UPPSALA UNIVERSITET
Matematiska Institutionen
Topias Tolonen-Weckström

Assignment 2 solutions
1MA255, VT25
2025/05/12

Partial Differential Equations with Applications to Finance

Instructions: There are three problems worth a total of 24 points. A score 12 points yield 2 bonus points. Your answer should be well motivated in order to receive full credit in each question. Collaboration is encouraged. The solutions should be submitted by 21 May to Studium or via email to me.

---

1. (8 points). Consider a geometric Brownian motion X_t

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t$$

with $X_0 = x_0$, μ and σ are constants (> 0) and W_t is the standard Brownian motion.

(i) Write down the Kolmogorov forward equation (also known as the Fokker-Planck or the KFE) satisfied by $\rho(t, x)$, the probability density function of X_t.

(ii) Denote the $n-$th moment of X_t by $M_n(t)$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} M_1(t) = \mu M_1(t),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} M_n(t) = \left(\mu n + \frac{\sigma^2}{2} n(n-1)\right) M_n(t), \quad n \geq 2.$$

You may assume that the density function decays sufficiently fast at infinity.

Solution. *(i) The purpose of the problem is to use the Fokker-Planck for probability densities:*

$$p_t(t, x) - \mathcal{L}^* p(t, x) = 0,$$

where the adjoint operator $\mathcal{L}^*$ for a geometric Brownian motion can be calculated as

$$(\mathcal{L}^* p)(t, x) = -\frac{\partial}{\partial x}(\mu x p) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma^2 x^2 p).$$

A straightforward calculation then yields

$$\frac{\partial}{\partial x}(\mu x p) = \mu p + \mu x p_x$$

and

$$\frac{\partial^2}{\partial x^2}(\sigma^2 x^2 p) = \sigma^2 \left(2p + 4x p_x + x^2 p_{xx}\right).$$

Combining these with the Fokker-Planck then yields

$$p_t + (\mu - \sigma^2) p + (\mu - 2\sigma^2) x p_x - \frac{1}{2} \sigma^2 x^2 p_{xx} = 0.$$

(ii) *Both of the claims could be very straightforward to show using the notion of $M_1 = \mathbb{E}$, and then utilizing Itô's lemma and the fundamental theorem of calculus, where applicable. However, I present the solution using the Fokker-Planck. From the definition of the first moment, we get*

$$M_1(t) = \int_{\mathbb{R}} xp(t,x)\,\mathrm{d}x.$$

Differentiating with respect to t and combining with the equality derived in (i) then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(t) = \int_{\mathbb{R}} x\left(-(\mu-\sigma^2)p - (\mu-2\sigma^2)xp_x + \frac{1}{2}\sigma^2x^2p_{xx}\right)\,\mathrm{d}x.$$

Integrating the first term of the integrand yields (by definition)

$$\int_{\mathbb{R}} x\left(-(\mu-\sigma^2)p\right)\,\mathrm{d}x = -(\mu-\sigma^2)M_1(t).$$

For the second term, integration by parts gives

$$-(\mu-2\sigma^2)\left(\int_{\mathbb{R}} x^2p_x\,\mathrm{d}x\right) = -(\mu-2\sigma^2)\left(\left[x^2p\right]_{\mathbb{R}} - \int_{\mathbb{R}} 2xp\,\mathrm{d}x\right) = -(\mu-2\sigma^2)(-2M_1(t)).$$

Similarly, by double integration by parts, the third and the last term yields

$$\frac{1}{2}\sigma^2\int_{\mathbb{R}} x^3p_{xx}\,\mathrm{d}x = \frac{1}{2}\sigma^2\left(\left[x^3p_x - 3x^2p\right]_{\mathbb{R}} + 6M_1(t)\right),$$

and moreover, $\left[x^3p_x - 3x^2p\right]_{\mathbb{R}} = 0$ by the assumption that p decays fast near infinity. Finally, putting the Frankenstein together yields

