

UPPSALA UNIVERSITET

LECTURE NOTES

Complex Analysis

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CONTENTS

1. Intro	2
1.1. Operations over \mathbb{C}	2
1.2. Cartesian representation	4
1.3. Polar form	5
1.4. Exponential form	6
1.5. Logarithmic form	6
2. Elementary complex functions	8
2.1. Branches of the complex logarithm	8
2.2. Complex mappings	8
2.3. Complex powers	8
2.4. Trigonometric and Hyperbolic functions	10
2.5. Mapping properties of $\sin(z)$	11
3. Topology of \mathbb{C}	12
3.1. Limits and Continuity	14
3.2. The complex derivative	15
3.3. Analytic functions	17
4. Cauchy-Riemann's equations	18
4.1. Inverse mappings	19
5. Harmonic Functions	20

1. INTRO

In this course, we shall study functions $f : \mathbb{C} \rightarrow \mathbb{C}$ (or more generally, $f : D \rightarrow \mathbb{C}$ where $D \subseteq \mathbb{C}$)

Definition/Sats 1.1: Complex Number

A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$

Anmärkning:

The number x is called the *real part* ($\operatorname{Re}(z) = x$) of the complex number, and y is called the *imaginary part* ($\operatorname{Im}(z) = y$) of the complex number

Anmärkning:

The set of all complex numbers is denoted by \mathbb{C}

Anmärkning:

\mathbb{C} is the *smallest* field extension to \mathbb{R} that is algebraically closed.

Anmärkning:

$i^2 = -1$

1.1. Operations over \mathbb{C} .

We define the operations *addition* and *multiplication* of two complex numbers as follows:

Definition/Sats 1.2: Addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Definition/Sats 1.3: Multiplication of complex numbers

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations, \mathbb{C} forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Definition/Sats 1.4: Complex conjugate

The *complex conjugate* of a complex number $z = x + iy$, denoted by \bar{z} , is defined by $\bar{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$
- $\overline{\bar{z}} = z$
- $z \cdot \bar{z} = |z|^2$
- $z^{-1} = \frac{\bar{z}}{|z|^2}$
- $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$

Anmärkning:

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

Anmärkning:

Multiplication by i is simply rotation by $\frac{\pi}{2}$ counterclockwise.

Definition/Sats 1.5

Let $z \in \mathbb{C}$. Then there exists a $w \in \mathbb{C}$ such that $w^2 = z$ (where $-w$ also satisfies this equation)

Bevis 1.1

Let $z = a + bi$ and $w = x + iy$ such that $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then $a = x^2 - y^2$ and $b = 2xy$

We also know that $|z| = a^2 + b^2 = |x^2 + y^2|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore, $x^2 + y^2 = \sqrt{a^2 + b^2}$ and:

$$\left. \begin{array}{l} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left. \begin{array}{l} -x^2 + y^2 = -a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let $\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$ and $\beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$ and let $\sqrt{}$ denote the positive square root of positive real numbers.

If b is positive, then either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$

If b is negative, then either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$

Therefore, the equation has solutions $\pm(\alpha + \mu\beta i)$ where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$

□

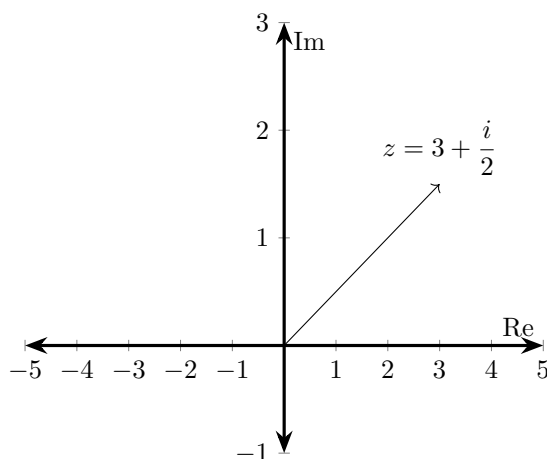
Anmärkning:

