Chapter 7

Stochastic Integration Techniques

Computing a stochastic integral starting from the definition of the Ito integral is not only difficult, but also rather inefficient. Like in elementary Calculus, several methods can be developed to compute stochastic integrals. We tried to keep the analogy with elementary Calculus as much as possible. The integration by substitution is more complicated in the stochastic environment and we have considered only a particular case of it, which we called *the method of heat equation*.

7.1 Notational Conventions

The intent of this section is to discuss some equivalent integral notations for a stochastic differential equation. Consider a process X_t whose increments satisfy the stochastic differential equation $dX_t = f(t, W_t)dW_t$. This can be written equivalently in the integral form as

$$\int_a^t dX_s = \int_a^t f(s, W_s)dW_s. \tag{7.1.1}$$

If we consider the partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, then the left side becomes

$$\int_a^t dX_s = \text{ms-}\lim_{n\to\infty} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j}) = X_t - X_a,$$

after canceling the terms in pairs. Substituting into formula (7.1.1) yields the equivalent form

$$X_t = X_a + \int_a^t f(s, W_s)dW_s.$$

This can also be written as $dX_t = d\Big(\int_a^t f(s, W_s)dW_s\Big)$, since X_a is a constant. Using $dX_t = f(t, W_t)dW_t$, the previous formula can be written in the following two equivalent ways:

(i) For any $a < t$, we have

$$\boxed{d\Big(\int_a^t f(s, W_s)dW_s\Big) = f(t, W_t)dW_t.} \tag{7.1.2}$$

(ii) If Y_t is a stochastic process, such that $Y_t dW_t = dF_t$, then

$$\boxed{\int_a^b Y_t\, dW_t = F_b - F_a.}$$

These formulas are equivalent ways of writing the stochastic differential equation (7.1.1), and will be useful in future computations. A few applications follow.

Example 7.1.1 *Verify the stochastic formula*

$$\int_0^t W_s\, dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

Let $X_t = \int_0^t W_s\, dW_s$ and $Y_t = \dfrac{W_t^2}{2} - \dfrac{t}{2}$. From Ito's formula

$$dY_t = d\Big(\frac{W_t^2}{2}\Big) - d\Big(\frac{t}{2}\Big) = \frac{1}{2}(2W_t\, dW_t + dt) - \frac{1}{2}dt = W_t\, dW_t,$$

and from formula (7.1.2) we get

$$dX_t = d\Big(\int_0^t W_s\, dW_s\Big) = W_t\, dW_t.$$

Hence $dX_t = dY_t$, or $d(X_t - Y_t) = 0$. Since the process $X_t - Y_t$ has zero increments, then $X_t - Y_t = c$, constant. Taking $t = 0$, yields

$$c = X_0 - Y_0 = \int_0^0 W_s\, dW_s - \Big(\frac{W_0^2}{2} - \frac{0}{2}\Big) = 0,$$

and hence $c = 0$. It follows that $X_t = Y_t$, which verifies the desired relation.

Example 7.1.2 *Verify the formula*

$$\int_0^t sW_s\, dW_s = \frac{t}{2}\Big(W_t^2 - \frac{t}{2}\Big) - \frac{1}{2}\int_0^t W_s^2\, ds.$$

Consider the stochastic processes $X_t = \int_0^t sW_s\,dW_s$, $Y_t = \dfrac{t}{2}\Big(W_t^2 - 1\Big)$, and $Z_t = \frac{1}{2}\int_0^t W_s^2\,ds$. Formula (7.1.2) yields

$$\begin{aligned} dX_t &= tW_t\,dW_t \\ dZ_t &= \frac{1}{2}W_t^2 dt. \end{aligned}$$

Applying Ito's formula, we get

$$\begin{aligned} dY_t &= d\Big(\frac{t}{2}\Big(W_t^2 - \frac{t}{2}\Big)\Big) = \frac{1}{2}d(tW_t^2) - d\Big(\frac{t^2}{4}\Big) \\ &= \frac{1}{2}\Big[(t + W_t^2)dt + 2tW_t\,dW_t\Big] - \frac{1}{2}tdt \\ &= \frac{1}{2}W_t^2 dt + tW_t\,dW_t. \end{aligned}$$

We can easily see that

$$dX_t = dY_t - dZ_t.$$

This implies $d(X_t - Y_t + Z_t) = 0$, i.e. $X_t - Y_t + Z_t = c$, constant. Since $X_0 = Y_0 = Z_0 = 0$, it follows that $c = 0$. This proves the desired relation.

