Course: Theory of Probability II
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Lecture 12
Uniform Integrability

Uniform integrability is a compactness-type concept for families of random variables, not unlike that of tightness.

Definition 12.1 (Uniform integrability)**.** A non-empty family $\mathcal{X} \subseteq \mathcal{L}^0$ of random variables is said to be **uniformly integrable (UI)** if

$$\lim_{K\to\infty} \left(\sup_{X\in\mathcal{X}} \mathbb{E}[|X| \mathbf{1}_{\{|X|\geq K\}}] \right) = 0.$$

Remark 12.2. It follows from the dominated convergence theorem (prove it!) that

$$\lim_{K\to\infty} \mathbb{E}[|X| \mathbf{1}_{\{|X|\geq K\}}] = 0 \text{ if and only if } X \in \mathcal{L}^1,$$

i.e., that for integrable random variables, far tails contribute little to the expectation. Uniformly integrable families are simply those for which the size of this contribution can be controlled uniformly over all elements.

We start with a characterization and a few basic properties of uniform-integrable families:

Proposition 12.3 (UI = $\mathcal{L}^1$-bounded + uniformly absolutely continuous)**.** *A family $\mathcal{X} \subseteq \mathcal{L}^0$ of random variables is uniformly integrable if and only if*

1. *there exists $C \geq 0$ such that $\mathbb{E}[|X|] \leq C$, for all $X \in \mathcal{X}$, and*
2. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{F}$, we have*

$$\mathbb{P}[A] \leq \delta \to \sup_{X\in\mathcal{X}} \mathbb{E}[|X| \mathbf{1}_A] \leq \varepsilon.$$

Proof. UI $\to$ 1., 2. Assume $\mathcal{X}$ is UI and choose $K > 0$ such that

$$\sup_{X\in\mathcal{X}} \mathbb{E}[|X| \mathbf{1}_{\{|X|>K\}}] \leq 1.$$

Since

$$\mathbb{E}[|X|] = \mathbb{E}[|X| \mathbf{1}_{\{|X|\leq K\}}] + \mathbb{E}[|X| \mathbf{1}_{\{|X|>K\}}] \leq K + \mathbb{E}[|X| \mathbf{1}_{\{|X|>K\}}],$$

for any X, we have $\sup_{X\in\mathcal{X}} \mathbb{E}[|X|] \leq K+1$, and 1. follows.

For 2., we take $\varepsilon > 0$ and use the uniform integrability of $\mathcal{X}$ to find a constant $K > 0$ such that $\sup_{X\in\mathcal{X}} \mathbb{E}[|X| \mathbf{1}_{\{|X|>K\}}] < \varepsilon/2$. For $\delta = \frac{\varepsilon}{2K}$ and $A \in \mathcal{F}$, the condition $\mathbb{P}[A] \leq \delta$ implies that

$$\begin{aligned}\mathbb{E}[|X| \mathbf{1}_A] &= \mathbb{E}[|X| \mathbf{1}_A \mathbf{1}_{\{|X|\leq K\}}] + \mathbb{E}[|X| \mathbf{1}_A \mathbf{1}_{\{|X|>K\}}] \\ &\leq K\mathbb{P}[A] + \mathbb{E}[|X| \mathbf{1}_{\{|X|>K\}}] \leq \varepsilon.\end{aligned}$$

1., 2. $\to$ UI. Let $C > 0$ be the bound from 1., pick $\varepsilon > 0$ and let $\delta > 0$ be such that 2. holds. For $K = \frac{C}{\delta}$, Markov's inequality gives

$$\mathbb{P}[|X| \geq K] \leq \tfrac{1}{K}\mathbb{E}[|X|] \leq \delta,$$

for all $X \in \mathcal{X}$. Therefore, by 2., $\mathbb{E}[|X| \mathbf{1}_{\{|X|\geq K\}}] \leq \varepsilon$ for all $X \in \mathcal{X}$. □

Remark 12.4. Boundedness in $\mathcal{L}^1$ is not enough for uniform integrability. For a counterexample, take $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$, and define

$$X_n(\omega) = \begin{cases} n, & \omega \leq \frac{1}{n}, \\ 0, & \text{otherwise.}\end{cases}$$

Then $\mathbb{E}[X_n] = n\frac{1}{n} = 1$, but, for $K > 0$, $\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n|\geq K\}}] = 1$, for all $n \geq K$, so $\{X_n\}_{n\in\mathbb{N}}$ is not UI.

Problem 12.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two uniformly-integrable families (on the same probability space). Show that the following families are also uniformly integrable:

1. $\{Z \in \mathcal{L}^0 : |Z| \leq |X| \text{ for some } X \in \mathcal{X}\}$.
2. $\{X + Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}$.

