Complex integration

We shall now study so-called contour integrals (or line integrals) of complex-valued fcns. This theory will teach us more about the properties of analytic fcns.

Contours

Def. A point set γ in $\mathbb{C}$ is said to be a smooth arc if it is the image of some continuous complex-valued fcn $z = z(t)$, $a \le t \le b$ s.t.

(i) $z(t)$ has a cont. derivative on $[a,b]$

(ii) $z'(t) := x'(t) + iy'(t) \neq 0$ on $[a,b]$

(iii) $z(t)$ is one-to-one on $[a,b]$

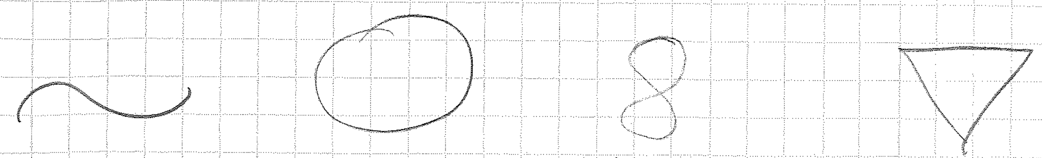
A point set γ in $\mathbb{C}$ is said to be a smooth closed curve if it is the range of some cont. fcn $z = z(t)$, $a \le t \le b$, satisfying (i), (ii) and

(iii') $z(t)$ is one-to-one on $[a,b)$, but

$$z(a) = z(b), \quad z'(a) = z'(b).$$

The phrase "γ is a smooth curve" means that γ is either a smooth arc or a smooth closed curve.

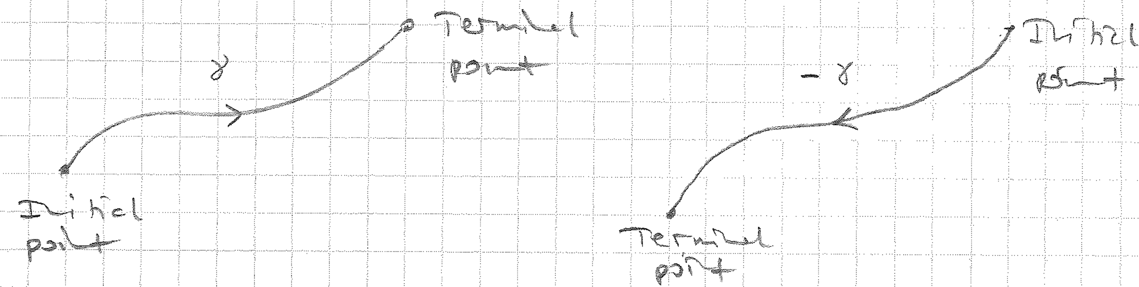
Ex.



smooth arc — smooth closed curve — Not smooth closed curve — Not smooth closed curve

There are (infinitely) many choices of "admissible" parametrizations. E.g. one can change "direction".

Obviously, there are two natural orientations of a smooth curve:



A smooth curve with a specified orientation is called a <u>directed</u> smooth curve

Ex. Give parametrizations of the following directed smooth curves:

(a) The straight line from z_1 to z_2

(b) The circle with radius r and center z_0 oriented counterclockwise

Sol. (a) $z(t) = z_1 + t(z_2 - z_1)$; $0 \le t \le 1$

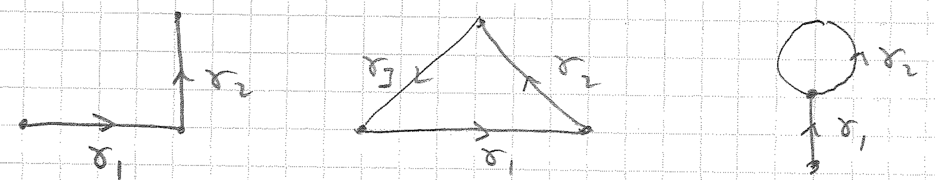
(b) $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

<u>Def.</u> A <u>contour</u> Γ is either a single point or a finite sequence $(\gamma_1, \dots, \gamma_n)$ of directed smooth curves s.t. the terminal point of γ_k coincides with the initial point of γ_{k+1}, $k=0,\dots,n-1$.

We write $\Gamma = \gamma_1 + \dots + \gamma_n$.

Γ is said to be a <u>closed contour</u> if the initial and terminal points coincide; if these are the only self-intersections we call Γ a <u>simple closed contour</u>.

<u>Ex.</u>



If γ is a smooth curve the length of γ, denoted $L(\gamma)$, is given by

$$L(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \left|\frac{dz}{dt}\right| dt.$$

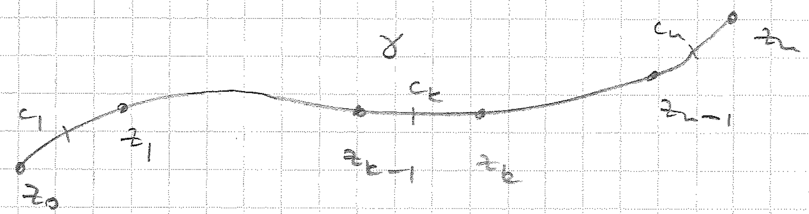
The length of $\Gamma = \gamma_1 + \dots + \gamma_n$ is given by

$$L(\Gamma) = L(\gamma_1) + \dots + L(\gamma_n).$$

Contour integrals (line integrals)

We shall now see how to define $\int_\Gamma f(z)\,dz$, the contour integral of a complex-valued fcn f over the contour Γ. We start by defining $\int_\gamma f(z)\,dz$, where γ is a directed smooth curve.

Consider the figure:



For each n we form a partition $\mathcal{P}_n = \{z_0, \dots, z_n\}$ of γ, as in the figure.

