Lecture 13

Multi-dimensional Models (Ch 14 in the book, but our presentation differs from the book.)

Model:
$$\begin{cases} dB_t = rB_t\,dt \\ dS_t^i = \mu_i S_t^i\,dt + S_t^i \sum_{j=1}^{n} \sigma_{ij}\,dW_t^j, \quad i = 1,\dots,n, \end{cases}$$

where r, μ_i, σ_{ij} are constants, and

$$\sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$$ is a non-singular matrix.

Remark: In the Meta-theorem, $R = M = n$ so we expect the market to be arbitrage-free and complete.

Question: What is the arbitrage-free price of a simple T-claim $\mathcal{X} = \Phi(S_T)$?

Idea (that we will not follow): We could construct a portfolio of $S^1, S^2, \dots, S^n, \Pi(\mathcal{X})$ which is locally risk-free (no dW-terms). Then, to avoid arbitrage, the drift of the portfolio must be r. This will give a PDE for the price.

Instead, we will take the following route.

We guess that the price is $\Pi_t(\mathcal{X}) = F(t, S_t^1, \dots, S_t^n)$ where $F(t, s_1, \dots, s_n)$ satisfies

$$(BS)\begin{cases} F_t + \frac{1}{2}\sum_{i,j=1}^{n} s_i s_j C_{ij} F_{s_i s_j} + r \sum s_i F_{s_i} - rF = 0 \\ F(T, s_1, \dots, s_n) = \Phi(s_1, \dots, s_n), \end{cases}$$

with $C = \sigma\sigma^*$.

To show that the guess is correct, we give a replication argument.

Theorem To avoid arbitrage, the price of $\mathcal{X} = \Phi(S_T)$ has to be $F(t, S_t)$ where $F(t,s)$ is given by equation (BS) above. Moreover, $\mathcal{X}$ is replicated by $h = (h^B, h^1, \dots, h^n)$ where

$$\begin{cases} h_t^B = \dfrac{F(t, S_t) - \sum_{i=1}^{n} S_t^i F_{s_i}(t, S_t)}{B_t} \\ h_t^i = F_{s_i}(t, S_t) \qquad (i = 1, \dots, n). \end{cases}$$

Proof: $V_t^h = h_t^B B_t + \sum_{i=1}^{n} h_t^i S_t^i = F(t, S_t)$,

so $V_T^h = F(T, S_T) = \Phi(S_T) = \mathcal{X}$ (correct terminal value!)

Is h self-financing?

We have

$$dV_t^h \overset{\text{Ito}}{=} F_t\,dt + \sum_{i=1}^n F_{s_i}\,dS_t^i + \frac{1}{2}\sum_{i,j=1}^n F_{s_i s_j}(dS_t^i)(dS_t^j)$$

$$= \left(F_t + \frac{1}{2}\sum_{i,j=1}^n S_t^i S_t^j C_{ij} F_{s_i s_j}\right)dt + \sum_{i=1}^n F_{s_i}\,dS_t^i$$

$$\overset{\text{(BS-eqn)}}{=} \left(rF - r\sum_{i=1}^n S_t^i F_{s_i}\right)dt + \sum_{i=1}^n F_{s_i}\,dS_t^i$$

$$= h_t^B\,dB_t + \sum_{i=1}^n h_t^i\,dS_t^i$$

Thus h is self-financing and it replicates $\mathcal{X}$. Any price different from $V_t^h = F(t, S_t)$ would lead to an arbitrage!

<u>Theorem</u> (Risk-neutral valuation)

The pricing function has the representation

$$F(t,s) = E_{t,s}^{Q}\left[e^{-r(T-t)}\Phi(S_T)\right]$$

where the Q-dynamics of S are

$$\begin{cases} dS_u^i = rS_u^i\,du + S_u^i\sum_{j=1}^n \sigma_{ij}\,dW_u^j \\ S_t^i = s_i \end{cases}$$

Reducing the state space

Let $n=2$, and assume that $\Phi(ks_1, ks_2) = k\Phi(s_1, s_2)$, $k>0$.

Then $\Phi(s_1, s_2) = s_2 \Phi(\frac{s_1}{s_2}, 1)$.

$$\boxed{\text{Ansatz: } F(t, s_1, s_2) = s_2 G(t, \tfrac{s_1}{s_2}) \text{ for some function } G(t,z).}$$

$F(T, s_1, s_2) = \Phi(s_1, s_2)$ translates into $G(T, z) = \Phi(z, 1)$.

