Lesson 1

Uppsala University
Department of Mathematics

Fourier Analysis, 5 credits
Fall 2022, Period 2
Course code: 1MA211

Background: complex exponentials and integrals

Use Exercises 1 - 7 to review important background material for this course. Please make sure that you understand how to use the methods that are needed to solve the exercises! You can skip the exercises that you already feel confident about.

Exercise 1

The power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for every $z \in \mathbb{C}$, and can be taken as the definition of the complex exponential e^z, for any $z \in \mathbb{C}$.

(a) Use Maclaurin series to prove Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$ for every $x \in \mathbb{R}$.

(b)* Prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$ for every $z_1, z_2 \in \mathbb{C}$.

Hint: For every $n \geq 0$, find all the terms in $e^{z_1}e^{z_2}$ for which the sum of the powers of z_1 and of z_2 is n.

(c) Use the previous two properties to find the usual expressions for $\cos(x+y)$ and $\sin(x+y)$.

Exercise 2

Compute

(a) i^i (b) $(1-i)^{28}$

Exercise 3

(a) $\displaystyle\int_{\pi/2}^{\pi} \cos^3(x)\sin(x)\,\mathrm{d}x$ (b) $\displaystyle\int \frac{\mathrm{d}x}{e^x + e^{-x}}$ (c) $\displaystyle\int \frac{\log x}{x}\,\mathrm{d}x$

(d) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$ (e) $\displaystyle\int_0^{\infty} x\,e^{-x}\,\mathrm{d}x$

Exercise 4

Show that (when the integrals make sense)

$$\int_0^t f(x-y)g(y)\,\mathrm{d}y = \int_0^t f(y)g(x-y)\,\mathrm{d}y.$$

and that

$$\int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y.$$

Exercise 5

Calculate the integrals

$$\text{(a)} \int_0^1 \int_0^1 x^2 + 2xy \,\mathrm{d}x\,\mathrm{d}y \qquad \text{(b)} \int_0^1 \int_0^y xy + 3y^2 \,\mathrm{d}x\,\mathrm{d}y$$

$$\text{(c)}^* \int_R \frac{y}{x} e^x \,\mathrm{d}x\,\mathrm{d}y,\ R = \big\{(x,y) \in \mathbb{R}^2 : y^2 \le x \le y, 0 \le y \le 1\big\}$$

Exercise 6

Show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x \int_{\mathbb{R}} g(y)\,\mathrm{d}y$$

for suitable functions f and g.

Exercise 7

Apply partial fraction decomposition to the following rational functions:

$$\text{(a)}\ \frac{1}{(x+2)(x-1)} \qquad \text{(b)}\ \frac{1}{(x-2)(x^2+2x+2)} \qquad \text{(c)}\ \frac{1}{x^3+3x^2-4}$$

Miscellaneous from the Introduction

Exercise 8

Computer exercise:

(a) We will prove later in the course that

$$f(x) = (x-1)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

for $0 \le x \le 2$, where

$$a_n = \int_0^2 (x-1)^2 \cos(n\pi x)\,\mathrm{d}x = \begin{cases} \frac{2}{3}, & n = 0 \\ \frac{4}{n^2\pi^2}, & n > 0 \end{cases}.$$

Use any program, e.g. Matlab, Python, GeoGebra, Desmos (`https://www.desmos.com/calculator`), to animate the convergence of the series.

Guide for Desmos: in slot 1, type

$$y = (x-1)^2.$$

In slot 2, type

$$y = \frac{1}{3} + \sum_{n=1}^{N} \frac{4}{n^2\pi^2} \cos(n\pi x)$$

(type *shift*+7 to get the fractions and *sum* to get the sum).

If you have done this correctly, you will get the opportunity to add N as a slider. Do this and take $1 \le N \le 20$ and step 1.

(b) Do the same thing for $f(x) = x$, $-1 < x < 1$ using

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

with

$$b_n = \int_{-1}^{1} x \sin(n\pi x)\,\mathrm{d}x$$

(calculate the integral above by hand).

Starting with the Laplace Transform

In Exercise 9 you are supposed to learn how to compute transforms (or inverse transforms) only by looking at the table of formulas.

Exercise 9

Use the table of formulas to find the function f that has Laplace transform

$$F(s) = \frac{2s}{(s-1)(s^2+2s+1)}$$

Hint: make the ansatz

$$F(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiply by the least common divisor.

Answers to some exercises

Exercise 2: (a) $e^{-\pi/2}$ (b) $-2^{14} = -16384$

Exercise 3: (a) $-1/4$ (b) $\arctan(e^x) + C$ (c) $\log^2(x)/2 + C$
(d) $\pi/2$ (e) 1

Exercise 5: (a) $5/6$ (b) $7/8$ (c) $e/2 - 1$

Exercise 7: (a) $\dfrac{1}{3(x-1)} - \dfrac{1}{3(x+2)}$ (b) $\dfrac{1}{10(x-2)} - \dfrac{x+4}{10(x^2+2x+2)}$
(c) $\dfrac{1}{9(x-1)} - \dfrac{1}{9(x+2)} - \dfrac{1}{3(x+2)^2}$

Exercise 9: $f(x) = \dfrac{1}{2}e^x + xe^{-x} - \dfrac{1}{2}e^{-x}$

Lesson 2

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The Laplace Transform

Exercise 1

Compute, using the definition, the Laplace transform of

(a) $f(t) = t^2$ **(b)** $f(t) = te^{2t}$ (c) $f(t) = e^t \sin t$

(d) $f(t) = \chi_{[0,1]}(t)$ (e) $f(t) = t\chi_{[0,1]}(t)$ (f) $f(t) = e^{2(t-1)}H(t-1)$

(h)* a periodic function $f(t)$ of period 1 and $f(t) = t$, $t \in [0, 1)$.

