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PROV I MATEMATIK
Markov Processes
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Permitted aids: Calculator

1. Let $(X_n)_{n=0}^{\infty}$ be a Markov chain with state space $S = \{1,2,3,4,5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Classify the states according to the canonical decomposition. (1p)

(b) Calculate $P(X_n \neq 2, \forall n \geq 1 | X_0 = 1)$. (2p)

(c) Let $T_{15} = \inf\{n : X_n = 5 | X_0 = 1\}$. Calculate $E(T_{15})$. (2p)

Solution:

(a) By drawing a transition diagram we see that states $1,2,3,4$ are transient, and state 5 is absorbing. Canonical decomposition: $S = C \cup T$, where is $C = \{5\}$ absorbing, and $T = \{1,2,3,4\}$ is the set of transient states.

(b) $P(X_n \neq 2, \forall n \geq 1 | X_0 = 1) = \sum_{n=1}^{\infty} P(X_n = 3, X_k = 1, 1 \leq k \leq n-1 | X_0 = 1) = \sum_{n=1}^{\infty} \frac{1}{4}(1/2)^{n-1} = 1/2$.

(c) Let $T_{ij} = \inf\{n : X_n = j | X_0 = i\}$. Since all paths from state 1 to state 5 goes via state 3 it follows that $E(T_{15}) = E(T_{13}) + E(T_{35})$.

By conditioning on the outcome of the first step we have

$$E(T_{13}) = 1 + \frac{1}{2}E(T_{13}) + \frac{1}{4}E(T_{23}),$$

and

$$E(T_{23}) = 1 + \frac{1}{2}E(T_{13}) + \frac{1}{2}E(T_{23})$$

so $E(T_{13}) = 6$, and $E(T_{35}) = E(T_{45}) = 1 + \frac{1}{2}E(T_{35}) + \frac{1}{4}E(T_{45})$, so $E(T_{35}) = 4$. Thus $E(T_{15}) = E(T_{13}) + E(T_{35}) = 6 + 4 = 10$.

2. A taxi moves between the airport (state 1) and two hotels A and B (state 2 and state 3 respectively) according to the following rules: If it is at the airport, then it will next drive to hotel A with probability 0.4 and hotel B with probability 0.6. If it is at a hotel, then it will next always drive to the airport.

The taxi driver needs to wait for customers at each location. Suppose the waiting times are independent exponentially distributed random variables with mean 12 minutes, 30 minutes and 10 minutes respectively for the 3 (ordered) states.

Let Y_t be the location of the taxi when the total accumulated waiting time is t (hours).

(a) Compute the generator $\mathbf{Q}$ of the Markov process $(Y_t)_{t\geq 0}$, and compute the long run distribution of Y_t. (3p)

(b) Suppose it takes 15 minutes for the taxi to drive between the airport and hotel A and 30 minutes for the taxi to drive between the airport and hotel B. Calculate the average time between arrivals of the taxi to hotel A. (3p)

Solution:

(a) The location of the taxi after n trips is a Markov chain $(X_n)_{n=0}^{\infty}$ with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0.4 & 0.6 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Markov chain $(X_n)_{n=0}^{\infty}$ is the jump chain for the the process $(Y_t)_{t\geq 0}$ with generator

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

and it therefore follows that $p_{ij} = -q_{ij}/q_{ii}$, if $i \neq j$.

Since the length of sojourns in states 1, 2, and 3 (measured in hours) have exponential distributions with intensity parameters $q_1 = -q_{11} = 60/12 = 5$, $q_2 = -q_{22} = 60/30 = 2$, $q_3 = -q_{33} = 60/10 = 6$, and $q_{ij} = -q_{ii}p_{ij}$ for $i \neq j$, it follows that

$$\mathbf{Q} = \begin{pmatrix} -5 & 2 & 3 \\ 2 & -2 & 0 \\ 6 & 0 & -6 \end{pmatrix}.$$

Since $(Y_t)_{t\geq 0}$ is an irreducible Markov process with finite state space it follows from the convergence theorem that

$$\lim_{t\to\infty} P(Y_t = j | Y_0 = i) = \pi_j,$$

where the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$ is the unique solution to $\pi\mathbf{Q} = \mathbf{0}$, with $\sum_{i=1}^{3} \pi_i = 1$, and $\pi_i \geq 0$.

