UPPSALA UNIVERSITET

FÖRELÄSNINGSATECKNINGAR

Finasiella Derivat

Rami Abou Zahra

Inlämningsdatum
December 10, 2024

Contents

1. Options

Motivating Discussion:
Say a Swedish company has signed a contract to buy a machine from a US company for 100000USD to be paid at delivery 6 months from now. $T = \frac{1}{2}$ years.
Current exchange rate is 11SEK/USD. The buyer is suject to currency risk. There are 3 possible strategies to implement:

1. Buy 100000USD today and deposit in the bank.
The risk is eliminated but money is tied up for a long time and the company may not have access to this money

2. Buy a *forward contract* from a bank, i.e the bank delivers the sum you need at $T = \frac{1}{2} = t$, in return, the company payes some constant $K \cdot 100000USD$ at $T = t$, where K is chosen at $t = 0$ such that no transfer of money is needed at $t = 0$. Here, the bank takes all of the risk, but if the exchange rate drops below K then we would have preffered to do nothing.

3. Buy a *European call option* on 100000USD, with strike price K and exercise date T. I.e, it gives the right but not the obligation to buy 100000USD at price $K \cdot 100000USD$ at time $T = t$. If exchange rate at T is $> K$, then we use the option. If its below at $t = T$ thne we do not use the option (right, not obligation)

The last one is a good choice, but not free. This leads to the 2 main problems in the course:

- How much is a fair price for an option?
- If you are the seller of an option, how to protect (hedge) from risk of exchange rate not going up?

Motivating Example in discrete time
At $t = 0$, we can trade in a market with 2 assets:

- *Bank account* (risk-free/non-risky asset)
  At $t = 0$ the value is 1 and at $t = 1$ the value is 1
- *Stock* (risky asset)
  At $t = 0$, $S_0 = 100$ then it either grows ($S_1 = 120$) or declines ($S_1 = 80$) with probability $p = 0.6$ and $p = 0.4$ respectively

Definition 1.1 Call option

A *call option* is a contract that gives its holder the right but not the obligation to buy one share of a stock at time T with predetermined price K. Thus, at time $t = 1$, the option is worth $S_1 - K$ if $S_1 > K$ and 0 else

What is a fair price of the option? The sensible thing to pay would be $p(S_1 - K)$. Assuming $K = 110$ in the above example, then $0.6(120 - 110) = 6$. But this is not the best price!

The idea is to replicate the option by finding a trading stategy using both the risk-free (B) and the risky asset (S) such that the value of the stock at $t = 1$ coincides with the value of hte option.
Is that possible? Yes. Let x = amount in the bank at $t = 0$ and y be the number of shares of stock. We want to pick x, y such that regardles if stock goes up or down we have increase.
At $t = 1$

$$\left.\begin{aligned} x + S_1 y &= S_1 - K \\ x + S_1 y &= 0 \end{aligned}\right\}$$

If $K = 110$ and $S_1 = \{120, 80\}$, then $x = -20$ and $y = \frac{1}{4}$ since

$$\begin{cases} x + 120y = 10 \\ x + 80y = 0 \end{cases}$$

At $t = 0$. Our strategy is therefore to borrow 20 from the bank and buy $\frac{1}{4}$ of a share. The cost is $25 - 20 = 5$ which is less than 6.

At time $t = 1$ our holdings are worth $\frac{1}{4}S_1 - 20 = \begin{cases} 10 & \text{if } S_1 = 120 \\ 0 & \text{if } S_1 = 80 \end{cases}$ which is exactly the same as the option.

Conclusion:
By the APT (Arbitrage pricing theory), the price of the call must be equal to the cost of setting up this portfolio.

Remark:
The probabilities do not influence the option value. They were never used in the calculation of the price.

Remark:
Let us change p into q such that $\mathbb{E}(S_1) = S_0 = 100$ in the example, which value of q satisfies this? It is symmetric in the example, so let $p = q = \frac{1}{2}$
Then $\mathbb{E}(\max\{S_1 - k, 0\}) = 10 \cdot \frac{1}{2} + 0 \cdot \frac{1}{5} = 5$
In general, the option price is $\mathbb{E}^Q\left(\frac{B_0}{B_1}\max\{S_1 - k, 0\}\right)$ where Q is chosen such that $\mathbb{E}^Q\left(\frac{B_0 S_1}{B_1}\right) = \frac{S_0}{B_0}$

Notation:
$a^+ = \max\{a, 0\}$. In particular,

$$(s - K)^+ = \begin{cases} s - K & \text{if } s \geq K \\ 0 & \text{if } s < K \end{cases}$$

Exercise:
- In the above example, find a replicating strategy for a put option (right but not obligated to sell one share) at price $K = 110$
- Find the vlaue of the option at $t = 0$

Answer:

$$\left.\begin{aligned} x &= 90 \\ y &= \frac{-3}{4} \end{aligned}\right\} \quad \text{option value of 15}$$

2. Continous time & Brownian Motion

2.1. Simple Random Walk.

Let X_i be i.i.d.r.v with $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \dfrac{1}{2}$
Let $S_n = \sum_{i=1}^{n} X_i$, then this is a stochastic process, still in discrete time. Do note that the expectation is 0 for the r.v. and that:

$$\mathbb{E}(S_n) = \sum_{k=1}^{n} \mathbb{E}(X_i) = 0$$
$$\operatorname{Var}(S_n) = \mathbb{E}(S_n^2) - \underbrace{(\mathbb{E}(S_n))^2}_{=0} = \sum_{k=1}^{n} \operatorname{Var}(X_i) = \sum_{k=1}^{n} 1 = n$$

Note that this was discrete time, how do we proceed to make this continuous?
We do this by scaling to finer time. Frist, fix a time interval:
Stage 1
Let $X_0^1 = 0$
At $t = 0$, toss a coin, $X_T^1 = \begin{cases} \sqrt{T} & \text{heads} \\ -\sqrt{T} & \text{tails} \end{cases}$.
Here $\mathbb{E}(X_T^1) = 0$ and $\operatorname{Var}\left(X_T^1\right) = T =$ elapsed time.

Stage 2

Add another time step. Let $X_0^2 = 0$, toss a coin, $X_{T/2}^2 = \begin{cases} \sqrt{\dfrac{T}{2}} & \text{heads} \\ -\sqrt{\dfrac{T}{2}} & \text{tails} \end{cases}$

Repeat at $t = \dfrac{T}{2}$, adding/subtracting $\sqrt{\dfrac{T}{2}}$

Stage n
Let $X_0^n = 0$, at each time $t_k = \dfrac{k}{n}T$, toss a coin.
Define $X_{t_{k+1}}^n = X_{t_k}^n + Y_k$ where $Y_k = \pm\sqrt{\dfrac{T}{2}}$ with prob. 1/2. Simulating our coin tosses.
Here

$$\mathbb{E}(X_{t_k}^n) = \mathbb{E}\left(\sum_{i=1}^{k-1} Y_i\right) = \sum_{i=1}^{k-1} \mathbb{E}(Y_i) = 0$$
$$\operatorname{Var}\left(X_{t_k}^n\right) = \operatorname{Var}\left(\sum_{i=1}^{n} Y_i\right) \overset{\text{indep}}{=} \sum_{i=1}^{k} = \frac{T}{n}k = t_k$$

Now the question becomes, what happens when $n \to \infty$? We obtain *Brownian Motion*, aka Weiner process.

Definition 2.2 Brownian Motion

Brownian Motion is a stochastic process W if:
- $W_0 = 0$
- Independent increments, i.e $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent (as long as they are not overlapping)
- $W_t - W_s \sim N(0, t-s)$
- $t \mapsto W_t$ is continuous

This is a nice definition and all, but does there even exists something which satsifies our definition?

Sats 2.1

$t \mapsto W_t$ is of infinite variation and nowhere differentiable
By infinite variation, it is meant

$$\lim_{n\to\infty} \sum_k \left| W_{t_{k+1}} - W_{t_k} \right| = \infty$$

A regular differentiable function has bounded variation. The next goal is to define the stochastic integral $\int_0^t g_s dW_s$, where g_t is a stochastic process determined by the Brownian motion W

Definition 2.3 Measurable w.r.t σ-algebra

Let X_t be a stochastic process. An event A is $\mathcal{F}_t^X$ measurable (denoted $A \in \mathcal{F}_t^X$) if it is possible to determine whether A has happened or not based on observations of $\{X_s : 0 \le s \le t\}$

Example:
$A = \{X_s \le 7 : \forall s \le 9\} \in \mathcal{F}_9^X$

Definition 2.4

If a random variable Z can be determined by observations of $\{X_s : 0 \le s \le t\}$, then $Z \in \mathcal{F}_t^X$

Example:

$$Z = \int_0^5 X_s d_s \in \mathcal{F}_5^X$$

If you only know X_5 up to 4, then you cannot determine Z

Definition 2.5

A stochastic process Y_t with $Y_t \in \mathcal{F}_t^X \quad \forall t$ is *adapted to the filtration* $\mathcal{F}_t^X$

Example:
$Y_t = \sup_{0 \le s \le t} W_s$ is adapted to $\mathcal{F}_t^W$

Definition 2.6

The process $g_t \in \mathcal{L}^2$ if

- g is adapted to $\mathcal{F}_t^W$
- $\int_0^t \mathbb{E}(g_s^2) ds < \infty$

Example:

Brownian motion $\in \mathcal{L}^2$, its adapted to $\mathcal{F}_t^W$ and $\int_0^t \mathbb{E}(\overbrace{W_s^2}^{\sim N(0,\sqrt{s})})ds = \int_0^t s ds = \dfrac{t^2}{2} < \infty$

2.2. **Stochastic integration.**
Assume $g \in \mathcal{L}^2$. If g is simple (i.e $g_s = g_{t_k}$ for $s \in [t_k, t_{k+1}]$), then we define

$$\int_0^t g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})$$

For egeneral $g \in \mathcal{L}^2$, we can approximate g using step functions which are simple such that

$$\int_0^t \mathbb{E}((g_s - g_s^n)^2) ds \to 0 \quad \text{as } n \to \infty$$

Then, one defines the stochastic integral as

$$\int_0^t g_s dW_s = \lim_{n\to\infty} g_s^n dW_s$$

Remark
One can show that the limit indeed exists and does not depend on the sequence used for approximation.

Remark:
Forward increments are used! The integrand is fixed at t_k, and we look at forward movements of the Brownian motion.

Remark:
Steiltjes integration si not possible since paths are not of unbounded variation.

Proposition:
Assume $g \in \mathcal{L}^2$ and adapted to a filtration, then:
1. $\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0$
2. $\mathbb{E}\left(\left(\int_0^t g_s dW_s\right)^2\right) = 0 = \int_0^t \mathbb{E}(g_s^2) dW_s$ (Ito isometry)
3. $X_t = \int_0^t g_s dW_s$, then X_t is $\mathcal{F}^W$-adapted

Bevis 2.1

Assume g is simple (if it was not, then approximate using step functions).

