SOLUTIONS to Final Exam

Fourier Analysis, 1MA211

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duration of the exam: 5 hours

There are 8 problems in this exam, and each one is worth 5 points. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5. **You need to motivate every step in your solution to get the full score on a question.** You can use the attached table of formulas. Good luck!

1. Use a technique that we studied in this course to find a function $y(x)$, with $x \geq 0$, that solves the initial value problem

$$\begin{cases} y''(x) - 2y'(x) - 3y(x) = 4e^{-x} \\ y(0) = -1, \quad y'(0) = 2 \end{cases}.$$

Solution: Applying the Laplace transform to both sides of the equation, we get

$$(s^2Y(s) + s - 2) - 2(sY(s) + 1) - 3Y(s) = \frac{4}{s+1}$$

so

$$(s^2 - 2s - 3)Y(s) = \frac{4}{s+1} - s + 4.$$

The fact that $(s^2 - 2s - 3) = (s+1)(s-3)$ implies that

$$Y(s) = \frac{-s^2 + 3s + 8}{(s+1)^2(s-3)}.$$

We can find a partial fraction decomposition for the right side:

$$\frac{-s^2 + 3s + 8}{(s+1)^2(s-3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s-3}. \tag{1}$$

To find C, multiply both sides by $s - 3$ and take the limit as $s \to 3$. We get

$$C = \lim_{s\to 3} \frac{-s^2 + 3s + 8}{(s+1)^2} = \frac{-9 + 9 + 8}{16} = \frac{1}{2}.$$

To find B, multiply both sides by $(s+1)^2$ and take the limit as $s \to -1$. We get

$$B = \lim_{s\to -1} \frac{-s^2 + 3s + 8}{s-3} = \frac{-1 - 3 + 8}{-4} = -1.$$

Now that we know B and C, we can find A by evaluating both sides of equation (1) above at a point that is not a zero of the denominator. For instance, at $s = 0$ we get

$$-\frac{8}{3} = A - 1 - \frac{1}{6} \Longrightarrow A = -\frac{3}{2}.$$

Therefore,

$$Y(s) = -\frac{3}{2}\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s-3}.$$

Taking the inverse Laplace transform, we conclude that

$$y(x) = -\frac{3}{2}e^{-x} - xe^{-x} + \frac{1}{2}e^{3x}.$$

$\square$

2. Find a function $u(x,t)$, where $0 \le x \le \pi$ and $t \ge 0$, that solves the boundary value problem

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = 2 \text{ and } u_x(\pi,t) = 2 & t > 0 \\ u(x,0) = 3x \text{ and } u_t(x,0) = 3\cos(2x) & 0 < x < \pi \end{cases}$$

Solution: We begin by homogenizing the problem by assuming $u(x,t) = v(x,t) + 2x$, where $v(x,t)$ satisfies

$$\begin{cases} v_{tt} = v_{xx} & 0 < x < \pi, \quad t > 0 \\ v_x(0,t) = v_x(\pi,t) = 0 & t > 0 \\ v(x,0) = x \text{ and } v_t(x,0) = 3\cos(2x) & 0 < x < \pi. \end{cases}$$

We now make the ansatz $v(x,t) = X(x)T(t)$ to separate the variables. By the equation and conditions we can conclude that the solution is on the form

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \cos(nx).$$

By the first initial condition we have

$$v(x,0) = \sum_{n=0}^{\infty} a_n \cos(nx) = x,$$

so the coefficients a_n are the coefficients of the Fourier series of the even extension of $f(x) = x$ on $0 < x < \pi$. We compute these to be $a_0 = \pi$ and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{2((-1)^n - 1)}{\pi n^2} \\ &= \begin{cases} \frac{-4}{\pi(2m+1)^2}, & \text{if } n = 2m+1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By the second initial condition, we have

$$v_t(x,0) = \sum_{n=0}^{\infty} nb_n \cos(nx) = 3\cos(2x),$$

whence $b_2 = \frac{3}{2}$ and $b_n = 0$ if $n \neq 2$. Lastly, putting everything together we find that

$$\begin{aligned} u(x,t) &= v(x,t) + 2x \\ &= \frac{\pi}{2} + \frac{-4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos((2m+1)t)\cos((2m+1)x) + \frac{3}{2}\sin(2t)\cos(2x) + 2x. \end{aligned}$$

□

3. Let V be the space of continuous functions $f : [0,1] \to \mathbb{C}$, with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)\,\overline{g(x)}\,e^x\,\mathrm{d}x.$$

Find two orthonormal elements of V.

