

UPPSALA UNIVERSITET

FÖRELÄSNINGSANTECKNINGAR

# Tillämpad Matematik

*Rami Abou Zahra*

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## CONTENTS

1. Introduction	2
1.1. Scaling	5
2. Perturbation Theory	8
2.1. Poincare-Lindstedt Method	10

## 1. INTRODUCTION

The topic of this course will vary a lot, since mathematics can be applied to physics, biology, etc. We will look into different ways to model real life, study it, and draw conclusions from it.

**Anmärkning:**

One could look at a mathematical models as a set of equations

**Example:** Planetary motion

- *Observation*: Keplers law  $\rightarrow$  elliptic orbits
- *Model*: Newtons gravitational law
- *Mistakes/Errors*: Mercury precession  $\rightarrow$  disalignment between model and observation
- *Rectify error*: Introducing relativistic effects in the model
- *Evaluation*: Is the old model useless? No, it is often easier to compute. It is better to keep it simple

We arrive at 2 models:

*Good model*  $\rightarrow$  Simple, general (not valid in a specific way)

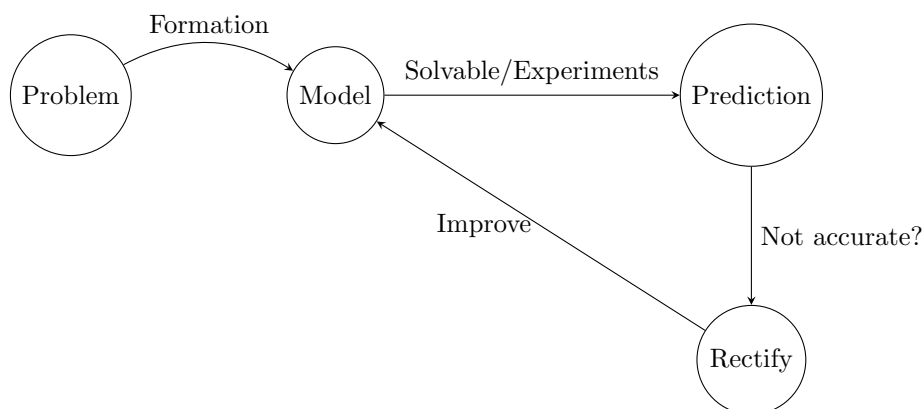


FIGURE 1.

**First step in the definition of a model:** Understand which variables are involved

Dimension	Unit	Derived	Dimension
Distance	$m$	$v$	$m/s$
Temperature	Degrees	$a$	$m/s$
Time	$s$		

**Definition/Sats 1.1: Physical law**

A physical law is  $f(q_1, \dots, 1_n) = 0$

$L_1, \dots, L_m$  are the dimensions

$$[q_i] = L_1 \cdots L_m$$

- $[q] = 1$  dimensions
- $[v] = L \cdot T^{-1}$

**Example:** Conservation of energy is an example of such physical law:

$$\frac{mp^2}{2} + V(q) = C \quad C \in \mathbb{R}$$

$$F(m, p, q) = \frac{mp^2}{2} + V(q) - C = 0$$

**Example:** Hooks law for springs:

$$F = \underbrace{k}_{\text{Not dimensionless}} \cdot L \quad f(F, k, L) = 0$$

### Definition/Sats 1.2: Unit free

A law is *unit free* if it is independent from the unit, in the sense that if we define a new system in the following way:

$$\overline{L}_i = \lambda_i L_i$$

Then  $\overline{L}_i$  is a new system of unit  $\lambda_i > 0$

$$[q_i] = L_1^{b_1} \cdots L_n^{b_n}$$

$$f(q, \cdots, q_n) = 0 \Leftrightarrow f(\overline{q}_1, \cdots, \overline{q}_n) = 0$$

**Example:**

$$f(x, t, g) = x - \frac{1}{2}gt^2 = 0$$

Describing a body falling. If we define the following units:

- $[x] = m$
- $[g] = ms^{-2}$
- $[t] = s$

We can check that if we use different units, say  $\overline{x} = 1000x$  (kilometers instead of meters) or  $\overline{t} = 3600t$  (hours instead of seconds), then we obtain the same law for  $f(\overline{x}, \overline{t}, g) = 0$

**Example:** Just looking at the dimension we can say something about the model. Take the pendulum and study the period of oscillation (is the mass or the length the one that defines the period?)

