Exam, Real analysis, 1MA226, 2019-06-15

Solutions.

1. Set $f(x) = x + \frac{1}{x}$; this is a function from $\mathbb{R} \setminus \{0\}$ to 0. Also set

$$E = \left\{x + \tfrac{1}{x} \ : \ \tfrac{1}{2} < x < \tfrac{3}{2}\right\} = f\big((\tfrac{1}{2}, \tfrac{3}{2})\big) = f\big((\tfrac{1}{2}, 1]\big) \cup f\big([1, \tfrac{3}{2})\big) \tag{1}$$

We have $f'(x) = 1 - x^{-2}$; hence $f'(x) < 0$ for all $x \in (0,1)$ and $f'(x) > 0$ for all $x \in (1, \infty)$. Therefore the function f is strictly decreasing on $(0,1]$ and strictly increasing on $[1, \infty)$. Furthermore f is continuous on $\mathbb{R} \setminus \{0\}$ (this follows from Rudin's Theorem 4.9, starting from the fact that the identity function $x \mapsto x$ from $\mathbb{R}$ to $\mathbb{R}$, is continuous).

Using the above facts about f, and the intermediate value theorem (=Rudin's Theorem 4.23), it now follows that

$$f\big((\tfrac{1}{2}, 1]\big) = \big[f(1), f(\tfrac{1}{2})\big) = \big[2, \tfrac{5}{2}\big) \tag{2}$$

and

$$f\big([1, \tfrac{3}{2})\big) = \big[f(1), f(\tfrac{3}{2})\big) = \big[2, \tfrac{13}{6}\big). \tag{3}$$

[Detailed proof of (2): Since f is strictly decreasing on $(0,1]$, we have for all $x \in (\frac{1}{2}, 1]$: $f(x) \geq f(1)$ and $f(x) < f(\frac{1}{2})$. Hence

$$f((\tfrac{1}{2}, 1]) \subset [f(1), f(\tfrac{1}{2})). \tag{4}$$

On the other hand, by the intermediate value theorem, for every $c \in (f(1), f(\frac{1}{2}))$ there exists some $x \in (\frac{1}{2}, 1)$ such that $f(x) = c$; hence $(f(1), f(\frac{1}{2})) \subset f((\frac{1}{2}, 1))$, and since $f((\frac{1}{2}, 1])$ also contains $f(1)$, we conclude that $f((\frac{1}{2}, 1])$ contains $f(1) \cup (f(1), f(\frac{1}{2})) = [f(1), f(\frac{1}{2}))$, i.e.

$$f((\tfrac{1}{2}, 1]) \supset [f(1), f(\tfrac{1}{2})). \tag{5}$$

Together, (4) and (5) prove that (2) holds.]

[The proof of (3) is completely similar.]

Together, (1), (2) and (3) give:

$$E = \big[2, \tfrac{5}{2}\big) \cup \big[2, \tfrac{13}{6}\big) = \big[2, \tfrac{5}{2}\big). \tag{6}$$

(In the last equality we used the fact that $\frac{13}{6} < \frac{5}{2}$.)

It is immediate from (6) that

$$\sup E = \sup \big[2, \tfrac{5}{2}\big) = \tfrac{5}{2}.$$

[Detailed proof: $\frac{5}{2}$ is an upper bound of $[2, \frac{5}{2})$, since $x \in [2, \frac{5}{2})$ implies $x < \frac{5}{2}$. Furthermore, for every $\gamma < \frac{5}{2}$ we have that the interval $(\gamma, \frac{5}{2})$ is non-empty, and every number $x \in (\gamma, \frac{5}{2})$ satisfies $x > \gamma$ and $x \in E$, showing that γ is *not* an upper bound of E. Done!]

Answer: $\frac{5}{2}$.

2. (a). For n *even* we have $x_n = 2n \to +\infty$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty} x_n = +\infty.$$

For all *odd* n we have $x_n = 0$. Note also that $x_n \geq 0$ for *all* n. Therefore,

$$\liminf_{n\to\infty} x_n = 0.$$

(b). Recall that $(1+\frac{1}{n})^n$ tends to e as $n \to \infty$. Also, the second term, $\sin\frac{2\pi n}{3}$, is periodic with period 3, taking the values $\frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}, \cdots$ as n runs through $1, 2, 3, \cdots$. It turns out that it is convenient to consider *six* subsequences:

(I) For $n = 1, 7, 13, 19, \ldots$ (i.e., $n = 6k+1$ with $k = 0, 1, 2, \ldots$), we have $x_n = -(1+\frac{1}{n})^n + \frac{1}{2}\sqrt{3} \to -e + \frac{1}{2}\sqrt{3}$ as $n \to \infty$.