Confirm your answer using the table of formulas whenever this is possible.

Exercise 2

Compute the inverse Laplace transform of

(a) $\dfrac{1}{s(s+1)}$ **(b)** $\dfrac{1}{s^2+4s+29}$ (c) $\dfrac{e^{-s}}{(s-1)(s-2)}$

(d) $\dfrac{se^{-s}}{(s^2+1)^2}$ (e) $\dfrac{6s^2+4s-2}{(s-2)^2(s^2+2s+2)}$ (f) $\dfrac{s^2}{(s^2+1)^2}$

Exercise 3

Calculate, using the Laplace transform, the convolution $f * g(t)$ for

(a) $f(t) = 1$, $g(t) = t$ (b) $f(t) = g(t) = t$ **(c)** $f(t) = t$, $g(t) = e^{2t}$

(d) $f(t) = 1$, $g(t) = \chi_{[0,1]}(t)$ (e) $f(t) = t$, $g(t) = \chi_{[0,1]}(t)$

Exercise 4

Give a function f that is defined on the interval $[0, \infty)$ and solves the integral equation

$$\int_0^t uf(t-u)\,\mathrm{d}u = t \sin t.$$

Exercise 5

Solve the system below using the Laplace transform:

$$\begin{cases} x + y' = 2e^t \\ x' - x - 2y' - y = \sin t \end{cases}$$

Use the initial values $x(0) = 2$ and $y(0) = 1$.

Exercise 6

Solve the system

$$\begin{cases} z'' + y = 5e^{2t} \\ y'' - z = 3e^{2t} \end{cases}$$

where $y(0) = z(0) = 1$, $y'(0) = z'(0) = 2$.

Exercise 7

Find a function f such that

$$\int_0^x e^{-y} \cos y \, f(x-y) \, \mathrm{d}y = x^2 e^{-x}$$

Exercise 8

Solve the following differential equations for $t > 0$.

(a) $\begin{cases} y''(t) + 2y'(t) + y(t) = 3, \\ y'(0) = y(0) = 0 \end{cases}$

(b) $\begin{cases} 4y''(t) - 2y(t) = 7, \\ y'(0) = y(0) = 0 \end{cases}$

(c) $\begin{cases} 3y''(t) + 2y'(t) - 5y(t) = 3, \\ y'(0) = 1, \ y(0) = 0 \end{cases}$

(d) $\begin{cases} y''(t) + 3y'(t) = -2, \\ y'(0) = 1, \ y(0) = 2 \end{cases}$

(e) $\begin{cases} y'' + 3y' + 2y = 10e^{-t}\sin(t), \\ y(0) = 1, y'(0) = -3. \end{cases}$

Answers to some exercises

Exercise 1:
(a) $2/s^3$
(b) $1/(s-2)^2$
(c) $1/((s-1)^2+1)$
(d) $(1-e^{-s})/s$
(e) $(1-se^{-s}-e^{-s})/s^2$
(f) $e^{-s}/(s-2)$
(g) $1/(s(1-e^s))-1/s^2$

Exercise 2:
(a) $1-e^{-t}$
(b) $e^{-2t}\sin(5t)/5$
(c) $e^{2(t-1)}H(t-1)-e^{t-1}H(t-1)$
(d) $\frac{1}{2}(t-1)\sin(t-1)H(t-1)$
(e) $e^{2t}-te^{2t}-e^{-t}\cos(t)$
(f) $(\sin t+t\cos t)/2$

Exercise 3:
(a) $t^2/2$
(b) $t^3/6$
(c) $e^{2t}/4-t/2-1/4$
(d) $\begin{cases} t, & t\le 1 \\ 1, & t>1 \end{cases}$
(e) $\begin{cases} t, & t^2/2\le 1 \\ t-1/2, & t>1 \end{cases}$

Exercise 4: $2\cos t - t\sin t$

Exercise 5: $x(t)=2e^t+\sin t$ $\quad$ $y(t)=\cos t$

Exercise 6: $y(t)=z(t)=e^{2t}$

Exercise 7: $f(x)=2xe^{-x}+x^3e^{-x}/3$

Exercise 8:
(a) $3-3(t+1)e^{-t}$
(b) $7e^{t/\sqrt{2}}/4+7e^{-t/\sqrt{2}}/4-7/2$
(c) $-\frac{3}{5}+\frac{3}{4}e^t-\frac{3}{20}e^{-\frac{5}{3}t}$
(d) $y(t)=(-6t-5e^{-3t}+23)/9$
(e) $9e^{-t}-3e^{-2t}-5e^{-t}\sin(t)-5e^{-t}\cos(t)$