Example 7.1.3 *Show that*

$$\int_0^t (W_s^2 - s)\,dW_s = \frac{1}{3}W_t^3 - tW_t.$$

Consider the function $f(t,x) = \frac{1}{3}x^3 - tx$, and let $F_t = f(t, W_t)$. Since $\partial_t f = -x$, $\partial_x f = x^2 - t$, and $\partial_x^2 f = 2x$, then Ito's formula provides

$$\begin{aligned} dF_t &= \partial_t f\,dt + \partial_x f\,dW_t + \frac{1}{2}\partial_x^2 f\,(dW_t)^2 \\ &= -W_t dt + (W_t^2 - t)\,dW_t + \frac{1}{2}2W_t\,dt \\ &= (W_t^2 - t)dW_t. \end{aligned}$$

From formula (7.1.2) we get

$$\int_0^t (W_s^2 - s)\,dW_s = \int_0^t dF_s = F_t - F_0 = F_t = \frac{1}{3}W_t^3 - tW_t.$$

Exercise 7.1.4 *Show that*

(*a*) $\displaystyle\int_0^t \ln W_s^2\,dW_s = W_t(\ln W_t^2 - 2) - \int_0^t \frac{1}{W_s}\,ds$;

(*b*) $\displaystyle\int_0^t e^{\frac{s}{2}}\cos W_s\,dW_s = e^{\frac{t}{2}}\sin W_t$;

$(c)\quad \displaystyle\int_0^t e^{\frac{s}{2}} \sin W_s \, dW_s = 1 - e^{\frac{t}{2}} \cos W_t;$

$(d)\quad \displaystyle\int_0^t e^{W_s - \frac{s}{2}} \, dW_s = e^{W_t - \frac{t}{2}} - 1;$

$(e)\quad \displaystyle\int_0^t \cos W_s \, dW_s = \sin W_t + \frac{1}{2}\int_0^t \sin W_s \, ds;$

$(f)\quad \displaystyle\int_0^t \sin W_s \, dW_s = 1 - \cos W_t - \frac{1}{2}\int_0^t \cos W_s \, ds.$

7.2 Stochastic Integration by Parts

Consider the process $F_t = f(t)g(W_t)$, with f and g differentiable. Using the product rule yields

$$\begin{aligned} dF_t &= df(t)\, g(W_t) + f(t)\, dg(W_t) \\ &= f'(t)g(W_t)dt + f(t)\big(g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt\big) \\ &= f'(t)g(W_t)dt + \frac{1}{2}f(t)g''(W_t)dt + f(t)g'(W_t)dW_t. \end{aligned}$$

Writing the relation in the integral form, we obtain the first integration by parts formula:

$$\boxed{\int_a^b f(t)g'(W_t)\, dW_t = f(t)g(W_t)\Big|_a^b - \int_a^b f'(t)g(W_t)\, dt - \frac{1}{2}\int_a^b f(t)g''(W_t)\, dt.}$$

This formula is to be used when integrating a product between a function of t and a function of the Brownian motion W_t, for which an antiderivative is known. The following two particular cases are important and useful in applications.

1. If $g(W_t) = W_t$, the aforementioned formula takes the simple form

$$\boxed{\int_a^b f(t)\, dW_t = f(t)W_t\Big|_{t=a}^{t=b} - \int_a^b f'(t)W_t\, dt.} \tag{7.2.3}$$

It is worth noting that the left side is a Wiener integral.

2. If $f(t) = 1$, then the formula becomes

$$\boxed{\int_a^b g'(W_t)\, dW_t = g(W_t)\Big|_{t=a}^{t=b} - \frac{1}{2}\int_a^b g''(W_t)\, dt.} \tag{7.2.4}$$

Application 7.2.1 *Consider the Wiener integral* $I_T = \int_0^T t\, dW_t$. *From the general theory, see Proposition 5.6.1, it is known that* I *is a random variable normally distributed with mean* 0 *and variance*

$$Var[I_T] = \int_0^T t^2\, dt = \frac{T^3}{3}.$$

Recall the definition of integrated Brownian motion

$$Z_t = \int_0^t W_u\, du.$$

Formula (7.2.3) yields a relationship between I *and the integrated Brownian motion*

$$I_T = \int_0^T t\, dW_t = TW_T - \int_0^T W_t\, dt = TW_T - Z_T,$$

and hence $I_T + Z_T = TW_T$. *This relation can be used to compute the covariance between* I_T *and* Z_T.

