UPPSALA UNIVERSITET
Matematiska Institutionen
Topias Tolonen-Weckström

Tenta i matematik
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Partial Differential Equations with Applications to Finance

Instructions: There are five problems giving a maximum of 40 points in total. The minimum score required in order to pass the course is 18 points. To obtain higher grades, the score has to be at least 25 or 32 points, respectively. Other than writing utensils and paper, no other materials are allowed. In the problems 4 and 5, you do not need to provide a proof to the respective verification theorems. **Good luck!**

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1. (8p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x) | t > 0, \quad x > 0\}$ with $u(0,x) = u_0(x)$ for $x > 0$, and $\frac{\partial u}{\partial x}(t,0) = 0$ for $t > 0$.

(a) (3p) Construct a suitable extension of the initial condition to the whole space.

(b) (5p) Show that

$$u(t,x) = \int_0^\infty u_0(y) h(t,x,y) \, \mathrm{d}y$$

for some function $h(t,x,y)$. Find the function h. *Note: h is not simply the fundamental solution!*

2. (8p) Let D denote a bounded interval $(a,b) \in \mathbb{R}$ with $a < b$ and let

$$\mathrm{d}X_t = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

with $X_0 = x \in D$ for $\mu, \sigma \neq 0$, and W_t being the standard Brownian motion. Calculate the expectation

$$\mathbb{E}_x\left[X_{\tau_D}^p + \tau_D\right],$$

where $p > 0$ is a constant and $\tau_D = \inf\{t > 0 | X_t \notin D\}$ is the so-called first exit time from D.

3. (8p) Consider the Itô diffusion X with

$$\mathrm{d}X_t = \mu(X_t) \, \mathrm{d}t + \sigma(X_t) \, \mathrm{d}W_t$$

for some drift and volatility coefficients μ, σ, and let $u(t,x,y) := \mathbb{P}_x(X_t \leq y)$ with $X_0 = x$ as the initial point.

(a) (6p) Use the Kolmogorov Forward Equation (Fokker-Planck) to derive a PDE satisfied by u in terms of the forward variables (t,y), and propose a suitable initial condition.

(b) (2p) Find a process X_t, for which

$$u(t,x,y) = 1 - u(t,y,x)$$

for all t, x, y.

4. (8p) Consider the Merton's asset allocation problem with one risky asset

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$$

and a risk-free rate $r = 0$. Here μ, σ are constants and W_t the standard Brownian motion. Let further X_t^u denote the wealth process where u_t is the amount of money invested in the risky asset at time t. Solve the $Merton's$ problem

$$v(t,x) = \sup_u \mathbb{E}_{t,x}\Big[\Phi(X_T^u)\Big]$$

for some termination time T, where $\Phi(x) = 1 - e^{-\gamma x}$ for some $gamma > 0$. *Note: An ansatz $v(t,x) = 1 - f(t)e^{-\gamma x}$ ought to be useful for some arbitrary f which will be solved further using the terminal conditions.*

5. (8p) Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_{0,x}[e^{-r\tau} X_\tau^+],$$

where $\mathrm{d}X_t = \mu \,\mathrm{d}t + \mathrm{d}W_t$ with $X_0 = x$; $X^+ := \max(X, 0)$, W_t is the standard Brownian motion, and $r > 0$.