ for some $i \in I$.
(3) Fix some $x \in C_i$. For any $y \in C_i$ we have by definition that $x \sim y$, so there is a connected subspace A_y containing both x and y. Therefore $A_y \subset C_i$, which means

$$C_i = \bigcup_{y \in C_i} A_y \,,$$

is connected, because all the A_y have the point $x \in C_i$ in common.

□

Remark 10.2.16. By (2) above, it means that X is connected if and only if X is the only connected component. Also, it follows that connected components are *maximal* in the sense that if there is any bigger subset $A \subset X$ such that

$$C \subsetneq A \subset X \,,$$

then A needs to be disconnected.

Remark 10.2.17. We can define path components of X by repeating Definition 10.2.14 but requiring A to be path-connected instead.

Then Proposition 10.2.15 is also true when we replace the word "connected" with "path-connected".

Example 10.2.18. (1) Consider $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology. Then each connected component consists of a single point. Note that each of the connected components are closed in $\mathbb{Q}$!

(2) We consider the topologists sine curve $\overline{S}$ again. Then it has one connected component (since it is connected), but it has 2 path components! One path-component is S, and the other is $V = \{0\} \times [-1, 1]$.

Proposition 10.2.19. *The connected components of a topological space are closed.*

Proof. Let C be a connected component. Since C is connected, so is $\overline{C}$. By Remark 10.2.16, it is the largest connected subspace containing C. Since $C \subset \overline{C}$ we must have $\overline{C} \subset C$ by maximality of C. Therefore $C = \overline{C}$ and C is closed. □

Remark 10.2.20. Note that this proof fails for path components, because C path-connected does not imply $\overline{C}$ path-connected (the topologists sine curve is a counterexample).

Definition 10.2.21 (Locally connected)**.** *A topological space X is* locally connected at x *if every neighborhood U of x contains a connected neighborhood V of x.*

If X is locally connected at each of its points, the space is called locally connected.

Remark 10.2.22. We define *locally path-connected* in the same way as above, but we replace the word "connected" with "path-connected".

Proposition 10.2.23. *If X is locally path-connected, then it is locally connected.*

Proof. Follows immediately from the definition and from Theorem 10.1.7. □

Example 10.2.24. (1) Consider $X = \{a, b, c\}$ and equip it with the topology

$$\mathcal{T} = \{\varnothing, X, \{a\}, \{a, b\}\}$$

By Example 10.1.9 (4), $(X, \mathcal{T})$ is connected and path-connected.

We have that $(X, \mathcal{T})$ is locally path-connected, because $\{a\} \subset X$ and $\{a, b\} \subset X$ are both path-connected when they are equipped with the subspace topology with respect to $\mathcal{T}_1$. Namely,

$$\begin{aligned} \gamma\colon\ [0,1] &\longrightarrow \{a\} \\ t &\longmapsto a \end{aligned}$$

is clearly continuous since $\gamma^{-1}(\{a\}) = [0, 1] \overset{\text{open}}{\subset} [0, 1]$. To show that $\{a, b\}$ is path-connected it suffices to write down a path from a to b (a path from b to a is obtained as the inverse). We define $\gamma_{a\to b}\colon [0, 1] \longrightarrow \{a, b\}$ as

$$\gamma_{a\to b}(t) = \begin{cases} a, & 0 \leq t < \frac{1}{2} \\ b, & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then it is continuous since $(\gamma_{a\to b})^{-1}(\{a\}) = [0, \frac{1}{2}) \overset{\text{open}}{\subset} [0, 1]$ and $(\gamma_{a\to b})^{-1}(\{a, b\}) = [0, 1] \overset{\text{open}}{\subset} [0, 1]$. Since all neighborhoods (of all points) are path-connected, it is locally path-connected.

(2) $(-1, 0) \cup (0, 1) \subset \mathbb{R}$ is not connected because we define it as a separation, but it is locally connected, since at any $x \in (-1, 0) \cup (0, 1)$, any neighborhood of x contains an arbitrarily small interval $(x - \varepsilon, x + \varepsilon)$ which is connected.

(3) The topologists sine curve $\overline{S}$ is connected. However it is *not* locally connected! Local connectedness fails at $\{0\} \times [-1, 1]$, because any small enough neighborhood will fail to contain a connected neighborhood.

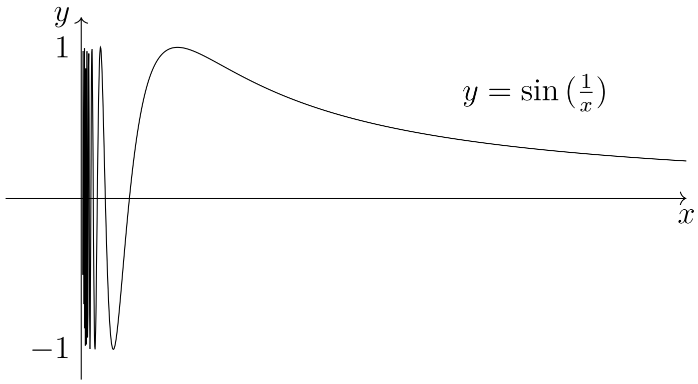
Exercises

10.1. Consider the topological space $(X, \mathcal{T})$ where $X = \{a, b, c\}$ and

$$\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} .$$

Show that $(X, \mathcal{T})$ is path-connected.

10.2. Consider the topologists sine curve $S = \{(x, \sin(\frac{1}{x})) \mid x \in (0, \infty)\} \subset \mathbb{R}^2$.



Explain using the above graph why

$$\overline{S} = S \cup (\{0\} \cup [-1, 1]) .$$

10.3. Show that if $f\colon X \longrightarrow Y$ is a function such that $f(\overline{A}) \subset \overline{f(A)}$ for any subset $A \subset X$, then f is continuous.

10.4. Show that any open and connected subset of $\mathbb{R}^n$ is path-connected.

(Hint: Use Theorem 10.1.12)

Lecture 11

11.1. Compactness

- Munkres: §26

The next topological property we are going to investigate is *compactness*. Connectedness has in many ways been natural, and its definition is somewhat clear. However, compactness is more difficult to really understand.

Definition 11.1.1 (Cover).
- *A collection $\mathcal{A}$ of subsets of X is said to* cover X *if* $X = \bigcup_{A\in\mathcal{A}} A$.
- *If every element in $\mathcal{A}$ is an open subset of X, then $\mathcal{A}$ is called an* open cover.
- *If $\mathcal{A}$ is a cover of X, and $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B}$ also is a cover of X, then $\mathcal{B}$ is called a* subcover *of $\mathcal{A}$.*

Definition 11.1.2 (Compactness). *A topological space X is called* compact *if every open cover has a finite subcover.*

Example 11.1.3.
(1) Every finite topological space X is compact, because all covers are already finite, because there are only finitely many open sets!
(2) Let X be an infinite set equipped with the discrete topology. Then X is not compact, because the open cover
$$X = \bigcup_{x\in X} \{x\} ,$$
does not have a finite subcover. Removing any set from the cover will leave at least one point uncovered.
(3) $\mathbb{R}$ is *not* compact. A cover without a finite subcover is
$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} \left(n, n+\frac{3}{2}\right) .$$
If we remove any interval from this cover, we leave a part of $\mathbb{R}$ uncovered! Hence, we can not find a finite subcover.
(4) $A := \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$ is not compact. Namely, if we for each $n \in \mathbb{Z}_+$ pick numbers ε_n such that $\varepsilon_n < \frac{1}{n+1}$, then we look at the interval $(\frac{1}{n} - \varepsilon_n, \frac{1}{n} + \varepsilon_n)$ and see that
$$A \cap \left(\frac{1}{n} - \varepsilon_n, \frac{1}{n} + \varepsilon_n\right) = \left\{\frac{1}{n}\right\} .$$
So we get a cover
$$A = \bigcup_{n=1}^{\infty} \left(\frac{1}{n} - \varepsilon_n, \frac{1}{n} + \varepsilon_n\right) ,$$
such that if we remove any of the intervals, we leave A uncovered, which means that A can not be compact.
(5) Let A be as above. Then $A \cup \{0\}$ is compact. Since $\mathbb{R}$ is equipped with the standard topology, 0 is the (unique) limit of the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$. So $0 \in A'$. Since the standard topology on $\mathbb{R}$ is T_1, we have that any neighborhood of 0 need to intersect A in infinitely many points (recall the statement of Theorem 5.1.2).

Therefore, pick any cover $U = \{U_i\}_{i\in I}$ of $A\cup\{0\}$, and let assume $j \in I$ is such that U_j is a neighborhood of 0. Then $\exists N > 0$ such that $\frac{1}{n} \in U_j$ for all $n \geq N$. Therefore

$$(A \cup \{0\}) \smallsetminus U_j = \left\{\frac{1}{n}\right\}_{n=1}^{N-1},$$

is a finite set which is covered by $U \smallsetminus \{U_j\}$. Since it is finite, we can always find a finite subcover. The conclusion is that $A \cup \{0\}$ is compact.

(6) $(0,1)$ is not compact, because the cover

$$(0,1) = \left(0, \frac{1}{2}\right) \cup \bigcup_{n=1}^{\infty} \left(\frac{n-1}{n}, \frac{n+1}{n+2}\right),$$

does not have a finite subcover for the same reason as in example (2). Namely, if we remove any of the open sets, we leave some part of $(0,1)$ uncovered.

(7) $(0,1]$ is not compact either. We can for example take the cover

$$(0,1] = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right].$$

This time we can not quite argue like we did in example (2) and (5). Instead we argue like follows: If we take *any* finite subcollection, lets say

$$\left\{\left(\frac{1}{n_i}, 1\right] \,\middle|\, n_i \in \mathbb{Z}_+\right\}_{i\in I}.$$

But then we have

$$\bigcup_{i\in I} \left(\frac{1}{n_i}, 1\right] = \left(\min_{i\in I} \frac{1}{n_i}, 1\right] \subsetneq (0,1],$$

so any finite subcollection can never cover $(0,1]$, so it is not compact.

(8) Consider $X = \mathbb{R}_{>0} \smallsetminus \mathbb{Z}_+$. Equip it with the topology $\mathcal{T}$ generated by the sets

$$S_n = \left(0, \frac{1}{n}\right) \cup (n, n+1),\ n \in \mathbb{Z}_+.$$

Then $(X, \mathcal{T})$ is not compact. It is an exercise to work out the details.

(9) $[0,1] \subset \mathbb{R}$ is compact. This is non-trivial, and we will show this in the next lecture!

Theorem 11.1.4. *If X is a compact topological space, and $A \overset{\text{closed}}{\subset} X$, then A is compact.*

Proof. Pick an open cover $U = \{U_i\}_{i\in I}$ of A. Then we can write $U_i = \widetilde{U}_i \cap A$ where $\widetilde{U}_i \overset{\text{open}}{\subset} X$, by definition of the subspace topology. Since $X \smallsetminus A$ is open, it follows that $\left\{\widetilde{U}_i\right\}_{i\in I} \cup \{X \smallsetminus A\}$ covers X. By compactness of X, there is a finite subcover $\left\{\widetilde{U}_{i_j}\right\}_{j=1}^{N} \cup \{X \smallsetminus A\}$. It follows that

$$A = \bigcup_{j=1}^{N} \widetilde{U}_{i_j} \cap A,$$

is a finite subcover of A. Therefore A is compact. □

Theorem 11.1.5. *If X is a Hausdorff topological space and $A \overset{\text{compact}}{\subset} X$, then A is closed.*

Proof. We will prove that $X \setminus A$ is open by showing that for any $x \in X \setminus A$, there is some neighborhood U of x, which is contained in $X \setminus A$, or equivalently that $U_x \cap A = \varnothing$. For every $a \in A$, we can find disjoint neighborhoods U_a of x and V_a of a. Then

$$\{V_a \cap A \mid a \in A\} \ ,$$

is an open cover of A. By compactness, we can find a finite subcover

$$A = \bigcup_{i=1}^{N} V_{a_i} \cap A \, .$$

Then $V := V_{a_1} \cup \cdots \cup V_{a_N} \supset A$, and it is disjoint from $U = U_{a_1} \cap \cdots \cap U_{a_N}$. Namely, if $z \in V_{a_j}$ for some $j \in \{1, \ldots, N\}$, then $z \notin U_{a_j}$ by construction, and hence $z \notin U$. Therefore U is a neighborhood of x contained in $X \setminus A$, so $A \overset{\text{closed}}{\subset} X$. ☐

Example 11.1.6. The standard topology on $\mathbb{R}$ is Hausdorff. By Example 11.1.3 (4) it follows that $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is a closed subset of $\mathbb{R}$ by Theorem 11.1.5. (We can also show that it is closed by explicitly writing the complement as a union of open sets.)

Theorem 11.1.7. *Let $f\colon X \longrightarrow Y$ be a continuous function. If $A \overset{\text{compact}}{\subset} X$, then $f(A) \overset{\text{compact}}{\subset} Y$.*

Proof. Pick an open cover $U = \{U_i\}_{i \in I}$ of $f(A)$. It follows that $\{f^{-1}(U_i) \cap A\}_{i \in I}$ is an open cover of A. By compactness we pick a finite subcover $\{f^{-1}(U_{i_j}) \cap A\}_{j=1}^{N}$ of A, and therefore $\{U_{i_j}\}_{j=1}^{N}$ is a cover of $f(A)$, which is a finite subcover of the original one. ☐

Corollary 11.1.8. *Suppose that $X \cong Y$. Then X is compact if and only if Y is compact.*

Proof. Exercise. ☐

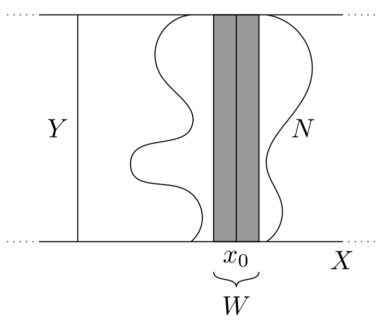
Theorem 11.1.9. *Let $f\colon X \longrightarrow Y$ be continuous and bijective. If X is compact and if Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to show that f is a closed function in order to conclude that it is a homeomorphism. Let $A \overset{\text{closed}}{\subset} X$. By Theorem 11.1.4, we have that $A \overset{\text{compact}}{\subset} X$. Then by Theorem 11.1.7, $f(A) \overset{\text{compact}}{\subset} X$. Finally by Theorem 11.1.5, $f(A) \overset{\text{closed}}{\subset} X$. ☐

We will now discuss compactness and products. We will first need a technical lemma.

Lemma 11.1.10 (The tube lemma)**.** *Let Y be a compact topological space. If $N \overset{\text{open}}{\subset} X \times Y$ contains the set $\{x_0\} \times Y$, then there is an open set $W \times Y$, where $x_0 \in W \overset{\text{open}}{\subset} X$ such that*

$$\{x_0\} \times Y \subset W \times Y \subset N \, .$$



Proof. We have $N \overset{\text{open}}{\subset} X \times Y$ and $\{x_0\} \times Y \subset N$. Then for any (x_0, y) there is some basis element $U_y \times V_y \subset N$ such that $x_0 \in U_y \overset{\text{open}}{\subset} X$ and $y \in V_y \overset{\text{open}}{\subset} Y$. In particular $\{V_y \mid y \in Y\}$ covers Y, so there is a finite subcover $\{V_{y_i}\}_{i=1}^N$. Then consider $W := U_{y_1} \cap \cdots \cap U_{y_N}$, which is still open in X and contains x_0. Then it follows

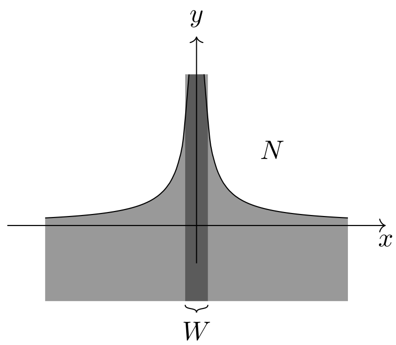
$$\{x_0\} \times Y \subset W \times Y = \bigcup_{i=1}^{N} W \times V_{y_i} \subset N\,.$$

The last inclusion holds because we have $W \times V_{y_i} \subset U_{y_i} \times V_{y_i} \subset N$ for every $i \in \{1, \dots, N\}$ and by assumption. □

Remark 11.1.11. The compactness of Y is crucial. If we take $X = Y = \mathbb{R}$ and consider N as follows

$$N := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y < \left|\frac{1}{x}\right|,\ x \neq 0\right\} \cup \{x = 0\}\ .$$

Then if we pick $x_0 = 0$, we have $\{0\} \times \mathbb{R} \subset N$ as in the figure below, but there is no tube $W \times \mathbb{R}$ contained in N, because N becomes thinner and thinner the further up on the y-axis we go.



We will now see a very powerful and important theorem. It is Tychonoff's theorem. It says that *any* product of compact topological spaces is compact. This is one reason that we prefer the product topology over the box topology as we previously discussed in Lecture 7.

Theorem 11.1.12 (Tychonoff's theorem). *The product $\prod_{i \in I} X_i$ equipped with the product topology is compact if and only if X_i is compact for each $i \in I$.*

Remark 11.1.13.
- Note that we allow I to be any set. Even uncountable!
- This theorem is false if we equip $\prod_{i \in I} X_i$ with the box topology.
- We will only prove the $\Leftarrow$ direction for finite products. The more general case is harder, and is prove in §37 in Munkres book.

Proof. **$\Rightarrow$:** We know that products come equipped with surjective continuous projections

$$\pi_i \colon \prod_{i \in I} X_i \longrightarrow X_i\,.$$

So by Theorem 11.1.7, $X_i = \pi_i\left(\prod_{i \in I} X_i\right)$ is compact.

