Lecture 11:

<u>Recall</u>: A martingale is a stochastic process $X_0, X_1, X_2, \ldots$ with respect to a filtration $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2, \ldots$ if X_n is adapted to $\tilde{\mathcal{F}}_n$ and $\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1}) = X_{n-1}$.

[Supermartingale $\leftrightarrow$ $\leq$
  Submartingale $\leftrightarrow$ $\geq$]

Usually we take $\tilde{\mathcal{F}}_n = \sigma(X_1, X_2, \ldots, X_n)$ but not always!

Remark: Let $m < n$. For every martingale, we have

$$\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_m) = \mathbb{E}\Big(\mathbb{E}\big(\ldots \mathbb{E}\big(\underbrace{\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1})}_{= X_{n-1}} \mid \tilde{\mathcal{F}}_{n-2}\big) \ldots \mid \tilde{\mathcal{F}}_{m+1}\big) \mid \tilde{\mathcal{F}}_m\Big)$$

$$= \mathbb{E}\Big(\mathbb{E}\big(\ldots \underbrace{\mathbb{E}(X_{n-1} \mid \tilde{\mathcal{F}}_{n-2})}_{X_{n-2}} \ldots \mid \tilde{\mathcal{F}}_{m+1}\big) \mid \tilde{\mathcal{F}}_m\Big)$$

$$\vdots$$

$$= \mathbb{E}(X_{m+1} \mid \tilde{\mathcal{F}}_m) = X_m .$$

<u>Definition</u> A pre-visible process is a sequence $C_1, C_2, \dots$ of random variables, such that C_n is $\mathcal{F}_{n-1}$-measurable for all n.

Let C_n be a pre visible process. The martingale transform of X by C is

$$(C \bullet X)_n = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

In particular, if $C_k = 1$ for all k, then

$$(C \bullet X)_n = X_n - X_0.$$

<u>Proposition</u>: If C is a bounded pre-visible process with $|C_n(\omega)| \leq K$ for all n and $\omega \in \Omega$, then $(C \bullet X)_n$ is a martingale if X_n is.
If C is also non-negative, then $(C \bullet X)_n$ is a sub-/super martingale whenever X_n is.

Proof: We have

$$\mathbb{E}\big((C\cdot X)_n - (C\cdot X)_{n-1} \mid \mathcal{F}_{n-1}\big)$$

$$= \mathbb{E}\big(C_n (X_n - X_{n-1}) \mid \mathcal{F}_{n-1}\big)$$

$$= C_n \big(\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - \mathbb{E}(X_{n-1} \mid \mathcal{F}_{n-1})\big)$$

$$= C_n \big(\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - X_{n-1}\big)$$

$$\begin{cases} = 0 & \text{if } X_n \text{ is a martingale} \\ \geq 0 & \text{if } C_n \geq 0 \text{ and } X_n \text{ is a submartingale} \\ \leq 0 & \text{if } C_n \geq 0 \text{ and } X_n \text{ is a supermartingale} \end{cases}$$

$\square$

Stopping times

A stopping time is a random variable T with values in $\{0, 1, 2, .., \infty\}$ and the property that $\{T \leq n\} = \{\omega \in \Omega : T(\omega) \leq n\} \in \mathcal{F}_n$ for all n. Equivalently, $\{T = n\} \in \mathcal{F}_n \ \forall n$.

This follows from $\{T \leq n\} = \{T \leq n-1\} \cup \{T = n\}$

Examples:

- All constants are stopping times
- "First occurrence": for example:

  $T = \min\{n : X_n = 0\}$ for an adapted process X_n

- If S, T are stopping times, then so are

  $\min\{S, T\} = S \wedge T$ "either stopped"

  and $\max\{S, T\} = S \vee T$ "both stopped"

- "Counting": for example, set

  N_n = number of indices $k \leq n$ with $X_k = 0$

  $T = \min\{n : N_n = 10\}$

Example: The following are (generally) __not__ stopping times

$T = \max\{n : N_n = 0\}$ (we cannot determine whether $N_k = 0$ for $k > n$)

Also

$T = \min\{n : X_n = \sup_k X_k\}$ ($\sup_k X_k$ not measurable w.r.t $\mathcal{F}_n$. Could be larger later)

Stopped processes:

Let X_n be an adapted process and T a stopping time with respect to a given filtration. The stopped process X^T is

$$X_n^T(\omega) = X_{n \wedge T(\omega)}(\omega) = \begin{cases} X_{T(\omega)}(\omega) & \text{if } n \geq T(\omega) \\ X_n(\omega) & \text{if } n < T(\omega) \end{cases}$$

Theorem: If X_n is a martingale / supermartingale / submartingale, then so is X_n^T. In particular, for every n, $\mathbb{E}(X_{T\wedge n}) \overset{\leq}{\underset{\geq}{=}} \mathbb{E}(X_0)$ $\begin{matrix}\text{super-} \\ \\ \text{sub-}\end{matrix}$ martingale.

