MATH3341:
Partial Differential Equations with Applications

Allan P. Fordy
Department of Applied Mathematics,
University of Leeds,
Leeds, LS2 9JT, UK

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Chapter 0

Introduction and Overview

This module is about *partial differential equations*, so we start by describing what these are. First, we have the distinction between *ordinary* and *partial* differential equations, abbreviated as ODEs and PDEs respectively.

Ordinary differential equations involve one or more functions of a *single independent variable*, which we call t (if interpreted as "time"). To be a *differential* equation, it must involve *derivatives* of these functions. The highest order derivative is called the *order* of the differential equation. Some examples for the single *dependent* function $x(t)$ are:

$$\frac{dx}{dt} + x^2 = 0, \quad \frac{d^2x}{dt^2} + 4x = \cos(t), \quad t^3 \frac{d^3x}{dt^3} - 3t\,\frac{dx}{dt} + 3x = 0, \tag{1}$$

with *general solutions*, respectively,

$$x(t) = \frac{1}{t + c_1}, \quad x(t) = \frac{1}{3}\cos t + c_1 \cos 2t + c_2 \sin 2t, \quad x(t) = \frac{c_1}{t} + c_2 t + c_3 t^3.$$

The ODEs listed in (1) are respectively first, second and third order, with general solutions depending upon one, two and three arbitrary constants c_i, respectively. One way of *fixing* these constants is to specify *initial conditions*, meaning that we give the values of $x(t)$ and as many of its derivatives as necessary at the the initial time $t = t_0$. When the equation is of order n, the "necessary number of derivatives" is $n - 1$.

Partial differential equations involve one or more *dependent* functions of *two or more independent* variables, $x_1,\ x_2,\ \cdots,\ x_n,\ n \geq 2$. We usually have $n = 2$, 3 or 4, so usually call the independent variables x, y, z, t (or a selection from this list). To be a *partial differential* equation, it must involve *partial derivatives* of these functions. The highest order partial derivative is called the *order* of the PDE. For example, the PDEs

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

are respectively first and second order. The second of these equations is first order in the $t-$derivative, but *second* order in the $x-$derivative. We often use the notation $u_x = \frac{\partial u}{\partial x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc., after which the above PDEs are written

$$u_t + uu_x = 0, \quad \text{and} \quad u_t = u_{xx}.$$

A solution of either of these PDEs is *any* function $u(x,t)$, which satisfies the equation (as in the solution of an ODE). However, whilst the general solution of an ODE depends upon only a *finite* number of parameters, that of a PDE depends upon a number of *arbitrary functions* of $n-1$ variables, where n is the number of *independent* variables. The specific number of arbitrary functions depends upon the order of the PDE (and the number of *dependent* functions in the case of a system). For instance, the *general solution* of the simple PDE $u_t - u_x = 0$ requires an arbitrary function of one variable:

$$u_t - u_x = 0 \quad \text{if and only if} \quad u(x,t) = \varphi(x+t).$$

This function can be *fixed* by specifying initial conditions. For our first order ODE of (1), this would amount to specifying a *single number* $x(0)$. For the above PDE we have to specify a *single function* of the variable x at $t=0$: $u(x,0) = u_0(x)$. This gives $\varphi(x) = u_0(x)$, so $u(x,t) = u_0(x+t)$ (φ was *arbitrary*, with u_0 *given*). For instance,

$$\left.\begin{aligned} u_t - u_x &= 0, \\ u(x,0) &= \frac{1}{x^2+1}, \end{aligned}\right\} \quad \Rightarrow \quad u(x,t) = \frac{1}{(x+t)^2+1}.$$

This particular equation is said to be *hyperbolic*, for which initial conditions are appropriate. The first order equation of (2) can be solved in a similar way, but the solution is rather more complicated. Such equations are the subject of Chapter 1.

The second order equation of (2) requires both an initial condition $u(x,0) = u_0(x)$ and some *boundary conditions*, meaning that we specify $u(a,t)$ and $u(b,t)$ at "end points" $x=a$ and $x=b$ (which may be either finite or infinite). For instance, it is easy to see that

$$u_n(x,t) = e^{-n^2 t}\sin(nx), \quad \text{for integer } n \quad \Rightarrow \quad \left\{\begin{aligned} &u_{nt} = u_{nxx}, \\ &u_n(0,t) = u_n(\pi,t) = 0. \end{aligned}\right.$$

Since the PDE is *linear*, we may take any linear combination of these functions and still satisfy the same equation *and* boundary conditions at $x=0$ and $x=\pi$. We can therefore write down an infinite parameter solution of this equation:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx), \tag{3}$$

satisfying the given boundary conditions. The remaining question would be whether or not we could find the constants c_n so that $u(x,0) = u_0(x)$, for some class of functions $u_0(x)$, satisfying $u_0(0) = u_0(\pi) = 0$. The answer for general $u_0(x)$ is given in terms of Fourier series (see MATH2431, Fourier Series, PDEs and Transforms). For simple examples, we do not need this general theory:

$$u_0(x) = \sin x + 3\sin 2x \quad \Rightarrow \quad u(x,t) = e^{-t}\sin x + 3e^{-4t}\sin 2x,$$

since $e^0 = 1$.

The precise formulation of just which initial and boundary values are appropriate for a given PDE is a difficult question in general. In Chapter 2 a classification of *linear, second order* PDEs is given. We find that these PDEs fall into three classes, called *hyperbolic, elliptic* and *parabolic*. The appropriate type of *initial boundary value problem* (IBVP) depends upon the class. Having determined the appropriate IBVP, we then have the problem of actually constructing a solution. This can be an impossibly difficult task, but there exist some general techniques for various classes of linear PDE. Some of these are discussed in Chapters 3 and 4.

Partial differential equations are important for modelling phenomena in the physical and biological sciences. The majority of well known "named" PDEs arose in *mathematical physics* and most of the general solution techniques first arose when solving specific problems (specific examples), related to these disciplines. In more recent times, PDEs have gained importance in the biological sciences. This is partly because some biological and physiological processes are direct consequences of chemical reactions, material transport, energy transfer, and so on, which themselves derive from the physical sciences. On the other hand, subjects such as ecology and population dynamics are not so closely related to the physical sciences, but are still modelled by ordinary and partial differential equations. Many of these modelling techniques can also be applied in the realm of economics and the social sciences, although these often include more complicated random processes.

Whilst *linear, second order* PDEs are very important, many equations which arise in applications are *nonlinear*. This means that we cannot just take linear combinations of simple solutions, such as (3), to build more complicated ones. Often, we can only obtain *qualitative* information, such as in *dynamical systems*, but there are some techniques available, such as finding *travelling wave solutions* or using *symmetry techniques*. These topics will be briefly discussed if time permits.

Chapter 1

First Order Partial Differential Equations.

In this chapter we introduce the basic theory of *first order partial differential equations*. The following topics are included:

- Method of characteristics
- Linear equations
- Semi-linear equations
- First integrals
- Quasi-linear equations
- Wave breaking and jumps
- The Cauchy problem
- Well posed problems

Basic Definitions

First discuss equations of the type

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u). \qquad (1)$$

Since the equation is **linear in** u_x **and** u_y, it is called **quasi-linear**.

Note that generally **all coefficients** depend upon u.

Solved by the **method of characteristics**, which essentially reduces the problem to solving a system of first order ODEs.

There are two very important special cases.

Linear Equations.

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = d(x, y).$$

Semi-linear Equations.

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = c(x, y, u).$$

$c(x, y, u)$ *is now allowed to be* **nonlinear in** u.

Method of Characteristics

Characteristics are special curves related to Equation (1).

For any curve $(x(t), y(t))$, the **directional derivative** of $u(x(t), y(t))$ is:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

If $(x(t), y(t))$ is chosen so that

$$\frac{dx}{dt} = a(x, y, u)\,, \quad \frac{dy}{dt} = b(x, y, u), \tag{2a}$$

then u satisfies the ODE:

$$\frac{du}{dt} = c(x, y, u). \tag{2b}$$

The ODEs (2a), (2b) are called the **characteristic system** for (1).

In theory (*subject to some conditions on the functions* a, b, c) we can **solve this system** as a function of t and **initial condition**

$$(x(0), y(0), u(0)) = (x_0, y_0, u_0).$$

This is just for **one characteristic**.

To solve the PDE we need to solve for **each characteristic** crossing some **initial curve** $(x_0(s), y_0(s), u_0(s))$.

We obtain the **solution** $(x(s, t), y(s, t), u(s, t))$, with initial condition

$$(x(s, 0), y(s, 0), u(s, 0)) = (x_0(s), y_0(s), u_0(s)).$$

Linear Equations

Consider the general linear equation

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y).$$

The characteristic system splits into two parts

$$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \tag{1.1}$$

$$\frac{du}{dt} + c(x,y)u = d(x,y). \tag{1.2}$$

The first pair of equations is a **self-contained** 2D system in the $x - y$ plane.

Generally, these equations **cannot** be explicitly solved. Our examples will be solvable.

Suppose we have a solution $(x(t), y(t))$. Then, equation (1.2) is a linear equation in u with **known** coefficients as functions of t.

This can be solved **in general** using an **integrating factor**. However, in many examples, even more elementary methods can be used.

Examples can be found in the supplementary pages.

Semi-linear Equations

Consider the general semi-linear equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u).$$

The characteristic system still splits into two parts

$$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \tag{1.3}$$

$$\frac{du}{dt} = c(x,y,u). \tag{1.4}$$

The first pair of equations is still a **self-contained** 2D system in the $x - y$ plane, and identical to the **linear case**.

As before, these equations **cannot** generally be explicitly solved, but our examples will be solvable.

Suppose we have a solution $(x(t), y(t))$. Then, equation (1.4) is now a **nonlinear** equation in u, but still with **known** coefficients as functions of t.

There is no longer a **general** way of solving this. However, there are examples which **can** be solved.

Examples can be found in the supplementary pages.

First Integrals

A function $F(x, y, u)$ is said to be a **first integral** for the characteristic system (2a), (2b) if

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial u}\frac{du}{dt} = (a, b, c) \cdot \nabla F = 0.$$

This means that the characteristic curves are tangent to the **level surfaces** of $F(x, y, u)$, so that

$$F(x, y, u) = \alpha \text{ along characteristics,}$$

for some constant α.

Any characteristic with initial conditions in this surface will remain in the surface for all t. The value of the constant α depends on the initial conditions, since

$$F(x(t), y(t), u(t)) = F(x_0, y_0, u_0) = \alpha.$$

Each first integral defines such a surface. Two first integrals are said to be **independent** if the normals to these surfaces are independent (we must check that ∇F and ∇G are linearly independent).

Any function of these first integrals is another first integral (but, of course, not independent of them, so giving no new information).

The intersection of two such surfaces is a curve, which must be characteristic.

Using First Integrals

Suppose we are given an **initial value problem**

$$a(x,y,u)\,\frac{\partial u}{\partial x} + b(x,y,u)\,\frac{\partial u}{\partial y} = c(x,y,u).$$
$$u = \varphi(s) \ \text{ on some initial curve.}$$

The first and most difficult problem is to actually **find** a first integral! In fact, for the calculations now described we need **two** first integrals.

Suppose we have found **two independent first integrals** $F(x,y,u)$ and $G(x,y,u)$.

On the initial curve we have

$$x = X(s), \quad y = Y(s), \quad u = \varphi(s),$$

so

$$F(X(s), Y(s), \varphi(s)) = f(s) \quad \text{and} \quad G(X(s), Y(s), \varphi(s)) = g(s).$$

Since F and G are first integrals, we have

$$F(x,y,u) = F(x(s,t), y(s,t), u(s,t)) = f(s)$$
$$G(x,y,u) = G(x(s,t), y(s,t), u(s,t)) = g(s).$$

Eliminating s between these two equations gives a relationship between x, y and u, which is an **implicit** formula for $u(x,y)$.