The goal is to find a law for the period. Suppose only the length and the mass are the only variables in our model, then we want to find  $P = f(l, m)$

Notice that we have an error in the dimension, since our period depends on time, so just looking at that we can see that there is something that is missing.

We could be interested in adding another term, the gravitational acceleration. We get:

$$T = kL^{\alpha_1} M^{\alpha_2} \frac{L^{\alpha_3}}{T^{-2\alpha_3}}$$

$$\begin{cases} \alpha_2 = 0 & \rightarrow \text{mass is not involved} \\ \alpha_1 + \alpha_3 = 0 \\ -2\alpha_3 = 0 \end{cases} \quad \alpha_3 = \frac{-1}{2} \quad \alpha_1 = \frac{1}{2}$$

$$\Rightarrow P \approx k \sqrt{\frac{L}{g}}$$

Another thing we may do is to introduce dimensionless variables:

### Definition/Sats 1.3: Pi's theorem

Let  $f(q_1, \cdots, q_m) = 0$  be a unit free law with the usual notation for dimension  $[q_i] = L_1^{\alpha_{1i}} \cdots L_n^{\alpha_{ni}}$   
 $n < m$

Define the dimension matrix  $A$

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix}$$

Let  $\pi$  be the rank( $A$ ). Then there exists  $m - r$  dimensionsless variables  $\Pi_1, \dots, \Pi_{m-r}$  (which can be formed from  $q_i$ )

We have an equivalent law  $F(\Pi_1, \dots, \Pi_{m-r}) = 0$

### Anmärkning:

When we have a law, it does not mean that we have the right law (only  $q_1, \dots, q_n$  are involved) but it is not meaningless

The usefulness of Pi-theorem:

- Case in which only one dimensionsless variable is involved  
 $F(\Pi_1) = 0 \rightarrow$  zeroes are discrete  
 $\Pi_1$  can assume discrete values and can be deduced from experiments

In the case of 2 dimensionsless quantities  $F(\Pi_1, \Pi_2) = 0$ , if we can invert the relationship then we can write one variable as a function of the other using implicit function theorem.

$$\Pi_1 = f(\Pi_2) \quad f \text{ is unknown} \rightarrow \text{deduced from observation}$$

**Example:** Allometry (Biology), the study of characteristics of living creatures change with their size. We look for a law that involves

- $q_1 = l = \text{length of the organism} \quad [q_1] = L$
- $q_2 = t = \text{time} \quad [q_2] = T$
- $q_3 = \rho = \text{density} \quad [q_3] = \frac{M}{L^3}$
- $q_4 = a = \text{resource assimilation rate} \quad [q_4] = \frac{M}{L^2 T}$
- $q_5 = b = \text{resource utilisation rate} \quad [q_5] = \frac{M}{L^3 T}$

We look for a law that involves 2 dimensionsless variables, so we apply the theorem:

$$A = \begin{pmatrix} 1 & 0 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} L \\ M \\ T \end{pmatrix}$$

(Look at the exponent of the respective variable)

The rank( $A$ ) = 3  $\rightarrow$  5 - 3 = 2 dimensionsless variables

We can try to express  $q_i$  as a linear combination of the others. We know the following:

$$\begin{cases} \alpha_1 - 3\alpha_3 = -2 \\ \alpha_3 = 1 \\ \alpha_2 = -1 \end{cases} \Rightarrow \alpha_1 = 1$$

This means that  $q_4$  can be expressed as  $q_4 = \frac{q_1 q_3}{q_2}$ , yielding:

$$\Pi_1 = \frac{q_1 q_3}{q_2 q_4} = \frac{l \rho}{t a} \rightarrow \text{dimensionsless}$$

We can do the same for  $q_5 \Rightarrow q_5 = \frac{q_3}{q_2}$  yielding another dimensionsless variable:

$$\Pi_2 = \frac{q_3}{q_2 q_5} = \frac{\rho}{t b}$$

Summa summarum:

$$F(\Pi_1, \Pi_2) = 0 = F\left(\frac{l \rho}{t a}, \frac{\rho}{t b}\right)$$

$$\pi_1 = f(\Pi_2)$$

### 1.1. **Scaling.**

The goal is to rescale variables to a quantity that is related to that specific problem. Measuring seconds when it comes to glaciers might be less useful as measuring with years, and seconds for a chemical reaction might be too little.

