Example $g(t,x) = \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma x\}$

$$Y_t = g(t, B_t) = e^{\mu t}\, e^{\sigma B_t - \frac{1}{2}\langle \sigma B \rangle_t}$$

By Itô's formula

$$dY_t = \dots = \mu Y_t\, dt + \sigma Y_t\, dB_t$$

In other words, $Y_t = e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$, $t \geq 0$

is a solution of the stoch. differential eqn

$\mu = 0 \Rightarrow$ $\cancel{E[e^{\sigma B_t - \frac{\sigma^2}{2}t}]}$ $Y_t = e^{\sigma B_t - \frac{\sigma^2 t}{2}}$, $t \geq 0$

is a martingale with mean

$$E[Y_t] = E[e^{\sigma B_t}]\, e^{-\sigma^2 t/2} = E[Y_0] = 1,$$

hence $E[e^{\sigma B_t}] = e^{\sigma^2 t/2}$, as is well known since $B_t \sim N(0,t)$.

11.97

$$g(t,x) = e^{\sigma B_t - \frac{1}{2}\sigma^2 t} \quad = e^{\sigma B_t - \frac{1}{2}\langle \sigma B\rangle_t}$$

F10:2

$$g_t(t,x) = -\frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2}t} e^{\sigma B_t}$$

$$g_x(t,x) = \sigma e^{\sigma B_t - \frac{\sigma^2}{2}t}$$

$$g_{xx}(t,x) = \sigma^2 e^{\sigma B_t - \frac{\sigma^2}{2}t}$$

$$Y_t = g(t,B_t) = 1 + \int_0^t -\frac{\sigma^2}{2} Y_s \, ds + \frac{1}{2}\int_0^t \sigma^2 Y_s \, ds + \int_0^t \sigma Y_s \, dB_s$$

$$dY_t = \sigma Y_t \, dB_t$$

We know, for 1-dim or multidimensional BM, that if $f \in \mathcal{H}$ then

$$X_t = \int_0^t f_s \, dB_s \quad \text{is a cont. } L^2\text{-martingale}$$

The <u>exponential process</u> of $\{X_t\}$ is

$$M_t = \mathcal{E}(X)_t = \exp\{X_t - \tfrac{1}{2}\langle X\rangle_t\}$$

$$= \exp\{\int_0^t f_s \, dB_s - \tfrac{1}{2}\int_0^t f_s^2 \, ds\}, \; t \geq 0.$$

By Itô's formula,

$$M_t = 1 + \int_0^t M_s f_s \, dB_s .$$

Thus, if $M \cdot f \in \mathcal{H}$ then $M = \{M_t\}$ is a martingale with mean $E(M_t) = 1$.

Sufficient for this is <u>Novikov's condition</u>, $E[e^{\frac{1}{2}\int_0^T f_s^2 ds}] < \infty$.

Under weaker cond's, M is a <u>local</u> martingale.

<u>Example</u> Take $f \in L^2$ <u>deterministic</u>.

Then the exponential martingale shows that

$$\boxed{E[e^{\int_0^t f_s \, dB_s}] = e^{\frac{1}{2}\int_0^t f_s^2 \, ds}, \quad 0 \leq t \leq T.}$$

$$\frac{dM_t}{M_t} = f_t\, dB_t$$

$$Y_t = e^{\mu t + \sigma B_t - \frac{\sigma^2 t}{2}}$$ Geometric BM

$$\frac{dY_t}{Y_t} = \mu\, dt + \sigma\, dB_t$$

BW Ch15

Let's try this with jumps:

$$\frac{dY_t}{Y_{t-}} = \frac{Y_t - Y_{t-}}{Y_{t-}} = c\, dN_t$$

$$dY_t = c\, Y_{t-}\, dN_t \qquad (*)$$

Or, if jump at t,

$$Y_t - Y_{t-} = c\, Y_{t-}$$

$$Y_t = (1+c)\, Y_{t-}, \quad dN_t = 1$$

Summing over the jumps,

$$Y_t = (1+c)^{N_t} \cdot Y_0, \quad t \geq 0$$

is the solution of $(*)$

Now, take $c = \{c_t\}$ time dependent:

$$dY_t = c_t Y_{t-} dN_t \qquad \text{at each jump time } T_k,\ k \geq 1$$

$$dY_{T_k} = Y_{T_k} - Y_{T_k-} = c_{T_k} Y_{T_k-}.$$

that is, $Y_{T_k} = (1 + c_{T_k}) Y_{T_k-}$.

