Lec 4

Cauchy-Riemann's equations, Harmonic functions

Cauchy-Riemann's eqns

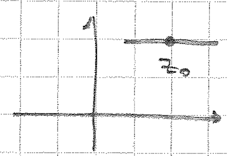
Suppose $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$. Then,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{f(x_0+\Delta x + i(y_0+\Delta y)) - f(x_0+iy_0)}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0) + iv(x_0,y_0))}{\Delta z}$$

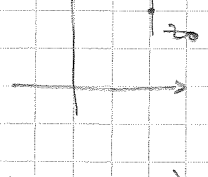
1) Let $\Delta z = \Delta x$, i.e. $\Delta y = 0$



$$\Rightarrow f'(z_0) = \lim_{\Delta x \to 0} \frac{(u(x_0+\Delta x, y_0) - u(x_0,y_0)) + i(v(x_0+\Delta x, y_0) - v(x_0,y_0))}{\Delta x}$$

$$= u_x(x_0,y_0) + iv_x(x_0,y_0)$$

2) Let $\Delta z = i\Delta y$, i.e. $\Delta x = 0$



$$\Rightarrow f'(z_0) = \lim_{\Delta y \to 0} \frac{(u(x_0, y_0+\Delta y) - u(x_0,y_0)) + i(v(x_0, y_0+\Delta y) - v(x_0,y_0))}{i\Delta y}$$

$$= -iu_y(x_0,y_0) + v_y(x_0,y_0)$$

It must therefore hold that

$$u_x + iv_x = -iu_y + v_y$$

$$\iff \boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

Cauchy-Riemann's eqs. (CR-eqns)

We have proven the following:

Thm A necessary condition for $f = u + iv$ to be differentiable at $z_0 = x_0 + iy_0$ is that the Cauchy-Riemann equations are satisfied at (x_0, y_0).

Remark: We also saw the following: If f is diff. at the point z_0, the derivative is given by

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

The following provides a sufficient cond. for diff.

Thm Suppose that $f = u + iv$ is defined in an open set G containing $z_0 = x_0 + iy_0$. Suppose also that u_x, u_y, v_x, v_y exist in "G", are continuous at (x_0, y_0), and satisfy the CR-equations at (x_0, y_0).

Then f is differentiable at z_0.

" CR-eqns + $u, v \in C^1 \Rightarrow f$ diff. "

Proof. In view of the cont. of the first partial derivatives at (x_0, y_0), it holds that

$$u(x_0+\Delta x, y_0+\Delta y) = u(x_0,y_0) + u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + $$
$$+ \sqrt{(\Delta x)^2+(\Delta y)^2}\, \rho_1(\Delta x, \Delta y)$$

and

$$v(x_0+\Delta x, y_0+\Delta y) = v(x_0,y_0) + v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + $$
$$+ \sqrt{(\Delta x)^2+(\Delta y)^2}\, \rho_2(\Delta x, \Delta y)$$

where $\rho_1, \rho_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

[see Calculus course ; $C^1 \Rightarrow$ diff]

$$\Rightarrow f(z_0+\Delta z) - f(z_0) =$$

$$= u_x(x_0,y_0)\Delta x + \underbrace{u_y(x_0,y_0)}_{= -v_x(x_0,y_0)}\Delta y + i\big(v_x(x_0,y_0)\Delta x + \underbrace{v_y(x_0,y_0)}_{= u_x(x_0,y_0)}\Delta y\big)$$
$$+ \sqrt{(\Delta x)^2+(\Delta y)^2}\,\big(\rho_1(\Delta x,\Delta y) + i\rho_2(\Delta x,\Delta y)\big)$$

$$= /\ \text{CR-equ}\ / =$$

$$= u_x(x_0,y_0)\Delta z + i\, v_x(x_0,y_0)\Delta z +$$
$$+ |\Delta z|\,\big(\rho_1(\Delta x,\Delta y) + i\rho_2(\Delta x,\Delta y)\big)$$

Since $\rho_1, \rho_2 \to 0$ as $\Delta z \to 0$ it follows that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists and equals $u_x(x_0,y_0) + i\, v_x(x_0,y_0)$. ∎

Ex. Show that e^z is entire and compute its derivative.

