Chapter 6

Laplace equation

In this chapter we consider Laplace equation in d-dimensions given by

$$u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_dx_d} = 0. \tag{6.1}$$

We study Laplace equation in $d = 2$ throughout this chapter (excepting Section 6.2), and most of the ideas can be generalized to general space dimensions $d > 2$. When $d = 2$, the independent variables x_1, x_2 are denoted by x, y, and write $\boldsymbol{x} = (x, y)$. Thus Laplace equation in two independent variables is

$$u_{xx} + u_{yy} = 0. \tag{6.2}$$

The non-homogeneous problem

$$u_{xx} + u_{yy} = f, \tag{6.3}$$

where f is a function of the independent variables x, y only is called the Poisson equation.

6.1 ▪ Green's identities

Green's Identities play an important role in the analysis of Laplace equation. They are derived from divergence theorem. Let us recall the divergence theorem now.

Theorem 6.1 (Divergence theorem). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded piecewise smooth domain. Let $\Psi : \Omega \to \mathbb{R}^2$ be a function. Let $\Psi = (\psi_1, \psi_2)$ where $\psi_i \in C^1(\overline{\Omega}) \cap C(\overline{\Omega})$ for $i = 1, 2$. Then*

$$\int_\Omega \nabla.\Psi(x,y)\,dx\,dy = \int_{\partial\Omega} \Psi.\boldsymbol{n}\,d\sigma, \tag{6.4}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. ∎

Theorem 6.2 (Green's identities). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded piecewise smooth domain, and let $\boldsymbol{n}$ denote the unit outward normal to $\partial\Omega$. For $u, v \in C^2(\overline{\Omega}) \cap C^1(\overline{\Omega})$, we have*

(i) **(Green's Identity-I)**

$$\int_\Omega \Delta u(x,y)\,dx\,dy = \int_{\partial\Omega} \partial_{\boldsymbol{n}} u\,d\sigma, \tag{6.5}$$

where $\partial_{\boldsymbol{n}} u := \nabla u.\boldsymbol{n}$ is called the normal derivative of u; as it represents the directional derivative of u in the dirction of the unit outward normal $\boldsymbol{n}$.

(ii) **(Green's Identity-II)**

$$\int_\Omega \big(v\Delta u - u\Delta v\big)(x,y)\,dx\,dy = \int_{\partial\Omega} \big(v\,\partial_{\boldsymbol{n}} u - u\,\partial_{\boldsymbol{n}} v\big)\,d\sigma \tag{6.6}$$

(iii) **(Green's Identity-III)**

$$\int_\Omega \nabla u.\nabla v\,dx\,dy = \int_{\partial\Omega} v\,\partial_{\boldsymbol{n}} u\,d\sigma - \int_\Omega v\Delta u\,dx\,dy. \tag{6.7}$$

Proof. Applying (6.4) with $\Psi = \nabla u$, $\Psi = v\nabla u - u\nabla v$, and $\Psi = v\nabla u$ yield Green's identities I, II, and III respectively. □

6.2 ▪ Fundamental solutions in $\mathbb{R}^d$

Since Lapalce equation is invariant under translations, and rotations (see Exercise 6.4), we look for solutions to Laplace equation having such symmetric properties.

Let us fix $\xi \in \mathbb{R}^d$, and look for solutions to $\Delta u = 0$ having the form $v_\xi(\boldsymbol{x}) = \psi(r)$, where

$$r = \|x - \xi\| = \sqrt{\sum_{i=1}^{d}(x_i - \xi_i)^2}.$$

Substituting the formula for v_ξ in the equation, we get

$$\Delta v_\xi(\boldsymbol{x}) = \psi''(r) + \frac{d-1}{r}\psi'(r) = 0.$$

Solving the last equation, we get

$$\psi'(r) = C\,r^{1-d} \tag{6.8}$$

Integrating we get

$$\psi(r) = \begin{cases} C\ln r & \text{if } d = 2, \\ \frac{C}{2-d}r^{2-d} & \text{if } d \geq 3. \end{cases} \tag{6.9}$$

Theorem 6.3. *Let $K(\boldsymbol{x},\xi)$ denote the function defined by $K(\boldsymbol{x},\xi) := \psi(\|\boldsymbol{x}-\xi\|)$.*

(i) *Let ξ be a point of Ω. For $u \in C^2(\overline{\Omega})$ the following identity holds*

$$u(\xi) = \int_\Omega K(\boldsymbol{x},\xi)\Delta u\,d\boldsymbol{x} - \int_{\partial\Omega} (K(\boldsymbol{x},\xi)\partial_{\boldsymbol{n}} u - u\,\partial_{\boldsymbol{n}} K(\boldsymbol{x},\xi))\,d\sigma. \tag{6.10}$$

(ii) *The following equality holds in the sense of distributions on $\mathbb{R}^d$:*

$$\Delta K(x,\xi) = \delta_\xi.$$

i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ the following equality holds.

$$\varphi(\xi) = \int_{\mathbb{R}^d} K(\boldsymbol{x},\xi)\Delta\varphi(\boldsymbol{x})\,d\boldsymbol{x}. \tag{6.11}$$

(iii) *If $u \in C^2(\overline{\Omega})$ and harmonic in Ω (i.e., $\Delta u = 0$ in Ω), then for $\xi \in \Omega$ we get*

$$u(\xi) = -\int_{\partial\Omega} (K(\boldsymbol{x},\xi)\partial_{\boldsymbol{n}} u - u\partial_{\boldsymbol{n}} K(\boldsymbol{x},\xi))\, d\sigma. \tag{6.12}$$

Proof. **Step 1: Proof of (i):**

Let $u \in C^2(\overline{\Omega})$ and ξ be a point of Ω. Note that we cannot apply Green's identity II (6.6) with $v = v_\xi(\boldsymbol{x}) = \psi(\|x-\xi\|)$ since v_ξ is singular at $\boldsymbol{x} = \xi$. Thus we cut out a ball $B(\xi,\rho)$ from Ω along with its boundary, and then apply Green's identity II. Note that the domain $\Omega_\rho ;= \Omega \setminus B[\xi,\rho]$ is bounded by $\partial\Omega$ and $S(\xi,\rho)$. Since $\Delta v_\xi = 0$ in Ω_ρ, we have

$$\int_{\Omega_\rho} v_\xi \Delta u \, d\boldsymbol{x} = \int_{\partial\Omega} \left(v_\xi \partial_{\boldsymbol{n}} u - u\partial_{\boldsymbol{n}} v_\xi\right) d\sigma + \int_{S(\xi,\rho)} \left(v_\xi \partial_{\boldsymbol{n}} u - u\partial_{\boldsymbol{n}} v_\xi\right) d\sigma. \tag{6.13}$$

Let us now compute the second term on RHS of the equation (6.13).

$$\int_{S(\xi,\rho)} \left(v_\xi \partial_{\boldsymbol{n}} u - u\partial_{\boldsymbol{n}} v_\xi\right) d\sigma = \int_{S(\xi,\rho)} v_\xi \partial_{\boldsymbol{n}} u \, d\sigma - \int_{S(\xi,\rho)} u\partial_{\boldsymbol{n}} v_\xi \, d\sigma \tag{6.14}$$

Note that on the circle $S(\xi,\rho)$, we have $v_\xi(\boldsymbol{x}) = \psi(\|x-\xi\|) = \psi(\rho)$. Using this information, and divergence theorem, we get

$$\int_{S(\xi,\rho)} v_\xi \partial_{\boldsymbol{n}} u \, d\sigma = \psi(\rho)\int_{S(\xi,\rho)} \partial_{\boldsymbol{n}} u \, d\sigma = -\psi(\rho)\int_{B(\xi,\rho)} \Delta u \, d\boldsymbol{x}. \tag{6.15}$$

Note that the outward unit normal $\boldsymbol{n}$ on the circle $S(\xi,\rho)$ points towards its center ξ. Also $\partial_{\boldsymbol{n}} v_\xi = -\psi'(\rho)$ holds at points on the circle $S(\xi,\rho)$. Thus we get

$$\int_{S(\xi,\rho)} u\partial_{\boldsymbol{n}} v_\xi \, d\sigma = -C\rho^{1-d}\int_{S(\xi,\rho)} u \, d\sigma. \tag{6.16}$$

Since both u and Δu are continuous at ξ, we have the following convergences as $\rho \to 0$:

$$\psi(\rho)\int_{B(\xi,\rho)} \Delta u \, d\boldsymbol{x} \to 0, \quad \rho^{1-d}\int_{S(\xi,\rho)} u \, d\sigma \to \omega_d u(\xi), \tag{6.17}$$

where ω_d denotes the surface area of the unit sphere in $\mathbb{R}^d$.

Now passing to the limit as $\rho \to 0$ in the equation (6.13), we get

$$\int_{\Omega} v_\xi \Delta u \, d\boldsymbol{x} = \int_{\partial\Omega} \left(v_\xi \partial_{\boldsymbol{n}} u - u\partial_{\boldsymbol{n}} v_\xi\right) d\sigma + C\omega_d u(\xi) \tag{6.18}$$

in view of (6.15), (6.16), (6.17).

Choosing $C = \frac{1}{\omega_d}$ in the formulae (6.9), the equation (6.18) yields (6.10).

Step 2: Proof of (ii): In the equation (6.10) if we take $u = \varphi \in C_0^\infty(\Omega)$, then we get (6.11).

Step 3: Proof of (iii): Equation (6.12) is an immediate consequence of (6.10). □

6.3 ▪ Boundary value problems associated to Laplace equation

We can study Laplace equation or Poisson equation on any open domain $\Omega \subseteq \mathbb{R}^2$. These problems will always have an infinite number of solutions. Thus it is interesting to know if there are solutions to these equations if some constraints are placed on the unknown function. When Ω is a bounded domain with piecewise smooth boundary, there are atleast three different kinds of restrictions are placed on the unknown function and its derivatives along the boundary $\partial\Omega$ of the domain Ω. Such restrictions are called boundary conditions.

Let us now describe three boundary value problems (BVP) for Poisson equation. They differ in the nature of boundary conditions. We also define the notions of solutions to these boundary value problems.

1. For given functions f, g, **Dirichlet Problem** consists of solving the boundary value problem

$$\Delta u = f(x) \quad \text{in} \quad \Omega, \tag{6.19a}$$

$$u = g \quad \text{on} \quad \partial\Omega. \tag{6.19b}$$

Definition 6.4 (Solution to Dirichlet BVP). *Let $f \in C(\Omega)$, and $g \in C(\partial\Omega)$. A function $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ is said to be a solution to Dirichlet BVP if*

(i) *the function φ is a solution to Poisson equation (6.19a), i.e., for each $x \in \Omega$,*

$$\Delta\varphi(x) = f(x)$$

holds, and

(ii) *for each $x \in \partial\Omega$, the equality $\varphi(x) = g(x)$ holds.*

2. For given functions f, g, **Neumann Problem** consists of solving the boundary value problem

$$\Delta u = f(x) \quad \text{in} \quad \Omega, \tag{6.20a}$$

$$\partial_{\boldsymbol{n}} u = g \quad \text{on} \quad \partial\Omega. \tag{6.20b}$$

Definition 6.5 (Solution to Neumann BVP). *Let $f \in C(\Omega)$, and $g \in C(\partial\Omega)$. A function $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is said to be a solution to Neumann BVP if*

(i) *the function φ is a solution to Poisson equation (6.20a), i.e., for each $x \in \Omega$,*

$$\Delta\varphi(x) = f(x)$$

holds, and

(ii) *for each $x \in \partial\Omega$, the equality $\partial_{\boldsymbol{n}}\varphi(x) = g(x)$ holds.*

3. For given functions f, g, and $\alpha \in \mathbb{R}$, **Third Boundary Value Problem a.k.a. Robin Problem** consists of solving the boundary value problem

$$\Delta u = f(x) \quad \text{in} \quad \Omega, \tag{6.21a}$$

$$u + \alpha\partial_{\boldsymbol{n}} u = g \quad \text{on} \quad \partial\Omega. \tag{6.21b}$$

Definition 6.6 (Solution to Robin BVP). *Let $f \in C(\Omega)$, and $g \in C(\partial\Omega)$. A function $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is said to be a solution to Robin BVP if*

(i) *the function φ is a solution to Poisson equation (6.21a), i.e., for each $x \in \Omega$,*

$$\Delta\varphi(x) = f(x)$$

holds, and

(ii) *for each $x \in \partial\Omega$, the equality $\varphi(x) + \alpha\partial_{\boldsymbol{n}}\varphi(x) = g(x)$ holds.*

Remark 6.7.

