UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
Jörgen Östensson

Complex Analysis
1MA022
Spring 2022

Answers to the suggested problems

A) COMPLEX PLANE, ELEMENTARY FUNCTIONS

1. a) circle, b) annulus, c) disk, d) $[-1,1]$, e) half-plane, f) horizontal strip, g) vertical strip, h) $\mathbb{C}\setminus\mathbb{R}$, i) half-plane, j) empty set.

3. a) $e^{\pi i/4} = \frac{1+i}{\sqrt{2}}, \quad e^{5\pi i/4} = -\frac{1+i}{\sqrt{2}}$,
   b) $e^{\pm\pi i/4} = \frac{1\pm i}{\sqrt{2}}, \quad e^{\pm 3\pi i/4} = \frac{-1\pm i}{\sqrt{2}}$,
   c) $-2, \quad 2e^{\pm\pi i/3} = 1 \pm i\sqrt{3}$,
   d) 16.

4. a) $e^2(\cos 1 + i\sin 1)$,
   b) $\frac{5\sqrt{2}}{2}(-1+i)$,
   c) $\frac{\sqrt{2}}{4}\Big((e+e^{-1}) - i(e-e^{-1})\Big)$,
   d) $\frac{\ln 2}{2} + i\frac{\pi}{4}$.

5. $n = 4k,\ k \in \mathbb{Z}_+$.

6. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \qquad \sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$

8. Zeros: $\dfrac{\pi}{2} + k\pi,\ k\in\mathbb{Z}.$ Periods: $2\pi k,\ k\in\mathbb{Z}.$

9. $\operatorname{Re} z^z = e^{x\ln|z| - y\arg z}\cos(y\ln|z| + x\arg z)$,
   $\operatorname{Im} z^z = e^{x\ln|z| - y\arg z}\sin(y\ln|z| + x\arg z)$.

10. a) $z = \pm\big(\frac{\pi}{2} - i\ln(2+\sqrt{5})\big) + 2k\pi, \quad k\in\mathbb{Z}$,
    b) $z = \ln(2k\pi) + i\big(\pm\frac{\pi}{2} + 2l\pi\big), \quad k\in\mathbb{Z}_+,\ l\in\mathbb{Z}$,
    c) $z = \frac{\pi}{8} - i\frac{\ln 2}{4} + k\pi, \quad k\in\mathbb{Z}$,
    d) $z = \pm\frac{\pi}{3} - i\ln 2 + 2k\pi, \quad k\in\mathbb{Z}$,
    e) $z = \pm i\ln\left((2k+\frac{1}{2})\pi + \sqrt{(2k+\frac{1}{2})^2\pi^2 - 1}\right) + 2l\pi, \quad k\in\mathbb{N},\ l\in\mathbb{Z}$, and
    $z = \pm i\ln\left((2k-\frac{1}{2})\pi + \sqrt{(2k-\frac{1}{2})^2\pi^2 - 1}\right) + (2l+1)\pi, \quad k\in\mathbb{Z}_+,\ l\in\mathbb{Z}.$

B) ANALYTIC AND HARMONIC FUNCTIONS

4. a) $v = 3x^2y - y^3 + y^2 + y - x^2 + C, \quad C\in\mathbb{R}$,
   b) $v = 2xy + C, \quad C\in\mathbb{R}$,
   c) $v = -\cosh x\cos y + C, \quad C\in\mathbb{R}$,
   d) $v = e^x(y\sin y - x\cos y) + C, \quad C\in\mathbb{R}$,
   e) $v = -\frac{1}{2}\ln(x^2+y^2) + C, \quad C\in\mathbb{R}$.

Updated: 17th Jan, 2022.

5. $f(z) = \frac{1-i}{4}(z^2 + C), \quad C \in \mathbb{R}.$

8. $f(z) = az + bz^2, \quad a, b \in \mathbb{C}.$

9. $f(z) = z^3 + iaz^2 + bz + ic, \quad a, b, c \in \mathbb{R}.$

10. $f(z) = ae^{-z} + ib, \quad a \in \mathbb{C},\ b \in \mathbb{R}.$

C) CONFORMAL MAPPINGS

1. $T(0) = -1.$

2. $T(z) = -i\dfrac{z+i}{z-3i}$ or $T(z) = -i\dfrac{z-3i}{z+i}.$

3. $T(z) = \dfrac{z}{2-z}$ or $T(z) = \dfrac{z-2}{z}.$
   The line is mapped onto the circle $|w+1-i| = 1$ or $|w-1-i| = 1$, respectively.

