UPPSALA UNIVERSITET

LECTURE NOTES

Complex Analysis

Rami Abou Zahra

Inlämningsdatum
January 25, 2023

Contents

1. Intro

In this course, we shall study functions $f : \mathbb{C} \to \mathbb{C}$ (or more generally, $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$)

Definition/Sats 1.1: Complex Number

A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$

Anmärkning:
The number x is called the *real part* ($\mathrm{Re}(z) = x$) of the complex number, and y is called the *imaginary part* ($\mathrm{Im}(z) = y$) of the complex number

Anmärkning:
The set of all complex numbers is denoted by $\mathbb{C}$

Anmärkning:
$\mathbb{C}$ is the *smallest* field extension to $\mathbb{R}$ that is algebraically closed.

Anmärkning:
$i^2 = -1$

1.1. Operations over $\mathbb{C}$.

We define the operations *addition* and *multiplication* of two complex unmebrs as follows:

Definition/Sats 1.2: Addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Definition/Sats 1.3: Multiplication of complex numbers

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations, $\mathbb{C}$ forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Definition/Sats 1.4: Complex conjugate

The *complex conjugate* of a complex number $z = x + iy$, denoted by $\overline{z}$, is defined by $\overline{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\dfrac{z_1}{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$
- $\overline{\overline{z}} = z$
- $z \cdot \overline{z} = |z|^2$
- $z^-1 = \dfrac{\overline{z}}{|z|^2}$
- $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$

Anmärkning:
$\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$
$\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$

Anmärkning:
Multiplication by i is simply rotation by $\dfrac{\pi}{2}$ counterclockwise.

Definition/Sats 1.5

Let $z \in \mathbb{C}$. Then there eixsts a $w \in \mathbb{C}$ such that $w^2 = z$ (where $-w$ also satisfies this equation)

Bevis 1.1

Let $z = a + bi$ and $w = x + iy$ such that $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then $a = x^2 - y^2$ and $b = 2xy$
We also know that $|z| = a^2 + b^2 = \left|x^2 + y^2\right|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore, $x^2 + y^2 = \sqrt{a^2 + b^2}$ and:

$$\left.\begin{aligned} x^2 - y^2 &= a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned}\right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left.\begin{aligned} -x^2 + y^2 &= -a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned}\right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let $\alpha = \sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}}$ and $\beta = \sqrt{\dfrac{-a + \sqrt{a^2 + b^2}}{2}}$ and let $\sqrt{\ }$ denote the positive square rroot of positive real numbers.

If b is positive, then either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$
If b is negative, then either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$

Therefore, the equation has solutions $\pm(\alpha + \mu\beta i)$ where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$

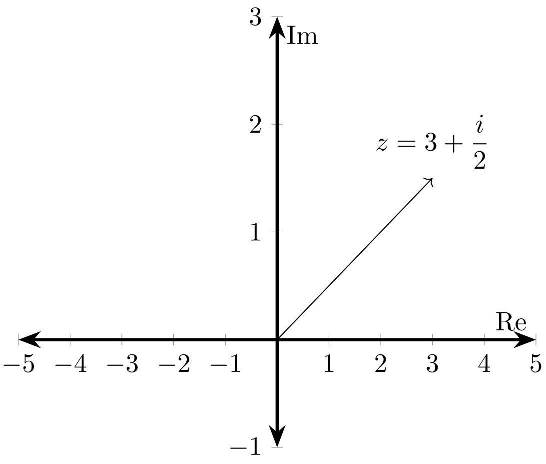
$\square$

Anmärkning:
From the proof above, we can conclude the following:

- The square roots of a complex number are real $\Leftrightarrow$ the complex number is real and positive
- The square roots of a complex number are purely imaginary $\Leftrightarrow$ the complex number is real and negative
- The two square roots of a number coincide $\Leftrightarrow$ the complex number is zero

1.2. **Cartesian representation.**

It is natural to represent a complex number $z = x + iy$ as a tuple (x, y), and we can therefore represent it in the standard cartesian plane:



Anmärkning:
This is sometimes called the *complex plane*

Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number $z = x+iy$ (geometrically the length of the vector), denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:
- $|z|^2 = z \cdot \overline{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Anmärkning:
Every $z \in \mathbb{C}$ such that $z \neq 0$ (that is, $x \neq 0$ or $y \neq 0$) has a multiplcative inverse $\frac{1}{z}$ given by:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Definition/Sats 1.7: Triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that $|z_1 + z_2| \leq |z_1| + |z_2|$

Lemma 1.1: Reversed triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Bevis 1.2

$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

So that $|z_1| - |z_2| \leq |z_1 - z_2|$ $\square$

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $-|z| \leq \mathrm{Re}(z) \leq |z|$
- $-|z| \leq \mathrm{Im}(z) \leq |z|$
- $|\overline{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \cdots + z_n w_n| \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \cdots + |w_n|^2}$

1.3. Polar form.

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, and hence there exists θ such that:

$$\frac{x}{|z|} = \cos(\theta) \qquad \frac{y}{|z|} = \sin(\theta)$$

Therefore $z = x + iy$ can be written as:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Where $r = |z|$ is uniquely determined by z, while θ is 2π-periodic. This is called the *polar form* of z and just as the cartesian representation requires a tuple of information $(|z|, \theta)$

Definition/Sats 1.8: Argument

The *argument* of a complex number z, denoted by $\arg(z)$, is the angle θ between z and the real number line in the complex plane

Anmärkning:
Since the argument is 2π periodic, the angle is usually given as $\theta + k2\pi \quad k \in \mathbb{Z}$, but we are only intersted in θ
This θ is called the *principal value* of $\arg(z)$, denoted by $\mathrm{Arg}(z)$ and belongs to $(-\pi, \pi]$

Anmärkning:
We are always allowed to change an angle by multiples of 2π, the principal value argument is the angle after changing the argument such that it lies between $(-\pi, \pi]$

Anmärkning:
A specifiaction of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that $\mathrm{Arg}(z)$ is "discontinous" along the negative real axis. This is called a *branch-cut*

Suppose $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$
Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))] \\ & r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