It therefore follows that the long run distribution of Y_t is the unique stationary distribution given by $\pi = (2/5, 2/5, 1/5)$.

(b) Let T be the time between arrivals of the taxi to hotel A (measured in hours). We have $T = T_W + T_D$ where T_W denotes the waiting time and T_D denotes the driving time between arrivals to hotel A. Since $\pi_2 = -1/(q_{22}E(T_W))$ i.e. $2/5 = 1/(2E(T_W))$ it follows that $E(T_W) = 5/4$.

Let T_{ij} denote the driving time in hours from state i until state j is reached. We have $E(T_D) = 1/4 + E(T_{12})$, since it takes 15 minutes (1/4 hours) to drive from hotel A to the airport and on average $E(T_{12})$ minutes drive from the airport to hotel A. (Note that the taxi starting at the airport may go via routes to hotel B before it drives to hotel A). By conditioning on the outcome of the first trip away from the airport we get $E(T_{12}) = 0.4 \cdot 1/4 + 0.6(1/2 + E(T_{32}))$, and $E(T_{32}) = 1/2 + E(T_{12})$, so

$$E(T_{12}) = 0.4 + 0.6E(T_{32}) = 0.4 + 0.6(1/2 + E(T_{12})) = 0.7 + 0.6E(T_{12}),$$

and thus $E(T_{12}) = 7/4$ and therefore $E(T_D) = 1/4 + 7/4 = 2$. **Alternatively**, by conditioning on the number of visits to state 3 (i.e. trips to hotel B) between two visits to state 2 (hotel A) for the jump-chain (and noting that a return trip to hotel B from the airport is a 1h drive), a direct calculation gives:

$$\begin{aligned} E(T_D) &= \sum_{k=0}^{\infty} \underbrace{E(T_D|k \text{ trips to state } 3)}_{k+0.5} \underbrace{P(k \text{ trips to state } 3)}_{0.4\cdot 0.6^k} \\ &= 0.5 + 0.6 \underbrace{\sum_{k=0}^{\infty} k \cdot 0.4 \cdot 0.6^{k-1}}_{\frac{1}{0.4}} = 0.5 + 1.5 = 2, \end{aligned}$$

(In the last line we used the fact that if a random variable has the first success distribution with parameter p then the expected value is $1/p$.)

Thus $E(T) = E(T_W) + E(T_D) = 5/4 + 2 = 3.25$ hours (=3 hours and 15 minutes).

3. Consider a homogeneous Markov chain, $(X_n)_{n=0}^{\infty}$, with state space $S = \{0, 1, ...\}$, with $P(X_{n+1} = 0 \mid X_n = j) = \frac{j+1}{j+2}$, and $P(X_{n+1} = j+1 \mid X_n = j) = \frac{1}{j+2}$, $j \geq 0$.

   (a) Prove that this Markov chain is positive recurrent. (3p)

   (b) Find the limit $\lim_{n\to\infty} P(X_n = 2 \mid X_0 = 2)$. (3p)

Solution: The Markov chain, $(X_n)_{n=0}^{\infty}$ is irreducible, and also aperiodic since $P(X_{n+1} = 0 \mid X_n = 0) = \frac{1}{2} > 0$. By irreducibility it suffices to prove that one state, e.g. state 0, is positive. Let $T_{00} = \inf\{n \geq 1 : X_n = 0 \mid X_0 = 0\}$ be the first return time to state 0 if the Markov chain starts in zero.

