UPPSALA UNIVERSITET
Matematiska Institutionen
Topias Tolonen-Weckström

Suggested Solutions
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Partial Differential Equations with Applications to Finance

Instructions: There are five problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no other materials are allowed. In the problems 4 and 5, you do not need to provide a proof to the respective verification theorems. **Good luck!**

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1. (8p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x)|t > 0, \quad x > 0\}$ with $u(0,x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t,0) = 0$ for $t > 0$.

(a) (3p) Construct a suitable extension of the initial condition to the whole space.

(b) (5p) Show that

$$u(t,x) = \int_0^\infty u_0(y) h(t,x,y)\,\mathrm{d}y$$

for some function $h(t,x,y)$. Find the function h. *Note: h is not simply the fundamental solution!*

Solution. *(a) A suitable extension would be to consider a function*

$$v(t,x) = \begin{cases} u(t,x), & x > 0, \\ u(t,-x), & x < 0. \end{cases}$$

Then, clearly

$$\frac{\partial v}{\partial x}(t,0) = 0, \qquad \text{and} \qquad v(0,x) = u(0,x) = u_0(x).$$

Therefore v solves for

$$\begin{cases} v_t - v_{xx} = 0, & \text{in } (0,\infty) \times (0,\infty, \\ v(0,x) = u_0(x), & x > 0, \\ v_x(t,0) = 0, & t > 0. \end{cases}$$

(b) Let

$$v_0(x) = \begin{cases} u_0(x), & x > 0, \\ u_0(-x), & x < 0, \end{cases}$$

be a continuous and bounded function. Then, based on our knowledge of the Cauhcy IVP problem, v is the convolution

$$v(t,x) = \int_{\mathbb{R}} g(t, x-y) v_0(y)\,\mathrm{d}y,$$

where g is the fundamental solution to the Cauchy IVP. Now,

$$\begin{aligned}
v(t,x) &= \int_{\mathbb{R}} g(t,x-y)v_0(y)\,\mathrm{d}y \\
&= \int_0^{\infty} g(t,x-y)u_0(y)\,\mathrm{d}y + \int_{-\infty}^{0} g(t,x-y)u_0(-y)\,\mathrm{d}y \\
&= \int_0^{\infty} g(t,x-y)u_0(y)\,\mathrm{d}y - \int_{\infty}^{0} g(t,x+z)u_0(z)\,\mathrm{d}y \\
&= \int_0^{\infty} \left(g(t,x-y)+g(t,x+y)\right)u_0(x)\,\mathrm{d}y.
\end{aligned}$$

The desired result follows by denoting $h(t,x,y) := g(t,x-y)+g(t,x+y)$.

2. (8p) Let D denote a bounded interval $(-a,b) \in \mathbb{R}$ with $a,b>0$ and let W_t be the standard Brownian motion. Propose a suitable boundary value problem so that u solves

$$u(x) := \mathbb{E}_x\left[W_{\tau_D}^p + \tau_D\right],$$

where $p>0$ is a constant and $\tau_D = \inf\{t>0 | W_t \notin D\}$ is the so-called first exit time from D, and calculate $u(0)$.

Solution. *We note that*

$$\mathbb{E}\left[W_{\tau_D}^p + \tau_D\right] = \mathbb{E}\left[W_{\tau_D}^p + \int_0^{\tau_D} 1\,\mathrm{d}s\right],$$

so the problem can be suitably formulated as a Dirichlet-Poisson problem

$$\begin{cases} \frac{1}{2}u_{xx} + 1 = 0 & \text{in } D, \\ u(-a) = (-a)^p & \text{at } -a, \\ u(b) = (b)^p & \text{at } b \end{cases}$$

by recalling that since W is the Standard Brownian motion, $\mathcal{L}u = \frac{1}{2}u_{xx}$.
Moreover, we note that $\frac{1}{2}\left(-x^2\right)_{xx} = -1$ so the general solution to the above ODE is given by

$$u(x) = -x^2 + Ax + B$$

for some constants A, B.
To find the particular solution, the boundary conditions then yield

$$\begin{cases} -a^2 - Aa + B = (-a)^p, \\ -b^2 + Ab + B = (b)^p. \end{cases} \implies \begin{cases} A = \frac{b^p-(-a)^p}{b+a} + (b-a), \\ B = ab + \frac{ab^p + b(-a)^p}{b+a}. \end{cases}$$

Finally we note that

$$\mathbb{E}\left[W_{\tau_D}^p + \tau_D\right] = u(0) = B = ab + \frac{ab^p + b(-a)^p}{b+a},$$

and we are done.

