Exam, Real analysis, 1MA226, 2015-03-21

Solutions.

1. Suppose that X contains more than one point. Then if x is any point in X, both the sets $A = \{x\}$ and $B = X \setminus \{x\}$ are nonempty, and $X = A \cup B$. Furthermore, *A and B are separated.* [Proof: It suffices to prove that $\overline{A} = A$ and $\overline{B} = B$, since then it follows that $A \cap \overline{B} = \overline{A} \cap B = A \cap B = \emptyset$. In fact we have $\overline{E} = E$ for *every* subset $E \subset X$, since X is discrete! Indeed, for every $E \subset X$ and every $p \in X$ we have $N_{1/2}(p) \cap E = \{p\}$; hence p is not a limit point of E; thus E has *no* limit points in X, and therefore $\overline{E} = E$.] Hence we have expressed X as the union of two nonempty separated sets. This contradicts the assumption that X is connected!

Hence X cannot contain more than one point; hence *the cardinality of X is 1.* (Let's agree that a metric space is always nonempty...)

Now it follows that X *is compact*, since every finite metric space is compact. $\square$

2. (Recall from my comments to the exam: We have to write, e.g., "$\sum_{n=2}^\infty$", *not* "$\sum_{n=1}^\infty$".)

By Theorem 3.29, the given series is absolutely convergent when $a > 1$, and *not* absolutely convergent when $a \leq 1$.

On the other hand we claim that the series is convergent for *every* $a \in \mathbb{R}$. To prove this, let $a \in \mathbb{R}$ be given. Note that the series is alternating, and that the terms of the series tend to 0. (Indeed, it is a "standard limit fact" that we have $\lim_{n\to\infty} \frac{1}{n(\log n)^a} = 0$. [1]) Hence by Theorem 3.43, *modified to the case of* $\sum_{n=N}^\infty$ *for some* $N \in \mathbb{Z}^+$, it now suffices to prove that there exists some $N \in \mathbb{Z}^+$ such that

(1)
$$\frac{1}{N(\log N)^a} \geq \frac{1}{(N+1)(\log(N+1))^a} \geq \frac{1}{(N+2)(\log(N+2))^a} \geq \cdots$$

We prove the stronger statement that there exists some $X > 0$ such that the function $f(x) := \frac{1}{x(\log x)^a}$ is decreasing on the interval $[X, \infty)$. We have for all $x > 1$:

$$f'(x) = -\frac{1}{x^2(\log x)^a} + \frac{1}{x}(-a)(\log x)^{-a-1}\frac{1}{x} = \frac{-a - \log x}{x^2(\log x)^{a+1}}.$$

We know that $\log x \to +\infty$ as $x \to +\infty$; hence there is some $X > 1$ such that $\log x > -a$ for all $x \geq X$; and then it follows from the above formula that $f'(x) < 0$ for all $x \geq X$, and so f is strictly decreasing on the interval $[X, \infty)$. [2] Hence, afortiori, (1) holds for $N = \lceil X \rceil$.

Answer: (a) For $a > 1$. (b) For all $a \in \mathbb{R}$. (c) For *no* $a \in \mathbb{R}$.

---

[1]If $a \geq 0$ then $\frac{1}{n(\log n)^a} \leq \frac{1}{n}$ for all $n \geq 3$ so that the statement is trivial. For $a < 0$ the task is to prove $\lim_{n\to\infty} \frac{(\log n)^b}{n} = 0$ where $b = |a| > 0$. Raising to $1/b$ this is equivalent to the statement that $\lim_{n\to\infty} \frac{\log n}{n^{1/b}} = 0$, and this holds by equation (45) on p. 181 in Rudin's book.

[2]Once you have studied Rudin's Chapter 5 you will know that this follows from the Mean Value Theorem (Theorem 5.10) by an analogous argument as in Theorem 5.11. Indeed, for any given $x_2 > x_1 \geq X$, Theorem 5.10 implies that there exists some $x \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Here $f'(x) < 0$ by what we have proved, and so $f(x_2) - f(x_1) < 0$, i.e. $f(x_2) < f(x_1)$.

3. We compute $x_2 = \frac{1}{2}$, $x_3 = \frac{2}{3}$, $x_4 = \frac{3}{4}$, $x_5 = \frac{4}{5}$. This leads us to conjecture

$$x_n = \frac{n-1}{n} \qquad \text{for all } \ n \geq 1.$$

Let us prove this claim by induction! The claim holds for $n = 1$ since $x_1 = 0$ according to the problem formulation. Assume that the claim holds for a certain $n \in \mathbb{Z}^+$. Then it follows that

$$x_{n+1} = \frac{1}{2 - x_n} = \frac{1}{2 - \frac{n-1}{n}} = \frac{1}{\frac{2n-(n-1)}{n}} = \frac{n}{n+1};$$

hence the claim holds for $n+1$ as well. Hence by the induction principle, the claim holds for all $n \geq 1$.

