UPPSALA UNIVERSITY
Department of Mathematics
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Markov Processes, 1MS012
Spring semester 2024

Lecture 15 Markov Processes, 1MS012

1 Brownian motion

1.1 History

Robert Brown (1827), english botanist: describe the motion of pollen particles moving in a gas or in a container of water.

Albert Einstein (1905): mathematical description using laws of physics as bombardment of molecules in the surrounding medium.
(Independent of Brown's work.)

Norbert Wiener (1923): Formulated Brown's observations with mathematical rigor.

Remark:
Louis Bachelier (1900) proposed a "Brownian motion model" for the movement of prices in the French bond market. Although ignored by academics for many decades it is now regarded as the first step in the mathematical theory of stock markets.

1.2 Construction of Brownian motion from a simple random walk

We can model the phenomena observed by Brown with simple random walk in 3 dimensions where the motions in each coordinate are independent symmetric simple random walks.

At each Δt time unit take a step of size Δx either to the left or to the right.

If X_t is the position at time t then

$$X_{n\Delta t} = S_n \Delta x,$$

where $S_n = \sum_{j=1}^{n} I_j$ is a simple random walk with (I_n) i.i.d. with $P(I_n = 1) = P(I_n = -1) = 0.5$.

Since

$$E(X_{n\Delta t}) = E(S_n \Delta x) = \Delta x E(\sum_{j=1}^{n} I_j) = \Delta x \sum_{j=1}^{n} \underbrace{E(I_j)}_{=0} = 0$$

and

$$Var(X_{n\Delta t}) = Var(S_n \Delta x) = (\Delta x)^2 Var(S_n) = (\Delta x)^2 Var(\sum_{j=1}^{n} I_j)$$

$$\underbrace{=}_{by\ independence} (\Delta x)^2 \sum_{j=1}^{n} \underbrace{Var(I_j)}_{=1} = n(\Delta x)^2,$$

and

$$X_t = (\Delta x) S_{\lfloor \frac{t}{\Delta t} \rfloor},$$

where $\lfloor t/\Delta t \rfloor$ is the largest integer less than or equal to $t/\Delta t$, we have $EX_t = 0$, $Var(X_t) = (\Delta x)^2 \lfloor \frac{t}{\Delta t} \rfloor$.

X_t is approximately normally distributed with these parameters, if Δt is small, since a sum of many i.i.d. random variables is approximately normally distributed by the central limit theorem.

Let Δx and Δt tend to 0 in a way resulting in a non-trivial limiting process:

$$(\Delta x)^2 = \sigma^2 \Delta t \Rightarrow Var(X_t) \to \sigma^2 t,$$

for any constant $\sigma^2 > 0$.

Then the limiting process is a called a Brownian motion.

Definition: A random process $(X_t)_{t \geq 0}$ is called a Brownian motion or Wiener process with parameter $0 < \sigma^2 < \infty$ if

(a) $X_0 = 0$

(b) (X_t) has
Independent increments: The increments $X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots,$ $X_{t_n} - X_{t_{n-1}}$ are independent random variables for any choice of timepoints $0 < t_1 < t_2 < \ldots < t_n$.

Stationary increments: The distribution of $X_{t+h} - X_t$ does not depend on t for any $h > 0$

(c) $X_{t+h} - X_t \sim N(0, \underbrace{\sigma^2 h}_{\text{(Variance)}})$, for all $h > 0$.

(d) (X_t) has continuous trajectories

Definition: A Brownian motion with $\sigma^2 = 1$ is called a **standard Brownian motion**.

The above arguments in deriving the Brownian motion as a limiting process uses a special case of the invariance principle/functional central limit theorem:
Suppose $\Delta t = 1/n$, and $\Delta x = 1/\sqrt{n}$ above and let

$$B_t^{(n)} = (\Delta x) S_{\lfloor \frac{t}{\Delta t} \rfloor} = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}.$$

The functional CLT states that $(B_t^{(n)})_{t\geq 0}$ converges (in distribution) to a standard Brownian motion, $(B_t)_{t\geq 0}$, as $n \to \infty$. (The central limit theorem is a special case of the the functional CLT obtained by letting $t = 1$.)

Example: Suppose (S_k) is a simple symmetric random walk with $(S_0 = 0)$. We have earlier calculated (Gamblers ruin, Lecture 4)

$$P(S_k \text{ reaches } -i < 0 \text{ before } j > 0) = \frac{j}{i+j},$$

if i and j are positive integers. Thus for large n

$$P(B_t^{(n)} \text{ reaches } -a < 0 \text{ before } b > 0) \approx \frac{b\sqrt{n}}{a\sqrt{n} + b\sqrt{n}} = \frac{b}{a+b},$$

for any real $a, b > 0$.

