# LINEAR ALGEBRA III

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# LECTURE 1

**Goal** Essentially go beyond previous courses on Linear Algebra by discussing vector spaces (possibly of infinite dimension) over arbitrary fields. Going further with the spectral theorem by considering Jordan normal form: given any $n \times n$ matrix $A$ with complex entries, we can find a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{C}^n$ such that

$$\begin{aligned} J &= C^{-1}AC \\ &= \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{bmatrix} \end{aligned}$$

where $C$ is the change of bases matrix, and each $J_i$ is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

where $\lambda_i$ are eigenvalues.

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## Basic Mathematical Structures

Let us recall a few basic structures. A **monoid** is a set with an associative binary composition and a unit. A **group** is a monoid in which every element has an inverse. A **ring** is a set $R$ with two binary compositions addition $+$ and multiplication $\cdot$ such that $(R,+)$ is a commutative group, $(R,\cdot)$ is a monoid and multiplication is distributive over addition. A **field** is a commutative ring $F$ such that $F \setminus \{0\}$ is a group. We may now define the structure of a vector space.

**Definition 1** A set $V$ is a **vector space over a field** $\mathbb{F}$ if we have defined two maps

$$V \times V \xrightarrow{+} V \quad \text{and} \quad \mathbb{F} \times V \xrightarrow{\cdot} V$$

such that $(V,+)$ is a commutative group and the following holds for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{v}, \mathbf{w} \in V$:

1. $0_{\mathbb{F}}\mathbf{v} = 0_V$
2. $1_{\mathbb{F}}\mathbf{v} = 1_V$
3. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
4. $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
5. $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

Note that when $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we often have an inner product. That is not the case for a general $\mathbb{F}$.

**Example 2**

- $\mathbb{F}$ is a vector space over $\mathbb{F}$
- $\mathbb{F}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{F}\}$
- $\mathbb{F}^\infty = \{(a_1, \dots) \mid a_i \in \mathbb{F}\}$
- $\mathscr{P}_n(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{F}\}$
- For any set $S$, $\mathbb{F}^S = \{f : S \to \mathbb{F}\}$

Let $K_1 \subset K_2$ be an inclusion of fields, preserving $0, 1, +, \cdot$. In other words, $K_1$ is a subfield of $K_2$, or that $K_2$ is a field extension of $K_1$. Then, $K_2$ is a vector space over $K_1$. In particular, $\mathbb{C}$ is a vector space over $\mathbb{R}$, and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

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From now on, let $(V, +, \cdot)$ be a vector space over a field $\mathbb{F}$.

**Definition 3** A subset $U \subset V$ is a **vector subspace** if it contains $0 \in V$, is closed under + and is closed under $\cdot$.

A **linear combination** is a vector of the form $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$, where $a_i \in \mathbb{F}$ and $\mathbf{v}_i \in V$. Importantly, the sum is finite. Given a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \in V$, their **span** is the vector subspace

$$\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots) = \{\text{linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \dots\} \subset V$$

A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \in V$ is called **linearly independent** if, for every linear combination $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = 0$, we have $a_1 = \cdots = a_k = 0$.

**Definition 4** A subset $B \subset V$ is a **basis** of $V$ if

1. $\operatorname{Span}(B) = V$
2. The subset is linearly independent

The number of elements in a basis for $V$ is called the **dimension** of $V$, denoted $\dim V$

# LECTURE 2

## Basics of Vector Spaces

We start this lecture with an important proposition.

**Proposition 5** Every linearly independent subset $S \subset V$, can be extended to a basis $B$ of $V$.

*Proof.*

If $\dim V = \infty$, need the Axiom of Choice. Consider the case when $\dim V = n < \infty$ and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, with $m \leq n$. If $m = n$, then $S$ is a basis and we are done. Otherwise, take any $\mathbf{v}_{m+1} \in V$ that is not in $\text{Span}(S)$. Clearly, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}\}$ is a linearly independent set if

$$a_1\mathbf{v}_1 + \cdots + a_{m+1}\mathbf{v}_{m+1} = 0$$

for which there are two possible cases. If $a_{m+1} \neq 0$, then $\mathbf{v}_{m+1} \in \text{Span}(S)$, which contradicts the assumption on $\mathbf{v}_{m+1}$. If $a_{m+1} = 0$, then

$$a_1\mathbf{v}_1 + \cdots + a_{m+1}\mathbf{v}_{m+1} = 0 \implies \text{all } a_i = 0$$

because $S$ is assumed to be linearly independent.

$\square$

**Theorem 6**
- Every vector space $V$ has a basis
- Every two bases of $V$ have the same number of elements.

Suppose we have a collection $\{U_i\}_{i \in I}$ of vector subspaces of $V$, can we use it to produce more vector subspaces of $V$; consider the following proposition.

**Proposition 7** The intersection of the subspaces is a subspace

$$\bigcap_{i \in I} U_i = \{\mathbf{v} \in V \mid \mathbf{v} \in U_i \text{ for all } i \in I\}$$

Frequently, we will consider finite intersection $U_1 \cap \cdots \cap U_m$. Do also note that the union is not necessarily a vector subspace. What is the smallest subspace of $V$ containing some union $U_1 \cup U_2$?

**Definition 8** Let $\{U_i\}_{i\in I}$ be a collection of subspaces of $V$. The **sum** of these subspaces is

$$\begin{aligned}\sum_{i\in I} U_i &= \{\sum_{i\in I} \mathbf{u}_i \in V \mid \mathbf{u}_i \in U_i \wedge \text{ finitely many } \mathbf{u}_i \neq 0\} \subset V \\ &= \mathrm{Span}(\bigcup_{i\in I} U_i)\end{aligned}$$

Again, we will mostly consider finite sums

$$U_1 + \cdots + U_m = \{\mathbf{u}_1 + \cdots + \mathbf{u}_m \in V \mid \mathbf{u}_i \in U_i\}$$

We tie things together with the following proposition.