It now follows that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n-1}{n} = 1$. Hence also $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 1$.

Answer: $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 1$.

4. Assume that $\int_0^1 f_n(x)\,dx \to 0$. Take N so large that $\int_0^1 f_n(x)\,dx < \frac{1}{2}$ for every $n \geq N$. Now fix (temporarily) some $n \geq N$. We must have $\inf\{f_n(x) \ : \ x \in [0,1]\} < \frac{1}{2}$, for otherwise $f_n(x) \geq \frac{1}{2}$ for all $x \in [0,1]$ which implies $\int_0^1 f_n(x)\,dx \geq \frac{1}{2}$, contradicting the choice of n and N. But a continuous function on a compact interval attains its supremum and its infimum; hence there exist $a, b \in [0,1]$ (which depend on n) such that $f_n(a) = 1$ and $f_n(b) < \frac{1}{2}$. We now define a sequence (x_n) of points in $[0,1]$ as follows: For every $n < N$ we set $x_n = 0$. For every *even* $n \geq N$ we take $x_n \in [0,1]$ such that $f_n(x_n) = 1$ (i.e. "$x_n = a$" above), and for every *odd* $n \geq N$ we take $x_n \in [0,1]$ such that $f_n(x_n) < \frac{1}{2}$ (e.g. "$x_n = b$" above). Then the sequence $(f_n(x_n))_{n=1,2,\ldots}$ does not converge (for example since it is not Cauchy; we have $|f_n(x_n) - f_{n+1}(x_{n+1})| > \frac{1}{2}$ for all $n \geq N$). $\square$

5. Consider the map $\phi : C([0,1]) \to C([0,1])$ given by

$$\phi(f)(x) = \frac{1}{2}\int_x^1 (x-y)f(y)\,dy + x^2 e^{x^2}.$$

We first have to prove that this is really a map from $C([0,1])$ to $C([0,1])$ as claimed. Thus let $f \in C([0,1])$ be given. We then have to prove that $\phi(f)(x)$ is a continuous function on $[0,1]$. Since $x^2e^{x^2}$ is a continuous function of x, it suffices to prove that $\int_x^1 (x-y)f(y)\,dy$ is a continuous function of $x \in [0,1]$. But we have

$$\int_x^1 (x-y)f(y)\,dy = x\int_x^1 f(y)\,dy - \int_x^1 yf(y)\,dy,$$

and by Theorem 6.20[3], both "$\int_x^1 f(y)\,dy$" and "$\int_x^1 yf(y)\,dy$" are continuous functions of x. Hence the above expression is a continuous function of x, completing the proof that ϕ maps $C([0,1])$ to $C([0,1])$.

Next, for any $f, g \in C([0,1])$ and any $x \in [0,1]$ we have

$$\begin{aligned}|\phi(f)(x) - \phi(g)(x)| &= \frac{1}{2}\left|\int_x^1 (x-y)(f(y)-g(y))\,dy\right| \\ &\leq \frac{1}{2}d(f,g)\int_x^1 (y-x)\,dy \leq \frac{1}{4}d(f,g).\end{aligned}$$

This proves that ϕ is a contraction on $C([0,1])$. Recall also that $C([0,1])$ is complete. Hence by the contraction principle, ϕ has a unique fixed point in $C([0,1])$. This is equivalent to saying that the integral equation in the problem formulation has a unique solution in $C([0,1])$. $\square$

[3]Pedantically, Thm. 6.20 is stated in the case where the *upper* integration limit is varying. To apply it to our situation, one may rewrite $\int_x^1 f(y)\,dy = \int_0^1 f(y)\,dy - \int_0^x f(y)\,dy$; here Thm. 6.20 implies that $\int_0^x f(y)\,dy$ is a continuous function of x; hence also $\int_x^1 f(y)\,dy$ is a continuous function of x. Similarly for $\int_x^1 yf(y)\,dy$.

