UPPSALA UNIVERSITET

Föreläsningstackneningar

Grafteori

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Contents

1. TODO

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2. Bridges of Köningsberg

This was the birth of graphtheory. The idea here is that the precise location of where the person is does not matter, only the placement of the bridges and mainland. Therefore, we can encode the position by an abstract point (*vertex*) and connect these to *edges* to represent bridges.

2.1. **Vocabulary.**

We therefore obtain the follwing:

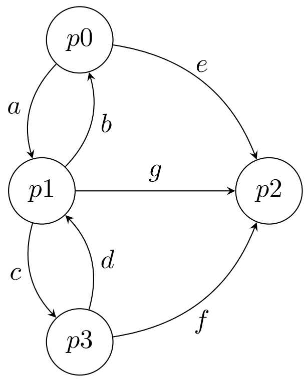

Figure 1.

Definition/Sats 2.1: Multigraph

A *multigraph* G is a tripple $G = (V, E, \iota)$ consisting of:
- A set V of vertices
- A set E of edges
- $\iota : E \to \{A \subseteq V \mid |A| = 1 \text{ or } |A| = 2\}$

Example:
$\iota(c) = \{2, 3\} = \iota(d)$
$\iota(e) = \{1, 4\}$

Anmärkning:
Notice that the graphical view (and the placement of the vertices) is not reflected in the tripple, therefore we can draw the same graph in a completely different manner.

Loops:
This is what happens when $|A| = 1$:

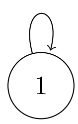

Figure 2.

Parallell edges:
$\iota(e) = \iota(e')$

Neighbours/adjacent:

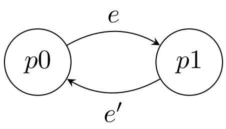

FIGURE 3.

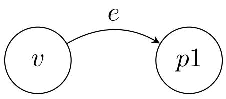

FIGURE 4. v is *incident* to e and a neighbour to w

Definition/Sats 2.2: Finite graph

We say that a graph G is finite if we have:

$$|V| + |E| < \infty$$

Definition/Sats 2.3: Walk

Let $G = (V, E, \iota)$ be a graph
A *walk* of length k is a sequence $v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$ where as the notation suggests, $e_1, \cdots, e_k$ are edges and $v_0, \cdots, v_k$ are vertices such that $\iota(e_i) = \{v_{i-1}, v_i\}$ for $i = 1, \cdots, k$

Definition/Sats 2.4: Trail

A *trail* is a walk that uses no edges twice. This is something we want in the Bridges of Köningsberg

Definition/Sats 2.5: Path

A *path* is a walk that uses no vertex twice.

Definition/Sats 2.6: Circuit

A *circuit* is a trail where first and last vertex coincide.
Meaning I start somewhere, dont repeat edges, and return at start place (**CHECK**)

Definition/Sats 2.7: Cycle

A *cycle* is a circuit where those are the only vertices coinciding

Example:
Using the bridges, an example of a trail and a circuit, but not a cycle because vertex 3 is visited twice $1a3g4f2c3b1$
An example of a cycle would be $1a3b1$

Anmärkning:
Every path is a trail.
Every cycle is a circuit.

Definition/Sats 2.8: Eulerian trails

A trail is called *Eulerian* if it uses every edge in the graph

Definition/Sats 2.9: Eulerian circuits

A circuit using every edge is called an *Eulerian circuit*

Anmärkning:
If a graph admits an Eulerian circuit, then the graph is called *simply Eulerian*

Definition/Sats 2.10: Connected vertex

Let $G = (V, E, \iota)$ be a graph
We say that a vertex $v \in V$ is *connected* to a vertex $w \in V$ if there exists walk (or equivalently a trail/path) starting in v and ending in w

If v is connected to w for all $v, w \in V$, then the graph G is *connected*

What we are saying here is that we call vertices that we can walk to connected.

Anmärkning:
v is conncted to v (every vertex is connected to itself)
Moreover, if v is connected to w, then w is connected to v.
v is connected to w and w connected to z, then v is connected to z.

Anmärkning:
Connection is an equivalence relation.

Definition/Sats 2.11: Connected components

Equivelance classes of the equivalence relation are called *connected components*.

Definition/Sats 2.12: Degree of vertex

Let $G = (V, E, \iota)$ be a graph and $v \in V$. The *degree* of v is $\deg(v)$ and is the number of half-edges incident to v.

