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Counting lattice paths in restricted planes

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Abstract

The number of lattice paths of fixed length consisting of unit steps in the north, south, east or west directions in the plane $\{(x,y)\in\mathbb{R}^2 \mid 0\leqslant y\leqslant x\}$ is shown. Also, the paths which do not cross the line $y=-x+a$ for a positive integer a, in the plane $\{(x,y)\in\mathbb{R}^2 \mid 0\leqslant y\leqslant x\}$ are enumerated. The proofs are purely combinatorial, using the bijections, the technique of the enumeration of noncrossing paths and the reflection principle. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

We consider lattice paths in the plane consisting of four unit steps in the north, south, east and west directions; we call such paths *NSEW-paths* for brevity. Such NSEW-paths have already been investigated [1,4–6,9,11]. The number of NSEW-paths with n steps from the origin to a point (a,b) in a plane $\mathbb{R}^2$ without restriction is known as

$$\binom{n}{\frac{n-a-b}{2}}\binom{n}{\frac{n+a-b}{2}}$$

in which the bionomial coefficient $\binom{n}{k}$ is asumed to be zero; k is nonnegative integer. Let Π be $\{(x,y)\in\mathbb{R}^2 \mid 0\leqslant y\leqslant x\}$. We want to enumerate the NSEW-paths starting from the origin and moving only interior to the plane Π. Our motivation comes from the work of Guy et al. [11] who enumerate the NSEW-paths in the plane limited by the x-axis or both x-axis and y-axis. We can also cite the work of Gouyou-Beauchamps [9,10], who enumerate the NSEW-paths to find the number of the standard Young tableaux with height bounded by 4 or 5 [9,10].

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We define Ω_a and Γ_a, as the sections of Π, for a positive integer a, as follows:

Ω_a, the set of points below the line $y=-x+a$ in Π except the points on the line,
Γ_a, the set of points on the left side of left the line $x=a$ in Π
except the points on the line.

In this paper, we provide the combinatorial proofs of those NSEW-paths of length n going from the origin to a point (a,b) in Π equal to

$$\frac{(b+1)(a+2)(a-b+1)(a+b+3)n!(n+2)!}{((n-a-b)/2)!((n-a+b)/2+1)!((n+a-b)/2+2)!((n+a+b)/2+3)!},$$

and we also give the number of the NSEW-paths lying in Ω_a or in Γ_a. We extend the plane Π to the quarter plane $\Xi=\{(x,y)\in\mathbb{R}^2 \mid x\geqslant 0,\text{ and } |y|\leqslant x\}$ and we can give the number of NSEW-paths of length n joining from the origin to a point (a,b) in the Ξ, by applying the work to be used for the number of NSEW-paths in the plane Π.

2. NSEW-paths in restricted planes

Let an alphabet be a finite set of letters. A word is finite sequence of letters and a empty word will be denoted by ε as a unit. Consider two alphabets $W=\{x,\bar{x},y,\bar{y}\}$ and $A=\{a,\bar{a}\}$. We denote by W^* the set of all words generated by letters in the alphabet W, and we denote by A^* the set of all words generated by letters in the alphabet A. The length of a word f, denoted by $|f|$, is the number of letters of f.

For a letter x, $|f|_x$ means the number of letters x in f. A word f' is a left factor of a word f in W^* if there exists a word $f''\in W^*$ such that $f=f'f''$.

For $f\in W^*$ we define the two mappings δ_x and δ_y from W^* to N by

$$\delta_x(f)=|f|_x-|f|_{\bar{x}} \quad\text{and}\quad \delta_y(f)=|f|_y-|f|_{\bar{y}}$$

In the same way, for $h\in A^*$, we define the mapping β from A^* to N by

$$\beta(h)=|h|_a-|h|_{\bar{a}}.$$

A set of words $\tilde{D}\subseteq A^*$ is called *Dyck language* if every word h of $\tilde{D}$ satisfies the following conditions:

(i) $\beta(h)=0$,
(ii) For h' left factor h, $\beta(h')\geqslant 0$.

The set of all left factors of words $\tilde{D}$, of length l and of image p by β, is denoted by $B_{l,p}$, where l and p have the same parity. If we code the North-East steps by a and South-East steps by $\bar{a}$, then the Dyck word codes the Dyck path. In the same way, we see that the word of $B_{l,p}$ also codes a left factor of a Dyck path of length l and reaching at height p, l and p having same parity [3].

