UPPSALA UNIVERSITET

Föreläsningsanteckningar

Tillämpad Matematik

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Contents

1. Introduction

The topic of this course will vary a lot, since mathematics can be applied to physics, biology, etc.
We will look into different ways to model real life, study it, and draw conclusions from it.

Anmärkning:
One could look at a mathematical models as a set of equations

Example: Planetary motion

- *Observation*: Keplers law → elliptic orbits
- *Model*: Newtons gravitational law
- *Mistakes/Errors*: Mercury precission → disalignment between model and observation
- *Rectify error*: Introducing relativistic effects in the model
- *Evaluation*: Is the old model useless? No, it is often easier to compute. It is better to keep it simple

We arrive at 2 models:

Good model→ Simple, general (not valid in a specific way)

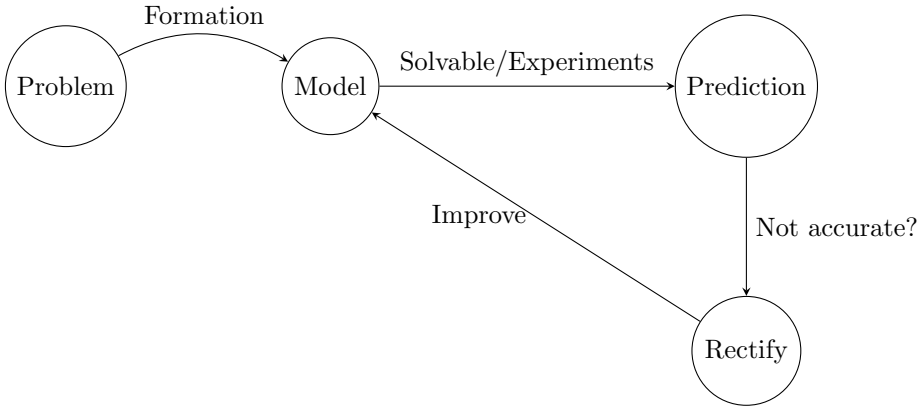


Figure 1.

First step in the definition of a model: Understand which variables are involved

Dimension	Unit	Derived	Dimension
Distance	m	v	m/s
Temperature	Degrees	a	m/s
Time	s		

Definition/Sats 1.1: Physical law

A physical law is $f(q_1, \cdots, 1_n) = 0$

$L_1, \cdots, L_m$ are the dimensions
$[q_i] = L_1 \cdots L_m$

- $[q] = 1$ dimensions
- $[v] = L \cdot T^{-1}$

Example: Conservation of energy is an example of such physical law:

$$\frac{mp^2}{2} + V(q) = C \qquad C \in \mathbb{R}$$

$$F(, m, p, q) = \frac{mp^2}{2} + V(q) - C = 0$$

Example: Hooks law for springs:

$$F = \underbrace{k}_{\text{Not dimensionless}} \cdot L \qquad f(F,k,L) = 0$$

Definition/Sats 1.2: Unit free

A law is *unit free* if it is independent from the unit, in the sense that if we define a new system in the following way:

$$\overline{L_i} = \lambda_i L_i$$

Then $\overline{L_i}$ is a new system of unit $\lambda_i > 0$

$$[q_i] = L_1^{b_1} \cdots L_n^{b_n}$$
$$f(q, \cdots, q_n) = 0 \Leftrightarrow f(\overline{1_1}, \cdots, \overline{q_m}) = 0$$

Example:

$$f(x,t,q) = x - \frac{1}{2}gt^2 = 0$$

Describing a body falling. If we define the following units:

- $[x] = m$
- $[g] = ms^{-2}$
- $[t] = s$

We can check that if we use different units, say $\overline{x} = 1000x$ (kilometers instead of meters) or $\overline{t} = 3600t$ (hours instead of seconds), then we obtain the same law for $f(\overline{x}, \overline{t}, g) = 0$

Example: Just looking at the dimension we can say something about the model. Take the pendulum and study the period of oscillation (is the mass or the length the one that defines the period?)

The goal is to find a law for the period. Suppose only the length and the mass are the only variables in our model, then we want to find $P = f(l,m)$
Notice that we have an error in the dimension, since our period depends on time, so just looking at that we can see that there is something that is missing.

We could be interested in adding another term, the gravitational acceleration. We get:

$$T = kL^{\alpha_1} M^{\alpha_2} \frac{L^{\alpha_3}}{T^{-2\alpha_3}}$$
$$\begin{cases} \alpha_2 = 0 \quad \rightarrow \text{mass is not involved} \\ \alpha_1 + \alpha_3 = 0 \\ -2\alpha_3 = 01 \qquad \alpha_3 = \dfrac{-1}{2} \qquad \alpha_1 = \dfrac{1}{2} \end{cases}$$
$$\Rightarrow P \approx k\sqrt{\frac{e}{g}}$$

Another thing we may do is to introduce dimensionless variables:

Definition/Sats 1.3: Pi:s theorem

Let $f(q_1, \cdots, q_m) = 0$ be a unit free law with the usual notation for dimension $[q_i] = L_1^{\alpha_{1i}} \cdots L_n^{alpha_{ni}}$
$n < m$

Define the dimension matrix A

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix}$$

> Let π be the rank(A). Then there exists $m-r$ dimensionsless variabless $\Pi_1, \cdots, \Pi_{m-r}$ (which can be formed from q_i)
>
> We hve an equivalent law $F(\Pi_1, \cdots, \Pi_{m-r}) = 0$

Anmärkning:
When we have a law, it does not mean that we have the right law (only $q_1, \cdots, q_n$ are involved) but it is not meaningless

The usefulness of Pi-theorem:

- Case in which only one dimensionsless variable is involved
  $F(\Pi_1) = 0 \rightarrow$ zeroes are discrete
  Π_1 can assume discrete values and can be deduced from experiments

In the case of 2 dimensionsless quantities $F(\Pi_1, \Pi_2) = 0$, if we can invert the relationship then we can write one variable as a function of the other using implicit function theorem.

$$\Pi_1 = f(\Pi_2) \qquad f \text{ is unknown} \rightarrow \text{deduced from observation}$$

Example: Allometry (Biology), the study of characteristics of living creatures change with their size.
We look for a law that involves

- $q_1 = l =$ length of the organism $\quad [q_1] = L$
- $q_2 = t =$ time $\quad [q_2] = T$
- $q_3 = \rho =$ density $\quad [q_3] = \dfrac{M}{L^3}$
- $q_4 = a =$ resource assimilation rate $\quad [q_4] = \dfrac{M}{L^2 T}$
- $q_5 = b =$ resource utilisation rate $\quad [q_5] = \dfrac{M}{L^3 T}$

We look for a law that involves 2 dimensionsless variables, so we apply the theorem:

$$A = \begin{pmatrix} 1 & 0 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} L \\ M \\ T \end{pmatrix}$$

(Look at the exponent of the respective variable)
The rank$(A) = 3 \rightarrow 5 - 3 = 2$ dimensionsless variables
We can try to express q_i as a linear combination of the others. We know the following:

$$\begin{cases} \alpha_1 - 3\alpha_3 = -2 \\ \alpha_3 = 1 \\ \alpha_2 = -1 \end{cases} \Rightarrow \alpha_1 = 1$$

This means that q_4 can be expressed as $q_4 = \dfrac{q_1 q_3}{q_4}$, yielding:

$$\Pi_1 = \frac{q_1 q_3}{q_2 q_4} = \frac{l\rho}{ta} \rightarrow \text{dimensionsless}$$

We can do the same for $q_5 \Rightarrow q_5 = \dfrac{q_3}{q_2}$ yielding another dimensionsless variable:

$$\Pi_2 = \frac{q_3}{q_2 q_5} = \frac{\rho}{tb}$$

Summa sumarum:

$$F(\Pi_1, \Pi_2) = 0 = F\left(\frac{l\rho}{ta}, \frac{\rho}{tb}\right)$$
$$\pi_1 = f(\Pi_2)$$

1.1. **Scaling.**

The goal is to rescale variables to a quantity that is related to that specific problem. Measuring seconds when it comes to glaciers might be less useful as measuring with years, and seconds for a chemical reaction might be too little.

For example, with time, $\bar{t} = \dfrac{t}{t_c}$. New rescaled time is 1 once it has passed the desired scale. c stands for characteristic
The same can be done for other quantities such as length $\overline{h} = \dfrac{h}{h_c}$

Example: Projectile problem where we only consider gravity. Using Newtons gravitational law:

$$\frac{md^2h}{dt^2} = -G \cdot \frac{mM}{(R+h)^2} \Rightarrow \frac{d^2h}{dt^2} = -G\frac{M}{(R+h)^2}$$

We know that for $h = 0$, $\frac{d^2h}{dt^2} = -g = \frac{-GM}{R^2} = \frac{-gR^2}{(h+R)^2}$
We also know $h(0) = 0$, $\frac{dh}{dt}(0) = v$ (initial velocity)

We can introduce dimensionsless variables:

- $[t] = T$
- $[h] = L$
- $[R] = L$
- $[v] = LT^{-1}$
- $[g] = LT^{-2}$

Since only L, T are involved, we have 2 rows:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\text{rank}(A) = 2 \Rightarrow$ 3 dimensionsless variables
We could for example do

$$\Pi_1 = \frac{h}{R} \qquad \Pi_2 = \frac{h}{vt} \qquad \Pi_3 = \frac{h}{gt^2}$$

Let us see what happens if we do some scaling for the time $\bar{t}$ and the length $\bar{h}$:

$$\bar{t} = \frac{t}{t_c} \qquad \bar{h} = \frac{h}{h_c}$$

With a dimension of time, we could pick $\frac{R}{v}$, or $\sqrt{\frac{R}{g}}$, $\frac{v}{g}$
The same for h, we could pick R, $\frac{v^2}{g}$

Usually only one choice is the one that helps us solve the problem:

$$\bar{t} = \frac{t}{R/v} \qquad \bar{h} = \frac{h}{R}$$

Now we need to express the laws that we have in terms of $\bar{t}$ and $\bar{h}$:

$$\frac{d^2h}{dt^2} = \frac{-gR^2}{(R\bar{h}+R)^2} = \frac{-g}{(\bar{h}+1)^2} \qquad h = \bar{h}R$$
$$\frac{dh}{dt} = \frac{d\bar{h}}{dt}R \qquad \frac{d\bar{h}}{d\bar{t}} = \frac{d\bar{h}}{dt}\frac{dt}{d\bar{t}} = \frac{R}{v}\frac{d\bar{h}}{dt}$$
$$\frac{d^2\bar{h}}{d\bar{t}^2} = \frac{d^2\bar{h}}{dt^2}\frac{R^2}{v^2} \rightarrow \frac{v^2}{Rg}\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}$$

We can call $\varepsilon = \frac{v^2}{Rg}$ (ε small)

The equation $\varepsilon\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}$ has no solution when $\varepsilon = 0$

With a different choice

$$\bar{t} = \frac{t}{vg^{-1}} \qquad \bar{h} = \frac{h}{v^2g^{-1}}$$
$$\Rightarrow \frac{d^2\bar{h}}{dt^2} = -\frac{1}{(1+\varepsilon\bar{h})^2} \qquad \bar{h}(0) = 0 \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1$$

Notice now that when $\varepsilon = 0$:

$$\overline{h}'' = -1 \qquad \overline{h}' = -\bar{t} + a = -\bar{t} + t$$

$$\overline{h} = -\frac{t^2}{2} + \bar{t} + b = -\frac{\bar{t}^2}{2} + \bar{t}$$

By substituting the old variables back, we get:

$$h = \frac{-t^2}{2} g + vt$$

The quantities that we used for t_c, h_c:

$$t_c = \frac{v}{g} \qquad h_c = \frac{v^2}{g}$$

$$h' = 0 \to -tg + v = 0 \Rightarrow t = \frac{v}{g}$$

Then h_c is the maximum height that the body reaches.