From the proof above, we can conclude the following:

- The square roots of a complex number are real \Leftrightarrow the complex number is real and positive
- The square roots of a complex number are purely imaginary \Leftrightarrow the complex number is real and negative
- The two square roots of a number coincide \Leftrightarrow the complex number is zero

1.2. Cartesian representation.

It is natural to represent a complex number $z = x + iy$ as a tuple (x, y) , and we can therefore represent it in the standard cartesian plane:



Anmärkning:

This is sometimes called the *complex plane*

Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number $z = x + iy$ (geometrically the length of the vector), denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:

- $|z|^2 = z \cdot \bar{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Anmärkning:

Every $z \in \mathbb{C}$ such that $z \neq 0$ (that is, $x \neq 0$ or $y \neq 0$) has a multiplicative inverse $\frac{1}{z}$ given by:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Definition/Sats 1.7: Triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that $|z_1 + z_2| \leq |z_1| + |z_2|$

Lemma 1.1: Reversed triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Bevis 1.2

$$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

So that $|z_1| - |z_2| \leq |z_1 - z_2|$ □

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
- $-\operatorname{Re}(z) \leq \operatorname{Re}(z) \leq |z|$
- $-\operatorname{Im}(z) \leq \operatorname{Im}(z) \leq |z|$
- $|\bar{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \dots + |w_n|^2}$

1.3. Polar form.

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, and hence there exists θ such that:

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta)$$

Therefore $z = x + iy$ can be written as:

$$z = r(\cos(\theta) + i \sin(\theta))$$

Where $r = |z|$ is uniquely determined by z , while θ is 2π -periodic. This is called the *polar form* of z and just as the cartesian representation requires a tuple of information $(|z|, \theta)$

Definition/Sats 1.8: Argument

The *argument* of a complex number z , denoted by $\arg(z)$, is the angle θ between z and the real number line in the complex plane

Anmärkning:

Since the argument is 2π periodic, the angle is usually given as $\theta + k2\pi$ $k \in \mathbb{Z}$, but we are only interested in θ

This θ is called the *principal value* of $\arg(z)$, denoted by $\operatorname{Arg}(z)$ and belongs to $(-\pi, \pi]$

Anmärkning:

We are always allowed to change an angle by multiples of 2π , the principal value argument is the angle after changing the argument such that it lies between $(-\pi, \pi]$

Anmärkning:

A specification of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that $\operatorname{Arg}(z)$ is "discontinuous" along the negative real axis. This is called a *branch-cut*

Suppose $z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1))$, $z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$

Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 [(\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))] \\ &\quad r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

Anmärkning:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

1.4. Exponential form.

Definition/Sats 1.9

For $z = x + iy \in \mathbb{C}$, let $e^z = e^x(\cos(y) + i \sin(y))$

Anmärkning:

$e^{iy} = \cos(y) + i \sin(y) \quad y \in \mathbb{R}$ (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$\begin{aligned} e^z &= e^{x+iy} = e^x \cdot e^{iy} \\ e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \\ \Rightarrow e^{iy} &= 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos(\theta)} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin(\theta)} \\ \Rightarrow e^z &= e^x(\cos(\theta) + i \sin(\theta)) \end{aligned}$$

Anmärkning:

One can through comparing see that $|e^z| = e^x$, and that $|e^{iy}| = 1$

Properties of the exponential form:

- $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$
- $e^z \neq 0 \quad \forall z \in \mathbb{C}$
- $x \in \mathbb{R} \Rightarrow e^x > 1$ if $x > 0$ and $e^x < 1$ if $x < 0$
- $|e^{x+iy}| = e^x$
- $e^{i\pi/2} = i \quad e^{i\pi} = -1 \quad e^{3i\pi/2} = -i \quad e^{2i\pi} = 1$
- e^z is 2π -periodic
- $e^z = 1 \Leftrightarrow z = 2\pi ki \quad k \in \mathbb{Z}$

Definition/Sats 1.10: deMoivre's formula

For $n \in \mathbb{Z}$, $(r(\cos(\theta) + i \sin(\theta)))^n = r^n(\cos(n\theta) + i \sin(n\theta))$

1.5. Logarithmic form.

In real analysis, we have defined the logarithm as the inverse of e^x . This has previously worked since for $x \in \mathbb{R}$, e^x is injective.