(a) By definition

$$P(T_{00} = k) = P(X_k = 0, X_{k-1} = k-1, \ldots, X_1 = 1 \mid X_0 = 0) = \frac{k}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}, \ k \geq 1,$$

so

$$E(T_{00}) = \sum_{k=1}^{\infty} kP(T_{00} = k) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} - \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = e - 1 < \infty.$$

(b) By the convergence theorem it follows that $\lim_{n\to\infty} P(X_n = 2 \mid X_0 = 2) = \pi_2$, where $\pi = (\pi_j)$ is the unique stationary distribution, satisfying $\pi_j \geq 0$ for all $j \geq 0$, and

$$\pi_{j+1} = \frac{\pi_j}{j+2}, \ j \geq 0,$$

so

$$\pi_k = \frac{\pi_0}{(k+1)!},$$

and since $\sum_{i=0}^{\infty} \pi_i = 1$, (or since $\pi_0 = 1/E(T_{00})$), it therefore follows that

$$\pi_k = \frac{1}{(k+1)!(e-1)}.$$

Thus

$$\lim_{n\to\infty} P(X_n = 2 \mid X_0 = 2) = \frac{1}{6(e-1)}.$$

4. A call center has n operators where each operator can serve only one call. Suppose the incoming calls form a Poisson process with rate λ and that the time it takes for an operator to serve a customer is exponential with mean $1/\mu$, independent of other customers and the process of incoming calls. If all operators are busy at the time of an incoming call, then that incoming call is rejected and will never be treated.

   (a) Find the long-run proportion of non-treated calls. (3p)

   (b) Suppose $\lambda = \mu$. How many operators are needed in order to make the probability of an incoming call being rejected at most 0.005? (3p)

Solution:

Let X_t be the number of busy operators at time t. $(X_t)_{t\geq 0}$ is a Birth-death process on $S = \{0, 1, ..., n\}$ with generator

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdot \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdot \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & n\mu & -n\mu \end{pmatrix}.$$

In order to find the stationary distribution we solve $\pi\mathbf{Q} = 0$, where $\pi = (\pi_0, ..., \pi_n)$. This gives the balance equations $\pi_k k\mu = \pi_{k-1}\lambda$, $1 \leq k \leq n$.

Thus $\pi_k = \frac{(\lambda/\mu)^k}{k!}\pi_0$, $0 \leq k \leq n$.

$$\sum_{k=0}^{n} \pi_k = 1 \Leftrightarrow \pi_0 \sum_{k=0}^{n} \frac{(\lambda/\mu)^k}{k!} = 1.$$

(a) A call is non-treated if all operators are busy at the time of an incoming call. Therefore the proportion of non-treated calls is

$$\pi_n = (\frac{(\lambda/\mu)^n}{n!})/(\sum_{k=0}^{n} \frac{(\lambda/\mu)^k}{k!}).$$

(b) Suppose $\lambda = \mu$. Then $\pi_n = (\frac{1}{n!})/(\sum_{k=0}^{n} \frac{1}{k!})$.
We want to find the smallest n making $(\frac{1}{n!})/(\sum_{k=0}^{n} \frac{1}{k!}) \leq 0.005$, or equivalently

$$200 \leq n! \sum_{k=0}^{n} \frac{1}{k!}.$$

The smallest n satisfying this equation is clearly > 4 since

$$n! \sum_{k=0}^{n} \frac{1}{k!} \leq (n+1)!,$$

and $5! = 120$.
Since

$$200 \leq 5! \sum_{k=0}^{5} \frac{1}{5!},$$

at least 5 operators are needed.

5. An internet sales company receives orders according to a Poisson process with intensity parameter $\lambda > 0$. All received orders are collected and manually treated at timepoints forming another independent Poisson process with parameter $\mu > 0$.

   (a) Calculate the long run distribution of the number of untreated orders. (3p)

   (b) Suppose there are no untreated orders at time $t = 0$. Prove that the probability that there are no untreated orders at time $t \geq 0$ is given by $p_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$. (3p)

Solution: Let X_t = the number of untreated orders at time t. Then (X_t) is a Markov process with state space $S = \{0, 1, 2, \ldots\}$ and generator

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \\ \mu & 0 & -(\lambda+\mu) & \lambda & \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

The process (X_t) is irreducible and positive recurrent and therefore there exists a unique stationary distribution $\pi = (\pi_0, \pi_1, ...)$.