3. (8p) Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function and $D>0$. Under suitable assumptions, use the Fokker-Planck equation (Kolmogov Forward) to show that the Gibbs distribution

$$p(x) = \frac{1}{Z}e^{-\frac{V(x)}{D}}$$

is the limiting density function for X_t, which follows the dynamics

$$\mathrm{d}X_t = -DV'(X_t)\,\mathrm{d}t + \sqrt{2}D\,\mathrm{d}W_t, \quad X_0 = 0,$$

and find the normalizing constant Z. Be careful in noting down possible assumptions on p as you go.

Solution. *The purpose of the problem is to recognize the concept of a steady-state distribution (a limiting density function), and to show that the required form for $p(x) = p_infty(x) := \lim_{t\to\infty} p(t,x)$ can be derived from the KFE for a time-homogeneous distributions:*

$$p_t - \mathcal{L}^* p = 0.$$

So, let us start with

$$\mathrm{d}X_t = -DV'(X_t)\,\mathrm{d}t + \sqrt{2}D\,\mathrm{d}W_t.$$

We note the following assumptions to solve the problem (listing these all is not required for full points):

- $\lim_{t\to\infty} p(t,x)$ *needs to exists,*
- p *smooth, i.e.* $p \in C_o^2(\mathbb{R})$, *and*
- *limit density is time-homogeneous, i.e.* $p_t = 0$.

Now, from the KFE and by assuming that $p_t = 0$ we get

$$\frac{\partial}{\partial x} DV'(x)p(x) + D^2\frac{\partial^2 p(x)}{\partial x^2} = 0.$$

Integrating this with respect to x yields

$$DV'(x)p(x) + D^2\frac{\partial p(x)}{\partial x} = B$$

for some constant B. Noting that for steady-state distributions we must have $\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$, which suggests that $B = 0$. Reshuffling the equation then yields

$$\frac{\frac{\partial p(x)}{\partial x}}{p(x)} = -\frac{V'(x)}{D}.$$

Noting that $\frac{\frac{\partial p(x)}{\partial x}}{p(x)}$ integrates to $\log p(x)$ with respect to x, we get

$$\log p(x) = -\frac{V(x)}{D} + C \quad \Leftrightarrow \quad p(x) = Ce^{-\frac{V(x)}{D}}$$

for some constant C. Utilising again the fact that for limiting densities we have $\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$, we get

$$C := \frac{1}{Z} := \frac{1}{\int_{\mathbb{R}} e^{-\frac{V(x)}{D}}\,\mathrm{d}x}.$$

4. (8p) Consider the Merton's asset allocation problem with one risky asset

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$$

and a risk-free rate $r = 0$. Here μ, σ are constants and W_t the standard Brownian motion. Let further X_t^α denote the wealth process where α_t is the share invested in the risky asset at time t. Write down the dynamics of X_t^α and find the value function and the optimal control α^* in the Merton's problem,

$$V(t,x) = \sup_\alpha \mathbb{E}_{t,x}\left[\Phi(X_T^\alpha)\right]$$

for some termination time T, where the utility function satisfies the so-called Kelly criterion: $\Phi(x) = \log x$. *Note: Use an ansatz $\hat{V}(t,x) = \Phi(x) + \lambda(t)$ for some suitable function λ with $\lambda(T) = 0$, which you have to solve from the corresponding HJB equation.*

Solution. *I'm Merton's allocation problem one wishes to find an optimal share of investment α_t between risky and risk-free assets. We begin by noting that since $r = 0$, we have for the risk-free asset B_t that $\mathrm{d}B_t = 0$. Then, we can describe the dynamics of X_t, namely*

$$\mathrm{d}X_t = \frac{\alpha_t}{S_t}\,\mathrm{d}S_t + \frac{(X_t - \alpha_t)}{B_t}\,\mathrm{d}B_t = \frac{\alpha_t}{S_t}\,\mathrm{d}S_t + 0 = \mu\alpha_t\,\mathrm{d}t + \sigma\alpha_t\,\mathrm{d}W_t.$$

By HJB, the candidate value function $\hat{V}$ then solves

$$\begin{cases} \hat{V}_t + \sup_\alpha\{\mathcal{L}^\alpha\hat{V}\} = 0, \\ \hat{V}(T,x) = \Phi(x) + \lambda(T) = \log x \end{cases}$$

for some function $\lambda(t)$ with $\lambda(T) = 0$. The generator for our process X_t can be calculated as

$$\mathcal{L}^\alpha\hat{V} = \mu\alpha\hat{V}_x + \frac{1}{2}\sigma^2\alpha^2\hat{V}_{xx}.$$