The problem is that for e^z where $z \in \mathbb{C}$, it is not injective and should therefore not have an inverse.

Given $z \in \mathbb{C} \setminus \{0\}$, we define $\ln(z)$ as the cut of all $w \in \mathbb{C}$ whose image under the exponential form is z , i.e $w = \ln(z) \Leftrightarrow z = e^w$.

Here, $\ln(z)$ is a *multivalued form*

We can use the fact that $|z| = r = e^x$ to derive some interesting properties of the logarithm:

$$\begin{aligned} z &= r e^{i\theta} & w &= u + iv \\ \text{If } z &= e^w \Leftrightarrow r e^{i\theta} = e^u \cdot e^{iv} \\ \Leftrightarrow u &= \ln(r) = \ln(|z|) & v &= \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z} \end{aligned}$$

Definition/Sats 1.11: Complex logarithm

For $z \neq 0$, we define the complex logarithm for $z \in \mathbb{C}$ as:

$$\begin{aligned}\ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\operatorname{Arg}(z) + k2\pi) \quad k \in \mathbb{Z}\end{aligned}$$

2. ELEMENTARY COMPLEX FUNCTIONS

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

2.1. Branches of the complex logarithm.

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of 2π).

If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

Definition/Sats 2.12: Principal logarithm

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\text{Ln}(z) = \ln(|z|) + i \cdot \text{Arg}(z)$$

Anmärkning:

We have essentially extended the "normal" logarithm, which is defined on $(0, \infty)$, to be defined on $\mathbb{C} \setminus \{0\}$

Anmärkning:

The principal logarithm is discontinuous for negative reals, since their principal value argument is $-\pi$, but the principal value argument is discontinuous at $-\pi$. This is the so called *branch-cut*

Anmärkning:

Even though the principal logarithm is discontinuous for negative reals, it is not undefined. Any negative real number z will have $\text{Arg}(z) = \pi$, which the logarithm very much is defined for.

Anmärkning:

When branching, we do not necessarily have to pick $(-\pi, \pi]$, we can pick any interval $(\alpha, \alpha + 2\pi]$. This is usually denoted by \arg_α .

2.2. Complex mappings.

One can think of a complex mapping $f: \mathbb{C} \rightarrow \mathbb{C}$ as $f(z) = f(x + iy) = w = u + iv$

Then it becomes clear which regions map to where by drawing them in their respective z -plane and w -plane.

2.3. Complex powers.

Given $z \in \mathbb{C}$, consider the following equation:

$$(1) \quad w^n = z$$

The set of all solutions w of (1) is denoted $z^{1/n}$ and is called the *n-th root of z*.

Anmärkning:

If $z = 0$, then $w = 0$

Suppose $z \neq 0$, then we may write $w = |w| e^{i\alpha}$ and $z = |z| e^{i\theta}$
 By deMoivre's formula, (1) becomes:

$$|w|^n e^{in\alpha} = |z| e^{i\theta}$$

Then, the following follows:

$$\left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ n\alpha &= \theta + k2\pi \quad k \in \mathbb{Z} \end{aligned} \right\} \Leftrightarrow \left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ \alpha &= \frac{\theta}{n} + k \frac{2\pi}{n} \quad k \in \mathbb{Z} \end{aligned} \right\}$$

Notice now that every $k \in \mathbb{Z}$ gives a solution to (1)

Since sine and cosine are both 2π -periodic, then only $k = 0, 1, \dots, n-1$ actually give *different* solutions
 (since $k = n \Rightarrow \alpha = \frac{\theta}{n} + n \frac{2\pi}{n}$)

Suppose $z \neq 0$. For $n \in \mathbb{Z}$ it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that $\ln(z)$ attains.

