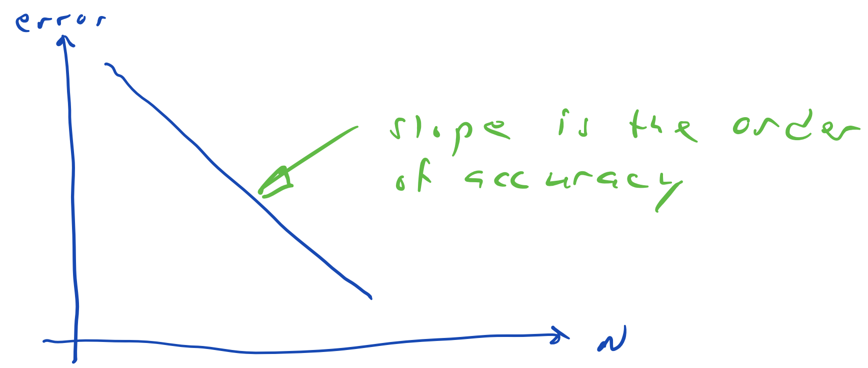
Example exam questions

1a) i) Order of accuracy

The rate in which the accuracy increases as N gets larger (or step size h gets smaller)



ii) Markov process

A stochastic process that fulfills the markov criteria, i.e. it is "memoryless". (the outcome at stage k only depend on the outcome at stage k-1

b) i) machie epsilon

ii) Least squares method

---

2)

T	100	200	300	400
C	1,3	0,9	0,8	0,7

Ansatz: $p_2(T) = a_0 + a_1(T-\bar{T}) + a_2(T-\bar{T})^2$, $\bar{T} = 250$

$$\Rightarrow \quad A = \begin{pmatrix} 1 & -150 & 22500 \\ 1 & -50 & 2500 \\ 1 & 50 & 2500 \\ 1 & 150 & 22500 \end{pmatrix}, \quad C = \begin{pmatrix} 1,3 \\ 0,9 \\ 0,8 \\ 0,7 \end{pmatrix}$$

Normal eq.: $A^T A\, x = A^T c$

$$A^T A = \begin{pmatrix} 4 & 0 & 50000 \\ 0 & 50000 & 0 \\ 50000 & 0 & 1{,}025\cdot 10^9 \end{pmatrix}, \quad A^T c = \begin{pmatrix} 3{,}7 \\ -95 \\ 49250 \end{pmatrix}$$

$$A^T A\, x = A^T c \Rightarrow x = \begin{pmatrix} 0{,}8313 \\ -0{,}0019 \\ 7{,}5\cdot 10^{-6} \end{pmatrix}$$

$$p_2(T) = 0{,}8313 + 0{,}0019\,(T-250) + 7{,}5\cdot 10^{-6}\,(T-250)^2$$

and

$$p_2(150) = 1{,}096$$

3/a/ Assume μ_A stored in mA, σ_A stored in sA, and so forth.

```
N = a large number
tot_price = zeros(N,1)
for i in range(N)
    A = mA * sA * randn( )
    B = mB * sB * randn( )
    C = mC * sC * randn( )
    tot_price[i] = A + B + C
price_average = mean(tot_price)
```

or better version: $mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}$, $S = \begin{pmatrix} \sigma_A \\ \sigma_B \\ \sigma_C \end{pmatrix}$

```
N = ...
for i in range(N)
    price = mu + S * randn(3,1)
    tot_price = sum(price)
price_average = mean(tot_price)
```

b/ Order of accuracy is $O(\frac{1}{\sqrt{N}})$

Reduce accuracy with factor $\frac{1}{2}$ $\Rightarrow$

$$\frac{1}{2} \cdot O\left(\frac{1}{\sqrt{N}}\right) \approx O\left(\frac{1}{\sqrt{4N}}\right)$$

$\therefore$ we need $4 \cdot N$ loop iterations (instead of N)

---

4 a/ $A = QR$, A $m \times n$, $m > n$

solve $Ax = b$ with normal equation

$A^T A x = A^T b$, $A = QR$ $\Rightarrow$

$$(QR)^T (QR) x = (QR)^T \cdot b$$

$$R^T \underbrace{Q^T Q}_{= I} R x = R^T Q^T b$$

$$R^T R x = R^T Q^T b \Rightarrow R x = Q^T b$$

(R must be non-singular $\Leftrightarrow$ A full rank)

