# Analysis of Time Series, L10

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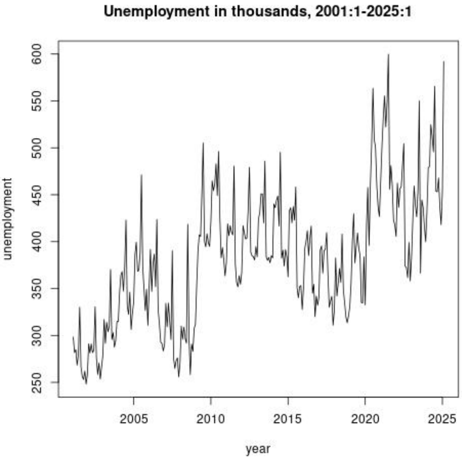
# Today

- 4.1: Cyclical behaviour and periodicity
- 4.2: The spectral density

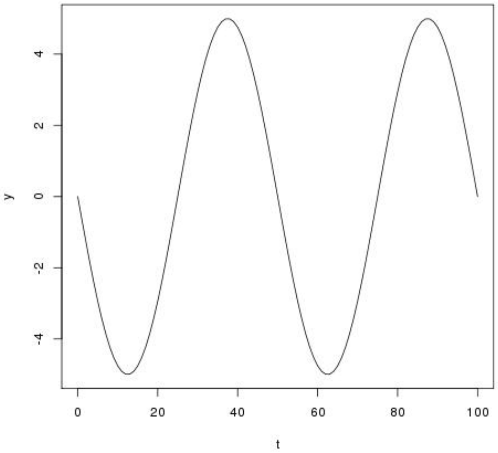
# Cyclical behaviour and periodicity




Season length 12 months. Business cycle 10 years?

# Cyclical behaviour and periodicity

Plot of the function $y = 5\cos(2\pi t/50 + 0.5\pi)$:



# Cyclical behaviour and periodicity

Fourier analysis:

- Let $x_t$ be deterministic, $t = 1, ..., n$, $n = 2m$ where $m$ is an integer.
- We may write

$$x_t = \sum_{k=0}^{m} \left\{ u_{k1} \cos(2\pi t\omega_k) + u_{k2} \sin(2\pi t\omega_k) \right\},$$

where $\omega_k = k/n$, $u_{02} = u_{m2} = 0$ and Fourier coefficients

$$u_{k1} = \begin{cases} \frac{1}{n} \sum_{t=1}^{n} x_t \cos(2\pi t\omega_k), & \text{if } k = 0 \text{ or } k = m, \\ \frac{2}{n} \sum_{t=1}^{n} x_t \cos(2\pi t\omega_k), & k = 1, 2, ..., m-1, \end{cases}$$

$$u_{k2} = \frac{2}{n} \sum_{t=1}^{n} x_t \sin(2\pi t\omega_k), \quad k = 1, 2, ..., m-1.$$

# Cyclical behaviour and periodicity

- Let $x_t$ be a mixture of periodic series

$$x_t = \sum_{k=1}^{q} \left\{ U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t) \right\},$$

  where $U_{k1}, U_{k2}$ for $k = 1, 2, ..., q$ are independent with zero mean and variances $\sigma_k^2$.
- Autocovariance function (why?)

$$\gamma(h) = \mathrm{cov}(x_{t+h}, x_t) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h).$$

- In particular,

$$\gamma(0) = \mathrm{var}(x_t) = \sum_{k=1}^{q} \sigma_k^2.$$

# The spectral density

Example 4.4:

- Let
$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t),$$
where $U_1$ and $U_2$ are independent random variables with mean zero and variance $\sigma^2$.
- The autocovariance function satisfies (why?)
$$\begin{aligned}\gamma(h) &= \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h} \\ &= \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega),\end{aligned}$$
where
$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2, & \omega_0 \le \omega. \end{cases}$$

# The spectral density

- By theorem C.1, for *any* autocovariance function for a stationary process, $\gamma(h)$, there is a non decreasing function $F$ with $F(-1/2) = 0$, $F(1/2) = \gamma(0)$, such that

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega).$$

- $F(\omega)$ is called *the spectral distribution function*.
- If $F(\omega)$ is differentiable with derivative $f(\omega)$, then

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} f(\omega) d\omega,$$

  where $f(\omega)$ is called *the spectral density function*.

# The spectral density

## Theorem (Property 4.2)

*If the autocovariance function $\gamma(h)$ for a stationary process satisfies*

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

*then for $h = 0, \pm 1, \pm 2, \ldots,$*

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \omega h} f(\omega) d\omega$$

*where the spectral density $f$ has the representation*

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}, \quad -\frac{1}{2} \leq \omega \leq \frac{1}{2}.$$

# The spectral density

Some properties:

1. For all $\omega$, $f(\omega) \geq 0$
2. $f(\omega) = f(-\omega)$
3. $f(\omega) = f(1-\omega)$
4. $\gamma(0) = \int_{-1/2}^{1/2} f(\omega) d\omega$

Calculate the spectral density function of a white noise process!

# The spectral density

Recall: If $x_t$ is ARMA$(p, q)$,

$$\phi(B)x_t = \theta(B)w_t,$$

where $\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$, $\theta(B) = 1 + \theta_1 B + ... + \theta_q B^q$.