Anmärkning:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

1.4. **Exponential form.**

Definition/Sats 1.9

For $z = x + iy \in \mathbb{C}$, let $e^z = e^x(\cos(y) + i\sin(y))$

Anmärkning:
$e^{iy} = \cos(y) + i\sin(y) \quad y \in \mathbb{R}$ (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$\Rightarrow e^{iy} = 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin(\theta)}$$

$$\Rightarrow e^z = e^x(\cos(\theta) + i\sin(\theta))$$

Anmärkning:
One can through comparing see that $|e^z| = e^x$, and that $\left|e^{iy}\right| = 1$

Properties of the exponential form:

- $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$
- $e^z \neq 0 \quad \forall z \in \mathbb{C}$
- $x \in \mathbb{R} \Rightarrow e^x > 1$ if $x > 0$ and $e^x < 1$ if $x < 0$
- $\left|e^{x+iy}\right| = e^x$
- $e^{i\pi/2} = i \quad e^{i\pi} = -1 \quad e^{3i\pi/2} = -1 \quad e^{2i\pi} = 1$
- e^z is 2π-periodic
- $e^z = 1 \Leftrightarrow z = 2\pi ki \quad k \in \mathbb{Z}$

Definition/Sats 1.10: deMoivre's formula

For $n \in \mathbb{Z}$, $(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta))$

1.5. **Logarithmic form.**

In real analysis, we have defined the logarithm as the inverse of e^x. This has previously worked since for $x \in \mathbb{R}$, e^x is injective.
The problem is that for e^z where $z \in \mathbb{C}$, it is not injective and should therefore not have an inverse.

Given $z \in \mathbb{C} \setminus \{0\}$, we define $\ln(z)$ as the cut of all $w \in \mathbb{C}$ whose image undre the exponential form is z, i.e $w = \ln(z) \Leftrightarrow z = e^w$.

Here, $\ln(z)$ is a *multivaled form*

We can use the fact that $|z| = r = e^x$ to derive some interesting properties of the logarithm:

$$z = re^{i\theta} \qquad w = u + iv$$

$$\text{If } z = e^w \Leftrightarrow re^{i\theta} = e^u \cdot e^{iv}$$

$$\Leftrightarrow u = \ln(r) = \ln(|z|) \qquad v = \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z}$$

Definition/Sats 1.11: Complex logarithm

For $z \neq 0$, we define the complex logarithm for $z \in \mathbb{C}$ as:

$$\begin{aligned} \ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\operatorname{Arg}(z) + k2\pi) \quad k \in \mathbb{Z} \end{aligned}$$

2. Elementary complex functions

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

2.1. **Branches of the complex logarithm.**

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of 2π).
If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

Definition/Sats 2.12: Principal logarithm

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\mathrm{Ln}\,(z) = \ln(|z|) + i \cdot \mathrm{Arg}(z)$$

Anmärkning:
We have essentially extended the "normal" logarithm, which is defined on $(0, \infty)$, to be defined on $\mathbb{C} \setminus \{0\}$

Anmärkning:
The principal logarithm is discontinous for negative reals, since their principal value argument is $= -\pi$, but the principal value argument is discontinous at $-\pi$. This is the so called *branch-cut*

Anmärkning:
Even though the principal logarithm is discontinous for negative reals, it is not undefined. Any negative real number z will have $\mathrm{Arg}(z) = \pi$, which the logarithm very much is defined for.

Anmärkning:
When branching, we do not necessarily have to pick $(-\pi, \pi]$, we can pick any interval $(\alpha, \alpha + 2\pi]$. This is usually denoted by $\arg_\alpha$.

2.2. **Complex mappings.**

One can think of a complex mapping $f : \mathbb{C} \to \mathbb{C}$ as $f(z) = f(x + iy) = w = u + iv$
Then it becomes clear which regions map to where by drawing them in their respective z-plane and w-plane.

2.3. **Complex powers.**

Given $z \in \mathbb{C}$, consider the following equation:

$$w^u = z \tag{1}$$

The set of all solutions w of (1) is denoted $z^{1/n}$m and is called the *n-th root of z*.

Anmärkning:
If $z = 0$, then $w = 0$

Suppose $z \neq 0$, then we may write $w = |w|\, e^{i\alpha}$ and $z = |z|\, e^{i\theta}$
By deMoivre's formula, (1) becomes:

$$|w|^n\, e^{in\alpha} = |z|\, e^{i\theta}$$

Then, the following follows:

$$\left.\begin{array}{ll} |w| = \sqrt[n]{|z|} & \\ n\alpha = \theta + k2\pi & k \in \mathbb{Z} \end{array}\right\} \Leftrightarrow \left.\begin{array}{ll} |w| = \sqrt[n]{|z|} & \\ \alpha = \dfrac{\theta}{n} + k\dfrac{2\pi}{n} & k \in \mathbb{Z} \end{array}\right\}$$

Notice now that every $k \in \mathbb{Z}$ gives a solution to (1)

Since sine and cosine are both 2π-periodic, then only $k = 0, 1, \cdots, n-1$ actually give *different* solutions (since $k = n \Rightarrow \alpha = \dfrac{\theta}{n} + n\dfrac{2\pi}{n}$)

Suppose $z \neq 0$. For $n \in \mathbb{Z}$ it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that $\ln(z)$ attains.

It is also true, that for $n = 1, 2, 3, \cdots$:

$$z^{\frac{1}{n}} = e^{\frac{1}{n} \ln(z)}$$

We can let $n \in \mathbb{C}$, and obtain the following definition:

Definition/Sats 2.13: Complex power

For $\alpha \in \mathbb{C}$, let:

$$z^\alpha = e^{\alpha \ln(z)} \qquad z \neq 0$$

Anmärkning:
This makes z^α a multivalued function, but it is possible to have a single-valued output from it.

Definition/Sats 2.14

Let $a, b \in \mathbb{C}$ where $a \neq 0$. Then a^b is single-valued (does not depend on the choice of branch for the logarithm) $\Leftrightarrow b \in \mathbb{Z}$

If $b \in \mathbb{Q}$ and is in lowest form (that is, $b = \dfrac{p}{q}$ where p, q have no common factors), then a^b has exactly q distinct values (the q:th roots of a^p)

If $b \in \mathbb{C}\backslash\mathbb{Q}$, then a^b has infinetly many values.