1.

$$\mathbb{E}\left(\int_0^t g_s dW_s\right) = 0 = \mathbb{E}\left(\sum_{k=1}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right) = \sum_{k=0}^{n-1} \mathbb{E}\left(\underbrace{g_{t_k}}_{\text{indep.}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep.}}\right)$$
$$= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}) \mathbb{E}\underbrace{(W_{t_{k+1}} - W_{t_k})}_{\sim N(0,\sigma^2)} = 0$$

2. This is the variance of a stochastic integral:

$$\mathbb{E}\left(\left(\sum_{k=0}^{n-1} g_{t_k}(W_{t_{k+1}} - W_{t_k})\right)^2\right) = \mathbb{E}\left(\sum_{k=0}^{n-1} g_{t_k}^2 (W_{t_{k+1}} - W_{t_j})\right)^2 + 2\sum_{j<k} \underbrace{g_{t_k} g_{t_j}}_{\in \mathcal{F}_{t_k}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep. of } \mathcal{F}_{t_k}} \underbrace{(W_{t_{j+1}} W_{t_j})}_{\in \mathcal{F}_{t_k}}$$
$$= \sum_{k=0}^{n-1} \mathbb{E}\left(g_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2\right) + 2\sum_{j<k} \mathbb{E}\left(g_{t_k} g_{t_j} (W_{t_{k+1}} - W_{t_k})(W_{t_{j+1}} - W_{t_j})\right)$$
$$= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}^2) \mathbb{E}\left(\underbrace{(W_{t_{k+1}} - W_{t_k})^2}_{t_{k+1} - t_k}\right) + 2\sum_{j<k} \mathbb{E}(\cdots) \underbrace{\mathbb{E}(W_{t_k+1} - W_{t_k})}_{=0}$$
$$= \int_0^t \mathbb{E}(g_{t_k}^2) dW_s$$

$\square$

2.3. **Properties of the stochastic integral.**
Examples:
$\int_0^t 1 dW_s = W_t - W_0 = W_t$, but that is $\int_0^t W_s dW_s$? W_s is not piecewise constant, but we may approximate it by letting $g_t^n = W_{t_k}$ for $t \in [t_k, t_{k+1})$. What happens here is essentially discretisation but for finer and finer time.
This yields the approximation

$$\int_0^t \mathbb{E}\left((g_s^n - W_s)^2\right) ds = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \underbrace{\mathbb{E}\left((W_s - W_{t_k})^2\right)}_{s-t_k} \leftarrow \quad \text{variance of increment of BM}$$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds = \sum_{k=0}^{n-1} \frac{1}{2}(t_{k+1} - t_k)^2 = \sum_{k=0}^{n-1} \frac{1}{2} \Delta t$$

$$\Delta t = \frac{t}{n} \Rightarrow \frac{1}{2}(\Delta t)^2 \frac{t}{\Delta t} = \frac{\Delta t}{2} t \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}} - W_{t_k}) = \frac{1}{2} \sum_{k=0}^{n-1} \left(W_{t_{k+1}}^2 - W_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2\right) = \frac{1}{2} W_{t_n} - \underbrace{\frac{1}{2} \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2}_{I_n}$$

We claim $I_n \to t$ as $n \to \infty$:

$$\mathbb{E}(I_n) = \underbrace{\mathbb{E}\left(\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2\right)}_{\text{2nd moment}} = \sum_{k=0}^{n-1} (t_{k+1} - t_k) = t_n = t$$

Need to check $\mathbb{E}((I_n - t)^2) = 0$:

$$\mathbb{E}\left((\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2 - \overbrace{(t_{k+1} - t_k)}^{\Delta t}) \right)^2$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left(((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)^2 \right) + \sum_{j \neq k} \mathbb{E}\left(((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)((W_{t_{j+1}} - W_{t_j}) - \Delta t) \right)$$

$$= \sum_{j \neq k} \mathbb{E}\left((W_{t_{k+1}} W_{t_k})^4\right) - (\Delta t)^2 = \sum_{k=0}^{n-1} 2(\Delta t)^2 \sim \Delta t \to 0$$

hus, $I_n \to t$ as $n \to \infty$, so

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}$$

Remark:
Lets prove if $X \sim N(0, \sigma)$, then $\mathbb{E}(X^4) = 3\sigma^2$

$$\mathbb{E}(X^4) = \int z^4 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-z^2}{2\sigma^2}\right\} \overset{\text{parts}}{\Rightarrow} - \left[z^3 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}\right]_{-\infty}^{\infty} - \int 3z^2 \frac{\sigma^2}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\pi\sigma^3\right\} dz$$

$$= 3\sigma^2 \cdot \underbrace{\int z^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\}}_{\sigma^2} = 3\sigma^4$$

3. Martingales

Let $\mathcal{F}_t$ be a filtration, "information generated by B; up to a time t".
If Y is a random variable, then $\mathbb{E}(Y \mid \mathcal{F}_t)$ is the conditional expectation given all information up to time t

Example:

$$\mathbb{E}(W_s \mid \mathcal{F}_t) = W_t$$

Definition 3.7 Martingale

A process X is a martingale if X is $\mathcal{F}_t$-adapted. X_t integrable, i.e

- $\mathbb{E}(|X_t|) < \infty \quad \forall t$
- $\mathbb{E}(X_s \mid \mathcal{F}_t) = X_t$ for $s > t$

Example:
W_t is a martingale, $W_t^2 - t$ is a martingale since

$$\begin{aligned} Y_t := W_t^2 - t \qquad \mathbb{E}(Y_t \mid \mathcal{F}_s) &= \mathbb{E}(W_t^2 - t \mid \mathcal{F}_s) \\ &= \mathbb{E}((W_t - W_s)^2 + 2W_sW_t - W_s^2 \mid \mathcal{F}_s) - t \\ &= t - s + 2\mathbb{E}(W_sW_t \mid \mathcal{F}_s) - \mathbb{E}(W_s^2 \mid \mathcal{F}_s) - t = 2W_s \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} W_s^2 - s \\ &= W_s^2 - s = Y_s \end{aligned}$$

$Y_t = \int_0^t g_u dW_u$ is a martingale since:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = \mathbb{E}\left(\int_0^s g_u dW_u \mid \mathcal{F}_s\right) + \mathbb{E}\left(\int_s^t g_u dW_u \mid \mathcal{F}_s\right) = \int_0^s g_u dW_u = Y_s$$

However, W_t^3 is *not* a martingale:

$$\begin{aligned} \mathbb{E}(W_t^3 \mid \mathcal{F}_s) &= \mathbb{E}(W_s^3 + (W_t - W_s)^3 - 3W_tW_s^2 + 3W_t^2W_s \mid \mathcal{F}_s) \\ &= W_s^3 + 0 - 3W_s^2 \underbrace{\mathbb{E}(W_t \mid \mathcal{F}_s)}_{W_s} + 3W_s \underbrace{\mathbb{E}(W_t^2 \mid \mathcal{F}_s)}_{t-s+W_s^2} \\ &= W_s^3 + 3(t-s)W_s \neq W_s^3 \end{aligned}$$

Remark: A martingale is a "fair game"

4. Itos formula

Assume

$$X_t = a + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

for some adapted process μ_t and σ_t. Short-hand notation $\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t \\ X_0 = a \end{cases}$

Let $f(t,x)$ be a $C^{1,2}$-function and define $Z_t = f(t, X_t)$, what does dZ_t look like?

Recall:

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$$

so $W_t^2 = t + 2\int_0^t W_s dW_s$, thus

$$d(W_t^2) = dt + 2W_t dW_t$$

Fix n and let $t_k = \frac{k}{n}t$
Let $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$ and consider

$$S_n = \sum_{k=0}^{n-1} (\Delta W_{t_k})^2$$

We have

$$\mathbb{E}(S_n) = \sum_{k=0}^{n-1} \mathbb{E}\left((\Delta W_{t_k})^2\right) = \sum_{k=0}^{n-1} \frac{t}{n} = t$$

and

$$\operatorname{Var}(S_n) \overset{\text{indep.}}{=} \sum_{k=0}^{n-1} \operatorname{Var}\left((\Delta W_{t_k})^2\right) = n\operatorname{Var}\left((\Delta W_{t_0})^2\right) = n \cdot 2\frac{t^2}{n^2} \to 0 \qquad \text{as } n \to \infty$$

Thus $S_n \to t$ as $n \to \infty$ (in $\mathcal{L}^2$). This motivates to write

$$\int_0^t (dW_s^2) = t$$
$$\Leftrightarrow dW_t^2 = dt$$

4.1. Taylor Expansion.

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2} + \frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t + \text{ higher order terms} \\ &= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x}dW + \text{ higher order terms} \end{aligned}$$

Sats 4.2: Itos formula

If $dX_t = \mu_t dt + \sigma_t dW_t$ and $Z_t = f(t, X_t)$, then

$$dZ_t = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x}dW_t$$

Here $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(t, X_t)$ and similarly for other derivatives of f

Alternative formulation:

$$dZ_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

Where $(dX_t)^2$ is calculated using

- $(dt)^2 = 0$

- $dtdW_t = 0$
- $(dW_t)^2 = dt$

Example:
Compute $\int_0^t W_s dW_s$. Let $Z_t = W_t^2$, then by Itos formula

$$\begin{aligned} dZ_t &= 2W_t dW_t + \frac{1}{2} \cdot 2(dW_t)^2 \\ &= dt + 2W_t dW_t \end{aligned}$$

Thus $W_t^2 = Z_t = t + 2\int_0^t W_s dW_s$, so $\int_0^t W_s dW_s = \dfrac{W_t^2}{2} - \dfrac{t}{2}$

Example:
Compute $\mathbb{E}(W_t^4)$
Let $Z_t = W_t^4$, then by Itos formula

$$\begin{aligned} dZ_t &= 4W_t^3 dW_t + \frac{1}{2} \cdot 12W_t^2 (dW_t)^2 \\ &= 6W_t^2 dt + 4W_t^3 dW_t \end{aligned}$$

Thus

$$W_t^4 = Z_t = 6\int_0^t W_s^2 ds + 4\int_0^t W_s^3 dW_s$$

Taking expectation yields

$$\begin{aligned} \mathbb{E}(W_t^4) &= 6\int_0^t \underbrace{\mathbb{E}(W_s^2)}_{s} ds + 4\underbrace{\mathbb{E}\left(\int_0^t W_s^3 dW_s\right)}_{=0} \\ &= 6\int_0^t s ds = 3t^2 \end{aligned}$$

Alternatively, without using Itos formula

$$\begin{aligned} \mathbb{E}(W_t^4) = \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx &\overset{\text{parts.}}{=} \left[x^3 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} \right]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \\ &= 3t\mathrm{Var}\,(W_t) = 3t^2 \end{aligned}$$

Example:
Compute $\mathbb{E}(e^{\alpha W_t})$
Let $Z_t = e^{\alpha W_t}$. Itos formula yields

$$\begin{aligned} dZ_t &= \alpha e^{\alpha W_t} dW_t + \frac{1}{2}\alpha^2 e^{\alpha W_t} (dW_t)^2 \\ &= \frac{\alpha^2}{2} e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t \\ &= \frac{\alpha^2}{2} Z_t dt + \alpha Z_t dW_t \end{aligned}$$

Integration yields

$$Z_t = 1 + \frac{\alpha^2}{2}\int_0^t Z_s ds + \alpha\int_0^t Z_s dW_s$$

So

$$\begin{aligned} \mathbb{E}(Z_t) &= 1 + \mathbb{E}\left(\frac{\alpha^2}{2}\int_0^t Z_s ds\right) + \underbrace{\mathbb{E}\left(\alpha\int_0^t Z_s dW_s\right)}_{=0} \\ &= 1 + \frac{\alpha^2}{2}\int_0^t \mathbb{E}(Z_s) ds \end{aligned}$$

Let $m(t) = \mathbb{E}(Z_t)$, then

$$\begin{cases} \dfrac{dm}{dt} = \dfrac{\alpha^2}{2} m(t) \\ m(0) = 1 \end{cases}$$

Which has the solution $m(t) = e^{\frac{\alpha^2}{2}t}$

4.2. **Multi-dimensional Ito formula.**
Assume $dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$ where W^i are d independent Brownian motions.
On a matrix form:

$$\underbrace{dX_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1} dt + \underbrace{\sigma_t}_{n\times d} \underbrace{dW_t}_{d\times 1}$$

Let $Z_t = f(t, X_t)$ where $f : [0, \infty] \times \mathbb{R}^2 \to \mathbb{R}$ is $C^{1,2}$

Sats 4.3: Itos multi-dimensional formula

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

Where

- $dW_t^i dW_t^j = 0$ if $i \neq j$
- $\left(dW_t^i\right) = dt$
- $(dt)^2 = dt dW_t = 0$

Alternatively

$$dZ_t = \left(\frac{\partial f}{\partial t} + \sum_{i=1}^n \mu_t^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_t^{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \sigma_t^i dW_t$$

Where $C = \sigma\sigma^*$ and σ^i is the i:th row of σ
Indeend,

$$\begin{aligned} dX_t^i dX_t^j &= \left(\sum_{j)1}^d \sigma^{ik} dW^k \right) \left(\sum_{l=1}^d \sigma^{jl} dWl \right) \\ &= \left(\sum_{k=1}^d \sigma^{ik} \sigma^{jl} \right) dt \\ &= (\sigma\sigma^*)^{ij} dt \end{aligned}$$

Example:
If $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ dY_t = \gamma Y_t dt + \delta Y_t dV_t \end{cases}$ and $Z_t = X_t Y_t$; find dZ_t

Itos formula yields

$$\begin{aligned} dZ_t &= Y_t dX_t + X_t dY_t + \frac{1}{2} \cdot 2 dX_t dY_t \\ &= (\alpha + \gamma) Z_t dt + Z_t (\sigma dW_t + \delta dV_t) \end{aligned}$$

Setting $\overline{W}_t = \dfrac{1}{\sqrt{\sigma^2 + \delta^2}} (\sigma W_t + \delta V_t)$, then $\overline{W}$ is a Brownian Motion and

$$dZ_t = (\alpha + \gamma) Z_t dt + \sqrt{\sigma^2 + \delta^2} Z_t d\overline{W}_t$$

5. Correlated Brownian Motions

Let $\overline{W} = \begin{bmatrix} \overline{W}^1 \\ \vdots \\ \overline{W}^d \end{bmatrix}$ where $\overline{W}^1, \cdots, \overline{W}^d$ are independent

Consider $W = \delta \overline{W}$ where

$$\delta = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1d} \\ \vdots & \vdots & \vdots \\ \delta_{d1} & \cdots & \delta_{dd} \end{bmatrix} = \underbrace{\begin{bmatrix} \delta_1 \\ \vdots \\ \delta_d \end{bmatrix}}_{\text{Row vectors with } ||\delta_i|| = 1}$$

Here $||\delta_i|| = \sqrt{\delta_{i1}^2 + \cdots + \delta_{id}^2}$.
So W^i is a Brownian motion.

