Le 15

Laurent series. Zeros and singularities

Thm (Laurent)

Suppose that f is analytic in $r < |z - z_0| < R$.

Then f can be expressed as the sum of two series:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \qquad (1)$$

where both series converge absolutely in $r < |z - z_0| < R$ and uniformly in any closed subannulus $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$.

The coeff. a_j are given by

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \qquad (j \in \mathbb{Z}) \qquad (2)$$

where C is any positively oriented circle $|z - z_0| = \rho$ with $r < \rho < R$.

Any pointwise convergent expansion of f in $r < |z - z_0| < R$ of the form (1) agrees with that above; in other words the coeff. a_j agree with those in (2) and the expansion is unique

Remarks

1) We allow that $r = 0$ and $R = +\infty$.

2) The above expansion of f is called the <u>Laurent series</u> for f in $r < |z - z_0| < R$ and is often written shortly as $\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$.

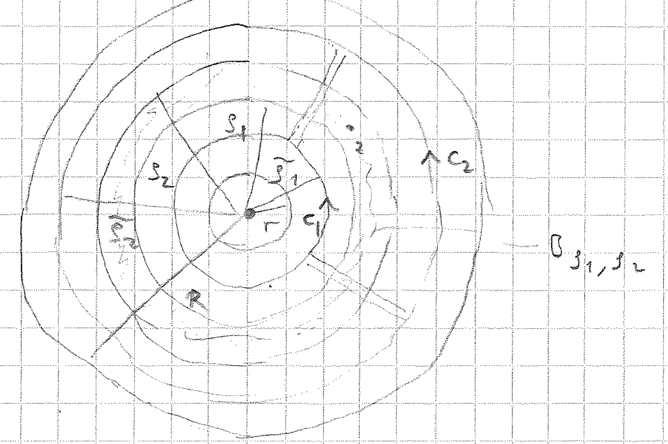
3) <u>If</u> f is analytic in $|z - z_0| < R$, then $a_j = 0$ for $j < 0$ by Cauchy's integral theorem, and the other terms reproduce the Taylor series (according to Cauchy's generalized integral formula)

<u>Proof</u>: We start by proving existence.

Fix z in $\{ \rho_1 \leq |z - z_0| \leq \rho_2 \} =: B_{\rho_1, \rho_2}$

Let C_1, C_2 be positively oriented circles of radius $\tilde{\rho}_1, \tilde{\rho}_2$ where $\tilde{\rho}_1 < \rho_1 < \rho_2 < \tilde{\rho}_2$.

See figure:



Remark: In fact, the first term is analytic off C_2 and the second is analytic off C_1

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (3)$$

As in the proof of Taylor's theorem:

For $\zeta \in C_2$:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0}\,\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} = \frac{f(\zeta)}{\zeta - z_0}\sum_{j=0}^{\infty} \underbrace{\left(\frac{z-z_0}{\zeta - z_0}\right)^j}_{|\cdot|<1}$$

$$= \sum_{j=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}(z-z_0)^j$$

with unif. conv. in the variable ζ on C_2.

$$\Rightarrow \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta\right]}_{= \frac{1}{2\pi i}\int_C \, d\zeta = a_j}(z-z_0)^j$$

$$= \sum_{j=0}^{\infty} a_j (z-z_0)^j \quad \text{with } a_j \text{ as in (2).}$$

In fact this series converges (pointwise) for any z inside C_2; hence abs. and uniformly on any strictly smaller disk (and so on B_{ρ_1,ρ_2}).

[Since ρ_2 was arbitrary the first term in (3) is analytic on $|z - z_0| < R$]

Similarly, for $\zeta \in C_1$:

$$-\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{z-z_0}\cdot\frac{1}{1-\frac{\zeta-z_0}{z-z_0}} = \frac{f(\zeta)}{z-z_0}\sum_{j=0}^{\infty} \underbrace{\left(\frac{\zeta-z_0}{z-z_0}\right)^{j}}_{|\cdot|<1}$$

$$= \sum_{j=1}^{\infty} \frac{f(\zeta)}{(\zeta-z_0)^{-j+1}}\,(z-z_0)^{-j}$$

with unif. conv. in the variable ζ on C_1.

$$\Rightarrow -\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \sum_{j=1}^{\infty} \Big[\underbrace{\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-j+1}}\,d\zeta}_{=\frac{1}{2\pi i}\int_{C} \cdots \, d\zeta \,=\, a_{-j}}\Big](z-z_0)^{-j}$$

$$= \sum_{j=1}^{\infty} \underbrace{a_{-j}\,(z-z_0)^{-j}}_{\text{analytic on } |z-z_0|>r} \quad \text{with } a_{-j} \text{ as in (2).}$$

The latter series converges (pointwise) for any z outside C_1. An argument similar to that in the power series lemma shows that the series converges abs. and uniformly outside any strictly larger circle (and so on B_{ρ_1,ρ_2}).

[Since $\tilde{\rho}_1$ was arbitrary the second term in (3) is analytic on $|z-z_0|>r$]

This proves existence.

Suppose now that we have an expansion of f

$$f(z) = \sum_{j=-\infty}^{\infty} a_j \, (z - z_0)^j$$

which converges pointwise in $\{ r < |z - z_0| < R \}$.

