Martingales

Setting: $X = \{X_t\}_{t\in T}$ on filtered prob. space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$$

where $(\mathcal{F}_t)$ is family of σ-algebras $\mathcal{F}_s \subset \mathcal{F}_t$, $s \le t$, which satisfies "usual conditions":

- each $\mathcal{F}_t$ is complete; contains all zero sets
- right-continuous $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, $t \ge 0$

The process is measurable, i.e.

the map
$$\begin{cases} (t,\omega) \longrightarrow X_t(\omega) \\ T \times \Omega \longrightarrow \mathbb{R} \\ \mathcal{B}(T) \otimes \mathcal{F} \rightarrow \mathcal{B}(\mathbb{R}) \end{cases}$$
is measurable

and adapted, i.e. $X_t \in \mathcal{F}_t$, all $t \in T$.

Sometimes this needs to be refined:

- X progressively measurable
- predictably —"—

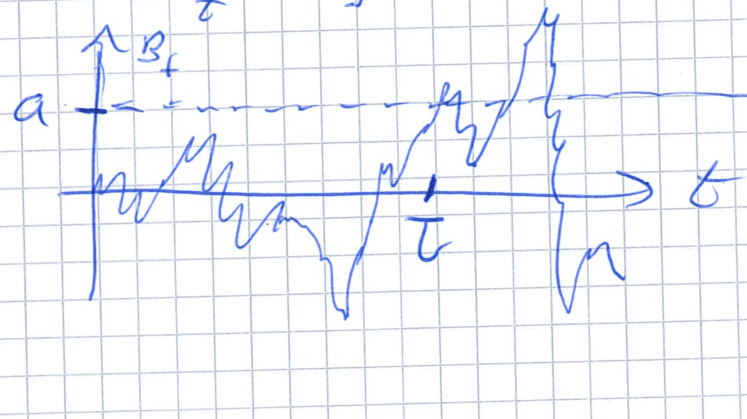
Def A random variable $\tau \geq 0$ on the same prob. space is a stopping time w.r.t $(\mathcal{F}_t)$ if $\{\tau \leq t\} \in \mathcal{F}_t$, $t \in T$

See Example 1.2.2-3

Then $t \to X_{t \wedge \tau}$, $t \geq 0$, the stopped process, is defined by
$$X_{t\wedge\tau}(\omega) = \begin{cases} X_t(\omega) & \text{if } t \leq \tau(\omega) \\ X_{\tau(\omega)}(\omega) & \text{if } t > \tau(\omega) \end{cases}$$
(we assume $P(\tau < \infty) = 1$)

Example $\{B_t\}_{t\geq 0}$ Brownian motion

$$\tau = \inf\{t \geq 0 : B_t \geq a\}$$



Def $\{X_t\}_{t\geq 0}$ def on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

is called a $\begin{cases} \underline{\text{martingale}} \\ (\mathbb{P}, (\mathcal{F}_t))-\text{martingale} \end{cases}$

if i) $\mathbb{E}[|X_t|] < \infty$, $t \geq 0$

ii) $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ $\mathbb{P}$-a.s. $0 \leq s \leq t$

iii) $\{X_t\}$ is $(\mathcal{F}_t)$-adapted

The properties of cond. expectation (F4: proposition) imply

- $\mathbb{E}[X_s] = \mathbb{E}[\mathbb{E}[X_t | \mathcal{F}_s]] = \mathbb{E}[X_t]$, $s \leq t$
- If $\{X_t\}_{t\geq 0}$ is an $(\mathbb{P}, (\mathcal{F}_t))$-martingale and adapted to a smaller σ-algebra (filtration) $(\mathcal{G}_t)$, $\mathcal{G}_t \subset \mathcal{F}_t$, then

$\{X_t\}_{t\geq 0}$ is a $(\mathbb{P}, (\mathcal{G}_t))$-martingale

Indeed, $X_s = \mathbb{E}[X_s | \mathcal{G}_s] = \mathbb{E}[\mathbb{E}[X_t | \mathcal{F}_s] | \mathcal{G}_s]$

$= \mathbb{E}[\mathbb{E}[X_t | \mathcal{G}_s] | \mathcal{F}_s]$

$= \mathbb{E}[X_t | \mathcal{G}_s]$, $s \leq t$.

