Sequences and series of fcns. Uniform convergence.

We defined during Le 3 what is meant by convergence of a sequence of complex numbers

[Recall; the sequence $\{A_n\}_{n=1}^{\infty}$ has limit A if for every given $\varepsilon > 0$ there exists N s.t.

$$n \geq N \rightarrow |A_n - A| < \varepsilon \;]$$

Def. A series is a formal expression of the form $c_0 + c_1 + c_2 + \ldots$, or equivalently $\sum_{j=1}^{\infty} c_j$, where $c_j \in \mathbb{C}$. The n:th partial sum of the series, denoted S_n, is the sum of the first $n+1$ terms, that is

$$S_n = \sum_{j=0}^{n} c_j .$$

If $\{S_n\}_{n=0}^{\infty}$ has limit S the series is said to converge to S and we write

$$S = \sum_{j=0}^{\infty} c_j$$

If the series does not converge it is said to diverge.

Ex. It holds that

$$\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}, \quad |c| < 1.$$

Proof: $(1-c)(1+c+\ldots+c^{n-1}+c^n) =$

$= 1+\cancel{c+\ldots+c^{n-1}+c^n}-(\cancel{c+c^2+\ldots+c^n}+c^{n+1})$

$= 1 - c^{n+1}$, i.e.

$$\frac{1}{1-c} - (1+c+\ldots+c^n) = \frac{c^{n+1}}{1-c}$$

The result follows since $\frac{c^{n+1}}{1-c} \to 0$ as $n \to \infty$

when $|c| < 1$.

$\boxtimes$

You can often show that a series converges

by comparing it with another series whose

convergence is known.

Thm (comparison test)

Suppose that

$$|c_j| \leq M_j$$

for all $j \geq J$. Then, if $\sum_{j=0}^{\infty} M_j$ converges,

so does $\sum_{j=0}^{\infty} c_j$.

The proof is rather straightforward if one

is familiar with the Cauchy convergence criterion —

$\left[\{A_n\}_{n=1}^{\infty} \text{ converges} \iff \{A_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence} \right]$

Also follows if you combine Thm 1 & Thm 3 in S.1.

Def. The series $\sum_{j=0}^{\infty} c_j$ is said to be <u>absolutely convergent</u> if $\sum_{j=0}^{\infty} |c_j|$ converges.

An absolutely convergent series is convergent, as seen by a trivial use of the comparison thm.

The following result can often be used to prove convergence of a series.

Thm (Ratio test)

Suppose that the terms of the series $\sum_{j=0}^{\infty} c_j$ have the property that the ratios $\left|\frac{c_{j+1}}{c_j}\right| \to L$ as $j \to \infty$. Then the series converges if $L < 1$ and diverges if $L > 1$.

The proof goes as in the Calculus course.

Ex. Show that the series $\sum_{j=0}^{\infty} \frac{4^j}{j!}$ converges.

Sol Put $c_j = \frac{4^j}{j!}$

$$\Rightarrow \left|\frac{c_{j+1}}{c_j}\right| = \frac{4^{j+1}}{(j+1)!} \cdot \frac{j!}{4^j} = \frac{4}{j+1} \to 0, \; j \to \infty$$

Convergence follows by the ratio test. $\square$

Def (Pointwise convergence)

Let $\{f_n\}_{n=0}^{\infty}$ be a seq. of fcns defined on some set $E \subseteq \mathbb{C}$. We say that the seq. $\{f_n\}_{n=0}^{\infty}$ converges pointwise to f on E if for each $z \in E$ the seq. of complex numbers $\{f_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$; i.e.

for every $z \in E$ and every $\varepsilon > 0$ there exists N (dep. on z and ε) such that

$$n \geq N \implies |f_n(z) - f(z)| < \varepsilon .$$

Def (Uniform convergence)

Let $\{f_n\}_{n=0}^{\infty}$ be a seq. of fcns def. on $E \subseteq \mathbb{C}$. We say that the seq. $\{f_n\}_{n=0}^{\infty}$ of fcns converges uniformly to f on E if

for every $\varepsilon > 0$ there exists N (dep. on ε) s.t.

$$n \geq N \implies |f_n(z) - f(z)| < \varepsilon \quad \forall z \in E.$$

$$\left[\iff \sup_{z \in E} |f_n(z) - f(z)| \to 0 \quad \text{as } n \to \infty \right]$$

Why care about uniform convergence ?

Because uniform conv. preserves properties of fcns under limit operations.

Thm Let $\{f_n\}_{n=0}^{\infty}$ be a seq. of fcns continuous on a set $E \subseteq \mathbb{C}$ and converging uniformly to f on E. Then f is continuous on E.

Proof Take $z_0 \in E$ and let $\varepsilon > 0$ be given.

Want to show that there is a $\delta > 0$ s.t.

$$z \in E, \ |z - z_0| < \delta \Rightarrow |f(z_0) - f(z)| < \varepsilon.$$

First choose N so big that $|f(z) - f_N(z)| < \frac{\varepsilon}{3}$ $\forall z \in E$; possible thanks to uniform convergence.