By the invariance principle it follows that

$$P(B_t \text{ reaches } -a < 0 \text{ before } b > 0) = \frac{b}{a+b}.$$

2 Further properties of Brownian motion

1. Recurrence/transience like a simple random walk:
Recall: Symmetric simple random walk in 1 or 2 dimensions is recurrent but symmetric simple random walk in 3 dimensions is transient.
Shizuo Kakutani stated this: "A drunk man will find his way home but a drunk bird may get lost forever."
This property is inherited by the Brownian motion where recurrence here means that any line segment (or ball) is re-visited with probability one in one or two dimensions, but balls need not be revisited in 3d.

2. "erratic" trajectories:
It can be proved that trajectories of a Brownian motion are nowhere differentiable.

3. A Brownian motion (X_t) can be standardized:

$$B_t = \frac{X_t}{\sigma}$$

is a standard Brownian motion.

Since a Brownian motion (X_t) has independent increments, and

$$X_{t_1} \sim N(0, \sigma^2 t_1),\ X_{t_{k+1}} - X_{t_k} \sim N(0, \sigma^2(t_{k+1} - t_k)),\ k = 1, ..., n-1,$$

it follows that the finite dimensional density functions of (X_t) has the form

$$f_{X_{t_1}, X_{t_2}, ..., X_{t_n}}(x_1, ..., x_n) = f_{X_{t_1}}(x_1) f_{X_{t_2}-X_{t_1}}(x_2-x_1) \cdots f_{X_{t_n}-X_{t_{n-1}}}(x_n-x_{n-1}),$$

where for $Z \sim N(0, \sigma^2)$, we use the notation $f_Z(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ for its density function.

3 Suggested exercises

Basic exercises:
29–32

Extra problems:
c3

Exercises Lawler:
8.4abc, 8.9

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4 Transformations of Brownian Motion

Many transformations of Brownian motion generates another Brownian motion:

Theorem: Suppose $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then the transformed processes

1. $B_1(t) = cB_{t/c^2}$, for fixed $c > 0$
2. $B_2(t) = tB_{1/t}$ for $t > 0$, $B_2(t) = 0$ for $t = 0$
3. $B_3(t) = B_{t+h} - B_h$, for fixed $h > 0$.

are all standard Brownian motions.

Roughly all these results says that Brownian motion is stochastically self-similar in various ways. Such properties are useful when studying properties of trajectories of Brownian motion.

Proofs: We need to check the defining properties of Brownian motion for each of the transformed processes.

1.
$$B_1(t) = cB_{t/c^2}$$

(a) $B_1(0) = cB_{0/c^2} = cB_0 = 0$.

(b)–(c) If $t_1 < t_2 < .. < t_n$, then $t_1/c^2 < t_2/c^2 \leq t_3/c^2 < ... < t_n/c^2$, and the corresponding increments B_{t_1/c^2}, and $B_{t_n/c^2} - B_{t_{k-1}/c^2}$, $k = 2, ..., n$ are thus independent. Multiples of each of these increments by c are also independent i.e. $B_1(t_1)$, and $B_1(t_k) - B_1(t_{k-1})$, $k = 2, ..., n$ are independent, so $B_1(t)$ has independent increments.

The increments

$$B_1(t+h) - B_1(t) = cB_{(t+h)/c^2} - cB_{t/c^2} = c(B_{(t+h)/c^2} - B_{t/c^2})$$

are clearly normally distributed as a constant multiple of a normally distributed random variable. Since the increment $B_{(t+h)/c^2} - B_{t/c^2}$ has mean zero, then

$$B_1(t+h) - B_1(t) = c(B_{(t+h)/c^2} - B_{t/c^2}))$$

must have mean zero. Since the mean is zero it follows that the variance is

$$\begin{aligned} E((B_1(t+h) - B_1(t))^2) &= E((cB_{(t+h)/c^2} - cB_{t/c^2})^2) \\ &= c^2 E((B_{(t+h)/c^2} - B_{t/c^2})^2) \\ &= c^2((t+h)/c^2 - t/c^2) = h, \end{aligned}$$

so $B_1(t+h) - B_1(t) \sim N(0,h)$. Thus B_1 has both stationary, and normally distributed increments with $\sigma^2 = 1$.