$\Leftarrow$: It suffices to prove it for the case when $|I| = 2$. Then by induction it holds for any finite product. Assume X and Y are compact, and let $\{U_i\}_{i \in I}$ be any open cover of $X \times Y$. For any $x \in X$, we have

$$Y \cong \{x\} \times Y \subset X \times Y\,,$$

which means that $\{x\}\times Y$ is compact. An open cover of $\{x\}\times Y$ is $\{U_{x,i} \cap (\{x\} \times Y)\}_{i\in I}$. By compactness there is a finite subcover

$$\{x\} \times Y = \bigcup_{j=1}^{N} U_{x,i_j} \cap (\{x\} \times Y) ,$$

such that

$$\{x\} \times Y \subset N_x := \bigcup_{j=1}^{N} U_{x,i_j} .$$

By the tube lemma, there is some $W_x \overset{\text{open}}{\subset} X$ such that

$$\{x\} \times Y \subset W_x \times Y \subset N_x .$$

Since $x \in W_x$, the collection $\{W_x \mid x \in X\}$ forms an open cover of X. We take another finite subcover

$$X = \bigcup_{i=1}^{M} W_{x_i} .$$

Then it follows that

$$X \times Y = \bigcup_{i=1}^{M} W_{x_i} \times Y \subset \bigcup_{i=1}^{M} N_{x_i} = \bigcup_{i=1}^{M}\bigcup_{j=1}^{N} U_{x_i,i_j} ,$$

which means $X \times Y = \bigcup_{i=1}^{M}\bigcup_{j=1}^{N} U_{x_i,i_j}$ and we found our finite subcover.

□

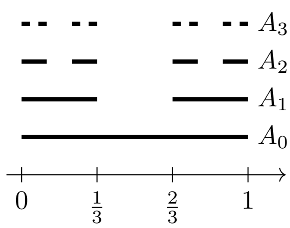
11.1.1. The Cantor set. Another important example is the Cantor set. It is defined inductively as follows.

$$A_0 := [0,1]$$

$$A_n := A_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) .$$

Each A_n is a union of closed intervals. We get A_n from A_{n-1} by dividing A_{n-1} into thirds, and then removing the middle third. The Cantor set is defined as

$$C := \bigcap_{n=0}^{\infty} A_n .$$



Since each A_n is a finite union of closed interval, each A_n is closed, and C is defined as an intersection of closed sets, which means that C is closed. Using that $[0,1]$ is compact (which we have not proved yet, but we will see it next time), and Theorem 11.1.4, we have that since $C \overset{\text{closed}}{\subset} [0,1]$, it follows that C is compact as well.

Another description of the Cantor set is that $\forall x \in C$, we can write x in base 3 using only the digits 0 and 2. Equivalently, we can write x as

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} ,$$

where $a_k \in \{0, 2\}$.

Exercises

11.1. Consider $X = \mathbb{R}_{>0} \smallsetminus \mathbb{Z}_+$. Equip it with the topology $\mathcal{T}$ generated by the sets

$$S_n = \left(0, \frac{1}{n}\right) \cup (n, n+1)\,,\ n \in \mathbb{Z}_+\,.$$

Show that $(X, \mathcal{T})$ is non-compact.

11.2. Suppose $X \cong Y$. Show that X is compact if and only if Y is compact.

Lecture 12

Recap.
- A topological space X is compact if and only if every open cover of X has a finite subcover.

12.1. Sequential compactness

- Munkres: §28

In analysis it is a common occurrence that we use compactness to show that sequences has convergent subsequences. We will now see a version of compactness that is well-suited when studying analysis.

Definition 12.1.1 (Sequential compactness)**.** *Let X be a topological space. We say that X is* sequentially compact *if every sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.*

Before we state an important theorem about how sequential compactness and compactness are related, we write down a definition which we have essentially already seen back in lecture 5.

Definition 12.1.2 (Metrizable)**.** *We say that a topological space $(X, \mathfrak{T})$ is* metrizable *if there exists a metric d on X such that the metric topology generated by d is equal to $\mathfrak{T}$.*

Theorem 12.1.3. *Suppose that X is a topological space.*

(1) Let X be metrizable. Then X compact $\Leftrightarrow$ X sequentially compact.
(2) Let X be first countable. Then X compact $\implies$ X sequentially compact.

Proof. **⇒:** Since metric spaces are first countable, we prove both (1) and (2) at the same time in this step.

Namely, let X be first countable and compact. We argue by contradiction. Hence assume $\{x_n\}_{n=1}^{\infty}$ is a sequence with no convergent subsequence.

Next, assume that $\{x_n\}_{n=1}^{\infty}$ is such that

$$\forall x \in X \; \exists U \overset{\text{open}}{\subset} X,\, x \in U \;:\; x_n \in U \text{ for at most finitely many } n \in \mathbb{Z}_+ . \tag{12.1.1}$$

Then for each $x \in X$, let U_x be a neighborhood of x such that $x_n \in U_n$ for at most finitely many $n \in \mathbb{Z}_+$. Then $\{U_x\}_{x \in X}$ is obviously a cover. By compactness, there is a finite subcover,

$$X = \bigcup_{i=1}^{N} U_{x_i} ,$$

which means $\{x_n\}_{n=1}^{\infty} \subset \bigcup_{i=1}^{N} U_{x_i}$. But each U_{x_i} only contains finitely many point from the sequence. This is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ needs to have a convergent subsequence.

To finish the proof, we need to prove that every sequence satisfies (12.1.1). We will argue by contradiction again! Namely, assume that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies

$$\exists x \in X \; \forall U \overset{\text{open}}{\subset} X,\, x \in U \;:\; x_n \in U \text{ for infinitely many } n \in \mathbb{Z}_+ . \tag{12.1.2}$$

Then we will derive a contradiction. Take such $x \in X$ and let $\{B_i\}_{i=1}^{\infty}$ be a countable basis at x. Define

$$U_k := \bigcap_{i=1}^{k} B_i \,.$$

Then $U_k \overset{\text{open}}{\subset} X$ and $x \in U_k$. So by assumption (12.1.2) we may pick an increasing sequence of numbers n_k such that $x_{n_k} \in U_k \;\forall k \in \mathbb{Z}_+$. We will now show $x_{n_k} \to x$. By definition of countable basis, any neighborhood U of x will contain some U_K for some $K > 0$. Therefore, when $k > K$ we have

$$x_{n_k} \in \bigcap_{i=1}^{k} B_i \subset B_K \subset U \,.$$

Therefore $x_{n_k} \to x$, which contradicts the assumption that $\{x_n\}_{n=1}^{\infty}$ had no convergent subsequence. Therefore (12.1.1) holds, and we are done.

$\Leftarrow$: Assume that X is a metrizable topological space that is sequentially compact. That is, the topology of X is induced by some metric d. We denote the ball centered at $x \in X$ with radius in this metric by $B(x, r)$.

Let $\{U_i\}_{i\in I}$ be an open cover of X.

Assume

$$\exists \delta > 0 \;\forall x \in X \;\exists i \in I \;:\; B(x, \delta) \subset U_i \,. \tag{12.1.3}$$

Then pick some $\delta > 0$ satisfying (12.1.3). If we can find a finite subcover of the cover $\{B(x, \delta)\}_{x\in X}$ we would be done, because for each $x \in X$ there is some $i \in I$ such that $B(x, \delta) \subset U_i$. That is, if

$$X = \bigcup_{k=1}^{N} B(x_k, \delta) \,,$$

then we also have

$$X = \bigcup_{k=1}^{N} U_{i_k} \,.$$

where $i_k \in I$ is such that $B(x_k, \delta) \subset U_{i_k}$. We argue by contradiction. Suppose that $\{B(x, \delta)\}_{x\in X}$ has no finite subcover. Then we can pick an infinite sequence $x_1, x_2, x_3, \ldots$ such that

$$x_{n+1} \notin \bigcup_{i=1}^{n} B(x_i, \delta) \,,$$

for every n. Therefore we have $d(x_m, x_n) > \delta$ for any $m \neq n$. For such a sequence it is impossible to find a convergent subsequence, which leads to a contradiction. It follows that $\{B(x, \delta)\}_{x\in X}$ has a finite subcover, and hence $\{U_i\}_{i\in I}$ has a finite subcover.

What is left to show is that every sequentially compact metrizable topological space satisfies (12.1.3). We prove it by contradiction. Therefore, assume

$$\forall \delta > 0 \;\exists x \in X \;\forall i \in I \;:\; B(x, \delta) \not\subset U_i \,.$$

Hence, for any $n \in \mathbb{Z}_+$ we can pick some x_n such that $B\left(x_n, \frac{1}{n}\right) \not\subset U_i$ holds for every $i \in I$. By sequentially compactness, there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. We assume $x_{n_k} \to x$ for some $x \in X$. Since $x \in U_j$ for some $j \in I$ and since $U_j \overset{\text{open}}{\subset} X$, there is some $m > 0$ for which $B\left(x, \frac{1}{m}\right) \subset U_j$. By convergence of $\{x_{n_k}\}_{k=1}^{\infty}$ there is some $K > 0$ such that $d(x_{n_k}, x) < \frac{1}{2m}$ for every $k > K$. By taking an even larger $k > K$, we may also suppose that $n_k > 2m$. Therefore, for any $y \in B\left(x_{n_k}, \frac{1}{n_k}\right)$ it now follows from the triangle inequality that

$$d(y,x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{1}{2m} < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}\,.$$

Hence $y \in B\left(x, \frac{1}{n_k}\right) \subset U_j$ which in turn shows that $B\left(x_{n_k}, \frac{1}{n_k}\right) \subset U_j$ which is a contradiction. Therefore (12.1.3) holds, and we are done.

□

12.2. Compactness in $\mathbb{R}^n$

- Munkres: §27, §43

We will now discuss compactness in $\mathbb{R}^n$. Some of this material may already be familiar to some of you from courses in analysis.

Definition 12.2.4 (Cauchy sequence)**.** *Let (X,d) be a metric space. A sequence $\{x_n\}_{n=1}^\infty$ is called a* Cauchy sequence *if it satisfies*

$$\forall \varepsilon > 0\, \exists N > 0 \ :\ m,n \geq N \implies d(x_n, x_m) < \varepsilon\,.$$

Definition 12.2.5 (Complete metric space)**.** *We call a metric space* complete *if every Cauchy sequence is convergent.*

Remark 12.2.6.

- Convergent sequences are easily seen to be Cauchy. Namely, if $x_n \to x$ it means that $\forall \varepsilon > 0\ \exists N > 0$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ whenever $n \geq N$. Then we use the triangle inequality to say that for any $n, m \geq N$
$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$
so $\{x_n\}_{n=1}^\infty$ is Cauchy.
- $\mathbb{R}$ equipped with the standard metric is complete. (This is actually by definition, if we define $\mathbb{R}$ to be the completion of $\mathbb{Q}$.)
- Completeness is *not* a topological property! It depends on the metric.

Example 12.2.7. Let $(0,1) \subset \mathbb{R}$ be equipped with the standard metric on $\mathbb{R}$ and the subspace topology. Then $(0,1)$ is *not* complete. Namely, the sequence $x_n = \frac{1}{n}$ for $n \in \mathbb{Z}_+$ is Cauchy, but it does not converge inside of $(0,1)$.

This example shows that even though $(0,1) \cong \mathbb{R}$, it does not mean that $(0,1)$ is complete.

We will now prove that any closed and bounded interval in $\mathbb{R}$ is compact.

THEOREM 12.2.8. *A closed and bounded interval $[a,b] \subset \mathbb{R}$ is sequentially compact.*

PROOF. We can without loss of generality prove it for the interval $[0,1]$ since $[0,1] \cong [a,b]$ whenever $a < b$. (If $a = b$, then $[a,a] = \{a\}$ is already sequentially compact.) Let $\{x_n\}_{n=1}^\infty$ be a sequence in $[0,1]$. The goal is to construct a convergent subsequence. Let

$$A_i^j := \left[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\right] \subset [0,1]\,,$$

for $i \geq 0$ and $0 \leq j \leq 2^{i+1} - 1$. We then have

$$A_{i+1}^{2j} \cup A_{i+1}^{2j+1} = \left[\frac{2j}{2^{i+2}}, \frac{2j+1}{2^{i+2}}\right] \cup \left[\frac{2j+1}{2^{i+2}}, \frac{2j+2}{2^{i+2}}\right] = \left[\frac{2j}{2^{i+2}}, \frac{2j+2}{2^{i+2}}\right] = \left[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\right] = A_i^j\,.$$

It is also clear that $[0,1] = \bigcup_{j=0}^{2^{i+1}-1} A_i^j$. Since $[0,1] = A_0^0 \cup A_0^1$, one of A_0^0 and A_0^1 must contain x_n for infinitely many $n \in \mathbb{Z}_+$. If we have $x_n \in A_i^j$ we have that since

$$A_i^j = A_{i+1}^{2j} \cup A_{i+1}^{2j+1},$$

one of A_{i+1}^{2j} and A_{i+1}^{2j+1} must contain x_n for infinitely many $n \in \mathbb{Z}_+$. By induction we can thus construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that if $k \geq i$, then $x_{n_k} \in A_i^{j_i}$ for some j_i. Therefore if $k, \ell \geq i$ we have

$$|x_{n_k} - x_{n_\ell}| \leq \left|A_i^{j_i}\right| = \frac{j_i - 1}{2^{i+1}} - \frac{j_i}{2^{i+1}} = \frac{1}{2^{i+1}}.$$

This means that $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, which gives by completeness that $x_{n_k} \to x$ for some $x \in \mathbb{R}$. But since $[0,1] \subset \mathbb{R}$ is closed, we have that $x \in [0,1]$. This means that $\{x_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence in $[0,1]$ which means precisely that $[0,1]$ is sequentially compact. □

This theorem gives that any closed and bounded interval $[a,b] \subset \mathbb{R}$ is compact as well by Theorem 12.1.3, because the subspace topology on $[a,b]$ is metrizable; it is generated by the standard metric on $\mathbb{R}$, restricted to $[a,b]$.

Remark 12.2.9. The proof can also be modified to work for boxes $[a_1,b_1] \times \cdots \times [a_n,b_n] \subset \mathbb{R}^n$. But compactness of $[a,b] \subset \mathbb{R}$ implies compactness of boxes by Tychonoff's theorem also.

Definition 12.2.10. *Let (X,d) be a metric space and let $A \subset X$ be a subset. Then we say that A is* bounded *if $A \subset B(x,r)$ for some $x \in X$ and $r > 0$.*

THEOREM 12.2.11. *Let $\mathbb{R}^n$ be equipped with the standard metric and let $A \subset \mathbb{R}^n$ be a subset. Then the following are equivalent.*

(1) A is compact
(2) A is sequentially compact
(3) A is closed and bounded

Remark 12.2.12.
- The equivalence (1) $\Leftrightarrow$ (3) is called the Heine–Borel theorem.
- The equivalence (2) $\Leftrightarrow$ (3) is called the Bolzano–Weierstrass theorem.
- This theorem fails in general for metric spaces. It is important that we work in $\mathbb{R}^n$.

PROOF. We already proved (1) $\Leftrightarrow$ (2) in Theorem 12.1.3.

(2) $\implies$ (3): Closedness: It suffices to show $\overline{A} \subset A$. Let $x \in \overline{A}$. Then by the sequence lemma (Lemma 5.2.7), there is a sequence $\{a_n\}_{n=1}^{\infty}$ in A such that $a_n \to x$ (convergence in $\mathbb{R}^n$). Since A is assumed to be sequentially compact, we can pick a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $a_{n_k} \to a$, where $a \in A$ (convergence in A). But since the inclusion map $i\colon A \longrightarrow \mathbb{R}^n$ is continuous, we have by Theorem 10.1.10, we also have $a_{n_k} \to a$ when it is regarded as convergence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff it means that $x = a \in A$ since limits are unique. Hence $\overline{A} \subset A$ and we are done.

Boundedness: Assume for a contradiction that A is not bounded, and consider the collection of balls $\{B(0,n)\}_{n=1}^{\infty}$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ by letting $x_n \in A \setminus B(0,n)$ be arbitrary. Then let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequnce. We then show that $\{x_{n_k}\}_{k=1}^{\infty}$ can not be convergent, which would contradict the sequential compactness of A. To show that $\{x_{n_k}\}_{k=1}^{\infty}$ can not be convergent we will show that it is not Cauchy. To prove that, we assume for a contradiction that $\{x_{n_k}\}_{k=1}^{\infty}$ is Cauchy. To that end, let $N > 0$ be such that

$$i, j \geq N \implies d(x_{n_i}, x_{n_j}) < 1 .$$

Then choose some $j > 0$ such that $j \geq N$ and $n_j \geq |x_{n_i}| + 1$. Then we note that $d(x_{n_j}, 0) \geq n_j$ by construction of our sequence, and so we obtain

$$1 \leq n_j - |x_{n_i}| \leq d(x_{n_j}, 0) - d(x_{n_i}, 0) \leq d(x_{n_i}, x_{n_j}) ,$$

where we used the triangle inequality in the last step. This contradicts the assumption that $\{x_{n_k}\}_{k=1}^{\infty}$ was Cauchy. Therefore $\{x_{n_k}\}_{k=1}^{\infty}$ can not be convergent which contradicts the assumption of sequential compactness of A. Hence A is bounded.