It is also true, that for $n = 1, 2, 3, \dots$:

$$\frac{1}{z^n} = e^{\frac{1}{n} \ln(z)}$$

We can let $n \in \mathbb{C}$, and obtain the following definition:

Definition/Sats 2.13: Complex power

For $\alpha \in \mathbb{C}$, let:

$$z^\alpha = e^{\alpha \ln(z)} \quad z \neq 0$$

Anmärkning:

This makes z^α a multivalued function, but it is possible to have a single-valued output from it.

Definition/Sats 2.14

Let $a, b \in \mathbb{C}$ where $a \neq 0$. Then a^b is single-valued (does not depend on the choice of branch for the logarithm) $\Leftrightarrow b \in \mathbb{Z}$

If $b \in \mathbb{Q}$ and is in lowest form (that is, $b = \frac{p}{q}$ where p, q have no common factors), then a^b has exactly q distinct values (the q :th roots of a^p)

If $b \in \mathbb{C} \setminus \mathbb{Q}$, then a^b has infinitely many values.

Bevis 2.1

Chose some interval (branch), say $[0, 2\pi)$, for the arg function and let $\ln(z)$ be the corresponding branch of the logarithm. If we chose another branch, we would have $\ln(a) + 2\pi kbi$ rather than $\ln(a)$ (where $k \in \mathbb{Z}$)

Therefore, $a^b = e^{b \ln(a) + 2\pi kbi} = e^{b \ln(a)} \cdot e^{2\pi ki}$

Notice that $e^{2\pi kbi}$ stays the same regardless of $b \in \mathbb{Z}$, as long as it is an integer.

In the same way, it can be shown that $e^{2\pi kip/q}$ has q distinct values if p, q have no common factor.

If b is irrational, and if $e^{2\pi kbi} = e^{2\pi mbi}$, then it follows that $e^{(2\pi bi)(k-m)} = 1$, and therefore $b(k-m)$ is an integer.

Since b is irrational, then $n - m = 0$

□

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up. We can chose to branch z^α :

$$z^\alpha = e^{\alpha \text{Ln}(z)}$$

2.4. Trigonometric and Hyperbolic functions.

We have the following:

$$\left. \begin{aligned} e^{iy} &= \cos(y) + i \sin(y) \\ e^{-iy} &= \cos(y) - i \sin(y) \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \cos(y) &= \frac{e^{iy} + e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy} - e^{-iy}}{2i} \end{aligned} \right\}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

Definition/Sats 2.15: Complex sine and cosine

For $z \in \mathbb{C}$, we define:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals. When defining them for complex numbers, we just extend their domain:

Definition/Sats 2.16: Complex hyperbolic functions

For $z \in \mathbb{C}$, we define:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

- **Sine:**

$$\begin{aligned} \sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &\Rightarrow \frac{e^{-y}(\cos(x) + i \sin(x)) - e^y(\cos(x) - i \sin(x))}{2i} = \frac{(e^{-y} - e^y) \cos(x) + i(e^y - e^{-y}) \sin(x)}{2i} \\ &= \frac{(e^{-y} - e^y) \cos(x)}{2i} + \frac{(e^y - e^{-y}) \sin(x)}{2} \\ &\stackrel{i^{-1} = -i}{\Rightarrow} \underbrace{\frac{(e^y - e^{-y})}{2}}_{\sinh(y)} i \cos(x) + \underbrace{\frac{(e^y + e^{-y})}{2}}_{\cosh(y)} \sin(x) \end{aligned}$$