Another useful characterization of uniform integrability uses a class of functions which converge to infinity faster than any linear function:

Definition 12.5 (Test function of UI)**.** A Borel function $\varphi : [0,\infty) \to [0,\infty)$ is called a **test function of uniform integrability** if

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty.$$

Proposition 12.6 (Characterization of UI via test functions)**.** *A nonempty family $\mathcal{X} \subseteq \mathcal{L}^0$ is uniformly integrable if and only if there exists a test function of uniform integrability φ such that*

$$\sup_{X\in\mathcal{X}} \mathbb{E}[\varphi(|X|)] < \infty. \tag{12.1}$$

Moreover, if it exists, the function φ can be chosen in the class of non-decreasing convex functions.

The proof of necessity rests on a simple fact from analysis:

Lemma 12.7 (Functions arbitrary close to integrability). *Let $f : [0,\infty) \to [0,\infty)$ be a non-increasing function with $f(x) \to 0$, as $x \to \infty$. Then, there exists a continuous function $g : [0,\infty) \to (0,\infty)$ such that*

$$\int_0^\infty g(x)\,dx = +\infty \text{ but } \int_0^\infty f(x)g(x)\,dx < \infty. \tag{12.2}$$

Moreover, g can be chosen so that the function $x \mapsto x\int_0^x g(\xi)\,d\xi$ is convex.

Proof. Let $\tilde{f}$ be a strictly positive continuously differentiable with $\tilde{f}(x) \geq f(x)$, for all $x \geq 0$, with $\tilde{f}(x) \to 0$, as $x \to \infty$. With such $\tilde{f}$ and $g = -\tilde{f}'/\tilde{f}$, we have

$$\int_0^\infty g(x)\,dx = \lim_{x\to\infty} \Big(\ln(\tilde{f}(0)) - \ln(\tilde{f}(x)) \Big) = \infty.$$

On the other hand $\int_0^\infty f(x)g(x)\,dx \leq \int_0^\infty \tilde{f}(x)g(x)\,dx = \lim_{x\to\infty} \Big(\tilde{f}(0) - \tilde{f}(x)\Big) = \tilde{f}(0) < \infty$.

We leave it to the reader to argue that a (perhaps even larger) $\tilde{f}$ can be constructed such that $x \mapsto -x\ln\tilde{f}(x)$ is convex. □

Proof of Proposition 12.6. Suppose, first, that (12.1) holds for some test function of uniform integrability and that the value of the supremum is $0 \leq M < \infty$. For $n > 0$, there exists $C_n \in \mathbb{R}$ such that $\varphi(x) \geq nMx$, for $x \geq C_n$. Therefore,

$$M \geq \mathbb{E}[\varphi(|X|)] \geq \mathbb{E}[\varphi(|X|)\mathbf{1}_{\{|X|\geq C_n\}}] \geq nM\mathbb{E}[|X|\,\mathbf{1}_{\{|X|\geq C_n\}}],$$

for all $X \in \mathcal{X}$. Hence, $\sup_{X\in\mathcal{X}} \mathbb{E}[|X|\,\mathbf{1}_{\{|X|\geq C_n\}}] \leq \frac{1}{n}$, and the uniform integrability of $\mathcal{X}$ follows.

Conversely, if $\mathcal{X}$ is uniformly integrable the function

$$f(K) = \sup_{X\in\mathcal{X}} \mathbb{E}[|X|\,\mathbf{1}_{\{|X|\geq K\}}],$$

satisfies the conditions of Lemma 12.7 and a function g for which (12.2) holds can be constructed. Consequently, the function $\varphi(x) = x\int_0^x g(\xi)\,d\xi$ is a test-function of uniform integrability. On the other hand, for $X \in \mathcal{X}$, we have

$$\begin{aligned}\mathbb{E}[\varphi(|X|)] &= \mathbb{E}[|X| \int_0^\infty \mathbf{1}_{\{K\leq|X|\}} g(K)\,dK] \\ &= \int_0^\infty g(K)\mathbb{E}[|X|\,\mathbf{1}_{\{|X|\leq K\}}]\,dK \\ &= \int_0^\infty g(K)f(K)\,dK < \infty.\end{aligned}$$ □

Corollary 12.8 ($\mathcal{L}^p$-boundedness implies UI for $p > 1$)**.** *For $p > 1$, let $\mathcal{X}$ be a nonempty family of random variables bounded in $\mathcal{L}^p$, i.e., such that $\sup_{X \in \mathcal{X}} ||X||_{\mathcal{L}^p} < \infty$. Then $\mathcal{X}$ is uniformly integrable.*

Problem 12.2. Let $\mathcal{X}$ be a nonempty uniformly-integrable family in $\mathcal{L}^0$. Show that $\operatorname{conv} \mathcal{X}$ is uniformly-integrable, where $\operatorname{conv} \mathcal{X}$ is the smallest convex set in $\mathcal{L}^0$ which contains $\mathcal{X}$, i.e., $\operatorname{conv} \mathcal{X}$ is the set of all random variables of the form $X = \alpha_1 X_1 + \cdots + \alpha_n X_n$, for $n \in \mathbb{N}$, $\alpha_k \geq 0$, $k = 1, \dots, n$, $\sum_{k=1}^n \alpha_k = 1$ and $X_1, \dots, X_n \in \mathcal{X}$.

