Applications to harmonic fcns

The mean-value property

Suppose that f is analytic inside and on a circle C_R of radius R centered at z_0

By Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0}\, dz$$

Parametrize C_R : $z(t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{\cancel{Re^{it}}}\, i\cancel{Re^{it}}\, dt$$

i.e.

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt} \qquad (*)$$

$(*)$ is called the mean-value property.

A direct consequence of $(*)$ is the following:

Lemma Suppose that f is analytic in a disk centered at z_0, say $D(z_0)$.

Suppose also that

$$\max_{z \in D(z_0)} |f(z)| = |f(z_0)|$$

Then, $|f(z)| = |f(z_0)| \quad \forall z \in D(z_0)$.

Proof: Suppose $|f(z)|$ not constant.

Then $\exists\, z_1 \in D(z_0)$ s.t. $|f(z_1)| < |f(z_0)|$.

Let C_R be the circle with center z_0 passing through z_1.

By assumption $|f(z)| \leq |f(z_0)|$ $\forall z \in C_R$.

Since f is continuous there $\exists$ a segment

of C_R containing z_1 in which $|f(z)| < |f(z_0)|$.

Say that $|f(z)| < |f(z_0)| - 2\pi\varepsilon$ on a

segment of opening angle δ

$$\Rightarrow |f(z_0)| \overset{(*)}{=} \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt \right| \leq$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})|\, dt < \frac{1}{2\pi}\left[(2\pi - \delta)|f(z_0)| + \delta(|f(z_0)| - 2\pi\varepsilon)\right]$$

$$= |f(z_0)| - \delta\varepsilon. \quad \text{Contradiction!}$$

$\square$

The Lemma implies the following thm.

Thm (Maximum modulus principle) If f is analytic in a domain D, and $|f(z)|$ attains its maximum value at a point $z_0 \in D$, then f is constant in D

<u>Proof</u>: We show that $|f|$ is constant in D.

The result then follows from exercise 2.2 on PS12.

Suppose $|f(z)|$ <u>not</u> constant. $\Rightarrow \exists\, z_1 \in D$

s.t. $|f(z_1)| < |f(z_0)|$. Let γ be a (polygonal) path from z_0 to z_1. We now consider the values of $|f(z)|$ for z on γ starting at z_0.

There exists a $w \in \gamma$ s.t.

(i) $|f(z)| = |f(z_0)|$ $\forall z$ preceeding w on γ

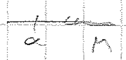
(ii) $\exists$ points z on γ arbitrary close to w s.t. $|f(z)| < |f(z_0)|$

[Follows from the supremum property of $\mathbb{R}$.

Parametrize γ with $z = z(t)$, $0 \le t \le 1$; $z(0) = z_0$, $z(1) = z_1$.

Let $M = \{ t \in [0,1]$ s.t. $|f(z(t))| < |f(z_0)| \}$

$M \neq \phi$ $(1 \in M)$ and M is bounded below (by 0).



Let $\alpha = \inf M$, $w := f(z(\alpha))$. Then

1) α is a lower bound for M

$\Rightarrow \forall\, t \in M$ it holds that $\alpha \le t$

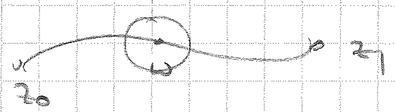
$\Rightarrow$ If $t < \alpha \Rightarrow t \notin M$, i.e. $|f(z(t))| = |f(z_0)|$, proving (i).

2) α is the greatest lower bound

$\Rightarrow \forall\, \beta > \alpha\ \exists\, t \in M$ s.t. $t < \beta$

i.e. there are points $z(t)$ with $|f(z(t))| < |f(z_0)|$ arbitrary close to w; proving (ii)]

Since f is continuous, (i) implies that $|f(w)| = |f(z_0)|$. There $\exists$ disk $D(w)$ centered at w contained in D. By the lemma above $|f|$ is constant in $D(w)$. But this contradicts (ii).



Thus, the assumption that $|f|$ is <u>not</u> constant must be false.

$\square$

Now suppose f analytic in a bounded domain D, and that f is continuous up to the boundary of D.

$\Rightarrow$ $|f(z)|$ attains a maximum in $\overline{D}$.

According to the above then the maximum can only be attained in D if f is constant. In any case we have the following variant of the maximum modulus principle

<u>Thm</u> A fcn which is analytic in a bounded domain and continuous up to the boundary attains its maximum modulus on the boundary.

Applications to harmonic fcns

Let's first give a new proof of the following

Thm Suppose ϕ harmonic in a simply connected domain D. Then there exists an analytic fcn f s.t. $\phi = \operatorname{Re} f$.

