Stochastic Processes Spring 2025

F1-F3

F1–F3 Background
What is a random process?
Foundations
Basic examples

F4 — Martingales
Brownian motion

Key concept and tool: Stochastic integral

Stochastic calculus

Generic class of "basic" stoch. processes:

Lévy processes

F9 — Markov processes
Diffusion — " —
Stoch. differential equations

FL-2

Elementary probability:

$$(\Omega, \mathcal{F}) \xrightarrow{X} \text{State space}$$

Ω: sample space

$\mathcal{F}$: σ-algebra of events

State space:
$\{0,1,2,\dots\}$
$[0,+\infty)$
$\mathbb{R}$
$\mathbb{R}^d$
; with Borel σ-algebra $\mathcal{B}$

A random variable X is a measurable function, that is,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \text{ all } B \in \mathcal{B},$$

and is not really a "random function":

put a measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ and let each event $\{X \in B\}$ have probability $\mathbb{P}(X \in B)$, $B \in \mathcal{B}$.

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow[X]{} (S, \mathcal{B})$$

The distribution of X is the (push-forward) measure $\mu_X = \mathbb{P} \circ X^{-1}$

s.t. $\mu_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\})$

$= \mathbb{P}(X \in B)$ using simplified notation

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (S, \mathcal{B}_X, \mu_X)$$

$$\mathcal{B}_X = \{B : X^{-1}(B) \in \mathcal{F}\}$$

F1:3

Suppose X is real-valued (cont. or discrete). Then, recall from measure theory

finite measure on $\mathbb{R}$ $\overset{1-1}{\longleftrightarrow}$ $\left\{\begin{array}{l}\text{functions } F_X \text{ on } \mathbb{R} \\ \bullet \text{ bounded} \\ \bullet \text{ increasing} \\ \bullet \text{ right-continuous} \\ \bullet\ F(x) \to 0,\ x \to -\infty \end{array}\right.$

μ_X

Connection: $F_X(x) = \mu_X((-\infty, x])$

Here, F_X is the distribution function

Probability theory develops measure theory in further directions:

- $(\mathbb{R}, \mathcal{B}_X, \mu_X)$ prob. space induced by X

  $(\mathbb{R}, \mathcal{B}_Y, \mu_Y)$ — " — Y

  $(\mathbb{R}^2, \mathcal{B}_{(X,Y)}, \mu_{(X,Y)})$ — " — (X,Y)

Def. X and Y are independent if $\mu_{(X,Y)} = \mu_X \otimes \mu_Y$ (product measure)

Compare: $\mathbb{P}(\{X \le x\} \cap \{Y \le y\}) = \mathbb{P}(\{\ \}) \mathbb{P}(\{\ \})$

$\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\, \mathbb{P}(Y \le y)$

- X and Y are <u>identically distributed</u> if $\mu_X = \mu_Y$
- If $X = Y$ $\mathbb{P}$-a.s. ($\mathbb{P}$-almost surely), that is, $\mathbb{P}(X \neq Y) = 0$, it is customary to identify them (identify the elements in the equivalence class)
- We pick up the Lebesgue integral but change notation/terminology:

  Expected value $\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$

  For real-valued X we then obtain $\mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mu_X(x)$

  with Lebesgue-Stieltjes form:

  $$\mathbb{E}(X) = \int_{\mathbb{R}} x \, dF_X(x)$$

- If μ_X is absolutely continuous w.r.t. Lebesgue measure m on $\mathbb{R}$, $\mu_X \ll m$ then F_X is an abs. cont. function such that $f_X = F_X'$ exists a.e.

FB-5

The X is a continuous random variable with prob. density f_X

and $$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx$$

(if this integral exists)

- Special distributions

$Re[0,1]$ $\mu_X = m$ on $[0,1]$

$Exp(\lambda)$ $f_X(x) = \lambda e^{-\lambda x} 1_{\{x \geq 0\}}$

$Po(\alpha)$ $\mathbb{P}(X=n) = \frac{\alpha^n}{n!} e^{-\alpha}$, $n = 0, 1, \ldots$

- $X_1, X_2, \ldots$ sequence of r.v.'s

e.g. independent and identically distributed (i.i.d)

$X_n \xrightarrow{a.s.} X$ $X_n(\omega) \to X(\omega)$

for all $\omega \in \Omega \setminus N$, some N where $\mathbb{P}(N) = 0$

$X_n \to X$ in mean $X_n \xrightarrow{L_1} X$

$\mathbb{E}|X_n - X| \to 0$, $n \to \infty$.

$X_n \to X$ in L^p

$\mathbb{X}_n \xrightarrow{\mathbb{P}} \mathbb{X}$ convergence in prob. | F/: 6

$\longrightarrow$ — in measure

$\forall \varepsilon > 0 \quad \mathbb{P}(|\mathbb{X}_n - \mathbb{X}| > \varepsilon) \to 0, \ n \to \infty$

$\mathbb{X}_n \xrightarrow{d} \mathbb{X}$ conv. in distribution
weak convergence

Real-valued case: $F_{\mathbb{X}_n}(x) \longrightarrow F_{\mathbb{X}}(x)$

for every x where $F_{\mathbb{X}}$ is continuous

$Var(\mathbb{X}) = \mathbb{E}[(\mathbb{X} - \mathbb{E}(\mathbb{X}))^2]$

$Cov(\mathbb{X}, \mathbb{Y}) =$ ——

$\varphi_{\mathbb{X}}(\theta) = \mathbb{E}[e^{i\theta \mathbb{X}}]$