# Lectures 18-19 (Interest Rate Theory)

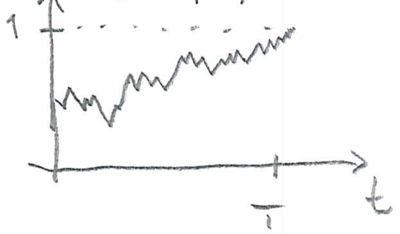


## Ch 19 Bonds and Interest Rates

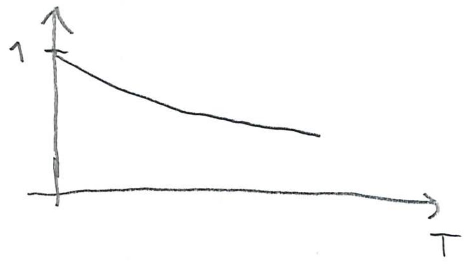
Def: A zero-coupon bond with maturity $T$ (or $T$-bond) gives its holder 1 SEK paid at $T$. The price is denoted $p(t,T)$.

Typical plots:

$t \mapsto p(t,T)$




$T \mapsto p(0,T)$ ("the term structure at time 0")




Note that $p(t,t) = 1$.

A strategy to obtain a deterministic rate of return over a future interval $[S,T]$

At time 0: Sell one $S$-bond
Buy $\frac{p(0,S)}{p(0,T)}$ $T$-bonds
} Cost = 0

At time $S$: Pay 1 SEK

At time $T$: Receive $\frac{p(0,S)}{p(0,T)}$

We have created a strategy which gives a riskless rate of return over the future interval $[S,T]$.

This is known as a forward rate.

Some different interest rates: 

The <u>LIBOR forward rate</u> $L(t;S,T)$ solves $\frac{p(t,S)}{p(t,T)} = 1+(T-S)L$

(i.e. $L(t;S,T) = -\frac{p(t,T)-p(t,S)}{(T-S)\,p(t,T)}$ )

The <u>continuously compounded forward rate</u> $R(t;S,T)$ solves

$$\frac{p(t,S)}{p(t,T)} = e^{(T-S)R} \qquad \left(\text{i.e. } R(t;S,T) = -\frac{\ln p(t,T)-\ln p(t,S)}{T-S}\right)$$

The <u>instantaneous forward rate</u> is

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}$$

The instantaneous <u>short rate</u> is $r_t = f(t,t)$

The <u>yield curve</u> at $t$ is the function

$$y(t,T) := -\frac{\ln p(t,T)}{T-t}, \quad T > t.$$

(it solves $p(t,T) = e^{-y(t,T)(T-t)}$ )

---

<u>Remark</u>: One could choose to model

1) the short rate $r_t$
2) bond prices $p(t,T)$
3) the instantaneous forward rate $f(t,T)$.

We will only model $r_t$ (the book is more extensive).

# Ch 20 Short Rate Models



**Model:** $\begin{cases} dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dW_t \\ dB_t = r_t B_t\,dt \end{cases}$

**Goal:** To price zero-coupon T-bonds for all T.

---

**Expectations:** $M$ = # traded assets excluding the bank account $= 0$

$R$ = # random sources $= 1$

The market is arbitrage-free but incomplete. Prices of T-bonds with different T should satisfy internal consistency relations.

---

Assume $p(t,T) = F^T(t, r_t)$ for some function $F^T$. Clearly, $F^T(T,r) = 1$. Fix S and T and form a locally risk-free portfolio $(w^S, w^T)$ of S-bonds and T-bonds.

$$dF^T(t, r_t) \underset{\text{Ito}}{=} \alpha_T F^T dt + \sigma_T F^T dW_t \qquad (*)\begin{cases} \alpha_T = \dfrac{F_t^T + \frac{\sigma^2}{2} F_{rr}^T + \mu F_r^T}{F^T} \\ \sigma_T = \dfrac{\sigma F_r^T}{F} \end{cases}$$

and $dF^S(t, r_t) = \alpha_S F^S dt + \sigma_S F^S dW_t$.

Then

$$dV_t^w = V_t^w(\alpha_T w^T + \alpha_S w^S)\,dt + (\sigma_T w^T + \sigma_S w^S) V_t^w\,dW_t,$$

and choosing $w$ such that

$$\begin{cases} w^S + w^T = 1 \\ \sigma_S w^S + \sigma_T w^T = 0 \end{cases} \Leftrightarrow \begin{cases} w^S = \dfrac{\sigma_T}{\sigma_T - \sigma_S} \\ w^T = \dfrac{-\sigma_S}{\sigma_T - \sigma_S} \end{cases} \quad \text{gives}$$

$$dV_t^w = \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} V_t^w \, dt .$$

No arbitrage yields $r_t = \dfrac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S}$, so

$$\underbrace{\frac{\alpha_S - r_t}{\sigma_S}}_{\text{expression involving } F^S\text{, not } F^T} = \underbrace{\frac{\alpha_T - r_t}{\sigma_T}}_{\text{expression involving } F^T\text{, but not } F^S} =: \lambda_t$$

market price of risk.