**Proposition 9** $\sum_{i\in I} U_i$ is a subspace of $V$.

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## The Internal Direct Sum

We now begin working towards direct sums, starting with internal direct sums.

**Definition 10** Let $\{U_i\}_{i\in I}$ be a collection of subspaces of $V$. If every element in $\sum_{i\in I} U_i$ can be written uniquely as

$$\sum_{i\in I} \mathbf{u}_i$$

with $\mathbf{u}_i \in U_i$ and finitely many $\mathbf{u}_i \neq 0$. Then, the sum is called an **internal direct sum** and denoted $\bigoplus_{i\in I} U_i$.

If the collection of subspaces is finite, we write $U_1 \oplus \cdots \oplus U_m$. Furthermore, as it turns it, there is a simple equivalence one can check.

**Proposition 11** $\sum_{i\in I} U_i = \bigoplus_{i\in I} U_i$ if and only if

$$0 = \sum_{i\in I} \mathbf{u}_i \implies \text{all } \mathbf{u}_i = 0$$

In other words, the sum is an internal direct sum if and only if 0 can be written uniquely as a sum. And in related fashion we have the following proposition.

**Proposition 12** $U_1 + U_2$ is an internal direct sum if and only if $U_1 \cap U_2 = \{0\}$.

Internal direct sums enjoy another peculiar result.

**Proposition 13** Let $U \subset V$ be a vector subspace. Then, there is another vector subspace $W \subset V$ such that $U \oplus W = V$.

*Proof.*

Let $B \subset U$ be a basis for $U$. Extend $B$ to a basis $\tilde{B}$ of $V$, such that $B \subset \tilde{B}$. Now, let $W = \mathrm{Span}(\tilde{B} \setminus B)$. Claim: $U \oplus W = V$.

- $U + W$ is understood to be a direct sum when the fact that

$$U \cap W = \mathrm{Span}(B) \cap \mathrm{Span}(\tilde{B} \setminus B) = \{0\}$$

  is considered with the previous result

- $U + W = V$ because, $U + W \subset V$ is a vector subspace containing the basis $\tilde{B}$

□

We end this section with a strong theorem.

**Theorem 14** Given $U_1, U_2 \subset V$ subspaces, with $\dim V < \infty$

$$\dim U_1 + U_2 = \dim U_1 + \dim U_2 - \dim U_1 \cap U_2$$

# LECTURE 3

## Homomorphism of Vector Spaces

In this section we will be working with linear maps - equivalently, homomorphisms of vector spaces.

**Definition 15** Given vector spaces $V, W$ over a field $\mathbb{F}$, a map $\varphi : V \to W$ is a **linear map** if

$$\varphi(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1\varphi(\mathbf{v}_1) + \cdots + a_n\varphi(\mathbf{v}_n)$$

for every $a_i \in \mathbb{F}$ and $\mathbf{v}_i \in V$

**Proposition 16** The set of linear maps

$$\mathscr{L}(V, W) = \{\varphi : V \to W \mid \varphi \text{ is linear}\}$$

is a vector space.

Given $\varphi \in \mathscr{L}(V, W)$, there are the following subspaces:

- The **Kernel**: $\ker\varphi = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = 0\} \subset V$
- The **Image**: $\operatorname{im}\varphi = \{\varphi(v) \in W \mid \mathbf{v} \in V\} \subset W$

Let us introduce some more terminology.

**Definition 17** If $\varphi \in \mathscr{L}(V, W)$ is a bijection, we call it an **isomorphism**. Then, we say that $V$ and $W$ are **isomorphic**, denoted $V \cong W$.

We may now present an interesting theorem.

**Theorem 18** If $V$ and $W$ are finite dimensional, then $V \cong W$ if and only if $\dim V = \dim W$.

And another theorem of importance.

**Theorem 19** Given $\varphi \in \mathscr{L}(V, W)$, then

$$\dim V = \dim(\ker\varphi) + \dim(\operatorname{im}\varphi)$$

*Proof.*

Recall that there is a subspace $U \subset V$ such that $\ker\varphi \oplus U = V$. Claim: The restriction of $\varphi$ to $U$, $\varphi_U : U \to \operatorname{im}\varphi$, is an isomorphism. Then, the claim implies the theorem.

$\square$

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## Matrices

Let $M(m \times n, \mathbb{F})$ denote the space of $m \times n$ matrices with entries in $\mathbb{F}$. We immediately have a peculiar property followed by a strong theorem.

**Proposition 20** $M(m \times n, \mathbb{F})$ is an $\mathbb{F}$-vector space of dimension $m \cdot n$.

**Theorem 21** Let $\dim V = n$ and $\dim W = m$ and pick ordered bases $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $V$ and $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ for $W$. The map

$$M : \mathscr{L}(V,W) \to M(m \times n, \mathbb{F})$$

such that

$$M(\varphi) = \begin{bmatrix} | & & | \\ \varphi(\mathbf{v}_1) & \ldots & \varphi(\mathbf{v}_n) \\ | & & | \end{bmatrix}$$

is an isomorphism of vector spaces.

Note that $M(\varphi)$ depends on the choice of ordered bases. Under such isomorphisms, composition

$$\mathscr{L}(V,W) \times \mathscr{L}(U,V) \to \mathscr{L}(U,W)$$

corresponds to matrix multiplication. More concretely, given

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

and choices of ordered bases for $U, V, W$, we get that

$$M(\psi \circ \varphi) = M(\psi) \cdot M(\varphi)$$

In other words, the map $M : \mathscr{L}(V,V) \to M(n \times n, \mathbb{F})$ is a ring isomorphism.

