UPPSALA UNIVERSITET

Lecture Notes

Complex Analysis

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Contents

1. Intro

In this course, we shall study functions $f : \mathbb{C} \to \mathbb{C}$ (or more generally, $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$)

Definition/Sats 1.1: Complex Number

A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$

Anmärkning:
The number x is called the *real part* ($\mathrm{Re}(z) = x$) of the complex number, and y is called the *imaginary part* ($\mathrm{Im}(z) = y$) of the complex number

Anmärkning:
The set of all complex numbers is denoted by $\mathbb{C}$

Anmärkning:
$\mathbb{C}$ is the *smallest* field extension to $\mathbb{R}$ that is algebraically closed.

Anmärkning:
$i^2 = -1$

1.1. Operations over $\mathbb{C}$.

We define the operations *addition* and *multiplication* of two complex unmebrs as follows:

Definition/Sats 1.2: Addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Definition/Sats 1.3: Multiplication of complex numbers

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations, $\mathbb{C}$ forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Definition/Sats 1.4: Complex conjugate

The *complex conjugate* of a complex number $z = x + iy$, denoted by $\overline{z}$, is defined by $\overline{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$
- $\overline{\overline{z}} = z$
- $z \cdot \overline{z} = |z|^2$
- $z^-1 = \frac{\overline{z}}{|z|^2}$
- $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$

Anmärkning:
$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$
$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$

Anmärkning:
Multiplication by i is simply rotation by $\frac{\pi}{2}$ counterclockwise.

Definition/Sats 1.5

Let $z \in \mathbb{C}$. Then there eixsts a $w \in \mathbb{C}$ such that $w^2 = z$ (where $-w$ also satisfies this equation)

Bevis 1.1

Let $z = a + bi$ and $w = x + iy$ such that $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then $a = x^2 - y^2$ and $b = 2xy$
We also know that $|z| = a^2 + b^2 = \left|x^2 + y^2\right|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore, $x^2 + y^2 = \sqrt{a^2 + b^2}$ and:

$$\left.\begin{aligned} x^2 - y^2 &= a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned}\right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left.\begin{aligned} -x^2 + y^2 &= -a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned}\right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let $\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$ and $\beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$ and let $\sqrt{}$ denote the positive square rroot of positive real numbers.

If b is positive, then either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$
If b is negative, then either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$

Therefore, the equation has solutions $\pm(\alpha + \mu\beta i)$ where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$

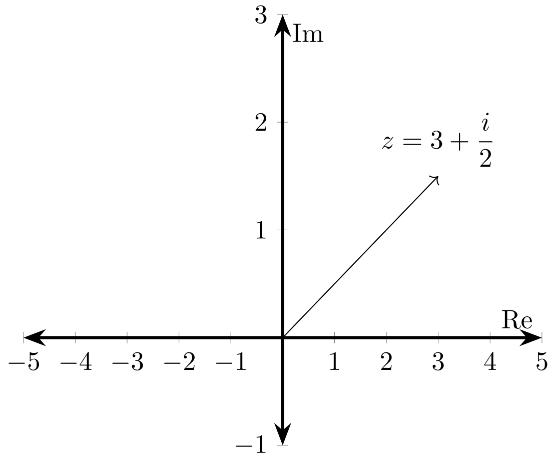
□

Anmärkning:
From the proof above, we can conclude the following:

- The square roots of a complex number are real $\Leftrightarrow$ the complex number is real and positive
- The square roots of a complex number are purely imaginary $\Leftrightarrow$ the complex number is real and negative
- The two square roots of a number coincide $\Leftrightarrow$ the complex number is zero

1.2. **Cartesian representation.**

It is natural to represent a complex number $z = x + iy$ as a tuple (x, y), and we can therefore represent it in the standard cartesian plane:



Anmärkning:
This is sometimes called the *complex plane*

Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number $z = x+iy$ (geometrically the length of the vector), denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:
- $|z|^2 = z \cdot \overline{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Anmärkning:
Every $z \in \mathbb{C}$ such that $z \neq 0$ (that is, $x \neq 0$ or $y \neq 0$) has a multiplcative inverse $\frac{1}{z}$ given by:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Definition/Sats 1.7: Triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that $|z_1 + z_2| \leq |z_1| + |z_2|$

Lemma 1.1: Reversed triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Bevis 1.2

