Heaps, Heapsort & Priority Queues

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(Based on previous material by Mohamed Faouzi Atig and Parosh Aziz Abdulla)

1. Introduction
2. Tree Definition
3. Heap Definition
4. MAX-HEAPIFY
5. BUILD-MAX-HEAP
6. HEAP-SORT
7. Priority Queues

Sorting Algorithms

- Problem: Sort an array A of n elements in non-decreasing order

Algorithm	Worst-Case	"Average-Case"	Best-Case	In place?
InsertionSort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	Yes
MergeSort	$\Theta(nlog(n))$	$\Theta(nlog(n))$	$\Theta(nlog(n))$	No
QuickSort	$\Theta(n^2)$	$\Theta(nlog(n))$	$\Theta(nlog(n))$	Yes

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QuickSort	$\Theta(n^2)$	$\Theta(nlog(n))$	$\Theta(nlog(n))$	Yes
HeapSort	$\Theta(nlog(n))$	$\Theta(nlog(n))$	$\Theta(nlog(n))$[1]	Yes

[1] Assuming all distinct elements; with n identical elements HeapSort is $\Theta(n)$.

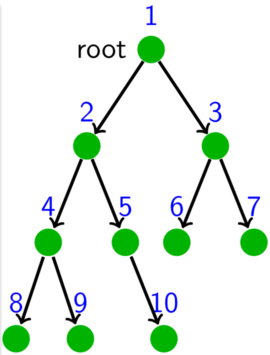
HeapSort: Introduction

- HeapSort was invented by J. W. J. Williams in 1964.
- Based on a useful of data structure called heap
- Sorting in place algorithm

Tree: Definition

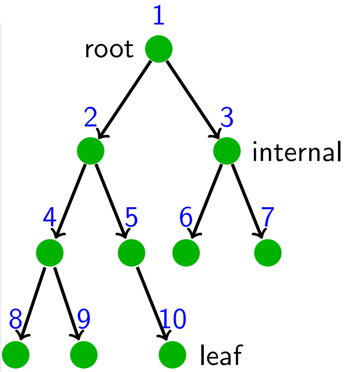
- A tree T is a directed graph (V, E) where:
  - V is a set of vertices (or nodes).
  - $E \subseteq V \times V$ is a finite set of edges (or arcs).

  such that the following properties hold:
  - T is an acyclic connected graph.
  - For each $(n_1, n_2) \in E$, the node n_1 is the parent of n_2
    - Each node of T has at most one parent.
    - There is exactly one node that does not have a parent called the root node.



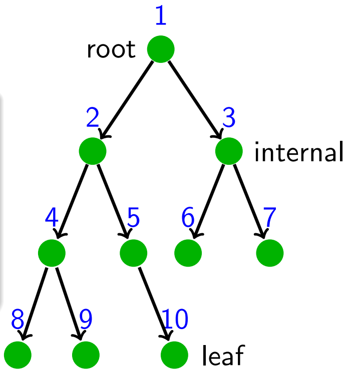
Tree: Notations

- If a node n_1 is a parent of a node n_2 then n_2 is a child of n_1
- If two nodes have the same parent then they are siblings.
- A node with at least one child is an internal node.
- A node with no children is a leaf.



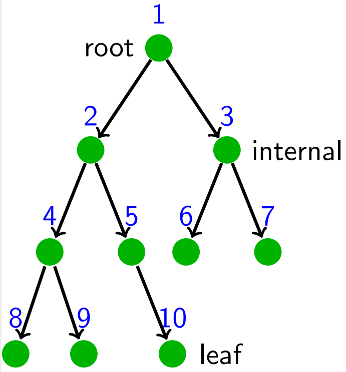
Tree: Notations

- The node 1 is a parent of the node 2
- The node 4 is a child of the node 2
- The nodes 4 and 5 are siblings



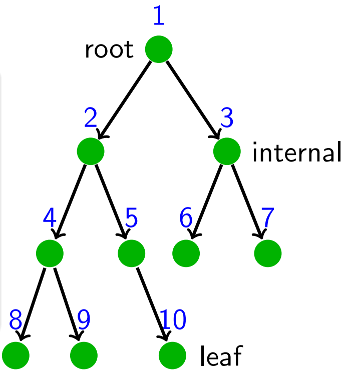
Tree: Notations

- A path is a sequence of nodes $n_1, n_2, \ldots, n_m$ such that for all $i : 1 \leq i < m$, (n_i, n_{i+1}) is an edge.
- The height of a node n is the number of edges of the longest path to a leaf from this node.
- The height of a tree is the height of its root node.
- The depth of a node n is the number of edges in the path from the root to n.



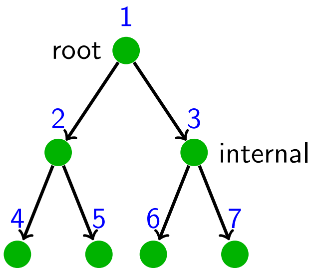
Tree: Notations

- The sequence $1, 2, 4, 8$ is path
- The height of node 2 is 2
- The height of the tree is 3
- The depth (or level) of the node 7 is 2



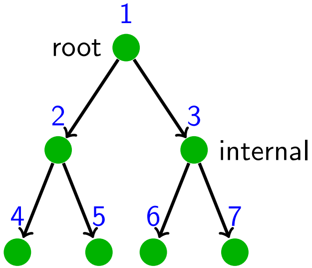
Binary Trees

- A binary tree is a tree such that:
  - Each node has at most two child nodes, distinguished by left and right.
  - The left child always precedes the right child
- A full binary tree is a binary tree in which each internal node has exactly two children.
- A perfect binary tree is a full binary tree in which all the leaves have the same depth.



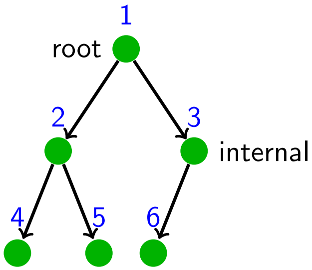
Full Binary Trees: Properties

- The number of leaves is equal to the number of internal nodes plus 1.
- The number of nodes at depth (or level) i is $\leq 2^i$



Complete Binary Tree

- A complete binary tree is a binary tree, which is completely filled at all the levels except possibly the highest, which is filled from left. Formally, we have
  - If h is the height of the tree, then:
    - For all $i : 0 \leq i < h$, there is exactly 2^i nodes at depth i
    - A leaf node has a depth h or $h - 1$
    - The leaves of depth h are filled from left to right.



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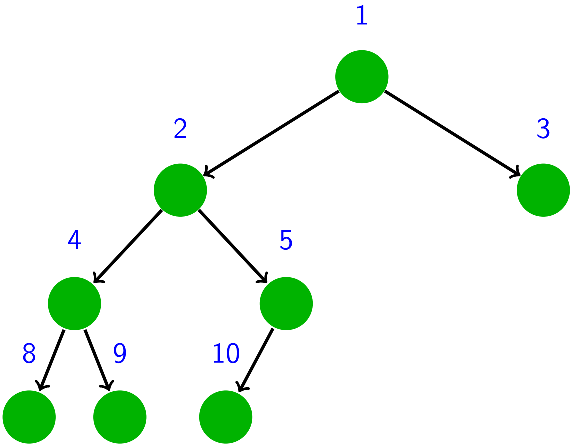

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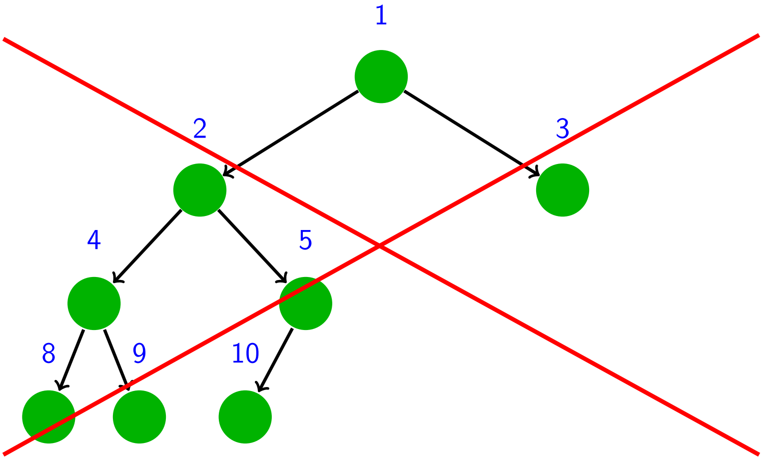

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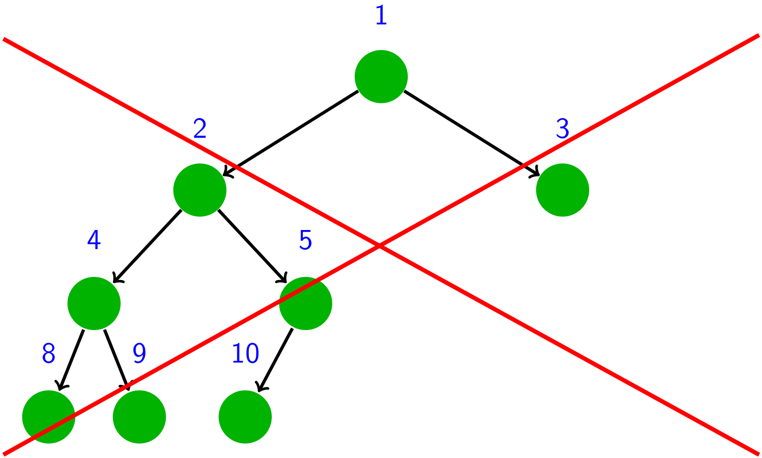
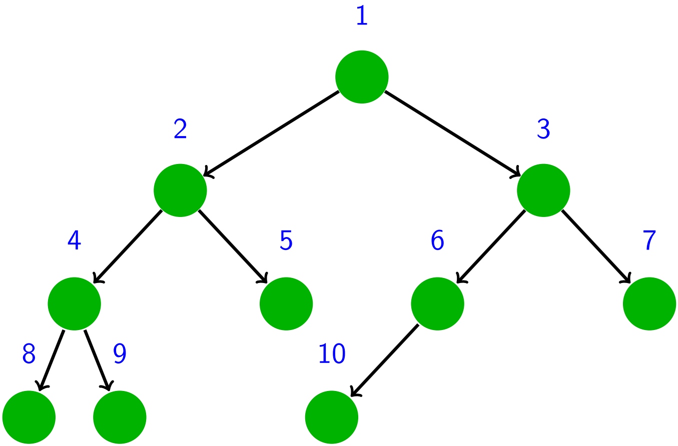


It is not completely filled at level 2

Example (2/2)



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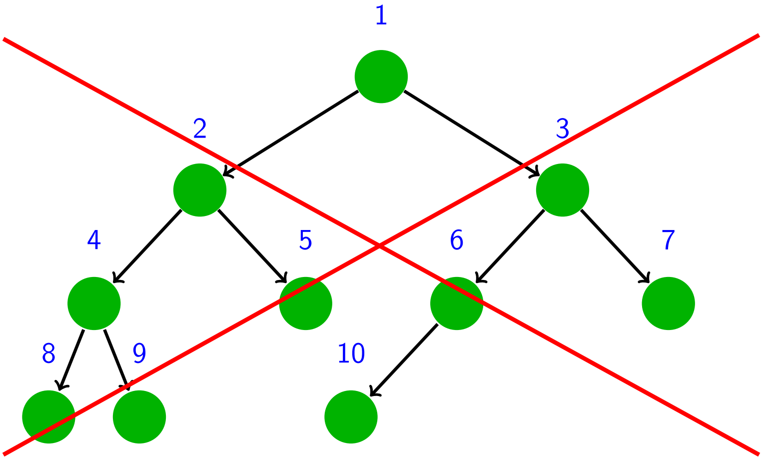

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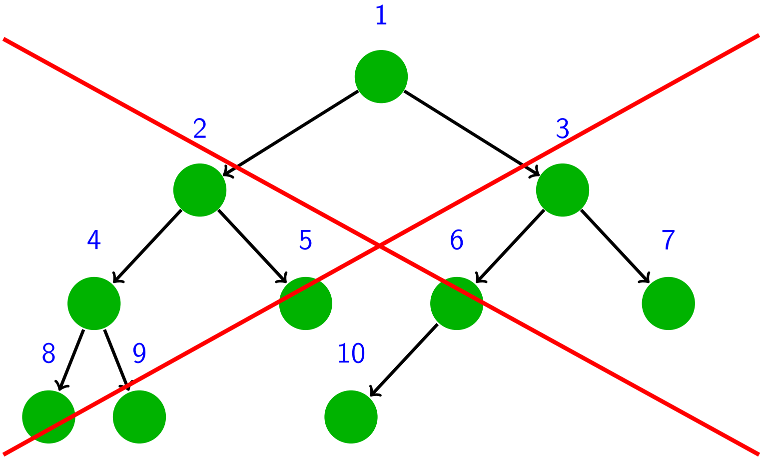


The leaves are not filled from left to right.

Complete binary tree: Properties

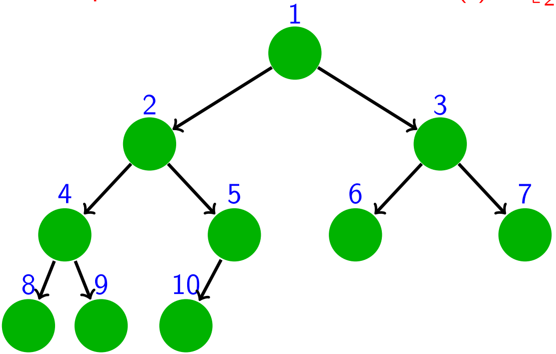
- Let T be a complete tree with n is the number of nodes and h is its height:
  - n is greater or equal to the number of nodes in the perfect tree of height $h-1$ plus one (i.e., $n \geq 2^h$)
  - n is less or equal than the number of nodes in the perfect tree of height h (i.e., $n \leq 2^{h+1}-1$)

$$
\begin{aligned}
2^h \leq n \leq 2^{h+1}-1 \quad &\Rightarrow \quad 2^h \leq n < 2^{h+1} \\
&\Rightarrow \quad h \leq log_2(n) < h+1 \\
&\Rightarrow \quad h \leq log_2(n) < h+1 \\
&\Rightarrow \quad h = \lfloor log_2(n) \rfloor
\end{aligned}
$$

Well-Indexed Tree

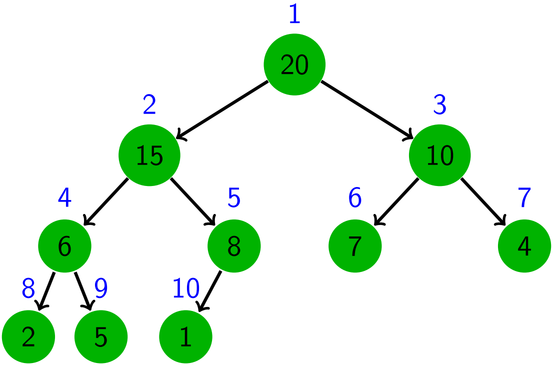
A well-indexed tree is a complete binary tree such that:

- The index of the root is 1
- The index of the left child of a node i is $\textsc{Left}(i) = 2i$
- The index of the right child of a node i is $\textsc{Right}(i) = 2i + 1$
- The index of the parent of a node i is $\textsc{Parent}(i) = \lfloor \frac{i}{2} \rfloor$



Max-Heap: Definition

- A max-heap is a well-indexed tree such that :
  - Each node is associated with a value.
  - The value of a node is at most the value of its parent.



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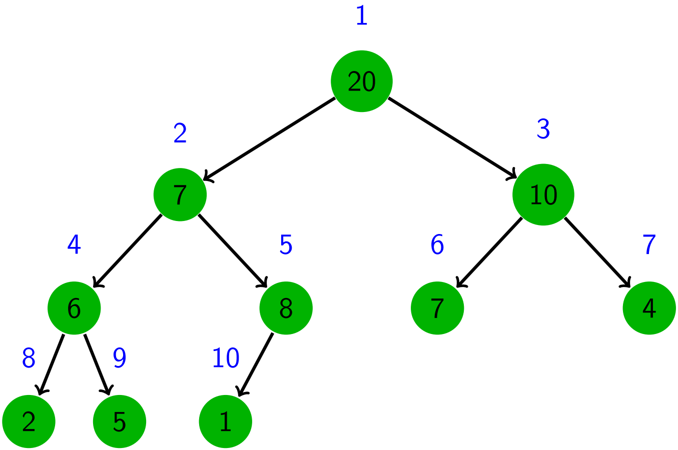

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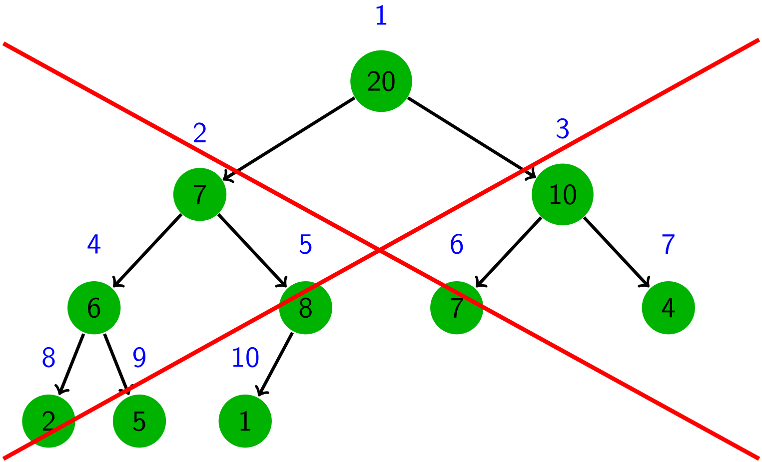
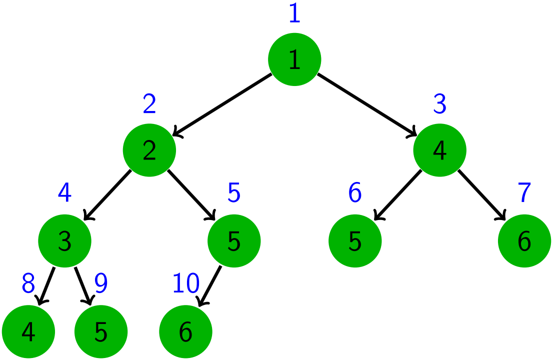


Min-Heap: Definition

- A min-heap is a well-indexed tree such that :
  - Each node is associated with a value.
  - The value of a node is at least the value of its parent.



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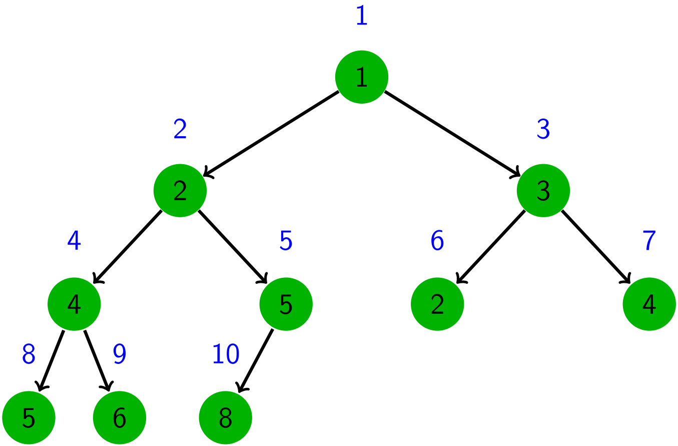

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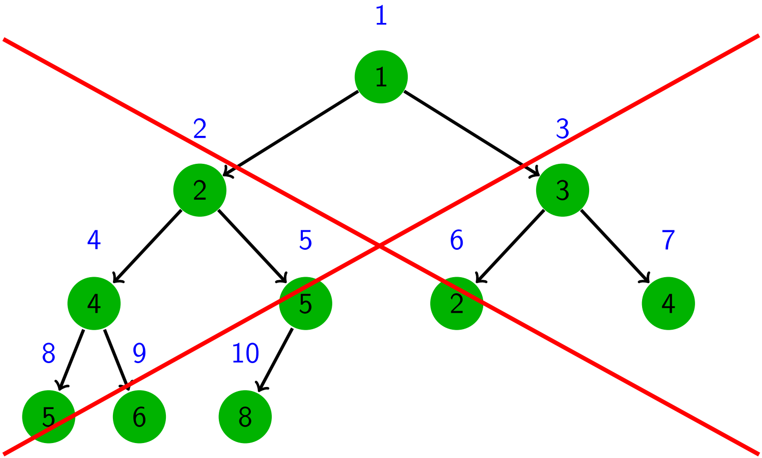
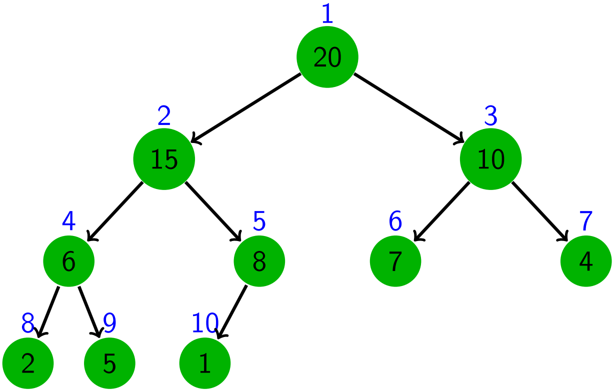


Implementation of a Heap

A heap can be represented as an array A such that the value of a node of index i is $A[i]$.



