Basic Exercises

Markov chain models

Discrete time

1. Per, Pål and Petter are playing with a ball. Per throws with probability 0.3 to Pål and with probability 0.7 to Petter. Pål throws the ball to Per with probability 0.6 and to Petter with probability 0.4. Petter throws the ball with equal probabilites to his two friends. All throws are independent of each other. This can be thought of as a Markov chain. Introduce appropriate states and set up the transition probability matrix.

2. On a library table is a stack of three volumes of an encyclopedia. Visitors to the library use the encyclopedia independently as follows. Each user takes with equal probability one of the volumes, looks in it and puts it back on top of the pile. Two users are never using the books simultaneously. Considering this as a Markov chain, what are the possible states of the chain? Find the transition probabilities.

3. A Markov chain has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

   Find the 2-step transition probabilities.

4. Suppose that the Markov chain in the previous example has initial distribution $\boldsymbol{\mu}_0 = (1/2, 1/2, 0)$. Find the distibution $\boldsymbol{\mu}_2$ two steps ahead.

5. Consider the weather during a number of days as a stochastic process (X_n) with the only possible states 0 : *sun* and 1 : *rain*. We assume for simplicity that the process is a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

   (a) Find the probability that a rainy day is followed by a sunny.
   (b) Formulate in words the event $\{X_{62} = 1\}$.
   (c) Find the conditional probability $P(X_{62} = 1 | X_{61} = 0)$.
   (d) Find the probability that a rainy day is followed by two sunny days.
   (e) Find $P(X_{62} = 1 | X_{60} = 0)$.
   (f) If Friday is sunny, what is the probability that the next following Sunday is also a sunny day?

6. (Continuation of the previous problem) Suppose we start the chain a sunny Friday, so that $X_0 = 0$.

   (a) What is the initial distribution vector $\boldsymbol{\mu}_0$?
   (b) Find the distibution $\boldsymbol{\mu}_1$ of X_1 and the probability that Saturday is sunny.

(c) What is the probability for rain on Sunday?

(d) What is $\boldsymbol{\mu}_1$ if $\boldsymbol{\mu}_0 = (1/3, 2/3)$?

7. A sequence of electrical impulses passes through a measuring device, which registers the largest value observed so far in the sequence. Assume that the impulse values at times $1, 2\ldots$ are described by independent random variables that are uniformly distributed on $\{1, 2, 3, 4, 5\}$. This means that if $X_1, X_2, \ldots$ are the registered values then

$$X_n = \max(Y_1, \ldots, Y_n), \quad n \geq 1.$$

   (a) Find the probability function for the random variable X_n.

   (b) Motivate that $\{X_n\}$ is a Markov chain.

   (c) Determine the transition probability matrix.

   (d) Check the result in (a) by computing the distribution of X_3 with the help of the initial distribution and the transition probability matrix.

8. A Markov chain has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

   Determine all stationary distributions of the chain.

9. The weather changes at a tourist resort from one day to the next can somewhat simplified be described as a Markov chain with the three states: E_1 : sun, E_2 : clouds, E_3 : rain. Using weather statistics of the area the following transition probability matrix has been estimated:

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.7 & 0 & 0.3 \end{pmatrix}$$

   A vacationer intends to visit the resort during 24-26 december. Under the assumption that there is still a lot of time before Christmas, derive the probability

   (a) that there will be three sunny days in a row;

   (b) of no rain at least during the first two days.

10. Assuming that the game in Exercise 1 has gone on for a long time, what are the probablities of possessing the ball for Per, Pål and Petter respectively?

11. A particle is placed uniformly at one of the nine points in a 3×3 square grid. The particle then performs a random walk such that at each step one of the adjacent points (to the right or left, upwards or downwards) is chosen with equal probabilities. This means that the particle never remains in a point or moves diagonally. Find the probability that the particle after three steps is at the central point.

12. A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

   Determine if the chain has a limiting distribution. If it does, derive this distribution.

13. A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Determine if the chain has an limiting distribution and if so, find it.

14. A Markov chain in discrete time with state space $S = (1, 2, 3, 4)$ is defined by the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.5 & 0.2 & 0 \\ 0.3 & 0.6 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, states 3 and 4 are absorbing. We wish to find the probability that the chain is absorbed in state 3 if it starts in state 1. Therefore, put

$$p_{3|j} = \text{probability of absorption in state 3, given } X_0 = j, \quad j = 1, 2,$$

and use the principle of conditioning on the first jump. In other words, derive two equations for the probabilities $p_{3|1}$ and $p_{3|2}$, then find these probabilities by solving the system of equations.