Lesson 3

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Sequences and Series

Exercise 1

Determine whether the following series are convergent

$$\text{(a)} \sum_{n=0}^{\infty} \frac{n^2+3n+1}{n^3+n+2} \quad \text{(b)} \sum_{n=0}^{\infty} \frac{n^2+n+1}{n^4+2n+2} \quad \text{(c)} \sum_{n=0}^{\infty} \frac{n^2+2n+1}{n^3+3^n+2}$$

$$\text{(d)} \sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}} \quad \text{(e)} \sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{n^2} \quad \text{(f)}^* \sum_{n=2}^{\infty} \frac{1}{n\left(\log n\right)^2}$$

Exercise 2

Determine if the series below are convergent

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n+2}} \quad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}} \quad \text{(c)} \sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$

Exercise 3

Determine if the series and sequences are uniformly convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}, x \in \mathbb{R}$ **(b)** $\displaystyle\sum_{n=0}^{\infty} n^2 e^{-nx} \cos(nx),\ x \in [3, \infty)$

(c) $f_n(x) = \dfrac{1}{(x-n)^2+1},\ x \in \mathbb{R}$ **(d)** $f_n(x) = n^x \left(1-x^4\right)^n,\ x \in [0,1]$

(e) $(-1)^n x, x \in [-1, 1]$

Exercise 4

Consider the series

$$\sum_{n=1}^{\infty} \frac{x \sin(3nx^2)}{5^n}$$

and let $I = [-10, 10]$.

(a) Show that the series is uniformly convergent in I

(b) Let $f(x)$ be the function defined by the series. Compute $\int_0^x f(t)dt$ for $x \in I$.

(c) If $I \subset \mathbb{R}$ was an arbitrary closed interval containing 0, could you still solve the previous questions?

Exercise 5

On which closed subintervals I of $[0,1]$ does the sequence of functions

$$4^n x^n (1-x)^n$$

converge uniformly?

Exercise 6*

Define the function $\psi_0 \in \mathcal{C}^\infty$ such that $\psi_0(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. Let $\psi_j(x) = \psi_0(2^{-j}x) - \psi_0(2^{-j+1}x)$, $j \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$. Calculate the value of the series

$$\sum_{n=0}^{\infty} \psi_n(x)$$

and determine if it is uniformly convergent or not. Hint: Simplify the partial sum $\sum_{n=0}^{N} \psi_n(x)$.

Exercise 7

Let $f_n : S \to \mathbb{C}$ be a sequence of functions defined on an interval $S \subset \mathbb{R}$. Show that the following two definitions of f_n converging uniformly to a function $f : S \to \mathbb{C}$ are equivalent:

$$\forall_{\epsilon>0} \exists_{n_0} \forall_{n>n_0} \forall_{x\in S} \, |f_n(x) - f(x)| < \epsilon$$

and

there is a sequence $(M_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} M_n = 0$ and $\forall_n \forall_{x\in S} |f_n(x) - f(x)| < M_n$.

Hint: For one of the implications, you may find it helpful to define

$$M_n = \sup_{x\in S} |f_n(x) - f(x)| + 1/n.$$

If for some n it turns out that $\sup_{x\in S} |f_n(x) - f(x)| = \infty$ (note that we are not assuming that the functions are bounded), then define $M_n = 1$ for such n.

Calculating Fourier Series

Exercise 8

Determine the Fourier series of

$$\text{(a) } \cos 2x \quad \text{(b) } \cos^2 x \quad \text{(c) } \sin^3 x$$

Hint: You don't need to calculate any integrals here!

Exercise 9

If $f(x) = (x+1)\cos x$ for $-\pi < x < \pi$ and is 2π-periodic, what is the sum of the Fourier series of f for $x = 3\pi$?
Hint: Note that you do not have to compute the series itself!

Answers to some exercises

Exercise 1:	(a) No	(b) Yes	(c) Yes	(d) No	(e) Yes	(f) Yes
Exercise 2:	(a) Yes	(b) No	(c) Yes			
Exercise 3:	(a) Yes	(b) Yes	(c) No	(d) No	(e) No	

Exercise 5: Every interval that does not contain the point $1/2$.

Exercise 6: Pointwise convergence is 1 and it is not uniformly convergent.

Exercise 8: (a) $a_2 = 1$, otherwise 0 (b) $a_0 = 1, a_2 = 1/2$, otherwise 0
(c) $b_1 = 3/4, b_3 = -1/4$, otherwise 0 **(using period 2π)**

Exercise 9: -1

Lesson 4

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Calculating Fourier Series and its Applications

Exercise 1

The function f is periodic and odd with period 2π and is given by $f(x) = x$ when $0 \le x \le \pi$.

(a) How does f depend on x in the interval $(-\pi, 0)$.

(b) Calculate the Fourier series of f on trigonometric form.

(c) Where on the interval $[\pi, 3\pi]$ does the Fourier series of f converge to f? What does the series converge to otherwise? Draw the graph of the Fourier series!

(d) Is the Fourier series uniformly convergent? Explain your answer.

(e) Calculate the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

Exercise 2

Let f be a 2π-periodic function that is given by $f(x) = e^x$ for $0 \le x < 2\pi$.

(a) Draw the graph of f on the interval $-2\pi \le x \le 2\pi$

(b) Calculate the Fourier series of f on complex form and draw the graph of the series.