$\therefore$ - Do $A = QR$

- solve $Rx = Q^T b$ (use backward substitution)

b/ "Preserve length" meaning Qx have same length (2-norm) as x $\Rightarrow$ $\|Qx\|_2 = \|x\|_2$

$$\|\underbrace{Qx}_{= y}\|_2 = \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2} = \sqrt{y^T y} = \sqrt{(Qx)^T (Qx)}$$

$$= \sqrt{x^T \underbrace{Q^T Q}_{= I} x} = \sqrt{x^T x} = \|x\|_2$$

5/ MCqueuesim(T,N) runs one MC-simulation using N realizations.
If we repeat M MC-simulation the solution will all differ slightly. The solution will be normal distributed (central limit theorem), and we can check how much the solution differ (=std)

```
T, N = ...
M = ...
for k in range(M)
    Q[k] = MCqueue(T,N)
Qmean = mean(Q)
Qstd = std(Q)
```

err $= 1.96 \cdot Qstd/\sqrt{M}$ (confidence interval)

meaning $Q = Qmean \pm err$ with 95 % probability

6/ - Normaleq $A^TAx = A^Tb$ cheapest, but A^TA high cond. number ⇒ losing accuracy (improve with scaling). Rank(A) must be $= n$

- $A = QR$, solve $Rx = Q^Tb$ (backw. subst) more expensive but no problem with cond. number. A must have rank $= n$

- pseudo invers $x^+ = A^+b$, based on SVD very expensive, but only method that works if rank(A) < n

7/ $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $x^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

it 1: $\bar{x}^{(1)} = A x^{(0)} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$x^{(1)} = \bar{x}^{(1)} / \|\bar{x}^{(1)}\|_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\|\bar{x}^{(1)}\|_2 = \sqrt{9+4} = \sqrt{13}$

it 2: $\bar{x}^{(2)} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{13}} \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

$x^{(2)} = \frac{1}{\sqrt{74}} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 0,8137 \\ 0,5812 \end{pmatrix}$, $\|\bar{x}^{(2)}\|_2 = \frac{1}{\sqrt{13}} \sqrt{49+25} = \frac{\sqrt{74}}{\sqrt{13}}$

$\lambda^{(2)} = x^{(2)T} A x^{(2)} = 2,4189$

(True value: $\lambda_1 = 2,4142$, $x_1 = \begin{pmatrix} 0,8165 \\ 0,5774 \end{pmatrix}$)

We will find the eigenvector corresponding to A's largest eigenvalue.

8/9 SVD:

$U = \begin{pmatrix} -0,48 & 0,58 & & & \\ -0,23 & -0,15 & & & \\ -0,76 & -0,44 & u_3 & u_4 & u_5 \\ -0,38 & 0,50 & & & \\ -0,28 & -0,45 & & & \end{pmatrix}$, $\Sigma = \begin{pmatrix} 30,81 & 0 \\ 0 & 4,1251 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} -0,62 & 0,78 \\ -0,78 & -0,62 \end{pmatrix}$

$\underbrace{u_3 \ u_4 \ u_5}_{\text{not needed}}$

$\Rightarrow A = 30,81 \cdot \begin{pmatrix} -0,48 \\ -0,23 \\ \vdots \\ -0,28 \end{pmatrix} (-0,62 \ -0,78) + 4,1251 \begin{pmatrix} 0,58 \\ \vdots \\ -0,45 \end{pmatrix} (0,78 \ -0,62)$

b/ $A_1 = 30{,}81 \cdot \begin{pmatrix} -0{,}48 \\ -0{,}23 \\ \vdots \\ -0{,}28 \end{pmatrix} \begin{pmatrix} -0{,}62 & -0{,}78 \end{pmatrix}$,

$$A - A_1 = 4{,}1251 \begin{pmatrix} 0{,}58 \\ \vdots \\ -0{,}45 \end{pmatrix} \begin{pmatrix} 0{,}78 & -0{,}62 \end{pmatrix}$$

A matrix with largest singular value $= 4{,}1251$

$$\Rightarrow \| A - A_1 \|_2 = 4{,}1251$$

c/ $A^+ = V \Sigma^+ U^T = \begin{pmatrix} -0{,}62 & 0{,}78 \\ -0{,}78 & -0{,}62 \end{pmatrix} \begin{pmatrix} 30{,}81 & 0 & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 4{,}1251 & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \begin{pmatrix} - \ ,48 \ \ldots \ -0{,}28 \\ 0{,}58 \ \ldots \ -0{,}45 \\ \cancel{u_3^T} \\ \cancel{u_4^T} \\ \cancel{u_5^T} \end{pmatrix}$

$$= \begin{pmatrix} 0{,}12 & -0{,}02 & -0{,}07 & 0{,}10 & -0{,}08 \\ -0{,}08 & 0{,}03 & 0{,}05 & -0{,}07 & 0{,}08 \end{pmatrix}$$

$$x^+ = A^+ \begin{pmatrix} 20 \\ 8 \\ 25 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 1{,}7392 \\ 0{,}2524 \end{pmatrix}$$

