Analysis of Time Series, L5

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Today

- 3.3: Difference equations (continued)
- 3.4:
  - ACF
  - PACF
  - Summary and examples
- Menti

Difference equations

Order one:

- Solve $u_n - \alpha u_{n-1} = 0$, $u_0 = c$.
- Recursion yields $u_n = \alpha^n c$.
- Equivalently, $\alpha(B)u_n = (1 - \alpha B)u_n = 0$, $u_0 = c$
  is solved by $u_n = (z_0^{-1})^n c$
  where $z_0 = \alpha^{-1}$ is a root of $\alpha(z) = 1 - \alpha z$.

Difference equations

Order two:

- Solve

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0 \tag{1}$$

- Equivalently, write $\alpha(B)u_n = (1 - \alpha_1 B - \alpha_2 B^2)u_n = 0$.
- Denote the roots of $\alpha(z)$ by z_1 and z_2.
- If $z_1 \neq z_2$, the general solution to (1) is

$$u_n = c_1 z_1^{-n} + c_2 z_2^{-n}.$$

- If $z_1 = z_2$, the general solution to (1) is

$$u_n = z_1^{-n}(c_1 + c_2 n).$$

Difference equations

Example 2: AR(2) (causal)

Let

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t,$$

i.e $\phi(B)x_t = w_t$, $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$.

1. Show that the autocorrelation function satisfies, for $h \geq 1$,

$$\rho(h) - \phi_1 \rho(h-1) - \phi_2 \rho(h-2) = 0.$$

2. Denote the roots of $\phi(z)$ by z_1 and z_2. Find an expression for $\rho(h)$ when
   a) $z_1 \neq z_2$ and real
   b) $z_1 = z_2$
   c) $z_1 = \bar{z}_2$ is a complex conjugate pair

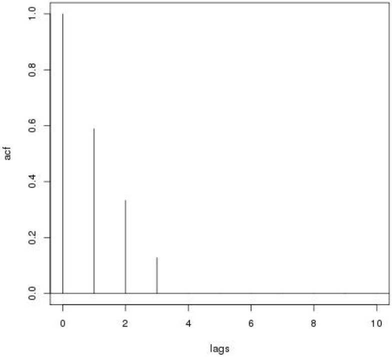
ACF

MA(q):

- Let
$$x_t = \theta_0 w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}, \quad \theta_0 = 1.$$
- The ACF is given by (why?)
$$\rho(h) = \begin{cases} \frac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{\sum_{j=0}^{q} \theta_j^2}, & 1 \leq h \leq q, \\ 0, & h > q \end{cases}$$
and $\rho(-h) = \rho(h)$.
- A good tool to identify q!

ACF

Theoretical ACF of $x_t = w_t + 0.6w_{t-1} + 0.4w_{t-2} + 0.2w_{t-3}$



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ACF

AR(p):

- Let
$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t.$$
- The ACF satisfies (why?)
$$\rho(h) - \phi_1\rho(h-1) - \phi_2\rho(h-2) - ... - \phi_p\rho(h-p) = 0, \quad h \geq p.$$
- What is the general form of the solution?

ACF

Recall the difference equation of order two:

- Solve

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0 \tag{2}$$

- Equivalently, write $\alpha(B)u_n = (1 - \alpha_1 B - \alpha_2 B^2)u_n = 0$.
- Denote the roots of $\alpha(z)$ by z_1 and z_2.
- If $z_1 \neq z_2$, the general solution to (2) is

$$u_n = z_1^{-n} c_1 + z_2^{-n} c_2.$$

- If $z_1 = z_2$, the general solution to (2) is

$$u_n = z_1^{-n}(c_1 + c_2 n).$$

ACF

In general:

- Solve

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} - \ldots - \alpha_p u_{n-p} = 0. \tag{3}$$

- Equivalently, write

$$\alpha(B)u_n = (1 - \alpha_1 B - \alpha_2 B^2 - \ldots - \alpha_p B^p)u_n = 0.$$

- Denote the roots of $\alpha(z)$ by z_1, z_2,..., z_r, each with multiplicity m_1, m_2,..., m_r.
- The general solution to (3) is of the form

$$u_n = z_1^{-n} P_1(n) + z_2^{-n} P_2(n) + \ldots + z_r^{-n} P_r(n),$$

where $P_1(n), \ldots, P_r(n)$ are polynomials of degrees $m_1 - 1, \ldots, m_r - 1$.

