1 Algorithm1_20230819

Compute $y(0.1)$ using the backward Euler's (Implicit Euler's) method for differential equation
$y'(t) = -10(y(t) - t^2), \;\; t \geq 0, \; y(0) = 1$
with steplength $h = 0.1$.
You can find the backward Euler's method in the formula sheet.
Perform the calculation on paper and choose your answer by selecting one of the options below (only one item is correct).

Select one alternative:

- 0.6734
- 0.3367
- 0.0000
- ✓ 0.5050
- 0.5500

Maximum marks: 2

$$y_{k+1} = y_k + h f(t_{k+1}, y_{k+1}) \Rightarrow y_1 = y_0 + 0.1\left[-10\left(y_1 - (0.1)^2\right)\right]$$

$$\Rightarrow 2y_1 = y_0 + 0.01 = 1 + 0.01 = 1.01 \Rightarrow y_1 = 0.5050$$

2 Algorithm2_20230819

The task is to approximate the value of integral

$$\int_0^\infty (1+x)\exp(-2x)\,dx$$

using the Monte Carlo method on five random points

$$0.0108,\ 0.0602,\ 0.3568,\ 0.8921,\ 1.7759$$

which are exponentially distributed according to probability density function (pdf) $f(x) = 2\exp(-2x)$. What is the approximate value?

Select one alternative:

- 1.61916
- ✓ 0.80958
- 0.29911
- 0.59823

Maximum marks: 2

$$\int_0^\infty (1+x)e^{-2x}dx = \frac{1}{2}\int_0^\infty \underbrace{(1+x)}_{h(x)}\underbrace{2e^{-2x}}_{f(x)}dx \approx \frac{1}{2}\,\frac{1}{N}\sum_{k=1}^{N} h(x_k)$$

$$= \frac{1}{2}\,\frac{1}{5}\left[(1+x_1)+(1+x_2)+\cdots+(1+x_5)\right]$$

$$= \frac{1}{10}\left[1.0108+1.0602+1.3568+1.8921+2.7759\right]$$

$$= 0.80958$$

	Deterministic Model	Stochastic Model	Deterministic Method	Stochastic Method
Gillespies algorithm (SSA)				✓
Monte Carlo Integration				✓
$\begin{aligned} dD_A/dt &= \theta_A D'_A - \gamma_A D_A A \\ dD_R/dt &= \theta_R D'_R - \gamma_R D_R A \\ dD'_A/dt &= \gamma_A D_A A - \theta_A D'_A \\ dD'_R/dt &= \gamma_R D_R A - \theta_R D'_R \\ dM_A/dt &= \alpha'_A D'_A + \alpha_A D_A - \delta_{M_A} M_A \\ dA/dt &= \beta_A M_A + \theta_A D'_A + \theta_R D'_R \\ &\quad - A(\gamma_A D_A + \gamma_R D_R + \gamma_C R + \delta_A) \\ dM_R/dt &= \alpha'_R D'_R + \alpha_R D_R - \delta_{M_R} M_R \\ dR/dt &= \beta_R M_R - \gamma_C AR + \delta_A C - \delta_R R \\ dC/dt &= \gamma_C AR - \delta_A C, \end{aligned}$	✓			
$I = \int_{-1}^{1} x^2 e^{\frac{-x^2}{2}} dx$	✓			
Assume all quantities to be integers. $\left.\begin{aligned} A + R &\xrightarrow{\gamma_c} C \\ A &\xrightarrow{\delta_a} \emptyset \\ C &\xrightarrow{\delta_a} R \\ R &\xrightarrow{\delta_r} \emptyset \\ D_a + A &\xrightarrow{\gamma_a} D'_a \\ D_r + A &\xrightarrow{\gamma_r} D'_r \end{aligned}\right\}$		✓		
$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = \delta xy - \gamma y,$	✓			
Classical Runge Kutta			✓	
Trapezoidal rule for integration			✓	

Maximum marks: 2

4 Concept2_20230819

When numerically solving differential equations, it is important to consider whether the equation is stiff. What are the characteristics of a stiff ODE?
One or more options may be correct, tick all correct items.

