Cont. of F7)

By the isometry,

$$\|I(H_n) - I(H_m)\|_{L^2} = \|I(H_n - H_m)\|_{L^2} = \|H_n - H_m\|_{L^2_T},$$

so $\{I(H_n)\}_{n\geq 1}$ is a Cauchy-sequence in L^2. By the completeness of L^2 (as a Banach space) there exists a limit element in L^2.

Denote this limit $I(\Phi)$

$$I(\Phi) := \int_0^T \Phi_s \, dB_s := \lim_{n\to\infty} \int_0^T H_n(s)\, dB_s$$

This is Theorem 4.2.5

Next, which random processes Φ can be approximated by simple ones?

Answer: Prop 4.2.6.

$$\Phi \in \mathcal{H} = \{\Phi : [0,T]\times\Omega \to \mathbb{R},$$
measurable
adapted
$$E[\int_0^T \Phi_s^2 \, ds] < \infty \}$$

$$I_t(\Phi) := \int_0^t \Phi_s \, dB_s := \int_0^T \Phi_s 1_{\{0 \le s \le t\}} \, dB_s, \quad 0 \le t \le T$$

Thm 4.3.1 The process $\{I_t(\Phi)\}_{t \in [0,T]}$ (well, a cont. modification of it) is a $(\mathbb{P}, (\mathcal{F}_t))$-martingale.

This can again be combined with the concept of stopping times

$$I_{t \wedge \tau}(\Phi) = \int \Phi_s 1_{\{0 \le s \le t \wedge \tau\}} \, dB_s$$

Example $(W_t)_{t\geq 0}$ ~~BM~~ Wiener process

$$T = \inf\{t>0 : W_t \notin [-a,b]\}, \quad a,b>0$$

exit time of $[-a,b]$

$W_t^2 - t$ is martingale

so $W_{t\wedge u}^2 - t\wedge u$, $t \geq 0$ is also mart. for each u

By optional stopping theorem

$$\cancel{E[W_{t\wedge T}^2 - t\wedge T] = 0}$$

$$E[W_{T\wedge u}^2 - T\wedge u] = 0$$

and so $E[W_{T\wedge u}^2] = E[T\wedge u]$

Let $u\to\infty$. By monotone convergence,

RHS: $E(T\wedge u) \nearrow E[T]$

LHS: $E[W_{T\wedge u}^2] \leq (a+b)^2 < \infty$

uniformly in u

By dominated convergence,

$$E[W_{T\wedge u}^2] \to E[W_T^2]$$

Hence $E(T) = E(W_T^2) < \infty$

and so $T < \infty$ a.s.

Hence $E(T) = E[W_T^2] = (-a)^2 \cdot P(W_T = -a) + b^2 P(W_T = b)$

F8:4

Also, $W_t, t>0$ is martingale

so $W_{t \wedge u}, t>0$ — " —

We have $|W_{T \wedge u}| < a+b$ and $E(W_{T \wedge u}) = E(W_0) = 0$

Let $u \to \infty$, use dominated convergence to get

$$0 = E[W_T] = -a \cdot P(W_T = -a) + b P(W_T = b)$$
$$= -a(1 - P(W_T = b)) + b P(W_T = b)$$

$$\Rightarrow P(W_T = b) = \frac{a}{a+b}$$
$$P(W_T = -a) = \frac{b}{a+b}$$

and $E(T) = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = ab$

$\square$

The entire construction can be generalized to hold for the wider class of integrands in

$$\mathcal{H}_{loc} = \left\{ \Phi : [0,T] \times \Omega \to \mathbb{R},\ \text{mble, adapted and } \int_0^T |\Phi_s|^2 ds < \infty \text{ a.s.} \right\}$$

Now, however, $\{I_t(\Phi)\}_{0 \le t \le T}$ is a continuous <u>local</u> $\mathcal{F}_t$ martingale.

See Section 4.4 and 2.4:

<u>Def</u> $\{X_t\}$ is a local martingale w.r.t. $(\mathcal{F}_t)$, if there exists a <u>sequence of stopping times</u> $\{\tau_n\}_{n \ge 0}$ such that 1) $\tau_n \nearrow \infty$ as $n \to \infty$ with prob 1, and 2) $X_{t \wedge \tau_n}$ is an $\mathcal{F}_t$-martingale, for each $n \ge 1$.

F8:7

- $\{M_t^2 - [M]_t\}_{t\geq 0}$ martingale

$0 \qquad t \qquad t+s$

$0 = u_0 < \cdots < u_m = t < u_{m+1} < \cdots < u_n = t+s$

$$\mathbb{E}[M_{t+s}^2 \mid \mathcal{F}_t]$$
$$= \mathbb{E}[(M_{t+s} - M_t + M_t)^2 \mid \mathcal{F}_t]$$
$$= \mathbb{E}[(M_{t+s} - M_t)^2 \mid \mathcal{F}_t] + 2\mathbb{E}[(M_{t+s} - M_t)M_t \mid \mathcal{F}_t] + \mathbb{E}[M_t^2 \mid \mathcal{F}_t]$$
$$= \mathbb{E}[(M_{t+s} - M_t)^2 \mid \mathcal{F}_t] + 0 + M_t^2$$
$$= \mathbb{E}\Big[\Big(\sum_{i=m}^{n-1} \Delta M_{u_i}\Big)^2 \Big| \mathcal{F}_t\Big] + M_t^2$$
$$= \mathbb{E}\Big[\sum_{i=m}^{n-1} (\Delta M_{u_i})^2 \Big| \mathcal{F}_t\Big] + M_t^2 \quad \text{as above}$$
$$\overset{|\pi|\to 0}{=} \mathbb{E}[[M]_{t+s} - [M]_t \mid \mathcal{F}_t] + M_t^2$$

F8:6

Thus,

$$\mathbb{E}[M_{t+s}^2 - [M]_{t+s} \mid \mathcal{F}_t] = M_t^2 - [M]_t, \quad t \geq 0$$

$\square$

Th. a) If $\{M_t\}_{t\geq 0}$ is a martingale [F8] 6

then $\{[M]_t\}_{t\geq 0}$ exists and is right continuous

b) If $\{M_t\}_{t\geq 0}$ is square integrable the limit exists in L^1,

~~then~~ $M_t^2 - [M]_t$, $t\geq 0$ is also a martingale

and $E[M_t^2] = E([M]_t)$

Proof of b)

$E[M_t^2] = E[[M]_t]$

Take $M_0 = 0$, write $M_t = \sum_{i=0}^{m-1} (M_{t_{i+1}} - M_{t_i})$,

where $0 = t_0 < \ldots < t_m = t$. Then

$$E[M_t^2] = E\left[\left(\sum_{i=0}^{m-1} \Delta M_{t_i}\right)^2\right]$$

$$= E\left[\sum_{i=0}^{m-1} (\Delta M_{t_i})^2 + \sum\sum_{i\neq j} (\Delta M_i)(\Delta M_j)\right]$$

$$= E\left[\sum_{i=0}^{m-1} (\Delta M_{t_i})^2\right] + 0$$ by the martingale property!

converges to $[M]_t$ in L^1, by a)

Hence $E[M_t^2] = E[[M]_t]$ as $|\pi| \to 0$