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Department of Mathematics
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Exam in Mathematical Statistics
Inference Theory II, 1MS037
2021–01–14

Time: 15.00-20.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Consider the random variable

$$X = \begin{cases} 0, & \text{with probability } 4\theta_1\theta_2, \\ 1, & \text{with probability } \theta_1^2, \\ 2, & \text{with probability } 4\theta_2^2, \end{cases}$$

   where $\theta_1 + 2\theta_2 = 1$. Suppose that we have an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X.

   Does the distribution belong to a strictly k-parametric family? In that case, determine k, the natural parameters(s) and the sufficient statistic(s). (5p)

2. A continuous random variable X is said to be Weibull distributed with parameters $\gamma > 0$ and $\beta > 0$ if it has density function

$$f(x) = \frac{\gamma}{\beta} x^{\gamma-1} \exp\left(-\frac{x^\gamma}{\beta}\right),$$

   for $x \geq 0$, and 0 otherwise.

   Suppose that we have an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X.

   You may without proof use that $E(X) = 3\sqrt{\pi\beta}/2$, $E(X^2) = \beta$, $E(X^4) = 2\beta^2$.

   (a) Assume that γ is fixed. Give a sufficient statistic for β. (2p)

   (b) Suppose that $\gamma = 2$. Consider the estimator

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

   Show that this is an unbiased estimator of β. (1p)

   (c) Is the estimator in (b) efficient for β? Motivate your answer. (3p)

3. Let X be a continuous random variable which is uniform on $(\theta, \theta + 1)$, where $-\infty < \theta < \infty$. Suppose that we have an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X.

   (a) Show that $(\min_i X_i, \max_i X_i)$ is a sufficient statistic for θ. (2p)

   (b) Show that $(\min_i X_i, \max_i X_i)$ is a *minimal* sufficient statistic for θ. (3p)

4. Suppose that X is Bernoulli distributed with parameter p, i.e. that $P(X = 1) = p = 1 - P(X = 0)$. Suppose that we have an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X.

   (a) Show that X_1 is an unbiased estimator of p. (1p)

   (b) Show that $T = \sum_{i=1}^{n} X_i$ is sufficient for p. (1p)

   (c) Use the Rao-Blackwell theorem to find an unbiased estimator of p with smaller variance that X_1. (2p)

   Hint: $\binom{n}{t} = \frac{n}{t}\binom{n-1}{t-1}$.

   (d) Is your estimator in (c) the best unbiased estimator (BUE) of p? Motivate your answer! (2p)

5. Consider testing that the observation x comes from a discrete distribution with probability function $p_0(x)$ vs the alternative that it comes from a discrete distribution with probability function $p_1(x)$, where these two probability functions are given in the following table:

x	1	2	3	4	5
$p_0(x)$	0.10	0.10	0.04	0.30	0.46
$p_1(x)$	0.10	0.30	0.20	0.15	0.25

   (a) Which is the most powerful (MP) test at level $\alpha = 0.05$? (2p)

   (b) Calculate the size of the type II error and the power for the MP test.(2p)

   (c) Calculate sizes of the errors of type I and II as well as the power for the test that rejects with probability 0.5 if $x = 2$, and otherwise does not reject. Compare to the power for the MP test. (2p)

6. Let X be normally distributed with expectation 0 and variance σ^2, and suppose that we have an independent sample $\mathbf{X} = (X_1, ..., X_n)$ where all X_i are distributed as X. The corresponding observations are $\mathbf{x} = (x_1, ..., x_n)$.

   (a) Consider testing H_0: $\sigma^2 \leq 1$ vs H_1: $\sigma^2 > 1$. Let $\chi^2_\alpha(n)$ be such that $P\{Y > \chi^2_\alpha(n)\} = \alpha$ for $Y \sim \chi^2(n)$.
   Let
   $$T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2.$$
   Show that the test that rejects H_0 if.f. $T(\mathbf{x}) > \chi^2_\alpha(n)$ is a UMP (uniformly most powerful) size α test for this situation. (3p)

   (b) Show that as $\sigma^2 \to 0$, the probability of rejecting H_0 in (a) tends to 0. (1p)

   Hint: You may use without proof that $\sum_i^n X_i^2/\sigma^2$ is $\chi^2(n)$.

   (c) Now, consider testing H_0: $\sigma^2 = 1$ vs H_1: $\sigma^2 \neq 1$. Is the test in (a) an unbiased size α test for this situation? Why or why not? (2p)

7. Suppose that we have a sample $\mathbf{X} = (X_1, X_2)$, where for $i = 1, 2$, X_i is Poisson with parameter μ_i. Moreover, suppose that $\mu_1 = \psi\mu_2$. Hence, the parameters of our model are ψ and μ_2.

   (a) Show that X_1 is sufficient for ψ and that
   $T = X_1 + X_2$ is sufficient for μ_2. (2p)

   (b) Show that $X_1 | T = t$ is Binomial with parameters t and $\psi/(1 + \psi)$. (1p)

   (c) Suppose that we want to test H_0: $\psi \geq 1$ vs H_1: $\psi < 1$, and that we have the observations $x_1 = 0$ and $x_2 = 4$.
   Does the UMP α similar test reject H_0 at the 10% level? (3p)

GOOD LUCK!