Analysis of Time Series, L7

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Today

3.6: Estimation

- Method of moments
- Maximum likelihood

Method of moments

AR(p):

- We observe $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + w_t$ for $t = 1, 2, \ldots, n$.
- Estimate $\phi_1, \phi_2, \ldots, \phi_p$ and $\sigma_w^2 = \mathrm{var}(w_t)$.

Definition (3.10)

The *Yule-Walker equations* are given by

$$\gamma(h) = \phi_1 \gamma(h-1) + \ldots + \phi_p \gamma(h-p), \quad h = 1, 2, \ldots, p,$$
$$\sigma_w^2 = \gamma(0) - \phi_1 \gamma(1) - \ldots - \phi_p \gamma(p).$$

Method of moments

- The system

$$\gamma(h) = \phi_1\gamma(h-1) + ... + \phi_p\gamma(h-p), \quad h = 1, 2, ..., p$$

is equivalent to

$$\begin{pmatrix} \gamma(0) & \gamma(1) & \dots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \dots & \gamma(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(p-1) & \gamma(p-2) & \dots & \gamma(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{pmatrix}$$

i.e. $\Gamma_p \boldsymbol{\phi} = \boldsymbol{\gamma}_p$.

- Method of moments estimators $\hat{\boldsymbol{\phi}} = \hat{\Gamma}_p^{-1}\hat{\boldsymbol{\gamma}}_p$.

Method of moments

Calculate the moment estimators of the AR/MA parameters for

1. The AR(1) process $x_t = \phi x_{t-1} + w_t$.
2. The AR(2) process $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$.
3. The MA(1) process $x_t = w_t + \theta w_{t-1}$. (Two solutions!)

Method of moments

Theorem (Property 3.8)

Let $\hat{\phi}$ be the vector of moment estimators for a causal AR process. Then, as $n \to \infty$,

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{d} N(\mathbf{0}, \sigma_w^2 \Gamma_p^{-1}).$$

Hence, $\hat{\phi} \approx N(\phi, n^{-1}\sigma_w^2 \Gamma_p^{-1})$ for large n. (Proof in Appendix B.)

For causal AR processes, the moment estimators are *asymptotically efficient* in the sense that they attain the minimal asymptotic variance.

Theorem (Property 3.9)

For a causal AR(p) process, the PACF fulfills, as $n \to \infty$,

$$\sqrt{n}\hat{\phi}_{hh} \xrightarrow{d} N(0,1), \quad \textit{for } h > p.$$

Method of moments

Calculate the asymptotic variances (and covariances) of the AR parameter moment estimators for

1. The AR(1) process $x_t = \phi x_{t-1} + w_t$.
2. The AR(2) process $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$.

Maximum likelihood

Causal AR(1) without constant: $x_t = \phi x_{t-1} + w_t$, $t = 1, 2, ..., n$, $w_t \sim N(0, \sigma_w^2)$, independent.

- Log likelihood (why?)

$$l(\phi, \sigma_w^2) = -\frac{n}{2}\log(2\pi\sigma_w^2) + \frac{1}{2}\log(1-\phi^2) - \frac{S(\phi)}{2\sigma_w^2},$$

where

$$S(\phi) = (1-\phi^2)x_1^2 + \sum_{t=2}^{n}(x_t - \phi x_{t-1})^2.$$

- No "simple" explicit expression for the MLE of ϕ.

Maximum likelihood

Causal AR(1) without constant: $x_t = \phi x_{t-1} + w_t$, $t = 1, 2, ..., n$, $w_t \sim N(0, \sigma_w^2)$, independent.

- *Conditional* log likelihood (the x_1 density 'disappears')

$$l(\phi, \sigma_w^2 | x_1) = -\frac{n-1}{2} \log(2\pi\sigma_w^2) - \frac{S_c(\phi)}{2\sigma_w^2},$$

  where

$$S_c(\phi) = \sum_{t=2}^{n} (x_t - \phi x_{t-1})^2.$$

- Conditional MLEs (give zero partial derivatives, cf linear regression)

$$\hat{\sigma}_w^2 = \frac{S_c(\hat{\phi})}{n-1}, \quad \hat{\phi} = \frac{\sum_{t=2}^{n} x_t x_{t-1}}{\sum_{t=2}^{n} x_{t-1}^2}.$$

Maximum likelihood

Causal AR(p) without constant:

$$x_t = \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} + w_t, \quad t = 1, 2, \dots, n,$$

$w_t \sim N(0, \sigma_w^2)$, independent.

