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Markov Processes, 2024

Extra problems (from old assignments)

a1. The total variation distance between two probability distributions π_1 and π_2 on a finite set S is defined by

$$d_{TV}(\pi_1, \pi_2) = \frac{1}{2} \sum_{x \in S} |\pi_1(x) - \pi_2(x)|.$$

Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain with state-space $S = \{1, 2, 3\}$ and $P(X_0 = 1) = P(X_0 = 2) = 1/2$ having transition probabilities $p_{11} = p_{13} = p_{22} = p_{23} = p_{31} = p_{32} = 1/2$. Determine the total variation distance between the distribution of X_4 and the stationary distribution $\pi = (1/3, 1/3, 1/3)$.

a2. A random walker moves at random between three states A, B, and C according to a Markov chain $\{X_n\}_{n=0}^{\infty}$. At each time-step she moves to one of the two other states. The probability of moving from A to B is the same as the probability of moving from B to A. Suppose this probability is $p < 1$.

Starting from state A, find the expected number of steps it takes to reach C.

a3. The probability that a commuter uses a warm shirt when he bikes to work in the morning is 0.9 if he used a warm shirt when he biked back from work the previous day and 0.6 if not. The probability that he uses a warm shirt when he bikes home from work in the evening is 0.5 if he used a warm shirt in the morning and 0.2 if he didn't.

What proportion of time will he bike with a warm shirt in the long run?

b1. Let $\{X_n\}$ be an irreducible aperiodic Markov chain with finite state space, S, transition matrix $\mathbf{P} = (p_{ij})$, and unique stationary distribution π. Let $f_s : S \to S$, $s \in [0, 1]$ be functions and $\{I_n\}$ a sequence of independent uniformly distributed random variables on the unit interval such that $P(f_{I_n}(i) = j) = p_{ij}$, for all i, $j \in S$, and n. Suppose that the smallest integer N such that

$$W = f_{I_N} \circ f_{I_{N-1}} \circ \cdots \circ f_{I_1}(x)$$

does not depend on $x \in S$ satisfies $P(N < \infty) = 1$. Show by constructing a counterexample, that W need not have the stationary distribution π.

Hint: Use Exercise 3.8 b).

b2. Consider a lone white king making random moves on a chessboard. At each move the king picks one of the possible squares to move to, uniformly at random.
(Recall the rule how a king is allowed to move on an 8×8 conventional size chessboard.)
Find the long run probability that the king will be in one of the 4 corners of the chessboard.

b3. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive according to a Poisson process with rate three customers per hour and the successive service times are independent exponential random variables with mean 10 minutes. An arriving customer leaves immediately if there is no available space in the barbershop. What proportion of time is the barber busy with customers?

c1. Suppose $\{X_t\}_{t\geq 0}$ is a birth process on $S = \{0, 1, 2, 3, ...\}$ starting at $X_0 = 0$ with birth-rates λ_k, $k \geq 0$ where $\lambda_0 = 0.5$. Give an example of λ_k, $k \geq 1$ making the process explode on average at time $t = 100$.

c2. Consider the following example of forward equations:

$$\begin{aligned} p'_{jk}(t) &= \lambda(k-1)p_{j,k-1}(t) - (\lambda+\mu)kp_{jk}(t) + \mu(k+1)p_{j,k+1}(t), \quad k \geq 1 \\ p'_{j0}(t) &= \mu p_{j1}(t), \end{aligned}$$

with initial conditions $p_{jj}(0) = 1$, $p_{jk}(0) = 0$, $j \neq k$, $j \geq 1$, where $\mu > \lambda > 0$ are parameters. The functions $p_{jk}(t)$ are the transition probability functions for an honest birth-and-death process $\{X_t\}_{t\geq 0}$. Which one - what are the jump rates? Next, consider the function $M(t) = E(X_t|X_0 = 1) = \sum_{k=1}^{\infty} kp_{1k}(t)$. Derive a differential equation for $M(t)$ using the forward equations and solve it for $M(t)$.

c3. Consider a standard Brownian motion $\{B_t\}_{t\geq 0}$ at times $0 < t_1 < t_1 + t_2 < t_1 + t_2 + t_3 < t_1 + t_2 + t_3 + t_4$. Determine the product moment

$$E(B_{t_1} B_{t_1+t_2} B_{t_1+t_2+t_3} B_{t_1+t_2+t_3+t_4}).$$

Hint: Use the independent increments property of a Brownian motion. You may also use the fact that $X \in N(0, 1)$ implies that $E(X^4) = 3$.

