Exam, Real analysis, 1MA226, 2019-01-14

Solutions.

1. Assume that X is finite, say $X = \{x_1, \ldots, x_n\}$. Let $\mathcal{U}$ be an open cover of X. Then for each $j \in \{1, \ldots, n\}$ there is at least one set in $\mathcal{U}$ which contains x_j; choose one such set and call it U_j. Now $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$. We have thus proved that every open cover of X has a finite subcover; therefore X is compact.

Conversely, assume that (X, d) is a discrete metric space which is compact. Note that the fact that (X, d) is discrete implies that *every* subset of X is open. Hence the family $\mathcal{U} = \{\{x\} \, : \, x \in X\}$ is an open cover of X. Now since (X, d) is compact, there exists a finite subcover of $\mathcal{U}$; in other words there exists a finite subset $F \subset X$ such that $X = \cup_{x \in F}\{x\}$. But the last relation implies $X = F$; hence X is finite. $\square$

2. (a). For n *even* we have $x_n = e^n \to +\infty$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty} x_n = +\infty.$$

For n *odd* we have $x_n = e^{-n} \to 0$ as $n \to \infty$. Note also that $x_n > 0$ for *all* n; hence there does not exist any subsequence of (x_n) which converges to a number < 0. Therefore,

$$\liminf_{n\to\infty} x_n = 0.$$

(b). For n *even* we have, using $\sqrt[n]{n} \geq 1$:

$$x_n \geq 0 + \log n + (-1) \to +\infty \quad \text{as } n \to \infty.$$

Hence

$$\limsup_{n\to\infty} x_n = +\infty.$$

For n *odd* the situation is more delicate. For all $n \geq 1$ we have $0 \leq n^{-1} \log n < 1$, and therefore by Taylor expansion,

$$\sqrt[n]{n} = e^{n^{-1}\log n} = 1 + \frac{\log n}{n} + O\Big(\Big(\frac{\log n}{n}\Big)^2\Big),$$

and so

$$n(\sqrt[n]{n} - 1) = \log n + O\Big(\frac{(\log n)^2}{n}\Big).$$

It follows that for *odd* n:

$$\begin{aligned} x_n = -\Big(\log n + O\Big(\frac{(\log n)^2}{n}\Big)\Big) + \log n + \sin\tfrac{2\pi n}{3} \\ = O\Big(\frac{(\log n)^2}{n}\Big) + \sin\tfrac{2\pi n}{3}. \end{aligned}$$

Here the first term, "$O(\frac{(\log n)^2}{n})$", tends to 0 as $n \to \infty$. On the other hand the second term, $\sin\frac{2\pi n}{3}$, is periodic with period 3, taking the values $\frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, \cdots$ as n runs through $1, 3, 5, \cdots$. Hence:

$$\liminf_{n\to\infty} x_n = -\frac{\sqrt{3}}{2}.$$

$\square$

3. Let us note that the series defining $F(x)$ is uniformly convergent on any interval $[a,b]$ with $1<a<b$. Indeed, $|n^{-x}\cos n\pi x|\leq n^{-a}$ for all $n\in\mathbb{Z}^+$ and $x\in[a,b]$, and it is known that the series $\sum_{n=1}^{\infty} n^{-a}$ is convergent for $a>1$; hence by Weierstrass' M-test, the series defining $F(x)$ is indeed uniformly convergent on $[a,b]$. Hence it follows that F is well-defined and continuous in the interval $[a,b]$. Since this is true for any $1<a<b$, it follows that F is well-defined and continuous in the whole interval $(1,\infty)$.

Next assume $2<a<b$. We claim that the series

$$-\sum_{n=1}^{\infty}\Big((\log n)n^{-x}\cos n\pi x+n^{-x}(n\pi)\sin n\pi x\Big) \tag{1}$$

is uniformly convergent on $[a,b]$. Note that for all $x\in[a,b]$ and $n\in\mathbb{Z}^+$, using $0\leq\log n<n$ we have:

$$\Big|(\log n)n^{-x}\cos n\pi x+n^{-x}(n\pi)\sin n\pi x\Big|\leq(1+\pi)n^{1-a},$$

and $\sum_{n=1}^{\infty} n^{1-a}$ is convergent since $a>2$. Hence by Weierstrass' M-test, the series in (1) is indeed uniformly convergent on $[a,b]$. Hence by Rudin's Thm. 7.17, we have that $F'(x)$ exists for all $x\in[a,b]$, and

$$F'(x)=-\sum_{n=1}^{\infty}\Big((\log n)n^{-x}\cos n\pi x+n^{-x}(n\pi)\sin n\pi x\Big).$$

