UPPSALA UNIVERSITET
Matematiska Institutionen
Topias Tolonen-Weckström

Solutions to Assignment 1
1MA255, VT25
2024/04/10

Partial Differential Equations with Applications to Finance

Instructions: There are three problems worth a total of 24 points. A score 12 points yield 2 bonus points. Your answer should be well motivated in order to receive full credit in each question. Collaboration is encouraged. The solutions should be submitted by April 27th to Studium or via email to me.

---

1. (10 points). For a given standard Brownian motion $W_t \in \mathbb{R}$, $W_0 = 0$, on a probability space $(\Omega, \mathcal{P}, \mathbb{P})$, consider the Ornstein-Uhlenbeck process which solves the stochastic differential equation

$$\mathrm{d}X_t = \theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t, \quad X_0 = x, \tag{1}$$

where $\theta, \sigma \in \mathbb{R}$.

(i) Show that the equation **??** admits a unique strong solution, and use the Itô's formula to find the solution to the above equation. (Hint: consider the transformation $e^{-\theta t} X_t$).

(ii) Fix some $T \geq 0$. Find $\mathbb{E}[X_T]$ and $\mathrm{Var}[X_T]$.

(iii) Compute the characteristic function ϕ

$$\phi_{X_T}(\xi) := \mathbb{E}\left[e^{i\xi X_T} | X_0 = x\right], \quad \text{for } \xi \in \mathbb{R}$$

of X_T in the following way. First, fix $\xi \in \mathbb{R}$, and use the Feynman-Kac theorem to show that the function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, $u(t,x) := \mathbb{E}\left[e^{i\xi X_T} | X_0 = x\right]$ satisfies the PDE

$$u_t + \theta x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Then, determine the terminal condition $u(T,x)$.

(iv) Then, propose a suitable terminal condition for **??** and use the ansatz

$$u(t,x) = \exp\{\beta(t) + i\xi\alpha(t)x\}$$

for some functions $\alpha(\cdot), \beta(\cdot)$ to show the identity

$$\phi_{X_T}(\xi) = \exp\left\{i\xi x e^{\theta T} - \frac{\sigma^2 \xi^2}{4\theta}\left(e^{2\theta T} - 1\right)\right\}.$$

(v) Recall that we can determine the kth moment of a random variable Y by computing the kth derivative of its characteristic function $\phi_Y(\xi)$:

$$\mathbb{E}\left[Y^k\right] = i^{-k}\phi_Y^{(k)}(0).$$

Use the identity in **??** to check your result in **??**.

Solution. *(i) Let $\mu(t,x) = \theta x$, $\sigma(t,x) = \sigma$. As they are a linear function and a constant, they satisfy the Lipschitz and linear growth conditions. That is, a strong solution exists.*

Moreover, let $Y_t := e^{-\theta x} X_t$ with $Y_0 = x$. Then by Itô's formula we have

$$
\begin{aligned}
\mathrm{d}Y_t &= -\theta Y_t \,\mathrm{d}t + e^{-\mu t}\,\mathrm{d}X_t \\
&= -\theta Y_t \,\mathrm{d}t + e^{-\theta t}\,\mathrm{d}\left(\theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t\right) \\
&= \sigma e^{-\theta t}\,\mathrm{d}W_t.
\end{aligned}
$$

Integrating gives us

$$
Y_t = x + \sigma \int_0^t e^{-\theta s}\,\mathrm{d}W_s,
$$

and moreover by plugging $Y_t = e^{-\theta x} X_t$ we get

$$
X_t = x e^{\theta t} + \sigma \int_0^t e^{\theta(t-s)}\,\mathrm{d}W_s.
$$

(ii) It's easy to see that by the known properties of expectation and Itô integral, for a fixed $T > 0$ we have

$$
\mathbb{E}[X_T] = x e^{\theta T} + \sigma \underbrace{\mathbb{E}\left[\int_0^T e^{\theta(T-s)}\,\mathrm{d}W_s\right]}_{=0}.
$$

