UPPSALA UNIVERSITET

Föreläsningsanteckningar

Tillämpad Matematik

Rami Abou Zahra

Inlämningsdatum
September 18, 2023

Contents

1. Introduction

The topic of this course will vary a lot, since mathematics can be applied to physics, biology, etc.
We will look into different ways to model real life, study it, and draw conclusions from it.

Anmärkning:
One could look at a mathematical models as a set of equations

Example: Planetary motion

- *Observation*: Keplers law $\rightarrow$ elliptic orbits
- *Model*: Newtons gravitational law
- *Mistakes/Errors*: Mercury precission $\rightarrow$ disalignment between model and observation
- *Rectify error*: Introducing relativistic effects in the model
- *Evaluation*: Is the old model useless? No, it is often easier to compute. It is better to keep it simple

We arrive at 2 models:

Good model$\rightarrow$ Simple, general (not valid in a specific way)

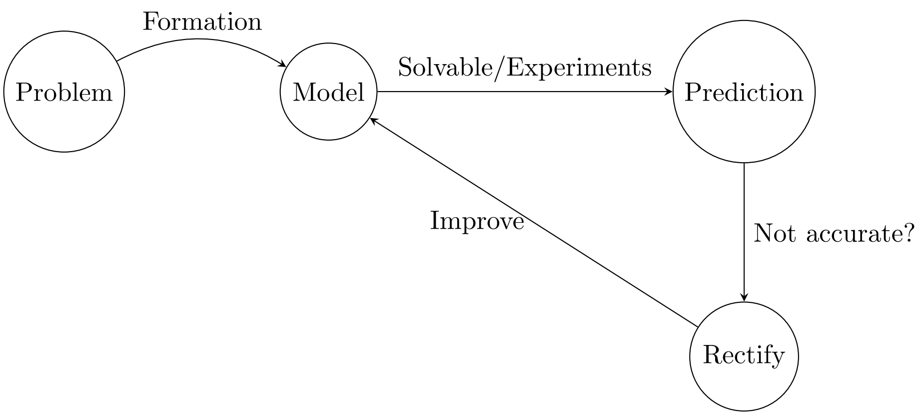

Figure 1.

First step in the definition of a model: Understand which variables are involved

Dimension	Unit	Derived	Dimension
Distance	m	v	m/s
Temperature	Degrees	a	m/s
Time	s		

Definition/Sats 1.1: Physical law

A physical law is $f(q_1, \cdots, 1_n) = 0$

$L_1, \cdots, L_m$ are the dimensions
$[q_i] = L_1 \cdots L_m$

- $[q] = 1$ dimensions
- $[v] = L \cdot T^{-1}$

Example: Conservation of energy is an example of such physical law:

$$\frac{mp^2}{2} + V(q) = C \qquad C \in \mathbb{R}$$

$$F(, m, p, q) = \frac{mp^2}{2} + V(q) - C = 0$$

Example: Hooks law for springs:

$$F = \underbrace{k}_{\text{Not dimensionless}} \cdot L \qquad f(F, k, L) = 0$$

Definition/Sats 1.2: Unit free

A law is *unit free* if it is independent from the unit, in the sense that if we define a new system in the following way:

$$\overline{L_i} = \lambda_i L_i$$

Then $\overline{L_i}$ is a new system of unit $\lambda_i > 0$

$$[q_i] = L_1^{b_1} \cdots L_n^{b_n}$$
$$f(q, \cdots, q_n) = 0 \Leftrightarrow f(\overline{1_1}, \cdots, \overline{q_m}) = 0$$

Example:

$$f(x, t, q) = x - \frac{1}{2} g t^2 = 0$$

Describing a body falling. If we define the following units:

- $[x] = m$
- $[g] = ms^{-2}$
- $[t] = s$

We can check that if we use different units, say $\overline{x} = 1000x$ (kilometers instead of meters) or $\bar{t} = 3600t$ (hours instead of seconds), then we obtain the same law for $f(\overline{x}, \bar{t}, g) = 0$

Example: Just looking at the dimension we can say something about the model. Take the pendulum and study the period of oscillation (is the mass or the length the one that defines the period?)

The goal is to find a law for the period. Suppose only the length and the mass are the only variables in our model, then we want to find $P = f(l, m)$
Notice that we have an error in the dimension, since our period depends on time, so just looking at that we can see that there is something that is missing.

We could be interested in adding another term, the gravitational acceleration. We get:

$$T = kL^{\alpha_1} M^{\alpha_2} \frac{L^{\alpha_3}}{T^{-2\alpha_3}}$$

$$\begin{cases} \alpha_2 = 0 \quad \rightarrow \text{mass is not involved} \\ \alpha_1 + \alpha_3 = 0 \\ -2\alpha_3 = 01 \qquad \alpha_3 = \dfrac{-1}{2} \qquad \alpha_1 = \dfrac{1}{2} \end{cases}$$

$$\Rightarrow P \approx k\sqrt{\frac{e}{g}}$$

Another thing we may do is to introduce dimensionless variables:

Definition/Sats 1.3: Pi:s theorem

Let $f(q_1, \cdots, q_m) = 0$ be a unit free law with the usual notation for dimension $[q_i] = L_1^{\alpha_{1i}} \cdots L_n^{alpha_{ni}}$
$n < m$

Define the dimension matrix A

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix}$$

Let π be the rank(A). Then there exists $m-r$ dimensionsless variabless $\Pi_1, \cdots, \Pi_{m-r}$ (which can be formed from q_i)

We hve an equivalent law $F(\Pi_1, \cdots, \Pi_{m-r}) = 0$

Anmärkning:
When we have a law, it does not mean that we have the right law (only $q_1, \cdots, q_n$ are involved) but it is not meaningless

The usefulness of Pi-theorem:

- Case in which only one dimensionsless variable is involved
  $F(\Pi_1) = 0 \rightarrow$ zeroes are discrete
  Π_1 can assume discrete values and can be deduced from experiments

In the case of 2 dimensionsless quantities $F(\Pi_1, \Pi_2) = 0$, if we can invert the relationship then we can write one variable as a function of the other using implicit function theorem.

$$\Pi_1 = f(\Pi_2) \qquad f \text{ is unknown} \rightarrow \text{deduced from observation}$$

Example: Allometry (Biology), the study of characteristics of living creatures change with their size. We look for a law that involves

- $q_1 = l =$ length of the organism $\quad [q_1] = L$
- $q_2 = t =$ time $\quad [q_2] = T$
- $q_3 = \rho =$ density $\quad [q_3] = \dfrac{M}{L^3}$
- $q_4 = a =$ resource assimilation rate $\quad [q_4] = \dfrac{M}{L^2 T}$
- $q_5 = b =$ resource utilisation rate $\quad [q_5] = \dfrac{M}{L^3 T}$

