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PROV I MATEMATIK
Markov Processes
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Permitted aids: Calculator

1. A Markov chain $\{X_n\}_{n\geq 0}$ with state space $S = \{1, 2, 3\}$ has transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \\ 1/6 & 2/3 & 1/6 \end{pmatrix}.$$

Let $T = \min\{n \geq 1 : X_n = 2\}$.

(a) Calculate $E(T|X_0 = k)$, for all $k \in \{1, 2, 3\}$. (2p)

(b) Find the limit $\lim_{n\to\infty} P(X_n \neq 2|X_0 = 1)$. (2p)

(c) Calculate $P(T \leq 7|X_0 = 1)$. (2p)

Solution:

(a) Let $m_k = E(T|X_0 = k)$. By conditioning on the outcome of the first step we get

$$\begin{cases} m_1 = 1 + \frac{1}{3}E(T|X_1 = 1, X_0 = 1) & = 1 + \frac{1}{3}m_1 \\ m_2 = 1 + E(T|X_1 = 3, X_0 = 2) & = 1 + m_3 \\ m_3 = 1 + \frac{1}{6}E(T|X_1 = 1, X_0 = 3) + \frac{1}{6}E(T|X_1 = 3, X_0 = 3) & = 1 + \frac{1}{6}m_1 + \frac{1}{6}m_3 \end{cases},$$

so $m_1 = m_3 = 3/2$ and $m_2 = 5/2$.

Alternative solution:

If $k = 1$ or $k = 3$ then $P(T = n|X_0 = k) = (1 - 2/3)^{n-1}(2/3)$ (so T then has the first success distribution with parameter $2/3$.

If X is a random variable having the first sucess distribution with parameter p, then $E(X) = 1/p$, so $E(T|X_0 = k) = 1/(2/3) = 3/2$, if $k \in \{1, 3\}$ and thus $E(T|X_0 = 2) = 1 + 3/2 = 5/2$.

(b) Since the Markov chain is irreducible and aperiodic it follows from the Markov chain convergence theorem that

$$\begin{aligned} \lim_{n\to\infty} P(X_n \neq 2|X_0 = 1) &= 1 - \lim_{n\to\infty} P(X_n = 2|X_0 = 1) \\ &= 1 - (E(T|X_0 = 2))^{-1} = 1 - (5/2)^{-1} = 1 - 2/5 = 3/5. \end{aligned}$$

(c) If X is a random variable having the first sucess distribution with parameter p, then $P(X > k) = (1 - p)^k$, $k \geq 0$, so

$$P(T \leq 7|X_0 = 1) = 1 - P(T > 7|X_0 = 1) = 1 - (1/3)^7 = \frac{2186}{2187} \approx 0.9995.$$

2. Let $\{X_n\}$ be a Markov chain with state space $S = \{1, ..., n\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

Suppose $\sum_{j=1}^n p_{ij} = 1$, for all i, and $\sum_{i=1}^n p_{ij} = 1$, for all j, i.e. suppose $\mathbf{P}$ is doubly stochastic, and suppose in addition that $p_{ij} > 0$ for any i and j. Find all stationary distributions π. (5p)

Solution: Since the column sums of $\mathbf{P}$ are one it follows that the uniform distribution $\pi = (1/n, ..., 1/n)$ is stationary i.e. satisfies $\pi\mathbf{P} = \pi$. Since the Markov chain is irreducible and aperiodic it follows from the Markov chain convergence theorem that this distribution is the only stationary distribution.

3. A minibar fridge at a hotel can store at most 4 soft drink cans. Soft drink cans are picked from the fridge according to a Poisson process with intensity 4 per day. When only one soft drink can remains in the fridge a hotel guest calls the hotel reception to fill the fridge. This service arrives after an exponentially distributed time with expectation one hour. It can happen that the fridge goes empty before the service arrives. At service the fridge is filled up to contain 4 soft drink cans.

   (a) The number of soft drink cans in the fridge can be described by a Markov process in continuous time. Find the intensity matrix for this process. (3p)

   (b) Find the long run probability that the fridge contains no soft drink cans. (2p)

   (c) How long time does it take on average between calls from a hotel guest to the hotel reception requesting to fill the fridge? (1p)

Solution:

(a) Cans are delivered with intensity 24 cans per day. The number of cans in the fridge at time t (measured in days) can be modelled by a Markov proceess (Y_t) with state space be $\{0, 1, 2, 3, 4\}$ where $Y_t = k$ means that there are k soft drink cans in the fridge at time t. The possible transitions $0 \mapsto 4$, $1 \mapsto 4$, corresponding to "filling up the fridge" occur with intensity 24 and the transitions $i \mapsto i - 1$, for $i = 1, 2, 3, 4$ occur with the "picking" intensity 4.