4. For example $T(z) = \dfrac{1+z}{1-z}.$
   Circles $|z| = r > 1$ are mapped onto circles in the left half-plane. Lines crossing the origin are mapped onto circles passing through ± 1, with the exception of the real axis which is mapped onto the real axis.

5. $T(z) = \dfrac{1+i}{2}\dfrac{3-z}{1+z}.$

6. $a \in (0, 1/3) \cup (1, \infty).$

7. $f(z) = (3+4i)\left(\dfrac{z-i}{z+i}\right)^2.$

8. $f(z) = \exp\left(\dfrac{2\pi i z}{z-2}\right).$

9. $f(z) = \dfrac{z^{1/2} - 2i}{z^{1/2} + 2i},$ where $\operatorname{Im} z^{1/2} > 0.$

10. $f(z) = \dfrac{2i}{\pi} \operatorname{Log}\left(i\left(\dfrac{z+i}{z-i}\right)^2\right).$

D) DIRICHLET PROBLEMS

1. $\phi(x, y) = \frac{1}{\pi} \arctan\left(\frac{2x}{1-x^2-y^2}\right) + \frac{1}{2}.$

2. $\phi(x, y) = \frac{1}{\pi} \arctan\left(\frac{x^2+y^2-2\sqrt{2}\,x+1}{x^2+y^2-1}\right) + \frac{1}{2}.$

3. $\phi(x, y) = \frac{2}{\pi} \arctan\left(\frac{1-x^2-y^2}{2y}\right).$

4. $\phi(x, y) = \frac{2}{\pi} \arctan\left(\frac{\sin y}{\sinh x}\right).$

5. $\phi(x, y) = \frac{1}{\pi} \arctan\left(\frac{5x^2-5y^2-4-(x^2+y^2)^2}{6xy}\right) + \frac{1}{2}.$

E) INTEGRATION

1. $8.$

2. a) 0, b) π, c) 0.

3. $2\pi i.$

4. $\begin{cases} 2\pi i a e^{a^2}, & |a| < 1, \\ 0, & |a| > 1. \end{cases}$

5. $-2\pi i.$

6. $2\pi - \ln(e^{2\pi} + 1) + \ln 2.$

7. $\frac{1}{4}(\ln 3 + 2i \arctan \frac{1}{2}).$

8. $4\pi i/3.$

9. $|f^{(n)}(z)| \leq \frac{n! M R}{(R-r)^{n+1}}.$

F) SEQUENCES AND SERIES OF FUNCTIONS

1. a) Pointwise: $(-1, 1]$.
   Uniformly: $[a, b]$, if $-1 < a < b < 1$.

   b) Pointwise: $(-\sqrt{2}, \sqrt{2})$.
   Uniformly: $[a, b]$, if $0 < a < b < \sqrt{2}$ or $-\sqrt{2} < a < b < 0$.

   c) Pointwise: $\mathbb{R}$.
   Uniformly: $[a, b]$, if $-\infty < a < b < \infty$.

   d) Pointwise: $\mathbb{R} \setminus \{0\}$.
   Uniformly: $[a, \infty)$, if $a > 0$, and $(-\infty, b]$, if $b < 0$.

2. Converges for $|z| \neq 1$ and $z = 1$.

3. No.

4. a) No, Yes. b) Yes.

5. a) Uniformly convergent on $\mathbb{R}$.
   b) Uniformly convergent on $a \leq |x| \leq b$, if $0 < a < b < \sqrt{2}$.

G) POWER SERIES

1. a) $R = 1/2$,
   b) $R = 6$,
   c) $R = 1$,
   d) $R = 5/3$,
   e) $R = 1/\sqrt{2}$,
   f) $R = \infty$,
   g) $R = 2$,
   h) $R = 0$,
   i) $R = e$.

2. a) $|z - 1| < 1$,
   b) All $z \in \mathbb{C}$,
   c) $|z - 2| < 1/2$,
   d) $|z + i| \leq 1$,
   e) $z = 3$,
   f) $|z - 2 - i| \leq 1/2$.

3. a) $f(z) = \dfrac{z}{(1-z)^2}$, b) $f(z) = \dfrac{z^2 + z}{(1-z)^3}$.

