- (X_n) $(\mathcal{F}_n)$-martingale. If $g: \mathbb{R} \to \mathbb{R}$ is convex

s.t. $\mathbb{E}|g(X_n)| < \infty$ $\forall n$, then

$g(X_n)$ is $(\mathcal{F}_n)$-submartingale

FG!)

Proof $g(X_{n-1}) = \cancel{\mathbb{E}(g(X_n)|\mathcal{F}_{n-1})}\ g(\mathbb{E}(X_n|\mathcal{F}_{n-1}))$

$\leq \mathbb{E}(g(X_n)|\mathcal{F}_{n-1})$ $\square$

- $|X_n|$, X_n^2, ... are submartingales

Examples 1) $X_1, X_2, \ldots$ independent

$\mathbb{E}(X_n) = 0$, $n \geq 0$ $\mathcal{F}_n = \sigma(\ldots)$

$S_n = \sum_1^n X_k$ is $\mathcal{F}_n$-martingale

2) $M_n = S_n^2 - n$, $\mathbb{E}(X_n^2) = 1$

$\mathbb{E}(S_n^2) = \mathbb{E}(\sum_1^n \sum_1^n X_k X_l) = \mathbb{E}(\sum_1^n X_k^2) + 0 = n$

$\mathbb{E}(S_n^2|\mathcal{F}_{n-1}) = \mathbb{E}(S_{n-1}^2 + 2S_{n-1} \cdot X_n + X_n^2 | \mathcal{F}_{n-1})$

$= S_{n-1}^2 + 2S_{n-1} \underbrace{\mathbb{E}(X_n|\mathcal{F}_{n-1})}_{\mathbb{E}(X_n) = 0} + \underbrace{\mathbb{E}(X_n^2|\mathcal{F}_{n-1})}_{\mathbb{E}(X_n^2) = 1}$

$\mathbb{E}(S_n^2 - n|\mathcal{F}_{n-1}) = S_{n-1}^2 - (n-1)$

3) Assume $\mathbb{E}|X| < \infty$

Then $M_n = \mathbb{E}(X|\mathcal{F}_n)$ is $(\mathcal{F}_n)$-martingale

$\mathbb{E}(M_n|\mathcal{F}_{n-1}) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_n)|\mathcal{F}_{n-1}) = \mathbb{E}(X|\mathcal{F}_{n-1})$ a.s.

Stopping times

$(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ filtered prob. space

Def An integer-valued random variable $\tau : \Omega \to \{0, 1, \dots, +\infty\}$ (possibly $\mathbb{P}(\tau < \infty) < 1$) is called a stopping time or a $((\mathcal{F}_n), \mathbb{P})$-stopping time, if

$$\{\tau \le n\} \in \mathcal{F}_n \quad \text{for all } n.$$

[equiv: $\{\tau > n\} \in \mathcal{F}_n\}$ all n

or $\{\tau = n\} \in \mathcal{F}_n\}$ —"—]

Lemma
- Fixed times n are stopping times
- τ_1, τ_2 s.t. $\Rightarrow$ $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ are s.t.
- $\tau_1 \le \tau_2 \le \dots$ seq. of s.t. $\Rightarrow$ $\sup_n \tau_n$ is a s.t.
- $\tau_1 \ge \tau_2 \ge \dots$ —"— $\inf \tau_n$ —"—

Lemma (M_n) is $(\mathcal{F}_n)$-martingal

τ is $(\mathcal{F}_n)$-stopping time

Then $\{M_{n \wedge \tau}, n \ge 0\}$ is also an $(\mathcal{F}_n)$-martingale

Proof Take $C_n = 1_{\{\tau > n-1\}}$. Then $C_n \in \mathcal{F}_{n-1}$

and
$$\sum_{k=1}^{n} C_k (M_k - M_{k-1}) = \sum_{k=1}^{n} 1_{\{\tau > k-1\}} (M_k - M_{k-1})$$
$$= \sum_{k=1}^{n} (M_{k \wedge \tau} - M_{(k-1) \wedge \tau})$$
$$= M_{n \wedge \tau}$$
□

By the previous lemma

$$\mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_0)$$

<u>Question</u> $\lim_{n\to\infty} \mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_\tau) = \mathbb{E}(M_0)$?

Answer <u>No</u>, in general

<u>Example</u> $M_n = \sum_{k=1}^{n} Z_k$ $\{Z_k\}$ i.i.d

$$Z_n = \begin{cases} 1 & \text{prob } 1/2 \\ -1 & \text{—"—} \end{cases}$$

Then (M_n) is a martingale, $M_0 = 0$, and $\mathbb{E}(M_n) = 0$ (symmetric random walk)

Now take $\tau = \min\{n : M_n = 1\}$

Then $\{\tau > n\} = \{M_n \leq 0\} \in \mathcal{F}_n$, so τ is s.t.

