Solutions to five problems on tensor products of lattices and related matters

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SOLUTIONS TO FIVE PROBLEMS ON TENSOR PRODUCTS OF LATTICES AND RELATED MATTERS

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ABSTRACT. The notion of a *capped tensor product*, introduced by G. Grätzer and the author, provides a convenient framework for the study of tensor products of lattices that makes it possible to extend many results from the finite case to the infinite case. In this paper, we answer several open questions about tensor products of lattices. Among the results that we obtain are the following:

Theorem 2. *Let A be a lattice with zero. If $A \otimes L$ is a lattice for every lattice L with zero, then A is locally finite and $A \otimes L$ is a capped tensor product for every lattice L with zero.*

Theorem 5. *There exists an infinite, three-generated, 2-modular lattice K with zero such that $K \otimes K$ is a capped tensor product.*

Here, 2-modularity is a weaker identity than modularity, introduced earlier by G. Grätzer and the author.

1. INTRODUCTION

For $\langle \vee, 0 \rangle$-semilattices A and B, the tensor product $A \otimes B$ may be defined, in a fashion formally similar to the tensor product of vector spaces in linear algebra, as a universal object with respect to the notion of *bimorphism*, see [6, 8, 9, 10, 11].

The notion of tensor product of $\langle \vee, 0 \rangle$-semilattices becomes interesting for *lattices*, and the tensor product of two lattices is not always a lattice. This phenomenon involves, among others, the study of *transferability* (see [4]), or of *lower bounded lattices* (see [3]). More precisely, we say that a finite lattice satisfies the *condition* $(T_{\vee})$, if the relation D_A of join-dependency on the set $J(A)$ of all join-irreducible elements of A has no cycle. This condition is equivalent to saying that A is a lower bounded homomorphic image of a free lattice. For lattices A and B with zero, if $A \otimes B$ is a so-called *capped tensor product*, then $A \otimes B$ is a lattice (see Section 2, and also [9, 10]). The problem whether the converse holds is still open. We say that a lattice A with zero is *amenable*, if $A \otimes L$ is a capped tensor product, for every lattice L with zero.

The following statement summarizes some of the results obtained in [9].

Theorem 1. *For a lattice A with zero, the following conditions are equivalent:*

  (i) *A is amenable.*
  (ii) *A is locally finite and $A \otimes B$ is a lattice, for every lattice B with zero.*
  (iii) *A is locally finite and $A \otimes F_{\mathbf{L}}(3)$ is a lattice.*
  (iv) *A is locally finite and every finite sublattice of A satisfies $(T_{\vee})$.*

It would be nice to be able to replace amenability of A by the more straight-forward condition "$A \otimes L$ is a lattice, for every lattice L with zero", that we shall call *weak amenability* of A. However, the problem whether weak amenability is equivalent to amenability was still open at the time where Theorem 1 was stated, as Problem 1 in [9]. We solve this problem here in the affirmative:

Theorem 2. *Every weakly amenable lattice is amenable.*

In particular, the local finiteness assumption can be removed from (ii) and (iii) in the statement of Theorem 1.

One of the reasons why capped tensor products were introduced in [10] was to provide a wide context in which the so-called *Isomorphism Theorem* (see [10]) would be valid, thus extending the finite case established in [6]. The question whether, for lattices A and B with zero, $A \otimes B$ capped implies that either A or B is locally finite is stated in Problem 1 in [10]. We answer this question negatively, thus, at the same time, showing the relevance of the notion of capped tensor product:

Theorem 5. *There exists an infinite, three-generated, 2-modular lattice K such that $K \otimes K$ is a capped tensor product.*

The lattice K of Theorem 5 enjoys some additional properties. For a variety $\mathbf{V}$ of lattices, we say that a lattice A with zero is $\mathbf{V}$-*amenable*, if $A \otimes L$ is a capped tensor product, for any lattice L with zero in $\mathbf{V}$. For a positive integer h, let $\mathbf{M}^h$ denote the variety of h-*modular lattices*, as introduced in [8], see Section 4; in particular, $\mathbf{M}^1 = \mathbf{M}$ is the variety of all modular lattices.

Theorem 4. *The lattice K is $\mathbf{M}^h$-amenable for all $h > 0$, although it is not locally finite.*

Hence the lattice K provides a negative solution for Problem 5 in [9]. We also prove in Theorem 3 that K satisfies the 2-*modular identity* introduced in [8], hence easily solving the first half of Problem 6 in [8] asking whether the free 2-modular lattice with three generators is finite (*answer*: no).

Finally, in Section 6, we show, in particular, that Problem 2 in [9], that asks whether there exists a simple, nontrivial, amenable lattice has an easy negative answer.

2. Basic concepts

We first recall some basic definitions about tensor products of $\langle \vee, 0 \rangle$-semilattices, stated, for example, in [10]. Let A and B be $\langle \vee, 0 \rangle$-semilattices. We introduce a partial binary operation, the *lateral join*, on $A \times B$: let $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B$; the *lateral join* $\langle a_0, b_0 \rangle \vee \langle a_1, b_1 \rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case, it is the join, $\langle a_0 \vee a_1, b_0 \vee b_1 \rangle$. A hereditary subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it contains the subset

$$\perp_{A,B} = (A \times \{0_B\}) \cup (\{0_A\} \times B),$$

and it is closed under lateral joins.

