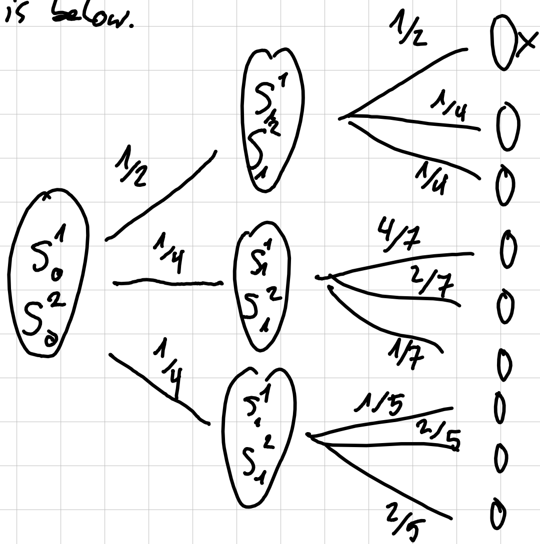
Q1 This is solving several linear equations. Only the computational result is below.

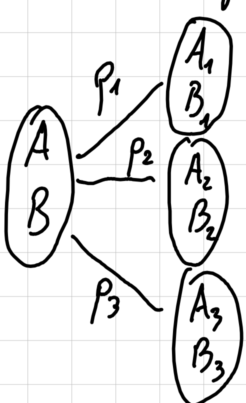


The measure $Q(\omega)$ then given by the path to obtain ω. For instance

$$Q(x) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

where $x = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$.

They all follow the scheme



$$A = p_1 A_1 + p_2 A_2 + p_3 A_3$$

$$B = p_1 B_1 + p_2 B_2 + p_3 B_3$$

$$1 = p_1 + p_2 + p_3$$

Q2 We first determine an EMM

$$\begin{tikzcd}
 & & S_2 = 15 & 0 \\
 & S_1 = 16 \arrow[ur, "2/3"] \arrow[r, "1/3"'] & S_2 = 18 & 0 \\
S_0 = 20 \arrow[ur, "1/3"] \arrow[dr, "2/3"'] & & & \\
 & S_1 = 22 \arrow[r, "2/3"] \arrow[dr, "1/3"'] & S_2 = 21 & 0 \\
 & & S_2 = 24 & 1
\end{tikzcd}$$

Payoff

Assuming we are in the bottom (blue) branch and we hold θ_0^1 stock we have

$$\begin{tikzcd}
 & \theta_1^1 \cdot 21 \\
\theta_0^1 \cdot 22 \mapsto \theta_1^1 \arrow[ur] \arrow[dr] & \\
 & \theta_1^1 \cdot 24
\end{tikzcd}
\qquad \text{So} \quad \begin{cases} \theta_1^1 \cdot 24 + \theta_1^0 \cdot 1 = 1 \\ \theta_1^1 \cdot 21 + \theta_1^0 \cdot 1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \theta_1^1 = \frac{1}{3} \\ \theta_1^0 = -7 \end{cases}$$

While in the top branch we may take $\theta_1^1 = \theta_1^0 = 0$.

This gives

$$\begin{cases} \theta_0^1 \cdot 16 + \theta_0^0 \cdot 1 = \theta_1^1 \cdot 16 + \theta_1^0 \cdot 1 = 0 \\ \theta_0^1 \cdot 22 + \theta_0^0 \cdot 1 = \theta_1^1 \cdot 22 + \theta_1^0 \cdot 1 = \frac{1}{3} \end{cases}$$

$$6 \cdot \theta_0^1 = 1/3 \Rightarrow \theta_0^1 = 1/18 \Rightarrow \frac{16}{18} + \theta_0^0 = 0 \Rightarrow \theta_0^0 = -\frac{16}{18}$$

Hence the value at time 0 is $\theta_0^1 \cdot 20 + \theta_0^0 = \frac{20}{18} - \frac{16}{18} = 2/9$

which is the fair price.

Q3 We have, from the lectures

$$\mathbb{E}_Q(\beta_T(S_T-K)^+) \quad \left[\text{with } Q \text{ defined by } q = \frac{r-b}{a-b}\right]$$

$$= \frac{1}{(1+r)^T}\sum_{i=0}^{T}\binom{T}{i} q^i (1-q)^{T-i}\left((1+a)^i(1+b)^{T-i}S_0 - K\right)^+$$

$$= \frac{1}{(1+r)^6}\sum_{i=0}^{6}\binom{6}{i}\left(\frac{b-r}{b-a}\right)^i\left(\frac{r-a}{b-a}\right)^{6-i}\left((1+a)^i(1+b)^{T-i}\cdot 10 - 12\right)^+$$

$$= \frac{1}{1.05^6}\sum_{i=0}^{3}\binom{6}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{6-i}\left(0.95^i\, 1.2^{T-i}\cdot 10 - 12\right)$$

