Le 7

Dirichlet's problem

[Recall: A real-valued fcn $\phi(x,y)$ is said to be harmonic in a domain D if $\phi \in C^2(D)$ and

$$\Delta \phi = \phi_{xx} + \phi_{yy} = 0 \quad \text{in } D.$$

We showed the following:

<u>Thm</u> If $f = u + iv$ is analytic in a domain D, then u and v are harmonic in D.

<u>Thm</u> If u is harmonic in a simply connected domain D, there exists a harmonic conjugate v to u in D.

<u>Ex.</u> Since $\log z = \ln |z| + i \arg z$ is analytic off the branch cut (any given branch)

- $\log |z|$ is harmonic in $\mathbb{C} \setminus \{0\}$
- $\arg z$ is harmonic off the cut.]

Many applications involve solving <u>Dirichlet's problem</u>:

Find a fcn $\phi(x,y)$ continuous on $D \cup \partial D$ (D domain), of class C^2 in D, s.t.

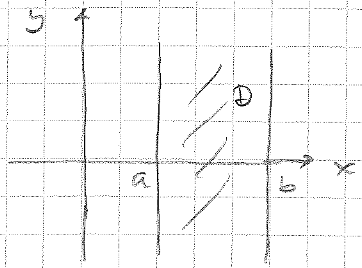
1) $\Delta \phi = \phi_{xx} + \phi_{yy} = 0$ in D

2) ϕ = some given fcn on ∂D.

This can be easily solved in some

Standard cases

(1)



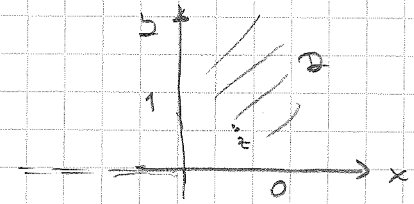
$$\begin{cases} \Delta\phi = 0 \text{ in } D \\ \phi(a,y) = A, \quad \phi(b,y) = B \end{cases}$$

Let $\phi(x,y) = \alpha x + \beta$; choose α, β s.t.

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \iff \begin{cases} \alpha = \dfrac{B-A}{b-a} \\ \beta = A - \dfrac{a(B-A)}{b-a} = \dfrac{Ab-aB}{b-a} \end{cases}$$

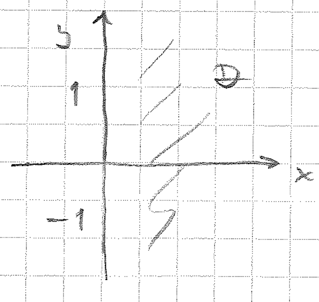
$$\Rightarrow \phi(x,y) = \frac{(B-A)x + Ab - aB}{b-a}$$

2)



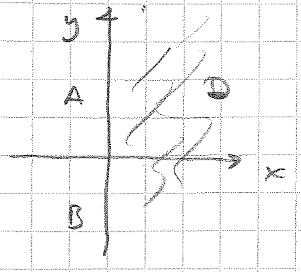
$$\phi(x,y) = \frac{2}{\pi} \operatorname{Arg} z = \frac{2}{\pi} \arctan \frac{y}{x}$$

3)



$$\phi(x,y) = \frac{2}{\pi} \operatorname{Arg} z = \frac{2}{\pi} \arctan \frac{y}{x}$$

4)

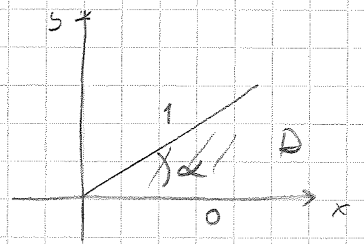


$$\phi(x,y) = \alpha \operatorname{Arg} z + \beta \quad \text{leads to}$$

$$\begin{cases} \alpha \frac{\pi}{2} + \beta = A \\ -\alpha \frac{\pi}{2} + \beta = B \end{cases} \iff \begin{cases} \alpha = \frac{A-B}{\pi} \\ \beta = \frac{A+B}{2} \end{cases}$$

i.e. $\phi(x,y) = \frac{A-B}{\pi} \operatorname{Arg} z + \frac{A+B}{2}$.

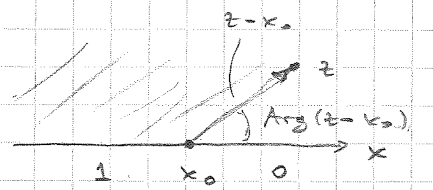
5)



$(0 < \alpha \leq \pi)$

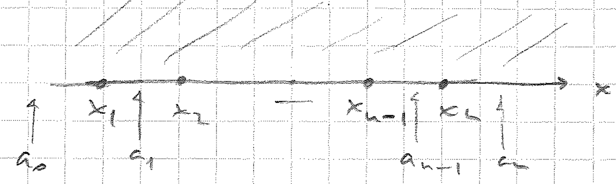
$$\phi(x,y) = \frac{1}{\alpha} \operatorname{Arg} z$$

