Zeros and singularities. The residue theorem

Zeros and singularities

Recall: Suppose f analytic at z_0.

We call z_0 a zero of order m for f if

$$f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0.$$

Last time we proved the following

Thm Suppose f analytic at z_0.

Then f has a zero of order m at z_0

if and only if f can be written as

$$f(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

Corollary Suppose f analytic at z_0 and that $f(z_0) = 0$.

Then either f is identically zero in a

neighborhood of z_0, or there exists a

punctured disk about z_0 in which f has no zeros.

Proof: Let $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ be the Taylor

series for f in a neighborhood of z_0.

If $a_j = 0 \ \forall j \geq 0$ then f must be

identically zero in a neighborhood of z_0.

Otherwise, let $m = \min\{j : a_j \neq 0\}$.

Clearly then f has a zero of order m at z_0.

By the theorem above

$$f(z) = (z - z_0)^m \, g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$

Then g is continuous at z_0 and $g(z_0) \neq 0$,

so there exists a disk $|z - z_0| < \delta$ in which

$g(z) \neq 0$. Thus, $f(z) \neq 0$ for $0 < |z - z_0| < \delta$.

∎

If f is analytic in a domain D and vanishes

in some disk in D, in fact f must vanish

identically in D. This can be proven by an

argument similar to the one used in the proof

of the maximum principle. We therefore have:

Thm (Uniqueness principle)

If f and g are analytic on a domain D,

and if $f(z) = g(z)$ for z belonging to

a set that has a nonisolated point,

then $f(z) = g(z)$ for all $z \in D$

Remark: A point $z_0 \in E$ is called an isolated point of E if there exists a $\delta > 0$ s.t. the punctured disk $0 < |z - z_0| < \delta$ contains no points of E

<u>Def</u> A point z_0 is called an <u>isolated singularity</u> of f if f is analytic in some punctured neighborhood $0 < |z - z_0| < R$ of z_0, but not analytic at z_0.

Let z_0 be an isolated sing. of f.

Then f has a Laurent series expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \, , \quad 0 < |z - z_0| < R. \quad (*)$$

(Note: $r = 0$)

<u>Def.</u> Let z_0 be an isolated singularity of f, and let $(*)$ be its Laurent expansion in $0<|z-z_0|<R$.

(i) If $a_j = 0 \ \forall j < 0$, we say that z_0 is a <u>removable singularity</u> of f.

(ii) If $a_{-m} \neq 0$ for some positive integer m, but $a_j = 0 \ \forall j < -m$, we say that z_0 is a <u>pole of order m</u> for f.
(a pole of order 1 is called a simple pole)

(iii) If $a_j \neq 0$ for infinitely many negative j, we say that z_0 is an <u>essential singularity</u> of f.

If f has a removable sing. at z_0, its Laurent series takes the form:

$$f(z) = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots, \quad 0 < |z - z_0| < R.$$

Ex. $\frac{\sin z}{z}$ at $z=0$

By putting $f(z_0) := a_0$, f becomes analytic in $|z - z_0| < R$, and hence bounded in a neighborhood of z_0.

Conversely, we have

<u>Thm</u> (Riemann's thm on removable singularities)

Let z_0 be an isolated singularity of f.

If f is bounded in a punctured neighborhood of z_0, then f has a removable singularity at z_0.

<u>Proof</u>. From Laurent's theorem

$$a_j = \frac{1}{2\pi i} \oint_{|z - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \qquad (j \in \mathbb{Z})$$

for any ρ with $0 < \rho < R$.

So, if $|f(\zeta)| \le M$ near z_0, by the ML-ineq.

$$\Rightarrow \quad |a_j| \le \frac{1}{2\pi} \cdot \frac{M}{\rho^{j+1}} \cdot 2\pi\rho = \frac{M}{\rho^j}.$$

So if $j < 0$, letting $\rho \to 0$ we see $a_j = 0$. ∎

⑤

Ex. $\frac{1}{(z-1)^2}$ at $z=1$

If f has a pole of order m at z_0, the Laurent series (*) takes the form

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-(m-1)}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{z-z_0}}_{\text{principal part of } f \text{ at } z_0} +$$

$$+ a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots ; \quad a_{-m} \neq 0.$$

One easily obtains the following:

<u>Thm</u> Let z_0 be an isolated singularity of f. Then z_0 is a pole of order m for f iff f in a punctured neighborhood of z_0 can be written as

$f(z) = \frac{g(z)}{(z-z_0)^m}$, where g is analytic at z_0 and $g(z_0) \neq 0$.

<u>Proof</u>: $\Rightarrow$ Let $g(z) = a_{-m} + a_{-(m-1)}(z-z_0) + \dots$

$\Leftarrow$ Taylor expand g near z_0

From our characterization of zeros and poles of order m one also easily proves (exercise!):

<u>Thm</u> If f has a zero of order m at z_0, then $\frac{1}{f}$ has a pole of order m at z_0.

