Financial Theory – Lecture 8

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Agenda

- Factor models.
- The arbitrage pricing theory (APT).

The lecture is based on

- Chapter 11 in the course book.

Factor models

Again, we have a market of N risky assets with returns $r_1, \ldots, r_N$.

A factor model is a model that explains the risk premium on every asset by it's exposure to a (low) number of factors.

In a factor model the two major questions are

1) Which factors should be used?
2) How large is the risk premium with respect to each of the factors?

Factor models

Examples of potential priced factors are:

- Macroeconomic variables such as GDP growth rate.
- Prices on raw materials such as oil and metals.
- Rates of return of indexes, specific portfolios or individual assets.

One-factor models

A one-factor model is defined as follows.

There exists a random variable F such that

$$r_i = E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i,\ \ i = 1, 2, \ldots, N,$$

and where for every $i, j = 1, 2, \ldots, N$

(i) $\text{Cov}[F, \varepsilon_i] = 0$.

(ii) $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0$ when $i \neq j$.

Here F is the factor affecting all returns and ε_i represents asset-specific return uncertainty.

Note that

$$E\left[\varepsilon_i\right] = 0.$$

One-factor models

In this model

$$\begin{aligned}
\text{Cov}[r_i, F] &= \text{Cov}\Big[E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i, F\Big] \\
&= \beta_i \text{Var}[F] \quad \Leftrightarrow \quad \beta_i = \frac{\text{Cov}[r_i, F]}{\text{Var}[F]}
\end{aligned}$$

$$\begin{aligned}
\text{Var}[r_i] &= \text{Var}\Big[E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i\Big] \\
&= \beta_i^2 \text{Var}[F] + \text{Var}[\varepsilon_i]
\end{aligned}$$

$$\begin{aligned}
\text{Cov}[r_i, r_j] &= \text{Cov}\Big[E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i, \\
&\qquad\qquad E\left[r_j\right] + \beta_j\big(F - E\left[F\right]\big) + \varepsilon_j\Big] \\
&= \beta_i \beta_j \text{Var}[F]
\end{aligned}$$

One-factor models

The variance

$$\mathrm{Var}[r_i] = \beta_i^2 \mathrm{Var}[F] + \mathrm{Var}[\varepsilon_i]$$

consists of two terms.

- $\beta_i^2 \mathrm{Var}[F]$ is called the systematic risk.
- $\mathrm{Var}[\varepsilon_i]$ is called the non-systematic risk, the idiosyncratic risk or the asset(firm)-specific risk.

Remark. Sometimes $\beta_i \mathrm{Std}[F]$ and $\mathrm{Std}[\varepsilon_i]$ are used in these definitions.

One-factor models

The equation

$$r_i = E\left[r_i\right] + \beta_i\big(F - E\left[F\right]\big) + \varepsilon_i$$

can be written

$$r_i = a_i + \beta_i F + \varepsilon_i,$$

where

$$a_i = E\left[r_i\right] - \beta_i E\left[F\right].$$

Alternatively, it can be written

$$r_i - r_f = a_i' + \beta_i F + \varepsilon_i,$$

where $a_i' = a_i - r_f$.

When estimating and/or testing these type of models, we need to run regressions. Depending on what we are interested in, we can use any of the model specifications above.

The Single-Index model

In the Single-Index model, also called the Market Model, the only factor is equal to the return on the market portfolio:

$$F = r_m.$$

It is usually expressed using excess returns:

$$r_i - r_f = \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i.$$

The condition $E[\varepsilon_i] = 0$ can be written

$$0 = E[r_i - r_f - \alpha_i - \beta_i(r_m - r_f)] \quad \text{or} \quad \alpha_i = E[r_i - r_f - \beta_i(r_m - r_f)].$$

If all $\alpha_i = 0$, then

$$E[r_i] - r_f = \beta_i\big(E[r_m] - r_f\big),$$

which is the same equation as in CAPM.

The Single-Index model

In the Single-Index Model with $\alpha_i = 0$ for all i it is not possible to find any portfolio that has a higher Sharpe ratio than the market portfolio.

To see this, consider the portfolio with weights $\boldsymbol{\pi}$ and return $r_p = \sum_{i=1}^{N} \pi_i r_i$:

$$
\begin{aligned}
E[r_p] - r_f &= \sum_{i=1}^{N} \pi_i E[r_i] - r_f \\
&= \sum_{i=1}^{N} \pi_i \big(E[r_i] - r_f\big) \\
&= \big\{\text{The Single-Index Model with all } \alpha_i = 0.\big\} \\
&= \sum_{i=1}^{N} \pi_i \beta_i \big(E[r_m] - r_f\big) \\
&= \big(E[r_m] - r_f\big) \sum_{i=1}^{N} \pi_i \beta_i .
\end{aligned}
$$

