Lecture 3

Recap:

An event is said to occur almost surely if its probability is 1.

Borel - Cantelli Lemma:

Let $E_1, E_2, \ldots$ be a seq. of events s.t. $\sum_{i=1}^{\infty} \mathbb{P}(E_i) < \infty$. Then, almost surely, only finitely many occur

$$\mathbb{P}(\limsup_{n\to\infty} E_n) = \mathbb{P}(E_n \text{ occurs for i.m. } n) = 0$$

Example: Given a seq of (fair) coin tosses, let E_n be the event that the first n were heads.

Then, $\mathbb{P}(E_n) = \left(\frac{1}{2}\right)^n$ and

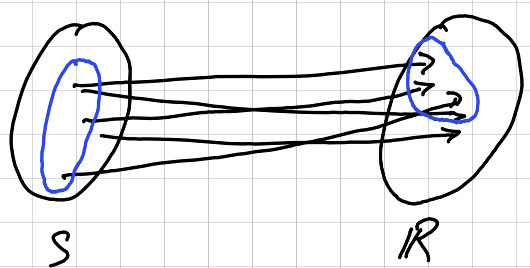
$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 < \infty.$$

Hence the probability that E occurs i.m. times is 0, so the probability that we get just heads is 0.

Random Variables

Def^n Let (S, Σ, μ) be a measure space. We say that a function $f : S \to \mathbb{R}$ (or $\mathbb{R} \cup \{\pm\infty\}$) is measurable if, for all Borel sets $A \in \sigma(\mathbb{R})$, the pre-image satisfies

$$f^{-1}(A) = \{ s \in S : f(s) \in A \} \in \Sigma .$$



We write $m\Sigma$ for the measurable with respect to Σ and $(m\Sigma)^+ \subseteq m\Sigma$ for the non-negative measurable functions and $b\Sigma$ for the bounded measurable functions.

Remark: This can be generalised, replacing $\mathbb{R}, \mathcal{B}(\mathbb{R})$ by any arbitrary measurable space T and σ-algebra Σ' on T.

Lemma: We have

1) $f^{-1}(A^c) = f^{-1}(A)^c$

2) $f^{-1}(\bigcup_i A_i) = \bigcup f^{-1}(A_i)$

3) $f^{-1}(\bigcap_i A_i) = \bigcap f^{-1}(A_i)$

Proof: We prove 2), the others are similar.

$$
\begin{aligned}
x \in f^{-1}\left(\bigcup_i A_i\right) &\iff f(x) \in \bigcup_i A_i \\
&\iff f(x) \in A_i \text{ for some } i \\
&\iff x \in f^{-1}(A_i) \text{ for some } i \\
&\iff x \in \bigcup_i f^{-1}(A_i) \qquad \square
\end{aligned}
$$

Prop If $f: \mathbb{R} \to \mathbb{R}$ is continuous, then it is measurable w.r.t. the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

This follows from the topological fact that $f^{-1}(\mathcal{O})$ is open whenever $\mathcal{O}$ is open and:

Prop: If $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$ is a collection such that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, then $f: S \to \mathbb{R}$ is measurable wrt Σ if and only if $f^{-1}(A) \in \Sigma$ for all $A \in \mathcal{C}$.

Proof: Let $\mathcal{H}$ be the set of all $A \subseteq \mathbb{R}$ such that $f^{-1}(A) \in \Sigma$. Clearly, $\mathcal{C} \subseteq \mathcal{H}$.

We show that $\mathcal{H}$ is a σ-algebra.

- $f^{-1}(\mathbb{R}) = S$, so $\mathbb{R} \in \mathcal{H}$
- If $A \in \mathcal{H}$, then $f^{-1}(A^c) = (\underbrace{f^{-1}(A)}_{\in \Sigma})^c \in \Sigma$
- If $A_i \in \mathcal{H}$ for all $i \in \mathbb{N}$, then

  $$f^{-1}\Big(\bigcup_i A_i\Big) = \bigcup_i \underbrace{f^{-1}(A_i)}_{\in \Sigma} \in \Sigma$$

  $$\Rightarrow \bigcup_i A_i \in \mathcal{H}.$$

Then $\mathcal{H}$ is a σ-algebra that contains $\mathcal{C}$.

Hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}) \subseteq \mathcal{H}$ and

$f^{-1}(A) \in \Sigma$ for all $A \in \mathcal{B}(\mathbb{R})$. $\square$

Examples: In order to show that f is measurable, it suffices to show

- $f^{-1}(A) \in \Sigma$ for all open A
- ——— " ——— closed A
- $f^{-1}((-\infty, x]) \in \Sigma$ for all $x \in \mathbb{R}$

...

<u>Lemma:</u> If f_1, f_2 are measurable functions $f_1, f_2 : S \to \mathbb{R}$, then so are $f_1 + f_2$ and $f_1 \cdot f_2$.

<u>Proof:</u> We want to show that

$$(f_1 + f_2)^{-1}((x, \infty)) \in \Sigma \quad \text{for all } x \in \mathbb{R},$$

given $f_1^{-1}((x, \infty)), f_2^{-1}((x, \infty)) \in \Sigma$.

We use:

$$f_1(s) + f_2(s) > x \iff \exists q \in \mathbb{Q} \text{ s.t. } f_1(s) > q,\ f_2(s) > x - q.$$

($\Leftarrow$) is trivial: we have $f_1(s) + f_2(s) > q + x - q = x$.

($\Rightarrow$) Let $\varepsilon > 0$ be s.t. $f_1(s) + f_2(s) = x + \varepsilon$.

i.e. $\varepsilon = f_1(s) + f_2(s) - x$.

$f_1(s) - q < \varepsilon$

$\Updownarrow$

Then exists $q \in \mathbb{Q}$ s.t. $f_1(s) - \varepsilon < q < f_1(s)$

and so $f_2(s) = x + \varepsilon - f_1(s) > x + f_1(s) - q - f_1(s) = x - q$

Using the fact gives

$$(f_1 + f_2)^{-1}((x, \infty)) = \bigcup_{q \in \mathbb{Q}} \Big(\underbrace{\overset{\in \Sigma}{f_1^{-1}((q, \infty))}}_{f_1(s) > q} \cap \underbrace{\overset{\in \Sigma}{f_2^{-1}((x - q, \infty))}}_{f_2(s) > x - q} \Big) \in \Sigma.$$

The proof for products is similar. $\square$

Lemma: The composition of measurable functions is measurable.

Proof: This follows from the fact that

$$(f_1 \circ f_2)^{-1}(A) = \{x : f_1(f_2(x)) \in A\}$$
$$= \{x : f_2(x) \in f_1^{-1}(A)\}$$
$$= f_2^{-1}(f_1^{-1}(A)) \qquad \square$$

Lemma If $f_n : S \to \mathbb{R}$ is measurable for every $n \in \mathbb{N}$ then so are

$$\inf_n f_n \quad \text{and} \quad \sup_n f_n,$$
$$\liminf_{n\to\infty} f_n \quad \text{and} \quad \limsup_{n\to\infty} f_n$$

Moreover, the set

$$\{s \in S : \lim_{n\to\infty} f_n(s) \text{ exists and is finite}\}$$

is in the σ-algebra Σ.

Proof: Note that $\left(\inf_n f_n\right)^{-1}([x,\infty))$

$$= \{s \in S : \inf_n f_n(s) \geq x\} = \bigcap_n \{s \in S : f_n(s) \geq x\}$$
$$= \bigcap_n f_n^{-1}([x,\infty)) \in \Sigma \quad \text{since } f_n \text{ measurable.}$$

So $\inf_n f_n$ is measurable. The proof for $\sup_n f_n$ is similar. Note that

$$\liminf_{n\to\infty} f_n(s) = \sup_n \inf_{m\geq n} f_m(s)$$ and so

is measurable by the previous facts. Similarly,

$$\limsup_{n\to\infty} f_n(s) = \inf_n \sup_{m\geq n} f_m(s)$$ is measurable.

