

1. (a) The exponential distribution is an exponential family and, therefore, has monotone likelihood ratio property in $T = \sum_{i=1}^n X_i$. So, in this case, the most powerful test will reject $H_0 : \theta = 2$ in favor of $H_1 : \theta = 1$ iff T is less than a cutoff. To find the cutoff, we must use the fact that, if H_0 is true, then T is a **Gamma**($n, 2$) random variable. Therefore, the size- α most powerful test rejects H_0 iff $T \leq \gamma_{n,2,\alpha}$, where $\gamma_{n,2,\alpha}$ is such that $P_{\theta=2}\{T \leq \gamma_{n,2,\alpha}\} = \alpha$. For example, if $n = 10$ and $\alpha = 0.05$, then you can find $\gamma_{10,2,0.05}$ in R with the command `qgamma(0.05, shape=10, scale=2)`.
- (b) The likelihood for $\text{Unif}(0, \theta)$ is $L(\theta) = \theta^{-n} I_{[0,\theta]}(X_{(n)})$ where $X_{(n)}$ is the largest order statistic. According to Neyman–Pearson, the most powerful tests rejects $H_0 : \theta = 2$ in favor of $H_1 : \theta = 1$ iff

$$\frac{L(2)}{L(1)} = \left(\frac{1}{2}\right)^n \frac{I_{[0,2]}(X_{(n)})}{I_{[0,1]}(X_{(n)})} \quad \text{is too small.}$$

Careful inspection of this expression reveals that it can only be two different values:

$$\frac{L(2)}{L(1)} = \begin{cases} 2^{-n} & \text{if } X_{(n)} \leq 1 \\ \infty & \text{if } X_{(n)} > 1. \end{cases}$$

Clearly, H_1 is wrong if $X_{(n)} > 1$, so the infinite value makes sense. But the fact that there's only one finite value means we have (essentially) no control over the size of the test. That is, the most powerful test rejects H_0 iff $X_{(n)} \leq 1$, and its size is

$$P_2(X_{(n)} \leq 1) = P_2(X_1 \leq 1)^n = 2^{-n}.$$

If n is five or more, then this is less than 0.05. But the good thing is that the power of this test when $\theta = 1$ is $P_1(X_{(n)} \leq 1) = 1$!

2. (a) In this case, $L(\theta) = \theta^n (\prod_{i=1}^n X_i)^{\theta-1} = \theta^n e^{(\theta-1) \sum_{i=1}^n \log X_i}$. Neyman–Pearson says the most powerful test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ rejects H_0 iff

$$L(1)/L(2) = 2^{-n} e^{-\sum_{i=1}^n \log X_i} \quad \text{is too small.}$$

This quantity is too small iff $-\sum_{i=1}^n \log X_i$ is, itself, too small. So the test can be expressed in terms of $T = -\sum_{i=1}^n \log X_i$. To figure out what “too small” means in this case, follow the hint which says that, if $H_0 : \theta = 1$ is true, then T has a **Gamma**($n, 1$) distribution. Therefore, the most powerful size- α test rejects H_0 iff $T \leq \gamma_{n,1,\alpha}$, where this cutoff is as defined in Problem #1.

- (b) Let $\theta_1 > 1$. The likelihood ratio is given by

$$L(1)/L(\theta_1) = \theta_1^{-n} e^{-(\theta_1-1) \sum_{i=1}^n \log X_i}.$$

Since $\theta_1 - 1 > 0$, we see, just as in part (a), that the likelihood ratio is increasing in $T = -\sum_{i=1}^n \log X_i$. Therefore, it has the monotone likelihood ratio property in T and, hence, the test described in part (a) is actually uniformly most powerful for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

3. Problem 8.2.6 HMC. By the monotone likelihood ratio property of $N(0, \theta)$, where θ is the variance, we know that (i) for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$, the uniformly most powerful test rejects H_0 iff $\sum_{i=1}^n X_i^2$ is too big, and (ii) for H_0 versus $H_1 : \theta < \theta_0$, the uniformly most powerful test rejects if $\sum_{i=1}^n X_i^2$ is too small. Test (i) has the largest possible power function on the interval (θ_0, ∞) and test (ii) has the largest possible power function on $(0, \theta_0)$. Since there can be no size- α test which has power function greater than the maximum of these two power functions on $(0, \infty)$, there can be no uniformly most powerful test of H_0 versus $H_1 : \theta \neq \theta_0$.
4. (Graduate only) Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Beta}(\theta, 1)$; that's the distribution in Problem #2. We're still interested in the quantity $T = -\sum_{i=1}^n \log X_i$, but when $\theta > 1$. In this case, we need to be a bit more careful about the distribution of T . First, it turns out that $-\log X_1$ has an exponential distribution with mean θ^{-1} .¹ In other words, $-\theta \log X_1 \sim \text{Exp}(1)$. Then it follows from before that $\theta T \sim \text{Gamma}(n, 1)$. Recall that the test from Problem #2(a) rejects H_0 iff $T \leq \gamma_{n,1,\alpha}$. So then the power function at θ is

$$\text{pow}(\theta) = P_\theta(T \leq \gamma_{n,1,\alpha}) = P_\theta(\theta T \leq \theta \gamma_{n,1,\alpha}) = P\{\text{Gamma}(n, 1) \leq \theta \gamma_{n,1,\alpha}\},$$

that is, the probability that a $\text{Gamma}(n, 1)$ random variable is no more than $\theta \gamma_{n,1,\alpha}$. In R, you may obtain a graph of this function with the following code:

```
n <- 10
alpha <- 0.05
pow <- function(theta) {
  g <- qgamma(alpha, shape=n, scale=1)
  out <- pgamma(theta * g, shape=n, scale=1)
  return(out)
}
curve(pow, xlim=c(1,4), lwd=2, xlab=expression(theta), ylab="Power")
```

¹To see this, define $Y = -\log X_1$ and use the transformation formula to find the PDF of Y ; you'll see that it's of the form of an exponential distribution.