Lecture 6

Prop. 5.8 (Feynman-Kac in higher dimensions + discounting)

Assume that $F : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{n} C_{ij}(t,x) \dfrac{\partial^2 F(t,x)}{\partial x_i \partial x_j} + \displaystyle\sum_{i=1}^{n} \mu_i(t,x) \dfrac{\partial F}{\partial x_i} - rF(t,x) = 0 \\ F(T,x) = \Phi(x) \end{cases}$$

where $C(t,x) = \sigma(t,x)\sigma^*(t,x)$ for some matrix σ $(n \times d)$.

Then
$$F(t,x) = e^{-r(T-t)} E_{t,x}\left[\Phi(X_T)\right] \quad \text{where}$$

$$\begin{cases} dX_s = \mu(s,X_s)\,ds + \sigma(s,X_s)\,dW_s \\ X_t = x. \end{cases}$$

Proof: Let $Z_s = e^{-r(s-t)} F(s,X_s)$. Then

$$dZ_s \overset{\text{Ito}}{=} e^{-r(s-t)} \underbrace{\left(\frac{\partial F}{\partial s} + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i \frac{\partial F}{\partial x_i} - rF \right)}_{0} ds + e^{-r(s-t)} \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sigma_i \, dW_s$$

so

$$\underset{\substack{\| \\ e^{-r(T-t)}\Phi(X_T)}}{Z_T} = \underbrace{Z_t}_{F(t,x)} + \int_t^T \ldots \, dW_s$$

Thus
$$F(t,x) = e^{-r(T-t)} E[\Phi(X_T)].$$

Exercise 5.13 (deluxe, in the book $r=0$)

Solve the PDE
$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\delta^2}{2}\dfrac{\partial^2 F}{\partial y^2} - rF = 0 \\ F(T,x,y) = xy \end{cases}$$

Solution: Here $C = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \delta^2 \end{pmatrix}$ so $\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \delta \end{pmatrix}$.

satisfies $C = \sigma\sigma^*$.

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \quad \text{so} \quad \begin{cases} X_T = x + \sigma(W_T^1 - W_t^1) \\ Y_T = y + \delta(W_T^2 - W_t^2) \end{cases}$$

Feynman-Kac gives
$$F(t,x,y) = E_{t,x,y}\left[e^{-r(T-t)} X_T Y_T\right] =$$
$$= e^{-r(T-t)} E\left[\left(x + \sigma(W_T^1 - W_t^1)\right)\left(y + \delta(W_T^2 - W_t^2)\right)\right]$$
$$\underset{\text{indep}}{=} e^{-r(T-t)} E\left[x + \sigma(W_T^1 - W_t^1)\right] E\left[y + \delta(W_T^2 - W_t^2)\right]$$
$$= e^{-r(T-t)} xy$$

Answer: $F(t,x,y) = e^{-r(T-t)} xy$

<u>Notation</u> The differential operator

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i}$$

is called the <u>infinitesimal operator</u> of X.

<u>Ito's formula</u>: If $Z_t = f(t, X_t)$ then

$$dZ_t = \left(\frac{\partial f}{\partial t} + \mathcal{A} f\right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i \, dW_t$$