Q1.

Let $\varepsilon > 0$. Then

$$\mathbb{P}(\min\{X_1, \dots, X_n\} > \varepsilon)$$

$$= \mathbb{P}(\{X_1 > \varepsilon\} \cap \{X_2 > \varepsilon\} \cap \dots \cap \{X_n > \varepsilon\})$$

$$= \mathbb{P}(\{X_1 > \varepsilon\}) \cdot \mathbb{P}(\{X_2 > \varepsilon\}) \cdots \mathbb{P}(\{X_n > \varepsilon\}) \quad \text{(independence)}$$

$$= (1-\varepsilon)^n \to 0 \quad \text{as } n \to \infty.$$

As ε was arbitrary $\min\{X_1, \dots, X_n\} \xrightarrow{P} 0$.

Q2

Since $\{X_n\}$ and $\{Y_n\}$ are UI,

$\forall \varepsilon > 0 \quad \exists K > 0$

$$\mathbb{E}(|X_n| ; |X_n| > \tfrac{K}{2}) < \varepsilon/2$$

and $\mathbb{E}(|Y_n| ; |Y_n| > \tfrac{K}{2}) < \varepsilon/2$.

a) Fix $\varepsilon > 0$, and let K be as above

Then, $\mathbb{E}(|X_n + Y_n| ; |X_n + Y_n| > K)$

$$\leq \mathbb{E}(|X_n + Y_n| ; \max\{|X_n|, |Y_n|\} > K/2)$$

$$\leq \mathbb{E}(2\max\{|X_n|, |Y_n|\} ; \max\{|X_n|, |Y_n|\} > K/2)$$

$$\leq 2\max\{\mathbb{E}(|X_n| ; |X_n| > \tfrac{K}{2}), \mathbb{E}(|Y_n| ; |Y_n| > \tfrac{K}{2})\}$$

$\leq \varepsilon$ as required.

b) No. let $X_n = Y_n = Z$

where $Z \in L^1$ but $Z \notin L^2$.

For example Z arising from pdf

$$f(x) = \begin{cases} 2x^{-3} & x \geq 1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{Then } \int f = 1,$$

$$\mathbb{E}(Z ; |Z| > K) = \int_K^\infty 2x^{-2}\, dx = 2/K \to 0 \text{ as } K \to \infty.$$

but $\mathbb{E}(|X_n Y_n| ; |X_n Y_n| > K) = \int_K^\infty 2x^{-1}\, dx = \infty$.

Q3 First assume $\mu_n, \sigma_n^2 \leq K$ for some $K > 0$. Then $\|X_n\|_2^2 = \mathbb{E}(|X_n|^2) = Var(X_n) + \mathbb{E}(X_n)^2$

$$\leq K + K^2 \leq 2K^2 < \infty.$$

Hence the family $\{X_n\}$ is uniformly bounded.

Now for the converse assume that μ_n or σ_n^2 is unbounded. If μ_n is unbounded $\exists\, n_k$ s.t. $|\mu_{n_k}| \geq k \geq 1$ But then $\exists\, n_k$ s.t.

$$\mathbb{E}(|X_{n_k}| ; |X_{n_k}| > k) \geq \tfrac{1}{2}k \geq \tfrac{1}{2}.$$

Hence $\{X_n\}$ is not UI. So, it suffices to check $\sigma_{n_k}^2 \to \infty$ but μ_n bounded.

Let $K = \frac{\sqrt{2\sigma_{n_k}^2}}{3}$ and assume WLOG that $|\mu_n| < K$ and $K \geq 1$.

for all n.

Then $\mathbb{E}\left(X_{n_k} ; |X_{n_k}| > K\right) \geq \int_K^\infty \frac{y}{\sqrt{2\pi\sigma_{n_k}^2}} e^{-\frac{(y-\mu_{n_k})^2}{2\sigma_{n_k}^2}} dy$

$$\geq \frac{1}{\sqrt{2\pi\sigma_{n_k}^2}} \int_K^{2K} e^{-\frac{(y+K)^2}{2\sigma_{n_k}^2}} dy \qquad 3K = \sqrt{2\sigma_{n_k}^2}$$

$$\geq \frac{1}{\sqrt{2\pi\sigma_{n_k}^2}} K e^{-\frac{(3K)^2}{2\sigma_{n_k}^2}} = \frac{K}{\sqrt{\pi}\, 3K} e^{-1} = \frac{1}{3e\sqrt{\pi}} > 0.$$

Hence $\{X_n\}$ is not UI.

[Q4 further below]

Q5 We first show that X_n is a martingale.

$$\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = \frac{1}{2} X_{n-1}^2 + \frac{1}{2}\left(2X_{n-1} - X_{n-1}^2\right) = X_{n-1}.$$

Hence also $\mathbb{E}(X_n) = \mathbb{E}(X_0) = a$.

