A1

1. A Markov chain on the state space $S = \{1, \dots, 9\}$ has transition matrix

$$\begin{pmatrix} 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.5 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{pmatrix}$$

Draw a state transition diagram (model graph). Classify the states according to the canonic decomposition. By finding the period of each state, determine the set of aperiodic states.

A2

2. Consider the Markov chain on the states $S = \{0, \dots, 5\}$ with probability transition matrix

$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/10 & 0 & 1/2 & 3/10 & 0 & 1/10 \\ 0 & 0 & 0 & 7/10 & 3/10 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Which states are transient, which are recurrent? What is the canonical decomposition of the state space?

b) If the chain starts in state 1, what is the absorbtion probability for this Markov chain?

c) Starting in state 1, what is the probability to reach state 5 before state 0, given that one of them is reached?

d) Find all stationary distributions.

A3

3. A binary source in a communication system generates a sequence $X_1, X_2, \dots$ of zeros and ones according to a Markov chain with transition matrix

$$\begin{pmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{pmatrix}$$

The sequence of symbols is transmitted over a binary symmetric channel, such that the symbol k is correctly received with probability $1 - \varepsilon$ and received in error with probability ε, independent of earlier transmissions. Denote by $Y_1, Y_2, \dots$ the received sequence of zeros and ones. When the system is in equilibrium, find the conditional probability $P(X_k = 1 | Y_k = 1)$ as a function of ε.

A4

4. A particle jumps randomly on the vertices A, B, C, D, E, and F of a hexagon. The particle starts in vertex A. Each time point the particle jumps one step clockwise (for the first jump this is A $\to$ B) with probability 1/3 and counter-clockwise with probability 2/3. Find the mean number of jumps until the particle visits vertex C for the first time.

B1 1. Cat and Mouse move between two rooms along separate paths. Their movements are independent and occur at discrete time points according to the transition matrices

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}, \qquad \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix},$$

respectively. Cat starts from room 1, Mouse from room 2. If they meet in the same room Cat eats Mouse. For how long does Mouse survive on average?

B2 2. Consider a pure birth process $X(t)$ in continuous time on the state space $E = \{0, 1, 2\}$ with non-zero intensities λ_0 och λ_1 for upward jumps from the states 0 and 1. Suppose $X_0 = 0$. Obviously the process will sooner or later be absorbed in state 2.

a) Find the transient state probabilities $p_i(t) = P(X_t = i)$, $i = 0, 1, 2$, $t \geq 0$.

b) Put

$$q_i(t) = P(X_t = i | \text{absorption has not yet occurred at time } t), \quad i = 0, 1.$$

In certain applications it is of interest to find out whether the limit $\lim_{t \to \infty} q_i(t)$ exists. If this is the case we call $q = (q_0, q_1)$ a *quasi-stationary distribution.* Show that for the case studied here we have a quasi-stationary distribution, and determine q.

B3 3. In a simplified model for DNA substitutions the nucleotides A och G are separate states but the two so called pyrimidines C and T are merged into a state CT. Substitutions between the ordered states $E = \{A, CT, G\}$ are supposed to occur according to a continuous time Markov chain with generator matrix

$$Q = \begin{pmatrix} -(\alpha+\beta) & \alpha & \beta \\ \beta & -2\beta & \beta \\ \beta & \alpha & -(\alpha+\beta) \end{pmatrix}$$

Find the asymptotic probabilities for the three states. If the Markov chain at time $t = 0$ has the stationary distribution as initial distribution, what is the probability that no substitution has yet occurred at time $t = 1$?

B4 4. Vehicles arrive at a village gas station according to a Poisson process with an average arrival rate of 30 automobiles per hour. The gas station is equipped with two pumps. On average a customer spend three minutes at one of the pumps. The service times are assumed to be exponential and independent of each other and of the customer arrival patterns. At the station there is only room for one vehicle at each pump plus one more waiting. If all three spots are occupied at the time of arrival of a fourth potential customer, the new car continues to another gas station. Find the steady-state distribution of the number of vehicles at the station. Compute the expected value and the standard deviation in this distribution. What is the proportion of lost customers?

