Seminars on Continuous Time Finance

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Contents

3 Stochastic Integrals

Exercise 3.1

(a) Since $Z(t)$ is determinist, we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha t} dt \\ &= \alpha Z(t) dt. \end{aligned}$$

(b) By definition of a stochastic differential

$$dZ(t) = g(t) dW(t)$$

(c) Using Itô's formula

$$\begin{aligned} dZ(t) &= \frac{\alpha^2}{2} e^{\alpha W(t)} dt + \alpha e^{\alpha W(t)} dW(t) \\ &= \frac{\alpha^2}{2} Z(t) dt + \alpha Z(t) dW \end{aligned}$$

(d) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha x} dX(t) + \frac{\alpha^2}{2} e^{\alpha x} (dX(t))^2 \\ &= Z(t) \left[\alpha\mu + \frac{1}{2}\alpha^2\sigma^2 \right] dt + \alpha\sigma Z(t) dW(t). \end{aligned}$$

(e) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= 2X(t)dX(t) + (d(X(t))^2 \\ &= Z(t) \left[2\alpha + \sigma^2 \right] dt + 2Z\sigma dW(t). \end{aligned}$$

Exercise 3.3 By definition we have that the dynamics of $X(t)$ are given by $dX(t) = \sigma(t) dW(t)$.

Consider $Z(t) = e^{iuX(t)}$. Then using the Itô's formula we have that the dynamic of $Z(t)$ can be described by

$$dZ(t) = \left[-\frac{u^2}{2}\sigma^2(t) \right] Z(t) dt + [iu\sigma(t)] Z(t) dW(t)$$

From $Z(0) = 1$ we get,

$$Z(t) = 1 - \frac{u^2}{2} \int_0^t \sigma^2(s) Z(s) ds + iu \int_0^t \sigma(s) Z(s) dW(s).$$

Taking expectations we have,

$$\begin{aligned} E\left[Z(t)\right] &= 1-\frac{u^2}{2}E\left[\int_0^t \sigma^2(s)Z(s)ds\right]+iuE\left[\int_0^t \sigma(s)Z(s)dW(s)\right] \\ &= 1-\frac{u^2}{2}\left[\int_0^t \sigma^2(s)E\left[Z(s)\right]ds\right]+0 \end{aligned}$$

By setting $E\left[Z(t)\right] = m(t)$ and differentiating with respect to t we find an ordinary differential equation,

$$\frac{\partial m(t)}{\partial t} = -\frac{u^2}{2}m(t)\sigma^2(t)$$

with the initial condition $m(0) = 1$ and whose solution is

$$\begin{aligned} m(t) &= \exp\left\{-\frac{u^2}{2}\int_0^t \sigma 2(s)ds\right\} \\ &= E\left[Z(t)\right] \\ &= E\left[e^{iuX(t)}\right] \end{aligned}$$

So, $X(t)$ is normally distributed. By the properties of the normal distribution the following relation

$$E\left[e^{iuX(t)}\right] = e^{iuE[X(t)]-\frac{u^2}{2}V[X(t)]}$$

where $V\left[X(t)\right]$ is the variance of $X(t)$, so it must be that $E\left[X(t)\right] = 0$ and $V\left[X(t)\right] = \int_0^t \sigma^2(s)ds$.

Exercise 3.5 We have a sub martingale if $E\left[X(t)|\mathcal{F}_s\right] \geq X(s)\forall, t \geq s$. From the dynamics of X we can write

$$X(t) = X(s) + \int_s^t \mu(z)dz + \int_s^t \sigma(z)dW(z).$$

By taking expectation, conditioned at time s, from both sides we get

$$\begin{aligned} E\left[X(t)|\mathcal{F}_s\right] &= E\left[X(s)|\mathcal{F}_s\right] + E\left[\left.\int_s^t \mu(z)dz\right|\mathcal{F}_s\right] \\ &= X(s) + E^s\left[\left.\underbrace{\int_s^t \mu(z)dz}_{\geq 0}\right|\mathcal{F}_s\right] \\ &\geq X(s) \end{aligned}$$

so X is a sub martingale.

Exercise 3.6 Set $X(t) = h(W_1(t), \cdots, W_n(t))$.

We have by Itô that

$$dX(t) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)$$

where $\frac{\partial h}{\partial x_i}$ denotes the first derivative with respect to the i-th variable, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ denotes the second order cross-derivative between the i-th and j-th variable and all derivatives should be evaluated at $(W_1(s), \cdots, W_n(s))$.

Since we are dealing with independent Wiener processes we know

$$\forall u: \quad dW_i(u)dW_j(u) = 0 \text{ for } i \neq j \quad \text{and} \quad dW_i(u)dW_j(u) = du \text{ for } i = j,$$

so, integrating we get

$$\begin{aligned}
X(t) &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(u) dW_j(u) \\
&= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \left[dW_i(u)\right]^2 \\
&= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du.
\end{aligned}$$

Taking expectations

$$\begin{aligned}
E\left[X(t) \middle| \mathcal{F}_s \right] &= E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right] + E\left[\frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\
&= \underbrace{\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^s \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du}_{X(s)} \\
&\quad + \underbrace{E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right]}_{0} + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\
&= X(s) + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right].
\end{aligned}$$

- If h is *harmonic* the last term is zero, since $\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} = 0$, we have

$$E\left[X(t) \middle| \mathcal{F}_s \right] = X(s) \quad \text{so } X \text{ is a martingale.}$$

- If h is *subharmonic* the last term is always nonnegative, since $\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \geq 0$ we have

$$E\left[X(t)\middle|\mathcal{F}_s\right] \geq X(s) \quad \text{so } X \text{ is a submartingale.}$$

Exercise 3.8

(a) Using the Itô's formula we find the dynamics of $R(t)$,

$$\begin{aligned} dR(t) &= 2X(t)(dX(t)) + 2Y(t)(dY(t)) + \frac{1}{2}\left[2(dX(t))^2 + 2(dY(t))^2\right] \\ &= (2\alpha+1)\left[X^2(t) + Y^2(t)\right]dt \\ &= (2\alpha+1)R(t)dt \end{aligned}$$

From the dynamics we can see immediately that $R(t)$ is deterministic (it has no stochastic component!).

(b) Integrating the SDE for $X(t)$ and taking expectations we have

$$X(t) = x_0 + \alpha \int_0^t E\left[X(s)\right] ds$$

Which once more can be solve setting $m(t) = E\left[X(t)\right]$,taking the derivative with respect to t and using ODE methods, to get the answer

$$E\left[X(t)\right] = x_0 e^{\alpha t}$$

4 Differential Equations

Exercise 4.1 We have:

$$dY(t) = \alpha e^{\alpha t} x_0 dt, \quad dZ(t) = \alpha e^{\alpha t}\sigma dt, \quad dR(t) = e^{-\alpha t} dW(t).$$

Itô's formula then gives us (the cross term $dZ(t)\cdot dR(t)$ vanishes)

$$\begin{aligned} dX(t) &= dY(t) + Z(t)\cdot dR(t) + R(t)\cdot dZ(t) \\ &= \alpha e^{\alpha t} x_0 dt + e^{\alpha t}\cdot\sigma\cdot e^{-\alpha t} dW(t) + \int_0^t e^{-\alpha s} dW(s)\cdot \alpha e^{\alpha t}\sigma dt \\ &= \alpha\left[e^{\alpha t} x_0 + \sigma\int_0^t e^{\alpha(t-s)} dW(s)\right] dt + \sigma dW(t) \\ &= \alpha X(t) dt + \sigma dW(t). \end{aligned}$$

Exercise 4.5 Using the dynamics of $X(t)$ and the Itô formula we get

$$\begin{aligned} dY(t) &= \left[\alpha\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right] Y(t)dt + \sigma\beta Y(t)dW(t) \\ &= \mu Y(t)dt + \delta Y(t)dW(t) \end{aligned}$$

where $\mu = \alpha\beta + \frac{1}{2}\beta(\beta-1)\sigma^2$ and $\delta = \sigma\beta$ so Y is also a GBM.

Exercise 4.6 From the Itô formula and using the dynamics of X and Y

$$\begin{aligned} dZ(t) &= \frac{1}{Y(t)}dX(t) - \frac{X(t)}{Y(t)^2}dY(t) - \frac{1}{Y(t)^2}dX(t)dY(t) + \frac{X(t)}{Y(t)^3}(dY(t))^2 \\ &= Z(t)\left[\alpha - \gamma + \delta^2\right] dt + \sigma Z(t)dW(t) - \delta Z(t)dV(t). \end{aligned}$$

Exercise 4.9 *From Feyman-Kac we have* We have

$$F(t,x) = E^{t,x}\left[2\ln[X(T)]\right],$$

and

$$\begin{aligned} dX(s) &= \mu X(s)ds + \sigma X dW(s), \\ X(t) &= x. \end{aligned}$$

Solving the SDE, we obtain (check the solution of the GBM in th extra exercises if you do not remmeber)

$$X(T) = \exp\left\{\ln x + (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma[W(T) - W(t)]\right\},$$

and thus

$$F(t,x) = 2\ln(x) + 2(\mu - \frac{1}{2}\sigma^2)(T-t).$$

Exercise 4.10 Given the dynamics of $X(t)$ any $F(t,x)$ that solves the problem has the dynamics given by

$$\begin{aligned} dF(t,x) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t,x)dt + \sigma(t,x)dW(t)\right] + k(t,x)dt - k(t,x)dt \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left[\sigma^2(t,x)dW(t)\right] \\
= & \left\{\underbrace{\frac{\partial F}{\partial t}+\mu(t,x)\frac{\partial F}{\partial x}+\frac{1}{2}\sigma^2(t,x)+k(t,x)}_{0}\right\}dt-k(t,x)dt \\
& +\frac{\partial F}{\partial x}\sigma(t,x)dW(t) \\
= & -k(t,x)dt+\frac{\partial F}{\partial x}\sigma(t,x)dW(t)
\end{aligned}
$$

We now write $F(T,X(T))$ in terms of $F(t,x)$ and the dynamics of F during the time period $t\dots T$(recall that we defined $X(t)=x$)

$$
\begin{aligned}
F(t,X(T)) &= F(t,x)-\int_t^T k(s,X(s)ds+\int_t^T \frac{\partial F}{\partial x}\sigma(s,X(s))dW(s) \\
&\Leftrightarrow \\
F(t,x) &= F(T,X(T))+\int_t^T k(s,X(s)ds-\int_t^T \frac{\partial F}{\partial x}\sigma(s,X(s))dW(s)
\end{aligned}
$$

Taking expectations $E_{t,x}[.]$ from both sides

$$
\begin{aligned}
F(t,x) &= E_{t,x}\left[F(T,X(T))\right]+E_{t,x}\left[\int_t^T k(s,X(s)ds\right] \\
&= E_{t,x}\left[\Phi(T)\right]+\int_t^T E_{t,x}\left[k(s,X(s)\right]ds
\end{aligned}
$$

Exercise 4.11 Using the representation formula from Exercise 4.10 we get

$$
F(t,x)=E_{t,x}\left[2\ln[X^2(T)]\right]+\int_t^T E_{t,x}\left[X(s)\right]ds,
$$

Given

$$
dX(s)=X(s)dW(s).
$$

The first term is easily computed as in the exercise 4.9 above. Furthermore it is easily seen directly from the SDE (how?)that $E_{t,x}[X(s)]=x$. Thus we have the result

$$
\begin{aligned}
F(t,x) &= 2\ln(x)-(T-t)+x(T-t) \\
&= \ln(x^2)+(x-1)(T-t)
\end{aligned}
$$

6 Arbitrage Pricing

Exercise 6.1

(a) From standard theory we have

$\Pi(t) = F(t, S(t))$, where F solves the Black-Scholes equation.

Using Itô we obtain

$$d\Pi(t) = \left[\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}\right] dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t).$$

Using the fact that F satisfies the Black-Scholes equation, and that $F(t, S(t)) = \Pi(t)$ we obtain

$$d\Pi(t) = r\Pi(t)\, dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t)$$

and so $g(t) = \sigma S(t)\frac{\partial F}{\partial s}$.

(b) Apply Itô's formula to the process $Z(t) = \frac{\Pi(t)}{B(t)}$ and use the result in (a).

$$\begin{aligned} dZ(t) &= \frac{1}{B(t)}(d\Pi(t)) - \frac{\Pi(t)}{B^2(t)}(d(B(t))) \\ &= \frac{g(t)}{B(t)} dW(t) \\ &= Z(t)\frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s} dW(t) \end{aligned}$$

Z is a martingale since $E_t[Z(T)] = Z(t)$ for all $t < T$ and its diffusion coefficient is given by $\sigma_Z(t) = \frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s}$.

