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Complex Analysis
1MA022
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Suggested exercises for the problem sessions

Exercises with **bold** enumeration are especially recommended for the problem session.

A) COMPLEX PLANE, ELEMENTARY FUNCTIONS (SESSION 1)

1. Identify and sketch the set of points satisfying:

   a) $|z-1-i|=1$,
   b) $1<|2z-6|<2$,
   c) $|z-1|+|z+1|\leq 2$,
   d) $|z-1|^2+|z+1|^2<8$,
   e) $|z-\sqrt{3}|^2+|z+\sqrt{3}|^2<12$,
   f) $|z-1|<|z|$,
   g) $0<\operatorname{Im} z<\pi$,
   h) $-\pi<\operatorname{Re} z<\pi$,
   i) $|\operatorname{Re} z|<|z|$,
   j) $\operatorname{Re}(iz+2)>0$,
   k) $|z-i|^2+|z+i|^2<2$.

2. Show that the equation $|z|^2-2\operatorname{Re}(\overline{a}z)+|a|^2=\rho^2$ represents a circle centered at a with radius ρ.

3. Express all values of the following expressions in both polar and cartesian coordinates, and plot them.

   a) $\sqrt{i}$,
   b) $(-1)^{1/4}$,
   c) $(-8)^{1/3}$,
   d) $(1+i)^8$,
   e) $(1-i)^{3/7}$.

4. Write in cartesian coordinates the following complex numbers:

   a) e^{2+i},
   b) $e^{\ln 5+\frac{3\pi i}{4}}$,
   c) $\cos(\frac{\pi}{4}+i)$,
   d) $\operatorname{Log}(1+i)$.

5. For which $n\in\mathbb{N}$ is i an nth root of unity?

6. Show that $\cos 2\theta=\cos^2\theta-\sin^2\theta$ and $\sin 2\theta=2\cos\theta\sin\theta$ using the complex exponential. Find formulas for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

7. Show that $e^{\overline{z}}=\overline{e^z}$.

8. Show that $|\cos z|^2=\cos^2 x+\sinh^2 y$, where $z=x+iy$. Find all zeros and periods of $\cos z$.

9. Compute the real and imaginary parts of z^z.

10. Find all the solutions of the equations:

    a) $\cos z=2i$,
    b) $e^{e^z}=1$,
    c) $\cot z=2+i$,
    d) $5\cos z-3i\sin z=2$,
    e) $\sin(\cos z)=1$.

Updated: 20th Mar, 2024.

B) HOLOMORPHIC AND HARMONIC FUNCTIONS (SESSION 2)

1. Show that if f and $\overline{f}$ are both holomorphic on a domain D, then f is constant.

2. Show that if f is holomorphic on a domain D and $|f|$, $\operatorname{Re} f$, $\operatorname{Im} f$ <u>or</u> $\arg f$ is constant in D, then f is also constant in D.

3. Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v.

4. Show that the following functions are harmonic, and find all harmonic conjugates:

a) $u = x^3 - 3xy^2 + 2xy + x$,
b) $u = x^2 - y^2 + 5$,
c) $u = \sinh x \sin y$,
d) $u = e^x(y\cos y + x\sin y)$,
e) $u = \arctan\left(\frac{y}{x}\right)$, $\quad x > 0$.

5. Find all holomorphic functions f such that $\operatorname{Re} f + \operatorname{Im} f = xy$.

6. Suppose that u is a harmonic function and that v is a harmonic conjugate of u. Show that
$$\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right).$$

7. Suppose that $f = u + iv$ is holomorphic and not identically constant.
a) Show that uv is the real part of a holomorphic function.
b) Show that $u^2 + v^2$ cannot be the real part of any holomorphic function.

8. Let $f\colon U \to \mathbb{C}$ be a function defined on an open set $U \subset \mathbb{C}$. Write $f = u + iv$, and assume that the functions u, v are C^1. Define
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right).$$
These are called the *Wirtinger derivatives* of f.
a) Show that $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left((u_x - v_y) + i(u_y + v_x)\right)$ and $\frac{\partial f}{\partial z} = \frac{1}{2}\left((u_x + v_y) + i(-u_y + v_x)\right)$.
b) Show that f is holomorphic iff $\frac{\partial f}{\partial \overline{z}} = 0$.
c) Show that $\frac{\partial(\overline{z}^k)}{\partial \overline{z}} = k\overline{z}^{k-1}$.
d) Suppose that f is holomorphic. Show that $\frac{\partial f}{\partial z} = f'(z)$.

