Q1 a) We have $T = \min\{n : S_n \le 0 \text{ or } S_n \ge K\}$

Since $S_n = a + \sum_{k=1}^{n} X_k$ we have $\{S_n < a\}, \{S_n > a\}$ in $\sigma(X_1, X_2, .. X_n) = \tilde{F}_n$ and so

$\{0 < S_n\} \cap \{S_n < K\} = \{0 < S_n < K\} \in \tilde{F}_n$

and so is its complemented $A_n = \{S_n \le 0 \text{ or } S_n \ge K\} \in \tilde{F}_n$.

Hence $\{T \le k\} = A_1 \cup A_2 \cup .. \cup A_k \in \tilde{F}_k$ and T is a stopping time.

b) We compute, for $\tilde{F}_n = \sigma(X_1, .., X_n)$

$$\mathbb{E}(M_n - M_{n-1} \mid \tilde{F}_{n-1}) = \mathbb{E}\left(S_n - \tfrac{1}{3}S_n^3 + \tfrac{1}{3}S_{n-1}^3 \mid \tilde{F}_{n-1}\right)$$

$$= \mathbb{E}\left(S_{n-1} + X_n - \tfrac{1}{3}(S_{n-1} + X_n)^3 + \tfrac{1}{3}S_{n-1}^3 \mid \tilde{F}_{n-1}\right)$$

$$= S_{n-1} + \tfrac{1}{3}S_{n-1}^3 + \mathbb{E}(X_n \mid \tilde{F}_{n-1}) - \tfrac{1}{3}\mathbb{E}\big(S_{n-1}^3 + 3S_{n-1}^2 X_n + 3S_{n-1}X_n^2 + X_n^3 \mid \tilde{F}_{n-1}\big)$$

$$= S_{n-1} + \tfrac{1}{3}S_{n-1}^3 + 0 - \tfrac{1}{3}S_{n-1}^3 - S_{n-1}^2 \overbrace{\mathbb{E}(X_n)}^{0} - S_{n-1}\mathbb{E}(X_n^2) + \frac{\mathbb{E}(X_n^3)}{3}$$

$$= S_{n-1} - S_{n-1}\left(\tfrac{1}{2}(-1)^2 + \tfrac{1}{2}(1)^2\right) + \tfrac{1}{3}\left(\tfrac{1}{2}(-1)^3 + \tfrac{1}{2}(1)^3\right)$$

$$= S_{n-1} - S_{n-1} \cdot 1 + \tfrac{1}{3} \cdot 0 = 0.$$

Since further $\mathbb{E}(M_n) < \infty$ and M_n is $\tilde{F}_n$-measurable, M_n is a martingale.

c) Assume the DOST holds. Then

$$\mathbb{E}(M_T) = \mathbb{E}(M_0) = S_0 - \frac{1}{3}S_0^3 = a - \frac{a^3}{3}$$

and $\mathbb{E}(M_T) = \mathbb{E}\left(\sum_{r=0}^{T} S_r - \frac{1}{3}S_T^3\right) = \mathbb{E}\left(\sum_{r=0}^{T} S_r\right) - \frac{1}{3}\mathbb{E}(S_T^3)$

$= \mathbb{E}\left(\sum_{r=0}^{T} S_r\right) - \frac{1}{3}\mathbb{P}(S_T = K)\, K^3.$

From the course, we have seen that the probability that a symetric standard random walk starting at 0 hits $b = K-a$ before $c = -a$ is $\frac{-c}{b+(-c)}$ i.e. $\frac{a}{K}$. Hence,

$$a - \frac{a^3}{3} = \mathbb{E}\left(\sum_{r=0}^{T} S_r\right) - \frac{1}{3}\frac{a}{K}K^3$$

$$\Rightarrow \mathbb{E}\left(\sum_{r=0}^{T} S_r\right) = a + \frac{aK^2}{3} - \frac{a^3}{3} = \frac{aK^2 - a^3}{3} + a$$

as required. It remains to check that DOST is satisfied. However, this is a priori not true: while $\mathbb{E}(T) < \infty$ (from course), we don't have $|M_{n+1} - M_n| = |S_{n+1} - S_{n+1}^3/3 + \frac{S_n^3}{3}|$ bounded.

Consider instead $\tilde{M}_n = M_{n \wedge T}$. Here,

$$|\tilde{M}_{n+1} - \tilde{M}_n| = \left| S_{(n+1)\wedge T} - \frac{S^3_{(n+1)\wedge T}}{3} + \frac{S^3_{n\wedge T}}{3} \right|$$

$$\leq K + \frac{K^3}{3} + \frac{K^3}{3} < \infty.$$

Since further, $\tilde{M}_T = M_T$ we may apply DOST to $\tilde{M}$ and hence M.