Exercise 6.4 We have as usual

$$\Pi(t) = e^{-r(T-t)} E^Q_{t,s}\left[S^\beta(T)\right].$$

We know from earlier exercises (check exercises 3.4 and 4.5) that $Y(t) = S^\beta(t)$ satisfies the SDE under Q

$$dY(t) = \left[r\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right] Y(t)dt + \sigma\beta Y(t)dW(t).$$

Using the standard technique, we can integrate, take expectations, differentiate with respect to time and solve by ODE techniques, to obtain

$$E^Q_{t,s}\left[S^\beta(T)\right] = s^\beta e^{\left[r\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right](T-t)},$$

So,

$$\Pi(t) = s^{\beta} e^{\left[r(\beta-1)+\frac{1}{2}\beta(\beta-1)\sigma^2\right](T-t)}.$$

Exercise 6.6 We consider only the case when $t < T_0$. The other case is handled in very much the same way. We have to compute $E_{t,s}^{Q}\left[\frac{S(T_1)}{S(T_0)}\right]$. Define the process X on the time interval $[T_0, T_1]$ by

$$X(u) = \frac{S(u)}{S(T_0)}.$$

We now want to compute $E_{t,s}^{Q}\left[X(T_1)\right]$. The stochastic differential (under Q) of X is easily seen to be

$$\begin{aligned} dX(u) &= rXdu + \sigma X dW(u), \\ X(T_0) &= 1. \end{aligned}$$

From this SDE it follows at once (the same technique of integrating, taking expectations, differentiate with respect to time and solve by ODE techniques) that

$$E_{t,s}^{Q}\left[X(T_1)\right] = e^{r(T_1-T_0)},$$

and thus the price, at t of the contract is given by

$$\Pi\left(t\right) = e^{-r(T_0-t)}.$$

Exercise 6.7 The price in SEK of the ACME INC., Z, is defined as $Z(t) = S(t)Y(t)$ and by Itô has the following dynamics under Q

$$dZ(t) = rZ(t)dt + \sigma Z(t)dW_1(t) + \delta Z(t)dW_2(t)$$

We also have, by using Itô once more, that the dynamics of $\ln Z^2$ are

$$d\ln Z^2(t) = \left[2r - \sigma^2 - \delta^2\right]dt + 2\sigma dW_1(t) + 2\delta dW_2(t)$$

which integrating and taking conditioned expectations give us

$$E_{t,z}^{Q}\left[\ln[Z^2(T)]\right] = \ln z^2 + \left[2r - \sigma^2 - \delta^2\right](T-t)$$

Since we know that

$$\Pi(t) = F(t,s) = e^{-r(T-t)}E_{t,z}^{Q}\left[\ln[Z^2(T)]\right],$$

the arbitrage free pricing function Π is

$$\begin{aligned}\Pi(t) &= e^{-r(T-t)}\left\{\ln z^2 + \left[2r-\sigma^2-\delta^2\right](T-t)\right\}\\ &= e^{-r(T-t)}\left\{2\ln(sy) + \left[2r-\sigma^2-\delta^2\right](T-t)\right\},\end{aligned}$$

where, as usual, $z = Z(t)$, $s = S(t)$ and $y = Y(t)$.

Exercise 6.9 The *forward price*, i.e. the amount of money to be payed out at time T, but decided at the time t is

$$F(t,T) = E_t^Q\left[\mathcal{X}\right].$$

Note that the forward price *is not* the *price of the forward contract* on the T-claim $\mathcal{X}$ which is what we are looking for.

Take for instance the long position: at time T, the buyer of a forward contract receives $\mathcal{X}$ and pays $F(t,T)$. Hence, the price at time t of that position is

$$\Pi(t;\mathcal{X}-F(t,T)) = E_t^Q\left[e^{-r(T-t)}\left(\mathcal{X}-\underbrace{F(t,T)}_{E_t^Q[\mathcal{X}]}\right)\right] = 0.$$

At time $s > t$, however, the underlying asset may have changed in value, in a way different from expectations, so then the price of a forward contract can be defined as

$$\begin{aligned}\Pi(s;\mathcal{X}-F(t,T)) &= E_s^Q\left[e^{-r(T-s)}\left(\mathcal{X}-F(t,T)\right)\right]\\ &= e^{-r(T-s)}\left[E_s^Q\left[\mathcal{X}\right] - \overbrace{E_t^Q\left[\mathcal{X}\right]}^{F(t,T)}\right].\end{aligned}$$

Remark: For the special case where the contract is on one share S we get:

$$\Pi(s) = e^{-r(T-s)}\left[E_s^Q\left[S(T)\right] - \underbrace{S(t)e^{r(T-t)}}_{E_t^Q[S(T)]}\right].$$

We can also easily calculate $E_s^Q\left[S(T)\right]$ since

$$E_s^Q\left[S(T)\right] = \underbrace{S(t) + r\int_t^s S(u)du}_{S(s)} + r\int_s^T E_s^Q\left[S(u)\right]du$$

so,

$$E_s^Q\left[S(T)\right] = S(s)e^{r(T-s)}$$

and, therefore, the free arbitrage pricing function at time $s > t$ is

$$\Pi(s) = S(s) - S(t)e^{r(s-t)}.$$

7 Completeness and Hedging

Exercise 7.2 We have $F(t, s, z)$ be defined by

$$\begin{aligned} F_t + r \cdot s \cdot F_s + \frac{1}{2}\sigma^2 s^2 F_{ss} + gF_z &= rF \\ F(T, s, z) &= \Phi(s, z) \end{aligned}$$

and the dynamics under Q for S and Z

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ dZ(u) &= g(u, S(u))du \end{aligned}$$

We want to show that $F(t, S(t), Z(t)) = e^{-r(T-t)}E_{t,s,z}^Q\left[\Phi(S(T), Z(T))\right]$.

For that we find , by Itô, the dynamics of $\Pi(t) = F(t, S(t), Z(t))$, the arbitrage free pricing process

$$\begin{aligned} d\Pi(t) &= F_t dt + F_s\left[(rS(t)dt + \sigma S(t)dW(t)\right] + F_z \cdot g(t, S(t))dt + \frac{1}{2}F_{ss}\sigma^2 S^2(t)dt \\ &= \underbrace{\left[F_t + r \cdot S(t) \cdot F_s + \frac{1}{2}\sigma^2 S^2(t)F_{ss} + g(t, S(t))F_z\right]}_{r\Pi(t)} + \sigma S(t)F_s dW(t) \end{aligned}$$

Integrating we have

$$\Pi(T) = \Pi(t) + r\int_t^T \Pi(u)du + \sigma\int_t^T S(u)F_s dW(u)$$

Hence

$$E_{t,z,s}^Q\left[\Pi(T)\right] = \Pi(t) + r\int_t^T E_{t,z,s}^Q\left[\Pi(u)\right]du$$

So, using the usual "trick" of setting $m(u) = E_{t,z,s}^Q\left[\Pi(u)\right]$ and using techniques of ODE we finally get

$$\Pi(t) = F(t, S(t), Z(t)) = e^{-r(T-t)}E_{t,s,z}^Q\left[\Phi(S(T), Z(T))\right].$$

(Remember that $\Pi(T) = F(T, S(T), Z(T)) = \Phi(S(T), Z(T))$.)

Exercise 7.3 The price arbitrage free price is given by (note that this time our claim is *not* simple, i.e. it is not of the form $\mathcal{X} = \Phi(S(T))$).

$$\begin{aligned} \Pi(t) &= e^{-r(T_2-t)} E_t^Q[\mathcal{X}] \\ &= e^{-r(T_2-t)} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E_t^Q[S(u)]\, du \end{aligned}$$

We know that under Q

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ S(t) &= s \end{aligned}$$

So,

$$\Rightarrow E_t^Q[S(u)] = se^{r(u-t)}$$

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} se^{r(u-t)} du = \frac{1}{T_2 - T_1} \frac{s}{r} \left[e^{r(T_2-t)} - e^{r(T_1-t)} \right]$$

The price to the "mean" contract is thus

$$\Pi(t) = \frac{s}{r(T_2 - T_1)} \left[1 - e^{-r(T_2-T_1)} \right].$$

8 Parity Relations and Delta Hedging

Exercise 8.1 The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} K, & \text{if } S(T) \le A \\ K + A - S(T), & \text{if } A < S(T) < K + A \\ 0, & \text{otherwise.} \end{cases},$$

has then following contract function (recall that $\mathcal{X} = \Phi S(T))$

$$\Phi(x) = \begin{cases} K, & \text{if } x \le A \\ K + A - x, & \text{if } A < x < K + A \\ 0, & \text{otherwise.} \end{cases},$$

which can be decomposed into the following "basic" contract functions written

$$\Phi(x) = K \cdot \underbrace{1}_{\Phi_B(x)} - \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - A - K]}_{\Phi_{c,A+K}(x)}.$$

Having this T-claim $\mathcal{X}$ is then equivalent to having the following (replicating) portfolio at time T:

* K in monetary units

* short (position in) a call with strike A

* long (position in) a call with strike $A + K$

Given the decomposition of the contract function Φ into basic contract functions, we immediately have that the arbitrage free pricing process Π is

$$\Pi(t) = K \cdot \overbrace{e^{-r(T-t)}}^{B(t)} - c(s, A, T) + c(s, A + K, T)$$

where $c(s, A, T)$ and $c(s, A+K, T)$ stand for the prices of European call options on S and maturity T with strike prices A and $A + K$, respectively. The Black-Scholes formula give us both $c(s, A, T)$ and $c(s, A + K, T)$.

The hedge portfolio thus consists of a reverse position in the above components, i.e., borrow $e^{-r(T-t)}K$, buy a call with strike K and sell a call with strike $A+K$.

Exercise 8.4 We apply, once again, the exact same technique. The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} 0, & \text{if } S(T) < A \\ S(T) - A, & \text{if } A \leq S(T) \leq B \\ C - S(T), & \text{if } B < S(T) \leq C \\ 0, & \text{if } S(T) > C. \end{cases}$$

where $B = \frac{A+C}{2}$, has a contract function Φ that can be written as

$$\Phi(x) = \underbrace{\max\left[0, x - A\right]}_{\Phi_{c,A}(x)} + \underbrace{\max\left[0, x - C\right]}_{\Phi_{c,C}(x)} - 2\underbrace{\max\left[0, x - B\right]}_{\Phi_{c,B}(x)}$$

Having this *butterfly* is then equivalent to having the following constant(replicating) portfolio at time T:

* long (position in) a call option with strike A

* long (position in) a call option with strike C

* short (position in) a call option with strike B

The arbitrage free pricing process Π follows immediately from the decomposition of the contract function Φ and is given by

$$\Pi(t) = c(s, A, T) + c(s, C, T) - 2c(s, B, T)$$

where $c(s, A, T)$, $c(s, B, T)$ and $c(s, C, T)$ stand for the prices of European call options on S, with maturity T and strike prices A, B and C, respectively, and can be computed using the Black-Scholes formula.

The hedge portfolio consists of a reverse position in the above components, i.e., sell two call options one with strike A and other with strike B and buy other two both with strike B.

Exercise 8.10 From the put-call parity we have that

$$p(t, s) = Ke^{-r(T-t)} + c(t, s) - S$$

where $p(t, s)$ and $c(t, s)$ stand for the price of a put and a call option on S with maturity T and strike price K.

The *delta* measures the variation in the price of a derivative with respect to changes in the value of the underlying. Differentiating the put-call parity w.r.t. S we have

$$\begin{aligned} \underbrace{\frac{\partial p(t, s)}{\partial S}}_{\Delta_{put}} &= \frac{\partial}{\partial S}\left(Ke^{-r(T-t)}\right) + \underbrace{\frac{\partial c(t, s)}{\partial S}}_{\Delta_{call}} - \frac{\partial S}{\partial S} \\ \Delta_{put} &= \Delta_{call} - 1 \end{aligned}$$

Since,$\Delta_{call} = N[d1] \Rightarrow \Delta_{put} = N[d1] - 1$.

To find the result on the *gamma* we differentiate one more time (so two times) w.r.t. S the put-call parity and get

$$\underbrace{\frac{\partial^2 p(t, s)}{\partial S^2}}_{\Gamma_{call}} = \underbrace{\frac{\partial^2 c(t, s)}{\partial S^2}}_{\Gamma_{put}}$$

From the fact that $\Gamma_{call} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$ it follows that $\Gamma_{put} = \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$.

9 Several Underlying Assets

Exercise 9.3

Consider the T-claim $\mathcal{X} = \max\left[S_1(T) - S_2(T); 0\right]$ where S_1 and S_2 are defined by

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)d\bar{W}_1$$

$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)d\bar{W}_2$$

and

$$d\bar{W}_1.d\bar{W}_2 = \rho dt.$$

The contract function Φ associated with it, is homogeneous of degree 1, and thus, it can be rewritten as a contract function Ψ on the normalized variable $z = \frac{s_1}{s_2}$.

$$\begin{aligned}
\Phi(s_1, s_2) &= \max\left[s_1 - s_2; 0\right] = s_2 \max\left[\underbrace{\frac{s_1}{s_2}}_{z} - 1; 0\right] \\
&= s_2 \underbrace{\max\left[z - 1; 0\right]}_{\Psi(z)}.
\end{aligned}$$

So, the pricing function $F(t, s_1, s_2) = s_2 G(t, z)$ and from

$$\begin{aligned}
F_t &= s_2 G_t \\
F_1 &= s_2 G_z \frac{\partial z}{\partial s_1} = s_2 G_z \frac{1}{s_2} = G_z \\
F_2 &= G + s_2 G_z \frac{\partial z}{\partial s_2} = G + s_2 G_z \left(-\frac{s_1}{s_2^2}\right) = G - z G_z \\
F_{11} &= G_{zz} \frac{\partial z}{\partial s_1} = \frac{1}{s_2} G_{zz} \\
F_{22} &= G_z \frac{\partial z}{\partial s_2} - \left(-\frac{\partial z}{\partial s_2} G_z + z G_{zz} \frac{\partial z}{\partial s_2}\right) = \frac{1}{s_2} z^2 G_{zz} \\
F_{12} &= G_{zz} \frac{\partial z}{\partial s_2} = -\frac{1}{s_2} z G_{zz}
\end{aligned}$$

we realize that G satisfies the (much easier) PDE,

$$\begin{aligned}
G_t + \frac{1}{2} z^2 G_{zz}(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) &= 0 \\
G(T, z) &= \max\left[z - 1, 0\right],
\end{aligned}$$

that we recognize as the Black-Scholes equation and, therefore we have that G is the price of a call with underlying Z, exercise price $K = 1$ and with, $r = 0$, and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

So, we can use Black-Scholes formula to concretize the pricing function $\Pi(t) = F(t, S_1(t), S_2(t))$:

$$\begin{aligned}\Pi(t) &= S_2(t)G(t, Z(t)) \\ &= S_2(t)\left\{\overbrace{\frac{S_1(t)}{S_2(t)}}^{Z(t)} N\left[d_1'\right] - N\left[d_2'\right]\right\} \\ &= S_1(t)N\left[d_1'\right] - S_2(t)N\left[d_2'\right]\end{aligned}$$

where

$$d_1' = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\sqrt{T-t}}\left\{\ln\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)(T-t)\right\}$$

and $d_2' = d_1' - \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\sqrt{T-t}$.