Bevis 2.1

Chose some interval (branch), say $[0, 2\pi)$, for the arg function and let $\ln(z)$ be the corresponding branch of the logarithm. If we chose another branch, we would have $\ln(a) + 2\pi kbi$ rather than $\ln(a)$ (where $k \in \mathbb{Z}$)
Therefore, $a^b = e^{b \ln(a) + 2\pi kbi} = e^{b \ln(a)} \cdot e^{2\pi ki}$

Notice that $e^{2\pi kbi}$ stays the same regardles of $b \in \mathbb{Z}$, as long as it is an integer.

In the same way, it can be shown that $e^{2\pi kip/q}$ has q distinct values if p, q have no common factor.

If b is irrational, and if $e^{2\pi kbi} = e^{2\pi mbi}$, then it follows that $e^{(2\pi bi)(k-m)} = 1$, and therefore $b(k-m)$ is an integer.
Since b is irrational, then $n - m = 0$ $\square$

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up. We can chose to branch z^α:

$$z^\alpha = e^{\alpha \mathrm{Ln}(z)}$$

2.4. **Trigonometric and Hyperbolic functions.**

We have the following:

$$\left.\begin{aligned} e^{iy} &= \cos(y) + i\sin(y) \\ e^{-iy} &= \cos(y) - i\sin(y) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos(y) &= \frac{e^{iy}+e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy}-e^{-iy}}{2i} \end{aligned}\right\}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

Definition/Sats 2.15: Complex sine and cosine

For $z \in \mathbb{C}$, we define:

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals.
When defining them for complex numbers, we just extend their domain:

Definition/Sats 2.16: Complex hyperbolic functions

For $z \in \mathbb{C}$, we define:

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z}-e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

- **Sine**:

$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$
$$\Rightarrow \frac{e^{-y}(\cos(x)+i\sin(x)) - e^{y}(\cos(x)-i\sin(x))}{2i} = \frac{(e^{-y}-e^{y})\cos(x) + i(e^{y}-e^{-y})\sin(x)}{2i}$$
$$= \frac{(e^{-y}-e^{y})\cos(x)}{2i} + \frac{(e^{y}-e^{-y})\sin(x)}{2}$$
$$\overset{i^{-1}=-i}{\Longrightarrow} \underbrace{\frac{(e^{y}-e^{-y})}{2}}_{\sinh(y)} i\cos(x) + \underbrace{\frac{(e^{y}+e^{-y})}{2}}_{\cosh(y)} \sin(x)$$

- **Cosine**:

$$\cos(x+iy) = \frac{e^{i(x+iy)+}e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2}$$
$$= \frac{e^{-y}(\cos(x)+i\sin(x)) + e^{y}(\cos(x)-i\sin(x))}{2} = \frac{\cos(x)(e^{y}+e^{-y}) + i(e^{-y}-e^{y})\sin(x)}{2}$$
$$= \underbrace{\frac{e^{y}+e^{-y}}{2}}_{\cosh(y)}\cos(x) - \underbrace{\frac{e^{y}-e^{-y}}{2}}_{\sinh(y)} i\sin(x)$$

This leads us to the following:

Definition/Sats 2.17: Addition formulas for complex trigonometric functions

- $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$
- $\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

Anmärkning:
Both sine and cosine can be defined as the unique solution to an ODE, namely:

$$\begin{array}{lll} f''(x)+f(x)=0 & f(0)=0,\ f'(0)=1 & f(x)=\sin(x) \\ f''(x)+f(x)=0 & f(0)=1,\ f'(0)=0 & f(x)=\cos(x) \end{array}$$

2.5. **Mapping properties of** $\sin(z)$**.**

Let $f(z) = \sin(z)$ in $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$, let A be the set of points allowed with respect to the above constraint and let B be the mapping of those points by $\sin(A)$

Claim: $f : A \to B$ is a bijective mapping

Bevis 2.2

Take a $z \in \mathbb{C}$ $\quad z = x+iy \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.
Then:

$$f(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$f(z) \in \mathbb{R} \Leftrightarrow \cos(x)\sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0$$

If $y = 0$, then:

$$f(z) = \sin(x)\cosh(y) = \sin(x) \in (-1,1)$$

Therefore, if $z \in A \Rightarrow f(z) \in B$. Now we need to show that for any $z \in B$, there is a u such that $f(u) = z$

Let $u = \sin(x)\cosh(y)$ $\quad, v = \cos(x)\sinh(y)$ and pick a vertical line at $x = a \neq 0$
We will now consider the images of these lines:

$$\cosh(y) = \frac{u}{\sin(a)} \qquad \sinh(y) = \frac{v}{\cos(a)}$$

$$(\cosh(y))^2 - (\sinh(y))^2 = 1 \Rightarrow \left(\frac{u}{\sin(a)}\right)^2 - \left(\frac{v}{\cos(a)}\right)^2 = 1$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line $y = b \neq 0$

$$\sin(x) = \frac{u}{\cosh(b)} \qquad \cos(x) = \frac{v}{\sinh(b)}$$

$$\cos^2(x) + \sin^2(x) = 1 \Rightarrow \left(\frac{u}{\cosh(b)}\right)^2 + \left(\frac{v}{\sinh(b)}\right)^2 = 1$$

This isa half-eclipse. Note that $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

$\square$

Definition/Sats 3.18: Open disc

The set $D_r(z_0) = \{z \in \mathbb{C} \ : \ |z - z_0| < r\}$ is called the *open-disc* with center z_0 and radius r

Anmärkning:
Since we have a strict inequality, it is open. If we had $\leq$, it would be a closed disc.

