Problem Session 2

Probability and Martingales, 1MS045

11 October 2024

Note: If not specified otherwise, all random variables are finite and real-valued, with the usual σ-algebra of Borel sets.

1. Prove: for every real-valued random variable X and every positive real number a, we have
$$\int_{-\infty}^{\infty} P(x < X \leq x + a)\, dx = a.$$

2. Let $X_1, X_2, \ldots, X_n$ be integrable (i.e., $E|X_k| < \infty$), and let $Y = \max_{1\leq k\leq n} X_k$.
   (a) Prove that Y is also integrable.
   (b) Prove that $E[X_k] \leq E[Y]$ for all $k \in \{1, 2, \ldots, n\}$.
   (c) Show by means of a counterexample that $E[|X_k|] \leq E[|Y|]$ does not necessarily hold.

3. Let $X_1, X_2, \ldots, X_n$ be random variables, and set $Y = \sup_n |X_n|$. Prove that Y is integrable if and only if there exists an integrable random variable Z such that $|X_n| \leq Z$ (almost surely) for all n.

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with $E[X_i^4] < \infty$, and set $\mu = E[X_i]$, $\sigma^2 = \mathrm{Var}(X_i)$ and $\mu_4 = E[(X_i - \mu)^4]$. Set
$$X_n = \frac{1}{n}\sum_{k=1}^{n} X_k \quad \text{and} \quad m_n^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - X_n)^2.$$
Prove that
$$E[m_n^2] = \frac{n-1}{n}\sigma^2,$$
$$\mathrm{Var}(m_n^2) = \frac{\mu_4 - \sigma^4}{n} - \frac{2\mu_4 - 4\sigma^4}{n^2} + \frac{\mu_4 - 3\sigma^4}{n^3}.$$

5. Let X be a non-negative random variable. Show that
$$\lim_{n\to\infty} nE\left(\frac{1}{X} I_{X>n}\right) = 0$$
and
$$\lim_{n\to\infty} \frac{1}{n} E[X I_{X<n}] = 0.$$
Here, I is an indicator function: $I_{X>n} = 1$ if $X > n$, and $I_{X>n} = 0$ otherwise.

6. Let X be a non-negative random variable and write $M_X(t) = \mathbb{E}(\exp(tX))$ (the function M is called the **moment generating function of** X). Note that $M_X(t)$ exists for all t, but may be infinite.

    (a) Assume that $M_X(\theta), M(-\theta) < \infty$ for some $\theta > 0$. Show that there exists a constant $C > 0$ such that $M_X(s) \leq C$ for all $s \in [-\theta, \theta]$.

    (b) Assume that $M_X(t) < \infty$ for all $t \in \mathbb{R}$. Show that order k derivatives of the moment generating exist and
    $$\frac{d^k}{dt^k} M_X(t) = \mathbb{E}(X^k e^{tX}).$$

    (c) Conclude that if $M_X(t) < \infty$ for all $t \in \mathbb{R}$ then all moments of X exist, i.e. $\mathbb{E}(|X|^k) < \infty$.

7. A stick of length 1 is broken at a random point, which means that the length of the remaining piece follows the uniform distribution on $[0, 1]$. The remaining piece is broken again at a random point, and so on. Let X_n be the length of the piece that remains after the stick has been broken n times. Prove that $\frac{\log X_n}{n}$ converges almost surely to a constant, and determine that constant.

8. Let X be a positive random variable with finite variance, and let $\lambda \in (0, 1)$. Prove that
$$P(X \geq \lambda \cdot E[X]) \geq \frac{(1-\lambda)^2}{(E[X])^2} \frac{E[X^2]}{E[X]}.$$
Hint: Use the identity $X = XI_{X \geq \lambda E[X]} + XI_{X < \lambda E[X]}$ to show that $(1 - \lambda)E[X] \leq E[XI_{X \geq \lambda E[X]}]$.

9. For every positive integer n, let p_n be a real number in the interval $(0, 1)$. Consider a sequence of n dots. Each of them is coloured red with probability p_n and green with probability $1 - p_n$, and the colours of the dots are independent of each other.

    (a) Let P_n be the number of pairs of consecutive dots that are both red. Determine $E[P_n]$.

    (b) Use Markov's inequality to prove: if $\lim_{n\to\infty} \sqrt{np_n} = 0$, then the probability $q_n = P(P_n > 0)$ that there are two consecutive red dots tends to 0 as $n \to \infty$.

    (c) Prove the converse of (b): if $\sqrt{np_n}$ does not tend to 0, then the probability q_n does not tend to 0 either.

10. Let $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ represent the outcome of three sequential coin tosses (in order either Heads or Tails). Assume that the coin tosses are independent of each other and that the coin is fair. Let X_H = number of heads, let E be the event that there are an odd number of heads in $\omega \in \Omega$, and write $Z_H = (X_H - 1)^2$

    Compute

    (a) $\mathbb{E}(X \mid E)$,

    (b) $\mathbb{E}(X \mid Y)$,

    (c) $\mathbb{E}(Y \mid X)$.