Problem 12.3. Let $\mathcal{C}$ be a non-empty family of sub-σ-algebras of $\mathcal{F}$, and let X be a random variable in $\mathcal{L}^1$. The family

$$\mathcal{X} = \{\mathbb{E}[X|\mathcal{F}] \, : \, \mathcal{F} \in \mathcal{C}\},$$

is uniformly integrable.

Hint: Argue that it follows directly from Proposition 12.6 that $\mathbb{E}[\varphi(|X|)] < \infty$ for some test function of uniform integrability. Then, show that the same φ can be used to prove that $\mathcal{X}$ is UI.

First properties of uniformly-integrable martingales

When it is known that the martingale $\{X_n\}_{n \in \mathbb{N}}$ is uniformly integrable, a lot can be said about its structure. We start with a definitive version of the dominated convergence theorem:

Proposition 12.9 (Improved dominated-convergence theorem)**.** *Suppose that $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of random variables in $\mathcal{L}^p$, where $p \geq 1$, which converges to $X \in \mathcal{L}^0$ in probability. Then, the following statements are equivalent:*

1. *the sequence $\{|X|_n^p\}_{n \in \mathbb{N}}$ is uniformly integrable,*
2. *$X_n \stackrel{\mathcal{L}^p}{\to} X$, and*
3. *$||X_n||_{\mathcal{L}^p} \to ||X||_{\mathcal{L}^p} < \infty$.*

Proof. 1. $\to$ 2.: Since there exists a subsequence $\{X_{n_k}\}_{k \in \mathbb{N}}$ such that $X_{n_k} \stackrel{a.s.}{\to} X$, Fatou's lemma implies that

$$\mathbb{E}[|X|^p] = \mathbb{E}[\liminf_k |X_{n_k}|^p] \leq \liminf_k \mathbb{E}[|X_{n_k}|^p] \leq \sup_{X \in \mathcal{X}} \mathbb{E}[|X|^p] < \infty,$$

where the last inequality follows from the fact that uniformly-integrable families are bounded in $\mathcal{L}^1$.

Now that we know that $X \in \mathcal{L}^p$, uniform integrability of $\{|X_n|^p\}_{n \in \mathbb{N}}$ implies that the family $\{|X_n - X|^p\}_{n \in \mathbb{N}}$ is UI (use Problem 12.1, 2.). Since $X_n \stackrel{\mathbb{P}}{\to} X$ if and only if $X_n - X \stackrel{\mathbb{P}}{\to} 0$, we can assume without loss of generality that $X = 0$ a.s., and, consequently, we need to show that

$\mathbb{E}[|X_n|^p] \to 0$. We fix an $\varepsilon > 0$, and start by the following estimate

$$\begin{aligned}\mathbb{E}[|X_n|^p] &= \mathbb{E}[|X_n|^p \mathbf{1}_{\{|X_n|^p \le \varepsilon/2\}}] + \mathbb{E}[|X_n|^p \mathbf{1}_{\{|X_n|^p > \varepsilon/2\}}] \\ &\le \varepsilon/2 + \mathbb{E}[|X_n|^p \mathbf{1}_{\{|X_n|^p > \varepsilon/2\}}].\end{aligned} \tag{12.3}$$

By uniform integrability there exists $\rho > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^p \mathbf{1}_A] < \varepsilon/2 \text{ whenever } \mathbb{P}[A] \le \rho.$$

Convergence in probability now implies that there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, we have $\mathbb{P}[|X_n|^p > \varepsilon/2] \le \rho$. It follows directly from (12.3) that for $n \ge n_0$, we have $\mathbb{E}[|X_n|^p] \le \varepsilon$.

2. $\to$ 3.: $\Big| ||X_n||_{\mathcal{L}^p} - ||X||_{\mathcal{L}^p} \Big| \le ||X_n - X||_{\mathcal{L}^p} \to 0$

3. $\to$ 1.: For $M \ge 0$, define the function $\psi_M : [0,\infty) \to [0,\infty)$ by

$$\psi_M(x) = \begin{cases} x, & x \in [0, M-1] \\ 0, & x \in [M, \infty) \\ \text{interpolated linearly}, & x \in (M-1, M). \end{cases}$$

For a given $\varepsilon > 0$, dominated convergence theorem guarantees the existence of a constant $M > 0$ (which we fix throughout) such that

$$\mathbb{E}[|X|^p] - \mathbb{E}[\psi_M(|X|^p)] < \frac{\varepsilon}{2}.$$

Convergence in probability, together with continuity of ψ_M, implies that $\psi_M(X_n) \to \psi_M(X)$ in probability, for all M, and it follows from boundedness of ψ_M and the bounded convergence theorem that

$$\mathbb{E}[\psi_M(|X_n|^p)] \to \mathbb{E}[\psi_M(|X|^p)]. \tag{12.4}$$

By the assumption 3. and (12.4), there exists $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}[|X_n|^p] - \mathbb{E}[|X|^p] < \varepsilon/4 \text{ and } \mathbb{E}[\psi_M(|X|^p)] - \mathbb{E}[\psi_M(|X_n|^p)] < \varepsilon/4,$$

for $n \ge n_0$. Therefore, for $n \ge n_0$.