Select one or more alternatives:

- [x] Implicit methods are appropriate to solve it
- [x] It requires a small steplength for stability in explicit methods
- [ ] The RK4 method can solve it with a relatively large steplength
- [ ] Explicit methods are appropriate to solve it
- [x] There may be some fast transients (rapid variations) in the solution
- [x] The coefficients on the right-hand side of the ODE are significantly different in magnitude.

Maximum marks: 2

5 Analysis1_20230819

Assume that the solution of an initial value problem (IVP) is calculated using a second order method (say Heun's method) with step length $h = 0.05$. It is estimated that the global error is ≈ 0.4. How small does h need to be for obtaining the approximate error ≈ 0.025?

Calculate the new h and enter it here to four decimal places*: 0.0125.

* for example: 0.1234

Maximum marks: 2

$$e = O(h^2) \Rightarrow e = ch^2$$

$$\Rightarrow \frac{e_1}{e_2} = \frac{h_1^2}{h_2^2} \Longrightarrow \frac{0.4}{0.025} = \frac{(0.05)^2}{h_2^2} \Longrightarrow h_2 = 0.0125$$

6 Analysis2_20230819

Suppose that we have estimated the integral $\int_a^b f(x)\,dx$ using the Monte Carlo method with $N = 1000$ random numbers. Alongside, we have assessed the error through the length of the confidence interval (or standard deviation), which we found to be $\varepsilon = 0.2$. Now, if we seek to enhance the accuracy by increasing the number of random points to $N = 16000$, what would the resulting length of the confidence interval (or standard deviation) be?

Calculate and enter it here: 0.05 .

$$\varepsilon = O\left(\frac{1}{\sqrt{N}}\right) \Rightarrow \frac{\varepsilon_1}{\varepsilon_2} = \frac{\sqrt{N_2}}{\sqrt{N_1}} \Rightarrow \frac{0.2}{\varepsilon_2} = \sqrt{\frac{16000}{1000}} \Rightarrow \varepsilon_2 = 0.05$$

Maximum marks: 2

7 Argumentation1_20230819

Specify the more suitable method (column) for each application (row).

	Implicit method	Explicit method
Stability is crucial	✓	
Non-stiff equation		✓
Low complexity per timestep is important		✓
$y_1' = y_2$ $y_2' = (1 - y_1^2)y_2 - y_1$		✓
$y_1' = y_2$ $y_2' = 1000(1 - y_1^2)y_2 - y_1$	✓	
Stiff equation	✓	
Reactions with fast transients	✓	

Maximum marks: 2

8 Argumentation2_20230819

Specify the more suitable method (column) for each application (row).

Please match the values:

	Stochastic method	Deterministic method
ODE		✓
Solution is continuous (concentration, velocity, ...)		✓
10D Integral	✓	
Scenarios in epidemic models with limited number of individuals	✓	
2D integral		✓
Stochastic differential equation (SDE)	✓	
Solution is discrete (individuals, number of molecules, ...)	✓	

Maximum marks: 2

9 Grade45_1_20230819

Consider the following linear system of ODEs:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad t > 0,$$

with initial conditions $y_1(0) = 5$, $y_2(0) = 2$. Apply the forward Euler's method with a steplength of h to solve this ODE, and determine the permissible range of values for h that guarantee the absolute stability of the method for this particular ODE.

Fill in your answer here or write in the answer sheet and hand it in.