The Single-Index model

We now use that the beta for a portfolio is

$$\beta_p = \frac{\text{Cov}[r_p, r_m]}{\text{Var}[r_m]} = \frac{\text{Corr}[r_p, r_m]\text{Std}[r_p]}{\text{Std}[r_m]} =: \frac{\rho_{pm}\sigma_p}{\sigma_m}$$

and

$$SR_m = \frac{E\left[r_m\right] - r_f}{\sigma_m} \quad \Leftrightarrow \quad E\left[r_m\right] - r_f = SR_m \sigma_m .$$

We get

$$\begin{aligned} E\left[r_p\right] - r_f &= \left(E\left[r_m\right] - r_f\right) \underbrace{\sum_{i=1}^{N} \pi_i \beta_i}_{=\beta_p} \\ &= SR_m \sigma_m \beta_p \\ &= SR_m \sigma_m \frac{\rho_{pm}\sigma_p}{\sigma_m} \\ &= SR_m \rho_{pm} \sigma_p . \end{aligned}$$

The Single-Index model

This last relationship can be written

$$\frac{E\left[r_p\right] - r_f}{\sigma_p} = \rho_{pm} SR_m,$$

or

$$SR_p = \rho_{pm} SR_m.$$

Since $\rho_{pm} \leq 1$ the highest Sharpe ratio is achieved when $\rho_{pm} = 1$.

One way of getting this is to choose $p = m$.

The Single-Index model

Assume that there is an asset i with $\alpha_i > 0$.

How can we benefit from this?

If we invest 1 unit of currency in asset i, then we get the payoff

$$1 + r_i = 1 + r_f + \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i.$$

But we can do better.

Short β_i units of currency of the market portfolio and invest the money in the risk-free asset. This results in the payoff

$$\beta_i(1 + r_f) - \beta_i(1 + r_m) = \beta_i(r_f - r_m).$$

The Single-Index model

By adding the zero net investment portfolio $\beta_i(r_f - r_m)$ we get the total payoff

$$\begin{aligned} 1 + r_i + \beta_i(r_f - r_m) &= 1 + r_f + \alpha_i + \beta_i(r_m - r_f) + \varepsilon_i + \beta_i(r_f - r_m) \\ &= 1 + r_f + \alpha_i + \varepsilon_i. \end{aligned}$$

Now we are only exposed to the asset-specific ε_i – the cash flow does not depend on the movement of the market.

The attractiveness of the position can be measured using the information ratio

$$IR = \frac{\alpha_i}{\mathrm{Std}[\varepsilon_i]}.$$

Multi-factor models

One factor is probably not enough in order to get a reasonable model.

When we have a multi-factor model with K factors, we say that we have a K-factor model.

For $i = 1, 2, \ldots, N$:

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big) + \varepsilon_i,$$

where for every i and k

$$\mathrm{Cov}[F_k, \varepsilon_i] = 0 \quad \text{and} \quad \mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0 \text{ when } i \neq j.$$

Note that we again have

$$E[\varepsilon_i] = 0.$$

Multi-factor models

With the vectors

$$\boldsymbol{\beta}_i = \begin{pmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{iK} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_K \end{pmatrix} \text{ and } E\left[\mathbf{F}\right] = \begin{pmatrix} E\left[F_1\right] \\ E\left[F_2\right] \\ \vdots \\ E\left[F_K\right] \end{pmatrix}$$

we can write the K-factor model as

$$r_i = E\left[r_i\right] + \boldsymbol{\beta}_i \cdot \Big(\mathbf{F} - E\left[\mathbf{F}\right]\Big) + \varepsilon_i.$$

The variance of r_i now becomes

$$\mathrm{Var}[r_i] = \boldsymbol{\beta}_i \cdot \Sigma_F \boldsymbol{\beta}_i + \mathrm{Var}[\varepsilon_i],$$

where

$$\Sigma_F = \mathrm{Var}[\mathbf{F}].$$

Multi-factor models

One can further show that

$$\text{Cov}[r_i, \mathbf{F}] = \Sigma_F \boldsymbol{\beta}_i$$

and

$$\text{Cov}[r_i, r_j] = \boldsymbol{\beta}_i \cdot \Sigma_F \boldsymbol{\beta}_j.$$

From the first equation we see that

$$\Sigma_F^{-1} \text{Cov}[r_i, \mathbf{F}] = \Sigma_F^{-1} \Sigma_F \boldsymbol{\beta}_i = \boldsymbol{\beta}_i.$$

Compare this with multivariate OLS.

Note that the moments $\text{Cov}[r_i, r_j]$ and $\text{Var}[r_i]$ are determined by Σ_F and the beta vectors.

Multi-factor models

One reason multi-factors are popular, is that it reduces the number of parameters that has to be estimated.

With N assets there are in general N means and $\frac{(N+1)N}{2}$ elements in the variance-covariance to estimate.

