i Information page

How?
The main option is to write your answers in the boxes in Inspera. As an option, if you need to, you can submit hand-written solutions on paper "the normal way". In part A you are hopefully able to answer in the Inspera only, whereas there might a little bit more math to write in part B. If you submit paper-answers, make a note about it in Inspera in the corresponding question. Please write your answers in english.

Grades
The exam is divided into two parts, part A and part B. Part A is related to grade 3 and part B to grade 4 and 5. On Part A there are 2 question related to each goal and maximum 2 points per question. In part B there are maximum 6 point per question. The grades are:

- Grade 3: You must at least solve one question on each goal. This corresponds minimum 6 points, distributed among the three goals (minimum 2 points per goal).
- Grade 4: You must fullfill grade 3 + at least completely solve one of the two questions in part B. This corresponds to minimum 6 points on part B.
- Grade 5: You must fullfill grade 3 + completely or with minor errors solve the two questions in part B. This corresponds to minimum 10-11 points on part B.

Tools available

The tools available in Inspera are

- **Online Python:** The idea is that you are able to use Python as a pocket calculator, very much in the same way as on the problem solving sessions. It will be enough to be able to use numpy.)
- **Numpy Cheat Sheet** is available as a link
- **Numpy reference manual** is available as a link
- Formula sheet

You should be able to find these resources at the bottom part of Inspera.

Tools to take with you
Pocket calculator is allowed, so you can bring it to the exam if you want to. It's not necessary though, you can use Python instead.

Good Luck!

1 Concept-1

Classify the algorithms and methods

	Deterministic Model	Stochastic Model	Deterministic Method	Stochastic Method
Gillespies algorithm				✓
Monte Carlo Integration				✓
$\begin{aligned} dD_A/dt &= \theta_A D'_A - \gamma_A D_A A \\ dD_R/dt &= \theta_R D'_R - \gamma_R D_R A \\ dD'_A/dt &= \gamma_A D_A A - \theta_A D'_A \\ dD'_R/dt &= \gamma_R D_R A - \theta_R D'_R \\ dM_A/dt &= \alpha'_A D'_A + \alpha_A D_A - \delta_{M_A} M_A \\ dA/dt &= \beta_A M_A + \theta_A D'_A + \theta_R D'_R \\ &\quad - A(\gamma_A D_A + \gamma_R D_R + \gamma_C R + \delta_A) \\ dM_R/dt &= \alpha'_R D'_R + \alpha_R D_R - \delta_{M_R} M_R \\ dR/dt &= \beta_R M_R - \gamma_C A R + \delta_A C - \delta_R R \\ dC/dt &= \gamma_C A R - \delta_A C, \end{aligned}$	✓			
$I = \int_{-1}^{1} x^2 e^{\frac{-x^2}{2}} dx$	✓			
Assume all quantities to be integers. $\left.\begin{array}{rcl} A + R & \xrightarrow{\gamma_c} & C \\ A & \xrightarrow{\delta_a} & \emptyset \\ C & \xrightarrow{\delta_a} & R \\ R & \xrightarrow{\delta_r} & \emptyset \\ D_a + A & \xrightarrow{\gamma_a} & D'_a \\ D_r + A & \xrightarrow{\gamma_r} & D'_r \end{array}\right\}$		✓		
$\frac{dx}{dt} = \alpha x - \beta x y, \quad \frac{dy}{dt} = \delta x y - \gamma y,$	✓			
QR-iteration			✓	
Backwards Substitution			✓	

Maximum marks: 2

2 Concepts-2

The QR-method (QR-iteration)
Select all statements that are true and complete the sentence

- ☐ is used in the SSA algorithm
- ☐ finds the Schur decomposition of a matrix ✔
- ☐ is an iterative method ✔
- ☐ is a method to solve linear system
- ☐ finds a low rank approximation of a matrix
- ☐ finds the eigenvalue decomposition of a real symmetric matrix ✔

Which of the following statements are true about 1D Brownian motion $B(t)$ with initial condition $B(0) = 0$

Select all statements that are true

- ☐ B(t) is normally distributed with standard deviation t
- ☐ B(t) is normally distributed with variance t ✔
- ☐ The probablity of B(t)>0 for all t>0 is 0.5 ✔
- ☐ Can be simulated by the SSA algorithm
- ☐ B(t) is continuous in time (almost surely) ✔

Maximum marks: 2

3 Algorithm 1 - A

The molecular weight (molar mass, g/mol) of nitrogen oxides are tabulated below (three decimal places accuracy):

NO	N_2O_3	N_2O	N_2O_5	NO_2	N_2O_4
30.006	76.012	44.013	108.010	46.006	92.001

Based on the data, do a **least squares** approximation to **calculate** the molecular weight for nitrogen and oxygen, all data must be used. (Hint: The molecule N_2O_3 contains 2 nitrogen atoms and 3 oxygen atom.)

Fill in your answer here

Maximum marks: 2

4 Algorithm 2 - A

You are running a lumber mill in Krokom and you are trying to **estimate the production price** of a single piece of your standard framing timber. The cost to produce your standard framing timber includes labor, energy and trees. Assume the cost of labor is constant at 3 SEK per piece of framing timber. The price of tree needed to make a single piece of timber is normally distributed with mean μ_T and standard deviation σ_T. The cost of energy needed to make a single piece of framing timber is distributed according to the pdf $f_{el}(x)$.