(d) B_1 has continuous trajectories, since compositions of continuous functions are continuous.

2.

$$B_2(t) = tB_{1/t}$$

(a) By definition, $B_2(0) = 0$.

(b)–(c)

$$B_2(t) - B_2(s) = tB_{1/t} - sB_{1/s} = (t-s)B_{1/t} - s(B_{1/s} - B_{1/t}),\ s < t$$

is the difference of independent normally distributed random variables each with mean 0, so the difference is normally distributed with mean 0. We need to check that the normal random variable has the correct variance:

$$\begin{aligned} E((B_2(t+h) - B_2(t))^2) &= E((B_2(t) - B_2(t+h))^2) \\ &= E((tB_{1/t} - (t+h)B_{1/(t+h)})^2) \\ &= E(\Big(tB_{1/t} - tB_{1/(t+h)} + tB_{1/(t+h)} \\ &\quad -(t+h)B_{1/(t+h)} + hB_0\Big)^2) \\ &= t^2 E((B_{1/t} - B_{1/(t+h)})^2) \\ &\quad +h^2 E((B_{1/(t+h)} - B_0)^2) \\ &= t^2(1/t - 1/(t+h)) \\ &\quad +h^2/(t+h) \\ &= h \end{aligned}$$

(In the 4:th equality we used independence of the increments $B_{1/t} - B_{1/(t+h)}$ and $B_{1/(t+h)} - B_0$. Thus $B_2(t)$ has stationary and normally distributed increments with the correct variance.

To prove that B_2 has independent increments is hard and beyond the scope of this course. One can rely on the fact that a Gaussian process with mean 0 and covariance function $\min(s,t)$ is a standard Brownian motion, and thus prove it indirectly.
Note that

$$\begin{aligned} Cov(B_s, B_t) &= E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t) \\ &= E(B_s(B_t - B_s) + B_s^2) \\ &= E(B_s)E(B_t - B_s) + E(B_s^2) = E(B_s^2) = s, \end{aligned}$$

if $s < t$, so

$$Cov(B_2(s), B_2(t)) = st \min(1/s, 1/t) = \min(s,t),$$

is a direct consequence of the same property for standard Brownian motion.

(d) The argument that $\lim_{t\to 0} B_2(t) = 0$ is equivalent to showing that $\lim_{t\to\infty} B_t/t = 0$. To show this is beyond the scope of this course. The translation property in the third statement of this theorem proves continuity at every value of t.

3.

$$B_3(t) = B_{t+h} - B_h$$

(a)

$$B_3(0) = B_{0+h} - B_h = 0.$$

(b)–(c) The increment

$$B_3(t+h) - B_3(t) = (B_{t+2h} - B_h) - (B_{t+h} - B_h) = B_{t+2h} - B_{t+h}$$

is by definition normally distributed with mean 0 and variance h.

If $t_1 < t_2 < ... < t_n$ then the increments

$$B_3(t_k) - B_3(t_{k-1}) = B_{t_k+h} - B_{t_{k-1}+h}$$

are independent by the property of independent increments of B_t.

(d) B_3 is continuous since it is a difference of continuous functions.

5 The Markov property

Consider a stochastic process (X_t) where X_t are continuous random variables.

Note that $P(X_t = a) = 0$, for all $a \in \mathbb{R}$, so expressions like $P(X_{t+h} = j \mid X_t = i)$ do not make sense.

We want to express the Markov property in a way that is useful in all cases. The following definition can then be used;

Definition: We say that a random process $(X_t)_{t\geq 0}$ is a Markov process if

$$P(X_{t_n} \leq x \mid X_{t_1}, ..., X_{t_{n-1}}) = P(X_{t_n} \leq x \mid X_{t_{n-1}})$$

for all $x \in \mathbb{R}$ and $t_1 < t_2 < ... < t_{n-1} < t_n$.

The transition density

$$f(t, y|s, x) = \frac{\partial}{\partial y} P(X_t \leq y \mid X_s = x),$$

is an alternative way to characterize Markov processes with continuous time and (uncountable) state space $\mathbb{R}$. The reason for writing this function of 4 variables like this is to stress that it is to be interpreted as the conditional density that $X_t = y$ given that $X_s = x$.

