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Suggested Solutions
1MA255, VT24
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Partial Differential Equations with Applications to Finance

Instructions: There are five problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no other materials are allowed. In the problems 4 and 5, you do not need to provide a proof to the respective verification theorems. **Good luck!**

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1. (8p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x)|t>0, \quad x>0\}$ with $u(0,x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t,0) = 0$ for $t > 0$.

(a) (3p) Construct a suitable extension of the initial condition to the whole space.

(b) (5p) Show that

$$u(t,x) = \int_0^\infty u_0(y) h(t,x,y)\,\mathrm{d}y$$

for some function $h(t,x,y)$. Find the function h. *Note: h is not simply the fundamental solution!*

Solution. *(a) A suitable extension would be to consider a function*

$$v(t,x) = \begin{cases} u(t,x), & x > 0, \\ u(t,-x), & x < 0. \end{cases}$$

Then, clearly

$$\frac{\partial v}{\partial x}(t,0) = 0, \qquad \textit{and} \qquad v(0,x) = u(0,x) = u_0(x).$$

Therefore v solves for

$$\begin{cases} v_t - v_{xx} = 0, & \textit{in } (0,\infty) \times (0,\infty, \\ v(0,x) = u_0(x), & x > 0, \\ v_x(t,0) = 0, & t > 0. \end{cases}$$

(b) Let

$$v_0(x) = \begin{cases} u_0(x), & x > 0, \\ u_0(-x), & x < 0, \end{cases}$$

be a continuous and bounded function. Then, based on our knowledge of the Cauhcy IVP problem, v is the convolution

$$v(t,x) = \int_{\mathbb{R}} g(t,x-y) v_0(y)\,\mathrm{d}y,$$

where g is the fundamental solution to the Cauchy IVP. Now,

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}} g(t, x-y) v_0(y)\, \mathrm{d}y \\ &= \int_0^\infty g(t,x-y)u_0(y)\,\mathrm{d}y + \int_{-\infty}^0 g(t,x-y)u_0(-y)\,\mathrm{d}y \\ &= \int_0^\infty g(t,x-y)u_0(y)\,\mathrm{d}y - \int_\infty^0 g(t,x+z)u_0(z)\,\mathrm{d}y \\ &= \int_0^\infty (g(t,x-y)+g(t,x+y))\, u_0(x)\,\mathrm{d}y. \end{aligned}$$

The desired result follows by denoting $h(t,x,y) := g(t,x-y)+g(t,x+y)$.

2. (8p) Let D denote a bounded interval $(a,b) \in \mathbb{R}$ with $a<b$ and let

$$\mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma W_t$$

with $X_0 = x \in D$ for $\mu, \sigma \neq 0$, and W_t being the standard Brownian motion. Calculate the expectation

$$\mathbb{E}_x\left[X^p_{\tau_D} + \tau_D\right],$$

where $p>0$ is a constant and $\tau_D = \inf\{t>0 | X_t \notin D\}$ is the so-called first exit time from D.

Solution. *Let $u(x) = \mathbb{E}_x\left[X^p_\tau + \tau\right]$. Then u solves for*

$$\begin{cases} \frac{1}{2}\sigma^2 u_{xx} + \mu u_x + 1 = 0 & \textit{in } D, \\ u(a) = a^p, \\ u(b) = b^p. \end{cases}$$

General solution of such PDE is given by

$$u(x) = C_1 e^{\frac{-2\mu x}{\sigma^2}} - 2x + C_2$$

for some constants C_1, C_2. Utilizing the boundary conditions at a and b, we arrive with

$$C_1 e^{\frac{-2\mu a}{\sigma^2}} - 2a + C_2 = a^p, C_1 e^{\frac{-2\mu b}{\sigma^2}} - 2b + C_2 = b^p.$$

Solving this pair of equations for C_1 and C_2 we arrive with

$$C_1 = \frac{2(a+b)+b^p+a^p}{e^{\frac{-2\mu a}{\sigma^2}} + e^{\frac{-2\mu b}{\sigma^2}}}, \quad C_2 = \frac{e^{\frac{-2\mu b}{\sigma^2}}}{e^{\frac{-2\mu a}{\sigma^2}} + e^{\frac{-2\mu b}{\sigma^2}}}\left(2(a+b)+b^p+a^p\right) - 2b - b^p.$$

Finally, we conclude by noting that by F-K

$$\mathbb{E}_x\left[X^p_\tau + \tau\right] = C_1 e^{\frac{-2\mu x}{\sigma^2}} - 2x + C_2$$

with C_1, C_2 as above. (Note: this is slightly a not-successful exam question: either I should've given the general solution formula given, or to have a driftless BM as the driver.)