$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

So that $|z_1| - |z_2| \leq |z_1 - z_2|$ $\square$

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $-|z| \leq \mathrm{Re}(z) \leq |z|$
- $-|z| \leq \mathrm{Im}(z) \leq |z|$
- $|\overline{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \cdots + z_n w_n| \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \cdots + |w_n|^2}$

1.3. **Polar form.**

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, and hence there exists θ such that:

$$\frac{x}{|z|} = \cos(\theta) \qquad \frac{y}{|z|} = \sin(\theta)$$

Therefore $z = x + iy$ can be written as:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Where $r = |z|$ is uniquely determined by z, while θ is 2π-periodic. This is called the *polar form* of z and just as the cartesian representation requires a tuple of information $(|z|, \theta)$

Definition/Sats 1.8: Argument

The *argument* of a complex number z, denoted by $\arg(z)$, is the angle θ between z and the real number line in the complex plane

Anmärkning:
Since the argument is 2π periodic, the angle is usually given as $\theta + k2\pi \quad k \in \mathbb{Z}$, but we are only intersted in θ
This θ is called the *principal value* of $\arg(z)$, denoted by $\mathrm{Arg}(z)$ and belongs to $(-\pi, \pi]$

Anmärkning:
We are always allowed to change an angle by multiples of 2π, the principal value argument is the angle after changing the argument such that it lies between $(-\pi, \pi]$

Anmärkning:
A specifiaction of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that $\mathrm{Arg}(z)$ is "discontinous" along the negative real axis. This is called a *branch-cut*

Suppose $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$
Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))] \\ & r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

Anmärkning:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

1.4. **Exponential form.**

Definition/Sats 1.9

For $z = x + iy \in \mathbb{C}$, let $e^z = e^x(\cos(y) + i\sin(y))$

Anmärkning:
$e^{iy} = \cos(y) + i\sin(y) \quad y \in \mathbb{R}$ (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$\Rightarrow e^{iy} = 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin(\theta)}$$

$$\Rightarrow e^z = e^x(\cos(\theta) + i\sin(\theta))$$

Anmärkning:
One can through comparing see that $|e^z| = e^x$, and that $\left|e^{iy}\right| = 1$

Properties of the exponential form:

- $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$
- $e^z \neq 0 \quad \forall z \in \mathbb{C}$
- $x \in \mathbb{R} \Rightarrow e^x > 1$ if $x > 0$ and $e^x < 1$ if $x < 0$
- $\left|e^{x+iy}\right| = e^x$
- $e^{i\pi/2} = i \quad e^{i\pi} = -1 \quad e^{3i\pi/2} = -1 \quad e^{2i\pi} = 1$
- e^z is 2π-periodic
- $e^z = 1 \Leftrightarrow z = 2\pi k i \quad k \in \mathbb{Z}$

Definition/Sats 1.10: deMoivre's formula

For $n \in \mathbb{Z}$, $(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta))$

1.5. **Logarithmic form.**

In real analysis, we have defined the logarithm as the inverse of e^x. This has previously worked since for $x \in \mathbb{R}$, e^x is injective.
The problem is that for e^z where $z \in \mathbb{C}$, it is not injective and should therefore not have an inverse.

Given $z \in \mathbb{C} \setminus \{0\}$, we define $\ln(z)$ as the cut of all $w \in \mathbb{C}$ whose image undre the exponential form is z, i.e $w = \ln(z) \Leftrightarrow z = e^w$.

Here, $\ln(z)$ is a *multivaled form*

We can use the fact that $|z| = r = e^x$ to derive some interesting properties of the logarithm:

$$z = re^{i\theta} \qquad w = u + iv$$

$$\text{If } z = e^w \Leftrightarrow re^{i\theta} = e^u \cdot e^{iv}$$

$$\Leftrightarrow u = \ln(r) = \ln(|z|) \qquad v = \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z}$$

Definition/Sats 1.11: Complex logarithm

For $z \neq 0$, we define the complex logarithm for $z \in \mathbb{C}$ as:

$$\begin{aligned} \ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\operatorname{Arg}(z) + k2\pi) \quad k \in \mathbb{Z} \end{aligned}$$

2. Elementary complex functions

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

2.1. **Branches of the complex logarithm.**

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of 2π).
If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

Definition/Sats 2.12: Principal logarithm

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\mathrm{Ln}\,(z) = \ln(|z|) + i \cdot \mathrm{Arg}(z)$$

Anmärkning:
We have essentially extended the "normal" logarithm, which is defined on $(0, \infty)$, to be defined on $\mathbb{C} \setminus \{0\}$

Anmärkning:
The principal logarithm is discontinous for negative reals, since their principal value argument is $= -\pi$, but the principal value argument is discontinous at $-\pi$. This is the so called *branch-cut*

Anmärkning:
Even though the principal logarithm is discontinous for negative reals, it is not undefined. Any negative real number z will have $\mathrm{Arg}(z) = \pi$, which the logarithm very much is defined for.

