Complex analysis

In this course we shall study fcns

$$f : \mathbb{C} \to \mathbb{C}$$

(or, more generally, $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$)

I expect that you are all familiar with the basic algebra of complex numbers, but start with some reminders.

<u>Def.</u> A <u>complex number</u> is a number of the form $x+iy$, where $x, y \in \mathbb{R}$. Two complex numbers x_1+iy_1 and x_2+iy_2 are said to be equal ($x_1+iy_1 = x_2+iy_2$) iff $x_1 = x_2$ and $y_1 = y_2$.

The number x is called the <u>real part</u> of $x+iy$, and the number y the <u>imaginary part</u> of $x+iy$.

We write: $x = \mathrm{Re}\,(x+iy)$, $y = \mathrm{Im}\,(x+iy)$.

The set of all complex numbers is denoted $\mathbb{C}$

We define <u>addition</u> and <u>multiplication</u> of two complex numbers as follows:

$$(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$$

$$(x_1+iy_1) \cdot (x_2+iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Complex numbers are often denoted by z or w.

Most algebraic laws of real numbers hold true also for complex numbers; e.g.

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
- $z_1 z_2 = z_2 z_1$
- $z_1 (z_2 z_3) = (z_1 z_2) z_3$
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

In other words, you compute with complex numbers "as usual", but replace i^2 by -1.

<u>Def</u> The <u>complex conjugate</u> of $z = x+iy$, denoted $\bar{z}$, is defined by $\bar{z} = x - iy$.

It holds that

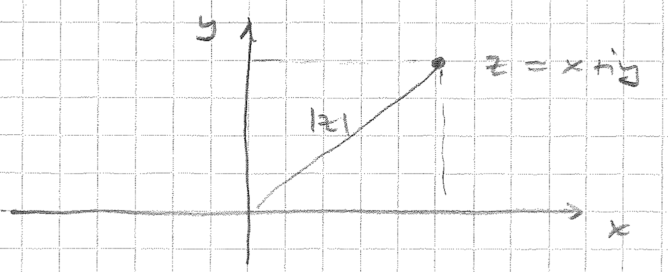
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Note also that

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

It is natural to represent a complex number $z = x+iy$ as a point (x,y) in a cartesian coordinate system.



This geometrization of $\mathbb{C}$ is called the <u>complex plane</u>

<u>Def.</u> The absolute value of $z = x+iy$, denoted $|z|$, is defined by

$$|z| = \sqrt{x^2+y^2}$$

It holds that

- $|z|^2 = z\bar{z}$
- $|z_1 z_2| = |z_1||z_2|$

Note also that every number $z \in \mathbb{C}$, $z \neq 0$, has a multiplicative inverse $\frac{1}{z}$ given by

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

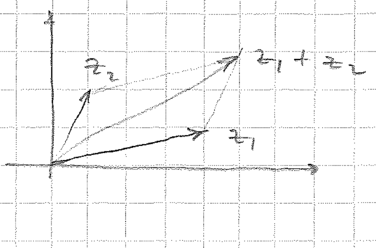
The following result is important

<u>Thm</u> (Triangle ineq.)

For $z_1, z_2 \in \mathbb{C}$ it holds that

$$|z_1+z_2| \leq |z_1|+|z_2|$$

Geometric interpretation:



Corollary (Reversed triangle ineq.)

For $z_1, z_2 \in \mathbb{C}$ it holds that

$$\big|\,|z_1| - |z_2|\,\big| \leq |z_1 - z_2|$$

Proof: $|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

so that $|z_1| - |z_2| \leq |z_1 - z_2|$. Now let $z_1 \leftrightarrow z_2$ $\square$

Polar form Let $z = x + iy \neq 0$.

The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle,

and hence there exists θ s.t.

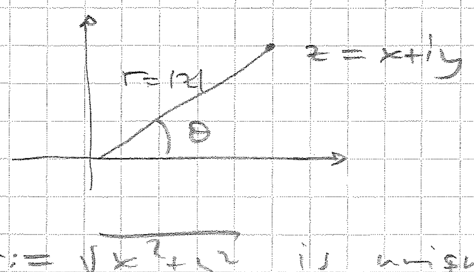
$$\frac{x}{|z|} = \cos\theta \quad , \quad \frac{y}{|z|} = \sin\theta$$

Therefore $z = x + iy$ can be written as

$$z = |z| \left(\cos\theta + i \sin\theta\right)$$

This is called the polar form of z.

