Exam
Graph Theory, 1MA170, August 2018

Johanna Strömberg
Uppsala Universitet

The exam consists of 5 questions worth 10 points each. Choose any 4 questions to answer. If you attempt all 5, the best 4 will count for your final grade. Answer each question carefully and with attention to details, citing any results from the lecture notes that you use. You may write your solutions in Swedish or English. Calculators are not allowed.

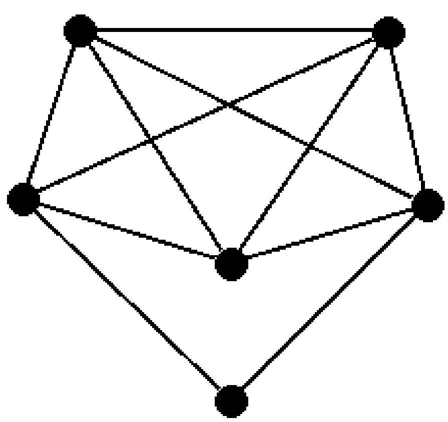
Good luck!

1. (Euler cycles)

   (a) Define *Eulerian graph.* (1)

   (b) State an algorithm for finding Euler cycles in a graph. (2)

   (c) Using the algorithm you stated find an Euler cycle in the graph below, clearly indicating each step of the algorithm: (2)

   (d) Prove Euler's theorem: a simple graph G is Eulerian iff it is connected and every vertex has even degree. (5)

2. (Hamiltonicity)

   (a) Define *Hamiltonian graph.* (1)

   (b) Explain how the computational problem of finding Hamilton cycles in a graph can be considered a special case of the Travelling Salesperson problem. (3)

   (c) Prove that $K_{n,n}$ is Hamiltonian for $n \geq 2$. Assuming the vertices are labelled, how many Hamilton cycles does it have? (4)

(d) Find a Hamilton path in the Petersen graph. (2)

3. (Planarity)

   (a) State Euler's formula for planar connected graphs. (1)

   (b) State and prove a generalisation of Euler's formula for planar graphs with k components. (4)

   (c) Prove or disprove: the dual of a planar graph is independent of its embedding. (3)

   (d) Which Platonic solids are duals of each other? (2)

4. (Spectral graph theory)

   (a) What is the *spectrum* of a graph G? (2)

   (b) Determine the spectrum of $K_{2,1}$. (3)

   (c) Let G be a graph with adjacency matrix A. Show that the entry (i, j) in A^k counts the number of walks of length k from vertex i to vertex j. (5)

5. (Extremal graph theory)

   (a) Prove that in any 2-edge colouring of K_6, there is a monochromatic K_3. (2)

   (b) Prove that if G is a simple graph on n vertices with $\delta(G) \geq (n-1)/2$, then G is connected. (4)

   (c) Let G be a simple graph on n vertices with chromatic number k. How many edges can G have? (4)