Anmärkning:
When branching, we do not necessarily have to pick $(-\pi, \pi]$, we can pick any interval $(\alpha, \alpha + 2\pi]$. This is usually denoted by $\arg_\alpha$.

2.2. **Complex mappings.**

One can think of a complex mapping $f : \mathbb{C} \to \mathbb{C}$ as $f(z) = f(x + iy) = w = u + iv$
Then it becomes clear which regions map to where by drawing them in their respective z-plane and w-plane.

2.3. **Complex powers.**

Given $z \in \mathbb{C}$, consider the following equation:

$$w^u = z \tag{1}$$

The set of all solutions w of (1) is denoted $z^{1/n}$m and is called the *n-th root of z*.

Anmärkning:
If $z = 0$, then $w = 0$

Suppose $z \neq 0$, then we may write $w = |w|\, e^{i\alpha}$ and $z = |z|\, e^{i\theta}$
By deMoivre's formula, (1) becomes:

$$|w|^n\, e^{in\alpha} = |z|\, e^{i\theta}$$

Then, the following follows:

$$\left.\begin{array}{l} |w| = \sqrt[n]{|z|} \\ n\alpha = \theta + k2\pi \quad k \in \mathbb{Z} \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} |w| = \sqrt[n]{|z|} \\ \alpha = \dfrac{\theta}{n} + k\dfrac{2\pi}{n} \quad k \in \mathbb{Z} \end{array}\right\}$$

Notice now that every $k \in \mathbb{Z}$ gives a solution to (1)

Since sine and cosine are both 2π-periodic, then only $k = 0, 1, \cdots, n-1$ actually give *different* solutions (since $k = n \Rightarrow \alpha = \dfrac{\theta}{n} + n\dfrac{2\pi}{n}$)

Suppose $z \neq 0$. For $n \in \mathbb{Z}$ it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that $\ln(z)$ attains.

It is also true, that for $n = 1, 2, 3, \cdots$:

$$z^{\frac{1}{n}} = e^{\frac{1}{n}\ln(z)}$$

We can let $n \in \mathbb{C}$, and obtain the following definition:

Definition/Sats 2.13: Complex power

For $\alpha \in \mathbb{C}$, let:

$$z^\alpha = e^{\alpha \ln(z)} \qquad z \neq 0$$

Anmärkning:
This makes z^α a multivalued function, but it is possible to have a single-valued output from it.

Definition/Sats 2.14

Let $a, b \in \mathbb{C}$ where $a \neq 0$. Then a^b is single-valued (does not depend on the choice of branch for the logarithm) $\Leftrightarrow b \in \mathbb{Z}$

If $b \in \mathbb{Q}$ and is in lowest form (that is, $b = \dfrac{p}{q}$ where p, q have no common factors), then a^b has exactly q distinct values (the q:th roots of a^p)

If $b \in \mathbb{C}\backslash\mathbb{Q}$, then a^b has infinetly many values.

Bevis 2.1

Chose some interval (branch), say $[0, 2\pi)$, for the arg function and let $\ln(z)$ be the corresponding branch of the logarithm. If we chose another branch, we would have $\ln(a) + 2\pi kbi$ rather than $\ln(a)$ (where $k \in \mathbb{Z}$)
Therefore, $a^b = e^{b\ln(a) + 2\pi kbi} = e^{b\ln(a)} \cdot e^{2\pi ki}$

Notice that $e^{2\pi kbi}$ stays the same regardles of $b \in \mathbb{Z}$, as long as it is an integer.

In the same way, it can be shown that $e^{2\pi kip/q}$ has q distinct values if p, q have no common factor.