Example $\{N_t\}_{t\geq 0}$

Poisson process, intensity $\lambda > 0$

we can take $\mathcal{F}_t = \mathcal{F}_t^N = \sigma(\{N_s \in (a,b]\}_{\substack{0\leq s\leq t \\ a,b\in\mathbb{R}}})$

"the σ-algebra generated by N itself"

or a general filtration $(\mathcal{F}_t)_{t\geq 0}$

$s\leq t$, $\mathbb{E}[N_t \mid \mathcal{F}_s] = \mathbb{E}[N_t - N_s + N_s \mid \mathcal{F}_s]$

$= \mathbb{E}[N_t - N_s \mid \mathcal{F}_s] + \mathbb{E}[N_s \mid \mathcal{F}_s]$

independence

known at time s

$= \mathbb{E}[N_t - N_s] + N_s$

$= \mathbb{E}[N_{t-s}] + N_s = \lambda(t-s) + N_s$

Thus, $\mathbb{E}[N_t - \lambda t \mid \mathcal{F}_s] = N_s - \lambda s$, $s \leq t$

Of course, $\tilde{N}_t \in \mathcal{F}_t$, where $\tilde{N}_t = N_t - \lambda t$

and $\mathbb{E}(|\tilde{N}_t|) \leq \mathbb{E}(N_t + \lambda t) = 2\lambda t < \infty$

So, $\{\tilde{N}_t\}$ is a $(\mathbb{P}, (\mathcal{F}_t))$-martingale

Compare BM $\{B_t\}_{t\geq 0}$; $\mathcal{F}_t = \mathcal{F}_t^B$.

$\boxed{F5:5}$

$$E(B_t \mid \mathcal{F}_s) = E(B_t - B_s \mid \mathcal{F}_s) + E(B_s \mid \mathcal{F}_s)$$

$$= E(B_t - B_s) + B_s = B_s \, , \quad s \leq t$$

so $\{B_t\}_{t\geq 0}$ is a martingale!

But $Y_t = B_t^2 - t$, $t \geq 0$; integrable and adapted ok

$$E(Y_t \mid \mathcal{F}_s) = E(B_t^2 - t \mid \mathcal{F}_s)$$

$$\cancel{= E(B_t^2 - B_s^2 \mid \mathcal{F}_s) + E(B_s^2 - t \mid \mathcal{F}_s)}$$

$$\cancel{= E((B_t - B_s)(B_t + B_s) \mid \mathcal{F}_s) + B_s^2 - t}$$

$$\cancel{= E(B_t(B_t - B_s) \mid \mathcal{F}_s) - B_s \underbrace{E(B_t - B_s \mid \mathcal{F}_s)}_{= 0} + B_s^2 - t}$$

$$= E((B_t - B_s + B_s)^2 \mid \mathcal{F}_s) - t$$

$$= E((B_t - B_s)^2 \mid \cancel{\mathcal{F}_s}) + 2E(B_s(B_t - B_s) \mid \mathcal{F}_s) + E(B_s^2 \mid \mathcal{F}_s) - t$$

$$= t - s + 2B_s \cdot 0 + B_s^2 - t$$

$$= B_s^2 - s = Y_s$$

so $\{Y_t\}$ is a martingale

Now let's take

$$\begin{cases} \{M_t\}_{t\geq 0} & (\mathcal{F}_t)\text{-martingale} \\ \tau & \text{stopping time w.r.t } (\mathcal{F}_t) \end{cases}$$

Under suitable assumptions, the stopped process $\{M_{t\wedge\tau}\}_{t\geq 0}$ is again a $(\mathcal{F}_t)$-martingale.

Thm 2.3.1 $\{M_t\}$ continuous
~~& bounded~~ (?)

This gives in part. $\mathbb{E}[M_{t\wedge\tau}] = \mathbb{E}[M_0]$

Is it also true that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] \; ?$$

Thm 2.3.1, yes, if τ bounded
$(\mathbb{P}(\tau \leq c) = 1)$

This is called optimal stopping