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Today

Chap. 5. Testing hypotheses (continued): Conditional tests

Conditional tests

Example 1:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be an i.i.d. sample from $N(\mu, \sigma^2)$ with *unknown* σ^2.
- Test H_0: $\mu \leq \mu_0$ vs H_1: $\mu > \mu_0$.
- Is there any test which in some sense is optimal in this situation?
- In particular, how should a test statistic for tests on μ with distribution not depending on σ^2 be found?

Example 2:

The same questions for the test of H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$.

Conditional tests

More general:

- Suppose we have the model $\mathcal{P} = \{\mathrm{P}_\theta : \theta \in \Theta \subseteq \mathcal{R}^k\}$.
- Suppose P_θ belongs to a k-parameter exponential family:

$$p(\mathbf{x};\theta) = A(\theta)\exp\left\{\sum_{j=1}^{k}\zeta_j(\theta)R_j(\mathbf{X})\right\}h(\mathbf{x}).$$

- Define $\beta_j = \zeta_j(\theta)$.
- Suppose the **parameter of interest** is $\lambda = \beta_1 = \zeta_1(\theta)$.
- $\vartheta = (\beta_2, ..., \beta_k)^{\mathrm{T}}$ is called the **nuisance parameter**.
- Write

$$p(\mathbf{x};\theta) = A(\theta)\exp\left\{\lambda U(\mathbf{x}) + \vartheta^{\mathrm{T}} T(\mathbf{x})\right\}h(\mathbf{x}).$$

Conditional tests

Example 1:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be a i.i.d. sample from $N(\mu, \sigma^2)$ with unknown σ^2.
- We want to test hypotheses about μ.
- The likelihood $p(\mathbf{x}; \theta) = L(\theta; \mathbf{x})$, where $\theta = (\mu, \sigma^2)$, may be written on a form as above (why?):

$$L(\theta) = A(\theta) \exp\left\{\lambda U(\mathbf{x}) + \vartheta T(\mathbf{x})\right\},$$

where $\lambda = \mu/\sigma^2$, $U(\mathbf{x}) = n\bar{x}$, $\vartheta = -1/(2\sigma^2)$, $T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$.

Conditional tests

- Recall: $p(\mathbf{x};\theta) = A(\theta)\exp\left\{\lambda U(\mathbf{x}) + \vartheta^{\mathrm{T}} T(\mathbf{x})\right\} h(\mathbf{x})$.
- Here, $U(\mathbf{x})$ is the suff. stat. for λ and $T(\mathbf{x})$ is the suff. stat. for ϑ.
- Rewrite $p(\mathbf{x};\theta)$ as the joint probability (density) function of (U, T):

$$\begin{aligned} p^{(U,T)}(u,t;\theta) &= A(\theta)\exp(\lambda u + \vartheta^{\mathrm{T}} t)h(u,t) \\ &= c(\lambda,t)\exp(\lambda u)h(u,t)\cdot A(\theta)\exp(\vartheta^{\mathrm{T}} t)c(\lambda,t)^{-1} \\ &= p^{U|T=t}(u|t;\lambda) \qquad\qquad \cdot\, p^{T}(t;\theta) \end{aligned}$$

- The trick is to concentrate on **the conditional model** $\mathcal{P}_t = \left\{\mathrm{P}_\lambda^{U|T=t} : \lambda \in A \subseteq \mathcal{R}\right\},\ t \in \mathcal{R}^{k-1}$.
- Test H_0: $\lambda \geq \lambda_0$ vs H_1: $\lambda < \lambda_0$.
- Let $\mathcal{Z}$ be the parameter space after transformation into (λ, ϑ).
- Write $\mathcal{Z} = \mathcal{Z}_0 \cup \mathcal{Z}_1$ where $\mathcal{Z}_0 = \{(\lambda,\vartheta) : \lambda \geq \lambda_0\}$ and $\mathcal{Z}_1 = \{(\lambda,\vartheta) : \lambda < \lambda_0\}$.
- Define the boundary set $\mathcal{Z}_{\mathrm{bound}} = \{(\lambda_0,\vartheta) : (\lambda_0,\vartheta) \in \mathcal{Z}\}$.

Conditional tests

Definition (5.13)

A test φ is said to be **α-similar** on $\mathcal{Z}_{\text{bound}}$ if.f. $\mathrm{E}_{(\lambda_0,\vartheta)}\varphi(\mathbf{X}) = \alpha$. (Independent on ϑ.)