(3) $\implies$ (1): Since A is bounded, we have $A \subset B(x, r)$ for some $x \in \mathbb{R}^n$ and $r > 0$. Take $N = |x| + r$. Then we have that if $y \in B(x, r)$, then by triangle inequality

$$|y| \leq |x| + |y - x| \leq N ,$$

and also $|y_i| \leq |y| = \sqrt{\sum_{i=1}^{n} y_i^2}$ for every $i \in \{1, \dots, n\}$, it follows that

$$B(x, r) \subset [-N, N]^n .$$

Since $[-N, N] \subset \mathbb{R}$ is compact it follows that $[-N, N]^n$ is compact by Tychonoff's theorem. Therefore, since A is closed in $\mathbb{R}^n$, it is also closed in $[-N, N]$ (since it is contained in the box $[-N, N]^n$), which means that it is compact by Theorem 11.1.4.

□

12.3. Compactness and completeness in general metric spaces

- Munkres: §43, §45

We will now consider general metric spaces, and not only $\mathbb{R}^n$. First we consider a useful lemma.

Lemma 12.3.13. *A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.*

PROOF. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. We will show that if $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence, then the sequence itself will converge. First, $\{x_n\}_{n=1}^{\infty}$ is Cauchy, so given $\varepsilon > 0$, choose $N > 0$ so that $n, m \geq N \implies d(x_n, x_m) < \frac{\varepsilon}{2}$. Next assume $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence converging to x. Therefore Choose $i > 0$ large enough so that $n_i \geq N$ and $d(x_{n_i}, x) < \frac{\varepsilon}{2}$. Then we use the triangle inequality to get

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \varepsilon ,$$

which means $x_n \to x$. □

Corollary 12.3.14. *Any compact metric space (X, d) is complete.*

PROOF. Exercise. □

Definition 12.3.15 (Bounded)**.** *A metric space (X, d) is called* bounded *if $\exists M > 0$ such that $d(x, y) \leq M$ $\forall x, y \in X$.*

Remark 12.3.16. Note that the above definition is equivalent with Definition 12.2.10 for subsets $A \subset X$.

Definition 12.3.17 (Totally bounded)**.** *A metric space (X, d) is called* totally bounded *if $\forall \varepsilon > 0$ there is a finite covering of X by balls of radius ε.*

Proposition 12.3.18. *If (X, d) is totally bounded, then (X, d) is bounded.*

PROOF. Exercise. ☐

Example 12.3.19.
(1) $(0, 1) \subset \mathbb{R}$ is totally bounded. For any $\varepsilon > 0$, we need $\lceil \frac{1}{2\varepsilon} \rceil$ number of balls to cover it. (Note that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.)

Also take note that $(0, 1)$ is neither complete nor compact as we have seen before.
(2) $[0, 1] \subset \mathbb{R}$ is totally bounded for the same reason as above, but instead we need two extra balls to cover the points $\{0\}$ and $\{1\}$. This time $[0, 1]$ is both complete and compact.
(3) $\mathbb{R}$ is not totally bounded because it is not bounded. It is not bounded because $d(x, -x) = |x - (-x)| = 2|x|$ can be made arbitrarily large by making $|x|$ arbitrarily large.

We also note that $\mathbb{R}$ is complete.

We will now see a result which gives a necessary and sufficient condition for compactness of metric spaces.

THEOREM 12.3.20. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

PROOF. **⇒:** This is partly Corollary 12.3.14. To show that compactness implies total boundedness we do the following.

Let $\varepsilon > 0$ be arbitrary. Then $X = \bigcup_{x \in X} B(x, \varepsilon)$ is an open cover. By compactness we pick a finite subcover $X = \bigcup_{i=1}^{N} B(x_i, \varepsilon)$, which shows that X is totally bounded.

⇐: Assume that (X, d) is complete and totally bounded. We will prove that X is sequentially compact, which suffices by Theorem 12.1.3. Pick a sequence $\{x_n\}_{n=1}^{\infty}$. We will prove that it has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which is Cauchy (hence convergent by the completeness assumption).

First, cover X by a finite number of balls of radius 1. One of the balls, call it B_1, contains x_n for infinitely many $n \in \mathbb{Z}_+$. Let $J_1 \subset \mathbb{Z}_+$ be such that $\{x_n\}_{n \in J_1} \subset B_1$.

Secondly, cover X by a finite number of balls of radius $\frac{1}{2}$ and repeat the above argument. Because J_1 is infinite, at least one of the balls, call it B_2, of radius $\frac{1}{2}$ must contain x_n for infinitely many $n \in J_1$. Let $J_2 \subset J_1$ be such that $\{x_n\}_{n \in J_2} \subset B_2$.

In general, given an infinite subset $J_k \subset \mathbb{Z}_+$, choose J_{k+1} to be an infinite subset of J_k such that there is a ball B_{k+1} of radius $\frac{1}{k+1}$ that contains x_n for all $n \in J_{k+1}$. This process gives a sequence

$$J_1 \supset J_2 \supset \cdots,$$

of infinite subsets. Now pick numbers $n_k \in J_k$ for every $k \in \mathbb{Z}_+$ (by construction we have $n_1 < n_2 < \cdots$). Then we claim that $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy subsequence. Namely, for any $i, j \geq N$ we have $x_{n_i} \in B_i \subset B_N$ and $x_{n_j} \in B_j \subset B_N$, so

$$d(x_{n_i}, x_{n_j}) \leq \frac{2}{N},$$

since both of them are contained in a ball B_N of radius $\frac{1}{N}$. Therefore $\{x_{n_k}\}_{k=1}^{\infty}$ is Cauchy and the result follows.

☐

Exercises

12.1. Show that any compact metric space is complete.

12.2. Let (X, d) be a totally bounded metric space. Show that there is some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in X$.

Lecture 13 (Problem Session 2)

Problem §23.2. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup_{i=1}^{\infty} A_n$ is connected.

Problem §23.5. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Problem §23.6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \smallsetminus A$, then C intersects ∂A.

Problem §24.1.

(1) Show that no two of the spaces $(0, 1)$, $(0, 1]$ and $[0, 1]$ are homeomorphic.
Hint: What happens if you remove a point from each of these spaces?

(2) Let $f\colon X \longrightarrow Y$ be an injective continuous map. If the map obtained by restricting the range, $f'\colon X \longrightarrow f(X)$ is a homeomorphism, then f is called an *embedding.*
Suppose that there exist embeddings $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$. Show by means of an example that X and Y need not be homeomorphic.

(3) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Problem §24.2. Let $f\colon S^1 \longrightarrow \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Problem §24.3. Let $f\colon X \longrightarrow X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f. What happens if X equals $[0, 1)$ or $(0, 1)$?

Problem §24.8.

(1) Is a product of path-connected spaces necessarily path-connected?

(2) If $A \subset X$ A is path-connected, is $\overline{A}$ necessarily path-connected?

(3) If $f\colon X \longrightarrow Y$ is continuous and X is path-connected, is $f(X)$ necessarily path-connected?

(4) If $\{A_\alpha\}_{\alpha \in I}$ is a collection of path-connected subspaces of X and if $\bigcap_{\alpha \in I} A_\alpha \neq \varnothing$, is $\bigcup_{\alpha \in I} A_\alpha$ necessarily path-connected?

Problem §25.4. Let X be locally path-connected. Show that every connected open set in X is path-connected.

Problem §25.5. Let X denote the rational points of the interval $[0, 1] \times \{0\}$ of $\mathbb{R}^2$. Let T be denote the union of all line segments joining the point $p = (0, 1)$ to points of X.

(1) Show that T is path-connected, but is locally connected only at the point p.

(2) Find a subset of $\mathbb{R}^2$ that is path-connected but is locally connected at none of its points.

Part 3

Important spaces, metrizability and homotopy

Lecture 14

Recap.
- A topological space X is compact if and only if every open cover of X has a finite subcover.
- A topological space X is sequentially compact if and only if every sequence has a convergent subsequence.
- If X is metrizable then X compact $\Leftrightarrow$ X sequentially compact.
- If $X = \mathbb{R}^n$ then $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

14.1. Local compactness

- Munkres: §29

Much like local connectedness, compactness also has a local counterpart, called local compactness.

Definition 14.1.1 (Local compactness)**.** *A topological space X is called* locally compact *at x if there is some compact subset $C \subset X$ that contains a neighborhood of x.*

If X is locally compact at each of its points, then X is called locally compact*.*

Remark 14.1.2. Note that this definition has a slightly different flavor to that of local connectedness. We have seen that connectedness does not imply local connectedness for example. This is no longer true for compactness, see Proposition 14.1.4.

Example 14.1.3.
(1) $\mathbb{R}$ is locally compact. Any point x is contained in some closed interval $[x-2\varepsilon, x+2\varepsilon]$ where $\varepsilon > 0$, which contains a neighborhood $(x-\varepsilon, x+\varepsilon) \subset [x-2\varepsilon, x+2\varepsilon]$ of x.
(2) Any open or closed subset of $\mathbb{R}^n$ is locally compact. The argument is the same as above, but we can enclose any point in a small closed and bounded box.
(3) $\mathbb{Q} \subset \mathbb{R}$ equipped with the subspace topology is not locally compact.

Suppose $x \in \mathbb{Q}$ and that U is any subset of $\mathbb{Q}$ containing x such that U contains a neighborhood U_x of x. By definition we have $U_x = \widetilde{U}_x \cap \mathbb{Q}$ where $\widetilde{U}_x = \bigcup_{i \in I_x}(a_i, b_i)$. In particular, this means that U_x (and hence U) contains a set of the form $(a_i, b_i) \cap \mathbb{Q} \subset U$. But now we claim that U can not be compact, which would then by definition prove that $\mathbb{Q}$ is not locally compact.

To show this, we show that $(a_i, b_i) \cap \mathbb{Q}$ is not compact. Note that $\mathbb{Q}$ is metrizable (the metric on $\mathbb{R}$ restricts to a metric on $\mathbb{Q}$ and the metric topology on $\mathbb{Q}$ is equal to the subspace topology). Next, pick any $c \in (a_i, b_i) \in (\mathbb{R} \smallsetminus \mathbb{Q})$. Then we can find a strictly increasing sequence in $(a_i, b_i) \cap \mathbb{Q}$ converging to c. This means that $(a_i, b_i) \cap \mathbb{Q}$ is not sequentially compact, because this sequence can not have any convergent subsequences since it is strictly increasing. Hence, by Theorem 12.1.3, $(a_i, b_i) \cap \mathbb{Q}$ can not be compact, which means that U can not be compact.

Proposition 14.1.4. *If X is a compact topological space, then X is locally compact.*

Proof. Exercise. $\square$

We will now state a theorem without proof, which will motivate our next big construction.

THEOREM 14.1.5. *Let X be a topological space. Then X is Hausdorff and locally compact if and only if there exists a topological space Y satisfying the following conditions:*

(1) X is a subspace of Y,
(2) The set $Y \smallsetminus X$ consists of a single point, and
(3) Y is Hausdorff and compact.

Moreover, the Y is unique *in the following sense: If Y and Y' are two topological spaces satisfying these conditions, then there is a homeomorphism $f\colon Y \longrightarrow Y'$ such that f is the identity function on X.*

14.1.1. One-point compactification.

Definition 14.1.6 (One-point compactification)**.** *Let X be a locally compact Hausdorff topological space. The* one-point compactification *of X is a topological space*

$$\widehat{X} := X \cup \{*\} ,$$

obtained by adding one extra point to X (sometimes this extra point $*$ is rather denoted by ∞). The topology on $\widehat{X}$ is defined as follows: $U \overset{\text{open}}{\subset} \widehat{X}$ if and only if

(1) $U \overset{\text{open}}{\subset} X$, or
(2) $U = (X \smallsetminus C) \cup \{*\}$ where $C \overset{\text{compact}}{\subset} X$

Proposition 14.1.7. *The above open sets of $\widehat{X}$ satisfies the axioms for a topology.*

PROOF. **(T1):** We have $\varnothing \overset{\text{open}}{\subset} \widehat{X}$ since $\varnothing \overset{\text{open}}{\subset} X$. Also $\widehat{X} \overset{\text{open}}{\subset} \widehat{X}$ since $\widehat{X} = (X \smallsetminus \varnothing) \cup \{*\}$ and $\varnothing \overset{\text{compact}}{\subset} X$.

(T2): Suppose $\{U_i\}_{i\in I}$ is a collection of open sets in $\widehat{X}$.

- If all U_i are of type (1) above in the definition, then $\bigcup_{i\in I} U_i \overset{\text{open}}{\subset} X$ and so $\bigcup_{i\in I} U_i \overset{\text{open}}{\subset} \widehat{X}$.
- If all U_i are of type (2), then $\bigcup_{i\in I} U_i = \bigcup_{i\in I} (X \smallsetminus C_i) \cup \{*\} = X \smallsetminus (\bigcap_{i\in I} C_i) \cup \{*\}$. Since each C_i is compact, and we have $\bigcap_{i\in I} C_i \subset C_j$ for any $j \in I$. Since X is Hausdorff, each C_j is also closed, and so $\bigcap_{i\in I} C_i$ is compact.
- Otherwise if $U_i \overset{\text{open}}{\subset} X$ for $i \in I'$ and $U_i = (X \smallsetminus C_i) \cup \{*\}$ for $i \in I''$ where $I = I' \cup I''$, we have

$$\begin{aligned}\bigcup_{i\in I} U_i &= \left(\bigcup_{i\in I'} U_i\right) \cup \left(\bigcup_{i\in I''} (X \smallsetminus C_i) \cup \{*\}\right)\\ &= \left(\bigcup_{i\in I'} X \smallsetminus F_i\right) \cup \left(\bigcup_{i\in I''} (X \smallsetminus C_i) \cup \{*\}\right)\\ &= \bigcup_{\substack{i\in I'\\ j\in I''}} (X \smallsetminus (F_i \cap C_j)) \cup \{*\} = X \smallsetminus \left(\bigcap_{\substack{i\in I'\\ j\in I''}} F_i \cap C_j\right) \cup \{*\} ,\end{aligned}$$

and $\bigcap F_i \cap C_j$ is compact since it is closed and contained in some C_j which is compact.

(T3): Suppose $\{U_i\}_{i=1}^N$ is a collection of open sets in $\widehat{X}$. Then we do a similar case study.

- If all U_i are of type (1), then $\bigcap_{i=1}^N U_i \overset{\text{open}}{\subset} X$ and so $\bigcap_{i=1}^N U_i \overset{\text{open}}{\subset} \widehat{X}$.

- If all U_i are of type (2) then
$$\bigcap_{i=1}^{N} U_i = \bigcap_{i=1}^{N} (X \smallsetminus C_i) \cup \{*\} = X \smallsetminus \left(\bigcup_{i=1}^{N} C_i \right) \cup \{*\} ,$$
and a finite union of compact sets is compact. (It is an exercise to show this!)
- Otherwise if $U_i \overset{\text{open}}{\subset} X$ for $i \in I' := \{1, \ldots, N'\}$ and $U_i = (X \smallsetminus C_i) \cup \{*\}$ for $i \in I'' := \{N'+1, \ldots, N\}$, we have
$$\begin{aligned} \bigcap_{i=1}^{N} U_i &= \left(\bigcap_{i=1}^{N'} U_i \right) \cup \left(\bigcap_{i=N'+1}^{N} (X \smallsetminus C_i) \cup \{*\} \right) \\ &= \left(\bigcap_{i=1}^{N'} X \smallsetminus F_i \right) \cup \left(\bigcap_{i=N'+1}^{N} (X \smallsetminus C_i) \cup \{*\} \right) \\ &= \bigcap_{\substack{i \in I' \\ j \in I''}} (X \smallsetminus (F_i \cap C_j)) \cup \{*\} = X \smallsetminus \left(\bigcup_{\substack{i \in I' \\ j \in I''}} F_i \cap C_j \right) \cup \{*\} , \end{aligned}$$
For each $i \in I'$ and $j \in I''$ we have $F_i \cap C_j$ is closed and contained in C_j which is compact, which means that $F_i \cap C_j$ is compact. Again we then use that a finite union of compact subsets is compact, and we have shown $\bigcap_{i=1}^{N} U_i \overset{\text{open}}{\subset} \widehat{X}$.

$\square$

Proposition 14.1.8. *Let X be a locally compact Hausdorff topological space and consider its one-point compactification $\widehat{X}$. Then the subspace topology $X \subset \widehat{X}$ coincides with the original topology on X.*

Proof. Exercise. $\square$

Proposition 14.1.9. *If X is a locally compact Hausdorff topological space, then $\widehat{X}$ is compact.*

Proof. Let $\{U_i\}_{i \in I}$ be any cover of $\widehat{X}$. Then since $* \in U_j$ for some $j \in I$, we know that $U_j = (X \smallsetminus C) \cup \{*\}$. Then we need to cover the rest of $\widehat{X}$, namely the compact set C. Since $\{U_i \cap C\}_{i \in I}$ is a cover of C, we pick a finite subcover, $\{U_{i_k} \cap C\}_{k=1}^{N}$. Therefore
$$\widehat{X} = U_j \cup \bigcup_{k=1}^{N} (U_{i_k} \cap C) = U_j \cup \bigcup_{k=1}^{N} U_{i_k} ,$$
and so $\{U_j\} \cup \{U_{i_k}\}_{k=1}^{N}$ is a finite subcover and hence $\widehat{X}$ is compact. $\square$

Proposition 14.1.10. *Suppose X is Hausdorff and locally compact. Then $\widehat{X}$ is Hausdorff.*

Proof. Pick two distinct points $x, y \in \widehat{X}$. We have two cases.

- If $x, y \in X$, then we can find two disjoint neighborhoods since X is Hausdorff.
- Without loss of generality assume $x = *$ and $y \in X$. By local compactness of X, pick a compact subset $C_y \overset{\text{compact}}{\subset} X$ containing y such that there is a neighborhood U_y of y such that $U_y \subset C_y$. Then $U_x := (X \smallsetminus C_y) \cup \{*\}$ is a neighborhood of $x = *$ such that $U_x \cap U_y = \varnothing$ by construction.