(c) Calculate the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2}$$

Hint: Use that $\frac{1}{1-in} + \frac{1}{1+in} = \frac{2}{1+n^2}$.

Exercise 3

The function f is periodic and even with period 2 and is given by $f(t) = 1 - x$ when $0 \le x \le 1$.

(a) Calculate its Fourier series on trigonometric form.

(b) To which function does the Fourier series converge? Draw the graph!

(c) Is the convergence uniform? Explain your answer.

(d) Calculate the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Exercise 4

Calculate the Fourier series for the even function with period 4, that for $0 < x < 2$ is given by

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}.$$

Sketch the graph of the series on the interval $[-2, 4]$.

Exercise 5

Use the function $f(x) = \cos \alpha x, x \in [-\pi, \pi]$, α real but not an integer, to prove that

$$\pi \cot \alpha\pi = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha + n} \quad \left(=: \text{p. v.} \sum_{n=-\infty}^{\infty} \frac{1}{\alpha + n} \right)$$

Exercise 6

Take $f(x) = x \cos 2x$ for $-\pi < x < \pi$, and assume f to have period 2π. Calculate it's Fourier series on trigonometric form and then on complex form.

Riemann–Lebesgue Lemma

Exercise 7

Show that if $f : [a, b] \to \mathbb{C}$ is integrable in the finite interval $[a, b]$, then it is also absolutely integrable in $[a, b]$.

Exercise 8

Use the Riemann–Lebesgue Lemma to show that for integrable f in $\mathbb{T}$

(a) $\lim_{n\to\infty} \int_{\mathbb{T}} f(x) \sin nx \, \mathrm{d}x = 0$ (b) $\lim_{n\to\infty} \int_{\mathbb{T}} f(x) \cos nx \, \mathrm{d}x = 0$

(c) $\lim_{n\to\infty} \int_{\mathbb{T}} f(x) \sin \left(n + \frac{1}{2} \right) x \, \mathrm{d}x = 0.$

Exercise 9

Explicitly verify the Riemann–Lebesgue lemma for the three functions

$$\text{(a) } f_1(x) = x(1-x), x \in [0,1] \quad \text{(b) } f_2(x) = \begin{cases} x, x \in \left[0, \frac{1}{2}\right) \\ 1-x, x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

$$\text{(c) } f_3(x) = \begin{cases} 0, x \in \left[0, \frac{1}{2}\right) \\ 1, x \in \left[\frac{1}{2}, 1\right) \end{cases} \quad ,$$

namely show that

$$\lim_{a \to \infty} \int_{\mathbb{T}} f_{1,2,3}(x) e^{-iax} \, \mathrm{d}x = 0.$$

Theory about Fourier Series

Exercise 10

The Fourier series of a function f can be written in two ways, either

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{or} \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

What is the relation between a_n, b_n and c_n?

Exercise 11*

Show that if f has period T, then

$$\int_I f(x) \, \mathrm{d}x$$

is independent of the choice of the interval I with length T.

Exercise 12

A function f that has period 2π also has period 4π. Show that the Fourier series of f is the same independently of whether you consider it to be 2π or 4π periodic.

Exercise 13

Prove the following relations for a function f and its complex Fourier coefficients $\hat{f}(n)$:

(a) If f is even, then $\hat{f}(n) = \hat{f}(-n)$ for all n.

(b) If f is odd, then $\hat{f}(n) = -\hat{f}(-n)$ for all n.

(c) If f is real-valued, then $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n (where the overline denotes complex conjugation).

Exercise 14

Let f, g be absolutely integrable functions in $\mathbb{T}$. Prove the following rules:

(a) Let $a \in \mathbb{Z}$. If $g(t) = e^{iat} f(t)$, then $\hat{g}(n) = \hat{f}(n-a)$.

(b) Let $b \in \mathbb{R}$. If $g(t) = f(t-b)$, then $\hat{g}(n) = e^{-inb} \hat{f}(n)$.

Exercise 15

Show the improved version of Riemann–Lebesgue Lemma: If $f \in \mathcal{C}^k(\mathbb{T})$, then $\lim_{n\to\infty} n^k \hat{f}(n) = 0$.

Exercise 16*

Prove converse statements to the assertions in Exercise 13, i.e. show that if f is continuous, then

(a) if $\hat{f}(n) = \hat{f}(-n)$ for all n, then f is even,

(b) if $\hat{f}(n) = -\hat{f}(-n)$ for all n, then f is odd,

(c) if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n, then f is real-valued.

(d) Show that if the continuity condition is omitted, then (a)-(c) are not true in general.

Hint: In (a)-(c), it is helpful to use the uniqueness theorem for Fourier series, which will be seen in the lectures.

Convolutions

Exercise 17

Show that

$$\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n)$$

Dirichlet Kernel

Exercise 18

Compute the Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ using the formula for geometric sums:

$$\sum_{k=m}^{n} a^k = \begin{cases} \frac{a^{n+1}-a^m}{a-1}, & a \neq 1 \\ n-m+1, & a = 1 \end{cases}.$$

Show that $D_N(x)$ is a bounded function, and that its convolution with a function $f(x) \in C(\mathbb{T})$ is indeed the partial sum of the Fourier series of f.