9. Determine all the holomorphic functions of the form
$$f(z) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 y^3,$$
where $a_1, \ldots, a_7 \in \mathbb{C}$.
Hint: Write x, y and f as functions of $z, \overline{z}$. Use the previous exercise.

10. Determine the holomorphic functions $f = u + iv$, for which $u(x,y) = x^3 + xg(y)$, where g is a twice continuously differentiable function.

11. Find all holomorphic functions such that its real part $u = u(x,y)$ satisfies the differential equation $\frac{\partial u}{\partial x} = -u$.

12. Let $f\colon D \to \mathbb{C}$ be a function defined on a subset $D \subset \mathbb{C}$. Show that $\lim_{z\to z_0} f(z) = L \in \mathbb{C}$ is equivalent to the following condition: for every sequence (z_n) in $D\backslash\{z_0\}$, if $\lim_{n\to\infty} z_n = z_0$ then $\lim_{n\to\infty} f(z_n) = L$.

C) CONFORMAL MAPPINGS (SESSION 3, 4)

1. Consider the function defined by $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ are constants.
   a) Show that if $ad - bc \neq 0$, then f is conformal on its domain.
   b) Show that if $ad - bc = 0$, then f is constant.

2. Find the image of $z = 0$ under the Möbius transformation which maps i, ∞ and 1 to 0, 1 and $-i$, respectively.

3. Find a Möbius transformation which maps the region $|z - i| < 2$ onto the upper half plane, the imaginary axis onto itself, and which fixes the point i.
Note: in the previous sentence, *onto* means that the image of the region $|z - i| < 2$ is the whole upper half plane.

4. A Möbius transformation T maps the upper half plane onto itself, and the circle $|z-1| = 1$ onto the imaginary axis such that the point $1 + i$ maps to i. Compute T. Is T uniquely determined by the given conditions? What is the image of the line $\operatorname{Im} z = 1$?

5. Find a Möbius transformation which maps the region outside the unit circle onto the left–half plane. What are the images of circles $|z| = r > 1$? And the images of lines passing through the origin?

6. Show that there exists a Möbius transformation which maps the region given by

$$|z - 1 + 2i| < 2\sqrt{2}, \quad |z - 1 - 2i| < 2\sqrt{2}, \quad |z| > 1,$$

onto the interior of the triangle with vertices at 0, 1 and i.

7. Find a conformal mapping which maps the region between $|z + 3| < \sqrt{10}$ and $|z - 2| < \sqrt{5}$ onto the interior of the first quadrant.

8. Find a conformal mapping which maps the region between $|z - 1| > 1$ and $|z| < 2$ onto the upper half plane.

9. Find a conformal mapping such that the complex plane minus the positive x-axis is transformed onto the interior of the unit circle, so that the point -4 is mapped to the origin.

10. Find a conformal mapping which maps the half-circle $\Omega_1 = \{z : |z| < 1,\ \operatorname{Re} z > 0\}$ onto the strip $\Omega_2 = \{w : |\operatorname{Re} w| < 1\}$.

D) DIRICHLET PROBLEMS (SESSION 4, 5)

1. Determine a function ϕ, which is harmonic in $D(0,1)$ and has boundary values 1 on $\partial D(0,1) \cap \{\operatorname{Re} z > 0\}$ and 0 on $\partial D(0,1) \cap \{\operatorname{Re} z < 0\}$.

2. Determine a function ϕ, harmonic in the interior of the unit circle, with boundary value 1 on $\partial D(0,1) \cap \{|\arg z| < \pi/4\}$ and boundary value 0 on $\partial D(0,1) \cap \{\pi/4 < |\arg z| \leq \pi\}$.

3. Find a function ϕ which is harmonic in $\Omega = D(0,1) \cap \{\operatorname{Im} z > 0\}$, that takes the value 1 on the straight line portion of the boundary and the value 0 on the circle part of the boundary.

4. Find a function ϕ, harmonic in $\Omega = \{\operatorname{Re} z > 0\} \cap \{0 < \operatorname{Im} z < \pi\}$, with boundary values 1 for $z = iy$, $0 < y < \pi$; 0 for $z = x$, $x > 0$; and 0 for $z = x + i\pi$, $x > 0$.

