Problem session 3

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1. Give examples or claim non-existence (with brief motivations) of:

a) A function $\mathbb{R}^2 \to \mathbb{R}$ which is differentiable but not continuously differentiable.

b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

2. Let the function $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \cdot 4^{-n} \leq x \leq 4^{-n} \text{ for some } n \in \{0,1,2,3,\ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable on $[0,1]$, and determine $\int_0^1 f(x)dx$.

3. Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Find a particular (potentially noninteger) value for a so that

a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0,1]$.

b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbf{R}$, but such that g_a'' is not continuous at zero.

4. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

5. Let $g : [0,a] \to \mathbf{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0,a]$. Show $|g(x)| \leq Mx$ for all $x \in [0,a]$

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

a) Compute $D_1 f(0,0)$ and $D_2 f(0,0)$.

b) Prove that f is not differentiable at $(0,0)$.

7. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

8. Compute the upper and lower integrals of the function $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 2 - x, & x \in [0, 1] \backslash \mathbb{Q} \\ x, & x \in \mathbb{Q} \cap [0, 1] \end{cases}$$

and conclude that it is not Riemann integrable.

Also, one must look at the following exercises 5.1-5.10 and 6.1-6.15 in Rudin's book.