Using First Integrals continued. . .

For some simple examples it is possible to find a function $\psi(s)$, such that $g(s) = \psi(f(s))$.

If we can find such a function ψ, then

$$G(x, y, u) - \psi(F(x, y, u)) = g(s) - \psi(f(s)) = 0,$$

since $G(x, y, u) - \psi(F(x, y, u))$ is a first integral.

The formula

$$G(x, y, u) = \psi(F(x, y, u))$$

is **not an identity**, since this would mean that F and G are **not independent**.

The choice of ψ depends explicitly on the initial conditions, so is **different for different solutions**.

It is, in fact, an **implicit formula for the particular solution**.

Quasi-Linear Equations

When $a(x,y,u)$ and/or $b(x,y,u)$ **actually depend upon** u, then the system

$$\dot{x} = a(x,y,u), \quad \dot{y} = b(x,y,u)$$

is no longer self contained, so **cannot be solved independently**.

We **must** treat the **full characteristic system**

$$\frac{dx}{dt} = a(x,y,u)\,, \quad \frac{dy}{dt} = b(x,y,u), \quad \frac{du}{dt} = c(x,y,u).$$

This is more difficult, but possible for some simple cases, often using **first integrals**.

Examples can be found in the supplementary pages.

A more detailed analysis of: $u_y + uu_x = 0$ with $u(x,0) = \varphi(x)$

The characteristic equations are:

$$\dot{x} = u\,, \quad \dot{y} = 1\,, \quad \dot{u} = 0,$$

with

$$x = s, \quad y = 0, \quad u = \varphi(s) \quad \text{when} \quad t = 0.$$

We have

$u = \varphi(s)$, (a **constant**, depending upon the initial conditions).

Then

$$\frac{dx}{dt} = \varphi(s) \;\Rightarrow\; x(s,t) = \varphi(s)t + \alpha(s) = \varphi(s)t + s,$$

$$\frac{dy}{dt} = 1 \;\Rightarrow\; y = t + \beta(s) = t,$$

The above are the values of x, y, u a "distance" t **along the characteristic** emanating from the point s on the $x-$axis.

The characteristics **project down** to straight lines in the $x - y$ plane:

$$y = \frac{x - s}{\varphi(s)}.$$

The gradients depend upon initial conditions.

$u_y + uu_x = 0$ **with** $u(x, 0) = \varphi(x)$ continued. . .

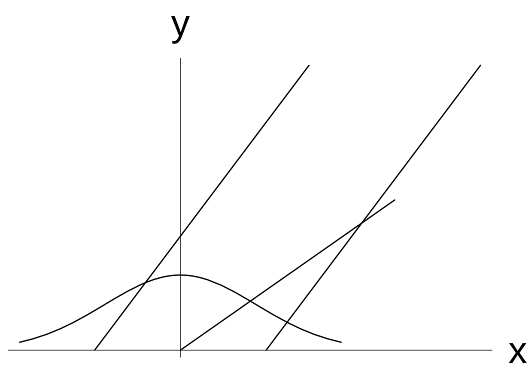

Figure 1.1: The projection of characteristics into the $x - y$ plane.

Along these lines, the solution is a constant, given by

$$u(s, t) = \varphi(s),$$

giving the unexpected result that $u(s, t)$ can have **more than one value** at any point at which two or more characteristics intersect!

However, we must be careful when we consider u as a function of x and y, since we need to **invert the formulae** for $x(s, t)$ and $y(s, t)$.

We must check the Jacobian of the "transformation" is non-zero.

$u_y + uu_x = 0$ with $u(x, 0) = \varphi(x)$ continued...

The Jacobian of

$$x = \varphi(s)t + s, \quad y = t \qquad \text{is} \qquad 1 + t\varphi'(s),$$

which is zero when

$$t = -\frac{1}{\varphi'(s)}.$$

This only happens for $t > 0$, when $\varphi'(s) < 0$.

This potential breakdown of our solution $u(x, y)$ is therefore entirely determined by the initial condition.

The first time t_B ("breaking time") at which this occurs is on the characteristic emanating from the point on the initial curve at which $\varphi'(s)$ is **maximally negative**.

$\varphi(s) = e^{-s^2}$ has **maximally negative gradient** at the point of inflection ($s = \frac{1}{\sqrt{2}}$), giving $t = \sqrt{\frac{2}{e}} = 1.16582$.

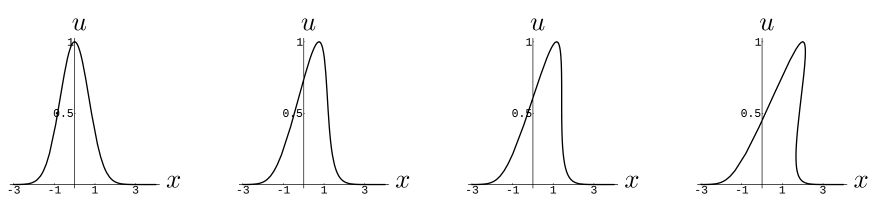

Figure 1.2: The evolution of $\varphi(s) = e^{-s^2}$. The values of t are 0, 0.75, $\sqrt{\frac{e}{2}}$, 2.

Left slope diminishes and right one steepens (**vertical** when $t = \sqrt{\frac{2}{e}}$).

Equation $u_y + uu_x = 0$ with $u(x,0) = \varphi(x) =$ The Tent Function

Consider the initial condition

$$\varphi(s) = \begin{cases} s & \text{for} \quad 0 \le s \le \frac{1}{2}, \\ 1-s & \text{for} \quad \frac{1}{2} \le s \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is the **tent function** which is the first plot of the Figure.

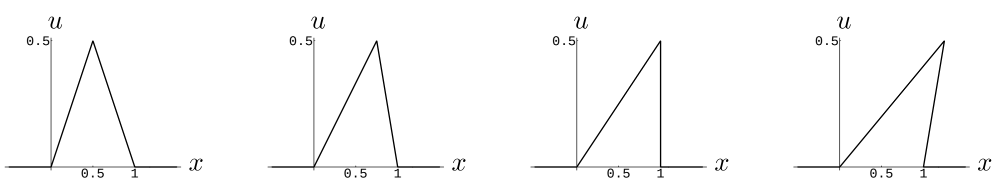

Figure 1.3: The evolution of the tent function. The values of y are 0, 0.5, 1, 1.5.

We can think of this as a simplified form of the bell curve of the previous Figure, but we can now **explicitly calculate** all formulae.

The **horizontal parts** of the graph remain fixed (amplitude is zero).

Each part of the graph with non-zero amplitude moves steadily to the right with speed equal to its height.

The two sides of the "tent" remain as straight lines, but with changing gradients, intersecting at the height of $\frac{1}{2}$.

Equation $u_y + uu_x = 0$ with tent function continued...

We use the formulae

$$x = \varphi(s)t + s, \quad y = t, \quad u = \varphi(s)$$

for each part of the tent.

In the $x-y$ plane, the characteristics project to intersecting lines, with equations: $x = s$ for $s \leq 0$ or $s \geq 1$, and

$$y = \begin{cases} \dfrac{x-s}{s} & \text{for} \quad 0 < s \leq \dfrac{1}{2} \\ \dfrac{x-s}{1-s} & \text{for} \quad \dfrac{1}{2} \leq s < 0. \end{cases}$$

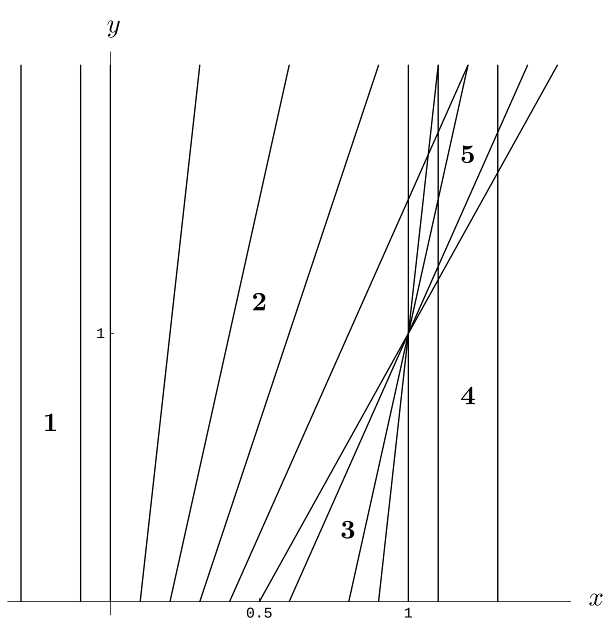

Figure 1.4: The characteristics projected into the $x - y$ plane. Five distinct regions are marked.

Equation $u_y + uu_x = 0$ with tent function continued...

For the left line: $0 \leq s \leq \frac{1}{2}$:

$$\left.\begin{array}{l} x = s(1+t), \\ y = t, \\ u = s, \end{array}\right\} \quad \Rightarrow \quad u(x,y) = \frac{x}{1+y},$$

For the right line: $\frac{1}{2} \leq s \leq 1$:

$$\left.\begin{array}{l} x = s(1-t)+t, \\ y = t, \\ u = 1-s, \end{array}\right\} \quad \Rightarrow \quad u(x,y) = \frac{1-x}{1-y},$$

These are to be thought of as graphs above the $x-$axis for each value of y.

These **lines intersect** when

$$\frac{x}{1+y} = \frac{1-x}{1-y} \quad \Rightarrow \quad x = \frac{y+1}{2}.$$

From these formulae, we see that $y = 1$ (meaning $t = 1$) is a bad value at which the **right line becomes vertical**.

Local Conservation Laws and jump conditions

We have seen that for the equation $u_y + uu_x = 0$ with rather general initial conditions $u(x, 0) = \varphi(x)$ (only requiring that the function $\varphi(x)$ should have a region with **negative gradient**), the solution will develop an infinite gradient and then become multi-valued.

This is quite a general phenomenon in the context of **quasi-linear** equations, but we will only explore this in the context of our specific equation $u_y + uu_x = 0$.

In Example 10 we plotted the characteristics (when projected onto the $x - y$ plane). The multi-valued solution corresponded to region 4, where these characteristics intersected.

For water waves, being **multi-valued** has a reasonable physical interpretation in terms of **wave breaking**.

However, if u represents a density or pressure (as in a gas) it makes no sense to have multi-valued functions.

We can avoid this problem if we allow our solution to have a discontinuity. Such solutions are called **weak solutions** of the PDE, since they cannot really be solutions of a PDE as some derivative fails to exist. Such solutions can physically represent **shock waves**.

Equation $u_y + uu_x = 0$ with a jump

We write the PDE as a **divergence** and then replace the **differential** equation by an **integral conservation law**:

$$u_y + \left(\frac{1}{2}u^2\right)_x = 0 \quad \rightarrow \quad \partial_y \int_a^b u(x,y)dx + \left[\frac{1}{2}u^2\right]_a^b = 0$$

where integration is taken over any interval $[a, b]$ of the $x-$axis.

In **region** 4, where there is an ambiguity in the definition of u, we place a curve, which is to be the location of a **jump** between the two values of $u(x, y)$.

We extend region 1 into the part of region 4 to the left of this curve and likewise, region 3 to the right.