Definition/Sats 3.19: Open subset

A subset M of $\mathbb{C}$ is called *open* if for every $z_0 \in M$ there exists an $r > 0$ such that $D_r(z_0) \subseteq M$

Definition/Sats 3.20: Interior point

A point $z_0 \in M$ is called an *interior-point* of M if there exists an $r > 0$ such that $D_r(z_0) \subseteq M$

Definition/Sats 3.21: Boundary point

A point$z_0 \in \mathbb{C}$ is called a *boundary point* of M if $\forall r > 0$ it holds that:

$$D_r(z_0) \cap M \neq \phi \quad \wedge \quad D_r(z_0) \cap M^c \neq \phi$$

Anmärkning:
The set of all interior points of M is denoted by $\mathrm{int}(M)$ and the set of all boundary points of M is denoted by ∂M

The following equivelances hold:

- M is closed $\Leftrightarrow \partial M \subseteq M$
- M is open $\Leftrightarrow \partial M \subseteq M^c$
- $\mathbb{C}$ is clopen
- $\emptyset$ is clopen
- The union of any collection of open subsets of $\mathbb{C}$ is open
- The intersection of any finite collection of open subsets of $\mathbb{C}$ is open

Definition/Sats 3.22: Closed set

We say that a set $X \subseteq \mathbb{C}$ is closed if its complement X^c is open

Definition/Sats 3.23: Polygonal path

A polygonal path P (sometimes called piecewise linear curve) is a curve specified by a sequence of points $(A_1, A_2, \cdots, A_n)$.
The curve itself consists of line segments connecting the consecutive points.

Definition/Sats 3.24: polygonal-path-connected open set

An open set M is called *polygonal-path-connected* if eveyr pair of points $z_1, z_2 \in M$ can be connected by a polygonal path contained in M

Anmärkning:
Some would call this just path-connected, or even just connected. This works in $\mathbb{R}^n$ (recall that $\mathbb{C} \cong \mathbb{R}^2$). Topologically speaking, polygonal-path-connectedness $\implies$ path-connectedness

Anmärkning:
A set X is connected $\Leftrightarrow$ the only subsets of X which are clopen are $\emptyset$ and X

Anmärkning:
One can assume the polygonal paths to have segments parallell to the ordinale ones.

Anmärkning:
An open connected set is called a *domain*

Definition/Sats 3.25

Suppose that $u(x, y)$ is a real-valued function defined in a domain $D \subseteq \mathbb{R}$
Also suppose that:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} =$$

in all of D. Then u is contained in D

Definition/Sats 3.26: Simply connected

A domain $D \subseteq \mathbb{C}$ is called *simply connected* if ever closed curve in D can be, within D, continously deformed to a point

Anmärkning:
Topologically speaking, D is homeomorphic to a point.

Definition/Sats 3.27: Non-connectedness

A set $A \subseteq \mathbb{C}$ is *not connected* if there are open sets U and V such that:

- $A \subseteq U \cup V$
- $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$

3.1. **Limits and Continuity.**

Definition/Sats 3.28: Complex limit

A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to have the limit z_0 (*converges to* z_0) if for every given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$|z_n - z_0| < \varepsilon \quad \forall n \geq N$$

We write this as:

$$\lim_{n\to\infty} z_n = z_0$$

Anmärkning:
Every cauchy sequence in $\mathbb{C}$ converges.

Anmärkning:
$z_n \to z_0 \Leftrightarrow \text{Re}(z_n) \to \text{Re}(z_0)$ and $\text{Im}(z_n) \to \text{Im}(z_0)$
This follows from $|x|, |y| \leq \sqrt{x^2+y^2} \leq |x| + |y|$

Definition/Sats 3.29

Let f be a function defined ina punctured neighborhood of z_0

We say that f has the limit w_0 as $z \to z_0$, if for every given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

We write this as:

$$\lim_{z\to z_0} f(z) = w_0$$

Anmärkning:
If a limit exists, it is unique.

Definition/Sats 3.30

For $z = x + iy$, let:

$$u(x,y) = \text{Re}(f(z)) \qquad v(x,y) = \text{Im}(f(z))$$

Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$
Then the following holds:

$$\lim_{z\to z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Definition/Sats 3.31: Continous function

Let f be a function defined in a neighborhood of z_0.

f is said to be continous at z_0 if:

$$\lim_{z\to z_0} f(z) = f(z_0)$$

A function f is said to be *continous on the (open) set* M if it is continous at each point of M

Anmärkning:
The following statements are equivalent (for $f : A \to \mathbb{C}$):

- f is continous
- The inverse image of every closed set is closed relative to A
- The inverse image of every open set is open relative to A
- The image set $f(A)$ is connected

Assume $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$
The following properties from the real limit hold for the complex limit:

- $\lim_{z\to z_0}(f(z) \pm g(z)) = A \pm B$
- $\lim_{z\to z_0} f(z)g(z) = AB$
- $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B} \qquad B \neq 0$

Anmärkning:
If f, g are continous at z_0, then so are $f \pm g$ and fg. The quotient is only continous if $g(z_0) \neq 0$

Anmärkning:
Constant functions, polynomials, and rational functions (whenever the denominator is non-zero) are all continous in $\mathbb{C}$

3.2. **The complex derivative.**

Analogous to the real case, we also have the following:

Definition/Sats 3.32: Differentiability

Let f be a complex-valued function defined in a neighborhood of z_0.
We say that f is differentiable at z_0 if the limit:
$$\lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
exists.
The limit is called the *derivative* of f at z_0, and is denoted by $f'(z_0)$ or $\dfrac{df}{dz}(z_0)$

Anmärkning:
Since Δz is a complex unmber, it can approach 0 from different directions. In order for the derivative to exist, the results must be independent of the direction of which Δz approaches 0 (i.e, approaches 0 from all directions)

Anmärkning:
If X is an open connceted set and $a, b \in X$, then there is a differntiable path $\gamma : [0, 1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$

Example:
The function $f(z) = \overline{z}$ is nowhere differentiable since:
$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{z_0 + \Delta z} - \overline{z_0}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$
As $\Delta z \to 0$ from the x-direction (real-line), the limit becomes $\dfrac{\overline{x}}{x} = 1$
However, as we approach from the y-direction (complex axis), the limit becomes $\dfrac{\overline{iy}}{iy} = \dfrac{-y}{y} = -1$
Since x, y were chosen arbitrarily, this applies to all x, y. Since the limits did not match, it is not differentiable and at no point.

Of course, all the properties from the real case hold here as well.
Suppose f, g are differentiable at z, then:

- $(f \neq g)'(z) = f'(z) \neq g'(z)$

- $(cf)'(z) = cf'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f \circ g)'(z) = f'(g(z))g'(z)$

3.3. **Analytic functions.**

Definition/Sats 3.33: Analytic function

A complex-valued function f is said to be *analytic* in an open set G if f is differentiable at every point in G.

