Exam Information:
Probability and Martingales, 1MS045

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Last Updated: December 20, 2024

1 General Comments

This document lists the main topics, definitions, and results that we have seen in the course. You can consider this a "revision guide" for the exam. No guarantee is made that this document is exhaustive. It complements the reading of the lecture notes, course material, problem classes, and assignments.

Topics marked with a (*) constitute material that will not (directly) be asked about in the exam. These results tend to come from other backgrounds in e.g. functional analysis or linear algebra.

While it will help to understand the derivation of all the results in the course, I have highlighted the theorems and lemmas I would expect you to be able to prove in **bold** type.

2 Allowed Material

Allowed material at the exam:

- non-programmable calculator,
- pens,
- paper,
- 1 one-sided A4 sheet of notes.

The question do not, as such, require extensive computations and should be doable without a calculator. However, you are allowed to bring a non-programmable calculator for the exam. You are further allowed to bring one A4 page (one-sided) of notes about the course with you. There is no restriction on the content of these notes only that they should be legible. They will also need to be attached to the exam submission.

3 Main results

3.1 General Measure and Probability Theory

- σ-Algebras. Definition and basic properties.
- Measures. Definition through σ-additivity and σ-algebras as measureable sets. probability measure, finite measure.

- Measure Spaces. Sets together with σ-algebra and measure. It is a Probability Space if the measure is a probability measure.
- Properties of measures. Inclusion exclusion principle, bounds of unions of sets.
- Monotonicity and measures of monotone sequences of sets.
- Borel σ-algebras, σ-algebras generated by sets of subsets, π-systems.
- (*) Measure uniquely defined on π systems.
- (*) Caratheodory's extension theorem.
- $\limsup$ and $\liminf$ of sets.
- **Fatou's lemma.**
- **First Borel–Cantelli lemma.**
- Random variables, measurability.
- Measurability under addition, multiplication, composition.
- Measurability of $\limsup$ $\liminf$ and similar notions.
- σ algebras generated by random variables.
- Law and distribution function.
- Properties of distribution functions.
- Independence of events, random variables, σ-algebras.
- (*) Independence through π systems.
- **Second Borel-Cantelli lemma.**
- Tail events and the Kolmogorov $0-1$ law.
- Integration with respect to a measure through step functions.
- Linearity, monotonicity, triangle inequality of integration.
- Integrability of measurable functions.
- Monotone convergence theorem.
- Dominated convergence theorem.
- Fatou's lemma.
- Densities and Radon-Nikodym theorem.
- Expectation as integration.
- **Markov's inequality**
- **Jensen's inequality** and convex functions.
- L^p norms for $p \geq 1$.
- (*) Cauchy-Schwartz inequality.
- Variance, covariance, and correlation.
- (*) Hölder inequality.
- (*) Minkowski inequality.
- **Law of large numbers** (assuming $\mathbb{E}(X^4) < \infty$).
- Chebyshev's inequality.
- Conditional expectation, definition, uniqueness.
- Basic properties of conditional expectations.
- Monotone convergence, dominated convergence, Fatou's lemma, Jensen's inequality for conditional expectations.
- Simplification rules for conditional expectations.
- Product spaces, product σ-algebra, product measure, marginals.
- Fubini's theorem.
- Conditional expectations of product space with common density.

3.2 Martingale Theory

- Stochastic process, filtration, process adapted to a filtration.
- Martingale, supermartingale, submartingale.
- Basic martingale properties (e.g. $\mathbb{E}(M_n|\mathcal{F}_k) = M_k$ for $k \leq n$.).
- Previsible processes and martingale transforms.
- **Martingale transforms are martingales.**
- Stopping times.
- **Doob's optional stopping theorem.**
- Computing expectations of stopping times.
- Upcrossings.
- **Doob's upcrossing lemma.**
- **Doob's martingale convergence theorem.**
- (*) Doob decomposition.
- Uniform integrability, sufficient conditions for uniform integrability.
- Convergence in probability, almost sure convergence, convergence in mean (i.e. in L^1), convergence in L^p and their interactions.
- Convergence in mean is equivalent to convergence in probability and uniform integrability.
- **UI martingales converge almost surely and in mean,** $M_n = \mathbb{E}(M_\infty|\mathcal{F}_n)$.
- L^2 martingales, **Law of large numbers** (assuming $\mathrm{Var}(X) < \infty$).
- Kolmogorov truncation lemma, law of large numbers (assuming finite expectation).
- Doob's submartingale inequality.
- Kolmogorov's inequality.

3.3 Martingales in Mathematical Finance

- Derivatives, in particular options (call/put/European/American).
- Arbitrage.
- Portfolios / replicating strategies.
- **Call-put parity** of European options.
- Strategies replicating claims.
- Equivalent martingale measures pt. 1.
- Binomial, Trinomial model.
- Complete markets.
- Risk free rates, discounting factor.
- Cox–Ross–Rubinstein formula.
- **Bounds to European and American options. Pricing equality.**
- General discrete models: pricing process, gains process, discounting factor, portfolios, value process, self-financing.
- Admissible strategies, viable markets, definition of arbitrage and weak arbitrage.
- **Equivalency of (strong) arbitrage and weak arbitrage.**
- All generating strategies of a claim have the same value process, provided the model is viable (arbitrage free).
- Strategies as previsible (predictable) processes.

- Equivalent martingale measures pt. 2.
- Fair price of a claim.
- Martingale measures are unique in complete markets.
- Attainable claims and value processes.
- Binomial model to Black-Scholes through taking limits.
- Black-Scholes formula.
- (*) The separating hyperplane theorem.
- Finitely generated probability spaces.
- First fundamental theorem of asset pricing.
- Completeness and representability of martingales through martingale transforms.
- Second fundamental theorem of asset pricing.