

UPPSALA UNIVERSITET

FÖRELÄSNINGSATECKNINGAR

# Finansiella Derivat

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Inlämningsdatum  
November 28, 2024

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## 1. OPTIONS

**Motivating Discussion:**

Say a Swedish company has signed a contract to buy a machine from a US company for 100000USD to be paid at delivery 6 months from now.  $T = \frac{1}{2}$  years.

Current exchange rate is 11SEK/USD. The buyer is subject to currency risk. There are 3 possible strategies to implement:

1. Buy 100000USD today and deposit in the bank.  
The risk is eliminated but money is tied up for a long time and the company may not have access to this money
2. Buy a *forward contract* from a bank, i.e the bank delivers the sum you need at  $T = \frac{1}{2} = t$ , in return, the company pays some constant  $K \cdot 100000USD$  at  $T = t$ , where  $K$  is chosen at  $t = 0$  such that no transfer of money is needed at  $t = 0$ . Here, the bank takes all of the risk, but if the exchange rate drops below  $K$  then we would have preferred to do nothing.
3. Buy a *European call option* on 100000USD, with strike price  $K$  and exercise date  $T$ . I.e, it gives the right but not the obligation to buy 100000USD at price  $K \cdot 100000USD$  at time  $T = t$ . If exchange rate at  $T$  is  $> K$ , then we use the option. If its below at  $t = T$  thne we do not use the option (right, not obligation)

The last one is a good choice, but not free. This leads to the 2 main problems in the course:

- How much is a fair price for an option?
- If you are the seller of an option, how to protect (hedge) from risk of exchange rate not going up?

**Motivating Example in discrete time**

At  $t = 0$ , we can trade in a market with 2 assets:

- *Bank account* (risk-free/non-risky asset)  
At  $t = 0$  the value is 1 and at  $t = 1$  the value is 1
- *Stock* (risky asset)  
At  $t = 0$ ,  $S_0 = 100$  then it either grows ( $S_1 = 120$ ) or declines ( $S_1 = 80$ ) with probability  $p = 0.6$  and  $p = 0.4$  respectively

**Definition 1.1 Call option**

A *call option* is a contract that gives its holder the right but not the obligation to buy one share of a stock at time  $T$  with predetermined price  $K$ . Thus, at time  $t = 1$ , the option is worth  $S_1 - K$  if  $S_1 > K$  and 0 else

What is a fair price of the option? The sensible thing to pay would be  $p(S_1 - K)$ . Assuming  $K = 110$  in the above example, then  $0.6(120 - 110) = 6$ . But this is not the best price!

The idea is to replicate the option by finding a trading strategy using both the risk-free (B) and the risky asset (S) such that the value of the stock at  $t = 1$  coincides with the value of the option.

Is that possible? Yes. Let  $x$  = amount in the bank at  $t = 0$  and  $y$  be the number of shares of stock. We want to pick  $x, y$  such that regardless if stock goes up or down we have increase.

At  $t = 1$

$$\left. \begin{aligned} x + S_1 y &= S_1 - K \\ x + S_1 y &= 0 \end{aligned} \right\}$$

If  $K = 110$  and  $S_1 = \{120, 80\}$ , then  $x = -20$  and  $y = \frac{1}{4}$  since

$$\begin{cases} x + 120y = 10 \\ x + 80y = 0 \end{cases}$$

At  $t = 0$ . Our strategy is therefore to borrow 20 from the bank and buy  $\frac{1}{4}$  of a share. The cost is  $25 - 20 = 5$  which is less than 6.

At time  $t = 1$  our holdings are worth  $\frac{1}{4}S_1 - 20 = \begin{cases} 10 & \text{if } S_1 = 120 \\ 0 & \text{if } S_1 = 80 \end{cases}$  which is exactly the same as the option.

**Conclusion:**

By the APT (Arbitrage pricing theory), the price of the call must be equal to the cost of setting up this portfolio.

**Remark:**

The probabilities do not influence the option value. They were never used in the calculation of the price.

**Remark:**

Let us change  $p$  into  $q$  such that  $\mathbb{E}(S_1) = S_0 = 100$  in the example, which value of  $q$  satisfies this? It is symmetric in the example, so let  $p = q = \frac{1}{2}$

Then  $\mathbb{E}(\max\{S_1 - k, 0\}) = 10 \cdot \frac{1}{2} + 0 \cdot \frac{1}{5} = 5$

In general, the option price is  $\mathbb{E}^Q\left(\frac{B_0}{B_1} \max\{S_1 - k, 0\}\right)$  where  $Q$  is chosen such that  $\mathbb{E}^Q\left(\frac{B_0 S_1}{B_1}\right) = \frac{S_0}{B_0}$

**Notation:**

$a^+ = \max\{a, 0\}$ . In particular,

$$(s - K)^+ = \begin{cases} s - K & \text{if } s \geq K \\ 0 & \text{if } s < K \end{cases}$$

**Exercise:**

- In the above example, find a replicating strategy for a put option (right but not obligated to sell one share) at price  $K = 110$
- Find the value of the option at  $t = 0$

**Answer:**

$$\left. \begin{array}{l} x = 90 \\ y = \frac{-3}{4} \end{array} \right\} \text{ option value of 15}$$

## 2. CONTINUOUS TIME &amp; BROWNIAN MOTION

## 2.1. Simple Random Walk.

Let  $X_i$  be i.i.d.r.v with  $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}$

Let  $S_n = \sum_{i=1}^n X_i$ , then this is a stochastic process, still in discrete time. Do note that the expectation is 0 for the r.v. and that:

$$\mathbb{E}(S_n) = \sum_{k=1}^n \mathbb{E}(X_i) = 0$$

$$\text{Var}(S_n) = \mathbb{E}(S_n^2) - \underbrace{(\mathbb{E}(S_n))^2}_{=0} = \sum_{k=1}^n \text{Var}(X_i) = \sum_{k=1}^n 1 = n$$

Note that this was discrete time, how do we proceed to make this continuous?

We do this by scaling to finer time. Frist, fix a time interval:

**Stage 1**

Let  $X_0^1 = 0$

At  $t = 0$ , toss a coin,  $X_T^1 = \begin{cases} \sqrt{T} & \text{heads} \\ -\sqrt{T} & \text{tails} \end{cases}$ .

Here  $\mathbb{E}(X_T^1) = 0$  and  $\text{Var}(X_T^1) = T = \text{elapsed time}$ .

**Stage 2**

Add another time step. Let  $X_0^2 = 0$ , toss a coin,  $X_{T/2}^2 = \begin{cases} \sqrt{\frac{T}{2}} & \text{heads} \\ -\sqrt{\frac{T}{2}} & \text{tails} \end{cases}$

Repeat at  $t = \frac{T}{2}$ , adding/subtracting  $\sqrt{\frac{T}{2}}$

**Stage n**

Let  $X_0^n = 0$ , at each time  $t_k = \frac{k}{n}T$ , toss a coin.

Define  $X_{t_{k+1}}^n = X_{t_k}^n + Y_k$  where  $Y_k = \pm \sqrt{\frac{T}{n}}$  with prob. 1/2. Simulating our coin tosses.

Here

$$\mathbb{E}(X_{t_k}^n) = \mathbb{E}\left(\sum_{i=1}^{k-1} Y_i\right) = \sum_{i=1}^{k-1} \mathbb{E}(Y_i) = 0$$

$$\text{Var}(X_{t_k}^n) = \text{Var}\left(\sum_{i=1}^n Y_i\right) \stackrel{\text{indep}}{=} \sum_{i=1}^k = \frac{T}{n}k = t_k$$

Now the question becomes, what happens when  $n \rightarrow \infty$ ? We obtain *Brownian Motion*, aka Wiener process.

**Definition 2.2 Brownian Motion**

*Brownian Motion* is a stochastic process  $W$  if:

- $W_0 = 0$
- Independent increments, i.e  $W_{t_4} - W_{t_3}$  and  $W_{t_2} - W_{t_1}$  are independent (as long as they are not overlapping)
- $W_t - W_s \sim N(0, t - s)$
- $t \mapsto W_t$  is continuous

This is a nice definition and all, but does there even exists something which satisfies our definition?

