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Matematiska institutionen
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Financial Derivatives
2018-01-11

Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points. No means of assistance other than pencil and paper are allowed. Motivate your answers carefully!

1. Let the stochastic process Y be the solution of

$$\begin{cases} dY(t) = a\,dt + \sqrt{Y(t)}\,dW(t) \\ Y(0) = y, \end{cases}$$

where $y > 0$ and a is a constant. Determine $\mathbb{E}[Y(t)]$ and $Var(Y(t))$.

2. Solve the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) - 2u(t,x) = 0 \\ u(T,x) = \cos x. \end{cases}$$

3. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time 0 of a contract which at time T pays the holder the amount $\mathcal{X} = \frac{1}{T}\int_0^T S^2(t)\,dt$.

4. Explain briefly the following notions:
(a) model-independent arbitrage;
(b) implied volatility;
(c) completeness of a model;
(d) in-out parity for barrier options;
(e) forward price.

5. A certain T-claim written on a stock S pays its holder the amount

$$\mathcal{X} = \begin{cases} S(T) & \text{if } S(T) \leq 50 \\ 100 - S(T) & \text{if } 50 \leq S(T) \leq 100 \\ 0 & \text{if } S(T) > 100 \end{cases}$$

at time T. If the current value of the underlying stock is 80, a call option with strike 50 on the same stock trades at 32, and a call on S with strike 100 trades at the price 2, determine the arbitrage-free price of $\mathcal{X}$.

6. The Swedish short rate is 1%, the American short rate is 2%, and the current exchange rate is 8.2 SEK/USD (i.e. krona per US dollar). Consider a contract that 6 months from now pays the holder the larger amount of 800 SEK and 100 US dollar. If the volatility of the exchange rate is 0.4, determine the arbitrage-free price of the contract.

7. Consider the model

$$\begin{cases} dr(t) = -ar(t)\,dt + \sigma\,dW(t), & t \geq 0 \\ r(0) = r_0 \end{cases}$$

for the short rate under the pricing measure, where a and σ are given constants. Determine the term structure, i.e. calculate bond prices at time $t = 0$ in this model for all possible maturities T.

8. Consider a market consisting of a bank account with constant interest rate $r \geq 0$ and a stock S with constant volatility. The stock pays a proportional discrete dividend of size $\delta S(T_0-)$ at time T_0 (here $0 < \delta < 1$ and $T_0 \in [0, T]$). Consider a T-claim that pays $\mathcal{X} = S(T)$ at time T.

a) What is the arbitrage-free price of $\mathcal{X}$ at time 0?
b) Find a replicating strategy for $\mathcal{X}$.

GOOD LUCK!