Sol. $e^z = e^x(\cos y + i\sin y)$

$\Rightarrow u(x,y) = e^x \cos y \quad , \quad v(x,y) = e^x \sin y$

$\Rightarrow \quad u_x = e^x \cos y = v_y$

$\qquad u_y = -e^x \sin y = -v_x$

Thus, $u, v \in C^1$ and satisfy the CR-eqns

$\Rightarrow e^z$ is entire; moreover

$$\frac{d}{dz} e^z = u_x + i v_x = u + iv = e^z$$

$\boxtimes$

Remark: By uniqueness of analytic continuation the above def of e^z is the only one which makes e^z entire.

See the Uniqueness principle on page 156.

Ex. Since e^{iz} and e^{-iz} are entire, so are $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

Moreover,

$$\frac{d}{dz} \sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) =$$

$$\stackrel{\text{chain rule}}{=} \frac{ie^{iz}+ie^{-iz}}{2i} = \cos z \quad \text{etc}$$

$\boxtimes$

Inverse mappings

Suppose $f = u + iv$ is analytic in a domain D (with $f'(z)$ continuous). Consider the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

as a mapping of $D \subset \mathbb{R}^2 \to \mathbb{R}^2$

Its Jacobian matrix $J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ has determinant

$$\det J_f = u_x v_y - u_y v_x \overset{CR}{=} u_x^2 + v_x^2 = |f'(z)|^2$$

The inverse fcn thm leads to the following:

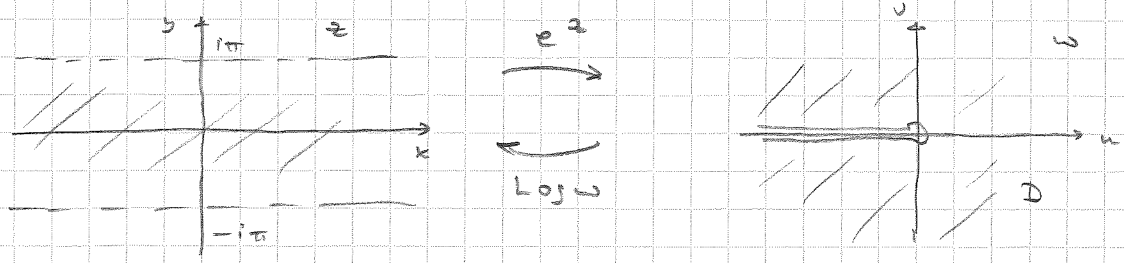
Thm (Inverse fcn thm)
Suppose $f(z)$ is analytic on a domain D (with $f'(z)$ continuous), and that $f'(z_0) \neq 0$.

Then there is a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ s.t. $f: U \to V$ is bijective, and the inverse fcn $f^{-1}: V \to U$ is analytic with derivative

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)}, \quad \text{where } w = f(z).$$

Proof: See book for analyticity of f^{-1}.

Ex. $f(z) := e^z$



$$f'(z) = e^z \neq 0 \quad \forall z \in \mathbb{C}$$

$f^{-1}(w) = \operatorname{Log} w$ is defined in $\mathbb{C} \setminus (-\infty, 0] =: D$
(not only locally)

$\Rightarrow$ $\operatorname{Log} w$ is analytic in D and

$$(\operatorname{Log} w)' = \frac{1}{f'(z)} = \frac{1}{e^z} = \frac{1}{w}$$

Clearly,

$$\boxed{\frac{d}{dw} \log w = \frac{1}{w}}$$

for any branch of $\log w$ (away from cut) $\square$

Ex. $f(z) = z^\alpha = e^{\alpha \log z}$; $\alpha \in \mathbb{C}$

$$\Rightarrow f'(z) = e^{\alpha \log z} \frac{d}{dz}(\alpha \log z) = e^{\alpha \log z} \frac{\alpha}{z} = z^\alpha \cdot \frac{\alpha}{z} = \alpha z^{\alpha - 1}$$ $\square$

Harmonic functions

Def. A real-valued fcn $\phi(x,y)$ is said to be harmonic in a domain D if $\phi \in C^2(D)$ and ϕ satisfies Laplace's equation

$$\Delta \phi = \phi_{xx} + \phi_{yy} = 0 \quad \text{in } D.$$

<u>Thm</u> Suppose $f = u + iv$ is analytic in a domain D.

