Analysis of Time Series, L14

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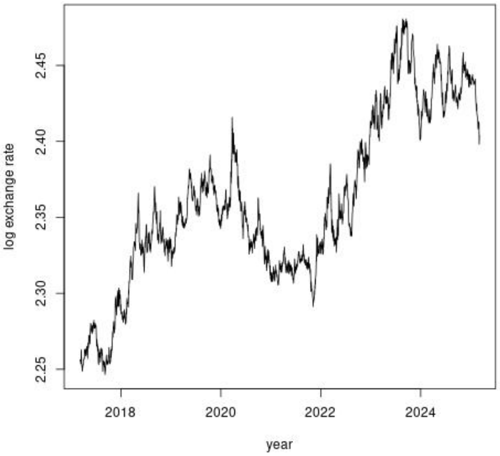
Today

5.2: Unit root testing

- DF test
- ADF test
- With deterministic terms
- KPSS test (not in book)
- Application
- Menti

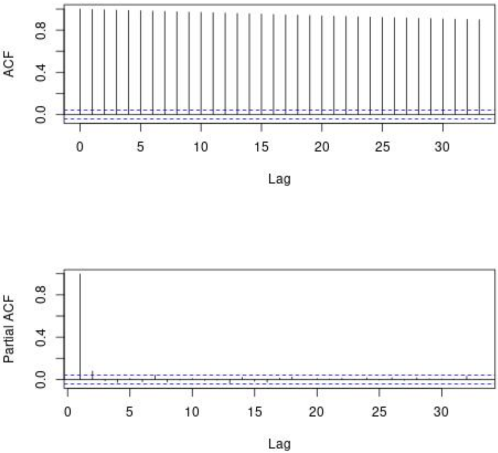
DF test

Log exchange rate, Skr/Euro, March 6 2017-March 5 2025:



DF test

Log Skr/Euro, ACF (slowly decaying) and PACF (cutting off).



```
> y1=log(y)
> arima(y1,order=c(1,0,0))

Call:
arima(x = y1, order = c(1, 0, 0))

Coefficients:
         ar1  intercept
      0.9979     2.3610
s.e.  0.0014     0.0347

sigma^2 estimated as 1.608e-05:  log likelihood = 8558.44,
aic = -17110.87
```

Estimated AR parameter very close to 1.

```
> arima(y1,order=c(2,0,0))

Call:
arima(x = y1, order = c(2, 0, 0))

Coefficients:
         ar1     ar2  intercept
      0.8848  0.1134     2.3610
s.e.  0.0217  0.0217     0.0389

sigma^2 estimated as 1.587e-05:  log likelihood = 8571.93,
aic = -17135.86
```

Sum of estimated AR parameters very close to 1.

DF test

- AR(1) process

$$x_t = \phi x_{t-1} + w_t, \quad t = 1, 2, ..., n.$$

- Causal if.f. $|\phi| < 1$.
- Stationary for a suitable choice of distribution for x_0 $(\mathrm{var}(x_0) = \frac{\sigma_w^2}{1-\phi^2})$ if.f. $|\phi| < 1$.
- Test H_0: $\phi = 1$ vs H_1: $|\phi| < 1$.
- A natural test statistic is $\hat{\phi} - 1$.
- The *Dickey-Fuller* (DF) test.

DF test

- Under H_0: $\phi = 1$, it follows that (why?)

$$\hat{\phi} - 1 = \frac{\sum_{t=1}^{n} x_t x_{t-1}}{\sum_{t=1}^{n} x_{t-1}^2} - 1 = \frac{\sum_{t=1}^{n} w_t x_{t-1}}{\sum_{t=1}^{n} x_{t-1}^2}.$$

- Moreover (why?), assuming $x_0 = 0$,

$$\hat{\phi} - 1 = \frac{x_n^2 - \sum_{t=1}^{n} w_t^2}{2\sum_{t=1}^{n} x_{t-1}^2}.$$

DF test

Definition (5.1)

A continuous time process $\{W(t);\ t \geq 0\}$ is called a *standard Brownian motion* if it satisfies

(*i*) $W(0) = 0$.

(*ii*) For any $0 \leq t_1 < t_2 < ... < t_n$ and integer n, $W(t_2) - W(t_1)$, $W(t_3) - W(t_2)$,...,$W(t_n) - W(t_{n-1})$ are independent.

(*iii*) $W(t + \Delta t) - W(t) \sim N(0, \Delta t)$ for $\Delta t > 0$.

One may show that for a white noise process $\{w_t\}$, as $n \to \infty$,

$$\frac{1}{\sigma_w \sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} w_j \xrightarrow{\mathcal{L}} W(t)$$

where $\lfloor a \rfloor$ is the integer part of a.

DF test

- As $n \to \infty$,

$$\frac{1}{\sigma_w \sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} w_j \xrightarrow{\mathcal{L}} W(t)$$

- It follows that (why?)

$$n(\hat{\phi} - 1) = n \frac{x_n^2 - \sum_{t=1}^n w_t^2}{2 \sum_{t=1}^n x_{t-1}^2} \xrightarrow{\mathcal{L}} \frac{W(1)^2 - 1}{2 \int_0^1 W(t)^2 dt}.$$

- No "standard" distribution!

