F12 $\quad (\Omega, \mathcal{F}, (\mathcal{F}_t), P)$

$t \to B_t : [0,\infty) \to \mathbb{R}$ Standard BM

$t \to B_t = (B^1_t, \ldots, B^d_t) \in \mathbb{R}^d, \ t > 0$

- $B_0 = 0$, cont. paths
- $B_{s+t} - B_s$ independent of $\mathcal{F}_s$
- $B_{s+t} - B_s \overset{d}{=} B_t$

"Brownian particle"

$t \to B^L_t \in \mathbb{R}, \ t \in \mathbb{R}^n$

Levy's ~~BM~~? Brownian function

iff

- Gaussian
- $B^L_0 = 0$
- $E(B^L_t) = 0, \ t \in \mathbb{R}^d$
- $E[(B^L_s - B^L_t)^2] = |t - s|, \quad s, t \in \mathbb{R}^d$

Def The process $X = \{X_t\}_{t \geq 0}$ is called a Levy process w.r.t. $(\mathcal{F}_t)$ if $X_t \in \mathcal{F}_t$, $t > 0$, and

a) $t \to X_t$ is right-continuous with left limit (CADLAG, RCLL) starting from $X_0 = 0$, almost surely, and

b) for every $s, t > 0$, the increment $X_{s+t} - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_t.

<u>Note</u> If $X=\{X_t\}$ is a Lévy process in R^d

~~then~~ cX, $c>0$, — " —

AX, A $m\times d$-matrix, — " — R^m

If $X_1, \dots, X_k$ are i.i.d Lévy process

then $\sum_1^k X_j$ is also a Lévy process

<u>Examples</u> a) $X_t = \mu t + \sigma B_t$; the most general

vector matrix <u>cont.</u> Lévy process

b) Poisson $\{N_t\}$ counting process

c) Compound Poisson $X_t = \sum_{k=1}^{N_t} Z_k$, $\{Z_k\}$ i.i.d

step function paths

d) Increasing Lévy processes, $t \to X_t \geq 0$

<u>Proposition</u>. Let $N(du, dv)$ be a Poisson

measure on $R_+ \times R_+$ with intensity measure

$$\mu(du, dv) = \lambda du\, \nu(dv),$$

where ν satisfies $\int_0^\infty (v \wedge 1)\, \nu(dv) < \infty$,

and let b be a constant, $b \geq 0$. Define

$$S_t = bt + \int_0^t \int_0^\infty v\, N(du, dv), \; t > 0.$$

Then, $S=\{S_t\}_{t\geq 0}$ is an increasing Lévy process, and

$$E[e^{-\theta S_t}]=e^{-\lambda t\{b\theta+\int_0^\infty(1-e^{-\theta v})\nu(dv)\}},\quad \theta>0.$$

<u>Proof</u> We have

$$S_t=bt+\int_{[0,\infty)^2} f(u,v)N(du,dv)\ ,\quad f(u,v)=v\,1_{[0,t]}(u).$$

Clearly, ~~...~~, S is increasing, right-continuous, and $S_0=0$.

Since $\int_{[0,\infty)^2}(f(u,v)\wedge 1)\mu(du,dv)$

$$=\lambda t\int_0^\infty(v\wedge 1)\nu(dv)<\infty,$$

it follows that $\int f\,dN<\infty$ and hence $S<\infty$, a.s.

disjoint + Poisson $\Rightarrow$ ind. increments!

Recall:

$$E[e^{-\theta\int f\,dN}]=e^{\int(1-e^{-\theta f})\mu(du,dv)}$$

We have

$$\int_0^\infty\int_0^\infty(1-e^{-\theta f(u,v)})\lambda du\,\nu(dv)$$

$$=\lambda\int_0^\infty(1-e^{-\theta v\cdot 1_{\{0<u<t\}}})\nu(dv)=\lambda\int_0^t(1-e^{-\theta v})\nu(dv)$$

Finally $E[e^{-\theta(S_{s+t}-S_s)}]=\dots=E[e^{-\theta S_t}]$. $\square$

This is a Lévy process with drift b and Lévy measure $\nu(dv)$

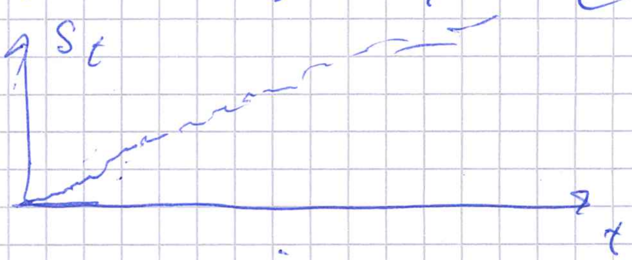
Example The Gamma process

Take $b=0$ and $\nu(dv) = \frac{e^{-cv}}{v}$, $v>0$.

[Singular as $v \downarrow 0$, infinite measure]

Now
$$\mathbb{E}[e^{-\theta S_t}] = e^{-\lambda t \underbrace{\int_0^\infty (1-e^{-\theta v})\frac{e^{-cv}}{v}\,dv}_{\text{"Frullani-integral"}}}$$
$$= e^{-\lambda t \cdot \ln\left(\frac{c+\theta}{c}\right)} = \left(\frac{c}{c+\theta}\right)^{\lambda t}.$$

Thus,
$$S_t \sim \Gamma(\lambda t, c), \quad \mathbb{E}[S_t] = \frac{\lambda t}{c}$$



Jump process with arbitrarily small jumps

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Frullani integral

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = \left(f(+\infty) - f(0)\right) \ln \frac{a}{b}$$

~ 1820

Example Increasing stable process

Take $b=0$, $\nu(dv)=\frac{a}{\Gamma(1-a)}\frac{1}{v^{1+a}}$, $v>0$

where $0<a<1$

Now

$$\mathbb{E}[e^{-\theta S_t}]=e^{-\lambda t\int_0^\infty(1-e^{-\theta v})\frac{a}{\Gamma(1-a)}v^{-(1+a)}dv}$$

$$=\_\_=e^{-\lambda t\theta^a}$$

In particular, $\mathbb{E}[e^{-\theta S_{ct}}]=e^{-\lambda t c\theta^a}$

$$=e^{-\lambda t(c^{1/a}\theta)^a}=\mathbb{E}[e^{-\theta(c^{1/a}S_t)}]$$

so $\underbrace{S_{ct}\overset{d}{=}c^{1/a}S_t}_{\text{"Stability"}}$, for all $t>0$

See Lalley-lectures Def 1.2

Now we return to the property of independent increments. In probability theory, a closely related notion is infinite divisibility:

A random variable X is inf. divisible if, for every integer $n\geq 1$, $X\overset{d}{=}\sum_{k=1}^n Z_k$, where $\{Z_k\}_1^n$ is i.i.d.

For Lévy process $X_t=\sum_{k=1}^n \underbrace{(X_{k\frac{t}{n}}-X_{(k-1)\frac{t}{n}})}_{\text{i.i.d}}$