$\square$

Remark 14.1.11. By Theorem 14.1.5 we have that if X is a locally compact Hausdorff topological space, then its one-point compactification $\widehat{X}$ is unique in the sense explained in Theorem 14.1.5.

Definition 14.1.12. *A continuous function $f\colon X \longrightarrow Y$ is called* proper *if*

$$C \overset{\text{compact}}{\subset} Y \implies f^{-1}(C) \overset{\text{compact}}{\subset} X\,.$$

Proposition 14.1.13. *Suppose $f\colon X \longrightarrow Y$ is continuous and proper. Then f induces a continuous function*

$$\widehat{f}\colon \widehat{X} \longrightarrow \widehat{Y}\,,$$

which is defined by $\widehat{f}|_X = f$ and $\widehat{f}(*_X) = *_Y$.

Proof. If $U \overset{\text{open}}{\subset} \widehat{Y}$ does not contain $*_Y$, then $U \overset{\text{open}}{\subset} Y$ by definition, and so $\widehat{f}^{-1}(U) = f^{-1}(U) \overset{\text{open}}{\subset} X$ by continuity of f. If $*_Y \in U$, then we know by definition that

$$U = (Y \smallsetminus C) \cup \{*_Y\}\ .$$

Then we have

$$\widehat{f}^{-1}((Y \smallsetminus C) \cup \{*_Y\}) = f^{-1}(Y) \smallsetminus f^{-1}(C) \cup \{*_X\} = X \smallsetminus f^{-1}(C) \cup \{*_X\}\ .$$

By the fact that f is proper, we have $f^{-1}(C) \overset{\text{compact}}{\subset} X$, which means that $\widehat{f}^{-1}((Y \smallsetminus C) \cup \{*_Y\}) \overset{\text{open}}{\subset} \widehat{X}$. ☐

Example 14.1.14. (1) Consider $X = \mathbb{R}$ with the standard topology. Then $\widehat{\mathbb{R}} \cong S^1$. Recall that the circle, S^1, is defined as

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2\,,$$

equipped with the subspace topology. Since $\mathbb{R}^2$ is equipped with the metric topology it is in particular Hausdorff, so S^1 is also Hausdorff. Another description of S^1 is via the continuous function

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto x^2 + y^2\,. \end{aligned}$$

Then note we have that $S^1 = f^{-1}(\{1\})$. Since $\{1\} \overset{\text{closed}}{\subset} \mathbb{R}$, we have that $f^{-1}(\{1\}) = S^1 \overset{\text{closed}}{\subset} \mathbb{R}^2$. It is also clear that S^1 is bounded in $\mathbb{R}^2$ (it is contained in a ball with radius 2 centered at the origin for instance). By the Heine–Borel theorem (Theorem 12.2.11), S^1 is compact.

Now remove the point $(-1,0) \in S^1$ from S^1. Then parametrize the rest of the circle by $(\cos(t), \sin(t))$ for $t \in (-\pi, \pi)$. Then the function

$$\begin{aligned} g\colon S^1 \smallsetminus \{(-1,0)\} &\longrightarrow (-\pi,\pi) \\ (\cos(t), \sin(t)) &\longmapsto t\,, \end{aligned}$$

is a homeomorphism. We already know from before that any open interval is homeomorphic to $\mathbb{R}$. So $S^1 \smallsetminus \{(-1,0)\} \cong \mathbb{R}$. By Theorem 14.1.5, we therefore get that the one-point compactification or $\mathbb{R}$ needs to be homeomorphic to S^1.

(2) Consider $X = \mathbb{R}^2$. What is $\widehat{X}$?

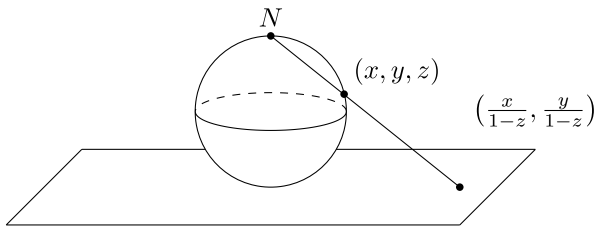
The above story with $\widehat{\mathbb{R}} \cong S^1$ generalizes. The sphere S^2, is defined as the set

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3\,,$$

or more generally, the n-sphere S^n is defined as

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^1 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}\,.$$

In fact, we will show that if $N := (0, \ldots, 0, 1)$ is the north pole, then $S^n \setminus \{N\} \cong \mathbb{R}^n$. We can in fact write down such a homeomorphism explicitly. This is known as *stereographic projection.*



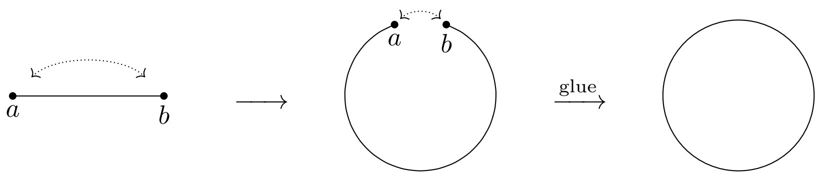
The stereographic projection is the function

$$\begin{aligned} p_{\text{stereo}} \colon S^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) &\longmapsto \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right). \end{aligned}$$

First note that since we removed the north pole, we have $x_{n+1} < 1$, so the denominator never vanishes. We claim that this function is a homeomorphism. So much like the argument when $n = 1$, we have that $\widehat{\mathbb{R}^n} \cong S^n$ for any $n \geq 1$.

We will now continue our discussion about some important spaces. We have by now seen the n-dimensional spheres S^n which are compact spaces that can be viewed as the one-point compactification of $\mathbb{R}^n$.

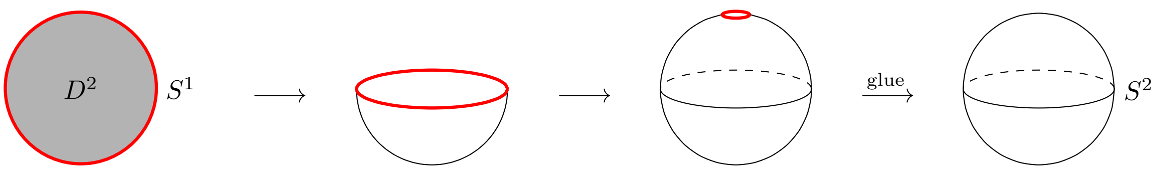
We can construct spheres in a different way. We recall our discussion in lecture 8 where we introduced the quotient topology.



We can realize S^1 as the quotient $[-1, 1]/\sim$ where $-1 \sim 1$ and $x \sim x$ for any other $x \in (-1, 1)$. Note that we may regard $[-1, 1]$ as the *closed unit disk* in $\mathbb{R}$, let us denote it by $D^1 := \overline{B(0, 1)} \subset \mathbb{R}$. By our above definition the zero dimensional sphere S^0 is the two points $S^0 = \{-1, 1\}$. So we can write

$$S^1 \cong D^1/S^0 \,.$$

This story also generalizes! With a little bit of imagination we see $S^2 \cong D^2/S^1$ where D^2 is the closed unit disk in $\mathbb{R}^2$ and S^1 is the boundary.



This is in fact true for any sphere. Before we see it, we will have the following proposition.

Proposition 14.1.15. *If X is a compact topological space, and $\sim$ is any relation on X, then $X/\sim$ is compact.*

PROOF. Recall that there is a projection

$$\begin{aligned} p\colon\ X &\longrightarrow X/{\sim} \\ x &\longmapsto [x]\,, \end{aligned}$$

and the quotient topology on $X/{\sim}$ is defined as follows:

$$U \overset{\text{open}}{\subset} X/{\sim} \ \Leftrightarrow\ p^{-1}(U) \overset{\text{open}}{\subset} X\,.$$

The rest of the proof is an exercise! □

Proposition 14.1.16. *Let D^n be the closed unit ball in $\mathbb{R}^n$, and consider its boundary $S^{n-1} = \partial D^n$. Then $S^n \cong D^n/S^{n-1}$.*

PROOF. Recall that we have a homeomorphism (see Example 7.3.13)

$$\begin{aligned} f\colon\ B(0,1) &\longrightarrow \mathbb{R}^n \\ \boldsymbol{x} &\longmapsto \left(\frac{x_1}{1-|\boldsymbol{x}|}, \dots, \frac{x_n}{1-|\boldsymbol{x}|}\right)\,. \end{aligned}$$

Using the inverse of the stereographic projection, which is given by the following formula:

$$\begin{aligned} p_{\text{stereo}}^{-1}\colon\ \mathbb{R}^n &\longrightarrow S^n \smallsetminus \{N\} \\ \boldsymbol{x} &\longmapsto \left(\frac{2x_1}{1+|\boldsymbol{x}|^2}, \dots, \frac{2x_n}{1+|\boldsymbol{x}|^2}, \frac{-1+|\boldsymbol{x}|^2}{1+|\boldsymbol{x}|^2}\right) \end{aligned}$$

we therefore get a homeomorphism $p_{\text{stereo}}^{-1} \circ f\colon\ B(0,1) \longrightarrow S^n \smallsetminus \{N\}$, which is given by the formula

$$\left(p_{\text{stereo}}^{-1} \circ f\right)(x_1, \dots, x_n) = \left(\frac{2x_1(1-|\boldsymbol{x}|)}{(1-|\boldsymbol{x}|)^2+|\boldsymbol{x}|^2}, \dots, \frac{2x_n(1-|\boldsymbol{x}|)}{(1-|\boldsymbol{x}|)^2+|\boldsymbol{x}|^2}, \frac{2|\boldsymbol{x}|-1}{(1-|\boldsymbol{x}|)^2+|\boldsymbol{x}|^2}\right)\,.$$

We then see that if $|\boldsymbol{x}| \to 1$, then $(p_{\text{stereo}}^{-1} \circ f)(x_1, \dots, x_n) \to (0, \dots, 0, 1)$. With this motivation we claim that we can extend $p_{\text{stereo}}^{-1} \circ f$ to a continuous function

$$p_{\text{stereo}}^{-1} \circ f\colon\ D^n \longrightarrow S^n\,,$$

which satisfies $\left(p_{\text{stereo}}^{-1} \circ f\right)(\partial D^n) = N = (0, \dots, 0, 1)$.

Therefore, we have the following diagram of functions

$$\begin{tikzcd} D^n \arrow[d, "p"] \arrow[dr, "p_{\text{stereo}}^{-1}\circ f"] & \\ D^n/S^{n-1} \arrow[r, "h"] & S^n\,, \end{tikzcd}$$

where $p_{\text{stereo}}^{-1} \circ f$ is a continuous surjection. The goal is to show that the induced continuous bijection $h\colon\ D^n/S^{n-1} \longrightarrow S^n$ is a homeomorphism. There are two approaches one might take.

(1) One may use Theorem 8.1.13 and prove that $p_{\text{stereo}}^{-1} \circ f$ is a quotient function.
(2) One may use Theorem 11.1.9 and prove that D^n/S^{n-1} is compact.

We will take approach 2. By Proposition 14.1.15, we have that D^n/S^{n-1} indeed is compact since D^n is compact. The closed unit disk D^n is is closed by definition, and contained in a ball of radius 2 centered at the origin, so it is bounded as well. Heine–Borel theorem then says that it is compact. Therefore any quotient of D^n is compact as well, and in particular D^n/S^{n-1}. So by Theorem 11.1.9, h is a homeomorphism. □

Exercises

14.1. Show that if X is compact, then it is locally compact.

14.2. Let X be a topological space. Show that a finite union of compact subsets of X is compact.

14.3. Let X be Hausdorff and let $\widehat{X}$ denote its one-point compactification. Then the subspace topology on $X \subset \widehat{X}$ coincides with the original topology on X.

14.4. Let X be a compact topological space and $\sim$ any relation on X. Show that $X/{\sim}$ is compact.

Lecture 15

Recap.
- Let $n \geq 1$. The n-sphere is defined as
$$S^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} .$$
- Another useful description of the sphere is that $S^n \cong D^n/\partial D^n$ where D^n is the closed unit ball in $\mathbb{R}^n$.

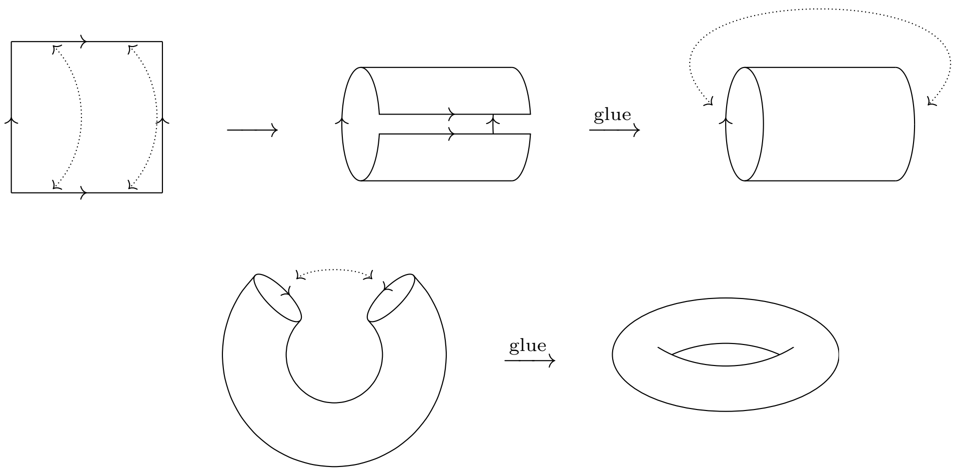
15.1. Some important spaces

In the last lecture we already discussed the sphere S^n and two ways of defining it. We also proved that it is compact. One way to prove is to study the continuous function

$$\begin{aligned} f\colon\ \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \\ (x_1, \ldots, x_{n+1}) &\longmapsto x_1^2 + \cdots + x_{n+1}^2 , \end{aligned}$$

and realizing that $f^{-1}(\{1\}) = S^n$, which means that S^n is closed. It is also contained in a ball of radius 2 centered at the origin in $\mathbb{R}^{n+1}$, which means it is closed and bounded, so by the Heine–Borel theorem it is compact. Another argument is to use the description $S^n \cong D^n/\partial D^n$ and use that D^n is compact, which means that S^n is compact since it is a quotient.

15.1.1. The torus and other surfaces. As above, we constructed spheres by gluing together the boundary of a closed unit ball to a point. Back in lecture 8 we talked about the torus, which also can be defined by gluing edges of a square.



We start with a square and glue together *opposite edges* so that the arrows on the edges match up.

The torus, denoted by T^2 is defined as

$$T^2 := [0,1]^2 /\!\sim ,$$

where $\sim$ is defined by the following complete set of relations

$$\begin{cases} (0,y) \sim (1,y) & \forall y \in [0,1] \\ (x,0) \sim (x,1) & \forall x \in [0,1] \\ (x,y) \sim (x,y) & \forall (x,y) \in (0,1)^2 \end{cases} .$$

Proposition 15.1.1. *T^2 is compact.*

PROOF. This follows from the fact that $[0,1] \subset \mathbb{R}$ is compact. Therefore $[0,1]^2$ is compact by Tychonoff's theorem, and hence $T^2 = [0,1]^2/{\sim}$ is compact by Proposition 14.1.15. (Any quotient of a compact space is compact.) $\square$

Proposition 15.1.2. $T^2 \cong S^1 \times S^1$

PROOF. We would like to show that there is a homeomorphism

$$\varphi\colon\ [0,1]^2/{\sim} \longrightarrow ([0,1]/{\approx})^2 ,$$

where $\approx$ is the relation defined by $0 \approx 1$ and $x \approx x$ for any $x \in (0,1)$ (and no other relations). If this is the case, we would be done, because $[0,1]/{\approx} = [0,1]/\{0,1\} \cong S^1$.

To prove that φ is a homeomorphism we consider the following diagram

$$\begin{tikzcd} {[0,1]^2} \arrow[d, "p"] \arrow[dr, "f"] & \\ {[0,1]^2/{\sim}} \arrow[r, "\varphi"] & ([0,1]/{\approx})^2 \end{tikzcd} ,$$

where f is defined as

$$\begin{aligned} f\colon\ [0,1]^2 &\longrightarrow ([0,1]/{\approx})^2 \\ (x,y) &\longmapsto ([x],[y]) , \end{aligned}$$

and where p is the quotient projection. We define φ so that the diagram commutes, that is, $p \circ \varphi = f$. We will show that f is continuous and surjective. Then we get that the induced map φ is continuous and bijective. Now, since $[0,1]^2/{\sim}$ is compact by Proposition 15.1.1, we have by Theorem 11.1.9 that φ is a homeomorphism.

To show that f is surjective and continuous we consider the following diagram

$$\begin{tikzcd} {[0,1]^2} \arrow[r, "\mathrm{id}"] \arrow[dr, "f"'] & {[0,1]^2} \arrow[d, "q\times q"] \\ & ([0,1]/{\approx})^2 \end{tikzcd}$$

Here $q\colon\ [0,1] \longrightarrow [0,1]/{\approx}$ is the quotient projection. The identity map id is a homeomorphism, and q is a continuous surjection, hence $q \times q$ is also continuous and surjective. Therefore, since $f = (q \times q) \circ \mathrm{id}$, we have that f is continuous and surjective. The result follows. $\square$

Similar to how we defined the sphere, we can generalize the construction with the torus to define the n-torus for $n \geq 1$ as

$$T^n := [0,1]^n/{\sim}$$

where $\sim$ is defined by the following complete set of relations:

$$\begin{cases} (0, x_2, \dots, x_n) \sim (1, x_2, \dots, x_n) & \forall (x_2, x_3, \dots, x_n) \in [0,1]^{n-1} \\ (x_1, 0, \dots, x_n) \sim (x_1, 1, \dots, x_n) & \forall (x_1, x_3, \dots, x_n) \in [0,1]^{n-1} \\ \vdots & \\ (x_1 \dots, x_{n-1}, 0) \sim (x_1, \dots, x_{n-1}, 1) & \forall (x_1, \dots, x_{n-1}) \in [0,1]^{n-1} \\ (x_1, \dots, x_n) \sim (x_1, \dots, x_n) & \forall (x_1, \dots, x_n) \in (0,1)^n \end{cases}.$$

It is formed by considering an n-dimensional cube and identifying all pairs of opposite sides! It is of course not possible to draw a picture or visualize it, but the case with $n = 2$ illustrates the situation well enough.