**Sats 2.1**

$t \mapsto W_t$  is of infinite variation and nowhere differentiable  
By infinite variation, it is meant

$$\lim_{n \rightarrow \infty} \sum_k |W_{t_{k+1}} - W_{t_k}| = \infty$$

A regular differentiable function has bounded variation. The next goal is to define the stochastic integral  $\int_0^t g_s dW_s$ , where  $g_t$  is a stochastic process determined by the Brownian motion  $W$

### Definition 2.3 Measurable w.r.t $\sigma$ -algebra

Let  $X_t$  be a stochastic process. An event  $A$  is  $\mathcal{F}_t^X$  measurable (denoted  $A \in \mathcal{F}_t^X$ ) if it is possible to determine whether  $A$  has happened or not based on observations of  $\{X_s : 0 \leq s \leq t\}$

**Example:**

$$A = \{X_s \leq 7 : \forall s \leq 9\} \in \mathcal{F}_9^X$$

### Definition 2.4

If a random variable  $Z$  can be determined by observations of  $\{X_s : 0 \leq s \leq t\}$ , then  $Z \in \mathcal{F}_t^X$

**Example:**

$$Z = \int_0^5 X_s ds \in \mathcal{F}_5^X$$

If you only know  $X_5$  up to 4, then you cannot determine  $Z$

### Definition 2.5

A stochastic process  $Y_t$  with  $Y_t \in \mathcal{F}_t^X \quad \forall t$  is *adapted to the filtration*  $\mathcal{F}_t^X$

**Example:**

$Y_t = \sup_{0 \leq s \leq t} W_s$  is adapted to  $\mathcal{F}_t^W$

### Definition 2.6

The process  $g_t \in \mathcal{L}^2$  if

- $g$  is adapted to  $\mathcal{F}_t^W$
- $\int_0^t \mathbb{E}(g_s^2) ds < \infty$

**Example:**

Brownian motion  $\in \mathcal{L}^2$ , its adapted to  $\mathcal{F}_t^W$  and  $\int_0^t \mathbb{E}(\overbrace{W_s^2}^{\sim N(0, \sqrt{s})}) ds = \int_0^t s ds = \frac{t^2}{2} < \infty$

## 2.2. Stochastic integration.

Assume  $g \in \mathcal{L}^2$ . If  $g$  is simple (i.e  $g_s = g_{t_k}$  for  $s \in [t_k, t_{k+1}]$ ), then we define

$$\int_0^t g_s dW_s = \sum_{k=0}^{n-1} g_{t_k} (W_{t_{k+1}} - W_{t_k})$$

For egeneral  $g \in \mathcal{L}^2$ , we can approximate  $g$  using step functions which are simple such that

$$\int_0^t \mathbb{E}((g_s - g_s^n)^2) ds \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Then, one defines the stochastic integral as

$$\int_0^t g_s dW_s = \lim_{n \rightarrow \infty} \int_0^t g_s^n dW_s$$

### Remark

One can show that the limit indeed exists and does not depend on the sequence used for approximation.

### Remark:

Forward increments are used! The integrand is fixed at  $t_k$ , and we look at forward movements of the Brownian motion.

### Remark:

Steiltjes integration si not possible since paths are not of unbounded variation.

### Proposition:

Assume  $g \in \mathcal{L}^2$  and adapted to a filtration, then:

1.  $\mathbb{E} \left( \int_0^t g_s dW_s \right) = 0$
2.  $\mathbb{E} \left( \left( \int_0^t g_s dW_s \right)^2 \right) = \int_0^t \mathbb{E}(g_s^2) ds$  (Ito isometry)
3.  $X_t = \int_0^t g_s dW_s$ , then  $X_t$  is  $\mathcal{F}^W$ -adapted

### Bevis 2.1

Assume  $g$  is simple (if it was not, then approximate using step functions).

1.

$$\begin{aligned} \mathbb{E} \left( \int_0^t g_s dW_s \right) &= 0 = \mathbb{E} \left( \sum_{k=0}^{n-1} g_{t_k} (W_{t_{k+1}} - W_{t_k}) \right) = \sum_{k=0}^{n-1} \mathbb{E} \left( \underbrace{g_{t_k}}_{\text{indep.}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep.}} \right) \\ &= \sum_{k=0}^{n-1} \mathbb{E}(g_{t_k}) \underbrace{\mathbb{E}(W_{t_{k+1}} - W_{t_k})}_{\sim N(0, \sigma^2)} = 0 \end{aligned}$$

**2.** This is the variance of a stochastic integral:

$$\begin{aligned}
\mathbb{E} \left( \left( \sum_{k=0}^{n-1} g_{t_k} (W_{t_{k+1}} - W_{t_k}) \right)^2 \right) &= \mathbb{E} \left( \sum_{k=0}^{n-1} g_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2 \right) + 2 \sum_{j < k} \underbrace{g_{t_k} g_{t_j}}_{\in \mathcal{F}_{t_k}} \underbrace{(W_{t_{k+1}} - W_{t_k})}_{\text{indep. of } \mathcal{F}_{t_k}} \underbrace{(W_{t_{j+1}} - W_{t_j})}_{\in \mathcal{F}_{t_k}} \\
&= \sum_{k=0}^{n-1} \mathbb{E} (g_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2) + 2 \sum_{j < k} \mathbb{E} (g_{t_k} g_{t_j} (W_{t_{k+1}} - W_{t_k}) (W_{t_{j+1}} - W_{t_j})) \\
&= \sum_{k=0}^{n-1} \mathbb{E} (g_{t_k}^2) \mathbb{E} \left( \underbrace{(W_{t_{k+1}} - W_{t_k})^2}_{t_{k+1} - t_k} \right) + 2 \sum_{j < k} \mathbb{E}(\dots) \underbrace{\mathbb{E}(W_{t_{k+1}} - W_{t_k})}_{=0} \\
&= \int_0^t \mathbb{E}(g_{t_k}^2) dW_s
\end{aligned}$$

□

### 2.3. Properties of the stochastic integral.

**Examples:**

$\int_0^t 1 dW_s = W_t - W_0 = W_t$ , but that is  $\int_0^t W_s dW_s$ ?  $W_s$  is not piecewise constant, but we may approximate it by letting  $g_t^n = W_{t_k}$  for  $t \in [t_k, t_{k+1})$ . What happens here is essentially discretisation but for finer and finer time.

This yields the approximation

$$\begin{aligned}
\int_0^t \mathbb{E}((g_s^n - W_s)^2) ds &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \underbrace{\mathbb{E}((W_s - W_{t_k})^2)}_{s - t_k} \leftarrow \text{variance of increment of BM} \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds = \sum_{k=0}^{n-1} \frac{1}{2} (t_{k+1} - t_k)^2 = \sum_{k=0}^{n-1} \frac{1}{2} \Delta t \\
\Delta t &= \frac{t}{n} \Rightarrow \frac{1}{2} (\Delta t)^2 \frac{t}{\Delta t} = \frac{\Delta t}{2} t \rightarrow 0 \quad \text{as } n \rightarrow \infty \\
\Rightarrow \sum_{k=0}^{n-1} W_{t_k} (W_{t_{k+1}} - W_{t_k}) &= \frac{1}{2} \sum_{k=0}^{n-1} (W_{t_{k+1}}^2 - W_{t_k}^2) = \frac{1}{2} W_{t_n}^2 - \underbrace{\frac{1}{2} \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2}_{I_n}
\end{aligned}$$

We claim  $I_n \rightarrow t$  as  $n \rightarrow \infty$ :

$$\mathbb{E}(I_n) = \mathbb{E} \left( \underbrace{\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2}_{\text{2nd moment}} \right) = \sum_{k=0}^{n-1} (t_{k+1} - t_k) = t_n = t$$

