Problem Session 1

Probability and Martingales, 1MS045

16 September 2024

Note: If not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

1. Show that the two operations Δ (symmetric difference), defined by $A\Delta B = (A\setminus B)\cup(B\setminus A)$, and $\cap$ (intersection) satisfy the following properties:
   (a) Δ and $\cap$ are both commutative and associative.
   (b) They satisfy the distributive law: $(A\Delta B)\cap C = (A\cap C)\Delta(B\cap C)$.

2. Which of the following statements are true for all possible sequences A_n, B_n of sets?
   (a) $\limsup_{n\to\infty}(A_n \cap B_n) = (\limsup_{n\to\infty} A_n) \cap (\limsup_{n\to\infty} B_n)$
   (b) $\limsup_{n\to\infty}(A_n \cup B_n) = (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n)$
   (c) $\liminf_{n\to\infty}(A_n \cap B_n) = (\liminf_{n\to\infty} A_n) \cap (\liminf_{n\to\infty} B_n)$
   (d) $\liminf_{n\to\infty}(A_n \cup B_n) = (\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n)$

3. Prove: if $f : S \to \mathbb{R}$ is a measurable function on some measure space S with σ-algebra Σ, then so is $|f|$. Show by means of a counterexample that the converse is not necessarily true.

4. Let $\{A_n, n \geq 1\}$ be a sequence of events in a probability space.
   (a) Suppose that $\lim_{n\to\infty} P(A_n) = 1$. Prove that there exists an increasing subsequence $\{n_k, k \geq 1\}$ such that
   $$P\left(\bigcap_{k\geq 1} A_{n_k}\right) > 0.$$
   (b) Give an example of a sequence of events (in a probability space of your choice) with $P(A_n) \geq \frac{1}{2}$ for all $n \geq 1$ for which there is no such subsequence.

5. Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Prove the following inequalities:
   (a) $P\left(\bigcup_{k=1}^n A_k\right) \geq \sum_{k=1}^n P(A_k) - \sum_{1\leq j<k\leq n} P(A_j \cap A_k)$.
   (b) $P\left(\bigcup_{k=1}^n A_k\right) \leq \sum_{k=1}^n P(A_k) - \sum_{1\leq j<k\leq n} P(A_j \cap A_k) + \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k)$.

6. Let X be a random variable. Show: for every $\epsilon > 0$, there exists a bounded random variable X_ϵ (i.e., there exists a constant M such that $|X_\epsilon| \leq M$ holds almost surely) such that $P(X \neq X_\epsilon) < \epsilon$.

7. Suppose that X is an integer-valued random variable, and let m be a positive integer. Prove that

$$\sum_{n=-\infty}^{\infty} P(n < X \leq n+m) = m.$$

8. Prove: if $\{X_n, n \geq 1\}$ is a sequence of independent random variables, then the following two statements are equivalent:

   - $P\left(\sup_{n\geq 1} X_n < \infty\right) = 1$,
   - there exists an $a > 0$ such that $\sum_{n=1}^{\infty} P(X_n > a) < \infty$.

9. Let $\{A_n, n \geq 1\}$ be a sequence of independent events in a probability space, and suppose that $P(A_n) < 1$ for all n. Prove that the following two statements are equivalent:

   - $P(A_n$ occurs for at least one $n) = 1$,
   - $P(A_n$ occurs for infinitely many $n) = 1$.

   Why is $P(A_n) = 1$ forbidden?

10. Let $X_1, X_2, \ldots$ be independent random variables, where X_n follows a uniform distribution on the interval $\left[0, \frac{1}{n}\right]$ (equivalently, $X_n = \frac{Y_n}{n}$, where Y_n follows a uniform distribution on $[0,1]$). Prove that $X = \sup_n X_n$ has the distribution function

$$F(x) = \lfloor \frac{1}{x} \rfloor ! \cdot x^{\lfloor \frac{1}{x} \rfloor}, \quad 0 < x \leq 1.$$

   (For $x \leq 0$, $F(x) = 0$, and for $x > 1$, $F(x) = 1$.) Here, $\lfloor a \rfloor$ is the greatest integer $\leq a$.