We look for a law that involves 2 dimensionsless variables, so we apply the theorem:

$$A = \begin{pmatrix} 1 & 0 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} L \\ M \\ T \end{pmatrix}$$

(Look at the exponent of the respective variable)
The rank$(A) = 3 \rightarrow 5 - 3 = 2$ dimensionsless variables
We can try to express q_i as a linear combination of the others. We know the following:

$$\begin{cases} \alpha_1 - 3\alpha_3 = -2 \\ \alpha_3 = 1 & \Rightarrow \alpha_1 = 1 \\ \alpha_2 = -1 \end{cases}$$

This means that q_4 can be expressed as $q_4 = \dfrac{q_1 q_3}{q_4}$, yielding:

$$\Pi_1 = \frac{q_1 q_3}{q_2 q_4} = \frac{l\rho}{ta} \rightarrow \text{dimensionless}$$

We can do the same for $q_5 \Rightarrow q_5 = \dfrac{q_3}{q_2}$ yielding another dimensionsless variable:

$$\Pi_2 = \frac{q_3}{q_2 q_5} = \frac{\rho}{tb}$$

Summa sumarum:

$$F(\Pi_1, \Pi_2) = 0 = F\left(\frac{l\rho}{ta}, \frac{\rho}{tb}\right)$$
$$\pi_1 = f(\Pi_2)$$

1.1. **Scaling.**

The goal is to rescale variables to a quantity that is related to that specific problem. Measuring seconds when it comes to glaciers might be less useful as measuring with years, and seconds for a chemical reaction might be too little.

For example, with time, $\bar{t} = \dfrac{t}{t_c}$. New rescaled time is 1 once it has passed the desired scale. c stands for characteristic
The same can be done for other quantities such as length $\overline{h} = \dfrac{h}{h_c}$

Example: Projectile problem where we only consider gravity. Using Newtons gravitational law:

$$\frac{md^2h}{dt^2} = -G \cdot \frac{mM}{(R+h)^2} \Rightarrow \frac{d^2h}{dt^2} = -G\frac{M}{(R+h)^2}$$

We know that for $h = 0$, $\dfrac{d^2h}{dt^2} = -g = \dfrac{-GM}{R^2} = \dfrac{-gR^2}{(h+R)^2}$

We also know $h(0) = 0$, $\dfrac{dh}{dt}(0) = v$ (initial velocity)

We can introduce dimensionsless variables:

- $[t] = T$
- $[h] = L$
- $[R] = L$
- $[v] = LT^{-1}$
- $[g] = LT^{-2}$

Since only L, T are involved, we have 2 rows:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\text{rank}(A) = 2 \Rightarrow 3$ dimensionless variables
We could for example do

$$\Pi_1 = \frac{h}{R} \qquad \Pi_2 = \frac{h}{vt} \qquad \Pi_3 = \frac{h}{gt^2}$$

Let us see what happens if we do some scaling for the time $\bar{t}$ and the length $\bar{h}$:

$$\bar{t} = \frac{t}{t_c} \qquad \bar{h} = \frac{h}{h_c}$$

With a dimension of time, we could pick $\dfrac{R}{v}$, or $\sqrt{\dfrac{R}{g}}$, $\dfrac{v}{g}$

The same for h, we could pick R, $\dfrac{v^2}{g}$

Usually only one choice is the one that helps us solve the problem:

$$\bar{t} = \frac{t}{R/v} \qquad \bar{h} = \frac{h}{R}$$

Now we need to express the laws that we have in terms of $\bar{t}$ and $\bar{h}$:

$$\frac{d^2h}{dt^2} = \frac{-gR^2}{(R\bar{h}+R)^2} = \frac{-g}{(\bar{h}+1)^2} \qquad h = \bar{h}R$$

$$\frac{dh}{dt} = \frac{d\bar{h}}{dt}R \qquad \frac{d\bar{h}}{d\bar{t}} = \frac{d\bar{h}}{dt}\frac{dt}{d\bar{t}} = \frac{R}{v}\frac{d\bar{h}}{dt}$$

$$\frac{d^2\bar{h}}{d\bar{t}^2} = \frac{d^2\bar{h}}{dt^2}\frac{R^2}{v^2} \rightarrow \frac{v^2}{Rg}\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}$$

We can call $\varepsilon = \dfrac{v^2}{Rg}$ (ε small)

The equation $\varepsilon\dfrac{d^2\bar{h}}{d\bar{t}^2} = -\dfrac{1}{(1+\bar{h})^2}$ has no solution when $\varepsilon = 0$

With a different choice

$$\bar{t} = \frac{t}{vg^{-1}} \qquad \bar{h} = \frac{h}{v^2g^{-1}}$$

$$\Rightarrow \frac{d^2\bar{h}}{dt^2} = -\frac{1}{(1+\varepsilon\bar{h})^2} \qquad \bar{h}(0) = 0 \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1$$

Notice now that when $\varepsilon = 0$:

$$\overline{h}'' = -1 \qquad \overline{h}' = -\overline{t} + a = -\overline{t} + t$$

$$\overline{h} = -\frac{t^2}{2} + \overline{t} + b = -\frac{\overline{t}^2}{2} + \overline{t}$$

By substituting the old variables back, we get:

$$h = \frac{-t^2}{2} g + vt$$

The quantities that we used for t_c, h_c:

$$t_c = \frac{v}{g} \qquad h_c = \frac{v^2}{g}$$

$$h' = 0 \rightarrow -tg + v = 0 \Rightarrow t = \frac{v}{g}$$

Then h_c is the maximum height that the body reaches.

2. Pertubation Theory

This applies to another class of problems that is known (in a sense that we know how to solve it, we can find the solution); and we consider a new problem that is made of a known problem + *a pertubation*:

$$\text{Problem} + \varepsilon \text{ Problem}_2 \qquad \varepsilon << 1$$

Example: Planetary motion
If we consider a 2-body problem (one planet & one star), this can be solved exactly
Consider now a 3-body problem, then this problem cannot be solved easily.

$$M_{\text{sun}} >> M_{p_1} M_{p_2}$$

$$\underbrace{F_{Sp_1} + F_{Sp_2}}_{\text{2-body}} + \overbrace{F_{p_2p_1}}^{F_{p_2p_1} << F_{Sp_{1,2}}}$$

Anmärkning: Here F_{Sp_i} denotes the gravitational pull from the sun to one planet.

In general, we apply pertubation theory to equation of the form $F(y, y', y'', \cdots, \varepsilon) = 0$.
We look for a solution $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 + \cdots$
We expect y_0 (leading term) to be the solution/approximation when $\varepsilon = 0$

We may ask ourselves if the approximation converges to the solution.