Solution: There are many possible solutions to this question. For example, if one can apply the Gram–Schmidt method starting with the linearly independent functions $v_1 = 1$ and $v_2 = e^{-x}$:

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = \int_0^1 e^x\,\mathrm{d}x = e^x\Big|_0^1 = e - 1$$

so the first orthonormal vector is

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{e-1}}.$$

Now,

$$\langle v_1, v_2 \rangle = \int_0^1 e^{-x}e^x\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x = 1$$

so

$$u_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = e^{-x} - \frac{1}{e-1}.$$

$$\begin{aligned} \|u_2\|^2 = \langle u_2, u_2 \rangle &= \int_0^1 (e^{-x} - \frac{1}{e-1})^2 e^x\,\mathrm{d}x = \int_0^1 (e^{-2x} - \frac{2}{e-1}e^{-x} + \frac{1}{(e-1)^2})e^x\,\mathrm{d}x = \\ &= \int_0^1 e^{-x} - \frac{2}{e-1} + \frac{1}{(e-1)^2}e^x\,\mathrm{d}x = \left[-e^{-x} - \frac{2x}{e-1} + \frac{e^x}{(e-1)^2}\right]_0^1 = \\ &= -e^{-1} + 1 - \frac{2}{e-1} + \frac{e-1}{(e-1)^2} = 1 - \frac{1}{e} + \frac{1}{1-e}. \end{aligned}$$

Therefore, the second orthonormal vector is

$$e_2 = \frac{e^{-x} - \frac{1}{e-1}}{\sqrt{1 - \frac{1}{e} + \frac{1}{1-e}}}.$$

□

4. Let $f : \mathbb{R} \to \mathbb{C}$ be a function of period 2π, such that

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, -\pi/2) \cup (\pi/2, \pi) \\ 1 & \text{if } x \in [-\pi/2, \pi/2] \end{cases}$$

(a) Find the Fourier series of f.

(b) Does this Fourier series converge pointwise? If yes, write the limit. Justify your answer.

(c) Does this Fourier series converge uniformly? If yes, write the limit. Justify your answer.

(d) Use the result of part (a) to compute

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Solution:

(a) Since f is even, we have $b_n = 0$ for all n and only need to compute a_n for $n = 0, 1, 2, \ldots$. We compute

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} dx = 1,$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(nx) dx = \frac{2}{\pi} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}, \quad n \neq 0. \end{aligned}$$

We obtain

$$\begin{aligned} f &\sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n} \cos(nx) \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} \cos((2m-1)x). \end{aligned}$$

(b) Yes. Since this function has lateral limits and lateral derivatives at every point, Dirichlet's theorem can be applied at every point x_0. If f is continuous at x_0 then the Fourier series converges to $f(x_0)$. If f has a jump discontinuity at x_0, then the Fourier series converges to $\frac{f(x_+)+f(x_-)}{2}$ (where $x_\pm$ are the lateral limits at x_0). Therefore, the pointwise limit is

$$\tilde{f}(x) = \begin{cases} 0, \text{ if } x \in \left[-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ \frac{1}{2}, \text{ if } x \in \{-\frac{\pi}{2}, \frac{\pi}{2}\} \\ 1, \text{ if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \end{cases}$$

(c) No. Since the partial sums are continuous functions and the limit $\tilde{f}$ is not continuous, the convergence can not be uniform.

