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Ch. 7 Arbitrage pricing

In this chapter $N=2$ (ie two assets):

$dB_t = rB_t\,dt$ (risk-free asset, think bank account)

constant interest rate

and $dS_t = \mu(t,S_t)S_t\,dt + \sigma(t,S_t)S_t\,dW_t$

(risky asset, think stock price)

Remarks

1. $B_t = B_0 e^{rt}$

2. μ and σ are functions of t and current stock price.

   "local mean rate of return" "volatility"

3. In the Black-Scholes model, μ and σ are constants.

Aim: To find a "fair" value of options written on S.

Terminology: Options are also called financial derivatives.

Def 7.6 A European call option with strike price K and maturity date T on the underlying asset S is a contract such that the holder (owner) at time T has the right, but not the obligation, to buy one share of S at price K from the option writer (seller).

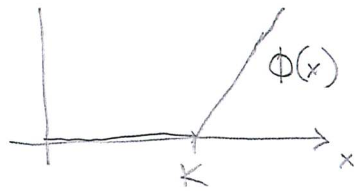
Remarks 1. A European put option gives the right (but not the obligation) to sell one share of S at time T at price K.

2. An American call/put gives the right to buy/sell at any time before T. (We will not treat American options in this course.)

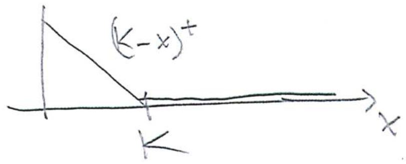
Def 7.7 A contingent claim with maturity T (or a T-claim) is a random variable $X \in \mathcal{F}_T^S$. A contingent claim is simple if $X = \Phi(S_T)$ for some contract function (or payoff function) Φ.

Ex: For a European call option, $\Phi(x) = (x-K)^+ = \max\{x-K, 0\}$.

Indeed, if $S_T \geq K$, then buy at price K and make profit $S_T - K$. If $S_T < K$, do not exercise the option.



For a European put option, $\Phi(x) = (K-x)^+$



We will determine the price $\Pi(t; \mathcal{X})$ of a T-claim $\mathcal{X}$ at time t by requiring the market to be arbitrage-free.

Def 7.8 A self-financing portfolio h is an arbitrage if
$$\begin{cases} V_0^h = 0 \\ P(V_T^h \geq 0) = 1 \\ P(V_T^h > 0) > 0 \end{cases}$$
The market is arbitrage-free if no arbitrage exists.

Ex: $\begin{cases} dS_t^1 = dt + dW_t \\ dS_t^2 = dW_t \\ dB_t = 0 \end{cases}$ is not arbitrage-free.

$\begin{cases} dS_t^1 = dt + dW_t^1 \\ dS_t^2 = dW_t^2 \\ dB_t = 0 \end{cases}$ indep. is arbitrage-free.

Assumption 7.3.1 The price process $\Pi_t(\mathcal{X})$ is such that $(B_t, S_t, \Pi_t(\mathcal{X}))$ is arbitrage-free.

We also assume that all assets (including the option) can be sold/bought with no market frictions (no transaction costs, no liquidity constraints).

Idea: Create a self-financing portfolio of options and the stock such that its value process is locally risk-free (has no dW-term). The drift of the value must then coincide with the interest rate (otherwise arbitrage). This will give a condition on the price of the option.

Assume $\mathcal{X} = \Phi(S_T)$ (simple T-claim) and that $\Pi_t(\mathcal{X}) = F(t, S_t)$ for some function F.

New notation: $F_t := \frac{\partial F}{\partial t}$, $F_s = \frac{\partial F}{\partial s}$, $F_{ss} = \frac{\partial^2 F}{\partial s^2}$

Then

$$dF(t,S_t) \overset{\text{Ito}}{=} F_t dt + F_s dS_t + \frac{1}{2} F_{ss} (dS_t)^2$$

$$= \underbrace{\frac{\left(F_t + \frac{\sigma^2 S_t^2}{2} F_{ss} + \mu S_t F_s\right)}{F}}_{=:\mu^F} F(t,S_t)dt + \underbrace{\frac{\sigma S_t F_s}{F}}_{=:\sigma^F} F\, dW_t$$

$$= \mu^F F dt + \sigma^F F dW_t$$

Let (w^S, w^F) be a self-financing relative portfolio of stocks and options $(w^S + w^F = 1)$, and let V be its value process. Then

$$dV_t = V_t \left(\frac{w^S}{S_t} dS_t + \frac{w^F}{F} dF_t \right)$$

$$= (\mu w^S + \mu^F w^F) V_t dt + (\sigma w^S + \sigma^F w^F) V_t dW_t$$

Let (w^S, w^F) be defined by

$$\begin{cases} w^S + w^F = 1 \\ \sigma w^S + \sigma^F w^F = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} w^S = \dfrac{\sigma^F}{\sigma^F - \sigma} \\ w^F = \dfrac{-\sigma}{\sigma^F - \sigma} \end{cases}$$

Then $dV_t = \dfrac{\mu\sigma^F - \mu^F\sigma}{\sigma^F - \sigma} V_t\, dt$.