Q2 a) Let $f_1(x) = \frac{5}{4}x + \frac{1}{4}x^2 - \frac{1}{4}x^3 - \frac{1}{4}x^4$

$$f_2(x) = \frac{5}{6}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{2}x^4$$

$$f_3(x) = \frac{11}{12}x + \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{1}{4}x^4$$

and note that $f_1(0) = f_2(0) = f_3(0) = 0$,

$f_1(-1) = f_2(-1) = f_3(-1) = -1$, and

$f_1(1) = f_2(1) = f_3(1) = 1$. Further

$$f_1'(x) = \frac{5}{4} + \frac{1}{2}x - \frac{3}{4}x^2 - x^3 \geq \frac{5}{4} + \frac{1}{2}x - x^2 - x^3$$

$$\geq \frac{5}{4} + \frac{1}{2}x - x^2 - x \qquad \text{for } x \in [-1,1]$$

$$= \frac{5}{4} - x\left(\frac{1}{2} + x\right) > 0.$$ Hence $f_1([-1,0]) = [-1,0]$

and $f_1([0,1]) = [0,1]$. A similar analysis establishes the same result for f_2 and f_3.

Given that $X_0 \in [-1,1]$, we have $X_n \in [-1,1]$

for all n. In particular X_n is bounded.

Since $\mathbb{E}(X_{n+1} \mid \tilde{\mathcal{F}}_n)$

$= \frac{1}{3} f_1(X_n) + \frac{1}{3} f_2(X_n) + \frac{1}{3} f_3(X_n) = X_n$ [calc omitted]

X_n is a bounded martingale. In particular,

it is a UI martingale and $X_n \to X_\infty$ a.s.

b) Since $X_\infty = \lim_{n\to\infty} X_n = \lim_{n\to\infty} X_{n+1}$ we have

$\lim_n |X_n - X_{n+1}| = 0$. Hence, $|f_i(X_n) - X_n| \to 0$

and
$$\begin{cases} \frac{1}{4} X_n + \frac{1}{4} X_n^2 - \frac{1}{4} X_n^3 - \frac{1}{4} X_n^4 \to 0 \\ -\frac{1}{6} X_n - \frac{1}{2} X_n^2 + \frac{1}{6} X_n^3 + \frac{1}{2} X_n^4 \to 0 \\ -\frac{1}{12} X_n + \frac{1}{4} X_n^2 + \frac{1}{12} X_n^3 - \frac{1}{4} X_n^4 \to 0 \end{cases}$$

These have common solutions $\{-1, 0, 1\}$ and so

$L = \{-1, 0, 1\}$.

c) Since X_n is a UI martingale we have

$\mathbb{E}(X_\infty) = \mathbb{E}(X_0) = 0$. This gives

$0 = (-1)\mathbb{P}(X_\infty = -1) + (1)\mathbb{P}(X_\infty = 1) + (0)\mathbb{P}(X_\infty = 0)$ and

$\mathbb{P}(X_\infty = -1) = \mathbb{P}(X_\infty = 1)$. Now consider the event $A = \{X_0 \geq 0\}$. We have $\mathbb{P}(A) = \frac{1}{2}$ and since $f_i[0,1] = [0,1]$, we get that

$$\mathbb{E}(X_\infty \mid A) = \mathbb{E}(X_0 \mid A)$$ by using UI.

Hence $\frac{1}{2} = \mathbb{E}(X_0 \mid A) = \mathbb{E}(X_\infty \mid A) = 1 \cdot \mathbb{P}(X_\infty = 1 \mid A)$

Hence $\mathbb{P}(X_\infty = 1 \mid A) = \frac{1}{2} = \mathbb{P}(X_\infty = 0 \mid A)$

and similarly $\mathbb{P}(X_\infty = -1 \mid A^c) = \frac{1}{2} = \mathbb{P}(X_\infty = 0 \mid A^c)$.

Thus $\mathbb{P}(X_\infty = 1) = \mathbb{P}(X_\infty = -1) = \frac{1}{2} \cdot \mathbb{P}(X_\infty = 1 \mid A) = \frac{1}{4}$

and $\mathbb{P}(X_\infty = 0) = \frac{1}{2}$.