Moreover,

$$\begin{aligned} dW_t^i dW_t^j &= \left(\sum_{k=1}^d \delta_{ik} d\overline{W}_t^k\right)\left(\sum_{l=1}^d \delta_{jl} d\overline{W}_t^l\right) \\ &= \sum_{k=1}^d \delta_{ik}\delta dt = (\delta\delta^*)_{ij} dt \end{aligned}$$

Definition 5.8 Correlated Wiener Process

W_t as constructed above is a d-dimensional *correlated Wiener process* with correlation matrix $\rho = \delta\delta^*$

Sats 5.4: Itos formula, correlated version

If W_t is a correlated Wiener process as above, and

$$\underbrace{dX_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1} dt + \underbrace{\sigma_t}_{n\times d} \underbrace{dW_t}_{d\times 1}$$

satisfies

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

where

- $(dt)^2 = dt dW^i = 0$
- $dW^i dW^j = \rho_{ij} dt$

Example:

Given $\overline{W} = \begin{bmatrix} \overline{W}^1 \\ \overline{W}^2 \end{bmatrix}$ (where $\overline{W}^1, \overline{W}^2$ are independent), construct $W = \begin{bmatrix} W^1 \\ W^2 \end{bmatrix}$ with correlation matrix $\rho = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix}$

Note that $\delta = \begin{bmatrix} 1 & 0 \\ \rho_0 & \sqrt{1-\rho_0^2} \end{bmatrix}$ satisfies $\rho\rho^* = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix} = \rho$

Thus $W = \begin{bmatrix} \overline{W}^1 \\ \rho_0 \overline{W}^1 + \sqrt{1-\rho_0^2}\overline{W}^2 \end{bmatrix}$ is a correlated Wiener process with correlated matrix δ

What other choices for δ are possible?

6. Stochastic Differential Equations

Let

- a d-dimensiona Brownian motion W
- $\mu : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$
- $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$
- $x_0 \in \mathbb{R}^n$

be given. A *stochastic differential equation* is an equation at the form

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{1}$$

Or, equivalently,

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

Sats 6.5

Assume

$$||\mu(t,x) - \mu(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K\, ||x - y||$$

and $||\mu(t,x)|| + ||\sigma(t,x)|| \leq K\, ||x||$ for some K

Then there exists a unique solution X_t to the SDE (1). Moreover,
1. X is $\mathcal{F}^W$-adapted
2. X_t has continuous trajectories
3. X is a Markov process

7. Geometric Brownian Motion

Consider

$$\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t & \alpha, \sigma \text{ constans} \\ X_0 = x \end{cases}$$

Anmärkning:
If $\sigma = 0$, then $dX_t = \alpha X_t dt$ so $X_t = x_0 e^{\alpha t}$
Let $Z_t = \ln(X_t)$. Then

$$dZ_t \overset{\text{Ito}}{=} \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)\, A^2 = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma W_t$$

so $Z_t = \ln(x_0) + \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t$ and $X_t = e^{Z_t} = x_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t}$

Moreover,

$$\mathbb{E}(X_t) = x_0 + \mathbb{E}\left[\int_0^t \alpha X_s ds\right] + \underbrace{\mathbb{E}\left[\int_0^t \sigma X_s dW_s\right]}_{=0}$$

So if $m(t) = \mathbb{E}(X_t)$, we find $\begin{cases} \frac{dm}{dt} = \alpha m(t) \\ m(0) = x_0 \end{cases}$

Thus $m(t) = x_0 e^{\alpha t}$

Results:
The solution of $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ X_0 = x_0 \end{cases}$ is $X_t = x_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}$
Moreover, $\mathbb{E}(X_t) = x_0 e^{\alpha t}$

Example:

Consider the SDE $\begin{cases} dX_t = -X_t dt + dW_t \\ X_0 = x \end{cases}$ (this is a mean-reverting Ornstein-Uhlenbeck process)
The trick here is to let $Y_t = e^t X_t$. Then

$$dY_t = e^t X_t dt + e^t dX_t = e^t dW_t$$
$$\Rightarrow Y_t = x + \int_0^t e^s dW_s$$

Thus $X_t = e^{-t} Y_t = xe^{-t} + e^{-t} \int_0^t e^s dW_s$
Moreover $\mathbb{E}(X_t) = xe^{-t}$

Definition 7.9 Diffusion process

The solution X of an SDE

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW \\ X_0 = x_0 \end{cases}$$

is called a *diffusion process*.
μ is called the *drift* and σ is the *diffusion coefficient*

8. Partial Differential Equtions

Consider the following *terminal value problem*:
Given function σ, μ, ϕ, find a function $F(t, x)$ such that

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) + \frac{\sigma^2(t,x)}{2} \frac{\partial^2 F}{\partial x^2} F(t,x) + \mu(t,x) \frac{\partial F}{\partial t}(t,x) = 0 \\ F(T,x) = \phi(x) \end{cases} \tag{2}$$

If $F(t, x)$ satisfies (2), define X_s by $\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t = x \end{cases}$ and let $Z_s = F(s, X_s)$. Then

$$dZ_s \overset{\text{Ito}}{=} \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x} dX_s + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dX_s)^2$$
$$= \underbrace{\left(\frac{\partial F}{\partial s} + \mu \frac{\partial F}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} \right)}_{=0} ds + \sigma \frac{\partial F}{\partial x} dW_s$$
$$= \sigma \frac{\partial F}{\partial x} dW_s$$

Integrate:

$$Z_T = Z_t + \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s$$

Take expectation:

$$\mathbb{E}(Z_T) = Z_t = F(t,x) = \mathbb{E}(F(T, X_T)) \overset{*}{=} \mathbb{E}(\phi(X_t))$$

We write $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$ (to indicate that $X_t = x$)

We have thus proved the following:

Sats 8.6: Feynman-Kac

If $F(t, x)$ satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2(t,x)}{2} \frac{\partial^2 F(t,x)}{\partial x^2} + \mu(t,x) \frac{\partial F}{\partial x} = 0 \qquad (t < T) \\ F(t,x) = \phi(x) \end{cases}$$

then $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$ where $\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t = x \end{cases}$

Example:

Solve the PDE $\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2} = 0 \\ F(T,x) = x^2 \end{cases}$

Solution:

Let X_s be the solution of $\begin{cases} dX_s = \sigma dW_s \\ X_t = x \end{cases}$ i.e $X_s = x + \sigma(W_s - W_t)$

By Feynman-Kac:

$$F(t,x) = \mathbb{E}_{t,x}(X_T^2) = \mathbb{E}((x + \sigma(W_T - W_t))^2)$$
$$= x^2 + 2x\sigma\mathbb{E}(W_t - W_t) + \sigma^2\mathbb{E}((W_T - W_t)^2)$$
$$= x^2 + \sigma^2(T-t)$$

$F(t,x) = x^2 + \sigma^2(T-t)$

Sats 8.7: Feynman-Kac in higher dimensions + discounting

Assume that $F : [0,T] \times R^n \to \mathbb{R}$ satisfies

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{1}{2}\sum_{i,j=1}^n C_{i,j}(t,x)\dfrac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i(t,x)\dfrac{\partial F}{\partial x_i} - rF(t,x) = 0 \\ F(T,x) = \phi(x) \end{cases}$$

Where $C(t,x) = \sigma(t,x)\sigma^*(t,x)$ for some matrix σ $(n \times d)$
Then $F(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}(\phi(X_T))$ where

$$\begin{cases} dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \\ X_t = x \end{cases}$$

Bevis 8.1

Let $Z_s = e^{-r(s-t)}F(s,X_s)$. Then

$$dZ_s \overset{\text{Ito}}{=} e^{-r(s-t)}\underbrace{\left(\frac{\partial F}{\partial s} + \frac{1}{2}\sum_{i,j=1}^n C_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i \frac{\partial F}{\partial x_i} - rF\right)}_{=0} ds + e^{-r(s-t)}\sum_{i=1}^n \frac{\partial F}{\partial x_i}\sigma_i dW_s$$

So

$$Z_T = \underbrace{Z_t}_{F(t,x)} + \int_t^T \cdots dW_s = e^{-r(T-t)}\phi(X_T)$$

Thus $F(t,x) = e^{-r(T-t)}\mathbb{E}(\phi(X_T))$ □

Example:

Solve the PDE $\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\delta^2}{2}\dfrac{\partial^2 F}{\partial y^2} - rF = 0 \\ F(T,x,y) = xy \end{cases}$

Solution:

Here $C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \delta^2 \end{bmatrix}$ so $\sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix}$ satisfies $C = \sigma\sigma^*$

$$d\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} \Rightarrow \begin{cases} X_t = x + \sigma(W_T^1 - W_t^1) \\ Y_T = y + \delta(W_T^2 - W_t^2) \end{cases}$$

Feynman-Kac gives

$$F(t,x,y) = \mathbb{E}_{t,x,y}\left(e^{-r(T-t)}X_TY_T\right) = e^{-r(T_t)}\mathbb{E}\left(\left(x+\sigma(W_T^1-W_t^1)\right)\left(y+\delta(W_T^2-W_t^2)\right)\right)$$
$$\overset{\text{indep}}{=} e^{-r(T-t)}\mathbb{E}\left(x+\sigma(W_T^1-W_t^1)\right)\mathbb{E}\left(y+\delta(W_T^2-W_t^2)\right) = e^{-r(T-t)}xy$$

par Answer is therefore $F(t,x,y) = e^{-r(T-t)}xy$

Definition 8.10 Infitesimal Operator

The differential operator

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i\frac{\partial}{\partial x_i}$$

is called the *infitesimal operator* of X

Itos formula:
If $Z_t = f(t,X_t)$, then $dZ_t = \left(\frac{\partial f}{\partial t} + \mathcal{A}f\right)dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sigma_i dW_t$

9. Portfolio Dynamics

Let the time axis be discrete

Definition 9.11

- N = the number of different assets
- S_n^i = the price of one unit of asset i at time n
- h_n^i = the number of units of asset i bought at time n
- $h_n = (h_n^1, h_n^2, \cdots, h_n^N)$ is a *portfolio*
- V_n = the value of a portfolio h_n at time $n = \sum_{i=1}^N h_n^i s_n^i = h_n \cdot S_n$

The interpretation:

- At time $n-$ we have an old portfolio h_{n-1} from the previous period
- At time n, S_n becomes observable
- At time n, after observing S_n, we chose h_n

Definition 9.12 Budget equation

$h_n \cdot S_{n+1} = h_{n+1} \cdot S_{n+1}$

Notation: If $\{x_n\}_{n=0}^\infty$ is a sequence of real numbers, let $\Delta x_n = x_{n+1} - x_n$.
The budget equation becomes $S_{n+1} \cdot \Delta h_n = 0$

Recall $Y_n = h_n \cdot S_n$
Since $\Delta V_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_{n+1} + h_n \cdot S_{n+1} - h_n \cdot S_n$
$= S_{n+1} \cdot \Delta h_n + h_n \cdot \Delta S_n$
we have $\Delta V_n = h_n \cdot \Delta S_n$ if the budget equation is fulfilled.