The uniform convergence pointed out above shows that this function is continuous in $[a,b]$. Hence F is C^1 in $[a,b]$. Since this is true for any $2<a<b$, we conclude that F is C^1 in the whole interval $(2,\infty)$. $\square$

4. For any $m \in \mathbb{Z}^+$ and any $\varepsilon \in (0, \frac{1}{2} \cdot 4^{-m})$, we let $P_{m,\varepsilon}$ be the partition of $[0,1]$ determined by the following numbers:

$$0 < \tfrac{1}{2} \cdot 4^{-m} - \varepsilon < 4^{-m} + \varepsilon < \tfrac{1}{2} \cdot 4^{-m+1} - \varepsilon < 4^{-m+1} + \varepsilon < \cdots < \tfrac{1}{2} \cdot 4^{0} - \varepsilon < 4^{0} = 1.$$

For this partition we find that:

$$\begin{aligned} U(P_{m,\varepsilon}, f) &= \sum_i M_i \Delta x_i \\ &= \left(\tfrac{1}{2} \cdot 4^{-m} - \varepsilon\right) + \left(\sum_{k=1}^{m} \left(\tfrac{1}{2} \cdot 4^{-k} + 2\varepsilon\right)\right) + \left(\tfrac{1}{2} \cdot 4^{0} + \varepsilon\right) \\ &= \tfrac{1}{2} \cdot 4^{-m} + \tfrac{1}{2} \cdot \frac{1 - 4^{-m-1}}{1 - 4^{-1}} + 2m\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ (for any fixed m) and then letting $m \to \infty$, we conclude that

$$\overline{\int_0^1} f(x)\, dx \le \inf\Big\{ U(P_{m,\varepsilon}, f) \ :\ m \in \mathbb{Z}^+,\ \varepsilon \in (0, \tfrac{1}{2} \cdot 4^{-m}) \Big\} \le \frac{1}{2} \cdot \frac{1}{3/4} = \frac{2}{3}.$$

On the other hand, for any $m \in \mathbb{Z}^+$, let $\widetilde{P}_m$ be the partition of $[0,1]$ determined by the following numbers:

$$0 < \tfrac{1}{2} \cdot 4^{-m} < 4^{-m} < \tfrac{1}{2} \cdot 4^{-m+1} < 4^{-m+1} < \cdots < \tfrac{1}{2} \cdot 4^{0} < 4^{0} = 1.$$

For this partition we find that:

$$\begin{aligned} L(\widetilde{P}_m, f) &= \sum_i m_i \Delta x_i \\ &= 0 + \left(\sum_{k=0}^{m} \tfrac{1}{2} \cdot 4^{-k}\right) \\ &= \tfrac{1}{2} \cdot \frac{1 - 4^{-m-1}}{1 - 4^{-1}}. \end{aligned}$$

Letting $m \to \infty$, we conclude that

$$\underline{\int_0^1} f(x)\, dx \ge \sup\Big\{ U(\widetilde{P}_m, f) \ :\ m \in \mathbb{Z}^+ \Big\} \ge \frac{1}{2} \cdot \frac{1}{3/4} = \frac{2}{3}.$$

Hence:

$$\frac{2}{3} \le \underline{\int_0^1} f(x)\, dx \le \overline{\int_0^1} f(x)\, dx \le \frac{2}{3}, \quad \text{i.e. } \underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x)\, dx = \frac{2}{3}.$$

This proves that f is Riemann integrable on $[0,1]$, and that $\int_0^1 f(x)\, dx = \frac{2}{3}$. $\square$

5. **NOTE: This problem lies outside the syllabus of the course, since the Stone-Weierstrass Theorem is no longer part of the syllabus (since 2019).**