$$\begin{aligned}\mathbb{E}[|X_n|^p \mathbf{1}_{\{|X_n|^p > M\}}] &\le \mathbb{E}[|X_n|^p] - \mathbb{E}[\psi_M(|X_n|^p)] \\ &\le \varepsilon/2 + \mathbb{E}[|X|^p] - \mathbb{E}[\psi_M(|X|^p)] \le \varepsilon.\end{aligned}$$

Finally, to get uniform integrability of the entire sequence, we choose an even larger value of M to get $\mathbb{E}[|X_n|^p \mathbf{1}_{\{|X_n|^p > M\}}] \le \varepsilon$ for the remaining $n < n_0$. □

Problem 12.4. For $Y \in \mathcal{L}^1_+$, show that the family $\{X \in \mathcal{L}^0 : |X| \le Y, \text{ a.s.}\}$ is uniformly integrable. Deduce the dominated convergence theorem from Proposition 12.9

Since convergence in $\mathcal{L}^p$ implies convergence in probability, we have:

Corollary 12.10 ($\mathcal{L}^p$-convergent $\to$ UI, for $p \geq 1$)**.** *Let $\{X_n\}_{n\in\mathbb{N}_0}$ be an $\mathcal{L}^p$-convergent sequence, for $p \geq 1$. Then family $\{X_n : n \in \mathbb{N}_0\}$ is UI.*

Since UI (sub)martingales are bounded in $\mathcal{L}^1$, they converge by Theorem 11.18. Proposition 12.9 guarantees that, additionally, convergence holds in $\mathcal{L}^1$:

Corollary 12.11 (UI (sub)martingales converge)**.** *Uniformly-integrable (sub)martingales converge a.s., and in $\mathcal{L}^1$.*

For martingales, uniform integrability implies much more:

Proposition 12.12 (UI martingales are Lévy martingales)**.** *Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a martingale. Then, the following are equivalent:*

1. *$\{X_n\}_{n\in\mathbb{N}_0}$ is a Lévy martingale, i.e., it admits a representation of the form $X_n = \mathbb{E}[X|\mathcal{F}_n]$, a.s., for some $X \in \mathcal{L}^1(\mathcal{F})$,*
2. *$\{X_n\}_{n\in\mathbb{N}_0}$ is uniformly integrable.*
3. *$\{X_n\}_{n\in\mathbb{N}_0}$ converges in $\mathcal{L}^1$,*

In that case, convergence also holds a.s., and the limit is given by $\mathbb{E}[X|\mathcal{F}_\infty]$, where $\mathcal{F}_\infty = \sigma(\cup_{n\in\mathbb{N}_0}\mathcal{F}_n)$.

Proof. 1. $\to$ 2. The representation $X_n = \mathbb{E}[X|\mathcal{F}_n]$, a.s., and Problem 12.3 imply that $\{X_n\}_{n\in\mathbb{N}_0}$ is uniformly integrable.

2. $\to$ 3. Corollary 12.11.

3. $\to$ 2. Corollary 12.10.

2. $\to$ 1. Corollary 12.11 implies that there exists a random variable $Y \in \mathcal{L}^1(\mathcal{F})$ such that $X_n \to Y$ a.s., and in $\mathcal{L}^1$. For $m \in \mathbb{N}$ and $A \in \mathcal{F}_m$, we have $|\mathbb{E}[X_n\mathbf{1}_A - Y\mathbf{1}_A]| \leq \mathbb{E}[|X_n - Y|] \to 0$, so $\mathbb{E}[X_n\mathbf{1}_A] \to \mathbb{E}[Y\mathbf{1}_A]$. Since $\mathbb{E}[X_n\mathbf{1}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_n]\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A]$, for $n \geq m$, we have

$$\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A], \text{ for all } A \in \cup_n\mathcal{F}_n.$$

The family $\cup_n\mathcal{F}_n$ is a π-system which generated the sigma algebra $\mathcal{F}_\infty = \sigma(\cup_n\mathcal{F}_n)$, and the family of all $A \in \mathcal{F}$ such that $\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A]$ is a λ-system. Therefore, by the $\pi - \lambda$ Theorem, we have

$$\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A], \text{ for all } A \in \mathcal{F}_\infty.$$

Therefore, since $Y \in \mathcal{F}_\infty$, we conclude that $Y = \mathbb{E}[X|\mathcal{F}_\infty]$. $\square$