(i) If Dirichlet BVP is posed on a bounded domain Ω, then the corresponding Dirichlet problem is called an **interior Dirichlet problem**. If Ω is the complement of a bounded domain, then corresponding Dirichlet problem is called **exterior Dirichlet problem**.

(ii) There are other kinds of boundary value problems possible. For example, on a part of the boundary $\partial\Omega$ one of the three boundary value problems described above is imposed and on the remaining part another one of the above three. We do not consider such boundary value problems here.

(iii) Cauchy-Kowalewski theorem guarantees that a solution of an analytic Cauchy problem for an elliptic equation exists and is unique (locally), but is not always well-posed. ∎

The following result says that for Neumann BVP to admit a solution, the data f, g must be compatible with each other.

Lemma 6.8 (Compatibility of data for Neumann problem). *Let $f \in C(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ is a solution to Neumann BVP on Ω, then*

$$\int_{\Omega} f(x)\,dx = \int_{\partial\Omega} g(y)\,d\sigma(y). \tag{6.22}$$

Proof. Integrating both sides of the equation $\Delta u = f$ on Ω yields

$$\int_{\Omega} f(x)\,dx = \int_{\Omega} \Delta u(x)\,dx. \tag{6.23}$$

Applying Green's identity-I (6.5), the integral on the right hand side of the equation (6.23) becomes

$$\int_{\Omega} \Delta u(x)\,dx = \int_{\partial\Omega} \partial_{\boldsymbol{n}} u(y)\,d\sigma(y). \tag{6.24}$$

This completes the proof. □

In particular, when u is a solution to Neumann BVP with $f = 0$, we get

$$\int_{\partial\Omega} g(y)\,d\sigma(y) = 0.$$

Theorem 6.9 (Uniqueness theorem). *Let Ω be a smooth domain. Then*

(a) *The Dirichlet problem has at most one solution.*

(b) *If u solves the Neumann problem, then any other solution is of the form $v = u + c$ for some real number c.*

(c) *If $\alpha \geq 0$, then the Robin problem has at most one solution.*

Proof. **Proof of (c):** Note that (a) is a special case of (c), hence it is enough to prove (c). Let u_1, u_2 be solutions of the Robin problem. Define $w := u_1 - u_2$. Observe that w is a harmonic function(why?), and w satisfies the boundary condtion

$$w + \alpha \partial_{\boldsymbol{n}} w = 0. \tag{6.25}$$

Using the Green's identity-III (6.7) with $u = v = w$, we get

$$\int_{\Omega} |\nabla w|^2 \, dx \, dy = -\alpha \int_{\partial\Omega} (\partial_{\boldsymbol{n}} w)^2 \, ds. \tag{6.26}$$

Since the left hand side of (6.26) is non-negative while the right hand side is non-positive, we conclude that both sides must be zero. This implies that $\nabla w = 0$ in Ω and also $\partial_{\boldsymbol{n}} w = 0$ on $\partial\Omega$. Since $\nabla w = 0$ in Ω, w must be a constant function. Since $\partial_{\boldsymbol{n}} w = 0$, in view of the equality (6.25), we conclude that $w = 0$ on the boundary $\partial\Omega$. Since $w \in C(\overline{\Omega})$, it follows that $w \equiv 0$ in Ω. This finishes the proof of (c).

Proof of (b): Let u_1, u_2 be solutions of the Neumann problem. Define $w := u_1 - u_2$. Observe that w is a harmonic function(why?), and w satisfies the boundary condtion

$$\partial_{\boldsymbol{n}} w = 0. \tag{6.27}$$

Using the Green's identity-III (6.7) with $u = v = w$, we get

$$\int_{\Omega} |\nabla w|^2 \, dx \, dy = 0. \tag{6.28}$$

This implies that w is a constant function. Thus $u_1 = u_2 + c$ for some constant $c \in \mathbb{R}$. $\square$

Remark 6.10 (On the formula (6.12)). (i) If Dirichlet problem (6.19) has a solution, then by Theorem 6.9 Dirichlet problem has exactly one solution. Let the solution to Dirichlet problem be denoted by u. The formula (6.12) gives a representation of the solution in terms of the boundary values of u and its normal derivative $\partial_{\boldsymbol{n}} u$ on $\partial\Omega$.

(ii) Note however that in Dirichlet problem, only the values of u are prescribed on $\partial\Omega$, and $\partial_{\boldsymbol{n}} u$ is an unknown function on $\partial\Omega$, and thus the formula (6.12) is not useful for computing the solution.

(iii) Note that boundary values of u already determine a solution to Dirichlet problem, and thus the quantity $\partial_{\boldsymbol{n}} u$ is already determined.

(iv) A related question concerning Laplace equation is whether Cauchy problem is well-posed for Laplace equation. The answer to this question is negative, and is made precise in Theorem 6.31

6.4 ▪ Poisson's formula and its applications

In this section, we consider Dirichlet boundary value problem for Laplace equation on the domain $\Omega = B(0,1) = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$ in $\mathbb{R}^2$. For a given function $f \in C(S(0,1))$, we consider the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad B(0,1), \tag{6.29a}$$
$$u = f \quad \text{on} \quad S(0,1). \tag{6.29b}$$

We are going to prove the following result.

Theorem 6.11 (Poisson's formula). *The solution to the BVP (6.29) is given by the Poisson's formula*

$$u(x) = \frac{1-\|x\|^2}{2\pi} \int_{S(0,1)} \frac{f(y)}{\|x-y\|^2}\, dy. \tag{6.30}$$

Defining $v(r,\theta) := u(r\cos\theta, r\sin\theta)$, and $F(\theta) := f(\cos\theta, \sin\theta)$, Poisson's formula in polar coordinates is given by

$$v(r,\theta) = \frac{1-r^2}{2\pi} \int_0^{2\pi} \left[\frac{F(\tau)}{1-2r\cos(\tau-\theta)+r^2} \right] d\tau \tag{6.31}$$

under the condition that $F \in C^1[0,2\pi]$ and is a 2π-periodic function.

Proof. Due to the geometry of disk, it is convenient to consider Laplace equation expressed in polar coordinates (see Exercise 6.1). Thus we consider the Dirichlet problem given by

$$\Delta_{(r,\theta)} v \equiv v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \quad \text{for } 0 < r < 1, 0 \le \theta \le 2\pi, \tag{6.32a}$$
$$v(1,\theta) = F(\theta) \qquad \text{for } 0 \le \theta \le 2\pi, \tag{6.32b}$$

The proof of theorem is divided into three main steps. In Step 1, we derive a formal solution to Dirichlet BVP. In Step 2, we justify that the formal solution derived in the first step is indded a solution. In Step 3, Poisson's formula will be obtained.

Step 1: Derivation of a formal solution

We obtain a solution of the equation (6.32a) using the method of separation of variables, in which solution is assumed to be of the form

$$v(r,\theta) = h(r) g(\theta), \tag{6.33}$$

where g is assumed to be periodic of period 2π *i.e.,* $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$.

Substituting the expression for v into (6.32a), and then dividing the resultant equation with $h(r)g(\theta)$ we get

$$\frac{h''(r)}{h(r)} + \frac{1}{r}\frac{h'(r)}{h(r)} + \frac{1}{r^2}\frac{g''(\theta)}{g(\theta)} = 0 \tag{6.34}$$

Re-arranging terms in (6.34) yields

$$r^2 \frac{h''(r)}{h(r)} + r\frac{h'(r)}{h(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda \tag{6.35}$$

Note that the LHS of the equation (6.35) is a function of r only, while the RHS is a function of θ only. Thus both of them must be equal to a constant, and let us denote it by λ. Thus we have

$$r^2\frac{h''(r)}{h(r)}+r\frac{h'(r)}{h(r)}=-\frac{g''(\theta)}{g(\theta)}=\lambda \tag{6.36}$$

Thus we get two ODEs from (6.36) which are given by

$$g''(\theta)+\lambda g(\theta)=0, \tag{6.37a}$$
$$r^2h''(r)+rh'(r)=\lambda h(r) \tag{6.37b}$$

Let us look for solutions of (6.37a) satisfying the periodic boundary conditions

$$g(0)=g(2\pi),\ \ g'(0)=g'(2\pi) \tag{6.38}$$

We are interested in finding non-trivial solutions to the boundary value problem (6.32), existence of which depend on the value of λ. Any λ for which the BVP (6.32) admits a non-trivial solution is called an eigenvalue and the corresponding non-trivial solutions are called eigenfunctions. Thus we are interested in finding non-trivial solutions to ODEs (6.37a) with boundary conditions (6.38).

(i) ($\lambda=0$) General solution of the ODE (6.37a) is given by $g(\theta)=a\theta+b$. Applying the boundary conditions (6.38), we get $a=0$, and b arbitrary. Thus $\lambda=0$ is a simple eigenvalue, and $g(\theta)=1$ is an eigenfunction. We need to solve ODE (6.37b) with $\mu=0$, which is given by

$$r^2h''(r)+rh'(r)=0. \tag{6.39}$$

The general solution of (6.39) is given by

$$h_0(r)=A\log r+B \tag{6.40}$$

where A,B are real numbers. Since we are interested in a solution of Laplace equation in the unit disk, we are looking for bounded solutions, and thus $A=0$. Thus we obtain the following solution of Laplace equation:

$$v_0(r,\theta)=h_0(r)g_0(\theta)=1 \tag{6.41}$$

(ii) ($\lambda>0$) When $\lambda>0$, we may write $\lambda=\mu^2$ where $\mu>0$. The ODE (6.37a) then becomes

$$g''(\theta)+\mu^2g(\theta)=0,$$

whose general solution is given by $g(\theta)=a\cos(\mu\theta)+b\sin(\mu\theta)$. Applying the boundary conditions (6.38), we get

$$a=a\cos(2\mu\pi)+b\sin(2\mu\pi)$$
$$b\mu=-a\mu\sin(2\mu\pi)+b\mu\cos(2\mu\pi).$$

The above system of linear equations may be written as

$$\begin{pmatrix} 1-\cos(2\mu\pi) & -\sin(2\mu\pi) \\ \sin(2\mu\pi) & 1-\cos(2\mu\pi) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.42}$$

Note that the system (6.42) possesses non-trivial solutions if and only if $\cos(2\mu\pi) = 1$. This means that $\mu \in \mathbb{N}$. Thus we get a sequence of eigenvalues $\lambda_n = n^2$ indexed by $n \in \mathbb{N}$, and a basis for the corresponding eigenspace is given by $\{\cos(n\theta), \sin(n\theta)\}$.