We now translate all derivatives in the BS-equation

$$F_t + \frac{1}{2} s_1^2 C_{11} F_{s_1 s_1} + \frac{1}{2} s_2^2 C_{22} F_{s_2 s_2} + s_1 s_2 C_{12} F_{s_1 s_2} + r s_1 F_{s_1} + r s_2 F_{s_2} - rF = 0$$

into derivatives of G:

$$F_t = s_2 G_t \qquad F_{s_1 s_1} = \frac{1}{s_2} G_{zz}$$

$$F_{s_1} = G_z \qquad F_{s_1 s_2} = \frac{-s_1}{s_2^2} G_{zz}$$

$$F_{s_2} = G - \frac{s_1}{s_2} G_z \qquad F_{s_2 s_2} = \frac{s_1^2}{s_2^3} G_{zz}$$

We get

$$s_2 G_t + \frac{1}{2} \frac{s_1^2}{s_2} C_{11} G_{zz} + \frac{1}{2} \frac{s_1^2}{s_2} C_{22} G_{zz} - \frac{s_1^2}{s_2} C_{12} G_{zz} + r s_1 G_z + r s_2 G - r s_1 G_z - r s_2 G = 0$$

which simplifies to

$$G_t + \frac{1}{2} \frac{s_1^2}{s_2^2} (C_{11} + C_{22} - 2C_{12}) G_{zz} = 0.$$

Since the argument of G and its derivatives is $(t, \frac{s_1}{s_2})$, we have the following.

<u>Proposition</u> ($n=2$) Assume $\phi(ks_1, ks_2) = k\phi(s_1, s_2)$. Then $F(t, s_1, s_2) = s_2 G(t, \frac{s_1}{s_2})$ where $G(t,z)$ solves

$$\begin{cases} G_t + \frac{1}{2}(C_{11} + C_{22} - 2C_{12}) z^2 G_{zz} = 0 \\ G(T, z) = \phi(z, 1). \end{cases}$$

<u>Example</u>:
$$\begin{cases} dS_t^1 = \mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t^1 \\ dS_t^2 = \mu_2 S_t^2 dt + \sigma_2 S_t^2 dW_t^2 \\ dB_t = r B_t dt \end{cases}$$
indep.

Let $X = (S_T^1 - S_T^2)^+$ (This is an <u>exchange option</u>. It gives the right to exchange one share of S^2 for one share of S^1.)

We have $\phi(s_1, s_2) = (s_1 - s_2)^+$ so $\phi(ks_1, ks_2) = k\phi(s_1, s_2)$.

By our recipe, $F(t, s_1, s_2) = s_2 G(t, \frac{s_1}{s_2})$ where $G(t,z)$ solves

$$\begin{cases} G_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) z^2 G_{zz} = 0 \\ G(T, z) = (z-1)^+. \end{cases}$$

Using the BS-formula, $G(t,z) = zN(d_1) - N(d_2)$, so

$$F(t, s_1, s_2) = s_2 G(t, \tfrac{s_1}{s_2}) = s_1 N(d_1) - s_2 N(d_2) \text{ where}$$

$$\begin{cases} d_1 = \dfrac{\ln \frac{s_1}{s_2} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t)}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}} \\ d_2 = d_1 - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}. \end{cases}$$

Ex: In the market
$$\begin{cases} dB_t = rB_t\,dt \\ dS_t^1 = \mu_1 S_t^1\,dt + \sigma_1 S_t^1\,dW_t^1 \\ dS_t^2 = \mu_2 S_t^2\,dt + \sigma_2 S_t^2\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^2\right) \end{cases}$$

find the price at $t=0$ of the T-claim $X = \dfrac{(S_T^1)^2}{S_T^2}$.

$\Phi(s_1,s_2) = \dfrac{s_1^2}{s_2}$ so $\Phi(ks_1,ks_2) = k\Phi(s_1,s_2)$.

Thus $F(t,s_1,s_2) = s_2 G(t,\frac{s_1}{s_2})$ where $\begin{cases} G_t + \frac{1}{2}z^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)G_{zz} = 0 \\ G(T,z) = z^2. \end{cases}$

Denoting $\sigma := \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$ we have

$G(0,z) = E_{0,z}\left[Z_T^2\right]$ where $dZ_t = \sigma Z\,dW_t$. With $Y_t := Z_t^2$

we find $dY_t = 2Z_t\,dZ_t + (dZ_t)^2 = \sigma^2 Y_t\,dt + 2\sigma Y_t\,dW_t$,

so $G(0,z) = E\left[Z_T^2\right] = z^2 e^{\sigma^2 T}$

Answer: $F(0,s_1,s_2) = s_2\,G(0,\frac{s_1}{s_2}) = \dfrac{s_1^2}{s_2}\,e^{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)T}$.