Problem Session 1.
[Selected Answers]

Q2 a) Not true.

Let $A_n = \begin{cases} \{0\} & \text{if } n \text{ even} \\ \{1\} & \text{if } n \text{ odd} \end{cases}$

$B_n = \begin{cases} \{0\} & \text{if } n \text{ odd} \\ \{1\} & \text{if } n \text{ even} \end{cases}$

Then $\limsup A_n = \limsup B_n = \{0, 1\}$.

However, $A_n \cap B_n = \{0\} \cap \{1\} = \emptyset$ for all n.

Hence $\limsup (A_n \cap B_n) = \emptyset \neq \{0, 1\} = \limsup A_n \cap \limsup B_n$.

b) True. Let $L = \limsup A_n \cup B_n$,

$\overline{A} = \limsup A_n$ and $\overline{B} = \limsup B_n$.

If $x \in \overline{A} \cup \overline{B}$ then $x \in \overline{A}$ or $\in \overline{B}$.

WLOG assume $x \in \overline{A}$. Then $x \in A_n$ for i.m. n

and so $x \in A_n \cup B_n$ for i.m. n.

Hence $x \in L$ and $\overline{A} \cup \overline{B} \subseteq L$.

For the other direction, assume $x \in L$. Then, $x \in A_n \cup B_n$ for i.m. $n \in \mathbb{N}$. But by the pigeonhole principle there must be i.m. n_k (subseq.) such that $x \in A_{n_k}$ or $x \in B_{n_k}$. Hence $x \in \bar{A}$ or $x \in \bar{B}$. We conclude $x \in \bar{A} \cup \bar{B}$ and $L \subseteq \bar{A} \cup \bar{B}$. This shows $L = \bar{A} \cup \bar{B}$.

3.) Assume $f : S \to \mathbb{R}$ is measurable. Let $A \in \mathcal{B}(\mathbb{R})$. Then $|f|^{-1}(A) = \{s \in S : |f(s)| \in A\}$

$$= \{s \in S : f(s) \leq 0 \text{ and } -f(s) \in A\}$$
$$\cup \{s \in S : f(s) > 0 \text{ and } f(s) \in A\}$$

$$= \left(\{s \in S : -f(s) \in A\} \cap \{s \in S : f(s) \leq 0\}\right)$$

$$\cup \left(\{s \in S : f(s) \in A\} \cap \{s \in S : f(s) > 0\}\right).$$

But these four events are in Σ as f is measurable. Hence $|f|^{-1}(A) \in \Sigma$.

For a counterexample, let $S = \{0,1\}$

$f(s) = \begin{cases} 1 & s \text{ even} \\ -1 & , \text{ odd.} \end{cases}$, let $\Sigma = \sigma(|f|)$

Then, $|f(s)| = 1$ for all $s \in S$ and

$\Sigma = \sigma(|f|) = \{\emptyset, S\}$.

However $f^{-1}(1) = \{0\} \notin \Sigma$.

4.) Let $\varepsilon_k = \left(\frac{1}{2}\right)^{k+1}$ and $n_k\ (> n_{k-1})$ large enough s.t. $\mathbb{P}(A_{n_k}) \geq 1 - \varepsilon_k$. Such n_k exists as $\mathbb{P}(A_n) \to 1$.

Then $\mathbb{P}(A_{n_1}) \geq 1 - \left(\frac{1}{2}\right)^2 = 3/4$.