6. We have $\left|\frac{\cos nx}{n(\log n)^2}\right| \leq \frac{1}{n(\log n)^2}$ for all $n \geq 2$, and the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is known to be convergent (Thm. 3.29). Hence, by Weierstrass' M-test (Theorem 7.10), the series $\sum_{n=2}^{\infty} \frac{\cos nx}{n(\log n)^2}$ converges uniformly on $\mathbb{R}$. Since each term of the series is a continuous function of x, it follows (by Theorem 7.12) that $f(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{n(\log n)^2}$ is a continuous function on $\mathbb{R}$. Note also that

$$|f(x)| = \left|\sum_{n=2}^{\infty} \frac{\cos nx}{n(\log n)^2}\right| \leq \sum_{n=2}^{\infty}\left|\frac{\cos nx}{n(\log n)^2}\right| \leq \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

for all $x \in \mathbb{R}$. This proves that the function f is bounded, namely $|f(x)| \leq B$ for all $x \in \mathbb{R}$ where $B := \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$. Finally we determine the $C((-\infty, \infty))$-norm of f, i.e.

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|.$$

The above shows that $\|f\| \leq B$. On the other hand we have $f(0) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} = B$. Hence $\|f\| = B$.

Answer: The $C((-\infty, \infty))$-norm equals $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

7.

Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(u,v) = (u^{17} + v^{17}, u^{18} + v^{18})$; this is a C^1 function and the differential $f'(u,v) \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ is represented by the matrix

$$[f'(u,v)] = \begin{pmatrix} (D_1 f_1)(u,v) & (D_2 f_1)(u,v) \\ (D_1 f_2)(u,v) & (D_2 f_2)(u,v) \end{pmatrix} = \begin{pmatrix} 17u^{16} & 17v^{16} \\ 18u^{17} & 18v^{17} \end{pmatrix}.$$

In particular

$$[f'(1,-1)] = \begin{pmatrix} 17 & 17 \\ 18 & -18 \end{pmatrix},$$

which is non-singular. Hence by the Inverse Function Theorem there exist open sets U and V in $\mathbb{R}^n$ such that $(1,-1) \in U$, $f|_U$ is injective, and $f(U) = V$, and the inverse function $g := f|_U^{-1}$ (which by definition is a bijection from V onto U) is also C^1. Writing $g = (g_1, g_2)$, this means that for every $(x,y) \in V$, $(g_1(x,y), g_2(x,y))$ is the unique solution (u,v) in U to the equation $f(u,v) = (x,y)$, i.e. exactly the system of equations given in the problem. Note also that $(0,2) = f(1,-1) \in f(U) = V$, and so since V is open, V contains a neighborhood of $(0,2)$. In this neighborhood, the given system of equations has a unique solution $u(x,y) = g_1(x,y)$, $v(x,y) = g_2(x,y)$, exactly as required. Also $g(0,2) = g(f(1,-1)) = (1,-1)$, i.e. $u(0,2) = 1$ and $v(0,2) = -1$. $\square$

8. Note that for all $(x,y,z) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ we have

$$x^2+y^2+z^2-(xy+yz+xz) = \frac{1}{2}\Big((x-y)^2+(x-z)^2+(y-z)^2\Big) \geq 0;$$

therefore $xy+yz+xz \leq x^2+y^2+z^2$ and

$$\frac{xy+yz+xz}{x^2+y^2+z^2} \leq 1;$$

and also

$$x^2+y^2+z^2+2(xy+yz+xz) = (x+y+z)^2 \geq 0;$$

and therefore $xy+yz+xz \geq -\frac{1}{2}(x^2+y^2+z^2)$ and so

$$\frac{xy+yz+xz}{x^2+y^2+z^2} \geq -\frac{1}{2}.$$

Hence the range of f is contained in $[-\frac{1}{2},1]$, and since this is a closed set, it follows that also *all limit points of f at $\mathbf{0}$ are contained in* $[-\frac{1}{2},1]$. On the other hand, note that $f(1,1,1)=1$ and $f(1,-1,0)=-\frac{1}{2}$, and hence the range of f *equals* the interval $[-\frac{1}{2},1]$. (Proof: For example we can consider the function $F(t) := f(1,-1+2t,t)$ for $t \in [0,1]$; this is a continuous function[4] with $F(0)=f(1,-1,0)=-\frac{1}{2}$ and $F(1)=f(1,1,1)=1$; hence by Theorem 4.23, for every $c \in [-\frac{1}{2},1]$ there exists some $t \in [0,1]$ such that $F(t)=c$, and hence there exists a point $(x,y,z)=(1,-1+2t,t) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $f(x,y,z)=c$.)

