Multi-dimensional Ito formula

Assume $dX_t^i = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j$, $i = 1, \dots, n$

where $W^1, \dots, W^d$ are d independent Brownian motions.

On a matrix form:

$$\underset{n\times 1}{dX_t} = \underset{n\times 1}{\mu_t}\, dt + \underset{n\times d}{\sigma_t}\, \underset{d\times 1}{dW_t}$$

Let $Z_t = f(t, X_t)$ where $f : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is $C^{1,2}$.

Theorem 4.19 (Ito's formula, multi-dim)

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

where
$$dW_t^i dW_t^j = 0 \quad \text{if } i \neq j$$
$$(dW_t^i)^2 = dt$$
$$(dt)^2 = dt\, dW_t^i = 0.$$

Alternatively,

$$dZ_t = \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_t^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C_t^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_t^i dW_t$$

where $C = \sigma\sigma^*$ and σ^i is the i:th row of σ.

Indeed, $dX_t^i dX_t^j = \left(\sum_{k=1}^{d} \sigma^{ik} dW^k \right)\left(\sum_{l=1}^{d} \sigma^{jl} dW^l \right)$

$$= \left(\sum_{k=1}^{d} \sigma^{ik} \sigma^{jk} \right) dt$$

$$= (\sigma\sigma^*)^{ij} dt$$

Exercise 5.8 If
$$\begin{cases} dX_t = \alpha X_t\,dt + \sigma X_t\,dW_t \\ dY_t = \gamma Y_t\,dt + \delta Y_t\,dV_t \end{cases}$$
indep. BM's

and $Z_t = X_t Y_t$, find dZ_t.

Solution: Ito's formula gives

$$dZ_t = Y_t\,dX_t + X_t\,dY_t + \frac{1}{2}\cdot 2\underbrace{dX_t\,dY_t}_{0}$$
$$= (\alpha+\gamma) Z_t\,dt + Z_t(\sigma\,dW_t + \delta\,dV_t)$$

Setting $\bar{W}_t := \frac{1}{\sqrt{\sigma^2+\delta^2}}(\sigma W_t + \delta V_t)$, $\bar{W}$ is a BM (why?),

and $dZ_t = (\alpha+\gamma) Z_t\,dt + \sqrt{\sigma^2+\delta^2}\, Z_t\,d\bar{W}_t$

Correlated Brownian motions

Let $\bar{W} = \begin{pmatrix} \bar{W}^1 \\ \vdots \\ \bar{W}^d \end{pmatrix}$ where $\bar{W}^1, \dots, \bar{W}^d$ are independent Brownian motions.

Consider $W = \delta \bar{W}$ where $\delta = \begin{pmatrix} \delta_{11} & \cdots & \delta_{1d} \\ \vdots & & \vdots \\ \delta_{d1} & \cdots & \delta_{dd} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_d \end{pmatrix}$

row vectors with $\|\delta_i\| = 1$
$\|$
$\sqrt{\delta_{i1}^2 + \dots + \delta_{id}^2}$

Then $E[(W_t^i)^2] = E\left[\left(\sum_{j=1}^d \delta_{ij} \bar{W}_t^j\right)^2\right]$

$$= \left(\sum_{j=1}^d \delta_{ij}^2\right) t = t$$

so W^i is a Brownian motion.

Moreover, $dW_t^i \, dW_t^j = \left(\sum_{k=1}^{d} \delta_{ik} \, d\bar{W}_t^k\right)\left(\sum_{l=1}^{d} \delta_{jl} \, d\bar{W}_t^l\right)$ (3)

$$= \sum_{k=1}^{d} \delta_{ik}\delta_{jk} \, dt = (\delta\delta^*)_{ij} \, dt$$

Def 4.20 W_t constructed above is a d-dimensional correlated Wiener process with correlation matrix $\rho = \delta\delta^*$.

Prop 4.21 (Ito's formula, correlated version)

If W_t is a correlated Wiener process as above, and

$$\underset{n\times 1}{dX_t} = \underset{n\times 1}{\mu_t} \, dt + \underset{n\times d}{\sigma_t} \, \underset{d\times 1}{dW_t},$$

then $Z_t = f(t, X_t)$ satisfies

$$dZ_t = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i \, dX_t^j$$

where $(dt)^2 = dt \, dW^i = 0$

$dW^i dW^j = \rho_{ij} \, dt$

Ex: Given $\bar{W} = \begin{pmatrix} \bar{W}^1 \\ \bar{W}^2 \end{pmatrix}$ (where $\bar{W}^1, \bar{W}^2$ are independent)

construct $W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$ with correlation matrix $\rho = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}$.

Note that $\delta = \begin{pmatrix} 1 & 0 \\ \rho_0 & \sqrt{1-\rho_0^2} \end{pmatrix}$ satisfies $\delta\delta^* = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix} = \rho$.

Thus $W = \begin{pmatrix} \bar{W}^1 \\ \rho_0 \bar{W}^1 + \sqrt{1-\rho_0^2} \, \bar{W}^2 \end{pmatrix}$ is correlated Wiener process with correlation matrix ρ.

What other choices for δ are possible?