$$\begin{aligned} Cov(I_T + Z_T, I_T + Z_T) &= Var[TW_T] \iff \\ Var[I_T] + Var[Z_T] + 2Cov(I_T, Z_T) &= T^2 Var[W_T] \iff \\ T^3/3 + T^3/3 + 2Cov(I_T, Z_T) &= T^3 \iff \\ Cov(I_T, Z_T) &= T^3/6, \end{aligned}$$

where we used that $Var[Z_T] = T^3/3$. *The processes* I_t *and* Z_t *are not independent. Their correlation coefficient is* 0.5 *as the following calculation shows*

$$\begin{aligned} Corr(I_T, Z_T) &= \frac{Cov(I_T, Z_T)}{\Big(Var[I_T]Var[Z_T]\Big)^{1/2}} = \frac{T^3/6}{T^3/3} \\ &= 1/2. \end{aligned}$$

Application 7.2.2 *If we let* $g(x) = \frac{x^2}{2}$ *in formula (7.2.4), we get*

$$\boxed{\int_a^b W_t\, dW_t = \frac{W_b^2 - W_a^2}{2} - \frac{1}{2}(b-a).}$$

It is worth noting that letting $a = 0$ *and* $b = T$, *we retrieve a formula that was proved by direct methods in chapter 3*

$$\boxed{\int_0^T W_t\, dW_t = \frac{W_T^2}{2} - \frac{T}{2}.}$$

Similarly, if we let $g(x) = \frac{x^3}{3}$ in (7.2.4) yields

$$\boxed{\int_a^b W_t^2\, dW_t = \frac{W_t^3}{3}\Big|_a^b - \int_a^b W_t\, dt.}$$

Application 7.2.3 *Choosing $f(t) = e^{\alpha t}$ and $g(x) = \sin x$, we shall compute the stochastic integral $\int_0^T e^{\alpha t} \cos W_t\, dW_t$ using the formula of integration by parts*

$$\begin{aligned}
\int_0^T e^{\alpha t} \cos W_t\, dW_t &= \int_0^T e^{\alpha t} (\sin W_t)'\, dW_t \\
&= e^{\alpha t} \sin W_t\Big|_0^T - \int_0^T (e^{\alpha t})' \sin W_t\, dt - \frac{1}{2}\int_0^T e^{\alpha t} (\sin W_t)''\, dt \\
&= e^{\alpha T} \sin W_T - \alpha \int_0^T e^{\alpha t} \sin W_t\, dt + \frac{1}{2}\int_0^T e^{\alpha t} \sin W_t\, dt \\
&= e^{\alpha T} \sin W_T - \Big(\alpha - \frac{1}{2}\Big) \int_0^T e^{\alpha t} \sin W_t\, dt.
\end{aligned}$$

The particular case $\alpha = \frac{1}{2}$ leads to the following exact formula of a stochastic integral

$$\boxed{\int_0^T e^{\frac{t}{2}} \cos W_t\, dW_t = e^{\frac{T}{2}} \sin W_T.} \tag{7.2.5}$$

In a similar way, we can obtain an exact formula for the stochastic integral $\int_0^T e^{\beta t} \sin W_t\, dW_t$ as follows

$$\begin{aligned}
\int_0^T e^{\beta t} \sin W_t\, dW_t &= -\int_0^T e^{\beta t} (\cos W_t)'\, dW_t \\
&= -e^{\beta t} \cos W_t\Big|_0^T + \beta \int_0^T e^{\beta t} \cos W_t\, dt - \frac{1}{2}\int_0^T e^{\beta t} \cos W_t\, dt.
\end{aligned}$$

Taking $\beta = \frac{1}{2}$ yields the closed form formula

$$\boxed{\int_0^T e^{\frac{t}{2}} \sin W_t\, dW_t = 1 - e^{\frac{T}{2}} \cos W_T.} \tag{7.2.6}$$

A consequence of the last two formulas and of Euler's formula

$$e^{iW_t} = \cos W_t + i \sin W_t,$$

is

$$\boxed{\int_0^T e^{\frac{t}{2} + iW_t}\, dW_t = i(1 - e^{\frac{T}{2} + iW_T}).}$$

The proof details are left to the reader.

