UPPSALA UNIVERSITET

Föreläsningsanteckningar

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1. σ-algebras & Measure spaces

1.1. σ-algebras.

Definition 1.1 σ-algebra

A collection of subsets Σ of a set S is called a *σ-algebra* if:
- $\varnothing \in \Sigma$
- Is an algebra:
  - Closed under complements such that for $A \in \Sigma \Rightarrow A^c = S \backslash A \in \Sigma$
  - Closed under unions such that $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$
- Closed under countably infinite unions $A_i \in \Sigma$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

Example:
$\Sigma = \{\varnothing, S\}$ is a σ-algebra on any set S.
Another example is $\mathcal{P}(S)$, which denotes the powerset.
Another example is $S = \mathbb{N}$, then $\Sigma = \{\varnothing, \mathbb{N}, \{2k : k \in \mathbb{N}\}, \{2k+1 : k \in \mathbb{N}\}\}$

Remark:
There exists many equivalent definitions of a σ-algebra. For example, instead of the first axiom of $\varnothing \in \Sigma$, an equivalent definition could be "Σ is non-empty", since then $\exists A \in \Sigma \Rightarrow A^c \in \Sigma \Rightarrow A \cup A^c = S \in \Sigma \Rightarrow (A \cup A^c)^c = \varnothing \in \Sigma$

Remark:
Closed under unions $\Rightarrow$ closed under finite unions since $A_1, \cdots, A_n \in \Sigma \Rightarrow A_1 \cup A_2 \in \Sigma, A_1 \cup A_2 \cup A_3 = \underbrace{(A_1 \cup A_2)}_{\in \Sigma} \cup A_3$, thus by induction $A_1 \cup \cdots \cup A_n \in \Sigma$

This does *not* imply Σ is closed under countable unions.

Counter-example:
Consider $S = [0, 1) \subseteq \mathbb{R}$. Let Σ be all finite unions of disjoint sets on the form $[a, b)$ such that $0 \leq a \leq b < 1$ (if $a = b \Rightarrow \varnothing$).
First and all algebra axioms are fulfilled, but the last one is not since we cvan consider $A_n = \left[\frac{1}{n}, 1\right)$.
Then $\bigcup_{i=2}^{\infty} = (0, 1) \notin \Sigma$

An algebra Σ is an algebra in an algebraic sense.
The symmetric difference $A \triangle B = (A \backslash B) \cup (B \backslash A)$. This behaves like "+" on Σ and intersections behave like multiplication.
Just like one would expect from an algebra, the multiplication is distributive over addition, eg. $C \cap (A \triangle B) = (C \cap A) \triangle (C \cap B)$

1.2. Measures.

Let Σ be a σ-algebra on S, and let μ_0 be a function from Σ_0 to $[0, \infty] = [0, \infty) \cup \{\infty\}$, essentially a function that assigns some value to subsets of Σ.
Intuitively, a measure should increase if we measure something bigger.

Definition 1.2 Additive and σ-additive measures

A measure μ_0 is called *additive* if $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$ where A, B are disjoint sets.

A measure μ_0 is called *σ-additive* if this holds for ocuntable unions, i.e if A_n are pairwise disjoint, then $\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n)$

Remark:
We say that μ_0 is a measure if μ_0 is σ-additive and $\mu_0(\varnothing) = 0$

Example:
$S = \{1, 2, \cdots, 6\}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A) = \dfrac{1}{6}|A|$. Note here that $\mu_0(S) = 1$

Definition 1.3 Probability measures

All measures that sum up to 1 are called *probability measures*

Example:
$S = \mathbb{N}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A \in \Sigma) = |A|$. Here $\mu_0(S) = \infty$

Example:
$S = \mathbb{N}$, $\Sigma = \mathcal{P}(S)$ and set $\mu_0(A \in \Sigma) = \begin{cases} 0 & \text{if} \quad |A| < \infty \\ \infty & \text{if} \quad |A| = \infty \end{cases}$
This is an example of an additive but not σ-additive measure, since if $A_n = \{n\}$, then $\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$, but $\sum_{n=1}^{\infty} \mu_0(A_n) = -1$

1.3. **Measure spaces.**

Definition 1.4 Measure space triplet

A *measure space* is a triplet (S, Σ, μ) where S is some set, Σ is a σ-algebra over S, and μ is a σ-additive function $\mu : \Sigma \to [0, \infty]$ such that $\mu(\varnothing) = 0$

Definition 1.5 Probability space

If $\mu(S) = 1$, then the triplet is called a *probability space*.