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## The External Direct Sum

It is now time for us to introduce the external direct sum and then show that it is isomorphic to the internal direct sum - whenever they both exist.

**Definition 22** Given a finite collection of vector spaces $V_1,\ldots,V_n$. Its **external direct sum** is

$$V_1 \times \cdots \times V_n = \{(\mathbf{v}_1,\ldots,\mathbf{v}_n) \mid \mathbf{v}_i \in V_i\}$$

with standard addition and scalar multiplication.
If we have a general collection of vector spaces $\{V_i\}_{i\in I}$, then the external direct sum is

$$\bigoplus_{i\in I} V_i = \{f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i \text{ and only finitely many } f(i) \neq 0\}$$

This has the structure of a vector space, and when the collection of vector spaces is finite we get

$$(\mathbf{v}_1,\ldots,\mathbf{v}_n) = (f(1),\ldots,f(n))$$

Finally, we may bridge the gap between internal direct sums and external direct sums.

**Theorem 23** Given a collection of subspaces $\{U_i\}_{i\in I} \subset V$, the internal and external direct sums of the $U_i$ are isomorphic.

*Proof.*

For the sake of clarity, suppose we have finitely many subspaces. The linear map

$$U_1 \times \cdots \times U_n \to U_1 + \cdots + U_n$$
$$(\mathbf{u}_1,\ldots,\mathbf{u}_n) \mapsto \mathbf{u}_1 + \cdots + \mathbf{u}_n$$

is an isomorphism: surjective by the definition of summing subspaces and injective by assumption of the internal direct sum.

□

Now, since the internal and external direct sum are isomorphic, we will just say direct sum and write $\oplus$.

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## Quotient Spaces

Let $U \subset V$ be a subspace. Define a relation on $V$:

$$\mathbf{v} \sim \mathbf{w} \iff \mathbf{v} - \mathbf{w} \in U$$

This relation can quite easily be shown to be an equivalence relation. Consider the set of equivalence classes, or cosets:

$$V/U = \{\mathbf{v} + U \mid \mathbf{v} \in V\}$$

**Proposition 24** $V/U$ is a vector space with

$$[\mathbf{v}]+[\mathbf{w}]=[\mathbf{v}+\mathbf{w}]$$
$$a\cdot[\mathbf{v}]=[a\cdot\mathbf{v}]$$

The proof is omitted, as it is tedious but easy. For the tireless; show that addition and multiplication are well-defined, and check the axioms of the vector space.

**Proposition 25** The map

$$\begin{aligned}\pi : V &\to V/U\\ \mathbf{v} &\mapsto [\mathbf{v}]\end{aligned}$$

is well-defined and linear.

**Theorem 26** For every $\varphi \in \mathscr{L}(V,W)$ there is a unique linear map

$$\psi : V/\ker\phi \to W$$

such that $\varphi = \psi\circ\pi$. Furthermore, $\psi$ is injective and defines an isomorphism

$$\psi : V/\ker\phi \to \operatorname{im}\varphi$$

Also, $V \cong \ker\varphi \oplus \operatorname{im}\varphi \cong \ker\varphi \oplus V/\ker\varphi$, where $\oplus$ is the external direct sum.

*Proof.*

Take $[\mathbf{v}]\in V/\ker\phi$. We have that $\psi([\mathbf{v}])=\psi(\pi(\mathbf{v}))=\varphi(\mathbf{v})$.

- $\psi$ is well-defined:

$$\begin{aligned}[\mathbf{v}]=[\mathbf{w}] &\implies \mathbf{v}-\mathbf{w}\in\ker\varphi\\ &\implies \psi([\mathbf{v}])-\psi([\mathbf{w}])\\ &=\varphi(\mathbf{v})-\varphi(\mathbf{w})\\ &=\varphi(\mathbf{v}-\mathbf{w})\end{aligned}$$

- $\psi$ is linear:

$$\begin{aligned}\psi(a[\mathbf{v}]+b[\mathbf{w}]) &= \psi([a\mathbf{v}+b\mathbf{w}])\\ &= a\varphi(\mathbf{v})+b\varphi(\mathbf{w})\\ &= a\psi([\mathbf{v}])+b\psi([\mathbf{w}])\end{aligned}$$

- $\psi$ is injective:

$$\begin{aligned}\psi([\mathbf{v}]) &= \varphi(\mathbf{v}) \\ &= 0 \\ &\implies \mathbf{v} \in \ker\varphi \\ &\implies [\mathbf{v}] = 0\end{aligned}$$

- $\psi$ is surjective onto $\operatorname{im}\varphi$: if $\mathbf{w} = \varphi(\mathbf{v}) \in \operatorname{im}\varphi$, then $\mathbf{w} = \psi([\mathbf{v}]) \in \operatorname{im}\psi$
- $V \cong \ker\varphi \oplus \operatorname{im}\varphi$: follows from the claim in the proof of the Rank-Nullity theorem.

$\square$

# LECTURE 4

## Tensor Product

Let $V, W$ be vector spaces over some field $\mathbb{F}$. We are going to construct a new vector space $V \otimes W$, consisting of linear combinations of terms of the form $\mathbf{v} \otimes \mathbf{w}$. Furthermore, we will make it bilinear by design, that is

1. $(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \otimes \mathbf{w} - a_1(\mathbf{v}_1 \otimes \mathbf{w}) - a_2(\mathbf{v}_2 \otimes \mathbf{w}) = 0$
2. $\mathbf{v} \otimes (a_1\mathbf{w}_1 + a_2\mathbf{w}_2) - a_1(\mathbf{v} \otimes \mathbf{w}_1) - a_2(\mathbf{v} \otimes \mathbf{w}_2) = 0$

Let us cook.