This yields

$$\begin{aligned}
Y_t &= Y_0 \prod_{k=1}^{N_t} (1 + c_{T_k}) \\
&= Y_0 \prod_{\substack{s \leq t \\ \Delta N_s = 1}} (1 + c_s) = Y_0 \prod_{0 \leq s \leq t} (1 + c_s \Delta N_s) \\
&= Y_0 \, e^{\sum_{s \leq t} \ln(1 + c_s) dN_s} \\
&= Y_0 \, e^{\int_0^t \ln(1 + c_s) dN_s}.
\end{aligned}$$

Can we find an "exponential martingale"

$M = \{M_t\}$, which solves $dM_t = \underbrace{c_t M_{t-}}_{\text{predictable?}} \cancel{dN_t} \underbrace{(dN_t - \lambda dt)}_{\text{mart?}}$

$$dM_t = -\lambda c_t M_{t} dt + c_t M_{t-} dN_t$$

$$\Rightarrow M_t = e^{-\lambda\int_0^t \varsigma_s\,ds}\, e^{\int_0^t \ln(1+\varsigma_s)\,dN_s} \cdot M_0$$

In particular, if (ς_s) is deterministic, $\int_0^t \varsigma_s\,ds<\infty$, and predictable (e.g. left-continuous) then with $M_0=1$,

$$1 = \mathbb{E}[M_t] = \mathbb{E}\left[e^{\int_0^t \ln(1+\varsigma_s)\,dN_s}\right]e^{-\lambda\int_0^t\varsigma_s\,ds}$$

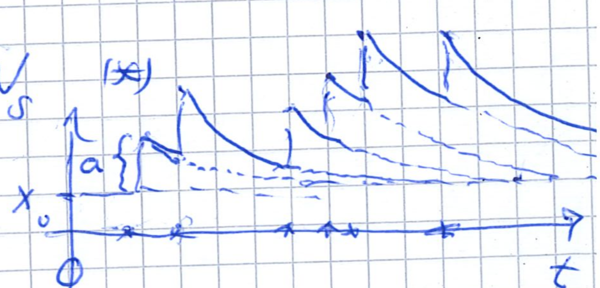
and so, taking $f_s = \ln(1+\varsigma_s)$,

$$\boxed{\mathbb{E}\left[e^{\int_0^t f_s\,dN_s}\right] = e^{\lambda\int_0^t(e^{f_s}-1)\,ds}}$$

compare BM.

Example Poisson shot noise process

$$X_t = X_0 + \int_0^t a e^{-b(t-s)}\,dN_s \qquad (*)$$



$$= X_0 + \int_0^t a e^{-b(t-s)}(dN_s - \lambda\,ds) + \int_0^t \lambda a e^{-bs}\,ds$$

From (*) we see that

$$dX_t = a\,dN_t - b(X_t - X_0)\,dt$$

$(X_0=0)$ $\quad X_t = -b\int_0^t X_s\,ds + aN_t \quad$ Sol'n of a SDE

$$= a(N_t-\lambda t) + b\int_0^t\left(\frac{a}{b}\lambda - X_s\right)ds, \quad t\ge 0$$

Variation

$$X_t = x_0 e^{-bt} + a\int_0^t e^{-b(t-s)}\,dN_s$$

$$dX_t = -b x_0 e^{-bt}dt - b\left(a\int_0^t e^{-b(t-s)}\,dN_s\right)dt + a\,dN_t$$

$$= a\,dN_t - b X_t\,dt$$

$$\Rightarrow X_t = x_0 - b\int_0^t X_s\,ds + aN_t$$

$$= \cancel{\ldots}\; x_0 + \underbrace{a(N_t - \lambda t)}_{\text{comp. martingale part}} + \underbrace{b\int_0^t \left(\frac{\lambda a}{b} - X_s\right)ds}_{\text{mean-reverting drift}}$$

For BM this construction is the Ornstein-Uhlenbeck process (OU-process) or Gaussian Shotnoise.

Def 5.2.1 Prop 5.2.2.

Weak Lemma 5.2.3

Sol'n of SDE Def 5.5.1 Thm 5.5.2

Example 5.5.3 BM on a circle, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$