The Chapman-Kolmogorov equations states that

$$f(t, y|s, x) = \int_{-\infty}^{\infty} f(u, z|s, x) f(t, y|u, z) dz, \qquad s < u < t$$

Example: Consider a standard Brownian motion, (B_t). Suppose we know that $B_s = x$ where $s \geq 0$ and $x \in \mathbb{R}$. Conditional on this $B_t \sim N(x, t - s)$, for $t \geq s$, i.e.

$$\underbrace{\frac{\partial}{\partial y} P(B_t \leq y \mid B_s = x)}_{f(t,y|s,x)} = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}, \qquad -\infty < y < \infty.$$

From the Chapman-Kolmogorov equations (following the calculations of Einstein), it is possible to derive the equations,

$$\frac{\partial f}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2} \qquad \text{(forward equation)}$$

$$\frac{\partial f}{\partial s} = -\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \quad \text{(backward equation)},$$

for the transition density of a Brownian motion with variance parameter σ^2.

Note the connection between partial differential equations (PDEs) and random walks.

6 Diffusion processes

Diffusion processes is a class of processes generalizing Brownian motion.

Definition: Suppose (X_t) is a Markov process with continuous sample paths and suppose there exist functions $a(t,x)$, $b(t,x)$ such that

$$P(|X_{t+h} - X_t| > \epsilon \mid X_t) = o(h), \text{ for all } \epsilon > 0$$

$$E(X_{t+h} - X_t \mid X_t) = \underbrace{a(t, X_t)h}_{\text{instantaneous mean}} + o(h)$$

$$E((X_{t+h} - X_t)^2 \mid X_t) = \underbrace{b(t, X_t)h}_{\text{instantaneous variance}} + o(h)$$

Then (X_t) is called a diffusion process.

The above defines a stochastic differential equation.

$$dX_t = a(t, X_t)dt + \sqrt{b(t, X_t)}dB_t$$

where dB_t corresponds to infinitesimal increments of a standard Brownian motion. (In order to solve such equations we need theory from stochastic calculus not covered by this course.)

Under some additional technical conditions it is possible to derive the following equations specifying the transition densities.

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}(a(t,y)f) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(b(t,y)f) \quad \text{(forward equation)}$$

$$\frac{\partial f}{\partial s} = -a(s,x)\frac{\partial f}{\partial x} - \frac{1}{2}b(s,x)\frac{\partial^2 f}{\partial x^2} \quad \text{(backward equation)}$$

The diffusion process is time homogeneous if $a(t,x) = a(x)$ and $b(t,x) = b(x)$.

Brownian motion:
$a(t,x) = 0$, $b(t,x) = \sigma^2$.

The coefficients for a Brownian motion thus do not depend on x and t. This reflects the fact that Brownian motion is homogeneous in space and time by the independent and stationary increments properties.

Brownian motion with drift: $a(t,x) = m$, $b(t,x) = \sigma^2$.

$$X_t = mt + \sigma B_t$$

Compare:
Markov chain (discrete time countable state space): "generated" by a transition matrix $\mathbf{P} = (p_{ij})$.
Markov process (continuous time, countable state space): No smallest time unit. Generator: matrix of derivatives in 0, $\mathbf{Q} = \mathbf{P}'(0) = (p'_{ij}(0))$.
The transition probabilities are given as the solutions to the forward and backward equations $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{Q}$.
Diffusion processes (continuous time, (uncountable) state space $\mathbb{R}$): The transition densities are given as the solutions to the forward and backward equations. The functions $a(t,x)$ and $b(t,x)$ play the role of $\mathbf{Q}$.

7 Geometric Brownian motion

$$X_t = e^{Y_t},$$

where Y_t is a Brownian motion with drift.

Used in modeling stock market prices. (e.g. in the Black-Scholes theory)
S_t asset price at time t (e.g. stock)
(The assumption that stock prices are log-normally distributed is doubtful. Real world data indicates some heavy-tailed distribution.)

Interpretation

$$dS_t = rS_t dt + \sigma S_t dB_t$$

$$\underbrace{\frac{dS_t}{S_t}}_{return} = \underbrace{rdt}_{mean\ return} + \underbrace{\underbrace{\sigma}_{volatility} \quad dB_t}_{Gaussian\ random\ disturbance}$$

By using stochastic calculus (not a part of this course) it can be proved that the solution is given by

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma B_t}.$$

8 Suggested exercises

Exercises Lawler:
8.7

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9 The reflection principle and the distribution of $\max_{0\le u\le t} B_u$ and $|B_t|$.

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Fix a state $x > 0$. Let τ_x be the first time at which B_t attains the value x.

