F2:0

To obtain Brownian motion we need the predicates in Einstein Gedankenexperiment 1905, namely

1. $B_0 = 0$
2. $B_t - B_s \sim N(0, t-s)$, $0 \leq s \leq t$; stationary Gaussian increments
3. Independent increments
4. Continuous paths

But "$\mathbb{P}(\{B_t\}_{t\in T}$ continuous)" is not defined

since $\{t \to X_t \text{ cont}\} \notin \mathcal{B}(\mathbb{R}^T)$

compare BW Remark 1.3

So, we take a different approach

Def Two processes $\{X_t\}_{t\in T}$, $\{Y_t\}_{t\in T}$ are

- modifications of each other

  if $\mathbb{P}(X_t = Y_t) = 1$, all $t \in T$

- versions of each other

  if they have the same f.d.d.'s

We have, if X and Y modifications,

$$\mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n, X_{t_1} = Y_{t_1}, \ldots, X_{t_n} = Y_{t_n})$$

$$= \mathbb{P}(Y_{t_1} \in B_1, \ldots, Y_{t_n} \in B_n, X_{t_1} = Y_{t_1}, \ldots, X_{t_n} = Y_{t_n})$$

$$= \mathbb{P}(Y_{t_1} \in B_1, \ldots, Y_{t_n} \in B_n),$$ so that X and Y are versions

Recall ~~th~~. Using Kolmogorov's extension theorem, we have constructed pre-Brownian motion which satisfies predicates 1, 2, 3.

Let's denote this $\{\tilde{B}_t, t\in T\}$.

Strategy: Find a version $\{B_t\}_{t\in T}$ of $\{\tilde{B}_t\}_{t\in T}$ (i.e. same f.d.d.'s) ~~who~~ with continuous sample paths. Then $\{B_t\}_{t\in T}$ satisfies all of 1-4) and is therefore the final Brownian motion. How?

Kolmogorov-Čentsov cont. theorem

(BW Thm 6.1) (Riedle, Thm 3.3)

If $\{X_t\}_{t\in T}$ is such that, for some $\alpha, \beta > 0$, we have

$$E[|X_t - X_s|^\alpha] \le \text{const} \cdot |t-s|^{1+\beta}, \quad s,t \in T,$$

then there exists a continuous version of $\{X_t\}_{t\in T}$.

E 3:2

Let's apply this result to pre-BM:

$$\mathbb{E}|B_t - B_s|^2 = |t-s|$$ not enough ---

Lemma 3.3.4

But $\mathbb{E}|B_t - B_s|^4 = 3|t-s|^2$ (~~Exercise 2.8~~)

Hence there is a cont. version of BM

So, yes BM exists!

What did we do?

$$\begin{tikzcd}
(\mathbb{R}^T, & \mathcal{B}(\mathbb{R}^T), & P) \arrow[r, shift left, "B"] & (C, \mathcal{B}(C), P_B) \arrow[l, shift left, "B^{-1}"] \\
\Omega \arrow[u] & \mathcal{F} \arrow[u] & \mathbb{P} \arrow[u] &
\end{tikzcd}$$

Canonical prob. space for BM

"Wiener space"

Brown	1827	Part's in water
Bachelier	1900	stock prices
Einstein	1905	gas physics, predicates
Perron	1909	microscope
Wiener	1923	existence
Lévy	1940	characterizat.
Itô	1946	stochastic integration SDE

Finance 1973

F3: 2½

Brownian motion / Wiener process

$\{X_t\}_{t\geq 0}$; $t \to X_t \in \mathbb{R}$,

1, $X_0 = x \in \mathbb{R}$

2, stationary, independent increments

3, Gaussian distr. $E(X_t) = x$, $\mathrm{Var}(X_t) = \sigma^2 t$

4, Continuous paths

$$X_t = x + \sigma B_t \quad , \text{ where } \{B_t\}_{t\geq 0} \text{ is}$$

standard BM.

$$P_x(X_t \in A) = \int_A p_{\sigma^2 t}(x,y)\,dy$$

where $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$, $t > 0$

is the transition density.

