Proof continues:

Note that $K \cap L = \emptyset$ since any attainable strategy with $\mathbb{E}(\bar{G}_T(\theta)) = 1$ would be arbitrage. We can apply the separating hyperplane theorem and there exists a linear functional φ that is zero on L and greater than $c > 0$ on K. By finiteness, we can express φ as

$$\varphi(X) = \sum_{i=1}^{n} q_i X(\omega_i) \quad \text{for some constants } q_i$$

In particular consider the rand. var. $\xi_i = \frac{1}{p_i} I_{\omega_i}$ then ξ_i is non-negative and $\mathbb{E}(\xi_i) = \frac{1}{p_i} \mathbb{E}(I_{\omega_i}) = \frac{\mathbb{P}(\omega_i)}{p_i} = 1$.

So $\xi_i \in K$ and $\varphi(\xi_i) = q_i \cdot \frac{1}{p_i} \geq c > 0$.

Hence $q_i > 0$ for all i. Define Q by

$$Q(\{\omega_i\}) = \frac{q_i}{\sum_j q_j} > 0$$

This is a probability measure as $\sum_i \frac{q_i}{\sum_j q_j} = \frac{\sum_i q_i}{\sum_j q_j} = 1$

We have $\mathbb{E}_Q(\bar{G}_T(\theta)) = \frac{1}{\sum_j q_j} \underbrace{\sum_i q_i \bar{G}_T(\theta)(\omega_i)}_{= \varphi(\bar{G}_T(\theta)) = 0} = 0$ for all attainable θ.

as $\bar{G}_T(\theta) \in L$

Now Q is an equivalent martingale measure

$\rightsquigarrow$ $\to$ equivalent to $\mathbb{P}$ since $Q(\{\omega_i\}) > 0$

$\to$ $\mathbb{E}_Q(\bar{G}_T(\theta)) = 0$ for all predictable processes, so $\mathbb{E}_Q(\Delta \bar{S}_t^i \mid \tilde{\mathcal{F}}_{t-1}) = 0$ for all t.

This completes the proof. $\square$

<u>Remark:</u> This holds in greater generality for non-finite models.

Completeness of Market Models

Recall that a market model is complete if every contingent claim X has a replicating (generating) strategy θ: a strategy with $V_T(\theta) = X$.

Recall that we are only considering finite models.

Proposition: Let a viable market model with an equivalent martingale measure Q be given. The model is complete if and only if every real-valued martingale (wrt Q) $\{M_t : 0 \leq t \leq T\}$ ha a representation of the form

$$M_t = M_0 + \sum_{u=1}^{t} \gamma_u \cdot \Delta \bar{S}_u$$

for some predictable $\gamma_u = \{\gamma_u^1, .., \gamma_u^d\}$

Proof: We first assume the model is complete. Without loss of generality, we assume it is non-negative writing the original as a diffrence of martingales. Set $C = M_T S_T^0$ and interprete it as a claim. It has a replicating strategy θ and

$$V_T(\theta) = C \iff \bar{V}_T(\theta) = M_T.$$

Since $\bar{V}_T(\theta)$ is a martingale transform of a Q martingale it is a Q martingale itself. So,

$$\bar{V}_t(\theta) = \mathbb{E}_Q(\bar{V}_T(\theta) \mid \mathcal{F}_t) = \mathbb{E}_Q(M_T \mid \mathcal{F}_t) = M_t$$

But then,

$$M_t = \bar{V}_t(\theta) = V_0(\theta) + \sum_{u=1}^{t} \theta_u \cdot \Delta \bar{S}_u = M_0 + \sum_{u=1}^{t} \theta_u \cdot \Delta \bar{S}_u ,$$

so we have a representation of M_t in the desired form.

For the converse, consider a claim C. Define a martingale by $M_t = \mathbb{E}(\beta_T C \mid \mathcal{F}_t)$.

There must be a representation of the form

$M_t = M_0 + \sum_{u=1}^{t} \gamma_u \cdot \Delta \bar{S}_u$ for some $\gamma_u = (\gamma_u^1, \dots, \gamma_u^d)$.

Define a strategy by $\theta_t^i = \gamma_t^i$ which uniquely determines θ_t^0 as well by the self-financing property. The precise choice is

$\theta_t^0 = M_t - \gamma_t \cdot \bar{S}_t$ which gives the required replicating strategy:

$$\begin{aligned} V_t(\theta) = \theta_t \cdot S_t &= \theta_t^0 S_t^0 + \sum_{j=1}^{d} \theta_t^j S_t^j \\ &= S_t^0 \left(\theta_t^0 + \sum_{j=1}^{d} \gamma_t^j \bar{S}_t^j \right) \\ &= S_t^0 \left(\theta_t^0 + \gamma_t \cdot \bar{S}_t \right) = S_t^0 M_t \end{aligned}$$

In particular, $C = S_T^0 M_T = V_T(\theta)$. $\square$

Second Fundamental Theorem of Asset Pricing

A finite market model is complete if and only if it has a unique equivalent martingale measure.

