UPPSALA UNIVERSITET
Matematiska institutionen
Erik Ekström

Financial Derivatives
2024-01-12

*Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points. No means of assistance other than pencil and paper are allowed. Motivate your answers carefully!*

**1.** Let $X$ be the solution of

$$\begin{cases} dX_t = (2 - X_t)\,dt + \frac{X_t}{1+X_t^2} dW_t \\ X_0 = x. \end{cases}$$

Determine $E[X_t]$.

**2.** Solve the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + 2\frac{\partial^2 u}{\partial x^2}(t,x) - 2u(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\ u(T,x) = 3x + \cos x. \end{cases}$$

**3.** In the standard Black-Scholes model with volatility $\sigma$ and interest rate $r$, determine the arbitrage-free price at time 0 of the contract that pays its holder the amount

$$\mathcal{X} = \begin{cases} S_T & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K \end{cases}$$

at time $T$ (here $K > 0$ is a given constant).

**4.** Explain briefly the following notions:

(a) Asian option;
(b) implied volatility;
(c) replication of a $T$-claim;
(d) delta of an option;
(e) martingale modeling of the short rate.

**5.** Consider a market consisting of a bank account with constant interest rate $r$ and a stock $S$ with constant volatility. Moreover, the stock pays a proportional discrete dividend of size $\delta S_{T_0-}$ at time $T_0$ (here $0 < \delta < 1$ and $T_0 \in (0,T)$). For this market, derive the put-call-parity.

**6.** A certain $T$-claim on an underlying asset $S$ pays its holder the amount

$$\mathcal{X} = \begin{cases} 100 - S_T & \text{if } S_T \leq 50 \\ S_T & \text{if } S_T > 50 \end{cases}$$

at time $T$. The claim $\mathcal{X}$ trades at price 61, a zero-coupon $T$-bond with face value 100 trades at 98, and a call option on $S$ with maturity $T$ and strike price 50 trades at 12. Show how to construct an arbitrage if the current stock price is 59.

**7.** Consider a standard Black-Scholes model with risk-free rate $r$ and volatility $\sigma$, and a $T$-claim that at time $T$ pays the holder

$$\mathcal{X} = \frac{S_{T_1} + S_T}{2},$$

where $0 < T_1 < T$.

a) Find a replicating strategy for $\mathcal{X}$.
b) What is the arbitrage-free price at time 0 of $\mathcal{X}$?
c) What is the arbitrage-free price at time $t \in (T_1, T)$.

**8.** Consider the model

$$\begin{cases} dr_t = b\,dt + \sigma(t)\,dW_t, & t \geq 0 \\ r_0 = r \end{cases}$$

for the short rate under the pricing measure, where $\sigma(\cdot)$ is a function of time. Determine the term structure, i.e. calculate bond prices at time $t = 0$ in this model for all possible maturities $T$.

GOOD LUCK!