For each $n \in \mathbb{N}$, we need to solve ODE (6.37b) with $\mu = n^2$, which is given by

$$r^2 h''(r) + r h'(r) - n^2 h(r) = 0. \tag{6.43}$$

Note that this is an ODE of Cauchy-Euler type, which reduces to a constant coefficient ODE by a change of variable $s = \log r$, and hence can be easily solved, which is left as an exercise to the reader. The solutions are: For each $n \in \mathbb{N}$, the general solution of (6.43) is given by

$$h_n(r) = A r^n + B r^{-n} \tag{6.44}$$

where A, B are real numbers. Since we are interested in a solution of Laplace equation in the unit disk, we are looking for solutions that remain bounded as $r \to 0$, and thus $B = 0$. For each $n \in \mathbb{N}$, we obtain the following solution of Laplace equation:

$$v_n(r,\theta) = h_n(r) g_n(\theta) = r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \tag{6.45}$$

(iii) ($\lambda < 0$) When $\lambda < 0$, we may write $\lambda = -\mu^2$ where $\mu > 0$. The ODE (6.37a) then becomes

$$g''(\theta) - \mu^2 g(\theta) = 0,$$

whose general solution is given by $g(\theta) = a e^{\mu\theta} + b e^{-\mu\theta}$. Applying the boundary conditions (6.38), we get

$$\begin{aligned} a + b &= a e^{2\mu\pi} + b e^{-2\mu\pi} \\ a\mu - b\mu &= a\mu e^{2\mu\pi} - b\mu e^{-2\mu\pi}. \end{aligned}$$

The above system of linear equations may be written as

$$\begin{pmatrix} 1 - e^{2\mu\pi} & 1 - e^{-2\mu\pi} \\ 1 - e^{2\mu\pi} & -1 + e^{-2\mu\pi} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.46}$$

Note that the determinant of the coefficient matrix in (6.46) is zero if and only if $e^{2\mu\pi} + e^{-2\mu\pi} = 2$. Note that this relation is of the form $\alpha + \frac{1}{\alpha} = 2$ for which $\alpha = 1$ is the only solutoin. Since $\mu > 0$, the equation $e^{2\mu\pi} + e^{-2\mu\pi} = 2$ has no positive solution for μ. Thus any λ with $\lambda < 0$ is not an eigenvalue.

Consider the function v defined by superposition of v_0 and v_n for $n \in \mathbb{N}$ (which were obtained above) as follows:

$$v(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right), \tag{6.47}$$

where a_n, b_n are real numbers. We choose these constants so that v satisfies the boundary condition $v(1,\theta) = F(\theta)$. Thus we are led to the relation

$$v(1,\theta) = F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right). \tag{6.48}$$

Let the constants a_n, b_n be chosen as the Fourier coefficients of $F(\theta)$, *i.e.,*

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(\theta)\cos(n\theta)\,d\theta,\ b_n = \frac{1}{\pi}\int_0^{2\pi} F(\theta)\sin(n\theta)\,d\theta. \tag{6.49}$$

This finishes the construction of a formal solution of the Dirichlet boundary value problem. The solution is given by (6.47), where the coefficients a_n, b_n are given by (6.49).

Step 2: Formal solution is indeed a solution

We are going to show that v defined by (6.47) is indeed a solution of the Dirichlet boundary value problem (6.32).

Substituting the values of a_n, b_n from (6.49) into (6.47), we get

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} F(\tau)\,d\tau + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n\left(\int_0^{2\pi} F(\tau)\cos(n(\tau-\theta))\,d\tau\right) \tag{6.50}$$

As in the proof of Theorem 4.21, from the definition of a_n, b_n it follows that there exists an $M > 0$ such that for all $n \in \mathbb{N}$ we have

$$|a_n| \leq M,\ n|a_n| \leq M;\ \text{and } |b_n| \leq M,\ n|b_n| \leq M,$$

which follow from the formulae (6.49) and using integration by parts once in them.

As a consequence of the above estimates on a_n, b_n, the Fourier series for $F(\theta)$ converges uniformly, and hence the formal series in (6.47) for all $r \leq R < 1$ where $0 < R < 1$ is an arbitrary but fixed number. It also follows that series in (6.47) can be differentiated term-by-term and the resultant series also converges uniformly for all $r \leq R < 1$. Since each term in the series is a solution to Laplace equation, so will be $v(r,\theta)$. This proves that the formal solution is indeed a solution.

Step 3: Poisson's formula

Since the series in (6.47) is uniformly convergent for $r \leq R < 1$, the summation and integral in (6.50) can be interchanged. Thus we get

$$\begin{aligned} v(r,\theta) &= \frac{1}{\pi}\int_0^{2\pi} F(\tau)\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos(n(\tau-\theta))\right]d\tau \\ &= \frac{1}{\pi}\int_0^{2\pi} F(\tau)\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n\left[\frac{e^{in(\tau-\theta)} + e^{-in(\tau-\theta)}}{2}\right]\right]d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} F(\tau)\left[1 + \frac{re^{i(\tau-\theta)}}{1-re^{i(\tau-\theta)}} + \frac{re^{-i(\tau-\theta)}}{1-re^{-i(\tau-\theta)}}\right]d\tau \end{aligned}$$

Simplifying the last equation, we get the following Poisson's summation formula for the solution of Laplace equation for $r < 1$:

$$v(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi}\left[\frac{F(\tau)}{1-2r\cos(\tau-\theta)+r^2}\right]d\tau. \tag{6.51}$$

In cartesian coordinates, the formula (6.51) becomes

$$u(x) = \frac{1-||x||^2}{2\pi} \int_{S(0,1)} \frac{f(y)}{||x-y||^2}\, dy. \tag{6.52}$$

Checking that $u(x)$ given by (6.52) satisfies the boundary condition (6.29b) is left as an exercise (see Exercise 6.13). □

6.4.1 ▪ Harnack's inequality

Theorem 6.12 (Harnack's inequality). *Let $B(0,R) \subset \mathbb{R}^d$ be the open ball with center at 0 and having radius R. Let $u : B(0,R) \to [0,\infty)$ be a harmonic function. Then for any $x \in B(0,R)$, the following inequalities hold:*

$$\frac{R^{d-2}(R-||x||)}{(R+||x||)^{d-1}} u(0) \le u(x) \le \frac{R^{d-2}(R+||x||)}{(R-||x||)^{d-1}} u(0). \tag{6.53}$$

Proof. Let us prove this for $d=2$. Recall from Poisson's formula (6.52)

$$u(x) = \frac{1-||x||^2}{2\pi} \int_{S(0,1)} \frac{u(y)}{||x-y||^2}\, dy. \tag{6.54}$$

As a consequence of triangle inequality, for $y \in S(0,1)$ the following inequalities hold:

$$1-||x|| \le ||x-y|| \le 1+||x||.$$

Using these inequalities in the Poisson's formula (6.54) yields

$$u(x) \le \frac{1+||x||}{1-||x||} \frac{1}{2\pi} \int_{S(0,1)} u(y)\, dy = \frac{1+||x||}{1-||x||} u(0),$$

and

$$u(x) \ge \frac{1-||x||}{1+||x||} \frac{1}{2\pi} \int_{S(0,1)} u(y)\, dy = \frac{1-||x||}{1+||x||} u(0).$$

This completes the proof of Harnack's inequalities for $d=2$. □

Theorem 6.13 (Liouville's theorem). *If $u : \mathbb{R}^d \to \mathbb{R}$ is a harmonic function that is bounded below, then u must be a constant function.*

Proof. Let $m \in \mathbb{R}$ be such that $u(x) \ge m$ for all $x \in \mathbb{R}^d$. Then the function $v(x) := u(x) - m$ is a non-negative harmonic function. Applying Harnack's inequality to the function v yields for every $R > 0$, and for every $x \in B(0,R)$

$$\frac{R^{d-2}(R-||x||)}{(R+||x||)^{d-1}} v(0) \le v(x) \le \frac{R^{d-2}(R+||x||)}{(R-||x||)^{d-1}} v(0). \tag{6.55}$$

Fix an $x \in \mathbb{R}^d$, and $R > 0$ such that $R > ||x||$. Passing to the limit in the inequalities (6.55), we get

$$v(0) \leq v(x) \leq v(0).$$

Thus we get $v(x) = v(0)$. Since x is arbitrary, we conclude that v is a constant function. As a consequence, u is a constant function. $\square$

6.5 ▪ Dirichlet problem on a rectangle

In this section, Dirichlet problem for Laplace equation is considered on a rectangle. A solution is obtained using the method of separation of variables.

$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < \pi,\ 0 < y < A, \tag{6.56a}$$

$$u(0,y) = u(\pi,y) = 0 \quad \text{for } 0 \leq y \leq A, \tag{6.56b}$$

$$u(x,A) = 0 \quad \text{for } 0 \leq x \leq \pi, \tag{6.56c}$$

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq \pi. \tag{6.56d}$$

We use the method of separation of variables to solve the boundary value problem (6.56). Substituting

$$u(x,y) = X(x)Y(y)$$

in the equation (6.56a), and re-arranging the terms, we get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \tag{6.57}$$

Since the LHS and RHS of the equation (6.57) are functions of the variables x and y respectively, each of them must be a constant function. Thus we have, for some constant λ,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda \tag{6.58}$$

From the equation (6.58), we get the following two ODEs

$$X''(x) + \lambda X(x) = 0, \tag{6.59a}$$

$$Y''(y) - \lambda Y(y) = 0. \tag{6.59b}$$

Since we are interested in finding a non-trivial solution, the boundary conditions (6.56b)-(6.56c) give rise to the following conditions on the functions X and Y:

$$X(0) = 0, \quad X(\pi) = 0, \tag{6.60a}$$

$$Y(A) = 0. \tag{6.60b}$$

The ODE (6.59a) has non-trivial solutions satisfying the boundary conditions (6.60a) if and only if $\lambda = n^2$ for some $n \in \mathbb{N}$. Thus $\lambda_n = n^2$ are eigenvalues and the corresponding eigenfunctions are given by $X_n(x) = \sin nx$.

For each $n \in \mathbb{N}$, the solution of ODE (6.59b) with $\lambda = \lambda_n = n^2$ satisfying the condition (6.60b) (upto a constant multiple) is given by $Y_n(y) = \sinh n(A-y)$.

Since for each $n \in \mathbb{N}$, the function $u_n(x,y) = X_n(x)Y_n(y)$ is a solution (6.56a), we propose a formal solution of (6.56a) by superposition of the sequence of solutions (u_n) as follows:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh n(A-y). \tag{6.61}$$

The coefficients b_n in the formula (6.61) will be determined using the condition (6.56d). Thus we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \sinh nA. \tag{6.62}$$

Choosing c_n to be the Fourier sine coefficients of f, we get $b_n = \frac{c_n}{\sinh nA}$. That is, if f is given by

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx, \tag{6.63}$$

then c_n are given by

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(s) \sin ns\, ds. \tag{6.64}$$

Thus the formal solution of (6.56) is given by

$$u(x,y) \approx \sum_{n=1}^{\infty} \left(\frac{2}{\pi \sinh nA} \int_0^{\pi} f(s) \sin ns\, ds \right) \sin nx \sinh n(A-y). \tag{6.65}$$

The formal solution (6.65) is indeed a solution to (6.56) under some conditions, and is proved in Theorem 6.21.