Moreover, the second (central) moment can be calculated as

$$
\begin{aligned}
\mathbb{E}\left[X_T^2\right] &= \mathbb{E}\left[\left(x e^{\theta T} + \sigma \int_0^T e^{\theta(T-s)}\,\mathrm{d}W_s\right)^2\right] \\
&= x^2 e^{2\theta T} + \sigma^2 \mathbb{E}\left[\int_0^T e^{2\theta(T-s)}\,\mathrm{d}s\right] \\
&= x^2 e^{2\theta T} + \sigma^2 e^{2\theta T} \frac{1}{2\theta}\left(1 - e^{-2\theta T}\right) \\
&= x^2 e^{2\theta T} + \frac{\sigma^2}{2\theta}\left(e^{-2\theta T} - 1\right).
\end{aligned}
$$

Variance is thus

$$
\operatorname{Var}[X_t] = \mathbb{E}\left[X_T^2\right] - \mathbb{E}[X_T]^2 = \frac{\sigma^2}{2\theta}\left(e^{-2\theta T} - 1\right).
$$

(iii) The claim follows directly as an application to the Feynman-Kac formula with using the terminal condition

$$
u(T,x) = e^{i\xi x}.
$$

(iv) We are now interested in finding u that satisfies

$$
\begin{cases}
u_t + \theta x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0, \\
u(T,x) = u(T,x) = e^{i\xi x}.
\end{cases}
$$

Following the ansatz given, we get

$$
\begin{cases}
u_t = \left(\beta(t) + i\xi x \alpha(t)\right) u, \\
u_x = i\xi\alpha(t) u, \\
u_{xx} = -\xi^2 \alpha^2(t) u.
\end{cases}
$$

Solving this system of equations yields

$$\begin{cases} \alpha(t) = Ce^{-\theta t}, \\ \alpha(T) = 1, \end{cases}$$

for some constant C, from which we get $\alpha(t) = e^{\theta(T-t)}$. Moreover, we get

$$\begin{cases} \beta'(t) = \frac{1}{2}\sigma^2\xi^2 e^{2\theta(T-T)}, \\ \beta(T) = 0, \end{cases}$$

from which we solve (by the fundamental theorem of calculus)

$$\beta(t) = -\frac{1}{4\theta}\left(\sigma^2\xi^2(e^{2\theta(T-t)} - 1)\right).$$

Plugging these into the ansatz given we get

$$u(t,x) = \exp\left\{i\xi e^{\theta(T-t)}x - \frac{1}{4\theta}\left(\sigma^2\xi^2(e^{2\theta(T-t)} - 1)\right)\right\}.$$

Recalling that $\phi_{X_T}(\xi) = u(0,x)$, we find that

$$\phi_{X_T}(\xi) = \exp\left\{i\xi e^{\theta(T)}x - \frac{1}{4\theta}\left(\sigma^2\xi^2(e^{2\theta(T)} - 1)\right)\right\}.$$

(v) By straightforward calculations its easy to check that

$$\mathbb{E}[X_T] = i^{-1}\phi'_{X_T}(0) = i^{-1}\left(ixe^{\theta T}\right) = xe^{\theta T},$$

and

$$\mathbb{E}[X_T^2] = i^{-2}\left(\left(ixe^{\theta T}\right)^2 + \left(-\frac{\sigma^2}{2\theta}(e^{2\theta T} - 1)\right)\right) = x^2e^{2\theta T} + \frac{\sigma^2}{2\theta}(e^{2\theta T} - 1).$$

2. (7 points). Let $W = (W_1, \dots, W_n)$ be a standard $n-$dimensional Brownian motion. Let $(c_1, \dots, c_n) \in \mathbb{R}^n$ and $\alpha > 0$ be a constant. Consider the process $Z(t) \in \mathbb{R}$:

$$Z(t) = \sum_{j=1}^{n} c_j W_j(\alpha t), \ \ t \geq 0.$$

Show that Z is a standard Brownian motion i.f.f. $\frac{1}{\alpha} = \sum_{j=1}^{n} c_j^2$.

