GRAPH THEORY: RETAKE EXAM

17 AUGUST 2021

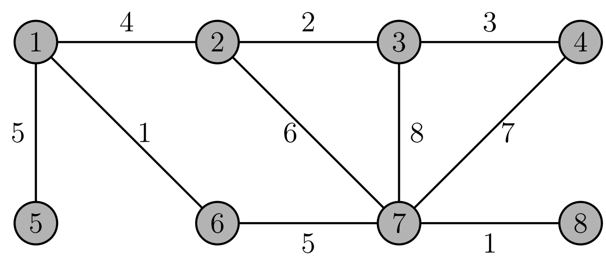
Throughout the entire exam, $G = (V, E)$ denotes a finite simple graph.

Problem 1. Recall from the lectures that a subset $S \subseteq V$ is a *vertex cover* of G if every edge $e \in E$ has an endpoint in S. The vertex covering number $\beta(G)$ is the minimum size of a vertex cover of G. Denote by $\alpha(G)$ the independence number of G, and let $\delta(G)$ be the minimum degree of G. (2p each)

(a) Show that $\beta(C_n) = \lceil n/2 \rceil$ for $n \geq 3$.
(b) Show that a set $S \subseteq V$ is a vertex cover if and only if $V \setminus S$ is independent. Conclude that $\alpha(G) + \beta(G) = |V|$.
(c) Show that, if G is a graph on n vertices, then $\beta(G) = n - 1$ if and only if G is complete.
(d) Determine $\beta(K_{a_1,\ldots,a_r})$. (You can assume without loss of generality that $a_1 \leq \cdots \leq a_r$)
(e) Show $\beta(G) \geq \delta(G)$.

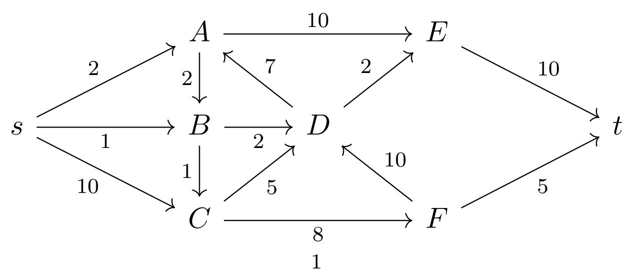
Problem 2. Consider the graph M_3 obtained by connecting all pairs of opposite vertices in a C_6. Use the matrix-tree-theorem to show that M_3 has $3^4 = 81$ spanning trees. (5p)

Problem 3. Consider the weighted graph



(a) Use Prim's or Kruskal's algorithm to find a minimum weight spanning tree. Your solution must contain all relevant steps in the algorithm. (2p)
(b) What is the Prüfer sequence of the tree obtained in (a)? (1p)
(c) Is the minimum spanning tree obtained in (a) unique? Why (not)? (2p)

Problem 4. Consider the flow network



(a) By applying the Ford-Fulkerson algorithm, find a maximal flow from s to t in the above network. Your solution must contain all relevant steps in the algorithm. (3p)

(b) Construct a minimum s-t-cut, and verify that the capacity of the cut equals the value of the flow in (a). (2p)

Problem 5. Show that if a (finite) tree T has a perfect matching M, then M is the unique perfect matching of T. (5p)

Problem 6. Let $n \geq 3$.

(a) Construct a graph G_n in the following way: Start with a cycle C_n with vertices labelled clockwise $1, 2, \ldots, n$ and a complete graph K_n whose vertices are labelled by $n+1, n+2, \ldots, 2n$. Then draw edges between vertices i and $n+i$ for all $1 \leq i \leq n$. Determine the chromatic number $\chi(G_n)$. (2p)

(b) Construct a graph H_n in the following way: Start with a cycle C_{2n} where the vertices are labelled clockwise $1, 2, \ldots, 2n$. Then draw an edge between any two vertices i, j with $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$ that are not already neighbours. Determine the chromatic number $\chi(H_n)$. (3p)

Problem 7. Let $n \geq 1$. Consider the complete bipartite graph $K_{n,n}$ and denote its two partition sets by A and B (each containing n vertices).

(a) For every edge of $K_{n,n}$, flip a fair coin to determine whether this edge should be red or blue. What is the probability that a fixed set consisting of a vertices from A and b vertices from B induces a monochromatic $K_{a,b}$? (1p)

(b) Let X denote the (random) number of such monochromatic $K_{a,b}$ in $K_{n,n}$. Determine $\mathbf{E}[X]$. (3p)

(c) Using (b), conclude that there always is a red/blue-colouring of $K_{n,n}$ such that the number of monochromatic $K_{a,b}$ is at most

$$\binom{n}{a}\binom{n}{b}2^{1-ab}.$$

(1p)

Please note: The only aids allowed are course lecture notes/videos and Diestel's book "Graph Theory". Aids not listed here are not permitted.

The problems above are to be solved and written down individually.

Upload your solutions as **one** pdf-file to studium, or send it by mail if uploading causes problems. Please take care that the solution is clearly readable, and that your name/anonymity code is written on it.