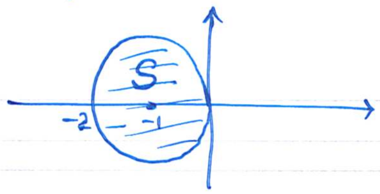
The Euler's method reads as $y_{k+1} = y_k + h f(t_k, y_k)$

For the above system then we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{k+1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k + h \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_k$$

where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

The Euler's method is absolutely stable for a system $y' = Ay$ if all eigenvalues of A times h fall into the stability region of the scalar Euler's method, i.e. inside the circle of radius 1 centered at $(-1, 0)$:



$\boxed{\lambda_k h \in S}$

Words: 0

Maximum marks: 10

To answer this question we must compute the eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. If we set $|A - \lambda I| = 0$, $\det \begin{bmatrix} -2-\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = 0$

then we have $\lambda^2 + 3\lambda = 0$ which gives $\lambda_1 = 0$, $\lambda_2 = -3$.

for stability we must have $\lambda_1 h \in S$ and $\lambda_2 h \in S$.

$\lambda_1 h = 0$ always belongs to S, and $\lambda_2 h \in S$ gives $-3h \in S$ or

$-2 \leq -3h \leq 0 \Rightarrow 0 \leq h \leq \frac{2}{3}$.

10 Grade45_2_20230819

Assume that you are running a lumber mill in Krokom and you are trying to **estimate the production price** of a single piece of your standard framing timber. The cost to produce your standard framing timber includes labor, energy and trees. Assume the cost of labor is constant at 3 SEK per piece of framing timber. The cost of energy needed to make a single piece of framing timber is normally distributed with mean $\mu_E = 0.5$ SEK and standard deviation $\sigma_E = 0.1$ SEK. The price of tree needed to make a single piece of timber is distributed according to Weibull pdf $f_T(x) = 5x^4 \exp(-x^5)$ for $x \in [0, \infty)$.

Design a Monte Carlo algorithm to estimate the mean production price p_{mean} and variance $p_{variance}$ of standard framing timber.

Assume the function `randn()` exists and that it returns one standard normal number (with mean 0 and variance 1) every time that it is called. However, it is necessary to provide a detailed formulation of how we can generate a random number from the given Weibull distribution and incorporate it into the Monte Carlo algorithm.

Fill in your answer here or write in the answer sheet and hand it in.

Labor cost is fixed at 3 SEK. Energy cost is sampled from $N(0.5, 0.01)$. We sample from standard normal distribution and use the fact that if z is a standard normal variable the $x = \mu + \sigma z$ is $N(\mu, \sigma^2)$.

For the tree cost, we use Inverse Transform Sampling to generate from the Weibull distribution:

1. Find cdf $F(x) = \int_0^x f_T(t)\,dt = \int_0^x 5t^4 e^{-t^5}\,dt = 1 - e^{-x^5}$
2. Find $F^{-1}(x)$ by setting $y = F(x)$, switch places of x and y and solve for x: $x = F(y) = 1 - e^{-y^5} \Rightarrow y^5 = \ln \frac{1}{1-x} \Rightarrow y = \sqrt[5]{\ln \frac{1}{1-x}} = F^{-1}(x)$
3. Generate $u \sim U(0,1)$
4. Set $X = F^{-1}(u)$

Words: 0

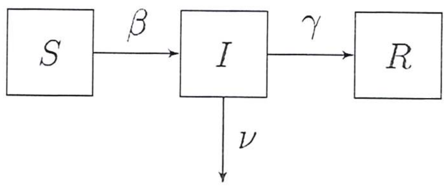
Maximum marks: 10

The MC algorithm is:

```
total_price = 0,  labor_cost = 3
for i = 1 : N
    u = rand()
    tree_cost = (ln(1/(1-u)))^(1/5)
    energy_cost = 0.5 + 0.1 · randn()
    total_price = total_price + (labor_cost + tree_cost + energy_cost)
end
Pmean = mean(total_price);   Pvariance = std(total_price)^2.
```

11 Grade45_3_20230819

Consider the following SIR diagram for spread of a virus within a population. The population is divided into three groups S, I and R standing for **Susceptible**, **Infected** and **Recovered** individuals, respectively. Here, β is the infection rate, γ is the recovery rate, and ν is the pathogen-induced mortality rate (death due to the virus).



