Computer Intensive Statistics and Applications Extension: SDE and Financial Applications

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Standard Brownian Motion

Definition

A stochastic process $\{W(t), 0 \leq t \leq T\}$ is a standard one-dimensional Brownian motion on $[0, T]$ if

1. $W(0) = 0$,
2. the mapping $t \to W(t)$ is a continuous function with probability 1,
3. the increments $W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, ..., $W(t_k) - W(t_{k-1})$ are independent for any k and for any $t_i \in [0, T]$,
4. $W(t) - W(s) \sim N(0, t - s)$ for any $0 \leq s < t \leq T$.

Black-Scholes Model

The Black-Scholes model is perhaps the first widely used mathematical model to model the asset price. Let $S(t)$ be the price of the asset at time t. Then,

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t),$$

where r can be interpreted as a riskless interest rate, i.e., each unit invested at time 0 grows to a value of $\exp(rt)$ at time t, and σ is the volatility. This model implies that

$$S(T) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W(T)\right\}.$$

Call Option and Present Value

A call option gives you the right to buy the stock at a strike price K by the expiration date T. Then, the present value of the option is

$$e^{-rT}\mathrm{E}\left[\max\left(S\left(T\right)-K,0\right)\right].$$

- For the standard Brownian motion, $W\left(t\right)\sim N\left(0,t\right)$. Hence, we can simulate $W\left(t\right)$ by $W\left(t\right)=\sqrt{t}Z$, where $Z\sim N\left(0,1\right)$.
- It is also easy to see that, by taking the log of $S\left(T\right)$,

$$\log S\left(T\right) \sim N\left(\log S\left(0\right)+\left(r-\frac{1}{2}\sigma^2\right)T,\ \sigma^2T\right).$$

Monte Carlo Evaluation of Present Value

We can approximate the present value in this example by

Algorithm 1: Monte Carlo evaluation of stock price

```
for i = 1 in 1 : n do
    Sample a candidate Z_i ~ N(0,1) ;
    Calculate S_i(T) = S(0) exp{ (r - (1/2)σ^2) T + σ√T Z_i } ;
    Calculate C_i = exp(-rT) max(S_i(T) - K, 0) ;
end
The approximated value is n^{-1} Σ_{i=1}^{n} C_i .
```

for $i = 1$ *in 1 : n* **do**

- Sample a candidate $Z_i \sim N(0,1)$;
- Calculate $S_i(T) = S(0) \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T} Z_i\right\}$;
- Calculate $C_i = \exp(-rT)\max(S_i(T) - K, 0)$;

end

The approximated value is $n^{-1}\sum_{i=1}^{n} C_i$.

Variance Reduction: Importance Sampling

Let

$$Y = \log S\left(T\right) \quad \sim \quad N\left(\log S\left(0\right) + \left(r - \frac{1}{2}\sigma^2\right)T,\ \sigma^2\right).$$

We can rewrite $\mathrm{E}\left[\max\left(S\left(T\right) - K, 0\right)\right]$ as

$$\begin{aligned} \mathrm{E}\left[\max\left(S\left(T\right) - K, 0\right)\right] &= \int_{-\infty}^{\infty} \left[\exp\left(y\right) - K\right] 1\left(y > \log K\right) p\left(y\right) dy \\ &= \int_{\log K}^{\infty} \left[\exp\left(y\right) - K\right] \frac{p\left(y\right)}{g\left(y\right)} g\left(y\right) dy. \end{aligned}$$

We can use importance sampling and let $g\left(y\right)$ be a density with support on $\left\{y > \log K\right\}$, e.g., a shifted exponential distribution.

