2021.01.08

1. $\begin{cases} dX(t) = -2X(t)\,dt + \sqrt{X(t)}\,dW(t) \\ X(0) = x \end{cases}$

$$E[X(t)] = x - 2E\left[\int_0^t X(s)\,ds\right] + E\left[\int_0^t \sqrt{X(s)}\,dW(s)\right]$$

$$= x - 2\int_0^t E[X(s)]\,ds$$

Set $m(t) = E[X(t)]$. Then $\begin{cases} \dot m(t) = -2m(t) \\ m(0) = x \end{cases}$ so $m(t) = xe^{-2t}$

For the variance, let $Y(t) := X^2(t)$. Then

$$dY(t) = 2X\,dX + (dX)^2 = (-4Y(t) + X(t))dt + 2X^{3/2}dW$$

Let $M(t) = E[X^2(t)] = E[Y(t)]$. Then

$$\begin{cases} \dot M(t) = -4M(t) + E[X(t)] \\ \quad\quad = -4M(t) + xe^{-2t} \\ M(0) = x^2 \end{cases}$$

so $M(t) = \underbrace{Ce^{-4t}}_{\text{hom. solution}} + \underbrace{M_p(t)}_{\text{part. solution}}$

M_p can be found on the form $M_p(t) = Be^{-2t}$. Plugging in gives $-2B = -4B + x$ so $B = \frac{x}{2}$

The initial condition gives C: $x^2 = C + \frac{x}{2}$ so $C = x^2 - \frac{x}{2}$

$$\mathrm{Var}(X(t)) = E[X^2(t)] - (E[X(t)])^2 = \left(x^2 - \frac{x}{2}\right)e^{-4t} + \frac{x}{2}e^{-2t} - x^2e^{-4t}$$

$$= \frac{x}{2}e^{-2t}(1 - e^{-2t})$$

<u>Answer:</u> $E[X(t)] = xe^{-2t}$

$\mathrm{Var}(X(t)) = \frac{x}{2}e^{-2t}(1 - e^{-2t})$

② $\begin{cases} u_t + 2u_{xx} - u = 0 \\ u(T,x) = x + \sin 2x \end{cases}$

Feynman-Kac $\Rightarrow$

$u(t,x) = e^{-(T-t)} E_{t,x}\left[X(T) + \sin 2X(T) \right]$

where $\begin{cases} dX(s) = 2\,dW(s) \\ X(t) = x \end{cases}$

We have $E_{t,x}\left[X(T)\right] = E\left[x + 2(W(T) - W(t))\right] = x.$

Also note that $E_{t,x}\left[\sin 2X(T)\right] = E_{0,x}\left[\sin 2X(T-t)\right]$

Let $Y(s) := \sin 2X(s)$ $\left(\text{where } \begin{cases} dX(s) = 2\,dW(s) \\ X(0) = x \end{cases} \right)$

Ito $\Rightarrow$ $dY(s) = 2\cos 2X(s)\,dX(s) - 2\sin 2X(s)\,(dX(s))^2$

$= -8Y(s)\,ds + 4\cos 2X(s)\,dW(s)$

Thus $m(s) := E\left[Y(s)\right]$ satisfies $\begin{cases} \dot{m}(s) = -8m(s) \\ m(0) = \sin 2x \end{cases}$

so $E\left[Y(s)\right] = m(s) = e^{-8s}\sin 2x.$

Thus $E_{t,x}\left[\sin 2X(T)\right] = e^{-8(T-t)}\sin 2x$

<u>Answer</u>: $u(t,x) = xe^{-(T-t)} + (\sin 2x)\,e^{-9(T-t)}$

(3) i) $\Pi(0; S^2(T)) = s^2 e^{(r+\sigma^2)T}$

$= 144\, e^{0.05}$

ii) $\Delta = 2s\, e^{(r+\sigma^2)T} = 24\, e^{0.05}$ shares of S

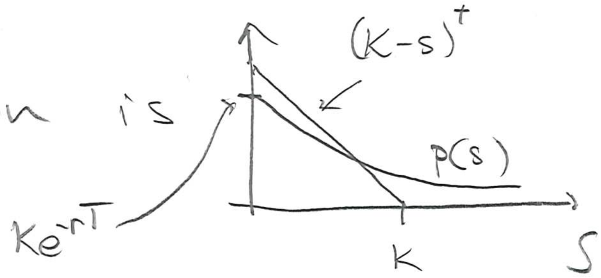
In the bank account: $144 e^{0.05} - 12 \cdot 24 e^{0.05} = -144 e^{0.05}$

(i.e. $144 e^{0.05}$ is borrowed from the bank)

(4) i) $E[W^7(t)] = \int_{\mathbb{R}} \underbrace{\underbrace{\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}}_{\text{even}} \underbrace{x^7}_{\text{odd}}}_{\text{odd}} dx = 0$ since the integrand is odd.