1. Given any set $S$, let $F(S)$ be a vector space for which the set $S$ forms a basis. An element in $F(S)$ is a linear combination

   $$a_1 s_1 + \cdots + a_n s_n$$

   Say that $F(S)$ is the **free vector space** on the set $S$. Note that we could have set $F(S) = \bigoplus_{i \in S} F$.
2. Take the vector space $F(V \times W)$ - step 1 with $S = V \times W$. The elements in $F(V \times W)$ are linear combinations

   $$a_1(\mathbf{v}_1, \mathbf{w}_1) + \cdots + a_n(\mathbf{v}_n, \mathbf{w}_n)$$

   This is a very large space.
3. Inspired by the earlier idea, we make the following construction. Let $U \subset F(V \times W)$ be the subspace spanned by the set of all elements of the forms
   (a) $(a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \mathbf{w}) - a_1(\mathbf{v}_1, \mathbf{w}) - a_2(\mathbf{v}_2, \mathbf{w}) = 0$
   (b) $(\mathbf{v}, a_1\mathbf{w}_1 + a_2\mathbf{w}_2) - a_1(\mathbf{v}, \mathbf{w}_1) - a_2(\mathbf{v}, \mathbf{w}_2) = 0$
4. Define $V \otimes W = F(V \times W)/U$. Denote the equivalence classes $[(\mathbf{v}, \mathbf{w})] \in V \otimes W$ by $\mathbf{v} \otimes \mathbf{w}$, we call this a **simple tensor**.

**Proposition 27** $V \otimes W$ is bilinear.

*Proof.*

Due to step 3 and step 4 in its construction.

□

**Example 28** Consider $\mathbb{R}^2 \otimes \mathbb{R}^2$, we have that

$$\begin{aligned}(1,0)\otimes(0,1)+(1,0)\otimes(1,1) &= (1,0)\otimes((0,1)+(1,1)) \\ &= (1,0)\otimes(1,2)\end{aligned}$$

And in general, we observe that

$$a(\mathbf{v}\otimes\mathbf{w}) = (a\mathbf{v}\otimes\mathbf{w}) = (\mathbf{v}\otimes a\mathbf{w})$$

**Theorem 29** Given a basis $B$ for $V$ and a basis $\tilde{B}$ for $W$, the set

$$\{\mathbf{v}\otimes\mathbf{w} \mid \mathbf{v}\in B, \mathbf{w}\in\tilde{B}\}$$

is a basis for $V\otimes W$.

**Corollary 30** $\dim(V\otimes W) = \dim V \cdot \dim W$

Observe that there is a bilinear map

$$\begin{aligned}\tau : V\times W &\to V\otimes W \\ (\mathbf{v},\mathbf{w}) &\mapsto \mathbf{v}\otimes\mathbf{w}\end{aligned}$$

Meaning

$$\begin{aligned}\tau(a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \mathbf{w}) &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2)\otimes\mathbf{w} \\ &= a_1(\mathbf{v}_1\otimes\mathbf{w}) + a_2(\mathbf{v}_2\otimes w) \\ &= a_1\tau(\mathbf{v}_1,\mathbf{w}) + a_2\tau(\mathbf{v}_2,\mathbf{w})\end{aligned}$$

This map gives rise to a useful theorem for constructing linear maps via $V\otimes W$.

**Theorem 31** Let $\varphi : V\times W \to U$ be a bilinear map. Then, there is a unique linear map $\tilde{\varphi} : \mathscr{L}(V\otimes W, U)$ such that $\tilde{\varphi}\circ\tau = \varphi$.

This gives us a nice slogan for the tensor product, namely: $\otimes$ converts bilinearity into linearity.

# LECTURE 5

## Dual Spaces

Let $V$ be a vector space over $\mathbb{F}$.

**Definition 32** The **dual space** of $V$ is the vector space

$$V' = \mathscr{L}(V, \mathbb{F})$$

An element $\varphi \in V'$ is called **linear functional** on $V$.

**Example 33**

- Given a set $S$ and a field $\mathbb{F}$, the set $F^S = \{\text{functions } f : S \to F\}$, is a vector space over $\mathbb{F}$. Given any $x_0 \in S$, there is a linear functional $\varphi_{x_0} \in (F^S)'$: $\varphi_{x_0}(f) = f(x_0)$.
- More concrete version of above: $\varphi_4 \in (\mathscr{P}_n(\mathbb{R}))'$ such that $\varphi_4(p) = p(4)$ for $p \in \mathscr{P}_n(\mathbb{R})$.
- Let $V = C([2,5], \mathbb{R})$ be the vector space of continuous functions $f : [2,5] \to \mathbb{R}$. The map $\varphi : V \to \mathbb{R}$ such that $\varphi(f) = \int_2^5 f(x)dx$ is in $V'$.

In these examples, a linear functional is a function of functions.

Suppose that $\dim V = n < \infty$ and fix a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Define $\varphi_1, \ldots, \varphi_n \in V'$ by $\varphi_i(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_i$.

**Proposition 34** The $\varphi_1, \ldots, \varphi_n$ form a basis of $V'$. In particular, $\dim V' = n$ and $V' \cong V$.

We call the $\varphi_1, \ldots, \varphi_n$ the **dual basis** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

*Proof.*

Linear independence: if $a_1\varphi_1 + \cdots + a_n\varphi_n = 0$, then

$$(a_1\varphi_1 + \cdots + a_n\varphi_n)(\mathbf{v}_i) = a_i = 0$$

hence, the $\varphi_i$ are linearly independent.