15. In a long DNA sequence, it has been recorded how often a certain base is followed by another. This is summarized in the matrix

$$\mathbf{P} = \begin{pmatrix} 0.300 & 0.205 & 0.285 & 0.210 \\ 0.322 & 0.298 & 0.078 & 0.302 \\ 0.248 & 0.246 & 0.298 & 0.208 \\ 0.177 & 0.239 & 0.292 & 0.292 \end{pmatrix},$$

where the rows and columns represent A, C, G, T in that order. For example, the number 0.078 in the second row gives the relative frequency with which a C is followed by a G. Note that the rows sum to 1.
A simplified model for the DNA sequence is to consider the matrix $\mathbf{P}$ as a transition model for a Markov chain with state space $S = (A, C, G, T)$.
For this model,

(a) find the limiting distribution; interpretation?,
(b) find the probability of observing an A followed by another A at a randomly chosen position in the sequence.

16. In certain parts of DNA so called CpG islands appear. In these, the letters C and G are more common than ususal. It has been suggested to model CpG islands using a Markovian dynamics with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.180 & 0.274 & 0.426 & 0.120 \\ 0.171 & 0.368 & 0.274 & 0.187 \\ 0.161 & 0.339 & 0.375 & 0.125 \\ 0.079 & 0.355 & 0.384 & 0.182 \end{pmatrix},$$

where the entries should be interpreted as in the previous exercise. Compute the limiting distribution.

17. Let (X_n) be a Markov chain on $\{0, 1\}$ generated by random iterations with the maps $f_1(x) = 0$ and $f_2(x) = 1 - x$ where the maps are chosen independently in each iteration-step with probabilities 0.3 and 0.7 resp. Find the transition probability matrix $\mathbf{P}$.

Continuous time

18. Suppose that $N(t)$ is a Poissonprocess with intensity λ, and put $p_n(t) = P(N(t) = n)$, $n \geq 0$. Verify that these probabilites solve the ordinary differential equations

$$\frac{d}{dt} p_n(t) = \lambda p_{n-1}(t) - \lambda p_n(t), \quad n \geq 1.$$

19. Consider the pure birth process $(X_t)_{t \geq 0}$ with $X_0 = 1$ and infinitesimal generator matrix given by

$$q_i = -\lambda i \qquad q_{i,i+1} = \lambda i, \quad i \geq 1,$$

where $\lambda > 0$ is a parameter. Motivate that the probabilities $p_{ik}(t) = P(X(t) = k | X(0) = i)$ satisfy, as $h \to 0$,

$$p_{1k}(t+h) = (1 - \lambda k h) p_{1k}(t) + \lambda(k-1) h p_{1,k-1}(t) + o(h).$$

Verify that these are the forward equations for the Markov chain. Show that the solution is given by $p_{1k}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}$.

20. For the birth-death process with state space $S = (1, 2, 3)$ and generator matrix

$$\mathbf{Q} = \begin{pmatrix} -\mu & \mu & 0 \\ 2\lambda & -(2\lambda + \mu) & \mu \\ 0 & 3\lambda & -3\lambda \end{pmatrix},$$

let

$$v_{ij} = \text{expected time to reach } j \text{ starting from } i, \quad 1 \leq i, j \leq 3.$$

Write down the system of equations satisfied by the quantities v_{ij}. If this chain with $\lambda = 1$ and $\mu = 10$ starts in state 1 what is the expected time to the first visit in state 3?

21. An automatic switch in a control system can be in one of the three states *on*, *off* or *stand-by*. When the switch is *on* it changes to *off* after an exponentially distributed time with expected value 2 seconds. If the switch is *off* it either turns *on*, with intensity 0.2 per second, or changes into *stand-by*, with intensity 0.4 per second. Finally, in state *stand-by* the switch turns to *on* after an exponential time with expected value 1 second. Draw a model graph of a Markov process that describes the system and indicate all transition intensities. In equiblibrium, what fraction of time is the switch in the three different states?

22. A tiger is always in one of the three states *sleeping*, *hunting* and *eating*. This tiger always goes from sleeping to hunting to eating and back to sleeping. On average, the tiger sleeps for 3 hours, hunts for 2 hours, and eats for 30 minutes. Assuming the tiger remains in each state for an exponential time, model tiger's life as a continuous time Markov chain. What are the stationary probabilities?