Exercise 9.4 For this "special" maximum option we have $\mathcal{X} = \max\left[aS_1(T); bS_2(T)\right]$ where S_1 and S_2 are defined by

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)d\bar{W}_1$$

$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)d\bar{W}_2$$

and $\bar{W}_1$ is independent from $\bar{W}_2$.

The contract function Φ is homogeneous of degree 1 and can be rewritten in terms of the normalized variable $z = \frac{s_1}{s_2}$:

$$\begin{aligned}\Phi(s_1, s_2) &= \max\left[as_1; bs_2\right] = bs_2 \max\left[\frac{as_1}{bs_2}; 1\right] \\ &= = bs_2\left\{\max\left[\frac{as_1}{bs_2} - 1; 0\right] + 1\right\} \\ &= bs_2\left\{\frac{a}{b}\max\left[\overbrace{\frac{s_1}{s_2}}^{z} - \frac{b}{a}; 0\right] + 1\right\} \\ &= bs_2 + as_2 \underbrace{\max\left[z - \frac{b}{a}; 0\right]}_{\Psi(z)}\end{aligned}$$

Thus, the pricing function F is given by $F(t, s_1, s_2) = bs_2 + as_2 G(t, z)$ where G solves the boundary problem So we have

$$\begin{aligned} G_t + \frac{1}{2} z^2 G_{zz}(\sigma_1^2 + \sigma_2^2) &= 0 \\ G(T, z) &= \max\left[z - \frac{b}{a}, 0\right]. \end{aligned}$$

Once again, the above expression is the Black-Scholes equation, G is the price of a call option on Z with strike price $K = \frac{b}{a}$, $r = 0$, and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ We can use Black-Scholes formula to concretize the pricing function $\Pi(t) = F(t, S_1(t), S_2(t)$:

$$\begin{aligned} \Pi(t) = bS_2(t) + aS_2(t)G(t, Z(t)) &= bS_2(t) + aS_2(t)\left\{\frac{S_1(t)}{S_2(t)} N\left[d_1'\right] - \frac{b}{a} N\left[d_2'\right]\right\} \\ &= bS_2(t) + aS_1(t)N\left[d_1'\right] - bS_2(t)N\left[d_2'\right] \\ &= (1 - N\left[d_2'\right])bS_2(t) + aN\left[d_1'\right]S_1(t) \end{aligned}$$

where

$$d_1' = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}}\left\{\ln\left(\frac{aS_1(t)}{bS_2(t)}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)(T-t)\right\}$$

and $d_2' = d_1' - \sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}$.

10 Incomplete Markets

Exercise 10.1 Given a claim $\mathcal{X} = \Pi(X(T)))$ and since the dynamics of X under the Q-measure are

$$dX(t) = \left[\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right]dt + \sigma(t, X(t))dW^Q(t),$$

we can find the Q-dynamics of the pricing function $F(t, X(t))$ using the Ito formula

$$\begin{aligned} dF(t, X(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t, X(t)) - \lambda(t, X(t))\sigma(t, X(t))\right]dt + \sigma(t, X(t))dW^Q(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t, X(t))dt \end{aligned}$$

$$\begin{aligned}
&= \underbrace{\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t,X(t)) - \lambda(t,X(t))\sigma(t,X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t,X(t))\right]}_{rF(t,X(t))} dt \\
&\quad + \sigma(t,X(t))dW^Q(t) \\
&= rF(t,X(t)) + \sigma(t,X(t))dW^Q(t)
\end{aligned}$$

where the last step results from the fact that F has to satisfy the pricing PDE:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\mu(t,X(t)) - \lambda(t,X(t))\sigma(t,X(t))\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t,X(t)) = rF.$$

11 Dividends

Exercise 11.2 We know that when there is a continuous dividend δ being paid we have the following dynamics under Q for the asset S and the dividen structure D

$$\begin{aligned}
dS(t) &= \left[r - \delta\left(S(t)\right)\right] S(t)dt + \sigma\left(S(t)\right) S(t)dW(t) \\
dD(t) &= S(t)\delta\left(S(t)\right) dt
\end{aligned}$$

Just by rewritting and rearranging terms we have

$$\begin{aligned}
dS(t) &= rS(t)dt - \underbrace{\delta\left(S(t)\right) S(t)dt}_{dD(t)} + \sigma\left(S(t)\right) S(t)dW(t) \\
e^{-r\cdot t}dS(t) - e^{-r\cdot t}rS(t)dt &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma\left(S(t)\right) S(t)dW(t) \\
d\left[e^{-r\cdot t}S(t)\right] &= -e^{-r\cdot t}dD(t) + e^{-r\cdot t}\sigma\left(S(t)\right) S(t)dW(t).
\end{aligned}$$

Integratting the last expression we get

$$\begin{aligned}
e^{-r\cdot t}S(t) &= e^{-r\cdot 0}S(0) - \int_0^t e^{-ru}dD(u) + \int_0^t e^{-r\cdot u}\sigma\left(S(u)\right)S(u)dW(u) \\
S(0) &= e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u) - \int_0^t e^{-r\cdot u}\sigma\left(S(u)\right)S(u)dW(u),
\end{aligned}$$

and taking $E_0^Q[.]$ expectations we finally get the results

$$S(0) = E_0^Q\left[e^{-r\cdot t}S(t) + \int_0^t e^{-ru}dD(u)\right].$$

Exercise 11.6 In the Black-Scholes model with a constant continuous dividend yield δ we have, under the Q-measure we have

$$dS(t) = (r - \delta)\, S(t)dt + \sigma S(t)dW^Q(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t, x) = c(t, x) - \Pi(t, S(T)) + e^{-r(T-t)}K.$$

But we also know that

$$\begin{aligned}
\Pi(t, S(T)) &= e^{-r(T-t)} E_t^Q\,[S(T)] \\
&= e^{-r(T-t)} S(t) e^{(r-\delta)(T-t)} \\
&= S(t)e^{-\delta(T-t)}.
\end{aligned}$$

So,

$$p(t, x) = c(t, x) - S(t)e^{-\delta(T-t)} + e^{-r(T-t)}K.$$

12 Currency Derivatives

Exercise 12.1 We have, under the objective probability measure, the following processes for the spot exchange rate X (units of domestic currency d per foreign currency unit f), and the domestic , B_d, and foreign, B_f, riskless assets:

$$\begin{aligned}
dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t) \\
dB_d(t) &= r_d B(t)_d dt \\
dB_f(t) &= r_f B(t)_d dt,
\end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. Hence the Q-dynamics of X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

From the "standard" call-put parity, which must be valid, we have the following relation between call and put options with the same maturity T and exercise price K:

$$p(t, x) = c(t, x) - \Pi(t, X(T)) + e^{-r_d(T-t)}K.$$

But we also know that

$$\begin{aligned}\Pi(t, X(T)) &= e^{-r_d(T-t)} E_t^Q [X(T)] \\ &= e^{-r_d(T-t)} X(t) e^{(r_d - r_f)(T-t)} \\ &= X(t) e^{-r_f(T-t)}.\end{aligned}$$

So,

$$p(t,x) = c(t,x) - X(t)e^{-r_f(T-t)} + e^{-r_d(T-t)}K.$$

(Remark: Compare this exercise with exercise 11.6 in the previous section, and see that the foreign risk-free rate can be treated as a continuous dividend-yield on the spot exchange rate.)

Exercise 12.2 The *binary option* on the exchange rate X is a T-claim, Z, of the form

$$\mathcal{Z} = 1_{[a,b]}\left(X(T)\right),$$

i.e. if $a \leq X(T) \leq b$ then the holder of this claim will obtain one unit of domestic currency, otherwise gets nothing. The dynamics under Q of the exchange rate X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_x X(t)dW(t).$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic.

Integrating the above expression we get

$$X(T) = X(t) + \int_t^T (r_d - r_f)X(u)du + \int_t^t \sigma_x X(u)dW(u)$$

and so we have (by solving the SDE)

$$X(T) = X(t)e^{\left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t) + \sigma_x(W(T)-W(t))},$$

and we see from taking the logarithm that

$$\ln(X(T)) \sim N\left[\ln\left(X(t)\right) + \left(r_d - r_f - \frac{\sigma_x^2}{2}\right)(T-t), \sigma_x\sqrt{T-t}\right].$$

So, the price, $\Pi(t)$ of the binary option is given by

$$\begin{aligned}\Pi(t) &= e^{-r_d(T-t)}E_t^Q[\mathcal{Z}] \\ &= e^{-r_d(T-t)}Q(a \le X(T) \le b) \\ &= e^{-r_d(T-t)}Q(\ln(a) \le \ln(X(T)) \le \ln(b)) \\ &= e^{-r_d(T-t)}Q(d_a \le z \le d_b)\end{aligned}$$

where $z \sim N[0,1]$ and $d_a = \frac{\ln\left(\frac{a}{X(t)}\right)-\left(r_d-r_f-\frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and $d_b = \frac{\ln\left(\frac{b}{X(t)}\right)-\left(r_d-r_f-\frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x\sqrt{T-t}}$ and so we have that the price of the binary exchange option $\mathcal{Z}$ is:

$$\Pi(t) = e^{-r_d(T-t)}\left[N(d_b) - N(d_a)\right].$$

Exercise 12.3 Under the objective probability measure we have the following dynamics for the domestic stock S_d, the foreign stock S_f, the exchange rate X and the domestic B_d and foreign B_f riskless assets

$$\begin{aligned}dS_d(t) &= \alpha_d S_d(t)dt + \sigma_d S_d(t)dW_d(t), \\ dS_f(t) &= \alpha_f S_f(t)dt + \sigma_f S_f(t)dW_f(t), \\ dX(t) &= \alpha_x X(t)dt + \sigma_x X(t)dW(t), \\ dB_d(t) &= r_d B_d(t)dt, \\ dB_f(t) &= r_f S_d(t)dt,\end{aligned}$$

where r_d and r_f are the domestic and foreign short rates which are assumed to be deterministic. The domestic stock denominated in terms of the foreign currency is given by $\tilde{S}_d = \frac{S_d}{X}$, then by Ito

$$\begin{aligned}d\tilde{S}_d(t) &= \frac{1}{X(t)}dS_d(t) - \frac{S_d(t)}{X^2(t)}dX(t) + \frac{S_d(t)}{X^3(t)}(dX(t))^2 - \frac{-1}{X^2(t)}\underbrace{dS_d(t)dX(t)}_{0 \text{ for } W_d \text{ and } W \text{ indep.}} \\ &= \alpha_d\tilde{S}_d(t)dt + \sigma_d\tilde{S}_d(t)dW_d(t) - \alpha_x\tilde{S}_d(t)dt - \sigma_x\tilde{S}_d(t)dW(t) + \sigma_x^2\tilde{S}_d(t)dt \\ &= \left(\alpha_d - \alpha_x + \sigma_x^2\right)\tilde{S}_d(t)dt + \sigma_d\tilde{S}_d(t)dW_d(t) - \sigma_x\tilde{S}_d(t)dW(t).\end{aligned}$$

The dynamics above are under the objective probability measure. Under Q_f the drift term of all assets denominated in the foreign currency must have r_f (the risk-free rate on the foreign economy) as the drift. Also, we can use the properties of the Wiener processes to normalize the diffusion part.

It follows that

$$d\tilde{S}_d(t) = r_f \tilde{S}_d(t)dt + \sqrt{\sigma_d^2 + \sigma_x^2}\tilde{S}_d(t)dW_f(t).$$

15 Bonds and Interest Rates

Exercise 15.1 *Forward Rate Agreement*

(a) Note that the cash flow to the lender's in a FRA ($-K$ at time S and $Ke^{R^*(T-S)}$ at time T) can be replicated by the following portfolio:

* sell K S-bonds

* buy $Ke^{R^*(T-S)}$ T-bonds

So, at time $t < S$, the value $\Pi(t)$, on the lender's cash flow in a FRA has to equal to the value of the replicating portfolio and is given buy

$$\Pi(t) = Ke^{R^*(T-S)}p(t,T) - Kp(t,S).$$

(b) If we have that $\Pi(0) = 0$ we must have

$$\begin{aligned}
Ke^{R^*(T-S)}p(0,T) - Kp(0,S) &= 0 \\
e^{R^*(T-S)}p(0,T) - p(0,S) &= 0 \\
R^*(T-S) &= \ln\left(\frac{p(0,S)}{p(0,T)}\right) \\
R^* &= \frac{\ln(p(0,S)) - \ln(p(0,T))}{T-S}
\end{aligned}$$

Since by definition the forward rate $R(t;S,T)$ is given by

$$R(t;S,T) = -\frac{\ln(p(t,T)) - \ln(p(t,S))}{T-S},$$

we have that $R^* = R(0;S,T)$.