Proof: Note that $C_n^T = \underbrace{I_{\{n \leq T\}} = 1 - I_{\{T \leq n-1\}}}_{\text{"not yet stopped at time } n-1\text{"}}$

is pre-visible.

We have $(C^T \bullet X)_n = \sum_{k=1}^{n} C_k^T (X_k - X_{k-1})$

$$= \sum_{k=1}^{n} I_{\{k \leq T\}} (X_k - X_{k-1}) = \sum_{k=1}^{T\wedge n} (X_k - X_{k-1})$$

$= X_{T\wedge n} - X_0$. So $X_{T \wedge n}$ is a martingale.

Since $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F})) = \mathbb{E}(X)$, the second conclusion follows $\square$

So for every fixed n, $\mathbb{E}(X_{T\wedge n}) = \mathbb{E}(X_0)$.
Is it true that $\mathbb{E}(X_T) = \mathbb{E}(X_0)$?

In general, NO!

Example: Consider the martingale

$$X_0 = 1 \quad , \quad X_n = \begin{cases} 2X_{n-1} & \text{prob. } \frac{1}{2} \\ 0 & \text{prob. } \frac{1}{2} \end{cases}$$

Let $T = \min\{n : X_n = 0\}$. Clearly, $\mathbb{E}(X_T) = 0 \neq \mathbb{E}(X_0)$.

Example: Consider the simple random walk

$$X_0 = 0 \quad , \quad X_n = \begin{cases} X_{n-1} + 1 & \text{prob } \frac{1}{2} \\ X_{n-1} - 1 & \text{prob } \frac{1}{2} \end{cases}$$

Define $T = \min\{n : X_n = 1\}$. This is a stopping time and one can show that $T < \infty$ a.s.

Hence $\mathbb{E}(X_T) = 1 \neq \mathbb{E}(X_0)$.

However, under simple conditions $\mathbb{E}(X_T) = \mathbb{E}(X_0)$

Doob's Optional Stopping Theorem

Let T be a stopping time, and let X be a $\substack{\text{super-} \\ \text{sub-}}$martingale. Suppose one of the following hold:

i) T is bounded (almost surely)

ii) X_n is bounded and $T < \infty$ for a.e. $\omega \in \Omega$.

iii) $\mathbb{E}(T) < \infty$ and $|X_n(\omega) - X_{n-1}(\omega)|$ for all n and almost every $\omega \in \Omega$.

Then,

$$\mathbb{E}(X_T) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0) \qquad \begin{matrix} \text{super-} \\ \text{martingale.} \\ \text{sub-} \end{matrix}$$

__Proof__: (i) If T is bounded by N (a.s.) we have $T \wedge N = T$ (a.s.) and so

$$\mathbb{E}(X_T) = \mathbb{E}(X_{T \wedge N}) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0).$$

(ii) We have $\mathbb{E}(X_{T \wedge n}) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0)$ for fixed n.

Since X is bounded we can use dominated convergence :

$$\mathbb{E}(X_0) = \lim_{n\to\infty} \mathbb{E}(X_{T\wedge n}) = \mathbb{E}(\lim_{n\to\infty} X_{T\wedge n}) = \mathbb{E}(X_T).$$

iii) We have

$$|X_{T\wedge n} - X_0| = \left|\sum_{k=1}^{T\wedge n} (X_k - X_{k-1})\right|$$

$$\leq \sum_{k=1}^{T\wedge n} |X_k - X_{k-1}| \leq K\cdot(T\wedge n) \leq KT.$$

So $\mathbb{E}(KT) = K\,\mathbb{E}(T) < \infty$ and we can apply DCT as above. $\square$

Corollary: If X_n is a nonnegative supermartingale and T an (almost surely) finite stopping time, then

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_0)$$

<u>Proof</u>: By Fatou's lemma,

$$\mathbb{E}(X_T) = \mathbb{E}(\lim_{n\to\infty} X_{T\wedge n}) \quad \text{(why does it exist?)}$$

$$= \mathbb{E}(\liminf_{n\to\infty} X_{T\wedge n})$$

$$\leq \liminf_{n\to\infty} \mathbb{E}(X_{T\wedge n}) \leq \mathbb{E}(X_0)$$

The following lemma is useful to show that $\mathbb{E}(T) < \infty$ for specific stopping times.