6)



$$\phi(x,y) = \frac{1}{\pi} \operatorname{Arg}(z - x_0)$$

7)



$$\phi(x,y) = a_n + \frac{1}{\pi} \sum_{k=1}^{n} (a_{k-1} - a_k) \operatorname{Arg}(z - x_k)$$

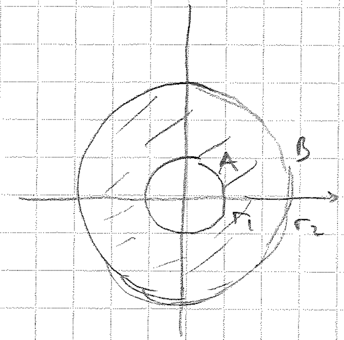
Indeed, since

$$\operatorname{Arg}(z - x_k) = \begin{cases} \pi , & x < x_k \\ 0 , & x > x_k \end{cases},$$

it follows that if $x_j < x < x_{j+1}$ then

$$\phi(x,0) = a_n + \sum_{k=j+1}^{n} (a_{k-1} - a_k) = a_j .$$

8)



$\phi(x,y) = \alpha \ln|z| + \beta$ leads to

$$\begin{cases} \alpha \ln r_1 + \beta = A \\ \alpha \ln r_2 + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{B-A}{\ln r_2 - \ln r_1} \\ \beta = \dfrac{A \ln r_2 - B \ln r_1}{\ln r_2 - \ln r_1} \end{cases}$$

so

$$\phi(x,y) = \frac{B-A}{\ln r_2 - \ln r_1} \ln|z| + \frac{A \ln r_2 - B \ln r_1}{\ln r_2 - \ln r_1} .$$

How about more complicated Dirichlet problems?

The idea is to "map" the complicated problem to an easier one using a conformal mapping.

We need the following simple:

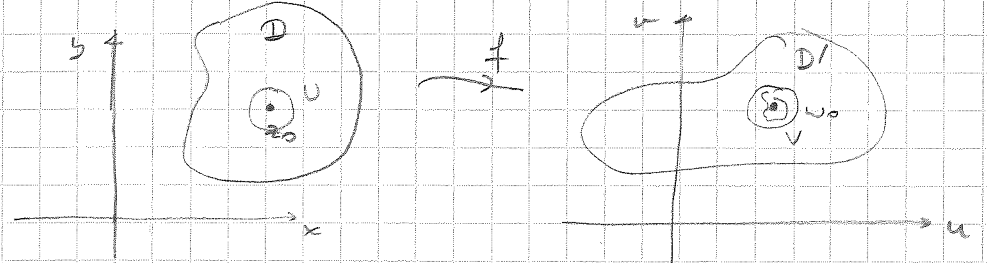
<u>Thm</u> Suppose $f: D \to D'$ analytic, $f = u + iv$. (D, D' domains)

If $\psi(u,v)$ is harmonic in D', then

$$\phi(x,y) := \psi(u(x,y), v(x,y)) \qquad (*)$$

is harmonic in D.

<u>Proof:</u> See figure:



Take $z_0 \in D$. Then $w_0 = f(z_0) \in D'$, and since D' is open there is a disk $V \ni w_0$ contained in D'. Since f is continuous, there is a disk $U \ni z_0$ in D s.t. $f(U) \subseteq V$. Since ψ is harmonic in V, which is simply connected, there is an analytic fcn g in V s.t. $\mathrm{Re}\, g = \psi$.

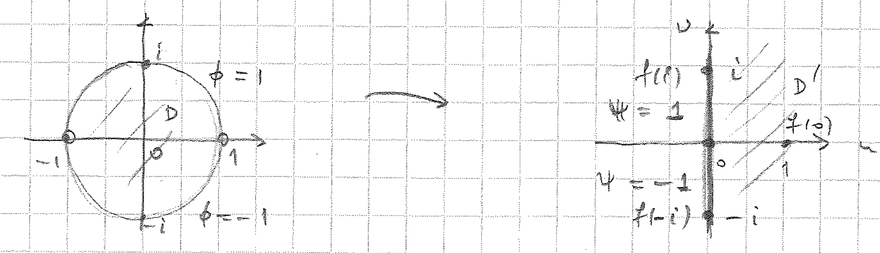
But then $g \circ f$ is an analytic fcn in U s.t. $\mathrm{Re}\,(g\circ f)(z) = \psi(u(x,y), v(x,y)) = \phi(x,y)$.

Hence ϕ is harmonic in U. Since z_0 was arbitrary, ϕ is harmonic in D. ∎

Suppose now that the analytic fn $f : D \to D'$ maps D bijectively onto D' and extends to a continuous bijection $f : \overline{D} \to \overline{D'}$. Suppose also that the boundary conditions for ψ in D' correspond to the boundary conditions for ϕ in D.
[I.e. (*) holds for $(x,y) \in \partial D$]
Then, if we can solve the Dirichlet problem for ψ, we can also solve it for ϕ.

Ex. Find a fn $\phi(x,y)$ harmonic inside the unit disk $|z| < 1$ s.t. $\phi(x,y) = +1$ on the upper half-circle and $\phi(x,y) = -1$ on the lower half-circle.

