Assignments

Question	Question title	Marks	Question type
i	Information		Information or resources
1	Wellposed (1 point)	1	Multiple Response
2	Wellposed (1 point)	1	Multiple Response
3	Iterative (1 point)	1	Inline Gap Match
4	FEM (1 point)	1	Inline Gap Match
5	Compare (1 point)	1	Multiple Response
6	Stability (1 point)	1	Inline Gap Match
7	Implementation (4 points)	4	Numeric Entry
8	Stability (4 points)	4	Multiple Response
9	Iterative (1 point)	1	Multiple Choice
10	Wellposed (3 points)	3	Multiple Response

i Information

This exam consists of 10 questions. Maximum number of points is 18. More advanced questions are worth 3 or 4 points. More basic questions are worth 1 point.

Grade requirements

- Grade 3: At least 7 points.
- Grade 4: At least 11 points.
- Grade 5: At least 14 points

You are allowed the following aids

- Two documents with formulas (pdf), available as a resource at the bottom of Inspera.
- Online python compiler (Programiz), available as a resource at the bottom of Inspera.
- NumPy documentation, available as a resource at the bottom of Inspera.
- Beta Mathematics Handbook
- Physics handbook
- Calculator
- Pen and paper

1 Wellposed (1 point)

For each IBVP below, determine whether it is well posed.

To score 1 point on this problem, you need 3/3 correct answers.

IBVP 1

$$\begin{cases} u_t + u_x = 0, & 0 < x < 1, \quad t > 0 \\ u(1,t) = 1, & t > 0 \\ u(x,0) = \sin(x), & 0 \le x \le 1 \end{cases}$$

Is IBVP 1 well posed?

- [ ] Well posed
- [ ] Not well posed ✔

IBVP 2

$$\begin{cases} u_t + u_x = \sin(x)\cos(t), & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, & t > 0 \\ u(x,0) = \sin(x), & 0 < x < 1 \end{cases}$$

Is IBVP 2 well posed?

- [ ] Well posed ✔
- [ ] Not well posed

IBVP 3

$$\begin{cases} u_t + u_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, & t > 0 \\ u(1,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x), & 0 \le x \le 1 \end{cases}$$

Is IBVP 3 well posed?

- [ ] Well posed
- [ ] Not well posed ✔

Maximum marks: 1

2 Wellposed (1 point)

Consider the following PDE and boundary conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in (0, L), \quad t > 0 \\ u = 0, & x = 0, \quad t > 0 \\ c^2 u_x = -\alpha u_t, & x = L, \quad t > 0 \end{cases}$$

where $c > 0$ and $\alpha > 0$ are real constants. We assume that the solution is real.

The initial data are assumed to be smooth functions that are compatible with the boundary conditions, but are otherwise considered unknown.

Which of the following relations does the solution $u(x, t)$ satisfy for **any** smooth and compatible initial data? Select all correct alternatives. Zero, one, or more than one alternative may be correct!

To score 1 point on this problem, you need to select **all** correct alternatives and **no** incorrect alternatives.

Select zero or more alternatives:

- ☐ $||u||^2 = 0$
- ☐ $||u||^2 = -\alpha u_t(L, t)^2$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u||^2 \right) = 0$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u||^2 \right) = -\alpha u_t(L, t)^2$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u_t||^2 + c^2 ||u_x||^2 \right) = 0$
- ☐ $\frac{1}{2} \frac{d}{dt} \left(||u_t||^2 + c^2 ||u_x||^2 \right) = -\alpha u_t(L, t)^2$ ✔
- ☐ $||u_t||^2 = 0$
- ☐ $||u_t||^2 = -\alpha u_t(L, t)^2$

Maximum marks: 1

3 Iterative (1 point)

Consider the time-independent advection-diffusion equation,

$$cu_x - \epsilon u_{xx} = F,$$

where $c > 0$ and $\epsilon > 0$ are real constants and $F = F(x)$ is a forcing function. Discretizing using the finite element method with piecewise linear basis functions leads to

$$B\text{\boldsymbol}\xi = \mathbf{F},$$

which is a linear system of equations for $\text{\boldsymbol}\xi$. The right-hand side $\mathbf{F}$ is an $n \times 1$ vector that depends on F. The matrix B is $n \times n$ and satisfies

$$B = cD + \epsilon A,$$

where

$$A = 1/h \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \end{bmatrix}.$$

We are interested in solving the advection-diffusion equation for many different forcing functions F. This will lead to a sequence of linear systems to solve, where B remains the same and $\mathbf{F}$ varies:

$$B\text{\boldsymbol}\xi_i = \mathbf{F}_i, \quad i = 1, 2, \ldots.$$

For this sequence of systems, which of the following three solution methods is the most suitable?