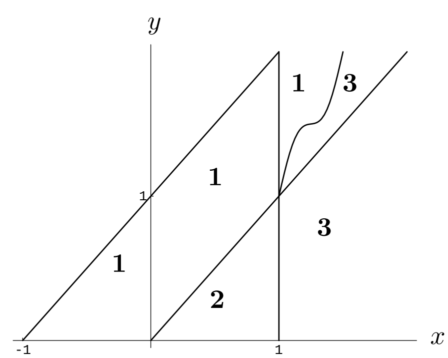

Figure 1.5: Regions 1 and 3 extended to the curve $x = \sigma(y)$, cutting the former region 4 into two parts.

For $0 \leq y < 1$ (fixed), $u(x, y)$ **changes smoothly** from $u = 1$ in region 1, through region 2 and into region 3 with value $u = 0$.

At $y = 1$ a **discontinuity** develops with $u(x, y)$ **jumping** from $u = 1$ to $u = 0$ across some curve $x = \sigma(y)$, which we must now determine.

Equation $u_y + uu_x = 0$. Using the integral conservation law

Choosing a and b respectively to the left and right of the curve $x = \sigma(y)$ in the **integral conservation law**

$$\partial_y \int_a^b u(x,y)dx + \left[\frac{1}{2}u^2\right]_a^b = 0,$$

we may split the integral as follows:

$$\partial_y \int_a^b u(x,y)dx = \partial_y \int_a^{\sigma(y)} u(x,y)dx + \partial_y \int_{\sigma(y)}^b u(x,y)dx.$$

We also have

$$\partial_y \int_a^{\sigma(y)} u(x,y)dx = \sigma'(y)u_- + \int_a^{\sigma(y)} u_y(x,y)dx$$

and

$$\partial_y \int_{\sigma(y)}^b u(x,y)dx = \int_{\sigma(y)}^b u_y(x,y)dx - \sigma'(y)u_+.$$

where $u_\pm$ represent the value of u to the left and right of the jump:

$$u_\pm = \lim_{\delta \to 0} u(\sigma(y) \pm \delta, y).$$

Equation $u_y + uu_x = 0$. The jump condition

We therefore have

$$\partial_y \int_a^b u(x,y)dx = \sigma'(y)(u_- - u_+) + \int_a^{\sigma(y)} u_y(x,y)dx + \int_{\sigma(y)}^b u_y(x,y)dx.$$

Here, we are integrating with respect to x, for each y, resulting in a function of y, which is **continuously differentiable on each side of the jump**.

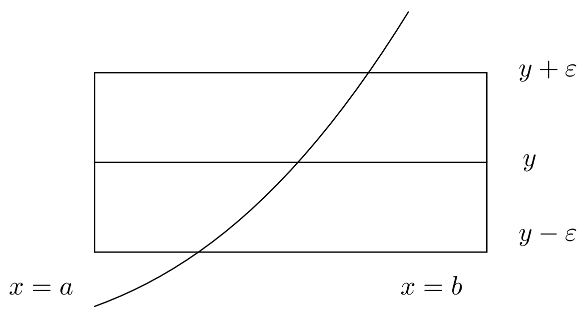

Figure 1.6: The calculation of the "jump conditions".

Our derivatives with respect to y are calculated on each side of the jump and then we take limits as $|b-a| \to 0$.

In this process, the two integrals have limit zero.

Thus we wind up with

$$\sigma'(y)(u_- - u_+) + \frac{1}{2}(u_+^2 - u_-^2) = 0.$$

The Cauchy Problem

Given an IVP, we can ask whether a solution exists, whether it is unique and what properties it has. This is referred to as a **Cauchy problem**, after A.L. Cauchy (1789 – 1857).

Cauchy Theorem

Consider the first order quasi-linear PDE:

$$au_x + bu_y = c,$$

where a, b and c are continuous functions of (x, y, u) and have continuous partial derivatives with respect to their arguments.

Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$, the initial values $u = u_0(s)$ are prescribed, x_0, y_0 and u_0 being continuously differentiable functions for $0 < s < 1$ (for example).

Furthermore, the initial curve should be nowhere characteristic:

$$\frac{dy_0}{ds}a(x_0(s), y_0(s), u_0(s)) - \frac{dx_0}{ds}b(x_0(s), y_0(s), u_0(s)) \neq 0.$$

Then **there exists a unique solution** *$u(x, y)$ defined in some neighbourhood of the initial curve, which satisfies the PDE and the initial conditions:*

$$u(x_0(s), y_0(s)) = u_0(s).$$

This is a **local** theorem. Indeed, we have seen that a smooth solution can cease to exist in finite time, developing a discontinuity.

The Characteristic Cauchy Problem

We have already seen the geometric picture of the solution, starting with an initial curve γ, which is **not characteristic**, and building up the solution surface by drawing characteristics through the points of γ.

The Cauchy problem is thus the problem of finding the solution surface which contains a given initial curve.

The theorem states that if γ is non-characteristic, then the solution is unique in some neighbourhood of γ.

We deduce that if two solution surfaces intersect, then the curve of intersection must be characteristic. Otherwise, we could use this as an initial curve γ and obtain a *unique* solution surface containing it.

If γ **is characteristic**, then **if there exists a solution at all, it will not be unique**.

In fact, when a solution does exist, there are **infinitely many** of them (**any** solution surface containing this particular characteristic).

In these circumstances, a solution will only exist if the initial function on γ is a solution of:

$$\dot{u} = c(x, y, u)$$

on this characteristic.

Well Posed Problems

Another important question is whether the solution is **continuously dependent upon initial data**.

In our notation, *"Is u a continuous function of s as well as of t?"*

In physical applications, this is very important, since we can only specify initial data with some limited accuracy (e.g. in an experiment).

A Cauchy problem *is said to be* **well posed** *if:*

1. *there exists a solution,*
2. *the solution is unique,*
3. *the solution is continuously dependent upon initial data.*

Otherwise, it is said to be **ill-posed**.

Example with circular characteristics

Consider the simple linear equation:

$$yu_x - xu_y = 0 \quad \text{with} \quad u(x,0) = \varphi(x) \quad \text{for} \quad -\infty < x < \infty,$$

whose characteristics are circles:

$$\dot{x} = y\,, \quad \dot{y} = -x \quad \Rightarrow \quad x^2 + y^2 = s^2, \text{ a constant.}$$

The solution is given by: $\dot{u} = 0$ on circles $\Rightarrow$ $u = f(x^2 + y^2)$.

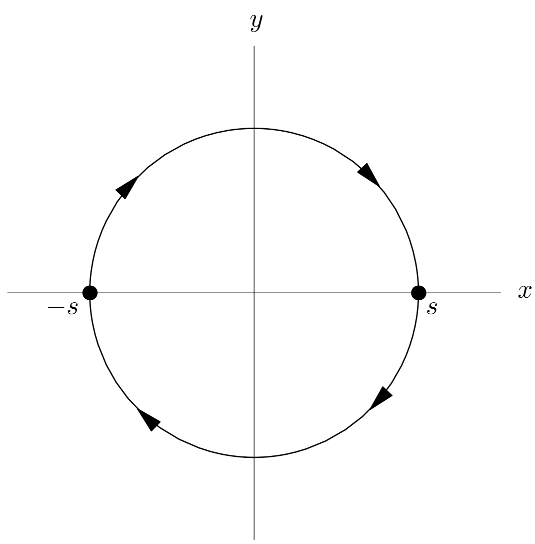

Figure 1.7: Characteristics are circles. φ cannot be chosen arbitrarily on the $x-$axis.

If we specify u at $x = s$, $y = 0$ with $s > 0$, then u is **determined** at $x = -s$, $y = 0$. We cannot **choose** u **arbitrarily** on both positive and negative $x-$axis. **Thus no solution exists**.

This does not contradict our existence theorem, since a and b vanish for $x = y = 0$.

We can resolve this problem by specifying $u(x,0)$ only for $x > 0$.

Chapter 2

The Classification of Second Order PDEs.

In this chapter we classify *linear, second order partial differential equations* by introducing a change in *independent coordinates* in order to reduce the equation to one of 3 *canonical forms.* The method of solution (discussed in later chapters) depends upon the equation type. The following topics are included:

- General change of coordinates
- Hyperbolic equations
- Parabolic equations
- Elliptic equations
- Complex variables
- Equations of mixed type

Basic Definitions

We now consider equations of the type

$$au_{xx} + 2bu_{xy} + cu_{yy} = d, \tag{1}$$

with **functional dependence**

$$a(x,y), \quad b(x,y), \quad c(x,y) \quad d(x,y,u,u_x,u_y).$$

This is **semi-linear**, since the **highest order part is linear**. Equation (1) is **linear** when

$$d(x,y,u,u_x,u_y) = e(x,y)u_x + f(x,y)u_y + g(x,y)u + h(x,y).$$

In the form (1), the equation is too complicated in general.

We show that new independent co-ordinates (ξ, η) can be found so that (1) takes one of 3 **canonical forms**.

1. $u_{\xi\eta} + \cdots = 0$,
2. $u_{\xi\xi} + \cdots = 0$,
3. $u_{\xi\xi} + u_{\eta\eta} + \cdots = 0$,

where the dots refer to terms in lower order derivatives.

These 3 canonical forms are respectively called **hyperbolic**, **parabolic** and **elliptic**.

The 3 most common examples are respectively

Linear wave equation: $u_{\xi\eta} = 0$, in characteristic co-ordinates,

The heat equation: $u_{xx} = u_t$

Laplace's equation: $u_{xx} + u_{yy} = 0$.

General Change of Co-ordinates

Define the **differential operator** L by

$$Lu \equiv \left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} \right) u. \qquad (2)$$

Introduce a general change of co-ordinates:

$$\xi = \xi(x, y)\,, \quad \eta = \eta(x, y),$$

with non-vanishing Jacobian:

$$\mathcal{J} = \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0.$$

The chain rule can be written as

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix},$$

with the appearance of the "Jacobian matrix" J, whose determinant is the Jacobian $\mathcal{J}$.

The second derivatives transform as

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \xi_x^2 \frac{\partial^2}{\partial \xi^2} + 2\xi_x \eta_x \frac{\partial^2}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2}{\partial \eta^2} + \xi_{xx}\frac{\partial}{\partial \xi} + \eta_{xx}\frac{\partial}{\partial \eta}, \\ \frac{\partial^2}{\partial x \partial y} &= \xi_x \xi_y \frac{\partial^2}{\partial \xi^2} + (\xi_x \eta_y + \xi_y \eta_x)\frac{\partial^2}{\partial \xi \partial \eta} + \eta_x \eta_y \frac{\partial^2}{\partial \eta^2} + \xi_{xy}\frac{\partial}{\partial \xi} + \eta_{xy}\frac{\partial}{\partial \eta}, \end{aligned}$$

and similarly for $\frac{\partial^2}{\partial y^2}$.

In fact, we can obtain the formulae for ∂_x^2 and ∂_y^2 from that of $\partial_x \partial_y$ by simply letting $y \to x$ or $x \to y$.

Change of Co-ordinates continued...

Substituting these formulae into (2) and collecting terms, leads to:

$$Lu = \hat{L}u + (L\xi)u_\xi + (L\eta)u_\eta,$$

where

$$\hat{L} = A\frac{\partial^2}{\partial\xi^2} + 2B\frac{\partial^2}{\partial\xi\partial\eta} + C\frac{\partial^2}{\partial\eta^2},$$

with

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \quad B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \quad L\xi = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy},$$

and similarly for $L\eta$.

Whenever ξ and η have **non-zero second partial derivatives**, some extra first order terms are introduced into the equation:

$$\hat{L}u = d - (L\xi)u_\xi - (L\eta)u_\eta,$$

where we must write d in terms of u_ξ and u_η, where appropriate.

Matrix Formula and Discriminant

A simple way of remembering these formulae is to write them as a single **matrix formula**:

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \quad \Rightarrow \quad \hat{M} = J^T M J,$$

where J is the Jacobian matrix and J^T its transpose.

