Lecture 1

Measure Spaces

σ - algebras

"universe"

A collection Σ of subsets of a set S is called a σ-algebra if:

1) It contains the empty set $\emptyset$
2) It is an algebra:
   a) If $A \in \Sigma$, then $A^c = S \setminus A \in \Sigma$
   b) If $A, B \in \Sigma$, then $A \cup B \in \Sigma$
3) It is also closed under countable unions. If $A_i \in \Sigma$ for all $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Example: The power set $\mathcal{P}(S) = \{A \subseteq S\}$

Example: $\{\emptyset, S\}$ is a σ-algebra

Example: $\{\emptyset, 2\mathbb{N}, 2\mathbb{N}-1, \mathbb{N}\}$
$= \{\emptyset, \{2,4,6,..\}, \{1,3,5,..\}, \{1,2,3,..\}\}$

Remark: Alternative (and equivalent) formulations exist. E.g. replacing 1) by "Σ is non-empty".
(Thus $A \in \Sigma \Rightarrow A^c \in \Sigma \Rightarrow A \cup A^c = S \in \Sigma \Rightarrow \emptyset = S^c \in \Sigma$)

Remark: Every algebra is closed under *finite* unions. Assume $A_1, \dots, A_k \in \Sigma$. Then,

$$A_1 \cup A_2 \in \Sigma,$$

$$A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3 \in \Sigma$$

$$\vdots$$

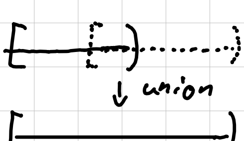
$$A_1 \cup \dots \cup A_k = (A_1 \cup \dots \cup A_{k-1}) \cup A_k \in \Sigma$$

but $\not\Rightarrow$ countable unions in Σ!

Example: Consider $S = [0\ 1)$, let Σ be finite unions of disjoint intervals of form $[a, b)$; $0 \leq a \leq b < 1$. We interpret $b = a$ as $[a, a) = \emptyset$. Then,

1) $\emptyset \in \Sigma$ ✓

2) $([a_1, b_1) \cup \dots \cup [a_k, b_k))^c = [b_1, a_2) \cup \dots \cup [b_k, 1) \in \Sigma$ ✓

Since 

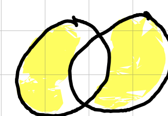
, $\bigcup_{i=1}^{k} [a_i, b_i) \cup \bigcup_{i=1}^{m} [c_i, d_i)$ is the union of at most $k+m$ intervals of form $[a,b)$ and hence in Σ. ✓

However, $\bigcup_{n=2}^{\infty} [\frac{1}{n}, 1) = (0,1) \notin \Sigma$ ✗

So, Σ is an algebra but <u>not</u> a σ-algebra.

<u>Remark</u>: Set algebras are algebras in the sense that

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



← (symmetric difference)

and $A \cap B$ take the role of "+" and "·" on Σ.

Measures

Let Σ_0 be a σ-algebra on S and let μ_0 be a function from Σ_0 to $[0,\infty] = [0,\infty) \cup \{\infty\}$; the extended real line.

We say that μ_0 is additive if for all disjoint $A, B \in \Sigma_0$ we have $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$.

We say that μ_0 is ==σ-additive== if further $\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$ for all collections $A_1, A_2, \ldots, \in \Sigma_0$ that are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$).

<u>Remark</u>: If μ_0 is additive we also have $\mu_0\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$ for all finite, pairwise disjoint $A_1, \ldots, A_k$. (Why?)

<u>Example</u>: Consider the σ-algebra $\Sigma = P(\mathbb{N})$ on $\{1, 2, 3, \ldots\}$. Define

$$\mu_0(A) = \# A \quad (= \text{number of elements in } A)$$

"counting measure"

This is (σ-) additive.

<u>Example</u>: Take $\Sigma_0 = P(\{1, 2, 3, 4, 5, 6\})$ and set $\mu_0(A) = \frac{\# A}{6}$. This represents the probability that the outcome of a fair die lies in A.

This is also σ-additive.

<u>Example</u>: Take $\Sigma_0 = P(\mathbb{N})$ and define

$$\mu_0(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

This is additive:

A	$\cup$	B	$\mu(A) + \mu(B) = \mu(A \cup B)$	
finite	finite	finite	$0 + 0 = 0$	✓
infinite	infinite	finite	$\infty + 0 = \infty$	✓
infinite	infinite	infinite	$\infty + \infty = \infty$	✓

But not σ-additive:

$\mu_0(\{k\}) = 0$ for all $k \in \mathbb{N}$

$$\mu_0\left(\bigcup_{k=1}^{\infty} \{k\}\right) = \mu_0(\mathbb{N}) = \infty \neq \sum_{k=1}^{\infty} \mu_0(\{k\}) = 0$$

Measure Spaces

A measure space consists of:

- A set S
- A σ-algebra Σ on S
- A σ-additive function $\mu : \Sigma \to [0, \infty]$

s.t. $\mu(\emptyset) = 0$. (which we call a measure)

A measure space (S, Σ, μ) is called a probability space if μ is a probability measure, i.e. $\mu(S) = 1$.