This is not the exact solution (it does not exist), but the least squares solution $\min \| b - Ax \|_2$.

can check $b - Ax^+ = \begin{pmatrix} -1{,}66 \\ -0{,}47 \\ -0{,}41 \\ 2{,}33 \\ 1{,}02 \end{pmatrix} \neq 0$

Part B

1)

	Det.	Stoch.
Model	- ODE - Integral, e.g. $v(t) = \int_0^t s(t)\,ds$	- Radioactive decay $X \xrightarrow{k\cdot\alpha} Z$ - Lotka-Volterra stoch. version
Num. method	- ODE-solver, e.g. Eulers method - trapezoidal rule for integration	SSA Monte-Carlo methods

solving high dim. integrals with MC. methods

No example of solving stoch. models with deterministic methos

---

B2)

$$G \xrightarrow{v\cdot G} G + M$$
$$M \xrightarrow{x\cdot M} M + P$$
$$M \xrightarrow{y\cdot M} \emptyset$$
$$P \xrightarrow{z\cdot P} \emptyset$$

$$\Rightarrow N = \begin{array}{c} \begin{array}{ccc} G & M & P \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

simulation with SSA

Assume
$$v \cdot G = 1,6$$
$$x \cdot M = 1,2 \quad \Rightarrow \quad p = \begin{pmatrix} 1,6 \\ 1,2 \\ 0,4 \\ 0,8 \end{pmatrix}$$
$$y \cdot M = 0,4$$
$$z \cdot p = 0,8$$

Random number $u = 0,55$

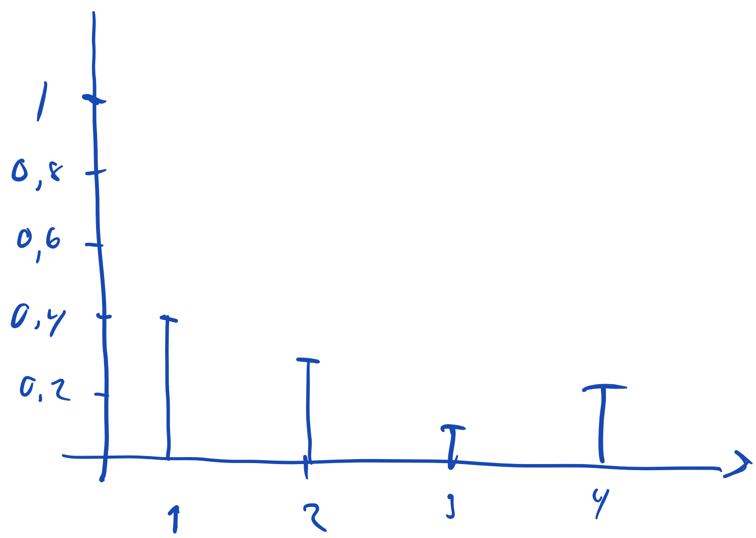
Choose which of the 4 eq. will happen next

$a_0 = 1,6 + 1.2 + 0.4 + 0.8 = 4$

probabilities:

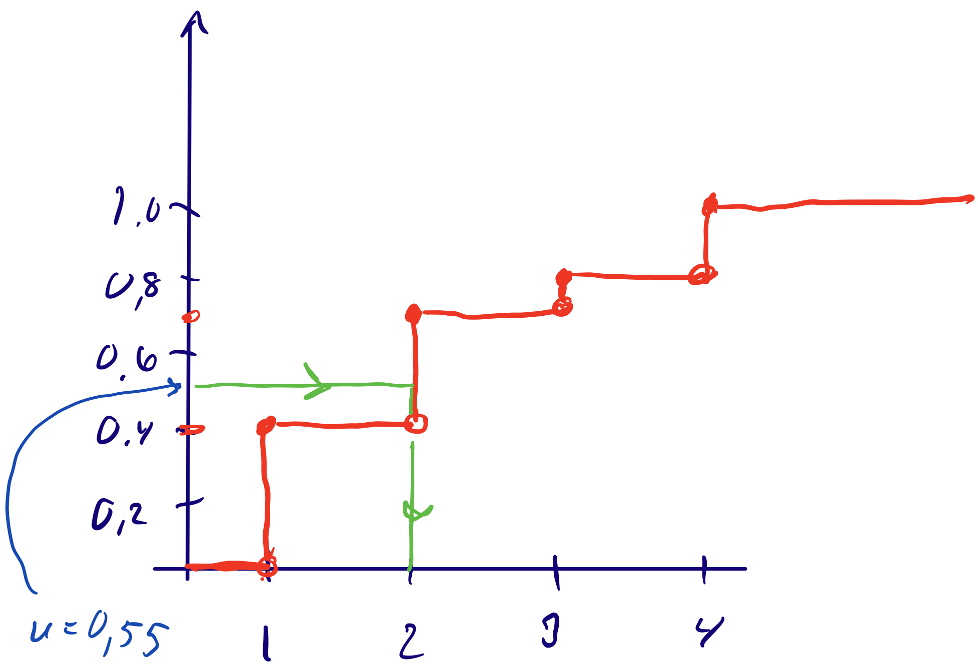
$$p/a_0 = \begin{pmatrix} 1,6 \\ 1.2 \\ 0,4 \\ 0,8 \end{pmatrix} = \begin{pmatrix} 0,4 \\ 0,3 \\ 0,1 \\ 0,2 \end{pmatrix}$$

