UPPSALA UNIVERSITET

Introduction to PDE

Lecture Notes

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Inlämningsdatum
March 24, 2024

Contents

Definition/Sats 1.1: Domain

An open connected set $\Omega \subset \mathbb{R}^n$ is called a *domain*

Definition/Sats 1.2: Bounded domain

A domain is *bounded* if its closure $\overline{\Omega}$ is compact

Definition/Sats 1.3: Smooth boundary

The boundary of a set (denoted $\partial\Omega$) is called *smooth* if it can be locally represented by a smooth function

1.1. **Examples of linear operators.**

- Gradient of $u \in C^1(\Omega)$, i.e $\nabla u = (u_{x_1}, \cdots, u_{x_n})$
- Laplacian of $u \in C^2(\Omega)$, i.e $\Delta u = u_{x_1 x_1} + \cdots u_{x_n x_n}$
- Divergence of vector field

Definition/Sats 1.4: Classification of PDEs

A PDE is said to be **Quasilinear**: $\sum_{k,l=1}^n a^{kl}(x, u(x), Du(x)) u_{x_k x_l} + b(x, u(x), Du(x)) = 0$

A PDE is said to be **Semilinear**: $\underbrace{\sum_{k,l=1}^n a^{kl}(x) u_{x_k x_l}}_{\text{principal term}} + b(x, u(x), Du(x)) = 0$

A PDE is said to be **linear**: $\sum_{k,l=1}^n a^{kl}(x) u_{x_k x_l} + \sum_{l=1}^n b^l(x) u_{x_l} + c(x)u(x) = f(x)$

Example: Heat equation

$$u_t - \Delta u = f \Leftrightarrow (\partial_t - \Delta)u$$

Take domain $\Omega \in \mathbb{R}^3$ and outward pointing unit $\overrightarrow{n}$
Let $u = u(x, y, z, t)$ be temperature at point $\overline{x} = (x, y, z)$ at time t, and let $q = (x, y, z, t, u)$ be the function describing the heat problem.
Heat production is dependent on temperature.
Let $\overline{Q} = \overline{Q}(x, y, z, t)$ be a vector field representing heat flux through $\partial\Omega$
The temperature change from t to $t + \Delta t$ corresponds to the flux in heat production as follows:

$$\int_\Omega (u(x, y, z, t + \Delta t) - u(x, y, z, t))dV = \int_t^{t+\Delta t} \int_\Omega q(x, y, z, t, u) dV dt - \int_t^{t+\Delta t} \int_{\partial\Omega} Q(x, y, z, t) \overrightarrow{n} dS dt$$

Divide both sides by Δt and take the limit as $\Delta t \to 0$ yields:

$$\int_\Omega u_t(x, y, z, t) dV = \int_\Omega q(x, y, z, t, u) dV - \int_{\partial\Omega} \overline{Q}(x, y, z, t) \overrightarrow{n} dS$$

We expect that in practice $\overline{Q} = -a\nabla u$ for $a > 0$
Note that the last term becomes

$$\int_{\partial\Omega} \overline{Q}(x, y, z, t) \overrightarrow{n} dS = \int_{\partial\Omega} -a \nabla u \overrightarrow{n} dS = -a \int_\Omega \nabla \bullet \nabla u dV$$

Move everything ot one side and integrating under same domain:

$$\int_\Omega (u_t(x, y, z, t) - q(x, y, z, t, y) - a\nabla \bullet \nabla u) dV = 0$$

This is precisely the heat equation.

The further study of the equation involves introducing/imposing further conditions:

- *Initial conditions*: At $t = 0$, $u(x, y, z, t) = \varphi(x, y, z)$
- *Dirichlet data*: Prescribes behaviour of boundary independent of time, i.e $u(x, y, z, t) = \psi(x, y, z) \quad \forall (x, y, z) \in \partial\Omega$
- *Neumann Conditions*: Prescribes heat production with boundary independent of time derivative: $u_{\overrightarrow{n}}(x, y, z, t) = \psi(x, y, z)$

There are other types, for example Robin conditions.
They can be mixed, and solutions can be determined if for example the combinations include stability etc.

