Problem Session 3

Probability and Martingales, 1MS045

28 October 2024

Note: If not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

Problems

1. Suppose that X is a random variable that has moments of all orders, i.e., $\mathbb{E}(|X|^p) < \infty$ for all $p > 0$. Prove that

$$\lim_{p\to\infty} (\mathbb{E}(|X|^p))^{1/p} = \inf\{K \geq 0 : \mathbb{P}(|X| > K) = 0\}.$$

   (If the set $\{K \geq 0 : \mathbb{P}(|X| > K) = 0\}$ is empty, the infimum is ∞).

2. Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(X^2) < \infty$. We define the conditional variance with respect to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ by

$$\mathrm{Var}(X|\mathcal{G}) = \mathbb{E}((X - \mathbb{E}(X|\mathcal{G}))^2|\mathcal{G}).$$

   Prove that

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\mathcal{G})) + \mathrm{Var}(\mathbb{E}(X|\mathcal{G})).$$

3. Let $Y_1, Y_2, \ldots$ be independent random variables with $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1-p$ ($p \in (0,1)$, $p \neq \frac{1}{2}$) for all i, and consider the simple biased random walk $X_n = \sum_{i=1}^n Y_i$.
   (a) Find a constant $\theta \neq 1$ such that θ^{X_n} is a martingale.
   (b) Find a (deterministic) function $f(n)$ such that $X_n - f(n)$ is a martingale.
   (c) Let a and b be positive integers. Determine the probability that X_n reaches the value a before the value $-b$.
   (d) Determine the expected number of steps until one of these two values is reached.

4. Prove: a previsible martingale X_n is almost surely constant, i.e., $X_n = X_0$ holds almost surely for all n.

5. Suppose that X and Y are integrable random variables such that $\mathbb{E}(X|\mathcal{G}) = Y$ and $\mathbb{E}(X^2|\mathcal{G}) = Y^2$. Prove that $X = Y$ almost surely.
   Hint: Consider $\mathbb{E}((X - Y)^2|\mathcal{G})$.

6. Let (X, Y) be a uniformly random point in the unit disk (centre at $(0, 0)$, radius 1). Determine $\mathbb{E}(X \mid Y)$, $\mathbb{E}(|X| \mid Y)$, $\mathbb{E}(X \mid |Y|)$, and $\mathbb{E}(|X| \mid |Y|)$.

7. Let $Y_1, Y_2, \ldots$ be independent random variables that follow a normal distribution with mean 0 and variance 1. Set $S_n = Y_1 + Y_2 + \cdots + Y_n$. Prove that

   (a) $X_n = e^{S_n - n/2}$ is a martingale.

   (b) $X_n \to 0$ almost surely as $n \to \infty$.

   (c) X_n^r is a supermartingale for $0 < r < 1$, and a submartingale for $r > 1$.