2019.01.09 (Solutions

(1) Let $X_t = e^{-2W_t}$. Then

$$dX_t = -2X_t\,dW_t + \frac{1}{2}2^2 X_t (dW)^2$$

$$= 2X_t\,dt - 2X_t\,dW_t.$$

Since X is a geometric Brownian motion,

$$E\left[e^{-2W_t}\right] = E[X_t] = X_0 e^{2t} = e^{2t}.$$ Answer: e^{2t}

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(2) Feynman-Kac gives

$$u(t,x) = E_{t,x}\left[X_T^3\right] \underset{\substack{\uparrow \\ \begin{cases} dX = dt + 2dW \\ X_t = x \end{cases}}}{=} E\left[\left(x + T - t + 2(W(T) - W(t))\right)^3\right]$$

$$= (x+T-t)^3 + 3(x+T-t)E\left[2^2(W(T)-W(t))^2\right]$$

$$+ 3(x+T-t)^2 \underbrace{E\left[2(W(T)-W(t))\right]}_{0} + 8\underbrace{E\left[(W(T)-W(t))^3\right]}_{0}$$

$$= (x+T-t)^3 + 12(x+T-t)(T-t)$$ Answer: $u(t,x) = (x+T-t)^3 + 12(x+T-t)(T-t)$

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(3) $$E^Q\left[e^{-rT}S(T)1_{\{S(T)\geq a\}}\right] = s\int_d^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}x} e^{-\frac{x^2}{2}}dx =$$

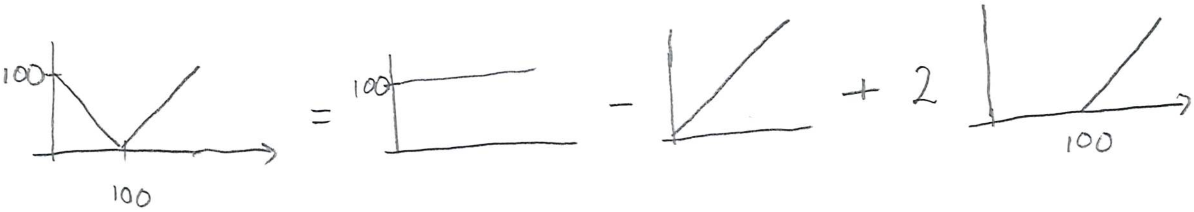
(where $d = \dfrac{\ln\frac{a}{s} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$)

$$= s\int_d^\infty \frac{e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}}dx = s\int_{d-\sigma\sqrt{T}}^\infty \varphi(y)dy = s\left(1 - N(-d+\sigma\sqrt{T})\right)$$

Answer: $s\left(1 - N\left(\dfrac{\ln\frac{s}{a} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right)$

(4) See the book.

(5)



but

$$10 < 2 \cdot 49 - 90 + 2 \cdot 4$$

At $t=0$:

- Buy one straddle
- Buy one share of S
- Sell two zero-coupon bonds
- Sell two call options

Receive $98 + 8 - 10 - 90 = 6$

At $t=T$: Receive the following.

If $S(T) < 100$, receive

$$\underbrace{100 - S(T)}_{\text{straddle}} + S(T) - 100 - 2 \cdot \underbrace{0}_{\text{call}} = 0$$

If $S(T) \geq 100$, receive

$$\underbrace{S(T) - 100}_{\text{straddle}} + S(T) - 100 - 2\underbrace{(S(T) - 100)}_{\text{call}} = 0$$

Thus we created an arbitrage!

⑥ a) Value is

$$V^h(t) = \underbrace{F_s(t,S(t))\cdot S(t)}_{\text{amount in stocks}} + \underbrace{F(t,S(t)) - S(t)F_s(t,S(t))}_{\text{amount in bank}}$$

$$= F(t,S(t))$$

b) h self-financing if $dV^h = F_s\,dS + \frac{F - SF_s}{B}\,dB$.

$$dV^h \underset{\text{by a) and Ito}}{=} F_t\,dt + F_s\,dS + \frac{1}{2}F_{ss}(dS)^2 =$$

$$= \left(F_t + \frac{1}{2}\sigma^2 S_t^2 F_{ss}\right)dt + F_s\,dS$$

$$\underset{\text{by PDE}}{=} (rF - rSF_s)\,dt + F_s\,dS$$

Equal!

and

$$F_s\,dS + \frac{F - SF_s}{B}\,dB \overset{dB = rB\,dt}{=} F_s\,dS + (rF - rSF_s)\,dt$$

Thus h is self-financing.

c) The claim $\mathcal{X} = \Phi(S(T))$ can be replicated using h.

Therefore the only possible arbitrage-free price of $\mathcal{X}$ is $F(t,s)$.

⑦ a) $F_\delta(0,s) = \underset{\text{price if no dividends}}{F_0(0, se^{-\delta T})} = se^{-\delta T}$

b) At $t=0$, buy $e^{-\delta T}$ shares of S, and re-invest all dividends in the stock. If stock holdings at t is $m(t)$, then one can buy new shares at rate δm.

$$\begin{cases} \dot{m}(t) = \delta m \\ m(0) = e^{-\delta T} \end{cases} \text{ gives } m(t) = e^{-\delta(T-t)}$$

In particular, $m(T) = 1$ so we have replicated $\mathcal{X}$.

(8) $\begin{cases} dr(t) = \sigma(t)\, dW(t) \\ r(0) = r_0 \end{cases}$

a) $p(t,T) = F(t, r(t))$ where $F(t,r)$ solves the term structure equation

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 F_{rr} - rF = 0 \\ F(T,r) = 1 . \end{cases}$$

The Ansatz $F(t,r) = \exp\{A(t,T) - B(t,T)r\}$ gives

$$\begin{cases} A_t + \frac{1}{2}\sigma^2 B^2 - (B_t + 1)r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$$

so $\begin{cases} B(t,T) = T - t \\ A(t,T) = \frac{1}{2}\int_t^T \sigma^2(s)(T-s)^2\, ds \end{cases}$

<u>Answer</u>: $p(0,T) = \exp\left\{\frac{1}{2}\int_0^T \sigma^2(s)(T-s)^2 ds - r_0 T\right\}$

b) Since $\frac{1}{2}\int_0^T \sigma^2(s)(T-s)^2 ds \geq 0$,

we have (from a) that $p(0,T) \geq e^{-r_0 T}$.

If observed prices $p^*(0,T) < e^{-r_0 T}$, the above model <u>cannot</u> be fitted against real data.