For example, with time,  $\bar{t} = \frac{t}{t_c}$ . New rescaled time is 1 once it has passed the desired scale.  $c$  stands for characteristic

The same can be done for other quantities such as length  $\bar{h} = \frac{h}{h_c}$

**Example:** Projectile problem where we only consider gravity. Using Newtons gravitational law:

$$\frac{md^2h}{dt^2} = -G \cdot \frac{mM}{(R+h)^2} \Rightarrow \frac{d^2h}{dt^2} = -G \frac{M}{(R+h)^2}$$

We know that for  $h = 0$ ,  $\frac{d^2h}{dt^2} = -g = \frac{-GM}{R^2} = \frac{-gR^2}{(h+R)^2}$

We also know  $h(0) = 0$ ,  $\frac{dh}{dt}(0) = v$  (initial velocity)

We can introduce dimensionsless variables:

- $[t] = T$
- $[h] = L$
- $[R] = L$
- $[v] = LT^{-1}$
- $[g] = LT^{-2}$

Since only  $L, T$  are involved, we have 2 rows:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\text{rank}(A) = 2 \Rightarrow 3$  dimensionsless variables

We could for example do

$$\Pi_1 = \frac{h}{R} \quad \Pi_2 = \frac{h}{vt} \quad \Pi_3 = \frac{h}{gt^2}$$

Let us see what happens if we do some scaling for the time  $\bar{t}$  and the length  $\bar{h}$ :

$$\bar{t} = \frac{t}{t_c} \quad \bar{h} = \frac{h}{h_c}$$

With a dimension of time, we could pick  $\frac{R}{v}$ , or  $\sqrt{\frac{R}{g}}$ ,  $\frac{v}{g}$

The same for  $h$ , we could pick  $R$ ,  $\frac{v^2}{g}$

Usually only one choice is the one that helps us solve the problem:

$$\bar{t} = \frac{t}{R/v} \quad \bar{h} = \frac{h}{R}$$

Now we need to express the laws that we have in terms of  $\bar{t}$  and  $\bar{h}$ :

$$\begin{aligned} \frac{d^2h}{dt^2} &= \frac{-gR^2}{(R\bar{h} + R)^2} = \frac{-g}{(\bar{h} + 1)^2} & h &= \bar{h}R \\ \frac{dh}{dt} &= \frac{d\bar{h}}{d\bar{t}} R & \frac{d\bar{h}}{d\bar{t}} &= \frac{d\bar{h}}{dt} \frac{dt}{d\bar{t}} = \frac{R}{v} \frac{d\bar{h}}{d\bar{t}} \\ \frac{d^2\bar{h}}{d\bar{t}^2} &= \frac{d^2\bar{h}}{dt^2} \frac{R^2}{v^2} \rightarrow \frac{v^2}{Rg} \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1 + \bar{h})^2} \end{aligned}$$

We can call  $\varepsilon = \frac{v^2}{Rg}$  ( $\varepsilon$  small)

The equation  $\varepsilon \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1 + \bar{h})^2}$  has no solution when  $\varepsilon = 0$

With a different choice

$$\begin{aligned} \bar{t} &= \frac{t}{vg^{-1}} & \bar{h} &= \frac{h}{v^2g^{-1}} \\ \Rightarrow \frac{d^2\bar{h}}{d\bar{t}^2} &= -\frac{1}{(1 + \varepsilon\bar{h})^2} & \bar{h}(0) &= 0 & \frac{d\bar{h}}{d\bar{t}}(0) &= 1 \end{aligned}$$