A general form of the integration by parts formula In general, if X_t and Y_t are two Ito diffusions, from the product formula

$$d(X_tY_t) = X_tdY_t + Y_tdX_t + dX_t\,dY_t.$$

Integrating between the limits a and b

$$\int_a^b d(X_tY_t) = \int_a^b X_tdY_t + \int_a^b Y_tdX_t + \int_a^b dX_t\,dY_t.$$

From the Fundamental Theorem

$$\int_a^b d(X_tY_t) = X_bY_b - X_aY_a,$$

so the previous formula takes the following form of integration by parts

$$\boxed{\int_a^b X_tdY_t = X_bY_b - X_aY_a - \int_a^b Y_tdX_t - \int_a^b dX_t\,dY_t.}$$

This formula is of theoretical value. In practice, the term $dX_t\,dY_t$ needs to be computed using the rules $dW_t^2 = dt$, and $dt\,dW_t = 0$.

Exercise 7.2.4 (a) *Use integration by parts to get*

$$\int_0^T \frac{1}{1+W_t^2}dW_t = \tan^{-1}(W_T) + \int_0^T \frac{W_t}{(1+W_t^2)^2}\,dt, \quad T > 0.$$

(b) *Show that*

$$\mathbb{E}[\tan^{-1}(W_T)] = -\int_0^T E\Big[\frac{W_t}{(1+W_t^2)^2}\Big]\,dt.$$

(c) *Prove the double inequality*

$$-\frac{3\sqrt{3}}{16} \le \frac{x}{(1+x^2)^2} \le \frac{3\sqrt{3}}{16}, \qquad \forall x \in \mathbb{R}.$$

(d) *Use part* (c) *to obtain*

$$-\frac{3\sqrt{3}}{16}T \le \int_0^T \frac{W_t}{(1+W_t^2)^2}\,dt \le \frac{3\sqrt{3}}{16}T.$$

(e) *Use part* (d) *to get*

$$-\frac{3\sqrt{3}}{16}T \le \mathbb{E}[\tan^{-1}(W_T)] \le \frac{3\sqrt{3}}{16}T.$$

(f) *Does part* (e) *contradict the inequality*

$$-\frac{\pi}{2} < \tan^{-1}(W_T) < \frac{\pi}{2}?$$

Exercise 7.2.5 *(a) Show the relation*

$$\int_0^T e^{W_t}\,dW_t = e^{W_T} - 1 - \frac{1}{2}\int_0^T e^{W_t}\,dt.$$

(b) Use part (a) to find $\mathbb{E}[e^{W_t}]$.

Exercise 7.2.6 *(a) Use integration by parts to show*

$$\int_0^T W_t e^{W_t}\,dW_t = 1 + W_T e^{W_T} - e^{W_T} - \frac{1}{2}\int_0^T e^{W_t}(1+W_t)\,dt;$$

(b) Use part (a) to find $\mathbb{E}[W_t e^{W_t}]$;

(c) Show that $Cov(W_t, e^{W_t}) = te^{t/2}$;

(d) Prove that $Corr(W_t, e^{W_t}) = \sqrt{\dfrac{t}{e^t - 1}}$, *and compute the limits as* $t \to 0$ *and* $t \to \infty$.

Exercise 7.2.7 *(a) Let* $T > 0$. *Show the following relation using integration by parts*

$$\int_0^T \frac{2W_t}{1+W_t^2}\,dW_t = \ln(1+W_T^2) - \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\,dt.$$

(b) Show that for any real number x *the following double inequality holds*

$$-\frac{1}{8} \le \frac{1-x^2}{(1+x^2)^2} \le 1.$$

(c) Use part (b) to show that

$$-\frac{1}{8}T \le \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\,dt \le T.$$

(d) Use parts (a) and (c) to get

$$-\frac{T}{8} \le \mathbb{E}[\ln(1+W_T^2)] \le T.$$

(e) Use Jensen's inequality to get

$$\mathbb{E}[\ln(1+W_T^2)] \le \ln(1+T).$$

Does this contradict the upper bound provided in (d)?