We define the **discriminant** of L and $\hat{L}$ respectively as:

$$\Delta = b^2 - ac \quad \text{and} \quad \hat{\Delta} = B^2 - AC,$$

which are just the negatives of the determinants of the matrices M and $\hat{M}$.

It follows that under the above change of co-ordinates we have:

$$\hat{\Delta} = \mathcal{J}^2 \Delta.$$

For non-vanishing Jacobian, $\mathcal{J} \neq 0$, the **discriminant does not change sign**.

We can, therefore, **classify the differential operator** L by the sign of Δ, which is **independent of co-ordinates**.

The operator L is given the name

1. **hyperbolic** when $\Delta > 0$,
2. **parabolic** when $\Delta = 0$,
3. **elliptic** when $\Delta < 0$.

Hyperbolic Equations

We show that when $\Delta > 0$, we can simultaneously set $A = C = 0$. We assume that either a or c are non-zero, otherwise there is nothing to do.

We seek ξ and η so that:

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0,$$

and similarly for η.

For non-zero a we get the following quadratic equation for ξ_x/ξ_y:

$$a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\left(\frac{\xi_x}{\xi_y}\right) + c = 0.$$

For any set of co-ordinates ξ, η the **co-ordinate curves** are given by:

$$\xi(x, y) = \text{ const. and } \quad \eta(x, y) = \text{ const.}$$

The **differential** $d\xi$ is given by:

$$d\xi = \xi_x dx + \xi_y dy,$$

so, on co-ordinate curves,

$$d\xi = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\xi_x}{\xi_y},$$

which is a differential equation for co-ordinate curve $y(x)$, and similarly for η.

This is an identity, since *it is true for* **any** *function* ξ.

Hyperbolic Equations continued...

However, substituting this expression into our formula for A gives an explicit differential equation for $y(x)$, with coefficients determined by the original differential operator L:

$$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0.$$

This is a quadratic equation for $\frac{dy}{dx}$, with the **same positive discriminant** Δ. It thus has **two real roots**:

$$\frac{dy}{dx} = \frac{1}{a}\left(b \pm \Delta^{1/2}\right).$$

Since ξ and η satisfy the **same** equation, we can take the two different roots for $\frac{dy}{dx}$ to define **two different families of curves.**

We take the "+" sign to define the level ξ curves and the "−" sign to define the level η curves.

We solve these equations to obtain $\xi(x, y)$ and $\eta(x, y)$ and then transform the equation to:

$$2Bu_{\xi\eta} + (L\xi)u_\xi + (L\eta)u_\eta = d.$$

These special curves are called **characteristics**, *since they play a role similar to those in first order PDEs.*

Parabolic Equations

We now have $\Delta = 0$, but try repeating our previous arguments.

We can make $A = 0$ by **choosing** ξ **as before**. With

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y},$$

we have:

$$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0,$$

as before, but since $\Delta = 0$, there is **only one solution** (a double root):

$$\frac{dy}{dx} = \frac{b}{a},$$

so that it is not possible to **simultaneously** set A and C to zero.

However,

$$\Delta = b^2 - ac = 0 \Rightarrow B^2 - AC = 0.$$

Thus $A = 0 \Rightarrow B = 0$.

Thus, **in the parabolic case, we have** $A = B = 0$ and ξ defined by $\frac{dy}{dx} = \frac{b}{a}$, with:

$$\hat{L}u = \left(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\right) u_{\eta\eta}.$$

We can **choose** η **in any convenient way**, subject only to

$$\mathcal{J} = \xi_x\eta_y - \xi_y\eta_x \neq 0.$$

Elliptic Equations

In this case $\Delta = b^2 - ac < 0$, so **there do not exist real co-ordinates** ξ and η which make $A = C = 0$.

We show that we can make:

$$A = C \quad \text{and} \quad B = 0.$$

We seek ξ and η satisfying:

$$a\left(\xi_x^2 - \eta_x^2\right) + 2b\left(\xi_x\xi_y - \eta_x\eta_y\right) + c\left(\xi_y^2 - \eta_y^2\right) = 0,$$

$$a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y = 0.$$

We write the second as:

$$\left(a\eta_x + b\eta_y\right)\xi_x + \left(b\eta_x + c\eta_y\right)\xi_y = 0,$$

so that:

$$\xi_x = \frac{b\eta_x + c\eta_y}{\delta}, \quad \xi_y = -\frac{a\eta_x + b\eta_y}{\delta},$$

for some δ.

After much manipulation the **first equation** gives:

$$\left(\frac{ac - b^2}{\delta^2} - 1\right)\left(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\right) = 0.$$

Elliptic Equations continued...

Since $b^2 - ac < 0$, the expression

$$a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

is **positive definite**, so **cannot be zero**.

This implies

$$\delta = \sqrt{ac - b^2}.$$

The functions ξ and η must therefore satisfy the first order **linear** system:

$$\xi_x = \frac{b\eta_x + c\eta_y}{\sqrt{ac - b^2}}, \quad \xi_y = -\frac{a\eta_x + b\eta_y}{\sqrt{ac - b^2}},$$

called the **Beltrami equations**.

The **integrability conditions** $\xi_{xy} = \xi_{yx}$ lead to:

$$\partial_y \left(\frac{b\eta_x + c\eta_y}{\sqrt{ac - b^2}} \right) + \partial_x \left(\frac{a\eta_x + b\eta_y}{\sqrt{ac - b^2}} \right) = 0,$$

which is a second order equation for η.

It may appear that we have gained nothing, since we now have an even more complicated second order equation to solve!

However, whilst we need the general solution in the case of u, **any solution** of the above will suffice and then the Beltrami equations give ξ.

Each pair of functions (ξ, η) thus obtained, gives rise to a co-ordinate system which puts our elliptic equation into canonical form. We choose the simplest solution we can find.

Complex Variables

We can derive Beltrami's equations by considering the **complex variable**

$$F = \xi + i\eta,$$

which satisfies:

$$aF_x^2 + 2bF_xF_y + cF_y^2 = 0,$$

whose real and imaginary parts are

$$a\left(\xi_x^2 - \eta_x^2\right) + 2b\left(\xi_x\xi_y - \eta_x\eta_y\right) + c\left(\xi_y^2 - \eta_y^2\right) = 0,$$

$$a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y = 0.$$

We now proceed as in the hyperbolic case:

$$-\frac{F_x}{F_y} = \frac{dy}{dx} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right) = \frac{1}{a}\left(b \pm i\delta\right).$$

Of course, there are no **real** characteristics, but the manipulations are still valid.

Taking just the '+' sign:

$$aF_x + \left(b + i\delta\right)F_y = 0,$$

with real and imaginary parts:

$$a\xi_x + b\xi_y - \delta\eta_y = 0 \quad \text{and} \quad a\eta_x + b\eta_y + \delta\xi_y = 0,$$

which can be re-arranged to give Beltrami's equations.

Chapter 3

Properties of the Laplace and Poisson Equations

In this chapter we turn to *elliptic equations*. These are no longer associated with *initial value problems*. There is no sense of "evolution in time". Elliptic equations arise in the context of *steady state* behaviour such as in *electrostatics*, *equilibrium states* in heat flow or *steady, irrotational flows* in fluid dynamics.

Elliptic equations are therefore associated with *boundary value problems*, in which data is given on an entire boundary $\partial\Omega$ of some region Ω in $\mathbb{R}^n$. We only consider the case of $n = 2$.

The following topics are included:

- The Laplace and Poisson equations
- Ill posed Cauchy problem
- Dirichlet and Neumann BVPs
- Green's identities and uniqueness theorems
- The mean value theorem and min/max property
- The Laplacian in different orthogonal coordinate systems
- Conformal Transformations

Introductory Comments

Elliptic equations are typified by Laplace's equation:

$$\Delta u = u_{xx} + u_{yy} = 0,$$

or its inhomogeneous version, **Poisson's equation**:

$$\Delta u = u_{xx} + u_{yy} = \rho(x, y),$$

in rectangular Cartesian co-ordinates.

The coordinates are to be interpreted as **spatial**, *with "time" playing no role.*

The **Cauchy Problem** *therefore plays no role. We need to consider* **Boundary Value Problems**.

However, we first show that the **Cauchy Problem** for Laplace's equation is **Ill Posed**.

Recall that:

A Cauchy problem *is said to be* **well posed** *if:*

1. *there exists a solution,*
2. *the solution is unique,*
3. *the solution is continuously dependent upon initial data.*

Otherwise, it is said to be **ill-posed**.

Ill Posed Cauchy Problem

The **Cauchy problem** *is not well posed for an elliptic equation. The following example fails on the third requirement of the solution being continuously dependent upon initial data.*

Consider the following Cauchy problems for Laplace's equation:

1.
$$\Delta u = 0 \quad \text{with} \;\; u(x,0) = u_y(x,0) = 0,$$

2.
$$\left.\begin{array}{l} \Delta u = 0 \;\text{ with }\; u(x,0) = 0\,, \\ u_y(x,0) = k^{-1}\sin kx, \end{array}\right\} \quad \text{for some fixed, but arbitrary } k.$$

The unique solution to the **first** *is* $u(x,y) \equiv 0$.

The second has solution

$$u(x,y) = \frac{\sin kx \, \sinh ky}{k^2}.$$

Therefore, we have constructed a unique solution to each of these Cauchy problems.

However, **by choosing** k **to be large**, the initial conditions can be made as close as we like, *whereas the second solution grows without bound for large* y*:*

$$\frac{\sin kx \sinh ky}{k^2} \quad \longrightarrow \quad \infty \quad \text{as} \quad y \to \infty,$$

so the Cauchy problem is not well posed.

Boundary Value Problems

In physical applications, the Laplace equation arises in the context of boundary value problems (BVP).

Let $\Omega \subset \mathbf{R}^2$ be some region in the plane with boundary $\partial\Omega$. Let $u(x, y)$ satisfy Laplace's equation in Ω.

We consider two types of boundary value problem, specifying either u or the normal derivative $\frac{\partial u}{\partial n}$ on $\partial\Omega$.

The Dirichlet Problem

Let $\Omega \subset \mathbf{R}^2$ have **piecewise smooth boundary** $\partial\Omega$, *and let $u(x, y)$ be continuous in $\Omega \cup \partial\Omega$.*

The **Dirichlet problem** *is to determine whether the following BVP is* **well posed** *(solution exists, is unique and is continuously dependent upon boundary data):*

$$\begin{aligned} &\Delta u = u_{xx} + u_{yy} = 0 && \textit{in} \quad \Omega, \\ &u = f && \textit{on} \quad \partial\Omega, \end{aligned}$$

for given function $f(s)$ on $\partial\Omega$. Here, each smooth part will be separately parameterised.

Typical applications are electrostatics in some bounded region or steady heat conduction in a region with the temperature specified on the boundary.

A piecewise smooth boundary is not just a mathematical artifact. A rectangular plate (in heat conduction) has 4 corners, so has piecewise smooth (but **not** *smooth) boundary.*

The Neumann Problem

Let $\Omega \subset \mathbf{R}^2$ *have* **piecewise smooth boundary** $\partial\Omega$*, and let* $u(x, y)$ *be* C^1 *(continuous, with continuous first derivatives) in* $\Omega \cup \partial\Omega$*.*

The **Neumann problem** *is to determine whether the following BVP is well posed:*

$$\begin{aligned} \Delta u = u_{xx} + u_{yy} = 0 & \quad in \quad \Omega, \\ \frac{\partial u}{\partial n} = f & \quad on \quad \partial\Omega, \end{aligned}$$

where $\frac{\partial u}{\partial n} = \hat{\mathbf{n}} \cdot \nabla u$ *is the outward normal derivative and* f *is given on* $\partial\Omega$*.*

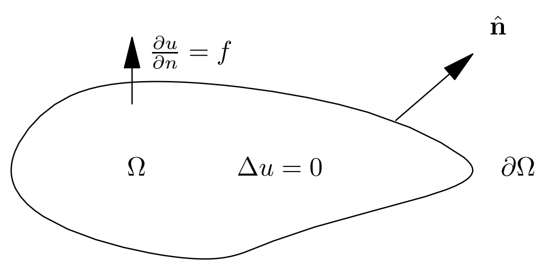

Figure 3.1: The Neumann problem.