Note that $x \mapsto x^2$, $x \mapsto 2x - x^2$ map $(0,1)$ into $(0,1)$

Hence $X_n \in (0,1)$ as $X_0 = a \in (0,1)$.

Further, X_n is uniformly bounded and thus X_n is a UI martingale and $X_n \to X_\infty$ a.s. with $\mathbb{E}(X_n) = \mathbb{E}(X_\infty) = a$.

Note that X_∞ cannot be in $(0,1)$ a.s.

We will show this by contradiction: Let

Assume that $\mathbb{P}(X_\infty \in (0,1)) > 0$. Then,

there exists a small region $(x, x+\varepsilon)$ for $x > 0$ and

$x+\varepsilon < 1$, with $\mathbb{P}(X_\infty \in (x, x+\varepsilon)) > 0$.

But then $X_n \in (x, x+\varepsilon)$ for large enough n.

However $X_{n+1} = X_n^2$ or $2X_n - X_n^2$ and for small

enough ε, only depending on x. But then $X_{n+1} \notin (x, x+\varepsilon)$

a contradiction. Hence $X_\infty \in \{0,1\}$.

Alternatively, note that X_n and X_{n-1}

converge to the same X_∞. Hence

$|X_{n+1} - X_n| \to 0$. Since

$$|X_{n+1} - X_n| = \begin{cases} |X_n^2 - X_n| & \text{with prob } \frac{1}{2} \\ |2X_n - X_n^2 - X_n| & \text{---''---} \end{cases} \underset{\text{a.s.}}{=} |X_n^2 - X_n|$$

We must have $|X_\infty^2 - X_\infty| = |X_\infty (X_\infty - 1)| \Rightarrow X_\infty \in \{0,1\}$.

Now $\mathbb{E}(X_\infty) = a$, which gives $\mathbb{P}(X_\infty = 1) = a$ and

$\mathbb{P}(X_\infty = 0) = 1 - a$. $\square$

Q4 a) Clearly $\mathbb{E}(Z_0) = 1$.

Inductively, if $\mathbb{E}(Z_n) = m^n$, we have

$$\mathbb{E}(Z_{n+1}) = \mathbb{E}(\mathbb{E}(Z_{n+1} | \tilde{\mathcal{F}}_n))$$
$$= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{Z_n} X_{i,n+1} \middle| \tilde{\mathcal{F}}_n\right)\right)$$
$$= \mathbb{E}\left(\sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n+1} | \tilde{\mathcal{F}}_n)\right)$$
$$= \mathbb{E}\left(Z_n \, \mathbb{E}(X)\right) = \mathbb{E}(m \cdot Z_n)$$
$$= m \cdot \mathbb{E}(Z_n) = m \cdot m^n = m^{n+1}.$$

This completes the inductive step.

b) Similar to above,

$$\mathbb{E}(M_{n+1} | \tilde{\mathcal{F}}_n) = \frac{1}{m^{n+1}} \mathbb{E}(Z_{n+1} | \tilde{\mathcal{F}}_n)$$
$$= \frac{1}{m^{n+1}} \mathbb{E}\left(\sum_{i=1}^{Z_n} X_{i,n+1} \middle| \tilde{\mathcal{F}}_n\right) = \frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n+1} | \tilde{\mathcal{F}}_n)$$
$$= \frac{1}{m^{n+1}} Z_n \cdot m = \frac{Z_n}{m^n} = M_n$$ as required.

Since $Z_n \geq 0$ and so $M_n \geq 0$, Doob's convergence theorem gives $M_n \to M_\infty$ a.s.

c) We compute $\mathbb{E}\left((M_{n+1}-M_n)^2\right)$

$$= \mathbb{E}\left(\left(\frac{1}{m^{n+1}}\sum_{i=1}^{Z_n}\lambda_{i,n+1} - \frac{1}{m^n}Z_n\right)^2\right)$$

$$= \frac{1}{m^{2(n+1)}}\mathbb{E}\left(\mathbb{E}\left(\left(\sum_{i=1}^{Z_n}X_{i,n+1} - m\cdot Z_n\right)^2 \,\middle|\, \tilde{\mathcal{F}}_n\right)\right)$$

$$= \frac{1}{m^{2(n+1)}}\mathbb{E}\left(\mathbb{E}\left(\left(\underbrace{\sum_{i=1}^{Z_n}(X_{i,n+1}-m)}_{\text{sum of } Z_n \text{ many iid r.v. with mean } 0 \text{ \& Variance } \sigma^2.}\right)^2 \,\middle|\, \tilde{\mathcal{F}}_n\right)\right)$$

$$= \frac{1}{m^{2(n+1)}}\mathbb{E}(Z_n\cdot\sigma^2)$$

$$= \sigma^2\frac{m^n}{m^{2n+2}} = \sigma^2 m^{-n-2}.$$

Thus $\mathbb{E}(M_n^2) = \sum_{k=1}^{n}\mathbb{E}\left((M_k-M_{k-1})^2\right) = \sum_{k=1}\sigma^2 m^{-k-2} < \infty.$