A.*heap-size*=10

A.*length*=13

1	2	3	4	5	6	7	8	9	10	11	12	13
20	15	10	6	8	7	4	2	5	1	9	14	13

Implementation of a Heap

An array A representing a heap has two attributes:

- A.length: The length of the array
- A.heap-size: length of the left subarray containing elements from the heap= number of nodes inside the heap.

A property of the array A representing a max-heap:

- For all $i : 2 \leq i \leq A.\mathit{heap\text{-}size}$, we have $A[\textsc{Parent}(i)] \geq A[i]$

A property of the array A representing a min-heap:

- For all $i : 2 \leq i \leq A.\mathit{heap\text{-}size}$, we have $A[\textsc{Parent}(i)] \leq A[i]$

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Problem: Sort an array A of n elements in non-decreasing order

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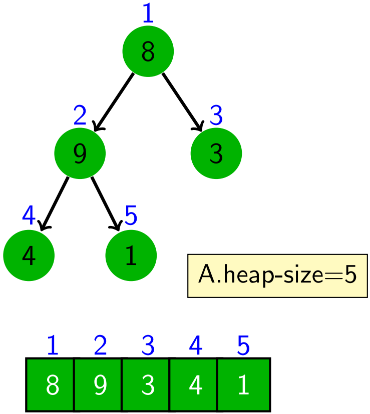

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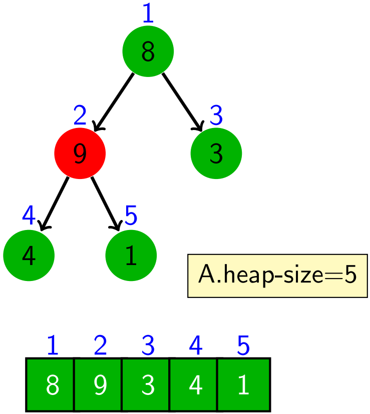

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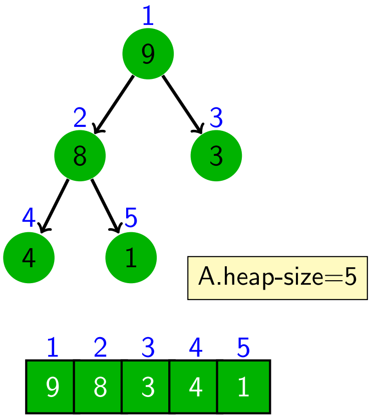

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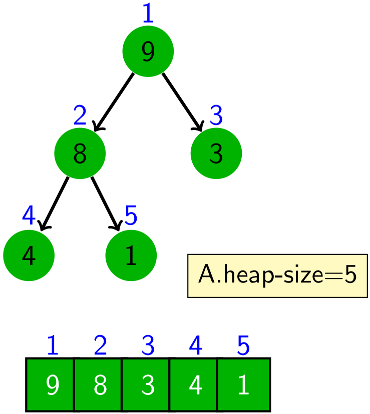

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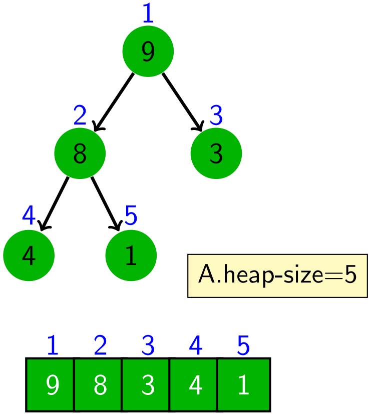

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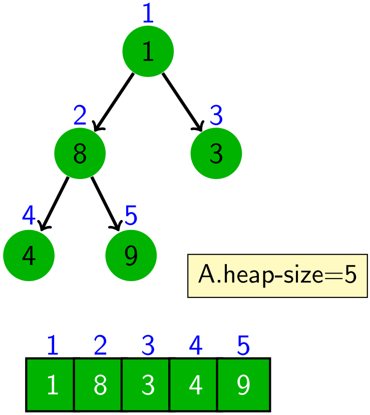

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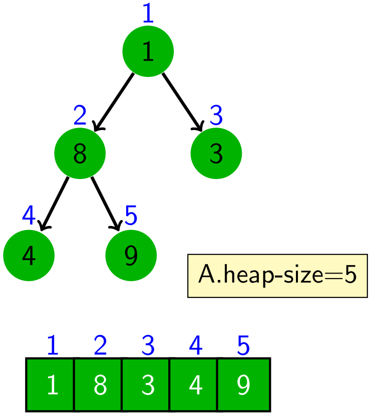

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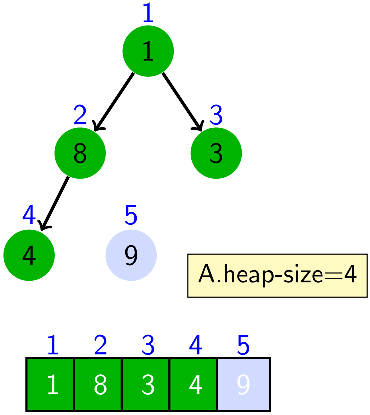

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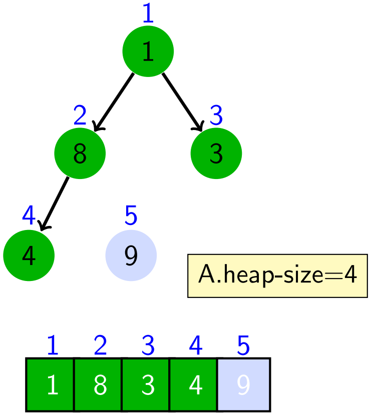

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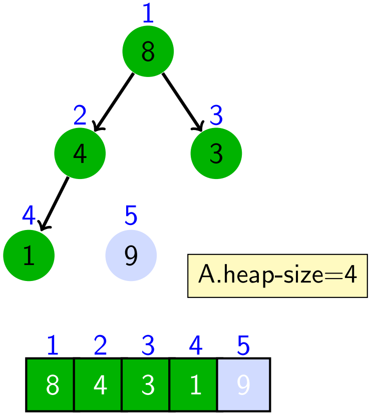

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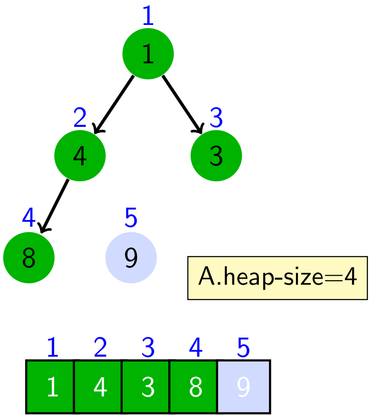

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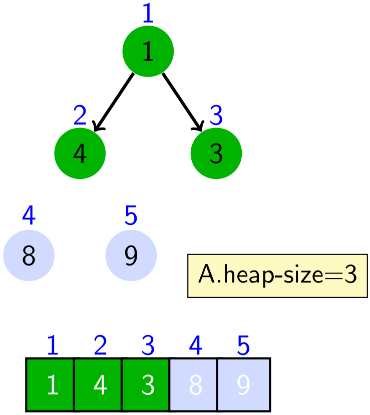

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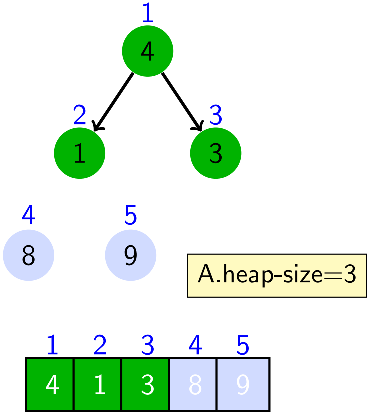

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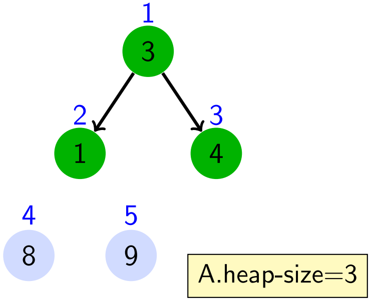

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3	1	4	8	9

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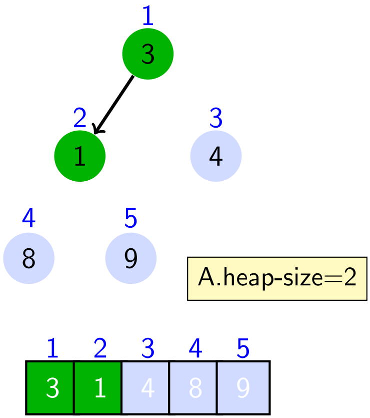

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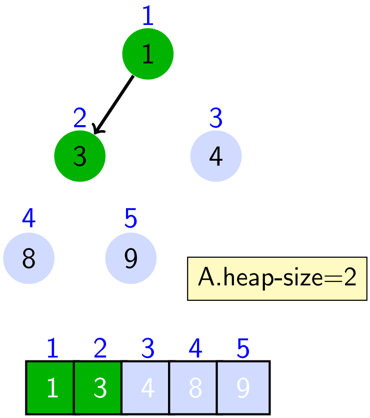

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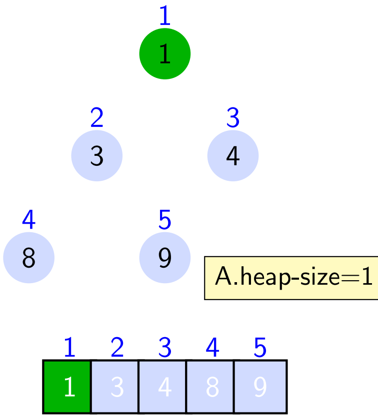

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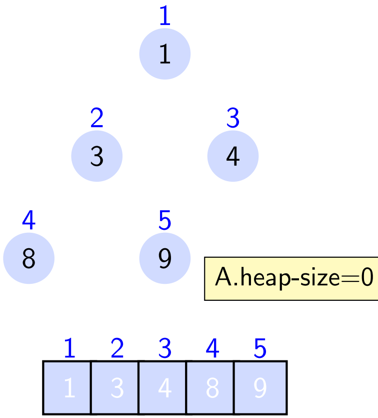
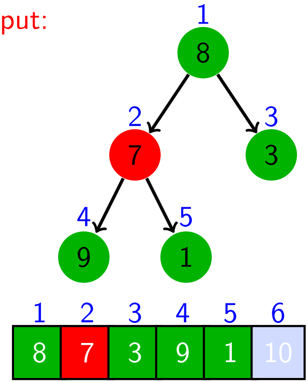


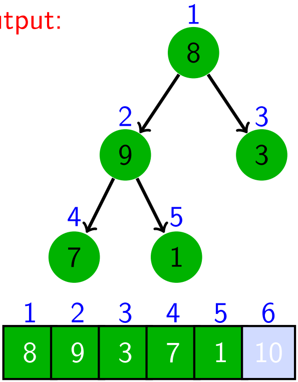
Max-Heapify

- Input: An array A and an index $i : 1 \leq i \leq A.heap-size$
- Assumption: The two sub-trees rooted at $\textsc{Left}(i)$ and $\textsc{Right}(i)$ are max-heaps
- Output: The sub-tree rooted at index i is a max-heap.

Input:



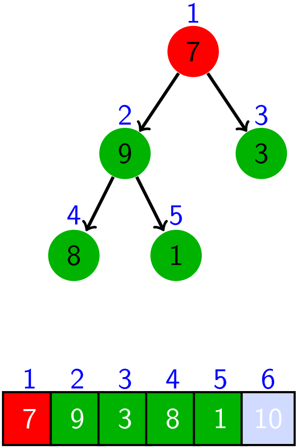
Output:



MAX-HEAPIFY: PRINCIPLE

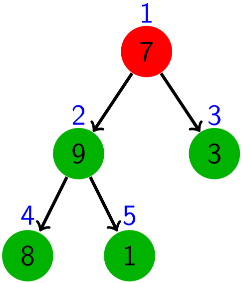
MAX-HEAPIFY

- Compare $A[i]$, $A[\textsc{Left}(i)]$, and $A[\textsc{Right}(i)]$



MAX-HEAPIFY

- Compare $A[i]$, $A[\textsc{Left}(i)]$, and $A[\textsc{Right}(i)]$
- If $A[i] < A[\textsc{Left}(i)]$ or $A[i] < A[\textsc{Right}(i)]$, swap $A[i]$ with the larger of the two children

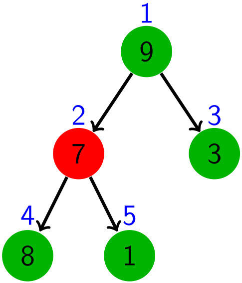


1	2	3	4	5	6
7	9	3	8	1	10

Max-Heapify: Principle

Max-Heapify

- Compare $A[i]$, $A[\text{Left}(i)]$, and $A[\text{Right}(i)]$
- If $A[i] < A[\text{Left}(i)]$ or $A[i] < A[\text{Right}(i)]$, swap $A[i]$ with the larger of the two children

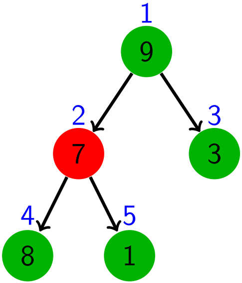


1	2	3	4	5	6
9	7	3	8	1	10

Max-Heapify: Principle

Max-Heapify

- Compare $A[i]$, $A[\text{Left}(i)]$, and $A[\text{Right}(i)]$
- If $A[i] < A[\text{Left}(i)]$ or $A[i] < A[\text{Right}(i)]$, swap $A[i]$ with the larger of the two children
- Continue this process of comparing and swapping down the heap, until subtree rooted at i is a max-heap

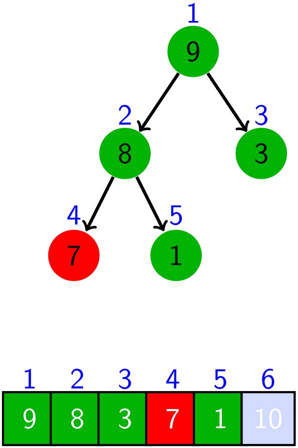


1	2	3	4	5	6
9	7	3	8	1	10

MAX-HEAPIFY: PRINCIPLE

MAX-HEAPIFY

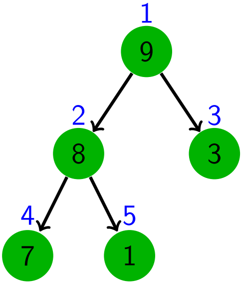
- Compare $A[i]$, $A[\textsc{Left}(i)]$, and $A[\textsc{Right}(i)]$
- If $A[i] < A[\textsc{Left}(i)]$ or $A[i] < A[\textsc{Right}(i)]$, swap $A[i]$ with the larger of the two children
- Continue this process of comparing and swapping down the heap, until subtree rooted at i is a max-heap



Max-Heapify: Principle

Max-Heapify

- Compare $A[i]$, $A[\textsc{Left}(i)]$, and $A[\textsc{Right}(i)]$
- If $A[i] < A[\textsc{Left}(i)]$ or $A[i] < A[\textsc{Right}(i)]$, swap $A[i]$ with the larger of the two children
- Continue this process of comparing and swapping down the heap, until subtree rooted at i is a max-heap

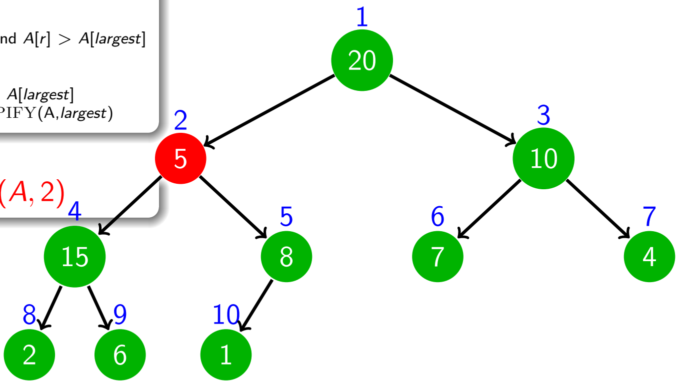


1	2	3	4	5	6
9	8	3	7	1	10

MAX-HEAPIFY(A, i)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY(A, 2)



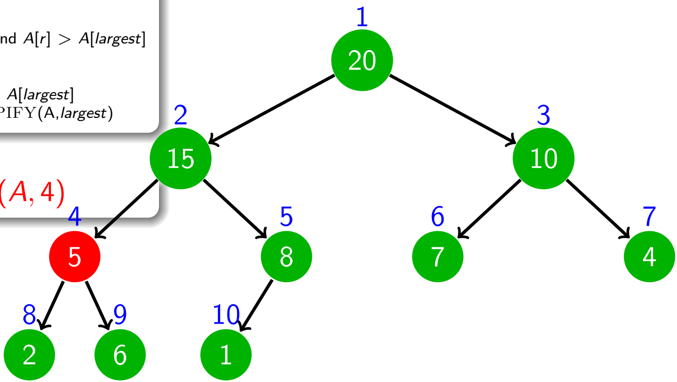
1	2	3	4	5	6	7	8	9	10	11	12	13
20	5	10	15	8	7	4	2	6	1	21	22	24

MAX-HEAPIFY

MAX-HEAPIFY(A, i)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY(A, 4)



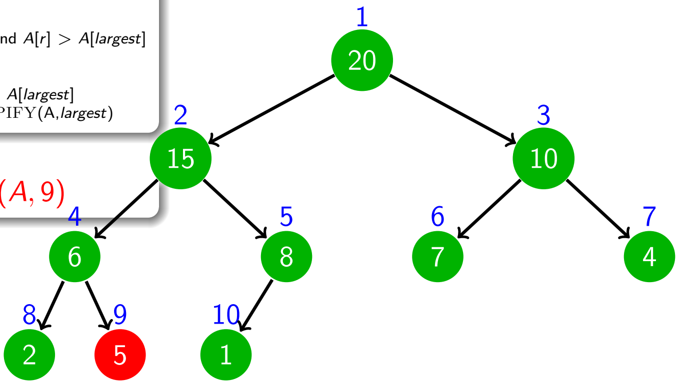
1	2	3	4	5	6	7	8	9	10	11	12	13
20	15	10	5	8	7	4	2	6	1	21	22	24

MAX-HEAPIFY

MAX-HEAPIFY(A, i)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY(A, 9)



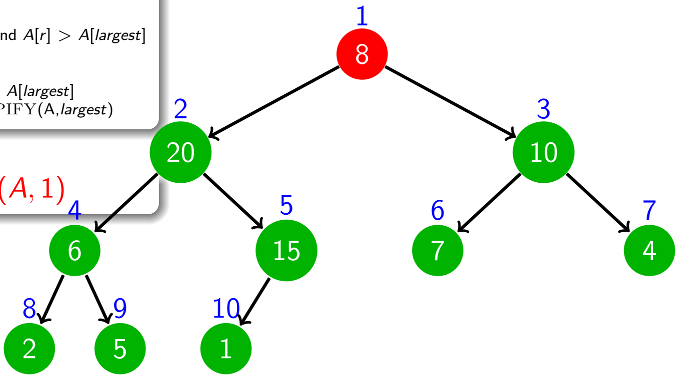
1	2	3	4	5	6	7	8	9	10	11	12	13
20	15	10	6	8	7	4	2	5	1	21	22	24

MAX-HEAPIFY

MAX-HEAPIFY(A, i)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY(A, 1)