True!

ii) The graph of a put option is



For s small,

$p(s) \approx Ke^{-rT} < K - s$.

False!

iii) $F_\delta(0,s) = F_0(0, se^{-\delta T})$

$\uparrow$ usual BS-formula.

Since the BS-formula is <u>increasing</u> in s,

we have $F_\delta(0,s) = F_0(0, se^{-\delta T}) \leq F(0,s)$

True!

(5) $C(K_1)+C(K_2) < 2C\left(\frac{K_1+K_2}{2}\right)$

At $t=0$: Buy a call with strike K_1

Buy — " — K_2

Sell two calls with strike $\frac{K_1+K_2}{2}$.

From this you receive $2C\left(\frac{K_1+K_2}{2}\right) - C(K_1) - C(K_2) > 0$.

At $t=T$: Receive $(S(T)-K_1)^+ + (S(T)-K_2)^+ - 2\left(S(T)-\frac{K_1+K_2}{2}\right)^+$

$=: Y$

If $S(T) \leq K_1$ then $Y = 0+0-0 = 0$

If $S(T) \in \left(K_1, \frac{K_1+K_2}{2}\right)$ then $Y = S(T)-K_1+0-0 \geq 0$

If $S(T) \in \left[\frac{K_1+K_2}{2}, K_2\right)$ then $Y = S(T)-K_1+0-2\left(S(T)-\frac{K_1+K_2}{2}\right)$

$= K_2 - S(T) \geq 0$

If $S(T) \geq K_2$ then $Y = S(T)-K_1+S(T)-K_2-2\left(S(T)-\frac{K_1+K_2}{2}\right)$

$= 0$

Thus we have a model-independent arbitrage!

(6) Price is $e^{-rT} E^Q\left[\frac{S_1^2(T)}{S_2(T)}\right] \underset{\text{indep.}}{=} e^{-rT} E[S_1^2(T)] \cdot E\left[\frac{1}{S_2(T)}\right]$

$= e^{-rT} s_1^2 e^{(2r+\sigma_1^2)T} \frac{1}{s_2} e^{(\sigma_2^2 - r)T} = \frac{s_1^2}{s_2} e^{(\sigma_1^2+\sigma_2^2)T}$

⑦ $\begin{cases} dr = ar(t)\,dt + \sigma(t)\,dW(t) \\ r(0) = r_0 \end{cases}$

Plug in $F^T(t,r) = e^{A(t,T) - B(t,T)r}$ into the term structure equation $\begin{cases} F_t^T + \frac{\sigma^2(t)}{2} F_{rr}^T + arF_r^T - rF^T = 0 \\ F^T(T,r) = 1 \end{cases}$

This gives $\begin{cases} A_t - B_t r + \frac{\sigma^2(t)}{2} B^2 - aBr - r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$

i.e $A_t + \frac{\sigma^2(t)}{2} B^2 = 0$ and $B_t + aB + 1 = 0$

Thus $B(t,T) = Ce^{-at} - \frac{1}{a} \underset{B(T,T)=0}{\uparrow}{=} \frac{1}{a}\left(e^{a(T-t)} - 1\right)$

so

$$A(t,T) = \int_t^T \frac{\sigma^2(s)}{2} B^2(s,T)\,ds = \frac{1}{2a^2}\int_t^T \sigma^2(s)\left(e^{a(T-s)} - 1\right)^2 ds$$

<u>Answer</u>: $p(0,T) = \exp\left\{\frac{1}{2a^2}\int_0^T \sigma^2(s)\left(e^{a(T-s)} - 1\right)^2 ds - r_0\frac{1}{a}\left(e^{aT} - 1\right)\right\}$

⑧ $F(s,0) = E^Q\left[e^{-rT} g(S(T))\right] = e^{-rT}\int_{\mathbb{R}} g\left(se^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$$\frac{\partial F}{\partial r} = Te^{-rT}\int_{\mathbb{R}} \underbrace{\left(g'(\ )\,se^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x} - g(\ )\right)}_{\geq 0}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \geq 0$$