The *extended tensor product* of A and B, denoted by $A \bar{\otimes} B$, is the lattice of all bi-ideals of $A \times B$. It is easy to see that it is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \bar{\otimes} B$ by

$$a \otimes b = \perp_{A,B} \cup \{\langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leqslant \langle a, b \rangle\}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal of $A \times B$. We denote by $A \otimes B$ the $\langle \vee, 0 \rangle$-semilattice of all compact elements of $A \bar{\otimes} B$. It is generated, as a $\langle \vee, 0 \rangle$-semilattice, by the pure tensors. We observe that if the semilattice S of compact elements of an algebraic lattice L forms a lattice in itself, then S is a sublattice of L. In particular, if $A \otimes B$ is a lattice, then it is a sublattice of $A \bar{\otimes} B$.

A *capping* of a bi-ideal I of $A \times B$ is a subset Γ of $A \times B$ such that I is the hereditary subset of $A \times B$ generated by $\Gamma \cup \perp_{A,B}$. We say that I is *capped*, if it has a *finite* capping. A tensor product $A \otimes B$ is *capped*, if all its elements are capped bi-ideals. It is easy to see that a capped tensor product is always a lattice.

A lattice A with zero is *amenable*, if $A \otimes L$ is a capped tensor product, for every lattice L with zero.

For a set X, we denote by $P \mapsto P^{\mathrm{d}}$ the *dualization* map on the free lattice $\mathrm{F_L}(X)$. We shall use several times the following result, see Lemma 2.2(iii) of [9]:

Lemma 2.1. *Let A and B be lattices with zero, let n be a positive integer, let a_0, ..., $a_{n-1} \in A$, b_0, ..., $b_{n-1} \in B$. Then*

$$\bigvee_{i<n} a_i \otimes b_i = \bigcup_{P \in \mathrm{F_L}(n)} P(a_0, \ldots, a_{n-1}) \otimes P^{\mathrm{d}}(b_0, \ldots, b_{n-1}).$$

3. WEAKLY AMENABLE LATTICES

Definition 3.1. A lattice A with zero is *weakly amenable*, if $A \otimes L$ is a lattice, for every lattice L with zero.

It is obvious that every amenable lattice is weakly amenable. The question of the converse is stated as Problem 1 in [9]. It was conjectured in [9] that this problem had a negative answer. Interestingly, this guess was too pessimistic, as we prove in Theorem 2. To prepare for this result, we first establish the following lemma.

Lemma 3.2. *Let m, n be positive integers, let U, V, U_0, ..., U_{n-1}, V_0, ..., V_{n-1} be elements of $\mathrm{F_L}(m)$. We let $\mathbf{x}_0$, ..., $\mathbf{x}_{m-1}$, $\mathbf{y}_0$, ..., $\mathbf{y}_{m-1}$ be the canonical generators of the free lattice with $2m$ generators, $\mathrm{F_L}(2m)$. For all $R \in \mathrm{F_L}(n)$, if the inequality*

$$U(\vec{\mathbf{x}}) \wedge V(\vec{\mathbf{y}}) \leqslant R(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n)$$

holds in $\mathrm{F_L}(2m)$ (where we put $\vec{\mathbf{x}} = \langle \mathbf{x}_i \mid i < m \rangle$ and $\vec{\mathbf{y}} = \langle \mathbf{y}_i \mid i < m \rangle$), then there exists a pure meet polynomial $R^* \leqslant R$ such that

$$U(\vec{\mathbf{x}}) \wedge V(\vec{\mathbf{y}}) \leqslant R^*(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n).$$

By a *pure meet polynomial*, we mean a polynomial of the form $\bigwedge_{i \in I} \mathbf{x}_i$, where I is a nonempty subset of $\{0, 1, \ldots, n-1\}$.

Proof. We argue by induction on the length of R. If R is a variable, we put $R^* = R$. If $R = R_0 \wedge R_1$, we put $R^* = R_0^* \wedge R_1^*$.

Now suppose that $R = R_0 \vee R_1$, for polynomials R_0 and R_1. So the inequality

$$U(\vec{\mathbf{x}}) \wedge V(\vec{\mathbf{y}}) \leqslant R_0(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n) \vee R_1(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n)$$

holds. Since the free lattice $\mathrm{F_L}(2m)$ satisfies Whitman's condition, one of the following inequalities holds:

$$(3.1) \qquad\qquad U(\vec{\mathbf{x}}) \leqslant R(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n)$$

$$(3.2) \qquad\qquad V(\vec{\mathbf{y}}) \leqslant R(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n)$$

$$(3.3) \qquad U(\vec{\mathbf{x}}) \wedge V(\vec{\mathbf{y}}) \leqslant R_\nu(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n), \qquad \text{for some } \nu < 2.$$

However, (3.1) never holds. Indeed, if $L = \mathrm{F_L}(m)^\circ$ is the lattice obtained by adding a new zero element (say, 0) to $\mathrm{F_L}(m)$, then there exists a unique lattice homomorphism that sends $\mathbf{x}_j$ to itself and $\mathbf{y}_j$ to 0 for all $j < m$, and applying that homomorphism to (3.1) gives the inequality $U(\vec{\mathbf{x}}) \leqslant 0$, which does not hold. Similarly, (3.2) does not hold. So only (3.3) remains, that is, there exists $\nu < 2$ such that the inequality

$$U(\vec{\mathbf{x}}) \wedge V(\vec{\mathbf{y}}) \leqslant R_\nu(U_j(\vec{\mathbf{x}}) \wedge V_j(\vec{\mathbf{y}}) \mid j < n)$$

holds. We put $R^* = R_\nu^*$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Theorem 2. *Every weakly amenable lattice is amenable.*

Proof. Let A be a weakly amenable lattice, we prove that A is amenable. If A is finite, then this follows from Theorem 3 of [9].