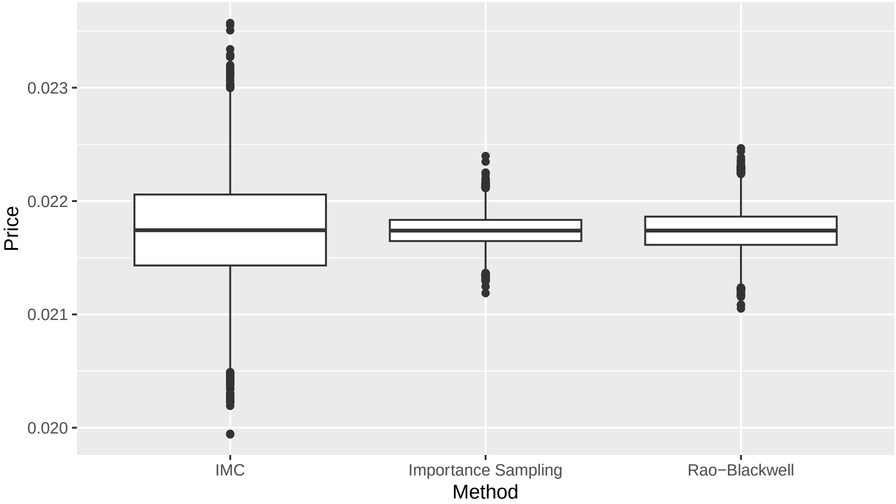
Variance Reduction: Rao-Blackwell

We can also condition $\max(S(T) - K, 0)$ on $S(T) > K$ as

$$\begin{aligned} \mathrm{E}\left[\max\left(S\left(T\right) - K, 0\right)\right] &= \mathrm{P}\left\{S\left(T\right) > K\right\} \mathrm{E}\left[S\left(T\right) - K \mid S\left(T\right) > K\right] \\ &= \mathrm{P}\left(Y > \log K\right) \mathrm{E}\left[\exp\left(Y\right) - K \mid Y > \log K\right]. \end{aligned}$$

- $\mathrm{P}(Y > \log K)$ is the normal probability.
- We can sample Y from the conditional distribution $Y \mid Y > \log K$ (a truncated normal distribution) to approximate $\mathrm{E}[\exp(Y) - K \mid Y > \log K]$.

Simulation: Present Value



Simulating Standard Brownian Motion

- Sometimes the asset value is path dependent.
- If we want to simulate W at multiple time points $0 < t_1 < \cdots < t_k$, then

$$W\left(t_{i+1}\right) = W\left(t_i\right) + \sqrt{t_{i+1} - t_i} Z_{i+1},$$

  where Z_1, ..., Z_k are iid $N\left(0, 1\right)$.
  - Because the increments $W\left(t_1\right) - W\left(t_0\right)$, $W\left(t_2\right) - W\left(t_1\right)$, ..., $W\left(t_k\right) - W\left(t_{k-1}\right)$ are independent.
  - The simulation is exact (i.e., correct distribution) at the time points t_1, ..., t_k.
  - But any deterministic interpolation between t_i will introduce discretization error.

Brownian Motion

If $X(t) = \mu t + \sigma W(t)$, then $X(t)$ is a Brownian motion with drift μ and diffusion coefficient σ^2.

- Since $W(t) \sim N(0, \sigma^2)$, we get $X(t) \sim N(\mu t, \sigma^2 t)$.
- Since $W(0) = 0$, we also have $X(0) = 0$.

Since $X(t) - X(s) = \mu(t-s) + \sigma[W(t) - W(s)]$, the increments are still normal and independent. Hence, we can simulate the Brownian motion by

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1}.$$

Geometric Brownian Motion

The stochastic process $S(t)$ is a geometric Brownian motion if $\log S(t)$ is a Brownian motion with initial value $\log S(0)$.

- The technical definition is that $S(t)$ is a geometric Brownian motion if

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t).$$

- The solution is that

$$S(t) = S(0) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right],$$

  where $S(0)$ is the initial value.

Simulate Geometric Brownian Motion

$$S(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right].$$

For any $u < t$,

$$S(t) = S(u)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t-u) + \sigma\left[W(t) - W(u)\right]\right\}.$$

Hence, we can simulate the geometric Brownian motion as

$$S(t_{i+1}) = S(t_i)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right\}.$$

The simulation is still exact at the time points $\{t_i\}$.

Barrier Option: Discrete Monitoring

Consider a barrier option where the option is knocked out if $S(t) > B$, where B is the barrier level. The payoff such option is

$$\max\{S(T) - K, 0\} \prod_{m=1}^{M} 1\{S(t_m) \le B\}$$

if the monitoring times are $\{t_m\}$, for $m = 1, ..., M$. The present value of the option is

$$\mathrm{E}\left[\max\{S(T) - K, 0\} \prod_{m=1}^{M} 1\{S(t_m) \le B\}\right].$$

It can be easily approximated by

$$\frac{1}{N}\sum_{i=1}^{n}\left[\max\{S(T_i) - K, 0\} \prod_{m=1}^{M} 1\{S(t_{m,i}) \le B\}\right].$$

Barrier Option: Continuous Monitoring

Consider the previous barrier option but monitoring is continuous. The present value equals to

$$\mathrm{E}\left[\max\left\{S(T)-K,0\right\}1\left(S(t)\leq B,\forall t\leq T\right)\right].$$