- **Cosine:**

$$\begin{aligned} \cos(x + iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} \\ &= \frac{e^{-y}(\cos(x) + i \sin(x)) + e^y(\cos(x) - i \sin(x))}{2} = \frac{\cos(x)(e^y + e^{-y}) + i(e^{-y} - e^y) \sin(x)}{2} \\ &= \underbrace{\frac{e^y + e^{-y}}{2}}_{\cosh(y)} \cos(x) - \underbrace{\frac{e^y - e^{-y}}{2}}_{\sinh(y)} i \sin(x) \end{aligned}$$

This leads us to the following:

Definition/Sats 2.17: Addition formulas for complex trigonometric functions

- $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$
- $\cos(x + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$

Anmärkning:

Both sine and cosine can be defined as the unique solution to an ODE, namely:

$$\begin{aligned} f''(x) + f(x) &= 0 & f(0) &= 0, f'(0) = 1 & f(x) &= \sin(x) \\ f''(x) + f(x) &= 0 & f(0) &= 1, f'(0) = 0 & f(x) &= \cos(x) \end{aligned}$$

2.5. Mapping properties of $\sin(z)$.

Let $f(z) = \sin(z)$ in $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$, let A be the set of points allowed with respect to the above constraint and let B be the mapping of those points by $\sin(A)$

Claim: $f : A \rightarrow B$ is a bijective mapping

Bevis 2.2

Take a $z \in \mathbb{C}$ $z = x + iy$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Then:

$$\begin{aligned} f(z) &= \sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y) \\ f(z) \in \mathbb{R} &\Leftrightarrow \cos(x) \sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0 \end{aligned}$$

If $y = 0$, then:

$$f(z) = \sin(x) \cosh(y) = \sin(x) \in (-1, 1)$$

Therefore, if $z \in A \Rightarrow f(z) \in B$. Now we need to show that for any $z \in B$, there is a u such that $f(u) = z$

Let $u = \sin(x) \cosh(y)$, $v = \cos(x) \sinh(y)$ and pick a vertical line at $x = a \neq 0$

We will now consider the images of these lines:

$$\begin{aligned} \cosh(y) &= \frac{u}{\sin(a)} & \sinh(y) &= \frac{v}{\cos(a)} \\ (\cosh(y))^2 - (\sinh(y))^2 &= 1 \Rightarrow \left(\frac{u}{\sin(a)} \right)^2 - \left(\frac{v}{\cos(a)} \right)^2 = 1 \end{aligned}$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line $y = b \neq 0$

$$\begin{aligned} \sin(x) &= \frac{u}{\cosh(b)} & \cos(x) &= \frac{v}{\sinh(b)} \\ \cos^2(x) + \sin^2(x) &= 1 \Rightarrow \left(\frac{u}{\cosh(b)} \right)^2 + \left(\frac{v}{\sinh(b)} \right)^2 = 1 \end{aligned}$$

This is a half-ellipse. Note that $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

□

3. TOPOLOGY OF \mathbb{C} **Definition/Sats 3.18: Open disc**

The set $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is called the *open-disc* with center z_0 and radius r

Anmärkning:

Since we have a strict inequality, it is open. If we had \leq , it would be a closed disc.

Definition/Sats 3.19: Open subset

A subset M of \mathbb{C} is called *open* if for every $z_0 \in M$ there exists an $r > 0$ such that $D_r(z_0) \subseteq M$

Definition/Sats 3.20: Interior point

A point $z_0 \in M$ is called an *interior-point* of M if there exists an $r > 0$ such that $D_r(z_0) \subseteq M$

Definition/Sats 3.21: Boundary point

A point $z_0 \in \mathbb{C}$ is called a *boundary point* of M if $\forall r > 0$ it holds that:

$$D_r(z_0) \cap M \neq \emptyset \quad \wedge \quad D_r(z_0) \cap M^c \neq \emptyset$$

Anmärkning:

The set of all interior points of M is denoted by $\text{int}(M)$ and the set of all boundary points of M is denoted by ∂M

The following equivalences hold:

- M is closed $\Leftrightarrow \partial M \subseteq M$
- M is open $\Leftrightarrow \partial M \subseteq M^c$
- \mathbb{C} is clopen
- \emptyset is clopen
- The union of any collection of open subsets of \mathbb{C} is open
- The intersection of any finite collection of open subsets of \mathbb{C} is open

Definition/Sats 3.22: Closed set

We say that a set $X \subseteq \mathbb{C}$ is closed if its complement X^c is open

Definition/Sats 3.23: Polygonal path

A polygonal path P (sometimes called piecewise linear curve) is a curve specified by a sequence of points (A_1, A_2, \dots, A_n) .