Need to check  $\mathbb{E}((I_n - t)^2) = 0$ :

$$\begin{aligned}
&\mathbb{E} \left( \left( \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2 - \overbrace{(t_{k+1} - t_k)}^{\Delta t} \right)^2 \right) \\
&= \sum_{k=0}^{n-1} \mathbb{E} \left( ((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)^2 \right) + \sum_{j \neq k} \mathbb{E} \left( ((W_{t_{k+1}} - W_{t_k})^2 - \Delta t) ((W_{t_{j+1}} - W_{t_j})^2 - \Delta t) \right) \\
&= \sum_{j \neq k} \mathbb{E} \left( (W_{t_{k+1}} W_{t_k})^4 \right) - (\Delta t)^2 = \sum_{k=0}^{n-1} 2(\Delta t)^2 \sim \Delta t \rightarrow 0
\end{aligned}$$

hus,  $I_n \rightarrow t$  as  $n \rightarrow \infty$ , so

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}$$

**Remark:**



Lets prove if  $X \sim N(0, \sigma)$ , then  $\mathbb{E}(X^4) = 3\sigma^2$

$$\begin{aligned}\mathbb{E}(X^4) &= \int z^4 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-z^2}{2\sigma^2}\right\} dz \stackrel{\text{parts}}{=} - \left[ z^3 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\} \right]_{-\infty}^{\infty} - \int 3z^2 \frac{\sigma^2}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\} dz \\ &= 3\sigma^2 \cdot \underbrace{\int z^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-z^2/2\sigma^2\right\} dz}_{\sigma^2} = 3\sigma^4\end{aligned}$$

### 3. MARTINGALES

Let  $\mathcal{F}_t$  be a filtration, "information generated by B; up to a time  $t$ ".

If  $Y$  is a random variable, then  $\mathbb{E}(Y | \mathcal{F}_t)$  is the conditional expectation given all information up to time  $t$

**Example:**

$$\mathbb{E}(W_s | \mathcal{F}_t) = W_t$$

#### Definition 3.7 Martingale

A process  $X$  is a martingale if  $X$  is  $\mathcal{F}_t$ -adapted.  $X_t$  integrable, i.e

- $\mathbb{E}(|X_t|) < \infty \quad \forall t$
- $\mathbb{E}(X_s | \mathcal{F}_t) = X_t$  for  $s > t$

**Example:**

$W_t$  is a martingale,  $W_t^2 - t$  is a martingale since

$$\begin{aligned}Y_t &:= W_t^2 - t \quad \mathbb{E}(Y_t | \mathcal{F}_s) = \mathbb{E}(W_t^2 - t | \mathcal{F}_s) \\ &= \mathbb{E}((W_t - W_s)^2 + 2W_s W_t - W_s^2 | \mathcal{F}_s) - t \\ &= t - s + 2\mathbb{E}(W_s W_t | \mathcal{F}_s) - \mathbb{E}(W_s^2 | \mathcal{F}_s) - t = 2W_s \underbrace{\mathbb{E}(W_t | \mathcal{F}_s)}_{W_s} W_s^2 - s \\ &= W_s^2 - s = Y_s\end{aligned}$$

$Y_t = \int_0^t g_u dW_u$  is a martingale since:

$$\mathbb{E}(Y_t | \mathcal{F}_s) = \mathbb{E}\left(\int_0^s g_u dW_u | \mathcal{F}_s\right) + \mathbb{E}\left(\int_s^t g_u dW_u | \mathcal{F}_s\right) = \int_0^s g_u dW_u = Y_s$$

However,  $W_t^3$  is *not* a martingale:

$$\begin{aligned}\mathbb{E}(W_t^3 | \mathcal{F}_s) &= \mathbb{E}(W_s^3 + (W_t - W_s)^3 - 3W_t W_s^2 + 3W_t^2 W_s | \mathcal{F}_s) \\ &= W_s^3 + 0 - 3W_s^2 \underbrace{\mathbb{E}(W_t | \mathcal{F}_s)}_{W_s} + 3W_s \underbrace{\mathbb{E}(W_t^2 | \mathcal{F}_s)}_{t-s+W_s^2} \\ &= W_s^3 + 3(t-s)W_s \neq W_s^3\end{aligned}$$

**Remark:** A martingale is a "fair game"

## 4. ITOS FORMULA

Assume

$$X_t = a + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

for some adapted process  $\mu_t$  and  $\sigma_t$ . Short-hand notation  $\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t \\ X_0 = a \end{cases}$

Let  $f(t, x)$  be a  $C^{1,2}$ -function and define  $Z_t = f(t, X_t)$ , what does  $dZ_t$  look like?

**Recall:**

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$$

so  $W_t^2 = t + 2 \int_0^t W_s dW_s$ , thus

$$d(W_t^2) = dt + 2W_t dW_t$$

Fix  $n$  and let  $t_k = \frac{k}{n}t$

Let  $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$  and consider

$$S_n = \sum_{k=0}^{n-1} (\Delta W_{t_k})^2$$

We have

$$\mathbb{E}(S_n) = \sum_{k=0}^{n-1} \mathbb{E}((\Delta W_{t_k})^2) = \sum_{k=0}^{n-1} \frac{t}{n} = t$$

and

$$\text{Var}(S_n) \stackrel{\text{indep.}}{=} \sum_{k=0}^{n-1} \text{Var}((\Delta W_{t_k})^2) = n \text{Var}((\Delta W_{t_0})^2) = n \cdot 2 \frac{t^2}{n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Thus  $S_n \rightarrow t$  as  $n \rightarrow \infty$  (in  $\mathcal{L}^2$ ). This motivates to write

$$\begin{aligned} \int_0^t (dW_s)^2 &= t \\ \Leftrightarrow dW_t^2 &= dt \end{aligned}$$

## 4.1. Taylor Expansion.

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} + \frac{\partial^2 f}{\partial x^2} (dX_t)^2 + \frac{\partial^2 f}{\partial t^2} (dt)^2 + \frac{\partial^2 f}{\partial t \partial x} dt dX_t + \text{higher order terms} \\ &= \left( \frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t + \text{higher order terms} \end{aligned}$$

**Sats 4.2: Itos formula**

If  $dX_t = \mu_t dt + \sigma_t dW_t$  and  $Z_t = f(t, X_t)$ , then

$$dZ_t = \left( \frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

Here  $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(t, X_t)$  and similarly for other derivatives of  $f$

**Alternative formulation:**

$$dZ_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2$$

Where  $(dX_t)^2$  is calculated using

- $(dt)^2 = 0$

- $dt dW_t = 0$
- $(dW_t)^2 = dt$

**Example:**

Compute  $\int_0^t W_s dW_s$ . Let  $Z_t = W_t^2$ , then by Itos formula

$$\begin{aligned} dZ_t &= 2W_t dW_t + \frac{1}{2} \cdot 2(dW_t)^2 \\ &= dt + 2W_t dW_t \end{aligned}$$

Thus  $W_t^2 = Z_t = t + 2 \int_0^t W_s dW_s$ , so  $\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$

**Example:**

Compute  $\mathbb{E}(W_t^4)$

Let  $Z_t = W_t^4$ , then by Itos formula

$$\begin{aligned} dZ_t &= 4W_t^3 dW_t + \frac{1}{2} \cdot 12W_t^2 (dW_t)^2 \\ &= 6W_t^2 dt + 4W_t^3 dW_t \end{aligned}$$

Thus

$$W_t^4 = Z_t = 6 \int_0^t W_s^2 ds + 4 \int_0^t W_s^3 dW_s$$

Taking expectation yields

$$\begin{aligned} \mathbb{E}(W_t^4) &= 6 \int_0^t \underbrace{\mathbb{E}(W_s^2)}_s ds + 4 \underbrace{\mathbb{E} \left( \int_0^t W_s^3 dW_s \right)}_{=0} \\ &= 6 \int_0^t s ds = 3t^2 \end{aligned}$$