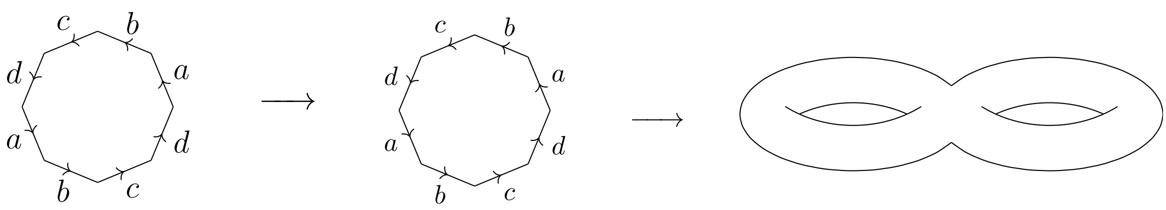
Remark 15.1.3. Take note that the 1-torus is the same as a circle! However, for $n \geq 2$, spheres and tori are not homeomorphic, but this is not trivial! It might be intuitively clear, because the torus has "a hole" in the middle of it, while the sphere does not. One way to prove it rigorously is to use *algebraic topology* and more precisely *homotopy groups*.

Tori and spheres for $n \geq 2$ have different homotopy groups, and hence they can not be homeomorphic, because homotopy groups are *invariants* of the topological spaces. We will get a small taste of this at the end of the course.

Proposition 15.1.4. $T^n \cong (S^1)^n$

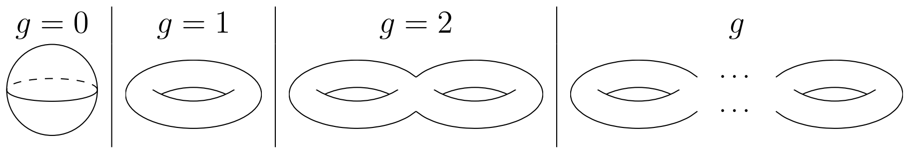
Proof. The proof is more or less identical to the proof of Proposition 15.1.2. □

We can generalize the torus in another direction than to generalize to higher dimensions. We can also construct a *double torus* (or *genus 2 surface*) by starting with an octagon, and identifying opposite sides as indicated:

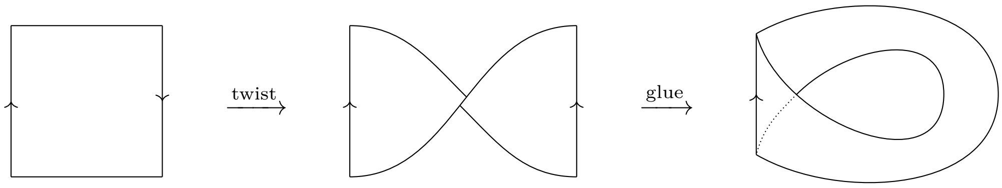


This is more difficult to visualize than the normal torus, but it is doable.

15.1.1.A. *Compact, orientable surfaces.* The sphere, the torus and the double torus are examples of *compat orientable surfaces*. A surface is *oriented* if a flatlander living on the surface can decide what *left and right* is. (See below for a description of some non-orientable surfaces.) In fact, *any* compact orientable surface is decided by a number g, which is called the *genus*. We can sloppily define *genus* as the number of "holes" in the surface. The sphere has genus 0, the torus has genus 1 and the double torus has genus 2. For each g, there is only one compact orientable surface, up to homeomorphism.



15.1.1.B. *Non-orientable surfaces.* Two examples of non-orientable surfaces include the Möbius strip and the Klein bottle.



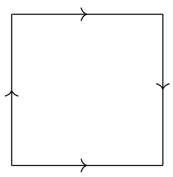
Similar to how we defined the torus, we can define the Möbius strip as the quotient of the square. Note how we have identified two opposite sides, but with a half twist. It is defined as

$$M := [0,1]^2/\sim\,,$$

where $\sim$ is defined by the following complete set of relations:

$$\begin{cases}(0,y)\sim(1,1-y) & \forall y\in[0,1]\\(x,y)\sim(x,y) & \forall(x,y)\in(0,1)\times[0,1]\end{cases}.$$

The Klein bottle is defined by further identifying edges in the above square defining the Möbius strip.



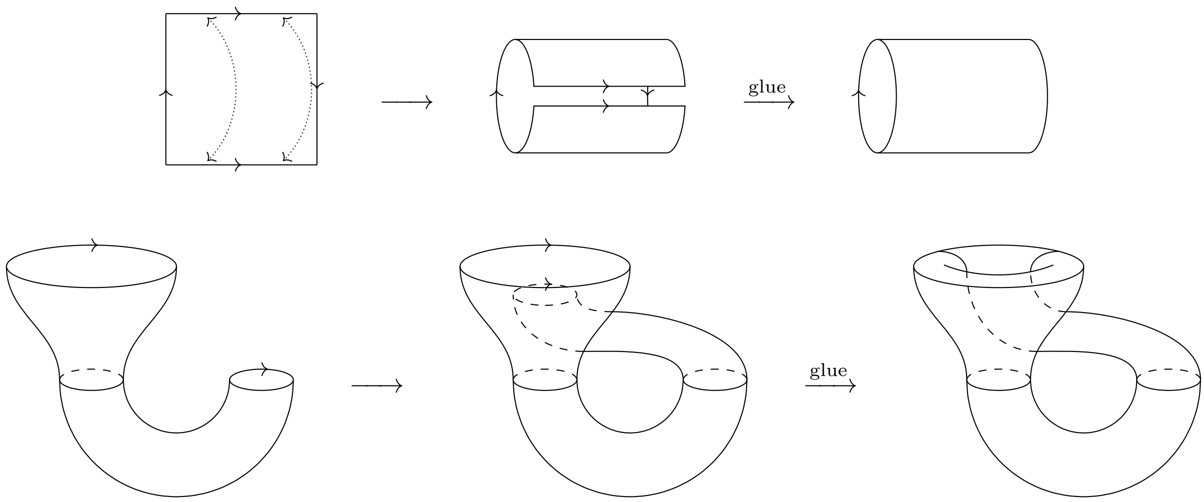
As before we can define the Klein bottle as a quotient of the square $[0,1]^2$

$$K := [0,1]^2/\sim\,,$$

where $\sim$ is defined by the following complete set of relations:

$$\begin{cases}(0,y)\sim(1,1-y) & \forall y\in[0,1]\\(x,0)\sim(x,1) & \forall x\in[0,1]\\(x,y)\sim(x,y) & \forall(x,y)\in(0,1)^2\end{cases}$$

We can also attempt to glue the edges and visualize the Klein bottle.



15.1.2. Real projective space. Consider $\mathbb{R}^{n+1} \smallsetminus \{0\}$ and define the relation $\sim$ as

$$\boldsymbol{x} \sim \boldsymbol{y} \overset{\text{def}}{\Leftrightarrow} \boldsymbol{x} = \lambda \boldsymbol{y} \quad \exists \lambda \neq 0 \,.$$

Equivalence classes are given by $[\boldsymbol{x}] = \{\lambda \boldsymbol{x} \mid \lambda \neq 0\}$. Note that $[\boldsymbol{x}]$ is the line in $\mathbb{R}^{n+1}$ that passes through the origin, and the point $\boldsymbol{x}$. The quotient is called *real projective space*

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \smallsetminus \{0\}) \,/\!\sim .$$

We can also describe $\mathbb{RP}^n$ with the help of the n-sphere S^n.

Proposition 15.1.5. *We have $\mathbb{RP}^n \cong S^n/\!\approx$, where $\boldsymbol{x} \approx \boldsymbol{y} \overset{\text{def}}{\Leftrightarrow} \boldsymbol{x} = \pm \boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in S^n$.*

PROOF. We consider a map

$$\varphi \colon\ S^n/\!\approx \longrightarrow \mathbb{RP}^n$$

which is defined so that the following diagram commutes

$$\begin{tikzcd} S^n \arrow[d, "p"] \arrow[dr, "f"] & \\ S^n/\!\approx \arrow[r, "\varphi"] & \mathbb{RP}^n \end{tikzcd} \tag{15.1.1}$$

where $f(\boldsymbol{x}) := [\boldsymbol{x}]$, and where $[\boldsymbol{x}]$ means the equivalence class of $\boldsymbol{x}$ with respect to the relation $\sim$ which defines $\mathbb{RP}^n$. Namely, each equivalence class in $S^n/\!\approx$ is of the form $\{\boldsymbol{x}, -\boldsymbol{x} \mid \boldsymbol{x} \in S^n\}$, and so we define $\varphi(\{\boldsymbol{x}, -\boldsymbol{x}\}) := [\boldsymbol{x}]$.

Another more geometrical way of thinking about it is that f sends a point on the sphere S^n to the line passing through the origin and that point. It is clear that f is surjective because every line through the origin in $\mathbb{R}^{n+1}$ intersects S^n exactly in two points $\boldsymbol{x}$ and $-\boldsymbol{x}$. Similar to the proof of Proposition 15.1.2, we show that f is continuous by looking at the following diagram

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^{n+1} \smallsetminus \{0\} \arrow[d, "q"] \\ & \mathbb{RP}^n \end{tikzcd} .$$

This diagram commutes, q is the quotient projection and i is the inclusion. Therefore $f = q \circ i$ is continuous. Hence we have that the induced map φ in (15.1.1) is continuous and bijective. Lastly we will show that φ is a homeomorphism by constructing an inverse. Namely, define

$$\begin{aligned} \psi \colon\ \mathbb{RP}^n &\longrightarrow S^n/\!\approx \\ [\boldsymbol{x}] &\longmapsto \left\{ \pm \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right\} . \end{aligned}$$

It is clearly inverse to φ since

$$(\psi \circ \varphi)\,(\{\pm \boldsymbol{x}\}) = \psi([\boldsymbol{x}]) = \{\pm \boldsymbol{x}\}$$

the last equality, because $|\boldsymbol{x}| = 1$ by assumption, since $\boldsymbol{x} \in S^n$. Similar to how we showed that φ is continuous, we have the following commutative diagram

$$\begin{tikzcd} \mathbb{R}^{n+1} \smallsetminus \{0\} \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dr, "g"] & S^n \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[r, "\psi"] & S^n/\!\approx . \end{tikzcd}$$

The maps p and q are the quotient projections. The map π is defined by $\boldsymbol{x} \longmapsto \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$, and g is defined by $\boldsymbol{x} \longmapsto \left\{\pm\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right\}$. The map π is continuous. By commutativity of the diagram we have that g and consequently ψ are continuous. Since ψ is a continuous inverse to φ, it shows that φ is a homeomorphism. $\square$

Corollary 15.1.6. $\mathbb{R}\mathrm{P}^n$ *is compact.*

Proof. This follows immediately from Proposition 14.1.13 and the previous proposition because S^n is compact. $\square$

In general, the Hausdorff property does not behave well with taking quotients, so we will have to prove the following proposition by hand.

Proposition 15.1.7. $\mathbb{R}\mathrm{P}^n$ *with the quotient topology is Hausdorff.*

Proof. Pick two distinct equivalence classes $[\boldsymbol{x}]$ and $[\boldsymbol{y}]$. That is $\forall\lambda \neq 0$ we have $\boldsymbol{x} \neq \lambda\boldsymbol{y}$. Let

$$f\colon\, S^n \longrightarrow \mathbb{R}\mathrm{P}^n\,,$$

be defined as $f = \varphi \circ p$ as in the proof of Proposition 15.1.5. Then we have $f^{-1}([\boldsymbol{x}]) = \{\pm\boldsymbol{x}\}$ and $f^{-1}([\boldsymbol{y}]) = \{\pm\boldsymbol{y}\}$. Since $\{\pm\boldsymbol{x}\}\cap\{\pm\boldsymbol{y}\} = \varnothing$, we can pick disjoint neighborhoods

$$U_x = U \cup (-U), \quad V_y = V \cup (-V)$$

of $\{\pm\boldsymbol{x}\}$ and $\{\pm\boldsymbol{y}\}$ respectively. Here we have

$$-U = \{-\boldsymbol{x} \mid \boldsymbol{x} \in U \subset S^n\}\ .$$

This is possible because S^n is Hausdorff (it is equipped with the subspace topology in $\mathbb{R}^{n+1}$).

We then claim that $f(U_x)$ and $f(V_y)$ are disjoint neighborhoods of $[\boldsymbol{x}]$ and $[\boldsymbol{y}]$. By part (1) of Theorem 8.1.13, and by the diagram (15.1.1) we have that $f\colon\, S^n \longrightarrow \mathbb{R}\mathrm{P}^n$ is a quotient map. We can then compute

$$f^{-1}(f(U_x)) = f^{-1}(\{[\boldsymbol{x}] \mid \boldsymbol{x} \in U_x\}) = U_x \cup (-U_x) = U_x \overset{\text{open}}{\subset} S^n$$
$$f^{-1}(f(V_y)) = f^{-1}(\{[\boldsymbol{y}] \mid \boldsymbol{y} \in V_y\}) = V_y \cup (-V_y) = V_y \overset{\text{open}}{\subset} S^n$$

which means by definition of a quotient map that $f(U_x), f(V_y) \overset{\text{open}}{\subset} \mathbb{R}\mathrm{P}^n$. To show that these neighborhoods are disjoint, assume that they are not and pick some $[\boldsymbol{z}] \in f(U_x) \cap f(V_y)$. Then we have

$$f^{-1}([\boldsymbol{z}]) \in f^{-1}\left(f(U_x) \cap f(V_y)\right) = f^{-1}\left(f(U_x)\right) \cap f^{-1}\left(f(V_y)\right) = U_x \cap V_y\,.$$

But by assumption $U_x \cap V_y = \varnothing$ which means $f^{-1}([\boldsymbol{z}]) = \varnothing$. But this contradicts the surjectivity of f. Hence such $[\boldsymbol{z}] \in \mathbb{R}\mathrm{P}^n$ can not exist and we have $f(U_x)\cap f(V_y) = \varnothing$. $\square$

Exercise 15.1.8. Show that $\mathbb{R}\mathrm{P}^1 \cong S^1$.

(*Hint: Try to explicitly construct a homeomorphism to* $\mathbb{R}\mathrm{P}^1$ *from the upper half circle* $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ y \geq 0\}$. *How does this fail? What can be done to "fix" it?*)

Exercises

15.1. Show that $\mathbb{R}\mathrm{P}^1 \cong S^1$.

(*Hint: Try to explicitly construct a homeomorphism to* $\mathbb{R}\mathrm{P}^1$ *from the upper half circle* $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ y \geq 0\}$. *How does this fail? What can be done to "fix" it?*)

Lecture 16

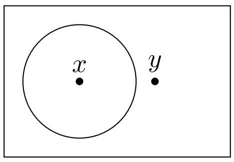
16.1. More about countability and separation axioms

- Munkres: §30, §31, §32

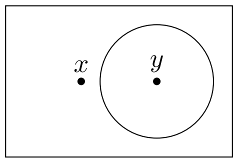
We will now discuss more about countability and separation axioms. We therefore start with a recap of what we have already discussed in this course previously.

Separation axioms: Let $(X, \mathcal{T})$ be a topological space. Let $x, y \in X$ be two distinct points. Then X is called

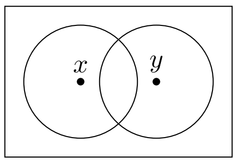
- T_0 if there is a neighborhood U_x of x or a neighborhood U_y of y not containing the other point.



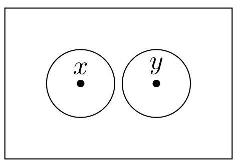
or



- T_1 if there are neighborhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$



- Hausdorff if there are disjoint neighborhoods U_x of x and U_y of y.



First countable: Let $(X, \mathcal{T})$ be a topological space, let $x \in X$ be any point. We say that X is first countable if there is a countable basis at x.

16.1.1. Second countability.

Recall that a countable basis at x is a countable collection $\{B_i\}_{i=1}^{\infty}$ of neighborhoods of x such that if $U \overset{\text{open}}{\subset} X$ is any neighborhood of x, then there is some i such that $B_i \subset U$.