Example 12.13. There exists a non-negative (and therefore a.s.-convergent) martingale which is not uniformly integrable (and therefore, not $\mathcal{L}^1$-convergent). Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a simple random walk starting from 1, i.e. $X_0 = 1$ and $X_n = 1 + \sum_{k=1}^{n} \xi_k$, where $\{\xi_n\}_{n\in\mathbb{N}}$ is an iid sequence with $\mathbb{P}[\xi_n = 1] = \mathbb{P}[\xi_n = -1] = \frac{1}{2}$, $n \in \mathbb{N}$. Clearly, $\{X_n\}_{n\in\mathbb{N}_0}$ is a martingale, and so is $\{Y_n\}_{n\in\mathbb{N}_0}$, where $Y_n = X_n^T$ and $T = \inf\{n \in \mathbb{N} : X_n = 0\}$. By convention, $\inf \emptyset = +\infty$. It is well known that a simple symmetric random walk hits any level eventually, with probability 1 (we will prove this rigorously later), so $\mathbb{P}[T < \infty] = 1$, and, since $Y_n = 0$, for $n \geq T$, we have $Y_n \to 0$, a.s., as $n \to \infty$. On the other hand, $\{Y_n\}_{n\in\mathbb{N}_0}$ is a martingale, so $\mathbb{E}[Y_n] = \mathbb{E}[Y_0] = 1$, for $n \in \mathbb{N}$. Therefore, $\mathbb{E}[Y_n] \not\to \mathbb{E}[X]$, which can happen only if $\{Y_n\}_{n\in\mathbb{N}_0}$ is *not* uniformly integrable.

Backward martingales

If, instead of $\mathbb{N}_0$, we use $-\mathbb{N}_0 = \{\dots, -2, -1, 0\}$ as the time set, the notion of a filtration is readily extended: it is still a family of sub-σ-algebras of $\mathcal{F}$, parametrized by $-\mathbb{N}_0$, such that $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, for $n \in -\mathbb{N}_0$.

Definition 12.14 (Backward (sub)martingale)**.** We say that a stochastic process $\{X_n\}_{n\in-\mathbb{N}_0}$, is a **backward submartingale** with respect to the filtration $\{\mathcal{F}_n\}_{n\in-\mathbb{N}_0}$, if

1. $\{X_n\}_{n\in-\mathbb{N}_0}$ is $\{\mathcal{F}_n\}_{n\in-\mathbb{N}_0}$-adapted,
2. $X_n \in \mathcal{L}^1$, for all $n \in \mathbb{N}_0$, and
3. $\mathbb{E}[X_n|\mathcal{F}_{n-1}] \geq X_{n-1}$, for all $n \in -\mathbb{N}_0$.

If, in addition to 1. and 2., the inequality in 3. is, in fact, an equality, we say that $\{X_n\}_{n\in-\mathbb{N}_0}$ is a **backward martingale**.

One of the most important facts about backward submartingales is that they (almost) always converge a.s., and in $\mathcal{L}^1$.

Proposition 12.15 (Backward submartingale convergence)**.** *Suppose that $\{X_n\}_{n\in-\mathbb{N}_0}$ is a backward submartingale such that*

$$\lim_{n\to-\infty} \mathbb{E}[X_n] > -\infty.$$

Then $\{X_n\}_{n\in-\mathbb{N}_0}$ is uniformly integrable and there exists a random variable $X_{-\infty} \in \mathcal{L}^1(\cap_n \mathcal{F}_n)$ such that

$$X_n \to X_{-\infty} \text{ a.s. and in } \mathcal{L}^1, \tag{12.5}$$

and

$$X_{-\infty} \leq \mathbb{E}[X_m | \cap_n \mathcal{F}_n], \text{ a.s., for all } m \in -\mathbb{N}_0. \quad (12.6)$$

Proof. We start by decomposing $\{X_n\}_{n\in-\mathbb{N}_0}$ in the manner of Doob and Meyer. For $n \in -\mathbb{N}_0$, set $\Delta A_n = \mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] \geq 0$, a.s., and $A_{-n} = \sum_{k=0}^{n} \Delta A_{-k}$, for $n \in \mathbb{N}_0$. The backward submartingale property of $\{X_n\}_{n\in\mathbb{N}_0}$ implies that $\mathbb{E}[X_n] \geq L = \lim_{n\to-\infty} \mathbb{E}[X_n] > -\infty$, so

$$\mathbb{E}[A_n] = \mathbb{E}[X_0 - X_n] \leq \mathbb{E}[X_0] - L, \text{ for all } n \in \mathbb{N}_0.$$

The monotone convergence theorem implies that $\mathbb{E}[A_{-\infty}] < \infty$, where $A_{-\infty} = \sum_{n=0}^{\infty} A_{-n}$. The process $\{M_n\}_{n\in-\mathbb{N}_0}$ defined by $M_n = X_n - A_n$ is a backward martingale. Indeed,

$$\mathbb{E}[M_n - M_{n-1}|\mathcal{F}_{n-1}] = \mathbb{E}[X_n - X_{n-1} - \Delta A_n|\mathcal{F}_{n-1}] = 0.$$

Since all backward martingales are uniformly integrable (why?) and the sequence $\{A_n\}_{n\in-\mathbb{N}_0}$ is uniformly dominated by $A_{-\infty} \in \mathcal{L}^1$ - and therefore uniformly integrable - we conclude that $\{X_n\}_{n\in-\mathbb{N}_0}$ is also uniformly integrable.

