Lecture 12 Parity Relations

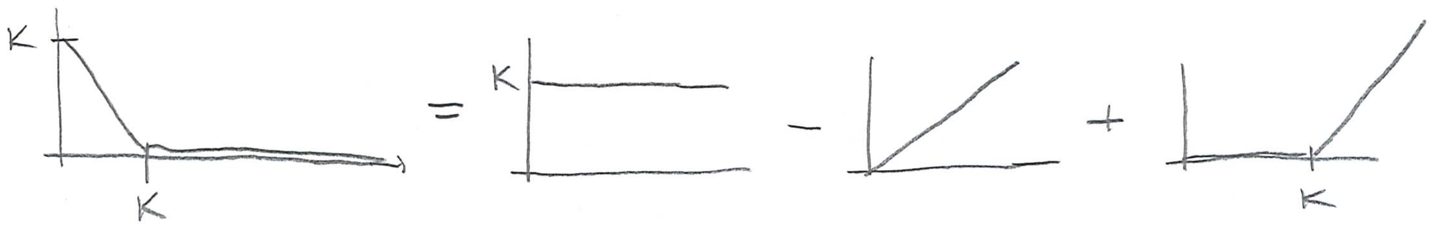
To replicate a T-claim in the BS-model we need continuous rebalancing of our portfolio. In reality, this is expensive (due to transaction costs).

Two approaches: 1. Static hedging

2. Delta and gamma hedging

10.1 Static Hedging

A put option can be replicated with a static portfolio of stocks, bonds and call options.



$$(s-K)^+ \quad = \quad K \quad - \quad s \quad + \quad (s-K)^+$$

Remark: A bond (or a zero-coupon T-bond) pays its owner a pre-determined fixed amount K at time T. If the interest rate is constant, the price of a T-bond is $Ke^{-r(T-t)}$. K is called the face value of the bond.

Prop 10.2 (put-call parity)

If $p(t,s)$ is the price at t of a put (strike K, maturity T) and $c(t,s)$ —— " —— call —— " ——, then
$$p(t,s) = Ke^{-r(T-t)} - s + c(t,s).$$
Moreover, the put can be replicated by a static portfolio consisting of a call, a short position in the stock, and a zero-coupon bond with face value K.

Ex: (What is the pricing formula for a put option in the standard BS-model?

Alternative 1: $p(t,s) = E^{Q}_{t,s}\left[e^{-r(T-t)}(K-S_T)^+\right] = e^{-r(T-t)}\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\left(K - se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,x}\right)dx$

$= \dots$ (long calculations)

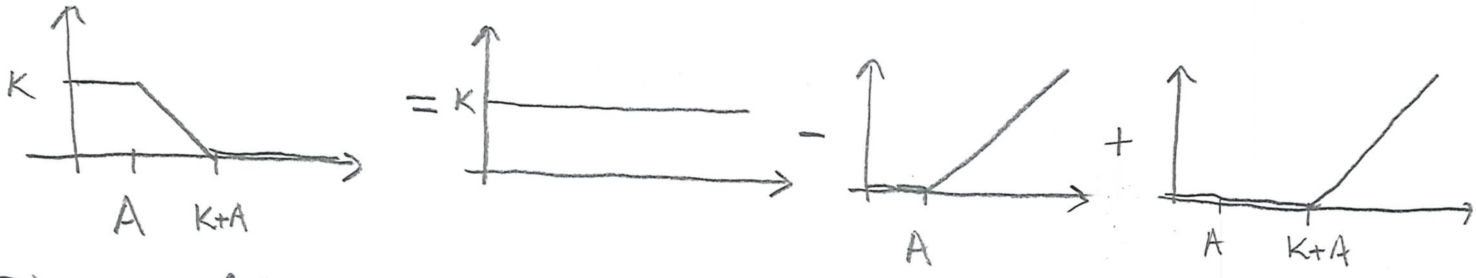
Alternative 2: Put-call parity gives

$$p(t,s) = Ke^{-r(T-t)} - s + c(t,s) = Ke^{-r(T-t)} - s + sN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$= KN(-d_2) - sN(-d_1), \quad \text{with } \begin{cases} d_1 = \dfrac{\ln\frac{s}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases}$$

Exercise 10.1

$$\mathcal{X} = \begin{cases} K & \text{if } S_T \le A \\ K + A - S_T & \text{if } A < S_T \le K+A \\ 0 & \text{if } K+A < S_T \end{cases}$$

Determine a static portfolio of stocks, bonds and call options that replicates $\mathcal{X}$.