Below we use this relation to *define* what is meant by a self-financing portfolio in continuous time.

Definition 9.13

Let $\{S_t \mid t \geq 0\}$ be an N-dimensional process

- A *portfolio* h is an $\mathcal{F}^s$-adapted N-dimensional process
- h is *Markovian* if $h_t = h(t, S_t)$ for some function h
- The *value process* V^h of h is

$$V_t^h = \sum_{i=1}^N h_t^i S_t^i = h_t \cdot S_t$$

- A portfolio h is *self-financing* if

$$dV_t^h = h_t \cdot dS_t$$

- For a given portfolio h, the corresponding *relative portfolio* w is

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h} \qquad i = 1, \cdots, N$$

Note that $\sum_{i=1}^N w_t^i = 1$.

Also, h is self-financing if and only if $dV_t^h = V_t^h \sum_{i=1}^N \frac{\partial w_t^i}{S_t^i} dS_t^i$

10. Arbitrage Pricing

In this chapter, $N = 2$ (two assets):

$$dB_t = rB_t dt$$

This is a risk-free asset, think bank account and r is a constant interest rate, and

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

is a risky asset, think stock price

Remarks:

1. $B_t = B_0 e^{rt}$
2. μ (local mean rate of return) and σ (volatility) are functions of t and current stock price
3. In the Black-Scholes model, μ and σ are constants

The aim is to find a "fair" value of options written on S
Options are also called *financial derivatives*

Definition 10.14 European Call Option

A *European call option* with strike price K and maturity date T on the underlying asset S is a contract such that the holder (owner) at time T has the right, but not the obligation to buy one share of S at price K from the option writer (seller)

Remarks:

- A *European put option* gives the right (but not the obligation) to *sell* one share of S at time T at price K
- An *American call/put* gives the right to buy/sell at *any* time before T

Definition 10.15

A *contingent claim with maturity* T (or a T*-claim*) is a random variable $X \in \mathcal{F}_T^S$
A contingent claim is *simple* is $X = \phi(S_T)$ for some *contract function* (or payoff function) ϕ

Example:

For a European call option, $\phi(x) = (x - K)^+ = \max\{x - K, 0\}$
Indeed, if $S_T \geq K$, then buy at price K and make profit $S_T - K$. If $S_T < K$, do not exercise the option.
For a European put option $\phi(x) = (K - x)^+$
We will determine the price $\pi(t, X)$ of a T-claim X at time t by requiring the market to be *arbitrage-free.*

Definition 10.16

A self-financing portfolio h is an *arbitrage* if $\begin{cases} V_0^h = 0 \\ \mathbb{P}(V_T^h \geq 0) = 1 \\ \mathbb{P}(V_T^h > 0) > 0 \end{cases}$

The market is *arbitrage-free* if no arbitrage exists.

Example:

$$\begin{cases} dS_t^1 = dt + dW_t \\ dS_t^2 = dW_t \\ dB_t = 0 \end{cases} \qquad \text{is not arbitrage free}$$

$$\begin{cases} dS_t^1 = dt + dW_t^1 \\ dS_t^2 = dW_t^2 \\ dB_t = 0 \end{cases} \qquad \text{is arbitrage free (first two lines indep)}$$

Assumption: The price process $\Pi_t(X)$ is such that $(B_t, S_t, \Pi_t(X))$ is arbitrage-free.
We also assume that all assets (including the option) can be sold/bought with no market frictions (no transaction consts, no liquidity constraints)

Idea: Create a self-financing portfolio of options and the sock such that its vlaue process is locally risk-free (has no dW-term). The drift of the valu emust then coincide with the interest rate (otherwise arbitrage). This will give a condition on the price of the option.

Assume $X = \phi(S_T)$ (simple T-claim) and that $\Pi_t(X) = F(t, S_t)$ for some function F.
New Notation: $F_t = \dfrac{\partial F}{\partial t}$, $F_s = \dfrac{\partial F}{\partial s}$, $F_{ss} = \dfrac{\partial^2 F}{\partial s^2}$

Then

$$
\begin{aligned}
dF(t, S_t) &\overset{\text{Ito}}{=} F_t dt + F_s dS_t + \frac{1}{2} F_{ss} (dS_t)^2 \\
&= \underbrace{\frac{\left(F_t + \dfrac{\sigma^2 S_t^2}{2} F_{ss} + \mu S_t F_s\right)}{F}}_{=\mu^F} F(t, S_t) dt + \underbrace{\frac{\sigma S_t F_s}{F}}_{=\sigma^F} F dW_t \\
&= \mu^F F dt + \sigma^F F dW_t
\end{aligned}
$$

Let (w^S, w^F) be a self financing relative portfolio of stocks and options ($w^S + w^F = 1$), and let V be its value process. Then

$$
\begin{aligned}
dV_t &= V_t \left(\frac{w^S}{S_t} dS_t + \frac{w^F}{F} dF_t\right) \\
&= \left(\mu w^S + \mu^F w^F\right) V_t dt + (\sigma w^S + \sigma^F w^F) V_t dW_t
\end{aligned}
$$

Let (w^S, w^F) be defined by

$$
\left.\begin{aligned}
w^S + w^F &= 1 \\
\sigma w^S + \sigma^F w^F &= 0
\end{aligned}\right\} \Leftrightarrow \begin{cases} w^s = \dfrac{\sigma^F}{\sigma^F - \sigma} \\ w^F = \dfrac{-\sigma}{\sigma^F - \sigma} \end{cases}
$$

Then $dV_t = \dfrac{\mu \sigma^F - \mu^F \sigma}{\sigma^F - \sigma} V_t dt$

By a no-arbitrage argument, we must have $r = \dfrac{\mu \sigma^F - \mu^F \sigma}{\sigma^F - \sigma}$

Here $\underbrace{r\sigma^F - r\sigma}_{=\frac{r\sigma S_t F_s}{F} - r\sigma} = \underbrace{\mu\sigma^F - \mu^F \sigma}_{=\frac{\mu\sigma S_t F_s}{F} - \frac{\sigma\left(F_t + \mu S_t F_s + \right) + \frac{-2S_t^2}{2} F_{ss}}{F}}$

$$
\begin{aligned}
r S_t F_s - r f &= \mu S_t F_s - F_t - \mu S_t F_s + \frac{\sigma^2}{2} S_t^2 F_{ss} \\
&= -F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} \\
F_t + \frac{\sigma^2 S_t^2}{2} F_{ss} &+ r S_t F_r - rF = 0
\end{aligned}
$$

Since S_t can take any value, F must satisfy the PDE

$$
F_t(t, s) + \frac{\sigma^2(t, s)}{2} s^2 F_{ss} + r s F_s(t, s) - r F(t, s) = 0
$$

Also, $\Pi_T(X) = F(T, S_T) = \phi(S_T)$, so we also have $F(T, S) = \phi(S_T)$

Sats 10.8: Black-Sholes equation

In the market $\begin{cases} dB_t = rB_t dt \\ dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \end{cases}$, the only arbitrage-free price of a T-claim $X = \phi(S_T)$ is $F(t, S_t)$, where $F(t, s)$ solves

$$\begin{cases} F_t(t, s) + \frac{\sigma^2(t, s)}{2} s^2 F_{ss}(t, s) + rsF_s(t, s) - rF(t, s) = 0 \\ F(T, s) = \phi(s) \end{cases}$$

The solution to the BS-equation is by Feynman-Kac

$$F(t, s) = \mathbb{E}_{t,s} \left(\exp\left\{-r(T - t)\right\} \phi(S_T) \right)$$

where

$$\begin{aligned} dS_u &= rS_u du + \sigma(u, S_u)S_u dW_u \\ S_t &= s \end{aligned} \tag{3}$$

we refer to

$$\begin{cases} dS_u = \mu(u, S_u)S_u du + \sigma(u, S_u)S_u dW_u \\ S_t = s \end{cases} \tag{4}$$

as the *P-dynamics* of S (the specification of S under the "physical measure" P). (3) is refered to as the *Q-dynamics* of S (Q is the *pricing measure*, or the *martingale measure*)

Sats 10.9

The arbitrage-free price of a simple T-claim $X = \phi(S_T)$ is $F(t, S_t)$ where

$$F(t, s) = \mathbb{E}^Q_{t,s} \left(\exp\left\{-r(T - t)\phi(S_T)\right\} \right)$$

and the Q-dynamics of S are as in (3)

Example:
In the standard BS-model (i.e constant σ), what is the arbitrage-free price of the T-claim $X = S_T^2$?
By risk-neutral valuation, $F(t, s) = \exp\left\{-r(T - t)\right\} \mathbb{E}^Q_{t,s}(S_T^2)$
Let $Y_u = S_u^2$, then

$$dY_u = 2S_u dS_u + (dS_u)^2 \overset{dS_u = rS_u du + \sigma S_u dW_u}{=} (2r + \sigma^2)Y_u du + 2\sigma Y_u dW_u$$

Y is a gBm and thus

$$\mathbb{E}^Q_{t,s}(S_T^2) = \mathbb{E}^Q(Y_T) = s^2 \exp\left\{(2r + \sigma^2)(T - t)\right\}$$

Which is the price of X at time t

Example:
What is the price of $X = S_t$?
By risk-neutral valuation

$$F(t, s) = \exp\left\{-r(T - t)\right\} \mathbb{E}^Q_{t,s}(S_T) = s$$

So the price at time t is S_t

Remark:
In time-homogenous models (such as the BS-model), the relevant quantity is time $T - t$ left to maturity.

Example: *Binary option*

In the standard BS-model, find the value of $X = \phi(S_T)$ where $\phi(x) = \begin{cases} 1 & \text{if } x \geq K \\ 0 & \text{if } x < K \end{cases}$

$$\begin{aligned}
F(0,s) &= \exp\{-rT\}\,\mathbb{E}^Q_{0,s}\left(I_{\{S_T\ge K\}}\right) = \exp\{-rT\}\,Q(S_T\ge K)\\
&= \exp\{-rT\}\,Q(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma W_T\right\}\ge K)\\
&= \exp\{-rT\}\,Q\left(\frac{1}{\sqrt{T}}W_T \ge \frac{\ln\left(\frac{K}{S}\right)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\\
&= \exp\{-rT\}\,Q\left(\frac{1}{\sqrt{T}}W_t \le \frac{\ln\left(\frac{S}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\\
&= \exp\{-rT\}\,N\left(\frac{\ln\left(\frac{S}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Where $N(x)\sim N(0,1)$, and the last line is the price at time t

Example:
What is the price of a European call option $X=(S_T-K)^+$? In the standard BS-model

$$F(0,s)=\exp\{-rT\}\,\mathbb{E}^Q_{0,s}\left((S_t-K)^+\right)=\exp\{-rT\}\,\mathbb{E}^Q\left(\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma W_T\right\}-K\right)^+\right)$$

$$=\exp\{-rT\}\int_a^\infty\left(s\exp\left\{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right\}-K\right)\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx \qquad a=\frac{\ln\left(\frac{K}{S}\right)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$\begin{aligned}
&s\int_a^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(x-\sigma\sqrt{T})^2}{2}\right\}dx-K\exp\{-rT\}\,N(-a)\\
&= s\int_{a-\sigma\sqrt{T}}^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx-K\exp\{-rT\}\,N(-a)\\
&= sN(\sigma\sqrt{T}-a)-K\exp\{-rT\}\,N(-a)
\end{aligned}$$

Here we used the fact that the normal-distribution has symmetric tails

Sats 10.10: Black-Scholes formula

In teh standard BS-model, the price of a European call option is $F(t,S_t)$, where

$$F(t,s)=sN(d_1)-K\exp\{-r(T-t)\}\,N(d_2)$$

and

$$\begin{cases} d_1=\dfrac{\ln\left(\frac{S}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\\ d_2=d_1-\sigma\sqrt{T-t}\end{cases}$$