We begin by expanding the notion of first countable to a strictly stronger one called *second countability.*

Definition 16.1.1 (Second countable)**.** *Let $(X, \mathcal{T})$ be a topological space. We say that X is* second countable *if it has a countable basis.*

Proposition 16.1.2. *If X is second countable, then it is first countable.*

PROOF. This follows immediately from the definition. If $\mathcal{B}$ is a countable basis of X, then $\{B \mid B \in \mathcal{B},\ x \in B\}$ is a countable basis at x for any $x \in X$. □

Example 16.1.3.
(1) $\mathbb{R}$ is second countable. Recall that a basis for the standard topology on $\mathbb{R}$ is $\{(a,b) \mid a < b\}$. However, this basis is of course not countable. We can make it countable by only allowing a and b to rational.
$$\mathcal{B} = \{(a,b) \mid a < b,\ a,b \in \mathbb{Q}\}\ ,$$
is a countable basis. It is easy to go through the definition of a basis to check that this indeed defines a basis. It is an exercise to show that the topology generated by $\mathcal{B}$ is the standard topology. (Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$, so for any $r \in \mathbb{R} = \overline{\mathbb{Q}}$ there exists a sequence $\{q_n\}_{n=1}^{\infty})$ such that $q_n \to r$.)
(2) $\mathbb{R}^n$ is second countable. A countable basis consists of balls centered at points with only rational components and rational radius
$$\mathcal{B} = \{B(q,r) \mid q \in \mathbb{Q}^n,\ r \in \mathbb{Q}_{>0}\}\ .$$
Again, to prove that $\mathcal{B}$ defines the standard topology on $\mathbb{R}^n$ it is important that $\mathbb{Q} \subset \mathbb{R}$ is dense.
(3) Let X be any set equipped with the discrete topology. If X is countable, then it is second countable. A countable basis is given by $\{\{x\} \mid x \in X\}$.
(4) Let X be any set equipped with the discrete topology. If X is uncountable, then X is *not* second countable. Any basis at $x \in X$ needs to contain the open set $\{x\} \overset{\text{open}}{\subset} X$. Hence any basis needs to contain the collection $\{\{x\} \mid x \in X\}$, but since X is uncountable, this collection is uncountable and hence the basis can not be countable.

Remark 16.1.4. Note that example (4) above shows that metric spaces are *not* in general second countable! The discrete topology on X is the metric topology with respect to the discrete metric
$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}\ .$$

Proposition 16.1.5. *Suppose X is a second countable topological space. Then*
(1) Any open cover of X has a countable subcover. (In this case we say that X is Lindelöf.*)*
(2) There is a countable dense subset of X. (In this case we say that X is separable.*)*

PROOF. (1) Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis of X and let $\{U_i\}_{i\in I}$ be an open cover of X. From this countable basis we will construct a countable subcover of $\{U_i\}_{i\in I}$. For every $n \in \mathbb{Z}_+$, define
$$V_n := \begin{cases} U_{i_n} & \text{for some } U_{i_n} \text{ such that } B_n \subset U_{i_n} \\ \varnothing & \text{otherwise} \end{cases}\ .$$
Then $X = \bigcup_{n=1}^{\infty} V_n$, since for any $x \in X$ there is some $n \in \mathbb{Z}_+$ such that $x \in B_n \subset U_{i_n}$.
(2) Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis of X. For each $n \in \mathbb{Z}_+$, pick any point $x_n \in B_n$. Then we claim that $\{x_n\}_{n=1}^{\infty}$ is dense. Namely, pick any point $x \in X$. Then let $U_x \subset X$ be any neighborhood of x. By definition of a basis we have that there is some $m \in \mathbb{Z}_+$ such that $B_m \subset U$. Hence $x_m \in B_m \subset U$ by definition of x_m. Therefore $U \cap \{x_n\}_{n=1}^{\infty} \neq \varnothing$, and so any neighborhood of any point in X intersects $\{x_n\}_{n=1}^{\infty}$ which means that $X = \overline{\{x_n\}_{n=1}^{\infty}}$.
□

Example 16.1.6. Consider $\mathbb{R}$ equipped with the lower limit topology which is generated by the basis

$$\{[a, b) \mid a < b\} .$$

This topology is first countable since for any $x \in \mathbb{R}$, we have that $\{[x, x + \frac{1}{n}) \mid n \in \mathbb{Z}_+\}$ is a countable basis. However, it is *not* second countable.

To prove this, let $\{B_i\}_{i\in I}$ be any basis. Then let $x \in X$. Then by definition of being a basis, we have that for any neighborhood $U_x \subset \mathbb{R}$ of x, there is some $i \in I$ such that $B_i \subset U_x$. This forces the existence of a basis elements of the form $[x, x + \varepsilon_x)$ for some $\varepsilon_x > 0$. Therefore

$$\{[x, x + \varepsilon_x) \mid \varepsilon_x > 0,\ x \in \mathbb{R}\} \subset \{B_i\}_{i\in I} ,$$

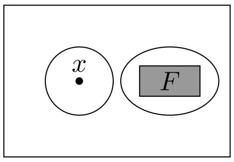
which forces $\{B_i\}_{i\in I}$ to be uncountable.

Proposition 16.1.7. *Let $f\colon X \longrightarrow Y$ be open and continuous. Let X be second countable. Then $f(X) \subset Y$ is second countable.*

Proof. Exercise. □

16.1.2. Regular and normal spaces. The T_3 and T_4 axioms. We have seen that in a Hausdorff space, we can separate points in the sense that we can find disjoint neighborhoods of the points. We will now state some seemingly stronger conditions.

Definition 16.1.8 (T_3)**.** *A topological space X is said to be T_3 if for any closed subset $F \subset X$ and any point $x \in X \setminus F$ there are open subsets $U_x, V_F \subset X$ such that $U_x \cap V_F = \varnothing$, $x \in U_x$ and $F \subset V_F$.*



Definition 16.1.9 (Regular)**.** *A topological space is called* regular *if it is T_1 and T_3.*

Remark 16.1.10.
(1) Warning! Our definition of regular aligns with the one given in Munkres book. Some authors choose to switch the definition of regular and T_3, so make sure you double check the definition of regular and T_3 whenever you encounter these concepts.
(2) It is clear from the definition that a regular topological space is Hausdorff (and hence T_1 and T_0), since points are closed in a T_1 space.
(3) However, the T_3 axiom alone does *not* imply Hausdorff, see Example 16.1.11 below!

Example 16.1.11. Consider the set of integers $\mathbb{Z}$. Equip it with the topology generated by the basis

$$\mathcal{B} = \{\{2k, 2k + 1\} \mid k \in \mathbb{Z}\} .$$

It is called the odd-even topology on $\mathbb{Z}$. Then any neighborhood of 0 contains the set $\{0, 1\}$, which means that any neighborhood of 0 needs to contain the point 1 as well. Therefore this topology is not T_0, hence it is not T_1 nor Hausdorff.

However, the odd-even topology is T_3. To see this, we note that any open set is of the form $U = \bigcup_{i\in I} \{2i, 2i + 1\}$ where $I \subset \mathbb{Z}$, and so

$$\mathbb{Z} \setminus U = \bigcup_{i\in\mathbb{Z}\setminus I} \{2i, 2i + 1\} ,$$

is also open. Hence U is closed. Because any set is open if and only if it is closed, we have that if $F \overset{\text{closed}}{\subset} \mathbb{Z}$, then

$$F = \bigcup_{j \in J} \{2j, 2j+1\} ,$$

where $J \subset \mathbb{Z}$. Then for any $x \in \mathbb{Z} \setminus F$, we can write down a separation

$$U_x = \mathbb{Z} \setminus F, \quad V_F = F .$$

Then clearly $U_x \cap V_F = \varnothing$ and $U_x, V_F \overset{\text{open}}{\subset} \mathbb{Z}$ are neighborhoods of $x \in \mathbb{Z}$ and $F \subset \mathbb{Z}$ respectively.

We will now see an example of a topological space which is Hausdorff but not regular.

Example 16.1.12. Consider $\mathbb{R}$ and let $K := \{\frac{1}{n}\}_{n=1}^{\infty}$. Equip $\mathbb{R}$ with the topology generated by the following basis

$$\mathcal{B} = \{(a,b) \mid a < b\} \cup \{(a,b) \setminus K \mid a < b\} .$$

It is clear that this topology is Hausdorff, because any two points $x, y \in \mathbb{R}$ can be separated by small disjoint intervals. We now prove that it is not regular. We choose $F = K$. It is closed since $\mathbb{R} \setminus K = \bigcup_{n=1}^{\infty} ((-n,n) \setminus K)$ is open.

We can then pick $x = 0 \in \mathbb{R} \setminus K$. Assume there are disjoint neighborhoods U_x and V_F of x and F respectively. Then note that any neighborhood of x needs to be of the form

$$U_x = \bigcup_{i \in I} (a_i, b_i) \setminus K ,$$

since if it was a union of intervals, then it would always intersect K and they would not be disjoint. We pick $n \in \mathbb{Z}_+$ large enough so that $\frac{1}{n} \in \bigcup_{i \in I} (a_i, b_i)$. Then we have $\frac{1}{n} \in (a_i, b_i)$ for some $i \in I$. By definition we also need to have

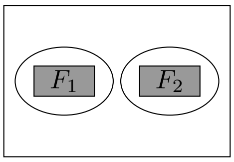
$$V_F = \bigcup_{j \in J} (c_j, d_j) ,$$

which means that $\frac{1}{n} \in (c_j, d_j)$ for some $j \in J$. Then we can pick any

$$\max\left(a_j, c_j, \frac{1}{n+1}\right) < z < \frac{1}{n} ,$$

and we have $z \in (c_j, d_j) \cap (a_i, b_i)$ which means $U_x \cap V_F \neq \varnothing$ which is a contradiction. Hence this topology is not regular.

Definition 16.1.13 (T_4)**.** *A topological space X is said to be T_4 if for any disjoint closed subsets $F_1, F_2 \subset X$ there are open subsets $U_1, U_2 \subset X$ such that $U_1 \cap U_2 = \varnothing$, $F_1 \subset U_1$ and $F_2 \subset U_2$.*



Definition 16.1.14 (Normal)**.** *A topological space is called* normal *if it is T_1 and T_4.*

Remark 16.1.15. We have the following chain of inclusions

$$\text{Normal} \implies \text{Regular} \implies \text{Hausdorff} \implies T_1 \implies T_0$$

Similar to the case with regularity, T_4 does not imply T_3, and it does not imply any other separation axiom either.

Example 16.1.16. Similar to the previous case. The integers $\mathbb{Z}$ equipped with the odd-even topology is T_4. It is an exercise to carry this out.

Example 16.1.17. Consider $X = (0,1)$ and equip it with the following topology:

$$\mathcal{T} = \{\varnothing, X\} \cup \left\{ \left(0, 1 - \frac{1}{n}\right) \,\middle|\, n \in \mathbb{Z}_+ \right\} .$$

Then any non-trivial open set is of the form $U_n = (0, 1 - \frac{1}{n})$. This topology is not T_0, since every non-empty open set contains both the points $\frac{1}{8}$ and $\frac{1}{4}$ for instance. Hence it it not T_0, T_1 nor Hausdorff. Since any non-trivial open set is of the form $U_n = (0, 1 - \frac{1}{n})$ it means that any non-trivial closed set is of the form

$$F_n = \left[1 - \frac{1}{n}, 1\right) .$$

We have that $(X, \mathcal{T})$ is not T_3, because if we pick $F = [\frac{1}{2}, 1)$, then the only open set containing F is $(0,1)$. Therefore if we pick $x = \frac{1}{4}$ for example, then it is impossible to find a neighborhood of x disjoint from $(0,1)$.

Now, $(X, \mathcal{T})$ is T_4, because the definition of being T_4 is vacuously true! There are no disjoint closed subsets, since $F_1 \subset F_2 \subset \cdots$.

Proposition 16.1.18. *Any metric space is normal.*

Proof. Let X be a metric space. Let $A, B \overset{\text{closed}}{\subset} X$ be disjoint. Then for every $a \in A$, choose $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \cap B = \varnothing$ (this is possible, because otherwise we would have $a \in \overline{B} = B$ which is impossible by our assumption that $A \cap B = \varnothing$). Similarly, for every $b \in B$, choose $\varepsilon_b > 0$ such that $B(b, \varepsilon_b) \cap A = \varnothing$. Then define

$$U_A := \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right), \quad U_B := \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{2}\right) .$$

Then U_A and U_B are neighborhoods of A and B respectively. We need to show that they are disjoint. If we assume that the are not disjoint, then we let $z \in U_A \cap U_B$. Then $\exists a \in A$ and $\exists b \in B$ such that

$$z \in B\left(a, \frac{\varepsilon_a}{2}\right) \cap B\left(b, \frac{\varepsilon_b}{2}\right) .$$

Then the triangle inequality gives

$$d(a,b) \le d(a,z) + d(z,b) < \frac{\varepsilon_a + \varepsilon_b}{2} \le \max(\varepsilon_a, \varepsilon_b) .$$

We now have two cases. Either $\max(\varepsilon_a, \varepsilon_b) = \varepsilon_a$ or ε_b. Depending on the case we would get either $a \in B(b, \varepsilon_b)$ or $b \in B(a, \varepsilon_a)$ neither of which is possible. $\square$

Proposition 16.1.19. *Every compact and Hausdorff space is normal.*

Proof. Let X be compact and Hausdorff. If $x \in X$ and $A \overset{\text{closed}}{\subset} X$, then we have by Theorem 11.1.4 that $A \overset{\text{compact}}{\subset} X$. Then by the proof of Theorem 11.1.5, we constructed two disjoint open subsets $U, V \overset{\text{open}}{\subset} X$ such that $x \in U$ and $A \subset V$. This already shows that X is regular.

We can generalize this proof slightly. Namely, let $A, B \overset{\text{closed}}{\subset} X$. Again by Theorem 11.1.4 we have $A \overset{\text{compact}}{\subset} X$ and $B \overset{\text{compact}}{\subset} X$. Then for each $a \in A$, choose disjoint open neighborhoods U_a and V_a of a and B respectively. The collection $\{U_a \cap A\}_{a \in A}$ clearly covers A. Since A is compact, we pick a finite subcover $\{U_{a_i} \cap A\}_{i=1}^N$. Then define

$$U := \bigcup_{i=1}^N U_{a_i}, \quad V := \bigcap_{i=1}^N V_{a_i} .$$

Then $U, V \overset{\text{open}}{\subset} X$ are neighborhoods of A and B respectively such that $U \cap V = \varnothing$. $\square$

Proposition 16.1.20. *Every regular and second countable space is normal.*

PROOF. See Munkres §32 for the proof. $\square$

Exercises

16.1. Show that $\mathbb{R}$ is second countable by showing that $\mathcal{B} = \{(a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$ is a basis that generates the standard topology on $\mathbb{R}$.

(*Hint: $\mathbb{Q}$ is dense in $\mathbb{R}$, so for any $r \in \mathbb{R} = \overline{\mathbb{Q}}$ there exists a sequence $\{q_n\}_{n=1}^{\infty}$ such that $q_n \to r$.*)

16.2. Let $f\colon X \longrightarrow Y$ be a continuous and open map, and let X be second countable. Show that $f(X) \subset Y$ is second countable.

16.3. Consider the integers $\mathbb{Z}$ equipped with the odd-even topology $\mathcal{T}$ which is generated by $\{\{2k, 2k+1\} \mid k \in \mathbb{Z}\}$. Show that $(\mathbb{Z}, \mathcal{T})$ is T_4.

(*Hint: Mimic Example 1.11, where we show that $(\mathbb{Z}, \mathcal{T})$ is T_3.*)

Lecture 17

17.1. Metrizability

- Munkres: §34, §40

We have already seen metrizability back in lecture 2. We recall the definition.

Definition 17.1.1 (Metrizable). *A topological space $(X, \mathfrak{T})$ is called* metrizable *if there is a metric d defined on X such that the metric topology with respect to d is $\mathfrak{T}$.*

Now that we have discussed regularity, second countability and normality we can actually state a theorem which provides us with a sufficient condition for when a topology is metrizable.

Before we can state and prove the *Urysohn metrization theorem*, we first have a deep theorem which we will not prove.

THEOREM 17.1.2 (Urysohn lemma). *Let X be a normal topological space, and let $A, B \overset{\text{closed}}{\subset} X$ be disjoint closed subsets of X. Consider the closed interval $[a, b]$. Then there exists a continuous map $f\colon X \longrightarrow [a, b]$ such that $f(x) = a \ \forall x \in A$ and $f(x) = b \ \forall x \in B$.*

Another lemma we need is the following.

Lemma 17.1.3. *Let X be regular and second countable. There exists a collection of continuous functions $\{f_n\colon X \longrightarrow [0, 1]\}_{n \in \mathbb{Z}_+}$ having the property that given any point $x_0 \in X$ and any neighborhood $U \subset X$ of x_0, there is some $m \in \mathbb{Z}_+$ such that $f_m(x_0) > 0$ and $f_m|_{X \smallsetminus U} = 0$.*

PROOF. Fix $x_0 \in X$. First we recall from Proposition 16.1.20 that X regular and second countable implies that X is normal. Then since $\{x_0\} \overset{\text{closed}}{\subset} X$ and $X \smallsetminus U \overset{\text{closed}}{\subset} X$, it follows from the Urysohn lemma that we can find a continuous function f_{x_0} with the desired property. However, if we choose one such function for each $x_0 \in X$ and each neighborhood $U \subset X$ we get a collection which in general is not countable.