Joint distribution of (B_s, B_t), $s \leq t$?

$$P(B_s \in A_1, B_t \in A_2) = \int_{A_1}\int_{A_2} p_s(0,x)\,p_{t-s}(x,y)\,dx\,dy$$

by def. of the f.d.d.'s, so

$$f_{(B_s,B_t)}(x,y) = p_s(0,x)\,p_{t-s}(x,y)$$

$$= \frac{1}{\sqrt{2\pi s}}\frac{1}{\sqrt{2\pi(t-s)}} e^{-x^2/2s} e^{-(y-x)^2/2(t-s)}$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}}\frac{1}{2\pi}\frac{1}{\sqrt{s(t-s)}} e^{-\{tx^2 - 2sxy + sy^2\}\frac{1}{2s(t-s)}}$$

~~$s(t-s) = (t-s)^2 + s^2$~~ $\Rightarrow \mathrm{Cov}(B_s, B_t) = s$!

Can we check this?

$$\mathrm{Cov}(B_s, B_t) = \mathbb{E}[B_s B_t] - \underbrace{\mathbb{E}(B_s)\mathbb{E}(B_t)}_{=0}$$

$$= \mathbb{E}[B_s(B_t - B_s) + B_s^2]$$

$$= \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}(B_s^2)$$

increments are independent!

$$= \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) + \mathrm{Var}(B_s)$$

$$= 0 + s \qquad \text{Ok!}$$

So $\boxed{\mathrm{Cov}(B_s, B_t) = \min(s,t)}$

Example Let $\{Y_t\}$ have the distribution of $\{B_t\}_{0 \le t \le 1}$ conditional on $B_1 = 0$.

$$f_{B_t | B_1}(x|y) = \frac{f_{(B_t, B_1)}(x,y)}{f_{B_1}(y)}, \quad 0 \le t \le 1$$

$$f_{Y_t}(x) = \frac{f_{(B_t, B_1)}(x, 0)}{f_{B_1}(0)} = \frac{p_t(x) p_{1-t}(0-x)}{p_1(0)}$$

$$= \frac{\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \frac{1}{\sqrt{2\pi(1-t)}} e^{-x^2/2(1-t)}}{\frac{1}{\sqrt{2\pi \cdot 1}}}$$

$$= \frac{1}{\sqrt{2\pi t(1-t)}} e^{-x^2 \frac{1}{2t(1-t)}}, \quad 0 < t < 1$$

$Y_t \sim N(0, t(1-t))$

$\frac{1}{t} + \frac{1}{1-t} = \frac{1}{t(1-t)}$

Let's construct a Poisson process $\{N_t, t \geq 0\}$:

a, $N_0 = 0$, integer-valued

b, The increments $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$,
$0 \leq t_1 \leq \ldots \leq t_n$, $n \geq 1$, are independent

c, For $0 \leq s \leq t$,

$$N_t - N_s \sim Po(\lambda(t-s)) \quad , \text{ some } \lambda > 0 .$$

d) The paths are right-continuous

~~Proof~~ Proof Kolmogorov extension theorem,

Compare BW page 52

with $p_t(x) = \dfrac{(\lambda t)^x}{x!} e^{-\lambda t}$, $x = 0, 1, 2, \ldots$

This gives a, b, c. Then ~~we can~~
pick a version ~~that~~ that satisfies d,

Remarks For more general setting, replace c) by "stationary increments":

$$N_{t+h} - N_{s+h} \overset{d}{=} N_t - N_s$$

for all $h > 0$ and $0 \leq s \leq t$.

BW Exercise 5.9

Direct construction:

Let $\{T_k\}_{k\geq 1}$ be i.i.d. sequence, $T_k \sim Exp(\lambda)$, $k \geq 1$. Put $S_n = \sum_{k=1}^{n} T_k$.

Define $N_t = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}}$, $t \geq 0$

Alternative: Define $N_t = \begin{cases} 0 & \text{if } t < T_1 \\ \max\{n \geq 1 : S_n \leq t\}, & \text{else} \end{cases}$

Note: $\{N_t \geq n\} = \{S_n \leq t\}$

Now one can check that these definitions are equivalent with 1)–4) above.

Moreover, $E[N_t] = \sum_{n=1}^{\infty} P(S_n \leq t)$

$$= \sum_{n=1}^{\infty} \int_0^t \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} dt$$

$$= \int_0^t \lambda \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} dt = \lambda t, \text{ etc.}$$

$N_t - N_0 = \sum_{s \leq t} \Delta N_s$, when $\Delta N_s = N_s - N_{s-}$

$= \sum_{k=1}^{N_t} 1$

If $\{Z_k\}_{k\geq 1}$ is i.i.d. sequence then $R_t = \sum_{k=1}^{N_t} Z_k$, $t \geq 0$ is called a <u>compound Poisson process</u>