(a) The distribution π satisfies $\pi\mathbf{Q} = \mathbf{0}$. Thus

$$-\lambda\pi_0 + \mu \underbrace{(\pi_1 + \pi_2 + ...)}_{1-\pi_0} = 0,$$

and

$$\lambda\pi_{k-1} - (\lambda+\mu)\pi_k = 0,$$

i.e. π is the geometric distribution with $\pi_k = \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^k$, $k \geq 0$.

(b) From the forward equations it follows that $p'_{00}(t) = \mu - (\lambda+\mu)p_{00}(t)$, and $p_{00}(0) = 1$, and thus $p_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$.
(Note that we get the same equation if we regard the state space as having two states corresponding to no untreated orders and at least one untreated order.)

6. Let $(B_t)_{t\geq 0}$ a standard Brownian motion.

   (a) Let $Y_t = B_{t/2} - B_{t/4}$. Is $(Y_t)_{t\geq 0}$ a Brownian motion?
   Motivate your answer by checking the defining properties of a Brownian motion. (3p)

   (b) Compute $P(\max_{2\leq t\leq 3} B_t < B_1)$. (3p)

Solution:

(a) $Y_t = B_{t/2} - B_{t/4}$ does not define a Brownian motion. E.g. the increments $Y_1 - Y_0 = B_{1/2} - B_{1/4} \sim N(0, 1/4)$, and $Y_2 - Y_1 = (B_1 - B_{1/2}) - (B_{1/2} - B_{1/4}) \sim N(0, 3/4)$, since $B_1 - B_{1/2} \sim N(0, 1/2)$ and $B_{1/2} - B_{1/4} \sim N(0, 1/4)$ are independent by the independent increments property of (B_t). Thus (Y_t) does not have stationary increments so $(Y_t)_{t\geq 0}$ is not a Brownian motion.
((Y_t) does also not have the independent increments property since $Y_1 - Y_0 \sim N(0, 1/4)$ and $Y_2 - Y_1 \sim N(0, 3/4)$ and $Y_2 = (Y_2 - Y_1) + (Y_1 - Y_0)$, so the increments $(Y_2 - Y_1)$ and $(Y_1 - Y_0)$ cannot be independent since by definition $Y_t \sim N(0, t/4)$ so $Y_2 \sim N(0, 1/2)$.)

(b) Since

$$\begin{aligned} P(\max_{2\leq t\leq 3} B_t < B_1) &= P(\max_{0\leq t\leq 1}(B_{t+2}-B_2)+B_2 < B_1) \\ &= P(\max_{0\leq t\leq 1}(B_{t+2}-B_2) < B_1 - B_2), \end{aligned}$$

and $\{B_{t+2}-B_2\}_{t\geq 0}$ is a standard Brownian motion independent of $B_2 - B_1$, and the random variable $\max_{0\leq t\leq 1}(B_{t+2}-B_2)$ has the same distribution as $|B_3 - B_2|$ by the reflection principle, and $X := B_3 - B_2$ and $Y := B_1 - B_2$ are independent standard normal random variables, it follows by symmetry of the 2 dimensional standard normal distribution, that

$$P(\max_{2\leq t\leq 3} B_t < B_1) = P(|X| < Y) = P(-Y < X < Y) = 1/4.$$

Alternative solution: Since $(B_{1+t}-B_1)_{t\geq 0}$ is a standard Brownian motion it follows that

$$\begin{aligned} P(\max_{2\leq t\leq 3} B_t < B_1) &= P(\max_{1\leq t\leq 2}(B_{1+t}-B_1) < 0) \\ &\underbrace{=}_{\text{symmetry}} \frac{1 - P((B_{1+t}-B_1)_{t\geq 0} \text{ has at least one zero in the interval } (1,2))}{2} \\ &= \frac{1-(1-\frac{2}{\pi}\arcsin\sqrt{1/2})}{2} \\ &= \frac{\arcsin\sqrt{1/2}}{\pi} = \frac{1}{4}. \end{aligned}$$

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