- Conditional log likelihood

$$l(\phi_1, \dots, \phi_p, \sigma_w^2 | x_1, \dots, x_p) = -\frac{n-p}{2} \log(2\pi\sigma_w^2) - \frac{S_c}{2\sigma_w^2},$$

  where

$$S_c = \sum_{t=p+1}^{n} (x_t - \phi_1 x_{t-1} - \dots - \phi_p x_{t-p})^2.$$

- Find the conditional MLEs!

Maximum likelihood

Causal and invertible ARMA(p, q) without constant:

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}, \quad t = 1, 2, \ldots, n,$$

$w_t \sim N(0, \sigma_w^2)$, independent.

- Not possible to find explicit conditional (or unconditional) MLEs (see p.118-119).
- Use numerical methods! (p.119-122).

Maximum likelihood

Causal and invertible ARMA(p, q) with constant, $t = 1, 2, \ldots, n$:

$x_t - \mu = \phi_1(x_{t-1} - \mu) + \ldots + \phi_p(x_{t-p} - \mu) + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}$,

i.e. $\phi(B)(x_t - \mu) = \theta(B) w_t$.

Let $\boldsymbol{\beta} = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)'$. Let $\hat{\boldsymbol{\beta}}$ be the vector of MLEs.

Theorem (Property 3.10)

1. *Under appropriate conditions, as $n \to \infty$,*

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \sigma_w^2 \Gamma_{p,q}^{-1}),$$

where

$$\Gamma_{p,q} = \begin{pmatrix} \Gamma_{\phi\phi} & \Gamma_{\phi\theta} \\ \Gamma_{\theta\phi} & \Gamma_{\theta\theta} \end{pmatrix}.$$

2. *$\hat{\boldsymbol{\beta}}$ is asymptotically efficient.*

Maximum likelihood

$$\Gamma_{p,q} = \begin{pmatrix} \Gamma_{\phi\phi} & \Gamma_{\phi\theta} \\ \Gamma_{\theta\phi} & \Gamma_{\theta\theta} \end{pmatrix},$$

where

$$\Gamma_{\phi\phi} = \begin{pmatrix} \gamma(0) & \gamma(1) & \dots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \dots & \gamma(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(p-1) & \gamma(p-2) & \dots & \gamma(0) \end{pmatrix},$$

where $\gamma(h)$ is the autocovariance function of $\phi(B)x_t = w_t$.

$\Gamma_{\theta\theta}$ is similarly constructed from $\theta(B)y_t = w_t$.

$\Gamma_{\phi\theta}$ is similarly constructed from the cross covariance function between $\phi(B)x_t = w_t$ and $\theta(B)y_t = w_t$, and $\Gamma_{\theta\phi}$ is analogous.

Maximum likelihood

Calculate the asymptotic variances (and covariances) of the ML parameter estimators for

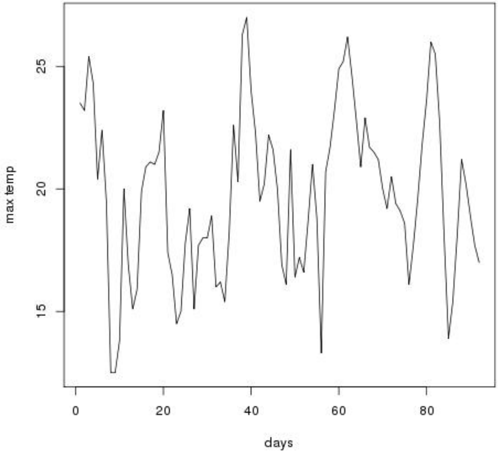
1. The MA(1) process $x_t = w_t + \theta w_{t-1}$.
2. The ARMA(1,1) process $x_t = \phi x_{t-1} + w_t + \theta w_{t-1}$.

Maximum likelihood

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

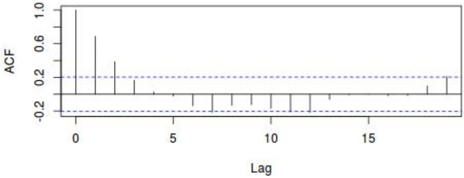
Maximum likelihood

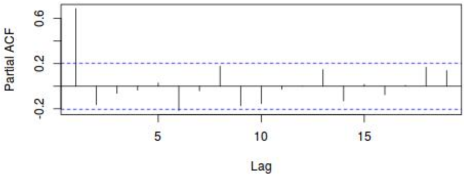
Daily temperature, Uppsala, summer 1984.



Maximum likelihood

Daily temperature, Uppsala, summer 1984, ACF (tails off) and PACF (cuts off after lag 1). Try AR(1)!