Example:

$$\hat{y} = -y + \varepsilon y^2 \qquad y(0) = 1$$

We try to do the easiest thing and plug in the approximation:

$$(\hat{y}_0 + \varepsilon\hat{y}_1 + \varepsilon^2\hat{y}_2 + \cdots) = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + \varepsilon(y_0 + y_1\varepsilon + \cdots)^2$$

Now we try to solve order by order (collect like terms and see which one equates):

$$\hat{y}_0 = -y_0$$

$$y_0 = Ae^{-t} \qquad y_0 = e^{-t} \qquad \text{since our initial condition}$$

$$\hat{y}_1 = -y_1 + y_0^2 = -y_1 + e^{-2t}$$

$$\hat{y}_1(0) = 0 \qquad \text{since we already used our initial condition}$$

$$y_1(t) = e^{-t} + Ae^{-2t} \Rightarrow y_1(t) = e^{-t} - e^{-2t}$$

$$\hat{y}_2 = -y_2 + 2y_0y_1 = -y_2 + 2e^{-t}(e^{-t} - e^{-2t})$$

$$\hat{y}_2(0) \Rightarrow y_2(t) = e^{-t} - 2e^{-2t} + e^{-3t}$$

We have found the first three terms. This problem can be solved exactly, and we can see if our construction solves the equation or not:

$$y(t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \cdots$$

This is a case where regular pertubation works really well, since the explicit solution is given by:

$$y(t) = \frac{e^{-t}}{1 - \varepsilon + \varepsilon e^{-t}} = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)}$$

$$\sum_n x^n = \frac{1}{1-x} \Rightarrow e^{-t}(1 - \varepsilon(e^{-t} - 1) + \varepsilon^2(e^{-t} - 1)^2 + \cdots)$$

This is not always the case, that it is the same solution. If we use the example from the last lecture (projectile problem), we get something different:

$$h'' = \frac{-1}{(1 + \varepsilon h)^2}$$

$$\varepsilon = 0 \Rightarrow h_0 = \frac{-t^2}{2} + t$$

We try the same technique, suppose $h = h_0 + \varepsilon h_1$:

$$(h_0' + h_1'\varepsilon)(1 + \varepsilon(h_0 + \varepsilon h_1))^2 = -1$$

We collect like terms:

$$h_1'' + 2h_0''h_0 = 0$$

$$h_1 = -(-1)\left(\frac{t^4}{4} + t^2 - t^3\right)$$

$$h_1' = \frac{t^5}{20} + \frac{t^3}{3} - \frac{t^4}{4} + C \qquad h_1'(0) = 0 \qquad h_1(0) = 0$$

$$h_1 = \frac{t^6}{100} + \frac{t^4}{12} - \frac{t^5}{20}$$

We have a polynomial in t which is greater than the one in h_0, and an exponential in the other.

Well, in the terms $\varepsilon(e^{-t} + \cdots)$, the size is dominated by ε even when t grows, while in the polynomial no matter how small ε we choose the polynomial can always grow bigger. So $h_0 + \varepsilon h_1$, h_1 term grows too much. This does not mean that the term is wrong, but it may not have a meaning in the problem that we are considering.

h_1 is growing faster than h_0 even though there is an ε in front of it. Recall that h_1 is just a correction, because we are adding a term that is bigger than our first approximation. We are essentially not writing a function that is adding smaller and smaller terms.

This method is called *regular pertubation*, and sometimes it works and sometimes it does not. In the case when it does not, we have to try a different technique.
We consider a different problem:

Example: Duffin Equation

$$\hat{\hat{u}} + u + \varepsilon u^3 = 0 \qquad t > 0$$

$$u(0) = 1 \qquad \hat{u}(0) = 0$$

In this case, we do not have an explicit formula for the solution. We can try to use regular pertubation and see if it has a meaning or not.

$$u = u_0 + \varepsilon u_1$$

$$\Rightarrow \hat{\hat{u}} + u_0 = 0 \qquad u_0(0) = 1 \qquad \hat{u}_0(0) = 0$$

$$u_0(t) = A\cos(t) + B\sin(t)$$

$$u_0(0) = 1 \Rightarrow A = 1$$

$$\hat{u}_0(0) = -\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$$

$$u_0 = \cos(t)$$

Notice that for u_0, we have an oscillatory solution (since the trig-functions are periodic).

We collect like terms and equate them:

$$\hat{\hat{u}}_1 + u_1 + u_0^3 = 0 \qquad u_1(0) = 0 \qquad \hat{u}_1(0) = 0$$

$$\hat{\hat{u}}_1 = -u_1 - \cos^3(t)$$

$$\cos^3(t) = \left(\frac{e^{i\pi} + e^{-\pi}}{2}\right)^3 = \frac{e^{3i\pi} + e^{-3i\pi} + 3d^{2i\pi - i\pi} + 3e^{i\pi - 2i\pi}}{8}$$

$$= \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$$

$$\Rightarrow \hat{\hat{u}}_1 + u_1 = \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$$

$$u_1(0) = 0 \qquad \hat{u}_1(0) = 0 \Rightarrow u_1(t) = A\cos(t) + B\sin(t) + C\cos(3t) + At\sin(t) + Bt\cos(t)$$

Notice that since one particular solution already included $\cos(t)$, we add another set of $A\sin(t) + B\cos(t)$, but multiplied with t.
With respect to the initial conditions, we get:

$$u_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \underbrace{\frac{3}{8}t\sin(t)}_{\text{Secular term}}$$

The secular term might be a problem, for exmaple in this case we are interested in an oscillatory solution, so we expect a correction that gives us oscillatory approximation. But the t term makes the correction explode when $t \to \infty$.
There is also another issue with this approximation. We can show that the solution to this equation is bounded, but with this solution it breaks when $t \to \infty$, so $u_0 + \varepsilon u_1$ is not good.

Bevis 2.1: Exact solution is bounded

Consider $\hat{\hat{u}} + u + \varepsilon u^3 = 0$, and multiply with $\hat{u}$:

$$\hat{u}\hat{\hat{u}} + \hat{u}u + \varepsilon \hat{u}u^3 = 0$$

$$= \frac{d}{dt}\left(\frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4}\right) = 0 \Rightarrow \frac{\hat{u}^2}{2} + \frac{u^2}{2} + \frac{\varepsilon u^4}{4} = \text{Constant} = \frac{1}{2} + \frac{\varepsilon}{4}$$

$$\Rightarrow u \text{ is bounded}$$

□

Sometimes having small errors may not seem like a big issue, but these small errors may explode further down as $t \to \infty$

2.1. **Poincare-Lindstedt Method.**

The idea is to do a rescaling, considering a pertubative correction of the frequencies of the oscillation.

We introduce a new variable (distorted time scale) $\tau = \omega t$ where $\omega = \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots$.
As in the other case, ω_0 is the leading term when $\varepsilon = 0$, which in the previous example is 1.