[since otherwise $(\ldots S_0 - K) \leq 0$]

$= 1.9135..$ [computer calculation]

Similarly, for a put,

$$\mathbb{E}_Q(\beta_T(K-S_T)^+) = \frac{1}{(1+r)^T}\sum_{i=0}^{T}\binom{T}{i}\frac{1}{2^T}\left(K-(1+a)^i(1+b)^{T-i}S_0\right)^+$$

$$= \frac{1}{(1+r)^T}\sum_{i=4}^{6}\binom{6}{i}\frac{1}{2^6}\left(12-(1+a)^i(1+b)^{6-i}\cdot 10\right)$$

$= 0.868..$ [computer calculation]

Q4) First $\left(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x} - e^{-rT}K\right)^+$

is positive for $x > x_0$, where

$$S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x_0} - e^{-rT}K = 0$$

$$\Rightarrow \log(S_0) - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x_0 = -rT + \log K$$

$$\Rightarrow x_0 = \frac{\log K - \log S_0 + \frac{1}{2}\sigma^2 T - rT}{\sigma\sqrt{T}}$$

and the integral becomes

$$\int_{x_0}^{\infty}\left(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x} - e^{-rT}K\right)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx$$

$$= S_0\int_{x_0}^{\infty} e^{-\frac{1}{2}\sigma^2 T} e^{\sigma\sqrt{T}x - \frac{1}{2}x^2}\cdot\frac{1}{\sqrt{2\pi}}\,dx - e^{-rT}K\int_{x_0}^{\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx$$

$$= S_0\int_{x_0}^{\infty}\frac{e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}}{\sqrt{2\pi}}\,dx - e^{-rT}K\int_{x_0}^{\infty} e^{-\frac{1}{2}x^2}/\sqrt{2\pi}\,dx$$

$$= S_0\int_{x_0-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}x^2}/\sqrt{2\pi}\,dx - e^{-rT}K\left(1-\Phi(x_0)\right)$$

$$= S_0\left(1-\Phi(x_0-\sigma\sqrt{T})\right) - e^{-rT}K\,\Phi(-x_0) = S_0\,\Phi(d_+) - e^{-rT}K\,\Phi(d_-)$$

Q5 Clearly $P(E) \geq 0$ since otherwise one could have an option on a stock with no risk. Using call-put parity we have

$$P_0(E) = C_0 - (S_0 - \beta^T K) \geq \beta^T K - S_0 .$$

This gives $P_0(E) \geq \max\{0, \beta^T K - S_0\}$.

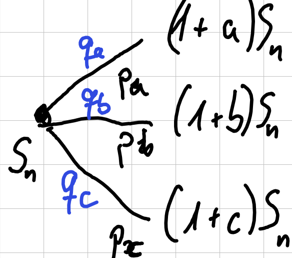
Now assume $P_0(E) > \beta^T K$.

Then sell a put and invest it at risk-free rate. Then, at maturity, the option is only used if $K \geq S_T$. If it is not used, we gain $\frac{1}{\beta^T} P_0(E)$. If it is used we gain $\frac{1}{\beta^T} P_0(E) + S_T - K$

$$> \frac{1}{\beta^T} \beta^T K + S_T - K = S_T \geq 0 .$$

That is, risk-free profit in either case.

Q6) a) Since for fixed maturity time T the model is finite, the model is viable if there exists an EMM. By symmetry, we must have



where q_a, q_b, q_c make this process a martingale.

Hence, we must have

$$0 = \mathbb{E}_Q(\beta_{n+1} S_{n+1} - \beta_n S_n \mid \tilde{\mathcal{F}}_n)$$

$$= \beta_{n+1}\big(q_a(1+a)S_n + q_b(1+b)S_n + q_c(1+c)S_n\big) - \beta_n S_n$$

$$= \left(\frac{q_a + a q_a + q_b + b q_b + q_c + c q_c}{1+r} - 1\right)\left(\frac{1}{1+r}\right)^n S_n$$

$$= \frac{1 + a q_a + b q_b + c q_c - 1 - r}{1+r}\left(\frac{1}{1+r}\right)^n S_n$$

and so $\dfrac{a q_a + b q_b + c q_c - r}{1+r} = 0$

and $a q_a + b q_b + c q_c - r = 0$

Under the standard assumption that $r \geq 0$.

This can be satisfied if one of a, b, c equals r or $\min\{a,b,c\} < r < \max\{a,b,c\}$.

