Financial Theory – Lecture 5

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Agenda

- Mean-variance analysis with only risky assets.
- Choice under uncertainty.

The lecture is based on

- Chapter 7 in the course book.

Mean-variance analysis with risky assets

We consider a market with N risky assets.

By a "risky asset" we mean an asset whose rate of return has a strictly positive standard deviation.

The rate of return of these assets are collected in the vector **r**.

Recall the notation

$$\boldsymbol{\mu} = E\left[\boldsymbol{r}\right]$$

and

$$\Sigma = \mathrm{Var}[\boldsymbol{r}].$$

Mean-variance analysis with risky assets

The first goal for us is to find the portfolio with mean $\bar{\mu}$ that has the smallest variance.

Mathematically we formulate this problem as

$$\begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \displaystyle\sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \overline{\mu}. \end{array} \qquad \Leftrightarrow \qquad \begin{array}{ll} \min\limits_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \overline{\mu}. \end{array}$$

The Lagrangian of this problem is

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda_1 (1 - \boldsymbol{\pi} \cdot \mathbf{1}) + \lambda_2 (\overline{\mu} - \boldsymbol{\pi} \cdot \boldsymbol{\mu}).$$

Here λ_1 and λ_2 are the (Lagrangian) multipliers.

Mean-variance analysis with risky assets

We need to be able to take the derivative of functions like

$$f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{a} \quad \text{and} \quad g(\boldsymbol{x}) = \boldsymbol{x} \cdot A\boldsymbol{x}$$

where $\boldsymbol{a}$ is a vector and A is a matrix.

Since

$$f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{a} = \sum_{i=1}^{N} x_i a_i$$

we have

$$\frac{\partial f}{\partial x_i} = a_i, \; i = 1, \ldots, N.$$

We write this as

$$\frac{\partial f}{\partial \boldsymbol{x}} = \boldsymbol{a}.$$

We also have

$$g(\boldsymbol{x}) = \boldsymbol{x} \cdot A\boldsymbol{x} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j A_{ij}.$$

If the matrix A is symmetric then

$$\frac{\partial g}{\partial x_i} = 2(A\boldsymbol{x})_i,$$

where $(A\boldsymbol{x})_i$ is the element of row i of $A\boldsymbol{x}$.

We write this as

$$\frac{\partial g}{\partial \boldsymbol{x}} = 2A\boldsymbol{x}.$$

Mean-variance analysis with risky assets

Let us return to the Lagrangian:

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma\boldsymbol{\pi} + \lambda_1(1 - \boldsymbol{\pi} \cdot \mathbf{1}) + \lambda_2(\overline{\mu} - \boldsymbol{\pi} \cdot \boldsymbol{\mu}).$$

The first-order condition with respect to $\boldsymbol{\pi}$ is

$$\frac{\partial L}{\partial \boldsymbol{\pi}} = 2\Sigma\boldsymbol{\pi} - \lambda_1\mathbf{1} - \lambda_2\boldsymbol{\mu} = 0.$$

This can be written

$$2\Sigma\boldsymbol{\pi} = \lambda_1\mathbf{1} + \lambda_2\boldsymbol{\mu} \quad \Leftrightarrow \quad \Sigma\boldsymbol{\pi} = \frac{\lambda_1}{2}\mathbf{1} + \frac{\lambda_2}{2}\boldsymbol{\mu}.$$

How do we solve for $\boldsymbol{\pi}$? Multiply both sides with Σ^{-1}!

Mean-variance analysis with risky assets

$$\underbrace{\Sigma^{-1}\Sigma}_{=I}\boldsymbol{\pi} = \Sigma^{-1}\left(\frac{\lambda_1}{2}\mathbf{1} + \frac{\lambda_2}{2}\boldsymbol{\mu}\right)$$

Since $I\boldsymbol{\pi} = \boldsymbol{\pi}$ we get

$$\boldsymbol{\pi} = \frac{\lambda_1}{2}\Sigma^{-1}\mathbf{1} + \frac{\lambda_2}{2}\Sigma^{-1}\boldsymbol{\mu}.$$

We need to find the Lagrange multipliers λ_1 and λ_2 $\longrightarrow$ use the constraints.