Substituting $x = -\log u$ gives, for each $n \in \{0, 1, 2, \ldots\}$:
$$0 = \int_0^1 f(x)e^{-nx}\,dx = \int_{e^{-1}}^1 \frac{f(-\log u)}{u}\,u^n\,du.$$
Let us write
$$g(u) := \frac{f(-\log u)}{u} \qquad (u \in [e^{-1}, 1]).$$
Then g is a continuous function from $[e^{-1}, 1]$ to $\mathbb{R}$ and
$$\int_{e^{-1}}^1 g(u)\,u^n\,du = 0, \qquad \forall n \in \{0, 1, 2, \ldots\}.$$
By linearity, this implies that
$$\int_{e^{-1}}^1 g(u)P(u)\,du = 0$$
for every *polynomial* $P(u)$. By the Stone-Weierstrass Theorem, there is a sequence (P_n) of polynomials which tend to g uniformly on $[e^{-1}, 1]$. Using this sequence we get:

$$\int_{e^{-1}}^1 g(u)^2\,du = \lim_{n\to\infty} \int_{e^{-1}}^1 g(u)P_n(u)\,du = \lim_{n\to\infty} 0 = 0. \tag{2}$$

Now assume that there is some $x \in [0, 1]$ such that $f(x) \neq 0$. Set $u_0 = e^{-x} \in [e^{-1}, 1]$; then $g(u_0) \neq 0$ and therefore $g(u_0)^2 > 0$. Hence, since the function $u \mapsto g(u)^2$ is continuous, there exists some $\delta_1, \delta_2 > 0$ such that
$$g(u)^2 > \delta_1, \qquad \forall u \in I := [e^{-1}, 1] \cap [u_0 - \delta_2, u_0 + \delta_2].$$
If $\delta_2 > 1 - e^{-1}$ then we may shrink δ_2 so that $\delta_2 = 1 - e^{-1}$. Then the above interval I has length $\geq \delta_2$. We also have, of course, $g(u)^2 \geq 0$ for *all* $u \in [e^{-1}, 1]$. Hence:
$$\int_{e^{-1}}^1 g(u)^2\,du \geq 0 + \int_I \delta_1\,du \geq \delta_2\delta_1 > 0.$$
This is a contradiction against (2)! Hence the assumption that there exists some $x \in [0, 1]$ such that $f(x) \neq 0$ must be *false*. We have thus proved that $f(x) = 0$ for all $x \in [0, 1]$. $\square$

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$F(u,v) = (u+v, e^u + e^v).$$

Note that F is C^1. We compute:

$$[F'(u,v)] = \begin{pmatrix} 1 & 1 \\ e^u & e^v \end{pmatrix}.$$

In particular

$$[F'(0,1)] = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix},$$

which is non-singular. Hence by the *Inverse Function Theorem* there exists an open set $V \subset \mathbb{R}^2$ which contains the point $(0,1)$, such that $F|_V$ is C^1, $U := F(V)$ is open, and $G := (F|_V)^{-1} : U \to V$ is C^1.

By the definition of $G = (F|_V)^{-1}$ we have $F(G(x,y)) = (x,y)$ for all $(x,y) \in U$. In other words:

$$\begin{cases} G_1(x,y) + G_2(x,y) = x \\ e^{G_1(x,y)} + e^{G_2(x,y)} = y, \end{cases} \qquad \forall (x,y) \in U.$$

Also $G(F(0,1)) = (0,1)$, i.e. $G(1,e+1) = (0,1)$. This means that if we write $u = G_1 : U \to \mathbb{R}$ and $v = G_2 : U \to \mathbb{R}$ then the functions u and v have all the properties required in the problem formulation!

By the chain rule we also have $F'(G(x,y)) \cdot G'(x,y) = I$ for all $(x,y) \in U$; thus in particular $F'(0,1) \cdot G'(1,e+1) = I$, or in other words:

$$[G'(1,e+1)] = [F'(0,1)]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix}^{-1} = \frac{1}{e-1}\begin{pmatrix} e & -1 \\ -1 & 1 \end{pmatrix}.$$

But we also know

$$[G'] = \begin{pmatrix} D_1G_1 & D_2G_1 \\ D_1G_2 & D_2G_2 \end{pmatrix} = \begin{pmatrix} D_1u & D_2u \\ D_1v & D_2v \end{pmatrix}.$$