2. Pertubation Theory

This applies to another class of problems that is known (in a sense that we know how to solve it, we can find the solution); and we consider a new problem that is made of a known problem + *a pertubation*:

$$\text{Problem} + \varepsilon \text{ Problem}_2 \qquad \varepsilon << 1$$

Example: Planetary motion
If we consider a 2-body problem (one planet & one star), this can be solved exactly
Consider now a 3-body problem, then this problem cannot be solved easily.

$$M_{\text{sun}} >> M_{p_1} M_{p_2}$$

$$\underbrace{F_{Sp_1} + F_{Sp_2}}_{\text{2-body}} + \overbrace{F_{p_2 p_1}}^{F_{p_2 p_1} << F_{Sp_{1,2}}}$$

Anmärkning: Here F_{Sp_i} denotes the gravitational pull from the sun to one planet.

In general, we apply pertubation theory to equation of the form $F(y, y', y'', \cdots, \varepsilon) = 0$.
We look for a solution $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 + \cdots$
We expect y_0 (leading term) to be the solution/approximation when $\varepsilon = 0$

We may ask ourselves if the approximation converges to the solution.

Example:

$$\hat{y} = -y + \varepsilon y^2 \qquad y(0) = 1$$

We try to do the easiest thing and plug in the approximation:

$$(\hat{y}_0 + \varepsilon\hat{y}_1 + \varepsilon^2\hat{y}_2 + \cdots) = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + \varepsilon(y_0 + y_1\varepsilon + \cdots)^2$$

Now we try to solve order by order (collect like terms and see which one equates):

$$\hat{y}_0 = -y_0$$

$$y_0 = Ae^{-t} \qquad y_0 = e^{-t} \qquad \text{since our initial condition}$$

$$\hat{y}_1 = -y_1 + y_0^2 = -y_1 + e^{-2t}$$

$$\hat{y}_1(0) = 0 \qquad \text{since we already used our initial condition}$$

$$y_1(t) = e^{-t} + Ae^{-2t} \Rightarrow y_1(t) = e^{-t} - e^{-2t}$$

$$\hat{y}_2 = -y_2 + 2y_0y_1 = -y_2 + 2e^{-t}(e^{-t} - e^{-2t})$$

$$\hat{y}_2(0) \Rightarrow y_2(t) = e^{-t} - 2e^{-2t} + e^{-3t}$$

We have found the first three terms. This problem can be solved exactly, and we can see if our construction solves the equation or not:

$$y(t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \cdots$$

This is a case where regular pertubation works really well, since the explicit solution is given by:

$$y(t) = \frac{e^{-t}}{1 - \varepsilon + \varepsilon e^{-t}} = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)}$$

$$\sum_n x^n = \frac{1}{1-x} \Rightarrow e^{-t}(1 - \varepsilon(e^{-t} - 1) + \varepsilon^2(e^{-t} - 1)^2 + \cdots)$$

This is not always the case, that it is the same solution. If we use the example from the last lecture (projectile problem), we get something different:

$$h'' = \frac{-1}{(1 + \varepsilon h)^2}$$

$$\varepsilon = 0 \Rightarrow h_0 = \frac{-t^2}{2} + t$$

We try the same technique, suppose $h = h_0 + \varepsilon h_1$:

$$(h_0' + h_1'\varepsilon)(1 + \varepsilon(h_0 + \varepsilon h_1))^2 = -1$$

We collect like terms:

$$h_1'' + 2h_0'' h_0 = 0$$
$$h_1 = -(-1)\left(\frac{t^4}{4} + t^2 - t^3\right)$$
$$h_1' = \frac{t^5}{20} + \frac{t^3}{3} - \frac{t^4}{4} + C \qquad h_1'(0) = 0 \qquad h_1(0) = 0$$
$$h_1 = \frac{t^6}{100} + \frac{t^4}{12} - \frac{t^5}{20}$$

We have a polynomial in t which is greater than the one in h_0, and an exponential in the other.

Well, in the terms $\varepsilon(e^{-t} + \cdots)$, the size is dominated by ε even when t grows, while in the polynomial no matter how small ε we choose the polynomial can always grow bigger. So $h_0 + \varepsilon h_1$, h_1 term grows too much. This does not mean that the term is wrong, but it may not have a meaning in the problem that we are considering.

h_1 is growing faster than h_0 even though there is an ε in front of it. Recall that h_1 is just a correction, because we are adding a term that is bigger than our first approximation. We are essentially not writing a function that is adding smaller and smaller terms.

This method is called *regular pertubation*, and sometimes it works and sometimes it does not. In the case when it does not, we have to try a different technique.
We consider a different problem:

Example: Duffin Equation

$$\hat{u} + u + \varepsilon u^3 = 0 \qquad t > 0$$
$$u(0) = 1 \qquad \hat{u}(0) = 0$$

In this case, we do not have an explicit formula for the solution. We can try to use regular pertubation and see if it has a meaning or not.

$$u = u_0 + \varepsilon u_1$$
$$\Rightarrow \hat{\hat{u}} + u_0 = 0 \qquad u_0(0) = 1 \qquad \hat{u}_0(0) = 0$$
$$u_0(t) = A\cos(t) + B\sin(t)$$
$$u_0(0) = 1 \Rightarrow A = 1$$
$$\hat{u}_0(0) = -\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$$
$$u_0 = \cos(t)$$

Notice that for u_0, we have an oscillatory solution (since the trig-functions are periodic).

We collect like terms and equate them:

$$\hat{\hat{u}}_1 + u_1 + u_0^3 = 0 \qquad u_1(0) = 0 \qquad \hat{u}_1(0) = 0$$
$$\hat{\hat{u}}_1 = -u_1 - \cos^3(t)$$
$$\cos^3(t) = \left(\frac{e^{i\pi} + e^{-\pi}}{2}\right)^3 = \frac{e^{3i\pi} + e^{-3i\pi} + 3d^{2i\pi - i\pi} + 3e^{i\pi - 2i\pi}}{8}$$
$$= \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$$
$$\Rightarrow \hat{\hat{u}}_1 + u_1 = \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$$
$$u_1(0) = 0 \qquad \hat{u}_1(0) = 0 \Rightarrow u_1(t) = A\cos(t) + B\sin(t) + C\cos(3t) + At\sin(t) + Bt\cos(t)$$

Notice that since one particular solution already included $\cos(t)$, we add another set of $A\sin(t) + B\cos(t)$, but multiplied with t.
With respect to the initial conditions, we get:

$$u_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \underbrace{\frac{3}{8}t\sin(t)}_{\text{Secular term}}$$

The secular term might be a problem, for exmaple in this case we are interested in an oscillatory solution, so we expect a correction that gives us oscillatory approximation. But the t term makes the correction explode when $t \to \infty$.
There is also another issue with this approximation. We can show that the solution to this equation is bounded, but with this solution it breaks when $t \to \infty$, so $u_0 + \varepsilon u_1$ is not good.

Bevis 2.1: Exact solution is bounded

Consider $\hat{\hat{u}} + u + \varepsilon u^3 = 0$, and multiply with $\hat{u}$:

$$\hat{u}\hat{\hat{u}} + \hat{u}u + \varepsilon \hat{u} u^3 = 0$$

$$= \frac{d}{dt}\left(\frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4}\right) = 0 \Rightarrow \frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4} = \text{Constant} = \frac{1}{2} + \frac{\varepsilon}{4}$$

$$\Rightarrow u \text{ is bounded}$$

□

Sometimes having small errors may not seem like a big issue, but these small errors may explode further down as $t \to \infty$

2.1. **Poincare-Lindstedt Method.**

The idea is to do a rescaling, considering a pertubative correction of the frequencies of the oscillation.

We introduce a new variable (distorted time scale) $\tau = \omega t$ where $\omega = \omega_0 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots$.
As in the other case, ω_0 is the leading term when $\varepsilon = 0$, which in the previous example is 1.