1. By analyzing the model graph and/or computing the recurrence probabilities, it follows that the two communication classes $\{1,2\}$ and $\{6,7\}$ together form the set of transient states $T = \{1,2,6,7\}$, and that the classes $C_1 = \{3,4,5\}$ and $C_2 = \{8,9\}$ are closed sets of positively recurrent states. Hence the canonical decomposition of the state space is given by $S = T \cup C_1 \cup C_2$. For example, the probability to return to state 1 is $f_1 = 0.9^2 = 0.81 < 1$, same for state 2. The probability to return to state 6 after n steps is $f_{66}(n) = (1/2)^{n-2} \cdot 0.4$ so that $f_6 = 0.4 \sum_{n=2}^{\infty} (1/2)^{n-2} = 0.8 < 1$, and so on. A1 (Solution)

Periodicity:

$$
\begin{aligned}
\{1,2\}: &\quad p_i^{(2n)} > 0,\ n \geq 1 \quad \Longrightarrow \quad d = 2 \\
\{3,4,5\}: &\quad p_i^{(3n)} > 0,\ n \geq 1 \quad \Longrightarrow \quad d = 3 \\
\{6\}: &\quad p_i^{(n)} > 0,\ n \geq 2 \quad \Longrightarrow \quad d = 1 \\
\{7,8,9\}: &\quad p_i^{(n)} > 0,\ n \geq 1 \quad \Longrightarrow \quad d = 1
\end{aligned}
$$

The set of aperiodic states is $\{6,7,8,9\}$.

2. a) We have the canonical decomposition $E = T \cup C_1 \cup C_2 \cup C_3$ where $T = \{1,2\}$ are the transient states, $C_1 = \{0\}$ and $C_2 = \{5\}$ are absorbing (irreducible closed classes of size one) and $C_2 = \{3,4\}$ is an irreducible closed communication class. A2 (Solution)

b) The principle of conditioning on the first event shows that the hitting probabilites $h_{ij} = P(H_{ij} < \infty)$ of reaching j starting in i, satisfy the equations

$$h_{10} = \frac{1}{2} + \frac{1}{2} h_{20} \quad \text{and} \quad h_{20} = \frac{1}{10} + \frac{1}{2} h_{20},$$

hence $h_{20} = 1/5$ and $h_{10} = 3/5$. Similarly,

$$h_{15} = \frac{1}{2} h_{25} \quad \text{and} \quad h_{25} = \frac{1}{10} + \frac{1}{2} h_{25},$$

gives $h_{25} = 1/5$ and $h_{15} = 1/10$. Let H be the event of absorption in $\{0\}$ or $\{5\}$. Then $P(H|X_0 = 1)) = h_{10} + h_{15} = 7/10$.

c) Let H_5 be the event of absorbtion in 5. Then

$$P(H_5|H, X_0 = 1) = P(H_5|X_0 = 1)/P(H|X_0 = 1) = \frac{h_{15}}{h_{10} + h_{15}} = \frac{1}{7}.$$

d) The sets C_1, C_2 and C_3 form irreducible classes with stationary distributions

$$\pi_1 = (1,0,0,0,0,0), \quad \pi_2 = (0,0,0,0,0,1), \quad \pi_3 = (0,0,0,10/19,9/19,0),$$

hence all stationary distributions are given by

$$\pi = a_1\pi_1 + a_2\pi_2 + a_3\pi_3, \quad 0 \leq a_1, a_2, a_3 \leq 1, \quad a_1 + a_2 + a_3 = 1.$$

A3
(Solution)

3. We want to find the conditional probability $P(X_k = 1|Y_k = 1)$ that a received digit one is indeed the result of sending a digit one. What we know is that a transmitted digit one is received correctly with probability $1-\varepsilon$, i.e., $P(Y_k = 1|X_k = 1) = 1-\varepsilon$, and incorrectly with probability ε, i.e., $P(Y_k = 1|X_k = 0) = \varepsilon$. According to Bayes rule, the probability we wish to compute equals

$$
\begin{aligned}
P(X_k = 1|Y_k = 1) &= \frac{P(X_k = 1, Y_k = 1)}{P(Y_k = 1)} \\
&= \frac{P(Y_k = 1|X_k = 1)P(X_k = 1)}{P(Y_k = 1|X_k = 1)P(X_k = 1) + P(Y_k = 1|X_k = 0)P(X_k = 0)} \\
&= \frac{(1-\varepsilon)P(X_k = 1)}{(1-\varepsilon)P(X_k = 1) + \varepsilon P(X_k = 0)}.
\end{aligned}
$$

