UPPSALA UNIVERSITET
Matematiska Institutionen
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Prov i matematik
1MA255, omtenta
2023/08/18

Partial Differential Equations with Applications to Finance

Writing time: 08:00 - 13:00.
Instructions: There are 5 problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, 4 or 5, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no help materials are allowed.

GOOD LUCK!

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1. (8p) Consider $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}^+ \to \mathbb{R}$ where $u(t,x) = v(\frac{x^2}{t})$.

i. Show that

$$u_t - u_{xx} = 0$$

if and only if

$$4zv_{zz} + (2+z)v_z = 0, \quad z > 0.$$

ii. Show that the general solution to the above ODE is

$$v(z) = C_1 \int_0^z e^{-x/4} x^{-1/2} dx + C_2$$

for some constants C_1, C_2.

iii. Differentiate $v(\frac{x^2}{t})$ w.r.t. x and select the proper constant to obtain the fundamental solution of the heat equation in dimension 1.

2. (8p) Let $D = (-a, b)$ where $a, b > 0$ and let X_t be a standard Brownian motion. Compute

$$\mathbb{E}[X_\tau^p + \tau],$$

where $p > 0$ and $\tau = \inf\{t > 0 : X_t \notin D\}$

3. (8p) Consider a stochastic process X_t with $X_0 = x$, with

$$dX_t = a(b - X_t)dt + \sigma dW_t,$$

where $a, \sigma > 0$. Use the Fokker-Planck equation to find the limiting distribution of X.

4. (8p) Solve the problem to minimize

$$\mathbb{E}\left[\exp\left\{\int_0^T u_t^2 dt + X_T^2\right\}\right]$$

given the scalar dynamics

$$dX_t = u_t\, dt + \sigma dW_t$$

where the control u is scalar and there are no control constraints.

Hint: Make the ansatz

$$V(t,x) = e^{A(t)x^2 + B(t)}.$$

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x\left[e^{-\beta\tau} X_\tau^+\right]$$

where $\beta > 0$, $x^+ = \max(x, 0)$, and X_t is a Brownian motion with drift μ:

$$X_t = x + \mu t + B_t,$$

where B_t is a standard Brownian motion.