Analysis of Time Series, L3

Rolf Larsson

Uppsala University

27 mars 2025

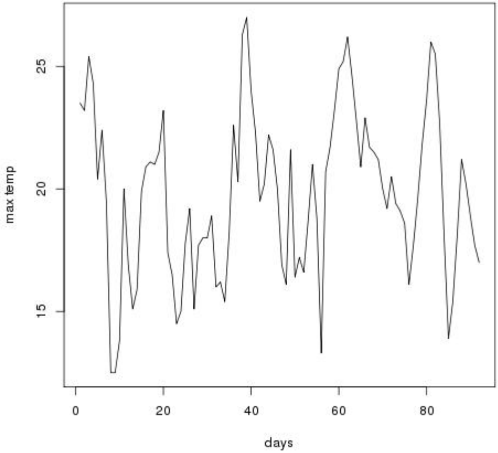
Today

3.1-3.2:

- Autoregressive (AR) models
- Moving average (MA) models
- Menti

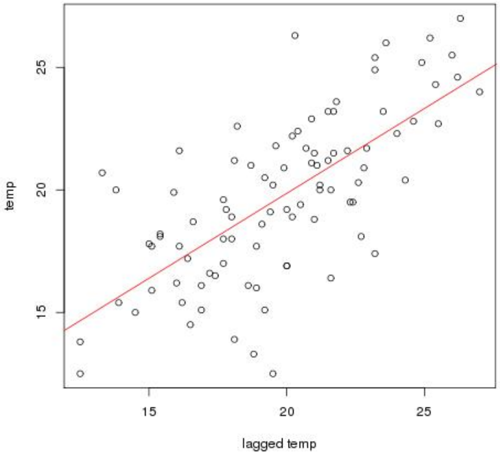
AR models

Daily temperatures in Uppsala, June-August 1984.



AR models

Regression on lagged temperatures: $x_t = 6.00 + 0.693x_{t-1} + w_t$



AR models

Autoregressive model of order 1, AR(1)

- without constant ($E(x_t) = 0$)

$$x_t = \phi x_{t-1} + w_t.$$

- with constant ($E(x_t) = \mu$)

$$x_t - \mu = \phi(x_{t-1} - \mu) + w_t$$

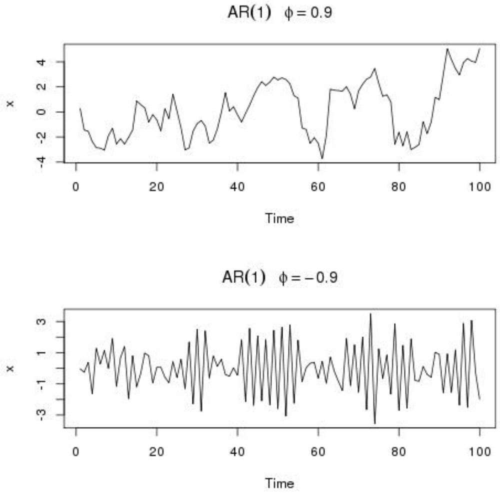
or equivalently

$$x_t = \mu + \phi(x_{t-1} - \mu) + w_t = \alpha + \phi x_{t-1} + w_t,$$

where $\alpha = \mu(1 - \phi)$.

AR models

Simulated series: $x_t = \phi x_{t-1} + w_t$



AR models

Definition (3.1)

An *autoregressive model* of order p, AR(p), with mean zero is given by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + w_t,$$

where w_t is white noise with mean zero and variance σ^2.

An *autoregressive model* of order p, AR(p) with mean μ is given by

$$x_t - \mu = \phi_1(x_{t-1} - \mu) + \phi_2(x_{t-2} - \mu) + \ldots + \phi_p(x_{t-p} - \mu) + w_t,$$

or equivalently

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + w_t,$$

where $\alpha = \mu(1 - \phi_1 - \ldots - \phi_p)$.

AR models

Recall: $Bx_t = x_{t-1}$, $B^k x_t = x_{t-k}$.

Definition (3.2)

The *autoregressive operator* is defined as

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p.$$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t$$

is equivalent to

$$\phi(B)x_t = w_t.$$

AR models

Definition (1.12)

A *linear process* x_t is given by

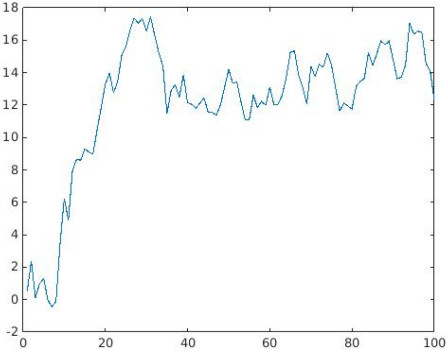
$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

If x_t is AR(1) with mean zero, show that

1. $$x_t = \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}.$$
2. If $|\phi| < 1$ and x_t is stationary, then x_t is a linear process.

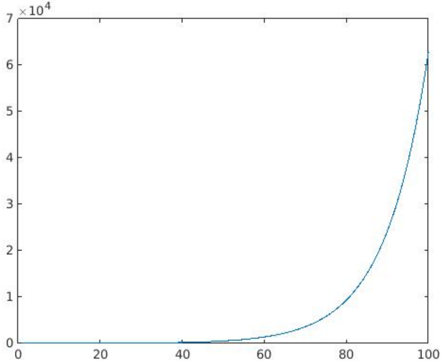
AR models

Simulated series: $x_t = x_{t-1} + w_t$



AR models

Simulated series: $x_t = 1.1x_{t-1} + w_t$ (observe the scale on the y axis)



By recursion: $x_t = 1.1^{t-1}x_1 + 1.1^{t-2}w_2 + ... + 1.1w_{t-1} + w_t$
The first term dominates!