3. (8p) Consider the Itô diffusion X with

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$$

for some drift and volatility coefficients μ, σ, and let $u(t,x,y) := \mathbb{P}_x(X_t \leq y)$ with $X_0 = x$ as the initial point.

(a) (6p) Use the Kolmogorov Forward Equation (Fokker-Planck) to derive a PDE satisfied by u in terms of the forward variables (t, y), and propose a suitable initial condition.

(b) (2p) Find a process X_t, for which

$$u(t, x, y) = 1 - u(t, y, x)$$

for all t, x, y.

Solution. *(a) Let $f(t, x, z)$ be the density function starting from $(0, x)$ at (t, z). Then $\mathbb{P}_x(X_t \le y) = \int_{-\infty}^{y} f(t, x, z)\,\mathrm{d}z$. Moreover, the Fokker-Planck states that f solves*

$$\begin{cases} f_t - \mathcal{L}^* f = 0, \\ f(t, x, z) = \delta(x - z) \text{ as } t \to \infty. \end{cases}$$

From this it directly follows that

$$f_t = (\mu(z) f)_z - \frac{1}{2}(\sigma^2 f)_{zz} = 0.$$

From the definition we used at the beginning for u, differentiating with respect to t yields

$$u_t(t, x, y) = -\mu(y) f(t, x, y) + \frac{1}{2}\left(\sigma^2(y) f(t, x, y)\right)_y.$$

Moreover, by noting that (fundamental theorem of calculus)

$$f(t, x, y) = u_y(t, x, y)$$

we get the result

$$\begin{cases} u_t = -\mu(y) u_y + \frac{1}{2}\left(\sigma^2(y) u_y\right)_y, \\ u(0, x, y) = \mathbb{1}_{x \le y}. \end{cases}$$

(b) The desired property holds, for example, for a traditional Brownian motion. Consider

$$\mathrm{d}X_t = \sigma\,\mathrm{d}W_t.$$

Then for some $Z \sim \mathrm{N}(0, 1)$ we have

$$\begin{aligned} u(t, x, y) = \mathbb{P}_x(X_t \le y) &= \mathbb{P}(x + \sigma W_t \le y) \\ &= \mathbb{P}\left(Z \ge \frac{y - x}{\sigma\sqrt{t}}\right) \\ &= \mathbb{P}\left(Z \le \frac{x - y}{\sigma\sqrt{t}}\right) \\ &= \mathbb{P}_y(X_t \le y) = 1 - u(t, y, x). \end{aligned}$$

4. (8p) Consider the Merton's asset allocation problem with one risky asset

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$$

and a risk-free rate $r = 0$. Here μ, σ are constants and W_t the standard Brownian motion. Let further X_t^u denote the wealth process where u_t is the amount of money invested in the risky asset at time t. Solve the $Merton's$ problem

$$v(t, x) = \sup_u \mathbb{E}_{t,x}\left[\Phi(X_T^u)\right]$$

for some termination time T, where $\Phi(x) = 1 - e^{-\gamma x}$ for some $\gamma > 0$. *Note: An ansatz $v(t, x) = 1 - f(t) e^{-\gamma x}$ ought to be useful for some arbitrary f which will be solved further using the terminal conditions.*

Solution. *We begin by noting that since $r = 0$, we have for the risk-free asset B_t that* $\mathrm{d}B_t = 0$.. *Then, we can describe the dynamics of X_t, namely*

$$\mathrm{d}X_t = \frac{u_t}{S_t}\,\mathrm{d}S_t + \frac{(X_t - u_t)}{B_t}\,\mathrm{d}B_t = \mu u_t\,\mathrm{d}t + \sigma u_t\,\mathrm{d}W_t.$$