**Theorem (Property 4.4)**

*If $x_t$ is ARMA$(p, q)$, its spectral density is given by*

$$f(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

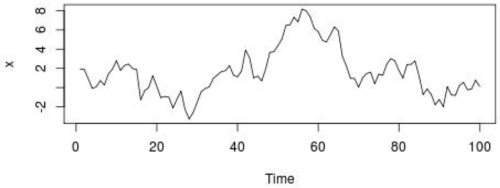
# The spectral density

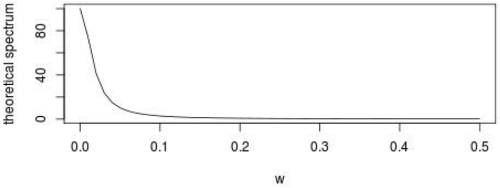
Calculate the spectral density of

1. An MA(1) process
2. An AR(1) process
3. An ARMA(1,1) process
4. A SARMA$(1,0) \times (1,0)_4$ process

# The spectral density

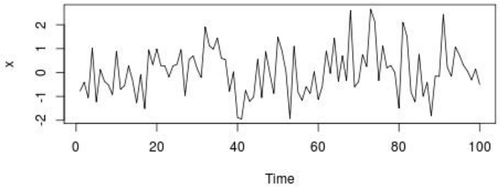
$x_t = 0.9x_{t-1} + w_t$ (smooth, high weight on low frequencies)

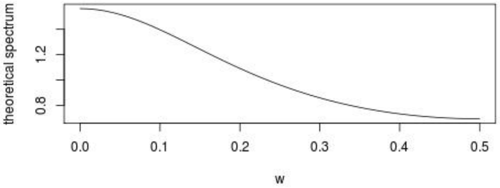


# The spectral density

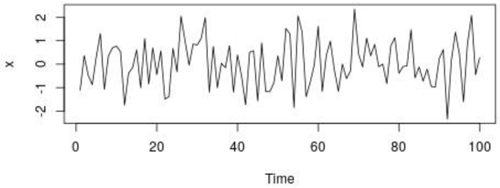
$x_t = 0.2x_{t-1} + w_t$ (less smooth)

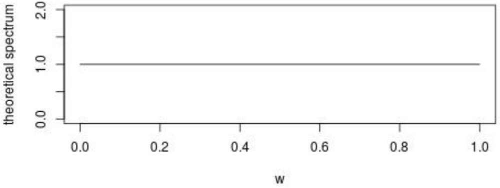


# The spectral density

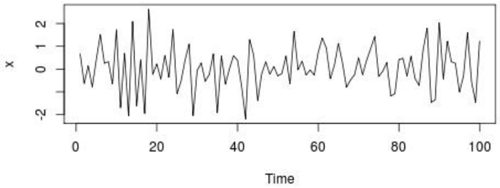
$x_t = w_t$ (all frequencies equally important)

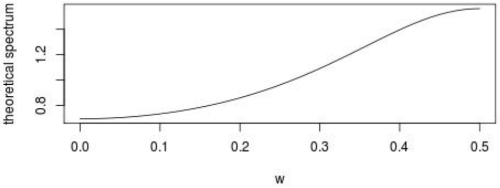


# The spectral density

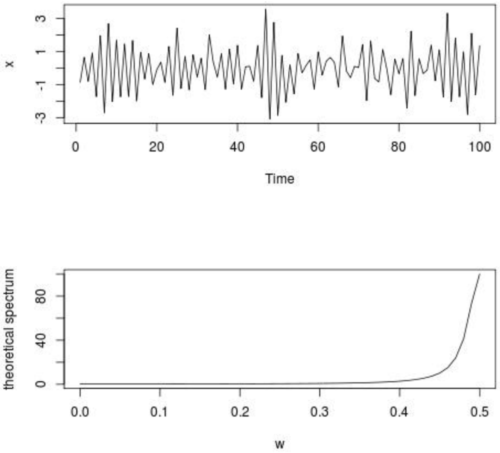
$x_t = -0.2x_{t-1} + w_t$ (more weight on high frequencies)



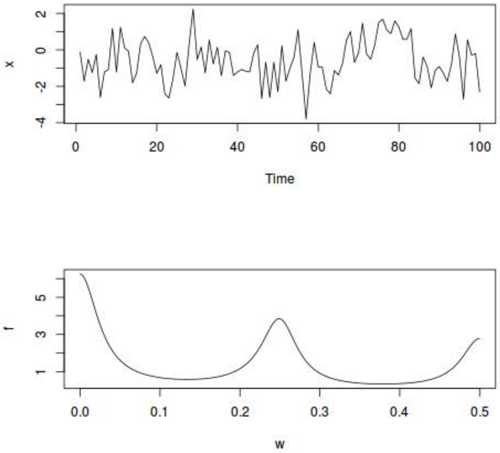
# The spectral density

$x_t = -0.9x_{t-1} + w_t$ (wiggly, high weight on high frequencies)



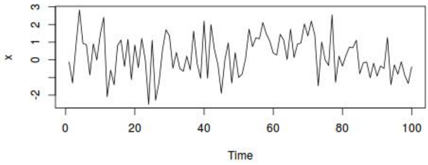
# The spectral density

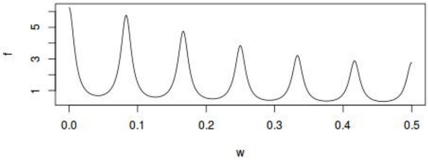
$(1 - 0.2B)(1 - 0.5B^4)x_t = w_t$ (high weight on frequency $1/4 = 0.25$)



# The spectral density

$(1 - 0.2B)(1 - 0.5B^{12})x_t = w_t$
(high weights on multiples of frequency $1/12 \approx 0.08$)



# News of today

- The spectral distribution
- The spectral density
- The spectral density for an ARMA process