Hence:

$$[u'(1,e+1)] = \frac{1}{e-1}\big(e, -1\big) \quad \text{and} \quad [v'(1,e+1)] = \frac{1}{e-1}\big(-1, 1\big).$$

$\square$

7. (This is very similar to Problem 6.2 in the lecture notes.)
Consider the map $\phi : \ell^\infty \to \ell^\infty$ given by

$$\phi((x_n)) = (y_n) \quad \text{with } \ y_n = \frac{1}{n^2} + \sum_{m=1}^{\infty} \frac{x_m}{n + 2^m}.$$

Note that ϕ indeed maps ℓ^∞ to ℓ^∞, since for any $(x_n) \in \ell^\infty$, if $\phi((x_n)) = (y_n)$ then for each n,

$$|y_n| \leq \frac{1}{n^2} + \sum_{m=1}^{\infty} \frac{|x_m|}{n + 2^m} \leq 1 + (\sup_m |x_m|) \sum_{m=1}^{\infty} \frac{1}{2^m} = 1 + \sup_m |x_m|,$$

and thus $\sup_n |y_n| \leq 1 + \sup_m |x_m|$ and in particular $(y_n) \in \ell^\infty$.

Note also that a sequence $(x_n) \in \ell^\infty$ satisfies the equation given in the problem iff (x_n) is a fixed point of ϕ. Hence if we prove that ϕ is a contraction, then the desired statement follows from Theorem 9.23 (the contraction principle, which can be applied since ℓ^∞ is complete).

To prove that ϕ is a contraction, let (x_n) and (x'_n) be arbitrary points in ℓ^∞, and set $(y_n) = \phi((x_n))$ and $(y'_n) = \phi((x'_n))$. Then

$$\begin{aligned} d((y_n), (y'_n)) &= \sup_n |y_n - y'_n| \\ &= \sup_n \left| \sum_{m=1}^{\infty} \Big(\frac{x_m}{n + 2^m} - \frac{x'_m}{n + 2^m} \Big) \right| \\ &\leq \sup_n \sum_{m=1}^{\infty} \frac{|x_m - x'_m|}{n + 2^m} \\ &\leq c \cdot d((x_n), (x'_n)), \end{aligned}$$

where

$$c := \sum_{m=1}^{\infty} \frac{1}{1 + 2^m} < \frac{1}{3} + \sum_{m=2}^{\infty} \frac{1}{2^m} = \frac{5}{6} < 1.$$

Hence ϕ is indeed a contraction of ℓ^∞. $\square$

8. Given any vector $h \in \mathbb{R}^2 \setminus \{0\}$, by the Mean Value Theorem applied to the function

$$g(t) := f(th)$$

(which is continuous on $[0,1]$ and differentiable on $(0,1)$), it follows that

$$\exists t \in (0,1): \quad f(h) - f(0) = g(1) - g(0) = g'(t) = f'(th)h.$$

(The last equality holds by the chain rule; note that $f'(th) \in L(\mathbb{R}^2, \mathbb{R})$ and $h \in \mathbb{R}^2$.)

Now let $\varepsilon > 0$ be given. Take $\delta > 0$ so small that

$$\|f'(x) - A\| < \varepsilon \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\} \text{ with } |x| < \delta.$$

Now for any fixed $h \in \mathbb{R}^2 \setminus \{0\}$ with $|h| < \delta$, by the above result there exists some $t \in (0,1)$ such that $f(h) - f(0) = f'(th)h$. Note also that $|th| < |h| < \delta$; hence $\|f'(th) - A\| < \varepsilon$, and thus

$$|f'(th)h - Ah| = \left|(f'(th) - A)h\right| \le \|f'(th) - A\| \cdot |h| < \varepsilon|h|.$$

Hence we have proved:

$$|f(h) - f(0) - Ah| = |f'(th)h - Ah| < \varepsilon|h|,$$

for all $h \in \mathbb{R}^2 \setminus \{0\}$ with $|h| < \delta$. Since such a δ can be found for any given $\varepsilon > 0$, we conclude that

$$\frac{|f(h) - f(0) - Ah|}{|h|} \to 0 \quad \text{as } h \to 0 \text{ in } \mathbb{R}^2.$$

Hence the differential $f'(0)$ exists, and equals A. $\square$