DF test

- An alternative is the t test

$$\hat{t} = \frac{\hat{\phi} - 1}{\sqrt{s^2 / \sum_{t=2}^{n} x_{t-1}^2}},$$

where

$$s^2 = \frac{1}{n-1} \sum_{t=2}^{n} (x_t - \hat{\phi} x_{t-1})^2.$$

- One may show that, as $n \to \infty$,

$$\hat{t} \xrightarrow{\mathcal{L}} \frac{W(1)^2 - 1}{2\sqrt{\int W(t)^2 dt}}.$$

ADF test

AR(1): $x_t = \phi x_{t-1} + w_t \Leftrightarrow \nabla x_t = \gamma x_{t-1} + w_t$, $\gamma = \phi - 1$.

Extension to AR(2):

- $$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$$
- Equivalent to (why?)
  $$\nabla x_t = \gamma x_{t-1} + \psi_1 \nabla x_{t-1} + w_t,$$
  where
  $$\gamma = \phi_1 + \phi_2 - 1, \quad \psi_1 = -\phi_2.$$

ADF test

Extension to AR(p):

- $$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t$$
- Equivalent to
$$\nabla x_t = \gamma x_{t-1} + \psi_1 \nabla x_{t-1} + \ldots + \psi_{p-1} \nabla x_{t-(p-1)} + w_t$$
where
$$\gamma = \sum_{j=1}^{p} \phi_j - 1, \quad \psi_j = -\sum_{i=j+1}^{p} \phi_i.$$
- The Augmented Dickey-Fuller (ADF) test:
Test H_0: $\gamma = 0$ vs H_1: $\gamma < 0$.
- Modified test statistics, but the same limit distributions as before.

With deterministic terms

Incorporating deterministic terms:

- $x_t = \beta_0 + \phi x_{t-1} + w_t$
- Test H_0: $(\beta_0, \phi) = (0, 1)$ vs H_1: $\neg H_0$.
- $x_t = \beta_0 + \beta_1 t + \phi x_{t-1} + w_t$
- Test H_0: $(\beta_1, \phi) = (0, 1)$ vs H_1: $\neg H_0$.
- Modified test statistics, other limit distributions.

KPSS test

- Test H_0: stationarity vs H_1: non stationarity.
- Model

$$x_t = r_t + \varepsilon_t,$$
$$r_t = r_{t-1} + u_t,$$

where $\{\varepsilon_t\}$ and $\{u_t\}$ are independent white noise sequences, $t = 1, 2, ..., n$.

- Test H_0: $\mathrm{var}(u_t) = \sigma_u^2 = 0$ vs H_1: $\sigma_u^2 > 0$.
- Under H_0, $x_t = r_0 + \varepsilon_t$ is stationary.
- Let $S_t = \sum_{j=1}^{t} e_j$, $e_j = (x_j - \bar{x})$ and estimate its variance by

$$s^2(l) = \frac{1}{n}\sum_{t=1}^{n} e_t^2 + \frac{2}{n}\sum_{s=1}^{l}\left(1 - \frac{s}{l+1}\right)\sum_{t=s+1}^{n} e_t e_{t-s}.$$

KPSS test

- Let $S_t = \sum_{j=1}^{t} e_j$, $e_j = (x_j - \bar{x})$ and estimate its variance by

$$s^2(l) = \frac{1}{n}\sum_{t=1}^{n} e_t^2 + \frac{2}{n}\sum_{s=1}^{l}\left(1 - \frac{s}{l+1}\right)\sum_{t=s+1}^{n} e_t e_{t-s}.$$

- Test statistic

$$\hat{\eta} = \frac{\sum_{t=1}^{n} S_t^2}{n^2 s^2(l)}.$$

- As $n, l \to \infty$ such that $l/\sqrt{n} \to 0$,

$$\hat{\eta} \xrightarrow{d} \int_0^1 \left\{W(r) - rW(1)\right\}^2 dr,$$

where $W(r)$ is the standard Brownian motion.

Unit root tests for log Skr/Euro in R:

```
> y1=log(y)
> library(tseries)
> adf.test(y1)

Augmented Dickey-Fuller Test

data:  y1
Dickey-Fuller = -2.6273, Lag order = 12, p-value = 0.3128
alternative hypothesis: stationary

> kpss.test(y1)

KPSS Test for Level Stationarity

data:  y1
KPSS Level = 15.517, Truncation lag parameter = 8, p-value = 0.01

Warning message:
In kpss.test(y1) : p-value smaller than printed p-value
```

The adf test does not reject non stationarity and the KPSS test rejects stationarity.

News of today

- Testing for non stationarity (unit root)
- Test statistics
- Limit distributions (non standard)
- Extensions:
  - Allowing for autocorrelation (ADF test)
  - Allowing for deterministic terms
- Testing the null of stationarity (KPSS test)