We have to rewrite the equation according to the new time:

$$\frac{du}{dt} = \frac{du}{dt}\frac{dt}{d\tau} = \hat{u}\frac{1}{\omega}$$

$$\Rightarrow \hat{u} = \omega u' \qquad u' = \frac{du}{d\tau}$$

In the Duffin equation with the new variables we get:

$$\omega^2 u'' + u + \varepsilon u^3 = 0 \qquad u(0) \overset{\tau=0}{=} 1 \qquad \frac{du}{d\tau}(0) = \frac{\hat{u}(0)}{\omega} = 0$$

We now study this equation when $\tau > 0$. Essentially what we do, the advantage in this expansion, is that we can kill the terms that in regular pertubation generates the problem terms (secular terms).
What we will do is choose ω_1 such that it kills the secular terms. The procedure is the same as in regular pertubation:

$$(\omega_0 + \omega_1\varepsilon)^2(u_0'' + \varepsilon u_1'') + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 = 0$$

Gather terms and equate:

$$w_0^2 + u_0'' + u_0 = 0 \qquad u_0(0) = 1 \qquad u_0'(0) = 0$$

$$u_0 = \cos(\tau) \qquad w_0 = 1$$

$$(1 + 2\omega_1\varepsilon + \omega_1^2\varepsilon^2)(u_0'' + \varepsilon u_1'') + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 = 0$$

$$\Rightarrow 2\omega_1 u_0'' + u_1'' + u_1 + u_0^3 = 0 \Rightarrow u_1'' + u_1 = -u_0^3 - 2\omega_1 u_0'' = -\cos^3(t) - 2\omega_1 - \cos(\tau)$$

$$\Rightarrow \frac{1}{4}\cos(3\tau) + \frac{3}{4}\cos(\tau) + 2\omega_1\cos(\tau)$$

$$\Rightarrow \frac{1}{4}\cos(3\tau) + \left(2\omega_1 - \frac{3}{4}\right)\cos(\tau)$$

Notice that the last $\cos(\tau)$ is the generator of our secular term, so we choose ω_1 so that $2\omega_1 - \frac{3}{4} = 0 \Rightarrow \omega_1 = \frac{3}{8}$

What we now get is an approximate term u_1 that does not have a secular term, but just sine and cosine (preserving oscillation). We can also use some of the previous calculations, but without the secular term since we have removed it.

$$u_1(\tau) = \frac{1}{32}\left(\cos(3\tau) - \cos(\tau)\right)$$
$$u = \cos(\tau) + \frac{\varepsilon}{32}\left(\cos(3\tau) - \cos(\tau)\right)$$
$$\tau = \omega t = (1 + \frac{3}{8}\varepsilon)t$$

We have so far covered *singular pertubation*. Let us look at an example when this method fails:

Example: $\varepsilon x^5 + x - 1 = 0 \qquad 0 < \varepsilon << 1$:

$$\varepsilon = 0 \qquad x = 1$$
$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$
$$x_0 = 1$$

From this equation we expect 5 different solutions since it is a polynomial of order 5.
The issue we have here is that the ε is in front of the term of highest order.

2.2. **Dominant Balancing method.**

Another example could therefore be $\varepsilon y'' + y' + y = 0$.

The strategy is to look for a scaling such that the leading term can remain big even though we multiply by ε. We can do this by defining a new variable $x = \dfrac{y}{f(\varepsilon)}$ that allows us to describe problem when the term that we are removing is not small anymore.

We solve using *dominant balancing*.
Case: Say $\varepsilon x^5 = O(X)$, then $\varepsilon x^4 = (1)$ and $x = O\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)$
We then get:

$$\varepsilon x^5 = \varepsilon O\left(\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)^4\right) = O\left(\varepsilon^{1-\frac{5}{4}}\right) = O\left(\varepsilon^{-\frac{1}{4}}\right)$$

With this choice, are these terms bigger than the remaining one? In this case the remaining one is 1, so it is true that when $\varepsilon << 1$ then $\dfrac{1}{\sqrt[4]{\varepsilon}} >> 1$

Case: Say $\varepsilon x^5 = (1)$, then $x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right)$. So for the second term (that is x), we get

$$x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right)$$

We see that this is not a good choice because the remaining term $x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right) >> 1$ when $\varepsilon \to 0$.
This is a problem because we want the leading term to be the bigger one.

It is this reasoning that we use when we determine $f(\varepsilon)$.

Since we saw that the choice $x = \frac{y}{\sqrt[4]{\varepsilon}} \Rightarrow f(\varepsilon) = \varepsilon^{-\frac{1}{4}}$, we substitute this variable in the equation and try to solve for y:

$$\frac{\varepsilon y^5}{(\sqrt[4]{\varepsilon})^5} + \frac{y}{\sqrt[4]{\varepsilon}} - 1 = 0$$
$$y^5 \varepsilon^{1-\frac{5}{4}} + y\varepsilon^{-\frac{1}{4}} - 1 = 0$$
$$\Rightarrow y^5 + y - \varepsilon^{\frac{1}{4}} = 0$$

Now ε is not in the term of the highest order and we can solve using regular pertubation.
We solve the equation for $\varepsilon = 0$:

$$y^5 + y = 0 \Leftrightarrow y(y^4 + 1) = 0$$

One solution is $y = 0$, but this is a false root because of our initial conditions (**CHECK**). The other 4 solutions are given by the roots of unity for $y^4 + 1 = 0$:

$$y^4 = -1 = e^{i\pi}$$
$$y_{1,2,3,4} = e^{\frac{i\pi + 2n\pi}{4}} \qquad n = 0, 1, 2, 3$$
$$\Rightarrow y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \qquad \text{leading term } y_0 \text{ are the roots } y_n$$

Going back to our original variable $x = \frac{y}{\sqrt[4]{\varepsilon}}$ has leading order terms $\frac{y_n}{\sqrt[4]{\varepsilon}}$.

This is for the leading order, you can of course compute for the other terms.

Examples:

- $\varepsilon x^2 + 2x + 1 = 0$
- $\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$

Try to understand which are the leading orders of the approximated solutions.
(**DO THESE**)

Example:

- $\varepsilon y'' + (1 + \varepsilon)y' + y = 0$
- $y(0) = 0$
- $y(1) = 1$

Here the problem is a bit different, due to the initial conditions, when $\varepsilon = 0$ the equation cannot be solved since

$$\varepsilon = 0 \Rightarrow y' + y = 0 \qquad y' = -y \qquad y(x) = Ae^{-x}$$
$$y(0) = A = 0 \Leftarrow \text{ not a solution}$$

We can compute the solution exactly, this will tell us what the problem is when approaching this in pertubative approach:

$$\varepsilon m^2 + (1+\varepsilon)m + 1 = 0$$
$$m_{1,2} = \frac{-(1+e) \pm \sqrt{(1+\varepsilon)^2 - 4\varepsilon}}{2\varepsilon}$$
$$m_1 = -1 \qquad m_2 = -\frac{1}{\varepsilon}$$
$$\Rightarrow y(x) = C_1 e^{-x} + C_2 e^{-\frac{x}{\varepsilon}}$$
$$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$
$$y(1) = C_2 \left(e^{-\frac{1}{\varepsilon}} - e^{-1} \right) = 1 \Leftrightarrow C_2 = \frac{1}{\left(e^{-\frac{1}{\varepsilon}} - e^{-1} \right)}$$
$$y(x) = \frac{1}{\left(e^{-(1/\varepsilon)} - e^{-1} \right)} \cdot \left(e^{-(x/\varepsilon)} - e^{-x} \right)$$

The problem is the size of the term $\varepsilon y''$, even if ε is very small the term can be very big.

Using the exact solution we can compute the second derivative:

$$y'' = \frac{1}{e^{-(1/\varepsilon)} - e^{-1}} \left(e^{-x} - \frac{1}{\varepsilon^2} e^{-(x/\varepsilon)} \right)$$

If we look at $y''(x)$ when $x \to 0$:

$$\frac{1}{e^{-(1/\varepsilon)} - e^{-1}} \left(C - \frac{1}{\varepsilon^2} D \right)$$

So $\varepsilon y''(x) = O\left(\frac{1}{\varepsilon}\right)$ which is not small when $\varepsilon = 0$
But when $x = O(1) \qquad x >> 1$:

$$y''(x) = 0$$

So in fact, $\varepsilon y''$ is small when x grows.

We can therefore say that our solution $y = Ae^{-x}$ is valid as long as x is not close to 0 since we have $\varepsilon y'' = O(\varepsilon)$ and we can use regular pertubation.
Using $y(1) = 1$ (and not 0, since we want x to be as far away from 0 as we can) we have:

$$y_{\text{outer}}(x) = e \cdot e^{-x} = e^{1-x}$$

Where y_{outer} denotes the outer domain ($x = O(1)$). This is only the leading order of the approximation (we are keeping the solution when $\varepsilon = 0$).

Just as we did with the polynomial, in order to fully solve this we introduce a rescaling to our problem to consider when $\varepsilon y''$ is small:

$$\tau = \frac{x}{f(\varepsilon)}$$
$$y' = \frac{dy}{dx} = \frac{dy}{d\tau}\frac{d\tau}{dx} = \hat{y}\frac{1}{f(\varepsilon)}$$

Inserting this in the equation gives us:

$$\varepsilon \frac{\hat{\hat{y}}}{f(\varepsilon)^2} + (1+\varepsilon)\frac{\hat{y}}{f(\varepsilon)} + y = 0$$

Now we compare cases:

Case 1:

$$\frac{\varepsilon}{f(\varepsilon)^2} = O\left(\frac{1+\varepsilon}{f(\varepsilon)}\right) \Rightarrow f(\varepsilon) = O\left(\frac{\varepsilon}{1+\varepsilon}\right) = O(\varepsilon)$$

$$\begin{cases} \frac{\varepsilon\hat{\hat{y}}}{\varepsilon^2} = \frac{\hat{\hat{y}}}{\varepsilon} \\ \frac{1+\varepsilon}{\varepsilon}\hat{y} \to O\left(\frac{1}{\varepsilon}\right) \\ y \text{ is } O(1) \end{cases}$$

$$\frac{1}{\varepsilon} >> 1$$

We can check that choosing:

$$\frac{\varepsilon}{f(\varepsilon)^2} = O(1) \qquad f(\varepsilon) = O(\sqrt{\varepsilon})$$

$$\begin{cases} \frac{\varepsilon}{f(\varepsilon)^2}\hat{\hat{y}} = O(1) \\ \frac{1+\varepsilon}{\sqrt{\varepsilon}} = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \end{cases}$$

In this case we see that the term in front of $\hat{y}$ is bigger than the one in front of $\hat{\hat{y}}$

The equation with the rescaling $\tau = \frac{x}{\varepsilon}$ becomes:

$$\frac{\varepsilon}{\varepsilon^2}\hat{\hat{y}} + \frac{1+\varepsilon}{\varepsilon}\hat{y} + y = 0$$
$$\hat{\hat{y}} + (1+\varepsilon)\hat{y} + \varepsilon y = 0$$

We solve using regular pertubation:

$$\varepsilon = 0 \qquad \hat{\hat{y}} + \hat{y} = 0 \qquad m^2 + m = 0 \Leftrightarrow m(m+1) = 0$$
$$y_{\text{inner}}(\tau) = A + Ae^{-\tau} = A(1 + e^{-\tau})$$

We have a set of equations to describe our solutions, one given by our outer approximation and one for our inner:

$$\begin{cases} y_{\text{outer}}(x) = e^{1-x} \\ y_{\text{inner}} = A(1 + e^{-(x/\varepsilon)}) \qquad x = O(\varepsilon) \end{cases}$$

We still have the parameter A. We can merge the two solutions in the part of the domain that is hared, for exmaple where $x \approx O(\sqrt{\varepsilon}) \qquad \varepsilon < \sqrt{\varepsilon} < 1$
We impose that when $x \cong O(\sqrt{\varepsilon})$, the limits of the two functions is the same.