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(t) = -(\mu-\sigma^2)M_1(t) - (\mu-2\sigma^2)(-2M_1(t)) + \frac{1}{2}\sigma^2 6M_1(t) = \mu M_1(t).$$

The second part follows a similar notion after noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_n(t) = \int_{\mathbb{R}} x^n p\,\mathrm{d}x.$$

Indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_n(t) = \int \mathbb{R}x^n\left(-(\mu-\sigma^2)p - (\mu-2\sigma^2)xp_x + \frac{1}{2}\sigma^2x^2p_{xx}\right)\,\mathrm{d}x.$$

Again, the first term yields straightforwardly

$$\int_{\mathbb{R}} -x^n(\mu-\sigma^2)p\,\mathrm{d}x = -(\mu-\sigma^2)M_n(t).$$

Reiterating the previous part, for the second term, integration by parts yields

$$\int_{\mathbb{R}} -x^{n+1}(\mu-2\sigma^2)p_x\,\mathrm{d}x = (\mu-2\sigma^2)(n+1)M_n(t),$$

and similarly

$$\int_{\mathbb{R}} x^{n+2}\frac{1}{2}\sigma^2p_{xx}\,\mathrm{d}x = \frac{1}{2}\sigma^2(n+1)(n+2)M_n(t).$$

The result follows by combining the three integrals.

2. (8 points). Solve the problem of minimizing

$$\mathbb{E}_x \left[\exp \left\{ \int_0^T u_t^2 \,\mathrm{d}t + X_T^2 \right\} \right]$$

given the dynamics

$$\mathrm{d}X_t = u_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t,$$

where W_t is the standard Brownian motion, and the control u_t is Markovian and there are no control constraints. *(Hint: Make the ansatz*

$$V(t,x) = e^{A(t)x^2 + B(t)}$$

and use the so-called exponential utility criterion[1]*).*

Solution. *Let*

$$V(t,x) = \inf_u \mathbb{E}_{t,x} \left[\exp\{ \int_t^T u_s^2 \,\mathrm{d}s + X_T^2 \} \right]$$

for some admissible set of controls. Let $\hat{V}$ be our candidate value function. Then, according to the exponential utility criterion (which is a form of HJB), we have

$$\begin{cases} \hat{V}_t + \inf_u \{ \hat{V} u^2 + \mathcal{L}\hat{V} \} = 0, \\ \hat{V}(T,x) = e^{x^2}, \end{cases}$$

where $\mathcal{L}\hat{V} = u\hat{V}_x + \frac{1}{2}\sigma^2 \hat{V}_{xx}$.

Following the hint, we use an Ansatz

$$\hat{V}(t,x) = \exp\left\{ A(t)x^2 + B(t) \right\}$$

for some functions A, B, which will be solved later. Then we have

$$\begin{cases} \hat{V}_t = (A_t x^2 + B_t)\hat{V}, \\ \hat{V}_x = 2Ax\hat{V}, \qquad \textit{and} \\ \hat{V}_{xx} = 2\hat{V}(2 + 2A^2x^2). \end{cases}$$

Moreover, we wish to find a u that minimizes $\hat{V}u^2 + \mathcal{L}\hat{V}$. From the first-order condition, we find

$$u^*(t,x) := -A(t)x.$$

Plugging in u^* to the HJB and rearranging, we arrive at

$$\left(A_t x^2 + B_t\right)\hat{V} + \left(A^2x^2 - 2A^2x^2 + \sigma^2(A + 2A^2x^2)\right)\hat{V} = 0.$$

This is true if and only if the two ODES with terminal conditions hold:

$$\begin{cases} A_t + (2\sigma^2 - 1)A^2 = 0, \\ A(T) = 1. \end{cases} \qquad \begin{cases} B_t + \sigma^2 A = 0, \\ B(T) = 0. \end{cases}$$