Now, for the Kelly criterion utility we use the candidate solution (ansatz) $\hat{V}(t,x) = \log x + \lambda(t)$, for which we have

$$\hat{V}_t = \lambda_t, \quad \hat{V}_x = \frac{1}{x}, \quad \text{and} \quad \hat{V}_{xx} - \frac{1}{x^2}.$$

Next, we solve for the optimal α. From the supremum inside the ODE, the First-Order condition suggests that

$$\frac{\mu}{x} - \frac{\sigma^2}{x^2}\alpha = 0 \Longleftrightarrow \alpha^* := \alpha = \frac{\mu x}{\sigma^2}.$$

We plug the candidate α^* to the ODE and find that

$$\lambda_t + \frac{1}{2}\frac{\mu^2}{\sigma^2} = 0.$$

Then, the original problem reduces to finding $\lambda(t)$ from

$$\begin{cases} \lambda_t = -\frac{1}{2}\frac{\mu^2}{\sigma^2}, \\ \lambda(T) = 0. \end{cases}$$

An general solution to this ODE is of form $-\frac{1}{2}\frac{\mu^2}{\sigma^2}t + C$ for some constant C, and the boundary condition yields $C = \frac{1}{2}\frac{\mu^2}{\sigma^2}T$. Thus the explicit solution is

$$\lambda(t) = \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t),$$

and moreover,

$$\hat{V}(t,x) = \log x + \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t).$$

Finally, by the Verification theorem, we have that $\hat{V} \equiv V$ and $\alpha^* = \frac{\mu x}{\sigma^2}$ is the optimal control.

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_x \left[e^{-r\tau} \left(e^{\frac{X_\tau}{4}} - 1 \right) \right],$$

where $\mathrm{d}X_t = r\,\mathrm{d}t + 2\sqrt{r}\,\mathrm{d}W_t$, W_t is the standard Brownian motion, and $r > 0$.

Solution. *For some boundary b, we expect a continuation region $C = (-\infty, b)$ and the stopping region $D = \mathbb{R} \setminus C$. For this stopping region, a stopping time of the form $\tau = \inf\{t \geq 0 | X_t \in D\}$ is a suitable candidate. Let $\hat{V}(x)$ be the candidate solution to the problem. For the process X_t, we then have, omitting arguments,*

$$\mathcal{L}\hat{V} = r\hat{V}_x + 4r\hat{V}_{xx}.$$

By dynamic programming, $\hat{V}$ should then solve the free-boundary problem

$$\begin{cases} r\hat{V}_x + 4r\hat{V}_{xx} - r\hat{V} = 0, & x < b, \\ \hat{V}(x) = e^{\frac{x}{4} - 1}, & x \geq b, \\ \hat{V}_x(x) = \frac{1}{4} e^{\frac{x}{4}}, & x \geq b, \\ \lim_{x \to \infty} \hat{V}(x) = 0 \end{cases}$$

for some boundary b. Noting that there's no 'x' terms in front of the different derivatives of $\hat{V}$, we use an ansatz $\hat{V}(x) = e^{\gamma x}$ for some γ. Plugging this in, the ODE becomes

$$r\gamma e^{\gamma x} + 4r\gamma^2 e^{\gamma x} - re^{\gamma x} = 0 \iff 4\gamma^2 + \gamma - 1 = 0.$$

Let us denote the solutions of this quadratic equation as $\gamma_+ > 0$ and $\gamma_- < 0$. Then, the general solution of the ODE would be of the form

$$\hat{V}(x) = Ce^{\gamma_+ x} + De^{\gamma_- x}$$

for some constants C, D.

From the last boundary condition we see that $C \equiv 0$ must hold. Remaining are two unknowns D, b and two equations: at the boundary b, one has

$$\begin{cases} \hat{V}(b) = e^{b/4} - 1, \\ \hat{V}_x(b) = \frac{1}{4} e^{b/4}. \end{cases}$$

The first equation, together with our ansatz $\hat{V}(x) = De^{\gamma_- x}$ yields

$$D = \frac{e^{b/4} - 1}{e^{\gamma_- b}}.$$

The remaining condition ('smooth' fit) implies then

$$b = 4 \log \left(\frac{\gamma_-}{\gamma_- - \frac{1}{4}} \right).$$

Stitching our partial arguments together, we have arrived at

$$\hat{V}(x) = \begin{cases} (e^{b/4} - 1)e^{\gamma_- (x - b)}, & x < b, \\ e^{x/4} - 1, & x \geq b. \end{cases}$$

Finally, by the Verification Theorem, we have that $\hat{V} \equiv V$ and the stopping time $\tau = \inf\{t \geq 0 | X_t \geq b\}$ is optimal.