1. Gaussian elimination,
2. LU factorization,
3. The conjugate gradient method.

Rank the three methods. Assume that n is of order 10^5.

To score 1 point, you need to rank all three methods correctly.

In the next problem, you will be asked to motivate your ranking.

Help

LU factorization | Gaussian elimination | The conjugate gradient method

Most suitable: LU factorization ✔

Second-most suitable: Gaussian elimination ✔

Least suitable: The conjugate gradient meth ✔

Maximum marks: 1

4 FEM (1 point)

Error displaying question "FEM (1 point)".
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Maximum marks: 1

5 Compare (1 point)

For each PDE below, select the spatial discretization method that would be the **most suitable**. Consider both how easy/difficult it would be to implement the method and what execution time would be required to provide a numerical solution with an error below the prescribed error tolerance.

To score 1 point, you must answer all three questions correctly.

<u>**PDE 1**</u>
$u_t + u_x = 0, \quad 0 < x < 1, \quad t > 0$
Relative error tolerance: 10^{-6}.
Which method is best for PDE 1?

- [ ] A fourth order finite difference method ✔
- [ ] The finite element method with piecewise linear basis functions

<u>**PDE 2**</u>
$\nabla \cdot a\nabla u = F, \quad (x, y) \in \Omega$
where $a = a(x, y)$ and $F = F(x, y)$ are known functions.
Relative error tolerance: 10^{-2}.
Which method is best for PDE 2?

- [ ] A sixth order finite difference method
- [ ] The finite element method with piecewise linear basis functions ✔

