Lec 19

Some more examples of integral calculations

- Finish example about integrands with branch points.

<u>Ex.</u> Compute $I = \displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)}$

$\uparrow$ not even

<u>Sol.</u> This takes a trick! (or real analysis...)

Let $\log z$ be the branch given by

$$\log z = \ln r + i\theta, \quad 0 < \theta < 2\pi \quad (z = re^{i\theta})$$

Put $f(z) = \dfrac{\log z}{(z+1)(z^2+2z+2)}$ and consider

the integral $\displaystyle\int_{\Gamma_{\varepsilon,R}} f(z)\,dz$, where $\Gamma_{\varepsilon,R}$ is the

same contour as in the previous example.

If we sum the contributions from the upper and

lower side of the segment $[\varepsilon, R]$ the

terms containing logarithms cancel:

$$\int_\varepsilon^R \frac{\ln x}{(x+1)(x^2+2x+2)}\,dx + \int_R^\varepsilon \frac{\ln x + i2\pi}{(x+1)(x^2+2x+2)}\,dx$$

$$= -2\pi i \int_\varepsilon^R \frac{dx}{(x+1)(x^2+2x+2)} \longrightarrow -2\pi i\, I,$$

as $R \to +\infty$, $\varepsilon \to 0^+$.

It holds that

or $\sqrt{(\ln R)^2 + (2\pi)^2}$

$$\left| \int_{C_R} \frac{\log z}{(z+1)(z^2+2z+2)} \, dz \right| \overset{ML}{\le} \frac{\ln R + 2\pi}{(R-1)(R^2-2R-2)} \cdot 2\pi R \to 0, \quad R \to +\infty$$

and that

$$\left| \int_{C_\varepsilon} \frac{\log z}{(z+1)(z^2+2z+2)} \, dz \right| \le \frac{|\ln \varepsilon| + 2\pi}{(1-\varepsilon)(2-2\varepsilon-\varepsilon^2)} \cdot 2\pi\varepsilon \to 0, \quad \varepsilon \to 0^+$$

Note that f has simple poles at $z = -1$

and when $z^2 + 2z + 2 = 0 \iff z = -1 \pm \sqrt{1-2} = -1 \pm i$,

and that

$$f(z) = \frac{\log z}{(z+1)(z-(-1+i))(z-(-1-i))}$$

The residue thm gives, after taking limits, that

$$-2\pi i \, I = 2\pi i \left(\operatorname{Res}(f,-1) + \operatorname{Res}(f,-1+i) + \operatorname{Res}(f,-1-i) \right)$$

that is

$$I = -\left(\frac{\log(-1)}{(-i)\cdot i} + \frac{\log(-1+i)}{i \cdot 2i} + \frac{\log(-1-i)}{(-i)\cdot(-2i)} \right)$$

$$= -\left(\frac{i\pi}{1} + \frac{\ln\sqrt{2} + i\frac{3\pi}{4}}{-2} + \frac{\ln\sqrt{2} + i\frac{5\pi}{4}}{-2} \right) = \underline{\underline{\frac{\ln 2}{2}}}$$

The same idea works if you would like to compute

Question → any integral of the type $\displaystyle\int_0^\infty \frac{P(x)}{Q(x)} \, dx$

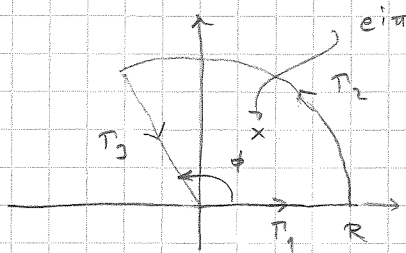
Turn page! where $\deg Q \geq \deg P + 2$ and $Q(x)$ has no zeros on $[0,\infty)$.

Sometimes one can proceed differently, see below.