Hence $\mathbb{E}(|M_n - M_\infty|) \to 0$ and

$\mathbb{E}(M_\infty) = \lim_n \mathbb{E}(M_n) = 1.$

Since $M_n \geq 0$, we must have $\mathbb{P}(M_\infty > 0) > 0$.

d) We have a.s. convergence, since

$M_n = Z_n$ is still a martingale (ident. proof as before).

Since $Z_n \geq 0$, we have $Z_n \to Z_\infty$ a.s.

Hence $Z_n, Z_{n-1} \to Z_\infty$ a.s and

$|Z_n - Z_{n-1}| \to 0$ a.s. as $n \to \infty$.

Now $|Z_n - Z_{n-1}| = \left| \sum_{i=1}^{Z_{n-1}} X_{i,n} - Z_{n-1} \right|$

$$= \left| \sum_{i=1}^{Z_{n-1}} (X_{i,n} - 1) \right| .$$

Since $X_{i,n} - 1 = \begin{cases} 1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$, we must

have, for n large, $|Z_k - Z_{k-1}| < 1$

for all $k \geq n$. But then Z_k must be

even for all $k \geq n$ and $\frac{Z_k}{2}$ many $X_{i,k+1}$

must be $+1$ & $\frac{Z_k}{2}$ must be -1. for all $k \geq n$

This can only happen (with positive probability)

if $Z_k = 0$.

Alternatively, We want to prove that

$\mathbb{P}(Z_k = 0$ for large enough $k) = 1$.

Write $q = \mathbb{P}(Z_k = 0$ for large enough $k)$.

Note that $q = \mathbb{P}(Z_1 = 0) + \mathbb{P}(Z_1 = 2$ and the desc. of both subtrees are eventually extinct$)$

So $q = \frac{1}{2} + \frac{1}{2} q^2 \Rightarrow q^2 - 2q + 1 = 0$

$\Rightarrow (q-1)^2 = 0$. Hence $q = 1$. $\square$

Q6 We show $\mathbb{E}(X_n) = 0$ from which $X_n = 0$ follows.

For a contradiction assume $\mathbb{E}(X_n) \geq c$ for some $c > 0$ and $n \in \mathbb{N}$

Then by the submartingale property $\mathbb{E}(X_k) \geq c > 0$ for all $k \geq n$. Let $\delta > 0$ be small enough s.t. $\mathbb{E}(X_n ; \frac{c}{8\delta}) \leq c/8$ (using UI) and let $k \geq n$ be large enough s.t.

$\mathbb{P}(X_k \leq \frac{c}{8}) \geq 1 - \delta$ which exists as $X_n \xrightarrow{a.s.} 0 \Rightarrow X_n \xrightarrow{p} 0$.

Now $c \leq \mathbb{E}(X_k) = \mathbb{E}(X_k ; X_k \leq \frac{c}{8}) + \mathbb{E}(X_k ; \frac{c}{8} \leq X_k \leq \frac{c}{\delta \cdot 8})$
$+ \mathbb{E}(X_k ; X_k \geq \frac{c}{\delta \cdot 8})$

$$\leq \frac{c}{8} + \delta \cdot \frac{c}{\delta \cdot 8} + c/8$$

$$= \frac{3}{8} c \text{ , a contradiction.}$$

Hence $\mathbb{E}(X_n) = 0 \ \forall n \in \mathbb{N}$ and as $X_n \geq 0$,

we must have $X_n = 0$ a.s.

Q7. At time n, the urn contains n balls. We have 4 cases:

Case	Probability
We pick red & add 2 red balls	$\frac{X_n}{n} \cdot p = a_1$
We pick green & add 2 green balls	$\left(1-\frac{X_n}{n}\right) p = a_2$
We pick red & add 1 red, 1 green ball	$\frac{X_n}{n} \cdot q = a_3$
We pick green & add 1 red, 1 green ball	$\left(1-\frac{X_n}{n}\right) q = a_4$

