Final Exam – Fourier Analysis, 1MA211

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duration of the exam: 5 hours

There are 8 problems in this exam, and each one is worth 5 points. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5. **You need to motivate every step in your solution to get the full score on a question.** You can use the attached table of formulas. Good luck!

1. Use a technique that we studied in this course to find a function $y(x)$, with $x \geq 0$, that solves the initial value problem

$$\begin{cases} y''(x) - 2y'(x) - 3y(x) = 4e^{-x} \\ y(0) = -1, \quad y'(0) = 2 \end{cases}.$$

2. Find a function $u(x,t)$, where $0 \leq x \leq \pi$ and $t \geq 0$, that solves the boundary value problem

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = 2 \text{ and } u_x(\pi,t) = 2 & t > 0 \\ u(x,0) = 3x \text{ and } u_t(x,0) = 3\cos(2x) & 0 < x < \pi \end{cases}$$

3. Let V be the space of continuous functions $f : [0,1] \to \mathbb{C}$, with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)\, \overline{g(x)}\, e^x \,\mathrm{d}x.$$

Find two orthonormal elements of V.

4. Let $f : \mathbb{R} \to \mathbb{C}$ be a function of period 2π, such that

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, -\pi/2) \cup (\pi/2, \pi) \\ 1 & \text{if } x \in [-\pi/2, \pi/2] \end{cases}$$

(a) Find the Fourier series of f.

(b) Does this Fourier series converge pointwise? If yes, write the limit. Justify your answer.

(c) Does this Fourier series converge uniformly? If yes, write the limit. Justify your answer.

(d) Use the result of part (a) to compute

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

5. Find a function $u(x,y)$ that solves the initial value problem

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + 6\,u\,y^2 & x \in \mathbb{R}, y > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}.$$

Hint: Recall that the ODE $f'(t) + p(t)\,f(t) = 0$, where $p(t)$ is a known function, can be solved either by using the integrating factor $I(t) = e^{\int p(t)\,\mathrm{d}t}$ or by separating variables.

6. Find a solution of the integral equation

$$f(x) + \int_{-\infty}^{\infty} e^{-|y|} f(x-y)\,\mathrm{d}y = 3e^{-|x|}.$$

7. (a) Compute the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ given by the function

$$f(x) = e^{iax},$$

where $a \in \mathbb{R}$ is a constant.

(b) Compute the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ given by the function

$$f(x) = \cos(x).$$

(c) Find a tempered distribution $g \in \mathcal{S}'(\mathbb{R})$ (that is not the zero distribution) which solves the equation

$$(x^2 - 1)\,g = 0.$$

Hint: Take the inverse Fourier transform.

8. Let $f_n : [a,b] \to \mathbb{C}$, with $n \geq 1$, be a sequence of integrable functions on the finite interval $[a,b]$, and assume that the sequence converges uniformly to the integrable function $f : [a,b] \to \mathbb{C}$.

(a) Show the existence of a constant $M > 0$ and of an integer n_0 such that, for every $n > n_0$, one has

$$|f_n(x)| \leq M \text{ for all } x \in [a,b]$$

(one says that the sequence f_n is *uniformly bounded*).

(b) Show that

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2\,\mathrm{d}x = 0$$

(one says that f_n *converges to* f *in mean-square*, or in $L^2([a,b])$).

Formulas for Fourier Analysis course

Triangle inequalities
Let $x, y \in \mathbb{R}$ and f, g be functions. Then

- $\big| |x| - |y| \big| \le |x \pm y| \le |x| + |y|$
- $\left| \int_\Omega f(x)\,\mathrm{d}x \right| \le \int_\Omega |f(x)|\,\mathrm{d}x$, for a subset $\Omega \subset \mathbb{R}$.

Some useful identities

- $e^{a+ib} = e^a(\cos(b) + i\sin(b))$
- $\int_{\mathbb{R}} x^n e^{-x^2/2}\,\mathrm{d}x = \begin{cases} \sqrt{2\pi}(n-1)(n-3)\ldots 5\cdot 3\cdot 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

Gram–Schmidt orthogonalisation
Let V be an inner product space and $\{v_1, \ldots, v_k\} \subset V$ be a linearly independent set of vectors. Then the Gram–Schmidt orthogonalisation is given by

$$\begin{aligned}
u_1 &= v_1, & e_1 &= \frac{u_1}{\|u_1\|} \\
u_2 &= v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle} u_1, & e_2 &= \frac{u_2}{\|u_2\|} \\
u_3 &= v_3 - \frac{\langle u_1, v_3\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle u_2, v_3\rangle}{\langle u_2, u_2\rangle} u_2, & e_3 &= \frac{u_3}{\|u_3\|} \\
&\vdots & &\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} \frac{\langle u_j, v_k\rangle}{\langle u_j, u_j\rangle} u_j, & e_k &= \frac{u_k}{\|u_k\|}.
\end{aligned}$$

Laplace transform

$f(t)$	$\tilde{f}(s) = F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
General formulas	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$e^{at} f(t)$	$F(s-a)$
$f(at), \quad a > 0$	$\frac{1}{a} F(\frac{s}{a})$
$f(t-a)H(t-a), \quad a > 0$	$e^{-as} F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0)$ $\quad -s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(u)\,du$	$s^{-1} F(s)$
$f * g(t) = \int_0^t f(u) g(t-u)\,du$	$F(s)\,G(s)$
Particular cases	
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$

Fourier Series

Functions of period 2π

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n})$$

Parseval's formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

Functions of period T

Let $\Omega = 2\pi/T$

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\Omega t + b_n \sin n\Omega t),$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\Omega t}\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\Omega t\,dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\Omega t\,dt.$$

Parseval's formula:

$$\frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

Some trigonometric identities

$$2\sin a \sin b = \cos(a-b) - \cos(a+b)$$

$$2\sin a \cos b = \sin(a-b) + \sin(a+b)$$

$$2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin^2 t = 1 - \cos 2t, \qquad 2\cos^2 t = 1 + \cos 2t$$

Fourier transform

$f(t)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$
General formulas	
$\alpha f(t) + \beta g(t)$	$\alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$
$e^{i\alpha t}f(t)$	$\hat{f}(\omega - \alpha)$
$f(t - t_0)$	$e^{-it_0\omega}\hat{f}(\omega)$
$f(-t)$	$\hat{f}(-\omega)$
$f(at) \quad (a \neq 0)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$
$tf(t)$	$i\,\frac{d\hat{f}}{d\omega}$
$f'(t)$	$i\omega\hat{f}(\omega)$
$\hat{f}(t)$	$2\pi f(-\omega)$
$f * g(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du$	$\hat{f}(\omega)\hat{g}(\omega)$
Particular cases	
$\chi_{[-a,a]}$	$\frac{2\sin a\omega}{\omega}$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert\omega\rvert}$
$e^{-t^2/2}$	$\sqrt{2\pi}e^{-\omega^2/2}$
δ	1
1	$2\pi\delta$

Plancherel's formulas:

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$$

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,d\omega$$