Problem session 4

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1. Let $g_n : [0,1] \to \mathbb{R}, n = 0, 1, 2, \ldots$, be a sequence of differentiable functions such that the sequence $g_n' : [0,1] \to \mathbb{R}$ is uniformly bounded.

a) Show that there is a sequence of constants $c_n \in \mathbb{R}$ such that the sequence of functions $h_n(x) = g_n(x) - c_n$ on $[0,1]$ has a uniformly convergent subsequence.

b) Show that if $\int_0^1 g_n$ is a bounded sequence in $\mathbb{R}$ then the sequence g_n has a uniformly convergent subsequence.

2. Let $a \leq b$. Prove, or disprove, that the space

$$M = \{f \in C([a,b]) : |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for every } x, y \in [a,b], f(a) = 0\}$$

is a compact space under the metric $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$.

3.Prove that the series $F(x) = \sum_{n=1}^{\infty} n^{-x} \cos n\pi x$ converges for all $x \in (1, \infty)$, and that the function $F(x)$ is differentiable in the interval $(2, \infty)$.

4. Give examples to illustrate that:

a) the pointwise limit of integrable functions is not necessarily integrable.

b) a bounded subset of $C([-1,1] \times [-1,1])$ is not equicontinuous.

5. Determine if the set $\mathscr{R}$ of Riemann integrable functions on the interval $[a,b], a < b$, is a closed subset of the space of bounded real-valued functions $\ell^{\infty}([a,b], \mathbb{R})$

a) In the uniform topology? (where $f_n \to f$ iff $\sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0$).

b) In the pointwise topology? (where $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in [a,b]$).