The curve itself consists of line segments connecting the consecutive points.

Definition/Sats 3.24: polygonal-path-connected open set

An open set M is called *polygonal-path-connected* if every pair of points $z_1, z_2 \in M$ can be connected by a polygonal path contained in M

Anmärkning:

Some would call this just path-connected, or even just connected. This works in \mathbb{R}^n (recall that $\mathbb{C} \cong \mathbb{R}^2$). Topologically speaking, polygonal-path-connectedness \implies path-connectedness

Anmärkning:

A set X is connected \Leftrightarrow the only subsets of X which are clopen are \emptyset and X

Anmärkning:

One can assume the polygonal paths to have segments parallel to the ordinate ones.

Anmärkning:

An open connected set is called a *domain*

Definition/Sats 3.25

Suppose that $u(x, y)$ is a real-valued function defined in a domain $D \subseteq \mathbb{R}$

Also suppose that:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} =$$

in all of D . Then u is contained in D

Definition/Sats 3.26: Simply connected

A domain $D \subseteq \mathbb{C}$ is called *simply connected* if ever closed curve in D can be, within D , continuously deformed to a point

Anmärkning:

Topologically speaking, D is homeomorphic to a point.

Definition/Sats 3.27: Non-connectedness

A set $A \subseteq \mathbb{C}$ is *not connected* if there are open sets U and V such that:

- $A \subseteq U \cup V$
- $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$

3.1. Limits and Continuity.

Definition/Sats 3.28: Complex limit

A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to have the limit z_0 (*converges to* z_0) if for every given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$|z_n - z_0| < \varepsilon \quad \forall n \geq N$$

We write this as:

$$\lim_{n \rightarrow \infty} z_n = z_0$$

Anmärkning:

Every cauchy sequence in \mathbb{C} converges.

Anmärkning:

$z_n \rightarrow z_0 \Leftrightarrow \operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z_0)$ and $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z_0)$

This follows from $|x|, |y| \leq \sqrt{x^2 + y^2} \leq |x| + |y|$

Definition/Sats 3.29

Let f be a function defined in a punctured neighborhood of z_0

We say that f has the limit w_0 as $z \rightarrow z_0$, if for every given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

We write this as:

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

Anmärkning:

If a limit exists, it is unique.

Definition/Sats 3.30

For $z = x + iy$, let:

$$u(x, y) = \operatorname{Re}(f(z)) \quad v(x, y) = \operatorname{Im}(f(z))$$

Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$

Then the following holds:

$$\lim_{z \rightarrow z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \end{cases}$$

Definition/Sats 3.31: Continuous function

Let f be a function defined in a neighborhood of z_0 .

f is said to be continuous at z_0 if:

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

A function f is said to be *continuous on the (open) set* M if it is continuous at each point of M

Anmärkning:

The following statements are equivalent (for $f : A \rightarrow \mathbb{C}$):

- f is continuous
- The inverse image of every closed set is closed relative to A
- The inverse image of every open set is open relative to A
- The image set $f(A)$ is connected

Assume $\lim_{z \rightarrow z_0} f(z) = A$ and $\lim_{z \rightarrow z_0} g(z) = B$

The following properties from the real limit hold for the complex limit:

- $\lim_{z \rightarrow z_0} (f(z) \pm g(z)) = A \pm B$
- $\lim_{z \rightarrow z_0} f(z)g(z) = AB$
- $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad B \neq 0$

Anmärkning:

If f, g are continuous at z_0 , then so are $f \pm g$ and fg . The quotient is only continuous if $g(z_0) \neq 0$

Anmärkning:

Constant functions, polynomials, and rational functions (whenever the denominator is non-zero) are all continuous in \mathbb{C}

3.2. The complex derivative.

Analogous to the real case, we also have the following:

Definition/Sats 3.32: Differentiability

Let f be a complex-valued function defined in a neighborhood of z_0 .