We have to rewrite the equation according to the new time:

$$\frac{du}{dt} = \frac{du}{dt}\frac{dt}{d\tau} = \hat{u}\frac{1}{\omega}$$

$$\Rightarrow \hat{u} = \omega u' \qquad u' = \frac{du}{d\tau}$$

In the Duffin equation with the new variables we get:

$$\omega^2 u'' + u + \varepsilon u^3 = 0 \qquad u(0) \overset{\tau=0}{=} 1 \qquad \frac{du}{d\tau}(0) = \frac{\hat{u}(0)}{\omega} = 0$$

We now study this equation when $\tau > 0$. Essentially what we do, the advantage in this expansion, is that we can kill the terms that in regular pertubation generates the problem terms (secular terms).
What we will do is choose ω_1 such that it kills the secular terms. The procedure is the same as in regular pertubation:

$$(\omega_0 + \omega_1\varepsilon)^2(u_0'' + \varepsilon u_1'') + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 = 0$$

Gather terms and equate:

$$w_0^2 + u_0'' + u_0 = 0 \qquad u_0(0) = 1 \qquad u_0'(0) = 0$$

$$u_0 = \cos(\tau) \qquad w_0 = 1$$

$$(1 + 2\omega_1\varepsilon + \omega_1^2\varepsilon^2)(u_0'' + \varepsilon u_1'') + u_0 + \varepsilon u_1 + \varepsilon(u_0 + u_1\varepsilon)^3 = 0$$

$$\Rightarrow 2\omega_1 u_0'' + u_1'' + u_1 + u_0^3 = 0 \Rightarrow u_1'' + u_1 = -u_0^3 - 2\omega_1 u_0'' = -\cos^3(t) - 2\omega_1 - \cos(\tau)$$

$$\Rightarrow \frac{1}{4}\cos(3\tau) + \frac{3}{4}\cos(\tau) + 2\omega_1\cos(\tau)$$

$$\Rightarrow \frac{1}{4}\cos(3\tau) + \left(2\omega_1 - \frac{3}{4}\right)\cos(\tau)$$

Notice that the last $\cos(\tau)$ is the generator of our secular term, so we choose ω_1 so that $2\omega_1 - \frac{3}{4} = 0 \Rightarrow \omega_1 = \frac{3}{8}$

What we now get is an approximate term u_1 that does not have a secular term, but just sine and cosine (preserving oscillation). We can also use some of the previous calculations, but without the secular term since we have removed it.

$$u_1(\tau) = \frac{1}{32}\left(\cos(3\tau) - \cos(\tau)\right)$$
$$u = \cos(\tau) + \frac{\varepsilon}{32}\left(\cos(3\tau) - \cos(\tau)\right)$$
$$\tau = \omega t = (1 + \frac{3}{8}\varepsilon)t$$

We have so far covered *singular pertubation*. Let us look at an example when this method fails:

Example: $\varepsilon x^5 + x - 1 = 0 \qquad 0 < \varepsilon << 1$:

$$\varepsilon = 0 \qquad x = 1$$
$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$
$$x_0 = 1$$

From this equation we expect 5 different solutions since it is a polynomial of order 5.
The issue we have here is that the ε is in front of the term of highest order.

2.2. **Dominant Balancing method.**

Another example could therefore be $\varepsilon y'' + y' + y = 0$.

The strategy is to look for a scaling such that the leading term can remain big even though we multiply by ε. We can do this by defining a new variable $x = \frac{y}{f(\varepsilon)}$ that allows us to describe problem when the term that we are removing is not small anymore.

We solve using *dominant balancing.*
Case: Say $\varepsilon x^5 = O(X)$, then $\varepsilon x^4 = (1)$ and $x = O\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)$
We then get:

$$\varepsilon x^5 = \varepsilon O\left(\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)^4\right) = O\left(\varepsilon^{1-\frac{5}{4}}\right) = O\left(\varepsilon^{-\frac{1}{4}}\right)$$

With this choice, are these terms bigger than the remaining one? In this case the remaining one is 1, so it is true that when $\varepsilon << 1$ then $\frac{1}{\sqrt[4]{\varepsilon}} >> 1$

Case: Say $\varepsilon x^5 = (1)$, then $x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right)$. So for the second term (that is x), we get

$$x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right)$$

We see that this is not a good choice because the remaining term $x = O\left(\frac{1}{\sqrt[5]{\varepsilon}}\right) >> 1$ when $\varepsilon \to 0$.
This is a problem because we want the leading term to be the bigger one.

It is this reasoning that we use when we determine $f(\varepsilon)$.

Since we saw that the choice $x = \frac{y}{\sqrt[4]{\varepsilon}} \Rightarrow f(\varepsilon) = \varepsilon^{-\frac{1}{4}}$, we substitute this variable in the equation and try to solve for y:

$$\frac{\varepsilon y^5}{(\sqrt[4]{\varepsilon})^5} + \frac{y}{\sqrt[4]{\varepsilon}} - 1 = 0$$
$$y^5 \varepsilon^{1-\frac{5}{4}} + y\varepsilon^{-\frac{1}{4}} - 1 = 0$$
$$\Rightarrow y^5 + y - \varepsilon^{\frac{1}{4}} = 0$$

Now ε is not in the term of the highest order and we can solve using regular pertubation.
We solve the equation for $\varepsilon = 0$:

$$y^5 + y = 0 \Leftrightarrow y(y^4 + 1) = 0$$

One solution is $y = 0$, but this is a false root because of our initial conditions (**CHECK**). The other 4 solutions are given by the roots of unity for $y^4 + 1 = 0$:

$$y^4 = -1 = e^{i\pi}$$
$$y_{1,2,3,4} = e^{\frac{i\pi + 2n\pi}{4}} \qquad n = 0, 1, 2, 3$$
$$\Rightarrow y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \qquad \text{leading term } y_0 \text{ are the roots } y_n$$

Going back to our original variable $x = \frac{y}{\sqrt[4]{\varepsilon}}$ has leading order terms $\frac{y_n}{\sqrt[4]{\varepsilon}}$.

This is for the leading order, you can of course compute for the other terms.

Examples:

- $\varepsilon x^2 + 2x + 1 = 0$
- $\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$

Try to understand which are the leading orders of the approximated solutions.
(**DO THESE**)

Example:

- $\varepsilon y'' + (1 + \varepsilon)y' + y = 0$
- $y(0) = 0$
- $y(1) = 1$

Here the problem is a bit different, due to the initial conditions, when $\varepsilon = 0$ the equation cannot be solved since

$$\varepsilon = 0 \Rightarrow y' + y = 0 \qquad y' = -y \qquad y(x) = Ae^{-x}$$
$$y(0) = A = 0 \Leftarrow \text{ not a solution}$$

We can compute the solution exactly, this will tell us what the problem is when approaching this in pertubative approach:

$$\varepsilon m^2 + (1+\varepsilon)m + 1 = 0$$

$$m_{1,2} = \frac{-(1+e) \pm \sqrt{(1+\varepsilon)^2 - 4\varepsilon}}{2\varepsilon}$$

$$m_1 = -1 \qquad m_2 = -\frac{1}{\varepsilon}$$

$$\Rightarrow y(x) = C_1 e^{-x} + C_2 e^{-\dfrac{x}{\varepsilon}}$$

$$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

$$y(1) = C_2 \left(e^{-\dfrac{1}{\varepsilon}} - e^{-1} \right) = 1 \Leftrightarrow C_2 = \frac{1}{\left(e^{-\dfrac{1}{\varepsilon}} - e^{-1} \right)}$$

$$y(x) = \frac{1}{\left(e^{-(1/\varepsilon)} - e^{-1}\right)} \cdot \left(e^{-(x/\varepsilon)} - e^{-x} \right)$$

The problem is the size of the term $\varepsilon y''$, even if ε is very small the term can be very big.

