Power series and Taylor series

Power series

Def. A series of the form $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ is called a power series. The constants a_j are called the coefficients of the power series.

Thm For any power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ there exists an $R \in [0, +\infty]$ (dep. on $\{a_j\}$) s.t.

(i) The series conv. absolutely for $|z - z_0| < R$

(ii) The series conv. unif. in any closed subdisk $|z - z_0| \leq r < R$.

(iii) The series diverges for $|z - z_0| > R$.

The number R is called the radius of convergence of the series.

The proof depends on the following

Lemma: Suppose that $\sum_{j=0}^{\infty} a_j w^j$ $(*)$ converges at some point having modulus $r_0 > 0$. Then the series $(*)$ converges absolutely and uniformly in any closed subdisk $|w| \leq r < r_0$.

<u>Proof</u>: Suppose that the series $(*)$ converges at some w_0 with $|w_0| = r_0$.

$\Rightarrow |a_j w_0^j| \to 0 \Rightarrow \exists M$ s.t. $|a_j w_0^j| = |a_j| r_0^j \leq M$ $\forall j \geq 0$.

For $|w| \leq r$ it follows that

$$|a_j w^j| = |a_j| r_0^j \left(\frac{|w|}{r_0}\right)^j \leq M \left(\frac{r}{r_0}\right)^j$$

Letting $M_j := M\left(\frac{r}{r_0}\right)^j$ the result follows by Weierstrass M-test. ☐

<u>Proof of theorem</u> Let $w = z - z_0$

If the series $\sum_{j=0}^{\infty} a_j w^j$ converges only for $w = 0$, or if it converges for all $w \in \mathbb{C}$, we are done. Otherwise, let R be the "greatest" r s.t. $(*)$ converges for some w with $|w| = r$. More precisely, let

$$M = \{ r \geq 0 : (*) \text{ converges for some } w \text{ with } |w| = r \}$$

Then M is non-empty and upper bounded.

Let $R = \sup M$. For any $r < R$ there exists an r_1 with $r < r_1 \leq R$ s.t. $r_1 \in M$.

By the lemma $(*)$ converges absolutely and uniformly for $|w| = |z - z_0| \leq r$. If $|w| > R$, then $(*)$ diverges. ☐

Ex The geometric series $\sum_{j=0}^{\infty} z^j$ has radius of convergence $R = 1$. The series does not converge if $|z| = 1$, since terms do not tend to zero.

Ex. The power series $\sum_{j=0}^{\infty} \frac{z^j}{j^2}$ converges uniformly for $|z| \leq 1$. This follows from the Weierstrass M-test with $M_j = \frac{1}{j^2}$.

It diverges if $|z| > 1$ since $\frac{r^j}{j^2} \to \infty$ if $r > 1$.

Hence $R = 1$.

Are there any formulae for R?

In practise the following is often useful

Proposition Consider a power series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$.

(a) Ratio test: If

$$L = \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right|$$

exists, then $R = \frac{1}{L}$.

(b) Root test: If

$$L = \lim_{j \to \infty} \sqrt[j]{|a_j|}$$

exists, then $R = \frac{1}{L}$.

<u>Remark</u>: 1) In both cases it is understood that $R = +\infty$ if $L = 0$ and $R = 0$ if $L = +\infty$.

2) In fact, the following formula, due to Hadamard, is true for any power series:

$$R = \frac{1}{\limsup_{j \to \infty} \sqrt[j]{|a_j|}} .$$

<u>Proof</u>: Follows directly from the standard ratio and root test. E.g. let

$$c_j(z) = a_j (z - z_0)^j .$$

$$\Rightarrow \left| \frac{c_{j+1}(z)}{c_j(z)} \right| = \left| \frac{a_{j+1}}{a_j} \right| |z - z_0| \to L |z - z_0| , \; j \to \infty$$

So, by the ratio test, if $L|z - z_0| < 1$ the series converges, and if $L|z - z_0| > 1$ the series diverges. Thus, $R = \frac{1}{L}$.

$\square$

<u>Ex.</u> (a) The series $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ has $R = +\infty$, since $a_j = \frac{1}{j!}$ and so $\left| \frac{a_{j+1}}{a_j} \right| = \frac{1}{j+1} \to 0$, $j \to \infty$

(b) The series $\sum_{j=0}^{\infty} j! \, z^j$ has $R = 0$, since $a_j = j!$ and so $\left| \frac{a_{j+1}}{a_j} \right| = j+1 \to +\infty$, $j \to \infty$.