Spanning: Let $\varphi \in V'$. Claim: $\varphi = \varphi(\mathbf{v}_1)\varphi_1 + \cdots + \varphi(\mathbf{v}_n)\varphi_n$. For every $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \in V$ we have that

$$\varphi(\mathbf{v}) = \varphi(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1\varphi(\mathbf{v}_1) + \cdots + a_n\varphi(\mathbf{v}_n)$$

and that

$$\begin{aligned}(\varphi(\mathbf{v}_1)\varphi_1 + \cdots + \varphi(\mathbf{v}_n)\varphi_n)(\mathbf{v}) &= \varphi(\mathbf{v}_1)\varphi_1(\mathbf{v}) + \cdots + \varphi(\mathbf{v}_n)\varphi_n(\mathbf{v}) \\ &= \varphi(\mathbf{v}_1)a_1 + \cdots + \varphi(\mathbf{v}_n)a_n\end{aligned}$$

$\square$

Importantly, we saw that if $\dim V = n < \infty$, then $V' \cong V$. Fixing a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, we have an isomorphism

$$\begin{aligned} f : V &\to V' \\ a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n &\mapsto a_1\varphi_1 + \cdots + a_n\varphi_n \end{aligned}$$

However, this isomorphism depends on the choice of basis and is thus non-canonical.

## Double Dual

Let $V'' = (V')'$ and define $f \in \mathscr{L}(V, V'')$ as follows: Given $\mathbf{v} \in V$, let $f(\mathbf{v}) = f_{\mathbf{v}} \in \mathscr{L}(V', \mathbb{F})$ such that for every $\varphi \in V'$, we have

$$f_{\mathbf{v}}(\varphi) = \varphi(\mathbf{v})$$

This $f$ is then linear because $\varphi$ is linear. As it turns out, this construction is rather neat.

**Proposition 35** $f \in \mathscr{L}(V, V'')$ is injective.

*Proof.*

We will show that the kernel of $f$ only contains 0. Take $\mathbf{v} \in V$ such that $f(\mathbf{v}) = f_{\mathbf{v}} = 0$. For every $\varphi \in V'$, we have $f_{\mathbf{v}} = \varphi(\mathbf{v}) = 0$. Claim: If $\mathbf{v} \neq 0$, then there is some $\varphi \in V'$ such that $\varphi(\mathbf{v}) \neq 0$. Take $U \subset V$ such that $\operatorname{Span} \mathbf{v} \oplus U = V$ and define $\varphi \in V'$ by $\varphi(a\mathbf{v} + \mathbf{u}) = a$, for all $a \in \mathbb{F}, \mathbf{u} \in U$.

$\square$

**Proposition 36** If $\dim V = n < \infty$, then $f \in \mathscr{L}(V, V'')$ is surjective.

*Proof.*

This essentially follows from the rank-nullity theorem.

$\square$

Thus, we get the following result.

**Theorem 37** If $\dim V < \infty$, then $f \in \mathscr{L}(V, V'')$ is an isomorphism.

This isomorphism is canonical, because $f$ does not depend on a choice of basis for $V$.

# LECTURE 6

## Forms

**Definition 38** Given vector spaces $V, W$ over some field $\mathbb{F}$. A **bilinear form** on $V \times W$ is a bilinear map

$$B : V \times W \to \mathbb{F}$$

that is

1. $B(a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \mathbf{w}) = a_1 B(\mathbf{v}_1, \mathbf{w}) + a_2 B(\mathbf{v}_2, \mathbf{w})$
2. $B(\mathbf{v}, a_1\mathbf{w}_1 + a_2\mathbf{w}_2) = a_1 B(\mathbf{v}, \mathbf{w}_1) + a_2 B(\mathbf{v}, \mathbf{w}_2)$

**Example 39**

- $\langle \cdot, \cdot \rangle_V : V \times V' \to \mathbb{F}$ where $(\mathbf{v}, \varphi) \mapsto \langle \mathbf{v}, \varphi \rangle_V = \varphi(\mathbf{v})$ is a bilinear form
- $B : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ where $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \mapsto \sum_{i=1}^n x_i y_i$ is a bilinear form. Note that when $\mathbb{F} = \mathbb{R}$, this becomes the standard inner product in $\mathbb{R}^n$.

By $\mathscr{B}(V, W)$ we denote the set of linear forms on $V \times W$. Quite naturally, $\mathscr{B}(V, W)$ is a vector space over $\mathbb{F}$. Furthermore, by the universal property of $\otimes$ we get that there exists a unique linear map

$$\tilde{B} \in \mathscr{L}(V \otimes W, \mathbb{F}) = (V \otimes W)'$$

This gives an isomorphism

$$\begin{aligned} \mathscr{B}(V, W) &\to \mathscr{L}(V \otimes W, \mathbb{F}) = (V \otimes W)' \\ B &\mapsto \tilde{B} \end{aligned}$$

If $\dim V = n < \infty$ and $\dim W = m < \infty$, then choices of ordered bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $V$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ of $W$ induce a matrix

$$A = (a_{ij}) \in M(n \times m, \mathbb{F}) \text{ such that } a_{ij} = B(\mathbf{v}_i, \mathbf{w}_j)$$

**Definition 40** $B \in \mathscr{B}(V)$ is **symmetric** if $B(\mathbf{v}, \mathbf{w}) = B(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$.