If b is irrational, and if $e^{2\pi kbi} = e^{2\pi mbi}$, then it follows that $e^{(2\pi bi)(k-m)} = 1$, and therefore $b(k-m)$ is an integer.
Since b is irrational, then $n - m = 0$ $\square$

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up. We can chose to branch z^α:

$$z^\alpha = e^{\alpha \mathrm{Ln}(z)}$$

2.4. **Trigonometric and Hyperbolic functions.**

We have the following:

$$\left.\begin{aligned} e^{iy} &= \cos(y) + i\sin(y) \\ e^{-iy} &= \cos(y) - i\sin(y) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos(y) &= \frac{e^{iy}+e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy}-e^{-iy}}{2i} \end{aligned}\right\}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

Definition/Sats 2.15: Complex sine and cosine

For $z \in \mathbb{C}$, we define:

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals.
When defining them for complex numbers, we just extend their domain:

Definition/Sats 2.16: Complex hyperbolic functions

For $z \in \mathbb{C}$, we define:

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z}-e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

- **Sine**:

$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$
$$\Rightarrow \frac{e^{-y}(\cos(x)+i\sin(x)) - e^{y}(\cos(x)-i\sin(x))}{2i} = \frac{(e^{-y}-e^{y})\cos(x) + i(e^{y}-e^{-y})\sin(x)}{2i}$$
$$= \frac{(e^{-y}-e^{y})\cos(x)}{2i} + \frac{(e^{y}-e^{-y})\sin(x)}{2}$$
$$\overset{i^{-1}=-i}{\Longrightarrow} \underbrace{\frac{(e^{y}-e^{-y})}{2}}_{\sinh(y)} i\cos(x) + \underbrace{\frac{(e^{y}+e^{-y})}{2}}_{\cosh(y)} \sin(x)$$

- **Cosine**:

$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2}$$
$$= \frac{e^{-y}(\cos(x)+i\sin(x)) + e^{y}(\cos(x)-i\sin(x))}{2} = \frac{\cos(x)(e^{y}+e^{-y}) + i(e^{-y}-e^{y})\sin(x)}{2}$$
$$= \underbrace{\frac{e^{y}+e^{-y}}{2}}_{\cosh(y)}\cos(x) - \underbrace{\frac{e^{y}-e^{-y}}{2}}_{\sinh(y)} i\sin(x)$$

This leads us to the following:

Definition/Sats 2.17: Addition formulas for complex trigonometric functions

- $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$
- $\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

Anmärkning:
Both sine and cosine can be defined as the unique solution to an ODE, namely:

$$f''(x) + f(x) = 0 \qquad f(0) = 0,\ f'(0) = 1 \qquad f(x) = \sin(x)$$
$$f''(x) + f(x) = 0 \qquad f(0) = 1,\ f'(0) = 0 \qquad f(x) = \cos(x)$$

2.5. **Mapping properties of** $\sin(z)$**.**

Let $f(z) = \sin(z)$ in $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$, let A be the set of points allowed with respect to the above constraint and let B be the mapping of those points by $\sin(A)$

Claim: $f : A \to B$ is a bijective mapping

Bevis 2.2

Take a $z \in \mathbb{C} \quad z = x + iy \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.
Then:

$$f(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$f(z) \in \mathbb{R} \Leftrightarrow \cos(x)\sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0$$

If $y = 0$, then:

$$f(z) = \sin(x)\cosh(y) = \sin(x) \in (-1, 1)$$

Therefore, if $z \in A \Rightarrow f(z) \in B$. Now we need to show that for any $z \in B$, there is a u such that $f(u) = z$

Let $u = \sin(x)\cosh(y) \quad , v = \cos(x)\sinh(y)$ and pick a vertical line at $x = a \neq 0$
We will now consider the images of these lines:

$$\cosh(y) = \frac{u}{\sin(a)} \qquad \sinh(y) = \frac{v}{\cos(a)}$$
$$(\cosh(y))^2 - (\sinh(y))^2 = 1 \Rightarrow \left(\frac{u}{\sin(a)}\right)^2 - \left(\frac{v}{\cos(a)}\right)^2 = 1$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line $y = b \neq 0$

$$\sin(x) = \frac{u}{\cosh(b)} \qquad \cos(x) = \frac{v}{\sinh(b)}$$
$$\cos^2(x) + \sin^2(x) = 1 \Rightarrow \left(\frac{u}{\cosh(b)}\right)^2 + \left(\frac{v}{\sinh(b)}\right)^2 = 1$$

This isa half-ecliplse. Note that $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

$\square$