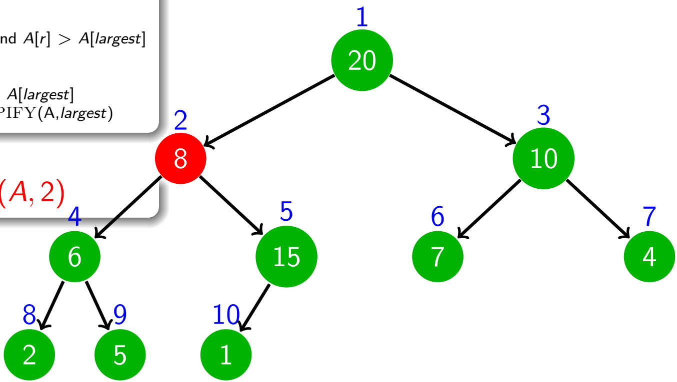


1	2	3	4	5	6	7	8	9	10	11	12	13
8	20	10	6	15	7	4	2	5	1	21	22	24

MAX-HEAPIFY

```
MAX-HEAPIFY(A, i)
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A. heap-size and A[l] > A[i]
    then largest ← l
    else  largest ← i
if r ≤ A. heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A,largest)
```

MAX-HEAPIFY$(A, 2)$

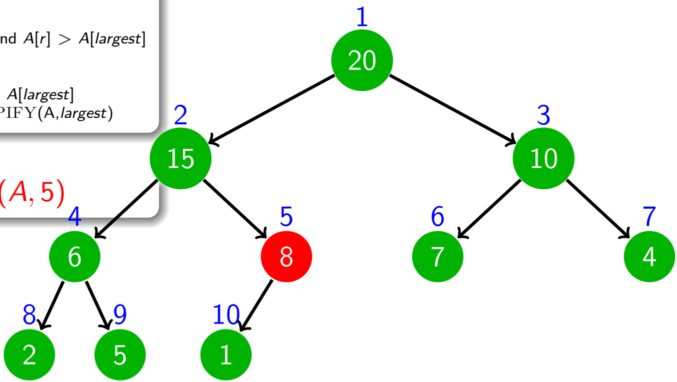


1	2	3	4	5	6	7	8	9	10	11	12	13
20	8	10	6	15	7	4	2	5	1	21	22	24

MAX-HEAPIFY

```
MAX-HEAPIFY(A, i)
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A. heap-size and A[l] > A[i]
    then largest ← l
    else  largest ← i
if r ≤ A. heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY$(A, 5)$



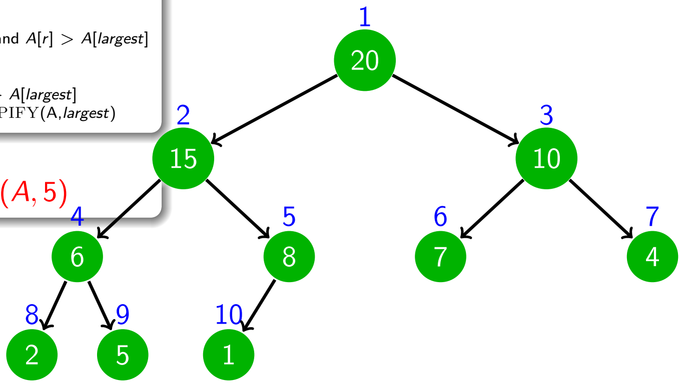
1	2	3	4	5	6	7	8	9	10	11	12	13
20	15	10	6	8	7	4	2	5	1	21	22	24

MAX-HEAPIFY

MAX-HEAPIFY(A, i)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then swap A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY(A, 5)



1	2	3	4	5	6	7	8	9	10	11	12	13
20	15	10	6	8	7	4	2	5	1	21	22	24

MAX-HEAPIFY: Runtime

- Let n be the number of nodes at sub-tree of the heap rooted at i
- Each of lines 1-9 takes constant time
- The number of calls at line 10 is bounded by the height $\lfloor \log_2(n) \rfloor$ of the sub-tree of the heap rooted at i

  $\Rightarrow$ Hence, $T(n) = O(log_2(n))$ since $\text{MAX-HEAPIFY}(A, i)$ should process $O(log_2(n))$ levels, with constant work at each level

$\text{MAX-HEAPIFY}(A, i)$

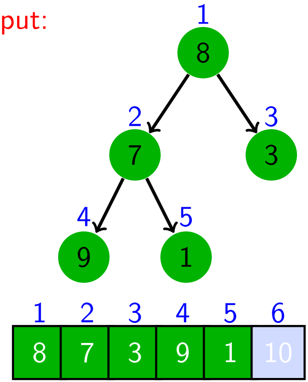
```
1  l ← LEFT(i)
2  r ← RIGHT(i)
3  if l ≤ A.heap-size and A[l] > A[i]
4     then largest ← l
5     else largest ← i
6  if r ≤ A.heap-size and A[r] > A[largest]
7     then largest ← r
8  if largest ≠ i
9     then swap A[i] ↔ A[largest]
10          MAX-HEAPIFY(A,largest)
```

$\Rightarrow$ $T(n)$ is linear in the height of the sub-tree of the heap rooted at i

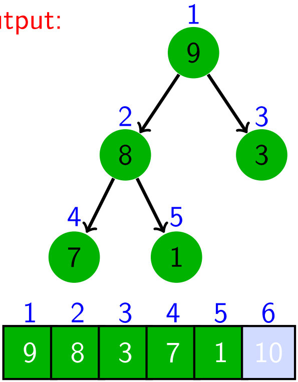
Build-Max-Heap

- Input: An array A
- Output: A max-heap from A

Input:



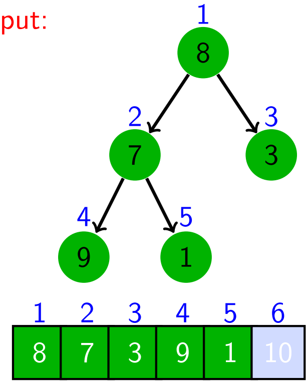
Output:



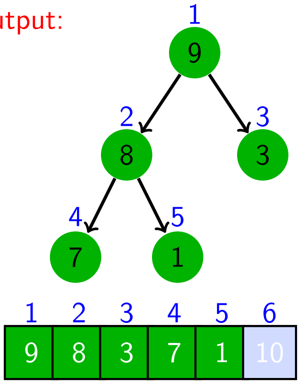
Build-Max-Heap

- Input: An array A
- Output: A max-heap from A
- Idea: Use Max-Heapify in a bottom-up manner

Input:



Output:



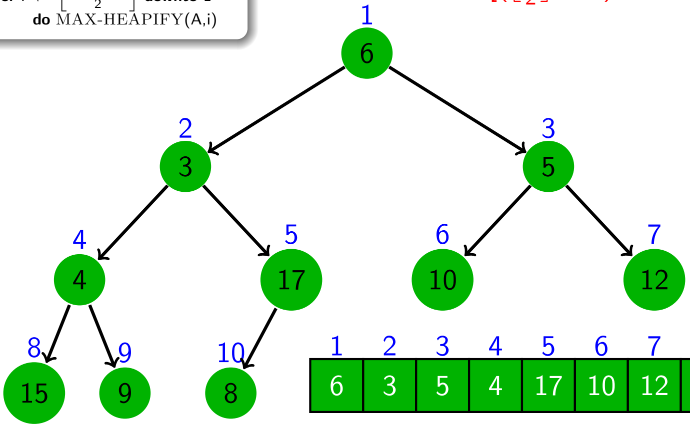
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

$A.\,heap\text{-}size \leftarrow A.\,length$
for $i \leftarrow \left\lfloor \frac{A.\,length}{2} \right\rfloor$ **downto** 1
  do MAX-HEAPIFY(A,i)

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 5$



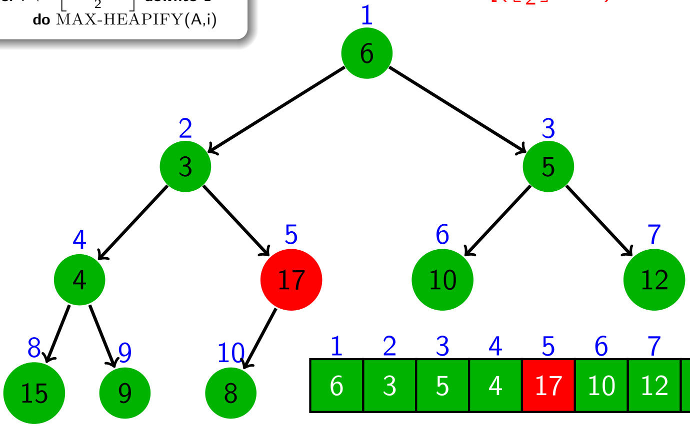
1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)
$A.\,heap\text{-}size \leftarrow A.\,length$
for $i \leftarrow \left\lfloor \frac{A.\,length}{2} \right\rfloor$ **downto** 1
  do MAX-HEAPIFY(A,i)

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 5$



1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

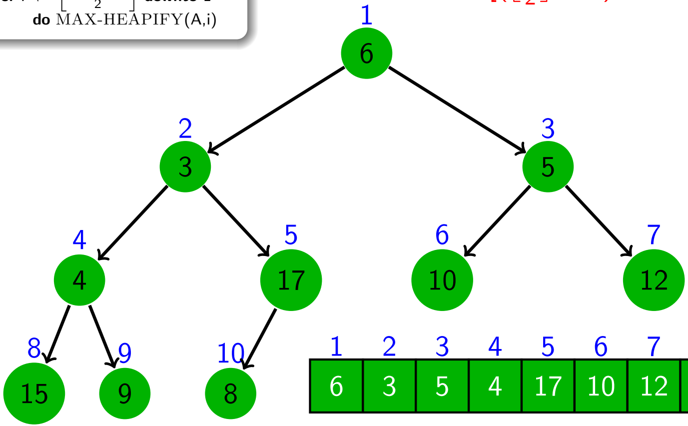
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
        do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\mathit{length}]$

$i = 4$



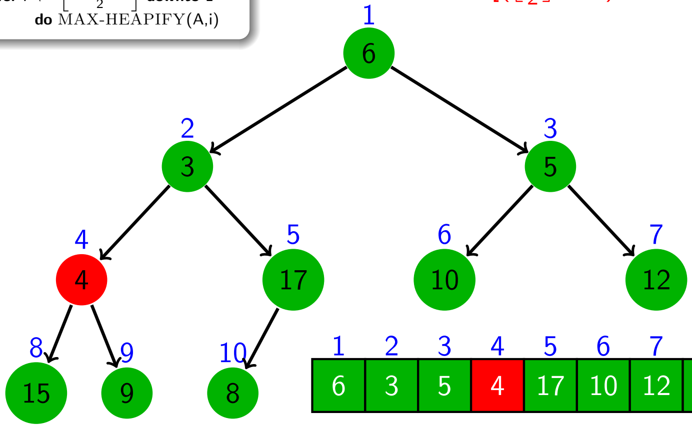
1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
  A.heap-size ← A.length
  for i ← ⌊A.length / 2⌋ downto 1
        do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 4$



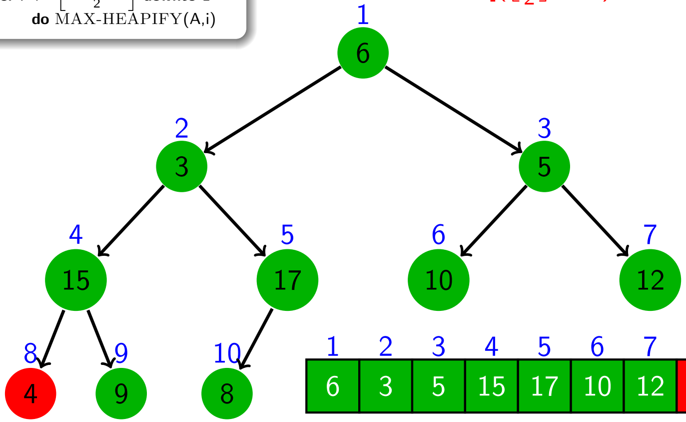
1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
    A.heap-size ← A.length
    for i ← ⌊A.length / 2⌋ downto 1
        do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 4$



1	2	3	4	5	6	7	8	9	10
6	3	5	15	17	10	12	4	9	8

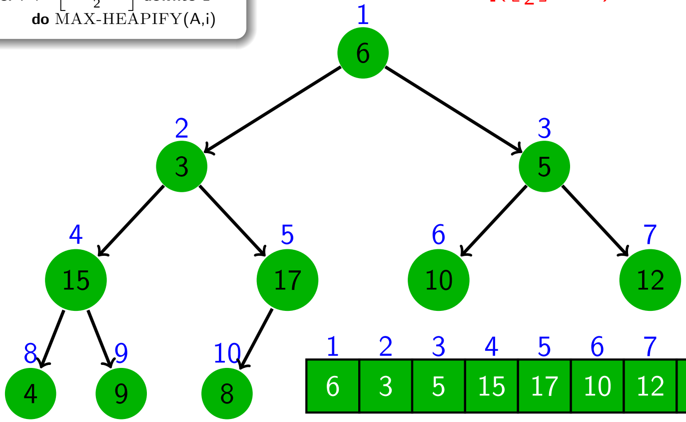
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.length]$

$i = 3$



1	2	3	4	5	6	7	8	9	10
6	3	5	15	17	10	12	4	9	8

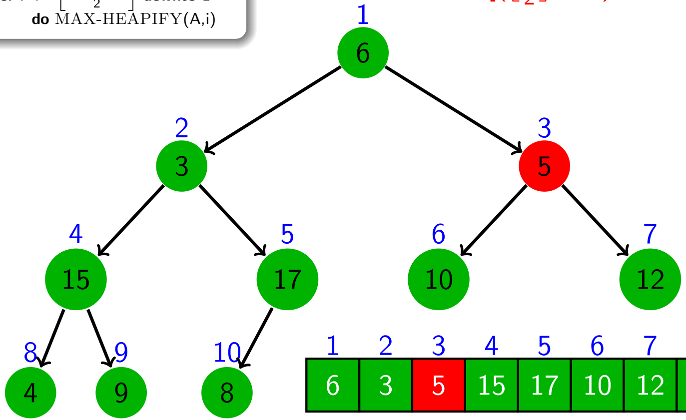
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
A.heap-size ← A.length
for i ← ⌊A.length/2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\mathit{length}]$

$i = 3$



1	2	3	4	5	6	7	8	9	10
6	3	5	15	17	10	12	4	9	8

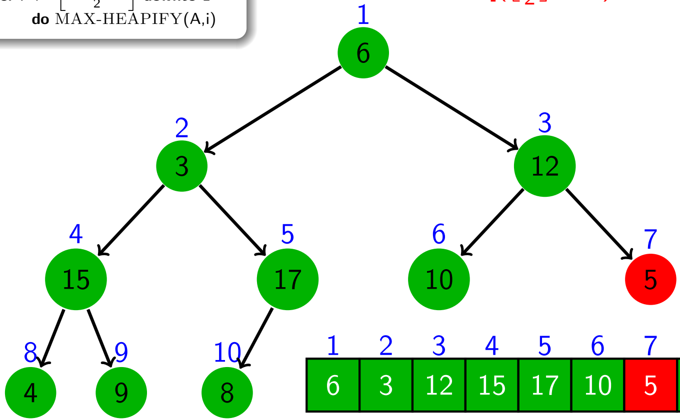
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 3$



1	2	3	4	5	6	7	8	9	10
6	3	12	15	17	10	5	4	9	8

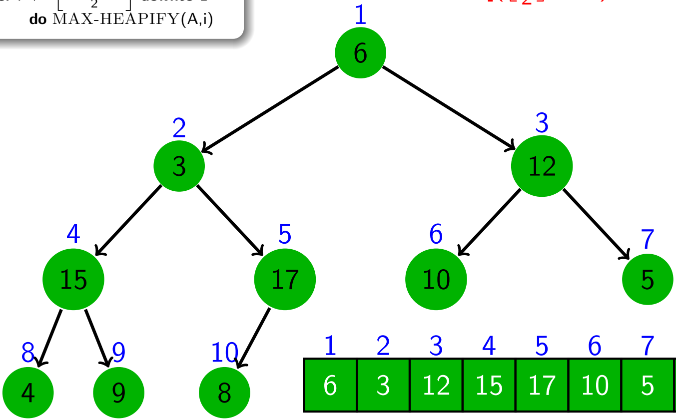
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
A. heap-size ← A. length
for i ← ⌊A. length / 2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\ length]$

$i = 2$



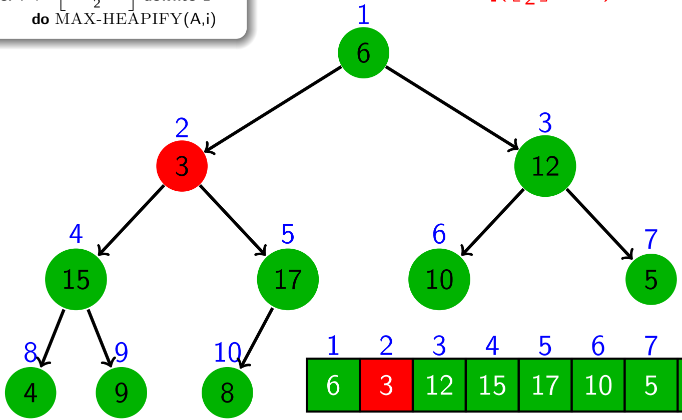
1	2	3	4	5	6	7	8	9	10
6	3	12	15	17	10	5	4	9	8

BUILD-MAX-HEAP(A)

```
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.length]$

$i = 2$



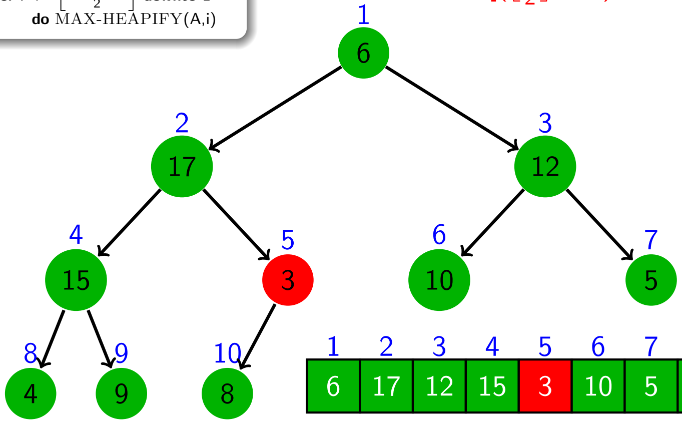
1	2	3	4	5	6	7	8	9	10
6	3	12	15	17	10	5	4	9	8

BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)
A. heap-size $\leftarrow$ *A. length*
for $i \leftarrow \left\lfloor \frac{A.\ length}{2} \right\rfloor$ **downto** 1
    do MAX-HEAPIFY(A,i)

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\ length]$

$i = 2$



1	2	3	4	5	6	7	8	9	10
6	17	12	15	3	10	5	4	9	8

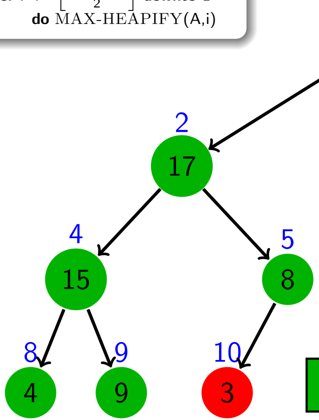
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

1 $A.\ heap\text{-}size \leftarrow A.\ length$
2 **for** $i \leftarrow \left\lfloor \frac{A.\ length}{2} \right\rfloor$ **downto** 1
3 **do** MAX-HEAPIFY(A,i)

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\ length]$

$i = 2$



1	2	3	4	5	6	7	8	9	10
6	17	12	15	8	10	5	4	9	3

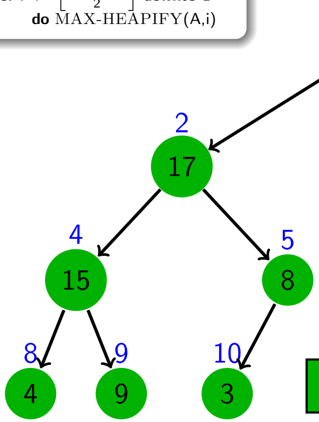
BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

```
1  A.heap-size ← A.length
2  for i ← ⌊A.length/2⌋ downto 1
3      do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\mathit{length}]$

$i = 1$



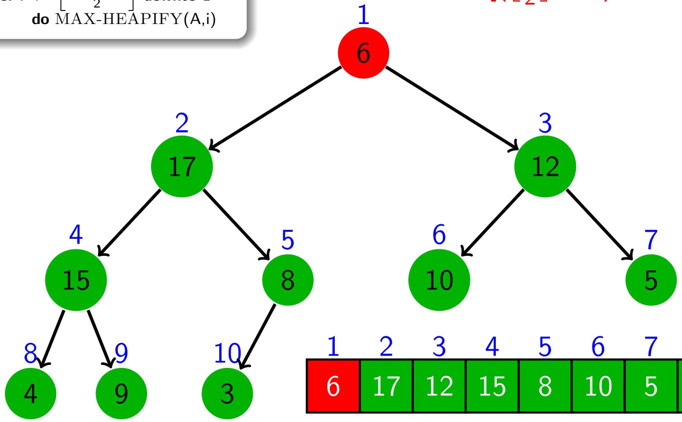
1	2	3	4	5	6	7	8	9	10
6	17	12	15	8	10	5	4	9	3

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
  A.heap-size ← A.length
  for i ← ⌊A.length / 2⌋ downto 1
        do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 1$

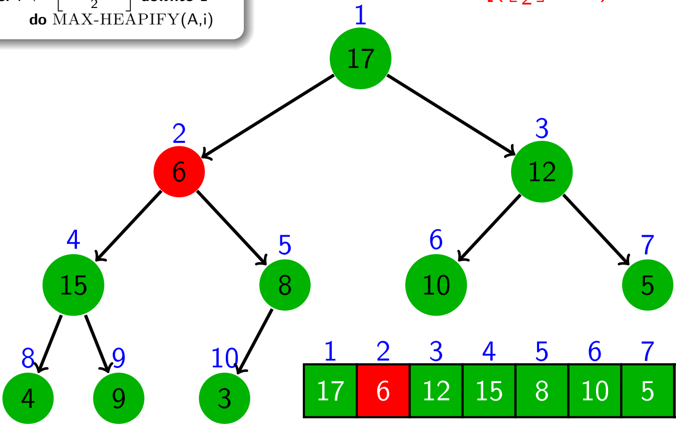


1	2	3	4	5	6	7	8	9	10
6	17	12	15	8	10	5	4	9	3