Example: (finite measure space)
Let $S = \{s_1, \cdots, s_k\}$ where $k \in \mathbb{N}$ be a set of outcomes. We also associate probabilities $p_1, \cdots, p_k$ to each $s_1, \cdots, s_k$ such that $\sum_i p_i = 1$. Let $\mu(A) = \sum_{s_i \in A} p_i \ \forall A \subseteq S$. If we let $\Sigma = \mathcal{P}(S)$, then (S, Σ, μ) is a measure and a probability space.

Example: (Lebesgue measure)
Let $S = \mathbb{R}$, $\Sigma = \mathcal{B}(\mathbb{R})$ be the Borel σ-algebra (smallest σ-algebra that makes open sets measureable, note that $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$) and let μ be something measuring length on finite unions of disjoint open intervals $A = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ such that $\mu(A) = |b_1 - a_1| + \cdots + |b_n - a_n|$

This μ is called the Lebesgue measure ($\mathcal{L}$)
Restricting S to $[0, 1]$, then we have a probability measure

$$\mu = \mathcal{L}\mid_{[0,1]}(A) = \mathcal{L}(A \cap [0, 1]) \Rightarrow ([0, 1], \mathcal{B}([0, 1], \mathcal{L}\mid_{[0,1]})) \quad \text{is a probability measure}$$

This is a formulation of uniform random numbers in $[0, 1]$

1.4. **Properties of measures.**
For a measure space, we have the following properties:
(1) $\mu(A \cup B) \leq \mu(A) + \mu(B)$
(2) $\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$
(3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) - \mu(A_1 \cap A_2) - \cdots - \mu(A_{n-1} \cap A_n) + \mu(A_1 \cap A_2 \cap A_3) \cdots + (-1)^{n+1}\mu(A_1 \cap A_2 \cdots \cap A_n)$

Note that for the first two points, we have previously assumed that A, B were disjoint. This would be the case for "joint" sets.

Bevis 1.1

Consider $\mu(A) = \mu(A \backslash B \cup (A \cap B)) = \mu(A \backslash B) + \mu(A \cap B)$ and proceed. $\square$

Remark:
For point 4, check Math Stackexchange

The idea is if we can consider some set that is measurable, we want to be able to say something about the compositions of those measurable sets so the idea is we include their subsets in the σ-algebra (in the space we set up) as well as keeping it closed in an algebraic sense.

1.5. **Monoticity of measure.**
Let (A_i) be a sequence of increasing sets in Σ such that $\varnothing \subseteq A_1 \subseteq \cdots \subseteq S$. Then:

$$\mu(A_i) = \mu(A_i \backslash A_{i-1} \cup (A_i \cap A_{i-1})) = \mu(A_i \backslash A_{i-1} \cup A_{i-1}) = \mu(A_i \backslash A_{i-1}) + \mu(A_{i-1}) \geq \mu(A_{i-1})$$

Thus, by induction, $\mu(A_1) \leq \mu(A_2) \leq \cdots$ and by monotone convergence the limit $\lim_{i\to\infty} \mu(A_i)$ exists in the extended positive real line.

Writing $A = \bigcup_{i=1}^{\infty} A_i$, we have $\mu(A) = \lim_{i\to\infty} \mu(A_i)$, this because:

$$A = A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash A_2) \cup \cdots$$

$$\mu(A) = \mu(A_1) + \mu(A_2 \backslash A_1) + \mu(A_3 \backslash A_2) + \cdots = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i \backslash A_{i-1})$$

where $A_0 = \varnothing = \lim_{n\to\infty} \mu(A_n)$

A similar result holds for decreasing sets, i.e $S \supseteq A_1 \supseteq A_2 \cdots \supseteq \varnothing$
We do the limit as $A = \bigcap_{i=1}^{\infty} A_i$ and by monotone convergence $\mu(A) = \lim_{i\to\infty} \mu(A_i)$ with similar proof.