Proof: Suppose first that the model is complete and assume that Q, Q' are equivalent martingale measures. Let X be a contingent claim with generating strategy θ. We have

$$\beta_T X = \bar{V}_T(\theta) = V_0(\theta) + \underbrace{\sum_{t=1}^{T} \theta_t \Delta \bar{S}_t}_{\text{martingale transform of the } Q,\, Q' \text{ martingale } \bar{S}_t}$$

By the martingale property, we have

$$\mathbb{E}_Q(\beta_T X) = V_0(\theta) = \mathbb{E}_{Q'}(\beta_T X)$$

This holds for all claims and in particular $X = I_A$ for all events A. Hence,

$$Q(A) = \mathbb{E}_Q(I_A) = \mathbb{E}_{Q'}(I_A) = Q'(A)$$

and $Q = Q'$. Hence the equivalent martingale measure is unique.

For the converse assume there exists a claim X that does not have a replicating strategy and let Q be an equivalent martingale measure. Define

$$L = \left\{ c + \sum_{t=1}^{T} \theta_k \cdot \Delta \bar{S}_t \;\middle|\; c \in \mathbb{R},\ \theta_t \text{ predictable} \right\}$$

This is a linear subspace of the vector space of all random variables in $(\Omega, \tilde{\mathcal{F}})$.
It is a proper subspace as $X \notin L$.
We assumed Ω is finite, so as $L \subsetneq \Omega$ there exists a random variable Z that is orthogonal to L, i.e. non-zero Z s.t.

$$\mathbb{E}_Q(YZ) = 0 \quad \text{for all } Y \in L$$

We define a new measure $Q' \neq Q$:
Set $Q'(\{\omega\}) = Q(\{\omega\})\left(1 + \frac{Z(\omega)}{2\|Z\|_\infty}\right)$ $\quad \|Z\|_\infty = \max_\omega \{|Z(\omega)|\}$
Note that $Z(\omega)/2\|Z\|_\infty \in (-\frac{1}{2}, \frac{1}{2})$ and Q' is positive whenever Q is. Further,

$$\sum_{\omega\in\Omega} Q'(\omega) = \underbrace{\sum_{\omega\in\Omega} Q(\omega)}_{\substack{=1 \text{ because} \\ Q \text{ is a prob. measure}}} + \frac{1}{2\|Z\|_\infty} \underbrace{\sum_{\omega\in\Omega} Q(\omega) Z(\omega)}_{\substack{= \mathbb{E}_Q(Z) = 0 \text{ as} \\ Y=1 \text{ is in } L.}} = 1$$

So Q' is a probability measure with $Q \sim Q'$.

We have

$$\mathbb{E}_{Q'}\left(\sum_{t=1}^{T} \theta_t \cdot \Delta \bar{S}_t\right) = \sum_{\omega} Q'(\omega) \sum_{i=1}^{T} \theta_t(\omega) \cdot \Delta \bar{S}_t(\omega)$$

$$= \sum_{\omega} Q(\omega) \sum_{i=1}^{T} \theta_t(\omega) \cdot \Delta \bar{S}_t(\omega)$$

$$+ \frac{1}{2\|Z\|_\infty} \underbrace{\sum_{\omega} Q(\omega) Z(\omega) \sum_{i=1}^{T} \theta_t(\omega) \cdot \Delta \bar{S}_t(\omega)}_{\substack{= \mathbb{E}_Q\left(Z \sum_{t=1}^{T} \theta_t \cdot \Delta \bar{S}_t\right) = 0 \\ \text{by orthagonality.}}}$$

$$= \mathbb{E}_Q\left(\sum_{t=1}^{T} \theta_t \cdot \Delta \bar{S}_t\right) = 0.$$

So $\mathbb{E}_{Q'}\left(\sum_{t=1}^{T} \theta_t \cdot \Delta \bar{S}_t\right) = 0$ for all choices of θ, which is only possible if $\bar{S}_t$ is a martingale w.r.t. Q'.

Now both Q, Q' are equivalent martingale measures and since Z is non-zero, there exists $\omega \in \Omega$ with $Q(\omega), Q'(\omega) > 0$ and $Q(\omega) \neq Q'(\omega)$. Hence Q is not unique, a contradiction. This proves the claim $\square$