```
BUILD-MAX-HEAP(A)
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
      do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 1$



1	2	3	4	5	6	7	8	9	10
17	6	12	15	8	10	5	4	9	3

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
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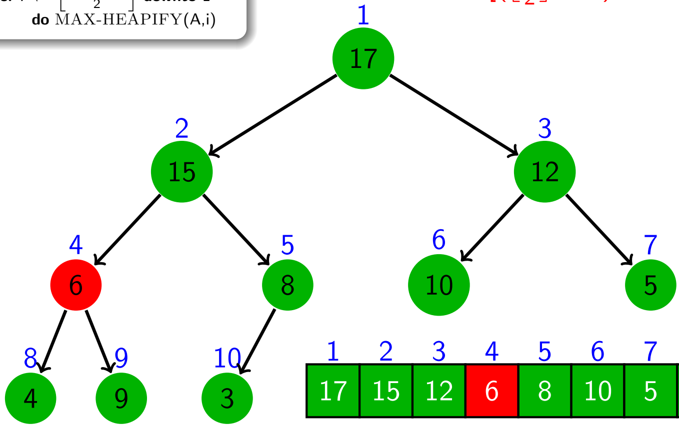
- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 1$



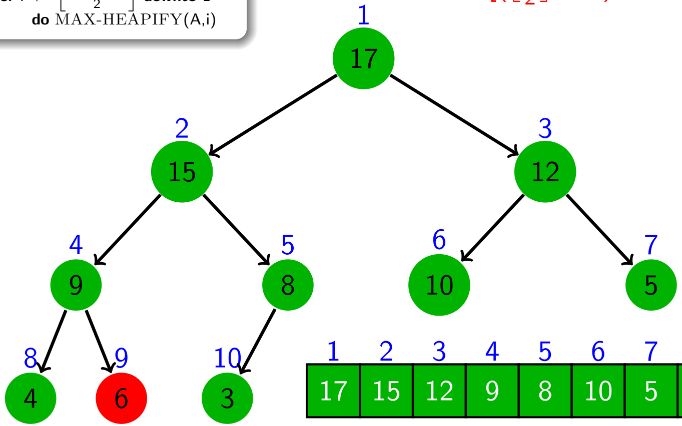
1	2	3	4	5	6	7	8	9	10
17	15	12	6	8	10	5	4	9	3

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
      do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.\,length]$

$i = 1$



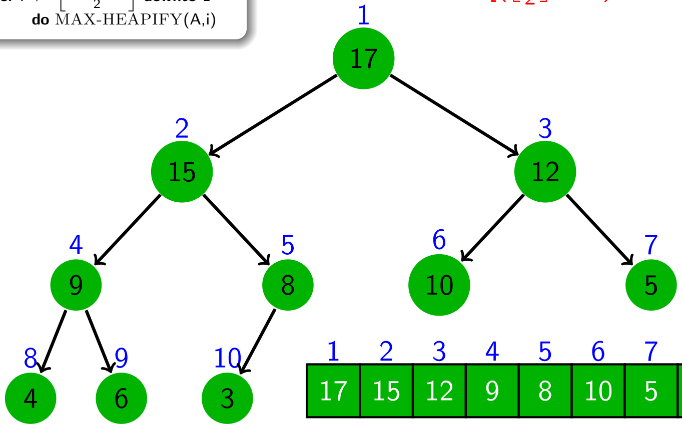
1	2	3	4	5	6	7	8	6	10
17	15	12	9	8	10	5	4	9	3

BUILD-MAX-HEAP

BUILD-MAX-HEAP(*A*)

```
A.heap-size ← A.length
for i ← ⌊A.length / 2⌋ downto 1
    do MAX-HEAPIFY(A,i)
```

- Remark: The leaves are the elements of $A[(\lfloor \frac{n}{2} \rfloor + 1)..A.length]$



1	2	3	4	5	6	7	8	6	10
17	15	12	9	8	10	5	4	9	3

BUILD-MAX-HEAP: Runtime

- Let $n = A.\,length$
- $h = \lfloor log_2(n) \rfloor$ be the height of the heap
- Simple bound:
  - $O(n)$ calls to MAX-HEAPIFY, each of which takes $O(log_2(n))$ time
    $\Rightarrow T(n) = O(n \cdot log_2(n))$

BUILD-MAX-HEAP(A)

```
1  A.heap-size ← A.length
2  for i ← ⌊A.length / 2⌋ downto 1
3      do MAX-HEAPIFY(A,i)
```

- Tight bound:
  - We have at most 2^i nodes at depth i (height $h - i$)
  - We call MAX-HEAPIFY for each node of depth $i \Rightarrow O(h - i)$
  - The runtime of BUILD-MAX-HEAP is:

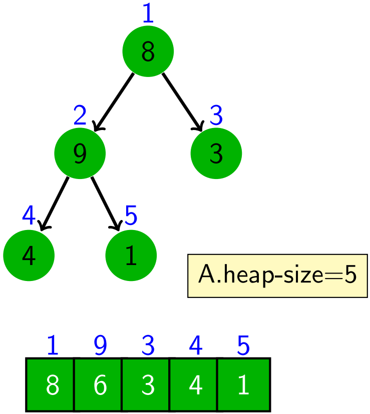
$$
\begin{aligned}
T(n) = \sum_{i=0}^{h-1} 2^i O(h-i) &= O\left(\sum_{i=0}^{h-1} 2^i (h-i)\right) \\
&= O\left(\sum_{j=1}^{h} \sum_{i=0}^{h-j} 2^i\right) \\
&= O(2^{h+1} - h - 2) \\
&= O(n)
\end{aligned}
$$

HeapSort: Principle

Problem: Sort an array A of n elements in non-decreasing order

HeapSort

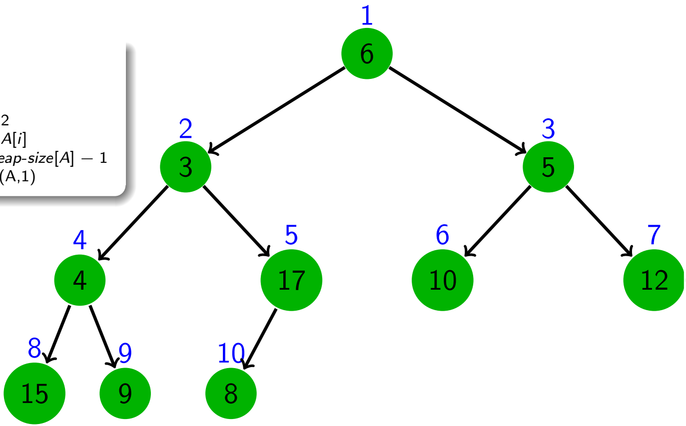
- Construct a max-heap from A (call BUILD-MAX-HEAP(A))
- Repeat until the heap is of size one:
  - Swap the values of the root and the right-most leaf of the heap (i.e., Swap A[1] and $A[A.heap - size]$)
  - Discard the right-most leaf from the heap by decreasing the heap size
  - Restore the max-heap property (call MAX-HEAPIFY(A,1))



HEAP-SORT

HEAP-SORT(A)

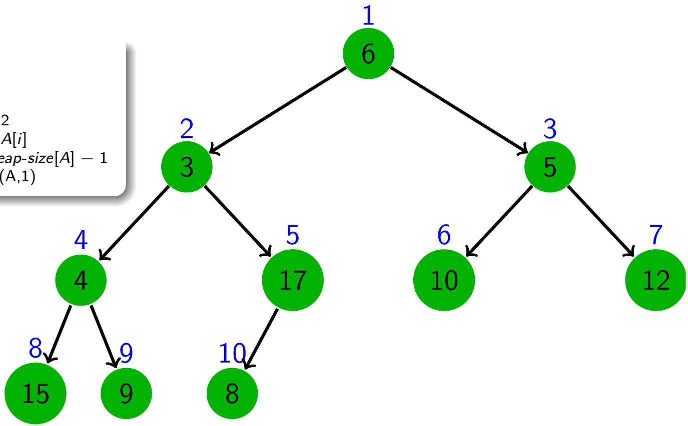
```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

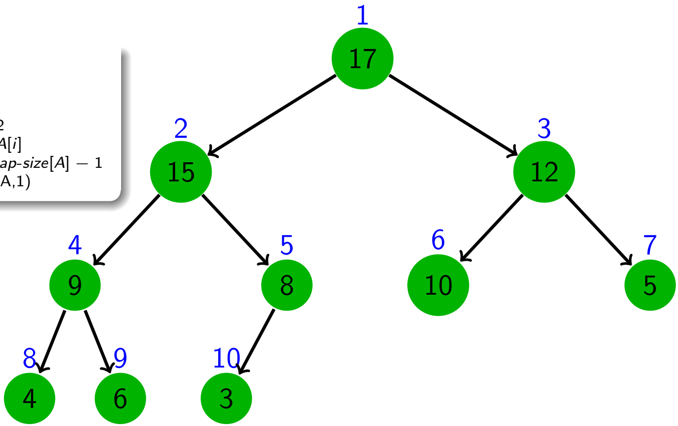


BUILD-MAX-HEAP(A)

1	2	3	4	5	6	7	8	9	10
6	3	5	4	17	10	12	15	9	8

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```

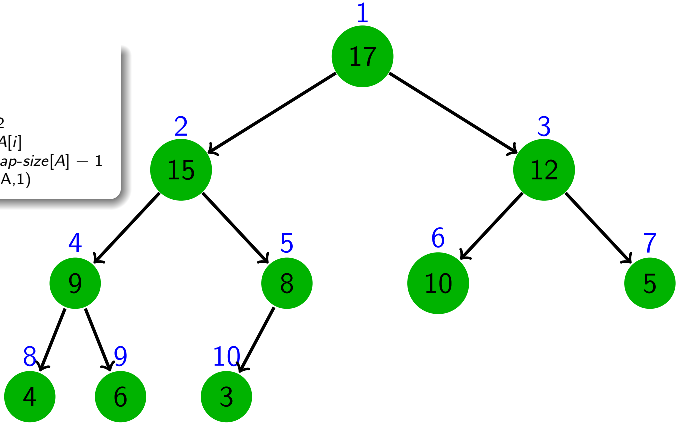


1	2	3	4	5	6	7	8	9	10
17	15	12	9	8	10	5	4	6	3

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
  do exchange $A[1] \leftrightarrow A[i]$
    $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
    MAX-HEAPIFY(A,1)



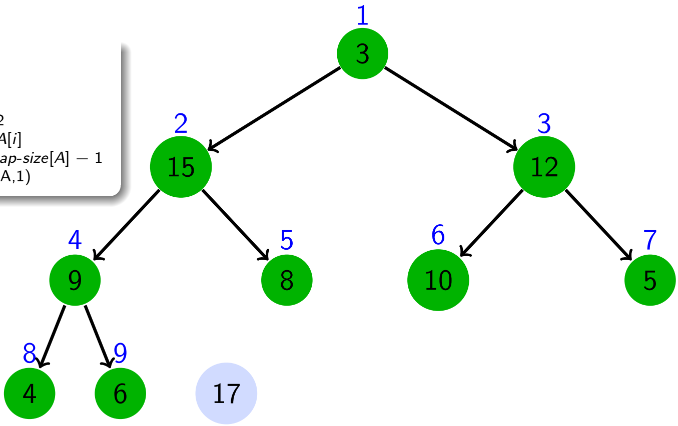
$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
17	15	12	9	8	10	5	4	6	3

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



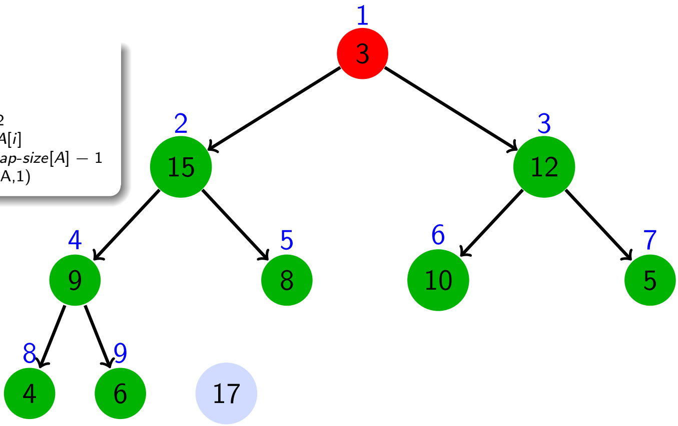
$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
3	15	12	9	8	10	5	4	6	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



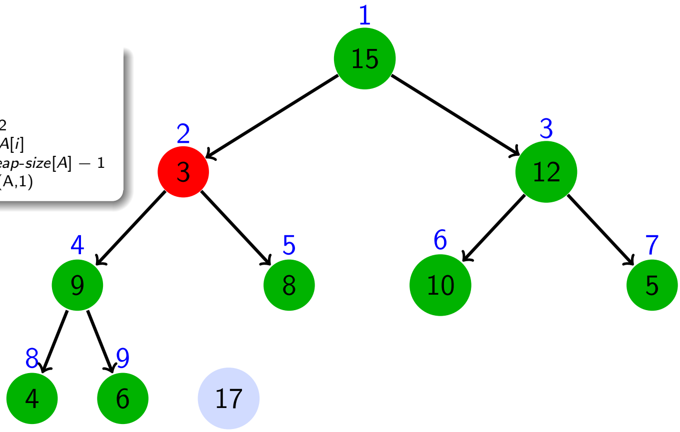
$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
3	15	12	9	8	10	5	4	6	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```

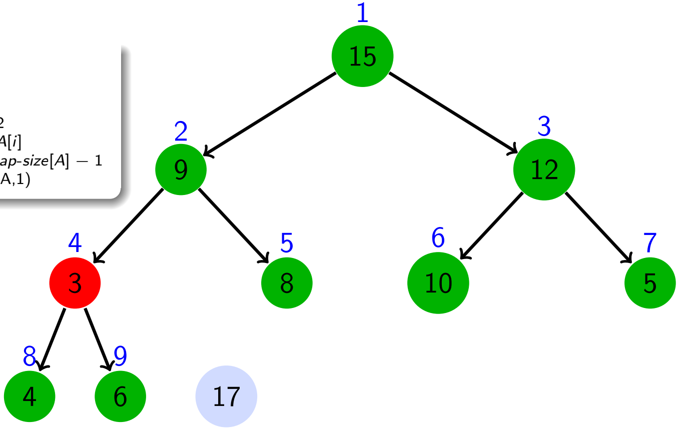


$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
15	3	12	9	8	10	5	4	6	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```

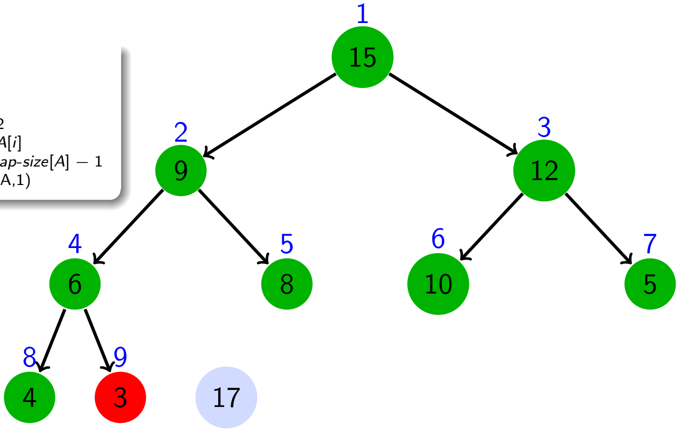


$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
15	9	12	3	8	10	5	4	6	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



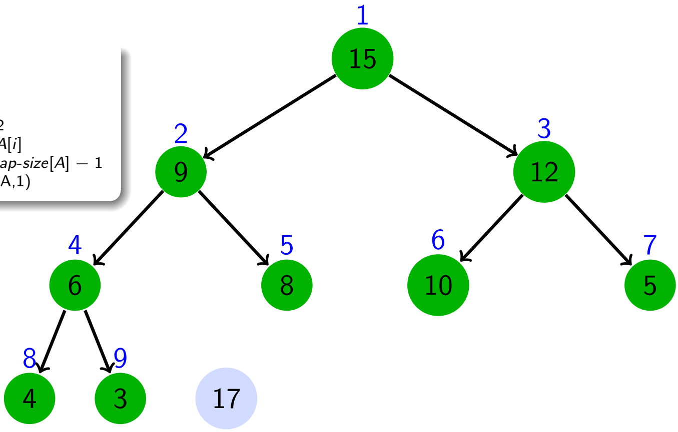
$\boxed{i = 10}$

1	2	3	4	5	6	7	8	9	10
15	9	12	6	8	10	5	4	3	17

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
    do exchange $A[1] \leftrightarrow A[i]$
        $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
        MAX-HEAPIFY(A,1)



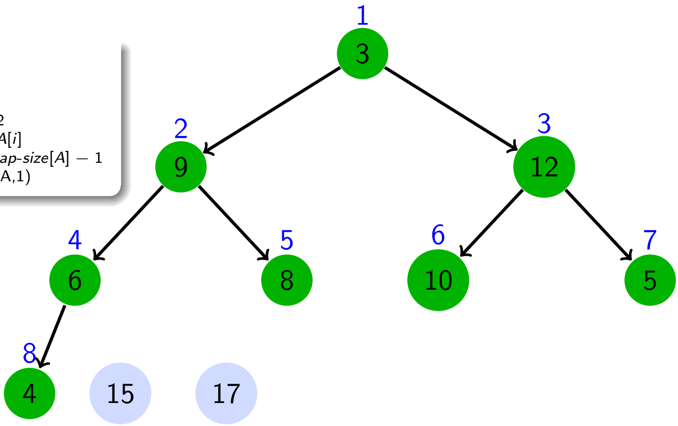
$\boxed{i = 9}$

1	2	3	4	5	6	7	8	9	10
15	9	12	6	8	10	5	4	3	17

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
  do exchange $A[1] \leftrightarrow A[i]$
    $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
    MAX-HEAPIFY(A,1)



$i = 9$

1	2	3	4	5	6	7	8	9	10
3	9	12	6	8	10	5	4	15	17

HEAP-SORT

HEAP-SORT(*A*)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
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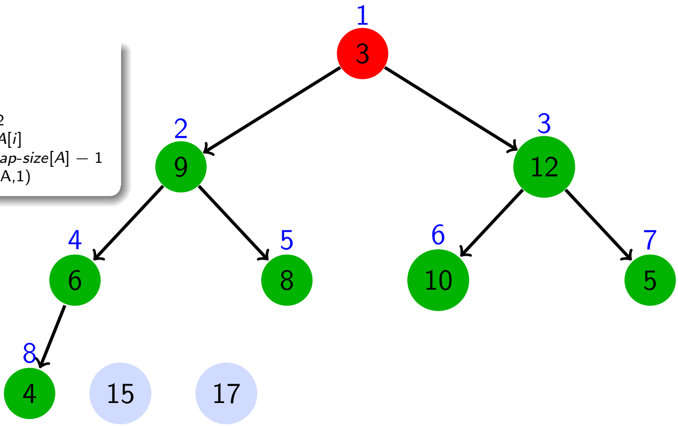
```