Alternatively, without using Itos formula

$$\begin{aligned} \mathbb{E}(W_t^4) &= \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \stackrel{\text{parts.}}{=} \left[ x^3 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} \right]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2 \frac{t}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \\ &= 3t \text{Var}(W_t) = 3t^2 \end{aligned}$$

**Example:**

Compute  $\mathbb{E}(e^{\alpha W_t})$

Let  $Z_t = e^{\alpha W_t}$ . Itos formula yields

$$\begin{aligned} dZ_t &= \alpha e^{\alpha W_t} dW_t + \frac{1}{2} \alpha^2 e^{\alpha W_t} (dW_t)^2 \\ &= \frac{\alpha^2}{2} e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t \\ &= \frac{\alpha^2}{2} Z_t dt + \alpha Z_t dW_t \end{aligned}$$

Integration yields

$$Z_t = 1 + \frac{\alpha^2}{2} \int_0^t Z_s ds + \alpha \int_0^t Z_s dW_s$$

So

$$\begin{aligned} \mathbb{E}(Z_t) &= 1 + \mathbb{E} \left( \frac{\alpha^2}{2} \int_0^t Z_s ds \right) + \underbrace{\mathbb{E} \left( \alpha \int_0^t Z_s dW_s \right)}_{=0} \\ &= 1 + \frac{\alpha^2}{2} \int_0^t \mathbb{E}(Z_s) ds \end{aligned}$$

Let  $m(t) = \mathbb{E}(Z_t)$ , then

$$\begin{cases} \frac{dm}{dt} = \frac{\alpha^2}{2} m(t) \\ m(0) = 1 \end{cases}$$

Which has the solution  $m(t) = e^{\frac{\alpha^2}{2} t}$

#### 4.2. Multi-dimensional Ito formula.

Assume  $dX_t^i = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j$  where  $W^i$  are  $d$  independent Brownian motions.

On a matrix form:

$$\underbrace{dX_t}_{n \times 1} = \underbrace{\mu_t}_{n \times 1} dt + \underbrace{\sigma_t}_{n \times d} \underbrace{dW_t}_{d \times 1}$$

Let  $Z_t = f(t, X_t)$  where  $f : [0, \infty] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^{1,2}$

#### Sats 4.3: Itos multi-dimensional formula

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

Where

- $dW_t^i dW_t^j = 0$  if  $i \neq j$
- $(dW_t^i)^2 = dt$
- $(dt)^2 = dt dW_t = 0$

**Alternatively**

$$dZ_t = \left( \frac{\partial f}{\partial t} + \sum_{i=1}^n \mu_t^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_t^{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \sigma_t^i dW_t$$

Where  $C = \sigma \sigma^*$  and  $\sigma^i$  is the  $i$ :th row of  $\sigma$

Indeend,

$$\begin{aligned} dX_t^i dX_t^j &= \left( \sum_{k=1}^d \sigma^{ik} dW_t^k \right) \left( \sum_{l=1}^d \sigma^{jl} dW_t^l \right) \\ &= \left( \sum_{k=1}^d \sigma^{ik} \sigma^{jl} \right) dt \\ &= (\sigma \sigma^*)^{ij} dt \end{aligned}$$

**Example:**

If  $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ dY_t = \gamma Y_t dt + \delta Y_t dV_t \end{cases}$  and  $Z_t = X_t Y_t$ ; find  $dZ_t$

Itos formula yields

$$\begin{aligned} dZ_t &= Y_t dX_t + X_t dY_t + \frac{1}{2} \cdot 2 dX_t dY_t \\ &= (\alpha + \gamma) Z_t dt + Z_t (\sigma dW_t + \delta dV_t) \end{aligned}$$

Setting  $\bar{W}_t = \frac{1}{\sqrt{\sigma^2 + \delta^2}} (\sigma W_t + \delta V_t)$ , then  $\bar{W}$  is a Brownian Motion and

$$dZ_t = (\alpha + \gamma) Z_t dt + \sqrt{\sigma^2 + \delta^2} Z_t d\bar{W}_t$$

## 5. CORRELATED BROWNIAN MOTIONS

Let  $\bar{W} = \begin{bmatrix} \bar{W}^1 \\ \vdots \\ \bar{W}^d \end{bmatrix}$  where  $\bar{W}^1, \dots, \bar{W}^d$  are independent

Consider  $W = \delta \bar{W}$  where

$$\delta = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1d} \\ \vdots & \vdots & \vdots \\ \delta_{d1} & \cdots & \delta_{dd} \end{bmatrix} = \underbrace{\begin{bmatrix} \delta_1 \\ \vdots \\ \delta_d \end{bmatrix}}_{\text{Row vectors with } \|\delta_i\| = 1}$$

Here  $\|\delta_i\| = \sqrt{\delta_{i1}^2 + \cdots + \delta_{id}^2}$ .  
So  $W^i$  is a Brownian motion.

Moreover,

$$\begin{aligned} dW_t^i dW_t^j &= \left( \sum_{k=1}^d \delta_{ik} d\bar{W}_t^k \right) \left( \sum_{l=1}^d \delta_{jl} d\bar{W}_t^l \right) \\ &= \sum_{k=1}^d \delta_{ik} \delta_{jk} dt = (\delta \delta^*)_{ij} dt \end{aligned}$$

**Definition 5.8 Correlated Wiener Process**

$W_t$  as constructed above is a  $d$ -dimensional *correlated Wiener process* with correlation matrix  $\rho = \delta \delta^*$

**Sats 5.4: Itos formula, correlated version**

If  $W_t$  is a correlated Wiener process as above, and

$$\underbrace{dX_t}_{n \times 1} = \underbrace{\mu_t}_{n \times 1} dt + \underbrace{\sigma_t}_{n \times d} \underbrace{dW_t}_{d \times 1}$$

satisfies

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

where

- $(dt)^2 = dt dW^i = 0$
- $dW^i dW^j = \rho_{ij} dt$

**Example:**

Given  $\bar{W} = \begin{bmatrix} \bar{W}^1 \\ \bar{W}^2 \end{bmatrix}$  (where  $\bar{W}^1, \bar{W}^2$  are independent), construct  $W = \begin{bmatrix} W^1 \\ W^2 \end{bmatrix}$  with correlation matrix

$$\rho = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix}$$

Note that  $\delta = \begin{bmatrix} 1 & 0 \\ \rho_0 & \sqrt{1 - \rho_0^2} \end{bmatrix}$  satisfies  $\rho \rho^* = \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix} = \rho$

Thus  $W = \begin{bmatrix} \bar{W}^1 \\ \rho_0 \bar{W}^1 + \sqrt{1 - \rho_0^2} \bar{W}^2 \end{bmatrix}$  is a correlated Wiener process with correlated matrix  $\delta$

What other choices for  $\delta$  are possible?

## 6. STOCHASTIC DIFFERENTIAL EQUATIONS

Let

- a  $d$ -dimensional Brownian motion  $W$
- $\mu : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
- $\sigma : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$
- $x_0 \in \mathbb{R}^n$

be given. A *stochastic differential equation* is an equation at the form

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0 \end{cases} \quad (1)$$

Or, equivalently,

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

**Sats 6.5**

Assume

$$\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K \|x - y\|$$

and  $\|\mu(t, x)\| + \|\sigma(t, x)\| \leq K \|x\|$  for some  $K$

Then there exists a unique solution  $X_t$  to the SDE (1). Moreover,

1.  $X$  is  $\mathcal{F}^W$ -adapted
2.  $X_t$  has continuous trajectories
3.  $X$  is a Markov process

## 7. GEOMETRIC BROWNIAN MOTION

Consider

$$\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t & \alpha, \sigma \text{ constants} \\ X_0 = x \end{cases}$$

**Anmärkning:**

If  $\sigma = 0$ , then  $dX_t = \alpha X_t dt$  so  $X_t = x_0 e^{\alpha t}$

Let  $Z_t = \ln(X_t)$ . Then

$$dZ_t \stackrel{\text{Ito}}{=} \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (dX_t)^2 = \left( \alpha - \frac{\sigma^2}{2} \right) dt + \sigma dW_t$$

so  $Z_t = \ln(x_0) + \left( \alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t$  and  $X_t = e^{Z_t} = x_0 e^{\left( \alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t}$