Hence, $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n)$

$$= (X_n+1)(a_1+a_4) + X_n(a_2+a_3)$$

$$= X_n(a_1+a_2+a_3+a_4) + (a_1+a_4)$$

$$= X_n + \frac{X_n}{n}p + \left(1-\frac{X_n}{n}\right)q$$

$$= X_n\left(1+\frac{1-q}{n}-\frac{q}{n}\right)+q = X_n\left(\frac{n+1-q-q}{n}\right)+q$$

$$= X_n\left(\frac{n+1-2q}{n}\right)+q$$

Hence,

$$\mathbb{E}(Y_{n+1} \mid \tilde{\mathcal{F}}_n) = \mathbb{E}\left(\binom{n+1-2q}{n}^{-1} \cdot \left(X_{n+1}-\frac{n+1}{2}\right) \Big| \tilde{\mathcal{F}}_n\right)$$

$$= \frac{n!\,(n+1-2q-n)!}{(n+1-2q)!}\left(\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) - \frac{n+1}{2}\right)$$

$$= \frac{n!\,(1-2q)!}{(n+1-2q)!}\left(X_n \cdot \frac{n+1-2q}{n} + q - \frac{n+1}{2}\right)$$

$$= \frac{n\cdot(n-1)!\,(1-2q)!}{(n+1-2q)\cdot(n-2q)!} \cdot \frac{n+1-2q}{n}\left(X_n + \frac{n}{n+1-2q}\left(q - \frac{n+1}{2}\right)\right)$$

$$= \frac{(n-1)!\,(1-2q)!}{(n-2q)!}\left(X_n - \frac{\frac{1}{2}n^2 + \frac{1}{2}n - qn}{n+1-2q}\right)$$

$$= \binom{n-2q}{n-1}^{-1}\left(X_n - \frac{n}{2}\right) = Y_n .$$

Hence Y_n is a martingale.

Q8 Lévy's upward theorem

a) By the tower property, $\mathbb{E}(M_n \mid \tilde{\mathcal{F}}_{n-1})$

$= \mathbb{E}(\mathbb{E}(X \mid \tilde{\mathcal{F}}_n) \mid \tilde{\mathcal{F}}_{n-1}) = \mathbb{E}(X \mid \tilde{\mathcal{F}}_{n-1}) = M_{n-1}$.

b) Further, $(M_n) \subseteq \mathcal{G} = \{\mathbb{E}(X \mid \mathcal{H}) : \mathcal{H} \subseteq \mathcal{F} \text{ sub } \sigma\text{-alg.}\}$

We have shown that $\mathcal{G}$ is UI and so $(M_n)_n$ is.

c) Let

$$\mu_1(F) = \mathbb{E}(Y ; F), \quad \mu_2(F) = \mathbb{E}(M_\infty ; F)$$

for $F \in \tilde{\mathcal{F}}_\infty$. Note that for all $F \in \tilde{\mathcal{F}}_\infty$ (and so $F \in \tilde{\mathcal{F}}_n$)

$$\mathbb{E}(Y; F) = \mathbb{E}(\mathbb{E}(X \mid \tilde{\mathcal{F}}_\infty); F)$$
$$= \int_F \mathbb{E}(X \mid \tilde{\mathcal{F}}_\infty)\, d\mathbb{P} = \mathbb{E}(X; F)$$

For $E \in \mathcal{F}_n$, we also have

$$\mathbb{E}(M_\infty \mid \mathcal{F}_n) = X_n \quad \text{and so} \quad \mathbb{E}(M_\infty; E) = \mathbb{E}(X; E).$$

Now $E \in \bigcup \mathcal{F}_n$ generate a π-system and so

$$\mathbb{E}(M_\infty; E) = \mathbb{E}(Y; E) \quad \forall E \in \bigcup \mathcal{F}_n$$
$$\Rightarrow \quad \mu_1(F) = \mu_2(F) \quad \forall F \in \sigma\left(\bigcup \mathcal{F}_n\right).$$

d) Finally, since M_∞ and Y are $\mathcal{F}_\infty$-measurable,
$F = \{ |M_\infty - Y| > 0 \} \in \tilde{\mathcal{F}}_\infty$ and $\mu_1(F) = \mu_2(F)$.

Hence $\int_F M_\infty \, d\mathbb{P} = \int_F Y \, d\mathbb{P} \Rightarrow \int_F M_\infty - Y \, d\mathbb{P} = 0$.

Hence $M_\infty = Y$ a.s., which proves the theorem $\square$

Q8 Kolmogorov 0-1 law

Consider I_F for $F \in \mathcal{T}$. This is clearly bounded and by Levy's Upward Thm,

$$I_F = \mathbb{E}(I_F \mid \mathcal{F}_\infty) = \lim \mathbb{E}(I_F \mid \tilde{\mathcal{F}}_n).$$

I_F is $\mathcal{T}_n$ measurable $\forall n$ and thus independent of $\tilde{\mathcal{F}}_n$.

So, $I_F = \lim \mathbb{E}(I_F \mid \tilde{\mathcal{F}}_n) = \lim \mathbb{E}(I_F) = P(F)$.

and can only take values in $\{0, 1\}$