UPPSALA UNIVERSITET

FÖRELÄSNINGSANTECKNINGAR

Tillämpad Matematik

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Contents

1. Introduction

The topic of this course will vary a lot, since mathematics can be applied to physics, biology, etc.
We will look into different ways to model real life, study it, and draw conclusions from it.

Anmärkning:
One could look at a mathematical models as a set of equations

Example: Planetary motion

- *Observation*: Keplers law $\to$ elliptic orbits
- *Model*: Newtons gravitational law
- *Mistakes/Errors*: Mercury precission $\to$ disalignment between model and observation
- *Rectify error*: Introducing relativistic effects in the model
- *Evaluation*: Is the old model useless? No, it is often easier to compute. It is better to keep it simple

We arrive at 2 models:

Good model$\to$ Simple, general (not valid in a specific way)

First step in the definition of a model: Understand which variables are involved

Dimension	Unit	Derived	Dimension
Distance	m	v	m/s
Temperature	Degrees	a	m/s
Time	s		

Definition/Sats 1.1: Physical law

A physical law is $f(q_1, \cdots, 1_n) = 0$

$L_1, \cdots, L_m$ are the dimensions
$[q_i] = L_1 \cdots L_m$

- $[q] = 1$ dimensions
- $[v] = L \cdot T^{-1}$

Example: Conservation of energy is an example of such physical law:

$$\frac{mp^2}{2} + V(q) = C \qquad C \in \mathbb{R}$$

$$F(, m, p, q) = \frac{mp^2}{2} + V(q) - C = 0$$

Example: Hooks law for springs:

$$F = \underbrace{k}_{\text{Not dimensionless}} \cdot L \qquad f(F, k, L) = 0$$

Definition/Sats 1.2: Unit free

A law is *unit free* if it is independent from the unit, in the sense that if we define a new system in the following way:

$$\overline{L_i} = \lambda_i L_i$$

Then $\overline{L_i}$ is a new system of unit $\lambda_i > 0$

$$[q_i] = L_1^{b_1} \cdots L_n^{b_n}$$

$$f(q, \cdots, q_n) = 0 \Leftrightarrow f(\overline{1_1}, \cdots, \overline{q_m}) = 0$$

Example:

$$f(x,t,q) = x - \frac{1}{2}gt^2 = 0$$

Describing a body falling. If we define the following units:

- $[x] = m$
- $[g] = ms^{-2}$
- $[t] = s$

We can check that if we use different units, say $\overline{x} = 1000x$ (kilometers instead of meters) or $\overline{t} = 3600t$ (hours instead of seconds), then we obtain the same law for $f(\overline{x}, \overline{t}, g) = 0$

Example: Just looking at the dimension we can say something about the model. Take the pendulum and study the period of oscillation (is the mass or the length the one that defines the period?)

The goal is to find a law for the period. Suppose only the length and the mass are the only variables in our model, then we want to find $P = f(l, m)$
Notice that we have an error in the dimension, since our period depends on time, so just looking at that we can see that there is something that is missing.

We could be interested in adding another term, the gravitational acceleration. We get:

$$T = kL^{\alpha_1}M^{\alpha_2}\frac{L^{\alpha_3}}{T^{-2\alpha_3}}$$

$$\begin{cases} \alpha_2 = 0 & \rightarrow \text{mass is not involved} \\ \alpha_1 + \alpha_3 = 0 \\ -2\alpha_3 = 01 & \alpha_3 = \dfrac{-1}{2} \qquad \alpha_1 = \dfrac{1}{2} \end{cases}$$

$$\Rightarrow P \approx k\sqrt{\frac{e}{g}}$$

Another thing we may do is to introduce dimensionless variables:

Definition/Sats 1.3: Pi:s theorem

Let $f(q_1, \cdots, q_m) = 0$ be a unit free law with the usual notation for dimension $[q_i] = L_1^{\alpha_{1i}} \cdots L_n^{alpha_{ni}}$ $n < m$

Define the dimension matrix A

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix}$$

Let π be the rank(A). Then there exists $m - r$ dimensionsless variabless $\Pi_1, \cdots, \Pi_{m-r}$ (which can be formed from q_i)

We hve an equivalent law $F(\Pi_1, \cdots, \Pi_{m-r}) = 0$

Anmärkning:
When we have a law, it does not mean that we have the right law (only $q_1, \cdots, q_n$ are involved) but it is not meaningless

The usefulness of Pi-theorem:

- Case in which only one dimensionsless variable is involved
  $F(\Pi_1) = 0 \rightarrow$ zeroes are discrete
  Π_1 can assume discrete values and can be deduced from experiments

In the case of 2 dimensionsless quantities $F(\Pi_1, \Pi_2) = 0$, if we can invert the relationship then we can write one variable as a function of the other using implicit function theorem.

$$\Pi_1 = f(\Pi_2) \qquad f \text{ is unknown} \to \text{deduced from observation}$$

Example: Allometry (Biology), the study of characteristics of living creatures change with their size. We look for a law that involves

- $q_1 = l =$ length of the organism $\quad [q_1] = L$
- $q_2 = t =$ time $\quad [q_2] = T$
- $q_3 = \rho =$ density $\quad [q_3] = \dfrac{M}{L^3}$
- $q_4 = a =$ resource assimilation rate $\quad [q_4] = \dfrac{M}{L^2T}$
- $q_5 = b =$ resource utilisation rate $\quad [q_5] = \dfrac{M}{L^3T}$

We look for a law that involves 2 dimensionsless variables, so we apply the theorem:

$$A = \begin{pmatrix} 1 & 0 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} L \\ M \\ T \end{pmatrix}$$