To prove convergence, we start by observing that the uniform integrability of $\{X_n\}_{n\in-\mathbb{N}_0}$ implies that $\sup_{n\in-\mathbb{N}_0} \mathbb{E}[X_n^+] < \infty$. A slight modification of the proof of the martingale convergence theorem (left to a very diligent reader) implies that $X_n \to X_{-\infty}$, a.s. for some random variable $X_\infty \in \cap_n \mathcal{F}_n$. Uniform integrability also ensures that the convergence holds in $\mathcal{L}^1$ and that $X_{-\infty} \in \mathcal{L}^1$.

In order to show (12.6), it is enough to show that

$$\mathbb{E}[X_{-\infty}\mathbf{1}_A] \leq \mathbb{E}[X_m\mathbf{1}_A], \quad (12.7)$$

for any $A \in \cap_n \mathcal{F}_n$, and any $m \in -\mathbb{N}_0$. We first note that since $X_n \leq \mathbb{E}[X_m|\mathcal{F}_n]$, for $n \leq m \leq 0$, we have

$$\mathbb{E}[X_n\mathbf{1}_A] \leq \mathbb{E}[\mathbb{E}[X_m|\mathcal{F}_n]\mathbf{1}_A] = \mathbb{E}[X_m\mathbf{1}_A],$$

for any $A \in \cap_n \mathcal{F}_n$. It remains to use the fact the $\mathcal{L}^1$-convergence of $\{X_n\}_{n\in-\mathbb{N}_0}$ implies that $\mathbb{E}[X_n\mathbf{1}_A] \to \mathbb{E}[X_{-\infty}\mathbf{1}_A]$, for all $A \in \mathcal{F}$. □

Remark 12.16. Even if $\lim \mathbb{E}[X_n] = -\infty$, the convergence $X_n \to X_{-\infty}$ still holds, but not in $\mathcal{L}^1$ and $X_{-\infty}$ may take the value $-\infty$ with positive probability.

Corollary 12.17 (Backward martingale convergence)**.** *If $\{X_n\}_{n\in-\mathbb{N}_0}$ is a backward martingale, then $X_n \to X_{-\infty} = \mathbb{E}[X_0| \cap_n \mathcal{F}_n]$, a.s., and in $\mathcal{L}^1$.*

Applications of backward martingales

We can use the results about the convergence of backward martingales to give a non-classical proof of the strong law of large numbers. Before that, we need a useful classical result.

Proposition 12.18 (Kolmogorov's 0-1 law). *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables, and let the* ***tail*** σ***-algebra*** $\mathcal{F}_{-\infty}$ *be defined by*

$$\mathcal{F}_{-\infty} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_{-n}, \text{ where } \mathcal{F}_{-n} = \sigma(\xi_n, \xi_{n+1}, \dots).$$

Then $\mathcal{F}_{-\infty}$ is $\mathbb{P}$-trivial, i.e., $\mathbb{P}[A] \in \{0,1\}$, for all $A \in \mathcal{F}_{-\infty}$.

Proof. Define $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$, and note that $\mathcal{F}_{n-1}$ and $\mathcal{F}_{-n}$ are independent σ-algebras. Therefore, $\mathcal{F}_{-\infty} \subseteq \mathcal{F}_{-n}$ is also independent of $\mathcal{F}_n$, for each $n \in \mathbb{N}$. This, in turn, implies that $\mathcal{F}_{-\infty}$ is independent of the σ-algebra $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. On the other hand, $\mathcal{F}_{-\infty} \subseteq \mathcal{F}_\infty$, so $\mathcal{F}_{-\infty}$ is independent of itself. This implies that $\mathbb{P}[A] = \mathbb{P}[A \cap A] = \mathbb{P}[A]\mathbb{P}[A]$, for each $A \in \mathcal{F}_{-\infty}$, i.e., that $\mathbb{P}[A] \in \{0,1\}$. □

Theorem 12.19 (Strong law of large numbers). *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an iid sequence of random variables in $\mathcal{L}^1$. Then*

$$\tfrac{1}{n}(\xi_1 + \dots + \xi_n) \to \mathbb{E}[\xi_1], \text{ a.s. and in } \mathcal{L}^1.$$

