EXERCISES FOR L7

16 NOVEMBER 2022

Exercise 1 (Leftovers from the lectures)**.** (a) In Lecture 1, we defined 2 vertices x, y to be connected if there is a walk (equivalently, a trail/path) from x to y. Verify that these are indeed equivalent. When using paths instead of walks, prove again that connectedness is an equivalence relation on vertices.

(b) Verify that $\varphi : G \to G'$ is an isomorphism if and only if φ is bijective on the underlying vertex sets, and the equivalence $\{\varphi(v), \varphi(w)\} \in E' \iff \{v, w\} \in E$ holds for all $v, w \in V$.

(c) The lecture notes for Lecture 2 contain a list of 11 non-isomorphic graphs on 4 vertices. Verify that indeed, no two of those graphs are isomorphic to each other. How would you go about showing that this list is complete, i.e. that there is no other isomorphism class on 4 vertices?

(d) Complete the proof of Cayley's theorem. That is, show that the two algorithms of transforming between labelled trees and Prüfer sequences are indeed inverse to each other. Convince yourself also that algorithm 2 is indeed well-defined – in particular, check the claim about the two remaining vertices in the end. (Hint: First do a small example – how does the first step in Algorithm 1 correspond to the first step in Algorithm 2? What about any later steps?).

Exercise 2 (Some counting)**.** What is the maximal number of edges a simple graph on n vertices with c components can have? How many vertex-/edge-induced supgraphs does a finite simple graph posess?

Exercise 3 (Morphisms)**.** Let G and G' be (finite) simple graphs.

(a) For which integers n, m is there a morphism $K_m \to K_n$?

(b) Show that there is a morphism $C_{2n} \to K_2$, but that there are no morphisms $C_{2n+1} \to K_2$.

(c) Show that, if $\varphi : G \to G'$ is an isomorphism, then $\deg_G(v) = \deg_{G'}(\varphi(v))$.

Exercise 4 (Trees)**.** Let $T = (V, E)$ be a finite tree.

(a) Show: If G is a connected graph on n vertices having $n - 1$ edges, then G is a tree.

(b) Show: If $\max_{v \in V} \deg(v) = \Delta$, then T has at least Δ leaves.

(c) Any isomorphism $T \to T$ either maps a vertex to itself, or it maps an edge to itself.

Exercise 5 (Counting spanning trees)**.** (a) Determine the number of spanning trees of a (labelled) $K_{m,n}$ for $m, n \in \mathbb{N}$.

(b) Let K_n' be a labelled complete graph where the edge $\{1, 2\}$ has been deleted. How many spanning trees does K_n' have?

Exercise 6 (MSTs)**.** Consider a weighted version of K_{10} with vertices labelled $1, \ldots, 10$ and edge weights $w(\{i,j\}) = i + j$. Find a minimum spanning tree. Is it unique?

Exercise 7 (Cycles in graphs)**.** Let $G = (V, E)$ be a finite simple graph. The *girth* g of G is the minimum length of a cycle in G. The minimum degree δ of G is $\delta = \min_{v \in V} \deg(v)$. Show:

(a) G contains a path of length δ.
(b) If $\delta \geq 2$ then G contains a cycle of length at least $\delta + 1$.
(c) If G contains a cycle, then $g \leq 2 \operatorname{diam}(G) + 1$.
(d) Suppose G contains a cycle C and there is a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.
(e) Assume that all vertices of G have degree at least 3. Show that G contains a cycle of even length.

Exercise 8 (Hamiltonicity)**.** Consider the graphs H^n constructed from the Towers of Hanoi on n disks (cf Lecture 2).

(a) Convince yourself that H^{n+1} consists of 3 copies of H^n, pairwise connected by edges.
(b) Show that H^n is Hamiltonian.

Exercise 9 (Petersen graph)**.** In this exercise, we will determine the spectrum and the number of spanning trees of the Petersen graph. Denote by A its adjacency matrix, by I the identity matrix and by J the matrix having all entries equal to 1 (all matrices of size 10×10).

(a) Show that $A^2 + A - 2I = J$.
(b) $(1, \ldots, 1)^t$ is an eigenvector for A. What is its eigenvalue? If $Av = \lambda v$ where v is an eigenvector linearly independent to $(1, \ldots, 1)^t$, then it can be chosen to be orthogonal to $(1, \ldots, 1)^t$. Verify that (a) now implies $\lambda \in \{-2, 1\}$.
(c) Using (b), determine the spectrum of A, and conclude the spectrum of the Laplacian matrix of the Petersen graph. Then apply the matrix-tree theorem.