UPPSALA UNIVERSITET
Matematiska Institutionen
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Prov i matematik
1MA255, VT22
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Partial Differential Equations with Applications to Finance

Writing time: 08:00 - 13:00.
Instructions: There are 5 problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, 4 or 5, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no help materials are allowed.

GOOD LUCK!

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1. (8p) Let $u(t, x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t, x) : t > 0, x > 0\}$ with $u(0, x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t, 0) = 0$ for $t > 0$.

i) (3p) Construct a suitable extension of the initial condition to the whole space.

ii) (5p) Show that

$$u(t, x) = \int_0^\infty u_0(y) h(t, x, y) dy$$

for some function $h(t, x, y)$. Find h.

2. (8p) Let $D = (-a, b)$ where $a, b > 0$ and let X_t be a standard Brownian motion in 1D with $X_0 = 0$.

i) (4p) Define

$$u(x) := \mathbb{E}_x[X_\tau^p + \tau],$$

where $p > 0$ and $\tau = \inf\{t > 0 : X_t \notin D\}$. Write down a suitable boundary value problem that u solves.

ii) (4p) Solve for u and calculate

$$\mathbb{E}_0[X_\tau^p + \tau]$$

3. (8p) Consider Ito diffusion X:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t,$$

and define $u(t,x,y) := \mathbb{P}_x(X_t \leq y)$ as the probability that starting from $X_0 = x$, the value of the process at time t is smaller than y.

i) (6p) Use the Kolmogorov forward (Fokker-Planck) equation to derive a PDE satisfied by u in terms of the forward variables (t, y), and propose a suitable initial condition.

ii) (2p) Find a process X, such that

$$u(t,x,y) = 1 - u(t,y,x)$$

for all t, x, y.

4. (8p) In the Merton's asset allocation problem, consider one risky asset

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

and risk-free rate $r = 0$. Let X_t^u denote the wealth process where u_t is the **amount** of money invested in the risky asset at time t.

i) (2p) Write down dynamics of X_t^u.

ii) (6p) Solve the Merton's problem

$$v(t,x) = \sup_u \mathbb{E}_{t,x}[\Phi(X_T^u)]$$

where $\Phi(x) = 1 - e^{-\gamma x}$ for some $\gamma > 0$. You **do not** need to prove the verification theorem for your solution.
Hint: *Use the ansatz* $v(t,x) = 1 - f(t)e^{-\gamma x}$.

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_{0,x}[e^{-\beta\tau} X_\tau^+]$$

where $\beta > 0$, $x^+ = \max(x, 0)$, and X_t is a Brownian motion with drift μ:

$$X_t = x + \mu t + B_t,$$

where B_t is a standard Brownian motion. You **do not** need to prove the verification theorem for your solution.