Proof. For notational reasons, backward martingales are indexed by $-\mathbb{N}$ instead of $-\mathbb{N}_0$. For $n \in -\mathbb{N}$, let $S_n = \xi_1 + \dots + \xi_n$, and let $\mathcal{F}_n$ be the σ-algebra generated by $S_n, S_{n+1}, \dots$. The process $\{X_n\}_{n\in-\mathbb{N}}$ is given by

$$X_{-n} = \mathbb{E}[\xi_1 | \mathcal{F}_n], \text{ for } n \in \mathbb{N}_0.$$

Since $\sigma(S_n, S_{n+1}, \dots) = \sigma(\sigma(S_n), \sigma(\xi_{n+1}, \xi_{n+2}, \dots))$, and the σ-algebra $\sigma(\xi_{n+1}, \xi_{n+2}, \dots)$ is independent of ξ_1, for $n \in \mathbb{N}$, we have

$$X_{-n} = \mathbb{E}[\xi_1 | \mathcal{F}_n] = \mathbb{E}[\xi_1 | \sigma(S_n)] = \tfrac{1}{n} S_n,$$

where the last equality follows from Problem 10.9 in Lecture 10. Backward martingales converge a.s., and in $\mathcal{L}^1$, so for the random variable $X_{-\infty} = \lim_n \frac{1}{n} S_n$ we have

$$\mathbb{E}[X_{-\infty}] = \lim_n \mathbb{E}[\tfrac{1}{n} S_n] = \mathbb{E}[\xi_1].$$

On the other hand, since $\lim_n \frac{1}{n} S_k = 0$, for all $k \in \mathbb{N}$, we have $X_{-\infty} = \lim_n \frac{1}{n}(\xi_{k+1} + \dots + \xi_n)$, for any $k \in \mathbb{N}$, and so $X_{-\infty} \in \sigma(\xi_{k+1}k, \xi_{k+2}, \dots)$. By Proposition 12.18, $X_{-\infty}$ is measurable in a $\mathbb{P}$-trivial σ-algebra, and is, thus, constant a.s. (why?). Since $\mathbb{E}[X_{-\infty}] = \mathbb{E}[\xi_1]$, we must have $X_{-\infty} = \mathbb{E}[\xi_1]$, a.s. □

Additional Problems

Problem 12.5 (A UI martingale not in H^1)**.** Set $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$, and $\mathbb{P}$ is the probability measure on $\mathcal{F}$ characterized by $\mathbb{P}[\{k\}] = 2^{-k}$, for each $k \in \mathbb{N}$. Define the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ by

$$\mathcal{F}_n = \sigma\Big(\{1\}, \{2\}, \dots, \{n-1\}, \{n, n+1, \dots\}\Big), \text{ for } n \in \mathbb{N}.$$

Let $Y : \Omega \to [1, \infty)$ be a random variable such that $\mathbb{E}[Y] < \infty$ and $\mathbb{E}[YK] = \infty$, where $K(k) = k$, for $k \in \mathbb{N}$.

1. Find an explicit example of a random variable Y with the above properties.
2. Find an expression for $X_n = \mathbb{E}[Y|\mathcal{F}_n]$ in terms of the values $Y(k)$, $k \in \mathbb{N}$.
3. Using the fact that $X^*_\infty(k) := \sup_{n\in\mathbb{N}} |X_n(k)| \geq X_k(k)$ for $k \in \mathbb{N}$, show that $\{X_n\}_{n\in\mathbb{N}}$ is a uniformly integrable martingale which is not in H^1.

Note: A martingale $\{X_n\}_{n\in\mathbb{N}}$ is said to be in H^1 if $X^*_\infty \in \mathbb{L}^1$.

Problem 12.6 (Scheffé's lemma)**.** Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a sequence of random variables in $\mathcal{L}^1_+$ such that $X_n \to X$, a.s., for some $X \in \mathcal{L}^1_+$. Show that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ if and only if the sequence $\{X_n\}_{n\in\mathbb{N}_0}$ is UI.

Problem 12.7 (Hunt's lemma)**.** Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a filtration, and let $\{X_n\}_{n\in\mathbb{N}_0}$ be a sequence in $\mathcal{L}^0$ such that $X_n \to X$, for some $X \in \mathcal{L}^0$, both in $\mathcal{L}^1$ and a.s.

1. *(Hunt's lemma)*. Assume that $|X_n| \leq Y$, a.s., for all $n \in \mathbb{N}$ and some $Y \in \mathcal{L}^1_+$. Prove that

$$\mathbb{E}[X_n|\mathcal{F}_n] \to \mathbb{E}[X|\sigma(\cup_n \mathcal{F}_n)], \text{ a.s.} \tag{12.8}$$

Hint: Define $Z_n = \sup_{m\geq n} |X_m - X|$, and show that $Z_n \to 0$, a.s., and in $\mathcal{L}^1$.

2. Find an example of a sequence $\{X_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^1$ such that $X_n \to 0$, a.s., and in $\mathcal{L}^1$, but $\mathbb{E}[X_n|\mathcal{G}]$ does not converge to 0, a.s., for some $\mathcal{G} \subseteq \mathcal{F}$. *Note:* The existence of such a sequence proves that (12.8) is not true without an additional assumption, such as the one of uniform domination in 1. It provides an example of a property which does not generalize from the unconditional to the conditional case.

Hint: Look for X_n of the form $X_n = \xi_n \frac{\mathbf{1}_{A_n}}{\mathbb{P}[A_n]}$ and $\mathcal{G} = \sigma(\xi_n; n \in \mathbb{N})$.