Moreover,

$$\mathbb{E}(X_t) = x_0 + \mathbb{E} \left[ \int_0^t \alpha X_s ds \right] + \underbrace{\mathbb{E} \left[ \int_0^t \sigma X_s dW_s \right]}_{=0}$$

So if  $m(t) = \mathbb{E}(X_t)$ , we find  $\begin{cases} \frac{dm}{dt} = \alpha m(t) \\ m(0) = x_0 \end{cases}$

Thus  $m(t) = x_0 e^{\alpha t}$

**Results:**

The solution of  $\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t \\ X_0 = x_0 \end{cases}$  is  $X_t = x_0 \exp \left\{ \left( \alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}$

Moreover,  $\mathbb{E}(X_t) = x_0 e^{\alpha t}$

**Example:**

Consider the SDE  $\begin{cases} dX_t = -X_t dt + dW_t \\ X_0 = x \end{cases}$  (this is a mean-reverting Ornstein-Uhlenbeck process)

The trick here is to let  $Y_t = e^t X_t$ . Then

$$\begin{aligned} dY_t &= e^t X_t dt + e^t dX_t = e^t dW_t \\ \Rightarrow Y_t &= x + \int_0^t e^s dW_s \end{aligned}$$

Thus  $X_t = e^{-t} Y_t = x e^{-t} + e^{-t} \int_0^t e^s dW_s$   
Moreover  $\mathbb{E}(X_t) = x e^{-t}$

### Definition 7.9 Diffusion process

The solution  $X$  of an SDE

$$\begin{cases} dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW \\ X_0 = x_0 \end{cases}$$

is called a *diffusion process*.

$\mu$  is called the *drift* and  $\sigma$  is the *diffusion coefficient*

## 8. PARTIAL DIFFERENTIAL EQUATIONS

Consider the following *terminal value problem*:

Given function  $\sigma, \mu, \phi$ , find a function  $F(t, x)$  such that

$$\begin{cases} \frac{\partial F}{\partial t}(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2 F}{\partial x^2}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) = 0 \\ F(T, x) = \phi(x) \end{cases} \quad (2)$$

If  $F(t, x)$  satisfies (2), define  $X_s$  by  $\begin{cases} dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s \\ X_t = x \end{cases}$  and let  $Z_s = F(s, X_s)$ . Then

$$\begin{aligned} dZ_s &\stackrel{\text{Ito}}{=} \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x} dX_s + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dX_s)^2 \\ &= \underbrace{\left( \frac{\partial F}{\partial s} + \mu \frac{\partial F}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} \right)}_{=0} ds + \sigma \frac{\partial F}{\partial x} dW_s \\ &= \sigma \frac{\partial F}{\partial x} dW_s \end{aligned}$$

Integrate:

$$Z_T = Z_t + \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s$$

Take expectation:

$$\mathbb{E}(Z_T) = Z_t = F(t, x) = \mathbb{E}(F(T, X_T)) \stackrel{*}{=} \mathbb{E}(\phi(X_T))$$

We write  $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$  (to indicate that  $X_t = x$ )

We have thus proved the following:

### Sats 8.6: Feynman-Kac

If  $F(t, x)$  satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2(t, x)}{2} \frac{\partial^2 F}{\partial x^2} + \mu(t, x) \frac{\partial F}{\partial x} = 0 & (t < T) \\ F(t, x) = \phi(x) \end{cases}$$

then  $F(t, x) = \mathbb{E}_{t,x}(\phi(X_T))$  where  $\begin{cases} dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s \\ X_t = x \end{cases}$

**Example:**

Solve the PDE 
$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} = 0 \\ F(T, x) = x^2 \end{cases}$$

**Solution:**

Let  $X_s$  be the solution of 
$$\begin{cases} dX_s = \sigma dW_s \\ X_t = x \end{cases} \quad \text{i.e. } X_s = x + \sigma(W_s - W_t)$$

By Feynman-Kac:

$$\begin{aligned} F(t, x) &= \mathbb{E}_{t,x}(X_T^2) = \mathbb{E}((x + \sigma(W_T - W_t))^2) \\ &= x^2 + 2x\sigma\mathbb{E}(W_T - W_t) + \sigma^2\mathbb{E}((W_T - W_t)^2) \\ &= x^2 + \sigma^2(T - t) \end{aligned}$$

$$F(t, x) = x^2 + \sigma^2(T - t)$$

### Sats 8.7: Feynman-Kac in higher dimensions + discounting

Assume that  $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n C_{i,j}(t, x) \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i(t, x) \frac{\partial F}{\partial x_i} - rF(t, x) = 0 \\ F(T, x) = \phi(x) \end{cases}$$

Where  $C(t, x) = \sigma(t, x)\sigma^*(t, x)$  for some matrix  $\sigma$  ( $n \times d$ )

Then  $F(t, x) = e^{-r(T-t)} \mathbb{E}_{t,x}(\phi(X_T))$  where

$$\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t = x \end{cases}$$

### Bevis 8.1

Let  $Z_s = e^{-r(s-t)} F(s, X_s)$ . Then

$$dZ_s \stackrel{\text{Ito}}{=} e^{-r(s-t)} \underbrace{\left( \frac{\partial F}{\partial s} + \frac{1}{2} \sum_{i,j=1}^n C_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i \frac{\partial F}{\partial x_i} - rF \right)}_{=0} ds + e^{-r(s-t)} \sum_{i=1}^n \frac{\partial F}{\partial x_i} \sigma_i dW_s$$

So

$$Z_T = \underbrace{Z_t}_{F(t,x)} + \int_t^T \dots dW_s = e^{-r(T-t)} \phi(X_T)$$

Thus  $F(t, x) = e^{-r(T-t)} \mathbb{E}(\phi(X_T))$  □

**Example:**

Solve the PDE 
$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} + \frac{\delta^2}{2} \frac{\partial^2 F}{\partial y^2} - rF = 0 \\ F(T, x, y) = xy \end{cases}$$

**Solution:**

Here  $C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \delta^2 \end{bmatrix}$  so  $\sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix}$  satisfies  $C = \sigma\sigma^*$

$$d \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} \Rightarrow \begin{cases} X_t = x + \sigma(W_T^1 - W_t^1) \\ Y_t = y + \delta(W_T^2 - W_t^2) \end{cases}$$



Feynman-Kac gives

$$\begin{aligned} F(t, x, y) &= \mathbb{E}_{t, x, y} \left( e^{-r(T-t)} X_T Y_T \right) = e^{-r(T-t)} \mathbb{E} \left( (x + \sigma(W_T^1 - W_t^1)) (y + \delta(W_T^2 - W_t^2)) \right) \\ &\stackrel{\text{indep}}{=} e^{-r(T-t)} \mathbb{E} (x + \sigma(W_T^1 - W_t^1)) \mathbb{E} (y + \delta(W_T^2 - W_t^2)) = e^{-r(T-t)} xy \end{aligned}$$

par Answer is therefore  $F(t, x, y) = e^{-r(T-t)} xy$

### Definition 8.10 Infinitesimal Operator

The differential operator

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^n C_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$$

is called the *infinitesimal operator* of  $X$

**Itos formula:**

If  $Z_t = f(t, X_t)$ , then  $dZ_t = \left( \frac{\partial f}{\partial t} + \mathcal{A}f \right) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \sigma_i dW_t$

## 9. PORTFOLIO DYNAMICS

Let the time axis be discrete

**Definition 9.11**

- $N$  = the number of different assets
- $S_n^i$  = the price of one unit of asset  $i$  at time  $n$
- $h_n^i$  = the number of units of asset  $i$  bought at time  $n$
- $h_n = (h_n^1, h_n^2, \dots, h_n^N)$  is a *portfolio*
- $V_n$  = the value of a portfolio  $h_n$  at time  $n = \sum_{i=1}^N h_n^i S_n^i = h_n \cdot S_n$