Since f_N is continuous at z_0 there $\exists \ \delta > 0$ s.t.

$$|f_N(z_0) - f_N(z)| < \frac{\varepsilon}{3} \quad \forall z \in E \text{ s.t. } |z - z_0| < \delta.$$

For such z it follows that

$$|f(z_0) - f(z)| = |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)|$$

$$\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

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We can therefore integrate the limit $f(z)$ over contours $\Gamma \subset E$.

The following holds

Thm Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of fcns continuous on $E \subseteq \mathbb{C}$ and converging uniformly to f on E. Suppose that the contour $\Gamma \subset E$. Then it holds that $\int_\Gamma f_n(z)\,dz \to \int_\Gamma f(z)\,dz$, $n \to \infty$

Proof: Let $L = L(\Gamma)$. Choose N s.t.

$$\left| f(z) - f_n(z) \right| < \frac{\varepsilon}{L} \quad \forall n \geq N \ \forall z \in E.$$

Then, for $n \geq N$,

$$\left| \int_\Gamma f(z)\,dz - \int_\Gamma f_n(z)\,dz \right| = \left| \int_\Gamma \left(f(z) - f_n(z) \right) dz \right|$$

$$\overset{ML}{<} \frac{\varepsilon}{L} \cdot L = \varepsilon$$

□

We now turn to series

$$\sum_{j=0}^{\infty} f_j(z)$$

of fcns.

Def. The series $\sum_{j=0}^{\infty} f_j(z)$ is said to converge pointwise resp. uniformly to $f(z)$ on E if the seq. $\{S_n\}_{n=0}^{\infty}$ of partial sums

$S_n(z) = \sum_{j=0}^{n} f_j(z)$ converges pointwise resp. uniformly to $f(z)$ on E

Ex. We saw that $\sum_{j=0}^{\infty} z^j$ converges

pointwise to $\frac{1}{1-z}$ on $|z| < 1$.

From the identity

$$\left| \frac{1}{1-z} - \sum_{j=0}^{n} z^j \right| = \left| \frac{z^{n+1}}{1-z} \right|$$

we see that the convergence is uniform

on any disk $|z| \le r$, $r < 1$. (But not on $|z| < 1$)

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Thm (Weierstrass M-test)

Suppose that $\sum_{j=0}^{\infty} M_j$ is a convergent series

with non-negative terms, and that

$$|f_j(z)| \le M_j \quad \forall\, z \in E \text{ and } j \ge J.$$

Then the series $\sum_{j=0}^{\infty} f_j(z)$ converges unif. on E.

Also easy if you know the Cauchy

convergence criterion.

We now turn to analytic fcns

Thm Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of analytic fcns in a domain D, which converges uniformly to f on D.

Then f is analytic in D

Proof Let $\tilde{D}$ be any disk in D. Then,

$$\int_{\Gamma} f(z)\,dz \underset{\text{above}}{\overset{\text{Thm}}{=}} \lim_{n\to\infty} \int_{\Gamma} f_n(z)\,dz \underset{\substack{\uparrow \\ \text{Cauchy} \\ \text{integral thm}}}{=} 0$$

for all closed contours Γ in $\tilde{D}$.

By Morera's thm it follows that f is analytic in $\tilde{D}$. But since D is a union of disks $\tilde{D}$ the result follows.

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We finally mention the following

Thm Suppose that $\{f_n\}_{n=0}^{\infty}$ is a sequence of fcns analytic in $|z-z_0| \le R$ which converges uniformly to $f(z)$ on $|z-z_0| \le R$.

Then, for each $r < R$ and each $m \ge 1$, the sequence of m:th derivatives $\{f_n^{(m)}(z)\}_{n=0}^{\infty}$ converges uniformly to $f^{(m)}(z)$ on $|z-z_0| \le r$.

Proof: Let $\varepsilon > 0$ be given.

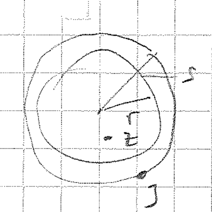
Choose N so large that

$$|f_n(z) - f(z)| < \varepsilon \quad \forall z : |z - z_0| \le R \quad \forall n \ge N.$$

Fix s s.t. $r < s < R$. By Cauchy's generalized integral formula

$$f_n^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}} \, d\zeta$$

for $|z - z_0| \le r$



But if $|z - z_0| \le r$ and $|\zeta - z_0| = s$, then

$|\zeta - z| \ge s - r$, so that

$$\left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}} \right| \le \frac{\varepsilon}{(s - r)^{m+1}}.$$

So by the ML-ineq.

$$\left| f_n^{(m)}(z) - f^{(m)}(z) \right| \le \frac{m!}{2\pi} \cdot \frac{\overset{\sup_{|z-z_0| \le R} |f_n(z) - f(z)| \underset{n\to\infty}{\to} 0}{\varepsilon}}{(s - r)^{m+1}} \cdot 2\pi s$$

for all z with $|z - z_0| \le r$.

This proves uniform conv of $\{f_n^{(m)}\}_{n=0}^{\infty}$ to $f^{(m)}$ on $|z - z_0| \le r$.

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