(Look at the exponent of the respective variable)
The rank$(A) = 3 \to 5 - 3 = 2$ dimensionsless variables
We can try to express q_i as a linear combination of the others. We know the following:

$$\begin{cases} \alpha_1 - 3\alpha_3 = -2 \\ \alpha_3 = 1 & \Rightarrow \alpha_1 = 1 \\ \alpha_2 = -1 \end{cases}$$

This means that q_4 can be expressed as $q_4 = \dfrac{q_1 q_3}{q_4}$, yielding:

$$\Pi_1 = \frac{q_1 q_3}{q_2 q_4} = \frac{l\rho}{ta} \to \text{dimensionsless}$$

We can do the same for $q_5 \Rightarrow q_5 = \dfrac{q_3}{q_2}$ yielding another dimensionsless variable:

$$\Pi_2 = \frac{q_3}{q_2 q_5} = \frac{\rho}{tb}$$

Summa sumarum:

$$F(\Pi_1, \Pi_2) = 0 = F\left(\frac{l\rho}{ta}, \frac{\rho}{tb}\right)$$
$$\pi_1 = f(\Pi_2)$$

1.1. **Scaling.**

The goal is to rescale variables to a quantity that is related to that specific problem. Measuring seconds when it comes to glaciers might be less useful as measuring with years, and seconds for a chemical reaction might be too little.

For example, with time, $\bar{t} = \dfrac{t}{t_c}$. New rescaled time is 1 once it has passed the desired scale. c stands for characteristic
The same can be done for other quantities such as length $\overline{h} = \dfrac{h}{h_c}$

Example: Projectile problem where we only consider gravity. Using Newtons gravitational law:

$$\frac{md^2h}{dt^2} = -G \cdot \frac{mM}{(R+h)^2} \Rightarrow \frac{d^2h}{dt^2} = -G\frac{M}{(R+h)^2}$$

We know that for $h = 0$, $\dfrac{d^2h}{dt^2} = -g = \dfrac{-GM}{R^2} = \dfrac{-gR^2}{(h+R)^2}$

We also know $h(0) = 0$, $\dfrac{dh}{dt}(0) = v$ (initial velocity)

We can introduce dimensionsless variables:

- $[t] = T$
- $[h] = L$
- $[R] = L$
- $[v] = LT^{-1}$
- $[g] = LT^{-2}$

Since only L, T are involved, we have 2 rows:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\mathrm{rank}(A) = 2 \Rightarrow 3$ dimensionless variables
We could for example do

$$\Pi_1 = \frac{h}{R} \qquad \Pi_2 = \frac{h}{vt} \qquad \Pi_3 = \frac{h}{gt^2}$$

Let us see what happens if we do some scaling for the time $\bar{t}$ and the length $\overline{h}$:

$$\bar{t} = \frac{t}{t_c} \qquad \overline{h} = \frac{h}{h_c}$$

With a dimension of time, we could pick $\dfrac{R}{v}$, or $\sqrt{\dfrac{R}{g}}$, $\dfrac{v}{g}$

The same for h, we could pick R, $\dfrac{v^2}{g}$

Usually only one choice is the one that helps us solve the problem:

$$\bar{t} = \frac{t}{R/v} \qquad \overline{h} = \frac{h}{R}$$

Now we need to express the laws that we have in terms of $\bar{t}$ and $\overline{h}$:

$$\frac{d^2h}{dt^2} = \frac{-gR^2}{(R\overline{h}+R)^2} = \frac{-g}{(\overline{h}+1)^2} \qquad h = \overline{h}R$$

$$\frac{dh}{dt} = \frac{d\overline{h}}{dt}R \qquad \frac{d\overline{h}}{d\bar{t}} = \frac{d\overline{h}}{dt}\frac{dt}{d\bar{t}} = \frac{R}{v}\frac{d\overline{h}}{dt}$$

$$\frac{d^2\overline{h}}{d\bar{t}^2} = \frac{d^2\overline{h}}{dt^2}\frac{R^2}{v^2} \to \frac{v^2}{Rg}\frac{d^2\overline{h}}{d\bar{t}^2} = -\frac{1}{(1+\overline{h})^2}$$

We can call $\varepsilon = \dfrac{v^2}{Rg}$ (ε small)

The equation $\varepsilon\dfrac{d^2\overline{h}}{d\bar{t}^2} = -\dfrac{1}{(1+\overline{h})^2}$ has no solution when $\varepsilon = 0$

With a different choice

$$\bar{t} = \frac{t}{vg^{-1}} \qquad \overline{h} = \frac{h}{v^2g^{-1}}$$

$$\Rightarrow \frac{d^2\overline{h}}{dt^2} = -\frac{1}{(1+\varepsilon\overline{h})^2} \qquad \overline{h}(0) = 0 \qquad \frac{d\overline{h}}{d\bar{t}}(0) = 1$$

Notice now that when $\varepsilon = 0$:

$$\overline{h}'' = -1 \qquad \overline{h}' = -\overline{t} + a = -\overline{t} + t$$

$$\overline{h} = -\frac{t^2}{2} + \overline{t} + b = -\frac{\overline{t}^2}{2} + \overline{t}$$

By substituting the old variables back, we get:

$$h = \frac{-t^2}{2} g + vt$$

The quantities that we used for t_c, h_c:

$$t_c = \frac{v}{g} \qquad h_c = \frac{v^2}{g}$$

$$h' = 0 \to -tg + v = 0 \Rightarrow t = \frac{v}{g}$$

Then h_c is the maximum height that the body reaches.