Preliminaries

Consider a uniform grid of $m+1$ points, with grid spacing $h = \frac{L}{m}$. Let $\mathbf{e}_\ell$ and $\mathbf{e}_r$ denote the following vectors in $\mathbb{R}^{m+1}$:

$$\mathbf{e}_\ell = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Definition of D_1

A difference operator D_1 is a first-derivative SBP operator with quadrature matrix H if $H = H^T > 0$ and

$$HD_1 = \mathbf{e}_r\mathbf{e}_r^T - \mathbf{e}_\ell\mathbf{e}_\ell^T - D_1^T H.$$

Definition of D_2

A difference operator D_2 approximating $\partial^2/\partial x^2$ is a second-derivative SBP operator if

$$HD_2 = \mathbf{e}_r\mathbf{d}_r^T - \mathbf{e}_\ell\mathbf{d}_\ell^T - M,$$

where $H = H^T > 0$, $M = M^T \geq 0$, and $\mathbf{d}_\ell^T v \simeq u_x|_{x=0}$, $\mathbf{d}_r^T v \simeq u_x|_{x=W}$ are finite difference approximations of the first derivatives at the left and right boundary points.

Discrete inner product

Let $(\cdot,\cdot)_H$ denote the discrete inner product, defined by

$$(\mathbf{u},\mathbf{v})_H = \mathbf{u}^* H\mathbf{v}.$$

Note that $(\cdot,\cdot)_H$ approximates the L^2 inner product $(\cdot,\cdot)$, since H is a quadrature operator. In the discrete inner product, the SBP operators satisfy

$$(\mathbf{u}, D_1\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{e}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{e}_\ell^T\mathbf{v}) - (D_1\mathbf{u},\mathbf{v})_H$$

and

$$(\mathbf{u}, D_2\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{d}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{d}_\ell^T\mathbf{v}) - \mathbf{u}^* M\mathbf{v}.$$

Note the similarities with the corresponding integration-by-parts formulas.

The projection operator

If L is the discrete boundary operator, such that the discretized boundary conditions can be formulated as $L\mathbf{v} = 0$, then the projection operator is defined as

$$P = I - H^{-1}L^T\left(LH^{-1}L^T\right)^{-1}L. \tag{1}$$

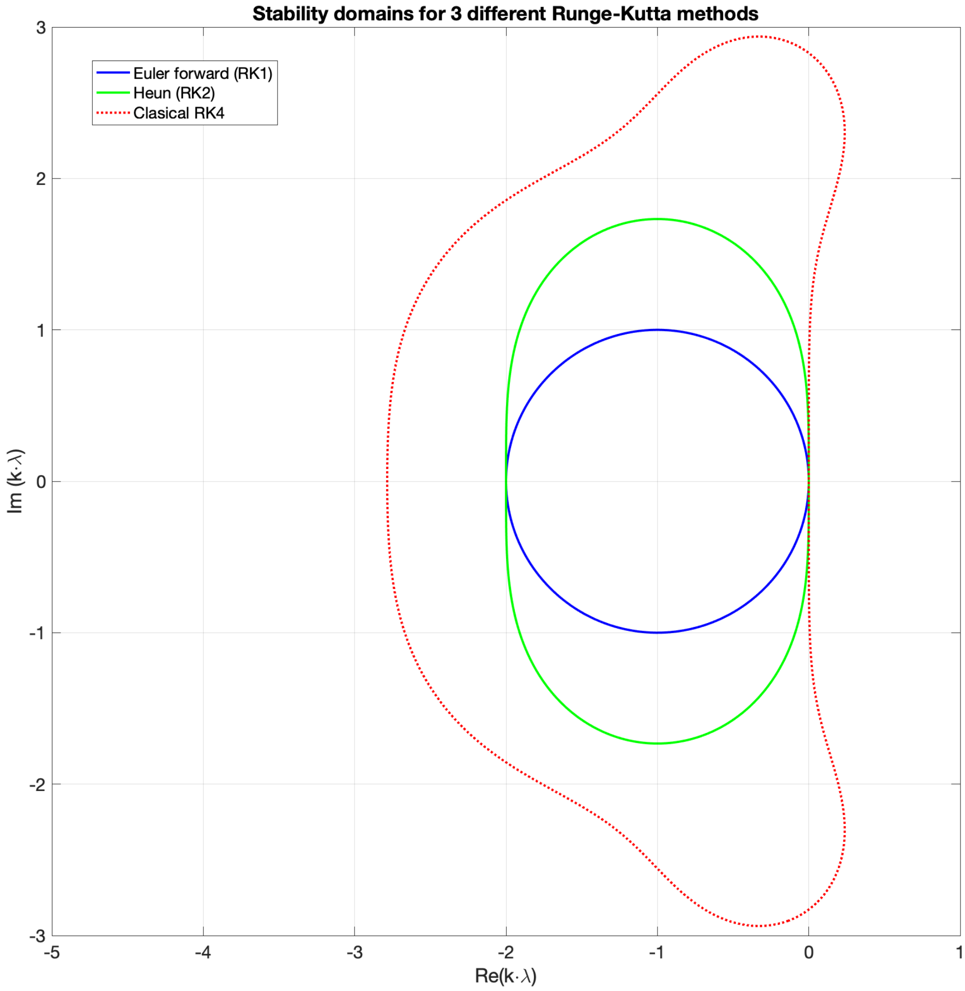

Figure 1: Stability regions for RK1 (Euler forward), RK2 (Heun's method), and RK4

Stability regions of Runge–Kutta methods

Figure 1 shows the stability regions for three different Runge–Kutta methods. Here k denotes the time step and λ the coefficient in the test equation:

$$y' = \lambda y.$$

Exercise 1

Consider the advection-diffusion IBVP,

$$
\begin{array}{lll}
u_t = au_x + bu_{xx}, & 0 < x < W, & t > 0, \\
au + 2bu_x = 0, & x = 0, & t > 0, \\
au + 2bu_x = 0, & x = W, & t > 0, \\
u = f, & 0 \leq x \leq W, & t = 0,
\end{array}
\tag{2}
$$

where a and $b > 0$ are real constants. You may assume that u is real too.

(a) Prove that the finite difference operator D_+D_- (obtained by stacking D_+ and D_-) is a second-order approximation of $\partial^2/\partial x^2$. The operator is given by

$$
(D_+D_-\mathbf{v})_j = \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2}.
$$

(b) Use the energy method to show that the IBVP (2) is well-posed.

(c) State a semi-discrete SBP-Projection approximation of the IBVP.

(d) Prove that your SBP-Projection approximation is stable.

(e) Derive a stable semi-discrete SBP-SAT approximation of the IBVP.

(f) Consider combining one of the spatial discretizations (it does not matter which one) with RK4 for time integration. Approximately how does the largest stable time step depend on the grid spacing h?
Hint: How do the largest eigenvalues of D_1 and D_2 depend on h?