Exercise 19

Show the following results for the Dirichlet kernel:

$$\textbf{(a)}\ \max_{\mathbb{T}} |D_N| = 2N+1 \quad \text{(b)* } C_1 \log N \le \int_{\mathbb{T}} |D_N(t)|\,\mathrm{d}t \le C_2 \log N.$$

It follows that $\int_{\mathbb{T}} |D_N(t)|\,\mathrm{d}t \to \infty$, and this is related to the existence of continuous functions with Fourier series that diverge at some points.

Answers to some exercises

Exercise 1:	(a) $f(x) = x$	(b) $a_n = 0, b_n = -\frac{2(-1)^n}{n}$
	(c) It converges to f in $(\pi, 3\pi)$ and to 0 at $\pi, 3\pi$	
	(d) No	(e) $\frac{\pi}{4}$
Exercise 2:	(b) $\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{2\pi}-1}{1-in} e^{inx}$	(c) $\frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1}$
Exercise 3:	(a) $\frac{1}{2} + \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos(\pi(2n+1)x)}{(2n+1)^2}$	(b) To f
	(c) Yes	(d) $\pi^2/8$
Exercise 10:	$a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$	
Exercise 18:	$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$	

Lesson 5

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Applications to Partial Differential Equations

Exercise 1

Find a solution $u = u(x,t)$ to the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < \pi, t > 0, \\ u(0,t) = 0, u(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin x + 5\sin 3x, & 0 < x < \pi \end{cases}$$

Exercise 2

Solve the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0 < x < 1, t > 0 \\ u(0,t) = 0, u(1,t) = 0, & t > 0 \\ u(x,0) = x, & 0 < x < 1. \end{cases}$$

Exercise 3

Find a function $u(x,t)$ that solves

$$\begin{cases} u_t = u_{xx} + 2x, & 0 < x < 3, t > 0, \\ u(0,t) = u(3,t) = 0, & t > 0, \\ u(x,0) = 8 - \frac{x^3}{3}, & 0 < x < 3 \end{cases}$$

Exercise 4

Solve the following boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0 < x < \pi, t > 0 \\ u_x(0,t) = -2, u_x(\pi,t) = -2, & t > 0 \\ u(x,0) = 1 - 2x + 3\cos(2x) - 2\cos(7x), & 0 < x < \pi \end{cases}$$

Exercise 5

Solve the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, t > 0, \\ u(0,t) = 0, u(\pi,t) = \pi, & t > 0, \\ u(x,0) = x - 5, \frac{\partial u}{\partial t}(x,0) = \sin 3x, & 0 < x < \pi. \end{cases}$$

Exercise 6

Solve the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin x, & 0 < x < \pi, t > 0, \\ u(0,t) = 0, u_x(\pi,t) = \pi, & t > 0, \\ u(x,0) = \pi x + \sin x, \frac{\partial u}{\partial t}(x,0) = x, & 0 < x < \pi. \end{cases}$$

Parseval's formula

Exercise 7

The 2π periodic function $f(x)$ is defined as $f(x) = x, 0 < x \leq \pi$ and we know that its Fourier series is given by

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

(a) Give a definition of $f(x)$ on the interval $-\pi < x \leq 0$ so that the Fourier series becomes the one above.

(b) Calculate the value of the series

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}.$$

Exercise 8

The function f is 2π periodic. Determine where the Fourier series converges and calculate its sum if

$$\text{(a) } f(x) = x^2 \text{ for } -\pi < x \leq \pi \quad \text{(b) } f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 \leq x \leq \pi \end{cases}$$

Use the Fourier series to calculate the following sums:

$$\text{(a')} \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{(b')} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}.$$

Exercise 9*

(a) Let $f : [a,b] \to \mathbb{C}$ be integrable in the finite interval $[a,b]$. Show that if

$$\int_a^b |f(x)|^2 \, \mathrm{d}x = 0,$$

then

$$\int_a^b |f(x)| \, \mathrm{d}x = 0.$$

Hint: Use the Cauchy–Schwarz inequality.

(b) Let $f : \mathbb{T} \to \mathbb{C}$ be integrable. Show that if $\hat{f}(n) = 0$ for all n, then

$$\int_0^{2\pi} |f(x)| \, \mathrm{d}x = 0.$$

Conclude that if f is also continuous, then $f = 0$.

(c) Let $f : \mathbb{R} \to \mathbb{C}$ be any function. Show that there is a unique even function $g(x)$ and a unique odd function $h(x)$ such that $f(x) = g(x) + h(x)$ for all $x \in \mathbb{R}$. Call g the *even part* of f and call h the *odd part* of f.

(d) Let $f : \mathbb{T} \to \mathbb{C}$ be integrable. Show that if $\hat{f}(n) = \hat{f}(-n)$ for all n, then

$$\int_0^{2\pi} |h(x)| \, \mathrm{d}x = 0,$$

where h is the odd part of f. Similarly, if $\hat{f}(n) = -\hat{f}(-n)$ for all n, then

$$\int_0^{2\pi} |g(x)| \, \mathrm{d}x = 0,$$

where g is the even part of f.