We say that f is differentiable at z_0 if the limit:

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

The limit is called the *derivative* of f at z_0 , and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$

Anmärkning:

Since Δz is a complex number, it can approach 0 from different directions. In order for the derivative to exist, the results must be independent of the direction of which Δz approaches 0 (i.e., approaches 0 from all directions)

Anmärkning:

If X is an open connected set and $a, b \in X$, then there is a differentiable path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = a$ and $\gamma(1) = b$

Example:

The function $f(z) = \bar{z}$ is nowhere differentiable since:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{z_0 + \Delta z} - \bar{z}_0}{\Delta z} = \frac{\bar{\Delta z}}{\Delta z} = \frac{\bar{\Delta x} + i\bar{\Delta y}}{\Delta x + i\Delta y}$$

As $\Delta z \rightarrow 0$ from the x -direction (real-line), the limit becomes $\frac{\bar{x}}{x} = 1$

However, as we approach from the y -direction (complex axis), the limit becomes $\frac{\bar{iy}}{iy} = \frac{-y}{y} = -1$

Since x, y were chosen arbitrarily, this applies to all x, y . Since the limits did not match, it is not differentiable and at no point.

Of course, all the properties from the real case hold here as well.

Suppose f, g are differentiable at z , then:

- $(f \neq g)'(z) = f'(z) \neq g'(z)$

- $(cf)'(z) = cf'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f \circ g)'(z) = f'(g(z))g'(z)$

3.3. Analytic functions.

Definition/Sats 3.33: Analytic function

A complex-valued function f is said to be *analytic* in an open set G if f is differentiable at every point in G .

We say that f is *analytic at z_0* if f is differentiable in a neighborhood of z_0 .

Anmärkning:

If f is analytic in all of \mathbb{C} , then f is said to be *entire* (or *holomorphic*).

Definition/Sats 3.34

If an entire function $f(z)$ has a root at w , then:

$$\lim_{z \rightarrow w} \frac{f(z)}{(z - w)}$$

is an entire function.

4. CAUCHY-RIEMANN'S EQUATIONS

Suppose $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$

Then:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}$$

1) Let $\Delta z = \Delta x$ (i.e $\Delta y = 0$):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

2) Let $\Delta z = i\Delta y$ (i.e $\Delta x = 0$):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

It must therefore hold that:

$$u_x + iv_x = -iu_y + v_y$$

This leads to the Cauchy-Riemann equations:

$$\left. \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \right\}$$

We have therefore arrived at the following:

Definition/Sats 4.35

A necessary condition for $f = u + iv$ to be differentiable at $z_0 = x_0 + iy_0$ is that the Cauchy-Riemann equations are satisfied at (x_0, y_0)

Anmärkning:

We also saw that if f is differentiable at the point z_0 , then the derivative is given by:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

The following provides a sufficient condition for Differentiability:

Definition/Sats 4.36

Suppose that $f = u + iv$ is defined in a open set G containing $z_0 = x_0 + iy_0$.

