UPPSALA UNIVERSITET

Lecture Notes

Complex Analysis

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Contents

1. Intro

In this course, we shall study functions $f : \mathbb{C} \to \mathbb{C}$ (or more generally, $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$)

Definition/Sats 1.1: Complex Number

A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$

Anmärkning:
The number x is called the *real part* ($\mathrm{Re}(z) = x$) of the complex number, and y is called the *imaginary part* ($\mathrm{Im}(z) = y$) of the complex number

Anmärkning:
The set of all complex numbers is denoted by $\mathbb{C}$

Anmärkning:
$i^2 = -1$

1.1. **Operations over $\mathbb{C}$.**

We define the operations *addition* and *multiplication* of two complex unmebrs as follows:

Definition/Sats 1.2: Addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Definition/Sats 1.3: Multiplication of complex numbers

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations, $\mathbb{C}$ forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Definition/Sats 1.4: Complex conjugate

The *complex conjugate* of a complex number $z = x + iy$, denoted by $\overline{z}$, is defined by $\overline{z} = x - iy$

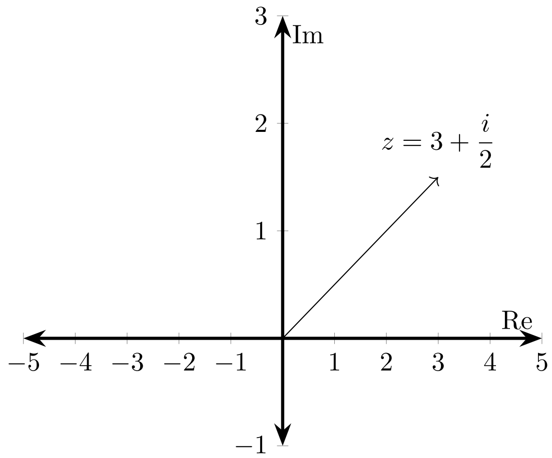
The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\overline{z}} = z$

Anmärkning:
$\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$
$\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2i}$

1.2. Cartesian representation.

It is natural to represent a complex number $z = x + iy$ as a tuple (x, y), and we can therefore represent it in the standard cartesian plane:



Anmärkning:
This is sometimes called the *complex plane*

Definition/Sats 1.5: Absolute value/Modulus

The absolute value of a complex number $z = x+iy$ (geometrically the length of the vector), denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:
- $|z|^2 = z \cdot \overline{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Anmärkning:
Every $z \in \mathbb{C}$ such that $z \neq 0$ (that is, $x \neq 0$ or $y \neq 0$) has a multiplcative inverse $\dfrac{1}{z}$ given by:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Definition/Sats 1.6: Triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that $|z_1 + z_2| \leq |z_1| + |z_2|$

Lemma 1.1: Reversed triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Bevis 1.1

$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

So that $|z_1| - |z_2| \leq |z_1 - z_2|$ □

1.3. **Polar form.**

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, and hence there exists θ such that:

$$\frac{x}{|z|} = \cos(\theta) \qquad \frac{y}{|z|} = \sin(\theta)$$

Therefore $z = x + iy$ can be written as:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Where $r = |z|$ is uniquely determined by z, while θ is 2π-periodic. This is called the *polar form* of z and just as the cartesian representation requires a tuple of information $(|z|, \theta)$

Definition/Sats 1.7: Argument

The *argument* of a complex number z, denoted by $\arg(z)$, is the angle θ between z and the real number line in the complex plane

Anmärkning:
Since the argument is 2π periodic, the angle is usually given as $\theta + k2\pi \quad k \in \mathbb{Z}$, but we are only intersted in θ
This θ is called the *principal value* of $\arg(z)$, denoted by $\text{Arg}(z)$ and belongs to $(-\pi, \pi]$

Anmärkning:
We are always allowed to change an angle by multiples of 2π, the principal value argument is the angle after changing the argment such that it lies between $(-\pi, \pi]$

Anmärkning:
One calls $\text{Arg}(z)$ a *branch* of $\arg(z)$. Also, note that $\text{Arg}(z)$ is "discontinous" along the negative real axis. This is called a *branch-cut*

Suppose $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$
Then:

$$z_1 \cdot z_2 = r_1r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$$
$$= r_1r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))]$$
$$r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Anmärkning:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

1.4. **Exponential form.**

Definition/Sats 1.8

For $z = x + iy \in \mathbb{C}$, let $e^z = e^x(\cos(y) + i\sin(y))$

Anmärkning:
$e^{iy} = \cos(y) + i\sin(y) \quad y \in \mathbb{R}$ (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$\Rightarrow e^{iy} = 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin(\theta)}$$

$$\Rightarrow e^z = e^x(\cos(\theta) + i\sin(\theta))$$

Anmärkning:
One can through comparing see that $|e^z| = e^x$, and that $\left|e^{iy}\right| = 1$

Definition/Sats 1.9: deMoivre's formula

For $n \in \mathbb{Z}$, $(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta))$

1.5. **Logarithmic form.**

In real analysis, we have defined the logarithm as the inverse of e^x. This has previously worked since for $x \in \mathbb{R}$, e^x is injective.
The problem is that for e^z where $z \in \mathbb{C}$, it is not injective and should therefore not have an inverse.

Given $z \in \mathbb{C}\setminus\{0\}$, we define $\ln(z)$ as the cut of all $w \in \mathbb{C}$ whose image undre the exponential form is z, i.e $w = \ln(z) \Leftrightarrow z = e^w$.

Here, $\ln(z)$ is a *multivaled form*

We can use the fact that $|z| = r = e^x$ to derive some interesting properties of the logarithm:

$$z = re^{i\theta} \qquad w = u + iv$$

$$\text{If } z = e^w \Leftrightarrow re^{i\theta} = e^u \cdot e^{iv}$$

$$\Leftrightarrow u = \ln(r) = \ln(|z|) \qquad v = \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z}$$

Definition/Sats 1.10: Complex logarithm

For $z \neq 0$, we define the complex logarithm for $z \in \mathbb{C}$ as:

$$\ln(z) = \ln(|z|) + i \cdot \arg(z)$$
$$= \ln(|z|) + i(\text{Arg}(z) + k2\pi) \quad k \in \mathbb{Z}$$