1. For Brownian motion $B = \{B_t\}_{t\geq 0}$ and in terms of the Brownian transition density $p_t(x) = (2\pi t)^{-1/2}e^{-x^2/2t}$, $t > 0$, $x \in \mathbb{R}$, the joint distribution of the pair (B_s, B_t), $s < t$, has the joint density function $p_s(x)p_{t-s}(y-x)\,dxdy$, and the conditional distribution of B_s given that $B_t = b$, has density

$$\frac{p_s(x)p_{t-s}(b-x)}{p_t(b)} = \frac{1}{\sqrt{2\pi s(t-s)/t}}\exp\Big\{-\frac{(x-sb/t)^2}{2s(t-s)/t}\Big\}$$

Let $Y = \{Y_t\}_{0<t<1}$ be the Brownian Bridge derived from B by putting $Y_t = B_t$ conditioned on $B_1 = 0$. It follows that Y is Gaussian with density function

$$f_{Y_t}(x) = \frac{1}{\sqrt{2\pi t(1-t)}}\exp\Big\{-\frac{x^2}{2t(1-t)}\Big\}.$$

In particular, $E(Y_t) = 0$, $\mathrm{Var}(Y_t) = t(1-t)$, $0 \leq t \leq 1$. Similarly as above, the two-dimensional joint density of (Y_s, Y_t), $0 < s < t < 1$, is obtained as

$$\frac{p_s(x)p_{t-s}(y)p_{1-t}(0)}{p_1(0)} = \frac{1}{2\pi\sqrt{s(t-s)(1-t)}}\exp\Big\{-\frac{t(1-t)x^2 - 2s(1-t)xy + s(1-s)y^2}{2s(t-s)(1-t)}\Big\}.$$

By comparing with the bivariate Gaussian distribution it is seen that the covariance and the correlation function of Y are given by

$$C(s,t) = s(1-t), \qquad \rho(s,t) = \sqrt{\frac{s(1-t)}{t(1-s)}}.$$

It can be shown that $\{Y_t\}_{0\leq t\leq 1}$ has the same distribution as the process $\{\widetilde{Y}_t\}_{0\leq t\leq 1}$ defined by $\widetilde{Y}_t = B_t - tB_1$. This is frequently used as an alternative definition of the Brownian bridge, which yields additional methods for deriving the joint density, the covariance, etc.

2. We know that $m_A = E(A_t) > 0$ is independent of t and that $\mathrm{Cov}(A_s, A_t)$ is a function $C_A(t-s)$ which only depends on $|t-s|$. By the independence of A and Φ,

$$E(X_t) = E(A_t)E[\sin(\omega t+\Phi)] = \frac{E(A_t)}{2\pi}\int_0^{2\pi}\sin(\omega t+y)\,dy = \frac{E(A_t)}{2\pi}\big[-\cos(\omega t+y)\big]_0^{2\pi} = 0,$$

so that $m(t) = 0$ and hence the mean is independent of t. The covariance function is therefore

$$C_X(s,t) = E[X_sX_t] = E[A_sA_t]\,\mathbb{E}[\sin(\omega s+\Phi)\sin(\omega t+\Phi)].$$

Here, $E[A_sA_t] = C_A(t-s) + m^2$ and, using a suitable trigonometric formula,

$$\begin{aligned}
\mathbb{E}[\sin(\omega s + \Phi)\sin(\omega t + \Phi)] &= \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega s + y)\sin(\omega t + y)\,dy \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\big(\cos(\omega|t-s|) - \cos(\omega(s+t)+2y)\big)\,dy = \frac{1}{2}\cos\omega|t-s|.
\end{aligned}$$

It follows that X is stationary, since

$$C_X(s,t) = \frac{1}{2}(C_A(t-s) + m^2)\cos\omega|t-s|,$$

which is a function that only depends on the difference of s and t.

3. Clearly, $Y_n \in \mathcal{F}_n$ and $E[|Y_n|] \le 1$ for all $n \ge 1$, so Y is adapted and integrable. Moreover,

$$E[Y_n|\mathcal{F}_{n-1}] = (-1)^n E[\cos(\pi X_{n-1} + \pi Z_n)|\mathcal{F}_{n-1}].$$

By independence,

$$\begin{aligned}
E[\cos(\pi X_{n-1} + \pi Z_n)|\mathcal{F}_{n-1}] &= \frac{1}{2}E[\cos(\pi X_{n-1} + \pi)|\mathcal{F}_{n-1}] + \frac{1}{2}E[\cos(\pi X_{n-1} - \pi)|\mathcal{F}_{n-1}] \\
&= \frac{1}{2}(\cos(\pi X_{n-1} + \pi) + \cos(\pi X_{n-1} - \pi)) \\
&= -\frac{1}{2}(\cos(\pi X_{n-1}) + \cos(\pi X_{n-1})),
\end{aligned}$$

and therefore $E[Y_n|\mathcal{F}_{n-1}] = Y_{n-1}$, as required.