Sample Solutions, Assigment 1.

Q1: a) We only need to consider pre-images of intervals in $\mathbb{R}$ as they generate the σ-algebra.

Hence $X_n^{-1}([a,b]) = \{s \in S : X_n(s) \in [a,b]\}$

$$= \{s \in \{1,2,..,n\} : s \in [a,b]\} \cup \begin{cases} \{n+1, n+2, ..\} & \text{if } 0 \in [a,b] \\ \emptyset \end{cases}$$

Hence $\sigma(X_n) = \sigma(P(A), \{n+1, n+2, ..\})$

$= P(\{1,2,..,n\}) \cup \{B \setminus A : A \in P(\{1,..,n\})\}$.

b) $\bigcup_{n \in \mathbb{N}} \sigma(X_n)$ is not a σ-algebra.

$A_k = \{2k\} \in \sigma(X_n)$ for all $n \geq 2k$ and so $A_k \in \bigcup_{n \in \mathbb{N}} \sigma(X_n)$ for all k. However,

$\bigcup_{k \in \mathbb{N}} A_k = \{2, 4, 6, ..\}$ is not contained in any $\sigma(X_n)$ and so $\bigcup_{n \in \mathbb{N}} \sigma(X_n)$ is not closed under countable unions.

Alternatively, note that any element in $\sigma(X_n)$ for any n is finite or co-finite (has finite complement). But then $\bigcup \sigma(X_n)$ only has finite or co-finite elements and any union (such as the even numbers) which is neither finite or co-finite cannot be in $\bigcup \sigma(X_n)$.

2) Recall that $F_X(t) \to 1$ as $t \to \infty$ and $F_X(-t) \to 0$ as $t \to \infty$. Hence,

$$F_X(t) - F_X(-t) \geq \mathbb{P}(|X| > t) \to 0 \quad \text{as} \quad t \to \infty.$$

Fix a_n large enough such that $\mathbb{P}(|X_n| > \frac{1}{n} a_n) \leq 2^{-n}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \tfrac{1}{n} a_n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Hence by the Borel-Cantelli lemma, $E_n := \{|X_n| > \frac{1}{n} a_n\} = \{\frac{|X_n|}{a_n} > \frac{1}{n}\}$ occurs finitely many times a.s.. We conclude $X_n / a_n \to 0$ with probability 1.

Q3

Let $Y_n = X_k - m$, then $\mathbb{E}(Y_n) = 0$ and

$$\mathbb{E}(e^{\theta Y_n}) = \mathbb{E}(e^{\theta(X_n - m)}) = e^{-m\theta}\,\mathbb{E}(e^{\theta X_n}) < \infty$$

by assumption. Then $A_n = \left\{ \sum_{k=1}^{n} Y_k > n^t \right\}$

a) Using Markov's inequality, we get [or using Chebyshev's ineq.]

$$\mathbb{P}(A_n) = \mathbb{P}\left(\sum_{k=1}^{n} Y_k > n^t\right)$$

$$= \mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_n > n^{t-\frac{1}{2}}\right)$$

$$= \mathbb{P}\left(e^{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k} > e^{n^{t-1/2}}\right)$$

$$\leq e^{-n^{t-1/2}}\,\mathbb{E}\left(e^{\frac{1}{\sqrt{n}}\sum Y_k}\right)$$

$$= e^{-n^{t-1/2}}\,\mathbb{E}\left(e^{Y_1/\sqrt{n}}\, e^{Y_2/\sqrt{n}} \cdots e^{Y_n/\sqrt{n}}\right)$$

$$= e^{-n^{t-1/2}}\,\mathbb{E}\left(e^{Y_1/\sqrt{n}}\right)^n \quad \text{by independence and same distribution.}$$

Now $e^{Y_1/\sqrt{n}} = 1 + \frac{Y_1}{\sqrt{n}} + \frac{Y_1^2}{2n} + \frac{Y_1^3}{3!\,n^{3/2}} + \dots$

and by Taylor's theorem,

$$e^{Y_1/\sqrt{n}} = 1 + \frac{Y_1}{\sqrt{n}} + \frac{Y_1^2}{2n} + o\left(Y_1^2/n\right)$$

Hence, $\mathbb{E}\left(e^{Y_1/\sqrt{n}}\right) = 1 + \frac{\mathbb{E}(Y_1)}{\sqrt{n}} + \frac{\mathbb{E}(Y_1^2)}{2n} + o\left(\frac{\mathbb{E}(Y_1^2)}{n}\right)$

$\leq 1 + \frac{c}{n}$ for some $c > 0$

since $\mathbb{E}(Y_1) = 0$, $\mathbb{E}(Y_1^2) = Var(X^2) < \infty$, and $o\left(\frac{\mathbb{E}(Y_1^2)}{n}\right) \leq \frac{C'}{n}$ for some $C' > 0$ as $\frac{1}{n} \to 0$.

