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Prov i matematik
Reell analys, 1MA226
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Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: Course material, lecture notes, old problems and solutions.

1. Consider the subset $A = \{(x,y) \in \mathbb{R}^2 \, : \, xy = 1\}$ of the metric space $\mathbb{R}^2$ (with its standard metric). Is A closed? Prove your claim.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:

(a). $x_n = \sum_{k=1}^{n} (-1)^k$.

(b). $x_n = \Big(1 + \dfrac{1}{n^{1/2}}\Big)^n \Big(1 + \dfrac{(-1)^n}{n^{3/2}}\Big)^{n^2}$.

3. Prove that the series $F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^3 + n}{x^2 + n^3}$ converges for all $x \in \mathbb{R}$, and that the function $F : \mathbb{R} \to \mathbb{R}$ is C^1.

4. Let f be a function from $[1,e]$ to $[0,1]$ satisfying

$$\big|f(x) - f(y)\big| \le \tfrac{1}{3}|x-y|, \qquad \forall x,y \in [1,e].$$

Prove that equation

$$\log x = f(x)$$

has a unique solution x in the interval $[1,e]$.

5. Let C be the closed unit square,

$$C = \{(x,y) \in \mathbb{R}^2 \, : \, x,y \in [0,1]\};$$

let f be a continuous function from C to $\mathbb{R}$, and let (a_n) be a sequence in $[0,1]$. Define the sequence of functions $f_1, f_2, \ldots$ from $[0,1]$ to $\mathbb{R}$ by $f_n(x) = f(x, a_n)$. Prove that this sequence (f_n) is equicontinuous.

6. Let f be the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by

$$f(x,y) = (x, (x+y)^3).$$

Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(0,0) \in U$ such that $f\big|_U$ is a bijection from U onto an open subset $V \subset \mathbb{R}^2$. Prove also that for any such open sets U and V, the inverse function $(f|_U)^{-1} : V \to U$ is *not* differentiable in all of V.

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7. Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = x$ if $x = 2^{-n}$ for some $n \in \mathbb{Z}^+$, otherwise $f(x) = 3$. Determine (with proof) the upper and lower Riemann integrals $\overline{\int_0^1} f(x)\,dx$ and $\underline{\int_0^1} f(x)\,dx$.

8. Let f be a continuous function from the real interval $[0,1]$ to a metric space (X,d), and assume that

$$\forall x, y \in [0,1] : \qquad \big[f(x) = f(y) \text{ or } d(f(x), f(y)) > \tfrac{1}{10}\big].$$

Prove that $f(0) = f(1)$.

LYCKA TILL / GOOD LUCK!