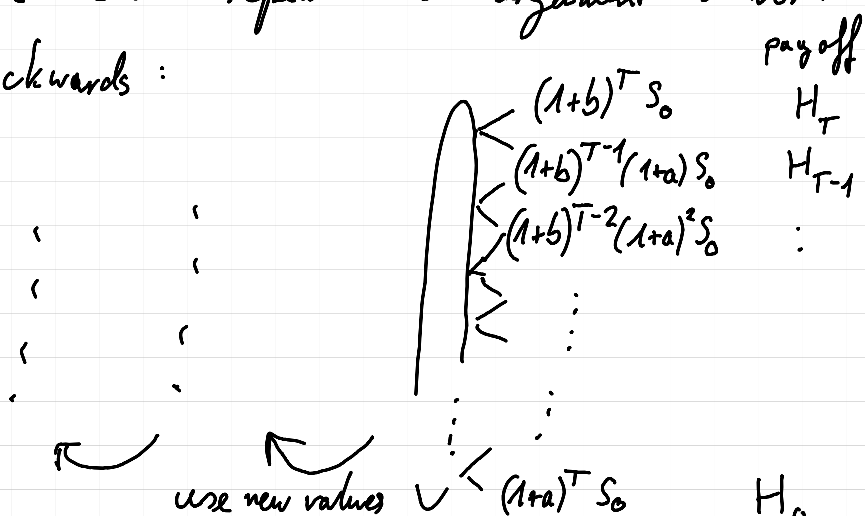
Recap: In the binomial model for one time step,

$$\begin{tikzcd}[row sep=small] & (1+a)S_0 \quad \text{payoff } H_a \\ S_0 \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\ & (1+b)S_0 \quad \text{payoff } H_b \end{tikzcd}$$

The value of a portfolio that replicates the payoff is $q\beta H_a + (1-q)\beta H_b$ at time 0, where β is the discounting rate/factor. $\frac{1}{1+r}$

and q is s.t. $\mathbb{E}(\beta S_1 \mid S_0) = S_0$

For a scenario with several time steps, we can repeat the argument & work backwards:



use new values as new payoffs to calculate $T-2$

Calculate at time $T-1$ with 1 time step argument.

After repeating this argument we get that the fair price at time 0 is $\mathbb{E}(\beta^T \text{payoff})$ where expectation is taken according to probabilities $q = \frac{b-r}{b-a}$, $1-q = \frac{r-a}{b-a}$ for factors $1+a$, $1+b$ respectively. They are chosen s.t. $\beta^n S_n$ is a martingale:

$$\mathbb{E}(\beta^n S_n \mid \beta^{n-1} S_{n-1}) = \beta^{n-1} S_{n-1}$$

The probability that we end with asset price $(1+b)^{T-k}(1+a)^k S_0$ is $\binom{T}{k}(1-q)^{T-k} q^k$.

Let $H(x)$ be the payoff if the asset price is x. Then,

$$\mathbb{E}(\beta^T \cdot \text{payoff}) = \beta^T \sum_{k=0}^{T} \binom{T}{k} (1-q)^{T-k} q^k H\left((1+b)^{T-k}(1+a)^k S_0\right)$$

For a European calloption, $H(x) = (x-K)^+$.

The fair price under the binomial model is:

$$\beta^T \sum_{k=0}^{T} \binom{T}{k} (1-q)^{T-k} q^k \left((1+b)^{T-k}(1+a)^k S_0 - K\right)^+$$

[Cox-Ross-Rubinstein formula]

Some General Bounds

- European & American options:

  Let $C_0(E)$, $C_0(A)$ price for a European / American call option with same parameters T, K.

  We have $0 \leq C_0(E) \leq C_0(A)$ since if $C_0(E) > C_0(A)$ buy American option, sell European option and gain difference.