Notice now that when  $\varepsilon = 0$ :

$$\bar{h}'' = -1 \quad \bar{h}' = -\bar{t} + a = -\bar{t} + t$$

$$\bar{h} = -\frac{t^2}{2} + \bar{t} + b = -\frac{\bar{t}^2}{2} + \bar{t}$$

By substituting the old variables back, we get:

$$h = \frac{-t^2}{2}g + vt$$

The quantities that we used for  $t_c$ ,  $h_c$ :

$$t_c = \frac{v}{g} \quad h_c = \frac{v^2}{g}$$

$$h' = 0 \rightarrow -tg + v = 0 \Rightarrow t = \frac{v}{g}$$

Then  $h_c$  is the maximum height that the body reaches.



## 2. PERTUBATION THEORY

This applies to another class of problems that is known (in a sense that we know how to solve it, we can find the solution); and we consider a new problem that is made of a known problem + a *perturbation*:

$$\text{Problem} + \varepsilon \text{ Problem}_2 \quad \varepsilon \ll 1$$

**Example:** Planetary motion

If we consider a 2-body problem (one planet & one star), this can be solved exactly. Consider now a 3-body problem, then this problem cannot be solved easily.

$$\begin{array}{c} M_{\text{sun}} \gg M_{p_1} M_{p_2} \\ F_{p_2 p_1} \ll F_{Sp_1,2} \\ \underbrace{F_{Sp_1} + F_{Sp_2}}_{\text{2-body}} + \underbrace{F_{p_2 p_1}} \end{array}$$

**Anmärkning:** Here  $F_{Sp_i}$  denotes the gravitational pull from the sun to one planet.

In general, we apply perturbation theory to equation of the form  $F(y, y', y'', \dots, \varepsilon) = 0$ .

We look for a solution  $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 + \dots$

We expect  $y_0$  (leading term) to be the solution/approximation when  $\varepsilon = 0$

We may ask ourselves if the approximation converges to the solution.

**Example:**

$$\hat{y} = -y + \varepsilon y^2 \quad y(0) = 1$$

We try to do the easiest thing and plug in the approximation:

$$(\hat{y}_0 + \varepsilon \hat{y}_1 + \varepsilon^2 \hat{y}_2 + \dots) = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) + \varepsilon (y_0 + y_1\varepsilon + \dots)^2$$

Now we try to solve order by order (collect like terms and see which one equates):

$$\begin{aligned} \hat{y}_0 &= -y_0 \\ y_0 &= Ae^{-t} \quad y_0 = e^{-t} \quad \text{since our initial condition} \\ \hat{y}_1 &= -y_1 + y_0^2 = -y_1 + e^{-2t} \\ \hat{y}_1(0) &= 0 \quad \text{since we already used our initial condition} \\ y_1(t) &= e^{-t} + Ae^{-2t} \Rightarrow y_1(t) = e^{-t} - e^{-2t} \\ \hat{y}_2 &= -y_2 + 2y_0 y_1 = -y_2 + 2e^{-t}(e^{-t} - e^{-2t}) \\ \hat{y}_2(0) &\Rightarrow y_2(t) = e^{-t} - 2e^{-2t} + e^{-3t} \end{aligned}$$

We have found the first three terms. This problem can be solved exactly, and we can see if our construction solves the equation or not:

$$y(t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots$$

This is a case where regular perturbation works really well, since the explicit solution is given by:

$$\begin{aligned} y(t) &= \frac{e^{-t}}{1 - \varepsilon + \varepsilon e^{-t}} = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)} \\ \sum_n x^n &= \frac{1}{1 - x} \Rightarrow e^{-t}(1 - \varepsilon(e^{-t} - 1) + \varepsilon^2(e^{-t} - 1)^2 + \dots) \end{aligned}$$

This is not always the case, that it is the same solution. If we use the example from the last lecture (projectile problem), we get something different:

$$\begin{aligned} h'' &= \frac{-1}{(1 + \varepsilon h)^2} \\ \varepsilon = 0 &\Rightarrow h_0 = \frac{-t^2}{2} + t \end{aligned}$$