PDF (probability density function)



For the CDF we will need the cumulative sum of p/a_0. cumulative sum $= \begin{pmatrix} 0,4 \\ 0,7 \\ 0,8 \\ 1,0 \end{pmatrix}$

CDF (cumulative distr. function)



Discrete inverse transform sampling (go from uniform distr. to another distr.)

Choose eg. number 2

The state vector:

$$\underline{Y}_{i+1} = \underline{Y}_i + (0 \quad 0 \quad 1)$$

eg. 2

$(G_i \; m_i \; p_i)$

3/ Ansatz, here a straight line, e.g.

$p_1 = a_0 + a_1 \cdot years \Rightarrow$

$$A = \begin{pmatrix} 1 & year_1 \\ 1 & year_2 \\ \vdots & \vdots \\ 1 & year_n \end{pmatrix} \cdot \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad y = \text{mean sea level}$$

Solve $Ax = y$. No solution, due to more eq. than unknowns (overdetermined system)

Find the solution that minimizes $\|y - Ax\|_2^2$ (Least squares solution).

Use normal eq. $A^T A x = A^T y$

Solve with Gaussian elimination. Problem that $A^T A$ often have high cond. number.

Better:

- scale the matrix A (with mean and std)
- or use QR-factorization $A = QR$, solve $Rx = Q^T y$
- Use $x^+ = A^+ b$, A^+ the pseudoinvers
  Better but more costly

Example

year, y	1994	1998	2000	2005	2011
sealevel	-18	0	5	22	38

Ansatz, v.1: $p_1(y) = a_0 + a_1 y \implies$

$$\begin{cases} a + 1994 \cdot a_1 = -18 \\ a + 1998 \cdot a_1 = 0 \\ a + 2000 \cdot a_1 = 5 \\ a + 2005 \cdot a_1 = 22 \\ a + 2011 \cdot a_1 = 38 \end{cases} \implies \underbrace{\begin{pmatrix} 1 & 1994 \\ 1 & 1998 \\ 1 & 2000 \\ 1 & 2005 \\ 1 & 2011 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} -18 \\ 0 \\ 5 \\ 22 \\ 38 \end{pmatrix}}_{b}$$

normaleq.: $A^T A x = A^T b$

(don't need to solve it in this question).

Alterative 2:

Ansatz, v. 2: $p_1(y) = a_0 + a_1(y - \bar{y})$, $\bar{y} = 2001{,}6$ (the mean)

$$\begin{cases} a_0 + (1994 - 2001{,}6) \cdot a_1 = -18 \\ a_0 + (1998 - 2001{,}6) \cdot a_1 = 0 \\ \quad\vdots \\ a_0 + (2011 - 2001{,}6) a_1 = 38 \end{cases} \implies \begin{pmatrix} 1 & -7{,}6 \\ 1 & -3{,}6 \\ 1 & -1{,}6 \\ 1 & 3{,}6 \\ 1 & 9{,}4 \end{pmatrix} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \begin{pmatrix} -18 \\ 0 \\ 5 \\ 22 \\ 38 \end{pmatrix}$$

Again, solve with normaleq. This version is better due to smaller cond. number in $A^T A$

$\implies$ more accurat solution

Here: Alt. 2 : $A^T A = \begin{pmatrix} 3 & 0 \\ 0 & 173{,}2 \end{pmatrix}$, $\mathrm{cond}_2(A^T A) = 34{,}6$

Alt. 1 : $\mathrm{cond}(A^T A) = 4{,}6 \cdot 10^{11}$

Big difference

Alternative 3: Use $p_1(y) = a_0 + \dfrac{(y - \bar{y})}{b} \cdot a_1$

$\bar{y} = 2001{,}6$ (the mean)

$b = 6{,}5809$ (standard deviation)

Again, solve with normal eq.

Here we get $A^T A = \begin{pmatrix} 5 & -2{,}6458 \\ -2{,}6450 & 17{,}1061 \end{pmatrix}$

and $\mathrm{cond}(A^T A) = 3{,}79$