Further,

$$\mathbb{P}\left(\bigcap_{j=1}^{k+1} A_{n_j}\right) = \mathbb{P}\left(\left(\bigcap_{j=1}^{k} A_{n_j}\right) \cap A_{n_{k+1}}\right)$$

$$= \mathbb{P}\left(\bigcap_{j=1}^{k} A_{n_j}\right) + \mathbb{P}(A_{n_{k+1}}) - \mathbb{P}\left(\left(\bigcap_{j=1}^{k} A_{n_j}\right) \cup A_{n_{k+1}}\right)$$

$$\geq \mathbb{P}\left(\bigcap_{j=1}^{k} A_{n_j}\right) + 1 - \varepsilon_{k+1} - 1 = \mathbb{P}\left(\bigcap_{j=1}^{k} A_{n_j}\right) - \varepsilon_{k+1}$$

$$= 1 - \varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_{k+1} \quad \text{(by induction)}$$

But then $\mathbb{P}\left(\bigcap_{j=1}^{\infty} A_{n_j}\right) = \lim_{k \to \infty} 1 - \varepsilon_1 - \varepsilon_2 - \dots - \varepsilon_k$

$$= 1 - \sum_{k=1}^{\infty} \varepsilon_k = 1 - \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2} > 0.$$

6) Let $F_X(t) = \mathbb{P}(X \leq t)$ be the cumulative distribution function of X.

Let $\varepsilon > 0$ be given and let t_1, t_2 be such that $F_X(t_1) = \varepsilon/3$ and $F_X(t_2) = 1 - \varepsilon/3$.

Define $X_\varepsilon(s) = \begin{cases} X(s) & \text{if } s \in X^{-1}([t_1, t_2]) \\ 0 & \text{otherwise.} \end{cases}$

This is measurable: Let $G = X^{-1}([t_1, t_2]) \in \Sigma$.

Then $X_\varepsilon^{-1}(A) = \{ s \in S : X_\varepsilon(s) \in A \}$

$$= \{ s \in S : X(s) \in A \cap G \}$$

$$\cup \{ s \in S : X(s) \in G^c \text{ if } 0 \in A \}$$

which is in Σ.

Since $|X_\varepsilon(s)| \leq \max\{|t_1|, |t_2|\}$ by construction, it is also bounded.

Finally

$$\mathbb{P}(X_\varepsilon \neq X) = \mathbb{P}(X(s) \notin [t_1, t_2])$$
$$= \mathbb{P}(X(s) < t_1 \text{ or } \mathbb{P}(X(s) > t_2)$$
$$\leq \mathbb{P}(X(s) < t_1) + 1 - \mathbb{P}(X(s) \leq t_2)$$
$$\leq \varepsilon/3 + \varepsilon/3 = \frac{2}{3}\varepsilon < \varepsilon$$

as required.

8) (2) => (1): Assume such an $a \in \mathbb{R}$ exists. Then, $\sum_{n=1}^{\infty} \mathbb{P}(X_n > a) < \infty$ combined with the Borel-Cantelli Lemma show that $X_n > a$ for at most finitely many n, almost surely. Say N is the largest such index. Then,

$$\sup X_n \leq \max\{a, \max_{n \leq N} X_n\} < \infty \text{ almost surely.}$$

(1) => (2) (equiv: not(2) => not(1)): Assume no such a exists. We can let

k_n be a seq. s.t. $k_n \nearrow \infty$,

and

$$\mathbb{P}(X_1 > 1) + \mathbb{P}(X_2 > 1) + \dots + \mathbb{P}(X_{k_1} > 1) \geq 1/2$$

and $\mathbb{P}(X_{k_1+1} > 2) + \dots + \mathbb{P}(X_{k_2} > 2) \geq 1/2$

$\vdots$

$k_{i-1} < j \leq k_i$

i.e. $\sum_{j=k_n+1}^{k_{n+1}} \mathbb{P}(X_j > n-1) = 1/2$ and $\sum_{j=1}^{\infty} \mathbb{P}(X_j > i :$

$$\sum_{j=1}^{\infty} \mathbb{P}(X_j > i : k_{i-1} < j < k_i) = \sum_{i=1}^{\infty} 1/2 = \infty$$

Thus, by independence and the second B.C. lemma infinitely many of $\{X_j > i : k_{i-1} < j \leq k_i\}$ occur. But since $i \to \infty$ as $j \to \infty$,

$\sup X_j = \infty$ almost surely.

Hence $\mathbb{P}(\sup X_j < \infty) \neq 1$ (in fact it's 0). $\square$

9) Clearly, (2) $\Rightarrow$ (1).