See Fig.



Note that $r := \sqrt{x^2+y^2}$ is uniquely determined

by z, whereas θ is only unique up to

integer multiples of 2π. I.e. if a particular θ suffices

so does $\theta + k2\pi$, $k \in \mathbb{Z}$. We let, given z,

all these be denoted by $\arg z$.

It is practical to have a notation for one of all values of $\arg z$.

The principal value of $\arg z$, denoted $\operatorname{Arg} z$, is specified as that value of $\arg z$ which belongs to $(-\pi, \pi]$.

Ex. $\arg(1+i) = \frac{\pi}{4} + k2\pi, \ k \in \mathbb{Z}$.

$$\operatorname{Arg}(1+i) = \frac{\pi}{4}.$$

Since $|1+i| = \sqrt{2}$, it holds that

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

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Remark: One calls $\operatorname{Arg} z$ a branch of $\arg z$.

Note that $\operatorname{Arg} z$ is "discontinuous" along the negative real axis, which is called a branch cut.

Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$

$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$

$= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

Correctly interpreted:

$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

The exponential fcn

Def For $z = x+iy \in \mathbb{C}$, let

$$e^z := e^x(\cos y + i\sin y)$$

Remark: Note that e^z agrees with the "usual" exponential fcn if $z \in \mathbb{R}$, i.e. the above constitutes an extension of this.

Note in particular that

$$e^{iy} = \cos y + i\sin y, \quad y \in \mathbb{R}$$

This is called Euler's formula.

I.e. $e^z = e^x e^{iy}$.

The polar form can now be written more compactly:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Moreover, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

In particular,

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

From this it follows that

$$\boxed{e^{z_1}e^{z_2} = e^{z_1+z_2}}$$

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Since $|e^{iy}| = 1$ it holds that

$$|e^z| = e^x.$$

Note finally that

$$(e^{i\theta})^n = \underbrace{e^{i\theta} \cdots e^{i\theta}}_{n \text{ times}} = e^{in\theta}, \quad n = 1, 2, 3, \ldots$$

i.e.

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 1, 2, 3, \ldots}$$

This is called deMoivre's formula (true for $n \in \mathbb{Z}$)

The logarithm fcn (as a multivalued fcn)

In real analysis one defines the logarithm ($\ln x$) as the inverse of e^x. The problem is that e^z is not an injective fcn (and hence has no inverse).

Given $z \in \mathbb{C}\setminus\{0\}$ one chooses to define $\log z$ as the set of all $w \in \mathbb{C}$ whose image under the exponential fcn is z,

i.e. $w = \log z \iff z = e^w$.

($\log z$ is a so-called multivalued fcn)

Write $z = re^{i\theta}$, $w = u + iv$. Then,

$$z = e^{w} \iff re^{i\theta} = e^{u} e^{iv}$$

$$\iff u = \ln r = \ln |z|$$

$$\text{and } v = \theta + k2\pi = \arg z \quad (k \in \mathbb{Z})$$

The explicit def. of $\log z$ is then:

$\boxed{\begin{aligned} &\underline{\text{Def}} \text{ For } z \neq 0 \text{ we def. } \log z \text{ as} \\ &\qquad \log z = \ln |z| + i \arg z = \\ &\qquad\qquad = \ln |z| + i\,(\operatorname{Arg} z + k2\pi), \ k \in \mathbb{Z} \end{aligned}}$

__Remark:__ Here $\operatorname{Arg} z$ denotes the principal value of $\arg z$, i.e. $\operatorname{Arg} z \in (-\pi, \pi]$.

__Ex__ Compute $\log(1+i)$

__Sol__ $\log(1+i) = \ln |1+i| + i \arg (1+i) =$

$$= \ln \sqrt{2} + i\,(\operatorname{Arg}(1+i) + k2\pi) =$$

$$= \frac{1}{2} \ln 2 + i\left(\frac{\pi}{4} + k2\pi\right).$$

□