Define

$$B_t^\star = \begin{cases} B_t & t \leq \tau_x \\ x - (B_t - x) & t \geq \tau_x \end{cases} \qquad \text{reflection of } B_t.$$

If $B_t > x$, then by continuity of trajectories of $(B_t)_{t\geq 0}$, it follows that $\tau_x < t$, i.e. B_u (and $B_u^\star$) attain the value x somewhere in the interval $0 \leq u < t$.

By symmetry,

$$\begin{aligned} P(B_t > x \mid \tau_x < t) &= P(B_t^\star > x \mid \tau_x < t) = P(B_t < x \mid \tau_x < t) \\ &= 1 - \underbrace{P(B_t \geq x \mid \tau_x < t)}_{P(B_t > x \mid \tau_x < t)} = 1/2. \end{aligned}$$

Therefore

$$P(B_t > x) = P(B_t > x,\ \tau_x < t) = \underbrace{P(B_t > x \mid \tau_x < t)}_{1/2} \quad \underbrace{P(\tau_x < t)}_{P(\max_{0\le u\le t} B_u > x)} \quad ,$$

and thus

$$P(\max_{0\leq u\leq t} B_u > x) = 2P(B_t > x) = 2(1 - P(B_t \leq x)) = 2(1 - \Phi(x/\sqrt{t})),$$

where Φ denotes the distribution-function of a $N(0,1)$-distributed random variable, i.e, $\phi(x) = \frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Since

$$P(|B_t| > x) = P(B_t > x) + P(B_t < -x) = 2P(B_t > x)$$

it follows that Brownian motion reflected at the origin $|B_t|$ has the same distribution as $\max_{0\le u\le t} B_u$ for any fixed t. The common density is given by

$$f(x) = \frac{d}{dx}(1 - (2(1 - \Phi(x/\sqrt{t})))) = \frac{d}{dx}(2\Phi(x/\sqrt{t}) - 1)$$

$$= 2\phi(x/\sqrt{t})(1/\sqrt{t}) = \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}}, \qquad x \geq 0.$$

These processes are continuous time Markov process whose sample paths are continuous. The mean and variance are given by

$$E(\max_{0\leq u\leq t} B_u) = E|B_t| = \sqrt{\frac{2}{\pi t}} \underbrace{\int_0^\infty x e^{-\frac{x^2}{2t}} dx}_{[te^{-\frac{x^2}{2t}}]_{x=\infty}^{x=0}} = \sqrt{\frac{2t}{\pi}},$$

and

$$Var(\max_{0\leq u\leq t} B_u) = Var|B_t| = E(B_t^2) - 2t/\pi = Var(B_t) - 2t/\pi = t - 2t/\pi = t(1-2/\pi).$$

10 Zeros of Brownian motion

Theorem: Let $(B_u)_{u\geq 0}$ be a standard Brownian motion and $0 \leq s < t$ be two fixed time points. Then

$$P(B_u \text{ has at least one zero in the interval}(s,t)) = 1 - \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}.$$

Proof: Suppose $0 < s < t$, and let E be the event that B_u has at least one zero in the interval (s,t), i.e. that there exists an u with $s < u < t$ such that $B_u = 0$. Let τ_x be the first time the Brownian motion takes the value x. By the reflection principle

$$P(\tau_x \leq t) \underbrace{=}_{\text{if } x>0} P(\max_{0\leq u\leq t} B_u \geq x) = 2(1-\Phi(\frac{x}{\sqrt{t}})), \qquad t > 0,$$

where Φ denotes the distribution function of a standard normal random variable thus satisfying $\frac{d\Phi(x)}{dx} = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Thus

$$\begin{aligned} P(E \mid B_s = \omega) &= P(\tau_{-\omega} < t-s) \underbrace{=}_{sym.} P(\tau_\omega < t-s) \\ &= \int_0^{t-s} \frac{d(P(\tau_\omega \leq y))}{dy} dy \\ &= \int_0^{t-s} 2\phi(\frac{\omega}{\sqrt{y}}) \frac{\omega}{2y^{3/2}} dy = \int_0^{t-s} \frac{\omega e^{-\omega^2/(2y)}}{\sqrt{2\pi y^3}} dy, \qquad \omega > 0. \end{aligned}$$