Since the partial sums of a power series are analytic (polynomials!), and converge uniformly in each closed subdisk of the "disk of convergence", the convergence theorems of Section V.2 (or Lecture 13) imply:

<u>Thm</u> Suppose $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ has radius of convergence $R>0$. Then the fcn

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad , \quad |z-z_0|<R$$

is analytic. The derivatives of f are obtained by termwise diff. of the series,

$$f'(z) = \sum_{k=1}^{\infty} k\, a_k (z-z_0)^{k-1}, \quad f''(z) = \sum_{k=2}^{\infty} k(k-1)\, a_k (z-z_0)^{k-2},$$

etc. In particular, the coeff. a_k are given by

$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

Indeed, note that

$$f(z_0) = a_0$$

$$f'(z_0) = \sum_{k=1}^{\infty} k\, a_k (z-z_0)^{k-1} \Big|_{z=z_0} = 1 \cdot a_1$$

$$f''(z_0) = \sum_{k=2}^{\infty} k(k-1)\, a_k (z-z_0)^{k-2} \Big|_{z=z_0} = 2 \cdot 1 \cdot a_2$$

$$\vdots$$

$$f^{(n)}(z_0) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, a_k (z-z_0)^{k-n} \Big|_{z=z_0} = n!\, a_n .$$

Ex. We have that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \quad , \quad |z|<1$$

Since we may diff termwise:

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1} \quad , \quad |z|<1.$$

Since we may also integrate termwise

$$\int_0^w \frac{1}{1-z}\,dz = \sum_{k=0}^{\infty} \int_0^w z^k\,dz = \sum_{k=0}^{\infty} \frac{w^{k+1}}{k+1} = \sum_{n=1}^{\infty} \frac{w^n}{n} \quad , \quad |w|<1$$

But $\int_0^w \frac{1}{1-z}\,dz = \Big[-\mathrm{Log}(1-z)\Big]_0^w = -\mathrm{Log}(1-w)$

$$\Rightarrow \quad \mathrm{Log}(1-w) = -\sum_{n=1}^{\infty} \frac{w^n}{n} \quad , \quad |w|<1.$$

Write $w=-z$

$$\Rightarrow \quad \boxed{\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \quad , \quad |z|<1.}$$

Ex. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Since $R=+\infty$, f is entire. Clearly,

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = f(z)$$

and $f(0) = 0$

$$\Rightarrow \quad \frac{d}{dz}\left(f(z)e^{-z}\right) = \underbrace{f'(z)}_{=f(z)} e^{-z} - f(z)e^{-z} = 0.$$

$$\Rightarrow \quad f(z)e^{-z} = c = \text{const}$$

Let $z\to 0 \Rightarrow c = 1. \Rightarrow f(z) = e^z$

$$\Rightarrow \quad \boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , \quad z\in\mathbb{C}}$$

Taylor series

We have seen that power series are analytic inside the disc of convergence $\{|z-z_0|<R\}$.

The following is an important converse.

Thm (Taylor's theorem)

Suppose that $f(z)$ is analytic for $|z-z_0|<R$.

Then,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \quad |z-z_0|<R.$$

Proof: Fix z and let r be s.t. $|z-z_0|<r<R$.

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Now,

$$\frac{f(\zeta)}{\zeta-z} = \frac{f(\zeta)}{\zeta-z_0}\,\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \frac{f(\zeta)}{\zeta-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{f(\zeta)(z-z_0)^k}{(\zeta-z_0)^{k+1}} \quad \text{for } |\zeta-z_0|=r,$$

where the convergence is uniform in the variable ζ on $|\zeta-z_0|=r$.

We can therefore interchange integration and summation.

$$\Rightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)(z-z_0)^k}{(\zeta - z_0)^{k+1}} \, d\zeta$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \right) (z-z_0)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k$$

where we used the generalized Cauchy integral formula in the last step.

<u>Remark:</u> The radius of convergence of the Taylor series is the largest number R such that $f(z)$ is (or can be extended to be) analytic on the disk $\{|z-z_0| < R\}$

<u>Ex.</u> 1) Let $f(z) = e^z \Rightarrow f^{(k)}(0) = 1 \quad \forall k \geq 0$

$$\Rightarrow \quad e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \qquad z \in \mathbb{C}.$$

2) Let $f(z) = \sin z$,

Then $f^{(2k)}(0) = 0$, $f^{(2k+1)}(0) = (-1)^k$

$$\Rightarrow \quad \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad z \in \mathbb{C}.$$

I finally mention the following

Thm If the power series $f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k$ and $g(z)=\sum_{k=0}^{\infty} b_k (z-z_0)^k$ converge for $|z-z_0|<R$, then

$$f(z)g(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

where $c_k = \sum_{j=0}^{k} a_{k-j} b_j$.

Proof: f, g are analytic in $|z-z_0|<R$

$\Rightarrow$ $h(z)=f(z)g(z)$ is analytic in $|z-z_0|<R$.

$\Rightarrow$ $h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(z_0)}{k!} (z-z_0)^k$, $|z-z_0|<R$.

Now,

$$h(z_0) = f(z_0)g(z_0)$$

$$h'(z_0) = f'(z)g(z)+f(z)g'(z)\Big|_{z=z_0} =$$

$$= f'(z_0)g(z_0)+f(z_0)g'(z_0)$$

$$h''(z_0) = f''(z_0)g(z_0)+2f'(z_0)g'(z_0)+f(z_0)g''(z_0)$$

$$h^{(k)}(z_0) = \sum_{j=0}^{k} \binom{k}{j} f^{(k-j)}(z_0)\, g^{(j)}(z_0)$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$

$$\Rightarrow \frac{h^{(k)}(z_0)}{k!} = \sum_{j=0}^{k} \frac{f^{(k-j)}(z_0)}{(k-j)!} \frac{g^{(j)}(z_0)}{j!} =$$

$$= \sum_{j=0}^{k} a_{k-j} b_j.$$

$\square$