Since "A_n occurs for at least one n" $= \bigcup_{n=1}^{\infty} A_n$

We have $1 = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n^c\right)$

$= 1 - \prod_{n=1}^{\infty} \mathbb{P}(A_n^c)$. So,

$$\prod_{n=1}^{\infty} \mathbb{P}(A_n^c) = \prod_{n=1}^{\infty} (1 - \mathbb{P}(A_n)) = 0.$$

Since none of the $\mathbb{P}(A_n)$ equal 1,

$$(*) \quad \prod_{n=1}^{N} (1 - \mathbb{P}(A_n)) \downarrow 0 \quad \text{as } N \to \infty.$$

We will prove that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ from which (2) follows by the second BC theorem.

We may assume there exists k s.t. $\mathbb{P}(A_n) < 1/2$ for all $n \geq k$. Otherwise $\mathbb{P}(A_n) \geq \frac{1}{2}$ for i.m n and $\sum \mathbb{P}(A_n) = \infty$. Note that $\log \frac{1}{2} = -\log 2$ and $\log 1 = 0$ and $\log x$ is concave. Hence $\log\left((1-p) + p \cdot \frac{1}{2}\right)$

$= \log\left(1 - \frac{1}{2}p\right) \geq (1-p)\log 1 + p \log \frac{1}{2} = -p \log 2$ for $p \in [0,1]$

Equivalently, $\log(1-x) \geq -x \, 2 \log 2$ for $x \in [0, \frac{1}{2}]$.

This and (*) gives that $\forall \varepsilon > 0 \ \exists N$ s.t.

$$(+) \quad \underbrace{\prod_{n=1}^{k-1} \mathbb{P}(A_n^c)}_{>0} \cdot \underbrace{\prod_{n=k}^{N} (1 - \mathbb{P}(A_n))}_{>0} \leq \left(\prod_{n=1}^{k-1} \mathbb{P}(A_n^c) \right) \cdot \varepsilon$$

Taking log gives

$$\log \varepsilon \geq \sum_{n=k}^{N} \log(1 - \mathbb{P}(A_n)) \geq \sum_{n=k}^{N} - \mathbb{P}(A_n) 2 \log 2$$

and $\displaystyle \sum_{n=k}^{N} \mathbb{P}(A_n) \geq \frac{\log 1/\varepsilon}{2 \log 2} \longrightarrow \infty$

as $\varepsilon \to 0$. Hence $\displaystyle \sum_{n=k}^{\infty} \mathbb{P}(A_n) = \infty$ as required. $\square$

$\mathbb{P}(A_n) = 1$ is forbidden as we need positivity in (+). Statement is not true otherwise.

E.g. Let $\mathbb{P}(A_1) = 1$, $\mathbb{P}(A_n) = 0$. Then one of A_n occurs almost surely (A_1) but also no other occurs.

10) Consider the distribution $F_X(t) = \mathbb{P}(X \leq t)$

$$= \mathbb{P}(\sup_n X_n \leq t) = \mathbb{P}(X_1 \leq t \;\&\; X_2 \leq t \;\& \ldots)$$

$$= \prod_{j=1}^{\infty} \mathbb{P}(X_j \leq t) \qquad \text{by independence.}$$

Now $\mathbb{P}(X_j \leq t) = F_{X_j}(t) = \begin{cases} 0 & t \leq 0 \\ jt & 0 < t \leq 1/j \\ 1 & t > 1/j \end{cases}$

Thus, for $t = 0$. $F_X(t) = 0$ and

for $\frac{1}{n+1} < t \leq \frac{1}{n}$, $F_X(t) = \prod_{k=1}^{n} F_{X_k}(t)$

$$= t \cdot 2t \cdot 3t \ldots nt$$

$$= n!\, t^n = \lfloor \tfrac{1}{t} \rfloor t^{\lfloor 1/t \rfloor} \quad \text{as } n+1 > \frac{1}{t} \geq n$$