Inserting (*) gives

$$F_t^T + \frac{\sigma^2}{2} F_{rr}^T + (\mu - \lambda\sigma) F_r^T - r F^T = 0$$

---

<u>Proposition 20.2</u> (The term structure equation)

The arbitrage-free price of a T-bond is $F^T(t, r_t)$ where $F^T(t,r)$ solves

$$\begin{cases} F_t^T + \frac{\sigma^2}{2} F_{rr}^T + (\mu - \lambda\sigma) F_r^T - r F^T = 0 \\ F^T(T,r) = 1 \end{cases}$$

Alternatively, $F^T(t,r) = E_{t,r}^Q \left[ e^{-\int_t^T r_s \, ds} \right]$

where $\begin{cases} dr_s = (\mu - \lambda\sigma)\, ds + \sigma \, dW_s^Q \\ r_t = r \end{cases}$ under $Q$.

---

Remarks: 1) For the stochastic representation of $F^T$, see exercise 5.12.

2) $T$-claims $\mathcal{X} = \phi(r_T)$ are priced similarly (replace the terminal condition by $F^T(T, r) = \phi(r)$).

3) The market price of risk $\lambda$ is not specified within the model, but needs to be estimated using market prices.

# Ch 21 Martingale Models for the Short Rate



Approach: Model $r$ directly under $Q$ as

$$dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t).$$

(From now on, $\mu$ is the drift under $Q$, not under $P$.)

Popular models

1. Vasiček $\quad dr_t = (b - ar_t)\,dt + \sigma\,dW_t$
2. Cox-Ingersoll-Ross (CIR) $\quad dr_t = (b - ar_t)\,dt + \sigma\sqrt{r_t}\,dW_t$
3. Dothan $\quad dr_t = ar_t\,dt + \sigma r_t\,dW_t$
4. Ho-Lee $\quad dr_t = \Theta(t)\,dt + \sigma\,dW_t$
5. Hull-White (extended Vasiček) $\quad dr_t = (b(t) - a(t)r_t)\,dt + \sigma(t)\,r_t\,dW_t$
6. Hull-White (extended CIR) $\quad dr_t = (b(t) - a(t)r_t)\,dt + \sigma(t)\sqrt{r_t}\,dW_t$

Remark: $\sigma$ can be estimated from historical data since $\sigma$ is the same under $P$ and $Q$. The drift $\mu$ cannot be estimated using historical data. Instead, $\mu$ is chosen so that the theoretical term structure $\{p(0,T), T \geq 0\}$ fits the observed term structure $\{p^*(0,T), T \geq 0\}$.
"Inversion of the yield curve".

# Affine Term Structures



If the term structure $\{p(t,T),\ 0 \le t \le T,\ T > 0\}$ has the form $p(t,T) = e^{A(t,T) - B(t,T) r_t}$ then the model admits an <u>affine term structure</u>.

<u>Question</u>: Which models admit an affine term structure?

To answer this, plug in $F^T(t,r) = e^{A(t,T) - B(t,T) r}$ into the term structure equation.

$$\begin{cases} F_t^T + \frac{\sigma^2}{2} F_{rr}^T + \mu F_r^T - r F^T = 0 \\ F^T(T,r) = 1 \end{cases}$$

We get
$$\begin{cases} A_t - B_t r + \frac{\sigma^2}{2} B^2 - \mu B - r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$$

Assume now that $\mu(t,r)$ and $\sigma^2(t,r)$ are both affine, i.e

$$(*) \quad \begin{cases} \mu(t,r) = \alpha(t) r + \beta(t) \\ \sigma^2(t,r) = \gamma(t) r + \delta(t) \end{cases}$$

We then get

$$A_t + \frac{\delta}{2} B^2 - \beta B - \left(B_t - \frac{\gamma}{2} B^2 + \alpha B + 1\right) r = 0$$

Prop. 21.2 (Affine term structure)

Assume that $\mu$ and $\sigma^2$ are affine as in (*) above.

Then bond prices are $p(t,T) = e^{A(t,T) - B(t,T) r_t}$

where
$$\begin{cases} B_t - \frac{\gamma}{2} B^2 + \alpha B + 1 = 0 \\ B(T,T) = 0 \end{cases}$$

and
$$\begin{cases} A_t + \frac{\delta}{2} B^2 - \beta B = 0 \\ A(T,T) = 0 \end{cases}$$

---

Ex (Vasiček model) $dr_t = (b - ar_t)dt + \sigma dW_t$.