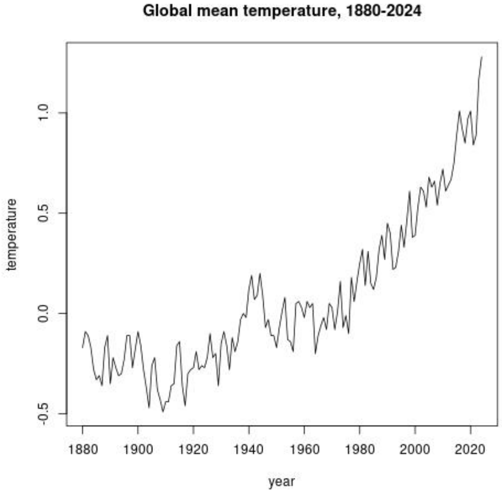




In R: (note: $\hat{\phi} \approx 0.69$ is outside the $\pm 2\cdot$ `s.e.` bound.)

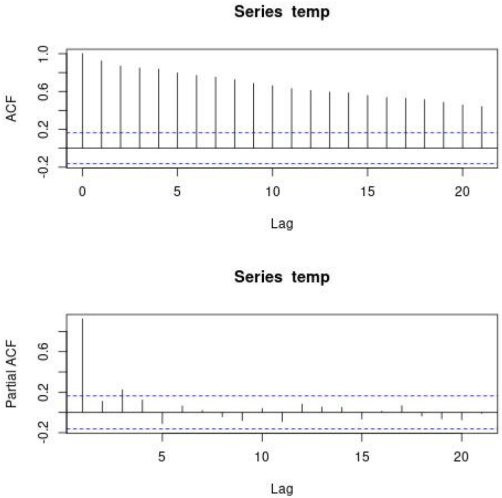
```
> x=read.table("tempUasom84.txt")$V1
> plot(x,type='l',xlab='days',ylab='max temp')
> par(mfrow=c(2,1))
> acf(x,main='')
> pacf(x,main='')
> arima(x,order=c(1,0,0))

Call:
arima(x = x, order = c(1, 0, 0))

Coefficients:
         ar1  intercept
      0.6946    19.7769
s.e.  0.0745     0.8053

sigma^2 estimated as 5.839:  log likelihood = -212.04,
aic = 430.08
```

Maximum likelihood

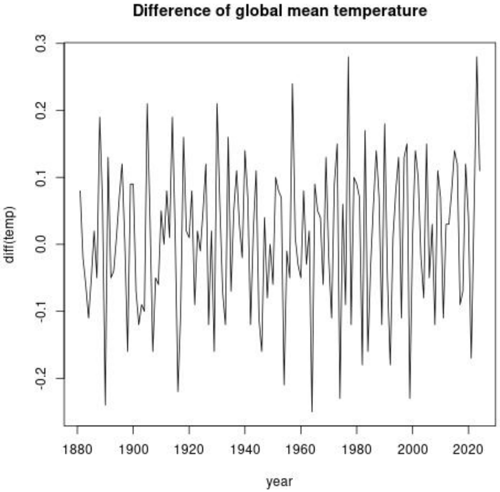


Maximum likelihood

Global mean temperature, ACF and PACF. (Typical signs of a trend.)

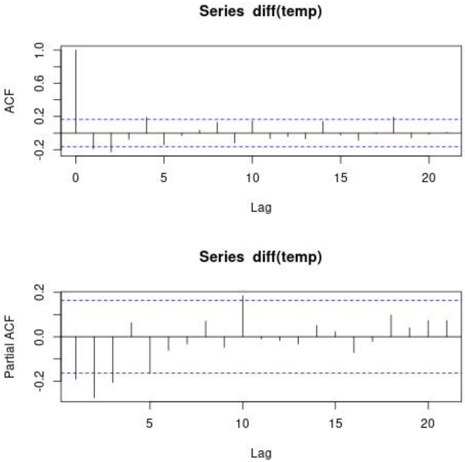


Maximum likelihood



Maximum likelihood

Difference of global mean temperature, ACF (cuts off after lag 2?) and PACF (tails off?). Try MA(2)!



Estimation in R with ML:

```
> arima(temp,order=c(0,1,2))

Call:
arima(x = temp, order = c(0, 1, 2))

Coefficients:
          ma1      ma2
      -0.3010  -0.2118
s.e.   0.0783   0.0702

sigma^2 estimated as 0.01106:  log likelihood = 119.84,
aic = -233.68
```

Estimation in R with CSS (similar results):

```
> arima(temp,order=c(0,1,2),method = "CSS")

Call:
arima(x = temp, order = c(0, 1, 2), method = "CSS")

Coefficients:
          ma1      ma2
      -0.3034  -0.2139
s.e.   0.0782   0.0703

sigma^2 estimated as 0.01106:  part log likelihood = 119.96
```

News of today

- Method of moments, Yule Walker equations
- Maximum likelihood/Least squares
  - Conditional on initial values
  - Unconditional