Exercise 15.2 Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t),$$

then we have, by Ito, that $\ln(p(t,T))$ has dynamics that are given by

$$\begin{aligned} d\ln(p(t,T)) &= m(t,T)dt + v(t,T)dW(t) - \frac{1}{2}v(t,T)v(t,T)^*dt \\ &= \left[m(t,T) - \frac{1}{2}v(t,T)v(t,T)^*\right]dt + v(t,T)dW(t), \end{aligned}$$

and integrating we have

$$\ln(p(t,T)) = \ln(p(0,T)) + \int_0^t m(s,T) - \frac{1}{2}v(s,T)v(s,T)^*ds + \int_0^t v(s,T)dW(s).$$

Since $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$, we also have

$$f(t,T) = \underbrace{-\frac{\partial \ln(p(0,T))}{\partial T}}_{f(0,T)} - \int_0^t \left(\underbrace{\frac{\partial m(s,T)}{\partial T}}_{m_T(s,T)} - v(s,T)\underbrace{\frac{\partial v(s,T)}{\partial T}}_{v_T(s,T)^*} \right) ds - \int_0^t \frac{\partial v(s,T)}{\partial T} dW(s)$$

and from its differential form we finally get the result

$$df(t,T) = \left[\underbrace{v(t,T)v_T(t,T)^* - m_T(t,T)}_{\alpha(t,T)}\right] dt - \underbrace{v_T(t,T)}_{\sigma(t,T)} dW(t).$$

Exercise 15.5 Let $\{y(0,T;T \geq 0\}$ denote the zero coupon yield curve.

(a) Then we have that

$$p(0,T) = e^{-y(0,T)\cdot T}.$$

Using the definition of the instantaneous forward rate $f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$ and the expression above we get the result

$$\begin{aligned} f(0,T) &= -\frac{\partial \ln(p(0,T))}{\partial T} \\ &= -\frac{\partial \ln\left(e^{-y(0,T)\cdot T}\right)}{\partial T} \\ &= \frac{\partial\left(y(0,T)\cdot T\right)}{\partial T} \\ &= y(0,T) + T\cdot\frac{\partial y(0,T)}{\partial T}. \end{aligned}$$

(b) If the zero coupon yield curve is an increasing function of T, then we know that $\frac{\partial y(0,T)}{\partial T} \geq 0$, and using the result from (a) we have that $f(0,T) \geq y(0,T)$ for any $T > 0$.

It remains to prove that $y_M(0,T) \leq y(0,T)$. This will follow from the price of a coupon bond is given by

$$p_T(t) = Kp(t,T_n) + \sum_{i=1}^{n} c_i p(t,T_i).$$

So in particular we have that

$$p_T(0) = K \underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)}.$$

but it can also we given, since y_M is the yield to maturity of a coupon bond, by

$$p_T(0) = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}.$$

So,

$$K \underbrace{e^{-y(0,T_n)\cdot T_n}}_{p(0,T_n)} + \sum_{i=1}^{n} c_i \underbrace{e^{-y(0,T_i)\cdot T_i}}_{p(0,T_i)} = Ke^{-y_M(0,T_n)\cdot T_n} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_n)\cdot T_i}$$

By comparing the LHS and the RHS and since $c_i \geq 0$, and $y(0,T_n) \geq y(0,T_{n-1}) \geq \cdots \geq y(0,T_1)$ (by assumption), we must have that $y(0,T_n) \geq y_M(0,T_n)$ since it is valid for any T_n we can write that $y(0,T) \geq y_M(0,T)$ for any T.

16 Short Rate Models

Exercise 16.2 The object of the exercise is to connect the forward rates to the risk neutral valuation of bond prices.

(a) Recall from the risk-neutral valuation of bond prices that

$$p(t,T) = E_t^Q \left[\exp\left\{ -\int_t^T r(s)ds \right\} \right]$$

and hence, using the definition of a instantaneous forward rate (and assuming that we can differentiate under the expectation sign) we get

$$
\begin{aligned}
f(t,T) &= -\frac{\partial \ln(p(t,T))}{\partial T} \\
&= -\frac{\partial}{\partial T}\ln\left(E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]\right) \\
&= \frac{E_t^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]}.
\end{aligned}
$$

(b) To check that indeed we have $f(t,t) = r(t)$ use the expression above

$$
f(t,T) = \frac{E_t^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]},
$$

and set $T = t$:

$$
\begin{aligned}
f(t,t) &= \frac{E_t^Q\left[r(t)\exp\left\{-\int_t^t r(s)ds\right\}\right]}{E_t^Q\left[\exp\left\{-\int_t^t r(s)ds\right\}\right]} \\
&= \frac{E_t^Q\left[r(t)\exp\{0\}\right]}{E_t^Q\left[\exp\{0\}\right]} = r(t).
\end{aligned}
$$

Exercise 16.3 Recall that in a *swap of a fixed rate vs. a short rate* we have:

- A invests K at time 0 and let it grow at a *fixed* rate of interest R over the time interval $[0,T]$. A has thus an amount K_A at time T and at that time (T) pays the surplus $K_A - K$ to B.
- B invests the principal at a *stochastic* short rate of interest over the interval $[0,T]$. B has thus an amount K_B at time T and at that time (T) pays the surplus $K_B - K$ to A.
- The *swap rate* for this contract is defined as the value, R, of the fixed rate which gives this contract value zero at $t = 0$.

At maturity party A has $\Phi(T) = K_B - Ke^{R\cdot T}$ where $K_B = Ke^{\int_0^T r(s)ds}$. Thus, the value of this contract at time 0 is given by

$$
\Pi(0) = E_{0,r}^Q\left[e^{-\int_0^T r(s)ds}\left(\underbrace{Ke^{\int_0^T r(s)ds} - Ke^{R\cdot T}}_{\Phi(T)}\right)\right]
$$

$$\begin{aligned}
&= E^Q_{0,r}\left[K - Ke^{R\cdot T - \int_0^T r(s)ds}\right] \\
&= K \cdot E^Q_{0,r}\left[1 - e^{R\cdot T} e^{-\int_0^T r(s)ds}\right] \\
&= K \cdot \left(1 - e^{R\cdot T} \underbrace{E^Q_{0,r}\left[e^{-\int_0^T r(s)ds}\right]}_{p(0,T)}\right)
\end{aligned}$$

Since we must have $\Pi(0) = 0$ we have

$$\begin{aligned}
K \cdot \left(1 - e^{R\cdot T} p(t,T)\right) &= 0 \\
e^{R\cdot T} &= \frac{1}{p(0,T)} \\
R \cdot T &= -\ln\left(p(0,T)\right) \\
R &= -\frac{\ln\left(p(0,T)\right)}{T}.
\end{aligned}$$

17 Martingale Models for the Short Rate

Exercise 17.1 In the Vasicek model we have

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

with $a > 0$.

(a) Using the SDE above and multiplying both sides by e^{at} we have that

$$\begin{aligned}
e^{at}dr(t) + r(t)ae^{at}dt &= e^{at}bdt + \sigma e^{at}dW(t) \\
d(e^{at}r(t)) &= e^{at}bdt + \sigma e^{at}dW(t) \\
e^{at}r(t) &= r(0) + \int_0^t e^{as}bds + \sigma\int_0^t e^{as}dW(s) \\
r(t) &= r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] + \sigma e^{-at}\int_0^t e^{a}(s)dW(s).
\end{aligned}$$

Looking at the solution of the SDE it is immediate that r is Gaussian, so it is enough to determine the mean and variance.

From the solution of the SDE we see that

$$E\left[r(t)\right] = r(0)e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right]$$

and

$$\begin{aligned} V[r(t)] &= \sigma^2 e^{-2at} \int_0^t e^{2as} ds \\ &= \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \end{aligned}$$

(b) As $t \to \infty$ we have

$$\lim_{t\to\infty} E[r(t)] = r(0) \lim_{t\to\infty} e^{-at} + \frac{b}{a}\left[1 - \lim_{t\to\infty} e^{-at}\right] = \frac{b}{a},$$

and

$$\lim_{t\to\infty} V[r(t)] = \frac{\sigma^2}{2a}\left(1 - \lim_{t\to\infty} e^{-2at}\right) = \frac{\sigma^2}{2a}.$$

So, $N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$ is the limiting distribution of r.

(c) The results in (a) and (b) are based on the assumption that the value $r(0)$ is known. If, instead we have that $r(0) \sim N\left[\frac{b}{a}, \frac{\sigma^2}{2a}\right]$, then for any t we have that

$$E[r(t)] = E[r(0)]\, e^{-at} + \frac{b}{a}\left[1 - e^{-at}\right] = \frac{b}{a},$$

and

$$\begin{aligned} V[r(t)] &= e^{-2at} V[r(0)] + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= \frac{\sigma^2 e^{-2at}}{2a} + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) = \frac{\sigma^2}{2a}. \end{aligned}$$

(d) To be done!

Exercise 17.2 From Exercise 17.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)e^{-a(u-t)} + \frac{b}{a}\left[1 - e^{-a(u-t)}\right] + \sigma \int_t^u e^{-a(u-s)} dW(s).$$

By definition,

$$p(t,T) = E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right].$$

Let us define $Z(t,T) = -\int_t^T r(u)du$, from exercise 17.1 we know that r is normally distributed, so Z is also normally distributed since

$$\begin{aligned} Z(t,T) &= -r(t)e^{at}\int_t^T e^{-au}du - \frac{b}{a}\left[1 - e^{at}\int_t^T e^{-au}du\right] + \sigma\int_t^T\int_t^u e^{-a(u-s)}dW(s)du \\ &= -\frac{r(t)}{a}\left[1 - e^{-a(T-t)}\right] - \frac{b}{a}(T-t) - \frac{b^2}{a}\left[1 - e^{-a(T-t)}\right] + \sigma\int_t^T\int_t^u e^{-a(u-s)}dW(s)du \end{aligned}$$

and has

$$E_{t,r}^{Q}\left[Z(t,T)\right] = \underbrace{\left(-\frac{1}{a}\left[1-e^{-a(T-t)}\right]\right)}_{\text{deterministic function of } t \text{ and } T} \quad r(t) \underbrace{-\frac{b}{a}(T-t)-\frac{b^2}{a}\left[1-e^{-a(T-t)}\right]}_{\text{deterministic function of } t \text{ and } T}$$

and

$$\begin{aligned} V_{t,r}^{Q}\left[Z(t,T)\right] &= V\left[\sigma\int_t^u\int_t^T e^{-a(u-s)}dudW(s)\right] \\ &= \underbrace{\sigma\int_t^u\left(\int_t^T e^{-a(u-s)}du\right)^2 ds}_{\text{deterministic function of } t \text{ and } T}. \end{aligned}$$

Since $Z(t,T)$ is normally distributed $\forall_{t,T}$ (from the properties of the normal distribution), we have that

$$\begin{aligned} p(t,T) &= E_{t,r}^{Q}\left[e^{Z(t,T)}\right] = e^{E_{t,r}^{Q}[Z(t,T)]+\frac{1}{2}V_{t,r}^{Q}[Z(t,T)]} \\ &\Rightarrow \\ \ln(p(t,T)) &= E_{t,r}^{Q}\left[Z(t,T)\right]+\frac{1}{2}V_{t,r}^{Q}\left[Z(t,T)\right] \\ &= \underbrace{-\frac{1}{a}\left[1-e^{-a(T-t)}\right]}_{B(t,T)} r(t) + \\ &+ \left(\underbrace{-\frac{b}{a}(T-t)-\frac{b^2}{a}\left[1-e^{-a(T-t)}\right]+\frac{1}{2}\sigma\int_t^u\left(\int_t^T e^{-a(u-s)}du\right)^2 ds}_{A(t,T)}\right). \end{aligned}$$

So, the Vasicek model has an affine term structure.

Alternative Solution

From Exercise 17.1 we know that for the *Vasicek model*, and $t < u$ we have

$$r(u) = r(t)\underbrace{e^{-a(u-t)}}_{D(u)} + \underbrace{\frac{b}{a}\left[1-e^{-a(u-t)}\right]+\sigma\int_t^u e^{-a(u-s)}dW(s)}_{F(u)}.$$

where $D(u)$ is deterministic and $F(u)$ is stochastic and both are non dependent on r.

By definition,

$$p(t,T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(u)du}\right]$$

$$
\begin{aligned}
&= E^Q_{t,r}\left[e^{-\int_t^T [r(t)D(u)+F(u)du]}\right] \\
&= \left[\underbrace{e^{\left(-\int_t^T D(u)du\right)r(t)}}_{B} \underbrace{E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right]}_{A}\right].
\end{aligned}
$$

In part A, since $-\int_t^T F(u)du$ is normally distributed (note it is a "sum" of Wiener increments), we know (from the properties of the normal distribution) that there exist a $A(t,T)$ such that

$$
E^Q_{t,r}\left[e^{-\int_t^T F(u)du}\right] = e^{A(t,T)}.
$$

From part B we immediatelly see that $B(t,T) = \int_t^T D(u)du$.

So, the Vasicek model has an affine term structure.

Exercise 17.3 The problem with the *Dothan's model* is that when r follows a GBM, like

$$
dr(t) = ar(t)dt + \sigma r(t)dW(t),
$$

the solution to the SDE is, for any $t < u$,

$$
r(u) = r(t)e^{\left(a-\frac{1}{2}\sigma^2\right)(u-t)+\sigma(W(u)-W(t))},
$$

i.e., in the Dothan's model r is lognormally distributed. Since,

$$
p(t,T) = E^Q_{t,r}\left[e^{-\int_t^T r(u)du}\right],
$$

to use the same procedure as in exercise 17.2 we would have to compute the above expected value, i.e., the expected value of an integral (that is a "sum") of lognormally distributed variables, which is a mess!