Consider $F(0,s)=sN(d_1)-K\exp\{-rT\}\,N(d_2)$ as above, then we have

$$F(0,s)=\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)^+\right)\le\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T)\right)=s$$

and

$$F(0,s)=\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)^+\right)\ge\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}\,(S_T-K)\right)=s-K\exp\{-rT\}$$

We shall see below that $F(0,s)=F(0,s;\sigma)$ is increasing in σ

Remark:

What about the put option?

$$\mathbb{E}^Q_{0,s}\left(\exp\{-rT\}(K-S_T)^+\right) = \text{ similar to above}$$

Alternatively, $(K-s)^+ = K - s + (s-K)^+$. We have priced $(s-K)^+$, and s, so $p(0,s) = K\exp\{-rT\} - s + c(0,s)$ where p is the put price and c is the call price. This relation is called the *put-call parity*
Thus,

$$\begin{aligned} p(0,s) &= K\exp\{-rT\} - s + sN(d_1) - K\exp\{-rT\}N(d_2) \\ &= K\exp\{-rT\}\underbrace{(1-N(d_2))}_{=N(-d_2)} - s\underbrace{(1-N(d_1))}_{=N(-d_1)} \end{aligned}$$

Sats 10.11

Let $F(t,s)$ be the pricing function f a simple T-claim $X = \phi(S_T)$ in the standard BS-model.
If ϕ is convex, then:
1. $F(t,s)$ is convex in s
2. $F(t,s)$ is increasing in σ

Bevis 10.1

$$F(0,s) = \exp\{-rT\}\int_{\mathbb{R}} \phi\left(s\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx$$

1.

$$F_{ss} = \exp\{-rT\}\int_{\mathbb{R}} \phi''\left(s\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\right)\exp2\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx \geq 0$$

2.

$$\begin{aligned} \frac{\partial F}{\partial \sigma} &= \int_{\mathbb{R}} \phi'\left(s\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\right)s\exp\left\{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}x\right\}\sqrt{T}(x-\sigma\sqrt{T})\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx \\ &= s\sqrt{T}\int_{\mathbb{R}} \phi'\left(\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\right)(x-\sigma\sqrt{T})\exp\left\{-\frac{(x-\sigma\sqrt{T})^2}{2}\right\}\frac{1}{\sqrt{2\pi}}dx \\ &\overset{\text{parts.}}{=} s\sqrt{T}\int_{\mathbb{R}} \phi''()s\exp\left\{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right\}\sigma\sqrt{T}\exp\left\{-\frac{(x-\sigma\sqrt{T})^2}{2}\right\}\frac{1}{\sqrt{2\pi}}dx \geq 0 \end{aligned}$$

$\square$

10.1. Drift estimation.

Assume $X_t = \mu_t + \sigma W_t$ and we want a confidence interval for μ. An estimare for μ is $\widehat{\mu} = \dfrac{X_t}{t} \in N\left(\mu, \dfrac{\sigma}{\sqrt{t}}\right)$ and a confidence interval is

$$\left(\widehat{\mu} - \frac{\sigma}{\sqrt{t}} \cdot 1.96, \widehat{\mu} + \frac{\sigma}{\sqrt{t}} \cdot 1.96\right)$$

If one wants a certain precision $\Delta\mu$ so that $\mathbb{P}(\mu \in (\widehat{\mu} - \Delta\mu, \widehat{\mu} + \Delta\mu)) = 0.95$, one needs

$$\frac{2\sigma}{\sqrt{T}} = \Delta\mu \quad \Leftrightarrow \quad t = \frac{4\sigma^2}{(\Delta\mu)^2}$$

Plug in reasonable values $\left.\begin{aligned} \sigma &= 0.3 \\ \Delta\mu &= 0.06 \end{aligned}\right\} \Rightarrow t = 100$ years!

Remark:
When pricing options, the drift of the stock needs not be estimated (since under the pricing measure Q, the drift is r)

11. Volatility

In the BS-formula, s, r, t are observable, T, K are specified in the contract and σ is not directly observable. All are needed.
There are 2 approaches, one using *historic volatility* and one using *implied volatility.*

11.1. Historic volatility.

If $dS_t = \mu S_t dt + \sigma S_t dW_t$, then sample S at $n+1$ time points and let

$$\xi_i = \ln\left(\frac{S_{ti}}{S_{t_{i-1}}}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}}) \sim N\left((\mu - \frac{\sigma^2}{2})\Delta t, \sigma\sqrt{\Delta t}\right)$$

An esimate of σ^2 is then $S^2 = \dfrac{\sum_{i=1}^n (\xi_i - \overline{\xi})^2}{(n-1)\Delta t}$ where $\overline{\xi} = \dfrac{1}{n}\sum_{i=1}^n \xi_i$

11.2. Implied volatility.

Let p be the price in the market of a certain call option (maturity T, with strike price K). Find σ such that $p = \mathrm{BS}(s, t, T, r, \sigma, K)$ where BS denotes the Black-Scholes formula
This σ is called *implied volatility*

Remark:
Recall that the BS-formula is increasing in σ

If gBm is the correct model (i.e option prices are calculated using the BS-formula), then the *same* implied volatility would be obtained for different K and T

12. Completeness and Hedging

Definition 12.17

A T-claim X can be *replicated* if there exists a self-financing portfolio h with $\mathbb{P}(V_T^h = X) = 1$.
If every T-claim can be replicated then the market is *complete*

Sats 12.12

Assume that a T-claim X can be replicated using h. Then the only possible arbitrage-free price of X is $\Pi_t(X) = V_t^h$

Bevis 12.1

If for example $\Pi_t(X) < V_t^h$ for some t; sell the portfolio and buy the claim $\Rightarrow$ arbitrage $\square$

We now specialize to the model

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \end{cases} \tag{5}$$

with $\sigma(t, s) > 0$

Sats 12.13

The model (5) is complete

We will prove a simpler result, namely that all *simple* T-claims can be replicated.
Recall that the value $\Pi_t(X)$ of a simple T-claim $X = \phi(S_T)$ is $F(t, S_t)$ where $F(t, s)$ is the pricing function. Thus

$$\begin{aligned} d\Pi_t &= F_t dt + F_s dS_t + \frac{1}{2}F_{ss}(dS_t)^2 \\ &= \left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss}\right)dt + F_s dS_t \end{aligned}$$

Moreover, a portfolio $h = (h^B, h^S)$ is self-financing if $dV_t^h = h_t^B dB_t + h_t^s dS_t$. Choose $h_t^S = F_s(t, S_t)$

Sats 12.14

Let $X = \phi(S_T)$ and define $F(t, s)$ by

$$\begin{cases} F_t + \dfrac{\sigma^2 S^2}{2}F_{ss} + rsF_s - rF = 0 \\ F(T, s)\phi(s) \end{cases}$$

Define $h = (h^B, h^S)$ by

$$\begin{cases} h_t^B = \dfrac{F(t, S_t) - S_t F_s(t, S_t)}{B_t} \\ h_t^S = F_s(t, S_t) \end{cases}$$

Then h replicates X and $\Pi_t(X) = V_t^h = F(t, S_t)$

Bevis 12.2

$V_t^h = h_t^B B_t + h_t^S S_t = F(t, S_t)$, so

$$\begin{aligned} dV_t^h &= F_t dt + F_s dS_t + \frac{1}{2}F_{ss}(dS_t)^2 \\ &= \left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss}\right)dt + F_s dS_t \\ &\overset{\text{BS PDE}}{=} r(F - S_t F_s)dt + F_s dS_t = h_t^B dB_t + h_t^S dS_t \end{aligned}$$

Thus h is self-financing. Since $V_T^h = F(T, S_t) = \phi(S_T) = X$, h replicates X.
By no-arbitrage $\Pi_t(X) = V_t^h = F(t, S_t)$ $\square$

Example:
If $X = S_T$, then $F(t, s) = s$, so $h_t^S = F_s = 1$

Example:

For a call option (in the standard BS-model), $F(0,s) = sN(d_1) - K\exp\{-rT\}N(d_2)$, thus

$$F_S(0,s) = N(d_1) + \frac{1}{\sqrt{2\pi}}\left(s\exp\left\{-\frac{d_1^2}{2}\right\} - K\exp\{-rT\}\exp\left\{-\frac{d_2^2}{2}\right\}\right)\frac{\partial d_1}{\partial s}$$

Moreoever,

$$s\exp\left\{-\frac{d_1^2}{2}\right\} - K\exp\{-rT\}\exp\left\{-\frac{d_2^2}{2}\right\} = \exp\left\{-\frac{d^2}{2}\right\}\left(s - K\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{\sigma\sqrt{T}d_1\right\}\right) = 0$$

so $F_s(0,s) = N(d_1)$

Remark:
The derivative $\Delta = F_s$ is called the *delta*.
In a replicating portfolio one should hold Δ shares of S at each time.

If the pricing function is convex in S, then in order to replicate it then Δ goes up then buy more stock. Conversely, sell off if the opposite.

Example:
For a call option in the standard BS-model

$$F(0,s) = sN(d_1) - K\exp\{-rT\}N(d_2)$$

$$\text{Where}\begin{cases} d_1 = \dfrac{\ln\left(\frac{s}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 = \dfrac{\ln\left(\frac{s}{K}\right) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{cases}$$

Thus

$$\begin{aligned}\Delta = F_s(0,s) &= N(d_1) + s\varphi(d_1)\frac{1}{s\sigma\sqrt{T}} - K\exp\{-rT\}\varphi(d_2)\frac{1}{s\sigma\sqrt{T}} \\ &= N(d_1) + \frac{1}{\sigma\sqrt{T}}\left(\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)\right)\end{aligned}$$

Where

$$N(x) = \int_{-\infty}^{x}\varphi(z)dz$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}$$

The claim is that we are left with 0 on the second term, we check:

$$\begin{aligned}\sqrt{2\pi}\frac{\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)}{} &\overset{d_2 = d_1 - \sigma\sqrt{T}}{=} \exp\left\{-\frac{d_1^2}{2}\right\} - \frac{K}{s}\exp\{-rT\}\exp\left\{-\frac{\left(d_1 - \sigma\sqrt{T}\right)^2}{2}\right\} \\ &= \exp\left\{-\frac{d_1^2}{2}\right\}\left(1 - \frac{K}{s}\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{d_1\sigma\sqrt{T}\right\}\right) \\ &= \exp\left\{-\frac{d_1^2}{2}\right\}\left(1 - \frac{K}{s}\underbrace{\exp\{-rT\}\exp\left\{-\frac{\sigma^2 T}{2}\right\}\exp\left\{\ln\left(\frac{s}{K}\right) + (r + \sigma^2/2)T\right\}}_{\frac{s}{K}}\right) = 0 \\ \Rightarrow N(d_1) + \frac{1}{\sigma\sqrt{T}}\left(\varphi(d_1) - \frac{K}{s}\exp\{-rT\}\varphi(d_2)\right) &= N(d_1)\end{aligned}$$

The Δ is simply the first derivative of the pricing function.

13. Volatility Mis-specification

Assume that a trader believes in

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

whereas the stock actually follows

$$d\,\widetilde{S}_t = \widetilde{\mu}\,(t, \widetilde{S}_t)\,\widetilde{S}_t\,dt + \widetilde{\sigma}\,(t, \widetilde{S}_t) d\,\widetilde{W}_t$$

What happens if the trader tries to replicate a simple T-claim $x = \phi(\widetilde{S}_T)$?

The trader solves $\begin{cases} F_t + \dfrac{\sigma^2}{2} s^2 F_{ss} + rsF_s - rF = 0 \\ F(T, s) = \phi(s) \end{cases}$ and constructs a portfolio $h = (h^B, h^S)$ with initial value $V_0^h = F(0, s)$ containing $F_s(t, \widetilde{\S}_t)$ shares of $\widetilde{S}$ at each time (and $V_t^h - \widetilde{S}_t\, F_s(t, S_t))$ in the bank account.