Design a Monte Carlo algorithm to estimate the mean production price p_{mean} and variance $p_{variance}$ of standard framing timber. Assume the function **`timber_el_pris()`** exists and that it returns one random number that is distributed according to the pdf $f_{el}(x)$ every time that it is called.

Fill in your answer here

Maximum marks: 2

5 Analysis 1 - A

You would like to **estimate the average production price + an uncertainty interval (error range)** of the form $p_{mean} \pm |e|$ after producing N pieces (a large quantity) of standard framing timber.

You can assume that the price of a single piece of framing timber is an independent and identically distributed random variable with mean p_{mean} and variance $p_{variance}$. (It is however not normally distributed.)

Using the **central limit theorem**, explain how you get **an estimate for** $|e|$ after producing N pieces of framing timber
(Hint: How does the **uncertainty interval** $\pm|e|$ of the **average production price** depend on N)

Fill in your answer here

Maximum marks: 2

6 Analysis 2 - A

A matrix has the eigenvectors

$$v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} v_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} v_4 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

and eigenvalues $\lambda_1 = 0.1, \lambda_2 = 2, \lambda_3 = 3, \text{ and } \lambda_4 = 4$.

What is the condition number $\text{cond}_2(A)$?

Select one alternative:

- 2
- 40 ✔
- 4
- 0.1

Let A_3 be the best rank 3 approximation to the matrix A in the 2 norm. What is $||A - A_3||_2$?

Select one alternative

- 0.1 ✔
- 3
- 5.3861
- 4

Which of the following statements is true for the specific matrix A

Select one alternative

- The eigenvalue decomposition of A is equal to it's singular value decomposition. ✔
- The matrix A is not symmetric.
- The matrix A is an orthonormal matrix.
- You do not have enough information to answer this question

Maximum marks: 2

7 Algorithm B - (for higher grade)

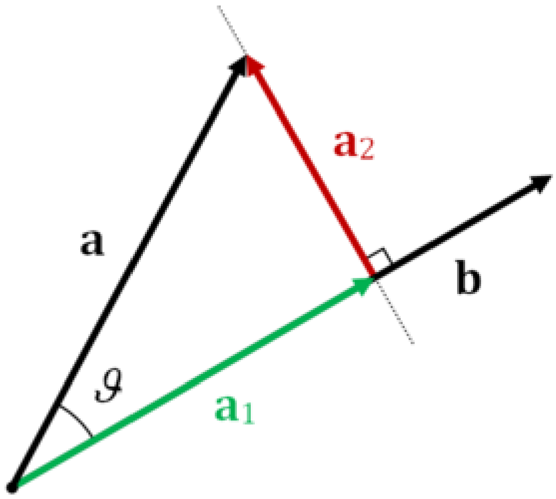
In this question we are taking a look at the QR decomposition of a matrix $A = QR$.

a) Assume you are trying to solve the least squares problem $min||Ax - b||_2$. Under what conditions is it **computationally optimal** you solve this problem using the QR decomposition.? (2 point)

Fill in your answer here

You are given two vectors $\mathbf{b}$ and $\mathbf{a}$ which are the columns of the matrix $A = [\mathbf{b}, \mathbf{a}]$. The QR decomposition is based on Gram Schmidt orthogonalization which uses orthogonal projection of vectors.

In class we derived, that the projection of the vector $\mathbf{a}$ onto $\mathbf{b}$ is given by $\mathbf{a}_1 = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$ where $\mathbf{a} \cdot \mathbf{b}$ is the dot product of $\mathbf{a}$ and $\mathbf{b}$. This is illustrated in the lower image.



b) Find an expression for $\mathbf{a}_2$ the vector orthogonal to $\mathbf{b}$ such that $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ (1 point). You expression should be in terms of $\mathbf{b}$ and$\mathbf{a}$

Fill in your answer here

c) Find the QR decomposition of the matrix $A = [\mathbf{b},\ \mathbf{a}] = [\mathbf{q}_1,\ \mathbf{q}_2]\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$. That is, find expressions for $r_{11}, r_{12}, r_{22}, \mathbf{q}_1,$ and $\mathbf{q}_2$ in terms of $\mathbf{b}$,$\mathbf{a}$,$\mathbf{a}_1$ and $\mathbf{a}_2$. (3 points)

Fill in your answer here

Maximum marks: 6

8 Analysis B (for higher grade)

In this question we will look into functions of a matrix. Assume that the matrix $A \in \mathbb{R}^{n \times n}$ has the eigenvalue decomposition $A = Q\Lambda Q^{-1}$, where Λ is the diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$.

a) Find the eigenvalue decomposition of $B = A^4$ (2 points)

Fill in your answer here

We can define functions of matrices using series expansions. Assume the function $f(X)$ has a convergent Taylor series around $X = 0,$ that is $f(X) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} X^k$.

b) Show that $f(A) = Qf(\Lambda)Q^{-1}$. (2 points)

Fill in your answer here

c) Let the matrix A be symmetric ($A = A^T$) and let the columns of the eigenvector matrix be $Q = [q_1, q_2, \ldots, q_n]$, that is q_i is the i-th eigenvector. Using your result in b) show that the following expansion of rank-1 matrices holds:

$f(A) = \sum_{i=1}^{n} q_i q_i^T f(\lambda_i) = q_1 q_1^T f(\lambda_1) + \cdots + q_n q_n^T f(\lambda_n)$ (2 points)

Fill in your answer here

Maximum marks: 6