A typical physical application is steady heat conduction, with **no heat flux over a boundary** (i.e. insulated), in which case $\frac{\partial u}{\partial n} = 0$.

Green's Theorem in the Plane

Consider a simple closed curve, which can be represented as the "sum" of two **graphs** above the $x-$axis. This will have an **inside** (to be our **domain**) and an **outside**.

We travel the **path** of the boundary in the **positive** direction if the domain is **on the left**.

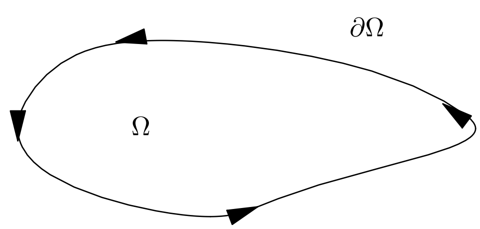

Figure 3.2: Domain and boundary for Green's Theorem.

Green's Theorem

Let Ω be a simply region of the plane whose boundary $\partial\Omega$ is as above. Let $M(x, y)$ and $N(x, y)$ be continuous functions with continuous first partial derivatives on Ω and $\partial\Omega$. Then

$$\int_{\partial\Omega} (M(x, y)dx + N(x, y)dy) = \int_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Remark

Green's Theorem is initially proved for such simple domains, but is easily extended to any **multiply connected domain**.

The Divergence Theorem in the Plane

We can write Green's Theorem in terms of a vector field and its divergence, which motivates Gauss's Theorem in 3 dimensions.

Consider the parameterised closed curve in the plane

$$t \mapsto (x(t), y(t)),\ 0 \le t \le 1, \quad \text{with} \quad (x(1), y(1)) = (x(0), y(0)).$$

The **tangent vector** and (outward pointing) **unit normal** are given by

$$\mathbf{t} = (\dot{x}, \dot{y}), \quad \hat{\mathbf{n}} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\,(\dot{y}, -\dot{x}),$$

and

$$ds = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\,dt \quad \Rightarrow \quad \hat{\mathbf{n}} ds = (\dot{y}, -\dot{x})\,dt.$$

Green's Theorem in the plane is re-written as:

Divergence Theorem

With Ω *and* $\partial\Omega$ *as before and* $\mathbf{F} = (N, -M)$*, with* $M(x, y)$ *and* $N(x, y)$ *as before, then*

$$\int_{\partial\Omega} \mathbf{F} \cdot \hat{\mathbf{n}}\,ds = \int_{\Omega} \nabla \cdot \mathbf{F}\,dxdy.$$

With $\mathbf{F} = \nabla u$ and noting that $\Delta u = \nabla \cdot (\nabla u) = \operatorname{div} \operatorname{grad} u$, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = \int_{\Omega} \Delta u\,dxdy, \quad \text{where} \quad \frac{\partial u}{\partial n} = \hat{\mathbf{n}} \cdot \nabla\,u.$$

Green's Identities

The importance of Laplace's equation stems from its origins in vector calculus:

$$\Delta u = \nabla \cdot (\nabla u) = \mathit{div}\ \mathit{grad}\, u,$$

valid in both 2 *and* 3 *dimensions.*

We have the following identity:

$$\nabla \cdot (v \nabla u) = (\nabla v) \cdot (\nabla u) + v \Delta u.$$

Thus, using the **divergence theorem** we obtain **Green's first identity** (hereafter "**Green 1**")

$$\begin{aligned} \int_\Omega (v\Delta u + (\nabla u) \cdot (\nabla v)) &= \int_\Omega \nabla \cdot (v \nabla u) \\ &= \int_{\partial\Omega} (v \nabla u) \cdot \hat{\mathbf{n}} ds = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds. \end{aligned}$$

Here Ω *could be* $3D$ *or* $2D$ *with* $\partial\Omega$ *being correspondingly* $2D$ *or* $1D$*, with* $\hat{\mathbf{n}}$ *the outward pointing normal. The* ds *is either a* **surface** *or* **line** *element, depending upon the dimension of* $\partial\Omega$*.*

Interchanging u **and** v and subtracting, we get **Green's second identity** (hereafter "**Green 2**"):

$$\int_\Omega (v\Delta u - u\Delta v) = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds,$$

since the $(\nabla u) \cdot (\nabla v)$ is symmetric.

Greens identities *hold for* **any** *functions* u, v, *but we will be particularly interested in solutions of* **Laplace's equation**.

Simple Consequences of Green's Identities

Solutions of Laplace's equation are called **harmonic functions**.

From **Green 1** we deduce the following:

1. $v = 1 \quad \Rightarrow \quad \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds, \qquad \text{(G1a)}$

2. $v = u \quad \Rightarrow \quad \int_\Omega (u\Delta u + \mid \nabla u \mid^2) = \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds. \qquad \text{(G1b)}$

Theorem

If a **harmonic function** *vanishes identically on $\partial\Omega$, then it vanishes identically throughout Ω.*

Proof

$$\text{(G1b)} \quad \Rightarrow \quad \int_\Omega \mid \nabla u \mid^2 = 0 \quad \Rightarrow \quad \nabla u = 0 \quad \text{in} \ \ \Omega.$$

Thus, u is constant in Ω. Since u is continuous, $u = 0$ (its value on $\partial\Omega$).

Remark.

The function u must not fail to be harmonic at **any point** *in Ω. For instance, if Ω is the interior of the unit circle, $u(x, y) = ln(x^2 + y^2)$ is harmonic everywhere* **except** $(x, y) = (0, 0)$ *and $u = 0$ on $\partial\Omega$.*

Uniqueness Theorems for Laplace's Equation

Theorem

If the normal derivative of a harmonic function u vanishes everywhere on $\partial\Omega$, then u is constant throughout Ω.

Proof

As above, we prove $u =$constant, but can no longer deduce that $u = 0$.

Theorem (Uniqueness of Solution of the Dirichlet Problem)

If the Dirichlet problem for region Ω has a solution, then the solution is unique.

Proof

Let u_1 and u_2 be 2 solutions of Laplace's equation, satisfying the boundary condition $u_i = f$ on $\partial\Omega$. Here the **same** f is specified for each u_i.

We define $w = u_1 - u_2$. Then,

$$\Delta w = \Delta u_1 - \Delta u_2 = 0 \quad \text{in} \quad \Omega$$

$$w = u_1 - u_2 = f - f = 0 \quad \text{on} \quad \partial\Omega.$$

Thus, by a previous Theorem,

$$w = 0 \quad \text{in} \quad \Omega \cup \partial\Omega, \quad \text{so} \quad u_1 = u_2.$$

This is a **uniqueness** *theorem and says nothing about* **existence***.*

Uniqueness Theorems for Laplace's Equation continued...

Theorem ((Almost) Uniqueness of Solution of the Neumann Problem)

If the Neumann problem for region Ω has a solution, then this solution is unique to within an additive constant.

Proof

Once again, we suppose u_1 and u_2 are 2 solutions of the Neumann problem:

$$\Delta u_i = 0 \quad \text{in} \;\; \Omega \quad \textbf{and} \quad \frac{\partial u_i}{\partial n} = f \quad \text{on} \quad \partial\Omega,$$

for the **same** f, and **define** $w = u_1 - u_2$.

Then:

$$\Delta w = 0 \quad \text{in} \;\; \Omega$$

$$\frac{\partial w}{\partial n} = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = f - f = 0 \quad \text{on} \;\; \partial\Omega.$$

Thus, by a previous Theorem, w=constant, so $u_1 = u_2$+ constant.

The Mean Value Property

Let Ω be the interior of the circle of radius r and centre (x, y) in the plane $\mathbf{R}^2$ with $\partial\Omega$ being the circle:

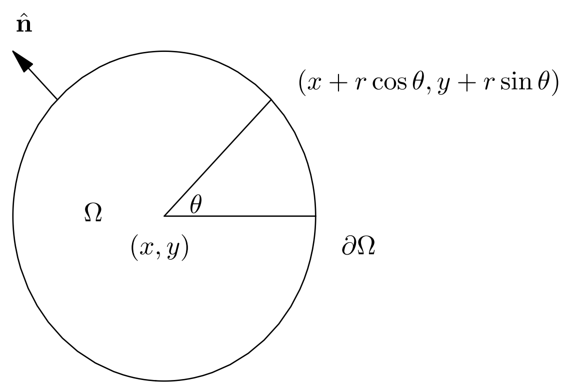

Figure 3.3: The mean value property.

The outward pointing normal is in the radial direction: $\hat{\mathbf{n}}\cdot\nabla = \frac{\partial}{\partial r}$.
If u is a harmonic function, then

$$\text{(G1a)} \quad \Rightarrow \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_{\Omega} \Delta u = 0.$$

We deduce

$$\begin{aligned}\frac{\partial}{\partial r}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)d\theta &= \int_0^{2\pi} \frac{\partial u}{\partial r}(x + r\cos\theta, y + r\sin\theta)d\theta \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} = 0.\end{aligned}$$

Thus,

$$\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)d\theta \quad \textbf{is independent of } r\textbf{.}$$

The Mean Value Property continued...

Allowing the circle to shrink, $r \to 0$, the integral remains constant, so

$$\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)d\theta = \int_0^{2\pi} u(x,y)d\theta = 2\pi u(x,y).$$

This gives the **mean value property**.

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)d\theta.$$

Mean Value Theorem

If $u(x,y)$ is continuous and has the mean value property for **every** *circle in domain Ω, then $u(x,y)$ is harmonic in Ω.*

Proof

Let $(x,y) \in \Omega$ and let $\mathcal{C}$ be the interior of a circle of radius r (denoted $\partial\mathcal{C}$), contained entirely within Ω.

Then

$$\iint_{\mathcal{C}} \Delta u = \int_{\partial\mathcal{C}} \frac{\partial u}{\partial n} d\theta = \int_0^{2\pi} \frac{\partial u}{\partial r} d\theta = 0, \quad \forall \mathcal{C}.$$

Letting $r \to 0$, we see that $\Delta u = 0$.

MinMax Theorem

If $u(x,y)$ is harmonic in $\Omega \subset \mathbf{R}^2$, then u attains its maxima and minima on the boundary $\partial\Omega$.

Proof Suppose maximum at an **interior** point $(x_0, y_0) \in \Omega$. Average of $u(x,y)$ around small circle centred on (x_0, y_0) **cannot** equal $u(x_0, y_0)$, **which is a contradiction**.

The Laplace Equation in Different Coordinate Systems

Any change of coordinates: *Apply chain rule to obtain formulae for second derivatives of u and thus obtain Laplace's equation in a new coordinate system.*

Vector calculus: Use identity $\Delta u = \operatorname{div} \operatorname{grad} u$ for **orthogonal** coordinate systems.

Polar Coordinates.

These are useful coordinates in circular regions:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{so} \quad r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x},$$

giving

$$\Delta u = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta}.$$

Cylindrical Coordinates.

These are useful coordinates when there is cylindrical symmetry in the BVP. Taking z to be a coordinate along the axis of the cylinder and r, θ as the coordinates in the plane of the circular cross section, we find:

$$\Delta u = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta} + u_{zz}.$$

The Laplace Equation in Different Coordinate Systems continued...