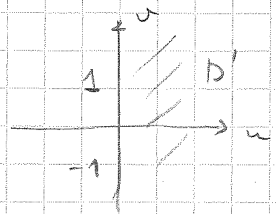
Sol. See figure:



The Möbius transformation taking $\begin{cases} -1 \longmapsto 0 \\ 0 \longmapsto 1 \\ 1 \longmapsto \infty \end{cases}$

is given by $w = f(z) = \dfrac{1+z}{1-z}$.

It maps the unit disk onto the right half-plane, the upper half-circle onto the positive imaginary axis and the lower half-circle onto the negative imaginary axis. The problem for ψ becomes:



As in 3) above, $\psi(u,v) = \frac{2}{\pi} \operatorname{Arg} w$

Thus,

$$\phi(x,y) = \frac{2}{\pi} \operatorname{Arg} \frac{1+z}{1-z} = \frac{2}{\pi} \operatorname{Arg} \frac{1+x+iy}{1-x-iy} =$$

$$= \frac{2}{\pi} \operatorname{Arg} \frac{(1+x+iy)(1-x+iy)}{(1-x)^2+y^2} = \frac{2}{\pi} \operatorname{Arg} \frac{1-x^2-y^2+i2y}{(1-x)^2+y^2} =$$

$$= \frac{2}{\pi} \arctan \frac{2y}{1-x^2-y^2}$$

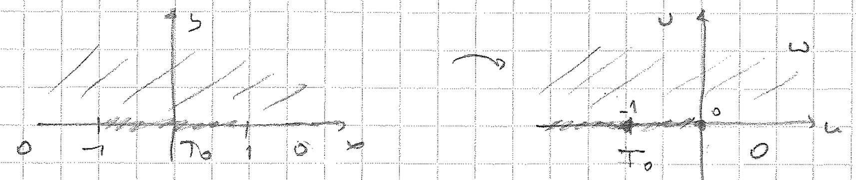
◻

[Ex. Find a fn ϕ harmonic in $\operatorname{Im} z > 0$ s.t.

$$\phi(x,0) = \begin{cases} T_0, & |x| < 1 \\ 0, & |x| > 0 \end{cases}$$

Sol. Alt. 1: Use 7).

Alt. 2



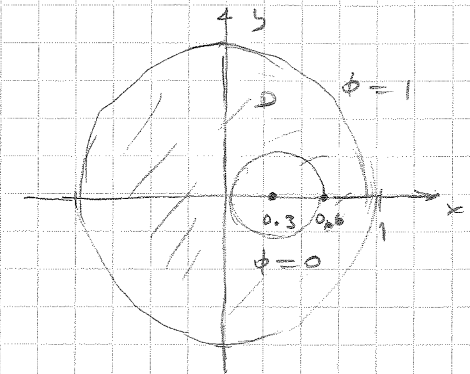
Let $w = T(z)$ map $\begin{cases} 1 \longmapsto 0 \\ 0 \longmapsto -1 \\ -1 \longmapsto \infty \end{cases}$

$$\frac{w-0}{w-\cancel{\infty}} \cdot \frac{-1-\cancel{\infty}}{-1-0} = \frac{z-1}{z+1} \cdot \frac{0+1}{0-1} \iff w = \frac{z-1}{z+1}$$

Thus, $\psi(u,v) = \frac{T_0}{\pi} \operatorname{Arg} w$

$\Rightarrow u(x,y) = \frac{T_0}{\pi} \operatorname{Arg} \frac{z-1}{z+1} = -$ Exercise $\square$]

Ex Find a harmonic fn ϕ in the region D with prescribed boundary values:



Sol There is a unique (real) pair of points α and α^* symmetric w.r.t. both circles. They are given by: $\alpha^* = \frac{1}{\alpha}$ and

$\alpha^* = 0.3 + \frac{(0.3)^2}{\alpha - 0.3}$. Solving gives

$$\alpha = \frac{1}{3}, \quad \alpha^* = 3$$

Let $w = T(z) = \frac{z - \frac{1}{3}}{z - 3}$. Both circles map to concentric circles with center 0. Since

$$|T(0)| = \frac{1}{9} \quad \text{and} \quad |T(1)| = \frac{1}{3}$$

the inner circle maps onto $|w| = \frac{1}{9}$ and the outer circle onto $|w| = \frac{1}{3}$.

As in 8) one finds that

$$\psi = \frac{\ln|w|}{\ln 3} + 2 = \frac{\ln|9w|}{\ln 3}$$

$$\Rightarrow \phi = \frac{\ln\left|9\left(\frac{z-\frac{1}{3}}{z-3}\right)\right|}{\ln 3} = \frac{\ln 3 + \ln\left|\frac{3z-1}{z-3}\right|}{\ln 3} =$$

$$= \frac{1}{\ln 3}\left\{\ln 3 + \frac{1}{2}\ln\left[(3x-1)^2+9y^2\right] - \frac{1}{2}\ln\left[(x-3)^2+y^2\right]\right)$$