Exercise 17.8 Take the follwoing CIR model

$$
dY(t) = \left(2aY(t) + \sigma^2\right)dt + 2\sigma\sqrt{Y(t)}dW(t), \quad Y(0) = y_0,
$$

Then by Ito $Z(t) = \sqrt{Y(t)}$ follows

$$
\begin{aligned}
dZ(t) &= \frac{1}{2}\left(Y(t)^{-\frac{1}{2}}\right)dY(t) + \frac{1}{2}\left(-\frac{1}{4}\left(Y(t)^{-\frac{3}{2}}\right)\right)(dY(t))^2 \\
&= \frac{1}{2}\left(2aY(t)^{\frac{1}{2}} + \sigma^2 Y(t)^{-\frac{1}{2}}\right)dt + \sigma dW(t) - \frac{1}{2}\sigma^2 Y(t)^{-\frac{1}{2}} \\
&= a\underbrace{Y(t)^{\frac{1}{2}}}_{Z(t)}dt + \sigma dW(t),
\end{aligned}
$$

which is a linear diffusion.

18 Forward Rate Models

Exercise 18.1 We know that the *Hull-White model*

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t)$$

has an affine term structure, i.e., that

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

and that in this model we have in particular, $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$.
Furthermore, $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t)$, so in this case we have

$$f(t,T) = -\frac{\partial A(t,T)}{\partial T} + e^{-a(T-t)}r(t).$$

By Itô we have that the dynamics under Q of the forward rates is given by

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right]dt + ae^{-a(T-t)}r(t)dt + e^{-a(T-t)}dr(t) \\ &= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right]dt + ae^{-a(T-t)}r(t)dt + e^{-a(T-t)}\left[(\Theta(t) - ar(t))dt + \sigma dW(t)\right] \\ &= \alpha(t,T)dt + e^{-a(T-t)}\sigma dW(t), \end{aligned}$$

where $\alpha(t,T) = -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] + e^{-a(T-t)}\Theta(t)$.

Exercise 18.2 Take as given an HJM model (under Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where the volatility $\sigma(t,T)$ is a deterministic function of t and T.

(a) By the HJM drift condition, $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma'(t,u)du \quad \forall t,T$, which for deterministic $\sigma(t,T)$, means that $\alpha(t,T)$ is also deterministic.

To see the distribution of the forward rates note that

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \\ &= \underbrace{f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma'(s,u)duds}_{\mu(t,T)} + \int_0^t \sigma(s,T)dW(s). \end{aligned}$$

Note that $f(0,T)$ is observable and that the double integral is deterministic, so the only stochastic part is $\int_0^t \sigma(s,T)dW(s)$.

Hence $f(t,T) \sim N\left[\mu(t,T), \int_0^t \sigma^2(s,T)ds\right]$.

Since $r(t) = f(t,t)$ we immediately have that $r(t)$ is also normally distributed.

(b) Since we have $p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$, and in (a) we have already shown that $f(t,s)$ is normally distributed, then $p(t,T)$ is lognormally distributed.

Exercise 18.3 Recall that in between the vectors of market-price of risks of the domestic market and the foreign market there is the following relation:

$$\lambda_f(t) = \lambda_d(t) - \sigma_x'(t).$$

where, $\sigma_x(t)$ is the vector of volatilities of in the SDE for the exchange-rate X(denoted in units of domestic currency per unit of foreign currency).

We also know that *any* process, so in particular forward rates, has under Q (the *domestic* martingale measure), the drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_d(t))$, where $\mu(t,T)$ is the drift term under the objective probability measure and that the volatility term remains the same. So, in particular, for the foreign forward rates must have under the domestic martingale measure, Q,

$$df_f(t,T) = (\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t))\,dt + \sigma_f(t,T)dW(t),$$

hence, $\alpha_f(t,T) = \mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)$.

Likewise, any process, under Q^f (the *foreign* martingale measure),has a drift term equal to $(\mu(t,T) - \sigma(t,T)\lambda_f(t))$.

So, in particular, for the foreign forward rates we have under Q^f

$$df_f(t,T) = \underbrace{(\mu_f(t,T) - \sigma_f(t,T)\lambda_f(t))}_{\tilde{\alpha}_f(t,T)}\,dt + \sigma_f(t,T)dW^f(t).$$

So, using the relation between λ_d and λ_f we can also establish a relation between α_f and $\tilde{\alpha}_f$,

$$\begin{aligned}
\tilde{\alpha}_f(t,T) &= \mu_f(t,T) - \sigma_f(t,T)\lambda_f(t) \\
&= \mu_f(t,T) - \sigma_f(t,T)\left[\lambda_d(t) - \sigma_x'(t)\right] \\
&= \underbrace{\mu_f(t,T) - \sigma_f(t,T)\lambda_d(t)}_{\alpha_f(t,T)} + \sigma_f(t,T)\sigma_x'(t).
\end{aligned}$$

We also have that, under the foreign martingale measure, Q^f, the coefficients of the foreign martingale measure must satisfy the standard HJM drift condition so:

$$\tilde{\alpha}_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f'(t,s)ds.$$

Using teh relation found between α_f and $\tilde{\alpha}_f$ and using the drift condition above we get

$$\begin{aligned}
\tilde{\alpha}_f(t,T) &= \sigma_f(t,T) \int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) + \sigma_f(t,T)\sigma_x'(t) &= \sigma_f(t,T) \int_t^T \sigma_f'(t,s)ds \\
\alpha_f(t,T) &= \sigma_f(t,T) \left[\int_t^T \sigma_f'(t,s)ds - \sigma_x'(t) \right].
\end{aligned}$$

19 Change of Numeraire

Exercise 19.1 In the *Ho-Lee model* we have under Q

$$dr(t) = \Theta(t)r(t)dt + \sigma dW(t)$$

And we know that on this model we have an affine term structure for bond-prices

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}$$

with $B(t,T) = T - t$.

If the Q-dynamics of $p(t,T)$ are given by

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

then by Ito we have

$$v(t,T) = -\sigma B(t,T) = -\sigma(T-t).$$

The project is to price an European call option with:

- date of maturity T_1
- strike price K
- where the underlying is a zero-coupon bond with date of maturity T_2

and $T_1 < T_2$.

Note that our Z-claim is given by

$$\mathcal{Z} = \max\left[p(T_1, T_2) - K; 0\right],$$

Hence, we have using a change of measure on the standard arbitrage pricing formula that

$$\begin{aligned}
\Pi(t; \mathcal{Z}) &= E_t^Q\left[e^{-\int_t^{T_1} r(s)ds} \max\left[p(T_1, T_2) - K; 0\right]\right] \\
&= p(t, T_1) E_t^{T_1}\left[\max\left[p(T_1, T_2) - K; 0\right]\right] \\
&= p(t, T_1) E_t^{T_1}\left[\max\left[\underbrace{\frac{p(T_1, T_2)}{p(T_1, T_1)}}_{Z(T_1)} - K; 0\right]\right].
\end{aligned}$$

Note that dividing $p(T_1, T_1)$ in the last step of the equation "changes nothing"(since $p(T, T) = 1$ for all T) but has the advantage of seeing our claim as a claim on the process Z, which we know is a martingale under Q^{T_1}, and as long as it has deterministic volatility the Black-Scholes formula can help us. To check this note that

$$Z(t) = \frac{p(t, T_2)}{p(t, T_1)}$$

can also be written as (just using the fact that we have an ATS)

$$Z(t) = \exp\left\{A(t, T_2) - A(t, T_1) - \left[B(t, T_2) - B(t, T_1)\right] r(t)\right\},$$

and, therefore, has the following dynamics under Q(applying Ito formula)

$$dZ(t) = \{\cdots\} Z(t)dt + Z(t)\sigma_z(t)dW(t)$$

where $\sigma_z(t) = -\sigma\left[B(t, T_2) - B(t, T_1)\right] = -\sigma\left(T_2 - T_1\right)$, will be the same as under Q^{T_1}, and is deterministic.

Since under Q^{T_1},

$$Z(T_1) = Z(t) - \sigma\left(T_2 - T_1\right) \int_t^{T_1} dW(s)$$

The conditional (on information at time t) distribution of Z is $Z(T_1) \sim N\left[Z(t), \sigma^2\left(T_2 - T_1\right)^2\left(T_1 - t\right)\right]$.

So, (from the BS formula)

$$\begin{aligned}
\Pi(t) &= p(t, T_1)\left\{Z(t)N\left[d_1(t, T_1)\right] - KN\left[d_2(t, T_1\right]\right\} \\
&= p(t, T_2)N\left[d_1(t, T_1)\right] - p(t, T_1)KN\left[d_2(t, T_1\right]
\end{aligned}$$

where

$$d_1(t,T_1) = \frac{\ln\frac{Z(t)}{K} + \frac{1}{2}\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}} = \frac{\ln\frac{p(t,T_2)}{Kp(t,T_1)} + \frac{1}{2}\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}{\sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}}$$

and $d_2(t,T_1) = d_1(t,T_1) - \sqrt{\sigma^2\left(T_2-T_1\right)^2\left(T_1-t\right)}$.

Exercise 19.2 Take as given an HJM model of the form

$$df(t,T) = \alpha(t,T) + \sigma(t,T)dW(t) + \sigma_2 e^{-a(T-t)}dW_2(t)$$

where $\sigma(t,T) = [\,\sigma_1(T-t) \quad \sigma_2 e^{-a(T-t)}\,]$, $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ and we have W_1 and W_2 independent Wiener processes and σ_1 and σ_2 constants.

(a) From the relation between the dynamics of bond prices and forward rates we know that

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)S(t,T)dW(t)$$

with $S(t,T) = -\int_t^T \sigma(t,s)ds$.

In this case we have then

$$\begin{aligned} S(t,T) &= -\int_t^T [\,\sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)}\,]\,ds \\ &= \Bigg[\underbrace{-\frac{\sigma_1}{2}(T-t)^2}_{\sigma_1^p(t,T)} \quad \underbrace{-\frac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)}_{\sigma_2^p(t,T)}\Bigg] \end{aligned}$$

So,

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)dt + p(t,T)S(t,T)dW(t) \\ &= r(t)p(t,T)dt + p(t,T)\,[\,-\tfrac{\sigma_1}{2}(T-t)^2 \quad -\tfrac{\sigma_2}{a}\left(1-e^{-a(T-t)}\right)\,]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \\ &= r(t)p(t,T)dt + p(t,T)\,[\,\sigma_1^p(t,T) \quad \sigma_2^p(t,T)\,]\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \end{aligned}$$

(b) We will use exactly the same technique as in exercise 19.1 to price an European call option with maturity T_0 on a T_1-bond.

Note that in this model the Z process $Z(t) = \frac{p(t,T_1)}{p(t,T_0)}$ is a martingale under Q^{T_0} and its dynamics has deterministic volatility given by

$$\sigma_{T_1,T_0}(t) = S(t,T_1) - S(t,T_0) = -\int_{T_0}^{T_1} [\,\sigma_1(s-t) \quad \sigma_2 e^{-a(s-t)}\,]\,ds$$

Hence (by the same arguments as in exercise 19.1)

$$Z(T_0) \sim N\left[Z(t), \underbrace{\int_t^{T_0} ||\sigma_{T_1,T_0}(s)||^2 ds}_{\Sigma^2_{T_0,T_1}}\right]$$

and the pricing formula is given by

$$\begin{aligned}\Pi(t) &= p(t,T_0)\left\{Z(t)N\left[d_1(t,T_0)\right] - KN\left[d_2(t,T_0\right]\right\} \\ &= \left\{p(t,T_1)N\left[d_1(t,T_0)\right] - p(t,T_0)KN\left[d_2(t,T_0\right]\right\}\end{aligned}$$

where

$$d_1(t,T_0) = \frac{\ln\frac{Z(t)}{K} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}} = \frac{\ln\frac{p(t,T_2)}{Kp(t,T_1)} + \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}}$$

and $d_2(t,T_1) = d_1(t,T_1) - \sqrt{\Sigma^2_{T_0,T_1}}$.

20 Extra Exercises

20.1 Exercises

Exercise 20.1 Exercise 3.4 in the book!

Exercise 20.2 Solve explictly the GMB SDE

$$\begin{aligned} dX(t) &= \mu X(t)ds + \sigma X dW(t), \\ X(t) &= x. \end{aligned}$$

Exercise 20.3 Let $\{Z_n\}$ be a sequence of i.i.d. (independent identically distributed) random variables with finite exponential moments of all orders. Define the function $\varphi : R \to R$ by

$$\varphi(\lambda) = E\left[e^{\lambda Z_n}\right],$$

and define the process X by

$$X_n = \frac{e^{\lambda S_n}}{[\varphi(\lambda)]^n}, \quad \text{where} \quad S_n = \sum_{k=1}^{n} Z_k.$$

Prove that X is an $\mathcal{F}_n$-martingale, where $\mathcal{F}_n = \sigma\{Z_i;\ i = 1, \ldots, n\}$.

Exercise 20.4 Compute the infitesimal generator of the following known processes

(a) Brownian Motion

$$dY(t) = dW(t)$$

(b) Geometric Brownian Motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

(c) Vasicek Model for the short rate

$$dr(t) = c\,(\mu - r(t))\,dt + \sigma dW(t)$$

(d) The Ornstein-Uhlenbeck process

$$dX(t) = -\alpha X(t)dt + \sigma dW(t)$$

(e) Graph of the Brownian Motion

$$\begin{cases} dX_1(t) = dt & X_1(0) = t_0 \\ dX_2(t) = dB(t) & X_2(0) = \epsilon \end{cases}$$

Exercise 20.5 Exercise 6.3 in the book.

Derive the Black-Scholes formula for the standard call option.

Exercise 20.6 Derive the formulas of the greeks of an European call with strike price K and time of maturity T. That is, prove that following relations, (with notation as in the Black–Scholes formula and where the letter φ denotes the density function of the $N[0,1]$ distribution,) holds

$$\begin{aligned} \Delta &= N(d_1), \\ \Gamma &= \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}, \\ \rho &= K(T-t)e^{-r(T-t)}N(d_2), \\ \Theta &= -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2), \\ \mathcal{V} &= s\varphi(d_1)\sqrt{T-t}. \end{aligned}$$

Hint: Prove that $s\varphi(d_1) = Ke^{-r(T-t)}\varphi(d_2)$, and use this to prove the results.