Exercise 7.2.8 *Use integration by parts to show*

$$\int_0^t \arctan W_s\,dW_s = W_t \arctan W_t - \frac{1}{2}\ln(1+W_t^2) - \frac{1}{2}\int_0^t \frac{1}{1+W_s^2}\,ds.$$

Exercise 7.2.9 (a) *Using integration by parts prove the identity*

$$\int_0^t W_s e^{W_s}\,dW_s = 1 + e^{W_t}(W_t - 1) - \frac{1}{2}\int_0^t (1+W_s)e^{W_s}\,ds;$$

(b) *Use part* (a) *to compute* $\mathbb{E}[W_t e^{W_t}]$.

Exercise 7.2.10 *Check the following formulas using integration by parts*

(a) $\displaystyle\int_0^t W_s \sin W_s\,dW_s = \sin W_t - W_t\cos W_t - \frac{1}{2}\int_0^t (\sin W_s + W_s\cos W_s)\,ds;$

(b) $\displaystyle\int_0^t \frac{W_s}{\sqrt{1+W_s^2}}\,dW_s = \sqrt{1+W_t^2} - 1 - \frac{1}{2}\int_0^t (1+W_s^2)^{-3/2}\,ds.$

7.3 The Heat Equation Method

In elementary Calculus, integration by substitution is the inverse application of the chain rule. In the stochastic environment, this will be the inverse application of Ito's formula. This is difficult to apply in general, but there is a particular case of great importance.

Let $\varphi(t,x)$ be a solution of the equation

$$\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = 0. \tag{7.3.7}$$

This is called the *heat equation without sources.* The non-homogeneous equation

$$\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = G(t,x) \tag{7.3.8}$$

is called the *heat equation with sources.* The function $G(t,x)$ represents the density of heat sources, while the function $\varphi(t,x)$ is the temperature at the point x at time t in a one-dimensional wire. If the heat source is time independent, then $G = G(x)$, i.e. G is a function of x only.

Example 7.3.1 *Find all solutions of the equation (7.3.7) of type* $\varphi(t,x) = a(t) + b(x)$.

Substituting into equation (7.3.7) yields

$$\frac{1}{2}b''(x) = -a'(t).$$

Since the left side is a function of x only, while the right side is a function of variable t, the only case where the previous equation is satisfied is when both sides are equal to the same constant C. This is called a separation constant. Therefore $a(t)$ and $b(x)$ satisfy the equations

$$a'(t) = -C, \qquad \frac{1}{2}b''(x) = C.$$

Integrating yields $a(t) = -Ct + C_0$ and $b(x) = Cx^2 + C_1x + C_2$. It follows that

$$\varphi(t,x) = C(x^2 - t) + C_1x + C_3,$$

with C_0, C_1, C_2, C_3 arbitrary constants.

Example 7.3.2 *Find all solutions of the equation (7.3.7) of the type* $\varphi(t,x) = a(t)b(x)$.

Substituting into the equation and dividing by $a(t)b(x)$ yields

$$\frac{a'(t)}{a(t)} + \frac{1}{2}\frac{b''(x)}{b(x)} = 0.$$

There is a separation constant C such that $\dfrac{a'(t)}{a(t)} = -C$ and $\dfrac{b''(x)}{b(x)} = 2C$. There are three distinct cases to discuss:

1. $C = 0$. In this case $a(t) = a_0$ and $b(x) = b_1x + b_0$, with a_0, a_1, b_0, b_1 real constants. Then

$$\varphi(t,x) = a(t)b(x) = c_1x + c_0, \qquad c_0, c_1 \in \mathbb{R}$$

is just a linear function in x.

2. $C > 0$. Let $\lambda > 0$ such that $2C = \lambda^2$. Then $a'(t) = -\frac{\lambda^2}{2}a(t)$ and $b''(x) = \lambda^2 b(x)$, with solutions

$$\begin{aligned} a(t) &= a_0 e^{-\lambda^2 t/2} \\ b(x) &= c_1 e^{\lambda x} + c_2 e^{-\lambda x}. \end{aligned}$$

The general solution of (7.3.7) is

$$\varphi(t,x) = e^{-\lambda^2 t/2}(c_1 e^{\lambda x} + c_2 e^{-\lambda x}), \quad c_1, c_2 \in \mathbb{R}.$$

3. $C < 0$. Let $\lambda > 0$ such that $2C = -\lambda^2$. Then $a'(t) = \frac{\lambda^2}{2}a(t)$ and $b''(x) = -\lambda^2 b(x)$. Solving yields

$$\begin{aligned} a(t) &= a_0 e^{\lambda^2 t/2} \\ b(x) &= c_1 \sin(\lambda x) + c_2 \cos(\lambda x). \end{aligned}$$