Definition (5.14)

Consider the test problem H_0: $\lambda \geq \lambda_0$ vs H_1: $\lambda < \lambda_0$. A test φ is **uniformly most powerful α-similar** for this problem if.f. φ is α-similar on $\mathcal{Z}_{\text{bound}}$ and

$$\mathrm{E}_{(\lambda,\vartheta)}\{\varphi(\mathbf{X})\} \geq \mathrm{E}_{(\lambda,\vartheta)}\{\psi(\mathbf{X})\}$$

for all $(\lambda, \vartheta) \in \mathcal{Z}_1$ and for all α-similar tests ψ on $\mathcal{Z}_{\text{bound}}$.

Conditional tests

We get the following generalization of the Blackwell "UMP theorem" for α-similar tests:

Theorem (5.7)

(One-sided conditional test) *Consider the test problem* H_0: $\lambda \geq \lambda_0$ *vs* H_1: $\lambda < \lambda_0$. *Assume that* $\mathcal{Z}$ *is convex and includes a* k*-dimensional interval. The test*

$$\varphi_I(u,t) = \begin{cases} 1 & \text{if} \quad u < c_0(t), \\ \gamma_0(t) & \text{if} \quad u = c_0(t), \\ 0 & \text{if} \quad u > c_0(t), \end{cases}$$

with $\gamma_0(t)$ *and* $c_0(t)$ *such that* $\mathrm{E}_{\lambda_0}\{\varphi_I(U,T)|T=t\} = \alpha$ *for all* t*, is a UMP* α*-similar test for this problem.*

Note: For H_0: $\lambda \leq \lambda_0$, H_1: $\lambda > \lambda_0$ the same result holds by switching $<$ and $>$ in the φ_I formula.

Conditional tests

Example 1:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be an i.i.d. sample from $N(\mu, \sigma^2)$ with unknown σ^2.
- Consider testing H_0: $\mu \leq \mu_0$ vs H_1: $\mu > \mu_0$.
- Show that the one-sided t test is UMP α-similar.

Conditional tests

Definition

Consider the test problem H_0: $\lambda = \lambda_0$ vs H_1: $\lambda \neq \lambda_0$. A test φ is **uniformly most powerful unbiased α-similar** for this problem if.f. φ is α-similar unbiased on $\mathcal{Z}_{\text{bound}}$ and

$$\mathrm{E}_{(\lambda,\vartheta)}\{\varphi(\mathbf{X})\} \geq \mathrm{E}_{(\lambda,\vartheta)}\{\psi(\mathbf{X})\}$$

for all $(\lambda, \vartheta) \in \mathcal{Z}_1$ and for all α-similar unbiased tests ψ on $\mathcal{Z}_{\text{bound}}$.

Conditional tests

Theorem (5.8)

(Two-sided conditional test) *Assume that $\mathcal{Z}$ is convex and includes a k-dimensional interval. Consider the test problem*
H_0: $\lambda = \lambda_0$ vs H_1: $\lambda \neq \lambda_0$. The test

$$\varphi_{II}(u,t) = \begin{cases} 1 & \text{if} \quad u < c_1(t),\ u > c_2(t), \\ \gamma_1(t) & \text{if} \quad u = c_1(t), \\ \gamma_2(t) & \text{if} \quad u = c_2(t), \\ 0 & \text{if} \quad c_1(t) < u < c_2(t), \end{cases}$$

with $\gamma_i(t)$ and $c_i(t)$ such that $\mathrm{E}_{\lambda_0}\{\varphi_{II}(U,T)|T=t\} = \alpha$ for all t and

$$\mathrm{E}_{\lambda_0}\{U\varphi_{II}(U,T)|T=t\} = \alpha \mathrm{E}_{\lambda_0}(U|T=t)$$

is a UMPU α-similar test for this problem.

Conditional tests

Example 2:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be a i.i.d. sample from $N(\mu, \sigma^2)$ with unknown σ^2.
- Consider testing H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$.
- Show that the two-sided t test is UMPU α-similar.

News of today

Generalization to conditional tests:

- The test concerns a parameter of interest, in the presence of nuisance parameters.
- The exponential family.
- Trick: Condition on the nuisance parameters!
- α-similar test on the boundary set
- UMP α-similar test
    - one-sided
    - two-sided (unbiased test)