Q3 We will show that $\{X_a\}_{a \in A}$ is UI iff $\inf A > 0$. To do so we first assume $\inf A > 0$. Notice that (using integ. by parts)

$$\|X_a\|_2^2 = \int_0^\infty a x^2 e^{-ax} \, dx = \left[x^2 (-e^{-ax})\right]_0^\infty + \int_0^\infty e^{-ax} 2x \, dx$$

$$= 0 + \left[2x \frac{-1}{a} e^{-ax}\right]_0^\infty + \int_0^\infty \frac{1}{a} e^{-ax} 2 \, dx$$

$$= 0 + \left[\frac{2}{a} \frac{-1}{a} e^{-ax}\right]_0^\infty = \frac{2}{a^2} \leq \frac{2}{(\inf A)^2} < \infty$$

and so $\|X_a\|_2 \leq K$ for some K. Thus $\{X_a\}$ is UI.

For the converse, first assume $\{X_a\}$ is U.I. Then, for all $\varepsilon > 0$, there exists $K > 0$ such that $\mathbb{E}(|X_a| ; |X_a| > K) < \varepsilon$.

We have

$$\mathbb{E}(|X_a| ; |X_a| > K) = \int_K^\infty a x e^{-ax} = \left(\frac{1}{a} + K\right) e^{-aK}$$

Thus $\frac{1}{a} e^{-aK} \leq \varepsilon$. Now assume for a contradiction that $\inf A = 0$. Then there exists a_n s.t. $a_n \to 0$ as $n \to \infty$, and $e^{-a_n K} \leq a_n$ (letting $\varepsilon = 1$). But the LHS tends to 1, whereas the RHS tends to 0, a contradiction.

We must have $\inf A > 0$. $\square$

Q4 a) First, we note that $\sigma(Y_n) \subseteq \tilde{\mathcal{F}}_n$ and as f is measurable, so is $f(Y_n)$.
By the boundedness of f, we further have $\sup_n \mathbb{E}(f(Y_n)) \leq K$. Finally,

$$\mathbb{E}(f(Y_{n+1}) \mid \tilde{\mathcal{F}}_n) = \mathbb{E}(f(Y_n + X_{n+1}) \mid \tilde{\mathcal{F}}_n)$$

$$\underset{a.s.}{=} \mathbb{E}(f(y + X_{n+1})) \quad \text{(where } y = Y_n \text{ is fixed)}$$

$$= \int_{\mathbb{R}} f(y+x)\, d\mu(x) = f(y) \quad \text{(by convolution eq.)}$$

$= f(Y_n)$ a.s.. Hence $f(Y_n)$ is a martingale that converges by Doob's convergence theorem.

That $f(Y_n)$ is UI follows immediately from f being bounded, e.g. using $\| f(Y_n) \|_2 \leq K$ and the results in lecture.

b) Since the limit M_∞ of $M_n = f(Y_n)$ is independent of any finite permutation of $X_1, \dots, X_n$, the Savage-Hewitt 0-1 law implies that the event $\{M_\infty > c\}$ (and its complement) have probability 0 or 1. Since M_∞ converges to some finite value, we must have some $c_0 \in \mathbb{R}$ s.t. $\mathbb{P}(M_\infty \leq c_0 - \varepsilon) = 0$ and $\mathbb{P}(M_\infty > c_0 + \varepsilon) = 1$ for all $\varepsilon > 0$. In other words M_∞ is constant almost surely.

The UI condition implies $\mathbb{E}(M_\infty) = \mathbb{E}(M_0) = \mathbb{E} f(x) = f(x)$. And since M_∞ is constant a.s. we have $f(x) = \mathbb{E}(M_\infty) \underset{a.s}{=} M_\infty = \lim_{n\to\infty} f(Y_n)$ as required.

We can now conclude the "$\Rightarrow$" direction of the theorem: Assume there exists positive probability that $f(x+y) \neq f(x)$, wlog we assume

$\mathbb{P}(\{y \in \mathbb{R} : f(x+y) > f(x) + \varepsilon_1\}) \geq \varepsilon_2$ for some $\varepsilon_1, \varepsilon_2 > 0$.

but then, with positive probability,

$$\lim_n f(x+y+Y_2+Y_3+\dots+Y_n) = f(x+y) \geq f(x)+\varepsilon_1$$

$$> f(x) = \lim_n f(x+Y_1+\dots+Y_n).$$

This is a contradiction as the LHS and RHS must be equal a.s.. The other implication of the theorem is trivial $\square$

c) If we can show that a tail event is symmetric we are done as then the σ-algebra of symmetric events is larger. (Not nec. strictly)

Recall that the tail algebra is defined by $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$ where $\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \dots)$.

Hence, any $E \in \mathcal{T}$ must be in all $\mathcal{T}_n$.

Let N be the largest index of a permutation.

