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PROBABILITY AND MARTINGALES
1MS045
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Each problem counts 5 points. Grades are awarded according to the following scale: 0–17 grade U; 18–24 grade 3; 25–31 grade 4; 32–40 grade 5. Allowed tools: pen, paper, calculator. All solutions should be clearly explained.

Note: if not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

1. (a) For a sequence of events $E_1, E_2, \ldots$ in a probability space, define $\limsup_{n\to\infty} E_n$ and $\liminf_{n\to\infty} E_n$. (1)
   (b) Prove *Fatou's lemma*: $P(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} P(E_n)$. (2)
   (c) State and prove the first *Borel–Cantelli lemma*. (2)
2. Let A be a subset of the interval $(1,\infty)$. For every $a \in A$, let X_a be a random variable with density $f_a(x) = (a-1)x^{-a} I_{[1,\infty)}(x)$. Under what condition(s) on A is the family $\{X_a : a \in A\}$ uniformly integrable? (5)
3. Let X and Y be integrable random variables. Prove the following identity: (5)
$$\int_{-\infty}^{\infty} \big(P(X < x \leq Y) - P(Y < x \leq X)\big)\, dx = \mathbb{E}Y - \mathbb{E}X.$$
4. State and prove *Kolmogorov's 0-1-law*. (5)
5. Let $X_1, X_2, \ldots$ be independent random variables with $P(X_i = 1) = P(X_i = 2) = \frac{1}{2}$ for all i, and set $P_n = X_1 X_2 \cdots X_n$ (with $P_0 = 1$).
   (a) Define *stopping time*. Which of the following is a stopping time with respect to the natural filtration? (2)
      - $\tau_1 = \sup\{n : P_n \leq 10\}$,
      - $\tau_2 = \inf\{n : P_n > 10\}$,
      - $\tau_3 = \inf\{n : P_n = P_{n-10}\}$.

   (b) Determine a constant $c > 0$ such that $R_n = c^{-n} P_n$ is a martingale. Show that R_n converges almost surely. What is its limit? (3)
6. Consider the following sequence of random variables: $X_0 = a$ for some $a \in (-1,1)$, and
$$X_n = \begin{cases} \frac{X_{n-1}^2 + 2X_{n-1} - 1}{2} & \text{with probability } \frac{1}{2}, \\ \frac{-X_{n-1}^2 + 2X_{n-1} + 1}{2} & \text{with probability } \frac{1}{2}, \end{cases}$$
for $n > 0$. Prove that the sequence $X_0, X_1, \ldots$ converges almost surely. What are the possible limits? For each of the possible limits L, determine (5)
$$P(\lim_{n\to\infty} X_n = L).$$

7. (a) Define *European call options* and *European put options*, and derive the *Call-Put parity* for their prices $C_0(E)$ and $P_0(E)$. (2)

   (b) Show that it implies the inequality $C_0(E) \geq S_0 - K$, where S_0 is the current value of the stock and K the strike price of the call option. (1)

   (c) Explain further how to deduce the following statement: an American call option on a stock that pays no dividends has the same fair price as the corresponding European option. (2)

8. (a) Define *viable* and *complete* market models. (1)

   (b) State the *first* and the *second fundamental theorem of asset pricing* (for finite market models). (2)

   (c) Provide an example of a finite market model that is viable, but not complete. (2)