We say that f is *analytic at* z_0 if f is differentiable in a neighborhood of z_0

Anmärkning:
If f is analytic in all of $\mathbb{C}$, then f is said to be *entire* (or *holomorphic*).

Definition/Sats 3.34

If an entire function $f(z)$ has a root at w, then:

$$\lim_{z \to w} \frac{f(z)}{(z - w)}$$

is an entire function.

4. Cauchy-Riemann's equations

Suppose $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$
Then:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}$$

1) Let $\Delta z = \Delta x$ (i.e $\Delta y = 0$):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

2) Let $\Delta z = i\Delta y$ (i.e $\Delta x = 0$):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

It must therefore hold that:

$$u_x + iv_x = -iu_y + v_y$$

This leads to the Cauchy-Riemann equations:

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$$

We have therefore arrived at the following:

Definition/Sats 4.35

A necessary condition for $f = u+iv$ to be differentiable at $z_0 = x_0+iy_0$ is that the Cauchy-Riemann equations are satisfied at (x_0, y_0)

Anmärkning:
We also saw that if f is differentiable at the point z_0, then the derivative is given by:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

The following provides a sufficient condition for Differentiability:

Definition/Sats 4.36

Suppose that $f = u + iv$ is defined in a open set G containing $z_0 = x_0 + iy_0$.
Suppose also that u_x, u_y, v_x, v_y exists in G and are continous at (x_0, y_0), and satisfy the Cauchy-Riemann equations at (x_0, y_0)

Then f is differentiable at z_0

Anmärkning:
Cauchy-Riemann equations + $u, v \in C^1 \Rightarrow f$ is differentiable

Bevis 4.1

In view of the continuity of the first parital derivates at (x_0, y_0) , it holds that:

$$u(x_0+\Delta x, y_0+\Delta y) = u(x_0,y_0) + u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \sqrt{(\Delta x)^2+(\Delta y)^2}\rho_1(\Delta x, \Delta y)$$
$$v(x_0+\Delta x, y_0+\Delta y) = v(x_0,y_0) + v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \sqrt{(\Delta x)^2+(\Delta y)^2}\rho_2(\Delta x, \Delta y)$$

Where $\rho_1, \rho_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

Then:

$$\begin{aligned} f(z_0+\Delta z) - f(z_0) &= u_x(x_0,y_0)\Delta x + \underbrace{u_y(x_0,y_0)}_{=-v_x(x_0,y_0)}\Delta y + i(v_x(x_0,y_0)\Delta x + \underbrace{v_y(x_0,y_0)}_{=u_x(x_0,y_0)}\Delta y) \\ &\quad + \sqrt{(\Delta x)^2+(\Delta y)^2}(\rho_1(\Delta x,\Delta y) + i\rho_2(\Delta x, \Delta y)) \\ &\overset{\text{CR-eq.}}{=} u_x(x_0,y_0)\Delta z + iv_x(x_0,y_0)\Delta z + |\Delta z|\left(\rho_1(\Delta x,\Delta y) + i\rho_2(\Delta x,\Delta y)\right) \end{aligned}$$

Since $\rho_1, \rho_2 \to 0$ as $\Delta z \to 0$, it follows that:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists and is equal to $u_x(x_0,y_0) + iv_x(x_0,y_0)$ □

4.1. **Inverse mappings.**

Suppose $f = u + iv$ is analytic in a domain D (with f' continous).
Consider the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

As a mappiong of $D \subset \mathbb{R}^2 \to \mathbb{R}^2$

Its Jacobian matrix:

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

has determinant:

$$\det(J_f) = u_x v_y - u_y v_x \overset{\text{CR-eq.}}{=} u_x^2 + v_x^2 = |f'(z)|^2$$

The inverse function then leads to the following:

Definition/Sats 4.37: Inverse function theorem

Suppose $f(z)$ is analytic on a domain D with $f'(z) \neq 0$ continous.
Then there is a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \to V$ is bijective, and the inverse function $f^{-1} : V \to U$ is analytic with derivative:

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \qquad w = f(z)$$

Definition/Sats 5.38: Harmonic function

A real-valued function $\phi(x, y)$ is said to be *harmonic* in a domain D if $\phi \in C^2(D)$ and ϕ satisfies Laplace's equations:

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

in D

Definition/Sats 5.39

Suppose $f = u + iv$ is analytic in a domain D. Then u, v are harmonic in D

Bevis 5.1

One can show that $u, v \in C^\infty$:

$$u_x = v_y \Rightarrow u_{xx} = v_{y_x}$$
$$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$$

As $v_{yx} = v_{xy}$, we have $u_{xx} + u_{yy} = 0$

Similarly, $v_{xx} + v_{yy} = 0$

□

Definition/Sats 5.40: Harmonic Conjugacy

If u is harmonic in a domain D and v is a harmonic function in D such that $u + iv$ is analytic in D, then we say that v is a *harmonic conjugate* of u in D

Definition/Sats 5.41

If u is harmonic in a simply connected domain $D \subseteq \mathbb{C}$, then there exists a harmonic conjugate v of u in D, and v is unique up to addition of a real constant

Bevis 5.2

Suppose u is harmonic in $D \subseteq \mathbb{C}$

Consider the vector-field $\overline{F} = (-u_y, u_x) \in C^1(0)$.
Note that:

$$\frac{\partial F_1}{\partial y} = -u_{yy} \overset{u \text{ harm.}}{=} u_{xx} = \frac{\partial F_2}{\partial x}$$

Since D is simply connected $\Rightarrow \overline{F}$ is conservative $\Rightarrow \exists v : \ \nabla v = \overline{F}$, i.e $(v_x, v_y) = (-u_y, u_x)$

$$\Rightarrow \left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \Rightarrow f = u + iv \text{ is analytic in } D$$

If $\overline{v}$ is another harmonic conjugate, then:

$$\overline{v}_x = -u_y = v_x$$
$$\overline{v}_y = u_x = v_y$$
$$\Rightarrow \nabla(v - \overline{v}) = \overline{0} \Rightarrow v - \overline{v} = c \in \mathbb{C}$$

$\square$

Anmärkning:
A vector field is conservative if it is the gradient of some function.
It has the property that its line integral is path independent.