Exercise 20.7 Exercise 11.5 in the book.

Exercise 20.8 Derive the formula for the value of a call option on a stock in a world were we *do not* assume deterministic interest rates.

20.2 Solution

Exercise 20.1 Integrating we get

$$X(t) = X(0) + \alpha \int_0^t X(s)ds + \int_0^t \sigma(s)dW(s)$$

Taking expectations we have

$$E[X(t)] = E[X(0)] + \alpha \int_0^t E[X(s)]\,ds$$

By setting $m(t) = E[X(t)]$ and after some calculations we obtain the answer

$$E[X(t)] = e^{\alpha t}E[X(0)]$$

Exercise 20.2 Take $Y(t) = \ln X(t)$, then by Itô we have

$$dY(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t)$$

Integrating we get for $T > t$:

$$\begin{aligned} Y(T) &= Y(t) + \int_t^T \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_t^T \sigma dW(t) \\ &= Y(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W(T) - W(t)) \end{aligned}$$

In terms of X we have

$$\begin{aligned} \ln X(T) &= \ln X(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W(T) - W(t)) \\ X(T) &= \underbrace{x}_{X(t)} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W(T) - W(t))\right\}. \end{aligned}$$

Exercise 20.3 We have

$$\varphi(\lambda) = E\left[e^{\lambda Z_n}\right], \quad X_n = \frac{e^{\lambda S_n}}{[\varphi(\lambda)]^n}, \quad S_n = \sum_{k=1}^n Z_k.$$

We want to show that

$$E\left[X_{n+1} \middle| \mathcal{F}_n\right] = X_n \underbrace{\Leftrightarrow}_{\text{Since } X_n \in \mathcal{F}_n} E\left[X_{n+1} - X_n \middle| \mathcal{F}_n\right] = 0.$$

From X definition we have

$$\begin{aligned} X_{n+1} - X_n &= \frac{e^{\lambda S_{n+1}}}{[\varphi(\lambda)]^{n+1}} - \frac{e^{\lambda S_n}}{[\varphi(\lambda)]^n} \\ &= \frac{e^{\lambda \sum_{k=1}^{n+1} Z_k} - \varphi(\lambda) e^{\lambda \sum_{k=1}^n Z_k}}{[\varphi(\lambda)]^n} \\ &= \underbrace{\frac{e^{\lambda \sum_{k=1}^n Z_k}}{[\varphi(\lambda)]^n}}_{X_n} \left(e^{\lambda Z_{n+1}} - \varphi(\lambda)\right). \end{aligned}$$

Hence

$$\begin{aligned} E\left[X_{n+1} - X_n \middle| \mathcal{F}_n\right] &= E\left[X_n\left(e^{\lambda Z_{n+1}} - \varphi(\lambda)\right) \middle| \mathcal{F}_n\right] \\ &= X_n E\left[e^{\lambda Z_{n+1}} - E\left[e^{\lambda Z_n}\right] \middle| \mathcal{F}_n\right] \\ &= X_n \underbrace{\left(E\left[e^{\lambda Z_{n+1}} \middle| \mathcal{F}_n\right] - E\left[e^{\lambda Z_n} \middle| \mathcal{F}_n\right]\right)}_{0, \text{ since } \{Z_n\} \text{ are iid}} \\ &= 0 \qquad \text{and we conclude that } X \text{ is an } \mathcal{F}_n \text{ martingale.} \end{aligned}$$

Exercise 20.4

(a) $\mathcal{A}f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x)$.

(b) $\mathcal{A}f(x) = \mu x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}(x)$.

(c) $\mathcal{A}f(x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x) + c\,(\mu - x)\,\frac{\partial f}{\partial x}$.

(d) $\mathcal{A}f(x) = -\alpha x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x)$.

(e) In vector notation we have $dX(t) = bdt + \sigma dW(t)$, where $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. And it follows that

$$\mathcal{A}f(x_1, x_2) = \frac{\partial f}{\partial x_1}(x_1, x_2) + \frac{1}{2}\frac{\partial^2 f}{\partial {x_2}^2}(x_1, x_2),$$

or since $X_1 = t$

$$\mathcal{A}f(t, x) = \frac{\partial f}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x).$$

Exercise 20.5 In the case of a standard call option on a stock with price process S with maturity T and exercise price K, we have that the option's price at time t and given $S(t) = s$, is

$$c(t, s) = e^{-r(T-t)} E^Q_{t,s}\left[\max\left(S(T) - K, 0\right)\right]. \tag{1}$$

where S has the following Q-dynamics

$$dS(t) = rS(t)dt + \sigma S(t)dW(t).$$

In order to compute the expectation in equation (1), recal that since S follows a GBM we get

$$S(T) = \underbrace{S(t)}_{s}\exp\left\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))\right\}.$$

So, $S(T)$ has the same distributions of the the normalized variable $\tilde{S(T)}$

$$\tilde{S}(T) = s\exp\left\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma\sqrt{T - t}\ Z\right\},$$

where $Z \sim N(0, 1)$.

And hence we have $\quad E^Q_{t,s}\left[\max\left(S(T)-K,0\right)\right] = E^Q_{t,s}\left[\max\left(\tilde{S}(T)-K,0\right)\right].$

Note also

$$E^Q_{t,s}\left[\max\left(\tilde{S}(T)-K,0\right)\right] = \underbrace{E^Q_{t,s}\left[\tilde{S}(T)|\tilde{S}(T)\geq K\right]}_{A} - K\,\underbrace{E^Q_{t,s}\left[1|\tilde{S}(T)\geq K\right]}_{B}.$$

Starting with part B we see that

$$\begin{aligned} B &= E^Q_{t,s}\left[1|\tilde{S}(T)\geq K\right] = K\;Q\left[\tilde{S}(T)\geq K\right] \\ &= K\;Q\left[Z\geq \underbrace{\frac{\ln\left(\frac{K}{s}\right)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{Z_0}\right] = K\;N\left[-Z_0\right] \end{aligned}$$

where N is the cumulative distribution function (CDF) of the standard normal distribution with density

$$\phi(z) = \frac{1}{\sqrt{2\Pi}}e^{-z^2/2}. \qquad \text{So, we have} \quad N\left[x\right] = \int_{-\infty}^{x}\phi(z)dz.$$

Taking now part A, we have

$$\begin{aligned} E^Q_{t,s}\left[\tilde{S}(T)|\tilde{S}(T)\geq K\right] &= \int_{Z_0}^{\infty} se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\;z}\phi(z)dz \\ &= \int_{Z_0}^{\infty} se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\;z}\frac{1}{\sqrt{2\Pi}}e^{-z^2/2}dz \\ &= se^{r(T-t)}\int_{Z_0}^{\infty}\frac{1}{\sqrt{2\Pi}}e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}\;z-z^2/2}dz \\ &= se^{r(T-t)}\int_{Z_0}^{\infty}\underbrace{\frac{1}{\sqrt{2\Pi}}e^{-\frac{(z-\sigma\sqrt{T-t})^2}{2}}}_{\text{density of a } N(\sigma\sqrt{T-t},1)}dz \\ &= \left\{\text{which using the CDF of a standard normal, } N\left[x\right]\text{, becomes}\right\} \\ &= se^{r(T-t)}\left(1-N\left[Z_0-\sigma\sqrt{T-t}\right]\right) \\ &= se^{r(T-t)}\left(N\left[-Z_0+\sigma\sqrt{T-t}\right]\right) \\ &= se^{r(T-t)}\left(N\left[d_1(t,T)\right]\right) \end{aligned}$$

where

$$\begin{aligned} d_1(t,T) = -Z_0+\sigma\sqrt{T-t} &= -\frac{\ln\left(\frac{K}{s}\right)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}+\sigma\sqrt{T-t} \\ &= \frac{\ln\left(\frac{s}{K}\right)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

And if also define

$$d_2(t,T) = -Z_0 = d_1(t,T) - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{s}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

we finally get the Black-Scholes formula (with d_1 and d_2 as just defined)

$$\begin{aligned} c(t,s) &= e^{-r(T-t)}\left\{se^{r(T-t)}N\left[d_1(t,T)\right] - K\ N\left[d_2(t,T)\right]\right\} \\ &= sN\left[d_1(t,T)\right] - K\ e^{-r(T-t)}N\left[d_2(t,T)\right]. \end{aligned}$$

Exercise 20.6 Take the Black-Scholes formula:

$$c(t,s) = sN\left[d_1(t,s)\right] - Ke^{-r(T-t)}N\left[d_2(t,s)\right]$$

where

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

and $d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}$

Using the hint let us first prove that

$$s\varphi(d_1) = Ke^{-r(T-t)}\varphi(d_2) \tag{2}$$

holds where φ is the density function of the $N[0,1]$ distribution. So we have $\varphi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}$ and

$$\begin{aligned} \varphi(d_2) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_1-\sigma\sqrt{T-t}\right)^2}{2}} \\ &= \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}}_{\varphi(d_1)}e^{-\frac{\sigma^2(T-t)}{2}}e^{-d_1\sigma\sqrt{T-t}} \end{aligned}$$

So it remains to show that

$$e^{-\frac{\sigma^2(T-t)}{2}}e^{-d_1\sigma\sqrt{T-t}} = \frac{K}{s}e^{-r(T-t)}$$

taking logs and rearranging we get

$$\begin{aligned} -\frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t} &= \ln\frac{K}{s}(-r(T-t)) \\ d_1 &= \frac{\ln\frac{K}{s} + (r(T-t)) - \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} \end{aligned}$$

which is true by definition of d_1 and concludes the proof.

With the relation (2) in mind we can now go for the task of calculating the *greeks* of a call option.

- *Delta*

  The *delta* measures the impact in the value of the derivative that results from price variations of the underlying, differentiating w.r.t s the Black-Scholes formula, we have

  $$\Delta_c = \frac{\partial c(t,s)}{\partial s} = N[d_1] + s\frac{\partial N[d_1]}{\partial d_1}\frac{\partial d_1(t,s)}{\partial s} - Ke^{-r(T-t)}\frac{\partial N[d_2]}{\partial d_2}\frac{\partial d_2(t,s)}{\partial s}.$$

  Note that $\frac{\partial N[d_1]}{\partial d_1} = \varphi(d_1)$ and likewise $\frac{\partial N[d_2]}{\partial d_2} = \varphi(d_2)$ and we also have that

  $$\frac{\partial d_1(t,s)}{\partial s} = \frac{\partial d_2(t,s)}{\partial s} = \frac{1}{s\sigma\sqrt{T-t}}$$

  Joining the pieces and using equation (2) we get the result:

  $$\begin{aligned}\Delta_c &= N[d_1] + s\varphi(d_1)\frac{1}{s\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\varphi(d_2)\frac{1}{s\sigma\sqrt{T-t}} \\ &= N[d_1] + \frac{1}{s\sigma\sqrt{T-t}}\underbrace{\left[s\varphi(d_1) - Ke^{-r(T-t)}\varphi(d_2)\right]}_{0}\end{aligned}$$

- *Gamma*

  The *gamma* measures the impact in the value of the *delta* that results from price variations of the underlying, differentiating w.r.t s Δ_c found above we get, immediately get the result:

  $$\frac{\partial \Delta_c}{\partial s} = \frac{\partial N[d_1]}{\partial d_1}\frac{\partial d_1(t,s)}{\partial s} = \varphi(d_1)\frac{1}{s\sigma\sqrt{T-t}}.$$

- *Rho*

  The *rho* measures the impact in the value of the derivative that results from interest rate variations, differentiating w.r.t r, the Black-Scholes formula we get

  $$\rho_c = S\varphi(d_1)\frac{\partial d_1}{\partial r} + K(T-t)e^{-r(T-t)}N[d_2] - Ke^{-r(T-t)}\varphi(d_2)\frac{\partial d_2}{\partial r}$$

  Since $\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}$ and from the relation (2) the first and third parcels on the RHS cancel the result follows

  $$\rho_c = K(T-t)e^{-r(T-t)}N[d_2].$$

- *Theta*

  The *theta* measures the impact in the value of the derivative that results from the decrease in time to maturity, differentiation w.r.t "time to maturity" is the same as differentiation w.r.t. $-T$. So, differentiating the Black-Scholes formula w.r.t. (-T) we get

  $$\Theta_c = -\left[S\varphi(d_1)\frac{\partial d_1}{\partial T} + Ke^{-r(T-t)}N[d_2] - Ke^{-r(T-t)}\varphi(d_2)\frac{\partial d_2}{\partial T}\right]$$

  Since $\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{1}{2\sqrt{T-t}}$ and using the relation (2) we get

  $$\begin{aligned}\Theta_c &= -\left[S\varphi(d_1)\frac{\partial d_1}{\partial T} + Ke^{-r(T-t)}N[d_2] - Ke^{-r(T-t)}\varphi(d_2)\left[\frac{\partial d_1}{\partial T} - \frac{1}{2\sqrt{T-t}}\right]\right] \\ &= -\frac{S\varphi(d_1}{2\sqrt{T-t}} - rKe^{-r(T-t)}N[d_2].\end{aligned}$$

- *Vega*

  The *vega* measures the impact in the value of the derivative that results from volatility variations, differentiating w.r.t σ, the Black-Scholes formula we get

  $$\nu_c = S\varphi(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}\varphi(d_2)\frac{\partial d_2}{\partial r}$$

  Since $\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r} - \sqrt{T-t}$ and from the relation (2) result follows

  $$\nu_c = S\varphi(d_1)\sqrt{T-t}.$$

Exercise 20.7 We have to show that $F_\delta(t,s) = F_0(t, se^{-\delta(T-t)})$, i.e, that the pricing function when we have a continuous dividend process δ is just the "standard" (no-dividend) price function where we replace s by $se^{-\delta(T-t)}$. To see this, let us denote S the price of a stock that *pays* a continuous dividend δ, and $\bar{S}$ the price of an otherwise equivalent stock but that does *not pay* any dividend.