6.6 ▪ Weak maximum principle

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function such that u can be extended to the closure of Ω as a continuous function. Such class of functions is denoted by $C(\overline{\Omega})$. When Ω is bounded every function in $C(\overline{\Omega})$ attains both its maximum and minimum values, somewhere in $\overline{\Omega}$. Weak Maximum principle says that if u happens to be a harmonic function, then the maximum and minimum values of u are definitely attained on the boundary of the domain Ω whether or not they are attained in Ω.

Theorem 6.14 (Weak maximum principle). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic function in Ω. Then the maximum value of u in $\overline{\Omega}$ is achieved on the boundary $\partial\Omega$.*

Proof. **Step 1:** Recall from differential calculus of two variables that at a point of interior maximum, $\frac{\partial^2 u}{\partial x^2} \leq 0$ and $\frac{\partial^2 u}{\partial y^2} \leq 0$. As a consequence, $\Delta u \leq 0$ at an interior maximum point. Thus if v is a function such that $\Delta v > 0$ in Ω, then the maximum value of v on $\overline{\Omega}$

cannot be attained in Ω. Hence v attains its maximum value only on the boundary $\partial\Omega$. The idea to prove Weak maximum principle is to find such a function v starting from the given harmonic function u.

Step 2: Define the function v_ϵ by

$$v_\epsilon(x,y) := u(x,y) + \epsilon(x^2+y^2). \tag{6.66}$$

Then $v_\epsilon \in C^2(\Omega)\cap C(\overline{\Omega})$. Note that $\Delta v_\epsilon > 0$ in Ω and thus v_ϵ attains its maximum only on the boundary $\partial\Omega$. Denoting

$$M := \max_{\partial\Omega} u,\ L := \max_{\partial\Omega}(x^2+y^2), \tag{6.67}$$

we have

$$v_\epsilon(x,y) \le M + \epsilon L,\ \forall (x,y)\in\Omega. \tag{6.68}$$

Since $u(x,y) \le v_\epsilon(x,y)$ for all $(x,y)\in\Omega$, we have

$$u(x,y) \le M + \epsilon L,\ \ \forall (x,y)\in\Omega. \tag{6.69}$$

Note that last inequality holds for every $\epsilon > 0$. Thus taking limits as $\epsilon \to 0$, we get

$$u(x,y) \le M,\ \ \forall (x,y)\in\Omega. \tag{6.70}$$

This finishes the proof. □

Corollary 6.15 (Weak minimum principle). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(\Omega)\cap C(\overline{\Omega})$ be a harmonic function in Ω. Then the minimum value of u in $\overline{\Omega}$ is achieved on the boundary $\partial\Omega$.*

Proof. Define by $v(x,y) = -u(x,y)$ on Ω. Then v is a harmonic function. Now apply the weak maximum principle to the harmonic function v and conclude. □

Remark 6.16. We have to understand clearly what the Weak maximum principle says and what it does not talk about. The weak maximum principle says that on bounded domains, any harmonic function takes its maximum value on the boundary surely. The weak maximum principle is silent on whether the harmonic function will take or will not take the maximumum value in the domain.

For example, consider Ω to be the unit disk centered at the origin. Then the constant function $u \equiv 1$ is definitely a harmonic function in Ω. It attains its maximum in Ω also, apart from attaining on $\partial\Omega$.

There is another maximum principle, called **Strong maximum priniciple**, which says that on bounded domains non-constant harmonic functions will never attain maxima in Ω and maxima are attained only on the boundary $\partial\Omega$. ■

6.6.1 ▪ Consequences of weak maximum principle

We already discussed uniqueness of solutions to all the three boundary value problems for Laplace equation posed on bounded domains using Green's identities. For Wave and Heat equations, we discussed uniqueness via Energy method and same analysis can be doen for Laplace equation as well.

We are now going to discuss uniqueness questions concerning the boundary value problems using another tool, namely, the weak maximum principle for harmonic functions on a bounded domain.

Theorem 6.17 (Uniqueness of solutions to Dirichlet problem). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and consider the Dirichlet problem on Ω:*

$$\begin{aligned} \Delta u &= f \text{ on } \Omega, \\ u &= g \text{ on } \partial\Omega, \end{aligned}$$

where g is a continuous function on $\partial\Omega$. Then the Dirichlet problem has at most one solution in the class $C^2(\Omega)\cap C(\overline{\Omega})$.

Proof. Assume that u and v belonging to the class $C^2(\Omega)\cap C(\overline{\Omega})$ solve the Dirichlet problem. Define $w := u - v$. Then w belongs to the class $C^2(\Omega)\cap C(\overline{\Omega})$ and solves the homogeneous Dirichlet problem:

$$\begin{aligned} \Delta w &= 0 \text{ on } \Omega, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

By weak maximum principle $w \leq 0$ and by weak minimum principle $w \geq 0$. Thus $w \equiv 0$. This finishes the proof of the theorem. □

Remark 6.18. (i) Note that uniqueness of solutions to Dirichlet problem was already proved in Theorem 6.9. Recall that its proof required that $\partial_{\mathbf{n}} u$ is defined on the boundary, and thus the uniqueness of solutions to Dirichlet problem could be proved only for $u \in C^1(\overline{\Omega})$. Thanks to maximum principle, uniqueness result is valid for any harmonic function which belongs to the space $C(\overline{\Omega})$.

(ii) In the above theorem, it is essential that Ω is a bounded domain. For unbounded domains the theorem may not hold good. For example, consider the following Dirichlet problem on the upper half plane:

$$\begin{aligned} \Delta u(x,y) &= 0 \text{ for } x \in \mathbb{R},\ 0 < y < \infty, \\ u(x,0) &= 1 \text{ for all } x \in \mathbb{R}. \end{aligned}$$

This problem has at least two solutions, namely $u_1(x,y) = xy$, and, $u_2(x,y) = 0$. ∎

Theorem 6.19. *Let $\Omega \subseteq \mathbb{R}$ be a bounded domain. For $i = 1,2$, let $u_i \in C^2(\Omega)\cap C(\overline{\Omega})$ solve the following Dirichlet problems on Ω:*

$$\begin{aligned} \Delta u_i &= f \text{ on } \Omega, \\ u &= g_i \text{ on } \partial\Omega, \end{aligned}$$

where g_i is a continuous function on $\partial\Omega$ for each $i = 1, 2$. Then u_1 and u_2 satisfy

$$\max_{\overline{\Omega}} |u_1(x,y) - u_2(x,y)| \leq \max_{\partial\Omega} |g_1(x,y) - g_2(x,y)|. \tag{6.71}$$

Proof. Define $w := u_1 - u_2$. Then w solves the following Dirichlet problem

$$\begin{aligned} \Delta w &= 0 \text{ on } \Omega, \\ w &= g_1 - g_2 \text{ on } \partial\Omega. \end{aligned}$$

Applying the weak maximum principle and the weak minimum principle, we get

$$\min_{\partial\Omega}(g_1 - g_2) \leq w(x,y) \leq \max_{\partial\Omega}(g_1 - g_2) \ \ \forall (x,y) \in \Omega, \tag{6.72}$$

which finishes the proof of the theorem. □

Theorem 6.20 (Harnack). *Let Ω be a bounded domain in $\mathbb{R}^2$. Let (u_n) be a sequence of functions in $C^2(\Omega) \cap C(\overline{\Omega})$ that are harmonic on Ω, such that $u_n = f_n$ on $\partial\Omega$. Let $f_n \to f$ in $C(\partial\Omega)$ i.e., $f_n \to f$ uniformly on $\partial\Omega$. Then*

(i) *The sequence (u_n) converges uniformly on $\partial\Omega$.*
(ii) *Let u denote the uniform limit of the sequence (u_n). Then u is harmonic on Ω, and $u \in C(\partial\Omega)$.*
(iii) *$u = f$ on $\partial\Omega$.*

Proof. For each $(m,n) \in \mathbb{N} \times \mathbb{N}$, the function $u_n - u_m$ is harmonic on Ω, and belongs to $C(\overline{\Omega})$. On the boundary $\partial\Omega$,

$$u_n - u_m = f_n - f_m$$

Since (f_n) converges uniformly on $\partial\Omega$, it is a uniformly Cauchy sequence. Thus for a given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $(x,y) \in \partial\Omega$, and for all $m \geq N$ and $n \geq N$, the following inequality holds.

$$|f_m(x,y) - f_n(x,y)| < \epsilon. \tag{6.73}$$

Applying Theorem 6.19, we get

$$\max_{\Omega} |u_m(x,y) - u_n(x,y)| \leq \max_{\partial\Omega} |f_m(x,y) - f_n(x,y)|. \tag{6.74}$$

From the inequalities (6.73) and (6.74), we get for all $m \geq N$ and $n \geq N$ the following inequality

$$\max_{\Omega} |u_m(x,y) - u_n(x,y)| \leq \max_{\partial\Omega} |f_m(x,y) - f_n(x,y)| < \epsilon. \tag{6.75}$$

Thus the sequence (u_n) is a Cauchy sequence in $C(\overline{\Omega})$, and hence converges uniformly, to $u \in C(\overline{\Omega})$. Since each member of the sequence u_n satisfies mean value property, u also satisfies mean value property being the uniform limit of the sequence (u_n). Since u is a continuous function and satisfies mean value property, it follows from Theorem 6.28 that u is harmonic in Ω. It is clear that $u = f$ on $\partial\Omega$. □

Theorem 6.21. *The formal solution (6.65) for the boundary value problem (6.56) is indeed a solution if the function $f : [0,\pi] \to \mathbb{R}$ is continuous, and satisfies $f(0) = f(\pi) = 0$, $\int_0^\pi (f')^2(s)\,ds < \infty$.*

Proof.
Recall the formal solution of (6.56) given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \frac{\sinh n(A-y)}{\sinh nA}, \tag{6.76}$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f(s)\sin ns\,ds$$

In order to establish that the formal solution given by (6.76) is indeed a solution to the BVP (6.56), we need to show that

(i) The series in (6.76) is twice continuously differentiable w.r.t. both the variables x and y, thus making sure that u_{xx} and u_{yy} are meaningful. Then we need to show that equation (6.56a) is satisfied.

(ii) We need to show that $u(x,y)$ given by (6.76) is continuous upto the boundary of $(0,\pi)\times(0,A)$ and that the boundary conditions (6.56b) - (6.56d) are satisfied.

Step 1: Proof of (i)
Note that

$$\frac{\sinh n(A-y)}{\sinh nA} = \frac{e^{n(A-y)} - e^{-n(A-y)}}{e^{nA} - e^{-nA}} \tag{6.77}$$

$$= e^{-ny}\frac{1-e^{-2n(A-y)}}{1-e^{-2nA}} \tag{6.78}$$

$$\leq \frac{e^{-ny}}{1-e^{-2A}}. \tag{6.79}$$

In view of the following inequalities

$$\left| b_n \sin nx \frac{\sinh n(A-y)}{\sinh nA}\right| \leq \frac{1}{1-e^{-2A}} e^{-ny},$$

and the fact that the series $\sum_{n=1}^{\infty} e^{-ny}$ converges uniformly for $y > y_0$, it follows that the series in (6.76) converges uniformly $y > y_0$. Since $y_0 > 0$ is arbitrary, it follows that $u(x,y)$ is a continuous function in $y > 0$. Thus boundary values at $x = 0$, $x = \pi$, $y = A$ are satisfied.

By a similar reasoning, it follows that the series in (6.76) can be differentiated term-by-term w.r.t. x and y twice and the resultant series define continuous functions.