The general solution of (7.3.7) in this case is

$$\varphi(t,x) = e^{\lambda^2 t/2}\big(c_1 \sin(\lambda x) + c_2 \cos(\lambda x)\big), \quad c_1, c_2 \in \mathbb{R}.$$

In particular, the functions x, $x^2 - t$, $e^{x-t/2}$, $e^{-x-t/2}$, $e^{t/2}\sin x$ and $e^{t/2}\cos x$, or any linear combination of them, are solutions of the heat equation (7.3.7). However, there are other solutions which are not of the previous type.

Exercise 7.3.3 *Prove that* $\varphi(t,x) = \frac{1}{3}x^3 - tx$ *is a solution of the heat equation (7.3.7).*

Exercise 7.3.4 *Show that* $\varphi(t,x) = t^{-1/2}e^{-x^2/(2t)}$ *is a solution of the heat equation (7.3.7) for* $t > 0$.

Exercise 7.3.5 *Let* $\varphi = u(\lambda)$, *with* $\lambda = \dfrac{x}{2\sqrt{t}}$, $t > 0$. *Show that* φ *satisfies the heat equation (7.3.7) if and only if* $u'' + 2\lambda u' = 0$.

Exercise 7.3.6 *Let* $erfc(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_x^\infty e^{-r^2}\,dr$. *Show that* $\varphi = erfc(x/(2\sqrt{t}))$ *is a solution of the equation (7.3.7).*

Exercise 7.3.7 (the fundamental solution) *Show that* $\varphi(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$, $t > 0$ *satisfies the equation (7.3.7).*

Sometimes it is useful to generate new solutions for the heat equation from other solutions. Below we present a few ways to accomplish this:

(i) by linear combination: if φ_1 and φ_2 are solutions, then $a_1\varphi_1 + a_1\varphi_2$ is a solution, where a_1, a_2 are constants.

(ii) by translation: if $\varphi(t,x)$ is a solution, then $\varphi(t-\tau, x-\xi)$ is a solution, where (τ,ξ) is a translation vector.

(iii) by affine transforms: if $\varphi(t,x)$ is a solution, then $\varphi(\lambda t, \lambda^2 x)$ is a solution, for any constant λ.

(iv) by differentiation: if $\varphi(t,x)$ is a solution, then $\dfrac{\partial^{n+m}}{\partial^n x \partial^m t}\varphi(t,x)$ is a solution.

(v) by convolution: if $\varphi(t,x)$ is a solution, then so are

$$\int_a^b \varphi(t, x-\xi)f(\xi)\,d\xi$$
$$\int_a^b \varphi(t-\tau, x)g(t)\,dt.$$

For more detail on the subject the reader can consult Widder [46] and Cannon [11].

Theorem 7.3.8 *Let $\varphi(t,x)$ be a solution of the heat equation (7.3.7) and denote $f(t,x) = \partial_x \varphi(t,x)$. Then*

$$\boxed{\int_a^b f(t,W_t)\,dW_t = \varphi(b,W_b) - \varphi(a,W_a).}$$

Proof: Let $F_t = \varphi(t,W_t)$. Applying Ito's formula we get

$$dF_t = \partial_x \varphi(t,W_t)\,dW_t + \Big(\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi\Big)dt.$$

Since $\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = 0$ and $\partial_x \varphi(t,W_t) = f(t,W_t)$, we have

$$dF_t = f(t,W_t)\,dW_t.$$

Writing in the integral form, yields

$$\int_a^b f(t,W_t)\,dW_t = \int_a^b dF_t = F_b - F_a = \varphi(b,W_b) - \varphi(a,W_a).$$

■

Application 7.3.9 *Show that*

$$\int_0^T W_t\,dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

Choose the solution of the heat equation (7.3.7) given by $\varphi(t,x) = x^2 - t$. Then $f(t,x) = \partial_x \varphi(t,x) = 2x$. Theorem 7.3.8 yields

$$\int_0^T 2W_t\,dW_t = \int_0^T f(t,W_t)\,dW_t = \varphi(t,x)\Big|_0^T = W_T^2 - T.$$

Dividing by 2 leads to the desired result.