Mean-variance analysis with risky assets

Portfolio weights sum to 1:

$$
\begin{aligned}
1 &= \boldsymbol{\pi} \cdot \mathbf{1} = \mathbf{1} \cdot \boldsymbol{\pi} \\
&= \mathbf{1} \cdot \left(\frac{\lambda_1}{2} \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \Sigma^{-1} \boldsymbol{\mu} \right) \\
&= \frac{\lambda_1}{2} \mathbf{1} \cdot \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu}
\end{aligned}
$$

Expected rate of return equal to $\bar{\mu}$:

$$
\begin{aligned}
\bar{\mu} &= \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \boldsymbol{\pi} \\
&= \boldsymbol{\mu} \cdot \left(\frac{\lambda_1}{2} \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \Sigma^{-1} \boldsymbol{\mu} \right) \\
&= \frac{\lambda_1}{2} \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} + \frac{\lambda_2}{2} \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu}
\end{aligned}
$$

Mean-variance analysis with risky assets

Now introduce the parameters

$$\begin{aligned} A &= \boldsymbol{\mu} \cdot \Sigma^{-1} \boldsymbol{\mu} \\ B &= \boldsymbol{\mu} \cdot \Sigma^{-1} \mathbf{1} = \mathbf{1} \cdot \Sigma^{-1} \boldsymbol{\mu} \\ C &= \mathbf{1} \cdot \Sigma^{-1} \mathbf{1} \\ D &= AC - B^2. \end{aligned}$$

Using these, the two equations above can be written

$$\begin{cases} 1 = \dfrac{\lambda_1}{2} C + \dfrac{\lambda_2}{2} B \\ \bar{\mu} = \dfrac{\lambda_1}{2} B + \dfrac{\lambda_2}{2} A \end{cases}$$

We want to solve for λ_1 and λ_2.

Mean-variance analysis with risky assets

The solution is given by

$$\begin{cases} \lambda_1 &= 2\dfrac{A - B\bar{\mu}}{D} \\[2ex] \lambda_2 &= 2\dfrac{C\bar{\mu} - B}{D}. \end{cases}$$

Inserting them in the expression for $\boldsymbol{\pi}$ results in

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{A - B\bar{\mu}}{D}\Sigma^{-1}\mathbf{1} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

These are the portfolio weights in the portfolio with mean rate of return $\bar{\mu}$ whose rate of return has the smallest variance.

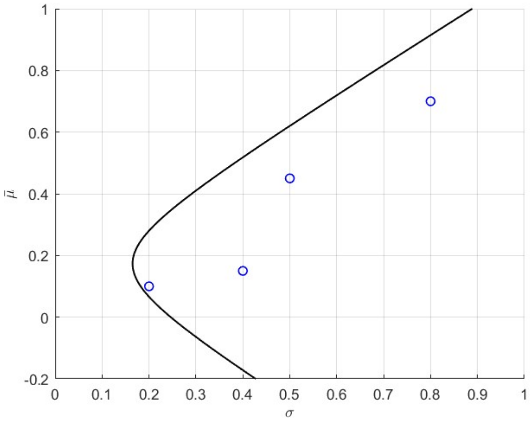
Mean-variance analysis with risky assets

Using the optimal weights from the previous slide we can calculate the standard deviation of this portfolio:

$$
\begin{aligned}
\sigma(\bar{\mu}) &= \mathrm{Std}[r(\boldsymbol{\pi}(\bar{\mu}))] \\
&= \sqrt{\boldsymbol{\pi}(\bar{\mu}) \cdot \Sigma \boldsymbol{\pi}(\bar{\mu})} \\
&= \ldots \\
&= \sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}.
\end{aligned}
$$

This is called the mean-variance frontier or the portfolio frontier.

Mean-variance analysis with risky assets



An example with four assets ($N = 4$).

Mean-variance analysis with risky assets

A portfolio whose mean and standard devition is on the mean-variance frontier is called a frontier portfolio.

We say that a portfolio is on the portfolio frontier if its standard deviation and mean is on the frontier.

The variance σ^2 for a given $\bar{\mu}$ is

$$\sigma^2 = \frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}.$$

This can be written

$$\frac{\sigma^2}{1/C} - \frac{\left(\bar{\mu} - B/C\right)^2}{D/C^2} = 1.$$

Mathematically this is an equation of a hyperbola in the $(\sigma, \bar{\mu})$-plane.

Mean-variance analysis with risky assets

Let us now turm to the problem of finding the portfolio with the smallest variance, not taking its expected rate of return into account.