6. Conformal mappings

Let D be a domain in $\mathbb{C}$, $z_0 \in D$.
Suppose $f : D \to \mathbb{C}$ is analytic with $f'(z_0) \neq 0$. Let $\gamma(t) = x(t) + iy(t)$ be a C^1-curve in D through $z_0 = \gamma(0)$ with $\gamma'(0) \neq 0$. Then $(f \circ \gamma)(t) = f(\gamma(t))$ is a C^1-curve through $(f \circ \gamma)(0) = f(z_0)$.

Moreover,

$$(f \circ \gamma)'(0) = \frac{d}{dt} f(\gamma(t))|_{t=0} = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}$$
$$= \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \cdot \frac{\gamma(t) - \gamma(0)}{t} = f'(z_0)\gamma'(0)$$

From this, we can conclude $(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$ is a tangent vector to $f \circ \gamma$ at $f(z_0)$

Note that $\arg(f \circ \gamma)'(0) = \arg(f'(z_0) + \arg(\gamma'(0)))$

If γ_1 and γ_2 are two C^1-curves which intersect at z_0, then the angle from $(f \circ \gamma_1)'(0)$ to $(f \circ \gamma_2)'(0)$ is the same as the angle from $\gamma_1'(0)$ to $\gamma_2'(0)$

Definition/Sats 6.42: Conformal C^1-mapping

A C^1-mapping $f : D \to \mathbb{C}$ is said to be *conformal* at z_0 if it satisfies the above paragraph.

If f maps D bijectively onto V, and if f is conformal at one point $z_0 \in D$, we call $f : D \to V$ a *conformal mapping*

Definition/Sats 6.43

If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0

Anmärkning:
One can in fact prove the converse of this theorem.

7. Stereographic projection

Consider the unit sphere $S \in \mathbb{R}^3$.
Given any point $P = (x_1, x_2, x_3) \in S$ other than the north pole $N = (0,0,1)$, we draw the line through N and P.
We define the *stereographic projection* of P to be the point $z = x + iy \in \mathbb{C} \sim (x, y, 0)$, where the line intersects the plane $x_3 = 0$. Then the following holds:

$$(x, y, 0) = (0,0,1) + t[(x_1, x_2, x_3) - (0,0,1)]$$

Where t is given by $1 + t(x_3 - 1) = 0 \Leftrightarrow t = \dfrac{1}{1 - x_3}$. We arrive at the following:

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$$

Conversely, given $z = x + iy \in \mathbb{C} \sim (x, y, 0)$ teh line through N and z is given by:

$$(x_1, x_2, x_3) = (0,0,1) + t[(x, y, 0) - (0,0,1)] \qquad t \in \mathbb{R}$$

Anmärkning:
The line intersects S when:

$$\begin{aligned} &x_1^2 + x_2^2 + x_3^2 = 1 \\ \Leftrightarrow\ &(tx)^2 + (ty)^2 + (1-t)^2 = 1 \\ \Leftrightarrow\ &t^2(x^2 + y^2 + 1) - 2t = 0 \\ \Leftrightarrow\ &t = 0 \vee t = \frac{2}{x^2 + y^2 + 1} = \frac{2}{|z|^2 + 1} \end{aligned}$$

This corresponds to $P = N$ or:

$$P = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

Thus, stereographic projections $s : S \backslash N \to \mathbb{C}$ define a bijection.

Letting $\widehat{C} = \mathbb{C} \cup \{\infty\}$ denote the *extended complex plane* and define $s(N) = \infty$, then s becomes a bijective map from S onto $\widehat{\mathbb{C}}$

Definition/Sats 7.44

Under stereographic projections, circles on S correspond to circles and lines in $\mathbb{C}$

Anmärkning:
We therefore call circles and lines in $\mathbb{C}$ "circles" in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where lines are considered as "circles through ∞"

Bevis 7.1

The general equation for a circle or line in the $z = x + iy$ plane is:

$$A(x^2 + y^2) + Cx + Dy + E = 0$$

Using $z = x + iy = \dfrac{x_1 + ix_2}{1 - x_3}$, we get:

$$A\left(\left(\frac{x_1}{1-x_3}\right)^2 + \left(\frac{x_2}{1-x_3}\right)^2\right) + \frac{Cx_1}{1-x_3} + \frac{Dx_2}{1-x_3} + E = 0$$

$$\Leftrightarrow A(x_1^2 + x_2^2) + Cx_1(1-x_3) + Dx_2(1-x_3)E(1-x_3)^2 = 0$$

Using $x_1^2 + x_2^2 + x_3^2 = 1$, we get:

$$A(1 - x_3^2) + Cx_1(1-x_3) + Dx_2(1-x_3) + E(1-x_3)^2 = 0$$

Dividing by $1 - x_3$ yields:

$$A(1 + x_3) + Cx_1 + Dx_2 + E(1 - x_3) = 0$$

$$\Leftrightarrow Cx_1 + Dx_2 + (A - E)x_3 + A + E = 0$$

This is the equation for a plane in $\mathbb{R}^3$, which intersects S in a circle $\square$

8. Möbius transformations

Definition/Sats 8.45: Moebius transformation

A *Möbius transformation* is a mapping of the form:

$$T(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C}$$

Where $ad - bc \neq 0$ (T is not constant)

Anmärkning:
If $c = 0$, we let $T(\infty) = \infty$. Then $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is bijective

If $c \neq 0$, then:

$$T : \mathbb{C} \backslash \left\{-\frac{d}{c}\right\} \to \mathbb{C} \backslash \left\{\frac{a}{c}\right\}$$

is a bijection. Letting $T\left(-\dfrac{d}{c}\right) = \infty$, and $T(\infty) = \dfrac{a}{c}$, we extend T to a bijective map $T : \widehat{\mathbb{C}} \to \widehat{C}$

The inverse is found by solving:

$$w = T(z)$$

which gives:

$$z = T^{-1}(w) = \begin{cases} \dfrac{-dw+b}{cw-a}, & \text{if } w \neq \dfrac{a}{c} \; w \neq \infty \\ \infty \;\; \text{if } w = \dfrac{a}{c} \\ \dfrac{-d}{c} \;\; \text{if } w = \infty \end{cases}$$

Anmärkning:
If we interpret $\dfrac{a}{c}$ and $-\dfrac{d}{c}$ as ∞, it also holds for $c = 0$