We can define an auxillary variable $\eta = \frac{x}{\sqrt{\varepsilon}}$:

$$\lim_{\varepsilon \to 0^+} y_{\text{outer}}(\eta\sqrt{\varepsilon}) = \lim_{\varepsilon \to 0^+} y_{\text{inner}}(\eta\sqrt{\varepsilon})$$
$$\lim_{\varepsilon \to 0^+} e^{1-\eta\sqrt{\varepsilon}} = e$$
$$\lim_{\varepsilon \to 0^+} A\left(1 - e^{-(\eta/\sqrt{\varepsilon})}\right) = A \Rightarrow A = e$$

You can also write this as one singular function:

$$y = y_{\text{outer}} + y_{\text{inner}} - \underbrace{\text{common limit}}_{e}$$
$$\Rightarrow e^{1-x} + e^{1-(x/\varepsilon)}$$

2.3. **WKB Methods (Wentzel, Kramers, Brillouin).**

This new method (still falling under pertubation theory) applies to a general class of problems express mainly through differential equations on the following forms (for $\varepsilon << 1$):

$$\varepsilon^2 y'' + q(x)y = 0$$
$$y'' + q(\varepsilon x)^2 y = 0$$
$$y'' + (\lambda^2 p(x) - q(x))y = 0 \qquad \lambda >> 1$$

These equations come up in problems related to quantum mechanics, where instead of Newtons law there are Schrödinger equations that involves a function $\psi(x,t)$ and satisfies:

$$i\hat{h}\psi_t = \frac{-\hat{h}^2}{2m}\psi_{xx} + V(x)\psi$$

Here ψ is the wave function and the equation tells us the probability that a particle is in a certain place. For example:

$$P(a < x \leq b) = \int_a^b |\psi|^2 \, dx$$

Let us look at $\phi(x,t)$ more deeply, specifically, let us assume it is separable.
Then we get $\phi(x,t) = y(x)\varphi(t)$. With this, we can rewrite the equation differently:

$$i\hat{h}y(x)\varphi'(t) = -\frac{\hat{h}^2}{2m}y''(x)\varphi(t) + V(x)y(x)\varphi(t) \frac{i\hat{h}\varphi'(t)}{\varphi(t)} = \frac{-\frac{\hat{h}^2}{2m}y''(x)\varphi(t)}{y(x)} + V(x)$$

$$\left.\begin{array}{l} \dfrac{i\hat{h}\varphi'}{\varphi} = E \\ -\dfrac{\hat{h}^2 y''}{y} + V(x) = E \end{array}\right\} \Rightarrow \varphi' = \frac{E}{i\hat{h}}\varphi = -i\frac{E\varphi}{\hat{h}} \qquad \varphi(t) = Ae^{-\frac{iEt}{\hat{h}}}$$

We then arrive at:

$$-\frac{\hat{h}^2}{2m}y'' + (V(x) - E)y = 0$$

Anmärkning:
$\hat{h} = \dfrac{h}{2\pi}$ where h is Plancks constant. We have arrived at our first equation since if we let $\varepsilon = \dfrac{\hat{h}}{\sqrt{2m}}$ and $q(x) = E - V(x)$ it is on the form $\varepsilon^2 y'' + q(x)y = 0$

In order to solve the first equation we make the ansatz

$$y = e^{\frac{u(x)}{\varepsilon}}$$
$$y' = e^{\frac{u(x)}{\varepsilon}} \cdot \frac{1}{\varepsilon}u'$$
$$y'' = e^{\frac{u(x)}{\varepsilon}} \frac{1}{\varepsilon^2}(u')^2 + e^{\frac{u(x)}{\varepsilon}} \cdot \frac{1}{\varepsilon}u''$$
$$\Rightarrow e^{u/\varepsilon}(u')^2 + \varepsilon e^{u/\varepsilon}u'' + q(x)e^{u/\varepsilon} = 0$$
$$\Rightarrow (u')^2 + u'' + q(x) = 0$$
$$\text{Let } f = u' \Rightarrow f^2 + \varepsilon f' + q(x) = 0$$

We can now solve with regular pertubation:

$$f = f_0 + \varepsilon f_1$$
$$(f_0 + \varepsilon f_1)^2 + \varepsilon(f_0' + f_1'\varepsilon) + q(x) = 0$$

Collect terms of the same order and compare:

$$f_0^2 + q(x) = 0 \Rightarrow f_0 = \pm\sqrt{-q(x)} \qquad q(x) < 0$$
$$2f_0 f_1 + f_0' = 0 \Rightarrow f_1 = -\frac{f_0'}{2f_0} = \frac{q'}{2 \cdot 2 \cdot \sqrt{-q(x)} \cdot \sqrt{-q(x)}} = -\frac{q'(x)}{4q(x)}$$

We arrive at:

$$f = \pm\sqrt{-q(x)} - \varepsilon\frac{q'}{4q(x)}$$

Since $f = u'$, we integrate:

$$u = \int_a^x \pm\sqrt{-q(s)}ds \underbrace{-\frac{\varepsilon}{4}\int_a^x \frac{q'(s)}{q(s)}ds}_{-\frac{\varepsilon}{4}\ln(|q(x)|)} + O(\varepsilon^2)$$

Summa sumarum, we have:

$$y = e^{\frac{u(x)}{\varepsilon}} = e^{\frac{\pm 1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds} e^{-\frac{\varepsilon}{4\varepsilon}\ln(-q(x))} e^{\frac{O(\varepsilon^2)}{\varepsilon}}$$
$$= \frac{1}{\sqrt[4]{-q(x)}} e^{\pm\frac{1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds} e^{O(s)}$$
$$= \underbrace{\frac{1}{\sqrt[4]{-q(x)}} e^{\pm\frac{1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds}}_{\text{Our approximation}} (1 + O(\varepsilon))$$

Anmärkning:
Here we have used the assumption that $q(x) < 0$. We can of course do this when $q(x) > 0$, but we change the ansatz to $y = e^{\frac{iu(x)}{\varepsilon}}$ (in order to avoid $\sqrt{-1}$).
It would look something like this:

$$y' = \frac{i}{\varepsilon} e^{\frac{iu(x)}{\varepsilon}} u'$$
$$y'' = \frac{-1}{\varepsilon^2}\frac{i}{\varepsilon} e^{\frac{iu(x)}{\varepsilon}} (u')^2 + \frac{i}{\varepsilon} e^{\frac{iu(x)}{\varepsilon}} u''$$

We try out the same calculations now:

$$-(u')^2 + i\varepsilon u'' + q(x) = 0$$
$$\text{Let } f = u' \Rightarrow -f^2 + i\varepsilon f' + q(x) = 0$$
$$f = f_0 + \varepsilon f_1$$

Collect like terms and equate:

$$-f_0^2 + q(x) = 0 \Rightarrow f_0 = \sqrt{q(x)}$$
$$-2f_0 f_1 + i f_0' = 0 \Rightarrow f_1 = \frac{i f_0'}{2f_0} = \frac{\pm i q'}{2 \cdot 2 \cdot \sqrt{q(x)} \cdot \sqrt{q(x)}} = \frac{iq'(x)}{4q(x)}$$

This is very similar to the previous computations and in fact it will be a similar solution but with an i appearing in some places:

$$y = e^{\frac{\pm i}{\varepsilon}\int_a^x \sqrt{q(s)}ds} \cdot \frac{1}{\sqrt[4]{q(x)}} (1 + O(\varepsilon))$$

Notice however that since e^{ik} can be written with the helps of trigonometric functions, we have an oscillatory (very fast oscillations) solution but we did not have this in the first case.

2.4. **Asymptotic Expansion of Integrals.**

This might happen when we solve a differential equation when we may need to compute an integral but we do not know how to do it explicitly, such as the following:

$$y'' + 2\lambda t y' = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

$$\text{Let } u = y' \Rightarrow u' + 2\lambda t u = 0 \Rightarrow u = Ce^{\lambda t^2}$$

$$y = \int_0^t e^{\lambda s^2} ds$$

These types of integrals are called *Laplace Integrals*

$$I(\lambda) = \int_a^b f(t)e^{-\lambda t} dt \qquad \lambda >> 1$$

They look like the Laplace transform:

$$\int_0^\infty f(t)e^{-\lambda t} dt$$

The term $e^{-\lambda t}$ is very small as $t \to \infty$. We *have* to assume that $f(t)$ is of exponential order in order to proceed pertubatively.

Lemma 2.1: Watson

Given the integral

$$I(\lambda) = \int_0^\infty t^\alpha h(t) e^{-\lambda t} dt$$

With $\alpha > -1$, $h(t)$ is analytic (has a Taylor expansion around 0), $h(0) \neq 0$.
Assume that h is of exponential order.

Then we can approximate the integral in the form of a series:

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)\Gamma(\alpha + n + 1)}{n!\lambda^{\alpha+n+1}}$$

with the Gamma function defined as:

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \qquad x > 0$$

Anmärkning:
This follows from Taylor expansion, but let us look at the proof:

Bevis 2.2

The first thing that we can do is to split the domain such that $t \in [0, T]$:

$$\int_0^T t^\alpha h(t) e^{-\lambda t} dt + \underbrace{\int_T^\infty t^\alpha h(t) e^{-\lambda t} dt}_{\text{Exponentially small}}$$

Since h is bounded, and t^α is of lower growth order than $e^{-\lambda t}$, and $e^{-\lambda t}$ is shrinking as $T \to \infty$, it can be regarded as a very small number.

In the first integral we use the fact that $h(t)$ is analytic:

$$\int_0^T t^\alpha \left(h(0) + h'(0)t + h''(0)t^2/2 + \cdots\right) e^{-\lambda t} dt + \text{Exp. small}$$

$$\text{Let } u = \lambda t \Rightarrow \int_0^{\lambda T} \left(\left(\frac{u}{\lambda}\right)^\alpha h(0) + h'(0) \left(\frac{u}{\lambda}\right)^{\alpha+1} + \frac{1}{2} h''(0) \left(\frac{u}{\lambda}\right)^{\alpha+2} + \cdots \right) e^{-u} \frac{du}{\lambda}$$

$$\frac{h(0)}{\lambda^{\alpha+1}} \int_0^{\lambda T} u^\alpha e^{-u} du = \frac{h(0)}{\lambda^{\alpha+1}} \int_0^{\infty} u^\alpha e^{-u} du \tag{1}$$

Since we add an exponentially small term, we can equate the integral going to λT with the integral going to ∞ by approximation.