Application 7.3.10 *Show that*

$$\int_0^T (W_t^2 - t)\,dW_t = \frac{1}{3}W_T^3 - TW_T.$$

Consider the function $\varphi(t,x) = \frac{1}{3}x^3 - tx$, which is a solution of the heat equation (7.3.7), see Exercise 7.3.3. Then $f(t,x) = \partial_x \varphi(t,x) = x^2 - t$. Applying Theorem 7.3.8 yields

$$\int_0^T (W_t^2 - t)\,dW_t = \int_0^T f(t,W_t)\,dW_t = \varphi(t,W_t)\Big|_0^T = \frac{1}{3}W_T^3 - TW_T.$$

Application 7.3.11 *Let $\lambda > 0$. Prove the identities*

$$\boxed{\int_0^T e^{-\frac{\lambda^2 t}{2} \pm \lambda W_t}\, dW_t = \frac{1}{\pm\lambda}\Big(e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1\Big).}$$

Consider the function $\varphi(t,x) = e^{-\frac{\lambda^2 t}{2} \pm \lambda x}$, which is a solution of the homogeneous heat equation (7.3.7), see Example 7.3.2. Then $f(t,x) = \partial_x \varphi(t,x) = \pm\lambda e^{-\frac{\lambda^2 t}{2} \pm \lambda x}$. Apply Theorem 7.3.8 to get

$$\int_0^T \pm\lambda e^{-\frac{\lambda^2 t}{2} \pm \lambda x}\, dW_t = \int_0^T f(t, W_t)\, dW_t = \varphi(t, W_t)\Big|_0^T = e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1.$$

Dividing by the constant $\pm\lambda$ ends the proof.

In particular, for $\lambda = 1$ the aforementioned formula becomes

$$\boxed{\int_0^T e^{-\frac{t}{2} + W_t}\, dW_t = e^{-\frac{T}{2} + W_T} - 1.} \tag{7.3.9}$$

Application 7.3.12 *Let $\lambda > 0$. Prove the identity*

$$\boxed{\int_0^T e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\, dW_t = \frac{1}{\lambda} e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T).}$$

From the Example 7.3.2 we know that $\varphi(t,x) = e^{\frac{\lambda^2 t}{2}} \sin(\lambda x)$ is a solution of the heat equation. Applying Theorem 7.3.8 to the function $f(t,x) = \partial_x \varphi(t,x) = \lambda e^{\frac{\lambda^2 t}{2}} \cos(\lambda x)$, yields

$$\begin{aligned}\int_0^T \lambda e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\, dW_t &= \int_0^T f(t, W_t)\, dW_t = \varphi(t, W_t)\Big|_0^T \\ &= e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\Big|_0^T = e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T).\end{aligned}$$

Divide by λ to end the proof.

If we choose $\lambda = 1$ we recover a result already familiar to the reader from section 7.2

$$\boxed{\int_0^T e^{\frac{t}{2}} \cos(W_t)\, dW_t = e^{\frac{T}{2}} \sin W_T.} \tag{7.3.10}$$

Application 7.3.13 *Let $\lambda > 0$. Show that*

$$\boxed{\int_0^T e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\, dW_t = \frac{1}{\lambda}\Big(1 - e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_T)\Big).}$$

Choose $\varphi(t,x) = e^{\frac{\lambda^2 t}{2}}\cos(\lambda x)$ to be a solution of the heat equation. Apply Theorem 7.3.8 for the function $f(t,x) = \partial_x\varphi(t,x) = -\lambda e^{\frac{\lambda^2 t}{2}}\sin(\lambda x)$ to get

$$\begin{aligned}\int_0^T (-\lambda)e^{\frac{\lambda^2 t}{2}}\sin(\lambda W_t)\,dW_t &= \varphi(t,W_t)\Big|_0^T \\ &= e^{\frac{\lambda^2 T}{2}}\cos(\lambda W_t)\Big|_0^T = e^{\frac{\lambda^2 T}{2}}\cos(\lambda W_T) - 1,\end{aligned}$$

and then divide by $-\lambda$.

Application 7.3.14 *Let $0 < a < b$. Show that*

$$\boxed{\int_a^b t^{-\frac{3}{2}} W_t e^{-\frac{W_t^2}{2t}}\,dW_t = a^{-\frac{1}{2}}e^{-\frac{W_a^2}{2a}} - b^{-\frac{1}{2}}e^{-\frac{W_b^2}{2b}}.} \tag{7.3.11}$$

From Exercise 7.3.4 we have that $\varphi(t,x) = t^{-1/2}e^{-x^2/(2t)}$ is a solution of the homogeneous heat equation. Since $f(t,x) = \partial_x\varphi(t,x) = -t^{-3/2}xe^{-x^2/(2t)}$, applying Theorem 7.3.8 yields the desired result. The reader can easily fill in the details.

