UPPSALA UNIVERSITET
Matematiska Institutionen
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Prov i matematik
1MA255, omtenta
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Partial Differential Equations with Applications to Finance

Writing time: 08:00 - 13:00.
Instructions: There are 5 problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, 4 or 5, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no help materials are allowed.

GOOD LUCK!

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1. (8p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x) : t > 0, x > x_0\}$ with $u(0,x) = u_0(x)$ for $x > x_0$, and $u(t,x_0) = 0$ for $t > 0$.

i) (3p) Construct a suitable extension of the initial condition to the whole space.

ii) (5p) Show that

$$u(t,x) = \int_{x_0}^{\infty} h(t,x,y)dy$$

for some function $h(t,x,y)$. Find h.

2. (8p) Let $D = (-a,a)$, where $a > 0$. Let X_t be a Brownian motion with drift:

$$dX_t = \mu dt + \sigma dB_t, \quad X_0 = 0,$$

where $\mu \neq 0$. Let $\tau := \inf\{t > 0 : X_t \notin D\}$.

i) Compute

$$\mathbb{P}(X_\tau = a).$$

ii) Compute

$$\mathbb{E}[X_\tau^+ + \tau]$$

where $x^+ = \max(x,0)$.

3. (8p) Consider a geometric Brownian motion X_t

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

starting at $X_0 = x$ a.s..

i) (2p) Write down the Kolmogorov forward (Fokker-Planck) equation satisfied by $\rho(t,x)$, the probability density function of X_t.

ii) (6p) Denote the n-th moment of X_t by $M_n(t)$, use the Fokker-Planck equation to show that

$$\frac{d}{dt}M_1(t) = \mu M_1(t),$$

and

$$\frac{d}{dt}M_n(t) = \left(\mu n + \frac{\sigma^2}{2}n(n-1)\right) M_n(t), \quad n \geq 2.$$

You may assume that the density function decays sufficiently fast at infinity.

4. (8p) Let X_t^α solve

$$dX_t^\alpha = (\mu X_t^\alpha + \alpha_t)dt + \sigma dB_t$$

where μ and σ are constants, and B_t is a standard BM. Solve the control problem

$$V(t,x) = \sup_\alpha \mathbb{E}_{t,x}[\exp\{\int_t^T -\alpha_s^2 ds + X_T^2\}].$$

Hint: use the ansatz $V(t,x) = e^{f(t)+h(t)x^2}$.

5. (8p) Solve the optimal stopping problem

$$v(x) = \sup_\tau \mathbb{E}_x[e^{-\beta\tau}(e^{\frac{X_\tau}{4}} - 1)]$$

where $\beta > 0$, and X_t has the following dynamics:

$$dX_t = \beta dt + 2\sqrt{\beta}dB_t,$$

where B_t is a standard Brownian motion. You **do not** need to prove the verification theorem for your solution.