1.2. **Other PDEs.**

- *Poisson equations*: $\Delta u = f$, if $f = 0$ then this is the Laplace equation (which has harmonic functions)
- *Wave equations*: $\Delta u - u_{tt}$
- *Minimal surface equation*: $\nabla \left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$
- Eikonal equation $|\nabla u| = 1$

2.

2.1. **Methods of solutions - Transport equation.**
A quick recall, a Quasi-linear PDE is on the form:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Example: Transport-equation

$$u_t + aux = 0$$
$$u(t=0,x) = h(x)$$

This can be solved with a number of methods.

Method 1 - Change of variables By letting $s = x - at$ and $v = x + at$, we get by the chain rule:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial u}{\partial s}(-a) + \frac{\partial u}{\partial v}a$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} + \frac{\partial u}{\partial v}$$

Then, by substitution into our original equation, we get:

$$0 = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 2a\frac{\partial u}{\partial v} \Rightarrow u = u(s) = u(x-at)$$

We assume $h \in C^1$

Method 2
Note that we can rewrite our problem as $(1,a) \bullet \nabla u = 0$, this means that the gradient of the solution must be orthogonal to the lines $(1,a)$ (where $a \neq 0$)
Thus, $u(x_0 + at, t) == c \quad \forall t \in \mathbb{R}$ and $u(x_0, 0) = 0$.
By initial condition, $u(x_0,0) = h(x_0)$. Since the lines can written as $x_0 + at = x$, we immediately obtain that by shifting:

$$u(x-at,0) = h(x-at) = u(x,t)$$

Method 2 (general)
For a general Quasi-linear PDE, we have $(a,b,c) \bullet (u_x, u_y, -1) = 0$
Some geometry; think of the graph S of the function $u = u(x,y)$ with $z = u(x,y)$. Since a,b,c are functions of x,y,u, they may be treated as functions in $\mathbb{R}^3$ and we may therefore define the vector field (a,b,c) on $\mathbb{R}^3$.

Definition/Sats 2.5: Integral surface of vector field

The graph (sruface) in the above mentioned paragraph is called the *integral surface* of $\overline{V}$

The equation will usually be given an initial condition, which we have to interpret geometrically. Naturallym, we prescribe a curve $\Gamma : S \to (x_0(s), y_0(s), z_0(s))$

Example
Consider the transport equation with initial conditions $u(x_0,0) = h(x_0)$. Think of t as y, then for $x = s$, $y = 0$, $z = h(s)$, we get:

$$\Gamma : S \to (s, 0, h(s))$$

Most Quasi-linear PDEs can be transformed this way

Method of characteristics
Idea is we want to define surface $S \subset \mathbb{R}^3$ representing the graph of a solution such that the vector field is tangent to S at any point and such that S is continuous along Γ
Take a poiunt $(x_0(s), y_0(s), z_0(s))$ and find a curve $\gamma_s(t) = (x(t), y(t), z(t))$ (integral curve of $\overline{V}$) along S

In this case, the curves equations are:

$$\begin{cases} x_t(t) = a(x(t), y(t), z(t)) \\ y_t(t) = b(x(t), y(t), z(t)) \\ z_t(t) = c(x(t), y(t), z(t)) \\ x(0) = x_0(s) \\ y(0) = y_0(s) \\ z(0) = z_0(s) \end{cases}$$

Potential problems
Only guaranteed existence for the system locally over t by the Picard-Lindelöf theorem.
The graph S may fold into itself so that $|\nabla u| = 0$
This happens during joining of local parametrization, where transitions may seem smooth.
Another problem is $\overline{V}$ may be paralel to Γ, the procedurew will not recover Γ

Example

$$uu_x + u_y = 0 \qquad y > 0$$
$$u(x, 0) = h(x)$$

Here we have $a = u = z$, $b = 1$, $c = 0$
Similar to before, $\Gamma : s \to (s, 0, h(s))$.