General Lévy process

$t \to X_t \in \mathbb{R}^d$

Let's state the result for $d=1$.

Thm Let $\{B_t\}$ be BM, and, independently, $M(ds,dx)$ a Poisson measure with intensity $ds\,\nu(dx)$, where $\int_{\mathbb{R}} (|x|^2 \wedge 1)\,\nu(dx) < \infty$.

~~Thm~~ Every Lévy process has the form

$$X_t = bt + \sigma B_t + \int_0^t \int_{|x|\le 1} x\,(M(ds,dx) - ds\,\nu(dx)) + \int_0^t \int_{|x|>1} x\,M(ds,dx)$$

Here

$$X = \underbrace{\sigma B_t}_{\text{cont. mart}} + \underbrace{\int_0^t \int_{|x|\le 1} x\,(M(ds,dx) - ds\,\nu(dx))}_{\text{discont. mart}} + \underbrace{bt + \int_0^t \int_{|x|>1} x\,M(ds,dx)}_{\text{finite variation process}}$$

$$= \underbrace{\underbrace{M_t^c + M_t^d}_{\text{martingale}} + \underbrace{V_t}_{\text{finite variation}}}_{\text{semi-martingale}}$$

!

The law of X_t is determined by the law of X_1, via the characteristic exponent:

$$\mathbb{E}[e^{i\theta X_t}] = e^{\Psi(\theta)\cdot t},$$

$$\Psi(\theta) = i\theta b - \frac{\sigma^2}{2}\theta^2 + \int_{|x|\le 1} (e^{i\theta x} - 1 - i\theta x)\,\nu(dx)$$

$$+ \int_{|x|>1} (e^{i\theta x} - 1)\,\nu(dx)$$

Characteristic triplet: (b, σ^2, ν)

Recall def of stable ~~process~~ Lévy process

$$X_{ct} \stackrel{d}{=} c^{1/\alpha} X_t, \quad t \ge 0 \quad \text{all } c > 0.$$

Can we have $\alpha = 1$? Yes! How?

$$\nu(dx) = \frac{1}{\pi x^2}\,dx,\ x \in \mathbb{R};\quad \int_{-\infty}^{\infty} (|x|^2 \wedge 1)\,\nu(dx) < \infty$$

ok!

$$X_t = \int_0^t \int_{-1}^{1} x\left(M(ds,dx) - ds\,\frac{dx}{\pi x^2}\right)$$

$$+ \int_0^t \int_{|x|>1} x\,M(ds,dx)$$

Then $\Psi(\theta) = |\theta|$, so yes, 1-stable

The corresponding distribution is Cauchy:

$$\mathbb{P}(X_t \in dx) = \frac{t}{\pi(t^2+x^2)}\,dx,$$

$$-\infty < x < \infty$$

Let's go to Lalley page 6:

We know that

$$\mathbb{P}(X_{s+t} - X_s \in A \mid \mathcal{F}_s) = \mathbb{P}(X_{s+t} - X_s \in A)$$

by ind. increments

In particular

$$\mathbb{P}(X_{s+t} - X_s \in A \mid \mathcal{F}_s) = \mathbb{P}(X_{s+t} - X_s \in A \mid \sigma(X_s))$$

$\uparrow$ $\sigma(X_u, u \leq s)$ $\quad$ $\uparrow$ only X_s

This is called the Markov property so X is a Markov process.

Thm (Thm 1 Lalley)

Lévy processes have the strong Markov property; For every stopping time τ, and the stopped σ-algebra

$$\mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \},$$

$X_{\tau+t} - X_\tau$ is independent of $\mathcal{F}_\tau$.