Now the general case. Since the class of amenable lattices with zero is closed under direct limits and sublattices, see, for example, Theorem 2 of [9], it suffices to prove that A is *locally finite*. Again by using Theorem 2 of [9], it suffices to consider the case where A is *finitely generated*, and then to prove that A is finite.

Let $\vec{a} = \langle a_i \mid i < m \rangle$ be a finite sequence of elements of A generating A as a lattice. Let $\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}, \mathbf{y}_0, \ldots, \mathbf{y}_{m-1}$ be the canonical generators of the free lattice with $2m$ generators, $\mathrm{F_L}(2m)$.

We define elements H and K of $A \otimes \mathrm{F_L}(2m)$ by putting

$$H = \bigvee_{i < m} a_i \otimes \mathbf{x}_i,$$

$$K = \bigvee_{i < m} a_i \otimes \mathbf{y}_i.$$

By Lemma 2.1, the following equalities hold:

$$H = \bigcup_{P \in \mathrm{F_L}(m)} P(\vec{a}) \otimes P^{\mathrm{d}}(\vec{\mathbf{x}}),$$

$$K = \bigcup_{Q \in \mathrm{F_L}(m)} Q(\vec{a}) \otimes Q^{\mathrm{d}}(\vec{\mathbf{y}}),$$

hence

$$H \cap K = \bigcup_{P,Q \in \mathrm{F_L}(m)} (P(\vec{a}) \wedge Q(\vec{a})) \otimes (P^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge Q^{\mathrm{d}}(\vec{\mathbf{y}})).$$

Since A is weakly amenable, $A \otimes \mathrm{F_L}(2m)$ is a lattice, hence $H \cap K$ is a compact bi-ideal of $A \times \mathrm{F_L}(2m)$. Thus there are a positive integer n and elements $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ of $\mathrm{F_L}(m)$ such that the following relation

$$(3.4) \quad (P(\vec{a}) \wedge Q(\vec{a})) \otimes (P^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge Q^{\mathrm{d}}(\vec{\mathbf{y}})) \subseteq \bigvee_{j < n} (P_j(\vec{a}) \wedge Q_j(\vec{a})) \otimes (P_j^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge Q_j^{\mathrm{d}}(\vec{\mathbf{y}}))$$

holds for all $P, Q \in \mathrm{F_L}(m)$. To conclude the proof, it suffices to prove that if $P(\vec{a})$ is nonzero, then it belongs to the join closure of $\{P_j(\vec{a}) \mid j < n\}$. Indeed, in that case, $|A| \leqslant 2^n$, so A is finite.

For an arbitrary $P \in \mathrm{F_L}(m)$ such that $P(\vec{a}) > 0$, we put $Q = P$ and we apply (3.4). By Lemma 2.1, there exists $R \in \mathrm{F_L}(n)$ such that the following system of inequalities is satisfied:

$$(3.5) \qquad P(\vec{a}) \leqslant R(P_j(\vec{a}) \wedge Q_j(\vec{a}) \mid j < n)$$

$$(3.6) \qquad P^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge P^{\mathrm{d}}(\vec{\mathbf{y}}) \leqslant R^{\mathrm{d}}(P_j^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge Q_j^{\mathrm{d}}(\vec{\mathbf{y}}) \mid j < n).$$

We observe that (3.5) holds in A, while (3.6) holds in $\mathrm{F_L}(2m)$. By applying Lemma 3.2 to R^{d} in (3.6), we obtain a pure *join* polynomial $R^* \geqslant R$ that may be substituted to R in the inequality (3.6) without affecting its validity. Since $R^* \geqslant R$, the inequality obtained by replacing R by R^* in (3.5) is obviously satisfied. Therefore, we may assume without loss of generality that R *is a pure join polynomial*, that is, $R = \bigvee_{j \in J} \mathbf{x}_j$ for some nonempty subset J of $\{0, 1, \dots, n-1\}$. By (3.6), the inequality

$$P^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge P^{\mathrm{d}}(\vec{\mathbf{y}}) \leqslant P_j^{\mathrm{d}}(\vec{\mathbf{x}}) \wedge Q_j^{\mathrm{d}}(\vec{\mathbf{y}})$$

holds for all $j \in J$. Therefore, by substituting a new unit element 1 for all the $\mathbf{y}_j$, we obtain that $P^{\mathrm{d}} \leqslant P_j^{\mathrm{d}}$, thus $P_j \leqslant P$. In particular, $P_j(\vec{a}) \leqslant P(\vec{a})$ for all $j \in J$. Therefore, by (3.5), we obtain

$$P(\vec{a}) \leqslant \bigvee_{j \in J} (P_j(\vec{a}) \wedge Q_j(\vec{a})) \leqslant \bigvee_{j \in J} P_j(\vec{a}) \leqslant P(\vec{a}),$$

so $P(\vec{a}) = \bigvee_{j \in J} P_j(\vec{a})$ belongs to the join closure of $\{P_j(\vec{a}) \mid j < n\}$. $\qquad \square$

4. An infinite, three-generated, 2-modular lattice with zero

Let K be the (infinite) lattice diagrammed on Figure 1.

We observe right away the following elementary properties of K:

Lemma 4.1. *The lattice K is generated by the three-element set $\{a, b, c\}$.*

For any lattice L, we define a map $u \mapsto u^{(1)}$ from L^3 to L^3 by the rule

$$\langle x, y, z \rangle^{(1)} = \langle x \vee (y \wedge z), y \vee (x \wedge z), z \vee (x \wedge y) \rangle, \quad \text{for all } x, y, z \in L.$$

Further, we put $u^{(0)} = u$ and $u^{(k+1)} = (u^{(k)})^{(1)}$, for all $k < \omega$.