We try the same technique, suppose  $h = h_0 + \varepsilon h_1$ :

$$(h'_0 + h'_1\varepsilon)(1 + \varepsilon(h_0 + \varepsilon h_1))^2 = -1$$

We collect like terms:

$$\begin{aligned} h_1'' + 2h_0''h_0 &= 0 \\ h_1 &= -(-1) \left( \frac{t^4}{4} + t^2 - t^3 \right) \\ h_1' &= \frac{t^5}{20} + \frac{t^3}{3} - \frac{t^4}{4} + C \quad h_1'(0) = 0 \quad h_1(0) = 0 \\ h_1 &= \frac{t^6}{100} + \frac{t^4}{12} - \frac{t^5}{20} \end{aligned}$$

We have a polynomial in  $t$  which is greater than the one in  $h_0$ , and an exponential in the other.

Well, in the terms  $\varepsilon(e^{-t} + \dots)$ , the size is dominated by  $\varepsilon$  even when  $t$  grows, while in the polynomial no matter how small  $\varepsilon$  we choose the polynomial can always grow bigger. So  $h_0 + \varepsilon h_1$ ,  $h_1$  term grows too much. This does not mean that the term is wrong, but it may not have a meaning in the problem that we are considering.

$h_1$  is growing faster than  $h_0$  even though there is an  $\varepsilon$  in front of it. Recall that  $h_1$  is just a correction, because we are adding a term that is bigger than our first approximation. We are essentially not writing a function that is adding smaller and smaller terms.

This method is called *regular perturbation*, and sometimes it works and sometimes it does not. In the case when it does not, we have to try a different technique.

We consider a different problem:

**Example:** Duffin Equation

$$\begin{aligned} \hat{u} + u + \varepsilon u^3 &= 0 \quad t > 0 \\ u(0) &= 1 \quad \hat{u}(0) = 0 \end{aligned}$$

In this case, we do not have an explicit formula for the solution. We can try to use regular perturbation and see if it has a meaning or not.

$$\begin{aligned} u &= u_0 + \varepsilon u_1 \\ \Rightarrow \hat{u} + u_0 &= 0 \quad u_0(0) = 1 \quad \hat{u}_0(0) = 0 \\ u_0(t) &= A \cos(t) + B \sin(t) \\ u_0(0) &= 1 \Rightarrow A = 1 \\ \hat{u}_0(0) &= -\sin(0) + B \cos(0) = 0 \Rightarrow B = 0 \\ u_0 &= \cos(t) \end{aligned}$$

Notice that for  $u_0$ , we have an oscillatory solution (since the trig-functions are periodic).

We collect like terms and equate them:

$$\begin{aligned} \hat{u}_1 + u_1 + u_0^3 &= 0 \quad u_1(0) = 0 \quad \hat{u}_1(0) = 0 \\ \hat{u}_1 &= -u_1 - \cos^3(t) \\ \cos^3(t) &= \left( \frac{e^{i\pi} + e^{-i\pi}}{2} \right)^3 = \frac{e^{3i\pi} + e^{-3i\pi} + 3e^{2i\pi-i\pi} + 3e^{i\pi-2i\pi}}{8} \\ &= \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t) \\ \Rightarrow \hat{u}_1 + u_1 &= \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t) \\ u_1(0) &= 0 \quad \hat{u}_1(0) = 0 \Rightarrow u_1(t) = A \cos(t) + B \sin(t) + C \cos(3t) + At \sin(t) + Bt \cos(t) \end{aligned}$$

Notice that since one particular solution already included  $\cos(t)$ , we add another set of  $A \sin(t) + B \cos(t)$ , but multiplied with  $t$ .

With respect to the initial conditions, we get:

$$u_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \underbrace{\frac{3}{8}t \sin(t)}_{\text{Secular term}}$$

The secular term might be a problem, for example in this case we are interested in an oscillatory solution, so we expect a correction that gives us oscillatory approximation. But the  $t$  term makes the correction explode when  $t \rightarrow \infty$ .