The minimum-variance portfolio (MVP) $\boldsymbol{\pi}_{\text{min}}$ solves the problem

$$\begin{array}{ll} \min_{\boldsymbol{\pi}} & \text{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} & \sum_{i=1}^{N} \pi_i = 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min_{\boldsymbol{\pi}} & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} & \boldsymbol{\pi} \cdot \mathbf{1} = 1. \end{array}$$

To solve this problem we again set up the Lagrangian

$$L(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda(1 - \boldsymbol{\pi} \cdot \mathbf{1})$$

and proceed as above

In this case we get (check this!)

$$\boldsymbol{\pi}_{\text{min}} = \frac{1}{C}\Sigma^{-1}\mathbf{1} = \frac{1}{\mathbf{1}\cdot\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}.$$

The standard deviation and mean of this portfolio is

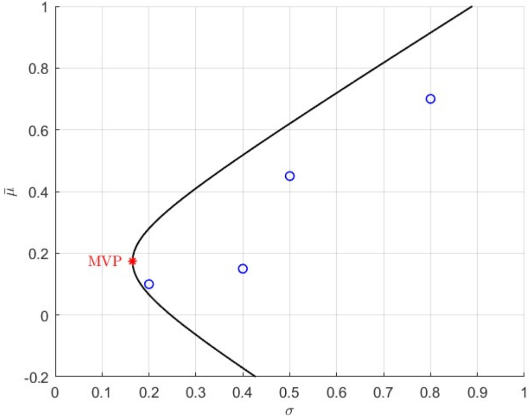
$$\sigma_{\text{min}} = \frac{1}{\sqrt{C}}$$

and

$$\mu_{\text{min}} = \frac{B}{C}$$

respectively.

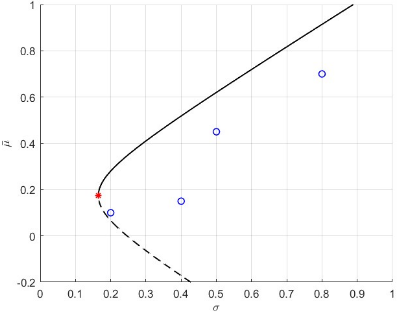
Mean-variance analysis with risky assets



Mean-variance analysis with risky assets

Looking at the mean-variance frontier, we see that for standard deviations larger than $\sigma_{\text{min}} = 1/\sqrt{C}$ there are two portfolios having this standard deviation.

Since an investor wants to maximise the expected return and minimise the standard deviation, no rational investor will hold a portfolio on the half of the mean-variance frontier that is below the MVP.



Mean-variance analysis with risky assets

Recall the general formula

$$\boldsymbol{\pi}(\bar{\mu}) = \underbrace{\frac{A - B\bar{\mu}}{D}}_{\text{Scalar}} \underbrace{\Sigma^{-1}\mathbf{1}}_{\text{Vector}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

Note that

$$\boldsymbol{\pi}_{\text{min}} = \frac{1}{C}\Sigma^{-1}\mathbf{1} \quad \Leftrightarrow \quad \Sigma^{-1}\mathbf{1} = C\boldsymbol{\pi}_{\text{min}}.$$

Conclusion: The vector $\Sigma^{-1}\mathbf{1}$ is up to a scaling equal to the MVP.

We can write the general optimal weight vector as

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu}.$$

Mean-variance analysis with risky assets

We have seen that $\boldsymbol{\pi}(\bar{\mu})$ can be written as a constant, depending on the chosen level $\bar{\mu}$, times the MVP plus another constant, also depending on $\bar{\mu}$, times the vector $\Sigma^{-1}\boldsymbol{\mu}$.

Define the portfolio

$$\boldsymbol{\pi}_{\text{slope}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{\mu} = \frac{1}{\mathbf{1}\cdot\Sigma^{-1}\boldsymbol{\mu}}\Sigma^{-1}\boldsymbol{\mu}.$$

Note: 1) We want a portfolio that is proportional to $\Sigma^{-1}\boldsymbol{\mu}$.

2) We multiply with $1/\mathbf{1}\cdot\Sigma^{-1}\boldsymbol{\mu}$ so that $\boldsymbol{\pi}_{\text{slope}}$ is a portfolio, i.e. its elements sum to 1.

Mean-variance analysis with risky assets

Using that

$$\boldsymbol{\pi}_{\text{slope}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{\mu} \quad \Leftrightarrow \quad \Sigma^{-1}\boldsymbol{\mu} = B\boldsymbol{\pi}_{\text{slope}}$$

we get

$$\begin{aligned}
\boldsymbol{\pi}(\bar{\mu}) &= \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\boldsymbol{\mu} \\
&= \frac{(A - B\bar{\mu})C}{D}\boldsymbol{\pi}_{\text{min}} + \frac{(C\bar{\mu} - B)B}{D}\boldsymbol{\pi}_{\text{slope}} \\
&= a_{\text{min}}(\bar{\mu})\boldsymbol{\pi}_{\text{min}} + a_{\text{slope}}(\bar{\mu})\boldsymbol{\pi}_{\text{slope}}.
\end{aligned}$$