Anmärkning:

$$T'(z) = \frac{d}{dt}\left(\frac{ax+b}{cz+d}\right) = \frac{a(cz+d)-(az+b)\cdot c}{(cz+d)^2}$$
$$= \frac{ad-bc}{(xz+d)^2} \neq 0$$

Thus $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is conformal

Anmärkning:
If:

$$T(z) = \frac{az+b}{cz+d} \qquad S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$
$$\Rightarrow (S \circ T)(z) = \frac{\alpha T(z)+\beta}{\gamma T(z)+\delta}$$
$$= \frac{\alpha\left(\frac{az+b}{cz+d}\right)+\beta}{\gamma\left(\frac{az+b}{cz+d}\right)+\delta} = \frac{(\alpha a+\beta c)z+(\alpha b+\beta d)}{(\gamma a+\delta c)z+(\gamma b+\delta d)}$$

This shows that compositions of Moebius transformations are Möbius transformations.

Anmärkning:

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\alpha a+\beta\delta & ab+\beta d\\ \gamma a+\delta c & \gamma b+\delta d\end{pmatrix}$$

Lemma 8.1

If a Moebius transformation T has more than two fixed points in $\widehat{\mathbb{C}}$ (z_0 is a fixpoint if $T(z_0) = z_0$), then $T(z) = z \ \forall z \in \widehat{\mathbb{C}}$

Bevis 8.1

If $c = 0$, then $T(z) = \dfrac{az+b}{d}$, so:

$$T(z) = z \Leftrightarrow \frac{az+b}{d} = z \Leftrightarrow (a-d)z+b = 0$$

So T has at most one fixed point in $\mathbb{C}$ unless $a = d$ and $b = 0 \Leftrightarrow T(z) = z \ \forall z \in \mathbb{C}$

So if $c = 0$, T has at most 2 fixed points in $\widehat{\mathbb{C}}$ ($T(\infty) = \infty$) unless $T(z) = z \ \forall z \in \mathbb{C}$

If $c \neq 0$, then:

$$T(z) = z \Leftrightarrow \frac{az+b}{cz+d} = z$$
$$\Leftrightarrow cz^2 + (d-c)z - b = 0$$

So T has at most 2 fixed points in $\mathbb{C}$ (and $T(\infty) = \dfrac{a}{c} \neq \infty$) unless $c = 0, a = d, b = 0$
This contradicts $c \neq 0$ □

Definition/Sats 8.46

If S, T are Möbius transformations such that $S(z_i) = T(z_i)$ at three different points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$, then $S = T$

Bevis 8.2

If $S(z_i) = T(z_i)$ for $i = 1,2,3$, then the Moebius transformation $T^{-1} \circ S$ has at least 3 fixed points. By the previous lemma:

$$T^{-1} \circ S(z) = z \quad \forall z \in \widehat{\mathbb{C}}$$

i.e $S(z) = T(z)$ $\forall z \in \widehat{\mathbb{C}}$ □

Anmärkning:
Particular cases of the Möbius transformation are:

- $T(z) = z + b$ (*translation*)
- $T(z) = az = |a|\, e^{i\arg(a)z}$ (*rotation & magnification*)
- $T(z) = \dfrac{1}{z}$ (*inversion*)

Anmärkning:
If $c \neq 0$:

$$T(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$$

This means that every Moebius transformation is a composition of translations, rotations, magnifications, and inversions.

Definition/Sats 8.47

Every Möbius transformation maps "circles" onto "circles"

Anmärkning:
Recall that a "circle" in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a circle or line in $\mathbb{C}$. A line in $\mathbb{C}$ is a "circle" through ∞ in $\widehat{\mathbb{C}}$

Bevis 8.3

It is easy to see that translations and rotations/magnifications map circles onto circles and line onto lines. This gives enough to prove that inversion:

$$T(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} 0 \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = u + iv$$

maps circles onto circles

A circle in $\widehat{\mathbb{C}}$ has the equation:

$$A(x^2+y^2) + Cx + Dy + E = 0$$
$$\Leftrightarrow A + C\frac{x}{x^2+y^2} + D\frac{y}{x^2+y^2} + E\frac{1}{x^2+y^2} = 0$$
$$\Leftrightarrow E(u^2+v^2) + Cu - Dv + A = 0$$

□

Given a "circle" C_z in the z-plane and a "circle" C_w in the w-plane, can one find a Moebius transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such tath $T(C_z) = C_w$? Yes!

8.1. **The cross-ratio.**

Definition/Sats 8.48: Cross-ratio

Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be distinct and put:

$$(z, z_1, z_2, z_3) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \in \widehat{\mathbb{C}}$$

If some of the z_i is ∞, the right hand side should be interpret as:

$$(z, z_1, z_2, z_3) = \begin{cases} \dfrac{z_2 - z_3}{z - z_3} & \text{if } z_1 = \infty \\ \dfrac{z - z_1}{z - z_3} & \text{if } z_2 = \infty \\ \dfrac{z - z_1}{z_2 - z_1} & \text{if } z_3 = \infty \end{cases}$$

(z, z_1, z_2, z_3) is called the *cross-ratio* of the four points

Anmärkning:
$S(z) = (z, z_1, z_2, z_3)$ is a Möbius transformation such that:

$$S(z_1) = 0 \qquad S(z_2) = 1 \qquad S(z_3) = \infty$$

By an earlier remark, this is the unique Moebius transformation mapping z_1, z_2, z_3 to $0, 1, \infty$

Definition/Sats 8.49

Given a tripple $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ of distinct points, and another tripple $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ of distinct points, then there is a unique Möbius transformation T such that $T(z_i) = w_i$

The mappings $w = T(z)$ is found by solving the cross-ratio equation:

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

Bevis 8.4

By an earlier remark, there is at most one such mapping. We now prove that there is exactly one by contradicting it.