Thus the intensity matrix for this Markov process is

$$\mathbf{Q} = \begin{pmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix} = \begin{pmatrix} -24 & 0 & 0 & 0 & 24 \\ 4 & -28 & 0 & 0 & 24 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

(b) Since this is an irreducible Markov process with finite state space we know that the long run probability that the fridge is empty is $P(Y_t = 0) \to \pi_0$, where $\pi = (\pi_0, .., \pi_4)$ is a probability vector solving the equation $\pi\mathbf{Q} = \mathbf{0}$.

This gives $-24\pi_0 + 4\pi_1 = 0$, $-28\pi_1 + 4\pi_2 = 0$, $-4\pi_2 + 4\pi_3 = 0$, $-4\pi_3 + 4\pi_4 = 0$, $24\pi_0 + 24\pi_1 - 4\pi_4 = 0$. Thus $\pi_2 = \pi_3 = \pi_4 = 7\pi_1 = 42\pi_0$.

$\sum_{i=0}^4 \pi_i = 1$ and $\pi_i \geq 0$ gives $\pi_0(1 + 6 + 42 + 42 + 42) = 1$, i.e $\pi_0 = 1/133$.

(c) If T_{11} denotes the time between calls to the reception from the guest (measured in days), then

$$\pi_1 = 6\pi_0 = \frac{6}{133} = \frac{1}{-q_{11}\pi_1},$$

so

$$E(T_{11}) = \frac{1}{-q_{11}\pi_1} = \frac{1}{28 \cdot (6/133)} = \frac{133}{168}.$$

Thus the average time between call from a hotel guest to the reception is $24E(T_{11}) =$ 19 hours.

4. Consider a population of N individuals. Let λ be a positive constant and assume that in a short time interval h any pair of individuals (i, j) may meet with probability $\frac{\lambda h}{N} + o(h)$, independent of all other interactions. The individuals may carry a virus. When a carrier meets a non-carrier, the virus is transferred and both individuals become carriers.

   (a) The number of carriers in the population can be modeled by a birth process, $\{X_t\}_{t\geq 0}$. Specify the birth-rates of this birth process. (3p)

   (b) Suppose $X_0 = 1$. Let T be the first time when all individuals in the population carries the virus. Calculate $E(T)$. (3p)

Solution:

Let X_t be the number of carriers in the population. Suppose $X_t = n$ and consider a short time interval $(t, t+h]$. There are n infected people and $N - n$ uninfected ones, so there are $n(N - n)$ different pairs (i, j) where i is infected and j is not. Each of these pairs has probability $\frac{h\lambda}{N} + o(h)$ of meeting in the time interval $(t, t+h]$. Thus the probability that one pair meets is $\frac{hn(N-n)\lambda}{N} + o(h)$ and the probability that more than one pair meet is $o(h)$.

Thus

(a) $\{X_t\}$ is a birth process on $S = \{0, 1, ..., N\}$ with

$$\lambda_n = n(N-n)\frac{\lambda}{N}, \quad 0 \leq n \leq N$$

(b) If T_n denotes the duration of time with n carriers in the population, then $E(T) = \sum_{n=1}^{N-1} T_n$, where $T_n \sim \text{Exp}(\lambda_n)$. Thus

$$E(T) = \sum_{n=1}^{N-1} E(T_n) = \sum_{n=1}^{N-1} \frac{1}{\lambda_n} = \frac{N}{\lambda}\sum_{n=1}^{N-1} \frac{1}{n(N-n)} = \frac{1}{\lambda}\sum_{n=1}^{N-1}(\frac{1}{n} + \frac{1}{N-n}) = \frac{2}{\lambda}\sum_{n=1}^{N-1}\frac{1}{n}.$$

5. Consider the Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with transition probabilities

$$p_{ij} = P(X_{n+1} = j | X_n = i) = \begin{cases} 1/4 & j \in \{0, i+1, i+2, i+3\} \\ 0 & \text{otherwise} \end{cases},$$

for any $i \in \mathbb{Z}$.