Here $\begin{cases} \mu = b - ar \\ \sigma^2 = \text{constant} \end{cases}$ so they are on the form (*).

The Ansatz $F^T(t,r) = e^{A(t,T) - B(t,T) r}$ gives (plug in the term structure eqn)

$$\begin{cases} A_t - B_t r + \frac{\sigma^2}{2} B^2 - (b - ar) B - r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$$

i.e $\begin{cases} B_t - aB + 1 = 0 \\ B(T,T) = 0 \end{cases}$ and $\begin{cases} A_t + \frac{\sigma^2}{2} B^2 - bB = 0 \\ A(T,T) = 0 \end{cases}$.

We get $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$ and

$$A(t,T) = \int_t^T \left( \frac{\sigma^2}{2} B^2(s,T) - bB(s,T) \right) ds$$

$$= \frac{\sigma^2}{2a^2} \int_t^T \left(1 - e^{-a(T-s)}\right)^2 ds - \frac{b}{a} \int_t^T 1 - e^{-a(T-s)} \, ds$$

$$= \left( \frac{\sigma^2}{2a^2} - \frac{b}{a} \right)(T-t) + \left( \frac{b}{a^2} - \frac{\sigma^2}{a^3} \right)\left(1 - e^{-a(T-t)}\right) + \frac{\sigma^2}{4a^3}\left(1 - e^{-2a(T-t)}\right)$$

<u>Remark</u> Alternatively, to see that the Vasiček model admits an affine term structure, use

$$r_t = re^{-at} + \frac{b}{a}(1-e^{-at}) + \sigma e^{-at}\int_0^t e^{as}\,dW_s \qquad \text{(Assignment 1)}.$$

Then

$$F^T(0,r) = E\left[e^{-\int_0^T r_t\,dt}\right] = E\left[e^{-r\int_0^T e^{-at}dt + \underbrace{\int_0^T \ldots dt}_{\text{no dependence on } r}}\right]$$

Risk-neutral valuation

$$= e^{-\frac{1}{a}(1-e^{-aT})r}\, E\left[e^{\int_0^T \ldots dt}\right],$$

so $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ for some $A$ and $B$.

<u>Remark</u> The same approach for the Dothan model gives a mess: If $dr_t = ar_t\,dt + \sigma r_t\,dW_t$ then

$$F^T(0,r) = E\left[e^{-r\int_0^T e^{(a-\frac{\sigma^2}{2})t+\sigma W_t}dt}\right] = ?$$

<u>Exercise 21.5</u> (Inversion of the yield curve, Ho-Lee model)

$dr_t = \theta(t)\,dt + \sigma\,dW_t$. Fit this to observed bond prices $\{p^*(0,T),\ T\geq 0\}$.

---

We first calculate theoretical bond prices $\{p(0,T),\ T\geq 0\}$.

Plug $F^T(t,r) = e^{A(t,T)-B(t,T)r}$ into the term structure equation

$$\begin{cases} F^T_t + \frac{\sigma^2}{2}F^T_{rr} + \theta F^T_r - rF^T = 0 \\ F^T(T,r) = 1. \end{cases}$$

We get $\begin{cases} A_t - B_t r + \frac{\sigma^2}{2} B^2 - \theta B - r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$,

so

$$\begin{cases} B_t + 1 = 0 \\ B(T,T) = 0 \end{cases} \quad \text{and} \quad \begin{cases} A_t + \frac{\sigma^2}{2} B^2 - \theta B = 0 \\ A(T,T) = 0 \end{cases}.$$

We get $B(t,T) = T-t$, so

$$A(t,T) = \int_t^T \frac{\sigma^2}{2}(T-s)^2 - \theta(s)(T-s)\, ds$$

Thus

$$p(0,T) = e^{\int_0^T \frac{\sigma^2}{2}(T-s)^2 - \theta(s)(T-s)\, ds - Tr}$$

Putting $p(0,T) = p^*(0,T)$ we must have

$$\frac{\sigma^2}{6} T^3 - \int_0^T \theta(s)(T-s)\, ds - rT = \ln p^*(0,T)$$

Differentiate:

$$\frac{\sigma^2}{2} T^2 - \int_0^T \theta(s)\, ds - r = \frac{\partial \ln p^*(0,T)}{\partial T}$$

Differentiate again:

$$\sigma^2 T - \theta(T) = \frac{\partial^2 \ln p^*(0,T)}{\partial T^2}$$

Conclusion: The drift should be chosen as

$$\theta(T) = \sigma^2 T - \frac{\partial^2 \ln p^*(0,T)}{\partial T}$$