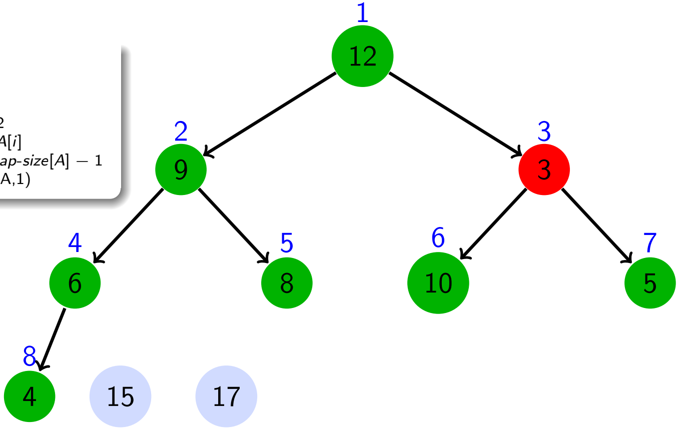


$\boxed{i = 9}$

1	2	3	4	5	6	7	8	9	10
3	9	12	6	8	10	5	4	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



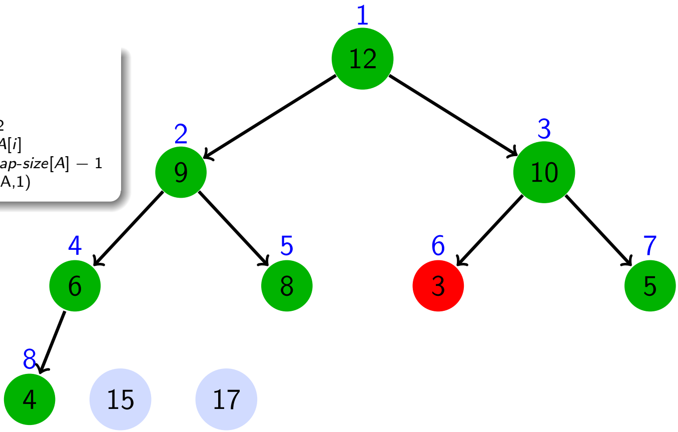
$\boxed{i = 9}$

1	2	3	4	5	6	7	8	9	10
12	9	3	6	8	10	5	4	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```



$\boxed{i = 9}$

1	2	3	4	5	6	7	8	9	10
12	9	10	6	8	3	5	4	15	17

HEAP-SORT

HEAP-SORT(A)

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BUILD-MAX-HEAP(A)
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    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
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```
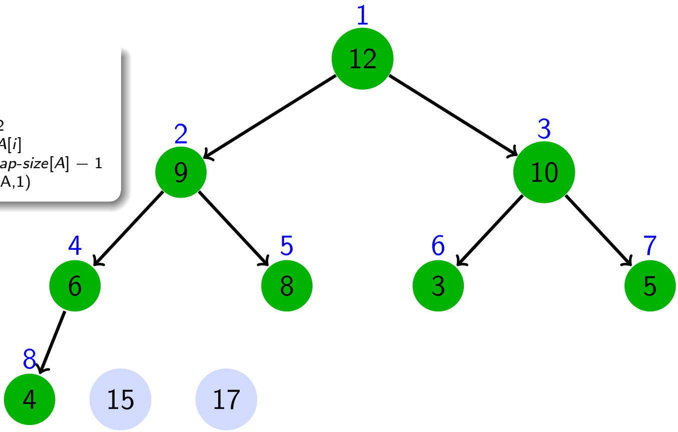
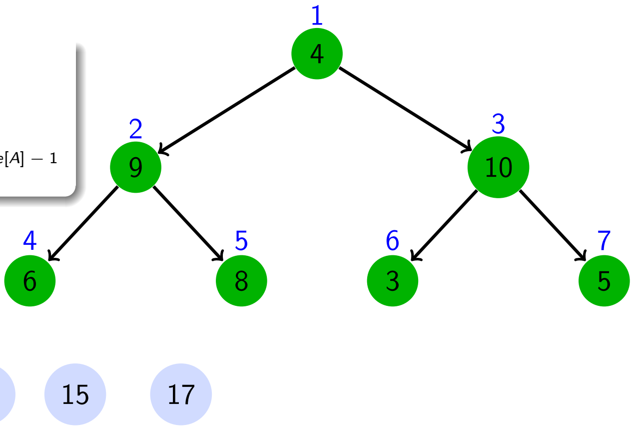



$\boxed{i = 8}$

1	2	3	4	5	6	7	8	9	10
12	9	10	6	8	3	5	4	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```

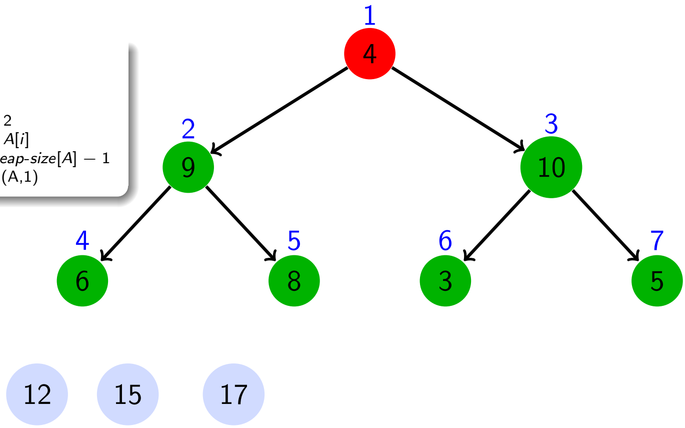


$\boxed{i = 8}$

1	2	3	4	5	6	7	8	9	10
4	9	10	6	8	3	5	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

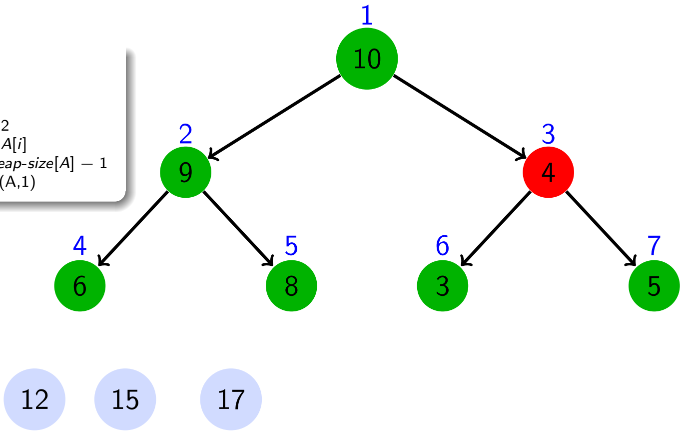


$\boxed{i = 8}$

1	2	3	4	5	6	7	8	9	10
4	9	10	6	8	3	5	12	15	17

HEAP-SORT

HEAP-SORT(A)
```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```

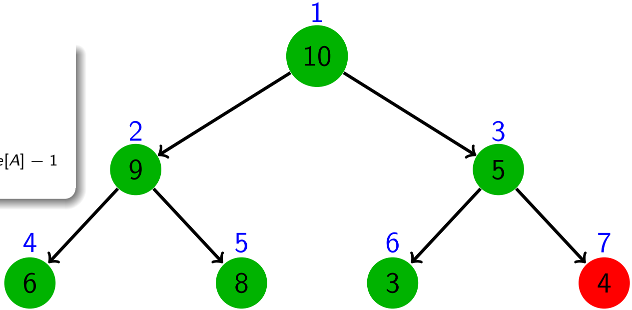


$\boxed{i = 8}$

1	2	3	4	5	6	7	8	9	10
10	9	4	6	8	3	5	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



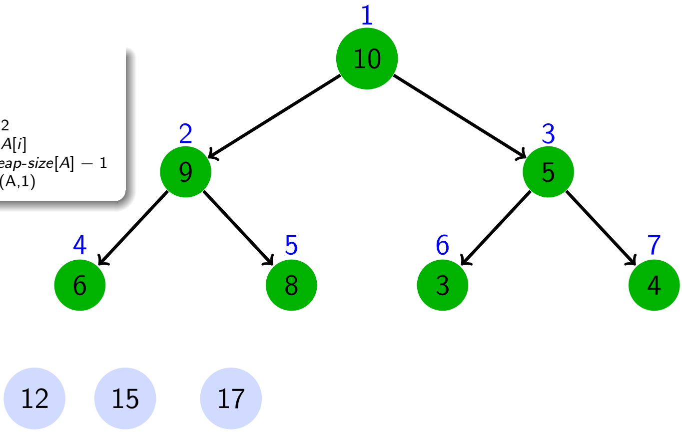
12 15 17

$\boxed{i = 8}$

1	2	3	4	5	6	7	8	9	10
10	9	5	6	8	3	4	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

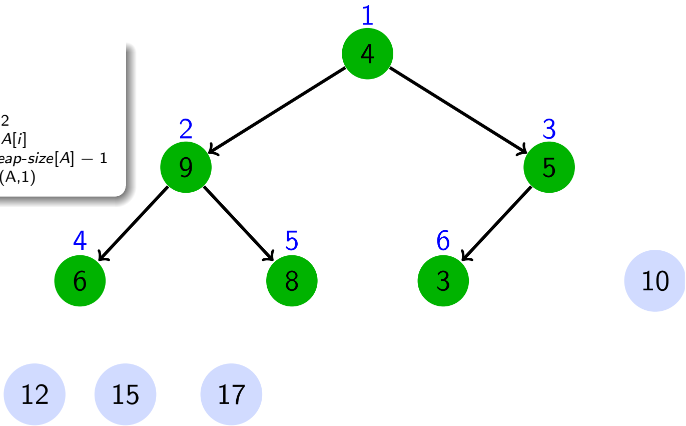


$\boxed{i = 7}$

1	2	3	4	5	6	7	8	9	10
10	9	5	6	8	3	4	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```

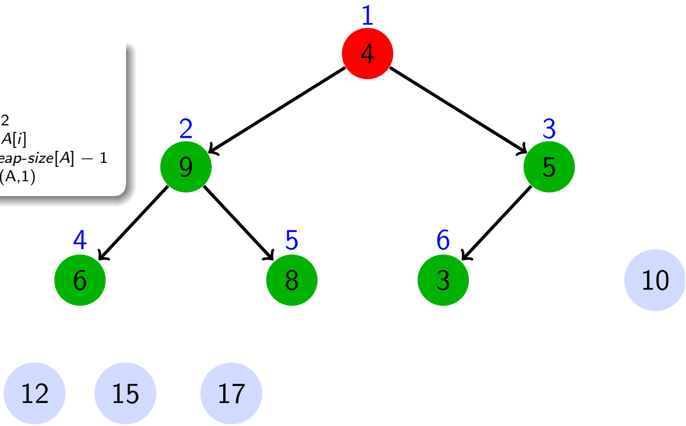


$\boxed{i = 7}$

1	2	3	4	5	6	7	8	9	10
4	9	5	6	8	3	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

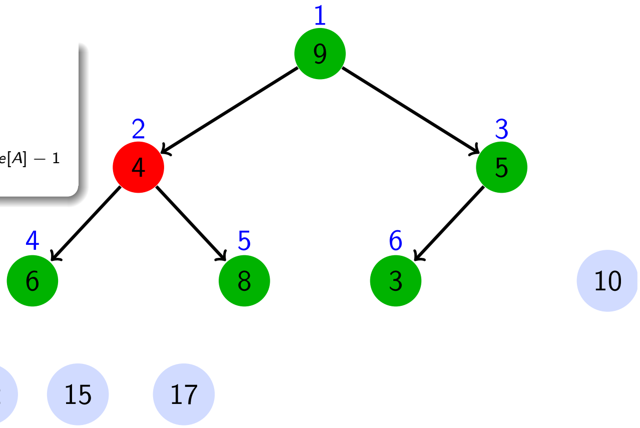


$\boxed{i = 7}$

1	2	3	4	5	6	7	8	9	10
4	9	5	6	8	3	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

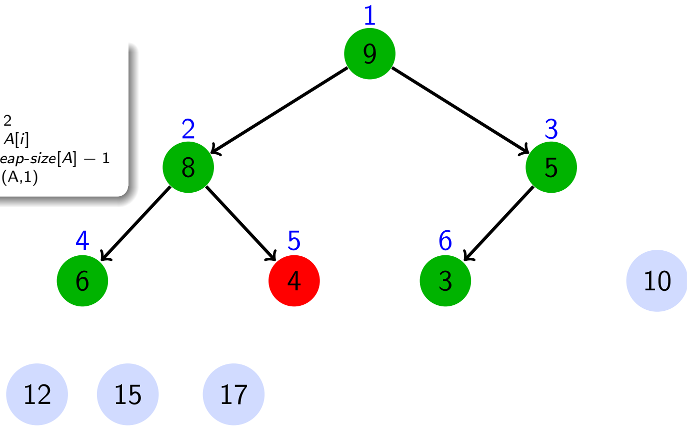


$\boxed{i = 7}$

1	2	3	4	5	6	7	8	9	10
9	4	5	6	8	3	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



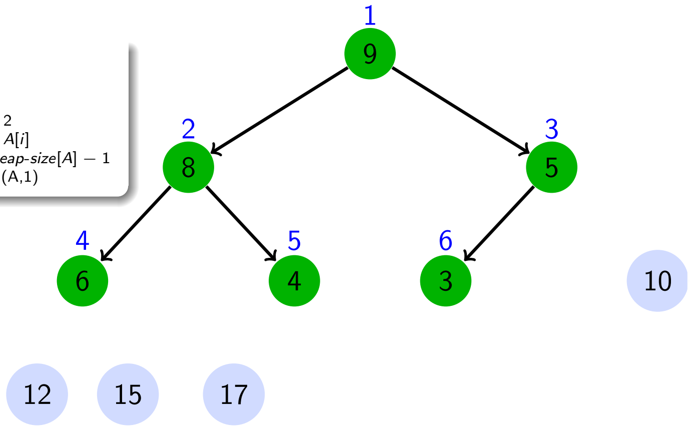
$\boxed{i = 7}$

1	2	3	4	5	6	7	8	9	10
9	8	5	6	4	3	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
    do exchange $A[1] \leftrightarrow A[i]$
        $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
        MAX-HEAPIFY(A,1)



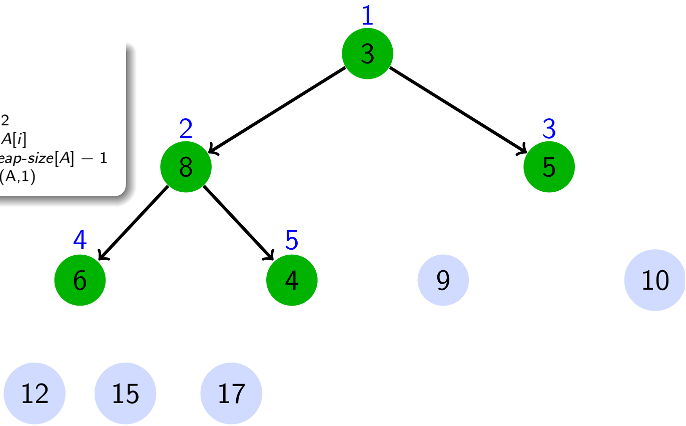
$\boxed{i = 6}$

1	2	3	4	5	6	7	8	9	10
9	8	5	6	4	3	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
    do exchange $A[1] \leftrightarrow A[i]$
        $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
        MAX-HEAPIFY(A,1)

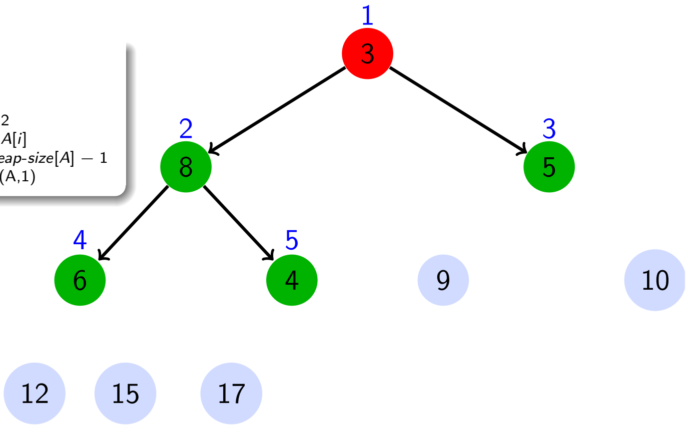


$\boxed{i = 6}$

1	2	3	4	5	6	7	8	9	10
3	8	5	6	4	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

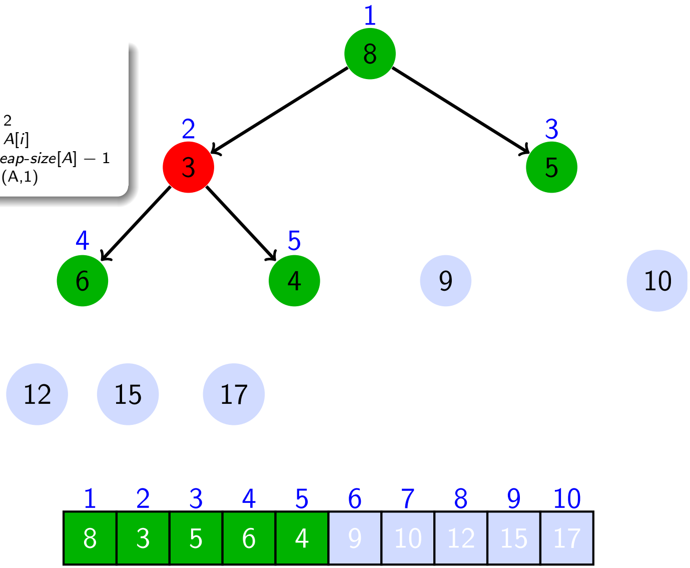


$\boxed{i = 6}$

1	2	3	4	5	6	7	8	9	10
3	8	5	6	4	9	10	12	15	17

HEAP-SORT

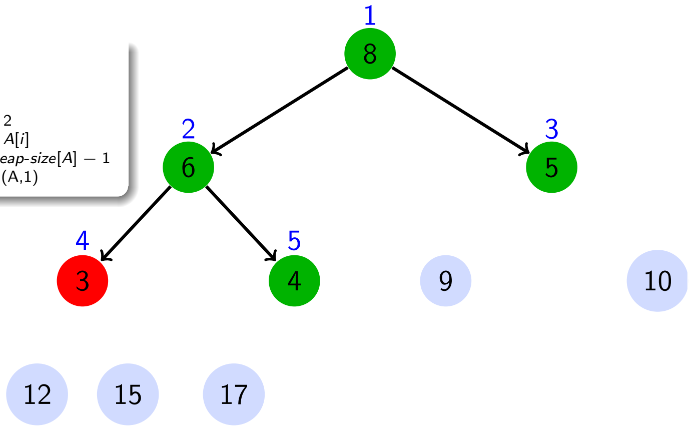
```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



$\boxed{i = 6}$

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



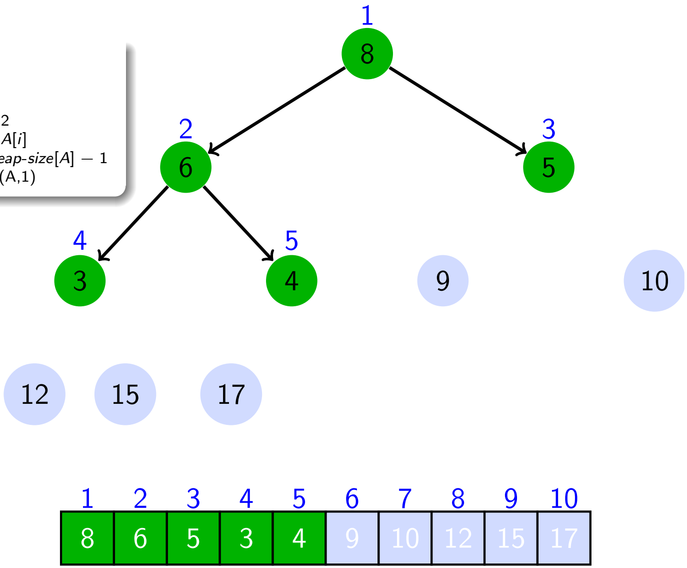
$\boxed{i = 6}$

1	2	3	4	5	6	7	8	9	10
8	6	5	3	4	9	10	12	15	17

HEAP-SORT

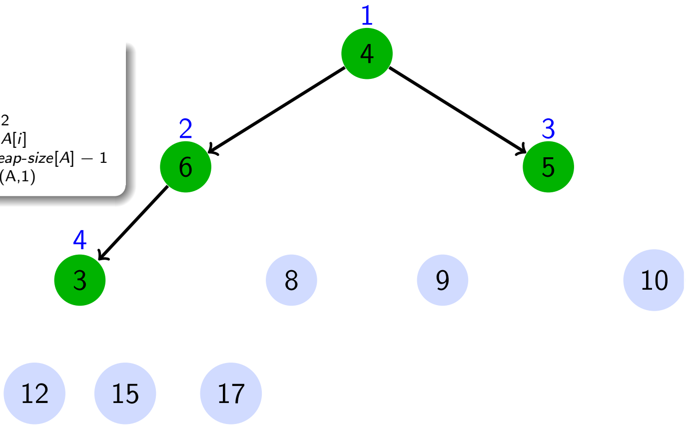
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for $i \leftarrow A.\,length$ **downto** 2
    do exchange $A[1] \leftrightarrow A[i]$
        $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
        MAX-HEAPIFY(A,1)