Hence,

$$\mathbb{E}\left(e^{Y_1/\sqrt{n}}\right)^n \leq \left(1 + \frac{c}{n}\right)^n \to e^c$$

and $\mathbb{E}\left(e^{Y_1/\sqrt{n}}\right)^n \leq K$ for some $K > 0$.

We get $\mathbb{P}(A_n) \leq e^{-n^{t-1/2}} \cdot K$ which is summable, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, and so by the Borel-Cantelli lemma $\mathbb{P}(\limsup A_n) = 0$ as required.

b) Since $X_1, X_2, ..$ are independent, we can Kolmogorov's 0-1 law to the tail algebra $\mathcal{T}$ generated by $X_1, X_2, ..$

Since Y_n is a random walk with mean 0, it is positive for i.m. n. But then

$$\limsup_n A_n \subseteq \left\{ \sum_{k=2}^{n} Y_k > n^t \text{ for i.m. } n \geq 2 \right\} (=: B_2)$$

$$\subseteq \left\{ \sum_{k=3}^{n} Y_k > n^t \text{ for i.m. } n \geq 3 \right\} (=: B_3)$$

And so $\limsup_n A_n \subseteq \bigcap_{n \in \mathbb{N}} B_n$ which is in the tail algebra $\mathcal{T}$. Hence $\mathbb{P}(\limsup_n A_n) = 0$ or 1 for all t, including $t < 1/2$.

[There was no need to be this detailed, any further valid conclusion would have given credit]

Q4)

a) We verify the measure axioms:

$\mu_n(\emptyset) = 0$: Since δ_x is a measure we have $\delta_x(\emptyset) = 0$ for all $x \in \mathbb{R}$. In particular,

$$\mu_n(\emptyset) = \sum_{k=1}^{n} 2^{-k} \delta_{1/k}(\emptyset) = \sum_{k=1}^{n} 0 = 0.$$

$\mu_n(A) \geq 0$: Again, since δ_x is a measure we have $\delta_x(A) \geq 0$, so $\mu_n(A) = \sum_{k=1}^{n} 2^{-k} \delta_{1/k}(A) \geq 0$.

σ-additivity: Let $A_1, A_2, \ldots$ be disjoint Borel sets. Since $\mu_n(A_i) = \sum_{k=1}^{n} 2^{-k} I_{\{1/k \in A_i\}}$ is a finite sum of indicator functions, and the A_i are disjoint, there are only finitely many A_i for which $\mu_n(A_i) > 0$.

So σ-additivity follows from additivity:

For A_1, A_2 disjoint, we have

$$\mu_n(A_1 \cup A_2) = \sum_{k=1}^{n} 2^{-k} I_{\{\frac{1}{k} \in A_1 \cup A_2\}}$$

$$= \sum_{k=1}^{n} 2^{-k} I_{\{\frac{1}{k} \in A_1\}} + \sum_{k=1}^{n} 2^{-k} I_{\{\frac{1}{k} \in A_2\}}$$

$$= \mu_n(A_1) + \mu_n(A_2).$$

b) For fixed $A \in \mathcal{B}(\mathbb{R})$ we have

$$\mu_n(A) = \sum_{k=1}^{n} 2^{-k} \delta_{1/k}(A) \leq \sum_{k=1}^{n+1} 2^{-k} \delta_{1/k}(A) = \mu_n(A).$$

Further, $\mu_n(A) = \sum_{k=1}^{n} 2^{-k} \delta_{1/k}(A) \leq \sum_{k=1}^{n} 2^{-k} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$

So μ_n is non-decreasing and bounded. By MCT

$\mu(A) = \lim_{n \to \infty} \mu_n(A)$ exists (and is finite).

c) The first two axioms are satisfied as in a) with an extra limit.

For σ-additivity notice that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$(\ddagger) \qquad = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) \quad (\text{since } \mu_n \text{ is a measure})$$

Let ν be the counting measure on $\mathbb{N}$.

Then $\displaystyle\sum_{i=1}^{\infty} \mu_n(A_i) = \int \mu_n(A_i) \, d\nu(i)$.