Mean-variance analysis with risky assets

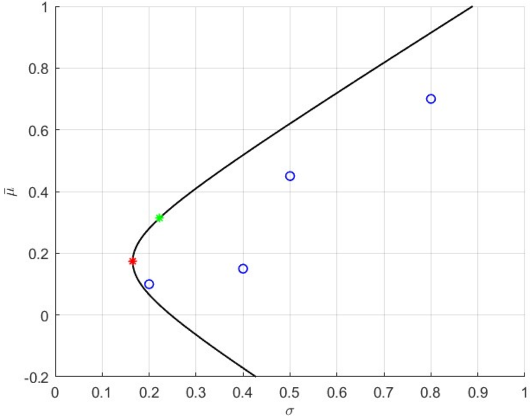
We have

$$\begin{aligned} a_{\min}(\bar{\mu}) + a_{\text{slope}}(\bar{\mu}) &= \frac{(A - B\bar{\mu})C}{D} + \frac{(C\bar{\mu} - B)B}{D} \\ &= \frac{AC - BC\bar{\mu} + BC\bar{\mu} - B^2}{D} \\ &= \frac{AC - B^2}{D} = 1. \end{aligned}$$

The interpretation is that for each $\bar{\mu}$, the optimal portfolio $\boldsymbol{\pi}(\bar{\mu})$ can be written as a combination of the two portfolios $\boldsymbol{\pi}_{\min}$ and $\boldsymbol{\pi}_{\text{slope}}$.

This is called two-fund separation: Any portfolio on the mean-variance frontier is the combination of the two portfolios $\boldsymbol{\pi}_{\min}$ and $\boldsymbol{\pi}_{\text{slope}}$.

Mean-variance analysis with risky assets



Mean-variance analysis with risky assets

More generally:

If the constant $B \neq 0$, then any two frontier portfolios $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ can be used to span the frontier, i.e. there exists a number w such that

$$\boldsymbol{\pi}(\bar{\mu}) = w\boldsymbol{\pi}_1 + (1 - w)\boldsymbol{\pi}_2.$$

Mean-variance analysis with risky assets

Is there any economic interpretation of $\boldsymbol{\pi}_{\text{slope}}$?

Yes!

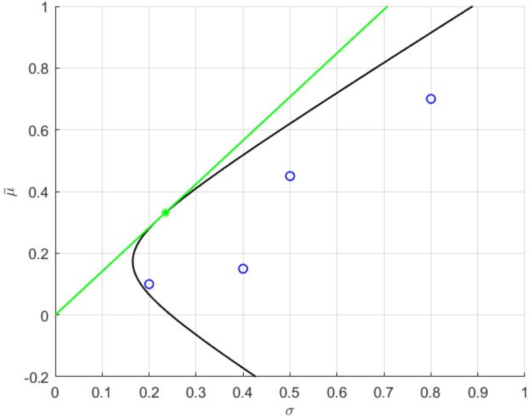
Consider a straight line

$$\bar{\mu} = k\sigma$$

in the $(\bar{\mu}, \sigma)$-plane for some k.

The portfolio $\boldsymbol{\pi}_{\text{slope}}$ represents the frontier portfolio with the largest slope of this line.

Mean-variance analysis with risky assets



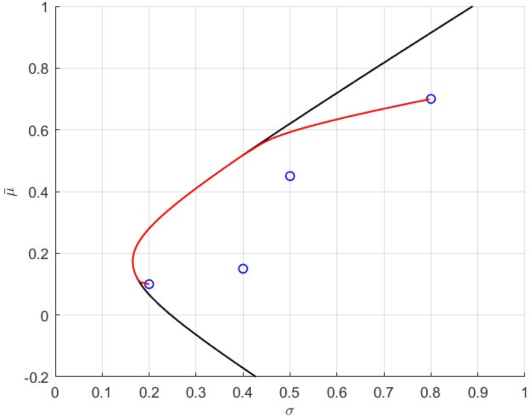
Mean-variance analysis with portfolio constraints

So far we have allowed short-selling. Let us look at the problem where short-selling is not allowed.

$$
\begin{aligned}
\min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\
\text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\
& \sum_{i=1}^{N} \pi_i \mu_i = \bar{\mu} \\
& \pi_i \geq 0, i = 1, 2, \ldots, N
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\
\text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\
& \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \bar{\mu} \\
& \boldsymbol{\pi} \geq \mathbf{0}.
\end{aligned}
$$

These type of problems are in general much harder to solve than when there are only equility constraints.