$$\begin{aligned}F_0(t,s) &= e^{-r(T-t)}E^Q_{t,s}\left[\Phi(\bar{S}(T))\right] \\ &= \left\{\text{with } d\bar{S}(t) = r\bar{S}(t)dt + \sigma\bar{S}(t)dW^Q(t)\right\} \\ &= e^{-r(T-t)}E^Q_{t,s}\left[\Phi\left(\overbrace{se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t)}}^{\bar{S}(T)}\right)\right] \qquad (3)\end{aligned}$$

$$
\begin{aligned}
F_\delta(t,s) &= e^{-r(T-t)}E^Q_{t,s}\left[\Phi(S(T))\right] \\
&= \left\{\text{with } dS(t) = (r-\delta)\,S(t)dt + \sigma S(t)dW^Q(t)\right\} \\
&= e^{-r(T-t)}E^Q_{t,s}\left[\Phi\left(\overbrace{se^{(r-\delta-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}}^{S(T)}\right)\right] \qquad (4)
\end{aligned}
$$

By comparing the arguments of the function $\Phi(\cdot)$ in (3) and (4) we immediately get the result since,

$$
\begin{aligned}
F_0(t, se^{-\delta(T-t)}) &= e^{-r(T-t)}E^Q_{t,s}\left[\Phi\left(e^{-\delta(T-t)}se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}\right)\right] \\
&= e^{-r(T-t)}E^Q_{t,s}\left[\Phi\left(\overbrace{se^{(r-\delta-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t)}}^{\bar{S}(T)}\right)\right] \\
&= F_\delta(t,s)
\end{aligned}
$$

Exercise 20.8

$$
\begin{aligned}
\mathcal{X} &= \max\left[S(T)-K;0\right] \\
&= \left[S(T)-K\right].\mathbf{I}\left\{S(T)\geq K\right\}
\end{aligned}
$$

where

$$
\mathbf{I}\left\{S(T)\geq K\right\} = \begin{cases} 1 & \text{if } S(T)\geq T \\ 0 & \text{if } S(T) < K \end{cases}
$$

We obtain,

$$
\begin{aligned}
\Pi(0,\mathcal{X}) &= E^Q_t\left[e^{-\int_0^T r(s)ds}\left[S(T)-K\right].\mathbf{I}\left\{S(T)\geq K\right\}\right] \\
&= E^Q_t\left[e^{-\int_0^T r(s)ds}S(T).\mathbf{I}\left\{S(T)\geq K\right\}\right] - E^Q_t\left[e^{-\int_0^T r(s)ds}K.\mathbf{I}\left\{S(T)\geq K\right\}\right] \\
&= \left\{\text{we change the measure on the }\underline{\text{first term}}\text{ to } Q^S\text{, where } S \text{ is the numeraire}\right\} \\
&\quad \left\{\text{for the }\underline{\text{second term}}\text{ we change measure but take the } T\text{-zero-coupon bond as the numeraire}\right\} \\
&= S(0)Q^S(S(T)\geq K) - Kp(0,T)Q^T(S(T)\geq K)
\end{aligned}
$$

Take first the <u>second term</u>. We have

$$
Q^T(S(T)\geq K) = Q^T\left(\underbrace{\frac{S(T)}{p(T,T)}}_{Z_{S,T}(T)}\geq K\right) = Q^T\left(Z_{S,T}(T)\geq K\right)
$$

But, under the T-forward measure we have

$$dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW^T$$

This is a basic GBM hence,

$$Z_{S,T}(T) = Z_{S,T}(0)\exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{S,T}\|^2(t)dt + \int_0^T \sigma_{S,T}(t)dW^T\right\}.$$

Which means that

$$\begin{aligned}
Q^T\left(Z_{S,T} \geq K\right) &= Q^T\left(\ln(Z_{S,T}(T)) \geq \ln(K)\right) \\
&= \left\{\ln(Z_{S,T}(T))|\mathcal{F}_t \sim N\left(\underbrace{\ln(Z_{S,T}(0)) - \frac{1}{2}\Sigma^2_{S,T}}_{\mu}, \Sigma^2_{S,T}\right) \quad \Sigma^2_{S,T} = \int_0^T \|\sigma_{S,T}\|^2(t)dt\right\} \\
&= 1 - N\left[\frac{\ln K - \mu}{\sqrt{\Sigma^2_{S,T)}}}\right] \\
&= N\left[\underbrace{\frac{-\ln K + \mu}{\sqrt{\Sigma^2_{S,T}}}}_{d_2}\right]
\end{aligned}$$

For the <u>first term</u>, we have

$$Q^S(S(T) \geq K) = Q^S\left(\frac{1}{S(T)} \leq \frac{1}{K}\right) = Q^S\left(\underbrace{\frac{p(T,T)}{S(T)}}_{Y_{S,T}(T)} \leq \frac{1}{K}\right) = Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right)$$

That is $Y_{S,T}(t) = \frac{1}{Z_{S,T}}(t)$. Since the process $Y_{S,T}$ is a normalized variable when S is the numeraire we have, under Q^S

$$dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}dW^S$$

and from it is easy to show (why?) that $\delta_{S,T} = -\sigma_{S,T}$, thus we have

$$Y_{S,T}(T) = Y_{S,T}(0)\exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{S,T}\|^2(t)dt - \int_0^T \sigma_{S,T}(t)dW^S\right\}.$$

So,

$$
\begin{aligned}
Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right) &= Q^S\left(\ln(Y_{S,T}(T)) \leq \ln(\frac{1}{K})\right) \\
&= \left\{ \ln(Y_{S,T}(T))|\mathcal{F}_t \sim N\left(\underbrace{\ln(Y_{S,T}(0)) - \frac{1}{2}\Sigma^2_{S,T}}_{\alpha}, \Sigma^2_{S,T}\right) \quad \Sigma^2_{S,T} = \int_0^T \|\sigma_{S,T}\|^2(t)dt \right\} \\
&= N\left[\frac{-\ln K - \alpha}{\sqrt{\Sigma^2_{S,T}}}\right] \\
&= \{\text{note that } \alpha = -\mu - \Sigma_{S,T}\} \\
&= N\left[\underbrace{\frac{-\ln K + \mu + \Sigma^2_{S,T}}{\sqrt{\Sigma^2_{S,T}}}}_{d_1}\right]
\end{aligned}
$$

and we see that actually we have $d_1 = d_2 + \sqrt{\Sigma^2_{S,T}}$.

The price of a call option in a stochastic interest rate world is thus,

$$\Pi(0) = S(0)N[d_1]\,Kp(0,T)N[d_2]$$

with d_1 and d_2 as defined above!

21 Exams

21.1 Exercises

21.1.1 March 26, 2003

Exercise 21.1 Let Y be given as the solution to the following SDE.

$$dX = \alpha X(t)dt + \sigma X^{\gamma} dW$$

here α, γ and σ are deterministic constants.

(a) Compute $E[X(t)]$.

(b) Compute $Var[X(t)]$ for the case when $\gamma = 1$.
Hint: Start by computing $E\left[X^2(t)\right]$

Exercise 21.2 The well known company *Cathy and Heathcliff INC.* are trading the new derivative, "the Fraction", which is defined by two fixed dates T_0 and T_1 with $T_0 < T_1$. The holder of a *Fraction* contract will, at the date T_1 obtain the amount

$$X = \frac{S(T_1)}{S(T_0)}$$

(a) Determine the arbitrage free price, at time $t < T_0$ of the *Fraction* contract. You live in a standard Black-Scholes world with short rate r.

(b) Try to construct the replicating portfolio for the *Fraction* contract. (It can be done almost by inspection).

Exercise 21.3 Consider the following SDE under the objective probability measure P.

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

where $\mu(x)$ and $\sigma(x)$ are deterministic given functions. We interpret X as a *non priced* underlying object, and now we want to price derivatives defined in terms of X. One possibility is to interpret X_t as the temperature (at time t) at a particular beach at Tylösand. Now consider the contingent claim

$$Z = \Phi(X_T)$$

where $\Phi(x)$ is some given deterministic function. In our interpetation this could be interpreted as a holidy insurance, which gives the holder an amount of money if the temperature is below some benchmark value.

(a) Is it possible to obtain a unique arbitrage free price process $\Pi(t; Z)$ for the derivative above? You must motivate your answer.

(b) Use the standard ideas from interest rate theory to analyze the situation above as completely as possible. In particular you should come up with a PDE for pricing of derivatives. Indicate clearly which objects in the PDE that are known to you, and how it in priciple would be possible to obtain further necessary information.

Exercise 21.4 Consider a standard Black-Scholes model under the objective probability measure P:

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW_t \\ dB(t) &= rB(t)dt. \end{aligned}$$

Consider two fixed points in time, T_1 and T_2 with $T_1 < T_2$ and define the T_2-claim X by

$$X = \max\left[S_{T_1}, S_{T_2}\right].$$

Compute the arbitrage free price $\Pi(t; X)$ of X at time t where $t \leq T_1$. You are allowed to use the standard Black-Scholes formula without proof (see below).

Exercise 21.5 Consider a standard Black-Scholes model. Show how it is possible to hedge (replicate) a given contingent claim X of the form $X = \Phi(S(T))$.

Black-Scholes formula:

$$F(t, s) = sN\left[d_1(t, s)\right] - e^{-r(T-t)}KN\left[d_2(t, s)\right].$$

Here N is the cumulative distribution function for the $N[0, 1]$ distribution and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

21.1.2 January 10, 10.00-15.00, 2003

All notation should be clearly defined.
Arguments should be complete and careful.
Each problem below will give you a maximum of 20 points.

Exercise 21.6 Let X and Y be given as solutions of the following system of SDEs

$$\begin{aligned} dX &= \alpha X dt + \sigma X dW, \quad X_0 = x_0 \\ dY &= \beta Y dt + \gamma Y dW, \quad Y_0 = y_0 \end{aligned}$$

Note that both equations are driven by the same **scalar** W.

(a) Define Z by $Z_t = X_t Y_t$. It turns out that Z satisfies an SDE itself. Which?

(b) Compute $E[Z_t]$.

Exercise 21.7 The rather unknown company *William S. INC* has blessed the market with the new derivative, "As You Like It" (AYLI). The AYLI is defined in terms of an underlying stock with price process S, and it is specified by two dates, T_0 and T_1 with $T_0 < T_1$ and an exercise price K. The holder of the AYLI may at time T_0 choose th have *either* a call option on S with strike price K and exercise date T_1 *or* a put option on S with strike price K and exercise date T_1. Derive a formula, for the price at time $t < T_0$ for the AYLI. The stock price is assumed to follow standard GBM

$$dS = \alpha S dt + \sigma S dW,$$

and the short rate r is assumed to be constant. If, for some reason, you want to use the Black-Scholes formula for a call option, you may do so without proof.

Exercise 21.8 Consider a bond market with bond prices as usual denoted by $p(t, T)$.

(a) Define the forward rates $f(t, T)$ in terms of bond prices, and derive a formula expressing bond prices in terms of forward rates.

(b) Assume that the forward rates under Q have the dynamics

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

and derive the relation which must hold between α and σ.

Exercise 21.9 Specify the assumptions in the Black-Scholes model, and derive the Black-Scholes partial differential equation for the price of a simple claim of the form $X = \Phi(S_T)$.

Exercise 21.10 Consider the following model for two stock prices (without dividends)

$$\begin{aligned} dS_1 &= \alpha_1 S_1 dt + S_1 \sigma_1 dW, \\ dS_2 &= \alpha_2 S_2 dt + S_2 \sigma_2 dW. \end{aligned}$$

Here α_1 and α_2 are known real numbers, whereas σ_1 and σ_2 are two-dimensional constant row vectors. W is a standard two dimensional Wiener process. The short rate is assumed to be constant. The models is assumed to be arbitrage free. Consider the T-claim X defined by the following points:

- X is specified in terms of the underlying stocks , a strike price K, and an exercise date T.
- The holder of the claim will obtain nothing if $S_2(T) \leq K$.
- If $S_2(T) > K$ then the holder of the claim will obtain $S_2(T) - K$ shares (i.e. units) of asset Number 1.

Compute the price $\Pi(t; X)$.

21.1.3 March 18, 2002

All notation should be clearly defined. Arguments should be complete and careful.

Exercise 21.11 Consider the following boundary value problem in the domain $[0, T] \times R$ for an unknown function $F(t, x)$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x)F(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

Here μ, σ, k and Φ are assumed to be known functions.

Derive a Feynman-Kač representation formula for this problem. In this formula it must be quite clear exactly at which points the various functions should be evaluated. In other words - if you suppress variables you must explain **very** clearly exactly what you mean.

If you think this problem is hard you may (with loss of 10 points) assume that $k(t, x) = 0$. ....(40p)

Exercise 21.12 Assume that the short rate under the objective measure P has the dynamics

$$dr = \mu(t,r)dt + \sigma(t,r)dW(t)$$

Derive the "term structure equation" for the determination of bond prices. (40p)

Exercise 21.13 Consider a standard Black-Scholes model under the objective probability measure P:

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW_t \\ dB(t) &= rB(t)dt. \end{aligned}$$

We want to price a European call option with exercise price K and exercise date T on the underlying stock. Let us also assume that the stock pays a continuous dividend yield δ, i.e. that the cumulative dividend stream has the structure

$$dD_t = \delta S_t dt$$

where $\delta > 0$.