(Semi-)inner products and orthogonalization

In the exercises below, let $C(I)$ be the space of continuous functions $f : I \to \mathbb{C}$, for some interval $I \subset \mathbb{R}$. Given a continuous function $w : I \to (0, \infty)$, equip $C(I)$ with the inner product

$$\langle f, g \rangle = \int_I f(x) \, \overline{g(x)} \, w(x) \, dx.$$

Denote this inner product space by $C(I, w)$. Recall that the norm induced by an inner product is $\|f\| = \sqrt{\langle f, f \rangle}$.
(Note that in these exercises we are using a convention where we do not divide the integral by the lenght of I, even if I is a finite interval.)

Exercise 10

Calculate

$$\text{(a) } \|e^x\|_{C(\mathbb{R},\, e^{-x^2})} \quad \text{(b) } \langle x, e^x \rangle_{C(\mathbb{R}_+, e^{-2x})}.$$

Exercise 11

Show that the first three Legendre polynomials, 1, x, $\frac{1}{2}(3x^2 - 1)$ are orthogonal in $C([-1, 1], 1)$.

Exercise 12

Show that the first three Laguerre polynomials, 1, $-x + 1$, $\frac{1}{2}(x^2 - 4x + 2)$ are orthonormal in $C([0, \infty), e^{-x})$.

Exercise 13

Apply the Gram–Schmidt orthogonalisation procedure to construct an orthogonal basis in the subspace of $C((0,1),1)$ spanned by the polynomials $1, x$ and x^2.

Exercise 14

Find the two first orthonormal polynomials with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\,x\,\mathrm{d}x,$$

by orthogonalising the polynomials $1, x$.

Exercise 15

Solve the same problem as the preceding, when

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,e^{-x^2}\,\mathrm{d}x.$$

Exercise 16*

Prove that the Cauchy–Schwarz inequality holds of semi-inner products (recall that these satisfy the properties in the definition of inner product, except that $\langle u, u\rangle = 0$ does not imply $u = 0$).
Hint: Adapt the proof in chapter 5 of Vretblad's book. Consider separately the case $\|u\| = \|v\| = 0$.

Answers to some exercises

Exercise 1: $2\sin(x)e^{-t} + 5\sin(3x)e^{-9t}$

Exercise 2: $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)e^{-n^2\pi^2 t}$

Exercise 5: $x + \frac{1}{3}\sin(3t)\sin(3x) - \sum_{k=0}^{\infty} \frac{20}{(2k+1)\pi}\cos((2k+1)t)\sin((2k+1)x)$

Answers to some exercises

Exercise 7: (a) $f(x) = -x$ (b) $\frac{\pi^4}{96}$

Exercise 8: (a') $\frac{\pi^4}{90}$ (b') $(\pi^2 - 8)/16$

Exercise 10: (a) $e^{1/2}\pi^{1/4}$ (b) 1

Exercise 13: $1, x - 1/2, x^2 - x + 1/6$

Exercise 14: $\sqrt{2}, 6(x - 2/3)$

Exercise 15: $\pi^{-1/4}, 2^{1/2}\pi^{-1/4}x$

Lesson 6

Uppsala University
Department of Mathematics

Fourier Analysis, 5 credits
Fall 2022, Period 2
Course code: 1MA211

Calculations involving the Fourier Transform

Exercise 1

Using the definition, calculate the Fourier transform of

(a) $\chi_{[0,1]}(x)$ (b) $xe^{-|x|}$ (c) $\sin(x).\chi_{[0,\pi]}(x)$ (d) $e^{-|x|}\cos x$

Double-check using the table of formulas.

Exercise 2

Find the Fourier transforms of the following functions:

(a) $\dfrac{1}{b^2+x^2}, b \neq 0$ (b) $\dfrac{1}{x^2+4x+5}$ **(c)** $\dfrac{x}{1+x^2}$

(d) e^{5ix-2x^2} **(e)** $e^{-2|x+3|}$ (f) $\dfrac{\sin(2x)}{x^2-2x+5}$

Exercise 3

Use the results in Exercises 1-2 to calculate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(2x)}{(1+x^2)^2}\,\mathrm{d}x$ (b) $\displaystyle\int_{-\infty}^{\infty} e^{i\pi x/2}\frac{\cos(\pi x/2)}{1-x^2}\,\mathrm{d}x$ **(c)** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\,\mathrm{d}x, \quad a,b>0$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{2+x^2}{x^4+4}\,\mathrm{d}x$ **(e)** $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,\mathrm{d}x$

Exercise 4

Solve the differential equation $f''(x) - f(x) = e^{-|x|}$.

Exercise 5

(a) Show that for every $\omega \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-x^2/2}e^{-\omega x}dx = \sqrt{2\pi}\, e^{\omega^2/2}.$$

(b) An important result in complex analysis says if two functions $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon \mathbb{C} \to \mathbb{C}$ are both differentiable in all of $\mathbb{C}$ and satisfy $f(x) = g(x)$ for every

$x \in \mathbb{R}$, then $f(z) = g(z)$ for every $z \in \mathbb{C}$. Use this fact and the previous result to give an alternative proof of the fact that

$$\mathcal{F}[e^{-x^2/2}] = \sqrt{2\pi}\, e^{-\omega^2/2}.$$

Exercise 6

Calculate the Fourier transform of

$$f(x) = \arctan(x+1) - \arctan(x-1).$$

Hint: What is the Fourier transform of f'?