— x —

Recall For an Itô process

$$dX_t = u\,dt + v\,dB_t$$

we have Itô's formula:

If $f(t,x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ $\left[\frac{\partial}{\partial t}f, \frac{\partial^2}{\partial x^2}f\right]$

then

$$df(t,X_t) = f_t'(t,X_t)\,dt + \frac{1}{2}f_{xx}''v^2\,dt + f_x'u\,dt + f_x'v\,dB_t$$

What if $\{X_t\}$ is a general Lévy process?

$$f(t,X_t) = f(0,X_0) + M_t + V_t,$$

where

$$M_t = \sigma\int_0^t f_x'(s,X_{s-})\,dB_s + \int_0^t\int_{|y|\le 1}\left(f(X_{s-}+y) - f(X_{s-})\right)\left(N(ds,dy) - ds\,\nu(dy)\right)$$

and

$$V_t = \sum_{s\le t}\left(f(s,X_s) - f(s,X_{s-})\right)1_{\{|\Delta X_s|>1\}} + \int_0^t \mathcal{L}f(s,X_s)\,ds,$$

with

$$\mathcal{L}f(s,x) = f_t'(s,x) + b f_x'(s,x) + \frac{\sigma^2}{2}f_{xx}''(s,x) + \int_{\mathbb{R}\setminus\{0\}}\left(f(s,x+y) - f(s,x) - y\,1_{\{|y|\le 1\}}f_x'(s,x)\right)\nu(dy)$$

being the infinitesimal generator.

5. Stochastic integration

Our goal in this section is to make sense of integration and differentiation of Lévy processes (or stochastic processes based on them). As we have seen, Lévy processes in general have jumps, hence differentiation must be thought of in a broader sense than usual. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function (in the usual sense). Then $f(t) - f(0) = \int_0^t f'(s)\mathrm{d}s$ or, equivalently, $f(t) - f(0)$ is the value on $[0,t]$ of a (signed) measure that is absolutely continuous w.r.t. to Lebesgue measure and the Radon-Nikodym derivative of the former w.r.t to the latter is a.e. equal to f'. Now, consider the Heaviside function g defined by $g(t) = 0$ if $t < 0$ and $g(t) = 1$ if $t \geq 0$. This function is càdlàg but it is not differentiable in the usual sense because of the jump at the origin. Yet, we may write that for all $t \in \mathbb{R}$, $g(t)$ is the value on $(-\infty, t]$ of the measure δ_0. Hence, the Dirac measure at 0 is in some sense a *weak* derivative of the Heaviside function. This lays the path to the notion of (random) distributions, which we will not pursue, but it is useful to keep this example in mind.

In what follows, $\mathcal{N}$ is a RPM on $E = \mathbb{R}^+ \times \mathbb{R}^d$ with intensity measure $\mathrm{d}t \otimes \nu$ (ν a Lévy measure) and B is a standard Brownian motion independent from $\mathcal{N}$. We denote by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $\mathcal{N}$ and B. In other words, $\mathcal{F}_t = \sigma(B_s,\ \mathcal{N}((0,s] \times A),\ s \leq t,\ A \in \mathcal{B}(\mathbb{R}^d))$.

5.1. **Stochastic integral with respect to a random Poisson measure.** In Section 2 we learnt how to integrate deterministic functions w.r.t. random Poisson measures (but the reader should keep in mind that even in this case the integral is a *random variable*). We will now treat the case of *random integrands.*

Definition 5.1. *Let $T > 0$. The* predictable σ-algebra[†], *denoted by $\mathcal{P}$, is the smallest σ-algebra w.r.t. which all mappings $F\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ that satisfy the two conditions below are measurable.*

(1) $\forall t \in [0,T]$, *the mapping $(z,\omega) \mapsto F(t,z,\omega)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable.*

(2) $\forall z \in \mathbb{R}^d, \omega \in \Omega$, *the mapping $t \mapsto F(t,z,\omega)$ is left-continuous.*

A process $F\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be predictable *if it is $\mathcal{P}$-measurable.*

Let us now introduce the space of random functions that we want to integrate. We define $\mathcal{H}^2(T)$ as the space of predictable processes that are square integrable on $[0,T] \times \mathbb{R}^d \times \Omega$ with respect to $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$.