Using the exact solution we can compute the second derivative:

$$y'' = \frac{1}{e^{-(1/\varepsilon)} - e^{-1}} \left(e^{-x} - \frac{1}{\varepsilon^2} e^{-(x/\varepsilon)} \right)$$

If we look at $y''(x)$ when $x \to 0$:

$$\frac{1}{e^{-(1/\varepsilon)} - e^{-1}} \left(C - \frac{1}{\varepsilon^2} D \right)$$

So $\varepsilon y''(x) = O\left(\frac{1}{\varepsilon}\right)$ which is not small when $\varepsilon = 0$

But when $x = O(1) \qquad x >> 1$:

$$y''(x) = 0$$

So in fact, $\varepsilon y''$ is small when x grows.

We can therefore say that our solution $y = Ae^{-x}$ is valid as long as x is not close to 0 since we have $\varepsilon y'' = O(\varepsilon)$ and we can use regular pertubation.

Using $y(1) = 1$ (and not 0, since we want x to be as far away from 0 as we can) we have:

$$y_{\text{outer}}(x) = e \cdot e^{-x} = e^{1-x}$$

Where y_{outer} denotes the outer domain ($x = O(1)$). This is only the leading order of the approximation (we are keeping the solution when $\varepsilon = 0$).

Just as we did with the polynomial, in order to fully solve this we introduce a rescaling to our problem to consider when $\varepsilon y''$ is small:

$$\tau = \frac{x}{f(\varepsilon)}$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\tau}\frac{d\tau}{dx} = \hat{y}\frac{1}{f(\varepsilon)}$$

Inserting this in the equation gives us:

$$\varepsilon \frac{\hat{\hat{y}}}{f(\varepsilon)^2} + (1+\varepsilon)\frac{\hat{y}}{f(\varepsilon)} + y = 0$$

Now we compare cases:

Case 1:

$$\frac{\varepsilon}{f(\varepsilon)^2} = O\left(\frac{1+\varepsilon}{f(\varepsilon)}\right) \Rightarrow f(\varepsilon) = O\left(\frac{\varepsilon}{1+\varepsilon}\right) = O(\varepsilon)$$

$$\begin{cases} \dfrac{\varepsilon\hat{\hat{y}}}{\varepsilon^2} = \dfrac{\hat{\hat{y}}}{\varepsilon} \\ \dfrac{1+\varepsilon}{\varepsilon}\hat{y} \to O\left(\dfrac{1}{\varepsilon}\right) \\ y \text{ is } O(1) \end{cases}$$

$$\frac{1}{\varepsilon} >> 1$$

We can check that choosing:

$$\frac{\varepsilon}{f(\varepsilon)^2} = O(1) \qquad f(\varepsilon) = O(\sqrt{\varepsilon})$$

$$\begin{cases} \dfrac{\varepsilon}{f(\varepsilon)^2}\hat{\hat{y}} = O(1) \\ \dfrac{1+\varepsilon}{\sqrt{\varepsilon}} = O\left(\dfrac{1}{\sqrt{\varepsilon}}\right) \end{cases}$$

In this case we see that the term in front of $\hat{y}$ is bigger than the one in front of $\hat{\hat{y}}$

The equation with the rescaling $\tau = \dfrac{x}{\varepsilon}$ becomes:

$$\frac{\varepsilon}{\varepsilon^2}\hat{\hat{y}} + \frac{1+\varepsilon}{\varepsilon}\hat{y} + y = 0$$

$$\hat{\hat{y}} + (1+\varepsilon)\hat{y} + \varepsilon y = 0$$

We solve using regular pertubation:

$$\varepsilon = 0 \qquad \hat{\hat{y}} + \hat{y} = 0 \qquad m^2 + m = 0 \Leftrightarrow m(m+1) = 0$$

$$y_{\text{inner}}(\tau) = A + Ae^{-\tau} = A(1 + e^{-\tau})$$

We have a set of equations to describe our solutions, one given by our outer approximation and one for our inner:

$$\begin{cases} y_{\text{outer}}(x) = e^{1-x} \\ y_{\text{inner}} = A(1 + e^{-(x/\varepsilon)}) \qquad x = O(\varepsilon) \end{cases}$$

We still have the parameter A. We can merge the two solutions in the part of the domain that is hared, for exmaple where $x \approx O(\sqrt{\varepsilon}) \qquad \varepsilon < \sqrt{\varepsilon} < 1$
We impose that when $x \cong O(\sqrt{\varepsilon})$, the limits of the two functions is the same.

We can define an auxillary variable $\eta = \dfrac{x}{\sqrt{\varepsilon}}$:

$$\lim_{\varepsilon\to 0^+} y_{\text{outer}}(\eta\sqrt{\varepsilon}) = \lim_{\varepsilon\to 0^+} y_{\text{inner}}(\eta\sqrt{\varepsilon})$$

$$\lim_{\varepsilon\to 0^+} e^{1-\eta\sqrt{\varepsilon}} = e$$

$$\lim_{\varepsilon\to 0^+} A\left(1 - e^{-(\eta/\sqrt{\varepsilon})}\right) = A \Rightarrow A = e$$

You can also write this as one singular function:

$$y = y_{\text{outer}} + y_{\text{inner}} - \underbrace{\text{common limit}}_{e}$$

$$\Rightarrow e^{1-x} + e^{1-(x/\varepsilon)}$$

2.3. **WKB Methods (Wentzel, Kramers, Brillouin).**

This new method (still falling under pertubation theory) applies to a general class of problems express mainly through differential equations on the following forms (for $\varepsilon << 1$):

$$\varepsilon^2 y'' + q(x)y = 0$$
$$y'' + q(\varepsilon x)^2 y = 0$$
$$y'' + (\lambda^2 p(x) - q(x))y = 0 \qquad \lambda >> 1$$