Step 2: Proof of (ii) In view of the assumptions on f, the fourier sine series of f converges to f uniformly. Denoting the k^{th} partial sum of the series (6.76) by s_k, we have

$$s_k(x,y) = \sum_{n=1}^{k} b_n \sin nx \frac{\sinh n(A-y)}{\sinh nA}. \tag{6.80}$$

We wish to show that $s_k(x,0) \to f$ uniformly on $[0,\pi]$, by showing that the sequence (s_k) is a Cauchy sequence in $C[0,\pi]$. Note that for $q > p$, the function $s_p(x,y) - s_q(x,y)$ solves the IBVP (6.56) with $s_p(x,0)-s_q(x,0)=\sum_{n=p}^{q} b_n \sin nx$. Since the fourier sine series of f converges uniformly to f, the corresponding sequence of partial sums $\sum_{n=1}^{q} b_k \sin nx$ is Cauchy in $C[0,\pi]$. The required conclusion follows on applying maximum principle. Theorem 6.19. □

6.7 ▪ Mean value property and its consequences

Definition 6.22. *Let $\Omega \subseteq \mathbb{R}^2$ be an open domain. A continuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to possess the* **Mean Value Property-I on** Ω *if for every point $P=(x,y)\in\Omega$ and for every $r>0$ such that the open disk $B_r(P) \subsetneq \Omega$, the function u satisfies the relation*

$$u(x,y)=\frac{1}{\pi r^2}\int\limits_{|\xi-x|^2+|\eta-y|^2\le r^2} u(\xi,\eta)\,d\xi\, d\eta. \tag{6.81}$$

Definition 6.23. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain. A continuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to possess the* **Mean Value Property-II on** Ω *if for every point $P=(x,y)\in\Omega$ and for every $r>0$ such that the open disk $B_r(P) \subsetneq \Omega$, the function u satisfies the relation*

$$u(x,y)=\frac{1}{2\pi r}\oint_{C_r} u(s)\,ds \tag{6.82}$$

where C_r denotes the circle centered at P having radius r.

On expanding the line integral in the equation (6.82), we get the following form of the equation (6.82):

$$u(x,y)=\frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta, y+r\sin\theta)\,d\theta. \tag{6.83}$$

Lemma 6.24. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function. Then the following statements are equivalent.*

(i) *u has the mean value property-I on Ω.*
(ii) *u has the mean value property-II on Ω.*

Proof. We will show (ii) implies (i) and all the steps in this proof can be reversed thereby proving that (i) implies (ii). Let $P=(x,y)\in\Omega$ be an arbitrary point. Further let $R>0$ be such that the open disk $B_R(P)\subsetneq\Omega$. Then if u satisfies mean value property-II, then for every $0<\tau\le R$ we have

$$u(x,y)=\frac{1}{2\pi}\int_0^{2\pi} u(x+\tau\cos\theta, y+\tau\sin\theta)\,d\theta. \tag{6.84}$$

Multiply the equation (6.84) with τ and then integrate w.r.t. τ over the interval $[0,r]$. This yields

$$\int_0^r \tau\, u(x,y)\,d\tau=\frac{1}{2\pi}\int_0^r\left(\int_0^{2\pi} u(x+\tau\cos\theta, y+\tau\sin\theta)\,d\theta\right)\tau\,d\tau. \tag{6.85}$$

The last equation simplifies to

$$\begin{aligned}\frac{r^2}{2}u(x,y) &= \frac{1}{2\pi}\int_0^r\int_0^{2\pi} u(x+\tau\cos\theta, y+\tau\sin\theta)\,\tau\,d\theta\,d\tau \\ &= \frac{1}{2\pi}\int_{|\xi-x|^2+|\eta-y|^2\le r^2} u(\xi,\eta)\,d\xi\,d\eta.\end{aligned}$$

Thus we get

$$u(x,y) = \frac{1}{\pi r^2}\int\limits_{|\xi-x|^2+|\eta-y|^2\le r^2} u(\xi,\eta)\,d\xi\,d\eta, \tag{6.86}$$

which proves that u has mean value property-I as well. □

Remark 6.25. In view of Lemma 6.24, we say that a continuous function u has the mean value property on a domain Ω if either of the mean value properties holds on Ω and as a consequence both the mean value properties hold on Ω. The mean value property-I may also be called the Solid mean value property (as the average is taken over the entire disk) and the mean value property-II may also be called Surface/Circle mean value property. This nomenclature is not standard. ∎

Theorem 6.26 (Mean value principle). *Let $\Omega \subseteq \mathbb{R}^2$ be a domain and u be a harmonic function in Ω. Then u has the mean value property on Ω.*

Proof. Let $P_0 = (x_0, y_0) \in \Omega$ and let $R > 0$ be such that $B(P_0,R) \subset B[P_0,R] \subset \Omega$.

We would like to prove that the function u has the surface mean value property, namely, for $0 < r < R$

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta, y+r\sin\theta)\,d\theta. \tag{6.87}$$

Note that the left hand side of (6.87) is a real number, whereas the right hand side of (6.87) depends on r. Thus to prove (6.87), the strategy is to prove that the right hand side is a constant function of the variable r. To this end, define a function $V : (0,R) \to \mathbb{R}$ by

$$V(r) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0+r\cos\theta, y_0+r\sin\theta)\,d\theta. \tag{6.88}$$

Let us compute $\frac{dV}{dr}$ for $0 < r < R$ and prove that it is zero. As a consequence, we will then have

$$V(r) = \lim_{\rho\to 0} V(\rho) = u(x_0,y_0), \tag{6.89}$$

which completes the proof of the mean value principle. It remain to show that $\frac{dV}{dr} = 0$ on $(0,R)$.

$$\begin{aligned}\frac{dV}{dr}(r) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial r}u(x_0+r\cos\theta, y_0+r\sin\theta)\,d\theta &= \int_{S(0,r)} \partial_\nu u\,d\sigma \\ &= \int_{B(0,r)} \Delta u\,d\boldsymbol{x} = 0.\end{aligned} \tag{6.90}$$

In proving the last equality, we have used the equation (6.24) and the fact that u is a harmonic function on Ω. □

Theorem 6.27. *If u is continuous and has the mean value property on a domain Ω, then it has continuous derivatives of all orders and all of them have the mean value property on Ω.*

Proof. **Step 1:** It is sufficient to show that the first order partial derivatives of u exist and are continuous. Once it is known that first order partial derivatives of u exist, it follows by computing partial derivatives from the equation

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\,d\theta \tag{6.91}$$

which holds as u satisfies mean value property on Ω to obtain

$$\begin{aligned} u_x(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} u_x(x + r\cos\theta, y + r\sin\theta)\,d\theta, \\ u_y(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} u_y(x + r\cos\theta, y + r\sin\theta)\,d\theta. \end{aligned}$$

By repeating the arguments by replacing u with its partial derivatives, we conclude that u has partial derivatives of all orders and all of them satisfy the mean value relation.

Step 2: Let us now prove that first order partial derivatives of u exist and are continuous. Since u has mean value property -I (6.81), we have

$$u(x,y) = \frac{1}{\pi r^2}\int\limits_{|\xi - x|^2 + |\eta - y|^2 \leq r^2} u(\xi,\eta)\,d\xi\,d\eta. \tag{6.92}$$

Writing the double integral on the right hand side as iterated integrals, the equation (6.92) takes the form

$$u(x,y) = \frac{1}{\pi r^2}\int\limits_{|\xi - x|^2 + |\eta - y|^2 \leq r^2} u(\xi,\eta)\,d\xi\,d\eta = \frac{1}{\pi r^2}\int_{y-r}^{y+r} d\eta \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} u(\xi,\eta)\,d\xi$$

The last equation tells us that $u_x(x,y)$ exists, in view of fundamental theorem of integral calculus. In fact $u_x(x,y)$ is given by

$$u_x(x,y) = \frac{1}{\pi r^2}\int_{y-r}^{y+r}\left[u(x + \sqrt{r^2 - (\eta - y)^2}, \eta) - u(x - \sqrt{r^2 - (\eta - y)^2}, \eta)\right] d\eta. \tag{6.93}$$

Since the right hand side of the equation (6.93) is a continuous function of (x,y), we conclude that u_x is a continuous function on Ω. Similarly one can deduce the existence of u_y at each point and continuity of u_y on Ω. □

Theorem 6.28. *If $u : \Omega \longrightarrow \mathbb{R}$ is continuous and u has the mean value property on Ω, then u is harmonic in Ω.*

Proof. If u has the mean value property, then by Theorem 6.27 we know that all partial derivatives of u exist and all of them are continuous on Ω. We want to show that $\Delta u = 0$. On the contrary, suppose that there exists a point $P = (x_0, y_0)$ in Ω such that $\Delta u(P) \neq 0$; and without loss of generality assume that $\Delta u(P) > 0$. By continuity of the second order partial derivatives, there exists a disk of radius $\epsilon > 0$ centered at P (denote it by $B(P,\epsilon)$ on which $\Delta u > 0$.

We will be using the following identity in the computation that follows.

$$\frac{d}{dr} u(x_0 + r\cos\theta, y_0 + r\sin\theta) = \nabla_{\boldsymbol{x}} u(x_0 + r\cos\theta, y_0 + r\sin\theta).(\cos\theta, \sin\theta)$$

Also note that the outward unit normal to the circle $S(P,r)$ is given by $\frac{1}{r}(x - x_0, -y_0)$. Thus for $0 < r < \epsilon$, we have

$$\begin{aligned} 0 < \int_{B(P,r)} \Delta u(x,y)\,dx\,dy &= \int_{S(P,r)} \partial_\nu u\, d\sigma \\ &= \int_0^{2\pi} \frac{\partial}{\partial r} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, r\,d\theta \\ &= r\frac{\partial}{\partial r}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,d\theta \\ &= r\frac{\partial}{\partial r}\big(2\pi u(x_0, y_0)\big) = 0. \end{aligned}$$

This contradiction means that our assumption that Δu at some point is non-zero is wrong. Thus u is a harmonic function. □

Theorem 6.29 (Derivative estimate). *Let $u : \Omega \to \mathbb{R}$ be a harmonic function, where $\Omega \subseteq \mathbb{R}^2$ is an open set. Let k be a non-negative integer. There exists a $C_k > 0$ such that for each multi-index α with $|\alpha| = k$, and for each ball $B(\boldsymbol{x}_0, r) \subseteq \Omega$,*

$$|D^\alpha u(\boldsymbol{x}_0)| \leq \frac{(16k)^k}{\pi r^{k+2}} \int_{B(\boldsymbol{x}_0,r)} |u(x,y)|\,dx\,dy. \tag{6.94}$$

Proof. Since the inequality (6.94) involves a non-negative integer k, we prove it by induction. For $k = 0$, the inequality (6.94) follows from mean value property of harmonic functions. We will prove the case $k = 1$ as the arguments used to prove that the required inequalities hold for $k = l + 1$, assuming that they hold for $k = l$ are similar, and when $k = 1$ the proof can be presented cleanly without extra clutter of notations.