Spherical Coordinates.

These are useful coordinates when there is spherical symmetry in the BVP. With

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \varphi = \tan^{-1}\frac{y}{x},$$

we get

$$\Delta u = u_{rr} + \frac{2}{r}\,u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\,u_\theta + \frac{1}{\sin^2\theta}\,u_{\varphi\varphi}\right).$$

We see that Laplace's equation $\Delta u = 0$ is **not usually constant coefficient***.*

This can make the solution more difficult to find, but in the above coordinate systems it is possible to use the **separation of variables***.*

The Cauchy–Riemann Equations

Complex characteristic coordinates:

$$z = x+iy, \quad \bar{z} = x-iy, \quad \Rightarrow \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Complex function

$$f(z, \bar{z}) = u(x, y) + iv(x, y)$$

is said to be **complex analytic** if it is **independent of** $\bar{z}$, leading to the **Cauchy–Riemann equations**:

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If we treat v as **given**, this is an **over-determined** first order system of equations for u, with **integrability conditions**

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

which is Laplace's equation for $v(x, y)$.

We similarly find that u is a solution of Laplace's equation.

These are called **conjugate harmonic functions**.

Conjugate Harmonic Functions

Example (Orthogonal Hyperbolae)

The function $v(x, y) = y^2 - x^2$ is harmonic, so $u(x, y)$ satisfies

$$u_x = v_y = 2y, \quad u_y = -v_x = 2x,$$

giving $u = 2xy$ as the conjugate harmonic function.

The curves $u = const.$ and $v = const.$ define two orthogonal families of hyperbolae.

In fact, it follows from the Cauchy–Riemann equations that **we always have**

$$\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) = (u_x, u_y) \cdot (-u_y, u_x) = 0,$$

so that the normals to the **level curves of conjugate harmonic functions** are orthogonal.

Starting with **any** *harmonic function $u(x, y)$, we can solve the Cauchy–Riemann equations to obtain the conjugate function.*

We can also easily compute conjugate pairs corresponding to **any given analytic function**.

The above example corresponds to the function $f(z) = z^2$.

Other simple examples are

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}, \qquad e^z = e^x \cos y + ie^x \sin y.$$

Conformal Mappings of the Plane

We seek a change of coordinates $\xi = \xi(x, y)$, $\eta = \eta(x, y)$, *which preserve the form of Laplace's equation in the plane with Cartesian coordinates* x, y.

We use our general formulae for $L \mapsto \hat{L}$:

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \quad C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$$

$$L\xi = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}, \quad L\eta = a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy},$$

to obtain

$$A = C \;\Rightarrow\; \xi_x^2 + \xi_y^2 = \eta_x^2 + \eta_y^2,$$

$$B = 0 \;\Rightarrow\; \xi_x\eta_x + \xi_y\eta_y = 0,$$

$$L\xi = L\eta = 0 \;\Rightarrow\; \xi_{xx} + \xi_{yy} = \eta_{xx} + \eta_{yy} = 0.$$

This set of equations is satisfied if ξ, η satisfy the simpler **Cauchy–Riemann equations**:

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x.$$

Any complex analytic function will give us an example of such a change of coordinates:

$$z = x + iy, \;\; \zeta = \xi + i\eta \quad \Rightarrow \quad \zeta = f(z).$$

With such transformations we map a bounded region of one shape into one of a **different** *shape.*

If we have solved Laplace's equation in the first, then this automatically gives the solution to a new BVP in the second.

Example of Conformal Mapping

Consider the change of coordinates

$$\zeta = e^z \quad \Rightarrow \quad \xi = e^x \cos y, \quad \eta = e^x \sin y.$$

This transforms the rectangular region with vertices at

$$A(0,0),\ B(0,\pi),\ C(a,\pi),\ D(a,0)$$

into a region between two semicircles, with vertices

$$A(1,0),\ B(-1,0),\ C(-e^a,0),\ D(e^a,0).$$

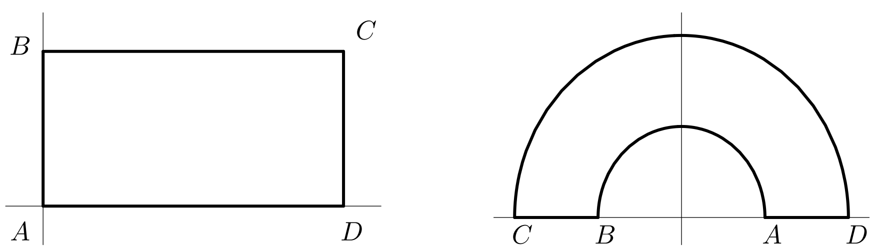

Figure 3.4: The rectangular region $ABCD$ is transformed onto a half annular region $ABCD$ by the conformal transformation.

$$u(x,y) \mapsto \bar{u}(\xi,\eta) = u\left(\ln\sqrt{\xi^2+\eta^2}, \tan^{-1}\left(\frac{\eta}{\xi}\right)\right).$$

Cartesian BVP (*previously solved by separation of variables*)

$$u_{xx} + u_{yy} = 0 \ \text{ for } \ 0 < x < a,\ 0 < y < \pi,$$

$$u(0,y) = u(a,y) = u(x,\pi) = 0,$$

$$u(x,0) = f(x) \ \text{ with } \ f(0) = f(a) = 0.$$

is transformed onto a BVP on the "horse shoe", with $u = 0$ on AB, BC and CD and $u(\xi,0) = f(\ln\xi)$ on AD.

Chapter 4

Separation of Variables

In this chapter we introduce the method of separation of variables, which enables us to solve a variety of Initial and Boundary Value Problems.

The method relies on the linearity of the equation as well as the BVP being set in a simple geometry, to which an orthogonal coordinate system is adapted.

The following topics are included:

- Separation of variables
- Use of the boundary conditions
- Finite Fourier series
- Application to the linear wave and heat equations and Laplace's equation
- Different geometries and coordinate systems
- Some higher dimensional examples

Separation of Variables for the Heat Equation

This technique is best described within the context of a simple example. We start with the heat equation.

Consider the following **initial boundary value problem** for $u(x,t)$ defined in the region $\Omega = \{(x,t) : 0 \le x \le 1,\ t > 0\}$:

$$u_t = u_{xx} \quad \text{in } \Omega \quad \text{and} \quad u(0,t) = u(1,t) = 0, \;\; u(x,0) = \varphi(x).$$

Boundary conditions. The conditions at $x = 0$ and $x = 1$,which hold for all $t > 0$.

Initial condition. The condition at $t = 0$.

The function $\varphi(x)$ should satisfy $\varphi(0) = \varphi(1) = 0$, to match the boundary conditions at $(0,0)$ and $(1,0)$.

Since the heat equation only has one $t-$derivative, there is only one initial condition.

Separated solution. Simple factorised solution:

$$u(x,t) = X(x)T(t),$$

where $X(x)$, $T(t)$ are functions of a **single variable**, as shown.

Substitute this into the heat equation and divide by $u(x,t)$

$$\frac{dT}{dt}T^{-1} = \frac{d^2X}{dx^2}X^{-1}.$$

The **left** *side of this equation is a function of t only, whilst the* **right** *side is a function of x only.*

This can only hold if each side is a constant.

The **boundary conditions** reduce to

$$X(0)T(t) = X(1)T(t) = 0, \;\; \forall t > 0 \quad \Rightarrow \quad X(0) = X(1) = 0.$$

The Heat Equation continued...

Ordinary differential equations for $X(x)$ and $T(t)$

$$\frac{dT}{dt} = \lambda T \quad \text{and} \quad \frac{d^2X}{dx^2} = \lambda X,$$

where λ is some constant. The first of these is easily solved:

$$T(t) = \alpha\, e^{\lambda t}, \quad \text{for arbitrary constant } \alpha.$$

For this to be well defined as $t \to \infty$, choose **negative** $\lambda = -k^2$.

We then have the *eigenvalue problem*

$$\frac{d^2X}{dx^2} + k^2 X = 0, \quad \text{so} \quad X(x) = A\cos(kx) + B\sin(kx),$$

for arbitrary constants A, B.

Question:

For what values of A, B, k do the **boundary conditions** *hold?*

The first boundary condition: $X(0) = 0 \;\Rightarrow\; A = 0$.

The second boundary condition (for arbitrary B) gives the eigenvalue:

$$X(1) = 0 \quad \Rightarrow \quad \sin(k) = 0 \quad \Rightarrow \quad k = n\pi.$$

Sequence of separated solutions. For each $n \geq 0$, we have a separated solution of the heat equation

$$u_n(x,t) = e^{-n^2\pi^2 t}\sin(n\pi x).$$

Generally, none of these will satisfy the **initial condition**.

The Initial Condition

Since the heat equation is **linear**, we can take linear combinations of the basic separated solutions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \, u_n(x,t),$$

which is no longer a separated solution.

The **initial condition** then gives

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x) = \varphi(x).$$

The next question is, "*For what sequence of numbers c_n does the series below converge (in some sense) to the given function $\varphi(x)$, so that the equality makes sense?*"

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x) = \varphi(x).$$

Remark 4.1 *The French mathematician/physicist Joseph Fourier (1768–1830) studied series of this form exactly in this context of the separation of variables for the heat equation. The general theory of these series was developed systematically by many later mathematicians and they now bear the name of "Fourier Series". We will not discuss the general theory of Fourier series (see the module MATH2431).*

Finite Fourier Series

In fact, the formula for c_n is straightforward, just relying on the **orthogonality relations** for the trigonometric functions:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{2} & \text{if } m = n, \end{cases}$$

so that

$$c_n = 2\int_0^1 \varphi(x)\sin(n\pi x)dx.$$

However a discussion of the convergence of this and other Fourier series would take us too far from the subject of this module.

We only consider functions $\varphi(x)$ which require a finite expansion, in which case the coefficients c_n can be determined by inspection.

Example: $\varphi(x) = \sin(\pi x)$

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sin(\pi x) \quad \Rightarrow \quad c_1 = 1, \;\; c_n = 0 \;\text{ for }\; n \neq 1,$$

so

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x)$$

Example: $\varphi(x) = 2\sin(\pi x) - 3\sin(2\pi x)$

$$\varphi(x) = 2\sin(\pi x) - 3\sin(2\pi x) \quad \Rightarrow \quad c_1 = 2, \; c_2 = -3, \; c_n = 0 \text{ otherwise}$$

so

$$u(x,t) = 2e^{-\pi^2 t}\sin(\pi x) - 3e^{-4\pi^2 t}\sin(2\pi x).$$

Linear Wave Equation

Consider the following **initial boundary value problem**, for $u(x,t)$ defined in the region $\Omega = \{(x,t) : 0 \leq x \leq \ell,\ t > 0\}$:

$$u_{tt} = c^2\, u_{xx} \quad \text{in} \quad \Omega \quad \text{and} \quad \begin{cases} u(0,t) = u(\ell,t) = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases}$$

This models a string of length ℓ, pegged at its endpoints.

The **boundary conditions** *are the same as for the heat equation, but now we have* **two initial conditions***, as a consequence of the double time derivative in the equation.*

Separated solution. Again, consider special solutions of the form $u(x,t) = X(x)T(t)$, which, as a result of the equation, must satisfy

$$\frac{d^2T}{dt^2}T^{-1} = c^2\frac{d^2X}{dx^2}X^{-1} = \lambda, \quad \text{a constant.}$$

With negative $\lambda = -c^2\omega^2$, X and T satisfy

$$\frac{d^2X}{dx^2} + \omega^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + c^2\omega^2 T = 0.$$

and the **boundary conditions** imply

$$X(0) = X(\ell) = 0.$$

Remark 4.2 (Why λ must be negative) *If $\lambda = c^2\mu^2 > 0$, then*

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

so the boundary conditions imply

$$A + B = 0, \quad Ae^{\mu\ell} + Be^{-\mu\ell} \quad \Rightarrow \quad A = B = 0.$$

Linear Wave Equation continued...