We then have

$$|\tilde{D} \cap A^{2n}| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where C_n is the nth Catalan number, and

$$|B_{l,p}| = (p+1)\frac{l!}{((l-p)/2)!((l+p)/2+1)!}.$$

Definition 1. Let $S_{n,p,q}$ be the language composed words f of W^* verifying the following conditions:

(i) $|f| = n$,
(ii) $\delta_x(f) = p$, $\delta_y(f) = q$,
(iii) $\delta_x(f') \geqslant \delta_y(f') \geqslant 0$, for every left factor f' of f.

If we code East step (or West step) by x (or $\bar{x}$) and if we code North step (or South) by y (or $\bar{y}$), then the language $S_{n,p,q}$ can be identical with the set of NSEW-paths of length n joining the origin with a point (p,q) lying on the plane $\Pi = \{(x,y) \in \mathbb{R}^2 | 0 \leqslant y \leqslant x\}$. The pair (g,h) of $B_{n,q} \times B_{n,p}$, $0 \leqslant q \leqslant p \leqslant n$, ($n, p$ and q of same parity), is a *noncrossing word* if, for every left factor h' (resp. g') of h (resp. g) such that $|h'| = |g'|$, we have $\beta(h') \leqslant \beta(g')$.

We denote by $R_{n,p,q}$ the set of pairs (g,h) which do not cross the words of $B_{n,p-q} \times B_{n,p+q}$ (n and $p+q$ of same parity).

Lemma 1. *There is a bijection between $S_{n,p,q}$ and $R_{n,p,q}$.*

Proof. We define the mapping I of Z^* in $Z^* \times Z^*$ in the following way: $I(x) = (a,a)$, $I(\bar{x}) = (\bar{a},\bar{a})$, $I(y) = (\bar{a},a), I(\bar{y}) = (a,\bar{a})$.

For a word f of $S_{n,p,q}$, let $I(f) = (g,h)$. We find that $|f| = |g| = |h|$, by the construction of mapping I. For left factors f', g', h' of j, g and h, respectively, we have $\beta(g') = \delta_x(f') - \delta_y(f')$ and $\beta(g') = \delta_x(f') + \delta_y(f')$. We then have $\beta(h') \geqslant \beta(g') \geqslant 0$, $\beta(g) = p - q$ and $\beta(h) = p + q$. So, (g,h) is a pair of noncrossing words.

Reciprocally, from a pair of words of $R_{n,p,q}$, we can construct a word f of length n by the inverse operation. Let f', g' and h' be the left factors of j, g and h. It is shown that $\delta_x(f') = (\beta(h') + \beta(g'))/2$ and that $\delta_x(f') = (\beta(h') - \beta(g'))/2$. We have that $\delta_x(f) = p$ and $\delta_y(f) = q$ and $\delta_x(f') \geqslant \delta_y(f') \geqslant 0$. $\square$

The pair (g,h) of $B_{n,p} \times B_{n+2,p+2}$ (n and p of same parity) forms *nontouching words* if, for all left factor h' (resp. g') of h (resp. g) such that $|h'| = |g'| + 2$, we have $\beta(h') > \beta(g')$.

We denote by $T_{n,p,q}$ the set of pairs of nontouching words of $B_{n,p-q} \times B_{n+2,p+q+2}$, $p \geqslant q \geqslant 0$ (n, p and q of same parity). We remark that $|T_{n,p,q}| = |R_{n,p,q}|$, because (g,h) is contained in $R_{n,p,q}$ if and only if (g, aah) is contained in $T_{n,p,q}$. To prove the following

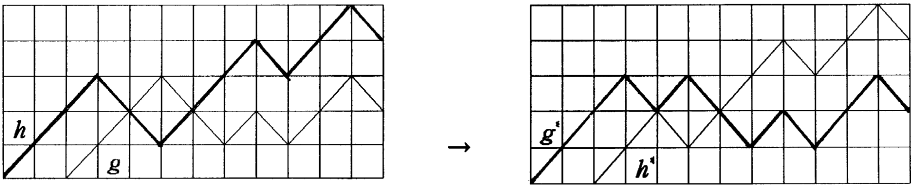

Fig. 1. The correspondence of two pairs of words which touch each other at least one point.

theorem, we use a technique developed by Gessel and Viennot [7] (see also [8,12]) for the enumeration of the number of noncrossing paths.

Theorem 1. *The cardinality of* $S_{n,p,q}$ *is equal to*

$$|S_{n,p,q}| = \frac{(q+1)(p+2)(p-q+1)(p+q+3)n!(n+2)!}{((n-p-q)/2)!((n-p+q)/2+1)!((n+p-q)/2+2)!((n+p+q)/2+3)!}.$$

Proof. $S_{n,p,q}$ is the set of noncrossing words of $B_{n,p-q} \times B_{n,p+q}$ (n and $p+q$ of same parity). We can say that $|S_{n,p,q}|$ is equal to the cardinality of the set of nontouching words of $B_{n,p-q} \times B_{n+2,p+q+2}$.

