Assignment 2 - Solutions

(1) (i) $X = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \ln S_u \, du$

By risk-neutral valuation,

$$\pi_0(X) = E^Q_{0,s}\left[e^{-rT_2} \frac{1}{T_2-T_1} \int_{T_1}^{T_2} \ln S_u \, du\right]$$

$$= \frac{e^{-rT_2}}{T_2-T_1} \int_{T_1}^{T_2} E^Q_{0,s}[\ln S_u] \, du$$

$$= \frac{e^{-rT_2}}{T_2-T_1} \int_{T_1}^{T_2} \left(\ln s + (r - \tfrac{\sigma^2}{2}) u\right) du$$

$$= e^{-rT_2}\left(\ln s + (r - \tfrac{\sigma^2}{2}) \frac{T_1+T_2}{2}\right)$$

(ii) Write $\frac{1}{T_2-T_1} \int_{T_1}^{T_2} \ln S_u \, du = \frac{1}{T_2-T_1}\left(\underbrace{\int_{T_1}^{t} \ln S_u \, du}_{\text{Known at } t.} + \int_{t}^{T_2} \ln S_u \, du\right)$

The price of $\frac{1}{T_2-T_1} \int_t^T \ln S_u \, du$ at time t is

$$E\left[\frac{e^{-r(T_2-t)}}{T_2-T_1} \int_t^{T_2} \ln S_u \, du\right] = \frac{e^{-r(T_2-t)}}{T_2-T_1} \int_t^{T_2} \underbrace{E_{t,s}[\ln S_u]}_{\ln s + (r - \frac{\sigma^2}{2})(u-t)} du$$

$$= \frac{e^{-r(T_2-t)}}{T_2-T_1}\left((T_2-t)\ln s + (r - \tfrac{\sigma^2}{2})\frac{(T_2-t)^2}{2}\right)$$

Answer: $\frac{e^{-r(T_2-t)}}{T_2-T_1}\left(\int_{T_1}^{t} \ln S_u \, du + (T_2-t)\ln S_t + (r - \tfrac{\sigma^2}{2})\frac{(T_2-t)^2}{2}\right)$

(2) Assume $c(K_1) - c(K_2) > e^{-rT}(K_2 - K_1)$. Find an arbitrage!

At $t=0$: Buy a call with strike K_2 and sell one call with strike K_1. Deposit $c(K_1) - c(K_2)$'s in the bank.

(Total cost at $t=0$ is then 0!)

At $t=T$: Receive

$$(S_T - K_2)^+ - (S_T - K_1)^+ + (c(K_1) - c(K_2))e^{rT} >$$

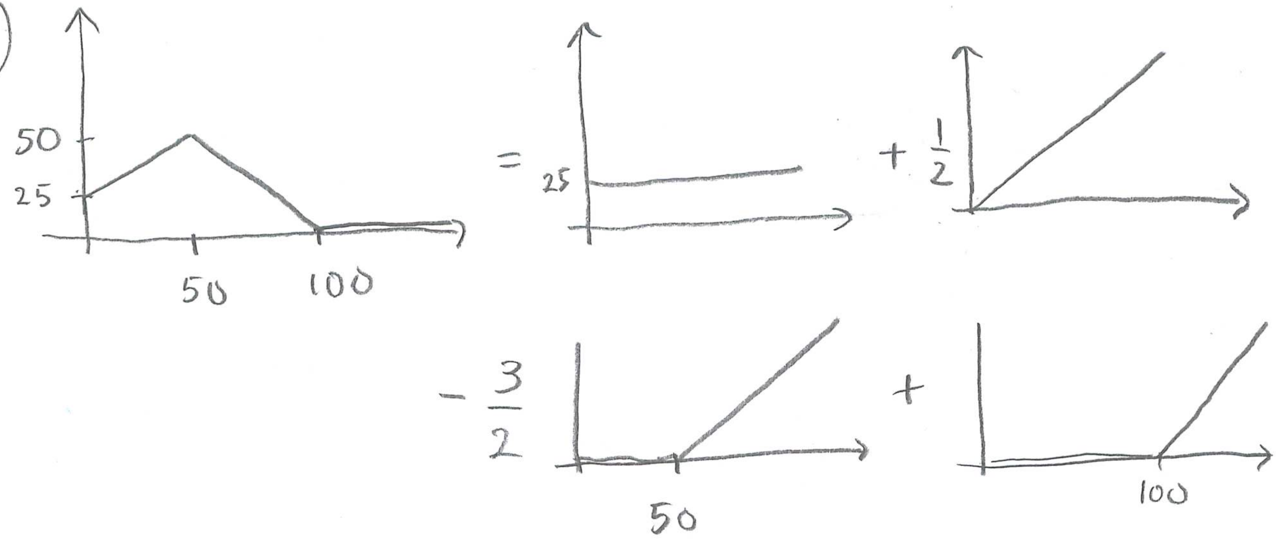
$$> (S_T - K_2)^+ - (S_T - K_1)^+ + K_2 - K_1$$

$$= \begin{cases} 0 & \text{if } K_2 < S_T \\ K_2 - S_T & \text{if } K_1 \leq S_T \leq K_2 \\ K_2 - K_1 & \text{if } S_T < K_1 \end{cases}$$

$$\geq 0.$$

Thus we have constructed an arbitrage!

(3)



The value of the T-claim coincides with the value of the following portfolio:

- A zero-coupon bond with face value 25
- $\frac{1}{2}$ share of S
- $-\frac{3}{2}$ call options with strike 50
- a call option with strike 100.

(4) We have $\phi(s_1, s_2) = \min\{s_1, s_2\}$ so $\phi(ks_1, ks_2) = k\phi(s_1, s_2)$.

Thus the pricing function F satisfies

$F(t, s_1, s_2) = s_2 G(t, \frac{s_1}{s_2})$, where $G(t,z)$ is given by

$$\begin{cases} G_t + \frac{1}{2}(C_{11} + C_{22} - 2C_{12}) z^2 G_{zz} = 0 \\ G(T,z) = \min\{z, 1\} \end{cases}$$

Here $C = \sigma\sigma^* = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

Since $\min\{z, 1\} = z - (z-1)^+$, we have

$G(t,z) = z - zN(d_1) + N(d_2)$ where

$$\begin{cases} d_1 = \dfrac{\ln z + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

Consequently,

$$F(t, s_1, s_2) = s_2 G(t, \frac{s_1}{s_2}) = s_1 N\left(\frac{\ln\frac{s_2}{s_1} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) + s_2 N\left(\frac{\ln\frac{s_1}{s_2} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)$$

<u>Answer</u>: The price is $F(t, S_t^1, S_t^2)$, where

$$F(t, s_1, s_2) = s_1 N\left(\frac{\ln\frac{s_2}{s_1} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) + s_2 N\left(\frac{\ln\frac{s_1}{s_2} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)$$

with $\sigma^2 := \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.