The candidate value function $\hat{V}$ then solves

$$\begin{cases} \hat{V}_t + \sup_u\{\mathcal{L}^u\hat{V}\} = 0, \\ \hat{V}(T,x) = \Phi(x). \end{cases}$$

The generator for our process can be calculated as

$$\mathcal{L}^u\hat{V} = \mu u\hat{V}_x + \frac{1}{2}\sigma^2 u^2\hat{V}_{xx}.$$

Moreover, for an Ansatz $\hat{V}(t,x) = 1 - f(t)e^{-\gamma x}$, we have

$$\hat{V}_t = -f_t e^{-\gamma x}, \quad \hat{V}_x = \gamma f e^{-\gamma x}, \quad \hat{V}_{xx} = -\gamma^2 f e^{-\gamma x}.$$

Plugging these in to the generator and denoting $k = \mu u\gamma - \frac{1}{2}\sigma^2 u^2\gamma^2$ yields

$$\mathcal{L}^u\hat{V} = kfe^{-\gamma x}.$$

We note that in finding the maximum of $\mathcal{L}\hat{V}$, the first order condition coincidences with

$$k = 0 \Leftrightarrow u^* := u = \frac{\mu}{\gamma\sigma^2}.$$

Plugging this in, we denote $k^* := k(u^*) = \frac{1}{2}\frac{\mu^2}{\sigma^2}$. The objective equation now becomes

$$-f_t e^{-\gamma x} + k^* f e^{-\gamma x} = 0.$$

This ODE has a general solution of

$$f(t) = Ce^{k^* t}$$

for some constant C. Recalling the desired terminal condition, we get

$$1 - f(T)e^{-\gamma x} = 1 - e^{-\gamma x} \Leftrightarrow f(T) = 1.$$

That is, $f(t) = \mathrm{e}^{-k^*(T-t)}$ and

$$\hat{V}(t,x) = 1 - \mathrm{e}^{-k^*(T-t)}e^{-\gamma x}.$$

By the Verification theorem, now $\hat{V} \equiv V$ and u^* is the optimal strategy. (Note also that for a well-defined solution, we need $V_x \geq 0$, i.e. $f(t) \geq 0$.)

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_{0,x}[e^{-r\tau}X_\tau^+],$$

where $\mathrm{d}X_t = \mu\,\mathrm{d}t + \mathrm{d}W_t$ with $X_0 = x$; $X^+ := \max(X,0)$, W_t is the standard Brownian motion, and $r > 0$.

Solution. *For some boundary b, we expect a continuous region $C = (-\infty, b)$ and the stopping region $D = \mathbb{R} \setminus C$. By dynamic programming, the candidate value function $\hat{V}$ should then solve*

$$\begin{cases} \mu \hat{V}_x + \frac{1}{2}\hat{V}_{xx} - r\hat{V} = 0, \quad x < b, \\ \hat{V}(x) = x, \quad x \geq b, \\ \hat{V}_x(x) = 1, \quad x \geq b, \\ \lim_{x\to\infty} \hat{V}(x) = 0. \end{cases}$$

We use an Ansatz $\hat{V} = e^{\gamma x}$, with which the above PDE reduces to

$$\mu\gamma\hat{V} + \frac{1}{2}\gamma^2\hat{V} - r\hat{V} = 0 \Rightarrow \frac{1}{2}\gamma^2 + \mu\gamma - r = 0.$$

Denote $\gamma_1 > 0$ the positive solution of this quadratic equation and, respectively $\gamma_2 < 0$ the negative solution. Then, our ODE has a general solution of the form

$$\hat{V}(x) = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x}.$$

The last boundary condition implies that $C_2 = 0$ (the term would explode otherwise). Moreover, the second boundary condition implies that

$$V(b) = b,$$

from which we find $C_1 = \frac{b}{e^{\gamma_1 b}}$. The smooth fit condition, on the other hand, gives us $b = \frac{1}{\gamma_1}$. To put the argument together, we now have a candidate solution

$$\hat{V}(x) = \begin{cases} \frac{b}{e^{\gamma_1 b}} e^{\gamma_1 x}, \quad x < b, \\ x, \quad x \geq b, \end{cases}$$

and by the Verification theorem we have $\hat{V} \equiv V$ with b being the optimal stopping time.