   (a) Express $\mathbb{Z}$ as a disjoint union of sets of transient states and closed irreducible sets of recurrent states. (2p)

   (b) Prove that this Markov chain has a unique stationary distribution. (You don't need to explicitly find the stationary distribution!) (2p)

   (c) Explain how we can obtain a non-biased (perfect) sample from the unique stationary distribution.
   Hint: Use the Propp-Wilson perfect sampling algorithm. (2p)

Solution:

(a), (b) We have $\mathbb{Z} = T \cup C$, where the set of negative integers $T = \{\ldots, -3, -2, -1\}$ is the set of transient states, and the set of non-negative integers $C = \{0, 1, \ldots\}$ is a closed irreducible set of positive-recurrent states. (The set C is clearly irreducible. State 0 is positive recurrent since if T_i denotes the first return time to state i then T_0 has the first success distribution with parameter $1/4$, so $E(T_0) = 4 < \infty$. It therefore follows from the class property of positive recurrence, that all states in C are positive recurrent, i.e. $E(T_i) < \infty$, for any $i \in C$.) The stationary distribution $\pi = (\pi_0, \pi_1, \ldots)$ with $\pi_i = 1/(E(T_i))$ for any $i \in C$, is therefore the unique stationary distibution for $\{X_n\}$.

(c) The Markov chain $\{X_n\}$ can be obtained by random (i.i.d.) iterations with the functions $f_1(x) = 0$ and $f_2(x) = x + 1$, $f_3(x) = x + 2$, $f_4(x) = x + 3$ chosen with equal probabilities independently in each iteration step. The Markov chain considered can thus be represented as $X_{n+1} = f_{I_{n+1}}(X_n)$, where $\{I_n\}_{n=1}^{\infty}$ is a sequence of independent random variables with $P(I_n = 1) = P(I_n = 2) = P(I_n = 3) = P(I_n = 4) = 1/4$ for each fixed n.
Let N be the smallest integer such that $I_n = 1$. We have $P(N = k) = (1/4)(3/4)^{k-1}$, for $k \geq 1$, and

$$Z := f_{I_1} \circ \cdots \circ f_{I_N}(x),$$

is a π-distributed random variable, so the corresponding realisation $z := f_{i_1} \circ \cdots \circ f_{i_N}(x)$ is a real number chosen according to π, by construction. (If we need more π-distributed random numbers then we just use the same construction using more independent random numbers uniformly distributed on $\{1, 2, 3, 4\}$.)

6. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

(a) Let

$$X_t = (1-t)B\left(\frac{t}{1-t}\right), \qquad 0 \leq t < 1.$$

Find the the infinitesimal drift function $a(t,x)$ of $\{X_t\}$, i.e. find $a(t,x)$ such that

$$E(X_{t+h} - X_t \mid X_t = x) = a(t,x)h + o(h).$$

(3p)

(b) Prove that

$$P(B(t) > 10 \text{ for some } 0 \leq t \leq 4) \leq 0.1.$$

(3p)

Solution:

(a) If $t < 1$ and $h > 0$ is small, then

$$\begin{aligned} E(X_{t+h} - X_t \mid X_t = x) &= E((1-t-h)B(\frac{t+h}{1-t-h}) - (1-t)B(\frac{t}{1-t})|X_t = x) \\ &= (1-t)\underbrace{E(B(\frac{t+h}{1-t-h}) - B(\frac{t}{1-t})|B(\frac{t}{1-t}) = x/(1-t))}_{=0} \\ &\quad -hE(B(\frac{t+h}{1-t-h})|B(\frac{t}{1-t}) = x/(1-t)) = -hx/(1-t), \end{aligned}$$

so

$$a(t,x) = -\frac{x}{1-t}.$$

(b) Since $|B(t)|$ has the same distribution as $\max_{0 \leq s \leq t} B(s)$, and $B(t) \sim N(0,t)$ and thus $Z = B(t)/\sqrt{t} \sim N(0,1)$ for each fixed $t > 0$ it follows that

$$\begin{aligned}
P(B(t) > 10 \text{ for some } 0 \leq t \leq 4) &= P(\max_{0 \leq t \leq 4} B(t) > 10) \\
&= P(|B(4)| > 10) \\
&= P(|Z| \geq 5) \\
&= 2P(Z \geq 5) = 2\int_5^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&\leq \int_5^\infty e^{-x} dx \leq 0.1.
\end{aligned}$$