Upon closer inspection of (1) , we see the following equality:

$$(1) = \frac{h(0)}{\lambda^{\alpha+1}} \Gamma(\alpha + 1)$$

The general term is given on teh following form:

$$\frac{h^{(n)}(0)}{n! \lambda^{\alpha+n+1}} \Gamma(\alpha + n + 1)$$

This is what we wanted to show □

Example:
Let us look at an application of this lemma. Suppose that we want to compute an approximation of the following integral:

$$I(\lambda) = \int_0^{+\infty} \frac{\sin(t)}{t} e^{-\lambda t} dt \qquad \lambda >> 1$$

Here, $h(t) = \dfrac{\sin(t)}{t}$, and $\alpha = 0$. We can split the term:

$$\int_0^T \frac{\sin(t)}{t} e^{-\lambda T} dt + \int_T^{+\infty} \frac{\sin(t)}{t} e^{-\lambda t} dt$$

We look at the behaviour of the function $|h(t)| = \left|\dfrac{\sin(t)}{t}\right|$ as $t \to \infty$. We know $h(t)$ is bounded, so the right integral is negligable as $t \to \infty$ since it is exponentially small.
We remain with the following:

$$\int_0^T \frac{\sin(t)}{t} e^{-\lambda t} dt \Rightarrow \int_0^T \frac{t - \dfrac{t^3}{3!} + \cdots}{t} e^{-\lambda t} dt$$

$$= \int_0^t \left(1 - \frac{t^2}{3!} + \cdots\right) e^{-\lambda t} dt$$

$$\text{Let } u = \lambda T \Rightarrow \int_0^{\lambda T} \left(1 - \left(\frac{u}{\lambda}\right)^2 + \cdots\right) e^{-\lambda t} \frac{du}{\lambda}$$

Yet again, we see what happens in the world of infinities for each one of the terms (recall that the integral is a linear operator)

$$\int_0^{+\infty} \frac{1}{\lambda} e^{-u} du = \frac{\Gamma(1)}{\lambda}$$

$$\frac{1}{3!\lambda} \int_0^{+\infty} \frac{u^2 e^{-u}}{\lambda^2} du = \frac{1}{3!\lambda^3} \Gamma(3)$$

$$\vdots$$

Anmärkning:

If we have an integral such as

$$I(\lambda) = \int_a^b f(t)e^{-\lambda g(t)}dt$$

We can study the problem in the same way by introducing the change of variable $s = g(t) - g(a)$

3. Calculus of Variations

From last time (**READ**):

$$\int_a^b L(x,y,y')dx \qquad y(a)=A \qquad y(b)=B$$

An extremal satisifies Euler-Lagranges equation:

$$L_y - \frac{d}{dx}(L_{y'}) = 0$$

One generalisation of this formula can be obtainedwhen $L = L(x,y,y',y'')$. In this case we have to assume some regularity for y, that is $y \in C^4[a,b]$, and two other initial condition for y'.

We compute the derivative of the functional $\delta J(y;v) = \frac{d}{d\varepsilon}J(y+\varepsilon v)|_{\varepsilon=0}$
We want $y+\varepsilon v$ to be an admissable function, so this means that $v(a)=v(b)=0$ and $v'(a)=v'(b)=0$

Using this when trying to compute the derivative of the functional:

$$\frac{d}{d\varepsilon}\int_a^b L(x,y+\varepsilon v, y'+\varepsilon v', y''+\varepsilon v'')dx|_{\varepsilon=0}$$
$$\Rightarrow \int_a^b L_y v + L_{y'}v' + L_{y''}v''dx$$

We solve this by integration by parts:

$$\int_a^b (L_y - \frac{d}{dx}L_{y'})vdx + \int_a^b L_{y''}v''dx = v'L_{y''}|_a^b - \int_a^b v'\frac{d}{dx}(L_{y''})dx$$

By our initial conditions, $v'L_{y''}|_a^b = 0$:

$$-v\left(\frac{d}{dx}L_{y''}\right)_a^b + \int_a^b v\left(\frac{d^2}{dx^2}L_{y''}\right)dx$$
$$= \int_a^b L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''}v(x)dx = 0 \qquad \forall v$$

The condition here becomes:

$$L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''} = 0$$

In this equation we will have the 4th derivative w.r.t y, which is why we need $y \in C^4[a,b]$

As an exercise, try to generalise the formula:
if $L = L(x,y,y',\cdots,y^{(n)}) \Rightarrow$ the extremals satisfy:

$$L_y + \sum_{k=1}^{n}(-1)^k\frac{d^k}{dx^k}L_{y^{(k)}} = 0$$

The reason we have a plus and minus is because whenever we integrate by parts we will get a plus and next time a minus etc...

Exercicse: Show that if L does not depend on y ($L = L(x,y',y'')$), then $L_{y'} - \frac{d}{dx}L_{y''} = C$ (some constant)

Exercicse: If L does not depend on x, show that $L - y'(L'_y - \frac{d}{dx}L_{y''}) - y''L_{y''} = C$ (some constant)

Another generalisation occurs when we have $L = L(x,y_1,y_2,y_1',y_2')dx$ with initial values:

$$y_1(a) = A_1 \qquad y_1(b) = B_1$$
$$y_2(a) = A_2 \qquad y_2(b) = B_2$$

Proceeding with the same strategy as before, we compute the derivative. Starting by imposing $\delta J(y;v) = 0$:

$$\int_a^b \frac{d}{d\varepsilon} L(x, y_1 + \varepsilon v_1, y_2 + \varepsilon v_2, y_1' + \varepsilon v_1', y_2' + \varepsilon v_2') dx|_{\varepsilon=0} = 0$$

In this case we have that $v_1, v_2(a) = 0 = v_1, v_2(b)$

$$\int_a^b \left(L_{y_1} - \frac{d}{dx} L_{y_1} \right) v_1(x) dx + \int_a^b \left(L_{y_2} - \frac{d}{dx} L_{y_2} \right) dx = 0 \qquad \forall v_1, v_2$$

The trick here is that since this equation has to be true for all v_1, v_2, so in particular $v_2 = 0$. This yields:

$$\int_a^b \left(L_{y_1} - \frac{d}{dx} L_{y_1} \right) v_1(x) dx = 0 \qquad \forall v_1 \Rightarrow L_{y_1} - \frac{d}{dx} L_{y_1'} = 0$$

and so

$$\int_a^b L_{y_2} - \frac{d}{dx} L_{y_2'} v_2(x) dx = 0 \qquad \forall v_2$$

$$\begin{cases} L_{y_1} - \dfrac{d}{dx} L_{y_1'} = 0 \\ L_{y_2} - \dfrac{d}{dx} L_{y_2'} = 0 \end{cases}$$

Exercicse: Show that if $L = L(y_i; y_i')$ (for $i = 1, \cdots, n$), then

$$L - \sum_{i=1}^{n} y_i' L_{y_i'} = C \qquad \text{if } y_i \text{ satisfies the Euler-Lagrange equations}$$

Let us see what happens if we leave one of the end-boundary conditions unsatisfied, how long will we come in our computation. This is a type of generalisation where the boundary can be free (aka a *free end-point problem*):

$$J(y) = \int_a^b L(x, y, y') dx \qquad y(a) = A \qquad y(b) \text{ is free}$$

We look for extremals. In all our previous computation, to have admissable functions, we took $v(a) = v(b) = 0$. When we looked at the derivative $y + \varepsilon v$, and for it to be admissable we have $y(a) + \varepsilon v(a) = A$, and $A + \varepsilon v(a) = A \Rightarrow v(a) = 0$.
This condition is not needed anymore in $v(b)$ because we are assuming that $y(b) + \varepsilon v(b)$ is free.

If that term is 0 $\forall v$, it must be 0 for $v(b) = 0$:

$$L_y - \frac{d}{dx} L_{y'} = 0$$

If $v(b) \neq 0$, then $v(b) L_{y'}(b, y(b), y'(b)) = 0$, and in that case:

$$\begin{cases} L_y - \dfrac{d}{dx} L_{y'} = 0 \\ L_{y'}(b, y(b), y'(b)) = 0 \end{cases}$$

Exercicse:

$$J(y) = \int_0^1 (y'^2 + y^2) dx \qquad y(0) = 1 \qquad y(1) = \text{ free}$$

Find extremals, so solve this equation

$$L_y - \frac{d}{dx}L_{y'} = 0$$
$$2y - \frac{d}{dx}(2y') = 0$$
$$2y - 2y'' = 0$$
$$y'' - y = 0 \qquad y(x) = Ae^{-x} + Be^{x} \qquad y(0) = A + B = 1 \Rightarrow A = 1 - B$$
$$y(x) = e^{-x} + B(e^{x} - e^{-x})$$
$$L_{y'} = (1, y(1), y'(1)) = 0 \qquad L_{y'} = 2y' \Rightarrow y'(1) = 0$$
$$y'(x) = -e^{-x} + B(e^{x} + e^{-x}) \Rightarrow y'(1) = -e^{-1} + B(e + e^{-1}) = 0$$
$$\Rightarrow B = \frac{1}{e}\frac{e}{e^2+1} = \frac{1}{e^2+1}$$

Another kind of exercise that can be computed with functionals is minimizing path distances.

Exercicse: Find the extremal paths that connect two points on a plane. This problem is equivalent to finding extremals to the functional $J(y) = \int_a^b \sqrt{1 + y'(x)^2}dx$ and $y(0) = a$, $y(1) = b$. Show that the solution is a line.

Exercicse: Find the extremal paths connecting two points on a cylinder.
Firstly, we switch to cylindrical coordinates: $(x, y, z) = (R\cos(\theta), R\sin(\theta), z)$ where R is the radius (for simplicity, we set $R = 1$)

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \qquad dx = -\sin(\theta)d\theta \qquad y = \cos(\theta)d\theta$$
$$\Rightarrow ds = \sqrt{\sin^2(\theta)d\theta^2 + \cos^2(\theta)d\theta^2 + dz^2} = \sqrt{d\theta^2 + dz^2} = d\theta\sqrt{1 + \left(\frac{dz}{d\theta}\right)^2}$$
$$S = \int_{P_1}^{P_1} ds = \int_{\theta_1}^{\theta_2} \sqrt{1 + z'(\theta)^2}d\theta$$

We now want to study the extremals with this functional:

$$L = L(z') = \sqrt{1 + z'^2(\theta)^2}d\theta$$

By the Euler-Lagrange equation:

$$L_z - \frac{d}{dx}L_{z'} = 0 \qquad L_{z'} = C \qquad \frac{1}{2\sqrt{1 + z'^2(\theta)}} \cdot 2z'(\theta) = C$$
$$z'^2 = C^2(1 + z'^2) \qquad z'^2(1 - C^2) = C^2 \Rightarrow z'^2 = \frac{C^2}{(1 - C^2)}$$
$$z' = K \text{ say for example } \frac{C}{\sqrt{1 - C^2}}$$
$$\Rightarrow z(\theta) = \theta K + a$$

Exercicse: Find the extremal paths connecting two points on a sphere. (You can use the radius 1 to help ya).