Now for any $c \in [-\frac{1}{2},1]$, let us choose some $\boldsymbol{x}=(x,y,z) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $f(\boldsymbol{x})=c$. Set $\boldsymbol{x}_k := k^{-1}\boldsymbol{x}$. Then $(\boldsymbol{x}_k)$ is a sequence in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ with $\boldsymbol{x}_k \to \mathbf{0}$, and for all $k \in \mathbb{Z}^+$ we have

(2)

$$f(\boldsymbol{x}_k) = f(k^{-1}x, k^{-1}y, k^{-1}z) = \frac{k^{-2}(xy+yz+xz)}{k^{-2}(x^2+y^2+z^2)} = f(x,y,z) = c.$$

Hence $f(\boldsymbol{x}_k) \to c$ as $k \to \infty$. This proves that c is a limit point of f at $\mathbf{0}$. We have thus proved that every $c \in [-\frac{1}{2},1]$ is a limit point of f at $\mathbf{0}$.

Combining this with our ealier finding we conclude that the set of limit points of f at $\mathbf{0}$ equals $[-\frac{1}{2},1]$. Hence also $\limsup_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = \sup[-\frac{1}{2},1] = 1$ and $\liminf_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = \inf[-\frac{1}{2},1] = -\frac{1}{2}$.

Answer: The set of limit points of f at $\mathbf{0}$ equals $[-\frac{1}{2},1]$. Also, $\limsup_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = 1$ and $\liminf_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) = -\frac{1}{2}$.

Comments: The above very short solution may appear to be "pulled out of a hat". However it can be found by some easy observations and

[4]Indeed note that $(1,-1+2t,t) \neq \mathbf{0}$ for all $t \in [0,1]$.

methods. The first key observation is that f is *homogeneous of degree zero*, i.e. $f(r\boldsymbol{x}) = f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and $r \in \mathbb{R} \setminus \{0\}$. (This is "obvious by inspection"; the detailed proof is as in (2) above, but with r in place of k^{-1}.) Therefore the range of f restricted to any punctured neighborhood of the origin equals the range of ("all of") f (and this is also equal to the range of f on any *sphere* with center at the origin). Hence the set of limit points of f at $\mathbf{0}$ is simply equal to *the closure of the range of f in $\mathbb{R} \cup \{+\infty, -\infty\}$*, and this is also equal to *the range of f restricted to any sphere with center at the origin.* This last set is automatically known to be compact, by Theorem 4.14, and connected, by Theorem 4.22 – if we take it as known that a sphere is connected. Hence the set of limit points of f at $\mathbf{0}$ (and also the range of all of f) equals a bounded closed interval!

It remains to find the end-points of that interval! Let us focus on the task of finding the left endpoint (the discussion for the right end-point is completely similar). This endpoint equals $-\alpha$, where α is the smallest real number which has the property that $f(x, y, z) \geq -\alpha$ for all $(x, y, z) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, i.e.

$$(3) \quad \alpha x^2 + \alpha y^2 + \alpha z^2 + xy + yz + xz \geq 0, \qquad \forall (x, y, z) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}.$$

Let us start by asking exactly *which* real numbers $\alpha > 0$ satisfy (3). Completing the square, the left hand side of (3) is seen to be equal to

$$\alpha\big(x + (2\alpha)^{-1}y + (2\alpha)^{-1}z\big)^2 + (\alpha - (4\alpha)^{-1})y^2 + (\alpha - (4\alpha)^{-1})z^2 + (1 - (2\alpha)^{-1})yz,$$

and by an easy inspection (using the fact that x only appears inside the first square in the last expression) it follows that (3) holds iff

$$(4) \qquad Ay^2 + Az^2 + Byz \geq 0, \qquad \forall (y, z) \in \mathbb{R}^2 \setminus \{\mathbf{0}\},$$

where $A = \alpha - (4\alpha)^{-1}$ and $B = 1 - (2\alpha)^{-1}$. Now for (4) to hold we must clearly have $A \geq 0$ (indeed take $(y, z) = (1, 0)$). This is equivalent with $\alpha \geq \frac{1}{2}$. Now one may simply note that if $\alpha = \frac{1}{2}$ then $A = B = 0$; hence (4) holds and thus also (3) holds! This of course implies that (3) also holds for all $\alpha > \frac{1}{2}$; and on the other hand the above argument shows that (3) *fails* for all $\alpha \in (0, \frac{1}{2})$. Hence: The smallest real number α for which (4) holds if $\alpha = \frac{1}{2}$; and thus the left endpoint of the interval which we are seeking to determine equals $-\frac{1}{2}$!