The interpretation:

- At time  $n-$  we have an old portfolio  $h_{n-1}$  from the previous period
- At time  $n$ ,  $S_n$  becomes observable
- At time  $n$ , after observing  $S_n$ , we chose  $h_n$

**Definition 9.12 Budget equation**

$$h_n \cdot S_{n+1} = h_{n+1} \cdot S_{n+1}$$

**Notation:** If  $\{x_n\}_{n=0}^\infty$  is a sequence of real numbers, let  $\Delta x_n = x_{n+1} - x_n$ .  
The budget equation becomes  $S_{n+1} \cdot \Delta h_n = 0$

Recall  $Y_n = h_n \cdot S_n$

Since  $\Delta V_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_n = h_{n+1} \cdot S_{n+1} - h_n \cdot S_{n+1} + h_n \cdot S_{n+1} - h_n \cdot S_n$   
 $= S_{n+1} \cdot \Delta h_n + h_n \cdot \Delta S_n$

we have  $\Delta V_n = h_n \cdot \Delta S_n$  if the budget equation is fulfilled.

Below we use this relation to *define* what is meant by a self-financing portfolio in continuous time.

**Definition 9.13**

Let  $\{S_t \mid t \geq 0\}$  be an  $N$ -dimensional process

- A *portfolio*  $h$  is an  $\mathcal{F}^s$ -adapted  $N$ -dimensional process
- $h$  is *Markovian* if  $h_t = h(t, S_t)$  for some function  $h$
- The *value process*  $V^h$  of  $h$  is

$$V_t^h = \sum_{i=1}^N h_t^i S_t^i = h_t \cdot S_t$$

- A portfolio  $h$  is *self-financing* if

$$dV_t^h = h_t \cdot dS_t$$

- For a given portfolio  $h$ , the corresponding *relative portfolio*  $w$  is

$$w_t^i = \frac{h_t^i S_t^i}{V_t^h} \quad i = 1, \dots, N$$

Note that  $\sum_{i=1}^N w_t^i = 1$ .

Also,  $h$  is self-financing if and only if  $dV_t^h = V_t^h \sum_{i=1}^N \frac{\partial w_t^i}{S_t^i} dS_t^i$

## 10. ARBITRAGE PRICING

In this chapter,  $N = 2$  (two assets):

$$dB_t = rB_t dt$$

This is a risk-free asset, think bank account and  $r$  is a constant interest rate, and

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

is a risky asset, think stock price

**Remarks:**

1.  $B_t = B_0 e^{rt}$
2.  $\mu$  (local mean rate of return) and  $\sigma$  (volatility) are functions of  $t$  and current stock price
3. In the Black-Scholes model,  $\mu$  and  $\sigma$  are constants

The aim is to find a "fair" value of options written on  $S$

Options are also called *financial derivatives*

**Definition 10.14 European Call Option**

A *European call option* with strike price  $K$  and maturity date  $T$  on the underlying asset  $S$  is a contract such that the holder (owner) at time  $T$  has the right, but not the obligation to buy one share of  $S$  at price  $K$  from the option writer (seller)

**Remarks:**

- A *European put option* gives the right (but not the obligation) to *sell* one share of  $S$  at time  $T$  at price  $K$
- An *American call/put* gives the right to buy/sell at *any* time before  $T$

**Definition 10.15**

A *contingent claim with maturity  $T$*  (or a  *$T$ -claim*) is a random variable  $X \in \mathcal{F}_T^S$   
A contingent claim is *simple* if  $X = \phi(S_T)$  for some *contract function* (or payoff function)  $\phi$

**Example:**

For a European call option,  $\phi(x) = (x - K)^+ = \max\{x - K, 0\}$

Indeed, if  $S_T \geq K$ , then buy at price  $K$  and make profit  $S_T - K$ . If  $S_T < K$ , do not exercise the option.

For a European put option  $\phi(x) = (K - x)^+$

We will determine the price  $\pi(t, X)$  of a  $T$ -claim  $X$  at time  $t$  by requiring the market to be *arbitrage-free*.

**Definition 10.16**

A self-financing portfolio  $h$  is an *arbitrage* if 
$$\begin{cases} V_0^h = 0 \\ \mathbb{P}(V_T^h \geq 0) = 1 \\ \mathbb{P}(V_T^h > 0) > 0 \end{cases}$$

The market is *arbitrage-free* if no arbitrage exists.

**Example:**

$$\begin{cases} dS_t^1 = dt + dW_t \\ dS_t^2 = dW_t \\ dB_t = 0 \end{cases} \quad \text{is not arbitrage free}$$

$$\begin{cases} dS_t^1 = dt + dW_t^1 \\ dS_t^2 = dW_t^2 \\ dB_t = 0 \end{cases} \quad \text{is arbitrage free (first two lines indep)}$$

*Assumption:* The price process  $\Pi_t(X)$  is such that  $(B_t, S_t, \Pi_t(X))$  is arbitrage-free.

We also assume that all assets (including the option) can be sold/bought with no market frictions (no transaction costs, no liquidity constraints)

*Idea:* Create a self-financing portfolio of options and the stock such that its value process is locally risk-free (has no  $dW$ -term). The drift of the value must then coincide with the interest rate (otherwise arbitrage). This will give a condition on the price of the option.

Assume  $X = \phi(S_T)$  (simple  $T$ -claim) and that  $\Pi_t(X) = F(t, S_t)$  for some function  $F$ .

*New Notation:*  $F_t = \frac{\partial F}{\partial t}$ ,  $F_s = \frac{\partial F}{\partial s}$ ,  $F_{ss} = \frac{\partial^2 F}{\partial s^2}$

Then

$$\begin{aligned} dF(t, S_t) &\stackrel{\text{Ito}}{=} F_t dt + F_s dS_t + \frac{1}{2} F_{ss} (dS_t)^2 \\ &= \underbrace{\left( F_t + \frac{\sigma^2 S_t^2}{2} F_{ss} + \mu S_t F_s \right)}_{= \mu^F F} F(t, S_t) dt + \underbrace{\frac{\sigma S_t F_s}{F}}_{= \sigma^F} F dW_t \\ &= \mu^F F dt + \sigma^F F dW_t \end{aligned}$$

Let  $(w^S, w^F)$  be a self financing relative portfolio of stocks and options ( $w^S + w^F = 1$ ), and let  $V$  be its value process. Then

$$\begin{aligned} dV_t &= V_t \left( \frac{w^S}{S_t} dS_t + \frac{w^F}{F} dF_t \right) \\ &= (\mu w^S + \mu^F w^F) V_t dt + (\sigma w^S + \sigma^F w^F) V_t dW_t \end{aligned}$$

Let  $(w^S, w^F)$  be defined by

$$\left. \begin{aligned} w^S + w^F &= 1 \\ \sigma w^S + \sigma^F w^F &= 0 \end{aligned} \right\} \Leftrightarrow \begin{cases} w^S = \frac{\sigma^F}{\sigma^F - \sigma} \\ w^F = \frac{-\sigma}{\sigma^F - \sigma} \end{cases}$$

Then  $dV_t = \frac{\mu \sigma^F - \mu^F \sigma}{\sigma^F - \sigma} V_t dt$

By a no-arbitrage argument, we must have  $r = \frac{\mu \sigma^F - \mu^F \sigma}{\sigma^F - \sigma}$

$$\begin{aligned} \text{Here } \underbrace{r \sigma^F - r \sigma}_{= \frac{r \sigma S_t F_s}{F} - r \sigma} &= \underbrace{\mu \sigma^F - \mu^F \sigma}_{= \frac{\mu \sigma S_t F_s}{F} - \frac{\sigma(F_t + \mu S_t F_s) + \frac{-2 S_t^2}{2} F_{ss}}{F}} \\ r S_t F_s - r F &= \mu S_t F_s - F_t - \mu S_t F_s + \frac{\sigma^2}{2} S_t^2 F_{ss} \\ &= -F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} \\ F_t + \frac{\sigma^2 S_t^2}{2} F_{ss} + r S_t F_s - r F &= 0 \end{aligned}$$