These equations come up in problems related to quantum mechanics, where instead of Newtons law there are Schrödinger equations that involves a function $\psi(x,t)$ and satisfies:

$$i\hat{h}\psi_t = \frac{-\hat{h}^2}{2m}\psi_{xx} + V(x)\psi$$

Here ψ is the wave function and the equation tells us the probability that a particle is in a certain place. For example:

$$P(a < x \leq b) = \int_a^b |\psi|^2 \, dx$$

Let us look at $\phi(x,t)$ more deeply, specifically, let us assume it is separable.
Then we get $\phi(x,t) = y(x)\varphi(t)$. With this, we can rewrite the equation differently:

$$i\hat{h}y(x)\varphi'(t) = -\frac{\hat{h}^2}{2m}y''(x)\varphi(t) + V(x)y(x)\varphi(t)\frac{i\hat{h}\varphi'(t)}{\varphi(t)} = \frac{-\frac{\hat{h}^2}{2m}y''(x)\varphi(t)}{y(x)} + V(x)$$

$$\left.\begin{aligned} &\frac{i\hat{h}\varphi'}{\varphi} = E \\ &-\frac{\hat{h}^2 y''}{y} + V(x) = E \end{aligned}\right\} \Rightarrow \varphi' = \frac{E}{i\hat{h}}\varphi = -i\frac{E\varphi}{\hat{h}} \qquad \varphi(t) = Ae^{-\dfrac{iEt}{\hat{h}}}$$

We then arrive at:

$$-\frac{\hat{h}^2}{2m}y'' + (V(x) - E)y = 0$$

Anmärkning:
$\hat{h} = \dfrac{h}{2\pi}$ where h is Plancks constant. We have arrived at our first equation since if we let $\varepsilon = \dfrac{\hat{h}}{\sqrt{2m}}$ and $q(x) = E - V(x)$ it is on the form $\varepsilon^2 y'' + q(x)y = 0$

In order to solve the first equation we make the ansatz

$$y = e^{\dfrac{u(x)}{\varepsilon}}$$
$$y' = e^{\dfrac{u(x)}{\varepsilon}} \cdot \frac{1}{\varepsilon}u'$$
$$y'' = e^{\dfrac{u(x)}{\varepsilon}}\frac{1}{\varepsilon^2}(u')^2 + e^{\dfrac{u(x)}{\varepsilon}} \cdot \frac{1}{\varepsilon}u''$$
$$\Rightarrow e^{u/\varepsilon}(u')^2 + \varepsilon e^{u/\varepsilon}u'' + q(x)e^{u/\varepsilon} = 0$$
$$\Rightarrow (u')^2 + u'' + q(x) = 0$$
$$\text{Let } f = u' \Rightarrow f^2 + \varepsilon f' + q(x) = 0$$

We can now solve with regular pertubation:

$$f = f_0 + \varepsilon f_1$$
$$(f_0 + \varepsilon f_1)^2 + \varepsilon(f_0' + f_1'\varepsilon) + q(x) = 0$$

Collect terms of the same order and compare:

$$f_0^2 + q(x) = 0 \Rightarrow f_0 = \pm\sqrt{-q(x)} \qquad q(x) < 0$$

$$2f_0 f_1 + f_0' = 0 \Rightarrow f_1 = -\frac{f_0'}{2f_0} = \frac{q'}{2 \cdot 2 \cdot \sqrt{-q(x)} \cdot \sqrt{-q(x)}} = -\frac{q'(x)}{4q(x)}$$

We arrive at:

$$f = \pm\sqrt{-q(x)} - \varepsilon\frac{q'}{4q(x)}$$

Since $f = u'$, we integrate:

$$u = \int_a^x \pm\sqrt{-q(s)}ds \underbrace{-\frac{\varepsilon}{4}\int_a^x \frac{q'(s)}{q(s)}ds}_{-\frac{\varepsilon}{4}\ln(|q(x)|)} + O(\varepsilon^2)$$

Summa sumarum, we have:

$$\begin{aligned} y = e^{\dfrac{u(x)}{\varepsilon}} &= e^{\dfrac{\pm 1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds} e^{-\dfrac{\varepsilon}{4\varepsilon}\ln(-q(x))} e^{\dfrac{O(\varepsilon^2)}{\varepsilon}} \\ &= \frac{1}{\sqrt[4]{-q(x)}} e^{\pm\dfrac{1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds} e^{O(s)} \\ &= \underbrace{\frac{1}{\sqrt[4]{-q(x)}} e^{\pm\dfrac{1}{\varepsilon}\int_a^x \sqrt{-q(s)}ds}}_{\text{Our approximation}} (1 + O(\varepsilon)) \end{aligned}$$

Anmärkning:
Here we have used the assumption that $q(x) < 0$. We can of course do this when $q(x) > 0$, but we change the ansatz to $y = e^{\dfrac{iu(x)}{\varepsilon}}$ (in order to avoid $\sqrt{-1}$).
It would look something like this:

$$y' = \frac{i}{\varepsilon} e^{\dfrac{iu(x)}{\varepsilon}} u'$$

$$y'' = \frac{-1}{\varepsilon^2}\frac{i}{\varepsilon} e^{\dfrac{iu(x)}{\varepsilon}} (u')^2 + \frac{i}{\varepsilon} e^{\dfrac{iu(x)}{\varepsilon}} u''$$

We try out the same calculations now:

$$-(u')^2 + i\varepsilon u'' + q(x) = 0$$

$$\text{Let } f = u' \Rightarrow -f^2 + i\varepsilon f' + q(x) = 0$$

$$f = f_0 + \varepsilon f_1$$

Collect like terms and equate:

$$-f_0^2 + q(x) = 0 \Rightarrow f_0 = \sqrt{q(x)}$$

$$-2f_0 f_1 + if_0' = 0 \Rightarrow f_1 = \frac{if_0'}{2f_0} = \frac{\pm iq'}{2 \cdot 2 \cdot \sqrt{q(x)} \cdot \sqrt{q(x)}} = \frac{iq'(x)}{4q(x)}$$

This is very similar to the previous computations and in fact it will be a similar solution but with an i appearing in some places:

$$y = e^{\dfrac{\pm i}{\varepsilon}\int_a^x \sqrt{q(s)}ds} \cdot \frac{1}{\sqrt[4]{q(x)}}(1 + O(\varepsilon))$$

Notice however that since e^{ik} can be written with the helps of trigonometric functions, we have an oscillatory (very fast oscillations) solution but we did not have this in the first case.

2.4. **Asymptotic Expansion of Integrals.**

This might happen when we solve a differential equation when we may need to compute an integral but we do not know how to do it explicitly, such as the following:

$$y'' + 2\lambda t y' = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

$$\text{Let } u = y' \Rightarrow u' + 2\lambda t u = 0 \Rightarrow u = Ce^{\lambda t^2}$$

$$y = \int_0^t e^{\lambda s^2} ds$$

These types of integrals are called *Laplace Integrals*

$$I(\lambda) = \int_a^b f(t)e^{-\lambda t} dt \qquad \lambda >> 1$$

They look like the Laplace transform:

$$\int_0^\infty f(t)e^{-\lambda t} dt$$

The term $e^{-\lambda t}$ is very small as $t \to \infty$. We *have* to assume that $f(t)$ is of exponential order in order to proceed pertubatively.