Put $S(z) = (z, z_1, z_2, z_3)$, $U(w) = (w, w_1, w_2, w_3)$:

$$\Rightarrow T(z) = (U^{-1} \circ S)(z) = U^{-1}(S(z))$$

$U^{-1}(S(z))$ is a Moebius transformation such that:

$$T(z_1) = U^{-1}(S(z_1)) = U^{-1}(0) = w_1$$
$$T(z_2) = \cdots$$
$$\vdots$$

Then:

$$w = T(z) \Leftrightarrow w = U^{-1}(S(z)) \Leftrightarrow U(w) = S(z)$$
$$\Leftrightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

□

This theorem can be used to construct a T as above, mappiong C_z to C_w

Let z_1, z_2, z_3 be three distinct points on a circle C_z in $\widehat{\mathbb{C}}$. Note that C_z is *oriented* by the order of these points, that is C_z aquires an orientation by proceeding through z_1, z_2, z_3 in succession

Since a Möbius transformation is conformed, it maps the region to the left of C_z, oriented by z_1, z_2, z_3, to the region left of $C_w = T(C_z)$ oriented by w_1, w_2, w_3

8.2. **Symmetry-preserving property.**

Two points z_1 and z_2 are said to be *symmetric* with respect to a line L if L is the perpendicular bisector of the line-segment joining z_1 and z_2

This means that every circle or line through z_1 and z_2 intersects L orthogonally

Definition/Sats 8.50

Two points z_1 and z_2 are said to be *symmetric* with respect to a circle C if every circle of line through z_1 and z_2 intersects C orthogonally

In particular, the center a of C and ∞ are symmetric with respect to C

Definition/Sats 8.51: Symmetry principle

Let C_z be a circle or line in the z-plane and $w = T(z)$ be any Moebius transformation. Then two points z_1 and z_2 are symmetric with respect to C_z if and only if their images $w_1 = T(z_1)$ and $w_2 = T(z_2)$ are symmetric with respect to the image $C_w = T(C_z)$ under T.

Bevis 8.5

"Two points are symmetric with respect to a given circle if and only if every circle containing the points intersects the given circle orthogonally" is a re-formulation of the theorem.

Möbius transformations preserve the class of circles, and they also preserve orthogonallity. Hence, they preserve the symmetric condition. $\square$

9. Dirichlets problem

We have previously discussed harmonic function over a domain D. These have many applications in solving Dirichlets problem:

Find a function $\phi(x,y)$ continous on $D \cup \partial D$ of class C^2 in D such that

- $\Delta\phi = \phi_{xx} + \phi_{yy} = 0$ in D
- $\phi =$ some given function on ∂D

9.1. **Standard cases.**

This can be easily solved in some standard cases:

- $\begin{cases} \Delta\phi = 0 \text{ in } D \\ \phi(a,y) = A \\ \phi(b,y) = B \end{cases}$ Let $\phi(x,y) = \alpha x + \beta$, choose α, β such that

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{B-A}{b-a} \\ \beta = A - \dfrac{a(B-A)}{b-a} = \dfrac{AB - aB}{b-a} \end{cases}$$

$$\Rightarrow \phi(x,y) = \frac{(B-A)x + Ab - aB}{b-a}$$

- $\phi(x,y) = \frac{2}{\pi}\mathrm{Arg}(z) = \frac{2}{\pi}\arctan\left(\frac{y}{x}\right)$
- $\phi(x,y) = \alpha\mathrm{Arg}(z) + \beta$ leads to:

$$\begin{cases} \alpha\frac{\pi}{2} + \beta = A \\ -\alpha\frac{\pi}{2} + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{A-B}{\pi} \\ \beta = \dfrac{A+B}{2} \end{cases}$$

i.e $\phi(x,y) = \frac{A-B}{\pi}\mathrm{Arg}(z) + \frac{A+B}{2}$

- $\phi(x,y) = \frac{1}{\alpha}\mathrm{Arg}(z)$
- $\phi(x,y) = \frac{1}{\pi}\mathrm{Arg}(z - z_0)$
- $\phi(x,y) = a_n + \frac{1}{\pi}\sum_{b=1}^{n}(a_{k-1} - a_k)\mathrm{Arg}(z - x_k)$:

$$\mathrm{Arg}(z - x_k) = \begin{cases} \pi \; x < x_k \\ 0 \; x > x_k \end{cases}$$

$\Rightarrow$ if $x_j < x < x_{j+1}$ then:

$$\phi(x,0) = a_n + \sum_{k=j+1}^{n}(a_{k-1} - a_k) = a_j$$

- $\phi(x,y) = \alpha\ln(|z|) + \beta$ leads to:

$$\begin{cases} \alpha\ln(r_1) + \beta = A \\ \alpha\ln(r_2) + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{B-A}{1 - r_2 - \ln(r_1)} \\ \beta = \dfrac{A\ln(r_2) - B\ln(r_1)}{\ln(r_2) - \ln(r_1)} \end{cases}$$

$$\Rightarrow \phi(x,y) = \frac{B-A}{1 - r_2 - \ln(r_1)}\ln(|z|) + \frac{A\ln(r_2) - B\ln(r_1)}{\ln(r_2) - \ln(r_1)}$$

How about more complicated Dirichlet problems?
The idea is to simplify the complicated problems to an easier one using a conformal mapping.

Definition/Sats 9.52

Suppose $f : D \to D'$ is analytic, $f = u + iv$.
If $\psi(u, v)$ is harmonic in D', then:

$$\phi(x, y) := \psi(u(x, y), v(x, y))$$

is harmonic in D

Bevis 9.1

Take $z_0 \in D$. Then $w_0 = f(z_0) \in D'$ and since D' is open, there is a disk $w_0 \in V$ contained in D'.

Since f is continous, there is a disk $z_0 \in U$ in D such that $f(U) \subseteq V$. Since ψ is harmonic in V, which is simply connected, there is an analytic function g in V such that $\mathrm{Re}(g) = \psi$

But then $g \circ f$ is an analytic function in U such that:

$$\mathrm{Re}(g \circ f)(z) = \psi(u(x, y), v(x, y)) = \phi(x, y)$$

Hence, ϕ is harmonic in U. Sinze z_0 wazs arbitrarily chosen, ϕ is harmonic in D $\square$

Suppose now that the analytic function $f : D \to D'$ maps D bijectively onto D' and extends to a continous bijection $f : \overline{D} \to \overline{D'}$.
Suppose also that the bohndary conditions for ψ in D' corresponds to the boundary conditions for ϕ in D.

Then, if we can solve the Dirichlet problem for ψ, we can also solve it for ϕ