Ex. Compute $I = \int_0^\infty \frac{1}{x^3+1}\,dx$

Sol Let $f(z) = \frac{1}{z^3+1}$ and consider $\int_{\Gamma_R} f(z)\,dz$

with Γ_R as in the figure below:



$\phi = \frac{2\pi}{3}$ $(R>1)$

Note that

$$\int_{\Gamma_3} f(z)\,dz = \big/\ z = re^{i\phi}\ ,\ r: R \to 0\ \big/$$

$$= \int_R^0 \frac{1}{r^3 \underbrace{e^{i3\phi}}_{=1} + 1}\, e^{i\phi}\,dr = -e^{i\phi}\int_0^R \frac{dr}{r^3+1}$$

The residue thm gives that

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \mathrm{Res}\left(f, e^{i\frac{\pi}{3}}\right) =$$

$$= 2\pi i \cdot \frac{1}{3z^2}\Big|_{z=e^{i\pi/3}} = \frac{2\pi i}{3e^{i(2\pi/3)}}$$

Standard estimate using the ML-ineq. and taking limits

$$\Rightarrow \left(1 - e^{i\frac{2\pi}{3}}\right) I = \frac{2\pi i}{3e^{i(2\pi/3)}}$$

$$\Rightarrow I = \frac{2\pi i}{3} \cdot \frac{1}{\left(1-e^{i\frac{2\pi}{3}}\right)e^{i(2\pi/3)}} = \frac{2\pi i}{3}\,\frac{1}{e^{i\frac{\pi}{3}}\left(e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}\right)e^{i\frac{2\pi}{3}}}$$

$$= \frac{2\pi i}{3} \cdot \frac{1}{e^{i\pi/3} - e^{-i\pi/3}} = \frac{\pi}{3}\,\frac{1}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$$

Exercise: Solve this using method from prev. example

The argument principle

Let C be a simple closed contour in $\mathbb{C}$.

Suppose that f is analytic and nonzero on C, and meromorphic inside C.

Then f has a finite number of zeros and poles inside C. Let $N_0(f)$ resp. $N_p(f)$ denote the number of zeros resp. poles of f inside C, both counted with multiplicity (or order).

Thm (Argument principle)

Let C be a simple closed, positively oriented, contour in $\mathbb{C}$. Suppose that f is analytic and nonzero on C, and meromorphic inside C.

Then,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = N_0(f) - N_p(f)$$

Proof: Let $G(z) := \dfrac{f'(z)}{f(z)}$.

The fn G is analytic on C, and has singularities inside C at points where f has zeros or poles.

Suppose that a point z_0 inside C is a zero of f of order m. Then f can be written as

$$f(z) = (z-z_0)^m g(z)$$

where g is analytic at z_0 with $g(z_0) \neq 0$

$$\Rightarrow \quad f'(z) = m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)$$

$$\Rightarrow \quad G(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

Since $\frac{g'(z)}{g(z)}$ is analytic at z_0, we see that G has a simple pole at z_0 with residue m.

If instead f has a pole of order k at z_p, then

$$f(z) = \frac{1}{(z-z_p)^k} g(z)$$

where g is analytic at z_p with $g(z_p) \neq 0$.

$$\Rightarrow \quad f'(z) = -\frac{k}{(z-z_p)^{k+1}} g(z) + \frac{1}{(z-z_p)^k} g'(z)$$

$$\Rightarrow \quad G(z) = -\frac{k}{z-z_p} + \frac{g'(z)}{g(z)}$$

Since g'/g is analytic at z_p, we see that G has a simple pole at z_p with residue $-k$.

The residue thm implies that

$$\int_C G(z)\,dz = \int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \left(N_0(f) - N_p(f) \right)$$

[3]

Remark: Note that $\int_C \frac{f'(z)}{f(z)}\,dz = i\,\Delta_C \arg f$

since, at least locally, we can introduce a branch of log such that

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = \frac{d}{dz}\left(\ln|f(z)| + i\arg f(z)\right).$$

It follows that

$$\int_C \frac{f'(z)}{f(z)}\,dz = \Delta_C \log|f| + i\,\Delta_C \arg f = i\,\Delta_C \arg f$$

The theorem can therefore be written

$$\boxed{\frac{1}{2\pi}\,\Delta_C \arg f = N_0(f) - N_p(f)}$$

and is therefore called the argument principle.

Corollary: Let C be a simple closed positively oriented contour in $\mathbb{C}$. Suppose that f is analytic inside and on C, and non-zero on C. Then,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_0(f)$$

(i.e. $\frac{1}{2\pi}\,\Delta_C \arg f = N_0(f)$)