Suppose also that u_x, u_y, v_x, v_y exists in G and are continous at (x_0, y_0) , and satisfy the Cauchy-Riemann equations at (x_0, y_0)

Then f is differentiable at z_0

Anmärkning:

Cauchy-Riemann equations + $u, v \in C^1 \Rightarrow f$ is differentiable

Bevis 4.1

In view of the continuity of the first partial derivatives at (x_0, y_0) , it holds that:

$$\begin{aligned} u(x_0 + \Delta x, y_0 + \Delta y) &= u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_1(\Delta x, \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) &= v(x_0, y_0) + v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_2(\Delta x, \Delta y) \end{aligned}$$

Where $\rho_1, \rho_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$

Then:

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= u_x(x_0, y_0)\Delta x + \underbrace{u_y(x_0, y_0)}_{= -v_x(x_0, y_0)}\Delta y + i(v_x(x_0, y_0)\Delta x + \underbrace{v_y(x_0, y_0)}_{= u_x(x_0, y_0)}\Delta y) \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \\ &\stackrel{\text{CR-eq.}}{=} u_x(x_0, y_0)\Delta z + iv_x(x_0, y_0)\Delta z + |\Delta z|(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \end{aligned}$$

Since $\rho_1, \rho_2 \rightarrow 0$ as $\Delta z \rightarrow 0$, it follows that:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is equal to $u_x(x_0, y_0) + iv_x(x_0, y_0)$ □

4.1. Inverse mappings.

Suppose $f = u + iv$ is analytic in a domain D (with f' continuous).

Consider the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

As a mapping of $D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Its Jacobian matrix:

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

has determinant:

$$\det(J_f) = u_x v_y - u_y v_x \stackrel{\text{CR-eq.}}{=} u_x^2 + v_x^2 = |f'(z)|^2$$

The inverse function then leads to the following:

Definition/Sats 4.37: Inverse function theorem

Suppose $f(z)$ is analytic on a domain D with $f'(z) \neq 0$ continuous.

Then there is a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \rightarrow V$ is bijective, and the inverse function $f^{-1} : V \rightarrow U$ is analytic with derivative:

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \quad w = f(z)$$

5. HARMONIC FUNCTIONS

Definition/Sats 5.38: Harmonic function

A real-valued function $\phi(x, y)$ is said to be *harmonic* in a domain D if $\phi \in C^2(D)$ and ϕ satisfies Laplace's equations:

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

in D

Definition/Sats 5.39

Suppose $f = u + iv$ is analytic in a domain D . Then u, v are harmonic in D

Bevis 5.1

One can show that $u, v \in C^\infty$:

$$\begin{aligned} u_x = v_y &\Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x &\Rightarrow u_{yy} = -v_{xy} \end{aligned}$$

As $v_{yx} = v_{xy}$, we have $u_{xx} + u_{yy} = 0$

Similarly, $v_{xx} + v_{yy} = 0$

□

Definition/Sats 5.40: Harmonic Conjugacy

If u is harmonic in a domain D and v is a harmonic function in D such that $u + iv$ is analytic in D , then we say that v is a *harmonic conjugate* of u in D

Definition/Sats 5.41

If u is harmonic in a simply connected domain $D \subseteq \mathbb{C}$, then there exists a harmonic conjugate v of u in D , and v is unique up to addition of a real constant

Bevis 5.2

Suppose u is harmonic in $D \subseteq \mathbb{C}$

Consider the vector-field $\overline{F} = (-u_y, u_x) \in C^1(0)$.

Note that:

$$\frac{\partial F_1}{\partial y} = -u_{yy} \stackrel{u \text{ harm.}}{=} u_{xx} = \frac{\partial F_2}{\partial x}$$

Since D is simply connected $\Rightarrow \overline{F}$ is conservative $\Rightarrow \exists v : \nabla v = \overline{F}$, i.e. $(v_x, v_y) = (-u_y, u_x)$

$$\Rightarrow \left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \Rightarrow f = u + iv \text{ is analytic in } D$$

If \bar{v} is another harmonic conjugate, then:

$$\begin{aligned} \bar{v}_x &= -u_y = v_x \\ \bar{v}_y &= u_x = v_y \\ \Rightarrow \nabla(v - \bar{v}) &= \bar{0} \Rightarrow v - \bar{v} = c \in \mathbb{C} \end{aligned}$$

□

Anmärkning:

A vector field is conservative if it is the gradient of some function.

It has the property that its line integral is path independent.