---

[1] For

$$V(t,x) = \mathbb{E}_{t,x}\left[\exp\left\{ \int_t^T \Psi(s, X_s^\alpha, \alpha_s)\,\mathrm{d}s + \Phi(X_T^\alpha) \right\} \right]$$

we know that V satisfies

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \sup_\alpha \{ V(t,x)\Psi(t,x,\alpha) + \quad \mathcal{L}^\alpha V(t,x) \} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = e^{\Phi(x)}. \end{cases}$$

For convenience's sake, let $k^* = 1 - 2\sigma^2$. Then, after moderate amount of calculations, the two ODEs yield

$$A(t) = \frac{1}{1 + k^*(T-t)}, \quad B(t) = \frac{\sigma^2 \log\{k^*(T-t) + 1\}}{k^*}.$$

We conclude that

$$\hat{V}(0,x) = \exp\{A(0)x^2 + B(0)\} = \exp\left\{\frac{x^2}{1 + (1-2\sigma^2)T} + \frac{\sigma^2}{1 - 2\sigma^2 \log\{(1-2\sigma^2)T + 1\}}\right\},$$

and

$$u^*(t,x) = -A(t)x = -\frac{x}{1 + (1-2\sigma^2)(T-t)}.$$

By the Verification theorem, $V \equiv \hat{V}$ must hold and u^* is the optimal control.

3. (8 points). Solve the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-\beta\tau} W_\tau^2],$$

where W_t is a standard Brownian motion. *(Hint: at some point, identity $e^x + e^{-x} = 2\cosh x$ might be useful.)*

Solution. *Here, based on our previous life experiences and/or lectures, we expect a continuation and stopping regions*

$$C := (-b, b), \quad D := \mathbb{R} \setminus C,$$

respectively, for some boundary b determined later. Moreover, we expec the optimal stopping rule to be of the form

$$\tau^* := \inf\{t \geq 0 | X_t \in D\}.$$

This is an infinite-horizon optimal stopping stopping problem (with discounting), so the candidate value function $\hat{V}$ should solve the following system of equations:

$$\begin{cases} \frac{1}{2}\hat{V}_{xx} - \beta\hat{V} = 0, & \text{("the PDE")} \\ \hat{V}(-b) = \hat{V}(b) = b^2, & \text{("continuous fit")} \\ \hat{V}_x(b) = 2b. & \text{("smooth fit")} \end{cases}$$

The PDE suggests that an ansatz of the form $e^{\gamma x}$ is useful for some constant γ. Indeed, plugging this in to the PDE yields

$$-\beta e^{\gamma x} + \frac{1}{2}\gamma^2 e^{\gamma x} = 0,$$

from which one solves $\gamma_1 := \sqrt{2\beta}$ and $\gamma_2 := -\sqrt{2\beta}$. That is, the general solution is of the form

$$\hat{V}(x) = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x}.$$

The continuous fit condition then suggests that $C_1 = C_2$ must hold. We denote $C = C_1 = C_2$, which results in

$$\hat{V}(b) = C\left(e^{\sqrt{2\beta}b} + e^{-\sqrt{2\beta}b}\right) = b^2$$

at $x = b$, and has an identity

$$\hat{V}(b) = 2C\cosh\sqrt{2\beta}b = b^2.$$

Moreover, the smooth fit condition suggests

$$\hat{V}_x(b) = 2C\sqrt{2\beta}\sinh\sqrt{2\beta}b = 2b,$$

from which we solve the optimal boundary

$$b^* := b = \sqrt{\frac{8}{\beta}}\left(\tanh\sqrt{2\beta}b\right)^{-1}.$$

That is, for the value function $\hat{V}$ *inside the continuous region we obtain*

$$\hat{V}(x) = \frac{(b^*)^2}{\cosh\sqrt{2\beta}b^*}\cosh\sqrt{2\beta}x,$$

and $\hat{V}(x) = x^2$ *in* D*.*

Finally, we conclude that by the Verification theorem, $\hat{V} \equiv V$ *and* τ^* *is an optimal stopping time.*