**Example 41**

- $B : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ such that $B(\mathbf{v}, \mathbf{w}) = x_1 y_1 + \cdots + x_n y_n$ is bilinear and symmetric.
- $\tilde{B} : \mathbb{F}^{2n} \times \mathbb{F}^{2n} \to \mathbb{F}$ such that $\tilde{B}(\mathbf{v}, \mathbf{w}) = x_1 y_2 - x_2 y_1 + \cdots + x_{2n-1} y_{2n} - x_{2n} y_{2n-1}$ is a bilinear form. Furthermore, $\tilde{B}$ is anti-symmetric/skew-symmetric/alternating.
- $B : C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \to \mathbb{R}$ such that $B(f, g) = \int_0^1 f(x) g(x) dx$ is bilinear and symmetric.

**Proposition 42** $B \in \mathscr{B}(V)$ is symmetric if and only if $B$ is represented by a symmetric matrix $A$.

---

**Definition 43** A **quadratic form** on $V$ is a map $q : V \to \mathbb{F}$ such that

1. $q(a\mathbf{v}) = a^2 q(\mathbf{v})$
2. $\bar{B}(\mathbf{v}, \mathbf{w}) := q(\mathbf{v} + \mathbf{w}) - q(\mathbf{v}) - q(\mathbf{w})$ is a bilinear form. Note that $\bar{B} : V \times V \to \mathbb{F}$ and that $\bar{B}$ is symmetric.

If the characteristic of $\mathbb{F}$ is not 2, we define $B(\mathbf{v}, \mathbf{w}) = \frac{1}{2} \bar{B}(\mathbf{v}, \mathbf{w})$.

**Example 44** If $V = \mathbb{R}^n$, then $q(\mathbf{v}) = ||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle$ is a quadratic form. And we get that

$$\begin{aligned} B(\mathbf{v}, \mathbf{w}) &= \frac{1}{2}(||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v}||^2 - ||\mathbf{w}||^2) \\ &= \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

**Theorem 45** If the characteristic of $\mathbb{F}$ is not 2, then there is a canonical isomorphism of vector spaces

$$\{\text{quadratic form } q : V \to \mathbb{F}\} \leftrightarrow \{\text{symmetric bilinear forms } B : V \times V \to \mathbb{F}\}$$

*Proof.*

Let the map from quadratic forms to symmetric bilinear forms be

$$q \mapsto \frac{1}{2}(q(\mathbf{v} + \mathbf{w}) - q(\mathbf{v}) - q(\mathbf{w})) =: B(\mathbf{v}, \mathbf{w})$$

and the other direction to be

$$B \mapsto B(\mathbf{v}, \mathbf{v}) =: q(\mathbf{v})$$

□

**Definition 46** If $\dim V = n < \infty$, then a bilinear form $B : V \times V \to \mathbb{F}$ can be **diagonalized** if there is a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $V$ such that $B(\mathbf{v}_i, \mathbf{v}_j) = 0$ if $i \neq j$.

In this basis, $B$ is represented by the diagonal matrix $(B(\mathbf{v}_i, \mathbf{v}_j))$. And if $B$ can be diagonalized, then $B$ is symmetric.

**Theorem 47** If the bilinear form $B : V \times V \to \mathbb{F}$ is symmetric, then it can be diagonalized.

---

Assume now that $\mathbb{F} = \mathbb{C}$.

**Definition 48** A map $h : V \otimes V \to \mathbb{C}$ is a **sesquilinear form** if:

1. $h(a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \mathbf{w}) = a_1 h(\mathbf{v}_1, \mathbf{w}) + a_2 h(\mathbf{v}_2, \mathbf{w})$
2. $h(\mathbf{v}, +a_1\mathbf{w}_1 + a_2\mathbf{w}_2) = \bar{a}_1 h(\mathbf{v}, \mathbf{w}_1) + \bar{a}_2 h(\mathbf{v}, \mathbf{w}_2)$

Meaning, that such a map is a sesquilinear form if it is linear in the first entry and conjugate-linear in the second entry. For the curios; "sesqui" means "one and a half".

**Definition 49** A sesquilinear form $h : V \times V \to \mathbb{C}$ is **hermitian** if it is conjugate symmetric, that is $h(\mathbf{w}, \mathbf{v}) = \overline{h(\mathbf{v}, \mathbf{w})}$.

**Example 50** The map $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that

$$h(\sum_{i=1}^{n} x_i \mathbf{e}_i, \sum_{j=1}^{n} y_j \mathbf{e}_j) = \sum_{i,j=1}^{n} x_i \cdot \bar{y}_j$$

is hermitian. This is the standard complex inner product in $\mathbb{C}^n$.

If $\dim_{\mathbb{C}} V = n < \infty$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an ordered basis for $V$, then a sesquilinear form $h : V \times V \to \mathbb{C}$ has an induced matrix $A = (h(\mathbf{e}_i, \mathbf{e}_j))_{i,j=1}^{n}$. For which we have the following.

**Proposition 51** A map $h : V \times V \to \mathbb{C}$ is hermitian if and only if $A$ is **self-adjoint**. Meaning that $A = \overline{A^t}$.

# LECTURE 7

## Inner Products

Let $\mathbb{F} = \mathbb{R}$ or let $\mathbb{F} = \mathbb{C}$.

**Definition 52** An **inner product** on a vector space $V$ over $\mathbb{F}$ is a map

$$\langle .,. \rangle : V \times V \to \mathbb{F}$$

such that

1. $\langle a_1\mathbf{v}_1 + a_2\mathbf{v}_2, \mathbf{w}\rangle = a_1\langle \mathbf{v}_1, \mathbf{w}\rangle + a_2\langle \mathbf{v}_2, \mathbf{w}\rangle$
2. $\langle \mathbf{v}, a_1\mathbf{w}_1 + a_2\mathbf{w}_2\rangle = \bar{a}_1\langle \mathbf{v}, \mathbf{w}_1\rangle + \bar{a}_2\langle \mathbf{v}, \mathbf{w}_2\rangle$
3. $\langle \mathbf{v}, \mathbf{w}\rangle = \overline{\langle \mathbf{w}, \mathbf{v}\rangle}$
4. $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$
5. $\langle \mathbf{v}, \mathbf{v}\rangle = 0$, then $\mathbf{v} = 0$

A vector space with a such map is called an **inner product space** .