We need also the absolute state probabilites $P(X_k = 0)$ and $P(X_k = 1)$. In general, these depend on the initial distribution. However, it is reasonable to assume that the system has been running for a sufficiently long time to attain a steady state approximately described by the asymptotic distribution. From general results we know that such a distribution exists and is given by the stationary distribution π which solves $\pi = \pi\mathbf{P}$. Hence, for large k, $P(X_k = 0) \approx \pi_0$ and $P(X_k = 1) \approx \pi_1$, with $\pi_0 = 1/3$, $\pi_1 = 2/3$. Thus,

$$P(X_k = 1|Y_k = 1) \approx \frac{2(1-\varepsilon)/3}{2(1-\varepsilon)/3 + \varepsilon/3} = \frac{2-2\varepsilon}{2-\varepsilon}.$$

A4
(Solution)

4. We have a Markov chain $\{X_n, n \geq 0\}$ in discrete time with state space $S = \{A, B, C, D, E, F\}$. Put

$$T_C = \inf\{n \geq 0 : X_n = C\} \quad \text{and} \quad m_i = E(T_C|X_0 = i), \quad i \in S.$$

By the principle of conditioning on the first jump starting in each of the states A, B, D, E and F, we obtain

$$
\begin{aligned}
m_A &= 1 + \frac{1}{3}m_B + \frac{2}{3}m_F \\
m_B &= 1 + \frac{2}{3}m_A \\
m_D &= 1 + \frac{1}{3}m_E \\
m_E &= 1 + \frac{1}{3}m_F + \frac{2}{3}m_D \\
m_F &= 1 + \frac{1}{3}m_A + \frac{2}{3}m_E
\end{aligned}
$$

To solve this system note, for example, that equations 1, 2, 5 yield $5m_E = 18 + m_A$, and equations 4, 5, 3 give $5m_A = 18 + 4m_E$. Hence $m_A = 54/7$, and so the mean time until absorbtion in vertex C given initial state A, is approximately 7.71 steps.

1. Introduce the states $1 = \{$Cat in room 1, Mouse in room 2$\}$, $2 = \{$Cat in room 2, Mouse in room 1$\}$ and $3 = \{$Cat and Mouse in same room$\}$. This gives a Markov chain (X_n) on the state space $E = \{1, 2, 3\}$ with $X_0 = 1$ and where state 3 is absorbing. The resulting transition probability matrix is

B1 (Solution)

$$\begin{pmatrix} 0.08 & 0.48 & 0.44 \\ 0.56 & 0.06 & 0.38 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now let $T = \min\{n > 0 : T_n = 3\}$ be the survival time and denote $m_i = E(T|X_0 = i)$. We want to find m_1, which is the average time Mouse survives. Condition on the first jump to get the relations $m_1 = 1 + 0.08m_1 + 0.48m_2$ and $m_2 = 1 + 0.56m_1 + 0.06m_2$, from which we find $m_1 = 355/149 \approx 2.38$ and $m_2 = 370/149 \approx 2.48$.

2. The continuous time pure birth process $\{X(t), t \geq 0\}$, $X(0) = 0$, has infinitesimal generator

B2 (Solution)

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 \\ 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) The transient state probabilities $p_i(t) = P(X_t = i)$ $i = 0, 1, 2,$ $t \geq 0$, are the solutions of the forward equation

$$\begin{cases} p_0'(t) = -\lambda_0 p_0(t) & p_0(0) = 1 \\ p_1'(t) = \lambda_0 p_0(t) - \lambda_1 p_1(t) & p_1(0) = 0 \\ p_2'(t) = \lambda_1 p_1(t) & p_2(0) = 0, \end{cases}$$

which is in explicit form the vector equation $p'(t) = p(t)Q$, $p(t) = (p_0(t), p_1(t), p_2(t))$. The solution is, for $\lambda_0 \neq \lambda_1$,

$$p_0(t) = e^{-\lambda_0 t} \quad p_1(t) = \frac{\lambda_0(e^{-\lambda_0 t} - e^{-\lambda_1 t})}{\lambda_1 - \lambda_0} \quad p_2(t) = \frac{\lambda_0 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_0 t}}{\lambda_1 - \lambda_0} + 1,$$

and for $\lambda_0 = \lambda_1 = \lambda$,

$$p_0(t) = e^{-\lambda t} \quad p_1(t) = \lambda t e^{-\lambda t} \quad p_2(t) = 1 - (1 + \lambda t)e^{-\lambda t}.$$

b) By definition of conditional expectation,

$$\begin{aligned} q_i(t) &= P(X_t = i | \text{absorption has not yet occurred at time } t) \\ &= \frac{p_i(t)}{1 - p_2(t)}, \quad i = 0, 1. \end{aligned}$$