Exercise 7

Assume that $f : \mathbb{R} \to \mathbb{C}$ is integrable, absolutely integrable and continuous and that

$$f(x-1) + f(x) + f(x+1) = 0$$

for all $x \in \mathbb{R}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 8

Suppose that $f : \mathbb{R} \to \mathbb{C}$ is integrable, absolutely integrable and that $\hat{f}$ has a finite number of zeros. Prove that there cannot exist a function $g : \mathbb{R} \to \mathbb{C}$ that is integrable and absolutely integrable, and a number $a \in \mathbb{R}$ such that

$$g(x+a) - g(x) = f(x), \text{ for } -\infty < x < \infty.$$

Exercise 9*

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!(x^2+n^2)}.$$

(a) Show that the series is uniformly convergent on $\mathbb{R}$ and that f is continuous on $\mathbb{R}$.

(b) Calculate the Fourier transform $\hat{f}(\xi)$.

(c) Calculate $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.

Riemann–Lebesgue Lemma

Exercise 10

The function $f \in C^1(\mathbb{R})$ is equal to 0 for $|x| \geq 5$. Prove, without the help of the Riemann–Lebesgue Lemma, that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \,\mathrm{d}x = 0$$

Convolutions

Exercise 11

Let $g(x) = \frac{1}{1+x^2}$. Calculate $g * g(x)$.

Exercise 12

Find a function f such that

$$\int_{-1}^{1} f(x-y)\,\mathrm{d}y = e^{-|x-1|} - e^{-|x+1|}, x \in \mathbb{R}.$$

Exercise 13

The function $f : \mathbb{R} \to \mathbb{C}$ is continuous, integrable and absolutely integrable. Moreover,

$$\int_{-1}^{1} f(x-y)\,\mathrm{d}y = f(x)$$

for all $x \in \mathbb{R}$. Show that $f(x) = 0$ for all x.

Exercise 14

Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin 5(x-y)\sin 6y}{y(x-y)}\,\mathrm{d}y$$

Exercise 15

Solve the integral equation

$$f(x) = e^{-|x|} + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} f(y)\,\mathrm{d}y$$

Exercise 16

For which values of $\alpha > 0$ does the equation

$$\int_{-\infty}^{\infty} f(x-y)e^{-y^2/2}\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}e^{-(x/\alpha)^2/2}$$

have a solution? Find all the solutions.

Exercise 17*

Find the Fourier transform of

$$f(x) = \frac{1-\cos x}{x^2}.$$

Exercise 18

Show that for every $A \neq 0$ the function

$$f(x) = \frac{\sin(Ax)}{x}$$

is not absolutely integrable (recall that this is one half of the Fourier transform of the function $\chi_{[-A,A]}$).
Hint: for every $k \in \mathbb{Z}$, study $\int_{I_k} |f(x)| \, dx$, where I_k is the interval $[\pi/2 + k\pi, \pi/2 + (k+1)\pi]$.

Exercise 19

Use the inverse Fourier transform to show that

$$\widehat{fg}(\omega) = \frac{1}{2\pi}(\widehat{f} * \widehat{g})(\omega).$$

Answers to some exercises

Exercise 1: (a) $e^{-i\omega/2}\dfrac{2\sin\omega/2}{\omega}$ (b) $-\dfrac{4i\omega}{(1+\omega^2)^2}$ (c) $\dfrac{e^{i\pi\omega}+1}{1-\omega^2}$ (d) $\dfrac{4+2\omega^2}{\omega^4+4}$

Exercise 2: (a) $\pi e^{-|b\omega|}/|b|$ (b) $\pi e^{2i\omega-|\omega|}$ (c) $-i\pi\,\mathrm{sgn}(\omega)e^{-|\omega|}$
(d) $\sqrt{\frac{\pi}{2}}e^{-(\omega-5)^2/8}$ (e) $4e^{3i\omega}/(4+\omega^2)$ (f) $\pi\left(e^{-i(\omega-1)-2|\omega-1|} - e^{-i(\omega+1)-2|\omega+1|}\right)/4i$

Exercise 3: (a) π/e^2 (b) 0 (c) $\pi e^{-ab}/b$ (d) π (e) $\pi/6$

Exercise 4: $-\frac{1}{2}|x|\,e^{-|x|} - \frac{1}{2}e^{-|x|}$

Exercise 5: $2\pi\sin(\omega)e^{-|\omega|}/\omega$

Exercise 8: (b) $-\pi + \pi e^{e^{-|\omega|}}$ (c) $\pi(e-1)$

Exercise 10: $\frac{\pi}{2}g\left(\frac{x}{2}\right)$

Exercise 11: $\mathrm{sgn}(x)e^{-|x|}$

Exercise 13: $\pi\sin(5x)/x$

Exercise 14: $(4e^{-x}H(x) + 4e^{x/2}(1-H(x))/3$

Exercise 15: $|\alpha| > 1$

Exercise 16: $\pi(1-|\omega|).\chi_{[-1,1]}(\omega)$

Lesson 7

Uppsala University
Department of Mathematics

Fourier Analysis, 5 credits
Fall 2022, Period 2
Course code: 1MA211

Plancherel's Theorem

Exercise 1

Calculate

$$\text{(a)} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 \, \mathrm{d}x \quad \text{(b)} \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^4} \, \mathrm{d}x \quad \text{(c)} \int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2} \, \mathrm{d}x.$$

$$\textbf{(d)} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} \, \mathrm{d}x \quad \textbf{(e)} \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2} \, \mathrm{d}x$$

You may use the results in Exercise 1 of the previous lesson if needed.