The solution of the first equation is

$$X(x) = \alpha \cos \omega x + \beta \sin \omega x.$$

Boundary conditions imply

$$X(0) = X(\ell) = 0 \quad \Rightarrow \quad \alpha = 0 \text{ and } \omega\ell = n\pi.$$

We then solve the T equation with $\lambda = -\frac{n^2\pi^2c^2}{\ell^2}$.

For each n we get a **separated solution** of the wave equation:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\left(a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right).$$

Generally, none of these will satisfy the **initial conditions**.

Again, we consider **linear combinations** of these separated solutions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t),$$

Note *that there is no need of the extra coefficient c_n here, since this can be absorbed into the definitions of the a_n, b_n.*

The **Initial conditions** then imply

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right) = \varphi(x), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n\pi c b_n}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) = \psi(x).$$

Again we can use the **orthogonality relations** *of the trigonometric functions to derive general formulae for the coefficients a_n, b_n for arbitrary initial conditions and thus obtain the general Fourier series for $u(x,t)$.*

However, we will only treat **finite expansions** *here.*

Finite Fourier Expansions

For **finite expansions** *we can determine the coefficients* a_n *and* b_n *"by inspection".*

For simplicity, consider the case for which

$$\psi(x) = 0, \quad \text{so} \quad b_n = 0 \ \text{ for all } \ n.$$

We thus use the basic functions

$$u_n^a(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right).$$

These are the **modes of vibration** *of the string.*

The mode with $n = 1$ is called the **fundamental mode** (*fundamental frequency*), whilst those with $n \geq 2$ are the **harmonics** (as in the harmonics of a note on the piano).

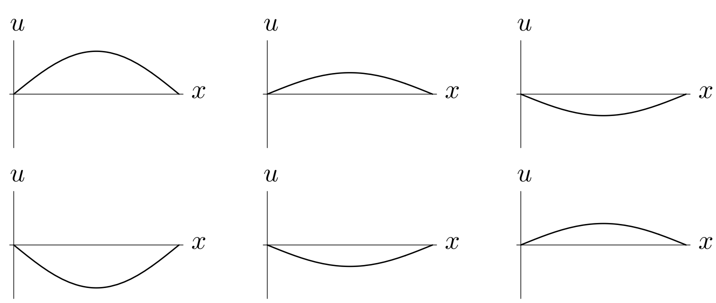

Figure 4.1: The fundamental mode, plotted at times $t_n = \frac{n\pi}{6}$, $n = 0, \cdots, 5$. The mode has (time) period $T = \pi$, so returns to its original position at t_6.

Plots of Further Modes

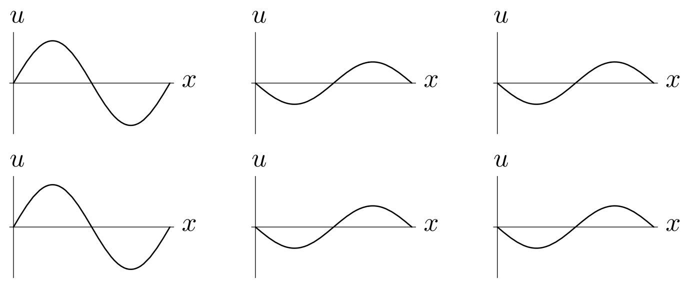

Figure 4.2: The $n = 2$ mode, plotted at times $t_n = \frac{n\pi}{6}$, $n = 0, \cdots, 5$. The mode has (time) period $T = \frac{\pi}{2}$, so returns to its original position at t_3, as can be seen.

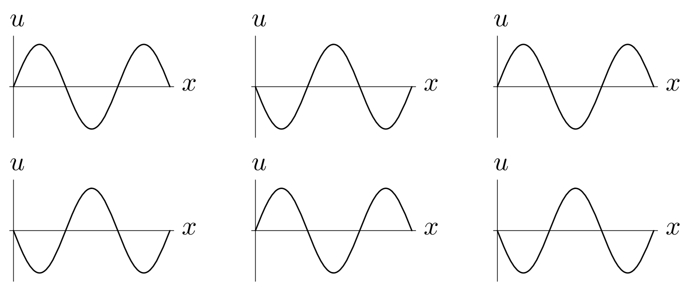

Figure 4.3: The $n = 3$ mode, plotted at times $t_n = \frac{n\pi}{6}$, $n = 0, \cdots, 5$. The mode has (time) period $T = \frac{\pi}{3}$, so returns to its original position at t_2, as can be seen.

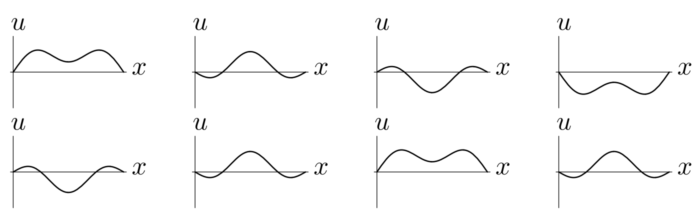

Figure 4.4: The **mixed** mode $u_1^a + 0.5u_3^a$, plotted at times $t_n = \frac{n\pi}{6}$, $n = 0, \cdots, 7$. The mode has (time) period $T = \pi$, so returns to its original position at t_6, as can be seen.

Laplace's equation in Cartesian Coordinates

Separation of Variables for the **Laplace equation** *depends upon the choice of coordinates.*

The **appropriate** *choice of coordinates depends upon the* **shape of the boundary** $\partial\Omega$*.*

In a *rectangular region in the plane*, we use Cartesian coordinates x, y. The location of the origin is chosen so that the **vertices of the rectangle** are at $(0,0)$, $(0,b)$, (a,b), $(a,0)$.

Boundary conditions should be **compatible** *at the vertices.*

Dirichlet Problem.

$$u_{xx} + u_{yy} = 0 \ \text{ for } \ 0 < x < a, \ 0 < y < b,$$

$$u(0,y) = u(a,y) = u(x,b) = 0,$$

$$u(x,0) = f(x) \ \text{ with } \ f(0) = f(a) = 0.$$

Separated Solution: $u(x,y) = X(x)Y(y)$ gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2.$$

and

$$u(0,y) = u(a,y) = 0 \quad \Rightarrow \quad X(0) = X(a) = 0,$$

$$u(x,b) = 0 \quad \Rightarrow \quad Y(b) = 0.$$

Note again *that choosing the constant to be* **positive** *would have lead to exponentials for $X(x)$ and the boundary conditions would then give the* **trivial solution**.

Laplace's Equation in a Rectangle

We have two ordinary differential equations

$$X''(x) + \lambda^2 X(x) = 0 \quad \text{and} \quad Y''(y) - \lambda^2 Y(y) = 0.$$

The first has general solution

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x),$$

but

$$X(0) = X(a) = 0 \quad \Rightarrow \quad B = 0, \;\; \lambda a = n\pi.$$

Sequence of eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n\pi}{a}, \quad X_n(x) = A_n \sin \lambda_n x.$$

The Y equation with $\lambda = \lambda_n$, has general solution:

$$Y_n(y) = B_n \cosh \lambda_n y + C_n \sinh \lambda_n y.$$

The **boundary condition** $Y_n(b) = 0$ gives

$$B_n = \sinh \lambda_n b, \;\; C_n = -\cosh \lambda_n b \quad \Rightarrow \quad Y_n(y) = \sinh \lambda_n (b - y).$$

Sequence of **separated solutions**

$$u_n(x, y) = \sin(\lambda_n x) \, \sinh \lambda_n (b - y).$$

Linear Superposition of $u_n(x, y)$

Once again we need to consider a series expansion

$$u(x, y) = \sum_{n=1}^{\infty} A_n u_n(x, y).$$

To match the **final boundary condition** we have

$$\sum_{n=1}^{\infty} A_n \sinh(\lambda_n b) \sin \lambda_n x = f(x).$$

For arbitrary $f(x)$ we need general Fourier techniques to calculate the coefficients A_n. If $f(x)$ is a **finite** *sum of functions* $\sin \lambda_n x$*, this is an elementary calculation.*

Example: $f(x) = \sin(\lambda_1 x) + 2\sin(\lambda_2 x)$ gives

$$A_1 = \operatorname{cosech}(\lambda_1 b), \quad A_2 = 2\operatorname{cosech}(\lambda_2 b).$$

Some Plots

The Figures below show 3D plots of the indicated functions in the region $\Omega = \{(x, y) : 0 \leq x, y \leq \pi\}$, with $a = b = \pi$. The "viewpoint" is looking down from some point above the $x - y$ plane and with negative y.

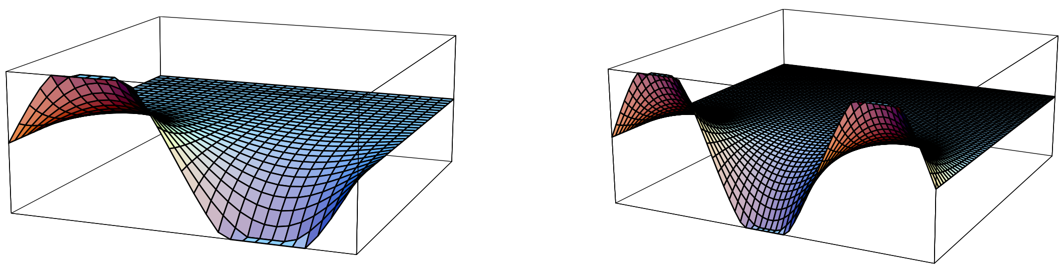

Figure 4.5: 3D plot of (a) $u_2(x, y)$ (b) $u_3(x, y)$.

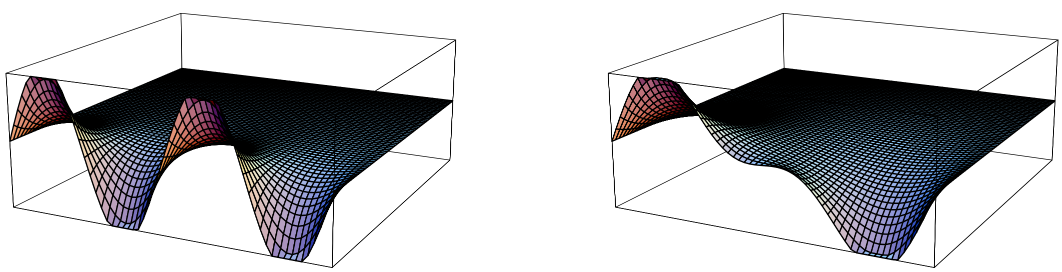

Figure 4.6: 3D plot of (a) $u_4(x, y)$ (b) $u(x, y) = u_2(x, y) + 0.001u_4(x, y)$.

Separation of Variables in a Circular Region

Using **polar coordinates**, we seek a **bounded** function $u(r, \theta)$, which satisfies:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } 0 < r < 1,\ 0 \leq \theta < 2\pi,$$
$$u(r, \theta + 2\pi) = u(r, \theta),$$
$$u(1, \theta) = f(\theta) \text{ with } f(\theta + 2\pi) = f(\theta).$$

$u(r, \theta)$ should be periodic in θ, since the points (r, θ) and $(r, \theta + 2\pi)$ represent the same point in the plane.