The reason we do half-edges is because we want loops to count twice (once for exit, and once on entry)

Definition/Sats 2.13: Euler; 1736

A finite connected graph is Eulerian iff all its vertex degrees are even.

Bevis 2.1

In $\Rightarrow$ direction. Any vertex on the circuit needs to have even degree because you need a half-edge to go into the vertex and another one to go out.
Since it is connected, if I visit every vertex I also visit every edge and these come in pairs

In $\Leftarrow$ direction. Assume $G = (V, E\iota)$ is finite, connected, and only has even degrees.
Assume G as no loops (convenience). We dont know if we can build an Eulerian circuit or even if we have a circuit, but we know that there is a trail (since it is connected)
Therefore, consider a circuit/trail $J = v_0 e_1 v_1 \cdots e_k v_k$. (**CHECK**)
Since the graph is finite, then there is a maximum trail, suppose J is a maximum trail (implying max length k). Then we cant possibly extend it, so any edge we see at k must already be on the trail.
What we want to show is that $v_0 = v_k$ because of this. Then we actually have a circuit.
Therefore, assume there are $2s \quad (s \in \mathbb{N})$ edges incident to v_k. We know there is an even number of edges (because we excluded loops).
If we look at our trail $v_0 e_1 v_1 \cdots v_{i-1} e_i \underbrace{v_i}_{v_k} e_{i+1} v_{i+1}$
Then e_i and e_{i+1} are incident to $v_k = v_i$, but so is v_k. But v_k only has one edge, therefore e_1 has to be incident to $v_k = v_0$
We have now shown we have a trail, we show it is Eulerian.
Assume for a contradiction that it is not Eulerian. This means that there are parts not in our trail.
There is $e \in E$ with endpoints $\iota(e) = \{v, w\}$ s.t e is not on J but one of v, w is.
WLOG v is on J. Say $v = v_j$ for some j.
Consider $wev_j e_{j+1} \cdots e_k \underbrace{v_k}_{v_0} e_1 v_1 e_2 v_2 \cdots e_j v_j$, we claim that this is a trail. Notice here that we have length $k+1$, which is longer than k. Contradiction. $\square$

Anmärkning:
A useful proof-tool in graphtheory is setting up a situation where we fix a maxlength and argue the contrary.

Anmärkning:
Notice how $\Rightarrow$ was "obvious", we call this *TONCAS* - The Obvious Necessary Conditions Are Sufficient

Anmärkning:
If we have loops, we can simply traverse these loops and add them to our trail. This will not affect the proof. **Corollary:**
A finite connected graph admits an Eulerian trail iff either 0 or 2 of its vertex degrees are odd

We can show this by retracing this back to the previous theorem. If we have 0 odd degrees, then the theorem holds.
If we have 2 vertices of odd degree, then we can draw an additional edge between v, w. This means that both of the vertices that had odd degrees have gotten their degrees bumped up by one, so they know have even degree, which implies the theorem (is an Eulerian circuit), so it visits all the edges (and especially the new edge). Then we can remove the new edge from the Eulerian circuit, which gives an Eulerian trail in the original graph.

If we look at the statement of the corollary, it leaves a graph. What happens if it has 1 odd vertex degree? We are gonna show that this is impossible.

Definition/Sats 2.14: Handshake lemma

Let $G = (V, E, \iota)$ be a finite graph.
Then $2\,|E| = \sum_{v \in V} \deg(v)$

In particular, G has even number of vertices of odd degree. (odd+odd = even, even + even = even)

Bevis 2.2: Handshake lemma

We use a trick from combinatorics (double counting). We identify a quantity and count it in 2 different ways.

We double count half-edges. Every edge gives 2 half-edges, so we $2|E|$ half-edges. On the other hand, every vertex gives $\deg(v)$ half-edges $\Rightarrow \sum_{v \in V} \deg(V)$ half-edges.
It does not matter how I count them, therefore these quantities have to be the same. $\square$

Anmärkning:
We can also use induction to show the Handshake lemma.
Start with 0 edges on V, which implies all the degrees are 0. Then add edges 1 by 1. And whenever you add an edge, the RHS increases by 2.

What happens if we have 4 vertices of odd degree?
We can partition $E = E_1 \cup E_2$ such that E_1 is a edge set of a trail and so is E_2.