$i = 5$



HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



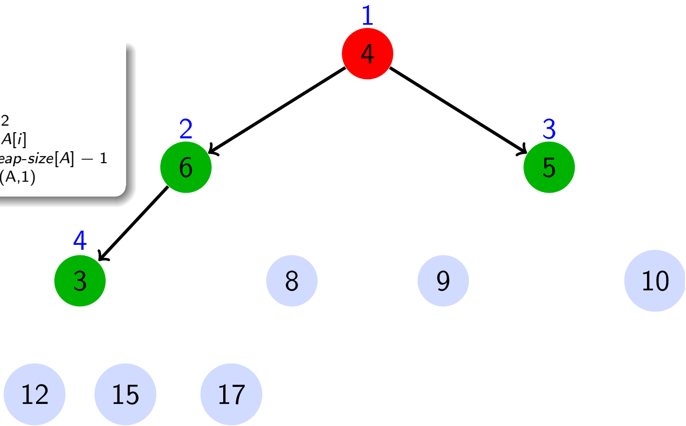
$\boxed{i = 5}$

1	2	3	4	5	6	7	8	9	10
4	6	5	3	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

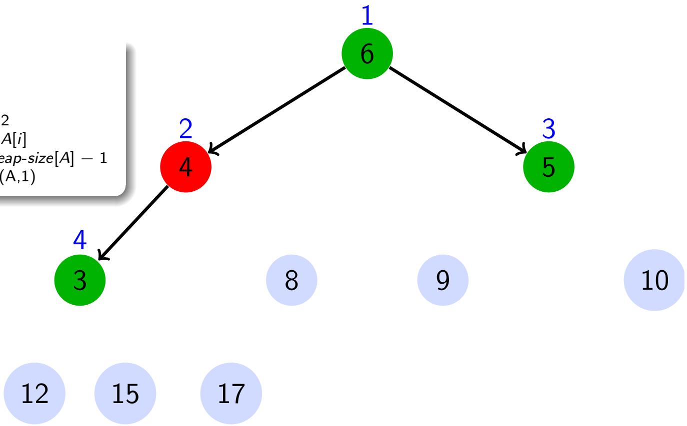


$\boxed{i = 5}$

1	2	3	4	5	6	7	8	9	10
4	6	5	3	8	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

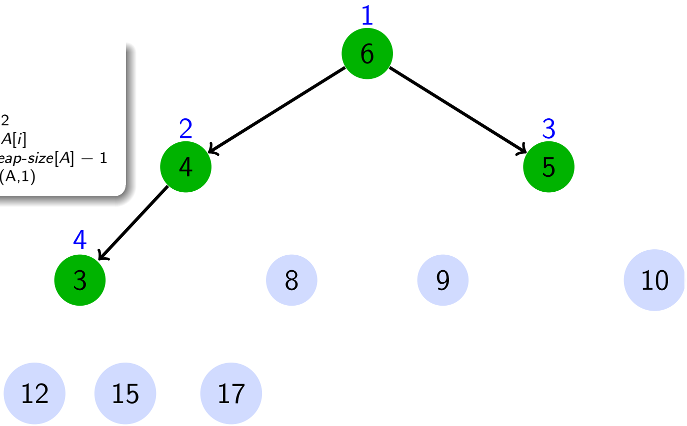


$\boxed{i = 5}$

1	2	3	4	5	6	7	8	9	10
6	4	5	3	8	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

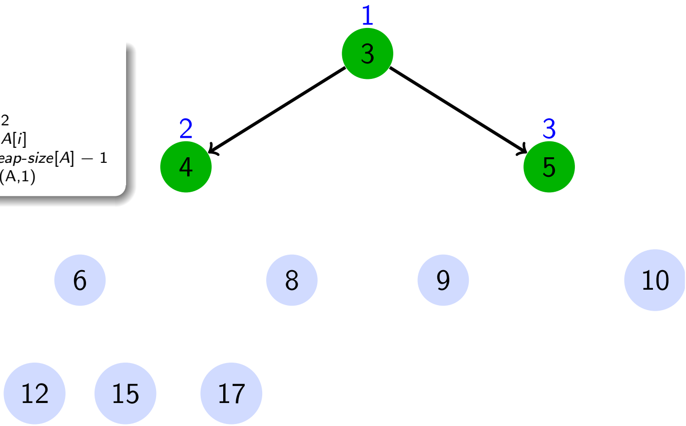


$\boxed{i = 4}$

1	2	3	4	5	6	7	8	9	10
6	4	5	3	8	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



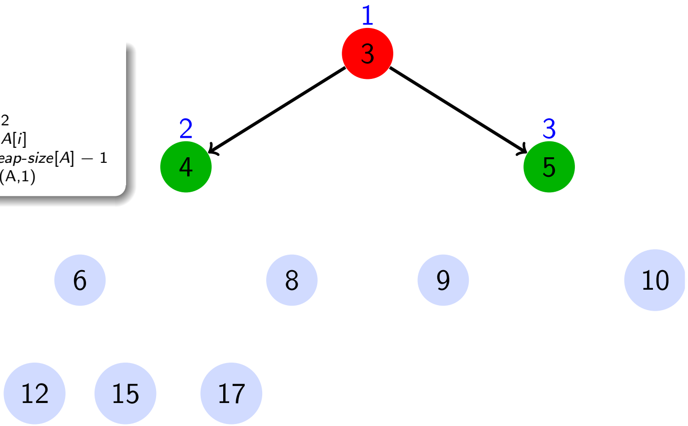
$\boxed{i = 4}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



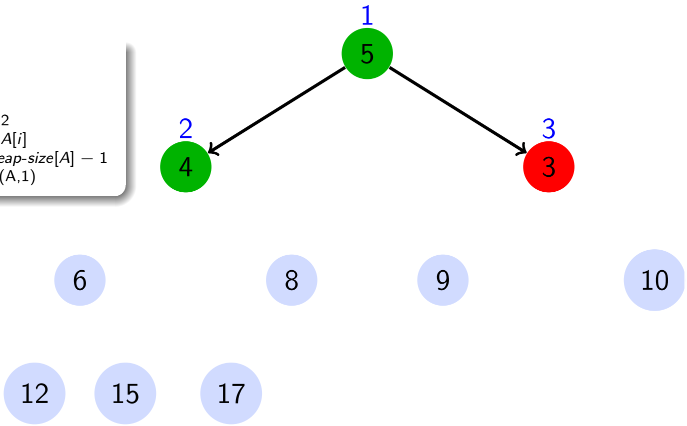
$\boxed{i = 4}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



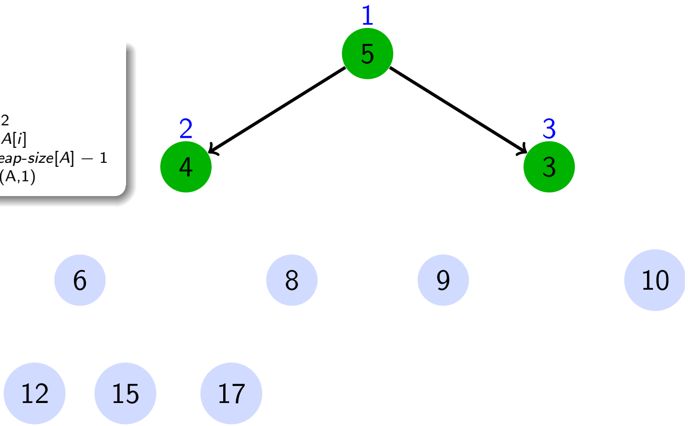
$\boxed{i = 4}$

1	2	3	4	5	6	7	8	9	10
5	4	3	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```



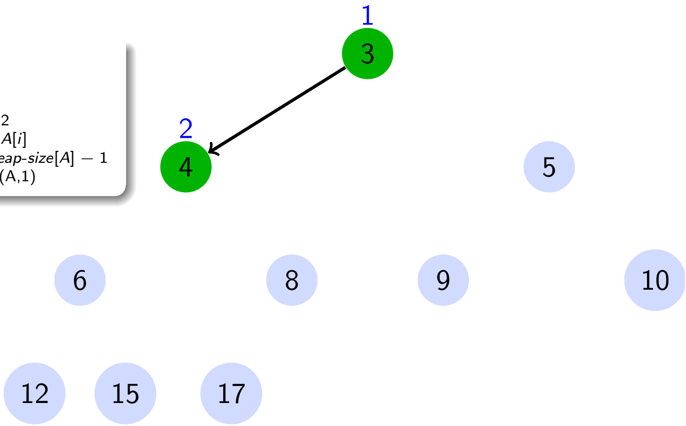
$\boxed{i = 3}$

1	2	3	4	5	6	7	8	9	10
5	4	3	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



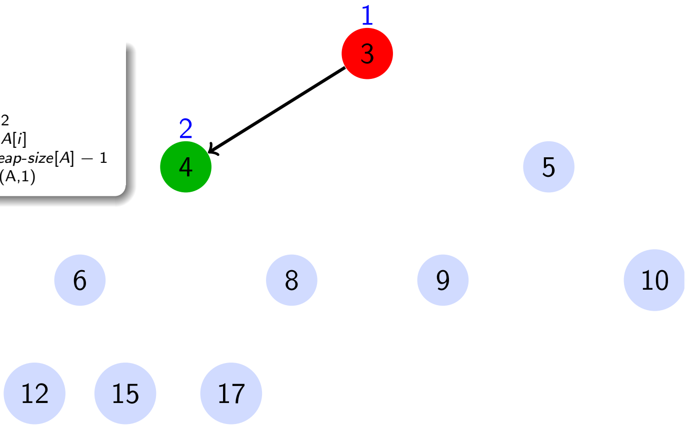
$\boxed{i = 3}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
        heap-size[A] ← heap-size[A] − 1
        MAX-HEAPIFY(A,1)
```

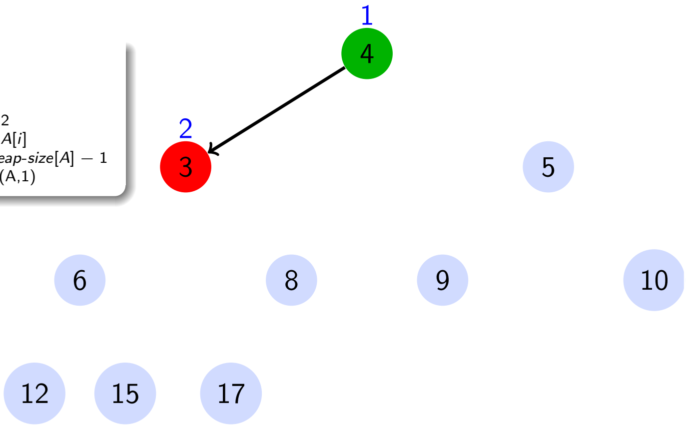


$\boxed{i = 3}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```



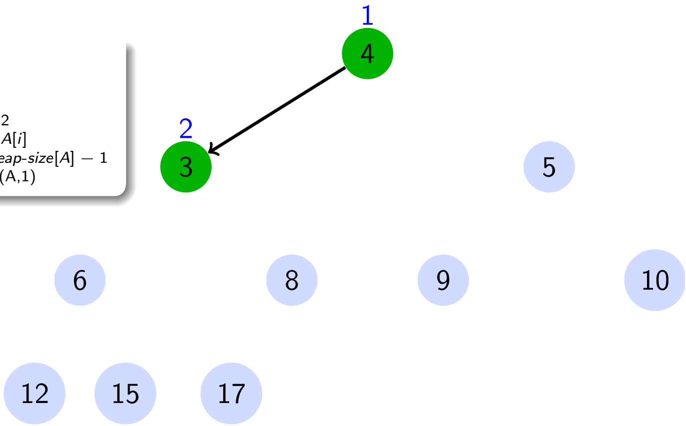
$\boxed{i = 3}$

1	2	3	4	5	6	7	8	9	10
4	3	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

BUILD-MAX-HEAP(A)
for $i \leftarrow A.\ length$ **downto** 2
  do exchange $A[1] \leftrightarrow A[i]$
    $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
    MAX-HEAPIFY(A,1)



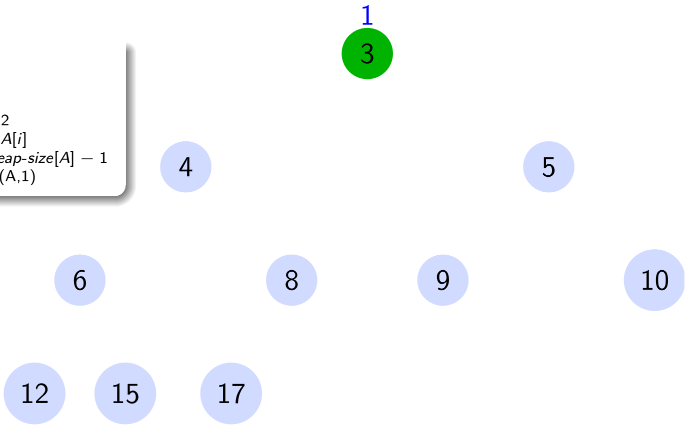
$\boxed{i = 2}$

1	2	3	4	5	6	7	8	9	10
4	3	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```

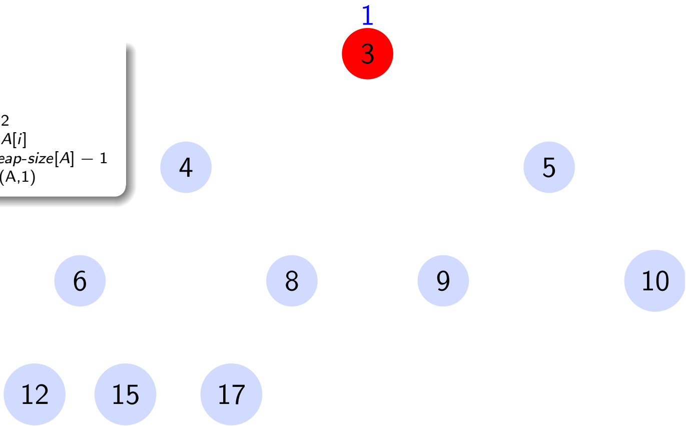


$\boxed{i = 2}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

```
HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
    do exchange A[1] ↔ A[i]
       heap-size[A] ← heap-size[A] − 1
       MAX-HEAPIFY(A,1)
```



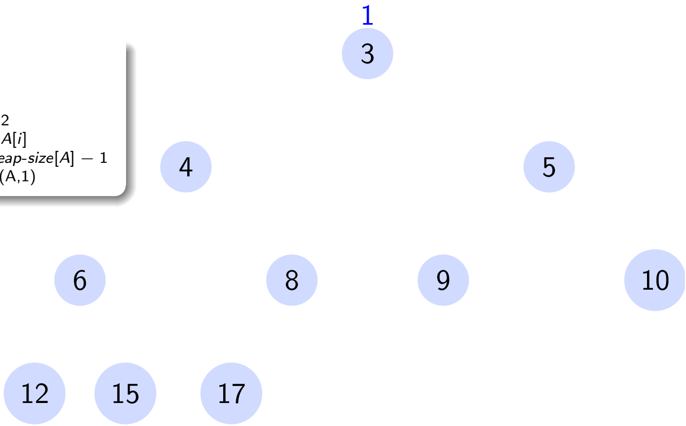
$\boxed{i = 2}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT

HEAP-SORT(*A*)

```
BUILD-MAX-HEAP(A)
for i ← A. length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```



$\boxed{i = 2}$

1	2	3	4	5	6	7	8	9	10
3	4	5	6	8	9	10	12	15	17

HEAP-SORT: Runtime

- Let $n = A.\mathit{length}$
- Cost of BUILD-MAX-HEAP: $O(n)$

HEAP-SORT(A)

```
BUILD-MAX-HEAP(A)
for i ← A.length downto 2
      do exchange A[1] ↔ A[i]
         heap-size[A] ← heap-size[A] − 1
         MAX-HEAPIFY(A,1)
```

- $(n-1)$ calls to MAX-HEAPIFY, each of which costs $O(log_2(n))$
- $$\begin{aligned} T(n) &= (n-1)O(log_2(n)) + O(n) \\ &= O(n \cdot log_2(n)) \end{aligned}$$

Priority Queues

- A priority queue is a data structure which represents a set S of elements.
- Each element has a key (an integer)
- A priority queue should support the following operations:
  - $\textsc{Insert}(S, x)$: inserts the element x into the set S (i.e., $S \leftarrow S \cup \{x\}$)
  - $\textsc{Maximum}(S)$: returns the element of S with largest key.
  - $\textsc{Extract-Max}(S)$: returns and removes the element of S with largest key.
  - $\textsc{Increase-Key}(S, i, x)$: finds the element with index i, and increases its value to x (x is assumed to be larger than the original value of the element)

Priority Queues

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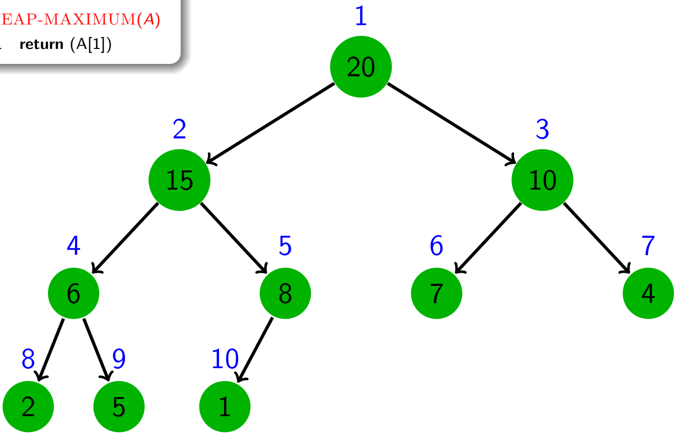
  - $\textsc{Increase-Key}(S, i, x)$: finds the element with index i, and increases its value to x (x is assumed to be larger than the original value of the element)

Max-Heaps efficiently implement priority queues.

HEAP-MAXIMUM

HEAP-MAXIMUM(A): returns the largest element of A: It is the root

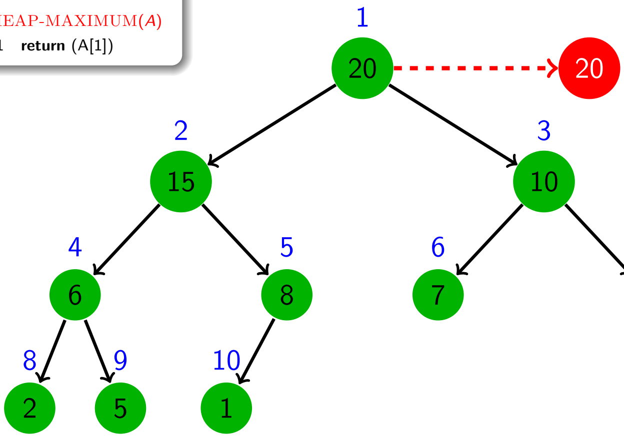
```
HEAP-MAXIMUM(A)
1  return (A[1])
```



HEAP-MAXIMUM

HEAP-MAXIMUM(A): returns the largest element of A: It is the root

HEAP-MAXIMUM(A)
1 **return** (A[1])

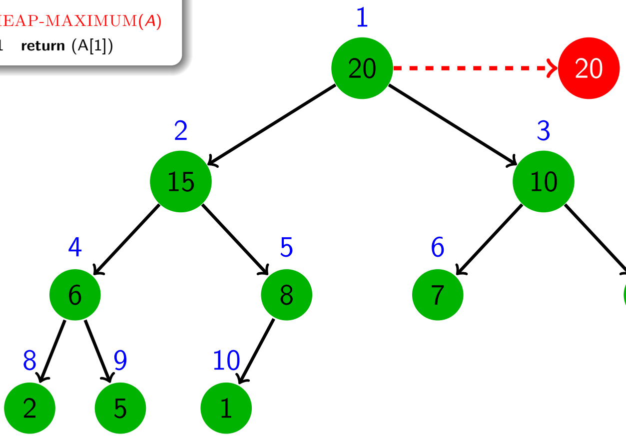


HEAP-MAXIMUM

HEAP-MAXIMUM(A): returns the largest element of A: It is the root

HEAP-MAXIMUM(A)
1 **return** (A[1])

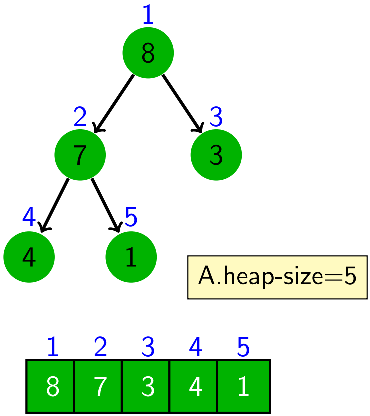
Time: $\Theta(1)$



EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

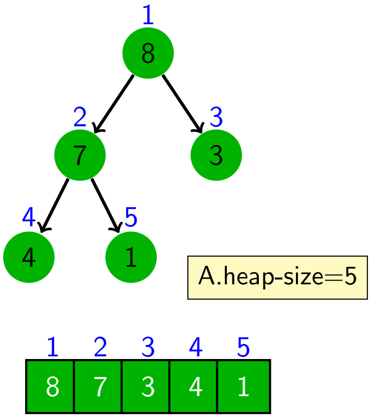


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.