But by disjointness,

$\displaystyle\sum_{i=1}^{\infty} \mu_n(A_i) = \mu_n\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu_n(\mathbb{R}) \leq 1$ and so

$$\mathbb{E}_\nu\left(|\mu_n(A_i)|\right) = \int |\mu_n(A_i)| \, d\nu(i) = \int \mu_n(A_i) \, d\nu(i) \leq 1.$$

Since also $\mu_n(A_i) \leq \mu_{n+1}(A_i)$ we can use MCT and get

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \int \mu_n(A_i) \, d\nu(i)$$

$$= \int \lim_{n\to\infty} \mu_n(A_i) \, d\nu(i)$$

$$= \int \mu(A_i) \, d\nu(i) = \sum_{i=1}^{\infty} \mu(A_i).$$

[Alternatively, one could use Fubini $(\ddagger)$].

5) a)

$$\mathbb{E}(X_1) = \mathbb{E}(1) = 1.$$

$$\mathbb{E}(X_2) = \frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\right) + \frac{1}{6}\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}\right) = \frac{319}{720}$$

$$\mathbb{E}(X_3) = \frac{1}{6}(1+3^2+5^2) = \frac{35}{6}$$

b) $\sigma(X_3) = \sigma(\underbrace{\{\{1\},\{3\},\{5\},\{2,4\}\}}_{\mathcal{H}})$

$\mathcal{H}$ forms the smallest generating set of $\sigma(X_3)$ and $Y = \mathbb{E}(X_2 | X_3)$ is constant on $H \in \mathcal{H}$ by $\mathcal{H}$-measurability

We get

$$\mathbb{P}(\omega=1)\, Y(1) = \int_{\{1\}} Y \, d\mathbb{P} = \int_{\{1\}} X_2 = \mathbb{P}(\omega=1)\, X_2(1)$$

So $Y(1) = X_2(1)$, and similarly $Y(3) = X_2(3)$, $Y(5) = X_2(5)$.

Father $\mathbb{P}(\omega \in \{2,4\})\, a = \int_{\{2,4\}} Y \, d\mathbb{P} = \int_{\{2,4\}} X_2 \, d\mathbb{P} = \frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\right)$

and $Y(2) = Y(4) = a = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right) = 3/8.$

That is $Y(\omega) = \begin{cases} 1 & \omega = 1 \\ 3/8 & \omega = 2 \\ 1/3 & \omega = 3 \\ 3/8 & \omega = 4 \\ 1/5 & \omega = 5 \end{cases}$.

c) $\sigma(X_2) = \mathcal{P}(\Omega)$ and

$$\mathbb{E}(X_3 | X_2) = X_3(\omega)$$

d) $\sigma(X_1) = \{\emptyset, \Omega\}$ and so

$$\mathbb{E}(X_2 | X_1) = \mathbb{E}(X_2) = \frac{319}{720} \quad \forall \omega \in \Omega$$

e) $\sigma(X_1) = \{\emptyset, \Omega\}$, that is no information is given by the outcome of X_1.

$\sigma(X_2) = \sigma(\{\{1\}, \{3\}, \{5\}, \{2, 4\}\})$, that is the outcome of X_2 determines whether ω is even or odd, and if odd determines the outcome ω.

$\sigma(X_3) = \mathcal{P}(\Omega)$, which is complete information on the outcome $\omega \in \Omega$.

Q6)

a) For X_n to be a martingale we require

$$X_{n-1} = \mathbb{E}(X_n \mid X_1, X_2, \dots, X_{n-1}) = \mathbb{E}(X_{n-1} Y_n \mid X_1, \dots, X_{n-1})$$

$$= X_{n-1} \mathbb{E}(Y_n \mid X_1, \dots, X_{n-1}) = X_{n-1} \mathbb{E}(Y_n) = \frac{a}{2} X_{n-1}$$

Hence $a = 2$.

b) Consider $\log X_n = \log Y_1 + \log Y_2 + \dots + \log Y_n$.

By the law of large numbers, $\frac{1}{n} \log X_n \to \mathbb{E} \log Y_1$.

Now $\mathbb{E} \log Y_1 = \frac{1}{a} \int_0^a \log y \, dy = \log(a) - 1$.

So, for $a < e$, $\mathbb{E} \log Y_1 < 0$ and

$\log X_n \to -\infty$. For $a > e$, $\mathbb{E} \log Y_1 > 0$ and

$\log X_n \to \infty$. Hence

$$\lim_{n \to \infty} X_n = \begin{cases} \infty & \text{if } a > e \\ 0 & \text{if } a < e \end{cases}$$

and the threshold is $a_0 = e$.