Proof: If such a fcn f exists, say that $f = \phi + i\psi$, then $f' = \phi_x + i\psi_x = \phi_x - i\phi_y$, so f would be a primitive of $\phi_x - i\phi_y$.

So, we expect that $\phi_x - i\phi_y$ has an antiderivative being the sought fcn f.

Therefore define $g(z) := \phi_x - i\phi_y$. Note that

$$(\phi_x)_x = (-\phi_y)_y \quad , \text{ since } \Delta\phi = 0$$

$$(\phi_x)_y = -(\phi_y)_x \quad , \text{ since } \phi \in C^2 .$$

$\Rightarrow$ Cauchy-Riemann equ's satisfied and g's real and imaginary parts are C^1, since $\phi \in C^2$.

$\Rightarrow$ g is analytic in a simply connected domain D

$\Rightarrow$ g has an antiderivative $G = u + iv$ in D.

Since $G' = g$, we have

$$u_x - i u_y = \phi_x - i \phi_y$$

$$\Rightarrow (\phi - u)_x = (\phi - u)_y = 0$$

$\Rightarrow \phi - u =$ constant, say $\phi = u + c$ $(c \in \mathbb{R})$

Put $f(z) = G(z) + c$; has sought properties

∎

Now, let ϕ be a harmonic fn in a simply connected domain D.

Let $f = \phi + i\psi$ be analytic in D (exists by thm)

$$\Rightarrow |\underbrace{e^f}_{\text{analytic}}| = |e^{\phi + i\psi}| = e^\phi$$

The exp. fn is monotonically increasing,

So if ϕ (and therefore $|e^f|$) attains a maximum in D, then e^f is constant.

Hence ϕ is constant.

Since ϕ attains its minimum precisely when $-\phi$ attains its maximum, we also have a corresponding minimum principle.

We have proven the following:

Thm If ϕ is harmonic in a simply connected domain D and if ϕ attains its maximum or minimum at some point $z_0 \in D$, then ϕ is constant.

Thm A fcn which is harmonic in a bounded simply connected domain, and continuous up to the boundary, attains its maximum and minimum on the boundary

Remark: The assumption of simple connectedness is not necessary in these two theorems.

[The first thm (or mean-value property) gives analog of Lemma. The argue as in the proof of the maximum modulus principle].

Recall:

Dirichlet's problem:

Find a fcn $\phi(x,y)$ which is harmonic in a domain D, and continuous in $\overline{D}$, with given values on the boundary of D

You ask: Does a solution of Dirichlet's problem exist? If so, is it unique?

The above thm implies uniqueness for __bounded__ D.

__Thm__ Let $\phi_1(x,y)$ and $\phi_2(x,y)$ be harmonic in a bounded domain D, and cont. on $\overline{D}$, s.t. $\phi_1(x,y) = \phi_2(x,y)$ on ∂D.

Then, $\phi_1 \equiv \phi_2$ in D

__Proof__: Let $\phi = \phi_1 - \phi_2$

By the (generalization) of the second thm ϕ attains both its maximum and minimum value on ∂D. But $\phi \equiv 0$ on $\partial D \Rightarrow \phi \equiv 0$. ∎

An explicit solution can be found if e.g. D is a disk.

Poisson's integral formula

Suppose $f = \phi + i\psi$ analytic inside and on the circle C_R centered at the origin of radius R. Then, by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (z \text{ inside } C_R)$$

Let z^*, z be symmetric w.r.t. C_R,

i.e. $(z^* - 0)\overline{(z - 0)} = R^2 \Leftrightarrow z^* = \dfrac{R^2}{\bar{z}}$

Then $|z^*| > R$, so by Cauchy's thm

$$f(z) = \frac{1}{2\pi i} \int_{C_R} f(\zeta) \Big(\frac{1}{\zeta - z} - \underbrace{\frac{1}{\zeta - z^*}}_{\to 0} \Big)\, d\zeta$$

Some simplification leads to

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{R^2 - |z|^2}{(\zeta - z)(R^2 - \bar{z}\zeta)} f(\zeta)\, d\zeta = /\zeta = Re^{it}/$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - \bar{z}Re^{it})} f(Re^{it})\, iRe^{it}\, dt$$

$$= \frac{R^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2}\, dt$$

Taking real parts and identifying (x,y) with $re^{i\theta}$

$$\Rightarrow \boxed{\phi(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(Re^{it})}{R^2 - 2Rr\cos(t - \theta) + r^2}\, dt}$$

This is Poisson's formula for the disk.