With a K-factor model there are N means and

$$\underbrace{\frac{(K+1)K}{2}}_{\text{Elements in } \Sigma_F} + \underbrace{(K+1)N}_{\beta\text{'s and } \alpha\text{'s}}$$

factor model parameters.

Multi-factor models

Number of factors K	$N = 10$	$N = 100$	$N = 500$
1	21	201	1 001
2	33	606	3 006
5	75	3 651	18 051
Without factors	55	5 050	125 250

Number of parameters in the variance-covariance matrix for different K-factor models.

See also Table 11.2 in Munk.

Arbitrage pricing theory

Consider the K-factor model

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big) + \varepsilon_i.$$

What can be said of $E[r_i]$?

It turns out that if we rule out arbitrage opportunities then there exists risk premia RP_k, $k = 1, 2, \ldots, K$ such that

$$E[r_i] = r_f + \beta_{i1}\mathrm{RP}_1 + \beta_{i2}\mathrm{RP}_2 + \ldots + \beta_{iK}\mathrm{RP}_K.$$

This is known as the Arbitrage pricing theory (APT).

Note that the risk premium RP_k is connected to factor F_k and that the β's are the same as the ones defining the model.

Arbitrage pricing theory

The general proof of the APT is surprisingly hard and technical from a mathematical point of view.

There is, however, a special case which is more straightforward to show.

Assume that

$$r_i = E[r_i] + \beta_{i1}\big(F_1 - E[F_1]\big) + \ldots + \beta_{iK}\big(F_K - E[F_K]\big).$$

These models are known as exact factor models since there is no ε_i.

Arbitrage pricing theory

In this model choose a portfolio $\boldsymbol{\pi}$ such that

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 0 \quad \text{and} \quad \sum_{i=1}^{N} \pi_i \beta_{ik} = \boldsymbol{\pi} \cdot \boldsymbol{\beta}^k = 0$$

for every $k = 1, 2, \ldots, K$.

For such a portfolio

$$\begin{aligned}
\boldsymbol{\pi} \cdot \mathbf{r} = \sum_{i=1}^{N} \pi_i r_i &= \sum_{i=1}^{N} \pi_i \left(E\left[r_i\right] + \sum_{k=1}^{K} \beta_{ik} (F_k - E\left[F_k\right]) \right) \\
&= \sum_{i=1}^{N} \pi_i E\left[r_i\right] + \sum_{k=1}^{K} \left(F_k - E\left[F_k\right]\right) \sum_{i=1}^{N} \pi_i \beta_{ik} \\
&= \sum_{i=1}^{N} \pi_i E\left[r_i\right] = \boldsymbol{\pi} \cdot E\left[\mathbf{r}\right].
\end{aligned}$$

Arbitrage pricing theory

A portfolio with

$$\sum_{i=1}^{N} \pi_i = 0$$

costs zero to buy.

A portfolio with

$$\sum_{i=1}^{N} \pi_i \beta_{ik} = 0 \text{ for every } k$$

has no risk.

A risk-free portfolio that costs zero must have zero payoff – otherwise we would have an arbitrage. Hence we have

$$\left.\begin{array}{l} \boldsymbol{\pi} \cdot \mathbf{1} = 0 \\ \boldsymbol{\pi} \cdot \boldsymbol{\beta}^k = 0 \text{ for } k = 1, \ldots, K \end{array}\right\} \quad \Rightarrow \quad \boldsymbol{\pi} \cdot E\left[\mathbf{r}\right] = 0.$$

Arbitrage pricing theory

Using basic linear algebra (I will spare you the details) it follows that there exists RP_0, $\mathrm{RP}_1, \ldots, \mathrm{RP}_K$ such that

$$E[r_i] = \mathrm{RP}_0 + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

This is the exact APT.

Although the conditions are strong, it shows the major principle of the APT.

If there exists a risk-free asset, then $\mathrm{RP}_0 = r_f$ and we get

$$E[r_i] = r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

Arbitrage pricing theory

For a general K-factor model the APT says that when the number N of assets is large, then

$$E[r_i] \approx r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

The exact mathematical result is that with $N = \infty$

$$\sum_{i=1}^{\infty} \left(E[r_i] - r_f - \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k \right)^2 < \infty.$$

From now on we follow the book and say that if APT holds, then

$$E[r_i] = r_f + \sum_{k=1}^{K} \beta_{ik} \mathrm{RP}_k.$$

Arbitrage pricing theory

In the Single-Index Model

$$r_i = E[r_i] + \beta_i\big(r_m - E[r_m]\big) + \varepsilon_i$$

we get

$$E[r_i] = r_f + \beta_i \mathrm{RP}_m$$

from APT.