Applications to PDE's

Exercise 2

Show that the function

$$E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

is a solution of the heat equation in the region $t > 0$. What is the initial condition as $t \to 0^+$?

Exercise 3

Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2uy = 0, \qquad x, y \in \mathbb{R}$$

with the initial condition $u(x,0) = f(x)$.

Exercise 4

For an infinite rod the units of length x and time t are chosen so that the heat equation takes the form

$$u_{xx} = u_t.$$

The temperature at time $t = 0$ is given by the function

$$e^{-x^2} + e^{-x^2/2}.$$

Determine the function that describes the temperature at every moment $t > 0$.

Exercise 5

A semi-infinite rod, materialised as the interval $[0, \infty)$, has at time $t = 0$ the temperature e^{x^2} for $0 < x < 1$ and 0 for $x > 1$. When $t > 0$, the end point (i.e., the point $x = 0$) is kept at a constant temperature of 0. Determine the temperature for every x at time $t = \frac{1}{4}$.

Hint: define initial condition $f(x)$ for $x < 0$ by $f(x) = -f(-x)$, to make f an odd function. Then solve the problem as if the rod were doubly infinite.

Exercise 6

In the unbounded plane sheet $\{(x, y) : y \geq 0\}$ there is a stationary and temperature distribution $u(x, y)$, such that the Fourier transform $\widehat{u}(\omega, y)$ is bounded. It is known that the temperature distribution on the line $y = 0$ is given by $\frac{1}{x^2+1}$. The problem can be described as

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0 \\ u(x, 0) = \frac{1}{x^2+1}, & x \in \mathbb{R} \end{cases}.$$

Determine $u(x, y)$ for all $y > 0$.

Answers to some exercises

Exercise 1: (a) π (b) $\pi/16$ (c) $3\pi/8$ (d) $\pi/2$ (e) $\pi/2$

Exercise 4: $u(x, t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}} + \frac{1}{\sqrt{1+2t}} e^{-\frac{x^2}{2+4t}}$

Lesson 8

Uppsala University
Department of Mathematics

Fourier Analysis, 5 credits
Fall 2022, Period 2
Course code: 1MA211

Distribution Theory

Exercise 1

Show that if $\varphi \in \mathscr{S}(\mathbb{R})$, then also $\varphi' \in \mathscr{S}(\mathbb{R})$. Is the converse true: i.e. must an antiderivative of a test function in $\mathscr{S}(\mathbb{R})$ also be in the same set?

Exercise 2

What is the action of the tempered distribution $x\delta'_a, a \in \mathbb{R}$ on a test function $\varphi \in \mathscr{S}(\mathbb{R})$.

Exercise 3

Show that $x^2\delta''' = 6\delta'$.

Exercise 4

Prove the formula:

$$g(x)\,(\delta')_a = g(a)\,(\delta')_a - g(a)\,\delta_a.$$

where $g \in C^\infty(\mathbb{R})$ has moderate increase (meaning: there are constants $M \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$ such that $|g(x)| \leq M(1+|x|)^n$ for all $x \in \mathbb{R}$). As for the notation, recall that if $f \in \mathcal{S}'(\mathbb{R})$, then there is $f_a \in \mathcal{S}'(\mathbb{R})$ with $f_a(\varphi) = f(\varphi_{-a}) = f(\varphi(x+a))$.

Exercise 5

Show that the distributional derivative of $f = \mathrm{p.v.}\,1/x$ is described by the formula

$$f'(\varphi) = -\lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi(x)-\varphi(0)}{x^2}\,\mathrm{d}x.$$

Exercise 6

Determine the Fourier transform of

$$\text{(a) } e^{-x}H(x) \quad \text{(b) } \frac{1}{1+ix} \quad \textbf{(c) } \frac{x}{1-ix} \quad \text{(d) } \frac{x^3}{1+x^2}$$

(H is the Heaviside function).

Exercise 7

Let $f_1 = \mathrm{p.v.}\,1/x$. Define f_n recursively for $n = 2, 3, \ldots$ by $f_{n+1} = -f'_n/n$. Prove that $x^n f_n(x) = 1$ for $n = 1, 2, 3\ldots$

Exercise 8

Let $f, g : \mathbb{R} \to \mathbb{C}$ be continuous functions with moderate increase. Show that if they define the same tempered distribution, then $f = g$.

Answers to some exercises

Exercise 1: No

Exercise 2: $-a\varphi'(a) - \varphi(a)$

Exercise 6: (a) $\frac{1}{1+i\omega}$ (b) $2\pi e^{-|\omega|}H(-\omega)$ (c) $2i\pi e^{-|\omega|}(\delta(\omega) - H(-\omega))$
(d) $i\pi e^{-|\omega|}(2\delta'(\omega) - \mathrm{sgn}(\omega))$