We say that a triple $u = \langle x, y, z \rangle$ of elements of L is

— *balanced*, if $u^{(1)} = u$, i.e., $x \wedge y = x \wedge z = y \wedge z$,

— *modular*, if $\{x, y, z\}$ generates a modular sublattice of L,

— *distributive*, if $\{x, y, z\}$ generates a distributive sublattice of L.

Of course, every distributive triple is modular.

We recall the following definition, introduced in [8]:

Definition 4.2. Let h be a positive integer. A lattice L is *h-modular*, if $u^{(h+1)} = u^{(h)}$, for any $u \in L^3$.

We shall denote by $\mathbf{M}^h$ the variety of all h-modular lattices.

In particular, it is proved in [8] that 1-modularity is equivalent to modularity. In relation to this, we recall the following classical lemma, that says, essentially, that for every modular lattice M with zero, the tensor product $M_3 \otimes M$ is capped, see [12], or also [13] or [7].
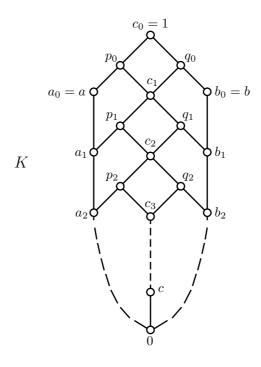
FIGURE 1

Lemma 4.3. *Let L be a lattice, let $u = \langle x, y, z \rangle$ be a modular triple of elements of L. Then $u^{(2)} = u^{(1)}$.*

More generally, the following is an immediate consequence of the definition of h-modularity:

Lemma 4.4. *Let h be a positive integer, let L be a h-modular lattice, let $u = \langle x, y, z \rangle$ be a triple of elements of L. Then $u^{(h+1)} = u^{(h)}$.*

Since the lattice K contains many copies of the pentagon N_5 (the five-element nonmodular lattice), it is not modular. However, it falls relatively short of modularity:

Theorem 3. *The lattice K is infinite, three-generated, and 2-modular.*

Hence, K provides an answer to the second part of Problem 6 in [8].

Proof. It remains to verify that K is 2-modular. Let $u = \langle x, y, z \rangle$ be a triple of elements of K, we must prove that $u^{(3)} = u^{(2)}$. If two of the elements x, y, and z are comparable, then, since N_5 is 2-modular and the sublattice generated by $\{x, y, z\}$ is a homomorphic image of N_5, $u^{(3)} = u^{(2)}$ and we are done. Hence it suffices to verify that the equality $u^{(3)} = u^{(2)}$ holds for u an *antichain* of K. If one component of u is c, then $u = u^{(1)}$. Otherwise, it is not hard to verify that $u^{(1)}$ is always a triple of elements of $K \setminus \mathrm{J}(K)$ (where $\mathrm{J}(K)$ denotes the set of all join-irreducible elements of K), in particular, $u^{(1)}$ is a distributive triple of elements of K. Hence, by Lemma 4.3, $(u^{(1)})^{(2)} = (u^{(1)})^{(1)}$, that is, $u^{(3)} = u^{(2)}$.   □

5. A NON LOCALLY FINITE LATTICE THAT IS $\mathbf{M}^h$-AMENABLE FOR ALL h

Now let h be a positive integer. We shall prove that the lattice K introduced in Section 4 is $\mathbf{M}^h$-amenable, *i.e.*, that $K \otimes L$ is a capped tensor product, for every h-modular lattice L with zero. The lattice L will be fixed throughout the present section. We denote by $\mathcal{A}$ the set of all maps $x \colon \mathrm{J}(K) \to L$ with finite range that are *antitone*, *i.e.*, $p \leqslant q$ implies that $x(p) \geqslant x(q)$, for all p, $q \in \mathrm{J}(K)$.

For any $x \in \mathcal{A}$, both sequences $\langle x(a_n) \mid n < \omega \rangle$ and $\langle x(b_n) \mid n < \omega \rangle$ are increasing, thus, since x has finite range, the two sequences are eventually constant. We denote by $x(a_\infty)$ and $x(b_\infty)$ their respective limits. We also denote by $d(x)$ the least nonnegative integer that satisfies the statement

$$x(a_n) = x(a_\infty) \quad \text{and} \quad x(b_n) = x(b_\infty), \qquad \text{for all } n < \omega \text{ such that } n \geqslant d(x).$$

Then we define a map $x^{(1)}$ from $\mathrm{J}(K)$ to L by the following equalities:

$$
\begin{aligned}
x^{(1)}(c) &= x(c) \vee (x(a_\infty) \wedge x(b_\infty)), \\
x^{(1)}(a_0) &= x(a_0), \\
x^{(1)}(b_0) &= x(b_0), \\
x^{(1)}(a_{n+1}) &= x(a_{n+1}) \vee (x(b_n) \wedge x(c)), \\
x^{(1)}(b_{n+1}) &= x(b_{n+1}) \vee (x(a_n) \wedge x(c)),
\end{aligned}
$$

(5.1)

for all $n < \omega$.