Let (g,h) be an element of $B_{n,p-q} \times B_{n+2,p+q+2}$. If g and h touch each other, then we denote by $g_3, g_4, \ldots, g_{n+2}$ the letters of A composing g and by $h_1, h_2, \ldots, h_{n+2}$ the letters of A composing h.

Let j $(3 \leqslant j \leqslant n+1)$ be the smallest index such that $\beta(h_1 h_2, \ldots, h_j) = \beta(g_3 g_4, \ldots, g_j)$. We make two words g^* and h^* in the following way:

$$g^* = h_1 h_2, \ldots, h_j g_{j+1} g_{j+2}, \ldots, g_{n+2},$$

$$h^* = g_3 g_4, \ldots, g_j h_{j+1} h_{j+2}, \ldots, h_{n+2}.$$

We have that $(g^*, h^*) \in B_{n+2,p-q} \times B_{n,p+q+2}$.

Reciprocally, from an element $(g^*, h^*) \in B_{n+2,p-q} \times B_{n,p+q+2}$ we can obtain $(g,h) \in B_{n,p-q} \times B_{n+2,p+q+2}$ where they touch each other. In fact, the Dyck words, which correspond to g^* and h^*, cross each other and so they touch each other at least one time. We invert the two paths from the first point where they meet. This correspondence is bijective (cf. Fig. 1).

So $S_{n,p,q}$ is the cardinality of $B_{n,p-q} \times B_{n+2,p+q+2}$ minus the cardinality of pairs of words $(g,h) \in B_{n+2,p-q} \times B_{n,p+q+2}$ that is, $|B_{n,p-q}||B_{n+2,p+q+2}| - |B_{n+2,p-q}||B_{n,p+q+2}|$.

By the definition of $B_{n,p}$, we can obtain the following equality:

$$\begin{aligned} |S_{n,p,q}| &= |R_{n,p,q}| \\ &= |B_{n,p-q}||B_{n+2,p+q+2}| - |B_{n+2,p-q}||B_{n,p+q+2}| \\ &= \frac{(q+1)(p+2)(p-q+1)(p+q+3)n!(n+2)!}{((n-p-q)/2)!((n-p+q)/2+1)!((n+p-q)/2+2)!((n+p+q)/2+3)!}. \qquad \square \end{aligned}$$

A generalised Young tableau of shape $(p_1, p_2, \ldots, p_m)$ is an array of $p_1 + p_2 + \cdots + p_m$ positive integer into m left-justified rows, with p_i elements in row i, and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$); the numbers in each column are in nondecreasing order from bottom to top, and the numbers in each row are strictly increasing order from left to right.

Corollary 1. *The cardinality of* $S_{2n+1,q+1,q}$ *is equal to the number of generalised Young tableaux with height bounded by* 4, *having* q *columns with odd height*

$$|S_{2n+1,q+1,q}| = \frac{\binom{n}{q}\binom{q+3}{q}}{\binom{n+q+4}{q}} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{4+i+j}{i+j}$$

and the number of generalised Young tableaux with height bounded by 4 *is equal to*

$$\sum_{q=0}^{n} |S_{2n+1,q+1,q}| = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{3+i+j}{i+j-1} = \frac{1}{2} C_{n+1} C_{n+2}.$$

Proof. By the correspondence used in the Theorem 1, $|B_{2n+1,1}||B_{2n+3,2q+3}| - |B_{2n+3,1}|\,|B_{2n+1,2q+3}|$ is equal to the number of configurations of two nontouching Dyck paths $\{w_1, w_2\}$, w_1 going from $(0,0)$ to $(2n+1, 1)$ and w_2 going from $(-2,0)$ to $(2n+1, 2q+3)$. We can then obtain the cardinality of $S_{2n+1,q+1,q}$ from the formula

$$\frac{\binom{n}{p}\binom{2k+p-1}{p}}{\binom{n+2k+p}{p}} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{2k+i+j}{i+j}$$

for the number of generalised Young tableaux, with entries between 1 and n, with p columns having an odd number of elements and having at most $2k$ rows [2].