- Call-put parity:

$$\underbrace{C_0(E)}_{\text{price for call option}} - \underbrace{P_0(E)}_{\text{price for put option}} = S_0 - \beta^T K$$

and so $C_0 \geq S_0 - \beta^T K \geq S_0 - K$ (assuming $\beta \leq 1$)

- We have $C_0(A) \geq C_0(E) \geq S_0 - K$. By the same argument $C_t(A) \geq S_t - K \;\; \forall\, 0 \leq t \leq T$.

  Hence $C_t(A)$ is always at least the current payoff $(S_t - K)^+$.

$\Rightarrow$ It is always better to keep option than to use it.

with a "perfect" strategy an American call option is only used at time T.

Thus $C_0(A) = C_0(E)$.

An American call option on a stock without dividents and with non-negative interest r has the same fair price as a European option.

General Discrete Models

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modelling the underlying market.
- Filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}$ modelling time and information
- price process : vector $S = (S^0, S^1, \dots, S^d)$ where
  - $\rightarrow$ S_t^0 is the risk-free (deterministic) investment ("cash in bank")
  - $\rightarrow$ S_t^i is the price of asset i at time t.
  - $\rightarrow$ We assume S_t^i is adapted to $\mathcal{F}_n$.
  - $\rightarrow$ At least one of S_t^i is strictly positive.

- Discounting factor $\beta_t = \frac{1}{S_t^0}$

Trading Strategies:

Portfolio at time t: vector $(\theta_t^0, .., \theta_t^d)$ describes how much we have of each asset.

θ_t^i is assumed to be pre-visible ($\tilde{F}_{t-1}$-meas.)

The value at time t is

$$V_t(\theta) = \theta_t \cdot S_t = \sum_{i=0}^{d} \theta_t^i S_t^i$$

A strategy is called self-financing if there are no withdrawals or additional funds.

$$\theta_{t+1} \cdot S_t = \theta_t \cdot S_t$$

Equivalently,

$$\begin{aligned} \Delta V_t(\theta) &= V_t(\theta) - V_{t-1}(\theta) \\ &= \theta_t \cdot S_t - \theta_{t-1} \cdot S_{t-1} \\ &= \theta_t \cdot S_t - \theta_t \cdot S_{t-1} \\ &= \theta_t \cdot (S_t - S_{t-1}) \\ &= \theta_t \cdot \Delta S_t \end{aligned}$$

- The **gains process** is defined by

$$G_0(\theta) = 0$$

$$G_t(\theta) = V_t(\theta) - V_0(\theta)$$

- To make prices / values at different times comparable, we define the **discounted version** of a random variable X_t at time t by

$$\overline{X}_t = \beta_t X_t = X_t / S_t^0 .$$

Discounting is always indicated by a bar.

- A portfolio is self-financing if and only if

$$\Delta \theta_t \cdot \overline{S}_{t-1} = (\theta_t - \theta_{t-1}) \cdot \overline{S}_{t-1}$$
$$= (\theta_t - \theta_{t-1}) \cdot \beta_{t-1} S_{t-1} = 0$$

for all t.

→ It is always possible to make portfolios self-financing by only changing θ_t^0 (amount in bank): this is solving a linear equation for θ_t^0.

A strategy is called admissible if
$V_t(\theta) \geq 0$ for all $t \geq 0$.

Suppose there was an admissible strategy such that:

$$V_0(\theta) = 0, \quad V_t(\theta) \geq 0 \quad \forall t, \quad \mathbb{E}(V_T(\theta)) > 0$$

This would constitute an arbitrage opportunity!
In a viable (arbitrage free) model, there are no such opportunities.

The following is called "weak" arbitrage

$$V_0(\theta) = 0, \quad V_T(\theta) \geq 0 \text{ at time } \underline{T}, \; \mathbb{E}(V_T(\theta)) > 0$$

Clearly arbitrage $\Rightarrow$ weak arbitrage
but we will also see that

weak arbitrage $\Rightarrow$ arbitrage.