The *tracking error* $Y_t = V_t^h - F(t, \widetilde{S}_t)$ satisfies $Y_0 = 0$ and

$$\begin{aligned} dY_t &= r(V_t^h - \widetilde{S}_t\, F_s)dt + F_s d\,\widetilde{S} - \left(F_t dt + F_s d\,\widetilde{S}_t + \frac{1}{2}\,\widetilde{\sigma}^2\,\widetilde{S}_t^{\,2}\, F_{ss} dt\right) \\ &= rV_t^h dt - \underbrace{\left(F_t + \frac{1}{2}\sigma^2\,\widetilde{S}^2\, F_{ss} + r\,\widetilde{S}_t\, F_s\right)}_{rF} dt + \frac{\sigma^2 - \widetilde{\sigma}^2}{2}\,\widetilde{S}_t^{\,2}\, F_{ss} dt \\ &= rY_t dt + \frac{\sigma^2 - \widetilde{\sigma}^2}{2}\,\widetilde{S}_t^{\,2}\, F_{ss} dt \end{aligned}$$

Thus, if $\sigma^2 \geq \widetilde{\sigma}^2$ and $F_\sigma \geq 0$, then $Y(T) = V(T) - \phi(\widetilde{S}_T) \geq 0$

A trader who overestimates volatility and who uses a model with a convex price will superreplicate the claim!

14. Asian Options

Asian options are option on the *average* of S.

An Asian call option pays $\chi = \left(\dfrac{1}{T}\int_0^T S_t dt - K\right)^+$ at T.

Note, it is not a simple T-claim!

Sats 14.15

Let $\chi = \phi(S_T, Z_T)$, where $Z_t = \int_0^t g(u, S_u)du$ for some function g.
Let $F(t,s,z)$ solve

$$\begin{cases} F_t + \dfrac{\sigma^2 s^2}{2} F_{ss} + rsF_s + g(t,s)F_z - rF = 0 \\ F(T,s,z) = \phi(s,Z) \end{cases}$$

and let $\begin{cases} h_t^B = \dfrac{F(t,S_t,Z_t) - S_t F_s(t,S_t,Z_t)}{B_t} \\ h_t^S = F_s(t,S_t,Z_t) \end{cases}$
Then h is self-financing and it replicates χ, with

$$\Pi_t(\chi) = V_t^h = F(t,S_t,Z_t)$$

Moreover, $F(t,s,Z) = \exp\{-r(T-t)\} \mathbb{E}_{t,s,z}^Q [\phi(S_T,Z_T)]$
where the Q-dynamics are

$$\begin{cases} dS_u = rS_u du + \sigma(u,S_u)S_u dW_u^Q \\ S_t = s \\ dZ_u = g(u,S_u)du \\ Z_t = z \end{cases}$$

Bevis 14.1

$$V_t^h = h_t^B B_t + h_t^S S_t = F(t,S_t,Z_t)$$

In particular, $V_T^h = F(T,S_T,Z_T) = \phi(S_T,Z_T) = \chi$
Moreover,

$$dV_t^h \overset{\text{Ito}}{=} F_t dt + F_s dS_t + \underbrace{F_z dZ_t}_{gdt} + \frac{1}{2}F_{ss}(dS_t)^2 + \underbrace{\frac{1}{2}F_{zz}(dZ)^2}_{=0} + F_{sz}\underbrace{dSdZ}_{=0}$$

$$= \underbrace{\left(F_t + \frac{\sigma^2}{2}S_t^2 F_{ss} + g(t,S_t)F_z\right)}_{=r(F-S_tF_s) \text{ by BS PDE}} dt + F_s dS_t$$

$$= r(F - S_t F_s)dt + F_s dS_t = h_t^B dB_t + h_t^S dS_t$$

So h is self-financing and replicates χ
Therefore, by no arbitrage, $\Pi_t(\chi) = V_t^h = F(t,S_t,Z_t)$
Finally, the stochastic representation follows from Feynman-Kac □

Example:
$\chi = \dfrac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_u du$ paid at T_2
What is the value of the T_2-claim χ at time 0?

$$\mathbb{E}_{t,s}^Q \left[\exp\{-r(T_2-t)\} \frac{1}{T_2-T_1}\int_{T_1}^{T_2} S_u du\right] = \frac{\exp\{-r(T_2-t)\}}{T_2-T_1}\int_{T_1}^{T_2} \underbrace{\mathbb{E}_{t,s}[S_u]}_{s\exp\{r(u-t)\}} du$$

$$= \frac{\exp\{-r(T_2-t)\}}{T_2-T_1}\frac{s}{r}\left(\exp\{r(T_2-t)\} - \exp\{r(T_1-t)\}\right)$$

$$= \frac{s}{r(T_2-T_1)}\left(1 - \exp\{-r(T_2-T_1)\}\right)$$

Which yields the answer, i.e the price is $\dfrac{S_t}{r(T_2-T_1)}\left(1 - \exp\{-r(T_2-T_1)\}\right)$

Remark:
All T-claims χ are priced as $\mathbb{E}^Q[\exp\{-rT\}\chi]$ (not only simple T-claims and Asian options)

Remark:
What is the value of χ in the previous exercise at $t \in [T_1, T_2]$?

$$\chi = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_u du = \underbrace{\frac{1}{T_2 - T_1}\int_{T_1}^{t} S_u du}_{\text{known at } t} + \underbrace{\frac{1}{T_2 - T_1}\int_{t}^{T_2} S_u du}_{y}$$

Price of y:

$$\begin{aligned}
&\mathbb{E}^Q_{t,s}\left[\exp\{-r(T_2 - t)\}\frac{1}{T_2 - T_1}\int_t^{T_2} S_u du\right] \\
&= \frac{\exp\{-r(T_2 - t)\}}{T_2 - T_1}\int_t^{T_2} s\exp\{r(u - t)\}\,du \\
&= \frac{s}{r(T_2 - T_1)}\left(1 - \exp\{-r(T_2 - t)\}\right)
\end{aligned}$$

The answer is $\frac{1}{T_2 - T_1}\left(\exp\{-r(T_2 - t)\}\int_{T_1}^t S_u du + \frac{S_t}{r}\left(1 - \exp\{-r(T_2 - t)\}\right)\right)$

14.1. **Completeness vs Absence of Arbitrage.**

1. The BS-model $\begin{cases} dB_t = rB_t dt \\ dS_t = \mu S_t dt + \sigma S_t dW_t \end{cases}$ is arbitrage-free and complete

2. The model

$$\begin{aligned}
dB_t &= rB_t dt \\
dS^1_t &= \mu_1 S^1_t dt + \sigma_1 S^1_t dW_t \\
dS^2_t &= \mu_2 S^1_t dt + \sigma_2 S^2_t dW_t
\end{aligned}$$

is complete, but (typically) *not* arbitrage free since one may construct a portfolio in S^1, S^2 with do dW term and with local mean rate of return $\neq r$

3. The model

$$\begin{aligned}
dB_t &= rB_t dt \\
dS_t &= \mu S_t dt + \sigma_1 S_t dW^1_t + \sigma_2 S_t dW^2_t
\end{aligned}$$

is arbitrage-free but *not* complete since $\chi = W^1_T$ cannot be replicated

Sats 14.16: Meta-theorem

Let M = the number of traded assets excluding B and R = the number random sources (BMs, Poisson processes) etc. Then:

- Absence of arbitrage $\Leftrightarrow M \leq R$
- Completeness $\Leftrightarrow M \geq R$
- Absence of arbitrage and completeness $\Leftrightarrow M = R$

15. Parity Relations

To replicate a T-claim in the BS-model, we need *continuous* rebalancing of our portfolio. In reality, this is expensive (due to transaction costs). There are two approaches to this:

1. Static hedging
2. Delta and gamma hedging

15.1. **Static Hedging.**

A put option can be replicated with a *static* portfolio of stocks, bonds and call options

Remark: A *bond* (or a *zero-coupan T-bond*) pays its owner a pre-determined fixed amount K at time T.
If the interest rate is constant, the price of a T-bond is $K\exp\{-r(T-t)\}$ where K is called the *face value* of the bond.

Lemma 15.1: Put-call parity

If $p(t,s)$ is the price at t of a put option (strike price K, maturity date T) and similarly $c(t,s)$ is the price of a call option, then

$$p(t,s) = K\exp\{-r(T-t)\} - s + c(t,s)$$

Moreover, the put can be replicated by a static portfolio consisting of a call, a short position in the stock, and a zero-coupon bond with face value K

Example:
What is the pricing formula for a put option in the standard BS-model?
Alternative 1:

$$\begin{gathered} p(t,s) = \mathbb{E}^Q_{t,s}\left[\exp\left\{-r(T-t)(K-S_T)^+\right\}\right] \\ = \exp\{-r(T-t)\}\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}\exp\left\{-x^2/2\right\}\left(K - s\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t x}\right\}\right)dx \\ = \cdots \end{gathered}$$

Alternative 2: Put-call parity yields

$$\begin{aligned} p(t,s) = K\exp\{-r(T-t)\} - s + c(t,s) &= K\exp\{-r(T-t)\} - s + sN(d_1) - K\exp\{-r(T-t)\}N(d_2) \\ &= KN(-d_2) - sN(d_1) \end{aligned}$$

where

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{s}{K}\right)+\left(r+\dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$

Example:

$$\chi = \begin{cases} K \quad \text{if } S_T \le A \\ K + A - S_T \quad \text{if } A < S_T \le K + a \\ 0 \quad \text{if } K + A < S_T \end{cases}$$

Determine a static portfolio of stocks, bonds, and call options that replicates χ
Here, χ can be graphed as the constant function K minus the linear function starting at A plus the linear function starting at $K+A$, so the portfolio consisting of:

- One zer-coupon bond with face value K
- One short position in a call with strike A
- One long position in a call with strike $K+A$

can be used to replicate χ

15.2. **The Greeks.**
Let $F(t,s)$ be the pricing function of a simple T-claim in the standard BS-model.

Definition 15.18

$$\Delta = \frac{\partial F}{\partial s} \quad \Gamma = \frac{\partial^2 F}{\partial s^2} \quad \rho = \frac{\partial F}{\partial r} \quad \theta = \frac{\partial F}{\partial t} \quad \nu = \frac{\partial F}{\partial \sigma}$$

15.3. **Delta and Gamma Hedging.**
The seller of an option would often try to replicate it to reduce risk. In discrete time, teh seller does as follows:

1. *At* $t = 0$: Sell the option, buy $F_s(0, S_0)$ shares of S, deposit $F(0, S_0) - F_s(0, S_0)$ in the bank
2. *At* $t = \Delta t$: Adjust stock holdings to $F_s(\Delta t, S_{\Delta T})$ shares (in a self-financing way, i.e adjust bank holdings accordingly)
3. *At* $t = k\Delta t$: Repeat until T

The Δ of the whole portfolio (option, stocks, bank account) is close to 0. If $\Gamma = \dfrac{\partial \Delta}{\partial s}$ is small, then chaning in Δ is small and then rebalancing can be made less frequently!

Let G be the pricing function of another lcaim χ_G on the same stock S. Modify the strategy as follows:

- Buy x_G units of χ_G (where $\dfrac{\partial^2 F}{\partial s^2} = x_G \dfrac{\partial^2 G}{\partial s^2}$)
- Buy x_s shares of S (where $\dfrac{\partial F}{\partial s} = x_s + x_G \dfrac{\partial G}{\partial s}$)
- Deposit $F(0, S_0) - x_G G(0, S_0) - x_s S_0$ in the bank account.

This portfolio is Δ-neutral *and* Γ-neutral. Rebalancing can be made less frequently!

16. Multi-dimensional Models

Definition 16.19 Multi Dimensional Model

A model $\begin{cases} dB_t = rB_t dt \\ dS_t^i = \mu_i S_t^i dt + S_t i \sum_{j=1}^n \sigma_{ij} dW_t^j \end{cases}$ where r, μ_i, σ_{ij} are constants and $\begin{pmatrix} \sigma_{11} & \cdots & \sigma_{in} \\ \vdots & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$ is a non-singular matrix is a *multi-dimensional* model

Remark:
In the meta-theorem, $R = M = n$, so we expect the market to be arbitrage-free and complete.

The question becomes, what is the arbitrage-free price of a simple T-claim $\chi = \phi(S_T)$?

The idea is that we could construct a portfolio of $S^1, S^2, \cdots, S^n, \Pi(\chi)$ which is locally risk-free (no dW-terms). Then, to avoid arbitrage, the drift of the portfolio must be r. This will yield a PDE for the price.