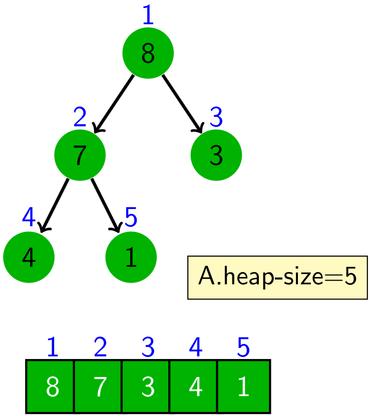


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)



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EXTRACT-MAX

- Make sure that the max-heap is not empty.
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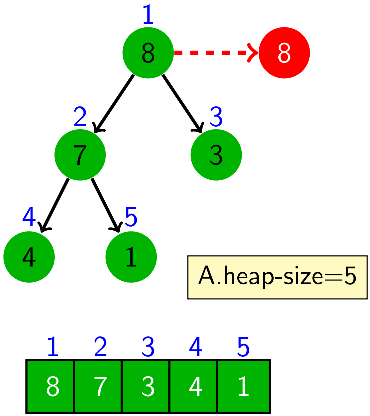
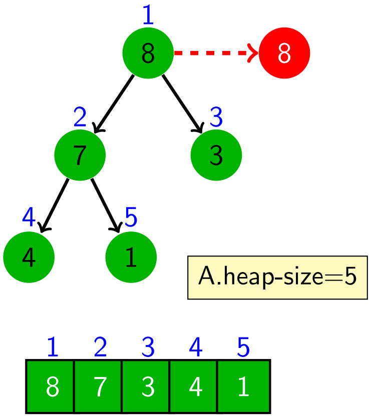


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)
- Make the last node of the heap the new root



EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

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- Make sure that the max-heap is not empty.
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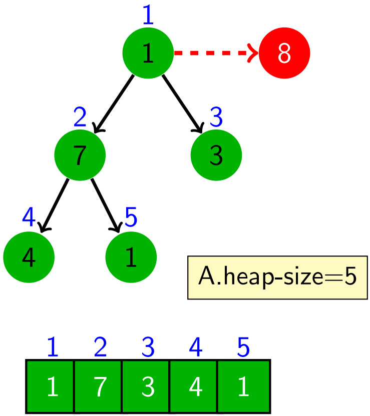
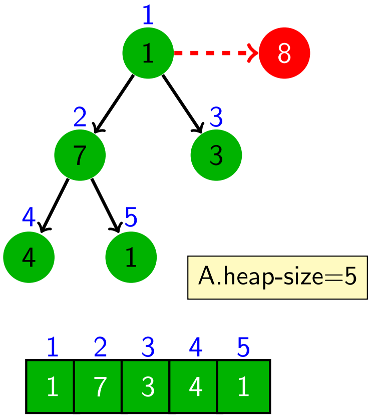


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)
- Make the last node of the heap the new root
- Discard the last node of the heap



EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

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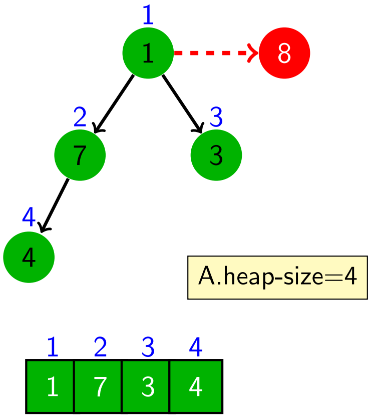
- Discard the last node of the heap

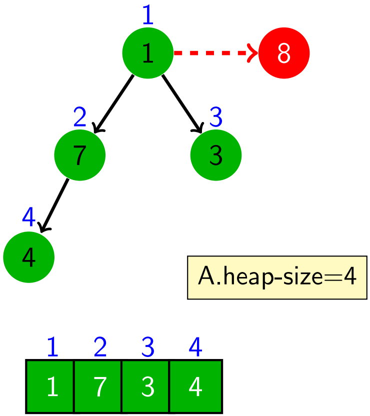


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)
- Make the last node of the heap the new root
- Discard the last node of the heap
- Restore the max-heap property

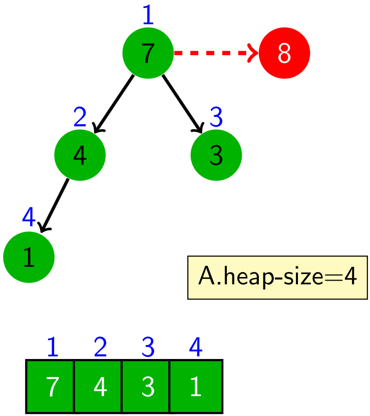


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)
- Make the last node of the heap the new root
- Discard the last node of the heap
- Restore the max-heap property

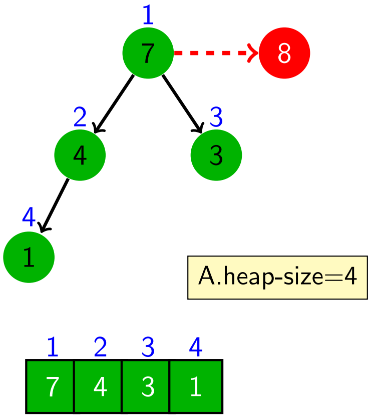


EXTRACT-MAX: Principle

Problem: returns and removes the largest element of the array

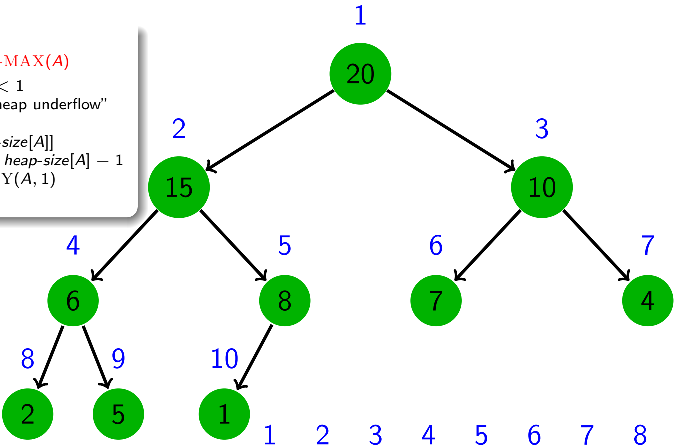
EXTRACT-MAX

- Make sure that the max-heap is not empty.
- Make a copy of the maximum element (the root)
- Make the last node of the heap the new root
- Discard the last node of the heap
- Restore the max-heap property
- Return the copy of the maximum element



HEAP-EXTRACT-MAX

```
HEAP-EXTRACT-MAX(A)
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

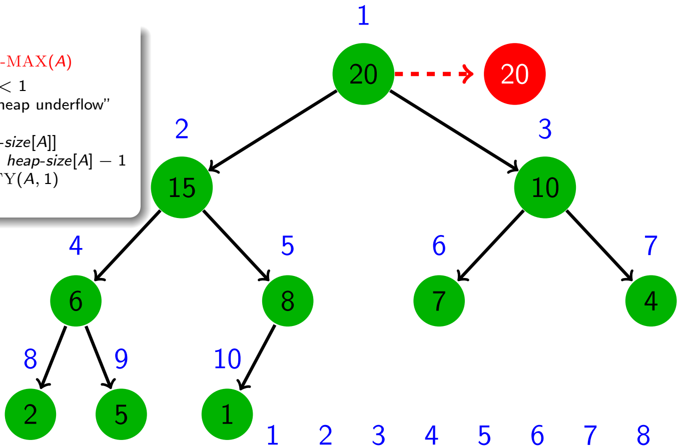


1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	1

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

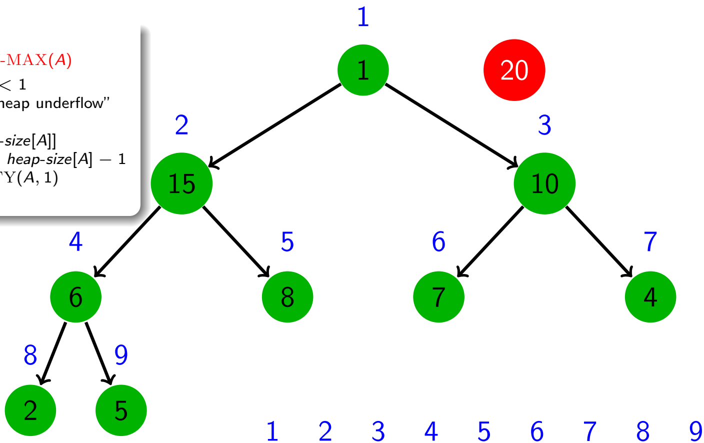


1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	1

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
   then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

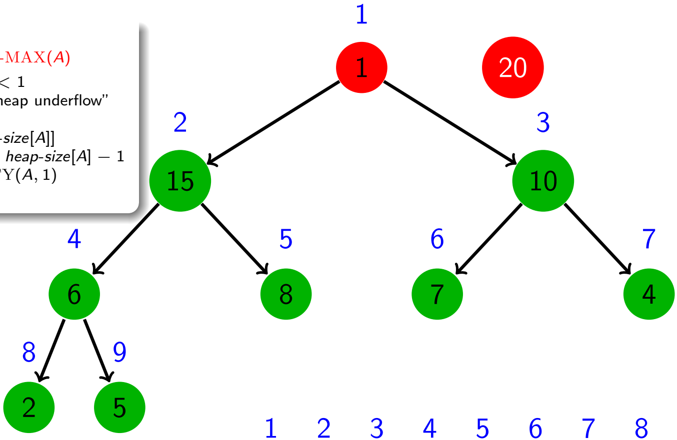


1	2	3	4	5	6	7	8	9
1	15	10	6	8	7	4	2	5

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

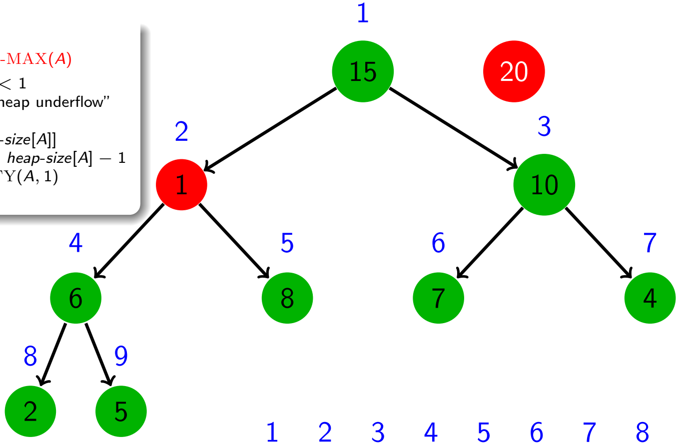


1	2	3	4	5	6	7	8	9
1	15	10	6	8	7	4	2	5

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

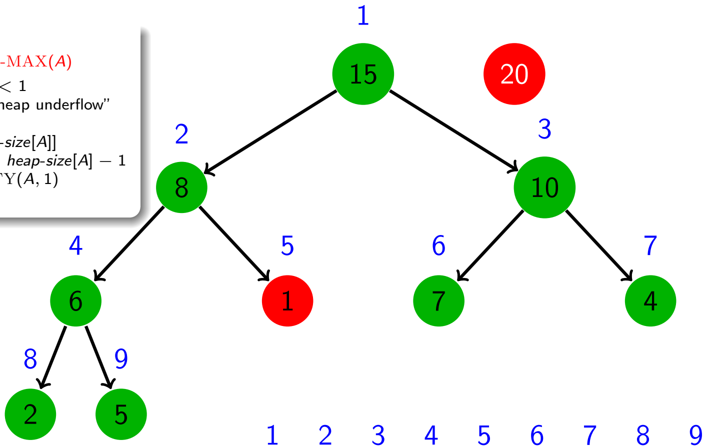


1	2	3	4	5	6	7	8	9
15	1	10	6	8	7	4	2	5

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```



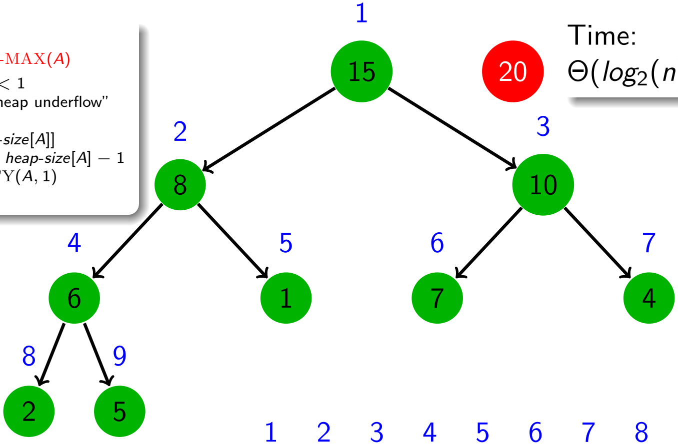
1	2	3	4	5	6	7	8	9
15	8	10	6	8	7	4	2	5

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

```
if heap-size[A] < 1
    then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] − 1
MAX-HEAPIFY(A, 1)
return max
```

Time:
$\Theta(log_2(n))$

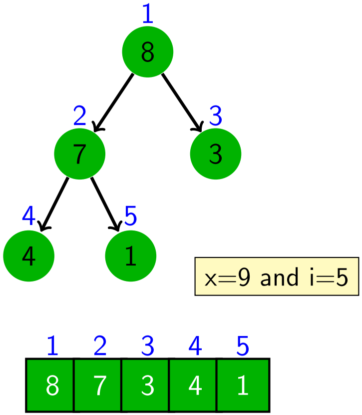


1	2	3	4	5	6	7	8	9
15	8	10	6	8	7	4	2	5

HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

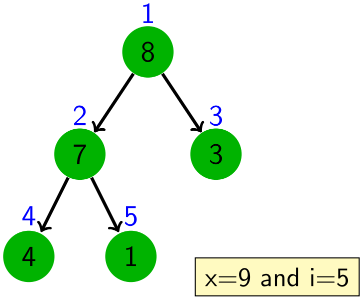


HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

- Make sure that x is larger than the original value at position i



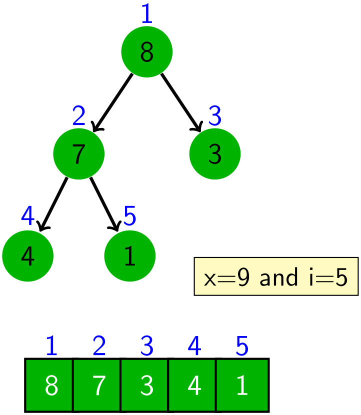
1	2	3	4	5
8	7	3	4	1

HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

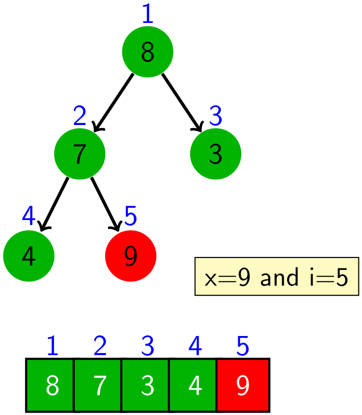
- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x



Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x

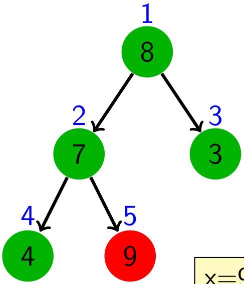


HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x
- Traverse the tree upward comparing x to its parent and swapping keys if necessary, until the max-heap property is restored



x=9 and i=5

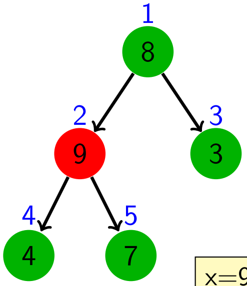
1	2	3	4	5
8	7	3	4	9

HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x
- Traverse the tree upward comparing x to its parent and swapping keys if necessary, until the max-heap property is restored



x=9 and i=5

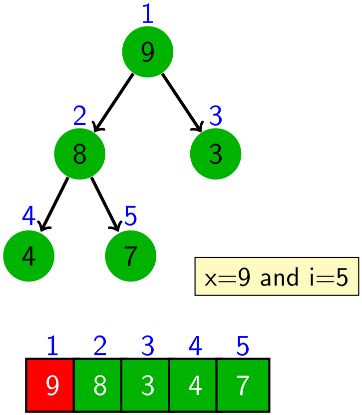
1	2	3	4	5
8	9	3	4	7

HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x
- Traverse the tree upward comparing x to its parent and swapping keys if necessary, until the max-heap property is restored

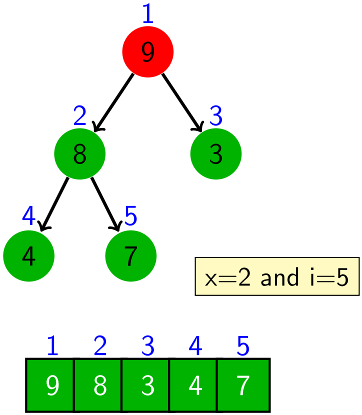


HEAP-INCREASE-KEY: Principle

Problem: Find the element with index i, and increase its value to x (x is assumed to be larger than the original value of the element)

HEAP-INCREASE-KEY

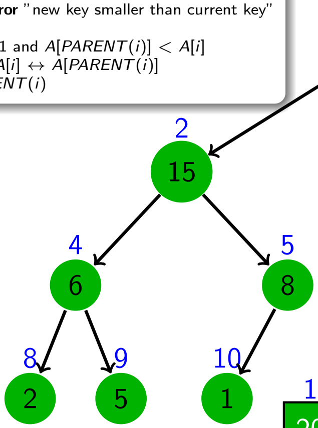
- Make sure that x is larger than the original value at position i
- Increase the value of the element with index i to x
- Traverse the tree upward comparing x to its parent and swapping keys if necessary, until the max-heap property is restored



HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```



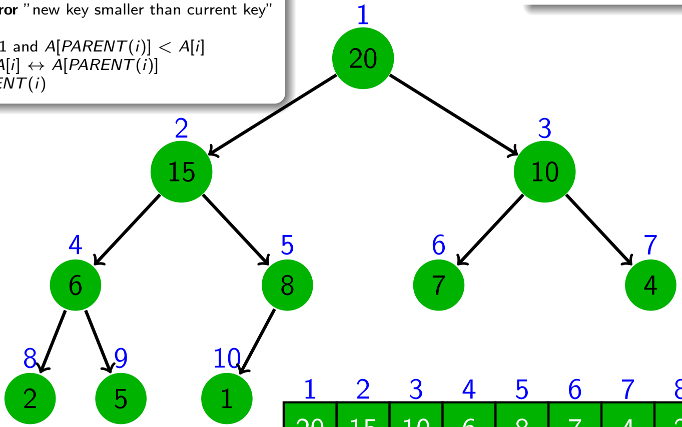
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	1

HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```

HEAP-INCREASE-KEY($A, 9, 18$)



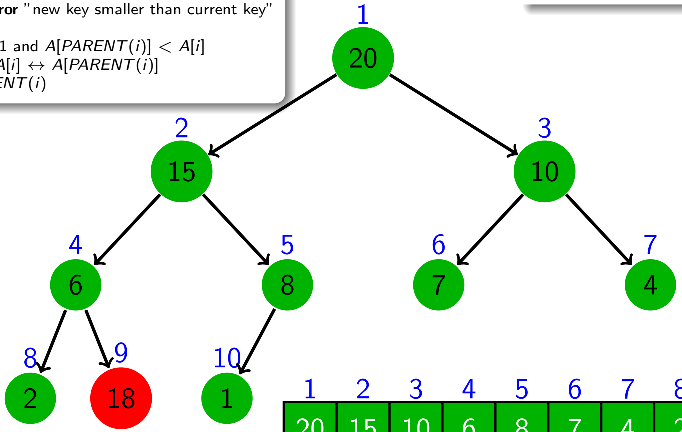
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	1

HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```

HEAP-INCREASE-KEY($A, 9, 18$)



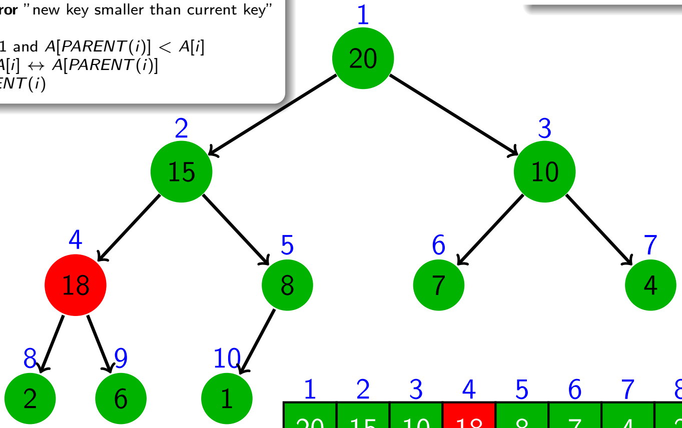
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	18	1

HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

if $x < A[i]$
    then error "new key smaller than current key"
$A[i] \leftarrow x$
while $i > 1$ and $A[PARENT(i)] < A[i]$
exchange $A[i] \leftrightarrow A[PARENT(i)]$
$i \leftarrow PARENT(i)$

HEAP-INCREASE-KEY($A, 9, 18$)



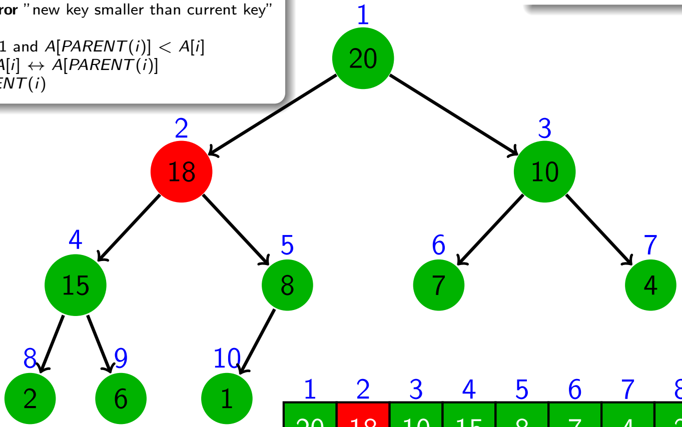
1	2	3	4	5	6	7	8	9	10
20	15	10	18	8	7	4	2	18	1

HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```

HEAP-INCREASE-KEY(A, 9, 18)



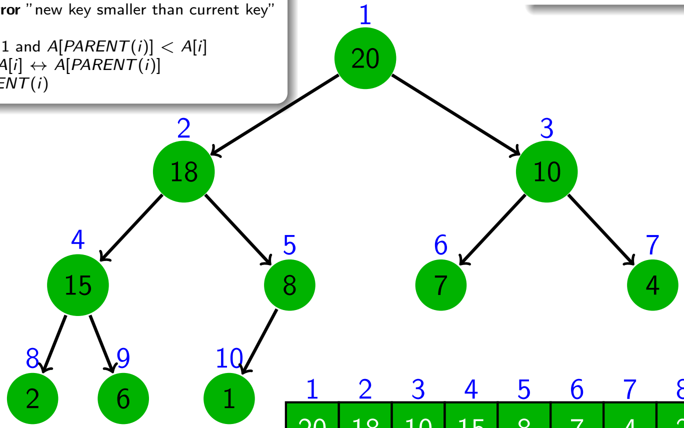
1	2	3	4	5	6	7	8	9	10
20	18	10	15	8	7	4	2	18	1

HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```

HEAP-INCREASE-KEY(A, 9, 18)



1	2	3	4	5	6	7	8	9	10
20	18	10	15	8	7	4	2	18	1

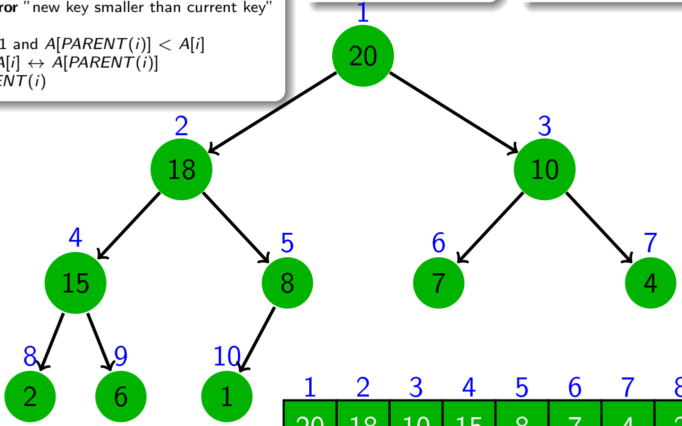
HEAP-INCREASE-KEY

HEAP-INCREASE-KEY(A, i, x)