Then, $E \in \mathcal{T}_{N+1}$ implies that the order of the first N indices did not matter. Since N is arbitrary, we must have E also symmetric.

Q5

a)

$$\begin{tikzcd}[column sep=large]
 & & S_2 = 6 \\
 & S_1 = 7 \arrow[ru, "q_{00}"] \arrow[r, "q_{01}"'] & S_2 = 8 \\
 & & S_2 = 8 \\
S_0 = 8 \arrow[ruu, "q_0"] \arrow[r, "q_1"] \arrow[rdd, "q_2"'] & S_1 = 9 \arrow[ru, "q_{10}"] \arrow[r, "q_{11}"'] & S_2 = 11 \\
 & & S_2 = 8 \\
 & S_1 = 11 \arrow[ru, "q_{20}"] \arrow[rd, "q_{21}"'] & \\
 & & S_2 = 12
\end{tikzcd}$$

Any EMM must satisfy $\mathbb{E}(S_n \mid \mathcal{F}_{n-1}) = S_{n-1}$

We get $\begin{cases} q_{00}\,6 + q_{01}\,8 = 7 \\ q_{00} + q_{01} = 1 \end{cases} \Rightarrow q_{00} = q_{01} = \frac{1}{2}$

$\begin{cases} q_{10}\,8 + q_{11}\,11 = 9 \\ q_{10} + q_{11} = 1 \end{cases} \Rightarrow q_{11} = \frac{1}{3},\ q_{10} = \frac{2}{3}$

$\begin{cases} q_{20}\,8 + q_{21}\,12 = 11 \\ q_{20} + q_{21} = 1 \end{cases} \Rightarrow q_{21} = \frac{2}{3} \quad q_{20} = \frac{1}{3}$

$\begin{cases} q_0\,7 + q_1\,9 + q_2\,11 = 8 \\ q_0 + q_1 + q_2 = 1 \end{cases} \Rightarrow \begin{cases} q_1 + q_2 = 1 - q_0 \\ 2q_1 + 4q_2 = 1 \end{cases} \Rightarrow \begin{cases} q_1 + q_2 = 1 - q_0 \\ q_1 = \frac{1}{2} - 2q_2 \end{cases}$

Thus, for $p \in (0, \frac{1}{4})$, all choices of $q_2 = p$, $q_1 = \frac{1}{2} - 2p$,

$q_0 = 1 - p - (\frac{1}{2} - 2p) = \frac{1}{2} + p$ give an EMM.

b & c) By the first & second fundamental theorem of asset pricing the model is viable since there exists an EMM. The model is not complete since the EMM is not unique.

Q6 Using Call-Put parity, and the formula from the problem class,

$$P_0(E) = e^{-rT} K \, \Phi(-d_-) - S_0 \, \Phi(-d_+),$$

where $d_\pm = \dfrac{\log S_0/K + (r \pm \sigma^2/2) T}{\sigma \sqrt{T}}$.

Note that $\dfrac{\partial d_+}{\partial S_0} = \dfrac{\partial d_-}{\partial S_0} = \dfrac{1}{\sigma \sqrt{T} S_0}$ and the chain rule

gives $\Delta = \dfrac{\partial P_0(E)}{\partial S_0} = -e^{-rT} K \, \Phi'(-d_-) \cdot \dfrac{1}{\sigma\sqrt{T} S_0} - \Phi(-d_+)$

$$+ S_0 \, \Phi'(-d_+) \cdot \frac{1}{\sigma \sqrt{T} S_0}$$

Since $\Phi'(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$ is just the density of the standard normal distribution, we can simplify:

$$\frac{e^{-rT} K \Phi'(-d_-)}{S_0 \Phi'(-d_+)} = \frac{e^{-rT} K}{S_0} e^{\frac{d_+^2 - d_-^2}{2}}$$

$$= e^{-rT} \frac{K}{S_0} e^{\frac{1}{2}(d_+ - d_-)(d_+ + d_-)}$$

$$= e^{-rT} \frac{K}{S_0} e^{\log S_0/K + rT} = 1.$$

Hence the first and last terms cancel and $\Delta = -\Phi(-d_+)$. Φ is increasing and d_+ is increasing in S_0. Hence Δ is increasing in S_0. Further $d_+ \to \infty$ as $S_0 \to \infty$ and $d_- \to -\infty$ as $S_0 \searrow 0$. We get

$$\lim_{S_0 \to \infty} \Delta = \lim_{d_+ \to \infty} -\Phi(-d_+) = 0$$

$$\lim_{S_0 \searrow 0} \Delta = \lim_{d_- \to -\infty} -\Phi(-d_+) = -1$$

as required.