Also $M_\tau = 1$. But then

$$1 = \mathbb{E}(M_\tau) \neq \mathbb{E}(M_0) = 0$$

The problem is $\mathbb{E}(\tau) = +\infty$.

Def A sequence $\{H_n\}_{n\geq 1}$ is $(\mathcal{F}_n)$-predictable, if

$$H_n \in \mathcal{F}_{n-1}, \text{ all } n \geq 1.$$

Ex 4/ If (M_n) is a martingale and (H_n) is predictable and bounded, then

$$J_n = \sum_{k=1}^{n} H_k (M_k - M_{k-1}), \text{ is a } (\mathcal{F}_n)\text{-martingale}$$

Proof $J_n - J_{n-1} = H_n(M_n - M_{n-1})$

$$E(J_n - J_{n-1} | \mathcal{F}_{n-1}) = E\{H_n(M_n - M_{n-1}) | \mathcal{F}_{n-1}\}$$

$$= H_n \underbrace{E(M_n - M_{n-1} | \mathcal{F}_{n-1})}_{= 0} \quad \square$$

$$E|J_n| \leq \sum \sup_k |H_k| |\Delta M_n|$$

(J_n) is called the martingale transform of H_n w.r.t. M_n or the discrete stochastic integral of H_n w.r.t. M_n:

$$J_n = \sum_{1}^{n} H_k \Delta M_k \sim \int_0^n H\,dM$$

$\cancel{H\,dM}$

Interpretation

$M_n - M_{n-1}$ = winnings game n per unit bet, $n \geq 1$

H_n = stake on game n

$\in \mathcal{F}_{n-1}$, i.e. decided based on outcome of games $1 \to n-1$

$H_n (M_n - M_{n-1})$ = your winnings, game n

$J_n = \sum_1^n H_k \Delta M_k$ your winnings game $1 \to n$

$\{J_n\}$ martingale $E(J_n) = E(J_0)$

'you cannot beat the system'

Poisson case

Let's write $N_t = \sum_{s\le t} \Delta N_s$, $\Delta N_s = N_s - N_{s-}$

and ~~also~~ use notation $N_t = \int_0^t dN_s = \sum_{s\le t} \Delta N_s$

F: 6-6

If φ is deterministic or stochastic ~~process~~ function, what is $\int_0^t \varphi_s dN_s$?

Example: $\int_0^t N_s dN_s = \sum_{s\le t} N_s \Delta N_s$

$$= \sum_{s\le t} N_s (N_s - N_{s-})$$

$$= \sum_{k=1}^{N_t} \underbrace{N_{S_k}}_{k} \underbrace{(N_{S_k} - N_{S_{k-1}})}_{=1}$$

$$= \sum_{k=1}^{N_t} k = \frac{N_t(N_t+1)}{2}$$

Also, $\int_0^t N_{s-} dN_s = \ldots = \sum_{k=1}^{N_t}(k-1)$

$$= \sum_{k=0}^{N_t-1} k = \frac{(N_t-1)N_t}{2}$$

!

Also observe $\int_0^t (N_s - N_{s-}) dN_s$ $\cancel{= N_t}$

$$= \int_0^t (dN_s)^2 = \int_0^t dN_s = N_t$$

$\int_0^1 f(x)(dx)^2 = 0$?

We have, for Poisson compound

- $E(Y_t) = E\left(\sum_{1}^{N_t} Z_k\right) = \sum_{n=0}^{\infty} \underbrace{E\left(\sum_1^{n} Z_k \mid N_t = n\right)}_{E\left(\sum_1^n Z_k\right) = nE(Z_1)} P(N_t = n)$

$$= \sum_{n=0}^{\infty} n P(N_t = n) E(Z_1)$$

$$= \lambda t E(Z_1)$$

- $E(e^{\theta Y_t}) = E\left(e^{\theta \sum_1^{N_t} Z_k}\right) = E\left(e^{\theta Z_1} \cdot e^{\theta Z_2} \cdots e^{\theta Z_{N_t}}\right)$

$$= \sum_{n=0}^{\infty} E(e^{\theta Z_1})^n P(N_t = n)$$

$$= \sum_0^{\infty} \frac{(\lambda t E(e^{\theta Z_1}))^n}{n!} e^{-\lambda t}$$

$$= e^{\lambda t E(e^{\theta Z_1})} e^{-\lambda t}$$

$$\left(\ln E(e^{\theta Y_t}) = \lambda t E[e^{\theta Z_1} - 1] \right)$$

- Exercise: Find $Var(Y_t)$

  Find $E(Y_t \mid \mathcal{F}_s)$

  Show that $\tilde{Y}_t = Y_t - \lambda t E(Z_1)$, $t \geq 0$ is a martingale