(II) For $n = 2, 8, 14, 20, \ldots$ (i.e., $n = 6k+2$ with $k = 0, 1, 2, \ldots$), we have $x_n = (1+\frac{1}{n})^n - \frac{1}{2}\sqrt{3} \to e - \frac{1}{2}\sqrt{3}$ as $n \to \infty$.

(III) For $n = 3, 9, 15, 21, \ldots$ (i.e., $n = 6k+3$ with $k = 0, 1, 2, \ldots$), we have $x_n = -(1+\frac{1}{n})^n + 0 \to -e$ as $n \to \infty$.

(IV) For $n = 4, 10, 16, 22, \ldots$ (i.e., $n = 6k+4$ with $k = 0, 1, 2, \ldots$), we have $x_n = (1+\frac{1}{n})^n + \frac{1}{2}\sqrt{3} \to e + \frac{1}{2}\sqrt{3}$ as $n \to \infty$.

(V) For $n = 5, 11, 17, 23, \ldots$ (i.e., $n = 6k+5$ with $k = 0, 1, 2, \ldots$), we have $x_n = -(1+\frac{1}{n})^n - \frac{1}{2}\sqrt{3} \to -e - \frac{1}{2}\sqrt{3}$ as $n \to \infty$.

(VI) For $n = 6, 12, 19, 24, \ldots$ (i.e., $n = 6k+6$ with $k = 0, 1, 2, \ldots$), we have $x_n = -(1+\frac{1}{n})^n - \frac{1}{2}\sqrt{3} \to -e - \frac{1}{2}\sqrt{3}$ as $n \to \infty$.

It follows from the above that

$$\limsup_{n\to\infty} x_n = \max\big(\pm e + \tfrac{1}{2}\sqrt{3}, \pm e - \tfrac{1}{2}\sqrt{3}, \pm e\big) = e + \tfrac{1}{2}\sqrt{3}$$

and

$$\liminf_{n\to\infty} x_n = \min\big(\pm e + \tfrac{1}{2}\sqrt{3}, \pm e - \tfrac{1}{2}\sqrt{3}, \pm e\big) = -e - \tfrac{1}{2}\sqrt{3}$$

□

3. Let us first prove that the series defining $F(x)$ is uniformly convergent on any interval $[a,b]$ with $0<a<b$. Indeed,

$$|e^{-nx}(\log n-\sin nx)|\leq e^{-na}(\log(n)+1) \qquad \text{for all } n\in\mathbb{Z}^+ \text{ and } x\in[a,b].$$

Furthermore, for any $a>0$ we have

$$\lim_{n\to\infty}\frac{e^{-(n+1)a}(\log(n+1)+1)}{e^{-na}(\log(n)+1)}=e^{-a}<1,$$

and hence by the ratio test, the series $\sum_{n=1}^{\infty}e^{-na}(\log(n)+1)$ converges. Using the above facts in combination with Weierstrass' M-test, we conclude that the series defining $F(x)$ is uniformly convergent on $[a,b]$, as claimed.

Hence it follows that F is well-defined and continuous in the interval $[a,b]$. Since this is true for any $0<a<b$, it follows that F is well-defined and continuous in the whole interval $(0,\infty)$.

Next consider the series obtained by formally differentiating the series for $F(x)$ term by term, i.e.:

$$\sum_{n=1}^{\infty}e^{-nx}\big((-n)(\log(n)-\sin nx)-n\cos nx\big). \tag{7}$$

We claim that this series is uniformly convergent on any interval $[a,b]$ with $0<a<b$. Indeed, for all $n\in\mathbb{Z}^+$ and $x\in[a,b]$ we have

$$\Big|e^{-nx}\big((-n)(\log(n)-\sin nx)-n\cos nx\big)\Big|\leq e^{-na}\,n\,(2+\log n).$$