Mean-variance analysis with portfolio constraints



Mean-variance analysis with portfolio constraints

Different types of constraints:

- No short-selling in one or several assets.
- Maximum fraction invested in one or several assets.
- Maximum fraction invested in a sector or country.

Choice under uncertainty

We know that a rational investor will choose a portfolio on the upper efficient frontier. But which portfolio will be chosen?

The choice depends on the investor's attitude towards risk.

An investor has initial wealth W_0 and invests in an asset with rate of return r. Then the future wealth is

$$W = W_0(1 + r).$$

Given axioms of investor behaviour, one can show the existence of a utility function representing the investor's risk preferences such that

$$E\left[u(W)\right]$$

is the utility of getting the cash flow W.

Choice under uncertainty

Let u be such a utility function.

In general it is

1) increasing, $u'(x) > 0$, and
2) concave, $u''(x) \leq 0$.

Examples of utility functions

- $u(x) = \ln x$.
- $u(x) = -e^{-x}$.
- $u(x) = \sqrt{x}$.
- $u(x) = x$.

Choice under uncertainty

How can we measure the level of attitude towards risk of an investor?

It is possible to show that the coefficient of absolute risk aversion

$$\mathrm{ARA}(x) = -\frac{u''(x)}{u'(x)}$$

is a good measure of this.

As an alternative the coefficient of relative risk aversion

$$\mathrm{RRA}(x) = -\frac{x\,u''(x)}{u'(x)}$$

can be used.

CARA utility functions

If

$$u(x) = -e^{-ax}$$

then

$$u'(x) = ae^{-ax} \quad \text{and} \quad u''(x) = -a^2 e^{-ax},$$

so

$$\text{ARA}(x) = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a.$$

These are known as CARA utility functions for constant absolute risk aversion.

CRRA utility functions

If

$$u(x) = \begin{cases} \dfrac{x^{1-\gamma} - 1}{1 - \gamma} & \text{if } \gamma > 0, \gamma \neq 1 \\ \\ \ln x & \text{if } \gamma = 1 \end{cases}$$

then

$$u'(x) = x^{-\gamma} \quad \text{and} \quad u''(x) = -\gamma x^{-\gamma-1},$$

so

$$\mathrm{RRA}(x) = -\frac{x \cdot \left(-\gamma x^{-\gamma-1}\right)}{x^{-\gamma}} = \gamma.$$

These are known as CRRA utility functions for constant relative risk aversion.

CARA utility and normally distributed rates of returns

Consider a market with N risky assets such that:

1) The rates of return vector $\boldsymbol{r}$ has a multivariate normal distribution.
2) The investor has a CARA utility function with parameter $a > 0$.

This means that

$$W = W_0\big((1 + r(\boldsymbol{\pi})\big) = W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}$$

is normally distributed with mean

$$E\left[W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}\right] = W_0 + W_0 E\left[\boldsymbol{\pi} \cdot \boldsymbol{r}\right] = W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{\mu}$$

and variance

$$\mathrm{Var}[W_0 + W_0\boldsymbol{\pi} \cdot \boldsymbol{r}] = W_0^2\mathrm{Var}[\boldsymbol{\pi} \cdot \boldsymbol{r}] = W_0^2\boldsymbol{\pi} \cdot \Sigma\boldsymbol{\pi}.$$

CARA utility and normally distributed rates of returns

One can show (see the book!) that in this model an investor is indifferent between portfolios that have the same value of

$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} - \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.$$

Hence, indifference curves are given by

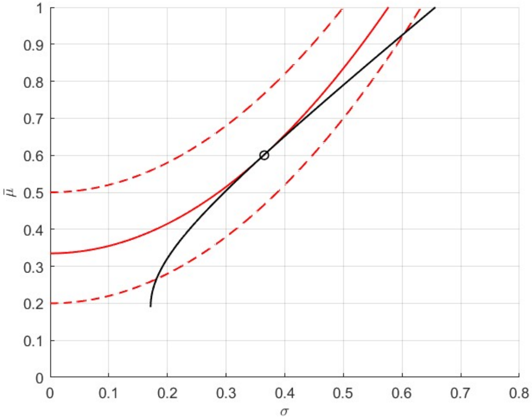
$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} - \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} = K$$

or

$$\boldsymbol{\pi} \cdot \boldsymbol{\mu} = K + \frac{a}{2} W_0 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi},$$

where we let the level of utility K vary.

CARA utility and normally distributed rates of returns



Utility increases in the north-west direction.