(a) Discuss verbally what relation you would expect to hold between the option price in the case of a dividend yield compared to the option price in the case when the dividend yield is zero. (Which would be greater and why?) (5p)

(b) Derive a pricing formula (as explicit as possible) for the call option in the case of a nonzero dividend yield. You are allowed to use and refer to the standard Black-Scholes formula without proof (see below). ....(35p)

Exercise 21.14 Consider two markets: a domestic market with short rate r_d, and a foreign market with short rate r_f. The exchange rate X is defined as the domestic price of one unit of the foreign currency.

We take as given the following dynamics (under the objective probability measure P)

$$\begin{aligned} dX &= X\alpha_X dt + X\sigma_X dW, && (5) \\ dB_d &= r_d B_d dt, && (6) \\ dB_f &= r_f B_f dt, && (7) \end{aligned}$$

where r_d, r_f,α_X, σ_X are deterministic constants, and W is a scalar Wiener process.

We want to price (in domestic terms) a European Call Option on one unit of the foreign currency, with strike price K and exercise date T. The contract Y is thus given by

$$Y = max[X(T) - K, 0]$$

Derive an explicit pricing formula for the option. The standard Black-Scholes formula for stock options (see below) can be used without further motivation. (40p)

Exercise 21.15 Consider a standard Black-Scholes model under the objective probability measure P:

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW_t \\ dB(t) &= rB(t)dt. \end{aligned}$$

Consider two fixed points in time, T_1 and T_2 with $T_1 < T_2$ and define the T_2-claim X by

$$X = \max\left[S_{T_1}, S_{T_2}\right].$$

Compute the arbitrage free price $\Pi(t;X)$ of X at time t where $t \le T_1$. You are allowed to use the standard Black-Scholes formula without proof (see below). (40p)

Black-Scholes formula:

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

21.2 Topics of solutions

21.2.1 March 26, 2003

Exercise 21.1

(a) Using standard technique and using the notation $m_t = E[X_t]$ we obtain

$$\begin{aligned} \frac{dm_t}{dt} &= \alpha m_t, \\ m_0 &= X_0 \end{aligned}$$

and we get

$$E[X_t] = m_t = X_0 e^{\alpha t}.$$

(b) There are several ways to solve this, but an easy one is to compute dX^2 as

$$dX^2 = X^2\left(2\alpha + \sigma^2\right)dt + 2\sigma X^2 dW$$

This gives us

$$E\left[X_t^2\right] = X_0^2 e^{(2\alpha+\sigma^2)t}$$

so

$$Var[X_t] = X_0^2 e^{(2\alpha+\sigma^2)t} - \left(X_0 e^{\alpha t}\right)^2 = X_0^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$$

Exercise 21.2

(a) Since we have a standard Black-Scholes model it is easily seen that

$$X = e^{(r-\frac{1}{2}\sigma^2)(T_1-T_0)+\sigma[W(T_1)-W(T_0)]},$$

The price is given as usual by

$$\Pi(t; X) = e^{-r(T_1-t)} E^Q_{t,s}[X]$$

and from the formula for X above we immediately have

$$E^Q_{t,s}[X] = e^{r(T_1-T_0)}$$

so the answer is

$$\Pi(t; X) = e^{-r(T_0-t)}.$$

(b) From (a) we know that $\Pi(0; X) = e^{-rT_0}$, and the replicating portfolio is as follows.

- At $t = 0$ put e^{-rT_0} SEK into the bank.
- At $t = T_0$ we have 1 SEK in the bank. For this amount we can buy $1/S(T_0)$ shares.
- At $t = T_1$ each share is worth $S(T_1)$ SEK so the value of the portfolio is $S(T_1)/S(T_0)$ SEK.

Exercise 21.3 See the textbook, Chapter 10.

Exercise 21.4 The hard part about this problem is that S_{T_1} is **not** the price of a traded asset at the time T_2. Therefore we cannot immediately apply the change of numeraire technique. In order to transform our problem to the standard setting we therefore introduce a self financing portfolio with value process V. This portfolio is defined as follows, for a fixed $t < T_1$:

- Start at t with one share of the stock.
- Hold this portfolio until T_1. At T_1 sell it, and put all the money in the bank.
- Keep the money in the bank until T_2.

The point is that V can be regarded as the price of a traded asset. An easy calculation shows that $V_{T_2} = e^{r\Delta} S_{T_1}$ where $\Delta = T_2 - T_1$. Thus we have $S_{T_1} = e^{-r\Delta} V_{T_2}$ and we can write

$$X = \max\left[e^{-r\Delta} V_{T_2}, S_{T_2}\right]$$

Using V as the numeraire we now obtain (see Ch. 19 for details)

$$\Pi(t; X) = V_t \cdot E^V_{t,s}\left[\max\left\{Z_{T_2}, e^{-r\Delta}\right\}\right]$$

where $Z_u = \frac{S_u}{V_u}$ and E^V denotes the martingale measure with V as the numeraire. An easy calculation gives us

$$\Pi(t; X) = V_t e^{-r\Delta} + V_t \cdot E^V_{t,s}\left[\max\left\{Z_{T_2} - e^{-r\Delta}, 0\right\}\right]$$

and in the last expectation we recognize a European call on Z with strike $e^{-r\Delta}$. The normailzed price system has (as usual) zero interest rate, so this option can be valued using the BS formula and we only have to compute the volatility σ_Z of Z. This volatility is time dependent, and from the definition of V it follows immediately that

$$\sigma_Z(u) = \begin{cases} 0 & \text{for } t \le u \le T_1 \\ \sigma & \text{for } T_1 < u \le T_2 \end{cases}$$

where σ is the original stock price volatility. Thus the integrated volatility to plug into the BS formula is

$$\hat{\sigma} = \int_t^{T_2} \sigma_Z(u) du = \sigma \cdot \Delta$$

Since, by construction, $V_t = S_t = s$, and $Z_t = 1$ we have the final answer (remember the zero interest rate)

$$\Pi(t; X) = s e^{-r\Delta} + s\left\{N[d_1] - e^{-r\Delta} N[d_2]\right\}$$

$$\begin{aligned} d_1(t) &= \frac{1}{\hat{\sigma}\sqrt{T_2 - t}}\left\{\ln\left(\frac{1}{K}\right) + \frac{1}{2}\hat{\sigma}^2(T_2 - t)\right\}, \\ d_2(t) &= d_1(t) - \hat{\sigma}\sqrt{T_2 - t}. \end{aligned}$$

Exercise 21.5 See the textbook, Chapter 7.

21.2.2 January 10, 2003

Exercise 21.6 With $f(x,y) = xy$ we apply the Ito formula

$$df = f_x dX + f_y dY + \frac{1}{2}\left\{ f_{xx}(dX)^2 + f_{yy}(dY)^2 + f_{xy} dX dY + f_{yx} dY dX \right\}$$

to obtain

$$dZ = Z\left(\alpha + \beta + \sigma\gamma\right)dt + Z\left(\sigma + \gamma\right)dW$$

Standard technique shows that $m_t = E[Z_t]$ satisfies

$$\frac{dm_t}{dt} = m_t\left(\alpha + \beta + \sigma\gamma\right)$$

so we obtain

$$E[Z_t] = x_0 y_0 e^{(\alpha+\beta+\sigma\gamma)t}$$

Exercise 21.7 The AYLI is a T_0-claim X defined by

$$X = \max\left[c(T_0; K, T_1), p(T_0; K, T_1)\right]$$

where $c(t; K, T)$ and $p(t; K, T)$ denotes the prices at time t of a call (put) with strike K and exercise date T on the given underlying. We now have put-call-parity

$$p(T_0; K, T_1) = c(T_0; K, T_1) - S_{T_0} + Ke^{-r(T_1 - T_0)}$$

so we obtain

$$\begin{aligned} X &= \max\left[c(T_0; K, T_1), c(T_0; K, T_1) - S_{T_0} + Ke^{-r(T_1-T_0)}\right] \\ &= c(T_0; K, T_1) + \max\left[Ke^{-r(T_1-T_0)} - S_{T_0}, 0\right] \end{aligned}$$

The last term is a put on S with exercise date T_0 and strike price $Ke^{-r(T_1-T_0)}$. The AYLI is thus equivalent to a sum of a put and a call (with different strikes and exercise dates) and we get the price

$$\Pi(t) = c(t; K, T_1) + p(t; Ke^{-r(T_1-T_0)}, T_0)$$

which, using put call parity and simplifying, can also be written

$$\begin{aligned} \Pi(t) &= c(t; K, T_1) + c(t; Ke^{-r(T_1-T_0)}, T_0) - S_t + Ke^{-r(T_1-t)} \\ &= \left\{S_t N[d_1] - Ke^{-r(T_1-t)} N[d_2]\right\} + \left\{S_t N[\tilde{d}_1] - Ke^{-r(T_1-T_0)} e^{-r(T_0-t)} N[\tilde{d}_2]\right\} \\ &\quad - S_t + Ke^{-r(T_1-t)} \\ &= S_t\left(N[d_1] + N[\tilde{d}_1] - 1\right) - Ke^{-r(T_1-t)}\left(N[d_2] + N[\tilde{d}_2] - 1\right) \end{aligned}$$

where

$$\begin{aligned}
d_1(t) &= \frac{1}{\sigma\sqrt{T_1 - t}}\left\{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2(T_1 - t)\right)\right\}, \\
d_2(t) &= d_1(t) - \hat{\sigma}\sqrt{T_1 - t} \\
\tilde{d}_1(t) &= \frac{1}{\sigma\sqrt{T_0 - t}}\left\{\ln\left(\frac{S_t}{Ke^{-r(T_1 - T_0)}}\right) + \left(r + \frac{1}{2}\sigma^2(T_0 - t)\right)\right\}, \\
\tilde{d}_2(t) &= \tilde{d}_1(t) - \hat{\sigma}\sqrt{T_0 - t}
\end{aligned}$$

Exercise 21.8 See textbook.

(a) Chapter 15

$$\begin{aligned}
f(t,T) &= -\frac{\partial \ln p(t,T)}{\partial T}, \\
p(t,T) &= e^{-\int_t^T f(t,s)ds}
\end{aligned}$$

(b) Chapter 18

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

Exercise 21.9 See the textbook, Chapter 6. Remember all institutional assumptions about no bid-ask spread, no tax effects, no transaction costs, liquid market for the stock and option etc.

Exercise 21.10 The T-claim X to be priced is easily seen to be given by

$$X = S_1(T)\max\{S_2(T) - K, 0\}$$

This is an obvious case of using S_1 as numeraire, with martingale measure denoted by Q^1 and expectation denoted by E^1. We obtain

$$\Pi(t;X) = S_1(t)E^1\left[\max\{S_2(T) - K, 0\}\middle|\mathcal{F}_t\right]$$

From general theory we know that the Q^1 dynamics of S_2 are

$$dS_2(t) = (r + \sigma_1\sigma_2)S_2(t)dt + S_2(t)\sigma_2 dW^1(t)$$

where $\sigma_1\sigma_2$ denotes inner product and W^1 is Q^1-Wiener. Solving this GBM it is easy to see that

$$S_2(T) = e^{(r+\sigma_1\sigma_2)(T-t)}S_2(t).$$

The expectation above is thus the price of a call on S with volatility $\|\sigma_2\|$ strike price K in a world with zero short rate. The total price is thus given by

$$\Pi\left(t;X\right)=S_1(t)\cdot\mathbf{c}\left(e^{(r+\sigma_1\sigma_2)(T-t)}S_2(t),\ t,\ K,\ \|\sigma_2\|,\ \underbrace{0}_{\text{short rate}},\ T\right)$$

where, with obvious notation, $\mathbf{c}(s,t,K,\sigma,r,T)$ as the Black-Scholes call option formula.

21.2.3 March 18, 2002

Exercise 21.11 The solution F to the PDE is given by

$$F(t,x)=E_{t,x}\left[e^{\int_t^T k(s,X_s)ds}\Phi(X_T)\right]$$

where

$$\begin{aligned}dX_t &= \mu(s,X_s)ds+\sigma(s,X_s)dW_s,\\ X_t &= x.\end{aligned}$$

This is most easily proved by defining X as above and showing that the process

$$Z_s=e^{\int_t^s k(u,X_u)du}F(s,X_s)$$

is a Q-martingale, and, thus

$$Z_t=E_t^Q\left[Z_T\right]\quad\Leftrightarrow\quad F(t,x)=E_{t,x}\left[e^{\int_t^T k(s,X_s)ds}\Phi(X_T)\right].$$

Alternatively, it can be solved like Exercise 4.10 was solved in the Seminars (check solutions given).

Exercise 21.12 The term structure equation has the form

$$\begin{aligned}F_t^T+\{\mu-\lambda\sigma\}F_r^T+\frac{1}{2}\sigma^2F_{rr}^T-rF^T &= 0,\\ F^T(T,T)=1\end{aligned}$$

See Chapter 16 in the textbook for details and discussion.

Exercise 21.13

(a) The stock price of a dividend paying stock should be lower than the corresponding price for a non dividend paying stock (see Chapter 11 for details).

Thus the price of call option should also be lower, recall that the delta of a call is always positive (chapter 8).

(b) The general formula is (see Ch. 11 and extra exercise on Seminar 4)

$$F^{\delta}(t,s) = F^{0}(t, se^{-\delta(T-t)})$$

Applied to the standard BS formula we obain

$$F(t,s) = se^{-\delta(T-t)}N\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

Exercise 21.14 See Chapter 12 in the textbook for the arguments. The final answer is

$$F(t,x) = xe^{-r_f(T-t)}N\left[d_1(t,x)\right] - e^{-r_d(T-t)}KN\left[d_2(t,x)\right].$$

$$\begin{aligned} d_1(t,x) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,x) &= d_1(t,x) - \sigma\sqrt{T-t}. \end{aligned}$$

Exercise 21.15 The same as Exercise 4 in the exam from March 2003.