Furthermore, for any $a>0$ we have

$$\lim_{n\to\infty}\frac{e^{-(n+1)a}\,(n+1)\,(2+\log(n+1))}{e^{-na}\,n\,(2+\log n)}=e^{-a}<1,$$

and hence by the ratio test, the series $\sum_{n=1}^{\infty}e^{-na}\,n\,(2+\log n)$ converges. Using the above facts in combination with Weierstrass' M-test, we conclude that the series in (7) is indeed uniformly convergent on $[a,b]$. Hence by Rudin's Thm. 7.17, we have that $F'(x)$ exists for all $x\in[a,b]$, and

$$F'(x)=\sum_{n=1}^{\infty}e^{-nx}\big((-n)(n-\sin nx)-n\cos nx\big).$$

The uniform convergence pointed out above shows that this function is continuous in $[a,b]$. Hence F is C^1 in $[a,b]$. Since this is true for any $0<a<b$, we conclude that F is C^1 in the whole interval $(0,\infty)$. $\square$

4. Example:

$$\mathcal{U} = \left\{ \left(\frac{1}{n}, 2 \right) \ : \ n \in \mathbb{Z}^+ \right\}.$$

Every element of $\mathcal{U}$ is an open interval, and for every $x \in (0, 1]$ we can choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < x$, and for this n we have $x \in \left(\frac{1}{n}, 2\right)$ and $(\frac{1}{n}, 2) \in \mathcal{U}$. Hence $\mathcal{U}$ is an open cover of $(0, 1]$.

Next let $\mathcal{V}$ be an arbitrary finite subset of $\mathcal{U}$; then

$$\mathcal{V} = \left\{ (\frac{1}{n}, 2) \ : \ n \in F \right\}$$

for some finite subset $F \subset \mathbb{Z}^+$. Since F is finite, there exists some $B \in \mathbb{Z}^+$ such that $n < B$ for all $n \in F$. Then for all $n \in F$ we have $B^{-1} < n^{-1}$ and so $B^{-1} \notin (\frac{1}{n}, 2)$; hence:

$$B^{-1} \notin \bigcup_{I \in \mathcal{V}} I.$$

But $B^{-1} \in (0, 1]$ since $B \in \mathbb{Z}^+$. Hence $\mathcal{V}$ is *not* a cover of $(0, 1]$. We have thus proved that $\mathcal{U}$ does not contain any finite subcover of $(0, 1]$. $\square$

5. Consider the map $\phi : C([0,1]) \to C([0,1])$ given by

$$\phi(f)(x) = \frac{1}{2}\int_x^1 (y-x)f(y)\,dy + xe^{x^2}.$$

We first have to prove that this is really a map from $C([0,1])$ to $C([0,1])$ as claimed. Thus let $f \in C([0,1])$ be given. We then have to prove that $\phi(f)(x)$ is a continuous function on $[0,1]$. Since $x^2e^{x^2}$ is a continuous function of x, it suffices to prove that $\int_x^1 (y-x)f(y)\,dy$ is a continuous function of $x \in [0,1]$. But we have

$$\int_x^1 (y-x)f(y)\,dy = \int_x^1 yf(y)\,dy - x\int_x^1 f(y)\,dy,$$

and by Theorem 6.20[1], both "$\int_x^1 yf(y)\,dy$" and "$\int_x^1 f(y)\,dy$" are continuous functions of x. Hence the above expression is a continuous function of x, completing the proof that ϕ maps $C([0,1])$ to $C([0,1])$.

Next, for any $f, g \in C([0,1])$ and any $x \in [0,1]$ we have

$$\begin{aligned}|\phi(f)(x) - \phi(g)(x)| &= \frac{1}{2}\left|\int_x^1 (y-x)(f(y)-g(y))\,dy\right| \\ &\leq \frac{1}{2}d(f,g)\int_x^1 (y-x)\,dy \leq \frac{1}{4}d(f,g).\end{aligned}$$

This proves that ϕ is a contraction on $C([0,1])$. Recall also that $C([0,1])$ is complete. Hence by the contraction principle, ϕ has a unique fixed point in $C([0,1])$. This is equivalent to saying that the integral equation in the problem formulation has a unique solution in $C([0,1])$. □

[1]Pedantically, Thm. 6.20 is stated in the case where the *upper* integration limit is varying. To apply it to our situation, one may rewrite $\int_x^1 yf(y)\,dy = \int_0^1 yf(y)\,dy - \int_0^x yf(y)\,dy$; here Thm. 6.20 implies that $\int_0^x yf(y)\,dy$ is a continuous function of x; hence also $\int_x^1 yf(y)\,dy$ is a continuous function of x. Similarly for $\int_x^1 f(y)\,dy$.