<u>Lemma</u> Suppose there exists $\varepsilon > 0$ & a positive integer N such that $\underbrace{\mathbb{P}(T \leq n + N \mid \mathcal{F}_n) \geq \varepsilon}$, for all n.
Then $\mathbb{E}(T) < \infty$.

"The probability of stopping at any point within the next N steps is at least $\varepsilon > 0$".

Proof: We have

$$\mathbb{P}(T > N) \leq 1 - \varepsilon \qquad \text{(first } N \text{ steps)}$$

$$\mathbb{P}(T > 2N \mid T > N) \leq 1 - \varepsilon \qquad \text{(steps } N+1, \ldots, 2N\text{)}$$

$$\mathbb{P}(T > 3N \mid T > 2N) \leq 1 - \varepsilon \qquad \ldots$$

So,
$$\mathbb{E}(T) \leq N \cdot \varepsilon + 2N\varepsilon(1-\varepsilon) + 3N\varepsilon(1-\varepsilon)^2 + \ldots$$
$$= N\varepsilon\left(1 + 2(1-\varepsilon) + 3(1-\varepsilon)^2 + \ldots\right)$$
$$= N\varepsilon \frac{1}{(1-(1-\varepsilon))^2} = \frac{N}{\varepsilon} < \infty \qquad \square$$

Example: Consider the simple random walk

$$X_n = \begin{cases} X_{n-1} + 1 & \text{with prob } \tfrac{1}{2} \\ X_{n-1} - 1 & \text{—"— } \tfrac{1}{2} \end{cases}, \quad X_0 = 0.$$

Take $T = \min\{n : |X_n| = a\}$, then $\mathbb{E}(T) < \infty$.

It follows by taking $N = a$, $\varepsilon = \frac{1}{2^a}$.

More generally, we can consider

$T = \min\{n : X_n \geq a \text{ or } X_n \leq -b\}$.

Since $|X_k - X_{k-1}| = 1$, the third (or second item) of Doob's optional stopping theorem applies.

This allows us to answer questions such as:

- What is the probability that we reach a before $-b$?
- What is the expected time for one of the two to happen.

We get $\mathbb{E}(X_T) = \mathbb{E}(X_0) = 0$ (from DOST)

$$\Leftrightarrow \quad a \cdot \mathbb{P}(X_T = a) + (-b)\,\mathbb{P}(X_T = -b) = 0$$

Further $\mathbb{P}(X_T = a) + \mathbb{P}(X_T = -b) = 1$

$$\Rightarrow \quad \mathbb{P}(X_T = a) = \frac{b}{a+b}, \quad \mathbb{P}(X_T = -b) = \frac{a}{a+b}$$

Now look at X_n^2:

$$\mathbb{E}(X_n^2 \mid \tilde{\mathcal{F}}_{n-1}) = \frac{1}{2}(X_{n-1}+1)^2 + \frac{1}{2}(X_{n-1}-1)^2$$

$$= X_{n-1}^2 + 1$$

It follows that

$$\mathbb{E}(X_n^2 - n \mid \tilde{\mathcal{F}}_{n-1}) = X_{n-1}^2 + 1 - n = X_{n-1}^2 - (n-1).$$

Hence $Y_n = X_n^2 - n$ is a martingale!!

The 2nd & 3rd item of DOST apply and

$$\mathbb{E}(Y_T) = \mathbb{E}(Y_0) = 0$$

$Y_T = X_T^2 - T =$ either $a^2 - T$ or $b^2 - T$.

$\Rightarrow \mathbb{E}(X_T^2) = \mathbb{E}(T)$ and

$a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = \mathbb{E}(T)$, so we find that

$$\mathbb{E}(T) = \frac{a^2 b + b^2 a}{a+b} = \frac{ab(a+b)}{a+b} = ab$$