Lemma 2.1: Watson

Given the integral

$$I(\lambda) = \int_0^\infty t^\alpha h(t) e^{-\lambda t} dt$$

With $\alpha > -1$, $h(t)$ is analytic (has a Taylor expansion around 0), $h(0) \neq 0$.
Assume that h is of exponential order.

Then we can approximate the integral in the form of a series:

$$I(\lambda) = \sum_{n=0}^\infty \frac{h^{(n)}(0)\Gamma(\alpha + n + 1)}{n! \lambda^{\alpha+n+1}}$$

with the Gamma function defined as:

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \qquad x > 0$$

Anmärkning:
This follows from Taylor expansion, but let us look at the proof:

Bevis 2.2

The first thing that we can do is to split the domain such that $t \in [0, T]$:

$$\int_0^T t^\alpha h(t) e^{-\lambda t} dt + \underbrace{\int_T^\infty t^\alpha h(t) e^{-\lambda t} dt}_{\text{Exponentially small}}$$

Since h is bounded, and t^α is of lower growth order than $e^{-\lambda t}$, and $e^{-\lambda t}$ is shrinking as $T \to \infty$, it can be regarded as a very small number.

In the first integral we use the fact that $h(t)$ is analytic:

$$\int_0^T t^\alpha \left(h(0) + h'(0)t + h''(0)t^2/2 + \cdots\right) e^{-\lambda t} dt + \text{Exp. small}$$

$$\text{Let } u = \lambda t \Rightarrow \int_0^{\lambda T} \left(\left(\frac{u}{\lambda}\right)^\alpha h(0) + h'(0)\left(\frac{u}{\lambda}\right)^{\alpha+1} + \frac{1}{2}h''(0)\left(\frac{u}{\lambda}\right)^{\alpha+2} + \cdots\right) e^{-u}\frac{du}{\lambda}$$

$$\frac{h(0)}{\lambda^{\alpha+1}} \int_0^{\lambda T} u^\alpha e^{-u} du = \frac{h(0)}{\lambda^{\alpha+1}} \int_0^{\infty} u^\alpha e^{-u} du \tag{1}$$

Since we add an exponentially small term, we can equate the integral going to λT with the integral going to ∞ by approximation.

Upon closer inspection of (1) , we see the following equality:

$$(1) = \frac{h(0)}{\lambda^{\alpha+1}}\Gamma(\alpha+1)$$

The general term is given on teh following form:

$$\frac{h^{(n)}(0)}{n!\lambda^{\alpha+n+1}}\Gamma(\alpha+n+1)$$

This is what we wanted to show □

Example:
Let us look at an application of this lemma. Suppose that we want to compute an approximation of the following integral:

$$I(\lambda) = \int_0^{+\infty} \frac{\sin(t)}{t} e^{-\lambda t} dt \qquad \lambda >> 1$$

Here, $h(t) = \dfrac{\sin(t)}{t}$, and $\alpha = 0$. We can split the term:

$$\int_0^T \frac{\sin(t)}{t} e^{-\lambda T} dt + \int_T^{+\infty} \frac{\sin(t)}{t} e^{-\lambda t} dt$$

We look at the behaviour of the function $|h(t)| = \left|\dfrac{\sin(t)}{t}\right|$ as $t \to \infty$. We know $h(t)$ is bounded, so the right integral is negligable as $t \to \infty$ since it is exponentially small.
We remain with the following:

$$\begin{aligned}
\int_0^T \frac{\sin(t)}{t} e^{-\lambda t} dt &\Rightarrow \int_0^T \frac{t - \frac{t^3}{3!} + \cdots}{t} e^{-\lambda t} dt \\
&= \int_0^t \left(1 - \frac{t^2}{3!} + \cdots\right) e^{-\lambda t} dt \\
\text{Let } u = \lambda T &\Rightarrow \int_0^{\lambda T} \left(1 - \left(\frac{u}{\lambda}\right)^2 + \cdots\right) e^{-\lambda t} \frac{du}{\lambda}
\end{aligned}$$

Yet again, we see what happens in the world of infinities for each one of the terms (recall that the integral is a linear operator)

$$\int_0^{+\infty} \frac{1}{\lambda} e^{-u} du = \frac{\Gamma(1)}{\lambda}$$

$$\frac{1}{3!\lambda} \int_0^{+\infty} \frac{u^2 e^{-u}}{\lambda^2} du = \frac{1}{3!\lambda^3}\Gamma(3)$$

$$\vdots$$

Anmärkning:

If we have an integral such as

$$I(\lambda) = \int_a^b f(t)e^{-\lambda g(t)}dt$$

We can study the problem in the same way by introducing the change of variable $s = g(t) - g(a)$

3. Calculus of Variations

From last time (**READ**):

$$\int_a^b L(x,y,y')dx \qquad y(a)=A \qquad y(b)=B$$

An extremal satisifies Euler-Lagranges equation:

$$L_y - \frac{d}{dx}(L_{y'}) = 0$$

One generalisation of this formula can be obtainedwhen $L = L(x,y,y',y'')$. In this case we have to assume some regularity for y, that is $y \in C^4[a,b]$, and two other initial condition for y'.

We compute the derivative of the functional $\delta J(y;v) = \frac{d}{d\varepsilon} J(y+\varepsilon v)|_{\varepsilon=0}$
We want $y+\varepsilon v$ to be an admissable function, so this means that $v(a)=v(b)=0$ and $v'(a)=v'(b)=0$

Using this when trying to compute the derivative of the functional:

$$\begin{aligned}
&\frac{d}{d\varepsilon}\int_a^b L(x, y+\varepsilon v, y'+\varepsilon v', y''+\varepsilon v'')dx|_{\varepsilon=0} \\
&\quad \Rightarrow \int_a^b L_y v + L_{y'}v' + L_{y''}v'' dx
\end{aligned}$$

We solve this by integration by parts:

$$\int_a^b (L_y - \frac{d}{dx}L_{y'})v dx + \int_a^b L_{y''}v'' dx = v'L_{y''}|_a^b - \int_a^b v' \frac{d}{dx}(L_{y''})dx$$

By our initial conditions, $v'L_{y''}|_a^b = 0$:

$$\begin{aligned}
&-v\left(\frac{d}{dx}L_{y''}\right)_a^b + \int_a^b v\left(\frac{d^2}{dx^2}L_{y''}\right)dx \\
&= \int_a^b L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''}v(x)dx = 0 \qquad \forall v
\end{aligned}$$

The condition here becomes:

$$L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''} = 0$$

In this equation we will have the 4th derivative w.r.t y, which is why we need $y \in C^4[a,b]$

As an exercise, try to generalise the formula:
if $L = L(x,y,y',\cdots,y^{(n)}) \Rightarrow$ the extremals satisfy:

$$L_y + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}L_{y^{(k)}} = 0$$

The reason we have a plus and minus is because whenever we integrate by parts we will get a plus and next time a minus etc...