Exercicse: Consider the surface generated by a rotation around the x-axis of a curve $y(x)$. Can we find extremals?

3.1. **Hamiltons Principal.**

Usually a system is described by a set of equations, and you want to know the evolution of the system given an initial condition.

Example: Newtons law

$$F = ma = mv''$$

This law can be deduced through a variational principle. To do so, we need to define a specific functional.

A mechanical system evolves in such a way that if we define the functional $\int(K - V)dt$ where K is the kinetic energy and V is the potential energy, the functional is stationary.
If we have a system we can compute the kinetic and the potential energy, and we define something called the Lagrangian $\mathcal{L} = K - V$, then we can study the integral:

$$\int_{t_0}^{t_1} \mathcal{L}(t, y, y')dt$$

Then the system evolves from $y(t_0)$ to $y(t_1)$ along a path $y(t)$ that is an extremal for the functional. So it means that y satisfies Euler-Lagrange equation.
If we write down the Euler-Lagrange equation for these conditions, we will get something equivalent to Newtons Law.

- Kinetic energy $= \frac{1}{2}m\hat{x}^2$
- Potential energy $= V(x)$
- Lagrangian $\mathcal{L}(x, \hat{x}) = \frac{1}{2}m\hat{x}^2 - V(x)$

$$\Rightarrow \mathcal{L}_x - \frac{d}{dx}(\mathcal{L}_{\hat{x}}) = -V'(x) - \frac{d}{dt}(m\hat{x}) = 0$$
$$-V'(x) = m\hat{\hat{x}} = F(x)$$

Another thing that is useful when studying these types of systems is to introduce *Hamiltonian formalisms*. We have seen that Euler-Lagrange equation define an ODE of second order. What we want to do here is to introduce a new formalism that allows us to deal with first order ODE.

We consider the funcitonal:

$$J(y) = \int_a^b L(t, y, y')dt$$

We define a new variable $p = \dfrac{\partial L(t, y, y')}{\partial y'} = L_{y'}(t, y, y')$ which we call canonical momentum.
We suppose that $L_{y'}$ is regular enough for us to invert it in the following manner:

$$y' = \Phi(t, p, y) \qquad L_{y'y'} \neq 0 \text{ can be a condition for the invertibility}$$

We define a new function (that is called the Hamiltonian):

$$H(t, y, p) = -L(t, y, \Phi(t, p, y)) + \Phi(t, p, y) \cdot p$$

Now we look at the equations satisified by $\dfrac{\partial H}{\partial y}, \dfrac{\partial H}{\partial p}$:

$$\frac{\partial H}{\partial y} = -L_y - L_{y'} \cdot \Phi_y + \Phi_y \cdot p = -L_y = \frac{-}{dt}L_{y'} = -\frac{d}{dt}p = -\hat{p}$$
$$\frac{\partial H}{\partial p} = -L_{y'}\Phi_p + \Phi_p p + \Phi(t, p, y) = y'$$

Now we have p, y on first order form in this new formalism
$\begin{cases} \hat{p} = -\dfrac{\partial H}{\partial y} \\ \hat{y} = \dfrac{\partial H}{\partial p} \end{cases}$. If we had $\mathcal{L}(t, y_1, \cdots, y_n, y_1', \cdots, y_n')$, and so extremals satisfy Euler-Lagrange equations
$L_{y_i} - \dfrac{d}{dt}L_{y'} = 0$,
With this procedure we would reduce to first order ODE.

For example if we look at this Lagrangian, $\mathcal{L}(x,\hat{x}) = \frac{1}{2}m\hat{x}^2 - V(x)$, we need to define $p = L_{\hat{x}} = m\hat{x}$.
Next step is to define Φ, that is inverting $p \Rightarrow \hat{x} = \frac{p}{m}$
Last step is to substitute this:

$$H(x,p) = -L\left(x, \frac{p}{m}\right) + \frac{p}{m}p = -\frac{1}{2}m\frac{p^2}{m^2} + V(x) + \underbrace{\frac{p^2}{m}}_{p\Phi = m\hat{x}^2 = 2K(\hat{x})}$$

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

$$\text{Hamilton equations} = \begin{cases} \hat{p} = -\dfrac{\partial H}{\partial x} = -V'(x) \\ \hat{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \end{cases}$$

In this set of equations we see that if both are satisfied and tried to compute the second derviative $\hat{\hat{x}}$, this would be the same as:

$$\hat{\hat{x}} = \frac{\hat{p}}{m} = \frac{-V'(x)}{m}$$

This is the same as Euler-Lagrange for the Lagrangian, or the Newton laws.

If the Hamiltonian does not depened explicitly on time, then it is constant:

$$\frac{dH}{dt}(x,p) = \frac{\partial H}{\partial x}\hat{x} + \frac{\partial H}{\partial p}\hat{p}$$

Using the Hamilton equation here, we get:

$$= -\hat{p}\hat{x} + \hat{x}\hat{p} = 0$$
$$H(x,p) = C$$

In fact, $H(x,p)$ represents the total energy of the system, so it makes sense that it is a constant since the total energy does not change ($H = K + V$), but the quotient of does.

If $H = H(t,x,p)$ (H depends on time):

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

And we can also show that $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$:

$$\frac{dH}{dt} = -\frac{dL}{dt} + \frac{d}{dt}(\Phi p)$$
$$\frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t} - \frac{\partial L}{\partial y}\frac{d}{dt}y - \frac{\partial L}{\partial y}\frac{d}{dt}(\Phi) + \left(\frac{d}{dt}\Phi\right)p + \Phi\cdot\hat{p}$$

Recall that $p = L_{y'}$, so:

$$= -\frac{\partial L}{\partial t} - \frac{\partial L}{\partial y}\hat{y} + \Phi\hat{p} = -\frac{\partial L}{\partial t} - \frac{\partial L}{\partial y}\Phi + \Phi\left(\frac{d}{dt}L_{y'}\right)$$
$$= \frac{\partial L}{\partial t} + \Phi\underbrace{\left(\frac{d}{dt}L_{y'} - L_y\right)}_{\text{E-L eq. } = 0} = -\frac{\partial L}{\partial t}$$

Example: Pendulum on a plane, want to find equation of motion using the Hamiltonian principle, so we try to write the Lagrangian for this problem and study the stationary solution.

$$\mathcal{L} = K - V$$

We can express the Kinetic energy as $K(\hat{\theta}) = \frac{1}{2}m(l\hat{\theta})^2$. The length of the arc is $l\theta$.
The potential is given by a function of θ:

$$V(\theta) = mgh = mg(l - l\cos(\theta))$$

$$\mathcal{L}(\theta,\hat{\theta}) = \frac{1}{2}ml^2\hat{\theta}\mathring{A}2 - mgl + mgl\cos(\theta)$$

Now we use Hamilton principle, that states that the motion is stationary for the functional:

$$\int \mathcal{L}(\theta, \hat{\theta}) dt$$

We look for Euler-Lagrange equation for $\mathcal{L}(\theta, \hat{\theta})$:

$$L_{\hat{\theta}} - \frac{d}{dt} L_{\hat{\theta}} = 0$$

$$-mgl\sin(\theta) - \frac{d}{dt}\left(ml^2\hat{\theta}\right) = 0$$

$$-mgl\sin(\theta) - ml^2\hat{\hat{\theta}} = 0$$

$$= -gl\sin(\theta) - l^2\hat{\hat{\theta}} = 0 \Rightarrow \hat{\hat{\theta}} + \frac{g}{l}\sin(\theta) = 0$$

When you consider small values of θ (small oscillations) around the equlibrium position ($\theta = 0$), then:

$$\hat{\hat{\theta}} + \frac{g}{l}\sin(\theta) = 0 \Leftrightarrow \hat{\hat{\theta}} + \frac{g}{l}\theta = 0$$

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{l}}t\right) + B\sin\left(\sqrt{\frac{g}{l}}t\right)$$

Harmonic oscillations with frequency $\sqrt{\frac{g}{l}}$

The same equations can be obtained through Hamiltonian formalism:
Define the canonical momentum $p = \dfrac{\partial L}{\partial \hat{\theta}} = ml^2\hat{\theta}$
Invert the equation to find an expression for $\hat{\theta}$: $\hat{\theta} = \dfrac{p}{ml^2} = \Phi(p)$

$$H(\theta, p) = -\mathcal{L}(\theta, \Phi(p)) + p \cdot \Phi(p) = \frac{-1}{2}ml^2\frac{p^2}{m^2l^4} + mgl - mgl\cos(\theta) + p\frac{p}{ml^2} = \frac{p^2}{2ml^2} - mgl\cos(\theta) + mgl$$

Hamilton equations are given by $\begin{cases} \hat{p} = \dfrac{\partial H}{\partial \theta} = -mg\sin(\theta) \\ \hat{\theta} = \dfrac{\partial H}{\partial p} = \dfrac{p}{ml^2} \end{cases}$

By computing $\hat{\hat{\theta}}$, we get the same equation that we fund for the Lagrangian.