ACF

AR(p):

- Solve

$$\rho(h) - \phi_1 \rho(h-1) - \phi_2 \rho(h-2) - ... - \phi_p \rho(h-p) = 0, \quad h \geq p. \quad (4)$$

- Equivalently, write

$$\phi(B)\rho(h) = (1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p)\rho(h) = 0.$$

- Denote the roots of $\phi(z)$ by z_1, z_2,..., z_r, each with multiplicity m_1, m_2,..., m_r.
- The general solution to (4) is of the form

$$\rho(h) = z_1^{-h} P_1(h) + z_2^{-h} P_2(h) + ... + z_r^{-h} P_r(h),$$

where $P_1(h), ..., P_r(h)$ are polynomials of degrees $m_1 - 1, ..., m_r - 1$.

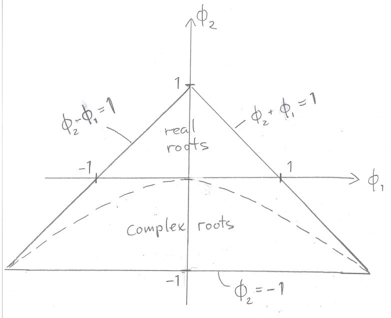
ACF

AR(p):

- $$\rho(h) = z_1^{-h} P_1(h) + z_2^{-h} P_2(h) + \ldots + z_r^{-h} P_r(h),$$
  where $P_1(h), \ldots, P_r(h)$ are polynomials of degrees $m_1 - 1, \ldots, m_r - 1$.
- Assume that the model is causal (all $|z_j| > 1$).
- If all roots are real, the ACF decays exponentially fast as $h \to \infty$.
- If some roots are complex, the ACF decays in a sinusoidal fashion.
- Not a good tool for identifying p!

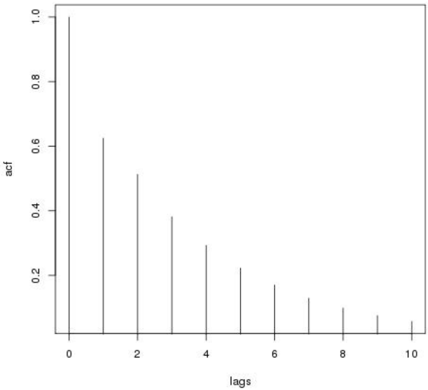
ACF

Causality region (inside the triangle) for AR(2):



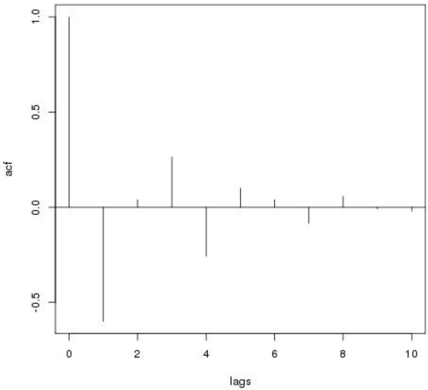
ACF

Theoretical ACF of $x_t = 0.5x_{t-1} + 0.2x_{t-2} + w_t$. (Real roots.)



ACF

Theoretical ACF of $x_t = -0.9x_{t-1} - 0.5x_{t-2} + w_t$. (Complex roots.)



PACF

Let $x_t = \phi x_{t-1} + w_t$ be causal.

1. Calculate $\mathrm{cov}(x_{t+2}, x_t)$. Is it zero?
2. a) Find a such that $E\{(x_{t+2} - ax_{t+1})^2\}$ is minimized.
   b) Find b such that $E\{(x_t - bx_{t+1})^2\}$ is minimized.
3. With such a, b, show that

$$\mathrm{cov}(x_{t+2} - ax_{t+1}, x_t - bx_{t+1}) = 0.$$

(uncorrelated 'projection errors')

PACF

- Let $\{x_t\}$ be a mean zero stationary process. Take any $h \geq 2$.
- Let
$$\hat{x}_{t+h} = a_1 x_{t+h-1} + a_2 x_{t+h-2} + \dots + a_{h-1} x_{t+1}$$
be such that $E\{(x_{t+h} - \hat{x}_{t+h})^2\}$ is minimized.
- Let
$$\hat{x}_t = b_1 x_{t+1} + b_2 x_{t+2} + \dots + b_{h-1} x_{t+h-1}$$
be such that $E\{(x_t - \hat{x}_t)^2\}$ is minimized.

Definition (3.9)

The *partial autocorrelation function* (PACF) is defined as

$$\phi_{hh} = \begin{cases} \mathrm{corr}(x_{t+1}, x_t), & h = 1, \\ \mathrm{corr}(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t), & h \geq 2. \end{cases}$$

PACF

- $$\phi_{hh} = \begin{cases} \operatorname{corr}(x_{t+1}, x_t), & h = 1, \\ \operatorname{corr}(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t), & h \geq 2. \end{cases}$$
- If $\{x_t\}$ is a normal (Gaussian) process,

$$\phi_{hh} = \operatorname{corr}(x_{t+h}, x_t | x_{t+1}, x_{t+2}, ..., x_{t+h-1}),$$

cf Appendix B.