Exercicse: Show that if L does not depend on y ($L = L(x,y',y'')$), then $L_{y'} - \frac{d}{dx}L_{y''} = C$ (some constant)

Exercicse: If L does not depend on x, show that $L - y'(L_y' - \frac{d}{dx}L_{y''}) - y''L_{y''} = C$ (some constant)

Another generalisation occurs when we have $L = L(x, y_1, y_2, y_1', y_2')dx$ with initial values:

$$\begin{aligned}
y_1(a) &= A_1 \qquad y_1(b) = B_1 \\
y_2(a) &= A_2 \qquad y_2(b) = B_2
\end{aligned}$$

Proceeding with the same strategy as before, we compute the derivative. Starting by imposing $\delta J(y;v) = 0$:

$$\int_a^b \frac{d}{d\varepsilon} L(x, y_1 + \varepsilon v_1, y_2 + \varepsilon v_2, y_1' + \varepsilon v_1', y_2' + \varepsilon v_2')dx|_{\varepsilon=0} = 0$$

In this case we have that $v_1, v_2(a) = 0 = v_1, v_2(b)$

$$\int_a^b \left(L_{y_1} - \frac{d}{dx}L_{y_1}\right) v_1(x)dx + \int_a^b \left(L_{y_2} - \frac{d}{dx}L_{y_2}\right) dx = 0 \qquad \forall v_1, v_2$$

The trick here is that since this equation has to be true for all v_1, v_2, so in particular $v_2 = 0$. This yields:

$$\int_a^b \left(L_{y_1} - \frac{d}{dx}L_{y_1}\right) v_1(x)dx = 0 \qquad \forall v_1 \Rightarrow L_{y_1} - \frac{d}{dx}L_{y_1'} = 0$$

and so

$$\int_a^b L_{y_2} - \frac{d}{dx}L_{y_2'} v_2(x)dx = 0 \qquad \forall v_2$$

$$\begin{cases} L_{y_1} - \dfrac{d}{dx}L_{y_1'} = 0 \\ L_{y_2} - \dfrac{d}{dx}L_{y_2'} = 0 \end{cases}$$

Exercicse: Show that if $L = L(y_i; y_i')$ (for $i = 1, \cdots, n$), then

$$L - \sum_{i=1}^{n} y_i' L_{y_i'} = C \qquad \text{if } y_i \text{ satisfies the Euler-Lagrange equations}$$

Let us see what happens if we leave one of the end-boundary conditions unsatisfied, how long will we come in our computation. This is a type of generalisation where the boundary can be free (aka a *free end-point problem*):

$$J(y) = \int_a^b L(x, y, y')dx \qquad y(a) = A \qquad y(b) \text{ is free}$$

We look for extremals. In all our previous computation, to have admissable functions, we took $v(a) = v(b) = 0$. When we looked at the derivative $y + \varepsilon v$, and for it to be admissable we have $y(a) + \varepsilon v(a) = A$, and $A + \varepsilon v(a) = A \Rightarrow v(a) = 0$.
This condition is not needed anymore in $v(b)$ because we are assuming that $y(b) + \varepsilon v(b)$ is free.

If that term is 0 $\forall v$, it must be 0 for $v(b) = 0$:

$$L_y - \frac{d}{dx}L_{y'} = 0$$

If $v(b) \neq 0$, then $v(b)L_{y'}(b, y(b), y'(b)) = 0$, and in that case:

$$\begin{cases} L_y - \dfrac{d}{dx}L_{y'} = 0 \\ L_{y'}(b, y(b), y'(b)) = 0 \end{cases}$$

Exercicse:

$$J(y) = \int_0^1 (y'^2 + y^2)dx \qquad y(0) = 1 \qquad y(1) = \text{ free}$$

Find extremals, so solve this equation

$$L_y - \frac{d}{dx}L_{y'} = 0$$
$$2y - \frac{d}{dx}(2y') = 0$$
$$2y - 2y'' = 0$$
$$y'' - y = 0 \qquad y(x) = Ae^{-x} + Be^{x} \qquad y(0) = A + B = 1 \Rightarrow A = 1 - B$$
$$y(x) = e^{-x} + B(e^{x} - e^{-x})$$
$$L_{y'} = (1, y(1), y'(1)) = 0 \qquad L_{y'} = 2y' \Rightarrow y'(1) = 0$$
$$y'(x) = -e^{-x} + B(e^{x} + e^{-x}) \Rightarrow y'(1) = -e^{-1} + B(e + e^{-1}) = 0$$
$$\Rightarrow B = \frac{1}{e}\frac{e}{e^2+1} = \frac{1}{e^2+1}$$

Another kind of exercise that can be computed with functionals is minimizing path distances.

Exercicse: Find the extremal paths that connect two points on a plane. This problem is equivalent to finding extremals to the functional $J(y) = \int_a^b \sqrt{1 + y'(x)^2}dx$ and $y(0) = a$, $y(1) = b$. Show that the solution is a line.

Exercicse: Find the extremal paths connecting two points on a cylinder.
Firstly, we switch to cylindrical coordinates: $(x, y, z) = (R\cos(\theta), R\sin(\theta), z)$ where R is the radius (for simplicity, we set $R = 1$)

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \qquad dx = -\sin(\theta)d\theta \qquad y = \cos(\theta)d\theta$$
$$\Rightarrow ds = \sqrt{\sin^2(\theta)d\theta^2 + \cos^2(\theta)d\theta^2 + dz^2} = \sqrt{d\theta^2 + dz^2} = d\theta\sqrt{1 + \left(\frac{dz}{d\theta}\right)^2}$$
$$S = \int_{P_1}^{P_1} ds = \int_{\theta_1}^{\theta_2} \sqrt{1 + z'(\theta)^2}d\theta$$

We now want to study the extremals with this functional:

$$L = L(z') = \sqrt{1 + z'^2(\theta)^2}d\theta$$

By the Euler-Lagrange equation:

$$L_z - \frac{d}{dx}L_{z'} = 0 \qquad L_{z'} = C \qquad \frac{1}{2\sqrt{1 + z'^2(\theta)}} \cdot 2z'(\theta) = C$$
$$z'^2 = C^2(1 + z'^2) \qquad z'^2(1 - C^2) = C^2 \Rightarrow z'^2 = \frac{C^2}{(1 - C^2)}$$
$$z' = K \text{ say for example } \frac{C}{\sqrt{1 - C^2}}$$
$$\Rightarrow z(\theta) = \theta K + a$$

Exercicse: Find the extremal paths connecting two points on a sphere. (You can use the radius 1 to help ya).