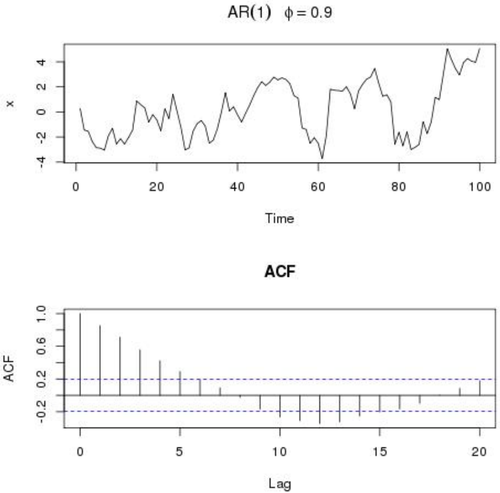
AR models

Let $x_t = \phi x_{t-1} + w_t$.

1. Derive the MA (moving average, linear process) representation using
   a) $\psi(B)\phi(B) = 1$ where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$
   b) $x_t = \phi^{-1}(B) w_t$
2. Prove that the autocorrelation function is $\rho(h) = \phi^{|h|}$.

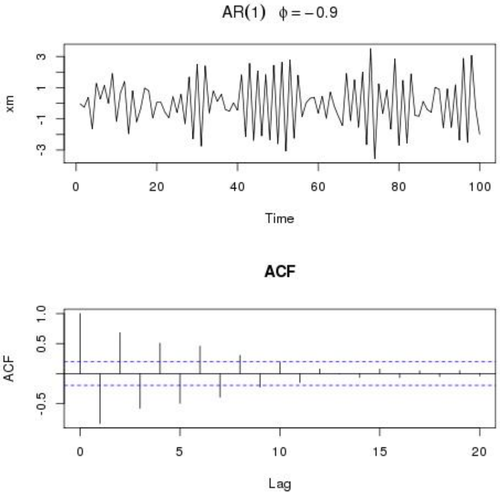
AR models

Simulated series: $x_t = 0.9x_{t-1} + w_t$ and estimated ACF



AR models

Simulated series: $x_t = -0.9x_{t-1} + w_t$ and estimated ACF



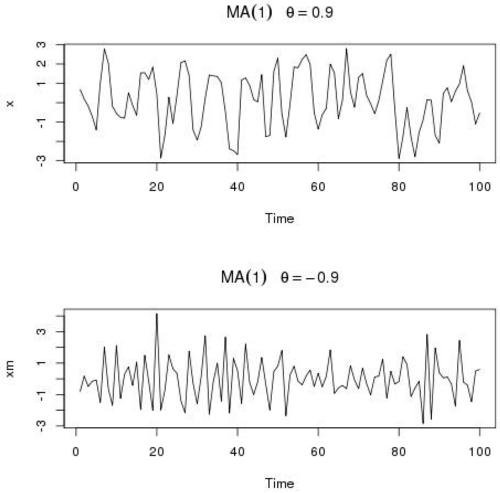
MA models

Moving average model of order 1, MA(1)

$$x_t = w_t + \theta w_{t-1}$$

MA models

Simulated series: $x_t = w_t + \theta w_{t-1}$



MA models

Definition (3.3)

A *moving average model* of order q, MA(q), is given by

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots + \theta_q w_{t-q},$$

where w_t is white noise with mean zero and variance σ^2.

MA models

Definition (3.4)

The *moving average operator* is defined as

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + ... + \theta_q B^q.$$

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + ... + \theta_q w_{t-q}$$

is equivalent to

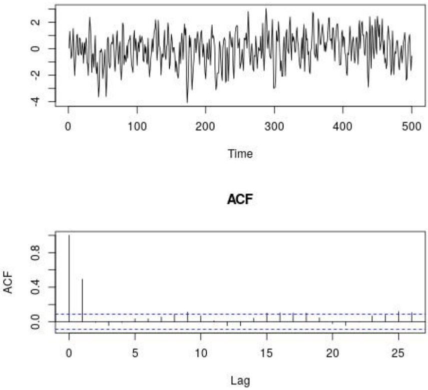
$$x_t = \theta(B) w_t.$$

MA models

1. Let $x_t = w_t + \theta w_{t-1}$.
   a) Derive the autocorrelation function $\rho(h)$, and show that it is the same if θ is replaced by $1/\theta$.
   b) Derive the AR representation.
2. Let x_t be an MA(q) process and show that $\rho(h) \neq 0$ only if $|h| \leq q$.

MA models

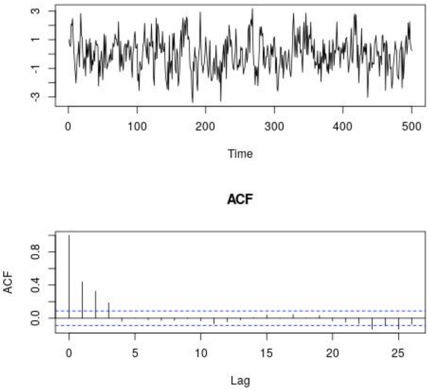
Simulation of $x_t = w_t + 0.8w_{t-1}$:



Theoretically, the ACF cuts off after lag 1.

MA models

Simulation of $x_t = w_t + 0.3w_{t-1} + 0.3w_{t-2} + 0.3w_{t-3}$:



Theoretically, the ACF cuts off after lag 3.

News of today

- AR processes
  - Definition
  - MA representation
  - Autocorrelation function
- MA processes
  - Definition
  - Autocorrelation function
  - AR representation