By a no-arbitrage argument, we must have

$$r = \frac{\mu\sigma^F - \mu^F\sigma}{\sigma^F - \sigma} \quad \text{(why?)}$$

$$r\sigma^F - r\sigma = \mu\sigma^F - \mu^F\sigma$$

$$\frac{r\sigma S_t F_s}{F} - r\sigma \qquad \frac{\mu\sigma S_t F_s}{F} - \frac{\sigma\left(F_t + \mu S_t F_s + \frac{\sigma^2 S^2}{2} F_{ss}\right)}{F}$$

$$r S_t F_s - rF = \mu S_t F_s - F_t - \mu S_t F_s + \frac{\sigma^2}{2} S_t^2 F_{ss}$$

$$= -F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} \qquad \text{(magic!)}$$

$$F_t + \frac{\sigma^2 S_t^2}{2} F_{ss} + r S_t F_s - rF = 0$$

Since S_t can take any value, F must satisfy the PDE

$$F_t(t,s) + \frac{\sigma^2(t,s)}{2} s^2 F_{ss} + rsF_s(t,s) - rF(t,s) = 0$$

Also, $\underset{\displaystyle \phi(S_T)}{\overset{\parallel}{\Pi_T(\mathcal{X})}} = F(T, S_T)$ so we also have $F(T,s) = \phi(s)$.

<u>Theorem 7.10</u> (Black-Scholes equation)

In the market $\begin{cases} dB_t = rB_t\,dt \\ dS_t = \mu(t,S_t) S_t\,dt + \sigma(t,S_t) S_t\,dW_t \end{cases}$

the only arbitrage-free price of a T-claim $\mathcal{X} = \phi(S_T)$ is $F(t, S_t)$, where $F(t,s)$ solves

$$\begin{cases} F_t(t,s) + \frac{\sigma^2(t,s)}{2} s^2 F_{ss}(t,s) + rsF_s(t,s) - rF(t,s) = 0 \\ F(T,s) = \phi(s) \end{cases}$$

Lecture 9

Recall the Black-Scholes equation

$$\begin{cases} F_t(t,s) + \frac{\sigma^2(t,s)}{2} s^2 F_{ss}(t,s) + rsF_s(t,s) - rF(t,s) = 0 \\ F(T,s) = \Phi(s). \end{cases}$$

The solution to the BS-equation is (by Feynman-Kac)

$$F(t,s) = E_{t,s}\left[e^{-r(T-t)}\Phi(S_T)\right]$$

where

$$(**) \quad \begin{cases} dS_u = rS_u\,du + \sigma(u,S_u)S_u\,dW_u \\ S_t = s. \end{cases}$$

We refer to
$$(*) \quad \begin{cases} dS_u = \mu(u,S_u)S_u\,du + \sigma(u,S_u)S_u\,dW_u \\ S_t = s \end{cases}$$

as the P-dynamics of S (the specification of S under the "physical measure" P).

$(**)$ is refered to as the Q-dynamics of S

(Q is the pricing measure, or the martingale measure).

Thm 7.11

The arbitrage-free price of a simple T-claim $\mathcal{X} = \Phi(S_T)$ is $F(t,S_t)$, where

$$F(t,s) = E^Q_{t,s}\left[e^{-r(T-t)}\Phi(S_T)\right]$$

and the Q-dynamics of S are as in $(**)$.

Ex: In the standard BS-model (i.e. constant σ), what is the arbitrage-free price of the T-claim $\mathcal{X} = S_T^2$?

By risk-neutral valuation,

$$F(t,s) = e^{-r(T-t)} E_{t,s}^Q [S_T^2].$$

Let $Y_u = S_u^2$. Then

$$dY_u = 2S_u dS_u + (dS_u)^2 \underset{\substack{\uparrow \\ dS_u = rS_u du + \sigma S_u dW_u}}{=} (2r+\sigma^2) Y_u du + 2\sigma Y_u dW_u$$

Y is a gBm and thus $E_{t,s}^Q[S_T^2] = E^Q[Y_T] = s^2 e^{(2r+\sigma^2)(T-t)}$.