Then convergence is uniform on C, so

$$\int_C \frac{f(z)}{(z-z_0)^{k+1}} \, dz = \sum_{j=-\infty}^{\infty} a_j \underbrace{\int_C (z - z_0)^{j-k-1} \, dz}_{= \begin{cases} 2\pi i, & j = k \\ 0, & \text{otherwise} \end{cases}}$$

$$= 2\pi i \, a_k ,$$

i.e. the coeff. a_j are uniquely determined by f

and are given by (2).

$\square$

In practise one does not calculate the

Laurent series for f in $r < |z - z_0| < R$ with

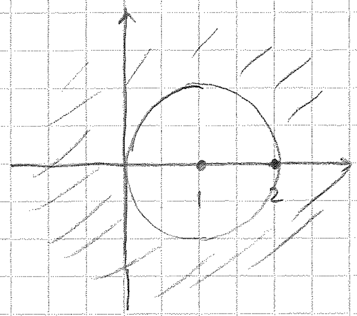
the expression for a_j in the theorem.

Instead, one uses known series expansions

and the uniqueness part of the theorem.

Ex Determine the Laurent series for $\dfrac{z^2-2z+3}{z-2}$ in the region $|z-1|>1$.

Sol See fig.



Note:

1) $\dfrac{z^2-2z+3}{z-2}$ is analytic in $\mathbb{C}\setminus\{2\}$, hence in $|z-1|>1$

2) So we have:

$z_0 = 1, \quad r = 1, \quad R = +\infty.$

$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{z-1}\cdot\frac{1}{1-\frac{1}{z-1}} = /|z-1|>1/$$

$$= \frac{1}{z-1}\sum_{j=0}^{\infty}\frac{1}{(z-1)^j} = \sum_{j=1}^{\infty}\frac{1}{(z-1)^j}, \quad |z-1|>1.$$

Now, $z^2-2z+3 = (z-1)^2+2$

$$\Rightarrow \frac{z^2-2z+3}{z-2} = \left[(z-1)^2+2\right]\cdot\left[\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\dots\right]$$

$$= (z-1)+1+\sum_{j=1}^{\infty}\frac{3}{(z-1)^j}, \quad |z-1|>1$$

B

Ex Determine the Laurent series for $e^{\frac{1}{z}}$ around $z=0$.

Sol We know that $e^{w} = \displaystyle\sum_{j=0}^{\infty}\frac{w^j}{j!}, \quad w\in\mathbb{C}.$

Clearly then

$$e^{\frac{1}{z}} = \sum_{j=0}^{\infty}\frac{1}{j!}\,\frac{1}{z^j}, \quad z\neq 0$$

(Here: $z_0 = 0, \quad r = 0, \quad R = +\infty$)

B

Zeros and singularities

Def Suppose that f is analytic at z_0. Then z_0 is called a zero of order m for the fcn f if $f^{(j)}(z_0) = 0$, $j = 0, \ldots, m-1$, but $f^{(m)}(z_0) \neq 0$. A zero of order 1 is called a simple zero.

Thm Suppose f is analytic at z_0. Then f has a zero of order m at z_0 if and only if f can be written as

$$f(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

Proof $\Rightarrow$ According to Taylor's theorem it then holds that

$$f(z) = a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \ldots$$

or $f(z) = (z - z_0)^m \left[a_m + a_{m+1}(z - z_0) + \ldots \right]$

where $a_m = \dfrac{f^{(m)}(z_0)}{m!} \neq 0$.

The bracketed series has the same radius of convergence as the Taylor series for f.

It defines a fcn, call it $g(z)$, analytic in a neighborhood of z_0. Clearly $g(z_0) = a_m \neq 0$.

$\Leftarrow$ By Taylor expanding g around z_0, we have that f is given by the power series

$$f(z) = g(z_0)(z-z_0)^m + g'(z_0)(z-z_0)^{m+1} + \dots$$

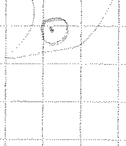
whose coeff. are given by $\dfrac{f^{(j)}(z_0)}{j!}$.

Hence $f^{(j)}(z_0) = 0$, $j = 0, \dots, m-1$, and $\dfrac{f^{(m)}(z_0)}{m!} = g(z_0) \neq 0$

(Or by direct differentiation ...)

<u>Corollary</u> Suppose f analytic at z_0 and that $f(z_0) = 0$. Then either f is identically zero in a neighb. of z_0, or there exists a punctured disk about z_0 in which f has no zeros.

<u>Proof</u>. Let $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ be the Taylor series for f in a neighborhood of z_0.



If $a_j = 0$ $\forall j \geq 0$ then f must be identically zero in a neighborhood of z_0.

Otherwise, let

$$m = \min\{j : a_j \neq 0\}.$$

Clearly then f has a zero of order m at z_0. By the theorem above

$$f(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$

Then g is continuous at z_0 and $g(z_0) \neq 0$, so there exists a disk $|z - z_0| < \delta$ in which $g(z) \neq 0$. Thus, $f(z) \neq 0$ for $0 < |z - z_0| < \delta$.

$\boxtimes$

If f is analytic in a domain D and vanishes in some disk in D, in fact f must vanish identically. This can be proven by an argument similar to the one used in the proof of the maximum modulus principle.

We therefore have the following.

<u>Thm</u> (Uniqueness principle)

If f and g are analytic on a domain D, and if $f(z) = g(z)$ for z belonging to a set that has a nonisolated point, then $f(z) = g(z)$ for all $z \in D$.