(d) By Parseval's formula

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx &= \frac{1}{4}+\frac{1}{2}\sum_{m=1}^{\infty}\frac{4}{\pi^2}\frac{1}{(2m-1)^2}\\ \iff \quad \frac{1}{2} &= \frac{1}{4}+\frac{2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\\ \iff \quad \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} &= \frac{\pi^2}{8}.\end{aligned}$$

□

5. Find a function $u(x,y)$ that solves the initial value problem

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}+6\,u\,y^2 & x\in\mathbb{R}, y>0\\ u(x,0)=f(x) & x\in\mathbb{R}\end{cases}.$$

Hint: Recall that the ODE $f'(t)+p(t)\,f(t)=0$, where $p(t)$ is a known function, can be solved either by using the integrating factor $I(t)=e^{\int p(t)\,\mathrm{d}t}$ or by separating variables.

Solution: Applying the Fourier transform with respect to the variable x, the problem becomes:

$$\begin{cases}i\omega\hat{u}=\frac{\partial \hat{u}}{\partial y}+6\,\hat{u}\,y^2 & \omega\in\mathbb{R}, y>0\\ \hat{u}(\omega,0)=\hat{f}(\omega) & \omega\in\mathbb{R}\end{cases} \tag{2}$$

The first line is an ODE in the variable y, which we can rewrite as

$$\frac{\partial\hat{u}}{\partial y}+(6\,y^2-i\omega)\hat{u}=0.$$

An integrating factor for this ODE is

$$I=e^{\int(6\,y^2-i\omega)\,\mathrm{d}y}=e^{2y^3-i\omega y}.$$

Multiplying the ODE by the integrating factor, we get that

$$\left(\frac{\partial\hat{u}}{\partial y}+(6\,y^2-i\omega)\hat{u}\right)I=\frac{\partial(\hat{u}\,I)}{\partial y}=0\Longrightarrow\hat{u}\,I=C(\omega)$$

where $C(\omega)$ depends only on ω. Therefore,

$$\hat{u}(\omega,y)=\frac{C(\omega)}{I}=C(\omega)e^{-2y^3+i\omega y}.$$

Evaluating at $y=0$, and using the second equation in (2), we get

$$\hat{u}(\omega,0)=C(\omega)=\hat{f}(\omega).$$

Therefore,

$$\hat{u}(\omega,y)=\hat{f}(\omega)e^{-2y^3+i\omega y}.$$

Taking the inverse Fourier transform in the variable ω, and using the table of formulas, we get

$$u(x,y) = e^{-2y^3} f(x+y).$$

$\square$

6. Find a solution of the integral equation

$$f(x) + \int_{-\infty}^{\infty} e^{-|y|} f(x-y)\,\mathrm{d}y = 3e^{-|x|}.$$

Solution: Note that we may write the equation as

$$f(x) + (e^{-|\cdot|} * f)(x) = 3e^{-|x|}.$$

Taking the Fourier transform on both sides yields

$$\hat{f}(\omega) + \frac{2}{1+\omega^2}\hat{f}(\omega) = 3\frac{2}{1+\omega^2},$$

and solving for $\hat{f}(\omega)$, we find

$$\hat{f}(\omega) = \frac{6}{3+\omega^2}.$$

It remains to find the inverse transform of $\hat{f}(\omega)$. By rewriting

$$\frac{6}{3+\omega^2} = \sqrt{3}\frac{1}{\sqrt{3}}\frac{2}{1+\left(\frac{\omega}{\sqrt{3}}\right)^2}$$

and using the table of formulas (in particular $\widehat{f(ax)} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{|a|}\right)$), we see that

$$f(x) = \sqrt{3}e^{-|\sqrt{3}x|}.$$

7. (a) Compute the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ given by the function

$$f(x) = e^{iax},$$

where $a \in \mathbb{R}$ is a constant.

Solution: Using the properties of the Fourier transform, we get

$$\mathcal{F}[e^{iax}] = \mathcal{F}[e^{iax}.1] = \mathcal{F}[1](\omega - a) = 2\pi\delta_a.$$

$\square$

(b) Compute the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ given by the function

$$f(x) = \cos(x).$$

Solution: By Euler's formula, we have that

$$\cos(x) = \frac{e^{ix} - e^{-ix}}{2}.$$

Using the linearity of the Fourier transform and the previous exercise, we conclude that

$$\mathcal{F}[\cos(x)] = \mathcal{F}[\frac{e^{ix} - e^{-ix}}{2}] = \frac{1}{2}(\mathcal{F}[e^{ix}] - \mathcal{F}[e^{-ix}]) = \pi(\delta_1 + \delta_{-1}).$$

$\square$

(c) Find a tempered distribution $g \in \mathcal{S}'(\mathbb{R})$ (that is not the zero distribution) which solves the equation

$$(x^2 - 1)\, g = 0.$$

Hint: Take the inverse Fourier transform.