Then u and v are harmonic in D.

<u>Proof</u> One can show that $u, v \in C^\infty$

(See thm on page 11a).

$u_x = v_y \Rightarrow u_{xx} = v_{yx}$

$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$

As $v_{yx} = v_{xy}$ $(v \in C^2)$ $\overset{\text{Add}}{\Longrightarrow}$ $u_{xx} + u_{yy} = 0$

Similarly, $v_{xx} + v_{yy} = 0$. $\square$

<u>Def</u> If u is harmonic in a domain D

and v is a harmonic fcn in D s.t.

$u + iv$ is analytic in D, then we say that

v is a <u>harmonic conjugate</u> of u in D.

<u>Ex.</u> Construct an analytic fcn whose real-part is

$u(x,y) = y^3 - 3x^2y$

<u>Sol</u> Note that u is harmonic in $\mathbb{R}^2$, since

$\Delta u = u_{xx} + u_{yy} = -6y + 6y = 0.$

If $f = u + iv$ is analytic, by the CR-eqns

$v_y = u_x = -6xy \qquad (1)$

and

$v_x = -u_y = -3y^2 + 3x^2 \qquad (2)$

Integrating (1):

$$u(x,y) = -3xy^2 + \phi(x)$$

Inserting into (2):

$$(*) \quad -\cancel{3}y^2 + \phi'(x) = -\cancel{3}y^2 + 3x^2 \iff \phi'(x) = 3x^2$$

$$\iff \phi(x) = x^3 + c \quad , \; c \in \mathbb{R} \text{ (real constant)}$$

Thus, $v = -3xy^2 + x^3 + c$

Since $u, v \in C^1$, conversely

$$f = u + iv = y^3 - 3x^2y + i\,(x^3 - 3xy^2 + c)$$

is analytic. Note that $f(x+i0) = i\,(x^3 + c)$

and that $g(z) = i(z^3 + c)$ is entire.

It follows by the Uniqueness principle that $f(z) = g(z)$.

I.e. $f(z) = i(z^3 + c)$

This is, of course, easy to verify.

<u>Note</u>: That all y's disappear in $(*)$ is a

consequence of u being harmonic. A general

proof in $\mathbb{R}^2$ works as in the example $\square$

<u>Thm</u> If u is harmonic in a simply connected domain $D \subseteq \mathbb{C}$ then there exists a harmonic conjugate v of u in D, and v is unique up to addition of a real constant.

<u>Proof</u> Suppose u harmonic in D.

Consider the vector-field $\bar{F} = (-u_y, u_x) \in C^1(D)$

Note that $\dfrac{\partial F_1}{\partial y} = -u_{yy} \underset{\substack{\uparrow \\ u \text{ harm.}}}{=} u_{xx} = \dfrac{\partial F_2}{\partial x}$.

D simply connected $\Rightarrow$ $\bar{F}$ is conservative

$\Rightarrow \exists\, v : \nabla v = \bar{F}$, i.e. $(v_x, v_y) = (-u_y, u_x)$

$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ (CR) ; also $u, v \in C^1$ (even C^2)

$\Rightarrow f = u + iv$ is analytic in D.

If $\tilde{v}$ is another harmonic conjugate, then

$$\tilde{v}_x = -u_y = v_x$$
$$\tilde{v}_y = u_x = v_y$$

$\Rightarrow \nabla(v - \tilde{v}) = \bar{0} \Rightarrow v - \tilde{v} = \text{const.}$

[5]