The portfolio consisting of

- One zero-coupon bond with face value K
- One short position in a call with strike A
- one long position in a call with strike $K+A$

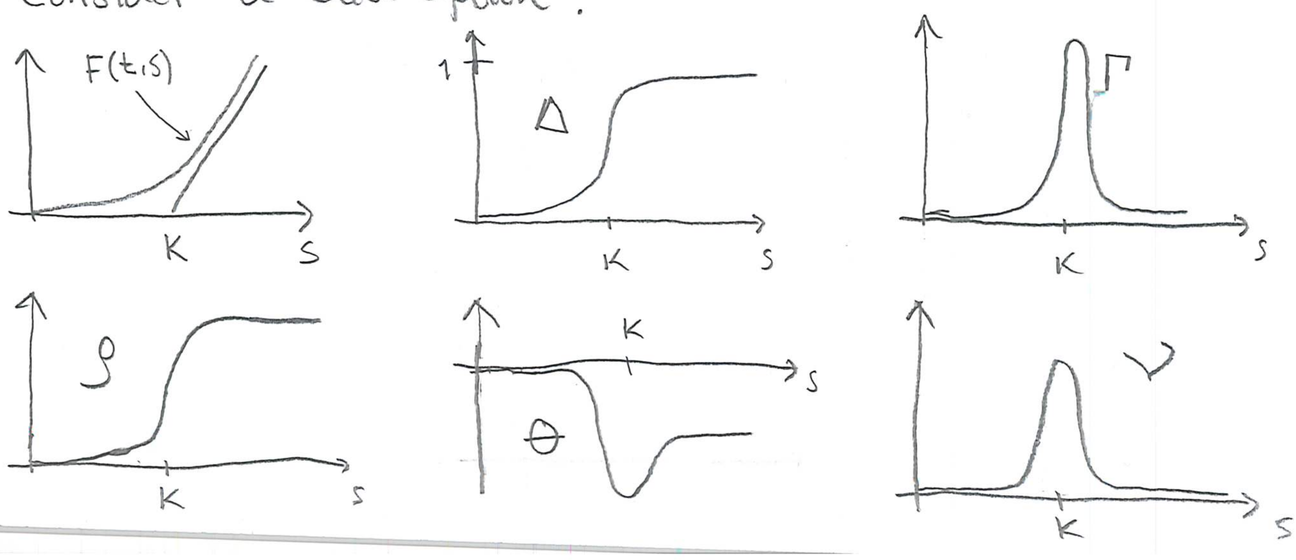
can be used to replicate $\mathcal{X}$.

10.2 The Greeks

Let $F(t,s)$ be the pricing function of a simple T-claim in the standard BS-model.

Def: $\Delta = \frac{\partial F}{\partial s}$ $\quad \Gamma = \frac{\partial^2 F}{\partial s^2}$ $\quad \rho = \frac{\partial F}{\partial r}$ $\quad \theta = \frac{\partial F}{\partial t}$ $\quad \nu = \frac{\partial F}{\partial \sigma}$ "vega"

Ex: Consider a call option.



10.3 Delta and Gamma Hedging

The seller of an option would often try to replicate it to reduce risk. In discrete time, the seller does as follows.

1. At $t=0$: sell the option, buy $F_s(0, S_0)$ shares of S, deposit $F(0, S_0) - S_0 F_s(0, S_0)$ in the bank.
2. At $t = \Delta t$: Adjust stock holdings to $F_s(\Delta t, S_{\Delta t})$ shares (in a self-financing way, i.e. adjust bank holdings accordingly).
3. At $t = k\Delta t$: Repeat the procedure until T.

The Δ of the whole portfolio (option, stocks, bank account) is close to 0. If $\Gamma = \frac{\partial \Delta}{\partial s}$ is small, then rebalancing can be made less frequently!

Let G be the pricing function of another claim $\mathcal{X}_G$ on the same stock S. Modify the strategy as follows:

Sell one option $\mathcal{X}_F$

Buy x_G units of $\mathcal{X}_G$ (where $\frac{\partial^2 F}{\partial s^2} = x_G \frac{\partial^2 G}{\partial s^2}$)

Buy x_s shares of S (where $\frac{\partial F}{\partial s} = x_s + x_G \frac{\partial G}{\partial s}$)

Deposit $F(0, S_0) - x_G G(0, S_0) - x_s S_0$ in the bank account.

This portfolio is Δ-neutral and Γ-neutral. Rebalancing can be made less frequently!