From the last statements,

$$\{s\in S : \lim_{n\to\infty} f_n(s) \text{ exists and is finite}\}$$

$$= \{s\in S : \limsup_n f_n(s) < \infty\} \cap \{s\in S : \liminf_n f_n(s) > -\infty\}$$

$$\cap \{s\in S : \liminf_n f_n(s) = \limsup_n f_n(s)\}$$

Since $\left\{\begin{matrix} \limsup_n f_n \\ \liminf_n f_n \\ \limsup_n f_n - \liminf_n f_n \end{matrix}\right\}$ are measurable

and the sets above are the preimages of $\left\{\begin{matrix} [-\infty,\infty) \\ (-\infty,\infty] \\ \{0\} \end{matrix}\right\}$

they, and their intersection, are in Σ. $\square$

Def: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ that is $\mathcal{F}$-measurable is called a random variable.

Example Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(A) = \#A/6$. (model of die cast)

and define
$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in \{1, 3, 5\} \\ 0 & \text{if } \omega \in \{2, 4, 6\} \end{cases}$$

X is called an indicator variable (here, of odd die rolls).

$Y(\omega) = \omega$ is also a (simple) random variable.

For any random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ we can define the σ-algebra generated by X to be the smallest (sub) σ-algebra that makes X measurable. This is

$\sigma(\{X^{-1}(A) : A \text{ is a Borel set}\})$ and we simply write $\sigma(X)$ here. (i.e. X is also a r.v. on $(\Omega, \sigma(X), \mathbb{P})$).

In the example above,

$$\sigma(Y) = \mathcal{P}(\{1,2,3,4,5,6\})$$

but $\sigma(X) = \{\emptyset, \{1,3,5\}, \{2,4,6\}, \{1,2,3,4,5,6\}\}$

Law and distribution function

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

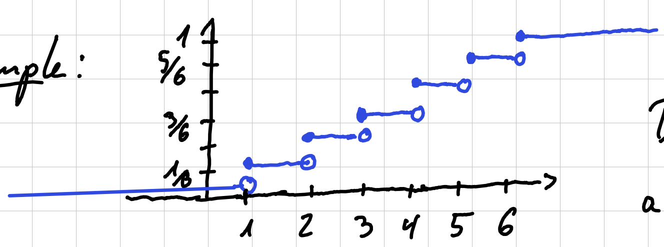
Define $\mathcal{L}_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$.

"probability that X is in A"

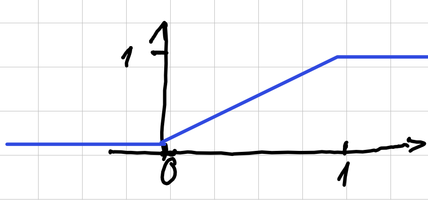
This is a probability measure on $\mathcal{B}(\mathbb{R})$ that is called the law of X. It is uniquely determined by the (cumulative) distribution function:

$$F_X(t) = \mathcal{L}_X((-\infty, t]) = \mathbb{P}(X \leq t)$$

Example:



Distribution of a die roll.



Distribution of a uniformly random variable on $[0,1]$.

Properties of distribution functions:

- non-decreasing $F_X(t) \leq F_X(s)$ if $t \leq s$.
- $\lim_{t \to -\infty} F_X(t) = 0$, $\lim_{t \to \infty} F_X(t) = 1$
- right-continuous : $\lim_{t \searrow a} F_X(t) = F_X(a)$.

The third can be proven as follows:

$$\lim_{t \searrow a} F_X(t) = \lim_{t \searrow a} \mathbb{P}(X \in (-\infty, t])$$

$$= \mathbb{P}\Big(\bigcap_{t > a} \{\omega : X(\omega) \in (-\infty, t]\}\Big)$$

$$= \mathbb{P}(\{\omega : X(\omega) \in (-\infty, a]\})$$

$$= F_X(a) .$$

Conversely, given a function F satisfying all the above, we can define a probability measure $\mathcal{L}$ with $\mathcal{L}((-\infty, t]) = F(t)$.