Lemma 5.1. *The set $\mathcal{A}$ is closed under the map $x \mapsto x^{(1)}$.*

Proof. For any subset X of L, we denote by $X^\wedge$ (resp., $X^\vee$) the meet-closure (resp., the join-closure) of X. Then $\mathrm{rng}\, x^{(1)}$ is a subset of $(\mathrm{rng}\, x)^{\wedge\vee}$, hence it is finite. To conclude the proof, it suffices to prove that $x^{(1)}(a_n) \leqslant x^{(1)}(a_{n+1})$ and $x^{(1)}(b_n) \leqslant x^{(1)}(b_{n+1})$, for all $n < \omega$. We verify for example the first inequality. It is trivial for $n = 0$. For $n > 0$, we compute:

$$
\begin{aligned}
x^{(1)}(a_n) &= x(a_n) \vee (x(b_{n-1}) \wedge x(c)) \\
&\leqslant x(a_{n+1}) \vee (x(b_n) \wedge x(c)) \qquad \text{(because x is antitone)} \\
&= x^{(1)}(a_{n+1}),
\end{aligned}
$$

which concludes the proof. $\square$

Lemma 5.1 makes it possible to define inductively an element $x^{(k)}$ of $\mathcal{A}$, for $x \in \mathcal{A}$ and $k < \omega$, by $x^{(0)} = x$ and $x^{(k+1)} = (x^{(k)})^{(1)}$, for all $k < \omega$. We further define a map $\ell \colon \mathcal{A} \to L^3$ by the rule

$$\ell(x) = \langle x(a_\infty), x(b_\infty), x(c) \rangle, \qquad \text{for all } x \in \mathcal{A}.$$

We proceed with the following easy observation:

Lemma 5.2. *The equality $\ell(x^{(1)}) = \ell(x)^{(1)}$ holds, for all $x \in \mathcal{A}$.*

Then we put $\mathcal{A}' = \{x \in \mathcal{A} \mid \ell(x^{(1)}) = \ell(x)\}$, a subset of $\mathcal{A}$. We get immediately from Lemmas 4.4 and 5.2, together with the h-modularity of L, the following statement:

Lemma 5.3. *The element $x^{(h)}$ belongs to $\mathcal{A}'$, for every $x \in \mathcal{A}$.*

For $x \in \mathcal{A}'$, the formula in (5.1) for computing $x^{(1)}$ simplifies:

$$x^{(1)}(c) = x(c),$$
$$x^{(1)}(a_0) = x(a_0),$$
(5.2)
$$x^{(1)}(b_0) = x(b_0),$$
$$x^{(1)}(a_{n+1}) = x(a_{n+1}) \vee (x(b_n) \wedge x(c)),$$
$$x^{(1)}(b_{n+1}) = x(b_{n+1}) \vee (x(a_n) \wedge x(c)),$$

for all $n < \omega$. In particular, we observe that for $n \geqslant d(x)$, the inequalities $x^{(1)}(a_n) \leqslant x(a_n) \vee (x(b_n) \wedge x(c)) = x(a_n) \vee (x(a_n) \wedge x(c)) = x(a_n)$ hold, thus $x^{(1)}(a_n) = x(a_n) = x(a_\infty)$. Similarly, $x^{(1)}(b_n) = x(b_n) = x(b_\infty)$. Also, $x^{(1)}(c) = x(c)$. Then an easy induction on k leads to the following result:

Lemma 5.4. *Let $x \in \mathcal{A}'$, let n, k be nonnegative integers. If $n \geqslant d(x)$, then $x^{(k)}(a_n) = x(a_\infty)$ and $x^{(k)}(b_n) = x(b_\infty)$. In particular, $d(x^{(k)}) \leqslant d(x)$.*

The following result deals with the values of $x^{(k)}$ on the a_n-s and b_n-s with small index n:

Lemma 5.5. *Let $x \in \mathcal{A}'$, let n, k be nonnegative integers. If $k \geqslant n$, then $x^{(k+1)}(a_n) = x^{(k)}(a_n)$ and $x^{(k+1)}(b_n) = x^{(k)}(b_n)$.*

Proof. We start by proving the following claim.

Claim. *For any $x \in \mathcal{A}'$ and positive integers n, k, the following equalities hold:*
$$x^{(k)}(a_n) = x(a_n) \vee (x^{(k-1)}(b_{n-1}) \wedge x(c)),$$
$$x^{(k)}(b_n) = x(b_n) \vee (x^{(k-1)}(a_{n-1}) \wedge x(c)).$$

Proof of Claim. We prove, for example, the first equality, by induction on k. It is trivial for $k = 1$. If it holds for k, then we compute, by using the induction hypothesis, together with the facts that $x^{(k)}(c) = x(c)$ and $x^{(k-1)}(b_{n-1}) \leqslant x^{(k)}(b_{n-1})$:

$$x^{(k+1)}(a_n) = x^{(k)}(a_n) \vee (x^{(k)}(b_{n-1}) \wedge x^{(k)}(c))$$
$$= x(a_n) \vee (x^{(k-1)}(b_{n-1}) \wedge x(c)) \vee (x^{(k)}(b_{n-1}) \wedge x(c))$$
$$= x(a_n) \vee (x^{(k)}(b_{n-1}) \wedge x(c)).$$

The proof for $x^{(k+1)}(b_n)$ is similar. $\square$ Claim.