Instead, we will take the following route. We *guess* that the price is $\Pi_t(\chi) = F(t, S_t^1, \cdots, S_t^n)$ where $F(t, S_1, \cdots, S_n)$ satisfies

$$\begin{cases} F_t + \frac{1}{2} \sum_{i,j=1}^n S_i S_j C_{ij} F_{s,S_j} + s \sum S_i F_{S_i} - rF = 0 \\ F(T, S_1, \cdots, S_n) = \phi(S_1, \cdots, S_n) \end{cases} \tag{6}$$

where $C = \sigma\sigma^*$
To show that the guess is correct, we give a replication argument.

Sats 16.17

To avoid arbitrage, the price of $\chi = \phi(S_T)$ has to be $F(t, S_t)$ where $F(t, s)$ is given by (6) above. Moreover, χ is replicated by $h = (h^B, h^1, \cdots, h^n)$ where

$$\begin{cases} h_t^B = \dfrac{F(t, S_t) - \sum_{i=1}^n S_t^i F_{S_i}(t, S_t)}{B_t} \\ h_t^i = F_{S_i}(t, S_t) \qquad (i = 1, \cdots, n) \end{cases}$$

Bevis 16.1

$$V_t^h = h_t^B B_t + \sum_{i=1}^n h_t^i S_t^i = F(t, S_t)$$

So $V_T^h = F(T, S_T) = \phi(S_T) = \chi$ which is the correct terminal value.
We have

$$\begin{aligned} dV_t^h &\overset{\text{Ito}}{=} F_t dt + \sum_{i=1}^n F_{S_i} dS_t^i + \frac{1}{2} \sum_{i,j=1}^n F_{S_i,S_j} (dS_t^i)(dS_t^j) \\ &= \left(F_t + \frac{1}{2} \sum_{i,j=1}^n S_t^i S_t^j C_{ij} F_{S_i,S_j} \right) dt + \sum_{i=1}^n F_{S_i} dS_t^i \\ &\overset{(6)}{=} \left(rF - r \sum_{i=1}^n S_t^j F_{S_i} \right) dt + \sum_{i=1}^n F_{S_i} dS_t^i \\ &= h_t^B dB_t + \sum_{i=1}^n h_t^i dS_t^i \end{aligned}$$

Thus h is self-financing and it replicates χ.
Any price different from $V_t^h = F(t, S_t)$ would lead to an arbitrage ∎

Sats 16.18: Risk Neutral Valuation

The prcing function has the representation

$$F(t,s) = \mathbb{E}^{Q}_{t,s}\left[\exp\left\{-r(T-t)\right\}\phi(S_T)\right]$$

Where the Q-dynamics of S are $\begin{cases} dS^i_u = rS^i_u du + S^i_u \sum_{j=1}^{n} \sigma_{ij} dW^j_u \\ S^i_t = S_i \end{cases}$

16.1. **Reducing the state space.**
Let $n = 2$, and assume that $\phi(kS_1, kS_2) = k\phi(S_1, S_2)$ for $k > 0$.
Then $\phi(S_1, S_2) = S_2 \phi\left(\dfrac{S_1}{S_2}, 1\right)$

Ansatz:

$$F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right)$$

For some function $G(t,z)$

The terminal condition $F(T, S_1, S_2) = \phi(S_1, S_2)$ translates into $G(T,z) = \phi(z, 1)$
We now translate all derivatives in the BS-equation:

$$F_t + \frac{1}{2}S_1^2 C_{11} F_{S_1S_1} + \frac{1}{2}S_2^2 C_{22} F_{S_2S_2} + S_1 S_2 C_{12} F_{S_1S_2} + rS_1 F_{S_1} + rS_2 F_{S_2} - rF = 0$$

Into derivatives of G :

$$F_t = S_2 G_t \qquad F_{S_1S_1} = \frac{1}{S_2} G_{zz}$$

$$F_{S_1} = G_z \qquad F_{S_1S_2} = \frac{-S_1}{S_2^2} G_{zz}$$

$$F_{S_2} = G - \frac{S_1}{S_2} G_z \qquad F_{S_2S_2} = \frac{S_1^2}{S_2^3} G_{zz}$$

We get:

$$S_2 G_t + \frac{1}{2}\frac{S_1^2}{2} C_{11} G_{zz} + \frac{1}{2}\frac{S_1^2}{S_2} C_{22} G_{zz} - \frac{S_1^2}{S_2} C_{12} G_{zz} + rS_1 G_z + rS_2 G - rS_1 G_z - rS_2 G = 0$$

which simplifies to

$$G_t + \frac{1}{2}\frac{S_1^2}{S_2^2}\left(C_{11} + C_{22} - 2C_{12}\right) G_{zz} = 0$$

Since the argument of G and its derivatives is $\left(t, \dfrac{S_1}{S_2}\right)$, we have the following:

Lemma 16.1

Assume $\phi(kS_1, kS_2) = k\phi(S_1, S_2)$, then $F(t, S_1, S_2) = S_2 G\left(t, \dfrac{S_1}{S_2}\right)$ where $G(t,z)$ solves

$$\begin{cases} G_t + \dfrac{1}{2}\left(C_{11} + C_{22} - 2C_{12}\right) z^2 G_{zz} = 0 \\ G(T,z) = \phi(z,1) \end{cases}$$

Example:

$$\begin{cases} dS^1_t = \mu_1 S^1_t dt + \sigma_1 S^1_t dW^1_t \\ dS^2_t = \mu_2 S^2_t dt + \sigma_2 S^2_t dW^2_t \\ dB_t = rB_t dt \end{cases}$$

Let $\chi = \left(S^1_T - S^2_T\right)^+$. This is an *exchange option.* It gives the right to exchange one share of S^2 for one share of S^1

We have $\phi(S_1, S_2) = (S_1 - S_2)^+$ so $\phi(kS_1, kS_2) = k\phi(S_1, S_2)$
By our recipe, $F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right)$ where $G(t, z)$ solves

$$\begin{cases} G_t + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right) z^2 G_{zz} = 0 \\ G(T, z) = (z-1)^+ \end{cases}$$

Using the BS-formula, $G(t, z) = zN(d_1) - N(d_2)$ so

$$F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right) = S_1 N(d_1) - S_2 N(d_2)$$

Where

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{S_1}{S_2}\right) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t)}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}} \\ d_2 = d_1 - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)} \end{cases}$$

Example:

In the market $\begin{cases} dB_t = rB_t dt \\ dS_t^1 = \mu S_t^1 dt + \sigma_1 S_t^1 dW_t^1 \\ dS_t^2 = \mu_2 S_t^2 dt + \sigma_2 S_t^2 \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right) \end{cases}$

Find the price at $t = 0$ of the T-claim $\chi = \dfrac{(S_T^1)^2}{S_T^2}$

To answer this, notet that $\phi(S_1, S_2) = \dfrac{S_1^2}{S_2}$, to $\phi(kS_1, kS_2) = k\phi(S_1, S_2)$
Thus, $F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right)$ where

$$\begin{cases} G_t + \frac{1}{2} z^2 \left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) G_{zz} = 0 \\ G(T, z) = z^2 \end{cases}$$

par Let $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, we have

$$G(0, z) = \mathbb{E}_{0,z}\left[Z_T^2\right] \qquad dZ_t = \sigma Z dW_t$$

Let $Y_t = Z_t^2$, then

$$dY_t = 2Z_t dZ_t + (dZ_t)^2 = \sigma^2 Y_t dt + 2\sigma Y_t dW_t$$

so $G(0, z) = \mathbb{E}\left[Z_T^2\right] = z^2 \exp\left\{\sigma^2 T\right\}$

Answer: $F(0, S_1, S_2) = S_2 G\left(0, \dfrac{S_1}{S_2}\right) = \dfrac{S_1^2}{S_2} \exp\left\{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) T\right\}$

Example:

$$\begin{cases} dS_t^1 = \mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t^1 \\ dS_t^2 = \mu_2 S_t^2 dt + \sigma_2 S_t^2 dW_t^2 \\ dB_t = rB_t dt \end{cases}$$

Here $dW^1 dW^2 = \rho dt$. Let $\chi = \left(S_T^1 - S_T^2\right)^+$.
By our recipe $F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right)$ where $G(t, z)$ satisfies

$$\begin{cases} G_t + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) z^2 G_{zz} = 0 \\ G(T, z) = (z-1)^+ \end{cases}$$

Using the BS formula

$$G(t, z) = zN(d_1) - N(d_2)$$

where

$$\begin{cases} d_1 = \dfrac{\ln(z) + \dfrac{\overbrace{\sigma^2}^{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \dfrac{\ln(z) - \dfrac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \end{cases}$$

Thus, the pricing function F is

$$\begin{aligned} F(t, S_1, S_2) &= S_2 G\left(t, \frac{S_1}{S_2}\right) = S_2\left(\frac{S_1}{S_1}N(d_1) - N(d_2)\right) \\ &= S_1 N(d_1) - S_2 N(d_2) \end{aligned}$$

Where d_1, d_2 are now equal to

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{S_1}{S_2}\right) + \dfrac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \dfrac{\ln\left(\dfrac{S_1}{S_2}\right) - \dfrac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \end{cases}$$

Remark:
In general, the payoff function ϕ could be something like $\min\{S_1(T), S_2(T)\}$, then according to the recipe we should plug in for the terminal condition $\min\{z, 1\} = \phi(z, 1)$.
This is a linear function minus a call option, so it is solvable. For the linear function the one-dimensional BS PDE is easy to solve.

17. Incomplete Markets

Assumption: Two objects are given:

- A risk-free asset $dB_t = rB_t dt$
- A stochastic process X which is *not* assumed to be the price of a traded assets, with

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Consider a T-claim $y = \phi(X_T)$, what is the price $\Pi_t(y)$ at $t < T$?

Example:
X_t is the temperature in Brighton at time g

$$\phi(x) = \begin{cases} 100 & \text{if } x \leq 20 \\ 0 & \text{if } x > 20 \end{cases}$$

The holder of the T-claim receives 100 if the temperature is below 20, 0 otherwise

Our expectations: In the meta-theorem, $R = 1$, $M = 0$ so the market is incomplete. The price of y is *not* uniquely determined. If the price of a benchmark derivative is given, however, then all other derivatives will have unique prices. Certain consistency relations between prices should hold!

Assume y and Z have price proesses

$$\Pi_t(y) = F(t, X_t) \qquad \Pi_t(Z) = G(t, X_t)$$

$$d\pi_t(y) = \mu_F F dt + \sigma_F F dW_t \qquad \begin{cases} \mu_F = \dfrac{F_t + \dfrac{\sigma^2}{2} F_{xx} + \mu F_x}{F} \\ \sigma_F = \dfrac{\sigma F_x}{F} \\ d\Pi_t(Z) = \alpha_G G dt + \sigma_G G dW_t \end{cases}$$

Let $w = (w^F, W^G)$ be a self-financing relative portfolio in F and G

$$\begin{aligned} dV_t^w &= V_t^w w^F \frac{dF}{F} + V_t^w w^G \frac{dG}{G} \\ &= \left(\mu_F w^F + \mu_G w^G\right) V_t^w dt + \left(\sigma_F w^F + \sigma_G w^G\right) V_t^w dW_t \end{aligned}$$

Chose w^F, w^G so that

$$\left.\begin{aligned} w^F + w^G = 1 \\ \sigma_F w^F + \sigma_G w^G = 0 \end{aligned}\right\} \Leftrightarrow \begin{cases} w^F = \dfrac{-\sigma_G}{\sigma_F - \sigma_G} \\ w^G = \dfrac{\sigma_F}{\sigma_F - \sigma_G} \end{cases}$$

Then $dV_t^w = \dfrac{\sigma_F \mu_G - \sigma_G \mu_F}{\sigma_F - \sigma_G} V_t^w dt$

By the no-arbitrage assumption, we must have $\dfrac{\sigma_F \mu_G - \sigma_G \mu_F}{\sigma_F - \sigma_G} = r$

Thus

$$\begin{aligned} \sigma_F \mu_G - \sigma_G \mu_F &= r\sigma_F - r\sigma_G \\ \Leftrightarrow \frac{\mu_F - r}{\sigma_F} &= \frac{\mu_G - r}{\sigma_G} \end{aligned}$$

Note that the LHS does not involve G and the RHS does not involve F

Lemma 17.1

Assume the market for derivatives is arbitrage-free. Then there exists a process λ such taht $\lambda(t, X_t) = \dfrac{\mu_F(t, X_t) - r}{\sigma_F(t, X_t)}$ for any pricing function F

Terminology: λ_t is called the *market price of risk*

We have $\lambda = \dfrac{\mu_F - r}{\sigma_F} = \dfrac{F_t + \frac{\sigma^2}{2} F_{xx} + \mu F_x - rF}{\sigma F_x}$

Lemma 17.2

The price of a T-claim $\phi(X_T)$ is $F(t, X_t)$ where $F(t,x)$ solves

$$\begin{cases} F_t + \frac{\sigma^2}{2} F_{xx} + (\mu - \sigma\lambda) F_x - rF = 0 \\ F(T,x) = \phi(x) \end{cases}$$

Moreover, $F(t,x) = \mathbb{E}^Q_{t,x}\left[\exp\{-r(T-t)\}\, \phi(X_T)\right]$

where $\begin{cases} dX_s = (\mu(s,X_s) - \lambda(s,X_s)\sigma(s,X_S))\,ds + \sigma(s,X_s) dW_s^Q \\ X_t = x \end{cases}$ under Q

Remark:
$\lambda(t,x)$ is *not* specified within the model. If we take the price of one derivative as given with price process $\Pi_t = G(t, X_t)$, then $\lambda(t,x) = \dfrac{\mu_G(t,x) - r}{\sigma_G(t,x)}$ can be calculated. This λ can then be used to price other derivatives.