Proposition 5.1. *The space $\mathcal{H}^2(T)$ is a Hilbert space equipped with the scalar product:*

$$\langle F, G\rangle_{\mathcal{H}^2(T)} := \int_0^T \int_{\mathbb{R}^d} \mathrm{E}\Big(F(t,z)G(t,z)\Big)\mathrm{d}t\nu(\mathrm{d}z). \tag{5.1}$$

We now aim at defining a notion of integral of $F \in \mathcal{H}^2(T)$ on $[0,T] \times \mathbb{R}^d$ w.r.t a compensated RPM denoted by $\widetilde{\mathcal{N}}$. The main idea is to define this integral for a set of *simple* functions that are dense in $\mathcal{H}^2(T)$. This defines an isometry between this space of simple functions and the space of square-integrable random variables, which can be extended to the whole space $\mathcal{H}^2(T)$ thanks to a density argument.

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† *tribu prévisible* in French

is called a *Lévy-type stochastic integral*, which we may also write, formally,

$$\mathrm{d}X_t = b(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \int_{|z|\le 1} H(t,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z) + \int_{|z|>1} K(t,z)\mathcal{N}(\mathrm{d}t,\mathrm{d}z). \tag{5.7}$$

Remark 5.2. *The last term in (5.6) is well-defined as ν is finite on $[-1,1]^{\mathrm{c}}$, so that it may be written as*

$$\int_0^t \int_{|z|>1} K(s,z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z) = \sum_{0<s\le t} K(s,\Delta X_s)1_{\{|\Delta X_s|>1\}}. \tag{5.8}$$

The sum above has an a.s. finite number of nonzero terms.

Exercise 26. *Check that a Lévy process is a Lévy-type stochastic integral.*

Theorem 5.1 (Itô's formula). *Let $X = (X_t)_{0\le t\le T}$ be a Lévy-type stochastic integral of the form (5.6) and $f \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$. Then, for all $0 \le t \le T$, we have a.s.*

$$f(t,X_t) - f(0,X_0) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}), \tag{5.9}$$

with

$$(\mathrm{I}) = \int_0^t \partial_t f(s,X_s)\mathrm{d}s + \int_0^t \partial_x f(s,X_s)b(s)\mathrm{d}s, \tag{5.10}$$

$$(\mathrm{II}) = \int_0^t \partial_x f(s,X_s)\sigma(s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \partial^2_{xx} f(s,X_s)\sigma^2(s)\mathrm{d}s, \tag{5.11}$$

and

$$\begin{aligned}(\mathrm{III}) = &\int_0^t \int_{|z|>1} [f(s^-,X_{s^-}+K(s,z)) - f(s^-,X_{s^-})]\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\\ &+ \int_0^t \int_{|z|\le 1} [f(s^-,X_{s^-}+H(s,z)) - f(s^-,X_{s^-})]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\\ &+ \int_0^t \int_{|z|\le 1} [f(s^-,X_{s^-}+H(s,z)) - f(s^-,X_{s^-}) - H(s,z)\partial_x f(s^-,X_{s^-})]\mathrm{d}s\nu(\mathrm{d}z).\end{aligned} \tag{5.12}$$

The first term is the deterministic part of the differentiation while the second term comes from the standard Itô's formula for stochastic calculus based on Brownian motion. We will thus focus on the third term, which is specific to this course. This last term itself is decomposed in three parts : the first part is due to the large jumps while the second and third ones, which can be somehow put in parallel with the two terms in (II) (first-order and second-order variations), are due to the small jumps.

5.3. A bit of practice: integration with respect to a Poisson counting measure.

Exercise 27 (Integration with respect to a Poisson counting measure). *Let $N = (N_t)_{t\ge 0}$ be a Poisson counting measure with intensity $\lambda > 0$ and $B = (B_t)_{t\ge 0}$ a standard Brownian motion.*

(1) *Give a meaning to $\mathrm{d}N_s$ and $\mathrm{d}\widetilde{N}_s$.*
(2) *Let $s \mapsto f(s)$ be a (deterministic) function. Give simple alternative expressions for the stochastic integrals*

$$\int_0^t f(s)\mathrm{d}N_s \quad \textit{and} \quad \int_0^t f(s)\mathrm{d}\widetilde{N}_s. \tag{5.13}$$