Answer: The price of $\mathcal{X}$ at time t is $S_t^2 e^{(r+\sigma^2)(T-t)}$.

Ex: What is the price of $\mathcal{X} = S_T$?

By risk-neutral valuation,

$$F(t,s) = e^{-r(T-t)} E_{t,s}^Q [S_T] = s.$$

Answer: The price at time t is S_t.

(Explain this using a self-financing portfolio in B and S!)

Remark: In time-homogeneous models (such as the standard BS-model), the relevant quantity is time $T-t$ left to maturity.

Ex (Binary option): In the standard BS-model, find the value of $\mathcal{X} = \Phi(S_T)$ where $\Phi(x) = \begin{cases} 1 & \text{if } x \geq K \\ 0 & \text{if } x < K \end{cases}$

$$F(0,s) = e^{-rT} E^Q_{0,s}\left[1_{\{S_T \geq K\}}\right] = e^{-rT} Q(S_T \geq K)$$

probability

$$= e^{-rT} Q\left(s e^{(r-\frac{\sigma^2}{2})T + \sigma W_T} \geq K\right)$$

$$= e^{-rT} Q\left(\frac{1}{\sqrt{T}} W_T \geq \frac{\ln\frac{K}{s} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= e^{-rT} Q\left(\frac{1}{\sqrt{T}} W_T \leq \frac{\ln\frac{s}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= e^{-rT} N\left(\frac{\ln\frac{s}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \quad \text{where } N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

is the distribution function of $N(0,1)$.

Answer: The price at time t is $e^{-r(T-t)} N\left(\frac{\ln\frac{S_t}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)$

What is the price of a European call option $\mathcal{X} = (S_T - K)^+$? In the standard BS-model,

$$F(0,s) = e^{-rT} E^Q_{0,s}\left[(S_T - K)^+\right] = e^{-rT} E^Q\left[\left(s e^{(r-\frac{\sigma^2}{2})T + \sigma W_T} - K\right)^+\right]$$

$$= e^{-rT} \int_a^{\infty} \left(s e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T} x} - K\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

$a = \frac{\ln\frac{K}{s} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$$= s \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx - Ke^{-rT} N(-a)$$

$$= s \int_{a-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - Ke^{-rT} N(-a)$$

$$= s N(\sigma\sqrt{T} - a) - Ke^{-rT} N(-a)$$

Prop 7.13 (Black-Scholes formula)

In the standard BS-model, the price of a European call option is $F(t, S_t)$, where

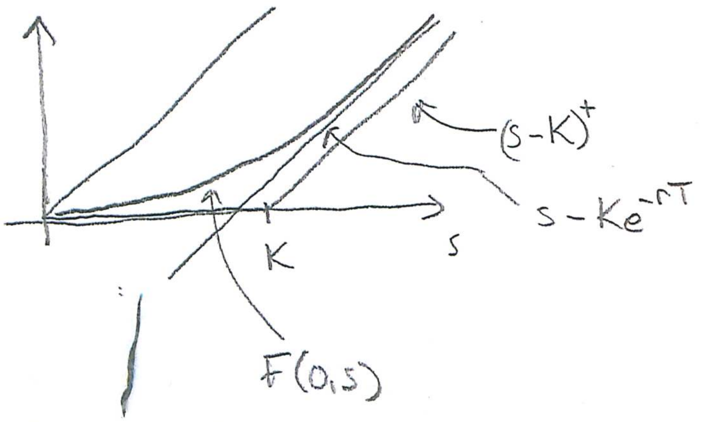
$$F(t,s) = sN(d_1) - Ke^{-r(T-t)}N(d_2)$$

and
$$\begin{cases} d_1 = \dfrac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases}$$

Consider $F(0,s) = sN(d_1) - Ke^{-rT}N(d_2)$ (as above).

We have $F(0,s) = E^Q_{0,s}\left[e^{-rT}(S_T - K)^+\right] \leq E^Q_{0,s}\left[e^{-rT}S_T\right] = s$

and $F(0,s) = E^Q_{0,s}\left[e^{-rT}(S_T - K)^+\right] \geq E^Q_{0,s}\left[e^{-rT}(S_T - K)\right] = s - Ke^{-rT}$.



We will see below that $F(0,s) = F(0,s;\sigma)$ is increasing in σ.