**Example 53** If $V = C([0,1],\mathbb{C})$ and $w : [0,1] \to (0,\infty)$ is continuous. Then $\langle .,. \rangle : V \times V \to \mathbb{C}$ defined by $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}w(x)dx$ is an inner product.

Let us list some important properties of the inner product:

**Properties 54**

1. Cauchy-Scwartz inequality:

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \leq ||\mathbf{v}|| \cdot ||\mathbf{w}||$$

2. Law of cosines:

$$||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 + 2\Re\langle \mathbf{v}, \mathbf{w}\rangle$$

   Note, if $\mathbb{F} = \mathbb{R}$, then $\langle \mathbf{v}, \mathbf{w}\rangle = ||\mathbf{v}||||\mathbf{w}||\cos\theta$

3. Triangle inequality:

$$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$$

4. Pythagoras Theorem: if the vectors $\mathbf{v}, \mathbf{w}$ are orthogonal we get that

$$||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$$

Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle .,. \rangle$.

**Proposition 55** If $\dim V < \infty$, then $V$ has an orthonormal basis.

*Proof.*

Gram-Schmidt.

□

**Definition 56** Let $U \subset V$ subspace.

- $U^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}$ is the **orthogonal complement** of $U$.
- If $\dim U < \infty$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal basis of $U$.

$$\begin{aligned} P_U &: V \to U \\ \mathbf{v} &\mapsto \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e}_i \rangle e_i \end{aligned}$$

is the **orthogonal projection** onto $U$. Call $\mathbf{v} - P_U(\mathbf{v})$ the **residual**.

**Lemma 57** If $\dim U < \infty$, then $\mathbf{v} - P_U(\mathbf{v}) \in U^{\perp}$.

**Proposition 58** $U + U^{\perp} = U \oplus U^{\perp}$

*Proof.*

We want to show that $U \cap U^{\perp} = \{0\}$. Let $\mathbf{v} \in U \cap U^{\perp}$. For which we get that

$$\langle \mathbf{v}, \mathbf{v} \rangle = 0 \implies \mathbf{v} = 0$$

□

**Proposition 59** If $\dim U < \infty$, then $V = U \oplus U^{\perp}$.

Note that the map

$$\begin{aligned} U^{\perp} &\to V/U \\ \mathbf{v} &\mapsto [\mathbf{v}] \end{aligned}$$

is an isomorphism if $\dim U < \infty$. Furthermore, if $\dim V < \infty$, then the map $V \to V'$ defined by $\mathbf{v} \mapsto \varphi_{\mathbf{v}}$ where $\varphi_{\mathbf{v}}(\mathbf{w}) = \langle \mathbf{w}, \mathbf{v} \rangle$ is an isomorphism.

**Theorem 60** Let $U \subset V$ be a finite dimensional vector subspace. Then, for every $\mathbf{v} \in V$, $P_U(\mathbf{v}) \in U$ minimizes the distance from $\mathbf{v}$ to vectors in $U$:

1. $||\mathbf{v} - \mathbf{u}|| \geq ||\mathbf{v} - P_U(\mathbf{v})||$ for all $\mathbf{u} \in U$
2. $||\mathbf{v} - \mathbf{u}|| = ||\mathbf{v} - P_U(\mathbf{v})||$ if and only if $\mathbf{u} = P_U(\mathbf{v})$ .

*Proof.*

Given $\mathbf{u} \in U$

$$\begin{aligned} ||\mathbf{v}-\mathbf{u}||^2 &= ||\mathbf{v}-P_U(\mathbf{v})+P_U(\mathbf{v})-\mathbf{u}||^2 \\ &= ||\mathbf{v}-P_U(\mathbf{v})||^2+||P_U(\mathbf{v})-\mathbf{u}||^2 \end{aligned}$$

□

# LECTURE 8

## Operators, Polynomials and Determinants

Let $V$ be a vector space over $\mathbb{F}$.

**Definition 61** An **operator** on $V$ is an element of $\mathscr{L}(V)$.

Observe that we can compose and take linear combinations of operators, given some $T \in \mathscr{L}(V)$ we have $T^n = T \circ \cdots \circ T$. Which inspires us to input operators to polynomials, given $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathscr{P}(F)$ and $T \in \mathscr{L}(V)$, we have $p(T) = a_0 \cdot I + a_1 \cdot T + \cdots + a_n \cdot T^n \in \mathscr{L}(V)$.

Recall, that if given a vector space $V$ of finite dimension and a choice of ordered basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $V$. Then we have an isomorphism $\mathbf{v} = a_1 \mathbf{v}_1, \ldots, a_n \mathbf{v}_n \mapsto (a_1, \ldots, a_n)$. Hence, we identify another isomorphism $f : \mathscr{L}(V) \cong \mathscr{M}(n \times n, \mathbb{F})$.

**Definition 62** Given $T \in \mathscr{L}(V)$, define $\det T = \det f(T)$.

## Eigenvalues and Generalized Eigenvalues

**Definition 63** $\lambda \in \mathbb{F}$ is an **eigenvalue** of $T \in \mathscr{L}(V)$ with **eigenvector** $\mathbf{v} \in V \setminus \{0\}$ if $T\mathbf{v} = \lambda \mathbf{v}$.