Solution. *There are two ways to tackle this problem. The first is to check all 4 requirements in the definition of the Brownian motion. The second is to recall that $Z(t)$ is clearly continuous and Gaussian as a linear combination of normal random variables, and*

$Z(0)=0$, and to note that we only need to examine if $\mathrm{Cov}[Z(s),Z(t)]=s\wedge t$*:*

$$\begin{aligned}
\mathrm{Cov}[Z(s),Z(t)] &= \mathbb{E}[Z(s)Z(t)] - \mathbb{E}[Z(s)]\mathbb{E}[Z(t)] \\
&= \mathbb{E}\left[\left(\sum_{j=1}^{n} c_j W_j(\alpha s)\right)\left(\sum_{i=1}^{n} c_i W_i(\alpha t)\right)\right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^{n} c_i c_j W_i(\alpha t) W_j(\alpha s)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} c_i^2 W_i(\alpha t) W_i(\alpha s)\right] \\
&= \sum_{i=1}^{n} c_i^2 \mathbb{E}\left[W_i(\alpha t) W_i(\alpha s)\right] \\
&= \sum_{i=1}^{n} c_i^2 \mathrm{Cov}\left[W_i(\alpha t), W_i(\alpha s)\right] \\
&= \sum_{i=1}^{n} c_i^2 \alpha (t\wedge s).
\end{aligned}$$

From this it's easy to see that $Z(t)$ is the Brownian motion if and only if

$$\sum_{i=1}^{n} c_i^2 \alpha(t\wedge s) = t\wedge s \quad \Longleftrightarrow \quad \sum_{i=1}^{n} c_i^2 \alpha = 1.$$

3. (7 points). Consider the $n-$dimensional hypercube:

$$H = \{(x_1,\dots,x_n)\in\mathbb{R}^n : |x_1|<1,\dots,|x_n|<1\}.$$

Let W be an $n-$dimensional BM with starting point $W_0 = (\frac{1}{2},\dots,\frac{1}{2}) \in H$. Define the exit time of W from H as τ_H.

(i) Use Dynkin's formula to show that $\mathbb{E}[\tau_H] < 1$.

(ii) Formulate a suitable Poisson problem and show that $\mathbb{E}[\tau_H] < 1$.

Solution. *The strategy here is to find a large enough bounded domain, which we can show to have a finite exit time.*

Assume that $W_0 = x$ and consider $S =: B_{\sqrt{n}}(0)$, the ball centered at 0 with a radius of $\sqrt{n}$. Clearly, $H\subset S$ and thus $\tau_H(\omega) < \tau_S(\omega)$ for all $\omega\in\Omega$. Since the ball is bounded, it follows that

$$\mathbb{E}[\tau_H] \le \mathbb{E}[\tau_S] < \infty.$$

(i) Let $f = ||x||^2$ and denote $\tau := \tau_s \wedge t$ for some finite time t. Then

$$\begin{aligned}
\mathbb{E}_x\left[f(W_\tau)\right] &= f(x) + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(W_s)\,\mathrm{d}s\right] \\
&= ||x||^2 + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2} 2n\,\mathrm{d}s\right] \\
&= ||x||^2 + n\mathbb{E}_x\left[\tau\right].
\end{aligned}$$

Note that at τ, the process W_τ is at the boundary of the ball. Thus

$$f(W_\tau) \le f(W_{\tau_s}) = n.$$

Combining what we've done so far, we get

$$\mathbb{E}_x[\tau] \leq \frac{1}{n}\left(n - ||x||^2\right) < 1.$$

(ii) *We proceed as in the lectures, and let $u(x) = \mathbb{E}_x[\tau_s]$. Then $u(x)$ solves a Poisson problem*

$$\begin{cases} \frac{1}{2}\Delta u + 1 = 0 & \textit{in } S, \\ u = 0 & \textit{on } \partial S. \end{cases}$$

Inspired by the previous part of the problem, we use an ansatz $u(x) = C\left(||x||^2 - n\right)$ for some constant C. Plugging the ansatz in to the PDE we get

$$\frac{1}{2}2Cn + 1 = 0 \Rightarrow C = -\frac{1}{n}.$$

From this we arrive with

$$u(x) \leq \frac{1}{n}\left(n - ||x||^2\right) < 1.$$

(iii) ***(Extra).*** *Note that the problem could also be solved with showing that $M_t = W_t^2 - nt$ is a martingale, and then using Doob's Optional Sampling to deduce that*

$$M_0 = \mathbb{E}[M_{\tau_s}] = n - n\mathbb{E}[\tau_s].$$