Now we prove the conclusion of Lemma 5.5, by induction on k. If $k = 0$, then $n = 0$ and the conclusion follows from the fact that $x^{(1)}(a_0) = x(a_0)$ and $x^{(1)}(b_0) = x(b_0)$. Now suppose that $k > 0$ (and $k \geqslant n$). In the nontrivial case where $n > 0$, we compute:

$$x^{(k+1)}(a_n) = x(a_n) \vee (x^{(k)}(b_{n-1}) \wedge x(c)) \qquad \text{(by the Claim above)}$$
$$= x(a_n) \vee (x^{(k-1)}(b_{n-1}) \wedge x(c)) \quad \text{(by the induction hypothesis)}$$
$$= x^{(k)}(a_n) \qquad\qquad\qquad\qquad \text{(by the Claim above)}.$$

Similarly, we could have proved that $x^{(k+1)}(b_n) = x^{(k)}(b_n)$. $\square$

Now, as an immediate consequence of Lemmas 5.4 and 5.5, we are able to state the following:

Corollary 5.6. *The equality $x^{(k)} = x^{(d(x))}$ holds, for all $x \in \mathcal{A}'$ and all $k < \omega$ such that $k \geqslant d(x)$.*

Now we put $\mathcal{A}^* = \{x \in \mathcal{A} \mid x^{(1)} = x\}$, and $d'(x) = d(x^{(1)}) + h$, for all $x \in \mathcal{A}$. Hence $\mathcal{A}^* \subseteq \mathcal{A}' \subseteq \mathcal{A}$. It follows from Lemmas 5.3 and 5.6 that $x^{(k)} = x^{(d'(x))}$, for all $x \in \mathcal{A}$ and all $k < \omega$ such that $k \geqslant d'(x)$. We shall denote this element by $\tilde{x}$. Hence, $\tilde{x}$ is the least element of $\mathcal{A}^*$ such that $x \leqslant \tilde{x}$, for any $x \in \mathcal{A}$, we shall call it the *closure* of x.

We denote by $\vee_{\mathrm{c}}$ the componentwise join on $\mathcal{A}$, *i.e.*, $(x \vee_{\mathrm{c}} y)(p) = x(p) \vee y(p)$, for $x, y \in \mathcal{A}$ and $p \in \mathrm{J}(K)$. It is clear that $\mathcal{A}$ is closed under $\vee_{\mathrm{c}}$, and that it is a semilattice under $\vee_{\mathrm{c}}$. Hence, $\mathcal{A}^*$ is also a join-semilattice under componentwise ordering, the join, that we shall denote by $\vee_*$, being given by $x \vee_* y = \tilde{z}$ where $z = x \vee_{\mathrm{c}} y$, for all $x, y \in \mathcal{A}^*$.

Lemma 5.7. *Let x be a map from $\mathrm{J}(K)$ to L with finite range. Then x belongs to $\mathcal{A}^*$ iff x extends to a homomorphism from $\langle K^-, \vee \rangle$ to $\langle L, \wedge \rangle$ (we put $K^- = K \setminus \{0_K\}$). Furthermore, such an extension is unique.*

Proof. For any $n < \omega$, the inequalities $c < a_n \vee b_n$, $a_{n+1} < b_n \vee c$, and $b_{n+1} < a_n \vee c$ hold in K. Therefore, if x extends to a homomorphism from K^- to L, then $x^{(1)} = x$ (see the formulas (5.1)).

Conversely, suppose that $x^{(1)} = x$. We prove that x extends to a unique homomorphism from K^- to L. The uniqueness assertion is obvious, because every element of K^- is a join of finitely many, and even at most two, elements of $\mathrm{J}(K)$. To prove the existence assertion, it suffices to prove that for any elements p, q, and r of $\mathrm{J}(K)$, $r < p \vee q$ implies that $x(p) \wedge x(q) \leqslant x(r)$. This is obvious if either $r \leqslant p$ or $r \leqslant q$, because x is antitone. Hence suppose that $r \nleqslant p$ and $r \nleqslant q$. We need to check the following cases:

- $c < a_m \vee b_n$, for $m, n < \omega$. Then $x(a_m) \wedge x(b_n) \leqslant x(a_\infty) \wedge x(b_\infty) \leqslant x^{(1)}(c) = x(c)$.
- $a_m < b_n \vee c$, for $m, n < \omega$ such that $m > n$. Then $x(b_n) \wedge x(c) \leqslant x(b_{m-1}) \wedge x(c) \leqslant x^{(1)}(a_m) = x(a_m)$.
- The case $b_m < a_n \vee c$, for $m, n < \omega$ such that $m > n$, is treated similarly.

The three cases above are sufficient to conclude the proof. $\qquad \square$

For an element x of $\mathcal{A}^*$, we shall denote by $\overline{x}$ the unique homomorphism of K^- to L that extends x. We observe that $\operatorname{rng} \overline{x} = (\operatorname{rng} x)^\wedge$, hence $\operatorname{rng} \overline{x}$ is finite.

Notation. For $x \in \mathcal{A}$ and $I \in K \bar{\otimes} L$, let $x \nearrow I$ abbreviate the following statement:
$$\langle p, x(p) \rangle \in I, \quad \text{for all } p \in \mathrm{J}(K).$$

Lemma 5.8. *Let $x, y \in \mathcal{A}$, let $I \in K \bar{\otimes} L$. Then the following assertions hold:*

(i) *$x, y \nearrow I$ implies that $x \vee_{\mathrm{c}} y \nearrow I$.*
(ii) *$x \nearrow I$ implies that $x^{(1)} \nearrow I$.*
(iii) *$x, y \in \mathcal{A}^*$ and $x, y \nearrow I$ implies that $x \vee_* y \nearrow I$.*

Proof. (i) is obvious.