5. Determine a function ϕ, harmonic in the first quadrant, with boundary values 1 on the interval $(1, 2)$ of the real axis and 0 otherwise.

E) INTEGRATION (SESSION 6)

1. Compute the integral $\int_\gamma |z-1|\,|dz| = \int_a^b |z(t)-1|.|z'(t)|\,dt$, where γ is the positively (= counterclockwise) oriented unit circle parametrized by a function $z\colon [a,b) \to \mathbb{C}$.

2. Compute $\int_\gamma \frac{dz}{1+z^2}$, where γ represents the positively oriented circle:

a) $|z| = 1/2$, b) $|z - i/2| = 1$, c) $|z| = 2$.

3. Compute the following integrals. All the curves are given the positive orientation.

a) $\int_{|z|=1} \frac{e^{2z}}{z^m}\,dz$, for every $m \in \mathbb{Z}$, b) $\int_{|z-3|=1} \frac{36\,\mathrm{Log}(z)}{z(z^2-9)^2}\,dz$.

4. Calculate for any complex number a, $|a| \neq 1$, the value of the integral $\int_\gamma \frac{ze^{z^2}}{z-a}\,dz$, where γ denotes the positively oriented unit circle.

5. Determine the value of the integral

$$\int_\gamma \left(z^2 \sin z + \left|z + \tfrac{3}{4}\right| + e^{\sin z} \cos z + \frac{1}{z(z+1)} \right) dz,$$

where γ is the curve defined by $z(t) = (2e^{2\pi i t} - 3)/4$, $0 \le t \le 1$.

6. Calculate $\int_\gamma \frac{dz}{z(z+1)}$, where γ is the curve defined by $z(t) = e^{(1+i)t}$, $0 \le t \le 2\pi$.

7. Compute the integral $\int_\gamma \frac{dz}{z^2-4}$, where γ is the curve defined by $z(t) = e^{it}$, $0 \le t \le 3\pi/2$.

8. Calculate

$$\int_\gamma \left(\cos^2 z \sin z + \frac{2}{2z^2+z-1} + e^{z^2} \right) dz,$$

where γ is the curve defined by $z(t) = (t^2 - t + 1)e^{2\pi i t}$, $0 \le t \le 1$.

9. Suppose that f is holomorphic and $|f(z)| \le M$, $|z| \le R$. Determine an upper bound for $|f^{(n)}(z)|$ for $|z| \le r < R$.

10. Show that if u is harmonic in the whole plane and bounded from above, then u is constant.

F) MAXIMUM MODULUS PRINCIPLE (SESSION 7)

1. Use the maximum modulus principle to prove the fundamental theorem of algebra.

2. State the maximum principle for a harmonic function on domain D. Prove it for any domain D assuming that it is true when D is simply connected.

3. Let $D \subset \mathbb{C}$ be a bounded domain with closure $\overline{D} = D \cup \partial D$. Let $f \colon \overline{D} \to \mathbb{C}$ be continuous and holomorphic on D. Show that if $f(z) \in \mathbb{R}$ for every $z \in \partial D$, then f is constant.
Slogan: if a holomorphic function on D only takes real values on ∂D, then it is constant.

G) SEQUENCES AND SERIES OF FUNCTIONS (SESSION 7)

1. Find the subsets of $\mathbb{R}$ where the following sequences of functions converge pointwise. Find intervals in $\mathbb{R}$ where they converge uniformly.

   a) $f_n(x) = x^n, \quad n \in \mathbb{N},$

   b) $f_n(x) = (1-x^2)^n, \quad n \in \mathbb{N},$

   c) $f_n(x) = n^3 \sin^3\left(\frac{x}{n}\right), \quad n \in \mathbb{N},$

   d) $f_n(x) = \dfrac{e^{n^2 x}+1}{e^{n^2 x}-1}, \quad n \in \mathbb{N}.$

2. For which $z \in \mathbb{C}$ does $\{f_n(z)\}_{n=1}^{\infty}$, where $f_n(z) = \dfrac{1}{1+z+z^2+\cdots+z^n}$, converge?

3. Show that $f_n(z) = e^{-nz}$, $n \in \mathbb{N}$, converges uniformly to 0 when $\operatorname{Re} z \geq a$, for each $a > 0$. Is the convergence uniform when $\operatorname{Re} z > 0$?

