Pricing & Arbitrage

Stock market: assets are modelled as random processes.

Derivatives: assets determined by other assets.

- Futures / Forward contracts.

  Buy / Sell an asset at time T at a fixed price K.

  If the value at time T is S_T, the win/loss ("payoff") is

  $$S_T - K \text{ (buyer)} \;/\; K - S_T \text{ (seller)}$$

- Swaps:

  Exchange of future cash flow, e.g. currency swaps

- **Options:**

  Right, but not obligation, to buy / sell an asset at a future time T for price K.

- call option (buying); payoff $(S_T - K)^+$
- put option (selling); payoff $(K - S_T)^+$
- European option: right to buy/sell can only be exercised at time T.
- American option: right can be exercised at any time up to T.

Options are "safe" (payoff non-negative), so we must have a cost.

Pricing: We want to know the fair price for an option.

We need some concepts to make this precise, and well-defined. The key concept is arbitrage (risk-free gains)

Example: Suppose Sweden plays Canada in the World Curling Championships Final.

One betting site offers:

A	Sweden wins:	$1{,}5 \cdot \text{bet}$
	Canada wins:	$2.0 \cdot \text{bet}$

Another offers:

B	Sweden wins:	$3{,}0 \cdot \text{bet}$
	Canada wins:	$1{,}2 \cdot \text{bet}$

We can do the following:

- Bet 3 units on Canada with A.
- Bet 2 units on Sweden with B.

We will always get 6 units, but only paid 5 units.

This situation, where we can make risk-free profit is called arbitrage. We will make the key assumption that markets are <u>arbitrage-free</u>.

More precisely, one assumes that there is a certain risk-free rate r at which money can be invested.

1 unit becomes $\begin{cases} (1+r)^T & \text{at time } T \\ e^{rT} & \text{at time } T \text{ with compound interest.} \end{cases}$

$r=0 \iff$ "no risk-free interest".

Absence of arbitrage means that we cannot do better than r without risks.

Another tool we will use are **hedging portfolios**.

$\to$ comparing assets by a "replicating strategy".

<u>Example</u>: **Call-Put parity**

It relates the prices of (European) call and put options on the same asset with the same parameters K, T.

Note that the difference in payoffs is

$$(S_T-K)^+ - (K-S_T)^+ = \begin{cases} S_T - K - 0 & \text{if } S_T \geq K \\ 0-(K-S_T) & \text{otherwise} \end{cases} = S_T - K.$$

Compare the following strategies:

1. Buy a call option, sell a put option.
   → payoff at time T is $S_T - K$.

2. Buy the asset at its current price S_0 and borrow $e^{-rT}K$ at risk-free interest.
   → at time T portfolio is worth $S_T - K$
   loan + interest.

Since both strategies are equal, their cost at time 0 must coincide. Otherwise, there is possibility for arbitrage. Hence,

$$\boxed{C_0 - P_0 = S_0 - e^{-rT}K} \quad \text{(call-put parity)}$$

put price at time 0

call price at time 0

We do not know C_0 / P_0 (yet), but one determines the other.

<u>Example</u>: Consider a model where there is only one time period $T=1$ and only two outcomes

$$S_1 = \begin{cases} 13 \\ 8 \end{cases} \quad \text{and} \quad S_0 = 10, \ K = 11, \ r = 0.$$

What is the fair price of a call option?

If we knew probabilities p, $1-p$ of outcomes, we'd have

$$\mathbb{E}\left((S_1 - K)^+\right) = p\,(13-10)^+ + (1-p)(8-10)^+$$
$$= 2p$$

But how to choose p?

We try to replicate the option with a portfolio of:

- η units cash
- θ units asset

At time $T=1$ this is worth

$$\begin{cases} \eta + 13\theta \\ \eta + 8\theta \end{cases}$$ which we want to set equal to the payoff of the option

$$\begin{cases} \eta + 13\theta = 2 \\ \eta + 8\theta = 0 \end{cases} \Rightarrow \quad \eta = -3.2, \ \theta = 0.4$$

So the fair price of the option has to be the value of the portfolio at time 0 :

$$\eta + 10\,\theta = 0.8.$$

Note that this price corresponds to the expected payoff with probability 0.4 that the price goes up. For such p, we get

$$S_1 = \begin{cases} 13 & \text{with prob } p = 0.4 \\ 8 & \text{with prob } 1-p \end{cases}, \quad S_0 = 10$$

and $\mathbb{E}(S_1 \mid S_0) = p \cdot 13 + (1-p)\,8 = 10 = S_0$

a <u>martingale</u> !!

This is not a coincidence. We will see that it holds in much greater generality.