Integration techniques will be used when solving stochastic differential equations in the next chapter.

Exercise 7.3.15 *Find the value of the following stochastic integrals*

(a) $\displaystyle\int_0^1 e^t\cos(\sqrt{2}W_t)\,dW_t$

(b) $\displaystyle\int_0^3 e^{2t}\cos(2W_t)\,dW_t$

(c) $\displaystyle\int_0^4 e^{-t+\sqrt{2}W_t}\,dW_t.$

Exercise 7.3.16 *Let $\varphi(t,x)$ be a solution of the following non-homogeneous heat equation with time-dependent and uniform heat source $G(t)$*

$$\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = G(t).$$

Denote $f(t,x) = \partial_x\varphi(t,x)$. Show that

$$\boxed{\int_a^b f(t,W_t)\,dW_t = \varphi(b,W_b) - \varphi(a,W_a) - \int_a^b G(t)\,dt.}$$

How does the formula change if the heat source G is constant?

7.4 Table of Usual Stochastic Integrals

Now we present a user-friendly table, which enlists integral identities developed in this chapter. This table is far too complicated to be memorized in full. However, the first couple of identities in this table are the most memorable, and should be remembered.

Let $a < b$ and $0 < T$. Then we have:

1. $\displaystyle\int_a^b dW_t = W_b - W_a;$

2. $\displaystyle\int_0^T W_t\,dW_t = \frac{W_T^2}{2} - \frac{T}{2};$

3. $\displaystyle\int_0^T (W_t^2 - t)\,dW_t = \frac{W_T^2}{3} - TW_T;$

4. $\displaystyle\int_0^T t\,dW_t = TW_T - \int_0^T W_t\,dt, \quad 0 < T;$

5. $\displaystyle\int_0^T W_t^2\,dW_t = \frac{W_T^3}{3} - \int_0^T W_t\,dt;$

6. $\displaystyle\int_0^T e^{\frac{t}{2}} \cos W_t\,dW_t = e^{\frac{T}{2}} \sin W_T;$

7. $\displaystyle\int_0^T e^{\frac{t}{2}} \sin W_t\,dW_t = 1 - e^{\frac{T}{2}} \cos W_T;$

8. $\displaystyle\int_0^T e^{-\frac{t}{2}+W_t}\,dW_t = e^{-\frac{T}{2}+W_T} - 1;$

9. $\displaystyle\int_0^T e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\,dW_t = \frac{1}{\lambda} e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T);$

10. $\displaystyle\int_0^T e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\,dW_t = \frac{1}{\lambda}\Big(1 - e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_T)\Big);$

11. $\displaystyle\int_0^T e^{-\frac{\lambda^2 t}{2} \pm \lambda W_t}\,dW_t = \frac{1}{\pm\lambda}\Big(e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1\Big);$

12. $\displaystyle\int_a^b t^{-\frac{3}{2}} W_t e^{-\frac{W_t^2}{2t}}\,dW_t = a^{-\frac{1}{2}} e^{-\frac{W_a^2}{2a}} - b^{-\frac{1}{2}} e^{-\frac{W_b^2}{2b}};$

13. $\displaystyle\int_0^T \cos W_t\,dW_t = \sin W_T + \frac{1}{2}\int_0^T \sin W_t\,dt;$

14. $\displaystyle\int_0^T \sin W_t\, dW_t = 1 - \cos W_T - \frac{1}{2}\int_0^T \cos W_t\, dt;$

15. $\displaystyle d\Big(\int_a^t f(s, W_s)\, dW_s\Big) = f(t, W_t)\, dW_t;$

16. $\displaystyle\int_a^b Y_t\, dW_t = F_b - F_a$, if $Y_t dW_t = dF_t$;

17. $\displaystyle\int_a^b f(t)\, dW_t = f(t)W_t|_a^b - \int_a^b f'(t)W_t\, dt;$

18. $\displaystyle\int_a^b g'(W_t)\, dW_t = g(W_t)\Big|_a^b - \frac{1}{2}\int_a^b g''(W_t)\, dt.$