Remark:
The last set in the decreasing sets does not necessarily have to be the empty set, recall that we are dealing with intersections instead of unions.

1.6. Generated σ-algebras.

Given any collection of subsets of $\mathfrak{A}\subseteq \mathcal{P}(S)$, the σ-algebra *generated* by $\mathfrak{A}$ is the smallest σ-algebra that cointains $\mathfrak{A}$ is denoted by $\sigma(\mathfrak{A}) = \bigcap_{\Sigma:\sigma\text{-alg}\&\mathfrak{A}\subseteq\Sigma}$
This is sometimes denoted by $< \mathfrak{A} >$

One can verify that this is indeed a σ-algbera:

(1) $\varnothing$ is contained in all σ-algebras, so $\varnothing$ is contained in all of the intersections
(2) If $A \in \sigma(\mathfrak{A})$, then $A \in \Sigma\ \forall\ \sigma$-algebras, but then $A^c \in \Sigma\ \forall\ \sigma$-algebras $\Rightarrow A^c \in \sigma(\mathfrak{A})$

The rest of the axioms for a σ-algebras are shown in an equivalent manner as in (2)

Example: (Borel σ-algebra)
Let $\mathcal{B}(S) = \sigma$(open subsets of S) (here we mean open in a topological sence since we need S to have a notion of opened-ness).
Since we mean open in a topological sense (which is defined as the complement of a closed set), we could have used the complement of a closed set to denote the open set, but since the complement is in the σ-algebra we may as well had the equivalent definition using the closed set all together.
This leads us to $\mathcal{B}(\mathbb{R}) = \sigma(\{(a, b) : a < b, a, b \in \mathbb{R}\})$. Instead of $\mathbb{R}$, any dense set could have worked as well (such as $\mathbb{Q}$)

Example:
Let $S = \{1, 2, 3 \cdots, 10\}$, and $\mathfrak{A} = \{\{1, 2\}, \{5\}\}$.
In order to generate a σ-algebra, we just need to recursively insert things that work with the axioms. For example, we need the empty set so we chuck in the empty set. We need the complement of the empty set so we chuck in the complement to the empty set. We need the complements to all the sets in $\mathfrak{A}$, so we add those as well, as well as their intersections.
We should then be left with just enough to call it a σ-algebra, and nothing more, hence the smallest σ-algebra:

$$\sigma(\mathfrak{A}) = \{\varnothing, S, \{1, 2\}, \{5\}, \{1, 2, 5\}, \{3, 4, 5, 6, 7, 8, 9, 10\}, \{1, 2, 3, 4, 6, 7, 8, 9, 10\}, \{3, 4, 6, 7, 8, 9, 10\}\}$$

Definition 1.6 π-system

A π-*system* on a set S is a collection of subsets π such that $\varnothing \in \pi$ and if $A, B \in \pi$ then $A \cap B \in \pi$

Definition/Sats 1.1

Suppose $\mathfrak{A} \subseteq \mathcal{P}(S)$ is a π-system and suppose that μ_1, μ_2 are measures on $(S, \sigma(\mathfrak{A}))$ such that $\mu_1(A) = \mu_2(A)\ \forall A \in \mathfrak{A}$

$\Rightarrow$ Then $\mu_1 = \mu_2$ on $(S, \sigma(\mathfrak{A}))$

In other words, π-systems uniquely determine a measure.

Example:
Let $S = \mathbb{R}$, $\mathfrak{A} = \{[-\infty, a) : a \in \mathbb{R}\}$, $\sigma(\mathfrak{A}) = \mathcal{B}(\mathbb{R})$
$\mathfrak{A}$ is a π-system and have any measure is uniquely defined on $\mathfrak{A}$.
note that $\mu([-\infty, a))$ is nothing but thte cumulative distribution function of the measure μ (in terms of a).
"Measure up to a point". The following gives justification to construct measures from small collections.

