In F7 we constructed the Itô-integral $I_t(\Phi) = \int_0^t \Phi_s \, dB_s$, $t \geq 0$,

w.r.t. BM $\{B_t\}_{t\geq 0}$, ~~and~~ for

$$\Phi \in \mathcal{H} = \left\{ \Phi : [0,T] \times \Omega \to \mathbb{R}, \text{ mble}, \Phi_t \in \mathcal{F}_t, \int_0^T \mathbb{E}[\Phi_s^2] \, ds < \infty \right\}.$$

Key property: $\mathbb{E}\left[\left(\int_0^T \Phi_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T \Phi_s^2 \, ds\right]$

the Itô-isometry

and $\mathbb{E}\left[\int_0^t \Phi_s \, dB_s\right] = 0$

Moreover (Thm 4.3.1), $\{I_t(\Phi)\}_{t\geq 0}$ "is" [has a modification which is] a continuous martingale.

Along the lines of F8, we see that the quadratic variation is

$$[I(\Phi)]_t = \int_0^t \Phi_s^2 \, ds, \quad t \geq 0,$$

and hence $L = \{L_t\}_{t\geq 0}$, where

$$L_t = \left(\int_0^t \Phi_s \, dB_s\right)^2 - \int_0^t \Phi_s^2 \, ds,$$

is a martingale.

Compare Example 4.6.7 later, based on Itô formula.

Example · $M_t = B_t$ BM

$$B_t^2 - [B]_t = B_t^2 - t,\ t \geq 0 \quad \text{martingale}$$

· $M_t = N_t - \lambda t$ Poisson

$$M_t^2 - [M]_t = (N_t - \lambda t)^2 - N_t$$

$$= \underbrace{\underbrace{(N_t - \lambda t)^2 - \lambda t}_{\text{mart}} - \underbrace{(N_t - \lambda t)}_{\text{mart}}}_{\text{mart.}}$$

For $\{M_t\}$ a martingale,

Def The <u>predictable</u> <u>quadratic variation</u> is a <u>predictable</u> process $\langle M \rangle_t$ (e.g. deterministic, left continuous, ...) such that

$$[M]_t - \langle M \rangle_t,\ t \geq 0,$$

is a martingale

In above example we see that

- $\langle B \rangle_t = [B]_t = t$
- $\langle N \rangle_t = \lambda t$.

Read <u>Poirat</u> pages 15-17

F9:3

If $\{X_t\}$ has bounded variation,

$$X_t - X_0 = \int_0^t dX_s \quad \left[= \int_0^t X_s' \, ds \right. \quad \text{if } X_t \text{ differentiable}$$

For f a differentiable function

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) \, dX_s$$

BM does not have bounded variation.
Then what?

Theorem 4.5.1 Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable twice with continuous derivatives. Then

$$f(B_t) - f(B_0) = \underbrace{\int_0^t f'(B_s) \, dB_s}_{\substack{\in \mathcal{H}_{loc} \text{ Rem. 4.5.2} \\ \in \mathcal{H} \subset \mathcal{H}_{loc} \text{ under additional assumption}}} + \frac{1}{2} \int_0^t f''(B_s) \underbrace{ds}_{d[B]_s}$$

Example $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$

$$B_t^3 = \int_0^t 3 B_s^2 \, dB_s + \frac{1}{2} \int_0^t 6 B_s \, ds$$

In particular, $B_t^3 - 3\int_0^t B_s \, ds = 3 \int_0^t B_s^2 \, dB_s$, $t \geq 0$
defines a martingale.

This is Exercise ~~Ex~~ 4.7

F9:4

Thm 4.5.6 is time-space version for $f(t, B_t)$.

Instead of BM, let us start with an Itô process:

$$X_t = x_0 + \int_0^t u_s\,ds + \int_0^t v_s\,dB_s,$$

where $v \in \mathcal{H}$ (or $v \in \mathcal{H}_{loc}$) and u is adapted such that e.g. $E\left[\left(\int_0^t u_s\,ds\right)^2\right] < \infty$.

Differential shorthand notation:

$$dX_t = u_t\,dt + v_t\,dB_t$$

Let $f \in C^2(\mathbb{R})$ (as above). Then

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s$$

$$= \int_0^t f'(X_s)u_s\,ds + \int_0^t f'(X_s)v_s\,dB_s + \frac{1}{2}\int_0^t f''(X_s)v_s^2\,ds$$

Thm, the Itô formula shows that $Y_t = f(X_t)$, $t \geq 0$, is itself an Itô-process!

$$dY_t = \left(f'(X_t)u_t + \frac{1}{2}f''(X_t)v_t^2\right)dt + f'(X_t)v_t\,dB_t$$

Sketch of proof:

Start with u and v simple.

Apply Taylor expansion up to order 2.

$$f(X_t)-f(x_0) = \underbrace{\sum_i f'(\cdot)\Delta X_i}_{\to \text{Itô integral}} + \frac{1}{2}\underbrace{\sum_i f''(\cdot)(\Delta X_i)^2}_{\text{similar analysis as for } [X]_t} + \text{Rem.}$$

BM in $\mathbb{R}^d$: $B_t = (B_t^1, \dots, B_t^d)$

$f \in C^2(\mathbb{R}^d \to \mathbb{R})$

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds$$

scalar product

Thus, if f is harmonic, that is $\Delta f(x) = 0$, then $f(B_t)$, $t \geq 0$ is a martingale

Example 4.6.9:

$X_t = (\cos B_t, \sin B_t)$, $t \geq 0$

"BM on a circle"