With the formula $\prod_{1 \leqslant i \leqslant j \leqslant n} (m+i+j-1)/(i+j-1)$, for the number of generalised Young tableaux having at most m rows [8], we find that the number of generalised Young tableaux with height bounded by 4 is equal to

$$\prod_{1 \leqslant i \leqslant j \leqslant n} \frac{3+i+j}{i+j-1} = \frac{1}{2} C_{n+1} C_{n+2}. \qquad \square$$

In particular case, Gouyou-Beauchamps [9] has given that $\sum_{p=0}^{2n} |S_{2n,p,0}| = C_n C_{n+1}$, and $\sum_{p=0}^{2n+1} |S_{2n+1,p,0}| = C_{n+1} C_{n+1}$, and he has shown that $\sum_{p=0}^{2n} |S_{2n,p,0}| = C_n C_{n+1}$ is equal to the number of standard Young tableaux with height bounded by 4, having $2n$ cells, and $\sum_{p=0}^{2n+1} |S_{2n+1,p,0}| = C_{n+1} C_{n+1}$ is equal to the number of standard Young tableaux with height bounded by 4, having $2n+1$ cells.

Proposition 1. *Let* $M_{\Omega_d}(n, p, q)$ *be the number of* NSEW-*path joining the origin with a point* (p, q), *in* Ω_d *for a positive integer* d. *Then we have*

$$M_{\Omega_d}(n, p, q) = \sum_{i \geqslant 0, (x_{2i} + y_{2i}) \leqslant n} |S_{n, x_{2i}, y_{2i}}| - \sum_{i \geqslant 0, (x_{2i+1} + y_{2i+1}) \leqslant n} |S_{n, x_{2i+1}, y_{2i+1}}|,$$

where

$$(x_0, y_0) = (p,q), (x_1, y_1) = (-q+d, -p+d)(x_2, y_2) = (-x_1 + 2d(d+1), y_1),$$

$$(x_{2i+1}, y_{2i+1}) = \left(-y_{2i} + 2^i(d+2) + \sum_{k=1}^{i-1} 2^k - 1, -x_{2i} + 2^i(d+2) \right.$$
$$\left. + \sum_{k=1}^{i-1} 2^k - 1\right) \quad \textit{for } i \geqslant 1$$

and

$$(x_{2i}, y_{2i}) = \left(-x_{2i-1} + 2^i(d+2) + \sum_{k=1}^{i-1} 2^k, -y_{2i-1}\right) \quad \textit{for } i \geqslant 2.$$

Proof. Let $A_0 = (p,q)$. The point $(-q+d, -p+d)$ is symmetric to (p,q) w.r.t. the line $y = -x + a$. So the number of NSEW-paths joining the origin with (p,q), touching the line $y = -x + a$ at a point (s,t) for the first time is equal to the number of NSEW-paths joining the origin with $(-q+d, -p+d)$ going through (s,t). Let $A_1 = (-q+d, -p+d)$. We can conclude that $M_{\Omega_d}(n, p, q) = |S_{n,p,q}|$ — (the number of NSEW-paths joining the origin with $A_1 = (-q+d, -p+d)$ without touching the line $x = d+1$).

The point $(q, -p+d)$ is symmetric to $(-q+d, -p+d)$ w.r.t. the line $x = d+1$. Let $A_2 = (q, -p+d)$. The number of NSEW-paths joing the origin with $A_1 = (-q+d, -p+d)$ touching the line $x = d+1$ at a point (s', t') for the first time is equal to the number of NSEW-paths joining the origin with $(q, -p+d)$, going through the point (s', t').

So we have $M_{\Omega_d}(n, p, q) = |S_{n,p,q}| - \{|S_{n,-q+d,-p+d}|$ – (the number of NSEW-paths joining the origin with $A_2 = (x_2, y_2) = (q, -p+d)$, nontouching the line $y = -x + 2(d+1))\}$. If we use the reflection principle recursively with the same procedure then we find that

$$\begin{aligned} M_{\Omega_d}(n, p, q) &= |S_{n,p,q}| - \{|S_{n,-q+d,-p+d}| - \{|S_{n,q,-p+d}| - \backslash \ldots \backslash\}\} \\ &= \sum_{i \geqslant 0, (x_{2i}+y_{2i}) \leqslant n} |S_{n,x_{2i-1},y_{2i-1}}| - \sum_{i \geqslant 0, (x_{2i+1}+y_{2i+1}) \leqslant n} |S_{n,x_{2i},y_{2i}}|, \end{aligned}$$

where

$$(x_0, y_0) = (p,q), (x_1, y_1) = (-q+d, -p+d), (x_2, y_2) = (-x_1 + 2d(d+1), y_1),$$

$$(x_{2i+1}, y_{2i+1}) = \left(-y_{2i} + 2^i(d+2) + \sum_{k=1}^{i-1} 2^k - 1, -x_{2i} + 2^i(d+2) + \sum_{k=1}^{i-1} 2^k - 1\right)$$
$$\text{for } i \geqslant 1$$

and

$$(x_{2i}, y_{2i}) = (-x_{2i-1} + 2^i(d+2) + \sum_{k=1}^{i-1} 2^k, -y_{2i-1}) \quad \text{for } i \geqslant 2. \qquad \square$$