Since  $S_t$  can take any value,  $F$  must satisfy the PDE

$$F_t(t, s) + \frac{\sigma^2(t, s)}{2} s^2 F_{ss} + r s F_s(t, s) - r F(t, s) = 0$$

Also,  $\Pi_T(X) = F(T, S_T) = \phi(S_T)$ , so we also have  $F(T, S) = \phi(S_T)$

### Sats 10.8: Black-Sholes equation

$$\text{In the market } \begin{cases} dB_t = rB_t dt \\ dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \end{cases}, \text{ the only arbitrage-free price of a } T\text{-claim } X = \phi(S_T) \text{ is } F(t, S_t), \text{ where } F(t, s) \text{ solves}$$

$$\begin{cases} F_t(t, s) + \frac{\sigma^2(t, s)}{2} s^2 F_{ss}(t, s) + r s F_s(t, s) - r F(t, s) = 0 \\ F(T, s) = \phi(s) \end{cases}$$

The solution to the BS-equation is by Feynman-Kac

$$F(t, s) = \mathbb{E}_{t,s} (\exp \{-r(T-t)\phi(S_T)\})$$

where

$$\begin{aligned} dS_u &= rS_u du + \sigma(u, S_u)S_u dW_u \\ S_t &= s \end{aligned} \tag{3}$$

we refer to

$$\begin{cases} dS_u = \mu(u, S_u)S_u du + \sigma(u, S_u)S_u dW_u \\ S_t = s \end{cases} \tag{4}$$

as the *P-dynamics* of  $S$  (the specification of  $S$  under the "physical measure"  $P$ ). (3) is referred to as the *Q-dynamics* of  $S$  ( $Q$  is the *pricing measure*, or the *martingale measure*)

### Sats 10.9

The arbitrage-free price of a simple  $T$ -claim  $X = \phi(S_T)$  is  $F(t, S_t)$  where

$$F(t, s) = \mathbb{E}_{t,s}^Q (\exp \{-r(T-t)\phi(S_T)\})$$

and the  $Q$ -dynamics of  $S$  are as in (3)

### Example:

In the standard BS-model (i.e constant  $\sigma$ ), what is the arbitrage-free price of the  $T$ -claim  $X = S_T^2$ ?

By risk-neutral valuation,  $F(t, s) = \exp \{-r(T-t)\} \mathbb{E}_{t,s}^Q (S_T^2)$

Let  $Y_u = S_u^2$ , then

$$dY_u = 2S_u dS_u + (dS_u)^2 \stackrel{dS_u = rS_u du + \sigma S_u dW_u}{=} (2r + \sigma^2)Y_u du + 2\sigma Y_u dW_u$$

$Y$  is a gBm and thus

$$\mathbb{E}_{t,s}^Q (S_T^2) = \mathbb{E}^Q (Y_T) = s^2 \exp \{(2r + \sigma^2)(T-t)\}$$

Which is the price of  $X$  at time  $t$

### Example:

What is the price of  $X = S_t$ ?

By risk-neutral valuation

$$F(t, s) = \exp \{-r(T-t)\} \mathbb{E}_{t,s}^Q (S_T) = s$$

So the price at time  $t$  is  $S_t$

### Remark:

In time-homogenous models (such as the BS-model), the relevant quantity is time  $T-t$  left to maturity.

### Example: Binary option

In the standard BS-model, find the value of  $X = \phi(S_T)$  where  $\phi(x) = \begin{cases} 1 & \text{if } x \geq K \\ 0 & \text{if } x < K \end{cases}$

$$\begin{aligned}
F(0, s) &= \exp \{-rT\} \mathbb{E}_{0,s}^Q (I_{\{S_T \geq K\}}) = \exp \{-rT\} Q(S_T \geq K) \\
&= \exp \{-rT\} Q(\text{sexp} \left\{ \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \right\} \geq K) \\
&= \exp \{-rT\} Q \left( \frac{1}{\sqrt{T}} W_T \geq \frac{\ln \left( \frac{K}{S} \right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}} \right) \\
&= \exp \{-rT\} Q \left( \frac{1}{\sqrt{T}} W_t \leq \frac{\ln \left( \frac{S}{K} \right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}} \right) \\
&= \exp \{-rT\} N \left( \frac{\ln \left( \frac{S}{K} \right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}} \right)
\end{aligned}$$

Where  $N(x) \sim N(0, 1)$ , and the last line is the price at time  $t$

**Example:**

What is the price of a European call option  $X = (S_T - K)^+$ ? In the standard BS-model

$$\begin{aligned}
F(0, s) &= \exp \{-rT\} \mathbb{E}_{0,s}^Q ((S_T - K)^+) = \exp \{-rT\} \mathbb{E}^Q \left( \left( \text{sexp} \left\{ \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \right\} - K \right)^+ \right) \\
&= \exp \{-rT\} \int_a^\infty \left( \text{sexp} \left\{ \left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}x \right\} - K \right) \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx \quad a = \frac{\ln \left( \frac{K}{S} \right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}} \\
&\quad s \int_a^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(x - \sigma \sqrt{T})^2}{2} \right\} dx - K \exp \{-rT\} N(-a) \\
&= s \int_{a - \sigma \sqrt{T}}^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx - K \exp \{-rT\} N(-a) \\
&= sN(\sigma \sqrt{T} - a) - K \exp \{-rT\} N(-a)
\end{aligned}$$

Here we used the fact that the normal-distribution has symmetric tails

**Sats 10.10: Black-Scholes formula**

In the standard BS-model, the price of a European call option is  $F(t, S_t)$ , where

$$F(t, s) = sN(d_1) - K \exp \{-r(T - t)\} N(d_2)$$

and

$$\begin{cases} d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma \sqrt{T - t}} \\ d_2 = d_1 - \sigma \sqrt{T - t} \end{cases}$$

Consider  $F(0, s) = sN(d_1) - K \exp \{-rT\} N(d_2)$  as above, then we have

$$F(0, s) = \mathbb{E}_{0,s}^Q (\exp \{-rT\} (S_T - K)^+) \leq \mathbb{E}_{0,s}^Q (\exp \{-rT\} (S_T)) = s$$

and

$$F(0, s) = \mathbb{E}_{0,s}^Q (\exp \{-rT\} (S_T - K)^+) \geq \mathbb{E}_{0,s}^Q (\exp \{-rT\} (S_T - K)) = s - K \exp \{-rT\}$$

We shall see below that  $F(0, s) = F(0, s; \sigma)$  is increasing in  $\sigma$

**Remark:**

What about the put option?

$$\mathbb{E}_{0,s}^Q (\exp \{-rT\} (K - S_T)^+) = \text{similar to above}$$

Alternatively,  $(K - s)^+ = K - s + (s - K)^+$ . We have priced  $(s - K)^+$ , and  $s$ , so  $p(0, s) = K \exp \{-rT\} - s + c(0, s)$  where  $p$  is the put price and  $c$  is the call price. This relation is called the *put-call parity*. Thus,

$$\begin{aligned} p(0, s) &= K \exp \{-rT\} - s + sN(d_1) - K \exp \{-rT\} N(d_2) \\ &= K \exp \{-rT\} \underbrace{(1 - N(d_2))}_{=N(-d_2)} - s \underbrace{(1 - N(d_1))}_{=N(-d_1)} \end{aligned}$$

### Sats 10.11

Let  $F(t, s)$  be the pricing function of a simple  $T$ -claim  $X = \phi(S_T)$  in the standard BS-model.

