UPPSALA UNIVERSITET
Matematiska institutionen — Financial Derivatives
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Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points. No means of assistance other than pencil and paper are allowed. Motivate your answers carefully!

1. Let $W(t)$ be a Brownian motion. Determine $\mathbb{E}[e^{-2W(t)}]$.

2. Solve the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + 2\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{\partial u}{\partial x}(t,x) = 0 \\ u(T,x) = x^3. \end{cases}$$

3. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time 0 of a contract which at time T pays the holder the amount $S(T)$ provided $S(T) \geq a$, and 0 otherwise.

4. Explain briefly the following notions:

(a) Asian option;
(b) implied volatility;
(c) delta of an option;
(d) martingale modeling of the short rate;
(e) forward price.

5. A *straddle* written on an underlying asset S pays its holder the amount the amount

$$\mathcal{X} = \begin{cases} 100 - S(T) & \text{if } S(T) \leq 100 \\ S(T) - 100 & \text{if } S(T) > 100 \end{cases}$$

at time T. The market price of the straddle is 10, the value of the underlying asset is 90, a call option with strike 100 and maturity T written on the same underlying asset trades at 4, and a zero-coupon with face value 50 trades at 49. Show how to construct an arbitrage.

6. Consider the standard Black-Scholes model and a simple T-claim $\mathcal{X} = \Phi(S(T))$. Let $F(t,s)$ be the solution of the Black-Scholes equation

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 s^2 F_{ss} + rsF_s = rF \\ F(T,s) = \Phi(s), \end{cases}$$

and let h be a portfolio which at time t consists of

$F_s(t, S(t))$ shares of the stock and the amount
$F(t, S(t)) - S(t)F_s(t, S(t))$ in the bank account.

(a) Show that the value of the portfolio h is $V^h(t) = F(t, S(t))$.
(b) Show that the portfolio is self-financing.
(c) What have we proved in (a) and (b)? Why is this important?

7. Consider a market consisting of a bank account with constant interest rate $r \geq 0$ and a stock S with constant volatility σ. The stock pays a continuous dividend yield δ, where $\delta > 0$ is a constant. Consider a T-claim that pays $\mathcal{X} = S(T)$ at time T.

a) What is the arbitrage-free price of $\mathcal{X}$ at time 0?
b) Find a replicating strategy for $\mathcal{X}$.

8. Consider the model

$$\begin{cases} dr(t) = \sigma(t)\, dW(t), \quad t \geq 0 \\ r(0) = r_0 \end{cases}$$

for the short rate under the pricing measure, where $\sigma(\cdot)$ is a function of time.

(a) Determine the term structure, i.e. calculate bond prices at time $t = 0$ in this model for all possible maturities T.
(b) Can any observed term structure $\{p^*(0,T); T > 0\}$ be fitted against this model?
(*Hint*: What if $p^*(0,T) < e^{-r_0 T}$ for some T?)

GOOD LUCK!