23. A birth-and-death process with states $0, 1, 2\ldots$ has birth intensities $\lambda_n = \lambda/\sqrt{n+1}$ and death intensities $\mu_n = \mu\sqrt{n}$, where λ and μ are positive parameters. Determine the asymptotic distribution.

24. The choice of birth and death intensities $\lambda_i = \lambda$, $i \geq 0$, and $\mu_i = \mu i$, $i \geq 1$, gives a continuous time birth-death process $\{X(t)\}$, which is known as the M/M/∞ service model. To interpret this Markov chain as a service model imagine Poisson events of intensity λ forming a stream of customer arrivals, and a service unit that provides service

for each customer immediately upon arrival. Each customer leaves after spending an exponential service time of mean $1/\mu$ in the system. Then the varying number of currently served customers in the system is given by the Markov process $\{X(t)\}$. Show that the process has a limit distribution for any values of the parameters λ and μ, and find the equilibrium distribution.

25. Let $\lambda_k = a^k$, $k \geq 0$, and $\mu_k = a^k$, $k \geq 1$, where a is a parameter, be the jump intensities for a birth-and-death process. For which parameters a does the process have a unique asymptotic distribution? For such a values, what is the equilibrium distribution?

26. A birth-death process has the jump intensities $\lambda_k = \lambda(N - k)$ and $\mu_k = \mu N k$. Here λ and μ are positive parameters, N is a positive integer, and k varies between 0 and N. Show that there is a stationary distribution, which is given by a particular standard distribution. What happens to the stationary distribution as $N \to \infty$?

27. At a factory there are two machines (A and B) that fail independently with intensity 2 (failures per day) each. There is one service man who only can repair one machine at the time. It takes an exponentially distributed time with mean 4 hours to repair a machine. These times are independent of each other and of the state of the other machine. Machine A is considered more important than B, so that when A fails, the service man always starts repairing it even if he already is working on B; in that case the work on B is resumed when the work on A is finished.

    (a) Describe the system with an appropriate Markov process.
    (b) Compute the asymptotic availability for the two machines.
    (c) Determine the proportion of time that the service man is busy with repairs.

28. A fisherman puts out his net in the evening and empties it the following morning. During the night, fishes are caught in the net according to a Poisson process with intensity λ (per hour). A caught fish escapes with intensity $\lambda/3$. Compute the probability that there are at least 4 fishes in the net when it is emptied in the morning. *(You can, just like the poor fishes, consider the night as very long.)*

Brownian motion

29. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/2t\}, \quad t \geq 0 \quad -\infty < x < \infty$$

its transition probability density. For $0 < t_1 < \cdots < t_n$,

$$f(x_1, \ldots, x_n) = p(t_1, x_1) p(t_2 - t_1, x_2 - x_1) \cdot \ldots \cdot p(t_n - t_{n-1}, x_n - x_{n-1})$$

is the joint transition probability density for $(B(t_1), \ldots, B(t_n))$ given that $B(0) = 0$. Why?

30. *Continuation* For $0 < t < 1$, what is the joint transition probability density for $B(t)$ and $B(1)$ given $B(0) = 0$?

31. *Continuation* Show that the conditional density function of $B(t)$, $0 < t < 1$, given that $B(0) = 0$ and $B(1) = 0$, is given by

$$\frac{1}{\sqrt{2\pi t(1-t)}} \exp\{-x^2/2t(1-t)\}, \quad -\infty < x < \infty.$$

32. Consider the Ornstein-Uhlenbeck process $U(t) = e^{-\beta t} B(e^{2\beta t}/2\beta)$, $t \geq 0$, where $\beta > 0$ is a parameter. What is the initial distribution of $U(0)$? Show that the infinitesimal drift of the process is given by

$$E(U(t+h) - U(t) | U(t) = u) = -\beta u h + o(h) \quad h \to 0.$$

Answers to exercises

Markov chain models

Discrete time

1 $S =$ (Per, Pål, Petter),

$$\mathbf{P} = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

2 $S = (123, 132, 213, 231, 312, 321)$ (the set of permutations of 3 elements), where ijk denotes the state when book i is on top, j is in the middle, and k is in the lowest position of the pile. Possible transitions: $ijk \to ijk$, $ijk \to jik$, $ijk \to kij$. Thus

$$\mathbf{P} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

3

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{pmatrix} 4/9 & 5/18 & 5/18 \\ 5/12 & 5/12 & 1/6 \\ 5/12 & 1/6 & 5/12 \end{pmatrix}$$