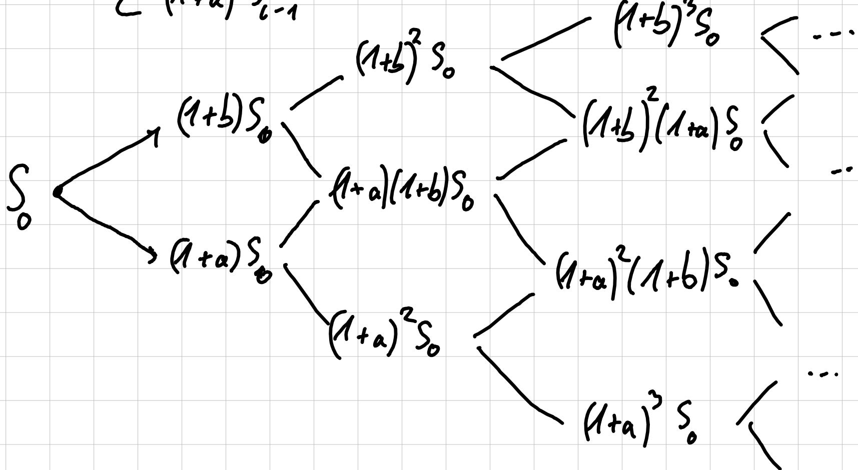
Fair option price = expected payoff assuming that the asset price is a martingale.

Finding a replicating strategy was possible here because there were only to possible outcomes. This might not be true in general. Models where every contingent claim (options) can be obtained by a hedging strategy are called **complete**.

Binomial Model

At time periods T, the asset price can change by a factor of $(1+a)$ or $(1+b)$; $a<b$.

$$S_i = \begin{cases} (1+b)\, S_{i-1} \\ (1+a)\, S_{i-1} \end{cases} \quad \text{for all time steps } i.$$



The risk-free rate satisfies $a<r<b$.

Consider a single time step

$$\begin{tikzcd}[row sep=small] & (1+b)S_b & \text{payoff } H_b \\ S_0 \arrow[ur, dash] \arrow[dr, dash] & & \\ & (1+a)S_a & \text{payoff } H_a \end{tikzcd}$$

A replicating strategy consists of

- η cash units $\longrightarrow$ $(1+r)\eta$ after 1 time step.
- θ units asset $\quad (1+b)\theta$ or $(1+a)\theta$

We want

$$H_a = \eta(1+r) + \theta(1+a)S_0 \iff \beta H_a = \eta + \beta\theta(1+a)S_0$$

$$H_b = \eta(1+r) + \theta(1+b)S_0 \iff \beta H_b = \eta + \beta\theta(1+b)S_0$$

, when $\beta = \frac{1}{1+r}$ is the discounting factor

Solving the linear equations gives

$$\theta = \frac{H_b - H_a}{S_b - S_a} = \frac{H_b - H_a}{(b-a)S_0}, \quad \eta = \beta\,\frac{(1+b)H_a - (1+a)H_b}{b-a}$$

The portfolio value at time 0 can be computed to be:

$$\eta + \theta S_0 = \beta \frac{(1+b)H_a - (1+a)H_b}{b-a} + \frac{H_b - H_a}{(b-a)S_0} S_0$$

$$= \beta \frac{1+b}{b-a} H_a - \beta \frac{1+a}{b-a} H_b + \frac{1}{b-a} H_b - \frac{1}{b-a} H_a$$

$$= \beta \left(H_a \left(\frac{1+b}{b-a} - \frac{1+r}{b-a} \right) + H_b \left(\frac{1+r}{b-a} - \frac{1+a}{b-a} \right) \right)$$

$$= \beta \left(H_a \frac{b-r}{b-a} + H_b \frac{r-a}{b-a} \right)$$

can be interpreted as probabilities

$$\underbrace{\frac{b-r}{b-a}}_{q} + \underbrace{\frac{r-a}{b-a}}_{1-q} = \frac{b-a}{b-a} = 1.$$

The probabilities turn S (when discounted) into a martingale:

$$\mathbb{E}(S_1 | S_0) = q(1+a)S_0 + (1-q)(1+b)S_0$$

$$= \frac{(b-r)(1+a)S_0}{b-a} + \frac{(r-a)(1+b)S_0}{b-a}$$

$$= \frac{b + \cancel{ab} - \cancel{r} - ar + \cancel{r} + rb - a - \cancel{ab}}{b-a} S_0$$

$$= \frac{b-a+rb-ra}{b-a} S_0 = (1+r) S_0$$

and $\mathbb{E}(\beta S_1 | S_0) = \beta(1+r) S_0 = S_0$.