Assignment 1 (solutions)

1 a) Let $Z_t = W_t^k$. Then

$$dZ_t = kW_t^{k-1}\,dW_t + \tfrac{1}{2}k(k-1)W_t^{k-2}\underbrace{(dW_t)^2}_{dt}$$

so

$$W_t^k = \underbrace{W_0^k}_{0} + \tfrac{1}{2}k(k-1)\int_0^t W_s^{k-2}\,ds + k\int_0^t W_s^{k-1}\,dW_s .$$

Taking expectations gives

$$E[W_t^k] = E\left[\tfrac{1}{2}k(k-1)\int_0^t W_s^{k-2}\,ds\right] + \underbrace{kE\left[\int_0^t W_s^{k-1}\,dW_s\right]}_{0}$$

$$= \tfrac{1}{2}k(k-1)\int_0^t E[W_s^{k-2}]\,ds$$

i.e.

$$\alpha_k(t) = \frac{k(k-1)}{2}\int_0^t \alpha_{k-2}(s)\,ds .$$

b) $\alpha_4(t) = \frac{4\cdot 3}{2}\int_0^t \alpha_2(s)\,ds = 6\int_0^t s\,ds = 3t^2$

c) $\alpha_6(t) = \frac{6\cdot 5}{2}\int_0^t \alpha_4(s)\,ds = 15\int_0^t 3s^2\,ds = 15t^3$

(2) a) Let $Y_t = e^{at}\left(X_t - \frac{b}{a}\right)$.

Then
$$dY_t = a e^{at}\left(X_t - \frac{b}{a}\right)dt + e^{at}\,dX_t$$
$$= \sigma e^{at}\,dW_t$$

so
$$Y_t = Y_0 + \sigma \int_0^t e^{as}\,dW_s \quad , \quad \text{where } Y_0 = x_0 - \frac{b}{a}.$$

Thus $X_t = \frac{b}{a} + e^{-at} Y_t = x_0 e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) + \sigma e^{-at} \int_0^t e^{as}\,dW_s$,

b) $E[X_t] = x_0 e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right)$

c) $\mathrm{Var}(X_t) = \mathrm{Var}\left(x_0 e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)}\,dW_s \right)$

$$= \mathrm{Var}\left(\sigma \int_0^t e^{-a(t-s)}\,dW_s\right) = \sigma^2 e^{-2at}\, E\left[\left(\int_0^t e^{as}\,dW_s\right)^2\right]$$

$$\overset{\text{Ito isometry}}{=} \sigma^2 e^{-2at} \int_0^t e^{2as}\,ds = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

③ $$\begin{cases} F_t + \frac{9}{2} F_{xx} + \frac{1}{2} F_{yy} + F_{xy} + F_x = 0 \\ F(T,x,y) = xy^2 \end{cases}$$

Let $C = \begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}$, and look for σ such that $\sigma\sigma^* = C$.

We try with $\sigma = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ so that $\sigma\sigma^* = \begin{pmatrix} a^2+b^2 & bc \\ bc & c^2 \end{pmatrix}$.

Thus $\sigma = \begin{pmatrix} \sqrt{8} & 1 \\ 0 & 1 \end{pmatrix}$ works!

(Remark: other choices of σ are also possible, e.g. $\sigma = \begin{pmatrix} 3 & 0 \\ \frac{1}{3} & \frac{\sqrt{8}}{3} \end{pmatrix}$)

Let X, Y solve

$$\begin{cases} d\begin{pmatrix} X_s \\ Y_s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds + \begin{pmatrix} \sqrt{8} & 1 \\ 0 & 1 \end{pmatrix} d\begin{pmatrix} V_s \\ W_s \end{pmatrix} \\ \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases}$$

where V, W are independent Brownian motions.

By Feynman-Kac,

$$F(t,x,y) = E_{t,x}\left[X_T Y_T^2\right] = E\left[\left(x + T - t + \sqrt{8}\, V_{T-t} + W_{T-t}\right)\left(y + W_{T-t}\right)^2\right]$$

$$= E\left[\left(x + T - t + \sqrt{8}\, V_{T-t}\right)\left(y + W_{T-t}\right)^2\right] + E\left[W_{T-t}\left(y + W_{T-t}\right)^2\right]$$

$$\overset{\text{indep.}}{=} E\left[x + T - t + \sqrt{8}\, V_{T-t}\right] E\left[\left(y + W_{T-t}\right)^2\right] + 2y(T-t)$$

$$= (x + T - t)(y^2 + T - t) + 2y(T-t)$$

<u>Answer</u>: $F(t,x,y) = (x + 2y + y^2)(T-t) + xy^2 + (T-t)^2$

(4) $$\begin{cases} F_t + 2x^2 F_{xx} - x F_x + 2x = 0 \\ F(T,x) = x^3 \end{cases}$$

Let X solve $$\begin{cases} dX_s = -X_s ds + 2X_s dW_s \\ X_t = x \end{cases}$$

By Exercise 5.10,

$$F(t,x) = E_{t,x}\left[X_T^3 + 2\int_t^T X_s \, ds\right].$$

We have $X_T = X_t e^{-3(T-t)+2(W_T-W_t)}$, so

$$E_{t,x}[X_T^3] = E_{t,x}\left[x^3 e^{-9(T-t)+6W_{T-t}}\right] =$$

$$= x^3 e^{-9(T-t)} e^{18(T-t)} = x^3 e^{9(T-t)}.$$

Also, $E_{t,x}[X_s] = x e^{-(s-t)}$, so

$$F(t,x) = x^3 e^{9(T-t)} + 2\int_t^T x e^{-(s-t)} ds$$

$$= x^3 e^{9(T-t)} + 2x\left(1 - e^{-(T-t)}\right)$$

Answer: $F(t,x) = x^3 e^{9(T-t)} + 2x\left(1 - e^{-(T-t)}\right)$