Example: Central motion in the space.
There is a force that is raidal and you have a potential $V(r)$ that is dependent on some distance r from your point.
In view of the radial symmetry, we can use spherical coordinates:

$$\begin{cases} x = r\cos(\varphi)\sin(\theta) \\ y = r\sin(\varphi)\sin(\theta) \\ z = r\cos(\theta) \end{cases} \Rightarrow (r, \varphi, \theta)$$

Want to write the kinetic energy, that is:

$$K = \frac{1}{2}m\left|\left|(\hat{x}, \hat{y}, \hat{z})\right|\right|^2 = \frac{1}{2}m(\hat{x}^2, \hat{y}^2, \hat{z}^2)$$

In spherical coordinates one can write:

$$K = \frac{1}{2}m\left(\hat{r}^2 + r^2\hat{\theta}^2 + r^2\hat{\varphi}^2\sin^2(\theta)\right)$$

$$\Rightarrow \mathcal{L} = K(r, \theta, \hat{r}, \hat{\theta}, \hat{\varphi}) - V(r)$$

Notice that the Lagrangian does not have an explicit dependance on φ, this can be used to define some constant of motion. To define the Hamiltonian, we will introduce the momenta:

$$p_r = L_{\hat{r}} = m\hat{r}$$

$$p_\theta = L_{\hat{\theta}} = mr^2\hat{\theta} p_\varphi = L_{\hat{\varphi}} = mr^2\hat{\varphi}\sin^2(\theta)$$

In the end, we will end up with a Hamiltonian that is dependant on $H(r, \theta, p_r, p_\varphi, p_\theta)$

Then, in the Hamiltonian equations:

$$\hat{p_\varphi} = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow p_\varphi = C$$

We can compute the constant and reduce the number of variables in our Hamiltonian

$$\Rightarrow H(r, \theta, p_r, p_\theta)$$

4. Dynamical Systems

In Dynamical Systems we have a set of all the possible status that a system can have, and a *rule* that tells us how the system evolves in time.

We can distinguish between discrete or continuous dynamical system. This can be due to the model itself, naturally we might be interested in a model where we are only interested in discrete time (height of someone), and sometimes it is more useful to have continuous cases (temperature).

Another case could be performing continuous operations on a computer, where we discretisise time when numerically computing integrals.

Example: We consider a particle that is moving with one degree of freedom (in one direction). One possible evolution can be described in the following way:

$$x_{n+1} = x_n - 1$$

Note that this is a discrete model. Let us look at a continuous model:

$$\hat{\hat{x}} = -1$$

What is the difference between the discrete and the continuous case then? When the time is discrete, the system is described by some map $F : X \to X$ where X is the space of the possible configurations. In the continuous case, we have a differential equation

$$\hat{x} = F(x, t)$$

When the system does not depend on time, it is called an *autonomous* system.

Looking at how a system is defined, we are of course interested in a few points. Sometimes these points could be fixed, sometimes they might be singularities, limit points, stationary points/sets, etc.
We are also intersted in what happens around our point of interest, namely the neighborhood. Are there solutions that converge to some point/set in that neighborhood?

Example: A growth-decay model is defined as

$$x_{n+1} = (1 + r)x_n$$

We have a discrete mapping. In this particular case we can write the map in a recursive way:

$$x_{n+1} = (1 + r)^{n+1} x_0$$

Where x_0 is the start condition. Depending on the value of r, the quantity could be increasing (growth model) or decreasing (decay model).

Definition/Sats 4.4: Fixed point

A *fixed point* (or *equilibrium solution*) is a point x_* such that its iteration is stationary (can almost say that it is mapped to itself).

$$x_{n+1} = f(x_n) \Rightarrow x_* = f(x_*)$$

Definition/Sats 4.5: Periodic orbit

A *periodic orbit* of period K is a point x_* such that after k iterations of the map we are, again at x_*:

$$f^K(x_*) = x_*$$

Example:

$$x_{n+1} = (1+r)x_n$$

A fixed point is $x_* = 0$

Example:
The map $x_{n+1} = -x_n$ has the origin as a fixed point and has a period orbit of period 2 since $f(x_n) = -x_n \Rightarrow f(f(x_n)) = f(-x_n) = x_n$, which is true $\forall x_n \neq 0$

Definition/Sats 4.6: Asymptotic Stability

A fixed point is *asymptotically stable* if $\forall y$ in the neighborhood of x_*,

$$\lim_{n\to\infty} d(f^n(y), x_*) = 0$$

To study the stability, one could consider the linearisation of the probelem.

Result: Consider a fixed point for a map $x_{n+1} = f(x_n)$. Then, if the spectrum of the linearisation of the map evaluated at the fixed point $Df(x_*)$ (Jacobian) is contained in the unit circle, then the point is asymptotically stable.

On the contrary, if there is an eigenvalue with absolute value greater than one, then the point is unstable. Recall from the lecture notes in ODE (P. 71):

Egenvärden	Typ	Stabilitet
$0 < \lambda_2 < \lambda_1$	Nod	Instabil
$\lambda_2 < 0 < \lambda_1$	Sadelpunkt	Instabil
$< \lambda_2 < \lambda_1 < 0$	Nod	Asymptotiskt stabil
$\alpha > 0, \beta \neq 0$	Spiral	Instabil
$\alpha = 0, \beta \neq 0$	Center	Stabil
$\alpha < 0, \beta \neq 0$	Spiral	Asymptotiskt stabil
$\lambda_1 = \lambda_2 > 0$	Nod (G.M = 2)	Instabil
	Improper node (G.M < 2)	Instabil
$\lambda_1 = \lambda_2 < 0$	Nod (G.M = 2)	Asymptotiskt stabil
	Improper nod (G.M < 2)	Asymptotiskt stabil

If we consider a point $x_* + y_n$ with y_n sall, so that the point is close to the fixed point and then look at the iteration of this point is:

$$\begin{aligned} f(x_* + y_n) &= f(x_*) + Df(x_*)y_n + \text{higher order terms} \\ &= x_* + y_{n+1} = x_* + Df(x_*)y_n + \text{higher order terms} \end{aligned}$$

At the linear approximation, y_n satisfies $y_{n+1} = \underbrace{Df(x_*)}_{\text{Matrix}} y_n$

For example, in the 1-dimensional case:

$$y_{n+1} = \underbrace{f'(x_*)}_{\lambda} y_n \Rightarrow \lambda y_n$$
$$\Rightarrow y_{n+1} = \lambda^{n+1} y_0$$

What does this formula tell us? Well, depending on λ we can tell if our model is growing or decaying. Considering a point close to our x_*, then if y_n is decaying then $x_* + y_n$ seems to converge to x_*. If it is exploding, then we do not have a stable point.

In the general m-dimensional case we deal with a problem in the form:

$$y_{n+1} = Ay_n = A^{n+1} y_0$$

Suppose the matrix is diagonalisable and we can compute the eigenvalues/eigenvectors (and the eigenvectors form a basis of $\mathbb{R}^m$).
We can then express our point as a linear combination of our vectors:

$$y_0 = c_1 v_1 + \cdots + c_m v_m$$
$$y_{n+1} = A^{n+1} y_0 = c_1 \lambda_1^{n+1} v_1 + \cdots + c_m \lambda_m^{n+1} v_m$$

We can reorder the eigenvalues such that the first one is the biggest:

$$y_{n+1} = \lambda_1^{n+1} \left(c_1 v_1 + \cdots + \left(\frac{\lambda_m}{\lambda_1} \right) c_m v_m \right)$$
$$\Rightarrow \left| \frac{\lambda_j}{\lambda_1} \right| < 1$$

The behaviour for $n \to \infty$ is given by the eigenvalue λ_1. The asymptotic is given by $c_1 \lambda_1^{n+1} v_1$.

If $|\lambda_1| < 1$, then the fixed point is stable. Otherwise, unstable.

Exercise: Find the fixed point and study the linear stability of the map

$$x_{n+1} = 2 + x_n - x_n^2$$

The function is given by $f(x) = 2 + x - x^2$, then fixed point are given by:

$$f(x) = x \Leftrightarrow 2 - x^2 = 0 \Leftrightarrow x_* = \pm\sqrt{2}$$

For the linear stability, we look at the value of $f'(x_*)$:

$$f'(x) = 1 - 2x \Rightarrow f'(\pm\sqrt{2}) = 1 \pm 2\sqrt{2}$$

Since $\left|1 \pm 2\sqrt{2}\right| > 1$, both points of equilibrium are unstable.

Example: $u' = f(u) = u^2(3 - u)$.
The equilibrium points are:

$$f(u) = 0 \Rightarrow \begin{cases} u_1 = 0 \\ u_2 = 3 \end{cases}$$

To study the behaviour of u, look at the sign of u:

$$f(u) > 0 \qquad u' > 0 \Rightarrow u \text{ increasing} f(u) < 0 \qquad u' < 0 \Rightarrow u \text{ decreasing}$$

If $u < 0$, then for $\lim_{t \to +\infty} u = 0$
If $0 < u < 3$, then solution converges to 3 (follows the arrows in the phase-plane)
If $u > 3$, then to 3.

We arrive at the following classifications:

Unstable

Asymptotically stable

We can determine the type of stability through f'. If $f' < 0$ then we can think of it as having "right"-facing arrows and the opposite for $f' > 0$

In case of asymptotical stability, is it just local? How does it look like in the global case?

In dimension > 1, we can request an equilibrium point x_* to be *stable* , that is if we fix an $\varepsilon-$ball around x_*, then $\exists \delta > 0$ such that all of the solutions with initial conditions in $B_\delta(x_*) \subseteq B_\varepsilon(x_*) \quad \forall t$ (see figure 2.7 on page 106)

Anmärkning: Asymptotically stable point $\Rightarrow$ stable solutions converge to x_* as $t \to +\infty$

4.1. **Linear Systems.**

Nice to study as tehy approximate behaviour of linear systems.

In this course, we will cover 2D systems $\mathbf{x}' = A\mathbf{x} \quad \mathbf{x} \in \mathbb{R}^2, A \in \mathcal{M}_{2\times 2}(\mathbb{R})$

We look for a solution on the form $x = ve^{\lambda t}$:

$$v\lambda e^{\lambda} t = Ave^{\lambda t} \Leftrightarrow v\lambda = Av \Rightarrow u \text{ is an eigenvector to } A \text{ with eigenvalue } \lambda$$

Imagine $\lambda_1, \lambda_2 \in \mathbb{C}$ are our eigenvalues. Then the solution can be written as a linear combination of the associated eigenvector:

$$c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = x(t) \quad \text{(general solution)}$$

Recall the classification of eigenvectors through the table above.