We now show that the above collection can be made into a countable one. To that end, let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for X. Then $\forall n, m \in \mathbb{Z}_+$ such that $\overline{B_n} \subset B_m$, apply the Urysohn lemma to get a continuous function

$$g_{n,m}\colon X \longrightarrow [0, 1]\,,$$

such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X \smallsetminus B_m) = \{0\}$. Then we claim that $\{g_{n,m}\}_{n,m=1}^{\infty}$ satisfies our requirements: Given $x_0 \in X$ and any neighborhood $U \subset X$ of x_0, we choose a basis element B_m containing x_0 such that $B_m \subset U$. Using regularity we can choose a neighborhood V of x such that $V \cap (X \smallsetminus B_m) = \varnothing$, which means $V \subset B_m$. Further we pick a basis element $n \in \mathbb{Z}_+$ such that $x_0 \in \overline{B_n} \subset B_m$. To see that this is possible, use Lemma 31.1 in Munkres to obtain a neighborhood V of x_0 such that $x_0 \in \overline{V} \subset B_m$. Then we may pick some $n \in \mathbb{Z}_+$ such that $x_0 \in B_n \subset V$. Taking closures gives us $x_0 \in \overline{B_n} \subset \overline{V} \subset B_m$ as desired. Then $g_{n,m}(x_0) = 1$ and $g_{n,m}|_{X \smallsetminus U} = 0$. $\square$

We now consider the topological space consisting of infinite sequences. Define the set

$$\mathbb{R}^{\infty} := \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \ \forall i \in \mathbb{Z}_+\}\ .$$

One way to give $\mathbb{R}^\infty$ a topology is by realizing that we can write this set as $\mathbb{R}^\infty = \prod_{i=0}^\infty \mathbb{R}$, and then we equip $\mathbb{R}^\infty$ with the product topology. Because of this, $\mathbb{R}^\infty$ comes equipped with continuous projections

$$\begin{aligned} \pi_i \colon \mathbb{R}^\infty &\longrightarrow \mathbb{R} \\ \boldsymbol{x} &\longmapsto x_i \end{aligned}$$

We will then use that $\mathbb{R}^\infty$ equipped with the product topology is in fact metrizable.

Lemma 17.1.4. *Let $\bar{d}(a,b) = \min(|a-b|, 1)$ be the standard bounded metric on $\mathbb{R}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^\infty$, define*

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sup_{i \in \mathbb{Z}_+} \frac{\bar{d}(x_i, y_i)}{i} .$$

Then D is a metric that induces the product topology on $\mathbb{R}^\infty$.

Proof. See Munkres, Theorem 20.5. □

We now state one of the main theorems for this lecture.

Theorem 17.1.5 (Urysohn metrization theorem). *If X is regular and second countable, then X is metrizable.*

Proof. We will prove that X is homeomorphic to a subspace of the metric space $\mathbb{R}^\infty$, which means that X is metrizable (because the subspace topology in a metric space is equal to the metric topology of the restricted metric to that subspace).

From Lemma 17.1.3, we get a countable collection of functions $\{f_n\}_{n=1}^\infty$ such that for every $x_0 \in X$ and any neighborhood $U \subset X$ of x_0 there is some $m \in \mathbb{Z}_+$ such that $f_m(x_0) = 1$ and $f_m|_{X \setminus U} = 0$. Define a function

$$\begin{aligned} F \colon X &\longrightarrow \mathbb{R}^\infty \\ x &\longmapsto (f_1(x), f_2(x), \ldots) . \end{aligned}$$

We will prove that F is a homeomorphism onto its image $F(X)$.

F is continuous: This follows from Munkres, Theorem 19.6, since each f_i is continuous.

F is injective: Given any two distinct points $x, y \in X$ we pick any neighborhood $U \subset X$ of x such that $y \in X \setminus U$. Then by construction there is some $m \in \mathbb{Z}_+$ such that $f_m(x) = 1$ and $f_m(y) = 0$. Therefore $F(x)$ and $F(y)$ differ at index m, so $F(x) \neq F(y)$ and hence F is injective.

F is open: Let $U \overset{\text{open}}{\subset} X$, and let $z_0 \in F(U) \subset F(X)$. We will show that there is some $W \overset{\text{open}}{\subset} F(X)$ such that $z_0 \in W \subset F(U)$, which will then show that $F(U) \overset{\text{open}}{\subset} F(X)$. Let $x_0 \in X$ be such that $F(x_0) = z_0$. Then choose some index $N \in \mathbb{Z}_+$ such that $f_N(x_0) = 1$ and $f_N(F \setminus U) = \{0\}$. Take $(0, \infty) \subset \mathbb{R}$, and let

$$V := \pi_N^{-1}((0,\infty)) \overset{\text{open}}{\subset} \mathbb{R}^\infty .$$

Let $W := V \cap F(X) \overset{\text{open}}{\subset} F(X)$. Then we claim that $W \subset F(U)$. But this is clear since if $z \in W$ then $\pi_N(z) \in (0,\infty)$. Furthermore we have $z = F(x)$ for some $x \in X$, and

$$\pi_N(z) = \pi_N(F(x)) = f_N(x) ,$$

Since f_N vanishes in $X \setminus U$ we must have $x \in U$ since we have $f_N(x) > 0$. Therefore $z \in F(U)$ which means $W \subset F(U)$ and we are done.

□

In fact, there is a stronger theorem, which gives both a necessary and sufficient condition for metrizability, called the *Nagata–Smirnov metrization theorem*. We will only state the theorem (see Theorem 17.1.9) without proof.

Definition 17.1.6. *Let X be a topological space and let $\mathcal{U}$ be any collection of subsets of X. We then say that $\mathcal{U}$ is* locally finite *if for every $x \in X$, there is a neighborhood $U \subset X$ of x that intersects only finitely many elements of $\mathcal{U}$.*

Similarly, we say that $\mathcal{U}$ is countably locally finite *if for every $x \in X$, there is a neighborhood $U \subset X$ of x that intersects only countably many elements of $\mathcal{U}$.*

Remark 17.1.7. Some other authors write *σ-locally finite* instead of *countably locally finite*.

Example 17.1.8.
(1) Trivially, any finite collection of subsets of X is locally finite, and any countable collection of subsets of X is countably locally finite.
(2) Consider $\mathbb{R}$ together with the collection $\mathcal{U} = \{(n, n+2) \mid n \in \mathbb{Z}\}$. Then $\mathcal{U}$ is locally finite. It is an exercise to sort out the details.
(3) The collection $\mathcal{V} = \{(0, \frac{1}{n}) \mid n \in \mathbb{Z}\}$ is not locally finite, since any neighborhood of $0 \in \mathbb{R}$ will contain infinitely many sets in $\mathcal{V}$.

THEOREM 17.1.9 (Nagata–Smirnov metrization theorem). *A topological space X is metrizable if and only if X is regular and has a basis that is countably locally finite.*

17.2. Baire category theorem

- Munkres: §48

We will now discuss a condition which might seem unnatural, but has applications in other fields such as analysis.

Definition 17.2.10 (Baire space)**.** *A topological space X is said to be a* Baire space *if the following condition holds: Given any countable collection $\{U_n\}_{n=1}^{\infty}$ of open dense subsets of X, then $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.*

Lemma 17.2.11. *Let X be a topological space. Then $B \subset X$ is dense if and only if* $\operatorname{int}(X \smallsetminus B) = \varnothing$

Corollary 17.2.12. *A topological space X is a Baire space if and only if the following condition holds: Given any countable collection $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X each of which has empty interior in X, then $\bigcup_{n=1}^{\infty} U_n$ also has empty interior in X.*

PROOF. Exercise. □

Remark 17.2.13. Munkres uses Corollary 17.2.12 as a definition of Baire space, but we choose to use the formulation involving density.

Example 17.2.14.
(1) The rational numbers $\mathbb{Q}$ is not Baire. Since $\mathbb{Q}$ is T_1 we have that each singleton $\{q\} \subset \mathbb{Q}$ is closed, and we also have $\operatorname{int}\{q\} = \varnothing$. But we have $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ which itself is open, hence $\operatorname{int}\mathbb{Q} = \mathbb{Q} \neq \varnothing$, so $\mathbb{Q}$ is not a Baire space.
(2) The integers $\mathbb{Z}$ is Baire. Any singleton in $\{z\} \subset \mathbb{Z}$ is open, since $\{z\} = (z-\varepsilon, z+\varepsilon) \cap \mathbb{Z}$ for $0 < \varepsilon < 1$. Therefore, the only set with empty interior in $\mathbb{Z}$ is the empty set. Therefore $\mathbb{Z}$ satisfies the Baire condition vacuously.

THEOREM 17.2.15 (Baire category theorem). *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Remark 17.2.16. More generally, locally compact Hausdorff spaces are Baire spaces.

PROOF. We only prove it for the case when X is compact and Hausdorff. For the case when X is a complete metric space is similar, see Munkres Theorem 48.2.

Suppose $\{A_n\}_{n=1}^{\infty}$ is a collection of closed subsets of X, each of which having empty interior. Then we want to show that $\operatorname{int}\left(\bigcup_{n=1}^{\infty} A_n\right) = \varnothing$. It suffices to prove that if we have a non-empty open set $U_0 \overset{\text{open}}{\subset} X$, then there is some point $x \in U_0$ such that $x \notin A_n$ $\forall n \in \mathbb{Z}_+$.

Consider first the set A_1. By assumption that $\operatorname{int} A_1 = \varnothing$ we have $U_0 \not\subset A_1$. Therefore we choose some $y \in U_0$ which does not belong to A_1. By Proposition 16.1.19, X is regular and A_1 is closed. Therefore we can choose a neighborhood U_1 of y such that

$$\overline{U_1} \cap A_1 = \varnothing$$
$$\overline{U_1} \subset U_0\,.$$

By induction, if we are given the non-empty open set U_{n-1}, we choose a point of U_{n-1} that is not in A_n. By regularity again we choose a neighborhood of this point such that

$$\overline{U_n} \cap A_n = \varnothing$$
$$\overline{U_n} \subset U_{n-1}\,.$$

Then we claim that $\bigcap_{n=1}^{\infty} \overline{U_n} \neq \varnothing$. From this the theorem follows.

To prove that this intersection is non-empty we need to use compactness of X. More precisely, we use Theorem 26.9 in Munkres which we have not discussed in the lectures. We use that our closed subsets $\overline{U_n}$ are nested

$$\overline{U_0} \supset \overline{U_1} \supset \cdots$$

and that X is compact. This will ensure that Theorem 26.9 from Munkres applies and hence $\bigcap_{n=1}^{\infty} \overline{U_n} \neq \varnothing$. $\square$

We will state one result which illustrates that the Baire property can be useful in analysis.

THEOREM 17.2.17. *Let X be a topological space and let (Y, d) be a metric space. Let $f_n\colon X \longrightarrow Y$ be a sequence of continuous functions such that $f_n(x) \longrightarrow f(x)$ for all $x \in X$, where $f\colon X \longrightarrow Y$.*

If X is a Baire space, the set of points at which f is continuous is dense in X.

Exercises

17.1. Consider the collection $\mathcal{U} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ of subsets of $\mathbb{R}$. Show that $\mathcal{U}$ is locally finite.

17.2. Prove that X is a Baire space if and only if the following condition holds: Given any countable collection $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X each of which has empty interior in X, then $\bigcup_{n=1}^{\infty} U_n$ also has empty interior in X.

Lecture 18

18.1. Homotopy of paths

- Munkres: §51

In this final lecture we will study some *algebraic topology*. Throughout this course we have studied different topological spaces and their properties. We have also studied the problem of deciding whether two topological spaces are homeomorphic. We have sometimes showed that two topological spaces are homeomorphic by constructing an explicit continuous function admitting a continuous inverse. We have also shown that two topological spaces (for example $(0,1)$ and $[0,1)$) are *not* homeomorphic by showing that they have different topological properties which are preserved under homeomorphisms. Among these properties are connectedness, path-connectedness and compactness.

Another way of studying topology is to assign some *algebraic object* associated to a topological space, which is *invariant under homeomorphisms*. This is precisely what the subject of algebraic topology is all about. To illustrate this in a diagram, suppose X is a topological space and suppose that $A(X)$ is an algebraic object (group, ring, field or some other object for example) associated to X such that

$$X \cong Y \implies A(X) \cong A(Y)\,.$$

Then, if we manage to show $A(X) \ncong A(Y)$ we can draw the conclusion that $X \ncong Y$. The philosophy and motivation behind such an assignment is that it is in some sense easier to decide whether two algebraic objects are non-isomorphic. Therefore we hope to find new ways to distinguish between non-homeomorphic topological spaces.

Before we talk about homotopies of paths, we make a quick recap of what we know about paths in topological spaces.

Recap.
- Let X be a topological space and $x, y \in X$. A *path* in X from x to y is a continuous map
$$\gamma\colon\ [0,1] \longrightarrow X\,,$$
such that $\gamma(0) = x$ and $\gamma(1) = y$.
- If γ is a path from x to y, then $\gamma^{-1}(t) := \gamma(1-t)$ is a path from y to x.
- If γ_1 and γ_2 are paths from x to y and y to z respectively, then
$$(\gamma_2\gamma_1)\,(t) := \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \leq x \leq 1 \end{cases}$$
is a path from x to z.

Definition 18.1.1 (Homotopy). *Let $f, g\colon\ X \longrightarrow Y$ be two continuous maps. A* homotopy *between f and g is a continuous map*

$$F\colon\ X \times [0,1] \longrightarrow Y\,,$$

such that

$$\begin{cases} F(x,0) = f(x) & \forall x \in X \\ F(x,1) = g(x) & \forall x \in X \end{cases}.$$

If there exists a homotopy between f and g, we say that f and g are homotopic*, and we write $f \simeq g$.*

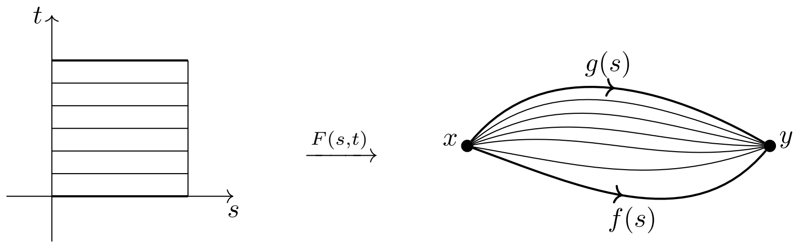
We are interested in the special case of homotopies when f and g are *paths* in a topological space. We will make a definition for this special case.

Definition 18.1.2. *Let $f, g\colon [0,1] \longrightarrow X$ be two paths from x to y. A* path homotopy *between f and g is a homotopy*

$$F\colon [0,1] \times [0,1] \longrightarrow X\,,$$

such that $F_t(s) := F(s,t)$ is a path from x to y for every $t \in [0,1]$.

If there exists a path homotopy F between two paths f and g we say that f and g are path homotopic*, and we write $f \simeq_p g$.*



Lemma 18.1.3. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. **Reflexivity:** It is clear that if $f\colon X \longrightarrow Y$ is a continuous function, then $F(x,t) := f(x)\ \forall t \in [0,1]$ is continuous and is a homotopy between f and itself.

Symmetry: Suppose $f \simeq g$ via a homotopy $F\colon X \times [0,1] \longrightarrow Y$. Then define $G(x,t) := F(x, 1-t)$. Then it is clear that G is continuous, and we have $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$, which shows $g \simeq f$.

Transitivity: Suppose $f \simeq g$ via the homotopy F and $g \simeq h$ via the homotopy G. Then define $H\colon X \times [0,1] \longrightarrow Y$ via

$$H(x,t) := \begin{cases} F(x,2t), & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then H is a homotopy from f to h. This is completely analogous to the proof that concatenation of paths is a path. This shows $f \simeq h$.

The proof that $\simeq_p$ is word-by-word the same as the proof above, except that we need to check that all involved homotopies are path homotopies.

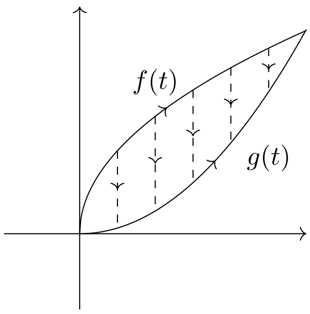
□

From now on, if $f\colon [0,1] \longrightarrow X$ is a path, we denote the path-homotopy equivalence class of f by $[f]$.

Example 18.1.4. (1) If $f, g\colon X \longrightarrow \mathbb{R}^2$ are any two continuous maps, then

$$F(x,t) := (1-t)f(x) + tg(x)\,,$$

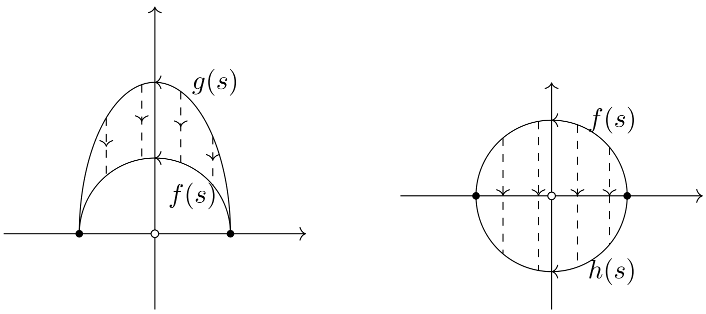
is a continuous map satisfying $F(x,0) = f(x)$ and $F(x,1) = g(x)$, hence it is a homotopy between f and g. It is called the *straight-line homotopy*, because it moves the point $f(x)$ to the point $g(x)$ along a straight-line segment joining them.



(2) We can similarly consider $\mathbb{R}^2 \smallsetminus \{0\}$. Now we consider two explicit paths

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$g(s) = (\cos(\pi s), 2\sin(\pi s)) .$$

Then $f \simeq g$ via the straight-line homotopy.



Similarly, we see that the straight-line homotopy can *not* take f to the path $h(s) = (\cos(\pi s), -\sin(\pi s))$, since at $t = \frac{1}{2}$, we have that $F(s, \frac{1}{2}) = \frac{1}{2}f(s) + \frac{1}{2}h(s) = (\cos(\pi s), 0)$ intersects the origin at $s = \frac{1}{2}$, which does not lie in $\mathbb{R}^2 \smallsetminus \{0\}$. In fact, it is impossible to find a path homotopy from $f(s)$ to $h(s)$ in $\mathbb{R}^2 \smallsetminus \{0\}$, so $f \not\simeq h$. This is intuitively clear, because we can not "jump over the hole at origin".

We will now define some operations on the set of equivalence classes of paths $[f]$.