Solution: Taking the inverse Fourier transform of the equation

$$x^2 g(x) = g(x)$$

one gets

$$-\frac{d^2 \check{g}(y)}{dy^2} = \check{g}(y).$$

This well-known ODE has solutions given by e^{iy} or $\cos(y)$, for example. Using the previous parts, we conclude that their Fourier transforms, which are respectively $2\pi\delta_1$ and $\pi(\delta_1 + \delta_{-1})$, solve the equation $(x^2 - 1)g = 0$.

More generally, observe that for all constants $c_1, c_2 \in \mathbb{C}$, the distribution $c_1\delta_1 + c_2\delta_{-1}$ solves the equation. Indeed, for every test function $\varphi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
\big((x^2 - 1)(c_1\delta_1 + c_2\delta_{-1})\big)(\varphi) &= (c_1\delta_1 + c_2\delta_{-1})\big((x^2 - 1)\varphi\big) = \\
&= c_1\delta_1\big((x^2 - 1)\varphi\big) + c_2\delta_{-1}\big((x^2 - 1)\varphi\big) = \\
&= c_1(1^2 - 1)\varphi(1) + c_2((-1)^2 - 1)\varphi(-1) = 0.
\end{aligned}$$

$\square$

8. Let $f_n : [a, b] \to \mathbb{C}$, with $n \geq 1$, be a sequence of integrable functions on the finite interval $[a, b]$, and assume that the sequence converges uniformly to the integrable function $f : [a, b] \to \mathbb{C}$.

(a) Show the existence of a constant $M > 0$ and of an integer n_0 such that, for every $n > n_0$, one has

$$|f_n(x)| \leq M \text{ for all } x \in [a, b]$$

(one says that the sequence f_n is *uniformly bounded*).

(b) Show that

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 \, \mathrm{d}x = 0$$

(one says that f_n *converges to* f *in mean-square*, or in $L^2([a, b])$).

Solution:

(a) Since f_n and f are assumed to be (Riemann) integrable, they are in particular by definition bounded on $[a, b]$, i.e. for all $a \leq x \leq b$, $|f_n(x)| \leq c_n$ and $|f(x)| \leq c$ for some constants c_n and c. Since $f_n \to f$ uniformly, given $\epsilon > 0$ there exists an integer N such that for any $n > N$, we have

$$|f_n(x) - f(x)| < \epsilon,$$

for all $a \leq x \leq b$. Hence, by using the triangle inequality and that $|f(x)| < c$, we get

$$|f_n(x)| \leq \epsilon + |f(x)| < \epsilon + c.$$

Now simply choose $M = \epsilon + c$ and $n_0 = N$.

(b) We have

$$\begin{aligned}
\int_a^b |f_n(x) - f(x)|^2 dx &\leq \sup_{a \leq x \leq b} |f_n(x) - f(x)| \int_a^b |f_n(x) - f(x)| dx \\
&\leq \sup_{a \leq x \leq b} |f_n(x) - f(x)| \int_a^b |f_n(x)| + |f(x)| dx \\
&\leq \sup_{a \leq x \leq b} |f_n(x) - f(x)| (b - a)(M + c),
\end{aligned}$$

where we used the result of (a) and the bound on $|f(x)|$ in the last inequality. Since $f_n \to f$ uniformly, we have $\sup_{a \leq x \leq b} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. We conclude that

$$\int_a^b |f_n(x) - f(x)|^2 dx \to 0,$$

as $n \to \infty$.