We find

$$q_0(t) = \frac{\lambda_1 - \lambda_0}{\lambda_1 - \lambda_0 e^{-(\lambda_1 - \lambda_0)t}} \qquad q_1(t) = \frac{\lambda_0(1 - e^{-(\lambda_1 - \lambda_0)t})}{\lambda_1 - \lambda_0 e^{-(\lambda_1 - \lambda_0)t}}, \quad \lambda_0 \neq \lambda_1,$$

and

$$q_0(t) = 1/(1 + \lambda t) \qquad q_1(t) = \lambda t/(1 + \lambda t), \quad \lambda_0 = \lambda_1 = \lambda.$$

The asymptotic probabilities obtained as $t \to \infty$ represent the quasi-stationary distribution, i.e. a *steady state conditional on non-absorption.* We obtain

$$\begin{aligned} &\text{If } \lambda_1 > \lambda_0: \quad q = (1 - \lambda_0/\lambda_1, \lambda_0/\lambda_1) \\ &\text{If } \lambda_1 \leq \lambda_0: \quad q = (0, 1) \end{aligned}$$

3. All states communicate since $\alpha > 0$ and $\beta > 0$. Thus, we have a finite, irreducible Markov chain in continuous time. Such a chain is positively recurrent, which is equivalent to saying that there exists a probability distribution which solves the balance equations $\pi\mathrm{Q} = 0$. This stationary distribution $\pi = (\pi_1, \pi_2, \pi_2)$ is unique and is also the asymptotic distribution. To find π, we need to solve the equations B3 (Solution)

$$\begin{aligned} -(\alpha+\beta)\pi_1 + \beta\pi_2 + \beta\pi_3 &= 0 \\ \alpha\pi_1 - 2\beta\pi_2 + \alpha\pi_3 &= 0 \\ \beta\pi_1 + \beta\pi_2 - (\alpha+\beta)\pi_3 &= 0 \end{aligned}$$

under the additional condition $\pi_1 + \pi_2 + \pi_3 = 0$. By symmetry, $\pi_1 = \pi_3$. It is then straightforward to see that

$$\pi = \left(\frac{\beta}{\alpha+2\beta}, \frac{\alpha}{\alpha+2\beta}, \frac{\beta}{\alpha+2\beta}\right).$$

If the chain starts in state 1 or state 3 the time until the first substitution occurs has the exponential distribution with parameter $\alpha+\beta$, and if the chain starts in state 2 this time is exponential with parameter 2β. It follows that the probability that the time until the first event is greater than t is $e^{-(\alpha+\beta)t}$ in the former case and $e^{-2\beta t}$ in the latter. Hence, we obtain the probability that no substitution has yet occurred at time t by giving these two choices the weight which corresponds to the stationary distribution. In particular, for $t = 1$ the required probability equals

$$\frac{\beta}{\alpha+2\beta}e^{-(\alpha+\beta)} + \frac{\alpha}{\alpha+2\beta}e^{-2\beta} + \frac{\beta}{\alpha+2\beta}e^{-(\alpha+\beta)} = \frac{2\beta}{\alpha+2\beta}e^{-(\alpha+\beta)} + \frac{\alpha}{\alpha+2\beta}e^{-2\beta}.$$

4. Let $X(t) \in \{0,1,2,3\}$, $t \geq 0$, denote the number of cars at the gas station, which is suitably modeled as the birth-death process with infinitesimal generator matrix B4 (Solution)

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda \\ 0 & 0 & 2\mu & -2\mu \end{pmatrix}.$$

The given information says that λ is 30 per hour and μ 20 per hour. Solve the balance equations to obtain the steady state solution $\pi = (32, 48, 36, 27)/143$. In particular, the expected number of cars and the variance of the number of cars in equiblirium are given by E(# cars) $= 201/143 \approx 1.406$ and V(# cars) $= 435/143 \approx 3.042$. The asymptotic fraction of lost customers equals the steady-state probability that the gas station is full, according to the ergodic theorem. Hence the loss probability becomes $\pi_3 \approx 0.189$, which means a loss of approximately 5 customers per hour.