Step 1: Case of $k = 1$: We prove the estimate for u_x, as the proof is similar for u_y. By Theorem 6.27, the partial derivative u_x possesses mean value property. As a consequence, we have

$$\begin{aligned} |u_x(\boldsymbol{x}_0)| &= \left| \frac{1}{\pi}\frac{4}{r^2} \int_{B(\boldsymbol{x}_0,\frac{r}{2})} u_x(x,y)\,dx\,dy \right| \\ &= \left| \frac{1}{\pi}\frac{4}{r^2} \int_{S(\boldsymbol{x}_0,\frac{r}{2})} u.n_x\,d\sigma \right| \\ &\leq \frac{4}{r} \max_{S(\boldsymbol{x}_0,\frac{r}{2})} |u|. \end{aligned} \tag{6.95}$$

Let us estimate the quantity $\max_{S(\boldsymbol{x}_0,\frac{r}{2})}|u|$. Let $\boldsymbol{x}\in S(\boldsymbol{x}_0,\frac{r}{2})$. Then

$$B\left(\boldsymbol{x},\frac{r}{2}\right)\subset B(\boldsymbol{x}_0,r)\subset\Omega.$$

Using (solid) mean value property, we get the estimate

$$|u(\boldsymbol{x})|\leq\frac{4}{\pi r^2}\int_{B(\boldsymbol{x}_0,r)}|u(x,y)|\,dx\,dy. \tag{6.96}$$

Thus we get

$$\max_{S(\boldsymbol{x}_0,\frac{r}{2})}|u|\leq\frac{4}{\pi r^2}\int_{B(\boldsymbol{x}_0,r)}|u(x,y)|\,dx\,dy. \tag{6.97}$$

The desired estimate (6.94) for $k=1$ follows from (6.95) and (6.97).

Step 2: Induction step: Let us assume that the result is true for $k=l$. Let us prove that the result is true for $k=l+1$. Let α be a multi-index with $|\alpha|=l+1$. Consider a ball $B(\boldsymbol{x}_0,r)\subset\Omega$. Note that $D^\alpha u$ will be a first order partial derivative of $D^\beta u$ for some multi-index β such that $|\beta|=l$. Proceeding as in Step 1 after writing down the (solid) mean value property for the function $D^\alpha u$ on the ball $B(\boldsymbol{x}_0,\frac{r}{l+1})$, we get

$$|D^\alpha u(\boldsymbol{x}_0)|\leq\frac{2(l+1)}{r}\max_{S(\boldsymbol{x}_0,\frac{r}{l+1})}|D^\beta u|. \tag{6.98}$$

Note that for $\boldsymbol{x}\in S(\boldsymbol{x}_0,\frac{r}{l+1})$, we have

$$B\left(\boldsymbol{x},\frac{l}{l+1}r\right)\subset B(\boldsymbol{x}_0,r)\subset\Omega.$$

Applying induction hypothesis, we have following estimate for $D^\beta u(\boldsymbol{x})$ for $\boldsymbol{x}\in S(\boldsymbol{x}_0,\frac{r}{l+1})$.

$$|D^\beta u(\boldsymbol{x})|\leq\frac{(16l)^l}{\pi r^{l+2}}\int_{B(\boldsymbol{x}_0,r)}|u(x,y)|\,dx\,dy. \tag{6.99}$$

The inequalities (6.98) and (6.99) together imply the estimate (6.94) for $k=l+1$. □

Recall that harmonic functions are infinitely differentiable (see Theorem 6.27). In fact harmonic functions are nicer than being infinitely differentiable, they have convergent Taylor series expansions at every point of Ω, and such functions are known as real-analytic functions.

Theorem 6.30 (Harmonic functions are analytic). *Let $u:\Omega\to\mathbb{R}$ be a harmonic function, where $\Omega\subseteq\mathbb{R}^2$ is an open set. Then u is analytic in Ω.*

Proof. Let $\boldsymbol{x}_0\in\Omega$ be fixed. Since u is an infinitely differentiable function, we may write down the Taylor series for u around the point $\boldsymbol{x}_0$ which is given by

$$u(\boldsymbol{x})\approx\sum_{\alpha\in(\mathbb{N}\cup\{0\})^2}\frac{D^\alpha u(\boldsymbol{x}_0)}{\alpha!}(\boldsymbol{x}-\boldsymbol{x}_0)^\alpha \tag{6.100}$$

In order to show that u is analytic, we need to show that the series in (6.100) converges to u. This is shown by expressing $u(\boldsymbol{x}) = T_N(\boldsymbol{x}) + R_N(\boldsymbol{x})$, where T_N is the Taylor polynomial of degree N associated with u and R_N is the remainder. We then show that $R_N \to 0$ as $N \to \infty$. Note that $T_N(\boldsymbol{x}$ is given by

$$T_N(\boldsymbol{x}) = \sum_{\alpha \in (\mathbb{N}\cup\{0\})^2, |\alpha| \leq N} \frac{D^\alpha u(\boldsymbol{x}_0)}{\alpha!} (\boldsymbol{x} - \boldsymbol{x}_0)^\alpha. \tag{6.101}$$

By mean value theorem, there exists $\boldsymbol{z}$ on the line joining $\boldsymbol{x}$ and $\boldsymbol{x}_0$ such that $R_N(\boldsymbol{x})$ has the following expression:

$$R_N(\boldsymbol{x}) = u(\boldsymbol{x}) - T_N(\boldsymbol{x}) = \sum_{\alpha \in (\mathbb{N}\cup\{0\})^2, |\alpha| = N+1} \frac{D^\alpha u(\boldsymbol{z})}{\alpha!} (\boldsymbol{x} - \boldsymbol{x}_0)^\alpha. \tag{6.102}$$

Let $r = \frac{1}{4} d(\boldsymbol{x}_0, \partial\Omega)$, and $M = \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_0, 2r)} |u(x,y)|\, dx\, dy$. For each $\boldsymbol{z} \in B(\boldsymbol{x}_0, r)$, we have

$$B(\boldsymbol{z}, r) \subset B(\boldsymbol{x}_0, 2r) \subset \Omega.$$

Applying Theorem 6.29 with $B(\boldsymbol{x}, r)$, and $|\alpha| = N+1$, we get

$$\begin{aligned} |D^\alpha u(\boldsymbol{z})| &\leq \frac{(16(N+1))^{N+1}}{\pi r^{N+1+2}} \int_{B(\boldsymbol{z}, r)} |u(x,y)|\, dx\, dy \\ &\leq \frac{(16(N+1))^{N+1}}{\pi r^{N+1+2}} \int_{B(\boldsymbol{x}_0, 2r)} |u(x,y)|\, dx\, dy \\ &\leq M \frac{(16(N+1))^{N+1}}{r^{N+1}} \end{aligned}$$

From the last string of inequalities, we conclude that

$$\sup_{\boldsymbol{z} \in B(\boldsymbol{x}_0, r)} |D^\alpha u(\boldsymbol{z})| \leq M \left(\frac{16}{r}\right)^{N+1} (N+1)^{N+1} \tag{6.103}$$

We now state two general inequalities. They are

(i) From Stirling's formula [3] which is concerned with growth rate of factorials, there exists a $C > 0$ such that for all $n \in \mathbb{N}$ the following inequality holds:

$$n^n \leq C e^n n! \tag{6.104}$$

(ii) This is an estimate on $n!$. By multinomial theorem, we have

$$d^k = (1 + 1 + \cdots + 1)^k = \sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!}.$$

Hence we get

$$|\alpha|! \leq d^{|\alpha|} \alpha! \tag{6.105}$$

In view of (6.104) and (6.105), from the estimate (6.103) we get

$$\sup_{\boldsymbol{z} \in B(\boldsymbol{x}_0, r)} |D^\alpha u(\boldsymbol{z})| \leq C M \left(\frac{32e}{r}\right)^{N+1} \alpha! \tag{6.106}$$

Let us now estimate the remainder term R_N given by (6.102).

$$|R_N(\boldsymbol{x})| \leq \sum_{\alpha\in(\mathbb{N}\cup\{0\})^2, |\alpha|=N+1} \frac{|D^\alpha u(\boldsymbol{z})|}{\alpha!} \, ||\boldsymbol{x}-\boldsymbol{x}_0||^{|\alpha|}$$
$$\leq \sum_{\alpha\in(\mathbb{N}\cup\{0\})^2, |\alpha|=N+1} CM\left(\frac{32e}{r}\right)^{N+1} ||\boldsymbol{x}-\boldsymbol{x}_0||^{N+1} \tag{6.107}$$

Let $s>0$ (to be chosen later) be such that $||\boldsymbol{x}-\boldsymbol{x}_0|| \leq s$. Then the estimate (6.107) yields

$$|R_N(\boldsymbol{x})| \leq \sum_{\alpha\in(\mathbb{N}\cup\{0\})^2, |\alpha|=N+1} CM\left(\frac{32es}{r}\right)^{N+1} \tag{6.108}$$

Choose an s such that $\frac{32es}{r} = \frac{1}{4}$. In other words, choose $s = \frac{r}{128e}$. With this choice of s, the estimate (6.108) takes the form

$$|R_N(\boldsymbol{x})| \leq \sum_{\alpha\in(\mathbb{N}\cup\{0\})^2, |\alpha|=N+1} CM\left(\frac{1}{4}\right)^{N+1} \leq CM2^{N+2}\left(\frac{1}{4}\right)^{N+1} \leq CM\frac{1}{2^N}. \tag{6.109}$$

Thus the remainder term $R_N \to 0$ as $N \to \infty$. Thus u is an analytic function on Ω. □

As a consequence of the last result, it follows that Cauchy problem for Laplace equation need not have solutions. This is the content of the next result.

Theorem 6.31. *Let $\Gamma \subset \mathbb{R}^2$ denote a segment of x-axis. Let g be a function defined on Γ. Then the Cauchy problem for $\Delta u = 0$ where Cauchy data is given by*

$$u(x,0)=0, \quad \frac{\partial u}{\partial y}(x,0)=g(x) \quad \textit{for} \quad (x,0)\in\Gamma. \tag{6.110}$$

has no solutions unless $x \mapsto g(x)$ is a real-analytic function.

Proof. Let $P(\xi,0) \in \Gamma$. Let B be an open disk with center at P such that $B\cap x$-axis is contained in Γ. Let B^+ denote the part of B which lies above x-axis, *i.e.,*

$$B^+ = \{(x,y)\in B : y \geq 0\}.$$

Let $u \in C^2(\overline{B^+})$ be a solution of Laplace equation corresponding to the Cauchy data (6.110). Let u be extended to whole of B by reflection by setting for $(x,y)\in B$,

$$u(x,y) := -u(x,-y) \text{ for } y<0.$$

Verifying that the extended function $u : B \to \mathbb{R}$ has the following properties

(i) $u \in C^2(B)$, and
(ii) $\Delta u = 0$ on B

is left as an exercise to the reader. By Theorem (6.30), the harmonic function u is analytic in B. In particular, u is analytic at P, and so is its derivative $\frac{\partial u}{\partial y}$. Thus g, which is the restriction of an analytic function to x-axis is itself real-analytic. This completes the proof of the theorem. □

Theorem 6.32 (Harnack's inequality). *Let $\mathcal{O}$ be a connected open subset of Ω such that $\mathcal{O} \subseteq \overline{\mathcal{O}} \subset \Omega$. There exists a constant $C > 0$ such that*

$$\sup_{\mathcal{O}} u \leq C \inf_{\mathcal{O}} u \tag{6.111}$$

holds for all non-negative harmonic functions $u : \Omega \to [0, \infty)$.

Proof. We will first prove the result on each of the balls contained in the domain Ω, and then deduce the result for $\mathcal{O}$ using connectedness of $\mathcal{O}$ and compactness of $\overline{\mathcal{O}}$.

