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Matematiska institutionen
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Tentamen i matematik
Reell Analys MA226
15-03-21

Tillåtna hjälpmedel: en ark papper (A4, båda sidor) med egna handskrivna anteckningar. Varje problem är värt 5 poäng. Tentamen får skrivas på svenska, eller engelska.

Allowed aids: one sheet of paper (A4, both sides) with own handwritten notes. Each problem is worth 5 points. The exam may be written in Swedish or English.

1. A metric space (X, d), where X is a set, is called discrete if $d(x, y) = 1$ whenever $x \neq y$. Prove that every connected discrete metric space is compact and determine its cardinality (e.g. number of elements if it is finite).

2. Determine for what values of parameter a the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\log n)^a}$$

is (a) absolutely convergent, (b) convergent, (c) divergent.

3. Let $x_1 = 0$ and let $x_{n+1} = \frac{1}{2-x_n}$, $n \in \mathbb{N}$. Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

4. Let f_n be a sequence of continuous positive functions on $[0, 1]$ such that $\max_{x\in[0,1]} f_n(x)=1$. Prove that if $\int_0^1 f_n(x)dx \to 0$, then there exists a sequence $x_n \in [0, 1]$ such that $f_n(x_n)$ does not converge.

5. Show that the equation

$$f(x) = \frac{1}{2}\int_x^1 (x - y)f(y)dy + x^2 e^{x^2}$$

has a unique solution $f \in C([0, 1])$.

6. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n(\log n)^2}$$

is convergent and defines a bounded continuous function f on $(-\infty, \infty)$. Find the $C((-\infty, \infty))$-norm of this function (the value may be defined implicitely, instead of a concrete number).

7. Prove that the following system of equations has in some neighborhood of $(x, y) = (0, 2)$ a unique solution $(u(x, y), v(x, y))$ satisfying $u(0, 2) = 1$, $v(0, 2) = -1$:

$$\begin{cases} u^{17} + v^{17} & = x \\ u^{18} + v^{18} & = y \end{cases}$$

8. Let $f(x, y, z) = \frac{xy+yz+xz}{x^2+y^2+z^2}$ whenever $(x, y, z) \neq (0, 0, 0)$. Find the set of limit points (=accumulation points) for f at $(0, 0, 0)$ and determine $\limsup_{(x,y,z)\to(0,0,0)} f(x, y, z)$ and $\liminf_{(x,y,z)\to(0,0,0)} f($

Comments:

Problem 2: Typo; replace "$\sum_{n=1}^{\infty}$" by (e.g.) "$\sum_{n=2}^{\infty}$". (This is needed since $\log 1 = 0$.)

Problem 6: Typo, as in problem 2; replace "$\sum_{n=1}^{\infty}$" by (e.g.) "$\sum_{n=2}^{\infty}$".

Problem 7: We have to require that u and v should be *continuous* functions of (x, y) in the neighborhood of $(0, 2)$, otherwise the stated uniqueness fails. ((Exercise: Prove that the uniqueness indeed fails in the original formulation of the problem.))

Problem 8: I have not seen that the concept of "limit points (=accumulation points) of f at $(0, 0, 0)$" is defined explicitly in Rudin's book. Anyway, one way to define it is in analogy with Rudin's Definition 3.16, namely: The set of *limit points* (or *accumulation points*) of f *at* $\mathbf{0} = (0, 0, 0)$ is the set of numbers a in the extended real number system such that $f(\boldsymbol{x}_k) \to a$ for some sequence $(\boldsymbol{x}_k)$ in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ with $\boldsymbol{x}_k \to \mathbf{0}$.

Similarly I have not seen the notations $\limsup_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x})$ and $\liminf_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x})$ defined in Rudin's book; however they can again be defined in analogy with his Definition 3.16. Namely: Let E be the set of limit points of f at $\mathbf{0}$. Then $\limsup_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) := \sup E$ and $\liminf_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x}) := \inf E$.

— A.S.