There is also another issue with this approximation. We can show that the solution to this equation is bounded, but with this solution it breaks when  $t \rightarrow \infty$ , so  $u_0 + \varepsilon u_1$  is not good.

### Bevis 2.1: Exact solution is bounded

Consider  $\hat{u} + u + \varepsilon u^3 = 0$ , and multiply with  $\hat{u}$ :

$$\begin{aligned} \hat{u}\hat{u} + \hat{u}u + \varepsilon\hat{u}u^3 &= 0 \\ = \frac{d}{dt} \left( \frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4} \right) &= 0 \Rightarrow \frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4} = \text{Constant} = \frac{1}{2} + \frac{\varepsilon}{4} \\ &\Rightarrow u \text{ is bounded} \end{aligned}$$

□

Sometimes having small errors may not seem like a big issue, but these small errors may explode further down as  $t \rightarrow \infty$

## 2.1. Poincare-Lindstedt Method.

The idea is to do a rescaling, considering a perturbative correction of the frequencies of the oscillation.

We introduce a new variable (distorted time scale)  $\tau = \omega t$  where  $\omega = \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \dots$ .

As in the other case,  $\omega_0$  is the leading term when  $\varepsilon = 0$ , which in the previous example is 1.

We have to rewrite the equation according to the new time:

$$\begin{aligned} \frac{du}{dt} &= \frac{du}{dt} \frac{dt}{d\tau} = \hat{u} \frac{1}{\omega} \\ \Rightarrow \hat{u} &= \omega u' \quad u' = \frac{du}{d\tau} \end{aligned}$$

In the Duffin equation with the new variables we get:

$$\omega^2 u'' + u + \varepsilon u^3 = 0 \quad u(0) \stackrel{\tau=0}{=} 1 \quad \frac{du}{d\tau}(0) = \frac{\hat{u}(0)}{\omega} = 0$$

We now study this equation when  $\tau > 0$ . Essentially what we do, the advantage in this expansion, is that we can kill the terms that in regular perturbation generates the problem terms (secular terms).

What we will do is choose  $\omega_1$  such that it kills the secular terms. The procedure is the same as in regular perturbation:

$$(\omega_0 + \omega_1\varepsilon)^2(u''_0 + \varepsilon u''_1) + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 = 0$$

Gather terms and equate:

$$\begin{aligned} w_0^2 + u''_0 + u_0 &= 0 \quad u_0(0) = 1 \quad u'_0(0) = 0 \\ u_0 &= \cos(\tau) \quad w_0 = 1 \\ (1 + 2\omega_1\varepsilon + \omega_1^2\varepsilon^2)(u''_0 + \varepsilon u''_1) + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 &= 0 \\ \Rightarrow 2\omega_1 u''_0 + u''_1 + u_1 + u_0^3 &= 0 \Rightarrow u''_1 + u_1 = -u_0^3 - 2\omega_1 u''_0 = -\cos^3(\tau) - 2\omega_1 - \cos(\tau) \\ &\Rightarrow \frac{1}{4} \cos(3\tau) + \frac{3}{4} \cos(\tau) + 2\omega_1 \cos(\tau) \\ &\Rightarrow \frac{1}{4} \cos(3\tau) + \left(2\omega_1 - \frac{3}{4}\right) \cos(\tau) \end{aligned}$$

Notice that the last  $\cos(\tau)$  is the generator of our secular term, so we choose  $\omega_1$  so that  $2\omega_1 - \frac{3}{4} = 0 \Rightarrow$

$$\omega_1 = \frac{3}{8}$$

What we now get is an approximate term  $u_1$  that does not have a secular term, but just sine and cosine (preserving oscillation). We can also use some of the previous calculations, but without the secular term since we have removed it.

$$u_1(\tau) = \frac{1}{32} (\cos(3\tau) - \cos(\tau))$$

$$u = \cos(\tau) + \frac{\varepsilon}{32} (\cos(3\tau) - \cos(\tau))$$

$$\tau = \omega t = \left(1 + \frac{3}{8}\varepsilon\right)t$$