Step 1: Let $r < d(\mathcal{O}, \partial\Omega)$. Let $\boldsymbol{x} \in \mathcal{O}$ and $\boldsymbol{y} \in \mathcal{O}$ be such that $||\boldsymbol{x} - \boldsymbol{y}|| < r$. By (solid) mean value property, we have

$$u(\boldsymbol{x}) = \frac{1}{4\pi r^2} \int_{B(\boldsymbol{x}, 2r)} u(z)\, dz. \tag{6.112}$$

Since $B(\boldsymbol{y}, r) \subseteq B(\boldsymbol{x}, 2r)$, from the equation (6.112) we get

$$u(\boldsymbol{x}) = \frac{1}{4\pi r^2} \int_{B(\boldsymbol{x}, 2r)} u(\boldsymbol{z})\, dz \geq \frac{1}{4\pi r^2} \int_{B(\boldsymbol{y}, r)} u(\boldsymbol{z})\, dz = \frac{1}{4} u(\boldsymbol{y}). \tag{6.113}$$

Thus we have $\frac{1}{4} u(\boldsymbol{y}) \leq u(\boldsymbol{x})$. Interchanging the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ we get $\frac{1}{4} u(\boldsymbol{x}) \leq u(\boldsymbol{y})$. Combining these two estimates, we get the following estimates, valid for all $\boldsymbol{x} \in \mathcal{O}$ and $\boldsymbol{y} \in \mathcal{O}$ satisfying $||\boldsymbol{x} - \boldsymbol{y}|| < r$:

$$\frac{1}{4} u(\boldsymbol{y}) \leq u(\boldsymbol{x}) \leq 4u(\boldsymbol{y}). \tag{6.114}$$

Step 2: Since $\mathcal{O}$ is connected and its closure is compact, there exists finitely many balls $B_1, B_2, \cdots, B_N$ having radius r, and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, 2, \cdots, N-1$. Let $\boldsymbol{x} \in \mathcal{O}$ and $\boldsymbol{y} \in \mathcal{O}$. Then there exists an i and a j such that $\boldsymbol{x} \in B_j$ and $\boldsymbol{y} \in B_k$. Without loss of generality, we may assume that $j \leq k$. For $i = 1, 2, \cdots, N-1$, let $\boldsymbol{z}_i \in B_i \cap B_{i+1}$ be fixed.

$$u(\boldsymbol{x}) \leq 4u(\boldsymbol{z}_j) \leq 4^2 u(\boldsymbol{z}_{j+1}) \leq \cdots \leq 4^{k-j} u(\boldsymbol{z}_{k-1}) \leq 4^{k+1-j} u(\boldsymbol{y}) \leq 4^N u(\boldsymbol{y}) \tag{6.115}$$

Using the first inequality in (6.114), and following a similar argument, we get

$$u(\boldsymbol{x}) \geq \frac{1}{4^N} u(\boldsymbol{y}) \tag{6.116}$$

Combining the inequalities (6.115) and (6.116), we get

$$\frac{1}{4^N} u(\boldsymbol{y}) \leq u(\boldsymbol{x}) \leq 4^N u(\boldsymbol{y}), \tag{6.117}$$

which is valid for every $\boldsymbol{x} \in \mathcal{O}$ and $\boldsymbol{y} \in \mathcal{O}$. The required conclusion (6.111) now follows. □

6.7.1 ▪ Strong maximum principle

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and $u : \Omega \longrightarrow \mathbb{R}$ be a harmonic function. The Weak maximum principle asserted that maximum value of u on the closure of Ω is attained on the boundary of Ω, and was silent about attaining the said maximum value in Ω. Strong maximum principle that we are going to prove says that if such a maximum is also attained inside Ω, then necessarily the harmonic function u is a constant function.

Theorem 6.33 (Strong maximum principle). *Let $\Omega \subseteq \mathbb{R}^2$ be a domain (domain need not be bounded) and $u : \Omega \longrightarrow \mathbb{R}$ be a harmonic function. If u attains its maximum in Ω, then u is constant.*

Proof. **Step 1:** Let u assume its maximum at $P_0 \in \Omega$ and let this maximum value be denoted by M. We want to prove that u is the constant function that takes the value M everywhere in Ω. Let P be an arbitrary point in Ω. We wil show that $u(P) = M$.

Step 2: Join P to P_0 by a smooth curve γ[12]. Since γ is a compact set, it maintains a positive distance from $\partial\Omega$ (in case $\partial\Omega$ is non-empty), let us denote this distance by d_γ. Take a disk of radius $\frac{d_\gamma}{2}$ with center at P_0 denoted by $B_{\frac{d_\gamma}{2}}(P_0)$. Since u is a harmonic function in Ω, it has the mean value property in Ω. Applying the mean value property[13] on this disk, we conclude that u equals M on the entire disk.

Step 3:(Continuation argument) We now take a point P_1 on γ which lies in the disk $B_{\frac{d_\gamma}{2}}(P_0)$ which is at least at a distance $\frac{d_\gamma}{4}$ from P_0. Note that $u(P_1) = M$. Now repeat the arguments of Step 2 with the disk $B_{\frac{d_\gamma}{2}}(P_1)$ and then find a P_2 in a similar way. Continuing this process till we get a $k \in \mathbb{N}$ with the property that $P \in B_{\frac{d_\gamma}{2}}(P_k)$[14]. We will then have $u(P) = M$. This finishes the proof of the theorem. □

Remark 6.34. This is a continuation of Remark 6.16. It is important to note that Strong maximum principle says that if a harmonic function attains its maximum in a domain Ω (bounded domain or otherwise), then it is necessarily a constant function. In particular Strong maximum principle does not talk about where the maximum will be attained; sometimes a harmonic function may not have a maximum value also. This happens when the domain Ω is not bounded, because in that case though u is continuous on $\overline{\Omega}$ we cannot assert the existence of a maximum value, since $\overline{\Omega}$ is not compact. See the next example in this context. ∎

Example 6.35. Let Ω be the domain exterior to the unit disk. The function $u(x, y) = \log(x^2 + y^2)$ is a harmonic function on Ω. It has neither a maximum value nor a minimum value in Ω. ∎

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[12] This is possible since Ω is open and connected and hence it is path-connected. Prove this.
[13] Mean Value Property-I or Mean Value Property-II ?
[14] Why is it possible to find such a k?

6.8 ▪ Dirichlet principle

Dirichlet principle characterises the solution of Dirichlet BVP in terms of minimizer of a functional. An analogous result in the context of systems of linear equations says that the solutoin of the linear system $A\boldsymbol{x} = \boldsymbol{b}$ (where A is a symmetric positive definite matrix) is the minimizer of the functional $\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b}$.

Theorem 6.36 (Dirichlet principle). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$, and $u \in C^2(\overline{\Omega})$. The following statements are equivalent.*

(i) *The function u is a solution to Dirichlet BVP*

$$-\Delta u = f(\boldsymbol{x}) \quad in \quad \Omega, \tag{6.118a}$$
$$u = g \quad on \quad \partial\Omega. \tag{6.118b}$$

(ii) *u is a minimizer of the functional $J : S \to \mathbb{R}$ where $S := \{v \in C^2(\overline{\Omega}) : v = g \text{ on } \partial\Omega\}$, and*

$$J(v) := \frac{1}{2}\int_\Omega ||\nabla v||^2\, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\, d\boldsymbol{x} \tag{6.119}$$

Proof. **Proof of (i)$\Longrightarrow$(ii):** Let u be a solution to Dirichlet BVP (6.118). Let $v \in S$. Multiplying the equation (6.118a) with $u - v$ and then integrating on Ω yields

$$-\int_\Omega \Delta u(\boldsymbol{x})(u(\boldsymbol{x}) - v(\boldsymbol{x}))\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})(u(\boldsymbol{x}) - v(\boldsymbol{x}))\, d\boldsymbol{x}$$

Integrating by parts on the LHS of the last equation gives

$$\int_\Omega \nabla u(\boldsymbol{x}).(\nabla u - \nabla v)(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})(u(\boldsymbol{x}) - v(\boldsymbol{x}))\, d\boldsymbol{x} \tag{6.120}$$

Note that the boundary terms resulting from integration by parts do no appear as $u = v$ on $\partial\Omega$. Re-arranging terms in the equation (6.120), we get

$$\int_\Omega ||\nabla u(\boldsymbol{x})||^2\, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\, u(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega \nabla u(\boldsymbol{x}).\nabla v(\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x} \tag{6.121}$$

Note that

$$\begin{aligned}\int_\Omega \nabla u.\nabla v\, d\boldsymbol{x} &\leq \int_\Omega ||\nabla u||\,||\nabla v||\, d\boldsymbol{x}\\ &\leq \left(\int_\Omega ||\nabla u||^2\, d\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_\Omega ||\nabla v||^2\, d\boldsymbol{x}\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\left(\int_\Omega ||\nabla u||^2\, d\boldsymbol{x} + \int_\Omega ||\nabla v||^2\, d\boldsymbol{x}\right)\end{aligned} \tag{6.122}$$

Using the inequality (6.122) in the equation (6.121), we get

$$\int_\Omega ||\nabla u(x)||^2\,dx - \int_\Omega f(x)\,u(x)\,dx \;\leq\; \frac{1}{2}\left(\int_\Omega ||\nabla u||^2\,dx + \int_\Omega ||\nabla v||^2\,dx\right) - \int_\Omega f(x)\,v(x)\,dx \tag{6.123}$$

Re-arranging the terms in (6.123), we get

$$\frac{1}{2}\int_\Omega ||\nabla u(x)||^2\,dx - \int_\Omega f(x)\,u(x)\,dx \;\leq\; \frac{1}{2}\int_\Omega ||\nabla v||^2\,dx - \int_\Omega f(x)\,v(x)\,dx \tag{6.124}$$

Note that the equation (6.123) is nothing but

$$J(u) \;\leq\; J(v)$$

Thus u is the minimizer of the functional J over the set S.

Proof of (ii)⟹(i): Let $v \in C_0^\infty(\Omega)$, and $t \in \mathbb{R}$. We have

$$J(u+tv) = J(u) + t\int_\Omega (\nabla u.\nabla v - f v)\;dx + \frac{t^2}{2}\int_\Omega ||\nabla v||^2\,dx$$

Rewriting the last equality, we get

$$J(u+tv) - J(u) = t\int_\Omega (\nabla u.\nabla v - f v)\;dx + \frac{t^2}{2}\int_\Omega ||\nabla v||^2\,dx. \tag{6.125}$$

Since the functional achieves its minimum at u, the real valued function $h:\mathbb{R}\to\mathbb{R}$ defined by

$$h(t) = J(u+tv) - J(u)$$

achieves its minimum at $t=0$. Thus $h'(0)=0$, which yields

$$\int_\Omega \nabla u(x).\nabla v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx \tag{6.126}$$

Integrating by parts on the LHS of the equation (6.126) gives

$$-\int_\Omega \Delta u(x)\,v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx \tag{6.127}$$

Since the equality (6.127) holds for every $v \in C_0^\infty(\Omega)$, we conclude that for each $x \in \Omega$,

$$-\Delta u(x) = f(x).$$

Note that u satisfies the boundary condition (6.118b) as $u \in S$. This proves that u is a solution to the BVP (6.118).

□

Exercises

General

6.1. [16] Show that the laplace operator in two space dimensions takes the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

in the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

6.2. [16] Show that the laplace operator in three space dimensions takes the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$$

in the cylindrical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$.

6.3. [16] Show that the laplace operator in three space dimensions takes the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}++\frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$$

in the spherical coordinates $x_1 = r\cos\theta\sin\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\phi$.

6.4. Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let $u : \Omega \longrightarrow \mathbb{R}$.