Conversely, if f has a pole of order m at z_0, then $\frac{1}{f}$ has a removable sing. at z_0, and if we define $\left(\frac{1}{f}\right)(z_0) = 0$, then $\frac{1}{f}$ is analytic with a zero of order m at z_0.

Thm Let z_0 be a isolated sing. of f.

Then z_0 is a pole of f if and only if

$|f(z)| \longrightarrow +\infty$ as $z \longrightarrow z_0$

Proof: $\Rightarrow$ If z_0 is a pole of order m, then

from $f(z) = \dfrac{g(z)}{(z-z_0)^m}$, g analy. at z_0 with $g(z_0) \neq 0$,

clearly $|f(z)| = |z-z_0|^{-m}\, |g(z)| \to +\infty$ as $z \to z_0$.

$\Leftarrow$ If $|f(z)| \to +\infty$ as $z \to z_0$, clearly $f(z) \neq 0$

near z_0, i.e. $h(z) = \dfrac{1}{f(z)}$ is analytic in a

punctured disk about z_0. Further $h(z) \to 0$, $z \to z_0$,

so by Riemann's thm h has a removable sing.

at z_0. So h extends to an analytic fcn near

z_0 with $h(z_0) = 0$. If m denotes the (finite!) order

of the zero of h at z_0, then f has a

pole of order m at z_0 ∎

Def A function f is said to be meromorphic

in a domain D if at every point of D it

is either analytic or has a pole.

The behavior of f near an essential singularity is much more complicated. The following holds:

Thm (Picard's (big) thm)

A fcn with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.

We prove the following weaker result.

Thm (Casorati – Weierstrass thm)

Suppose z_0 is an essential singularity of f.

Then, for every complex number w_0, there is a sequence $z_n \to z_0$ s.t. $f(z_n) \to w_0$.

Proof. If not, there is a w_0 and an $\varepsilon > 0$ s.t.

$|f(z) - w_0| \geq \varepsilon$ in a punctured disk about z_0.

Hence, $h(z) = \dfrac{1}{f(z) - w_0}$ is bounded in a punctured disk about z_0.

By Riemann's thm h has a removable sing. at z_0; so h can be defined at z_0 so as to become anal. at z_0.

Note that $f = \frac{1}{h} + w_0$. So if $h(z_0) \neq 0$ then f is bounded near z_0 and so f has a removable sing. at z_0, not an essential sing.

And if $h(z_0)=0$, then h has a zero of some (finite!) order $m \geq 1$ at z_0.

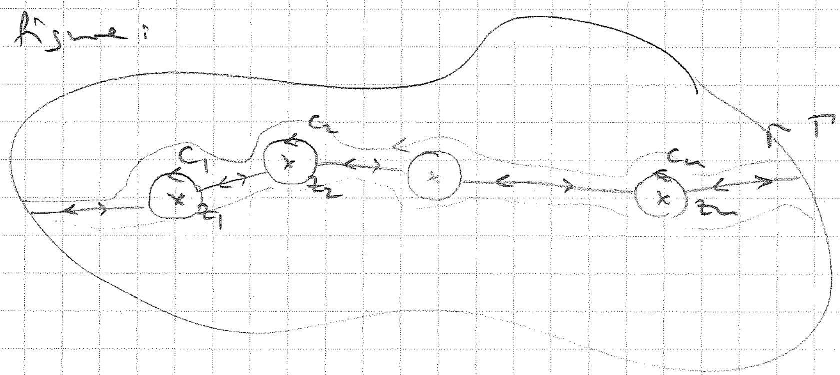
So then f has a pole of order m at z_0, not an essential sing.

$\square$

<u>The residue theorem</u>

Let Γ be a simple closed positively oriented contour in $\mathbb{C}$. Suppose f analytic inside and on Γ, with the exception of a finite number of isolated singularities $z_1, \ldots, z_n$ inside Γ.

See figure:



$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$

In a punctured neighb. of z_j the fcn f has a Laurent series expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z-z_j)^m . \qquad (a_m = a_m^{(j)})$$

According to Laurent's theorem, at z_j,

$$a_{-1} = \frac{1}{2\pi i} \int_{C_j} f(z)\, dz.$$

Def If f has an isolated sing. at the point z_0, then the coeff. a_{-1} of $\frac{1}{z-z_0}$ in the Laurent series expansion for f around z_0 is called the residue of f at z_0 and is denoted $\mathrm{Res}(f, z_0)$.

We therefore have that

$$\int_\Gamma f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

We have prove

Thm (Residue thm)

Let Γ be a simple closed positively oriented contour, and let f be analytic inside and on Γ with the exception of a finite number of isolated singularities $z_1, \ldots, z_n$ inside Γ. Then,

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$