```
if x < A[i]
    then error "new key smaller than current key"
A[i] ← x
while i > 1 and A[PARENT(i)] < A[i]
exchange A[i] ↔ A[PARENT(i)]
i ← PARENT(i)
```

Time: $\Theta(log_2 n)$

HEAP-INCREASE-KEY($A, 9, 18$)

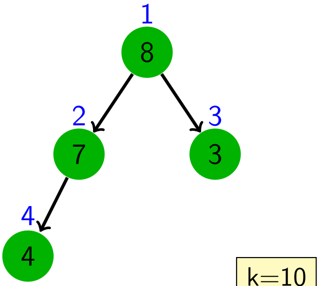


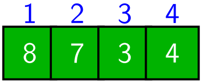
1	2	3	4	5	6	7	8	9	10
20	18	10	15	8	7	4	2	18	1

MAX-HEAP-INSERT: Principle

Problem: inserts the value *k* into the heap

MAX-HEAP-INSERT



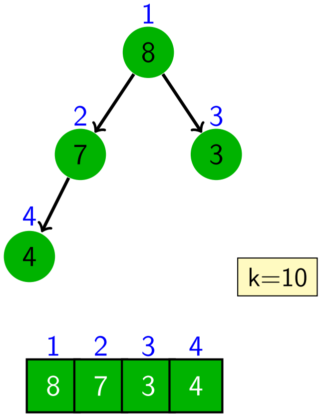


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$

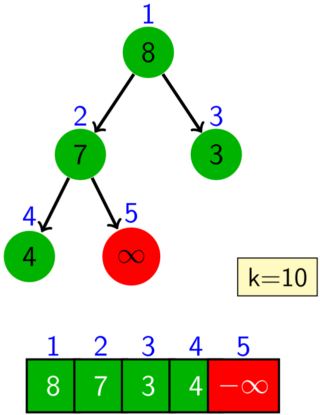


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$

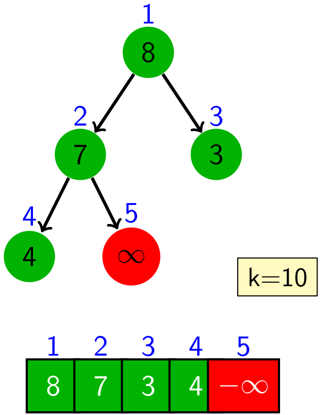


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$
- Increase the $-\infty$ value to k using the HEAP-INCREASE-KEY procedure

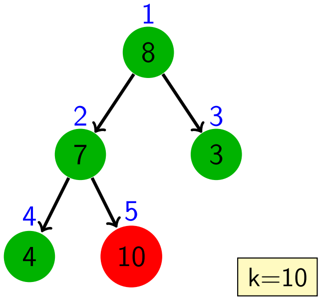


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$
- Increase the $-\infty$ value to k using the HEAP-INCREASE-KEY procedure



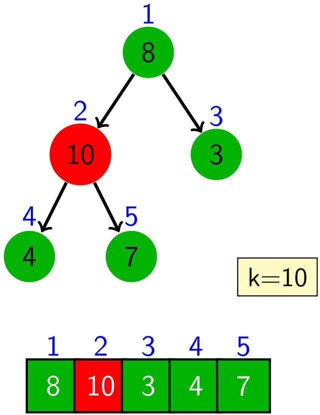
1	2	3	4	5
8	7	3	4	10

MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$
- Increase the $-\infty$ value to k using the HEAP-INCREASE-KEY procedure

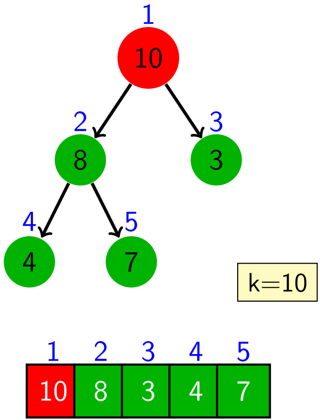


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$
- Increase the $-\infty$ value to k using the HEAP-INCREASE-KEY procedure

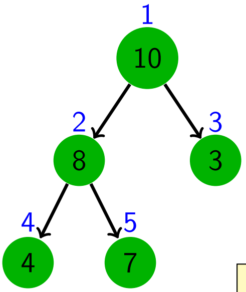


MAX-HEAP-INSERT: Principle

Problem: inserts the value k into the heap

MAX-HEAP-INSERT

- Insert a new node in the very last position in the tree with key $-\infty$
- Increase the $-\infty$ value to k using the HEAP-INCREASE-KEY procedure



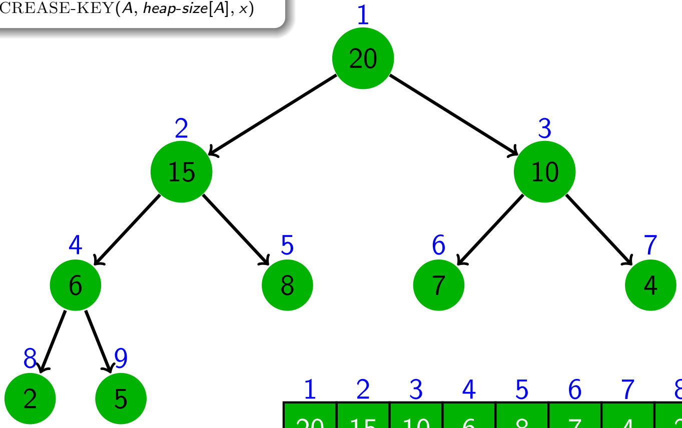
k=10

1	2	3	4	5
10	8	3	4	7

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY$(A, heap\text{-}size[A], x)$



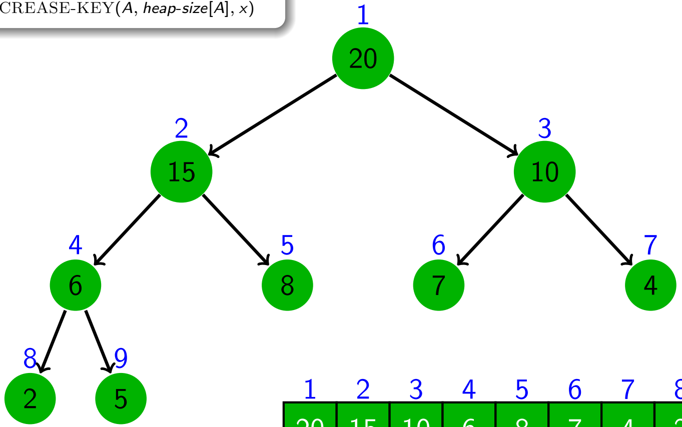
1	2	3	4	5	6	7	8	9
20	15	10	6	8	7	4	2	5

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

MAX-HEAP-INSERT$(A, 16)$



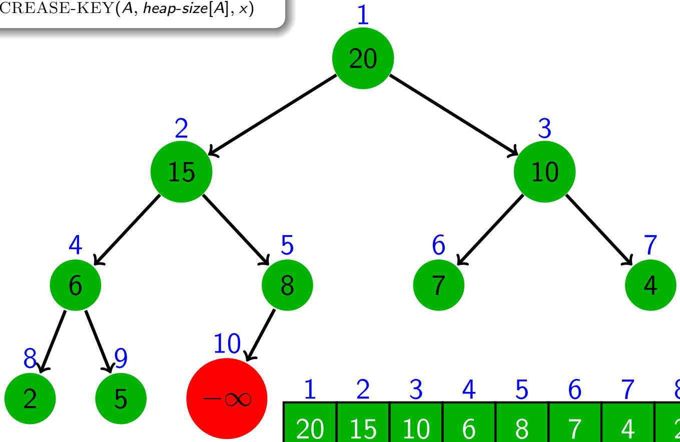
1	2	3	4	5	6	7	8	9
20	15	10	6	8	7	4	2	5

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

MAX-HEAP-INSERT(A, 16)



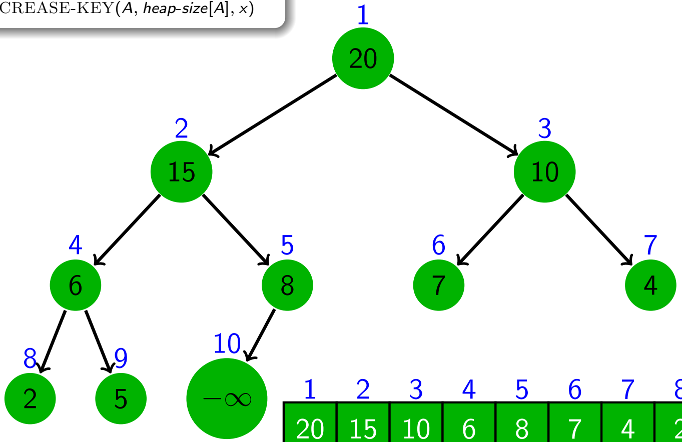
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	$-\infty$

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

MAX-HEAP-INSERT$(A, 16)$



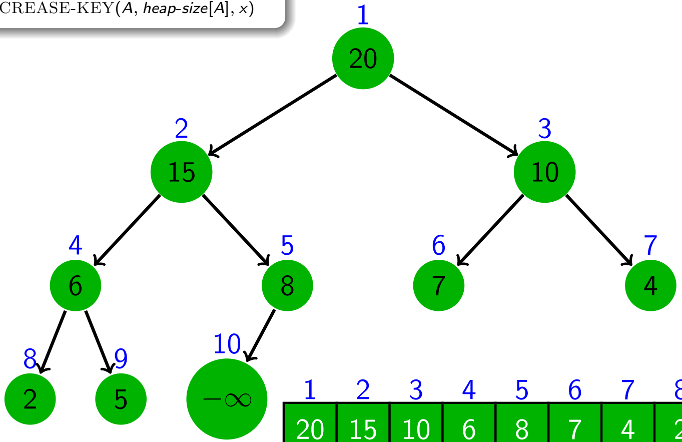
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	$-\infty$

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

HEAP-INCREASE-KEY$(A, 10, 16)$



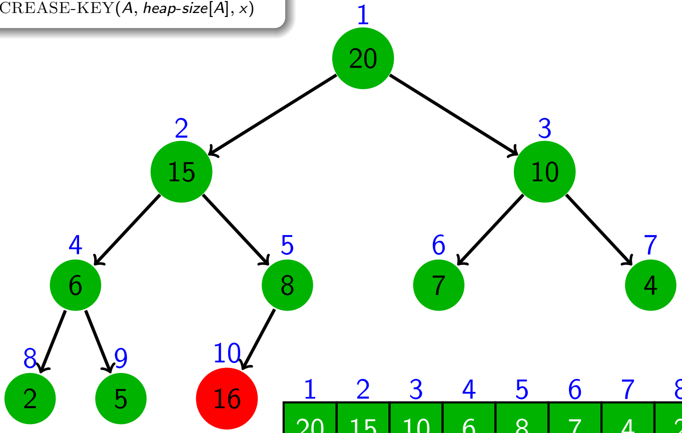
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	$-\infty$

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

HEAP-INCREASE-KEY(A, 10, 16)



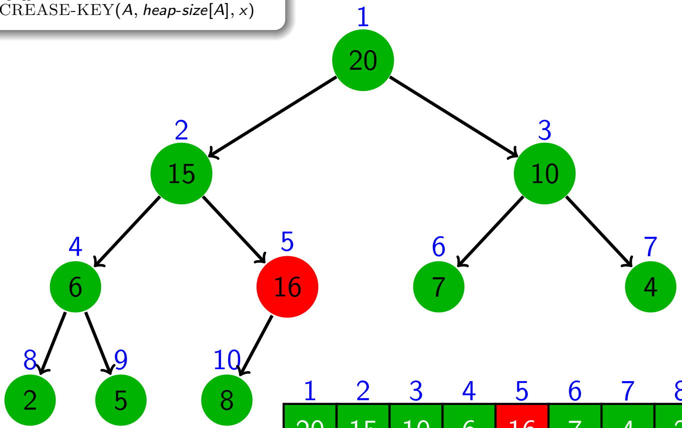
1	2	3	4	5	6	7	8	9	10
20	15	10	6	8	7	4	2	5	16

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

HEAP-INCREASE-KEY(A, 10, 16)



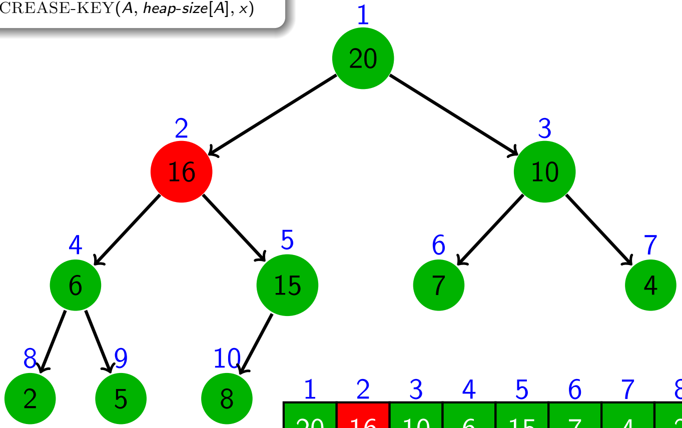
1	2	3	4	5	6	7	8	9	10
20	15	10	6	16	7	4	2	5	8

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY(A, $heap\text{-}size[A]$, x)

HEAP-INCREASE-KEY(A, 10, 16)

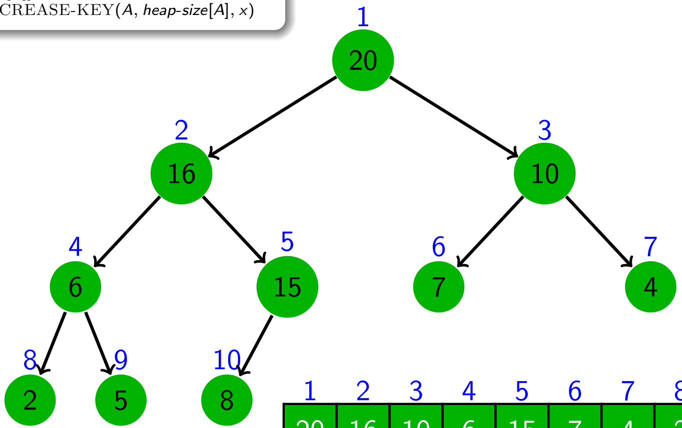


1	2	3	4	5	6	7	8	9	10
20	16	10	6	15	7	4	2	5	8

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```



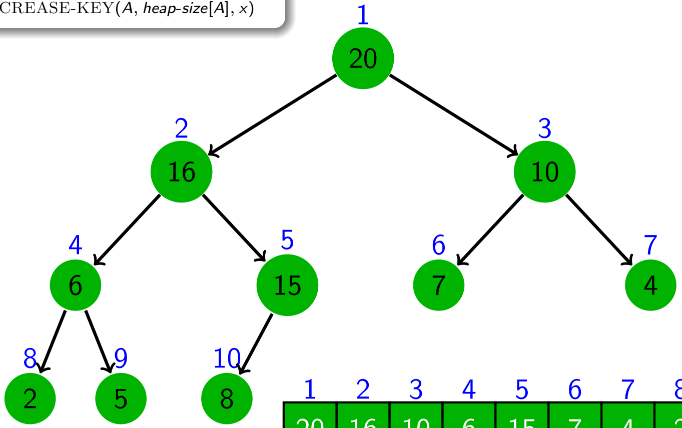
1	2	3	4	5	6	7	8	9	10
20	16	10	6	15	7	4	2	5	8

MAX-HEAP-INSERT

MAX-HEAP-INSERT(A, x)

```
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], x)
```

Time:
$\Theta(log_2 n)$



1	2	3	4	5	6	7	8	9	10
20	16	10	6	15	7	4	2	5	8