For Q to be equivalent to P, we must also have $q_i = 0$ iff $p_i = 0$. Hence, if $p_a, p_b, p_c > 0$ we must have $a = b = c = r$ or

$\min\{a,b,c\} < r < \max\{a,b,c\}$.

But $a = b = c = r$ gives a trivial (deterministic) model and we may exclude the case.

b) First assume the model is viable. To be complete we further need the uniqueness of Q. Any solution to

$$\begin{cases} a q_a + b q_b + c q_b = r \\ q_a + q_b + q_c = 1 \end{cases}$$

is an EMM. But this system of linear eq. has infinitely many solutions. For it to be unique we need one of p_a, p_b, p_c (and hence one of q_a, q_b, q_c) to be zero, to give the binomial model.

Q 7) We assume $p_a \in (0,1)$ as otherwise there is nothing to show. In each step we have

$$\begin{tikzcd}
 & \begin{pmatrix}(1+a)S_n^1 \\ (1+a')S_n^2\end{pmatrix} \\
\begin{pmatrix}S_n^1 \\ S_n^2\end{pmatrix} \arrow[ur, "q_a"', "p_a"] \arrow[dr, "p_b", "q_b"'] & \\
 & \begin{pmatrix}(1+b)S_n^1 \\ (1+b')S_n^2\end{pmatrix}
\end{tikzcd}$$

a) For such Q to be an EMM we need

$$0 = \mathbb{E}_Q\left(\beta_{n+1} S_{n+1}^1 - \beta_n S_n^1 \mid \mathcal{F}_n\right) = \left(\frac{1}{1+r}\left(q_a(1+a) + q_b(1+b)\right) - 1\right)\beta_n S_n$$

$$0 = \mathbb{E}_Q\left(\beta_{n+1} S_{n+1}^2 - \beta_n S_n^2 \mid \mathcal{F}_n\right) = \left(\frac{1}{1+r}\left(q_a(1+a') + q_b(1+b')\right) - 1\right)\beta_n S_n$$

as well as $q_a = 1 - q_b$. We get the system of lin. eq.

$$\begin{cases} q_a(1+a) + q_b(1+b) = 1+r \\ q_a(1+a') + q_b(1+b') = 1+r \\ q_a + q_b = 1 \end{cases} = \begin{cases} a\,q_a + b\,q_b = r \\ a'\,q_a + b'\,q_b = r \\ q_a + q_b = 1 \end{cases}$$

which has a solution if $a < r < b$ (or $b < r < a$) and $a = a'$, $b = b'$; since $q_a, q_b \neq 0$. Since it is a finite model, the conclusion follows from the existence of Q.

b) The model is also complete when viable as the solution is unique.

Q 8. Let Q, Q' be EMM. Fix $\lambda \in [0,1]$.

We need to show that the measure $R = \lambda Q + (1-\lambda) Q'$ is also an equivalent martingale measure.

We will ignore $\lambda = 0$ and $\lambda = 1$ since such R is equal to Q or Q'.

Let $A \in \mathcal{F}$. Then $R(A) = \lambda Q(A) + (1-\lambda) Q'(A) \geq \lambda Q(A)$

Hence R is positive whenever Q is. Now,

$Y = \mathbb{E}_R(S_t^i \mid \tilde{\mathcal{F}}_{t-1})$ satisfies $\int_F Y \, dR = \int_F S_t^i \, dR$

And $dR = \lambda dQ + (1-\lambda) dQ'$, so

$$\int_F Y \, dR = \int_F Y \cdot \lambda \, dQ + \int_F Y (1-\lambda) \, dQ'.$$

Since this holds for all $F \in \tilde{\mathcal{F}}$, we have

$$\mathbb{E}_R(S_t^i \mid \tilde{\mathcal{F}}_{t-1}) = \lambda \mathbb{E}_Q(S_t^i \mid \tilde{\mathcal{F}}_{t-1}) + (1-\lambda) \mathbb{E}_{Q'}(S_t^i \mid \tilde{\mathcal{F}}_{t-1})$$

and so $\mathbb{E}_R(\beta_t S_t^i \mid \tilde{\mathcal{F}}_{t-1}) = \lambda \mathbb{E}_Q(\beta_t S_t^i \mid \tilde{\mathcal{F}}_{t-1}) + (1-\lambda) \mathbb{E}_{Q'}(\beta_t S_t^i \mid \tilde{\mathcal{F}}_{t-1})$

$= \lambda \beta_{t-1} S_{t-1}^i + (1-\lambda) \beta_{t-1} S_{t-1}^i = \beta_{t-1} S_{t-1}^i$.

This holds for all i and hence R is an EMM.