Example (finite probability space)
Let $S = \{s_1, .., s_k\}$ be a finite set of outcomes (e.g. $S = \{1, .., 6\}$ for a die, $S = \{\text{"heads"}, \text{"tails"}\}$) and associate probabilities $p_1, .., p_k$ with $s_1, .., s_k$ such that $p_1 + p_2 + .. + p_k = 1$.
Set $\mu(A) = \sum_{i: s_i \in A} p_i$ for $A \in \Sigma := \mathcal{P}(S)$.

This defines a probability space (S, Σ, μ); $\mu(A)$ represents the probability that A occurs.

Example (Lebesgue Measure)
Let $S = \mathbb{R}$, $\Sigma = \mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$, i.e. the smallest σ-algebra that contains all open subsets of $\mathbb{R}$. Important: $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$!
But $\mathcal{B}(\mathbb{R})$ contains a lot more than just open sets.
For unions of open disjoint intervals
$A = (a_1, b_1) \cup .. \cup (a_n, b_n)$ we let
$\mathcal{L}(A) = (b_1 - a_1) + .. + (b_n - a_n)$. This can be

extended to $\mathcal{B}(\mathbb{R})$. (Details later) and is called the Lebesgue measure.

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$ is a measure space.

Restricting to $([0,1], \mathcal{B}([0,1]), \mathcal{L}|_{[0,1]})$ gives a probability space. $\mathcal{L}|_{[0,1]}(A) = \mathcal{L}(A \cap [0,1])$ represents a uniformly random number from $[0,1]$.

General Properties of Measures

Let (S, Σ, μ) be a measure space. We have:

1) $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in \Sigma$.

2) Generally, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ for all $A_i \in \Sigma$.

3) More precisely,

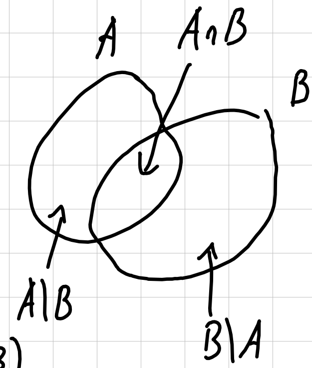
$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \quad (A, B \in \Sigma)$$

and generally, for $A_i \in \Sigma$,

$$\begin{aligned}\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = {} & \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n) \\ & - \mu(A_1 \cap A_2) - \mu(A_1 \cap A_3) - \ldots - \mu(A_{n-1} \cap A_n) \\ & + \mu(A_1 \cap A_2 \cap A_3) + \ldots + \mu(A_{n-2} \cap A_{n-1} \cap A_n) \\ & \vdots \\ & + (-1)^{n-1} \mu(A_1 \cap \ldots \cap A_n).\end{aligned}$$

"inclusion-exclusion principle"

Proof: Note that:



$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B)$$

$$\mu(B) = \mu(B \setminus A) + \mu(A \cap B)$$

$$\begin{aligned}
\mu(A \cup B) &= \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B) \\
&= \mu(A) - \mu(A \cap B) + \mu(B) \cancel{- \mu(A \cap B)} \cancel{+ \mu(A \cap B)} \\
&= \mu(A) + \mu(B) - \mu(A \cap B) \quad \text{as required} \quad \square
\end{aligned}$$

The general case follows by induction.

Monotonicity:

Let $(A_i)_{i=1}^{\infty}$ be an increasing sequence of sets in Σ, $\emptyset \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subseteq S$. Then,

$$\begin{aligned}
\mu(A_i) &= \mu((A_i \setminus A_{i-1}) \cup (A_i \cap A_{i-1})) \\
&= \mu(A_i \setminus A_{i-1} \cup A_{i-1}) \\
&= \mu(A_i \setminus A_{i-1}) + \mu(A_{i-1}) \\
&\geq \mu(A_{i-1}) \quad \text{and so} \quad 0 \leq \mu(A_1) \leq \mu(A_2) \leq \ldots
\end{aligned}$$

Since $(\mu(A_i))$ is an increasing sequence, the limit $L = \lim_i \mu(A_i)$ exists (but may be ∞).

Writing $A = \bigcup_{i=1}^{\infty} A_i$ we write $A_i \uparrow A$.

We have $\mu(A) = L$. This is because

$$A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \dots$$ disjoint union!

So
$$\mu(A) = \mu(A_1) + \mu(A_2 \setminus A_1) + \mu(A_3 \setminus A_2) + \dots$$
$$= \lim_{n\to\infty} \left(\mu(A_1) + \mu(A_2 \setminus A_1) + \dots + \mu(A_n \setminus A_{n-1})\right)$$
$$= \lim_{n\to\infty} \mu(A_n) = L.$$

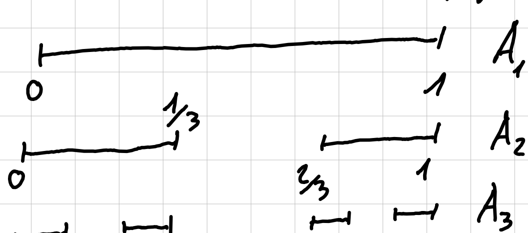
This also works for decreasing sequences:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \quad (A_i \in \Sigma).$$

Define $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$

We write $A_n \downarrow A$. In particular, if $\lim \mu(A_n) = 0$, then $\mu(A) = 0$.

Remark: Such null sets can be non-empty and even uncountable:



$$\mu(A_n) = 2^{n-1} \cdot 3^{-(n-1)} = \left(\tfrac{2}{3}\right)^{n-1} \to 0$$

but $A = \bigcap A_n$ is uncountable!