Problem 12.8 (Krickeberg's decomposition)**.** Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a martingale. Show that the following two statements are equivalent:

Hint: Consider $\lim_n \mathbb{E}[X^+_{m+n}|\mathcal{F}_m]$, for $m \in \mathbb{N}_0$.

1. There exists martingales $\{X^+_n\}_{n\in\mathbb{N}_0}$ and $\{X^-_n\}_{n\in\mathbb{N}_0}$ such that $X^+_n \geq 0$, $X^-_n \geq 0$, a.s., for all $n \in \mathbb{N}_0$ and $X_n = X^+_n - X^-_n$, $n \in \mathbb{N}_0$.
2. $\sup_{n\in\mathbb{N}_0} \mathbb{E}[|X_n|] < \infty$.

Problem 12.9 (Branching processes)**.** Let ν be a probability measure on $\mathcal{B}(\mathbb{R})$ with $\nu(\mathbb{N}_0) = 1$, which we call the **offspring distribution**. A population starting from one individual ($Z_0 = 1$) evolves as follows. The initial member leaves a random number Z_1 of children and dies. After that, each of the Z_1 children of the initial member, produces a random number of children and dies. The total number of all children of the Z_1 members of the generation 1 is denoted by Z_2. Each of the Z_2 members of the generation 2 produces a random number of children, etc. Whenever an individual procreates, the number of children has the distribution ν, and is independent of the sizes of all the previous generations including the present one, as well as of the numbers of children of other members of the present generation.

1. Suppose that a probability space and iid sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of random variables with the distribution μ is given. Show how you would construct a sequence $\{Z_n\}_{n\in\mathbb{N}_0}$ with the above properties.

   Hint: Z_{n+1} is a sum of iid random variables with the number of summands equal to Z_n.

2. For a distribution ρ on $\mathbb{N}_0$, we define the the **generating function** $P_\rho : [0,1] \to [0,1]$ of ρ by
$$P_\rho(x) = \sum_{k\in\mathbb{N}_0} \rho(\{k\})x^k.$$
   Show that each P_ρ is continuous, non-decreasing and convex on $[0,1]$ and continuously differentiable on $(0,1)$.

3. Let $P = P_\nu$ be the generating function of the offspring distribution ν, and for $n \in \mathbb{N}_0$, we define $P_n(x)$ as the generating function of the distribution of Z_n, i.e., $P_n(x) = \sum_{k\in\mathbb{N}_0} \mathbb{P}[Z_n = k]x^k$. Show that $P_n(x) = P(P(\dots P(x)\dots))$ (there are n Ps).

   Hint: Note that $P(x) = \mathbb{E}[x^{Z_n}]$ for $x > 0$ and use the result of Problem 10.6

4. Define the **extinction probability** p_e by $p_e = \mathbb{P}[Z_n = 0,$ for some $n \in \mathbb{N}]$. Prove that p_e is a fixed point of the map P, i.e., that $P(p_e) = p_e$.

   Hint: Show that $p_e = \lim_n P^{(n)}(0)$, where $P^{(n)}$ is the n-fold composition of P with itself.

5. Let $\mu = \mathbb{E}[Z_1]$, be the expected number of offspring. Show that when $\mu \le 1$ and $\nu(\{1\}) < 1$, we have $p_e = 1$, i.e., the population dies out with certainty if the expected number of offspring does not exceed 1.

   Hint: Draw a picture of the functions x and $P(x)$ and use (and prove) the fact that, as a consequence of the assumption $\mu \le 1$, we have $P'(x) < 1$ for all $x < 1$.

6. Assuming that $0 < \mu < \infty$, show that the process $\{X_n\}_{n\in\mathbb{N}_0}$, given by $X_n = Z_n/\mu^n$, is a martingale (with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$, where $\mathcal{F}_n = \sigma(Z_0, Z_1, \dots, Z_n)$).

7. Identify all probability measures ν with $\nu(\mathbb{N}_0) = 1$, and $\sum_{k\in\mathbb{N}_0} k\nu(\{k\}) = 1$ such that the branching process $\{Z_n\}_{n\in\mathbb{N}_0}$ with the offspring distribution ν is uniformly integrable.

Problem 12.10 (A bit of everything)**.** Given two independent simple symmetric random walks $\{\tilde{X}_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{Y}_n\}_{n\in\mathbb{N}_0}$, let $\{X_n\}_{n\in\mathbb{N}_0}$ denote $\{\tilde{X}_n\}_{n\in\mathbb{N}_0}$ stopped when it first hits the level 1, and let $\{Y_n\}_{n\in\mathbb{N}_0}$

be defined by

$$Y_0 = 0,\ Y_n = \sum_{k=1}^{n} 2^{-k}(\tilde{Y}_k - \tilde{Y}_{k-1}).$$

Identify the distribution of $\liminf_n (X_n + Y_n)$ and show that the sequence $\{X_n + Y_n\}_{n\in\mathbb{N}_0}$ is *not* uniformly integrable.