Thus

$$
\begin{aligned}
P(E) &= \int_{-\infty}^{\infty} P(E \mid B_s = \omega) P(B_s = d\omega) \\
&= 2\int_{0}^{\infty} P(E \mid B_s = \omega) P(B_s = d\omega) \\
&= 2\int_{0}^{\infty}\int_{0}^{t-s} \frac{\omega e^{-\omega^2/(2y)}}{\sqrt{2\pi y^3}} \frac{1}{\sqrt{2\pi s}} e^{-\omega^2/(2s)} dy d\omega \\
&= \frac{1}{\sqrt{s}\pi}\int_{0}^{t-s} y^{-3/2} \underbrace{\int_{0}^{\infty} \omega e^{\frac{-\omega^2}{2}(\frac{1}{y}+\frac{1}{s})} d\omega}_{\frac{ys}{y+s}} \, dy \\
&= \frac{\sqrt{s}}{\pi}\int_{0}^{t-s} \frac{1}{(y+s)\sqrt{y}} dy \\
&\quad \text{Substitute } v = \sqrt{y/s}, \text{ i.e. } y = v^2 s,\ dy = 2vsdv \\
&= \frac{1}{\pi}\int_{0}^{\sqrt{(t-s)/s}} \frac{1}{(v^2 s + s)v} 2vsdv = \frac{2}{\pi} \underbrace{\int_{0}^{\sqrt{(t-s)/s}} \frac{1}{1+v^2} dv}_{\arctan(\sqrt{(t-s)/s})} \\
&= \frac{2}{\pi}\arccos(\sqrt{\frac{s}{t}}) = \frac{2}{\pi}(\frac{\pi}{2} - \arcsin(\sqrt{\frac{s}{t}})) = 1 - \frac{2}{\pi}\arcsin(\sqrt{\frac{s}{t}}).
\end{aligned}
$$

Corollary: Let $V = \sup\{u < t : B_u = 0\}$, be the time for the last zero before time $t > 0$. Then

$$
\begin{aligned}
P(V > s) &= P(B_u \text{ has at least one zeros in } (s,t)) \\
&= 1 - \frac{2}{\pi}\arcsin(\sqrt{\frac{s}{t}}), \qquad 0 < s < t.
\end{aligned}
$$

Thus V has distribution function

$$
F_V(s) = P(V \le s) = \frac{2}{\pi}\arcsin(\sqrt{\frac{s}{t}}), \qquad 0 < s < t,
$$

and hence density function

$$
f_V(s) = \frac{d}{ds}F_V(s) = \frac{1}{\pi}\frac{1}{\sqrt{s(t-s)}}, \qquad 0 < s < t.
$$

Note that $f_V(s)$ is symmetric around $s = t/2$. Note also that $P(V > 0) = 1$, i.e. for any $t > 0$ the Brownian motion has a zero in the interval $(0,t)$ with probability one. It therefore follows that B_u has infinitely many zeros in $[0,t]$ for any $t > 0$.

11 Brownian bridge

Brownian bridge: B_t, $0 \leq t \leq 1$, conditional on the the event that $B_1 = 0$.

Let $(Z_t)_{t=0}^1$ be a Brownian bridge. We want to find the density function of Z_s for $0 < s < 1$.

More generally we find the conditional density of B_s given that $B_t = b$, $s < t$. (Thus Brownian bridge corresponds to $b = 0$ and $t = 1$.)

Conditional density:

$$
\begin{aligned}
f_{B_s|B_t=b}(x) &= \frac{f_{B_s,B_t}(x,b)}{f_{B_t}(b)} \underbrace{=}_{\text{indep. incr.}} \frac{f_{B_s}(x) f_{B_t-B_s}(b-x)}{f_{B_t}(b)} \\
&= \frac{\frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(b-x)^2}{2(t-s)}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}} \\
&= \frac{1}{\sqrt{2\pi \frac{s}{t}(t-s)}} e^{-\frac{x^2}{2s} - \frac{(b-x)^2}{2(t-s)} + \frac{b^2}{2t}} \\
&= \frac{1}{\sqrt{2\pi \frac{s}{t}(t-s)}} e^{-\frac{t}{2s(t-s)}(x - \frac{bs}{t})^2}
\end{aligned}
$$

Thus

$$
B_s|B_t = b \ \sim N\left(\frac{bs}{t}, \frac{s(t-s)}{t}\right).
$$

Therefore, in particular

$$
E(B_s|B_t) = \frac{s}{t} B_t,
$$

$$
Var(B_s|B_t) = \frac{s}{t}(t-s),
$$

and

$$
Z_t \sim N(0, t(1-t)), \quad 0 \leq t \leq 1.
$$

Proposition: If B_t is a standard Brownian motion, then

$$
Z_t = B_t - tB_1, \ 0 \leq t \leq 1,
$$

is a Brownian bridge.

12 Suggested exercises

Exercises Lawler:
8.4def, 8.11