Solutions Extra problems a1,a2,a3,b1,b2,b3,c1,c2,c3

a1. The Markov chain, $\{X_n\}_{n=0}^{\infty}$, with state-space $S = \{1, 2, 3\}$ has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Let $\mu^{\mathbf{n}} = (\mu_0^n, \mu_1^n, \mu_2^n) = (P(X_n = 0), P(X_n = 1), P(X_n = 2))$.

Since $\mu^{\mathbf{4}} = \mu^{\mathbf{0}}\mathbf{P}^4$ and by assumption $P(X_0 = 0) = P(X_0 = 1) = 1/2$ and therefore $\mu^{\mathbf{0}} = (1/2, 1/2, 0)$, we need to find $\mathbf{P}^4$;

$$\mathbf{P}^4 = \mathbf{P}^2\mathbf{P}^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/8 & 5/16 & 5/16 \\ 5/16 & 3/8 & 5/16 \\ 5/16 & 5/16 & 3/8 \end{pmatrix}.$$

Thus $\mu^{\mathbf{4}} = \mu^{\mathbf{0}}\mathbf{P}^4 = (11/32, 11/32, 10/32) \approx (0.3438, 0.3438, 0.3125)$, and therefore if $\pi = (1/3, 1/3, 1/3)$, then

$$d_{TV}(\mu^{\mathbf{4}}, \pi) = \frac{1}{2}(|11/32 - 1/3| + |11/32 - 1/3| + |10/32 - 1/3|) = 1/48 \approx 0.0208.$$

a2. The random walker moves at random between three states A, B, and C according to a Markov chain $\{X_n\}_{n=0}^{\infty}$. This Markov chain has transition probabilities $p_{AB} = p_{BA} = p < 1$ and $p_{AC} = p_{BC} = 1 - p$.

Let T_{AC} be the number of steps it takes to reach C from A, and T_{BC} be the number of steps it takes to reach C from B. By conditioning on the first step, we get

$$ET_{AC} = (1-p)1 + p(ET_{BC} + 1) = 1 + pET_{BC}$$

$$ET_{BC} = (1-p)1 + p(ET_{AC} + 1) = 1 + pET_{AC}.$$

Thus

$$ET_{AC} = 1 + p(1 + pET_{AC}) = 1 + p + p^2 ET_{AC}$$

$$\Leftrightarrow$$

$$ET_{AC} = \frac{1+p}{1-p^2} = \frac{1+p}{(1+p)(1-p)} = \frac{1}{1-p}$$

a3. Consider a Markov chain $\{X_n\}$ with 4 states $S = \{1, 2, 3, 4\}$ corresponding to the possible choices made a certain day, where
$X_n = 1$ corresponds to "no warm shirt used in the morning and no warm shirt used in the evening on day n",
$X_n = 2$ corresponds to "no warm shirt used in the morning and warm shirt used in the evening on day n",
$X_n = 3$ corresponds to "warm shirt used in the morning and no warm shirt used in the evening on day n",
$X_n = 4$ corresponds to "warm shirt used in the morning and warm shirt used in the evening on day n"

We have
$P(X_{n+1}=1|X_n=1)=P(X_{n+1}=1|X_n=3)=(1-0.6)\cdot(1-0.2)=0.32$
$P(X_{n+1}=2|X_n=1)=P(X_{n+1}=2|X_n=3)=(1-0.6)\cdot 0.2=0.08$
$P(X_{n+1}=3|X_n=1)=P(X_{n+1}=3|X_n=3)=0.6\cdot 0.5=0.30$
$P(X_{n+1}=4|X_n=1)=P(X_{n+1}=4|X_n=3)=0.6\cdot 0.5=0.30$
$P(X_{n+1}=1|X_n=2)=P(X_{n+1}=1|X_n=4)=(1-0.9)\cdot(1-0.2)=0.08$
$P(X_{n+1}=2|X_n=2)=P(X_{n+1}=2|X_n=4)=(1-0.9)\cdot 0.2=0.02$
$P(X_{n+1}=3|X_n=2)=P(X_{n+1}=3|X_n=4)=0.9\cdot 0.5=0.45$
$P(X_{n+1}=4|X_n=2)=P(X_{n+1}=4|X_n=4)=0.9\cdot 0.5=0.45$

i.e. $\{X_n\}$ has transition matrix

$$\mathbf{P}=\begin{pmatrix} 0.32 & 0.08 & 0.3 & 0.3 \\ 0.08 & 0.02 & 0.45 & 0.45 \\ 0.32 & 0.08 & 0.3 & 0.3 \\ 0.08 & 0.02 & 0.45 & 0.45 \end{pmatrix}$$

Solving $\pi\mathbf{P}=\pi$ for probability distributions $\pi=(\pi_1,\pi_2,\pi_3,\pi_4)$, gives $\pi=(20/91,5/91,33/91,33/91)$.