Definition 18.1.5 (Product of equivalence classes)**.** *Let $f, g\colon [0,1] \longrightarrow X$ be paths such that $f(1) = g(0)$, so that the concatenation gf is defined. Then we define*

$$[f] * [g] := [gf] .$$

Remark 18.1.6. Note that our notation for concatenation of paths differ slightly compared to Munkres. We write gf as the concatenation "first follow f, then follow g", while Munkres uses the notation $f * g$ to denote the same thing.

Lemma 18.1.7. *The operation $*$ defined on path-homotopy equivalence classes is well-defined.*

Proof. Recall that *well-defined* means that we need to show if $f \simeq f'$ and $g \simeq g'$, then we have $[f] * [g] = [f'] * [g']$. Or equivalently $[gf] = [g'f'] \Leftrightarrow gf \simeq g'f'$. Namely, if F is a path-homotopy between f and f', and G is a path-homotopy between g and g', then we have that

$$H(s,t) = \begin{cases} F(2s, t), & 0 \le s \le \frac{1}{2} \\ G(2s-1, t), & \frac{1}{2} \le s \le 1 \end{cases},$$

is a path-homotopy between gf and $g'f'$. We leave this verification as an exercise. $\square$

In fact, the operation $*$ satisfies some more properties.

THEOREM 18.1.8. *The operation $*$ on path-homotopy equivalence classes has the following properties:*

(1) *(Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$ and they are equal.*
(2) *(Right and left identities) Given $x \in X$, let $e_x\colon [0,1] \longrightarrow X$ denote the constant path defined by $e_x(t) = x \ \forall t \in [0,1]$. If f is a path in X from x to y, then*

$$[f] * [e_y] = [f] \quad \text{and} \quad [e_x] * [f] = [f]\,.$$

(3) *(Inverse) Given that path f in X from x to y, then*

$$[f] * [f^{-1}] = [e_x] \quad \text{and} \quad [f^{-1}] * [f] = [e_y]\,,$$

where f^{-1} is the inverse path of f defined by $f^{-1}(t) := f(1-t)$.

PROOF. See Munkres Theorem 51.2 for a proof. □

18.2. The fundamental group

- Munkres: §52

We now have almost everything we need in order to define the *fundamental group.*

Definition 18.2.9 (Based loops)**.** *Let X be a topological space, and fix a point $x_0 \in X$. A path in X that begins and ends at x_0 is called a* loop based at *x_0.*

Definition 18.2.10 (The fundamental group)**.** *Let X be a topological space, and fix a point $x_0 \in X$. Denote the set of path-homotopy equivalence classes of loops based at x_0 by $\pi_1(X, x_0)$. If we equip $\pi_1(X, x_0)$ with the operation $*$, it is called the* fundamental group.

Note that since any loop based at x_0 is a path starting and ending at x_0, it means that any two loops based at x_0 are composable. So if $[f], [g] \in \pi_1(X, x_0)$, then $[f] * [g]$ is always defined. Therefore, from the properties of the operation $*$ in Theorem 18.1.8 we have the following:

Proposition 18.2.11. *The set $\pi_1(X, x_0)$ together with the operation $*$ is a group.*

Recall that a *group* is an algebraic object defined as a set G together with an operation $*$ satisfying the following axioms:

(1) (Binary operation) $g, h \in G \implies g * h \in G$
(2) (Associativity) $(g * h) * k = g * (h * k) \ \forall g, h, k \in G$
(3) (Identity) $\exists e \in G$ such that $g * e = e * g = g \ \forall g \in G$
(4) (Inverses) $\forall g \in G \ \exists g^{-1}$ such that $g * g^{-1} = g^{-1} * g = e$

Furthermore recall that a *group homomorphism* is a function $\varphi\colon (G, *) \longrightarrow (H, \star)$ such that

$$\varphi(g * h) = \varphi(g) \star \varphi(h) \quad \forall g, h \in G\,.$$

A group homomorphism is called a *group isomorphism* if it is bijective.

Example 18.2.12. Consider $\mathbb{R}^n$. Then we consider $\pi_1(\mathbb{R}^n, 0)$. It is the set of all path-homotopy equivalence classes of loops based at the origin. In $\mathbb{R}^n$, all paths (and therefore also all loops) are homotopic via the straight-line homotopy. Therefore we *should* have that $\pi_1(\mathbb{R}^n, 0) = \{[e_0]\}$ is the trivial group with only the identity element.

To prove this, it suffices to pick any loop $f\colon [0,1] \longrightarrow \mathbb{R}^n$ based at the origin, and show that f is homotopic to the constant loop at the origin. But we can write down such a homotopy explicitly:

$$\begin{aligned} F\colon [0,1] \times [0,1] &\longrightarrow \mathbb{R}^n \\ (s,t) &\longmapsto (1-t)f(s)\,. \end{aligned}$$

This map is clearly continuous, and we have $F(0,t) = (1-t)f(0) = 0$, $F(1,t) = (1-t)f(1) = 0$, $F(s,0) = f(s)$ and $F(s,1) = 0 = e_0(s)$. Therefore $f \simeq e_0$ so $[f] = [e_0]$ for any loop f, hence

$$\pi_1(\mathbb{R}^n, 0) = \{[f] \mid f \text{ loop in } \mathbb{R}^n \text{ based at } 0\} = \{[e_0]\} \,.$$

So far the definition of the fundamental group is seemingly dependent on the choice of basepoint x_0. However, we will show that it is independent of the choice of basepoint if the topological space is path-connected.

Definition 18.2.13. *Let X be a topological space, and let α be a path from x_0 to x_1. Define a map*

$$\begin{aligned} \hat{\alpha}\colon\ \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_1) \\ [f] &\longmapsto [\alpha^{-1}] * [f] * [\alpha] \,. \end{aligned}$$

THEOREM 18.2.14. *The map $\hat{\alpha}$ is a group isomorphism.*

PROOF. First we show that $\hat{\alpha}$ is a group homomorphism. Namely, we need to show that for any $[f], [g] \in \pi_1(X, x_0)$ that $\hat{\alpha}([f]) * \hat{\alpha}([g]) = \hat{\alpha}([f] * [g])$. This is straightforward by the properties of $*$:

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\alpha^{-1}] * [f] * [\alpha]) * ([\alpha^{-1}] * [g] * [\alpha]) = [\alpha^{-1}] * [f] * [g] * [\alpha] = \hat{\alpha}([f] * [g])$$

To then show that $\hat{\alpha}$ is an isomorphism, we construct an explicit inverse to $\hat{\alpha}$. Namely, define

$$\begin{aligned} \hat{\beta}\colon\ \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [\alpha] * [f] * [\alpha^{-1}] \,. \end{aligned}$$

Then it suffices to show $\hat{\alpha} \circ \hat{\beta} = \mathrm{id}_{\pi_1(X, x_1)}$. Namely, let $[h] \in \pi_1(X, x_1)$. Then we compute

$$\begin{aligned} \hat{\alpha}(\hat{\beta}([h])) &= [\alpha] * \hat{\alpha}([h]) * [\alpha^{-1}] = [\alpha] * ([\alpha^{-1}] * [h] * [\alpha]) * [\alpha^{-1}] \\ &= ([\alpha] * [\alpha^{-1}]) * [h] * ([\alpha] * [\alpha^{-1}]) = [h] \,, \end{aligned}$$

which means $\hat{\alpha} \circ \hat{\beta} = \mathrm{id}_{\pi_1(X, x_1)}$ and we are done. ☐

Corollary 18.2.15. *If X is path-connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

This corollary says that if X is path-connected, then it does not matter in what point we base our loops at.

Definition 18.2.16. *Let X be a path-connected topological space. We say that X is* simply connected *if $\pi_1(X, x_0)$ is the trivial group for some (and hence every) $x_0 \in X$.*

Next we wish to study how the fundamental group changes when we start deforming the topological space.

Suppose that $h\colon X \longrightarrow Y$ is a continuous map and we have fixed two points (the basepoints) $x_0 \in X$ and $y_0 \in Y$. If we require $h(x_0) = y_0$ we can abbreviate this by simply writing that

$$h\colon\ (X, x_0) \longrightarrow (Y, y_0) \,,$$

is a continuous function. We will now see that such a continuous function induces a group homomorphisms of the fundamental groups.

Definition 18.2.17. *Let $h\colon (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map. Then define*

$$\begin{aligned} h_*\colon\ \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [f] &\longmapsto [h \circ f] \,. \end{aligned}$$

The map h_* is called the homomorphism induced by h*, relative to the basepoint x_0.*

Remark 18.2.18. We will not show it, but h_* is well-defined in the sense that if $[f] = [f']$ (or equivalently $f \simeq f'$), then $[h \circ f] = [h \circ f']$ (or equivalently $h \circ f \simeq h \circ f'$).

Now we will see that the homomorphism on fundamental groups induced by continuous maps behave well with respect to composition and identity maps.

THEOREM 18.2.19. *(1) If $h\colon (X, x_0) \longrightarrow (Y, y_0)$ and $k\colon (Y, y_0) \longrightarrow (Z, z_0)$ are continuous maps, then $(k \circ h)_* = k_* \circ h_*$.*

(2) If $\mathrm{id}\colon (X, x_0) \longrightarrow (X, x_0)$ is the identity map, then id_* is the identity homomorphism on $\pi_1(X, x_0)$.

PROOF. It follows almost immediately from using Definition 18.2.17. It is an exercise to write it out. □

We now come to the result we have been building up towards.

THEOREM 18.2.20. *If $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism, then $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is a group isomorphism.*

PROOF. Let $f^{-1}\colon (Y, y_0) \longrightarrow (X, x_0)$ be the inverse of f. Then we simply check that $(f^{-1})_*$ is the inverse of f_*.

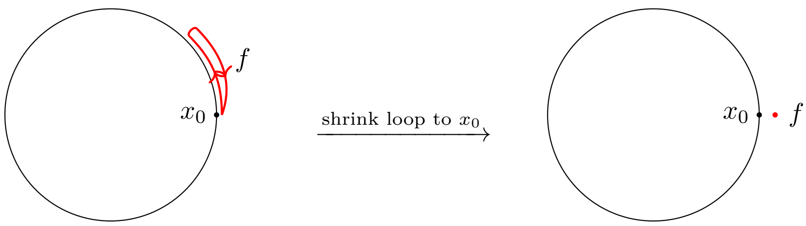
$$\begin{aligned} f_* \circ (f^{-1})_* &= (f \circ f^{-1})_* = \mathrm{id}_* \\ (f^{-1})_* \circ f_* &= (f^{-1} \circ f)_* = \mathrm{id}_* \,. \end{aligned}$$

□

This theorem gives us new ways to prove that topological spaces are non-homeomorphic! The idea is that if we compute the fundamental groups, and show that they are non-isomorphic it shows that the underlying topological spaces are non-homeomorphic.

This comes an extremely powerful tool later on, but to reach that point it is necessary to study the properties of the fundamental group even further. This is because right now the fundamental group might be hard to compute.

Example 18.2.21. (1) Consider $S^1 \subset \mathbb{R}^2$, and pick some basepoint $x_0 \in S^1$. Then we consider $\pi_1(X, x_0) =$ path-homotopy classes of loops based at x_0. It is not so far-fetched to believe that any loop that does not wrap around the circle a full turn is in fact path homotopic to the constant map at x_0.



Similarly, it is also not so far-fetched to believe that a loop that wraps exactly once around (going in the counterclockwise direction) is not path homotopic to the constant loop. The loop

$$f(s) = (\cos(2\pi s), \sin(2\pi s))\,,$$

is not homotopic to the constant loop id_{x_0}! A loop wrapping twice around is not homotopic to the loop wrapping only once around etc. A loop wrapping around once in the clockwise direction we count as being wrapped once around in the "negative direction". What we are describing is the fact that

$$\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)\,.$$

The integer tells us exactly how many turns the equivalence class of loops wrap around the circle, and there is exactly one such equivalence class for each integer. We do not have the tools to prove this formally, but we refer to §53 and §54 in Munkres for the interested reader.

(2) Consider $\mathbb{R}^2 \smallsetminus \{0\}$. In fact, this is very much similar to the circle. Intuitively we have a "hole" a the origin. Any loop wrapping once around are homotopic to all other loops wrapping exactly once around the hole. By the same reasoning as above, we have

$$\pi_1(\mathbb{R}^2 \smallsetminus \{0\}\,, x_0) \cong (\mathbb{Z}, +)\,.$$

It is also a fact that $S^1 \not\cong \mathbb{R}^2 \smallsetminus \{0\}$ because they have different connectivity properties. Removing any 2 points from S^1 makes it disconnected while removing any two points from $\mathbb{R}^2 \smallsetminus \{0\}$ will always keep it connected.

Note that even though they are non-homeomorphic they have isomorphic fundamental groups.

(3) We can now argue for why $\mathbb{R}^2 \not\cong \mathbb{R}^n$ when $n > 2$. We already argued for why $\mathbb{R} \not\cong \mathbb{R}^n$ when $n > 1$ by using connectivity arguments like we have done several times in this course.

But to prove that $\mathbb{R}^2 \not\cong \mathbb{R}^n$ when $n > 2$ we argue by simple connectivity! Or by using the fundamental group. If we remove any point from $\mathbb{R}^2$ we create a "hole". Of course, as we saw in the previous example, the formal way of expressing this is that

$$\pi_1(\mathbb{R}^2 \smallsetminus \{0\}\,, x_0) \cong (\mathbb{Z}, +)\,.$$

Now, if we remove any point from $\mathbb{R}^n$, we will not quite create a hole. Any loop in $\mathbb{R}^n \smallsetminus \{\boldsymbol{x}\}$ can be shrunk to a point, because if we meet the point we removed, we can simply use the other dimensions and "go around" the hole. Therefore

$$\pi_1(\mathbb{R}^n \smallsetminus \{\boldsymbol{x}\}\,, x_1) \cong \{\mathrm{id}_{x_0}\}\ ,$$

is the trivial group. Then we clearly have $\pi_1(\mathbb{R}^2 \smallsetminus \{0\}\,, x_0)) \not\cong \pi_1(\mathbb{R}^n \smallsetminus \{\boldsymbol{x}\}\,, x_1)$, which means $\mathbb{R}^2 \smallsetminus \{0\} \not\cong \mathbb{R}^n \smallsetminus \{\boldsymbol{x}\}$ and consequently $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n > 2$.

Exercises

18.1. Suppose $f, f', g, g' \colon [0,1] \longrightarrow X$ are paths in a topological space X such that the concatenations gf and $g'f'$ are defined. Furthermore assume $f \simeq f'$ and $g \simeq g'$ via the two path-homotopies

$$F\colon [0,1] \times [0,1] \longrightarrow X \quad \text{and} \quad G\colon [0,1] \times [0,1] \longrightarrow X$$

respectively. Show that $H\colon [0,1] \times [0,1] \longrightarrow X$ defined by

$$H(s,t) := \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2} \\ G(2s-1,t), & \frac{1}{2} \le s \le 1 \end{cases}\,,$$

is a path-homotopy between gf and $g'f'$.

18.2. (a) If $h\colon (X,x_0) \longrightarrow (Y,y_0)$ and $k\colon (Y,y_0) \longrightarrow (Z,z_0)$ are continuous maps, show that $(k \circ h)_* = k_* \circ h_*$.

(b) If $\mathrm{id}\colon (X,x_0) \longrightarrow (X,x_0)$ is the identity map, then show that id_* is the identity homomorphism on $\pi_1(X,x_0)$.

Lecture 19 (Problem Session 3)

Problem §26.2.

(a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact. (Recall that the finite complement topology is defined as $U \in \mathcal{T} \Leftrightarrow U = \varnothing$ or $\mathbb{R} \smallsetminus U$ finite.)

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} \smallsetminus A$ is either countable or all of $\mathbb{R}$, is $[0,1]$ a compact subspace?

Problem §26.7. Show that if Y is compact, then the projection $\pi_1 \colon X \times Y \longrightarrow X$ is a closed map.

Problem §26.8. Let $f \colon X \longrightarrow Y$; let Y be compact Hausdorff. Then f is continuous if and only if the graph of f,

$$G_f = \{(x, f(x)) \mid x \in X\} \ ,$$

is closed in $X \times Y$.

Problem §27.2. Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$. (By definition we have $d(x, A) \coloneqq \inf_{y \in A} d(x, y)$.)

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ε-neighborhood of A in X to be the set

$$U(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\} \ .$$

Show that $U(A, \varepsilon)$ equals the union of the open balls $B_d(a, \varepsilon)$ for $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

(e) Show that the result in (d) need not hold if A is closed but not compact.

Problem §27.6. Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Le tA_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} \smallsetminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \ .$$

The intersection $C = \bigcap_{n=1}^{\infty} A_n$ is called the Cantor set; it is a subspace of $[0,1]$.

(a) Show that C is totally disconnected.

(b) Show that C is compact.

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $\frac{1}{3^n}$; and show that the end points of these intervals lie in C.

(d) Show that C has no isolated points.

(e) Conclude that C is uncountable.

Problem §28.6. Let (X, d) be a metric space. If $f\colon X \longrightarrow X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)\,,$$

for all $x, y \in X$, then f is called an *isometry* of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism. [*Hint: If* $a \notin f(X)$, *choose* ε *so that the* ε*-neighborhood of* a *is disjoint from* $f(X)$. *Set* $x_1 = a$, *and* $x_{n+1} = f(x_n)$ *in general. Show that* $d(x_n, x_m) \geq \varepsilon$ *for* $n \neq m$.]

Problem §29.1. Show that the rationals $\mathbb{Q}$ are not locally compact.

Problem §29.3. Let X be a locally compact space. If $f\colon X \longrightarrow Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open? Justify your answer.

Problem §29.6. Show that the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.

Problem §29.8. Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.