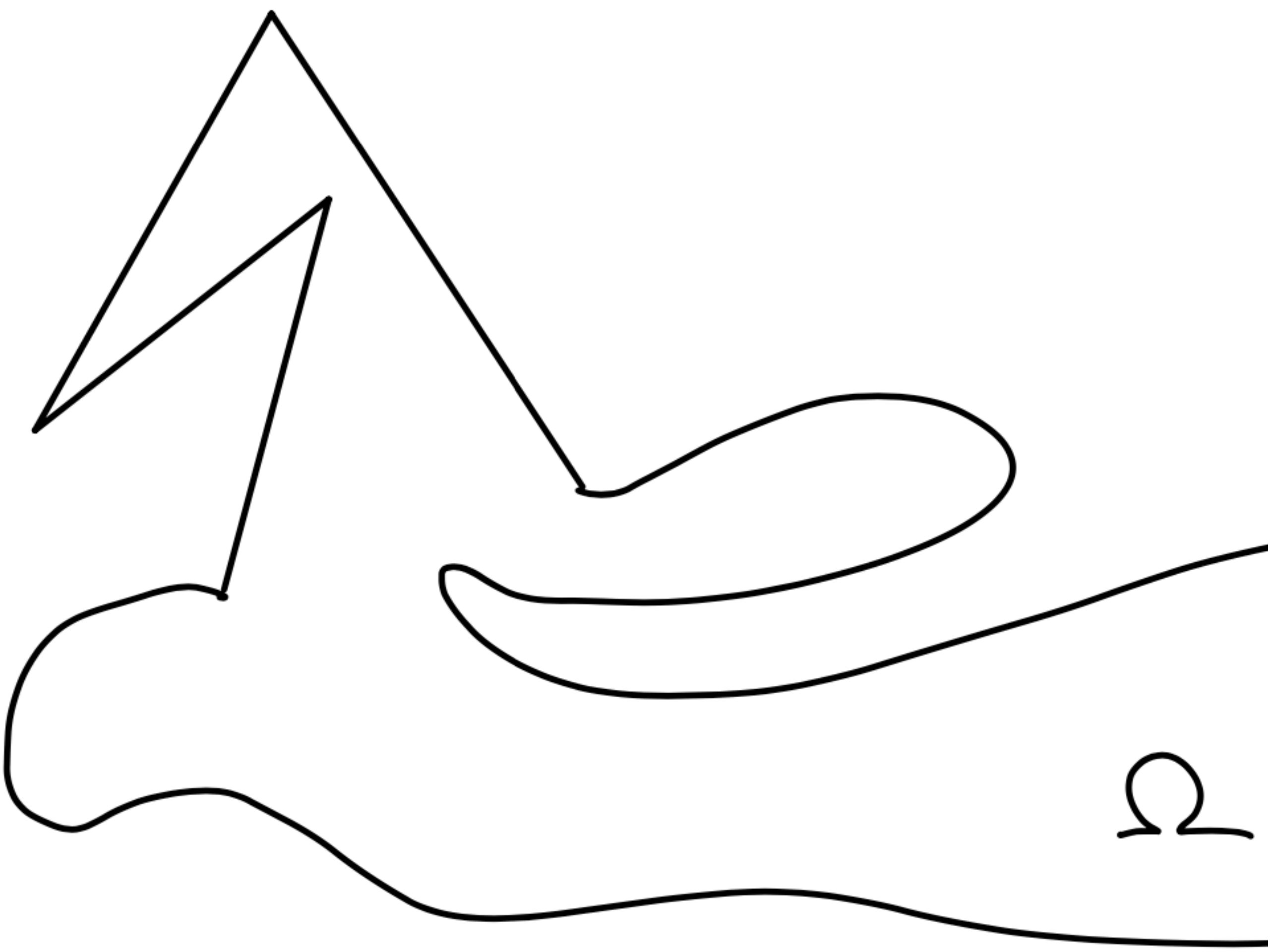

Figure: The domain Ω for PDE 2.

<u>**PDE 3**</u>
$u_t + i(u_{xx} + u_{yy}) = 0, \quad 0 < x < 1, \ 0 < y < 1, \ t > 0$

Relative error tolerance: 10^{-6}

Which method is best for PDE 3?

- [ ] A sixth order finite difference method ✔
- [ ] The finite element method with piecewise linear basis functions

Maximum marks: 1

6 Stability (1 point)

You are trying to discretize the heat equation,

$$u_t = u_{xx}, \quad x \in (0, L)$$

with the SBP-SAT method. There are three different sets of well-posed boundary conditions (BC) to consider. The BC at the left boundary ($x = 0$) has already been imposed correctly. Your task is to select **consistent and stable** SATs for the right boundary ($x = L$).

The SBP operator D_2 is defined as in the formula sheet that is available as a resource.

BC 1

$u(0, t) = 0$

$u_x(L, t) = 0$

Discretization of BC 1

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT1$

BC 2

$u(0, t) = 0$

$u(L, t) = 0$

Discretization of BC 2

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT2$

BC 3

$u(0, t) = 0$

$u_x(L, t) + \alpha u(L, t) = 0$

where $\alpha > 0$ is a real scalar.

Discretization of BC 3

$\mathbf{u}_t = D_2\mathbf{u} + H^{-1}\mathbf{d}_\ell(\mathbf{e}_\ell^T\mathbf{u} - 0) + SAT3$

What should SAT1, SAT2 and SAT3 be for the semi-discrete approximations to be **consistent and stable**? Drag and drop boxes below.

To score 1 point, you need to get all three answers right.

Help

$H^{-1}\mathbf{e}_r(D_1\mathbf{u} - 0)$

$H^{-1}\mathbf{e}_r(\mathbf{e}_r^T\mathbf{u} + \alpha\mathbf{d}_r^T\mathbf{u} - 0)$

$-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} - 0)$

$-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} + \alpha\mathbf{e}_r^T\mathbf{u} - 0)$

$-H^{-1}\mathbf{d}_r(\mathbf{e}_r^T\mathbf{u} - 0)$

SAT 1: $-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} - 0)$ ✔

SAT 2: $-H^{-1}\mathbf{d}_r(\mathbf{e}_r^T\mathbf{u} - 0)$ ✔

SAT 3: $-H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\mathbf{u} + \alpha\mathbf{e}_r^T\mathbf{u} - 0)$ ✔ .