In the same way, we can obtain the number $K_{\Gamma_a}(n,p,q)$ of NSEW-path of length n, joining the origin with a point (p,q), in Γ_a for a positive integer a, which is equal to

$$K_{\Gamma_a}(n,p,q)=\sum_{i\geqslant 0,(x_{2i}+y_{2i})\leqslant n}|S_{n,x_{2i},y_{2i}}|-\sum_{i\geqslant 0,(x_{2i+1}+y_{2i+1})\leqslant n}|S_{n,x_{2i+1},y_{2i+1}}|,$$

where

$$(x_0,y_0)=(p,q),(x_1,y_1)=(-p+2d,q),$$

$$(x_2,y_2)=(-q+2a+1,-x_1+2a+1),$$

$$(x_{2i},y_{2i})=\left(-y_{2i-1}+2^i(a+1)+\sum_{k=0}^{i-1}2^k,-x_{2i-1}+2^i(a+1)+\sum_{k=0}^{i-1}2^k\right)\quad \text{for } i\geqslant 2$$

and

$$(x_{2i+1},y_{2i+1})=\left(-x_{2i}+2^i(a+1)+\sum_{k=1}^{i-1}2^k,-y_{2i}\right)\quad \text{for } i\geqslant 1.$$

We can conclude that $M_{\Omega_a}(n,p,q)=|S_{n,p,q}|-K_{\Gamma_a}(n,p,q)$.

Now, we extend the plane Π to the quarter plane $\Xi=\{(x,y)\,|\,x\geqslant 0$, and $|y|\leqslant x\}$ and we can obtain the number of NSEW-paths in the Ξ in the similar way used in Lemma 1.

Let $V_{n,p,q}$ be the language composed of words of W^* verifying the following properties:

(i) $|f|=n$,
(ii) $\delta_x(f)=p$, and $\delta_y(f)=q$,
(iii) $\delta_x(f')\geqslant|\delta_y(f')|\geqslant 0$, for every left factor f' of f.

If we code the East step (or West step) by x (or $\bar{x}$) and if we code the North step (or South step) by y (or $\bar{y}$), then the language $V_{n,p,q}$ can be identical with the set of NSEW-paths of length n, lying on the plane Ξ and going from (0,0) to (p,q).

Let $Q_{n,p,q}$ be the set of pairs (g,h) of words of $B_{n,p-q}\times B_{n,p+q}$, where $-p\leqslant q\leqslant p$ (n and $p+q$ are of same parity). In the similar way used in Lemma 1, we can show that there exists a bijection between $V_{n,p,q}$ and $Q_{n,p,q}$. So $|V_{n,p,q}|$ is equal to $|Q_{n,p,q}|=|B_{n,p-q}||B_{n,p+q}|$, and we have the equality

$$|B_{n,p-q}||B_{n,p+q}|=\frac{(p-q+1)(p+q+1)n!n!}{((n+p-q)/2+1)!((n-p+q)/2)!((n+p+q)/2+1)!((n-p-q)/2)!}.$$

If $f=a_1a_2\ldots a_{k-1},a_k$ is a word in A^* for some positive integer k, then we denote $\bar{f}=\bar{a}_k\bar{a}_{k-1}\ldots\bar{a}_2\bar{a}_1$, where

$$\bar{a}_j=\begin{cases} a & \text{if } a_j=\bar{a},\\ \bar{a} & \text{if } a_j=a.\end{cases}$$

for each j with $1\leqslant j\leqslant k$.

Proposition 2. *We have that* $\sum_{p=0}^{n} |V_{n,p,0}| = \sum_{p=0}^{n} B_{n,p}^2 = C_n$, *where* C_n *is the nth Catalan number.*

Proof. For $(f_1, f_2) \in B_{n,p} \times B_{n,p}$ let $f = f_1 \bar{f}_2$. We can easily see that $f \in B_{2n,0}$. If $f \in B_{2n,0}$, then we can decompose f to $f'f''$, where f' is a left factor of length n of j. Obviously, if $\delta_x(f') = k$, then $\delta f'' = -k$, $0 \leqslant k \leqslant n$. We thus find $(f', \bar{f}'') \in B_{n,p} \times B_{n,p}$, and we deduce that $\sum_{p=0}^{n} |V_{n,p,0}| = |B_{2n,0}| = C_n$. $\square$

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