If  $\phi$  is convex, then:

1.  $F(t, s)$  is convex in  $s$
2.  $F(t, s)$  is increasing in  $\sigma$

### Bevis 10.1

$$F(0, s) = \exp \{-rT\} \int_{\mathbb{R}} \phi \left( s \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \right) \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx$$

1.

$$F_{ss} = \exp \{-rT\} \int_{\mathbb{R}} \phi'' \left( s \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \right) \exp 2 \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx \geq 0$$

2.

$$\begin{aligned} \frac{\partial F}{\partial \sigma} &= \int_{\mathbb{R}} \phi' \left( s \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \right) s \exp \left\{ -\frac{\sigma^2 T}{2} + \sigma \sqrt{T} x \right\} \sqrt{T} (x - \sigma \sqrt{T}) \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx \\ &= s \sqrt{T} \int_{\mathbb{R}} \phi' \left( \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \right) (x - \sigma \sqrt{T}) \exp \left\{ -\frac{(x - \sigma \sqrt{T})^2}{2} \right\} \frac{1}{\sqrt{2\pi}} dx \\ &\stackrel{\text{parts.}}{=} s \sqrt{T} \int_{\mathbb{R}} \phi''(\cdot) s \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right\} \sigma \sqrt{T} \exp \left\{ -\frac{(x - \sigma \sqrt{T})^2}{2} \right\} \frac{1}{\sqrt{2\pi}} dx \geq 0 \end{aligned}$$

□

### 10.1. Drift estimation.

Assume  $X_t = \mu t + \sigma W_t$  and we want a confidence interval for  $\mu$ . An estimate for  $\mu$  is  $\hat{\mu} = \frac{X_t}{t} \in N\left(\mu, \frac{\sigma}{\sqrt{t}}\right)$  and a confidence interval is

$$\left(\hat{\mu} - \frac{\sigma}{\sqrt{t}} \cdot 1.96, \hat{\mu} + \frac{\sigma}{\sqrt{t}} \cdot 1.96\right)$$

If one wants a certain precision  $\Delta\mu$  so that  $\mathbb{P}(\mu \in (\hat{\mu} - \Delta\mu, \hat{\mu} + \Delta\mu)) = 0.95$ , one needs

$$\frac{2\sigma}{\sqrt{t}} = \Delta\mu \quad \Leftrightarrow \quad t = \frac{4\sigma^2}{(\Delta\mu)^2}$$

Plug in reasonable values  $\left. \begin{array}{l} \sigma = 0.3 \\ \Delta\mu = 0.06 \end{array} \right\} \Rightarrow t = 100 \text{ years!}$

#### Remark:

When pricing options, the drift of the stock needs not be estimated (since under the pricing measure  $Q$ , the drift is  $r$ )

## 11. VOLATILITY

In the BS-formula,  $s, r, t$  are observable,  $T, K$  are specified in the contract and  $\sigma$  is not directly observable. All are needed.

There are 2 approaches, one using *historic volatility* and one using *implied volatility*.

### 11.1. Historic volatility.

If  $dS_t = \mu S_t dt + \sigma S_t dW_t$ , then sample  $S$  at  $n + 1$  time points and let

$$\xi_i = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}}) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma\sqrt{\Delta t}\right)$$

An estimate of  $\sigma^2$  is then  $S^2 = \frac{\sum_{i=1}^n (\xi_i - \bar{\xi})^2}{(n-1)\Delta t}$  where  $\bar{\xi} = \frac{1}{n} \sum_{i=1}^n \xi_i$

### 11.2. Implied volatility.

Let  $p$  be the price in the market of a certain call option (maturity  $T$ , with strike price  $K$ ). Find  $\sigma$  such that  $p = \text{BS}(s, t, T, r, \sigma, K)$  where BS denotes the Black-Scholes formula

This  $\sigma$  is called *implied volatility*

#### Remark:

Recall that the BS-formula is increasing in  $\sigma$

If gBm is the correct model (i.e option prices are calculated using the BS-formula), then the *same* implied volatility would be obtained for different  $K$  and  $T$

## 12. COMPLETENESS AND HEDGING

### Definition 12.17

A  $T$ -claim  $X$  can be *replicated* if there exists a self-financing portfolio  $h$  with  $\mathbb{P}(V_T^h = X) = 1$ .  
If every  $T$ -claim can be replicated then the market is *complete*

### Sats 12.12

Assume that a  $T$ -claim  $X$  can be replicated using  $h$ . Then the only possible arbitrage-free price of  $X$  is  $\Pi_t(X) = V_t^h$



**Bevis 12.1**

If for example  $\Pi_t(X) < V_t^h$  for some  $t$ ; sell the portfolio and buy the claim  $\Rightarrow$  arbitrage  $\square$

We now specialize to the model

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \end{aligned} \quad (5)$$

with  $\sigma(t, s) > 0$

**Sats 12.13**

The model (5) is complete

We will prove a simpler result, namely that all *simple*  $T$ -claims can be replicated.

Recall that the value  $\Pi_t(X)$  of a simple  $T$ -claim  $X = \phi(S_T)$  is  $F(t, S_t)$  where  $F(t, s)$  is the pricing function. Thus

$$\begin{aligned} d\Pi_t &= F_t dt + F_s dS_t + \frac{1}{2} F_{ss} (dS_t)^2 \\ &= \left( F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} \right) dt + F_s dS_t \end{aligned}$$

Moreover, a portfolio  $h = (h^B, h^S)$  is self-financing if  $dV_t^h = h_t^B dB_t + h_t^S dS_t$ . Choose  $h_t^S = F_s(t, S_t)$

**Sats 12.14**

Let  $X = \phi(S_T)$  and define  $F(t, s)$  by

$$\begin{cases} F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} + rS_t F_s - rF = 0 \\ F(T, s) = \phi(s) \end{cases}$$

Define  $h = (h^B, h^S)$  by

$$\begin{cases} h_t^B = \frac{F(t, S_t) - S_t F_s(t, S_t)}{B_t} \\ h_t^S = F_s(t, S_t) \end{cases}$$

Then  $h$  replicates  $X$  and  $\Pi_t(X) = V_t^h = F(t, S_t)$

**Bevis 12.2**

$V_t^h = h_t^B B_t + h_t^S S_t = F(t, S_t)$ , so

$$\begin{aligned} dV_t^h &= F_t dt + F_s dS_t + \frac{1}{2} F_{ss} (dS_t)^2 \\ &= \left( F_t + \frac{\sigma^2}{2} S_t^2 F_{ss} \right) dt + F_s dS_t \\ &\stackrel{\text{BS PDE}}{=} r(F - S_t F_s) dt + F_s dS_t = h_t^B dB_t + h_t^S dS_t \end{aligned}$$

Thus  $h$  is self-financing. Since  $V_T^h = F(T, S_T) = \phi(S_T) = X$ ,  $h$  replicates  $X$ .

By no-arbitrage  $\Pi_t(X) = V_t^h = F(t, S_t)$   $\square$

**Example:**

If  $X = S_T$ , then  $F(t, s) = s$ , so  $h_t^S = F_s = 1$

**Example:**

For a call option (in the standard BS-model),  $F(0, s) = sN(d_1) - K \exp\{-rT\} N(d_2)$ , thus

$$F_S(0, s) = N(d_1) + \frac{1}{\sqrt{2\pi}} \left( s \exp\left\{-\frac{d_1^2}{2}\right\} - K \exp\{-rT\} \exp\left\{-\frac{d_2^2}{2}\right\} \right) \frac{\partial d_1}{\partial s}$$

Moreover,

$$s \exp\left\{-\frac{d_1^2}{2}\right\} - K \exp\{-rT\} \exp\left\{-\frac{d_2^2}{2}\right\} = \exp\left\{-\frac{d_1^2}{2}\right\} \left( s - K \exp\{-rT\} \exp\left\{-\frac{\sigma^2 T}{2}\right\} \exp\left\{\sigma\sqrt{T}d_1\right\} \right) = 0$$

so  $F_s(0, s) = N(d_1)$

**Remark:**

The derivative  $\Delta = F_s$  is called the *delta*.

In a replicating portfolio one should hold  $\Delta$  shares of  $S$  at each time.