(a) Let $(a,b) \in \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x,y) = (x+a, y+b)$. Define $\tilde{\Omega}$ be given by

$$\tilde{\Omega} := \{(x-a, y-b) : (x,y) \in \Omega\}.$$

Let $\tilde{T}$ denote the restriction of T to $\tilde{\Omega}$. Show that $u \in C^2(\Omega)$ if and only if $u\circ\tilde{T} \in C^2(\tilde{\Omega})$ and that $\Delta(u\circ\tilde{T}) = (\Delta u)\circ\tilde{T}$. In particular,

$$\Delta u = 0 \Longleftrightarrow \Delta(u\circ\tilde{T}) = 0.$$

(b) Laplace equation is invariant under any real orthogonal transformation. Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be any real orthogonal transformation *i.e.,*

$$U(x,y) = M_U\begin{pmatrix} x \\ y \end{pmatrix} \forall (x,y) \in \mathbb{R}^2,$$

where M_U is a real 2×2 invertible matrix whose inverse is equal to its transpose. Define $\tilde{\Omega} := U^{-1}(\Omega)$ and let $\tilde{U}$ be the restriction of U to $\tilde{\Omega}$. Show that $u \in C^2(\Omega)$ if and only if $u\circ\tilde{U} \in C^2(\tilde{\Omega})$ and that $\Delta(u\circ\tilde{U}) = (\Delta u)\circ\tilde{U}$. In particular,

$$\Delta u = 0 \Longleftrightarrow \Delta(u\circ\tilde{U}) = 0.$$

6.5. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, which is also radial and harmonic. Show that u is a constant function.

Solutions on subdomains

6.6. [33] Let Ω be the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<4\}$. Consider the Neumann problem

$$\begin{aligned} \Delta u &= 0 \qquad \text{on } \Omega, \\ \partial_{\boldsymbol{n}} u &= \alpha x^2 + \beta y + \gamma \quad \text{for } (x,y) \in \partial\Omega, \end{aligned}$$

where α, β, γ are real numbers.

(i) Find the values of α, β, γ for which the problem is not solvable.

(ii) Solve the problem for those values of α, β, γ for which a solution exists.

6.7. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $A, B \in \mathbb{R}$. Solve $u_{xx} + u_{yy} = 0$ in the annular region $0 < a < \sqrt{x^2+y^2} < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$. (**Hint:** Look for a solution depending only on $r = \sqrt{x^2+y^2}$.)

6.8. Solve $u_{xx} + u_{yy} = 1$ in $\sqrt{x^2+y^2} < a$ with u vanishing on $x^2+y^2 = a^2$. (Ans: $u(x,y) = \frac{1}{4}(x^2+y^2-a^2)$.)

6.9. Solve $\Delta u = 0$ in the disk $D = \{(x,y) : x^2+y^2 < a^2\}$ with the boundary condition $u = 1 + 3\sin\theta$ on the circle $r = a$. (Ans: $u(r,\theta) = 1 + \frac{3r}{a}\sin\theta$.)

6.10. Solve the equation $\Delta u = 0$ in the domain $D = \{(x,y) : x^2+y^2 > a^2\}$ with the boundary condition $u = 1+3\sin\theta$ on the circle $r = a$ and with the condition that the solution is bounded for $r \to \infty$. (Ans: $u(r,\theta) = 1 + \frac{3a}{r}\sin\theta$.)

6.11. [33] Solve the problem

$$\begin{aligned} \Delta u &= 0 \quad \text{for } 0 < x < \pi,\ 0 < y < \pi, \\ u(x,0) = u(x,\pi) &= 0 \qquad \text{for } 0 \le x \le \pi, \\ u(0,y) &= 0 \qquad \text{for } 0 \le y \le \pi, \\ u(\pi,y) &= \sin y \qquad \text{for } 0 \le y \le \pi. \end{aligned}$$

Is there a point (x,y) in the square $(0,\pi)\times(0,\pi)$ such that $u(x,y) = 0$?

6.12. [33] Solve the Laplace equation $\Delta u = 0$ in the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 4\}$, subject to the boundary condition $u(x,y) = y$ on $\partial\Omega$, and the decay condition

$$\lim_{|x|+|y|\to\infty} u(x,y) = 0.$$

6.13. [38] This exercise is concerned with Poisson's formula, to show that the function defined by Poisson's formula satisfies the boundary conditions (see Theorem 6.11). Let $\xi \in S(0,1)$.

(a) Show that

$$\frac{1-||\boldsymbol{x}||^2}{2\pi} \int_{S(0,1)} \frac{1}{||\boldsymbol{x}-\boldsymbol{y}||^2}\, d\boldsymbol{y} = 1,$$

and then deduce

$$u(\boldsymbol{x}) - u(\xi) = \frac{1-||\boldsymbol{x}||^2}{2\pi} \int_{S(0,1)} \frac{f(\boldsymbol{y}) - f(\xi)}{||\boldsymbol{x}-\boldsymbol{y}||^2}\, d\boldsymbol{y}.$$

(b) For $\delta > 0$, write

$$u(\boldsymbol{x}) - u(\xi) = \frac{1 - ||\boldsymbol{x}||^2}{2\pi} \left(\int_{S(0,1)\cap B(\xi,\delta)} \frac{f(\boldsymbol{y}) - f(\xi)}{||\boldsymbol{x}-\boldsymbol{y}||^2}\, d\boldsymbol{y} + \int_{S(0,1)\setminus B[\xi,\delta]} \frac{f(\boldsymbol{y}) - f(\xi)}{||\boldsymbol{x}-\boldsymbol{y}||^2}\, d\boldsymbol{y} \right)$$
$$= I + II.$$

Fix $\epsilon > 0$. Using the continuity of f show that there exists a δ_0 such that for all $\delta < \delta_0$, $|I| < \epsilon$ holds.

(c) Show that if $||\boldsymbol{x} - \xi|| < \frac{\delta}{2}$ and $||\boldsymbol{y} - \xi|| > \delta$, then $||\boldsymbol{x} - \boldsymbol{y}|| > \frac{\delta}{2}$. Further deduce that

$$\lim_{\boldsymbol{x}\to\xi} II = 0.$$

Maxmimum principles

6.14. Formulate the strong minimum principle and prove it by following the method of proof of the strong maximum principle.

6.15. [23] Let $\Omega \subseteq \mathbb{R}^2$ be given by

$$\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}. \tag{6.130}$$

Let $u \in C^2(\overline{\Omega})$ be such that $\Delta u = 0$ in Ω, and $\lim\limits_{|(x,y)|\to\infty} u(x,y) = 0$. Show that

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{6.131}$$

6.16. [23] Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0,$$

where $a, b, c \in C(\overline{\Omega})$ and $c(x,y) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω. (**Hint:** Show that max $u \le 0$ and min $u \ge 0$.)

6.17. [23] Let $\Omega \subseteq \mathbb{R}^2$ be given by

$$\Omega := \{(x,y) \in \mathbb{R}^2 : y > 0\}. \tag{6.132}$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a harmonic function in Ω. Further assume that u is bounded above in $\overline{\Omega}$. Prove that

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u. \tag{6.133}$$

(**Hint:** Take for $\epsilon > 0$ the harmonic function

$$u(x,y) - \epsilon \log(\sqrt{x^2 + (y+1)^2}),$$

Apply the maximum principle to a region $\{(x,y) \in \mathbb{R}^2 : x^2 + (y+1)^2 < a^2\}$, $y \ge 0$ with large a. Let $\epsilon \to 0$.)

6.18. Check the validity of the maximum principle for the harmonic function $\frac{1-x^2-y^2}{1-2x+x^2+y^2}$ in the disk $\overline{D} = \{(x,y) : x^2 + y^2 \le 1\}$. Explain.

6.19. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that $\Delta u \geq 0$ in Ω. Prove that its maximum value is attained on the boundary $\partial\Omega$. What can you say about the minimum value?

6.20. Let u be a harmonic function on the whole plane such that $u = 3\sin(2\theta) + 1$ on the circle $x^2 + y^2 = 2$. Without finding the concrete form of the solution, find the value of u at the origin.

6.21. [33] Let u be a non-constant harmonic function in the disk $\{(x,y) : x^2+y^2 < R^2\}$. Define a function $M : (0,R) \to \mathbb{R}$ by

$$M(r) = \max_{x^2+y^2=r^2} u(x,y).$$

Show that M is a strictly increasing function on the interval $(0,R)$.

6.22. [33] Let $\Omega = \{(x,y) : 0 < x < \pi,\ 0 < y < \pi\}$.

(i) Assume that $v_{xx} + v_{yy} + xv_x + yv_y > 0$ in Ω. Prove that v has no local maximum in Ω.

(ii) Consider the problem

$$\begin{aligned} u_{xx} + u_{yy} + xu_x + yu_y &= 0 && \text{on } \Omega, \\ u(x,y) &= f(x,y) && \text{for } (x,y) \in \partial\Omega, \end{aligned}$$

where f is a given continuous function. Show that if u is a solution, then the maximum of u is achieved on the boundary $\partial\Omega$. (*Hint:* Use the auxiliary function $v_\epsilon(x,y) = u(x,y) + \epsilon x^2$)

(iii) Show that the problem in (ii) above has at most one solution.

6.23. [33] Let $v(x,y)$ be a smooth vector field defined on a domain Ω. Let u be a smooth solution of the Dirichlet problem

$$\begin{aligned} \Delta u + v.\nabla u &= F && \text{on } \Omega, \\ u(x,y) &= g(x,y) && \text{for } (x,y) \in \partial\Omega, \end{aligned}$$

where $F > 0$ in Ω, and $g < 0$ on $\partial\Omega$. Show that $u(x,y) < 0$ for $(x,y) \in \Omega$.

6.24. [33] Let Ω be the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 36\}$. Let u be a harmonic function on Ω which satisfies the Dirichlet boundary condition on $\partial\Omega$

$$u(x,y) = \begin{cases} x & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Prove that $u(x,y) < \min\{x, 0\}$ in Ω. (*Hint:* Prove that $u(x,y) < x$ and $u(x,y) < 0$ in Ω.)

(ii) Evaluate $u(0,0)$ using the mean value principle.

(iii) Using Poisson's formula, evaluate $u(0,y)$ for $0 \leq y < 6$.

(iv) Using separation of variables method, find the solution u.

(v) Is the solution u obtained in (iv) above classical?

6.25. Deduce the following version of Liouville's theoerm from Theorem 6.29: If $u : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function that is bounded, then u must be a constant function.

6.26. Derive mean value property-II starting from the formula (6.10) as follows:

(i) For every $w \in C^2(\overline{\Omega})$ satisfying $\Delta w = 0$, the function $G(\boldsymbol{x}, \xi)$ defined by

$$G(\boldsymbol{x}, \xi) = K(\boldsymbol{x}, \xi) + w(\boldsymbol{x})$$

is a fundamental solution of Lapalce equation.

(ii) Let ψ be as defined by (6.9). Now apply (6.10) when the domain Ω is the ball $B(\xi, \rho)$, and $G(\boldsymbol{x}, \xi) = \psi(\|\boldsymbol{x} - \xi\|) - \psi(\rho)$, to get

$$u(\xi) = \int_{B(\xi,\rho)} (\psi(\|\boldsymbol{x} - \xi\|) - \psi(\rho))\, \Delta u \, d\boldsymbol{x} + \frac{1}{\omega_d \rho^{d-1}} \int_{S(\xi,\rho)} u \, d\sigma.$$

for $\xi \in B(\xi, \rho)$, and $u \in C^2(B[\xi, \rho])$.

(iii) Deduce mean value property-II when the function u is harmonic.