Alternative proof of the fact that $\int_x^1 (y-x)f(y)\,dy$ is a continuous function of x: Take B such that $|f(y)| \leq B$ for all $y \in [0,1]$. We then have, for any $0 \leq x \leq x' \leq 1$:

$$
\begin{aligned}
&\left| \int_x^1 (y-x)f(y)\,dy - \int_{x'}^1 (y-x')f(y)\,dy \right| \\
&\leq \int_x^{x'} |y-x|\,|f(y)|\,dy + \int_{x'}^1 |x-x'||f(y)|\,dy \\
&\leq B \int_x^{x'} (y-x)\,dy + B(x'-x)(1-x') \\
&\leq B\Big((x'-x)^2 + (x'-x)\Big) \\
&\leq 2B(x'-x).
\end{aligned}
$$

Hence by symmetry we have

$$
\left| \int_x^1 (y-x)f(y)\,dy - \int_{x'}^1 (y-x')f(y)\,dy \right| \leq 2B|x'-x|
$$

for all $x, x' \in [0,1]$. In particular, for any $\varepsilon > 0$, we have

$$
\left| \int_x^1 (y-x)f(y)\,dy - \int_{x'}^1 (y-x')f(y)\,dy \right| < \varepsilon
$$

for all $x, x' \in [0,1]$ with $|x-x'| < \varepsilon/(2B)$. Hence $\int_x^1 (y-x)f(y)\,dy$ is a (uniformly continuous and hence a) continuous function of $x \in [0,1]$.
$\square$

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$F(u, v) = (e^u + v, u + e^v).$$

Note that F is C^1. We compute:

$$[F'(u, v)] = \begin{pmatrix} e^u & 1 \\ 1 & e^v \end{pmatrix}.$$

In particular

$$[F'(0, 1)] = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix},$$

which is non-singular. Hence by the *Inverse Function Theorem,* there exists an open set $V \subset \mathbb{R}^2$ which contains the point $(0, 1)$, such that $F|_V$ is C^1, $U := F(V)$ is open, and $G := (F|_V)^{-1} : U \to V$ is C^1.

By the definition of $G = (F|_V)^{-1}$ we have $F(G(x, y)) = (x, y)$ for all $(x, y) \in U$. In other words:

$$\begin{cases} e^{G_1(x,y)} + G_2(x, y) = x \\ G_1(x, y) + e^{G_2(x,y)} = y, \end{cases} \qquad \forall (x, y) \in U.$$

Also $G(F(0, 1)) = (0, 1)$, i.e. $G(2, e) = (0, 1)$. This means that if we write $u = G_1 : U \to \mathbb{R}$ and $v = G_2 : U \to \mathbb{R}$ then the functions u and v have all the properties required in the problem formulation!

By the chain rule we also have $F'(G(x, y)) \cdot G'(x, y) = I$ for all $(x, y) \in U$; thus in particular $F'(0, 1) \cdot G'(2, e) = I$, or in other words:

$$[G'(2, e)] = [F'(0, 1)]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix}^{-1} = \frac{1}{e - 1} \begin{pmatrix} e & -1 \\ -1 & 1 \end{pmatrix}.$$

But we also know

$$[G'] = \begin{pmatrix} D_1G_1 & D_2G_1 \\ D_1G_2 & D_2G_2 \end{pmatrix} = \begin{pmatrix} D_1u & D_2u \\ D_1v & D_2v \end{pmatrix}.$$

Hence:

$$[u'(2, e)] = \frac{1}{e - 1}\big(e, -1\big) \quad \text{and} \quad [v'(2, e)] = \frac{1}{e - 1}\big(-1, 1\big).$$

$\square$

7.

(a). We have $D_1 f(0,0) = 1$ since $f(x,0) = x$ for all $x \in \mathbb{R}$. Similarly, $D_2 f(0,0) = 0$ since $f(0,y) = 0$ for all $y \in \mathbb{R}$.