Separated Solution: $u(r, \theta) = R(r)\Theta(\theta)$ gives

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2,$$

The second of these implies

$$\Theta_n(\theta) = a\cos\lambda\theta + b\sin\lambda\theta.$$

Periodicity implies that $\lambda = n$, a positive integer. This means that exactly n oscillations fit into the circle (see Figure 4.7).

Sequence of eigenvalues and functions implies that there exists a sequence of eigenvalues $\lambda_n = n \neq 0$, with

$$\lambda_n = n \geq 1, \quad \Theta_n(\theta) = a_n\cos n\theta + b_n\sin n\theta.$$

Note again *that choosing the constant to be* **negative** *would have lead to exponentials in θ, so that it would have been impossible to achieve periodicity.*

The Equation for $R(r)$

This value of λ_n, gives **Euler's equation** for $R(r)$:

$$r^2 R'' + rR' - n^2 R = 0.$$

The **general solution** is

$$R_n(r) = c_n r^n + d_n r^{-n}.$$

Requiring our function to be bounded implies $d_n = 0$.
The case $n = 0$:

$$\Theta = a_0 + b_0\theta, \quad R = c_0 + d_0 \ln r.$$

Periodicity and boundedness imply

$$b_0 = d_0 = 0, \quad \text{so} \quad u_0(r,\theta) = a_0, \ \ \text{(absorbing } c_0\text{)}.$$

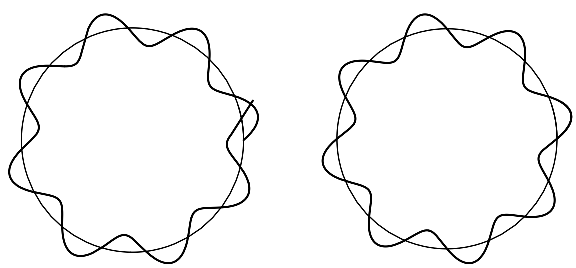

Figure 4.7: Function $\Theta(\theta)$ with (a) $\lambda = 7.8$ (b) $\lambda = 8$, showing the importance of λ being an integer.

The Boundary Condition on the Unit Circle

Take a series

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

satisfying

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = f(\theta).$$

Once again, for **arbitrary** $f(\theta)$ *we need general* **Fourier techniques** *to calculate the coefficients* a_n, b_n.

If $f(\theta)$ *is a* **finite** *sum of functions* $\sin n\theta$ *and* $\cos n\theta$, *this is an elementary calculation.*

Example:

$$f(\theta) = 1 - 2\sin 2\theta \quad \Rightarrow \quad \begin{cases} a_0 = 1, \quad b_2 = -2, \\ a_n = 0, \quad b_n = 0 \quad \text{otherwise.} \end{cases}$$

Some Higher Dimensional Equations

The heat and linear wave equations in 2 and 3 spatial dimensions are

$$u_t = \Delta u \quad \text{and} \quad \frac{1}{c^2} u_{tt} = \Delta u,$$

where Δ is either the 2 or 3 dimensional Laplacian operator in an appropriate coordinate system.

We first separate the time and spatial coordinates (labelled as $\mathbf{r}$):

$$u(t, \mathbf{r}) = T(t)\Phi(\mathbf{r}) \quad \Rightarrow \quad \begin{cases} \dfrac{T'(t)}{T(t)} = \dfrac{\Delta\Phi}{\Phi} = \lambda, \\ \dfrac{1}{c^2} \dfrac{T''(t)}{T(t)} = \dfrac{\Delta\Phi}{\Phi} = \lambda. \end{cases}$$

The equations for $T(t)$ are simple ordinary differential equations:

$$\begin{aligned} \frac{dT}{dt} &= \lambda T \qquad \text{for the heat equation,} \\ \frac{d^2T}{dt^2} &= c^2 \lambda T \quad \text{for the wave equation.} \end{aligned}$$

The physically interesting case is with $\lambda < 0$, so that $T(t)$ decays in time in the case of the heat equation or oscillates in time in the case of the wave equation.

Helmholtz Equation. In both cases the function $\Phi(\mathbf{r})$ satisfies the eigenvalue problem

$$\Delta\Phi = \lambda\Phi,$$

known as the *Helmholtz equation.*

The Helmholtz Equation

Given an **initial boundary value problem** *for the heat or wave equation in* 2 *spatial dimensions, separation of the* t *variable leads to a* **boundary value problem** *for Helmholtz's equation.*

Suppose we have arrived at the **boundary value problem**

$$\begin{aligned} &\Phi_{xx} + \Phi_{yy} = \lambda\Phi, && \text{for } \ \Omega = \{(x,y) : 0 < x < a, 0 < y < b\}, \\ &\Phi(x,y) = 0 && \text{for } \ x \in \partial\Omega. \end{aligned}$$

This is an **eigenvalue problem***, in which we have to find possible values of* λ*, together with the corresponding* **eigenfunctions**.

Separated solution. Again, choose

$$\Phi(x,y) = X(x)Y(y) \quad \Rightarrow \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \lambda,$$

with **boundary conditions**:

$$X(0) = X(a) = 0, \quad \text{and } \ Y(0) = Y(b) = 0.$$

The **eigenvalue problem** leads to

$$X''(x) + k_1^2 X(x) = 0, \quad \text{and } \ Y''(y) + k_2^2 Y(y) = 0,$$

where $\lambda = -k_1^2 - k_2^2$

Again, we have chosen constants so that $X(x)$ *and* $Y(y)$ *have trigonometric solutions, in order to match boundary conditions.*

We have reduced the problem to our previous 1−dimensional problems, with solutions (for arbitrary m and n):

$$k_1 = \frac{m\pi}{a}, \ \ X_m = \sin\left(\frac{m\pi x}{a}\right) \quad \text{and} \quad k_2 = \frac{n\pi}{b}, \ \ Y_n = \sin\left(\frac{n\pi y}{b}\right).$$

Helmholtz Eigenfunctions

For each m, n, we have eigenfunction u_{mn}, with eigenvalue λ_{mn}

$$\Phi_{mn} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \quad \lambda_{mn} = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Some of these functions are plotted below.

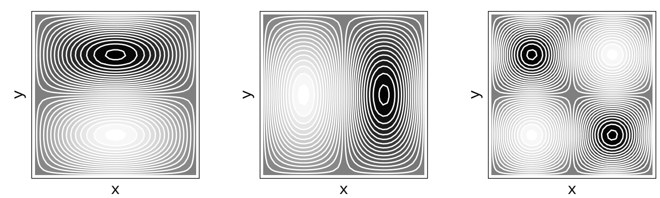

Figure 4.8: Case $a = b$ (a) Φ_{12} (b) Φ_{21}, (c) Φ_{22}.

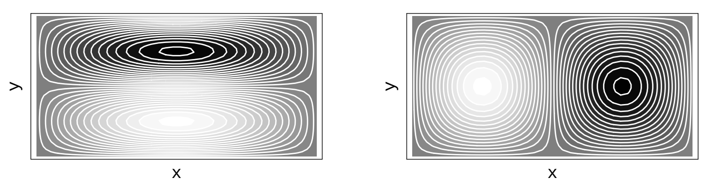

Figure 4.9: Case $a = 2b$ (a) Φ_{12} (b) Φ_{21}.

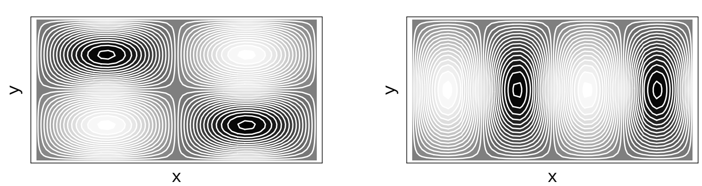

Figure 4.10: Case $a = 2b$ (a) Φ_{22} (b) Φ_{41}.

The Heat Equation in a Rectangular Region

Initial Boundary Value Problem for the heat equation

$$\boxed{\begin{aligned}
&\Omega = \{(x, y, t) : 0 < x < a, 0 < y < b, t > 0\},\\
&\partial\Omega = \text{rectangular boundary in } x - y \text{ plane},\\
&u_t = u_{xx} + u_{yy}, \quad \text{inside } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega, \quad t > 0,\\
&u(x, y, 0) = \varphi(x, y), \quad \varphi(x, y) = 0 \text{ on } \partial\Omega.
\end{aligned}}$$

Separate t: $u(x, y, t) = T(t)\Phi(x, y)$ implies

$$\frac{dT}{dt} = \lambda T, \quad \text{and} \quad \Delta\Phi = \lambda\Phi.$$

$\Phi(x, y)$ *satisfies the Helmholtz eigenvalue problem discussed above.*

Separate x, y: $\Phi(x, y) = X(x)Y(y)$ gives λ_{mn} and Φ_{mn} for the Helmholtz equation, already discussed, from which it follows that

$$T_{mn}(t) = e^{\lambda_{mn} t}, \quad \lambda_{mn} = -\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

Sequence of separated solutions:

$$u_{mn} = e^{\lambda_{mn} t} \sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi y}{b} \right).$$

Initial Conditions

To match the initial condition, we take the linear combination

$$u(x,y,t) = \sum_{m,n} c_{mn} u_{mn}.$$

Initial conditions:

$$\sum_{m,n} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = \varphi(x,y).$$

In general, coefficients c_{mn} are determined by Fourier analysis. *Once again, we'll only consider* **finite** *expansions.*

Example: $\varphi(x,y) = \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

$$\sum_{m,n} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right),$$

so $c_{21} = 1, \;\; c_{mn} = 0$, otherwise, and

$$u(x,y,t) = e^{\lambda_{21} t} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right), \quad \lambda_{21} = -\pi^2 \left(\frac{4}{a^2} + \frac{1}{b^2}\right).$$

Example: $\varphi(x,y) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)$

$$\sum_{m,n} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right),$$

so $c_{12} = 1, \;\; c_{mn} = 0$, otherwise, and

$$u(x,y,t) = e^{\lambda_{12} t} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right), \quad \lambda_{12} = -\pi^2 \left(\frac{1}{a^2} + \frac{4}{b^2}\right).$$

See Figures 4.8(a) and 4.8(b) for the case $a = b$ and Figures 4.9(a) and 4.9(b) for the case $a = 2b$.

The Wave Equation in a Rectangular Region

This models the patterns of vibration on a rectangular drum. More realistic boundary conditions would be on a circle.

Initial Boundary Value Problem for the wave equation

$$\begin{aligned}
&\Omega = \{(x, y, t) : 0 < x < a, 0 < y < b, t > 0\},\\
&\partial\Omega = \text{rectangular boundary in } x - y \text{ plane},\\
&\tfrac{1}{c^2}u_{tt} = u_{xx} + u_{yy}, \quad \text{inside } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega, \quad t > 0,\\
&u(x, y, 0) = \varphi(x, y), \quad \varphi(x, y) = 0 \text{ on } \partial\Omega.\\
&u_t(x, y, 0) = \psi(x, y), \quad \psi(x, y) = 0 \text{ on } \partial\Omega.
\end{aligned}$$

Separating variables: $u_{mn}(x, y, t) = T_{mn}(t)\Phi_{mn}(x, y)$ implies

$$T''_{mn}(t) + c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)T_{mn}(t) = 0,$$

so

$$T_{mn}(t) = a_{mn}\cos(k_{mn}t) + b_{mn}\sin(k_{mn}t), \quad k_{mn}^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Initial Conditions

To match the initial condition, we take the linear combination

$$u(x, y, t) = \sum_{m,n} u_{mn}.$$

Initial conditions:

$$\sum_{m,n} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = \varphi(x, y),$$
$$\sum_{m,n} b_{mn} k_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = \psi(x, y).$$

In general, coefficients a_{mn} and b_{mn} are determined by Fourier analysis.