Let $L(\gamma; z_{k-1}, z_k)$ denote the length of γ from z_{k-1} to z_k. Then $\mu(\mathcal{P}_n) = \max\limits_{1\le k\le n} L(\gamma; z_{k-1}, z_k)$ is a measure of the "fineness" of the partition.

Take, for $k = 1, \dots, n$, an arbitrary point c_k on γ between z_{k-1} and z_k. Then form the Riemann sum

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\,\Delta z_k .$$

<u>Def</u> We say that f is integrable along the directed smooth curve γ if there exists a complex number L s.t.

$$\lim_{n\to\infty} \mu(P_n) = 0 \Rightarrow \lim_{n\to\infty} S(P_n) = L.$$

(indep. of the choice of partition and Riemann sum)

The number L is called the <u>integral of f along γ</u>, and is denoted $\int_\gamma f(z)\,dz$.

<u>Properties</u> If f, g are integrable along $\gamma \Rightarrow$

- $\int_\gamma (f(z) \pm g(z))\,dz = \int_\gamma f(z)\,dz \pm \int_\gamma g(z)\,dz$
- $\int_\gamma c f(z)\,dz = c \int_\gamma f(z)\,dz$
- $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$

<u>Thm</u> If is cont. along γ, then f is integrable along γ.

How do we compute $\int_\gamma f(z)\,dz$?

First consider $\int_a^b f(t)\,dt$, where $f(t) = u(t) + iv(t)$

and u, v are continuous on $[a,b]$.

Let $F(t)$ be an antiderivative of $f(t)$, i.e.

$F(t) = U(t) + iV(t)$, where $U' = u$, $V' = v$

$$\Rightarrow \int_a^b f(t)\,dt = \int_a^b \big(u(t) + iv(t)\big)\,dt \overset{\text{Prop.}}{=} \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \quad (*)$$

$$= \Big[U(t)\Big]_a^b + i\Big[V(t)\Big]_a^b = F(b) - F(a)$$

Thm If f is conti. on $[a,b]$, $F'(t) = f(t)$ for $t \in [a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

The integral of f along an arbitrary directed smooth curve can be reduced to integrals as above by the parametrization $z(t)$, $a \le t \le b$.

Let

$$z_0 = z(t_0),\ z_1 = z(t_1), \dots, z_n = z(t_n),$$

where $a = t_0 < t_1 < \dots < t_n = b$.

$$\underset{c_k = z_k}{\Longrightarrow} \sum_{k=1}^{n} f(z_k)\,\Delta z_k = \sum_{k=1}^{n} f(z(t_k))\,\Delta z_k =$$

$$\simeq \Big/ \ \Delta z_k = z_k - z_{k-1} = z(t_k) - z(t_{k-1}) \simeq$$

$$\simeq z'(t_k)(t_k - t_{k-1}) = z'(t_k)\,\Delta t_k \ \Big/$$

$$\simeq \sum_{k=1}^{n} f(z(t_k))\, z'(t_k)\,\Delta t_k ,$$

which is a Riemann sum for the cont. fcn $f(z(t))\,z'(t)$ on $[a,b]$.

This suggests the following:

> **Thm** Let f be a continuous fcn on a directed smooth curve having admissible parametrization $z(t)$, $a \le t \le b$. Then
>
> $$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt \qquad (**)$$

Remark: One can take $(*)$ and $(**)$ as def. of $\int_\gamma f(z)\,dz$. One then only has to show that the integral is indep. of the choice of parametrization.

Ex. Compute $\int_{C_r} (z-z_0)^n \, dz$, $n \in \mathbb{Z}$, over

$C_r = \{ z : |z - z_0| = r \}$, oriented counterclockwise.

Sol. Let $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

Then $z'(t) = ire^{it}$, so

$$\int_{C_r} (z-z_0)^n \, dz = \int_0^{2\pi} r^n e^{int} \cdot ire^{it} \, dt =$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = \begin{cases} ir^{n+1} \left[\frac{1}{i(n+1)} e^{i(n+1)t} \right]_0^{2\pi} = 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

5)

Def Suppose $\Gamma = \gamma_1 + \dots + \gamma_n$ and let f be

cont. on Γ. Then we let

$$\int_\Gamma f(z) \, dz := \sum_{k=1}^{n} \int_{\gamma_k} f(z) \, dz$$

(and if $\Gamma = \{z_0\}$ we let $\int_\Gamma f(z) \, dz = 0$)

ML-inequality Suppose $|f(z)| \le M$ $\forall z \in \gamma$.

$$\Rightarrow \left| \sum_{k=1}^{n} f(c_k) \Delta z_k \right| \le \sum_{k=1}^{n} |f(c_k)| |\Delta z_k| \le M \sum_{k=1}^{n} |\Delta z_k| \le ML(\gamma)$$

Letting $\mu(P_n) \to 0$ implies: $\left| \int_\gamma f(z) \, dz \right| \le ML(\gamma)$

Thm Suppose f is continuous on Γ and

that $|f(z)| \le M$, $z \in \Gamma$. Then,

$$\left| \int_\Gamma f(z) \, dz \right| \le ML, \text{ where } L = L(\Gamma)$$

<u>Ex</u> Consider $\int_\Gamma \frac{e^z}{z^2+1}\,dz$, where Γ is the circle $|z|=2$ traversed counterclockwise.

$$|e^z| = e^x \leq e^2 \text{ on } \Gamma \text{ and}$$

$$|z^2+1| \geq |z|^2 - 1 = 4-1 = 3 \text{ on } \Gamma.$$

$$\Rightarrow \left| \int_\Gamma \frac{e^z}{z^2+1}\,dz \right| \leq \frac{e^2}{3} \cdot 4\pi$$

$\boxtimes$