4. Let $f_n(x) = \dfrac{nx}{nx+1}$, $n \in \mathbb{N}$.

   a) Does $\{f_n\}_{n=0}^{\infty}$ converge uniformly on $[0,1]$? What about on $[1,\infty)$?

   b) Is it true that $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$?

5. Determine when the following series of functions are uniformly convergent:

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^3+x^{2n}},$ b) $\displaystyle\sum_{n=1}^{\infty} x^2(1-x^2)^n.$

6. Show that each of the following series represents a holomorphic function in the right half-plane:

   a) $\displaystyle\sum_{n=1}^{\infty} e^{-n^2 z},$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^{z+1}}.$

H) POWER SERIES (SESSION 8)

1. Find the radius of convergence of the following power series:

   a) $\displaystyle\sum_{n=0}^{\infty} 2^n z^n,$

   b) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{6^n} z^n,$

   c) $\displaystyle\sum_{n=1}^{\infty} n^2 z^n,$

   d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n z^n}{4^n+5^n},$

   e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n z^{2n}}{n^2+n},$

   f) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{2n}}{4^n n^n},$

   g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{1+2^n n^n} z^n,$

   h) $\displaystyle\sum_{n=3}^{\infty} (\ln n)^{n/2} z^n,$

   i) $\displaystyle\sum_{n=1}^{\infty} \frac{n! z^n}{n^n}.$

2. Determine where the following series converge.

a) $\displaystyle\sum_{n=1}^{\infty}(z-1)^n$,
b) $\displaystyle\sum_{n=10}^{\infty}\frac{(z-i)^n}{n!}$,
c) $\displaystyle\sum_{n=0}^{\infty}2^n(z-2)^n$,
d) $\displaystyle\sum_{n=1}^{\infty}\frac{(z+i)^n}{n^2}$,
e) $\displaystyle\sum_{n=1}^{\infty}n^n(z-3)^n$,
f) $\displaystyle\sum_{n=3}^{\infty}\frac{2^n}{n^2}(z-2-i)^n$.

3. What functions are represented by the following series when $|z|<1$?

   a) $\displaystyle\sum_{n=1}^{\infty}nz^n$,
   b) $\displaystyle\sum_{n=1}^{\infty}n^2z^n$.

4. Calculate the first three (non-vanishing) coefficients of the power series expansion about the origin of the functions:

   a) $f(z)=\sin\left(\dfrac{1}{1-z}\right)$,
   b) $f(z)=e^{z/(1-z)}$,
   c) $f(z)=e^{z\sin z}$,
   d) $f(z)=\mathrm{Log}(1+e^z)$.

5. Determine the radius of convergence of the power series expansion of $\dfrac{z^2-1}{z^3-1}$ about $z=2$.

I) ZEROS AND UNIQUENESS; LAURENT SERIES EXPANSIONS AND ISOLATED SINGULARITIES (SESSION 9)

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1. Specify the order of the zero $z=0$ of the following functions:

   a) $f(z)=z^2(e^z-1)$,
   b) $f(z)=e^{\sin z}-e^{\tan z}$.

2. Find the zeros and orders of zeros of the following functions:

   a) $f(z)=\dfrac{z^2+1}{z^2-1}$,
   b) $f(z)=\dfrac{1}{z}+\dfrac{1}{z^5}$,
   c) $f(z)=z^2\sin z$,
   d) $f(z)=\cos z-1$,
   e) $f(z)=\sinh^2 z{+}\cosh^2 z$,
   f) $f(z)=\dfrac{\mathrm{Log}\, z}{z}$.

3. Show that $\sin^2 z+\cos^2 z=1$, $z\in\mathbb{C}$, assuming the corresponding identity for $z\in\mathbb{R}$ and using the uniqueness principle.

4. Show that if f and g are holomorphic on a domain D and $f(z)g(z)=0$ for all $z\in D$, then either f or g must be identically zero in D.