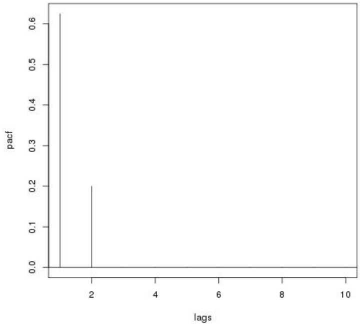
PACF

AR(p):

- Let
$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t.$$
- The PACF $\phi_{hh} = 0$ if $h > p$. (Why?)
- It is a good tool for identifying p!

PACF

Theoretical PACF of $x_t = 0.5x_{t-1} + 0.2x_{t-2} + w_t$.



Equals 0.2 at lag 2 and cuts off after this lag!

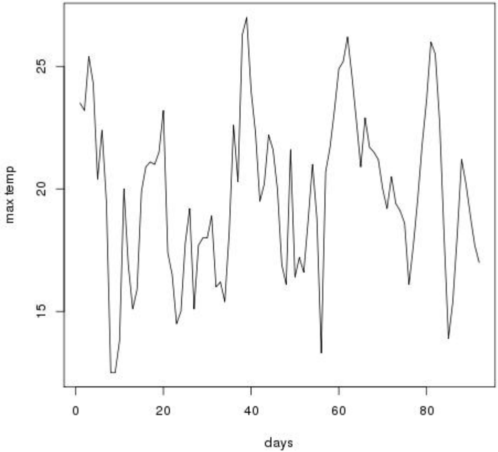
Summary and examples

Table 3.1:

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

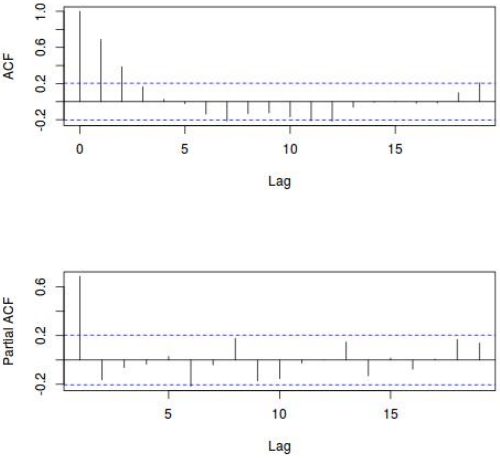
Summary and examples

Daily temperature, Uppsala, summer 1984.

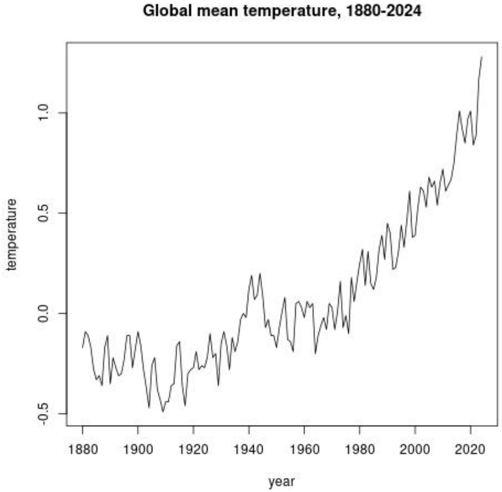


Summary and examples

Daily temperature, Uppsala, summer 1984, ACF and PACF. AR(1)?

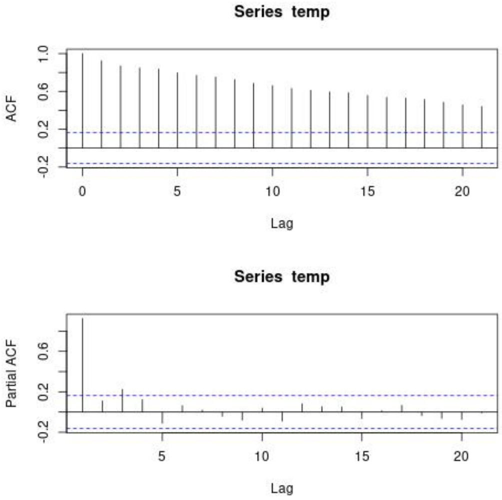


Summary and examples

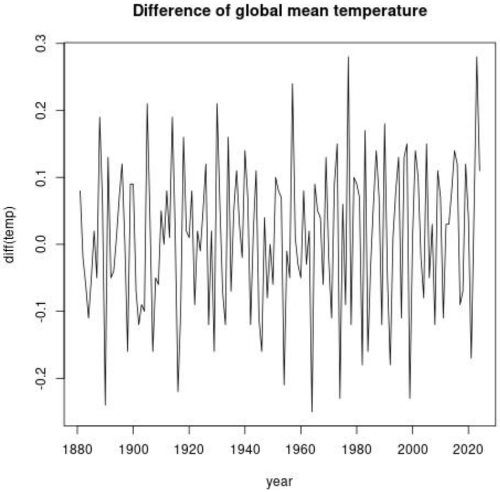


Summary and examples

Global mean temperature, ACF and PACF. (Indicates a trend.)