(ii) We assume that $x \nearrow I$, we prove that $\langle p, x^{(1)}(p) \rangle \in I$ for all $p \in \mathrm{J}(K)$. This amounts to verifying the following cases:

- $p = c$. From $\langle a_n, x(a_n) \rangle \in I$, $\langle b_n, x(b_n) \rangle \in I$, $c \leqslant a_n \vee b_n$, and the fact that I is a bi-ideal of $K \times L$ follows that $\langle c, x(a_n) \wedge x(b_n) \rangle \in I$. For $n \geqslant d(x)$,

we obtain that $\langle c, x(a_\infty) \wedge x(b_\infty) \rangle \in I$, hence, since $\langle c, x(c) \rangle \in I$, we obtain that $\langle c, x^{(1)}(c) \rangle \in I$.

- $p = a_n$, $n < \omega$. If $n = 0$, then $\langle a_n, x^{(1)}(a_n) \rangle = \langle a_n, x(a_n) \rangle \in I$. Now suppose that $n > 0$. From $\langle b_{n-1}, x(b_{n-1}) \rangle \in I$, $\langle c, x(c) \rangle \in I$, and $a_n < b_{n-1} \vee c$ follows that $\langle a_n, x(b_{n-1}) \wedge x(c) \rangle \in I$, hence, since $\langle a_n, x(a_n) \rangle \in I$, we obtain that $\langle a_n, x^{(1)}(a_n) \rangle \in I$.

- $p = b_n$, $n < \omega$. This case can be treated in a similar fashion as the previous one.

(iii) is an immediate consequence of (i) and (ii) above, together with the fact that $x \vee_* y = (x \vee_c y)^{(n)}$ for some $n < \omega$. $\qquad\square$

Notation. For $x \in \mathcal{A}^*$, we put $\varepsilon(x) = \{\langle u, \xi \rangle \in K \times L \mid u > 0 \Rightarrow \xi \leqslant \overline{x}(u)\}$.

Lemma 5.9. *The following assertions hold.*

(i) $\varepsilon(x) \subseteq I$ *iff* $x \nearrow I$, *for all* $x \in \mathcal{A}^*$ *and all* $I \in K \mathbin{\bar{\otimes}} L$.
(ii) $\varepsilon(x)$ *is a capped element of* $K \otimes L$, *for any* $x \in \mathcal{A}^*$.
(iii) ε *is a homomorphism from* $\langle \mathcal{A}^*, \vee_* \rangle$ *to* $\langle K \otimes L, \vee \rangle$.

Proof. (i) follows immediately from the fact that $\overline{x}$ is a homomorphism from $\langle K^-, \vee \rangle$ to $\langle L, \wedge \rangle$.

(ii) It follows, again, from the fact that $\overline{x}$ is a homomorphism from $\langle K^-, \vee \rangle$ to $\langle L, \wedge \rangle$ that $\varepsilon(x)$ is a bi-ideal of $K \times L$. It remains to verify that $\varepsilon(x)$ has a finite capping. To this end, for any $\xi \in \operatorname{rng} \overline{x}$, we denote by Γ_ξ the set of all maximal elements of $\overline{x}^{-1}\{\xi\}$. Furthermore, we put

$$\Gamma = \{\langle u, \xi \rangle \in K^- \times L \mid \xi \in \operatorname{rng} \overline{x} \text{ and } u \in \Gamma_\xi\}.$$

For any $\xi \in \operatorname{rng} \overline{x}$, Γ_ξ is an antichain of K, thus it has at most four elements. Hence, since $\operatorname{rng} \overline{x}$ is finite, Γ is finite. Now we prove that Γ is a capping of $\varepsilon(x)$. First, it is obvious that Γ is contained in $\varepsilon(x)$. Now let $\langle u, \xi \rangle \in \varepsilon(x)$, with $u > 0_K$ and $\xi > 0_L$. Put $\eta = \overline{x}(u)$. Then $u \in \overline{x}^{-1}\{\eta\}$, hence, since K is nœtherian (*i.e.*, every ascending chain of K is eventually constant), there exists $v \in \Gamma_\eta$ such that $u \leqslant v$. Hence, $\langle u, \xi \rangle \leqslant \langle v, \eta \rangle$, with $\eta \in \operatorname{rng} \overline{x}$ and $v \in \Gamma_\eta$, whence $\langle v, \eta \rangle \in \Gamma$, thus proving our assertion. Therefore, ε maps $\mathcal{A}^*$ to $K \otimes L$.

(iii) It is obvious that ε is an order-preserving map from $\mathcal{A}^*$ (with componentwise ordering) to $K \otimes L$ (with containment). It remains to prove that $\varepsilon(x \vee_* y) \subseteq \varepsilon(x) \vee \varepsilon(y)$, for all $x, y \in \mathcal{A}^*$ (the join in the right hand side is computed in $K \otimes L$). Put $I = \varepsilon(x) \vee \varepsilon(y)$. Then $\varepsilon(x)$, $\varepsilon(y)$ are contained in I, thus, by assertion (i) above, $x \nearrow I$ and $y \nearrow I$, whence, by Lemma 5.8(iii), $x \vee_* y \nearrow I$, *i.e.*, by assertion (i) above, $\varepsilon(x \vee_* y) \subseteq I$. $\qquad\square$

To conclude the proof, we now need nothing more than a short lemma:

Lemma 5.10. *The pure tensor* $u \otimes \xi$ *belongs to the range of* ε*, for all* $\langle u, \xi \rangle \in K \times L$.

Proof. Let $x\colon \mathrm{J}(K) \to L$ be the map defined by $x(p) = \xi$ if $p \leqslant u$, $x(p) = 0$ if $p \not\leqslant u$, for all $p \in \mathrm{J}(K)$. It is easy to compute that $\varepsilon(x) = u \otimes \xi$. $\qquad\square$

By Lemmas 5.9(iii) and 5.10, the range of ε contains $K \otimes L$, while by Lemma 5.9(ii), every element of the range of ε is capped. Hence, $K \otimes L$ is a capped tensor product. Hence we have proved the following theorem:

Theorem 4. *The lattice* K *has the following properties:*

(i) K *is infinite, three-generated,* 2-*modular.*
(ii) K *is* $\mathbf{M}^h$-*amenable for all* $h > 0$.