(b). Suppose that f is differentiable at $(0,0)$. Then by Rudin's Theorem 9.17,

$$[f'(0,0)] = \big[(D_1 f)(0,0)\ \ (D_2 f)(0,0)\big] = \big[1\ \ 0\big],$$

i.e. $f'(0,0)$ is the linear map from $\mathbb{R}^2$ to $\mathbb{R}$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a.$$

Hence the assumption that f is differentiable at $(0,0)$ means that

$$\lim_{(x,y)\to(0,0)} \frac{\big|f(x,y) - f(0,0) - x\big|}{\sqrt{x^2+y^2}} = 0,$$

or equivalently:

$$\lim_{(x,y)\to(0,0)} \frac{\big|-xy^2\big|}{(x^2+y^2)^{3/2}} = 0.$$

In particular this implies, by letting $(x,y) \to (0,0)$ along the line $y = x$:

$$\lim_{x\to 0} \frac{|-x^3|}{(2x^2)^{3/2}} = 0. \tag{8}$$

However for all $x > 0$ we have $\frac{|-x^3|}{(2x^2)^{3/2}} = \frac{x^3}{2^{3/2}x^3} = 2^{-3/2}$; hence

$$\lim_{x\to 0^+} \frac{|-x^3|}{(2x^2)^{3/2}} = 2^{-3/2}. \tag{9}$$

Together, (8) and (9) give a *contradiction!*

This proves that f is *not* differentiable at $(0,0)$. $\square$

8. For any $m \in \mathbb{Z}^+$ we let P_m be the partition of $[0,1]$ determined by the following numbers:

$$0 < \frac{1}{m} - \frac{1}{2m^2} < \frac{1}{m} + \frac{1}{2m^2} < \frac{1}{m-1} - \frac{1}{2m^2} < \frac{1}{m-1} + \frac{1}{2m^2}$$
$$< \cdots < \frac{1}{2} - \frac{1}{2m^2} < \frac{1}{2} + \frac{1}{2m^2} < 1 - \frac{1}{2m^2} < 1.$$

[Verification that all the above inequalities really hold: This is obvious except for the inequalities of the form $\frac{1}{j} + \frac{1}{2m^2} < \frac{1}{j-1} - \frac{1}{2m^2}$ for $j \in \{2,3,\ldots,m\}$. However we have $\frac{1}{j} + \frac{1}{2m^2} < \frac{1}{j-1} - \frac{1}{2m^2} \Leftrightarrow \frac{1}{m^2} < \frac{1}{j-1} - \frac{1}{j}$ $\Leftrightarrow \frac{1}{m^2} < \frac{1}{j(j-1)} \Leftrightarrow m^2 > j(j-1)$, and the last inequality is clearly true for every $j \in \{2,3,\ldots,m\}$. Done!]

Note that the function f is *identically zero* on every interval in the partition P_m except for the intervals

$$\Big[0, \frac{1}{m} - \frac{1}{2m^2}\Big] \quad \text{and} \quad \Big[\frac{1}{j} - \frac{1}{2m^2}, \frac{1}{j} + \frac{1}{2m^2}\Big] \qquad (j = m, m-1, \ldots, 2)$$
$$\text{and} \quad \Big[1 - \frac{1}{2m^2}, 1\Big].$$

On the other hand, on each interval J in the above list, the function f takes both the values 0 and 1, so that $\inf_J f = 0$ and $\sup_J = 1$. Hence we have

$$L(P_m, f) = \sum_i m_i \Delta x_i = \sum_i 0 \cdot \Delta x_i = 0,$$

and

$$U(P_m, f) = \sum_i M_i \Delta x_i = 1 \cdot \Big(\frac{1}{m} - \frac{1}{2m^2}\Big) + \Big(\sum_{j=2}^{m} 1 \cdot \frac{2}{2m^2}\Big) + 1 \cdot \frac{1}{2m^2}$$
$$= \frac{2m-1}{2m^2} + \frac{m-1}{m^2} + \frac{1}{2m^2} = \frac{4m-2}{2m^2}.$$

From this we conclude:

$$\lim_{m\to\infty} L(P_m, f) = 0 \qquad \text{and} \qquad \lim_{m\to\infty} U(P_m, f) = 0.$$

Hence by Rudin's Theorem 6.6 (and its proof) it follows that f is Riemann integrable on $[0,1]$, and that $\int_0^1 f(x)\,dx = 0$. $\square$