Thus by the Markov chain convergence theorem:

$$\begin{aligned} P(\text{warm shirt day } n) &= \sum_{k=1}^{4} P(\text{warm shirt day } n|X_n=k)P(X_n=k) \\ &= 0\cdot P(X_n=1)+0.5\cdot P(X_n=2)+0.5\cdot P(X_n=3)+1\cdot P(X_n=4) \\ &\to 0.5\cdot(5/91)+0.5\cdot(33/91)+33/91=52/91=4/7\approx 0.5714, \\ &\ \text{as } n\to\infty. \end{aligned}$$

b1. Let a be an arbitrary constant with $0<a<1$, and
$f_s(x)=1-x$, for $0\le s\le a$
$f_s(x)=1$, for $a<s\le 1$,
for $x\in\{0,1\}$ and $s\in[0,1]$.

If $\{I_n\}$ a sequence of independent random variables uniformly distributed on the unit interval then $P(f_{I_n}(0)=0)=0$, $P(f_{I_n}(0)=1)=1$, $P(f_{I_n}(1)=0)=a$, and $P(f_{I_n}(1)=1)=1-a$, i.e. random iteration with the functions f_s chosen uniformly at random generates a Markov chain on $\{0,1\}$ with transition matrix

$$\mathbf{P}=\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ a & 1-a \end{pmatrix}.$$

The unique stationary (reversible) distribution for $\mathbf{P}$ is $\pi=(\pi_0,\pi_1)=(\frac{a}{1+a},\frac{1}{1+a})$.

If N is the smallest integer such that

$$W=f_{I_N}\circ f_{I_{N-1}}\circ\cdots f_{I_1}(x)$$

does not depend on $x\in S$, then $W=1$ with probability one.

Thus W is not π-distributed. ($P(W=0)=0\neq\pi_0$, $P(W=1)=1\neq\pi_1$)

b2. We can regard the motion of the king as a random walk on a graph with 64 vertices (the possible squares) where there is an edge between 2 vertices if it is an allowed chess-move to go between the 2 squares. Since such a random walk forms an irreducible aperiodic (reversible) Markov chain the long run location of the king will follow the unique stationary distribution (independently of where it starts).

The vertices corresponding to the 4 corner-squares of the chessboard have 3 neighbors, the vertices corresponding to the middle 6×6 squares have 8 neighbors and the remaining (6+6+6+6) vertices have 5 neighbors. Totally there are $420 = 4 \cdot 3 + 36 \cdot 8 + 24 \cdot 5$ neighbors to all vertices. It follows that the long run probability of being in a fixed corner is $3/420$, the long run probability of being in a fixed middle square is $8/420$, and the long run probability of being in a fixed other square is $5/420$.

The long run probability that the king will be in one of the 4 corners of the chessboard is thus $4 \cdot (3/420) = 1/35 \approx 0.0286$.

b3. Let X_t be the number of customers in the barbershop at time t. $\{X_t\}$ is a birth-death process on $S = \{0, 1, 2\}$ with generator

$$\mathbf{Q} = \begin{pmatrix} -3 & 3 & 0 \\ 6 & -9 & 3 \\ 0 & 6 & -6 \end{pmatrix}$$

The proportion of time the barber is busy is the proportion of time X_t is not in state 0.

Since $\{X_t\}$ is an irreducible Markov chain it has a stationary distribution giving the long run proportion of time each state is visited.

We find the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2)$ by solving $\pi Q = 0$.

This gives the solution $\pi = (1/7)(4, 2, 1)$. The proportion of time spent in states 1 or 2 is thus $\pi_1 + \pi_2 = 3/7$.

c1. Let $\{X_t\}$ be a birth process with birthrates $\lambda_0 = 0.5$, and $\lambda_i = (99/98)^i$, $i \geq 1$ with $X_0 = 0$. Let U_j be the time spent in state j, and $J_n = \sum_{j=0}^{n-1} U_j$ be the time for the n:th jump. Let $J_\infty = \lim J_n$ be the explosion time. Since $U_j \in Exp(\lambda_j)$ we have

$$E(J_\infty) = E(\sum_{j=0}^{\infty} U_j) = \sum_{j=0}^{\infty} E(U_j) = 1/(0.5) + \sum_{j=1}^{\infty} (98/99)^j = 2 + 98 = 100.$$

Thus this process explodes on average at time $t = 100$.

c2. The process have forward equations

$$\begin{aligned} p'_{jk}(t) &= -(\lambda + \mu) k p_{jk}(t) + \lambda(k-1) p_{j,k-1}(t) + \mu(k+1) p_{j,k+1}(t), \quad k \geq 1 \\ p'_{j0}(t) &= \mu p_{j1}(t), \end{aligned}$$

From the these equations we see that $\{X_t\}$ is a birth-death process with linear growth $\lambda_k = \lambda k$, $k \geq 0$ and $\mu_k = \mu k$, $k \geq 1$.