This solves Problem 5 in [9] (the 2-modularity is an additional 'luxury'). Since K is 2-modular, we obtain the following consequence, which solves Problem 1 in [10]:

Theorem 5. *There exists an infinite, three-generated,* 2-*modular lattice* K *such that* $K \otimes K$ *is a capped tensor product.*

6. No simple nontrivial amenable lattices

It is proved in [7] that every nontrivial lattice L has a proper congruence-preserving extension, denoted there by $M_3\langle L \rangle$, a variant of E.T. Schmidt's $M_3[L]$ construction introduced in [13]. If L satisfies a certain axiom weaker than modularity, then $M_3[L] \cong M_3 \otimes L$, where M_3 is the modular lattice of height two with three atoms, see [8]. The construction $M_3 \otimes L$ cannot be used for general L to prove that L has a proper congruence-preserving extension, because it may happen that $M_3 \otimes L$ is not a lattice, see [8, 9]. The basic reason for this is, of course, that M_3 is not amenable. This motivated the following question:

Problem 2 in [9]**.** *Does there exist a simple, amenable lattice with more than two elements?*

In Proposition 9.1 of [9], we prove that no simple, amenable (or even join-semidistributive) lattice with more than two elements can have a largest element. (A lattice is said to be *join-semidistributive*, if it satisfies that $x \vee z = y \vee z$ implies that $x \vee z = (x \wedge y) \vee z$, for all $x, y, z \in L$.) It turns out that Problem 2 in [9] has a negative answer, that follows immediately from the following easy result:

Theorem 6. *There exists no simple, locally finite lattice* S *with more than two elements such that any finite sublattice of* S *has* $(\mathrm{T}_\vee)$.

Proof. Suppose, towards a contradiction, that S is as required. Then there are incomparable elements a, b of S. Since S is simple and locally finite, there exists a finite sublattice L of S such that $a, b \in L$ and $\Theta_L(a \wedge b, a) = \Theta_L(a \wedge b, b)$ ($\Theta_L(x, y)$ denotes the principal congruence of L generated by the pair $\langle x, y \rangle$). By assumption, L satisfies $(\mathrm{T}_\vee)$. Hence, L satisfies the statement, denoted in [1] by (DPT), that

$$\Theta(u_0, u) = \Theta(v_0, v) \quad \text{implies that} \quad u \wedge v \nleq u_0 \text{ and } u \wedge v \nleq v_0,$$

for all $u_0 < u$ and $v_0 < v$ in L, see [2, p. 73]. Putting $u = a$, $v = b$, and $u_0 = v_0 = a \wedge b$, we obtain a contradiction. $\square$

In contrast with Theorem 6, we observe the following example:

Example 6.1. *There exists an infinite, simple, locally finite, join-semidistributive lattice with zero.*

Proof. Consider the lattice S of all bounded intervals of the chain $\mathbb{Z}$ of all integers, partially ordered under containment. Then it is well-known (and easy to verify directly) that S is locally finite and join-semidistributive. Since S is atomistic (that is, every element of S is a join of finitely many—in fact, two—atoms), in order to prove that S is simple, it suffices to prove that $\Theta_S(\varnothing, a) = \Theta_S(\varnothing, b)$, for any atoms a and b of S such that $a \neq b$.

Observe that the atoms of L are exactly the singletons of the form $\{n\}$, for $n \in \mathbb{Z}$. So there are $u, v \in \mathbb{Z}$ such that $a = \{u\}$ and $b = \{v\}$. Without loss of generality,

we may assume that $u < v$. Pick x, $y \in \mathbb{Z}$ such that $x < u < v < y$. Then $\{u\} < \{v\} \vee \{x\}$ and $\{v\} < \{u\} \vee \{y\}$. Since $\{u\}$, $\{v\}$, $\{x\}$, and $\{y\}$ are distinct atoms of S, it follows that $\Theta_S(\varnothing, \{u\}) = \Theta_S(\varnothing, \{v\})$. Therefore, S is simple. $\qquad \square$

We observe a difference between these results and the easy observation that states that there is no nontrivial simple, join-semidistributive lattice with a largest element, see Proposition 9.1 of [9]. Namely, the proof of Theorem 6 requires amenability, which is necessary in view of Example 6.1, while Proposition 9.1 of [9] requires only join-semidistributivity.

7. New open problems

Problem 1. Let $\mathbf{V}$ be a variety of lattices, let A be a lattice with zero. If $A \otimes L$ is a lattice for any $L \in \mathbf{V}$, is A $\mathbf{V}$-amenable?

Solving Problem 1, even for a given variety $\mathbf{V}$ (for example, the variety $\mathbf{M}$ of all modular lattices), may also provide some insight towards a solution of the (still open) Problem 3 in [10], that asks whether every tensor product of lattices that is a lattice is capped.

By Theorem 2, Problem 1 has a positive solution for $\mathbf{V} = \mathbf{L}$, the variety of all lattices.

Problem 2. Let $\mathbf{V}$ be a nontrivial variety of lattices. Does there exist a non locally finite, $\mathbf{V}$-amenable lattice?

In Theorem 4, we prove that there exists a non locally finite lattice that is $\mathbf{M}^h$-amenable for all $h > 0$.

Acknowledgment

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