Furthermore, we assume that the infection rate depends on the proportion of infected individuals relative to the entire population, while the recovery and mortality rates are constants.

Part (1): By assuming that S, I and R are continuous and deterministic variables, write down a system of ODEs with initial conditions that governs this phenomenon. Then write a Matlab code for solving and plotting the solutions of the resulting ODE in time interval $[0, T]$, utilizing a suitable build in function (ODE solver), provided that all constant rates and initial values are given. It should be a detailed and executable program with justification for the choice of methods, e.g. why you choose a particular ODE solver.

Part (2): Consider a scenario where S, I and R are integer and stochastic variables. Write down all processes (reactions), propensity functions, probabilities, and state-change (stoichiometry) vectors pertaining to the model. Finally, provide a Matlab code that implements the Gillespies (SSA) algorithm for solving the model iteratively, and plot the solutions. Assume that the SSA subroutine has been given. It should be a detailed and executable program.

Note: minor errors in codes are acceptable.

Fill in your answer here or write in the answer sheet and hand it in.

Part (1):

Since the infection rate depends on the ratio of infected individuals and entire population, it is $\beta \frac{I}{S+I+R} = \beta \frac{I}{N}$. the system of odes is:

$$\begin{cases} \frac{dS}{dt} = -\beta \frac{I}{N} S \\ \frac{dI}{dt} = \beta \frac{I}{N} S - \nu I - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

with intial conditions $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$.

A matlab function for solving this ODE:

```
function [t, S, I, R] = SIR_determistic(S0, I0, R0, T, beta, nu, gamma)

    Initial = [S0; I0; R0];
    SIR_func = @(t,y) [-bata/sum(y) *y(1) *y(2);
                        beta/sum(y) *y(1) *y(2) - (nu+gamma)*y(2);
                        gamma*y(2)];

    [t,y] = ode45(SIR_func, [0,T], Initial);
    S = y(:,1); I = y(:,2); R = y(:,3);
    plot(t, S, '-k', t, I, '-b', t, R, '-r');
end% function
```

the system is not stiff, so ode45 is an appropriate function to solve it.

Part (2):

The processes (reactions) are (propensities are written on the arrows)

(1) $S \xrightarrow{\beta \frac{I}{N} S} I$

(2) $I \xrightarrow{\nu I} \phi$

(3) $I \xrightarrow{\gamma I} R$

Propensity functions are $\omega_1 = \beta \frac{I}{N} S$, $\omega_2 = \nu I$, $\omega_3 = \gamma I$

and if we define $a = \omega_1 + \omega_2 + \omega_2 = \beta \frac{I}{N} S + \nu I + \gamma I$ then probabilities are $P_1 = \frac{\omega_1}{a}$, $P_2 = \frac{\omega_2}{a}$, $P_3 = \frac{\omega_3}{a}$

In each time step to determine which reaction must occur, we sample from discrete distribution

m	1	2	3
P_k	P_1	P_2	P_3

and to know when the first reaction will occur we sample from exponential distribution with mean $\frac{1}{a(y)}$; i.e

$$\tau \sim Exp(a(y)).$$

Finally to update the state the state change vectors

$$v_1 = [-1, +1, 0]$$
$$v_2 = [0, -1, 0]$$
$$v_3 = [0, -1, +1]$$

are used.

the Malab code:

```
function [time, S, I, R] = SIR_stochastic(S0, I0, R0, T, beta, nu, gamma)
    Initial = [S0; I0; R0];
    N = 1000; hold on;
    for k = 1:N
        [t, s, i, r] = SSA(Initial, T, beta, nu, gamma);
        plot(t, s, 'k-', t, i, '-b', t, r, '-r');
        time{k} = t;  S{k} = s;  I{k} = i;  R{k} = r;
    end % for
end % function
```