4. a) $\sin 1,\ \cos 1,\ \cos 1 - (\sin 1)/2$,
   b) $1,\ 1,\ 3/2$,
   c) $1,\ 1,\ 1/3$,
   d) $\ln 2,\ 1/2,\ 1/8$.

5. $R = \sqrt{7}$.

H) ZEROS AND UNIQUENESS

1. a) 3, b) 3.

2. a) $z = \pm i$, order 1,
   b) $z = e^{i(\pi/4+k\pi/2)}$, $k = 0, 1, 2, 3$, order 1,
   c) $z = 0$, order 3;
   $z = k\pi$, $k \in \mathbb{Z} \setminus \{0\}$, order 1,
   d) $z = 2k\pi$, $k \in \mathbb{Z}$, order 2,
   e) $z = i(\pi/4 + k\pi/2)$, $k \in \mathbb{Z}$, order 1,
   f) $z = 1$, order 1.

5. No. It is impossible by the uniqueness principle.

6. $f(z) = \dfrac{z+1}{z^2+1}$.

I) LAURENT SERIES EXPANSIONS AND ISOLATED SINGULARITIES

1. a) $f(z) = \dfrac{1}{4z} + \dfrac{1}{5}\sum_{n=0}^{\infty}\Big((-1)^{n+1} + \dfrac{1}{4^{n+2}}\Big)z^{2n+1}$,
   b) $f(z) = \dfrac{1}{5}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{z^{2n+3}} + \dfrac{1}{5}\sum_{n=0}^{\infty}\dfrac{z^{2n-1}}{4^{n+1}}$,
   c) $f(z) = \dfrac{1}{5}\sum_{n=0}^{\infty}\big((-1)^n - 4^n\big)\dfrac{1}{z^{2n+3}}$.

2. $f(z) = \dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(-i)^n}{(z-i)^{n+1}} + \dfrac{1}{2}\sum_{n=0}^{\infty}\Big(\dfrac{-1}{2+i}\Big)^{n+1}(z-i)^n$.

3. $f(z) = 2i\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2n+1}\dfrac{1}{z^{2n+1}}$.

4. a) Poles at $z = \pm i$.
   b) Poles at $z = i2k\pi$, $k \in \mathbb{Z}$.
   c) Removable singularity at $z = 0$.
   d) Removable singularity at $z = 0$.
   e) Essential singularity at $z = 2$.
   f) Pole at $z = 1$.
   g) Essential singularity at $z = 0$.
   h) Poles at $z = e^{i(\pi/4+k\pi/2)}$, $k = 0, 1, 2, 3$.
   i) Poles at $z = 0, \pm 1$.

J) RESIDUE CALCULUS

1. $\dfrac{2\pi i}{(k+1)!}$.

2. $\dfrac{5\pi}{16}$.

3. $2\pi i\Big(1 - \cos 1 - \dfrac{1}{2}\sin 1\Big)$.

4. a) $\dfrac{\pi}{2}$, b) $\dfrac{3\pi}{16}$.

5. a) $\pi(2 - \sqrt{6})$,
   b) $\dfrac{4\pi}{3\sqrt{3}}$,
   c) $\sqrt{2}\pi$,
   d) $\dfrac{2\pi}{2^{2k}}\dbinom{2k}{k}$.

6. a) $\pi e^{-1/\sqrt{2}} \sin \frac{1}{\sqrt{2}}$,
   b) $\frac{\pi}{b} e^{-ab}$.

7. a) $\frac{\pi}{2}$,
   b) $\frac{\pi}{3}$,
   c) $\frac{\pi}{\sin(\pi a)}$,
   d) $\frac{2\pi \sin(\pi a/3)}{\sqrt{3}\sin(\pi a)}$.

8. a) $\frac{\pi}{6} \ln 3$,
   b) $\frac{\pi^2}{4}$,
   c) $\frac{\sqrt{2}\pi}{4}\left(\ln 4 + \frac{\pi}{2}\right)$.

K) THE ARGUMENT PRINCIPLE AND ROUCHÉ'S THEOREM

1. One zero has both negative real part and negative imaginary part.

2. Two zeros have positive real part.

4. Two roots are in the second quadrant and two roots are in the third quadrant.

5. 2.

6. 1.

7. 2.

8. a) 2, b) 5, c) 4.