4. Differential calculus for vector valued functions

We solve the problems together in the exercise sessions. Note that these problems are optional and for learning purposes: solving these does not provide extra points. Actual home assignments (giving you extra points) are given separately.

It is advised to take a look of the problems beforehand. Note that some of the problems might be very challenging, so do not feel bad if you are unable to solve them independently: we will go through the solutions together!

Problems for the session

4.1 Determine the values of a such that $f(x,y) = (x+ay, 2x+3y)$ is injective.

4.2 Is the function $f(x,y) = (u,v) = (x^2-y^2, 2xy)$ bijective in a neighbourhood of $(x_0,y_0) = (1,1)$? Determine $\frac{\partial u}{\partial x}(1,1)$ and $\frac{\partial x}{\partial u}(0,2)$.

4.3 Show that the equation $x^y + \sin y = 1$ defines a function $y = f(x)$ in a neighbourhood of $(x_0,y_0) = (1,0)$, and determine $f'(x)$.

4.4 Show that the equation $y^3 - y = x$ defines a function $y = f(x)$ in a neighbourhood of $(x_0,y_0) = (0,0)$. With implicit derivation, determine the coefficients a_0, a_1, a_2 in the second order Taylor approximation

$$f(x) \approx a_0 + a_1 x + a_2 x^2.$$

Problems for individual practice

In addition to the problems below, one can get routine by solving similar exercises from the exercise-book ”övningar i flerdimensionell analys”.

4.1 Let $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ and $g : \mathbb{R}^m \mapsto \mathbb{R}^n$. Verify the chain rule for the Jacobians

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

4.2 Determine all the points (x_0,y_0) such that $x^2+y^2 = 1$ defines a function $y = f(x)$ in the neighbourhood of (x_0,y_0). What about points for which $x = g(y)$?

4.3 Show that for $a < 1$ the equation $x^3 - 3ax + 2 = 0$ has a solution $x(a)$. Find $x'(0)$.

4.4 Let $f(x, y) = (u, v) = (x^2 + y, xy^2 - x^3)$. Determine the Jacobian of f and find the linear approximation at point $(x_0, y_0) = (1, 2)$.

4.5 Prove that, in the neighbourhood of $(0, 0, 0)$, the equation $x^3 + y^3 + z^3 + x^2 z - yz - z = 0$ defines a function $z = f(x, y)$. Find f'_x and f'_y.