Maximum marks: 1

7 Implementation (4 points)

Consider the following initial boundary-value problem

$$
\begin{array}{lll}
u_t = a\,u_x + b\,u_{xx}, & -1 \le x \le 1, & t \ge 0,\\
u = 0, & x = -1, & t \ge 0,\\
a\,u + 2\,bu_x = 0, & x = 1, & t \ge 0,\\
u = f(x), & -1 \le x \le 1, & t = 0,
\end{array}
$$

We discretize this using the SBP-SAT method

$$
\begin{array}{ll}
v_t = aD_1 v + bD_2 v + bH^{-1} d_l e_l^T v + \frac{a}{2} H^{-1} e_l e_l^T v - \frac{1}{2} H^{-1} e_r (a e_r^T + 2b d_r^T) v, & t \ge 0,\\
v = f, & t = 0,
\end{array}
$$

As initial data we use $f(x) = \exp\left(-(6x)^2\right)$ and set number of grid-points to 101. Use the 4th order accurate SBP operators (the SBP operators are found under "resurs") and time-integrate the solution using RK4 until $t = 1$ with time-step $k = 8 \cdot 10^{-5}$

When a=0 and b=0.5, the discrete l_2-norm of the solution (at t=0.4, with time step $k = 8 \cdot 10^{-5}$) is 0.196. This is something you can verify against your implementation.

The definition of the discrete l_2-norm: $\|v\|_h = \sqrt{h}\sqrt{\sum_{i=0}^{m-1} |v_i|^2}$

You are here asked to compute the discrete l_2-norm of the solution at t=0.7, when a=-1 and b=0.1. Give the answer as a decimal number (2 digits):

(0.25 - 0.27) .

Maximum marks: 4

8 Stability (4 points)

Consider the wave equation on a bounded domain,
$u_{tt} = u_{xx}\ , x \in [0,\ 1],\ t \geq 0$

A well-posed initial boundary value problem requires two boundary (and two initial) conditions. Someone implemented 5 different SBP-SAT approximations. Your job is to choose which of these lead to stable approximations.

Select one or more alternatives:

- $v_{tt} = D_2 v + H^{-1} e_l^T (e_l^T v_t + e_l^T v + d_l^T v) - H^{-1} e_r^T (e_r^T v_t + e_r^T v + d_r^T v)$
- $v_{tt} = D_2 v + H^{-1} e_l^T (d_l^T v) - H^{-1} e_r^T (d_r^T v)$ ✔
- $v_{tt} = D_2 v - H^{-1} e_l^T (e_l^T v_t + e_l^T v - d_l^T v) - H^{-1} e_r^T (e_r^T v_t + e_r^T v + d_r^T v)$ ✔
- $v_{tt} = D_2 v + H^{-1} e_r^T (e_r^T v) - H^{-1} e_r^T (e_r^T v_t + e_r^T v + d_r^T v)$
- $v_{tt} = D_2 v + H^{-1} e_l^T (d_l^T v) + H^{-1} e_r^T (d_r^T v)$

Maximum marks: 4

9 Iterative (1 point)

Consider a linear equation system $Ax = b$ where,

$$A = \begin{bmatrix} 6 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

The spectral radius of the Gauss-Seidel iteration matrix is

Select one alternative:

- 0.14
- 1.32
- 0.82 ✔
- 1.12
- 0.53

Maximum marks: 1

10 Wellposed (3 points)

Consider the following PDE and initial condition:

$$\begin{aligned} &u_t = u_{xx} - u_x, \quad 0 \le x \le 1, \quad t \ge 0\ , \\ &u = f, \qquad\qquad\quad 0 \le x \le 1, \quad t = 0\ . \end{aligned}$$

We assume that the solution is real.

Which of the following choices of boundary conditions lead to a well-posed initial-boundary value problem?

Select one or more alternatives:

- $\begin{cases} u = 0, & x = 0\ , \\ u_x - u = 0, & x = 1\ . \end{cases}$
- $\begin{cases} u_x = 0, & x = 0\ , \\ u_x = 0, & x = 1\ . \end{cases}$
- $\begin{cases} u_x - u = 0, & x = 0\ , \\ u_x + u = 0, & x = 1\ . \end{cases}$ ✔
- $\begin{cases} 2u_x - u = 0, & x = 0\ , \\ u_x = 0, & x = 1\ . \end{cases}$ ✔
- $\begin{cases} u = 0, & x = 0\ , \\ u = 0, & x = 1\ . \end{cases}$ ✔

Maximum marks: 3