By multiplying the k:th equation by k and summing up (with $j = 1$), we get the differential equation

$$\begin{aligned}
\underbrace{\sum_{k=1}^{\infty} kp'_{1k}(t)}_{M'(t)} &= -(\lambda+\mu)\sum_{k=1}^{\infty} k^2 p_{1k}(t) + \sum_{k=1}^{\infty} k\lambda(k-1)p_{1,k-1}(t) + \sum_{k=1}^{\infty} \mu k(k+1)p_{1,k+1}(t) \\
&= -(\lambda+\mu)\sum_{k=1}^{\infty} k^2 p_{1k}(t) + \sum_{k=1}^{\infty}(k+1)\lambda k p_{1,k}(t) + \sum_{k=2}^{\infty} \mu(k-1)kp_{1,k}(t) \\
&= -\lambda\sum_{k=1}^{\infty} k^2 p_{1k}(t) - \mu\sum_{k=1}^{\infty} k^2 p_{1k}(t) + \lambda\sum_{k=1}^{\infty} k^2 p_{1,k}(t) + \lambda\sum_{k=1}^{\infty} kp_{1,k}(t) \\
&\quad +\mu\sum_{k=2}^{\infty} k^2 p_{1,k}(t) - \mu\sum_{k=2}^{\infty} kp_{1,k}(t) \\
&= \lambda\sum_{k=1}^{\infty} kp_{1,k}(t) - \mu p_{1,1}(t) - \mu\sum_{k=2}^{\infty} kp_{1,k}(t) \\
&= \lambda\sum_{k=1}^{\infty} kp_{1,k}(t) - \mu\sum_{k=1}^{\infty} kp_{1,k}(t) = (\lambda-\mu)M(t),
\end{aligned}$$

with intitial condition $M(0) = \sum_{k=1}^{\infty} kp_{1,k}(0) = p_{1,1}(0) = 1$. This equation has solution $M(t) = e^{(\lambda-\mu)t}$, $t \geq 0$.

c3. Let $t_0 = 0$ and define the increments $Y_1 = B_{t_1} = B_{t_1} - B_{t_0}$, $Y_2 = B_{t_1+t_2} - B_{t_1}$, $Y_3 = B_{t_1+t_2+t_3} - B_{t_1+t_2}$, and $Y_4 = B_{t_1+t_2+t_3+t_4} - B_{t_1+t_2+t_3}$.

The random variables Y_i are independent and $Y_i \in N(0, \sqrt{t_i})$, for $i = 1, 2, 3, 4$. (Here the second parameter $\sqrt{t_i}$ denotes the standard deviation of Y_i). Thus $Var(Y_i) = E(Y_i^2) - (E(Y_i))^2 = E(Y_i^2) = t_i$, and since $Y_i/\sqrt{t_i} \in N(0,1)$ it follows from the hint that $EY_i^4 = 3t_i^2$.

Thus

$$\begin{aligned}
E(B_{t_1}B_{t_1+t_2}B_{t_1+t_2+t_3}B_{t_1+t_2+t_3+t_4}) &= E(Y_1(Y_1+Y_2)(Y_1+Y_2+Y_3)(Y_1+Y_2+Y_3+Y_4)) \\
&= EY_1^4 + 3E(Y_1^2Y_2^2) + E(Y_1^2Y_3^2) \\
&= EY_1^4 + 3E(Y_1^2)E(Y_2^2) + E(Y_1^2)E(Y_3^2) \\
&= 3t_1^2 + 3t_1t_2 + t_1t_3
\end{aligned}$$

where we in the second step used the fact that by independence terms of the form

$$E(Y_1^{n_1}Y_2^{n_2}Y_3^{n_3}Y_4^{n_4}) = \underbrace{E(Y_i^{n_i})}_{=0 \text{ if } n_i=1} E(\prod_{j\neq i} Y_j^{n_j}) = 0,$$

if $n_i = 1$ for some $i = 1, 2, 3, 4$.