We can use an idea similar to Brownian bridge to handle continuous monitoring. For a fixed $S(0)$,

$$\begin{aligned}
& \mathrm{E}\left[\max\left\{S(T)-K,0\right\}1\left(S(t)\leq B,\forall t\leq T\right)\right] \\
= \; & \mathrm{E}\left\{\mathrm{E}\left[\max\left\{S(T)-K,0\right\}1\left(S(t)\leq B,\forall t\leq T\right)\mid S(T)\right]\right\} \\
= \; & \mathrm{E}\left\{\max\left\{S(T)-K,0\right\}\mathrm{E}\left[1\left(S(t)\leq B,\forall t\leq T\right)\mid S(T)\right]\right\} \\
= \; & \mathrm{E}\left\{\max\left\{S(T)-K,0\right\}\mathrm{P}\left[S(t)\leq B,\forall t\leq T\mid S(T)\right]\right\}.
\end{aligned}$$

More General Stochastic Differential Equation

Now we consider a more general stochastic differential equation (SDE) of the form

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t),$$

where $a()$ and $b()$ depend on the stochastic process $X(t)$.

The stochastic process $X(t)$ is a solution, if $X(t)$ solves the integral

$$X(t) - X(0) = \int_0^t a(X(s))\,ds + \int_0^t b(X(s))\,dW(s).$$

We can approximate the solution using discretization methods.

Euler-Maruyama Method

We discretize the time interval $[0, T]$ into a time grid of step size Δt. Then, the Euler-Maruyama approximation is

$$\hat{X}(t_{n+1}) = \hat{X}(t_n) + a\left(\hat{X}(t_n)\right)\Delta t + b\left(\hat{X}(t_n)\right)\left[W(t_{n+1}) - W(t_n)\right],$$

where $\Delta t = t_{n+1} - t_n$ for any n.

- Consider the approximation

$$\begin{aligned} \int_{t_n}^{t_{n+1}} a(X(s))\,ds &\approx a(X(t_n))(t_{n+1} - t_n) = a(X(t_n))\Delta t, \\ \int_{t_n}^{t_{n+1}} b(X(s))\,dW(s) &\approx b(X(t_n))\left[W(t_{n+1}) - W(t_n)\right]. \end{aligned}$$

- These approximations yield the Euler-Maruyama approximation.

Milstein Method

A refinement is the Milstein approximation:

$$\begin{aligned}\hat{X}(t_{n+1}) &= \hat{X}(t_n) + a\left(\hat{X}(t_n)\right)\Delta t + b\left(\hat{X}(t_n)\right)[W(t_{n+1}) - W(t_n)] \\ &\quad + \frac{1}{2} b\left(\hat{X}(t_n)\right) b'\left(\hat{X}(t_n)\right)\left\{[W(t_{n+1}) - W(t_n)]^2 - \Delta t\right\},\end{aligned}$$

where $b'(x) = \partial b(x) / \partial x$.

- The Euler-Maruyama approximation satisfies

$$\sup_{0 \leq t_n \leq T} \mathrm{E}\left[\left|X(t_n) - \hat{X}(t_n)\right|\right] \leq O\left(\sqrt{\Delta t}\right).$$

- The Milstein approximation satisfies

$$\sup_{0 \leq t_n \leq T} \mathrm{E}\left[\left|X(t_n) - \hat{X}(t_n)\right|\right] \leq O(\Delta t).$$

Example: Geometric Brownian Motion

Example

Consider a geometric Brownian motion

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t).$$

1. The Euler-Maruyama approximation becomes

$$S(t_{n+1}) = S(t_n) + \mu S(t_n)\Delta t + \sigma S(t_n)\left[W(t_{n+1}) - W(t_n)\right].$$

2. The Milstein approximation becomes

$$\begin{aligned} S(t_{n+1}) &= S(t_n) + \mu S(t_n)\Delta t + \sigma S(t_n)\left[W(t_{n+1}) - W(t_n)\right] \\ &\quad + \frac{1}{2}\sigma S(t_n)\cdot\sigma\cdot\left\{\left[W(t_{n+1}) - W(t_n)\right]^2 - \Delta t\right\}. \end{aligned}$$

Example: Discretization Methods

Example

Consider the stochastic process

$$dX(t) = \theta(\mu - X(t))\,dt + \sigma dW(t).$$

1. The Euler-Maruyama approximation becomes

$$X(t_{n+1}) = X(t_n) + \theta[\mu - X(t_n)]\Delta t + \sigma[W(t_{n+1}) - W(t_n)].$$

2. The Milstein approximation becomes

$$\begin{aligned} X(t_{n+1}) &= X(t_n) + \theta[\mu - X(t_n)]\Delta t + \sigma[W(t_{n+1}) - W(t_n)] \\ &\quad + \frac{1}{2} b\left(\hat{X}(t_n)\right) \cdot 0 \cdot \left\{[W(t_{n+1}) - W(t_n)]^2 - \Delta t\right\}. \end{aligned}$$