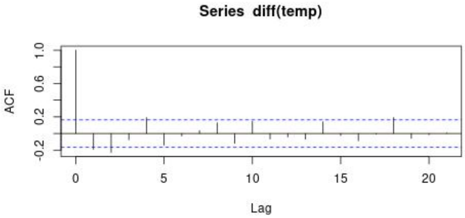


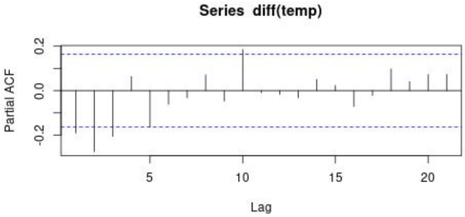
Summary and examples



Summary and examples

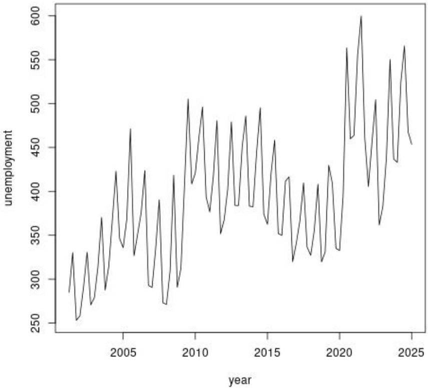
Difference of global mean temperature, ACF and PACF.
MA(2), MA(4), AR(3) or AR(5)...?





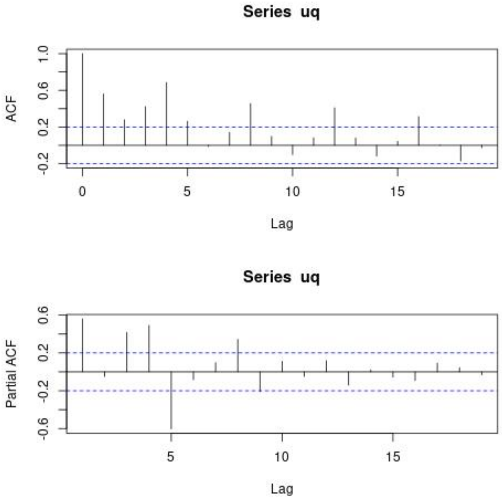
Summary and examples

Quarterly unemployment:

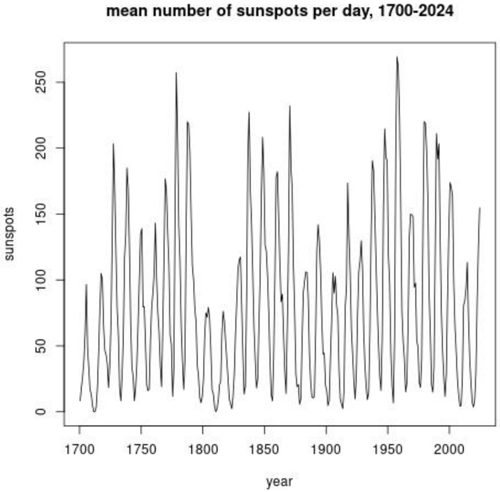


Summary and examples

Quarterly unemployment, ACF and PACF. MA(4) or AR(5)? Period of 4?

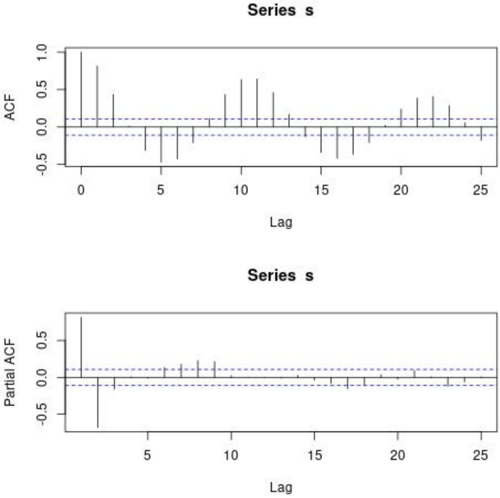


Summary and examples



Summary and examples

Sunspots, ACF and PACF. ARMA? Period of 10 or 11?



News of today

- Definition of the partial autocorrelation function (PACF).
- ACF and PACF for
  - MA processes
  - AR processes
  - ARMA processes
- Model identification, examples