If we can only find one eigenvalue, then the general solution is on the form $\mathbf{x}(t) = c_1 v_1 e^{\lambda t} + (w + v_1) c_2 e^{\lambda t}$, where w satisfies $(A - \lambda I)w = v$

If we have a new linear system $\mathbf{x}' = F(x) \quad \mathbf{x} \in \mathbb{R}^2$, we can define $A = DF(x_*)$ where x_* is an equilibrium point.
The system $\mathbf{x}' = A\mathbf{x}$ is the linearised problem around the equilibrium x_*

Definition/Sats 4.7

The origin $(0, 0)$ is a critical point for $\mathbf{x}' = A\mathbf{x}$ of the same type as x_* for $\mathbf{x}' = F(\mathbf{x})$ when:

- $\lambda_j \in \mathbb{R} \qquad \lambda_1 \cdot \lambda_2 > 0$ (same sign) (node)
- $\lambda_j \in \mathbb{R} \qquad \lambda_1 \cdot \lambda_2 < 0$ (opposite sign) (sadlepoint)
- $\mathrm{Re}(\lambda_j) = 0 \Rightarrow$ Spiral

Corollary: If $(0, 0)$ asymptotically stable for $\mathbf{x}' = A\mathbf{x}$, then x_* is asymptotically stable for $\mathbf{x}' = F(\mathbf{x})$

Example: Study the stability of the linear system:

$$\begin{cases} x' = x + \mu y \\ y' = x - y \end{cases} \Rightarrow A = \begin{pmatrix} 1 & \mu \\ 1 & -1 \end{pmatrix}$$

$$\text{Eigenvalues: } (1 - \lambda)(-1 - \lambda) - \mu = 0 \Leftrightarrow -1 + \lambda^2 - \mu = 0 \Rightarrow \lambda = \pm\sqrt{\mu + 1}$$

Note that we have different equilibrium points depending on the value of μ:

$$\mu > -1 \Rightarrow \text{ real eigenvalues with opposite signs } \Rightarrow \text{ saddle}$$
$$\mu < -1 \Rightarrow \text{ complex eigenvalues with } \mathrm{Re}(\lambda_j) = 0 \Rightarrow \text{ center}$$
$$\mu = -1 \Rightarrow \text{ critical value (*bifurcation value*)}$$

At the bifurcation value, we have:

$$\begin{cases} x' = x - y \\ y' = x - y \end{cases}$$

In this specific case, not only the origin is equilibrium, in this case $x' = y' = 0$, and $y = x$, every point in this line is an equilibrium point.

Example: Damped harmonic oscillator:

$$mx'' + ax' + kx = 0 \qquad m, a, k > 0$$

Note that for $a = 0$ we have a classical harmonical oscillator.

For the harmonic oscillator, the solutions are periodic oscillations aruond the equilibrium poisition. Do we have the same behaviour holding the damping term $y = x'$?

$$\left.\begin{matrix} my' + ay + kx = 0 \\ y = x' \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y = x' \\ y' = \dfrac{-ay}{m} - \dfrac{k}{m}x \end{matrix}\right\}$$

Here, the orign is an equilibrium point. In matrix form, the system is expressed as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{a}{m} \end{pmatrix}$$

The matrix A has eigenvalues:

$$+\lambda\left(\frac{a}{m} + \lambda\right) + \frac{k}{m} = 0 \Rightarrow \lambda^2 + \frac{a}{m}\lambda + \frac{k}{m} = 0$$
$$\Rightarrow \lambda = \frac{-a \pm \sqrt{a^2 - 4km}}{2m}$$

We know for $a = 0 \Rightarrow$ harmonic oscillation, we get the following eigenvalues: $\lambda = \dfrac{\pm\sqrt{-4km}}{2m}$
Note that we have pure imaginary roots,this implies we have perpetual oscillations around equilibrium.

Depending on the sign inside the square root, things can be a little different. It depends on $a^2 - 4km$:

$$a^2 - 4km < 0 \quad \text{real part is negative} \Rightarrow \text{ asymptotically stable with spiral}$$

From a physical point of view, oscillation is getting smaller and smaller as $t \to +\infty$
If we have a damping constant a such that $a^2 - 4km \geq 0$, then we have real eigenvalues who are both negative, so we are asymptotically stable without oscillation. It is a Node.

Example, (Lotka-Volteria system):

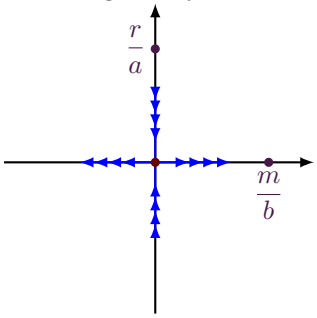
$$x = x(t) \text{ (prey population)}$$
$$y = y(t) \text{ (predator population)}$$

The purpose of this system is to study the evolution of time and how they interract with each other, which can be described by a dynamical system:

$$\left.\begin{matrix} x' = rx - axy \\ y' = -my + bxy \end{matrix}\right\}$$

- $r =$ rate of increase for prey
- $a =$ decrease in prey due to predator
- $m =$ mortality rate of predators
- $b =$ increase in the predator population due to preys killed ($b < a$, since each predator consumes more than 1 prey)

The equilibrium points are given by:



$$\left.\begin{aligned} x(r-ay)&=0\\ y(-m+bx)&=0\end{aligned}\right\} \quad (1)$$

We see that one point is $(0,0)$, which is the point where botb prey and predator are extinct. Another point is $\left(\frac{m}{b},\frac{r}{a}\right)$, which is the equilibrium of coexistance.

We study this non-linear system by looking at the stability of equilibrium:

$$A=\begin{pmatrix} r-ay & -ax\\ by & -m+bx\end{pmatrix}=DF(\overline{x}) \text{ where } F(\overline{x})=\overline{x}'$$

In the point $(0,0)$:

$$\begin{pmatrix} r & 0\\ 0 & -m\end{pmatrix}$$

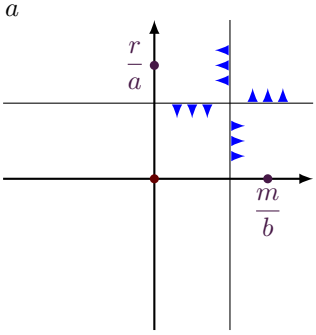
We have real eigenvalues with opposite signs, ergo a saddle point with eigenvectors $\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{r},\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{-m}$ (1)

In the point $\left(\frac{m}{b},\frac{r}{a}\right)$:

$$\begin{pmatrix} 0 & \frac{-am}{b}\\ \frac{br}{a} & 0\end{pmatrix}$$

Eigenvalues are given by $\lambda^2+mr=0 \Rightarrow= -mr$, which are pure imaginary and so we do not have enough information to conclude anything.

We can look at cases when $y',x'=0$ to get an idea at the behaviour of the system: $x(r-ay)=0\Rightarrow y=\frac{r}{a}\Rightarrow$ vector-field is vertical and arrows are determined by $y'=y(-mx+bx)$ (2)



(2)

Similarly, for $y'=0$, that is the line $x=\frac{m}{b}$, we try to express the solutions of the system by computing:

$$\frac{dy}{dx}=\frac{-my+bxy}{rx-axy}\Rightarrow (rx-axy)dy=(-my+bxy)dx=x(r-ay)dy=y(-m+bx)dx$$
$$\Rightarrow \int\frac{r-ay}{y}dy=\int\frac{-m+bx}{x}dx\Rightarrow r\ln(y)-ay=-\ln(x)+bx+c$$
$$\Rightarrow y^re^{-ay}=x^{-m}e^{bx}e^c\Rightarrow \text{ can't find closed form solution}$$

We want to figure out if we have a closed orbit or if we have a spiral.

Looking at our expression, in the point $x = \frac{m}{b}$:

$$y^r = Ce^{ay}$$

Where C is all the constants. We cannot cross the line $x = \frac{m}{b}$ many times, ergo we have no spiral behaviour.
Therefore, every cycle is a coexistance state, but the population will not approach the equation state $\left(\frac{m}{b}, \frac{r}{a}\right)$

Some general results about the existance of clsoed orbits in non-linear systems:

Definition/Sats 4.8: Bendixson-Dulan

Given a system $\left.\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y) \end{aligned}\right\}$, if $P_x + Q_y$ has the same sign in some region, then there are no closed orbits in that region

Example:

$$\left.\begin{aligned} x' &= x + x^3 - 2y \\ y' &= -3x + y^5 \end{aligned}\right\} \quad P_x + Q_y = 1 + 3x^2 + 5y^4$$

This is always positive for $(x, y) \in \mathbb{R}^2 \Rightarrow$ no closed orbits in $\mathbb{R}^2$

Definition/Sats 4.9

A closed orbit for the system $\left.\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y) \end{aligned}\right\}$ surrounds at least one equilibrium point. Of course, no equilibrium point implies no closed orbits.

Definition/Sats 4.10: Poincare-Bendixson

Let R be a closed bounded region without critical points.

If there exists an orbit Γ which is in R at t_0 and $\Gamma(t) \in R \quad \forall t > t_0$, then Γ is either a closed orbit, or it spirals towards a closed orbit for $t \to +\infty$

In a plane, an orbit can:

- Leave any bounded set as $t \to +\infty$
- Can be a closed orbit/equilibrium point
- Approaches a closed orbit/equilibrium point

Note that we cannot have chaotic behaviour in the 2D case.

Example:

$$\left.\begin{aligned} x' &= -y - x(x^2 + y^2 - \mu) \\ y' &= x - y(x^2 + y^2 - \mu) \end{aligned}\right\} \quad \text{Polar coordinates } \Rightarrow xx' + yy' = rr' \qquad xy' - yx' = r^2\theta'$$

$$\left.\begin{aligned} x' &= r'\cos(\theta) - r\sin(\theta)\theta' \\ y' &= r\sin(\theta) + r\cos(\theta)\theta' \end{aligned}\right\} xx' + yy' = r\cos(\theta(r'\cos(\theta) - r\sin(\theta)\theta') + r\sin(\theta)(r'\sin(\theta) + r\cos(\theta)\theta'))$$

$$= rr'$$

$$x(-y - x(x^2 + y^2 - \mu)) + y(x - y(x^2 + y^2 - \mu)) = -x^2(r^2 - \mu) - y^2(r^2 - \mu) = -r^2(r^2 - \mu) = rr'$$

$$\Rightarrow r' = r(r^2 - \mu)$$

We do the same for θ:

$$x(x - y(r^2 - \mu)) - y(-y - x(r^2 - \mu)) = r^2\theta'$$
$$= x^2 + y^2 - xy(r^2 - \mu) + xy(r^2 - \mu) = r^2\theta'$$
$$\Rightarrow \underbrace{x^2 + y^2}_{r^2} = r^2\theta' \Rightarrow \theta' = 1$$

We arrive at:

$$\left.\begin{aligned} r' &= r(r^2 - \mu) \\ \theta' &= 1 \quad \text{(constant angular velocity)} \end{aligned}\right\}$$

Equilibrium? $r' = 0 \Rightarrow r = 0 \wedge r = \sqrt{\mu}$ (closed period orbit).
If inside the circle, that is $r < \sqrt{\mu}$, then $r' < 0$ and we are moving away from the origin. Else, opposite behaviour, except where r becomes $< \sqrt{\mu}$, then it flips, ergo converging to the orbit.