We get the usual method of computing eigenvalues by observing that

$$T\mathbf{v} = \lambda \mathbf{v} \iff (\lambda I - T)\mathbf{v} = 0 \iff \det(\lambda I - T) = 0$$

**Definition 64** The **characteristic polynomial** of $T$ is

$$p_T(\lambda) = \det(\lambda I - T) \in \mathscr{P}_n(\mathbb{F})$$

and $\lambda$ is an eigenvalue of $T$ if and only if $p_T(\lambda) = 0$.

For now, let $\mathbb{F} = \mathbb{C}$ and $\dim V = n < \infty$.

**Theorem 65** Any $T \in \mathscr{L}(V)$ has an eigenvalue.

**Definition 66** If $\lambda \in \mathbb{C}$ is an eigenvalue of $T \in \mathscr{L}(V)$, the $\lambda$-**eigenspace** of $T$ is $E(\lambda, T) = \ker(\lambda I - T)$.

**Definition 67** Given $T \in \mathscr{L}(V)$ and an eigenvalue $\lambda \in \mathbb{C}$ of $T$, say that $\mathbf{v} \in V$ is a **generalized eigenvector** for $\lambda$ if $(T - \lambda I)^k(\mathbf{v}) = 0$ for some $k \geq 1$. Then $G(\lambda, T) = \{\mathbf{v} \in V \mid (T - \lambda I)^k(\mathbf{v}) = 0\}$ is called the **generalized $\lambda$-eigenspace**.

**Definition 68** If $\lambda \in \mathbb{C}$ is an eigenvalue of $T \in \mathscr{L}(V)$, its **geometric multiplicity** is $\dim E(\lambda, T)$ and its **algebraic multiplicity** is $\dim G(\lambda, T)$.

# LECTURE 9

**Lemma 69** If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of $T \in \mathscr{L}(V)$ and $\mathbf{v}_1, \dots, \mathbf{v}_m$ are the corresponding generalized eigenvectors, then the $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent.

*Proof.*

For simplicity, assume first that $\mathbf{v}_i \in E(\lambda_i, T) \setminus \{0\}$ are eigenvectors. We want to show that if

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = 0 \implies a_i = 0$$

Using $T\mathbf{v}_i - \lambda_i\mathbf{v}_i = 0$, we get

$$(T - \lambda_2 I)(T - \lambda_3 I)\cdots(T - \lambda_m I)(\mathbf{v}) = a_1(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_m)\mathbf{v}_1 = 0$$

which implies that $a_1 = 0$ as every other factor is nonzero. The same argument implies that $a_i = 0$ for all $1 \le i \le m$, as wanted.

The general case when $\mathbf{v}_i \in G(\lambda_i, T) \setminus \{0\}$ is dealt with in a similar manner.

□

**Corollary 70** If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of $T$, then

$$G(\lambda_1, T) + \cdots + G(\lambda_m, T) = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

**Theorem 71** If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of $T$, then

$$G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T) = V$$

*Proof.*

By induction on $n = \dim V$.

**$n = 1$:** In this case, $V$ has a single eigenvalue $\lambda \in \mathbb{C}$ and $\{0\} \neq E(\lambda, T) = G(\lambda, T) = V$.

**$n > 1$:** Assume that the theorem holds for any $T \in \mathscr{L}(W)$, with $\dim W < n$. Since $G(\lambda_1, T) = \ker(T - \lambda_1 I)^n$ and $\ker(T - \lambda_1 I)^n \cap \operatorname{im}(T - \lambda_1 I) = \{0\}$, we have $G(\lambda_1, T) + U = G(\lambda_1, T) \oplus U$ where $U = \operatorname{im}(T - \lambda_1 I)$. The rank-nullity theorem implies that

$$\dim\ker(T - \lambda_1 I)^n + \dim\operatorname{im}(T - \lambda_1 I)^n = n$$

which implies that $V = G(\lambda_1, T) \oplus U$.

Claim 1: $\mathbf{u} \in U \implies T(\mathbf{u}) \in U$. Hence, $T_U \in \mathscr{L}(U)$. Claim 2: The eigenvalues of $T_U$ are $\lambda_2, \ldots, \lambda_n$ and $G(\lambda_k, T_U) = G(\lambda_k m T)$ for all $2 \le k \le n$.

□

We may now observe that if we pick a basis for every subspace $G(\lambda_i, T) \subset V$ and take the union of all these bases, then we get a basis of $V$. In that basis, $T$ is represented by a block-diagonal matrix.

**Definition 72** $T \in \mathscr{L}(V)$ is **nilpotent** if $T^k = 0$ for some $k \ge 1$.

**Theorem 73** Let $N \in \mathscr{L}(V)$ be a nilpotent operator on a finite dimensional vector space. Then, there are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and integers $m_1, \ldots, m_k \ge 0$ such that

$$N^{m_1+1}(\mathbf{v}_1) = \cdots = N^{m_k+1}(\mathbf{v}_k) = 0$$

and

$$N^{m_1}(\mathbf{v}_1), \ldots, N(\mathbf{v}_1), \mathbf{v}_1, \ldots, N^{m_k}(\mathbf{v}_k), \ldots, N(\mathbf{v}_k), \mathbf{v}_k$$

is a basis of $V$.

**Theorem 74** If $V$ is a finite dimensional complex vector space and $T \in \mathscr{L}(V)$, then there is a basis for $V$ with respect to which $T$ is given by a Jordan matrix, that is

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

This theorem implies that for every $A \in M(n \times n, \mathbb{C})$, there is a Jordan matrix $J$ and a change of basis matrix $C$ such that $J = C^{-1}AC$. The following result follows from the existence of Jordan forms:

**Theorem 75** For every $T \in \mathscr{L}(V)$ we have that $p_T(T) = 0$.