5. Is there any function f, holomorphic in $|z|<1$, such that
$$f\Big(\frac{1}{2k}\Big)=\frac{1}{2k}\quad\text{and}\quad f\Big(\frac{1}{2k+1}\Big)=\frac{1}{2k},\qquad k=1,2,3,\ldots?$$

6. Determine all functions f holomorphic in $|z|<1$ and satisfying
$$f\Big(\frac{1}{k}\Big)=\frac{k+k^2}{1+k^2},\qquad k=2,3,4,\ldots$$

7. Determine the Laurent series expansions of $f(z)=\dfrac{1}{z(1+z^2)(4-z^2)}$ in the regions:

   a) $0<|z|<1$,
   b) $1<|z|<2$,
   c) $|z|>2$.

8. Expand $f(z)=\dfrac{1}{z^2+2z}$ in a Laurent series in the region $1<|z-i|<\sqrt{5}$.

9. Find the Laurent series of $f(z) = \operatorname{Log}\left(\frac{z-i}{z+i}\right)$ for $|z| > 1$.

10. Find the isolates singularities of the following functions, and determine whether they are removable, poles or essential.

a) $\dfrac{e^z}{1+z^2}$, d) $\dfrac{1-\cos z}{z}$, g) $z^2 \sin\left(\dfrac{1}{z}\right)$,

b) $\dfrac{e^z}{z(1-e^{-z})}$, e) $e^{z/(z-2)}$, h) $\dfrac{z^4}{1+z^4}$,

c) $\dfrac{z-\sin z}{z^3}$, f) $\dfrac{e^{2z}}{(z-1)^3}$, i) $\dfrac{1}{z^3-z^5}$.

J) RESIDUE CALCULUS (SESSION 10, 11)

1. Calculate $\displaystyle\int_\gamma z^k e^{1/z} dz$, $k \in \mathbb{N}$, where γ is any positively oriented circle centered at the origin.

2. Compute the integral $\displaystyle\int_\gamma \frac{dz}{(z^2+1)^4}$, where γ represents the positively oriented rectangle with vertices at 2, $2+2i$, $-2+2i$ and -2.

3. Determine the value of the integral $\displaystyle\oint_{|z|=4} \frac{e^{iz}}{z(z^2-1)^2} dz$.

4. Compute the following integrals of rational functions:

a) $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx$, b) $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^3}$.

5. Calculate the following trigonometric integrals:

a) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{\sqrt{3}+\cos\theta} d\theta$, c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta+\cos\theta}$,

b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$, d) $\displaystyle\int_0^{2\pi} \cos^{2k}(\theta) d\theta, \quad k \in \mathbb{N}$.

6. Compute the following integrals:

a) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^4} dx$, b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2} dx, \quad a, b > 0$.

7. Calculate the following integrals of functions with branch points:

a) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{x^2+4} dx$, c) $\displaystyle\int_0^{\infty} \frac{dx}{x^a(1+x)}, \quad 0 < a < 1$,

b) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{x^3+1} dx$, d) $\displaystyle\int_0^{\infty} \frac{x^a}{1+x+x^2} dx, \quad -1 < a < 1$.

8. Compute the following integrals:

a) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+9}$, c) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}\ln x}{x^2+16}$.

b) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2-1}$,

K) THE ARGUMENT PRINCIPLE AND ROUCHÉ'S THEOREM (SESSION 12)

1. Show that all zeros of the polynomial $p(z) = z^4 - 2iz^3 + 16$ are contained in the disk $|z| < 3$. How many of the zeros have both negative real part and negative imaginary part?

2. Show that all zeros of the polynomial $p(z) = z^5 - z + 16$ are contained in the annulus $1 < |z| < 2$. How many of the zeros have positive real part?

3. Show that the equation $2(z-1)^{17} = e^{-z}$ has exactly 17 distinct roots in the disk $|z-1| < 1$.

4. In which quadrants are the roots of the equation $z^4 + z^3 + 4z^2 + 2z + 3 = 0$?

5. Determine the number of zeros of the function $f(z) = z^2 + e^{z-1}$ in the region $|z| < 1$.

6. Determine the number of zeros of the function $z^2 + 4 - 3e^{iz}$ in the open square with vertices at 2, -2, $2 + 4i$ and $-2 + 4i$.

7. Find the number of zeros of the function $f(z) = 2 - 2z^2 + z^4 + e^{-z}$ in the right half-plane.

8. Calculate the number of zeros of the polynomial $p(z) = z^7 + 3z^5 - 6z^2 + 1$ in the regions:

   a) $|z| < 1$, b) $1 < |z| < 2$, c) $\operatorname{Re} z > 0$.