Incomplete Markets (Chapter 9)

Assumption: Two objects are given

- A risk-free asset $dB_t = rB_t dt$
- A stochastic process X, which is not assumed to be the price of a traded asset, with

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)\,dW_t .$$

Consider a T-claim $Y = \Phi(X_T)$. What is the price $\Pi_t(Y)$ at $t < T$?

Ex: X_t is the temperature in Brighton at time t.

$$\Phi(x) = \begin{cases} 100 & \text{if } x \leq 20 \\ 0 & \text{if } x > 20 \end{cases}$$

The holder of the T-claim receives 100 if the temperature at T is below 20, 0 otherwise.

Our expectations: In the Meta-Theorem, $R=1$, $M=0$ so the market is incomplete. The price of Y is not uniquely determined. If the price of a benchmark derivative is given, however, then all other derivatives will have unique prices. Certain consistency relations between prices should hold!

Assum $\mathcal{Y}$ and $\mathcal{Z}$ have price processes

$$\Pi_t(\mathcal{Y}) = F(t, X_t) \quad \text{and} \quad \Pi_t(\mathcal{Z}) = G(t, X_t).$$

$$d\Pi_t(\mathcal{Y}) = \mu_F F\,dt + \sigma_F F\,dW_t \quad \text{with} \quad \begin{cases} \mu_F = \dfrac{F_t + \frac{\sigma^2}{2} F_{xx} + \mu F_x}{F} \\ \sigma_F = \dfrac{\sigma F_x}{F} \end{cases}$$

and

$$d\Pi_t(\mathcal{Z}) = \alpha_G G\,dt + \sigma_G G\,dW_t.$$

Let $w = (w^F, w^G)$ be a self-financing relative portfolio in F and G.

$$dV_t^w = V_t^w w^F \frac{dF}{F} + V_t^w w^G \frac{dG}{G}$$

$$= (\mu_F w^F + \mu_G w^G) V_t^w\,dt + (\sigma_F w^F + \sigma_G w^G) V_t^w\,dW_t$$

Choose w^F, w^G so that

$$\begin{cases} w^F + w^G = 1 \\ \sigma_F w^F + \sigma_G w^G = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} w^F = \dfrac{-\sigma_G}{\sigma_F - \sigma_G} \\ w^G = \dfrac{\sigma_F}{\sigma_F - \sigma_G} \end{cases}$$

Then $dV_t^w = \dfrac{\sigma_F \mu_G - \sigma_G \mu_F}{\sigma_F - \sigma_G} V_t^w\,dt$

so by no-arbitrage we must have $\dfrac{\sigma_F \mu_G - \sigma_G \mu_F}{\sigma_F - \sigma_G} = r$.

Thus $\sigma_F \mu_G - \sigma_G \mu_F = r\sigma_F - r\sigma_G$ so

$$\frac{\mu_F - r}{\sigma_F} = \frac{\mu_G - r}{\sigma_G}$$

does not involve G

does not involve F

Proposition 9.1 Assume the market for derivatives is arbitrage-free. Then there exists a process λ such that $\lambda(t, X_t) = \frac{\mu_F(t, X_t) - r}{\sigma_F(t, X_t)}$ for any pricing function F.

Terminology: λ_t is called the market price of risk

We have $\lambda = \frac{\mu_F - r}{\sigma_F} = \frac{F_t + \frac{\sigma^2}{2} F_{xx} + \mu F_x - rF}{\sigma F_x}$

Propositions 9.2 + 9.3 The price of a T-claim $\Phi(X_T)$ is $F(t, X_t)$, where $F(t,x)$ solves

$$\begin{cases} F_t + \frac{\sigma^2}{2} F_{xx} + (\mu - \sigma \lambda) F_x - rF = 0 \\ F(T,x) = \Phi(x) \end{cases}$$

Moreover,

$$F(t,x) = E^Q_{t,x}\left[e^{-r(T-t)} \Phi(X_T)\right]$$

where $\begin{cases} dX_s = \left(\mu(s, X_s) - \lambda(s, X_s)\sigma(s, X_s)\right) ds + \sigma(s, X_s)\, dW_s^Q \\ X_t = x \end{cases}$

under Q.

Remark: $\lambda(t,x)$ is not specified within the model. If we take the price of one derivative as given with price process $\Pi_t = G(t, X_t)$, then $\lambda(t,x) = \frac{\mu_G(t,x) - r}{\sigma_G(t,x)}$ can be calculated. This λ can then be used to price other derivatives.

<u>Special case</u>: Assume that X <u>is</u> in fact a traded asset. The claim $\mathcal{Z} = X_T$ then has price $G(t, X_t) = X_t$ (why?), so

$$\lambda(t,x) = \frac{\mu_G - r}{\sigma_G} = \frac{G_t + \frac{\sigma^2}{2} G_{xx} + \mu G_x - rG}{\sigma G_x} \overset{G(t,x)=x}{=} \frac{\mu - rx}{\sigma}.$$

The factor $\mu - \lambda\sigma$ is then $\mu - \lambda\sigma = rx$.

Thus the usual BS-equation is recovered!