4

$$\boldsymbol{\mu}_2 = (31/72, 25/72, 2/9)$$

5 (a) 0.2
   (b) Day 62 rainy
   (c) 0.3
   (d) 0.14
   (e) 0.45
   (f) 0.55

6 (a) $\boldsymbol{\mu}_0 = (1, 0)$
   (b) $\boldsymbol{\mu}_1 = (0.7, 0.3)$, 0.7
   (c) 0.45
   (d) $(11/30, 19/30)$

7 (a)

$$P(X_n = k) = \frac{k^n - (k-1)^n}{5^n} \qquad k = 1, \ldots, 5$$

   (b) $X_n = \max(X_{n-1}, Y_n)$
   (c)

$$\mathbf{P} = \frac{1}{5} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

   (d) $\boldsymbol{\mu}_3 = \frac{1}{125}(1, 7, 19, 37, 61)$

8 $\pi_1 = \pi_2 = \pi_3 = 1/3$

9 (a) 1/5
(b) 28/45

10 $(0.352, 0.286, 0.361)$

11 4/27

12 $\pi = (1/3, 2/3, 0)$

13 $\pi = (0.3, 0.4, 0.3)$

14 $8/13 \approx 0.6154$, $6/13 \approx 0.4615$

15 (a) (0.262, 0.246, 0.239, 0.253),   (b) 0.079.

16 $(0.155, 0.341, 0.350, 0.154)$.

17 $\mathbf{P} = \begin{pmatrix} 0.3 & 0.7 \\ 1 & 0 \end{pmatrix}$

Continuous time

20 $\lambda = 1$ and $\mu = 10 \Rightarrow \nu_{13} = 0.22$.

21 $(6/13, 5/13, 2/13)$

22 $(6/11, 4/11, 1/11)$

23 $\mathrm{Po}(\lambda/\mu)$

24 $\mathrm{Po}(\lambda/\mu)$

25 $a > 1$, $\mathrm{Ge}(1 - 1/a)$

26 $\mathrm{Bin}(N, \frac{\lambda}{\mu N + \lambda}) \to \mathrm{Po}(\lambda/\mu)$ as $N \to \infty$.

27 (a) Introduce 4 states: $\{0, A, B, 2\}$ that show how many (and which) machines are non-working. Intensity matrix:

$$\mathbf{Q} = \begin{pmatrix} -4 & 2 & 2 & 0 \\ 6 & -8 & 0 & 2 \\ 6 & 0 & -8 & 2 \\ 0 & 0 & 6 & -6 \end{pmatrix}$$

(b) A: 3/4, B: $45/68 \approx 0.662$.
(c) $8/17 \approx 0.471$.

28 $1 - 13e^{-3} \approx 0.353$

Brownian motion

29 Using the properties of independent increments and stationarity

$$\begin{aligned}
&P(B(t_1) = x_1, \ldots, B(t_n) = x_n)\, dx_1 \ldots dx_n \\
&\quad = P(B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_{n-1})\, dx_1 \ldots dx_n \\
&\quad = P(B(t_1) = x_1)\, P(B(t_2) - B(t_1) = x_2 - x_1) \ldots P(B(t_n) - B(t_{n-1}) = x_n - x_{n-1})\, dx_1 \ldots dx_n \\
&\quad = P(B(t_1) = x_1)\, P(B(t_2 - t_1) = x_2 - x_1) \ldots P(B(t_n - t_{n-1}) = x_n - x_{n-1})\, dx_1 \ldots dx_n
\end{aligned}$$

30 Use 29 with $n = 2$, $t_1 = t$, and $t_2 = 1$.

31 Use 30 and form the required conditional density.

32 We have

$$
\begin{aligned}
&E[e^{-\beta(t+h)}B(e^{2\beta(t+h)}/2\beta) - e^{-\beta t}B(e^{2\beta t}/2\beta) \mid e^{-\beta t}B(e^{2\beta t}/2\beta) = u] \\
&\quad = e^{-\beta(t+h)}E[B(e^{2\beta(t+h)}/2\beta) - B(e^{2\beta t}/2\beta) \mid e^{-\beta t}B(e^{2\beta t}/2\beta) = u] \\
&\qquad +(e^{-\beta(t+h)} - e^{-\beta t})ue^{\beta t}.
\end{aligned}
$$

Since $E(B(s+h) - B(s)|B(s) = x) = 0$, it follows that

$$E(U(t+h) - U(t)|U(t) = u) = (e^{-\beta h} - 1)u = -\beta hu + o(h).$$