Alternatively, we can use that since f_n is integrable and converges uniformly to f on the compact set $[a, b]$, we have that $|f_n(x) - f(x)|^2$ is integrable and converges uniformly to 0 on $[a, b]$ (prove it). Hence,

$$\lim_{n \to \infty} \int_a^b |f_n(x) - f(x)|^2 dx = \int_a^b \lim_{n \to \infty} |f_n(x) - f(x)|^2 dx = 0.$$

$\square$

Formulas for Fourier Analysis course

Triangle inequalities
Let $x, y \in \mathbb{R}$ and f, g be functions. Then

- $\big| |x| - |y| \big| \leq |x \pm y| \leq |x| + |y|$
- $\left| \int_\Omega f(x)\,\mathrm{d}x \right| \leq \int_\Omega |f(x)|\,\mathrm{d}x$, for a subset $\Omega \subset \mathbb{R}$.

Some useful identities

- $e^{a+ib} = e^a(\cos(b) + i\sin(b))$
- $\int_\mathbb{R} x^n e^{-x^2/2}\,\mathrm{d}x = \begin{cases} \sqrt{2\pi}(n-1)(n-3)\ldots 5\cdot 3\cdot 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

Gram–Schmidt orthogonalisation
Let V be an inner product space and $\{v_1, \ldots, v_k\} \subset V$ be a linearly independent set of vectors. Then the Gram–Schmidt orthogonalisation is given by

$$
\begin{aligned}
u_1 &= v_1, & e_1 &= \frac{u_1}{\|u_1\|} \\
u_2 &= v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle} u_1, & e_2 &= \frac{u_2}{\|u_2\|} \\
u_3 &= v_3 - \frac{\langle u_1, v_3\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle u_2, v_3\rangle}{\langle u_2, u_2\rangle} u_2, & e_3 &= \frac{u_3}{\|u_3\|} \\
&\vdots & &\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} \frac{\langle u_j, v_k\rangle}{\langle u_j, u_j\rangle} u_j, & e_k &= \frac{u_k}{\|u_k\|}.
\end{aligned}
$$

Laplace transform

$f(t)$	$\tilde{f}(s) = F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
General formulas	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$e^{at} f(t)$	$F(s-a)$
$f(at), \quad a > 0$	$\frac{1}{a} F(\frac{s}{a})$
$f(t-a)H(t-a), \quad a > 0$	$e^{-as} F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0)$ $\quad -s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(u)\,du$	$s^{-1} F(s)$
$f * g(t) = \int_0^t f(u) g(t-u)\,du$	$F(s)\,G(s)$
Particular cases	
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$

Fourier Series

Functions of period 2π

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n})$$

Parseval's formula:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Functions of period T

Let $\Omega = 2\pi/T$

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t),$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\Omega t} \, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\Omega t \, dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\Omega t \, dt.$$

Parseval's formula:

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Some trigonometric identities

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

$$2 \sin a \cos b = \sin(a - b) + \sin(a + b)$$

$$2 \cos a \cos b = \cos(a - b) + \cos(a + b)$$

$$2 \sin^2 t = 1 - \cos 2t, \qquad 2 \cos^2 t = 1 + \cos 2t$$

Fourier transform

$f(t)$	$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$
General formulas	
$\alpha f(t)+\beta g(t)$	$\alpha\hat{f}(\omega)+\beta\hat{g}(\omega)$
$e^{i\alpha t}f(t)$	$\hat{f}(\omega-\alpha)$
$f(t-t_0)$	$e^{-it_0\omega}\hat{f}(\omega)$
$f(-t)$	$\hat{f}(-\omega)$
$f(at)\quad(a\neq 0)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$
$tf(t)$	$i\,\frac{d\hat{f}}{d\omega}$
$f'(t)$	$i\omega\hat{f}(\omega)$
$\hat{f}(t)$	$2\pi f(-\omega)$
$f*g(t)=\int_{-\infty}^{\infty} f(u)g(t-u)\,du$	$\hat{f}(\omega)\hat{g}(\omega)$
Particular cases	
$\chi_{[-a,a]}$	$\frac{2\sin a\omega}{\omega}$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert\omega\rvert}$
$e^{-t^2/2}$	$\sqrt{2\pi}e^{-\omega^2/2}$
δ	1
1	$2\pi\delta$

Plancherel's formulas:

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega$$

$$\int_{-\infty}^{\infty}f(t)\,\overline{g(t)}\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,d\omega$$