Special Case:
Assume that X *is* in fact a traded asset. The claim $\overline{Z} = X_T$ then has price $G(t, X_t) = X_t$, so

$$\lambda(t,x) = \frac{\mu_F - r}{\sigma_G} = \frac{G_t + \frac{\sigma^2}{2} G_{xx} + \mu G_x - rG}{\sigma G_x} \overset{G(t,x)=x}{=} \frac{\mu - rx}{\sigma}$$

The factor $\mu - \lambda\sigma$ is then $\mu - \lambda\sigma = rx$
Thus the usual BS-equation is recovered!

18. Discrete Dividends

Consider a stock S that pays dividends at times $T_1, \cdots, T_K$ where $0 < T_1 < T_2 \cdots T_K < T$.
In addition to S, there is also a bank account $dB_t = rB_t dt$
Between dividend dates, S follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

At each $t = T_i$, a dividend $\delta(S_{T_i})$ is paid out.
Here $\delta : [0, \infty) \to [0, \infty)$ is a continuous function with $\delta(S) \leq S$
To avoid arbitrage, we must have $S_{T_i} = S_{T_i} - \delta(S_{T_i})$

Question: What is the price of a T-claim $\chi = \phi(S_T)$?
Answer: For $t \in [T_i, T_{i+1}]$ we have $\Pi_t(\chi) = F^i(t, S_t)$ where $F^i(t, s)$ is constructed as follows:

- *Up to* T_{K-1}

$$\begin{cases} F_t^{K-2} + \dfrac{\sigma^2}{2} S^2 F_{ss}^{K-2} r S F_s^{K-2} - rF^{K-2} = 0 \\ F^{K-2}(T, S) = F^{K-1}(F, S - \delta(S)) \end{cases}$$

- *Up to* T_K

$$\begin{cases} F_t^{K-1} + \dfrac{\sigma^2}{2} S^2 F_{ss}^{K-1} + rSF_s^{k-1} = rF^{K-1} \\ F^{K_1}(T_K, S) = F^K(T_k, S - \delta(S)) \end{cases}$$

- *Up to* T

$$\begin{cases} F_T^K + \dfrac{\sigma^2}{2} S^2 F_{ss}^K + rSF_s^K = rF^k \\ F^K(T, S) = \phi(S) \end{cases}$$

Lemma 18.1: Risk-neutral valuation

The arbitrage-free price of a simple T-claim $\chi = \phi(S_T)$ in the presence of discrete dividends is $F(t, S_t)$ where

$$F(t, s) = \exp\{-r(T - t)\} \mathbb{E}_{t,s}^Q [\phi(S_T)]$$

Here, the following is under Q:

$$\begin{cases} dS_u = rS_u du + \sigma S_u dW_u^q \\ S_t = s \\ S_{T_i} = S_{T_i} - \delta(S_{T_i}) \end{cases}$$

Important special case:
$\delta(S) = \underbrace{\delta}_{\delta \in (0,1)} S$
Then

$$\begin{aligned} S_T &= S_{T_K} \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)(T - T_K) + \sigma(W_T^Q - W_{T_K}^Q) \right\} \\ &= (1 - \delta) S_{T_K^-} \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)(T - T_K) + \sigma(W_T^Q - W_{T_K}^Q) \right\} \\ &= (1 - \delta) S_{T_{K-1}} \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)(T - T_{K-1}) + \sigma(W_T^Q - W_{T_{K-1}}^Q) \right\} \\ &= (1 - \delta)^2 S_{T_{K_1}^-} \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)(T - T_{K-1}) + \sigma(W_T^Q - W_{T_{K-1}}^Q) \right\} \\ &= \cdots = (1 - \delta)^n S \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma(W_T^Q - W_t^Q) \right\} \end{aligned}$$

Where n is the number of dividends times in $[t, T]$
Therefore $F^\delta(t, s) = F^0(t, S(1 - \delta)^n)$, i.e pricing function in presence of dividends = pricing function with no dividends.

Example:
Assume $\delta(S) = \delta S$. What is the price of a call option $\chi = (S_T - K)^+$?
Answer:

$$F^\delta(t,s) = F^0(t, S(1-\delta)^n) = (1-\delta)^n SN(d_1)_K \exp\{r(T-t)\} N(d_2)$$

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{S(1-\delta)^n}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$

Example:
Find a replciating strategy for $\chi = S_T$ (assume n remaining dividends)
Answer:
The value of χ is $F^\delta(0,S) = F^0(0, S(1-\delta)^n) = S(1-\delta)^n$
At $t = 0$, buy $(1-\delta)^n$ shares of S
At $t = T_1$, receive $(1-\delta)^n \delta S_{T_1^-}$ in dividends.
New stock price is $S_{T_1} = (1-\delta)S_{T_1^-}$; so we can buy $\dfrac{(1-\delta)^n \delta S_{T_1^-}}{(1-\delta)S_{T_1^-}}$ new shares. Total holdings of $(1-\delta)^n + \delta(1-\delta)^{n-1} = (1-\delta)^{n-1}$
Contiue similarly at $T_2, \cdots, T_n$. After T_k; we have $(1-\delta)^{n-k}$ shares, so at $t = T$ we have $(1-\delta)^{n-n} = 1$ shares of S
Thus χ is replicated!

19. Continuous Dividends

The market admits the same model as previously, i.e

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \mu S_t dt + \sigma S_t dW_t \end{cases}$$

Dividend structure: $dD_t = \delta(S_t)S_t dt$ where δ is some continuous fucntion

Interpretation:
During an interval $[t_1, t_2]$, the holder of one share of S receives the amount

$$\int_{t_1}^{t_2} \delta(S_u) S_u du$$

To price a T-claim $\chi = \phi(S_T)$, we follow our usual approach.
Assume $\Pi_t(\chi) = F(t, S_t)$ and let (w^S, w^F) be a self-financing relative portfolio of S and F

$$\begin{aligned} dV_t^w &\overset{\text{self-fin}}{=} V_t^w w^S \frac{dS_t + dD_t}{S_t} + V_t^w w^F \frac{dF_t}{F_t} \\ &= V_t^w (w^S(\mu + \delta) + w^F \mu_F) dt + V_t^w (w^s \sigma + w^F \sigma_F) dW_t \end{aligned}$$

Where

$$\begin{cases} \mu_F = \dfrac{F_t + \mu S F_s + \dfrac{\sigma^2 S^2}{2} F_{ss}}{F} \\ \sigma_F = \dfrac{\sigma S F_s}{F} \end{cases}$$

Choose (w^S, w^F) such that

$$\left.\begin{aligned} w^S + w^F &= 1 \\ \sigma w^S + \sigma_F w^F &= 0 \end{aligned}\right\} \Leftrightarrow \begin{cases} w^S = \dfrac{-\sigma_F}{\sigma - \sigma_F} \\ w^F = \dfrac{\sigma}{\sigma - \sigma_F} \end{cases}$$

Comparing with the bank account to avoid arbitrage, we must have

$$w^S(\mu + \delta) + w^F \mu_F = r$$

Thus

$$-\sigma_F(\mu+\delta)+\mu_F\sigma = r(\sigma-\sigma_F) - SF_s(\mu+\delta) + F_t + \mu SF_S + \frac{\sigma^2 S^2}{2}F_{ss}$$
$$= rF - rSF_s$$

$$F_t + \frac{\sigma^2 S_t^2}{2}F_{ss} + (r-\delta)\, S_t F_s - rF = 0$$

Since S_t can take any value, the PDE must hold at all points (t,s)

Lemma 19.1

The pricing function $F(t,s)$ of $\chi = \phi(S_T)$ solves

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 S^2 F_{ss} + (r-\delta)SF_s - rF = 0 \\ F(T,S) = \phi(S) \end{cases}$$

Moreover, $F(t,s) = \mathbb{E}^Q_{t,s}\left[\exp\{-r(T-t)\}\,\phi(S_T)\right]$ where

$$\begin{cases} dS_u = (r-\delta)S_u du + \sigma S_u dW_u^Q \\ S_t = s \end{cases}$$

under Q

Remark:
If $\delta(s) = \delta$ (i.e constant), then

$$S_T = s\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right\}$$
$$= s\exp\{-\delta(T-t)\}\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T-W_t)\right\}$$

Thus $F^\delta(t,s) = F^0(t, s\exp\{-\delta(T-t)\})$
I.e the pricing function with continuous dividends is the same as the pricing function with no dividends

Example:
What is the price of $\chi = S_T$ if continuous dividends are paid (at a constant proportial to the rate δ)?
$F^\delta(0,s) = F^0(0, s\exp\{-\delta T\}) = s\exp\{-\delta T\}$

Can we find a replicating strategy?
At $t=0$; buy $\exp\{-\delta T\}$ shares of S. Use all dividends to buy new shares. If $f(t)$ shares are held at time t, then$\delta f(t)dt$ new shares can be bought during $(t, t+dt)$
Thus

$$\begin{cases} \frac{df(t)}{dt} = \delta f(t) \\ f(0) = \exp\{-\delta T\} \end{cases}$$

So $f(t) = \exp\{-\delta(T-t)\}$. In particular, $f(T) = 1$ so χ is replicated!

20. Forward Contracts

A forward contract is something where we get a delivery and payment at a later time. Very much like an option, but the payment is done at T. It is written on a T claim χ and contracted at some time t with delivery at time T is as follows

- At T, the holder receives χ (the T-claim) from the seller
- At T, the holder pays $f(t,T;\chi)$ to the seller
- The so-called *forward price* $f(t,T;\chi)$ is deterministic and is determined at the initial time t in such a way so that the forward contract value 0 at t

When you enter the agreement, the underlying market may fluctuate but you are still bounded by the contract. Therefore, at a later time point, the price could be non-zero.
We want the price

$$\begin{aligned} \Pi_t(\chi - f(t,T;\chi)) &= 0 \\ &= \Pi_t(\chi) - \Pi_t(f(t,T;\chi)) \\ &= \Pi_t(\chi)_{\exp}\{-r(T-t)\} f(t,T;\chi) \end{aligned}$$

So $f(t,T;\chi) = \exp\{r(T-t)\}\,\Pi_t(\chi)$

Example:
If $\chi = S_T$ (non-dividend paying asset, i.e in the standard BS model), what is its forward price?

$$f(t,T;\chi) = \exp\{r(T-t)\}\,S_t$$

Due to market fluctuations, once you have entered the contract its value may increase. So what is the value of a forward contract at time s $(t < s < T)$?
We will receive $\chi - f(t,T;\chi)$ at the end of time, so the value is

$$\Pi_s(\chi) - \exp\{-r(T-s)\}\,f(t,T;\chi)$$