Let the rate of return be r_m in this equation:

$$E[r_m] = r_f + \underbrace{\beta_m}_{=1} \mathrm{RP}_m = r_f + \mathrm{RP}_m \quad \Rightarrow \quad \mathrm{RP}_m = E[r_m] - r_f.$$

We recover the CAPM equation again:

$$E[r_i] = r_f + \beta_i\big(E[r_m] - r_f\big).$$

Arbitrage pricing theory

How should we choose the factors?

In order to be able to estimate the parameters in the model we need to have factors that are observable.

One way is to choose factors that are returns – such as the return r_m of the market portfolio.

If we want to use, say, inflation as a factor but want factors that we can observe more often than at the times the inflation is updated by then we can use a factor mimicking portfolio.

A factor mimicking portfolio is a portfolio of traded assets that closely follows a factor.

It is also common to choose factors that are excess returns.

Arbitrage pricing theory

Which factors should we use in order to get a good model to price assets?

Consider the K-factor model

$$r_i = \alpha_i + \beta_{i1}\big(F_1 - E\left[F_1\right]\big) + \ldots + \beta_{iK}\big(F_K - E\left[F_K\right]\big) + \varepsilon_i.$$

The pricing model implied by this factor model is

$$E\left[r_i\right] = r_f + \beta_{i1}\mathrm{RP}_1 + \beta_{i2}\mathrm{RP}_2 + \ldots + \beta_{iK}\mathrm{RP}_K.$$

When looking at data, the question is:

Is the risk premium RP_k different from zero?

The Fama-French model

A model that is good at pricing assets is the Fama-French model, also known as the Fama-French three factor model.

It is a model formulated in excess returns.

- The first factor is the market excess return: $r_m - r_f$.
- The second factor is the difference between the return on small stocks minus the return on large stocks: SMB
- The third factor is the return on stocks with a high book-to-market ratio minus the return on stocks of low book-to-market ratios: HML

The Fama-French model is

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \beta_{i,\mathrm{SMB}}\mathrm{SMB} + \beta_{i,\mathrm{HML}}\mathrm{HML} + \varepsilon_i.$$

The Fama-French model

The definition of a small/large firm is defined in terms of the firm's market cap ($=$ Number of stocks $\cdot$ Price of the stock).

The definition of a high/low book-to-market ratio firm is defined as the book value of the firm divided by it's market cap.

A firm with a low book-to-market value is known as a growth firm and a firm with a high book-to-market value is known as a value firm.

The Fama-French-Carhart model

The Fama-French model turned out to be succesful in explaining the expected rate of return of assets.

But more factors that explain the expected rate of return have been introduced after the Fama-French model was suggested in 1992.

One extension is the four factor model that was suggested by Carhart in 1997 and that added a momentum factor.

Momentum is the fact that a stock that has increased in value will continue to increase in the future.

The Fama-French-Carhart model

It turns out that this is in general true in the short-run, but that there is a reversal (the opposite of a momentum effect) in the long-run.

The four factor Fama-French-Carhart model is

$$\begin{aligned} r_i - r_f &= \alpha_i + \beta_{i,m}(r_m - r_f) + \beta_{i,\mathrm{SMB}}\mathrm{SMB} + \beta_{i,\mathrm{HML}}\mathrm{HML} \\ &\quad + \beta_{i,\mathrm{WML}}\mathrm{WML} + \varepsilon_i. \end{aligned}$$

Here WML is the excess return of winners-minus-losers.

Sometimes WML is called UMD for "up-minus-down".

The factor zoo

Which factors should be used?

In theory, every factor whose risk premium is different from zero is a "priced factor".

But we only have estimated values on the risk premia, so we are interested in the values that are significantly different from zero.

There has been a lot of research in trying to find different priced factors.

Over one hundred factors have been identified in the academic literature. This has lead to the term the factor zoo.

The factor zoo

Is it enough that a factor has a risk premium that is significantly different from zero?

Theoretically yes, but it would be nice to have an economic interpretation of the factor.

The rule of thumb is that if the "t-stat" if larger than 2, then the parameter is significantly different from zero.

But recently there have been a demand of an even higher t-stat in order for a factor to be accepted as a pricing factor.

In order to use factors in practise there are usually transaction costs involved. These should be included when testing if a factor is pricing assets.

The factor zoo

These are "the largest animals in the zoo".

- Value: Usually measured by book-to-market ratio, but there are other suggestions on how to measure value.
- Momentum: A portfolio of winners-minus-losers when looked at a given time period (e.g. measured over the last 12 months).
- Quality: Measures how well managed a company is (several measures exists). The excess return of quality-minus-junk (QMJ) can be used as a factor.
- Defensive: This is also known as a low-risk factor. Create an excess return of low risk stocks minus high risk stocks and use it as a factor.
- Size: The difference between the return on small companies minus the return on large companies.

See Section 11.6.2 in the course book for a more detailed description.