Definition 1.7 Caratheodorys extension theorem

If Σ_0 is an algebra and $\mu_0 : \Sigma \to [0, \infty]$ is a σ-additive, $\exists!\quad \mu$ on $\Sigma = \sigma(\Sigma_0)$ such that $\mu(A) = \mu_0(A) \quad \forall A \in \Sigma_0$

An important consequence is that the Lebesgue measure is unique (only one notion of length on $\mathcal{B}(\mathbb{R})$) defined through sets of the form $A = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ (disjoint union of open sets)

$$\mathcal{L}(A) = |b_1 - a_1| + \cdots + |b_n - a_n|$$

2. Probability Spaces

Probability spaces are normally denoted by $(\Omega, \mathcal{E}, \mathbb{P})$ where:

- Ω is the space of realisations
- $\mathcal{E}$ is the sets of events
- $\mathbb{P}$ is the probability measure

Example:
$\Omega = \mathbb{R}$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$, $\mathbb{P}(A = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2})dx$, $A = (a, b)$
This models a normally distributed real number.

2.1. Almost sure events.

We say that an event occurs *almost surely* if $\mathbb{P}(\mathcal{E}) = 1$ (equivalently $\mathbb{P}(\mathcal{E}^c) = 0$)

Proposition:
Let $E_1, \cdots \in \mathcal{E}$ be such that $\mathbb{P}(E_i) = 1 \quad \forall i \in \mathbb{N}$
Then, $\mathbb{P}\left(\bigcap_{i=1}^{\infty} E_i\right) = 1$

Bevis 2.1

Note that since each of them have probability measure 1, their complement must have measure 0 so:

$$\mathbb{P}\left(\bigcup_{i \in \mathbb{N}} E_i^c\right) \leq \sum_{i \in \mathbb{N}} \mathbb{P}(E_i^c) 0$$

However, since:

$$0 \leq \mathbb{P}\left(\bigcup_{i \in \mathbb{N}} E_i^c\right) \leq 0 \Rightarrow \mathbb{P}\left(\bigcup_{i \in \mathbb{N}} E_i^c\right) = 0$$

$$\Rightarrow \mathbb{P}\left(\left(\bigcup_{i \in \mathbb{N}} E_i^c\right)^c\right) = 1$$

But we have de-Morgans law, i.e $\bigcup_{i \in \mathbb{N}} E_i^c = \left(\bigcap_{i \in \mathbb{N}} E_i\right)^c$, which yields:

$$\left(\left(\bigcap_{i \in \mathbb{N}} E_i\right)^c\right)^c = \bigcap_{i \in \mathbb{N}} E_i$$

$\square$

Remark:
This applies only to countable unions. If uncountable, we could consider

$$\Omega = [0, 1], \quad \Sigma = \mathcal{B}([0, 1]), \quad \mathbb{P} = \mathcal{L}\mid_{[0,1]}$$

Then $\mathbb{P}(X = x) = 0$ (where X is some randomly chosen number and x is some fixed number). Taking the complement of this event yields $\mathbb{P}(X \neq x) = 1$ so $\mathbb{P}(X \neq x : x \in \mathbb{Q}) = 1$

2.2. **Liminf and limsup.**
Recall from real analysis:

$$\left.\begin{array}{l}\lim_{n\to\infty}\sup x_n = \lim_{n\to\infty}\sup_{m\geq n} x_n \\ \lim_{n\to\infty}\inf x_n = \lim_{n\to\infty}\inf_{m\geq n} x_n\end{array}\right\}\text{Limits exists in the extended reals and the limit exists iff limsup} = \text{liminf}$$

Recall that if $\lim_{n\to\infty}\sup x_n \geq x \Leftrightarrow \exists$ a subsequence $(x_n)_k$ with limit $\geq x$ and the opposite for liminf.

There exists a similar notion for sets.
Let $E_1, \cdots$ be events (sets)

$$\left.\begin{array}{l}\lim_{n\to\infty}\inf E_n = \bigcup_{n\geq 1}\bigcap_{m\geq n} E_n \\ \lim_{n\to\infty}\sup E_n = \bigcap_{n\geq 1}\bigcup_{m\geq n} E_n\end{array}\right\}$$

Some intuition here is definitely necessary.
For the first one, we are taking intersections of less and less sets (increasing sequence of sets), then finally unions. Think of this as events that eventually will appear

For the second one, it is decreasing (because of the intersection outside), all points will occur infinitely often.