Analysis of Categorical Data
Chapter 3: Inference for Contingency Table

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Intended Learning Outcome

Through this chapter, you should be able to

1. test independence in contingency table,
2. test monotone trend.

Odds Ratio

Suppose that we have observed a 2×2 table

	Y	
X	1	2
1	n_{11}	n_{12}
2	n_{21}	n_{22}

The sample odds ratio is

$$\hat{\theta} = \frac{n_{11}n_{22}}{n_{12}n_{21}} \geq 0.$$

If $\hat{\theta} > 0$, then we can consider

$$\log \hat{\theta} = \log n_{11} + \log n_{22} - \log n_{12} - \log n_{21}.$$

Wald Confidence Interval

An estimated standard error of $\log \hat{\theta}$ is

$$\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}.$$

Hence, a Wald confidence interval for $\log \theta$ is

$$\log \hat{\theta} \pm z_{\alpha/2} \sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}.$$

However,

$$\hat{\theta} = \frac{n_{11} n_{22}}{n_{12} n_{21}}$$

can be 0 (if $n_{11}n_{22} = 0$), ∞ ($n_{12}n_{21} = 0$), or undefined (if $n_{11}n_{22} = n_{12}n_{21} = 0$). Consequently, the Wald interval may not exist.

- An ad-hoc approach is to add 0.5 to n_{ij}.
- Use other approaches such as the score interval or the likelihood ratio confidence interval.

Example: Aspirin Use and Myocardial Infraction

Compute $\hat{\theta}$ and find a 95% confidence interval for θ

	Myocardial Infraction	
	Yes	No
Placebo	28	656
Aspirin	18	658

Independence

We have an $I \times J$ contingency table from multinomial sampling with probabilities $\{\pi_{ij}\}$.We want to test

$$\begin{aligned} H_0 : &\quad \text{independence as } \pi_{ij} = \pi_{i+}\pi_{+j} \text{ for all } i, j, \\ H_1 : &\quad H_0 \text{ is not true.} \end{aligned}$$

The log-likelihood under H_1 is

$$\ell_0\left(\pi_{i+}, \pi_{+j}\right) = \log\left(\frac{n!}{n_{11}!\cdots n_{IJ}!}\right) + \sum_i \sum_j n_{ij} \log\left(\pi_{i+}\pi_{+j}\right).$$

The log-likelihood under H_1 is

$$\ell_1\left(\pi_{ij}\right) = \log\left(\frac{n!}{n_{11}!\cdots n_{IJ}!}\right) + \sum_i \sum_j n_{ij} \log\left(\pi_{ij}\right).$$

Likelihood Ratio Test

The MLE under H_0 is

$$\hat{\pi}_{i+} = \frac{n_{i+}}{n},\ \hat{\pi}_{+j} = \frac{n_{+j}}{n}.$$

The MLE under H_1 is

$$\hat{\pi}_{ij} \quad = \quad \frac{n_{ij}}{n}.$$

The likelihood ratio test statistic is

$$G^2 = -2\left[\ell_0\left(\hat{\pi}_{i+}, \hat{\pi}_{+j}\right) - \ell_1\left(\hat{\pi}_{ij}\right)\right] \quad = \quad -2\sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}\log\left(\frac{n_{i+}n_{+j}/n}{n_{ij}}\right).$$

If H_0 holds, G^2 also converges in distribution to to chi-square with $(IJ-1)-(I-1)-(J-1) = (I-1)(J-1)$ degrees of freedom. A rule-of-thumb is that no more than 20% of $\hat{\mu}_{ij} < 5$.

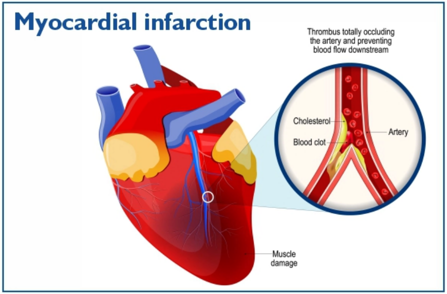
Pearson Chi-Square

The Pearson chi-square that tests H_0 : independence is

$$\begin{aligned} X^2 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left(\text{observed frequency}_{ij} - \text{expected frequency}_{ij}\right)^2}{\text{expected frequency}_{ij}} \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left(n_{ij} - n\hat{\pi}_{i+}\hat{\pi}_{+j}\right)^2}{n\hat{\pi}_{i+}\hat{\pi}_{+j}}. \end{aligned}$$

If H_0 holds, X^2 converges in distribution to to chi-square with $(I-1)(J-1)$ degrees of freedom. A rule-of-thumb is still that no more than 20% of $\hat{\mu}_{ij} < 5$.

Aspirin Use and Myocardial Infarction



Test independence

	Myocardial Infarction	
	Yes	No
Placebo	28	656
Aspirin	18	658

Fisher's Exact Test

For 2×2 tables, regardless of sampling, under the independence assumption, conditioning on both sets of marginal totals, the only free cell is n_{11}. It follows the hypergeometric distribution

$$P\left(n_{11} = t\right) = \frac{\binom{n_{1+}}{t}\binom{n_{2+}}{n_{+1} - t}}{\binom{n}{n_{+1}}}.$$

- For $H_0 : \theta = 1$ (independence) versus $H_1 : \theta > 1$, the Fisher's exact test uses the p-value $P\left(n_{11} \geq t_o\right)$ where t_o is the observed value of n_{11}.
- For $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$, there are different ways of computing the p-value. They lead to different p-values.

Example

Fisher's Tea Tasting Experiment

	Guess Poured First		
Poured First	Milk	Tea	Total
Milk	3	1	4
Tea	1	3	4
Total	4	4	8

Ordinality and Scoring

If our data are ordinal, using the above X^2 and G^2 are less ideal since they ignore ordinality of data.

To keep ordinality, many people choose to assign scores to the ordinal variables: $u_1 \leq u_2 \leq \cdots \leq u_I$ be the scores for the rows, and $v_1 \leq v_2 \leq \cdots \leq v_J$ be the scores for the columns. The scores are then treated as the values of the variables. However, this approach has several serious issues:

1. How shall we assign scores?
2. Are the distance between the assigned score actually reflect the "distance" between categories?

Ordinal Variables

Suppose that both X and Y are ordinal.

1. A pair of subjects is concordant if the subject ranked higher on X also ranks higher on Y.
2. A pair of subject is discordant if the subject ranking higher on X ranks lower on Y.

Age	Job satisfaction		
	1: Not satisfied	2: Satisfied	3: Very satisfied
1: < 30	34	53	88
2: $30 - 50$	80	174	304
3: > 50	29	75	172

Concordant/Discordant Pairs

Age	Job satisfaction		
	1: Not satisfied	2: Satisfied	3: Very satisfied
1: < 30	34	53	88
2: $30 - 50$	80	174	304
3: > 50	29	75	172

- Subject A belongs to $(1,1)$ and subject B belongs to $(2,2)$. The pair (A,B) is concordant.
- Subject A belongs to $(2,2)$ and subject B belongs to $(1,1)$. Also concordant.
- Subject A belongs to $(1,2)$ and subject B belongs to $(2,1)$. The pair (A,B) is discordant.
- Subject A belongs to $(2,1)$ and subject B belongs to $(1,2)$. Also discordant.

Probability of Concordant/Discordant

Suppose that we have two independent subjects A and B from a joint distribution $\{\pi_{ij}\}$.

1. The probability of a concordant pair is

$$\begin{aligned}\Pi_c &= \sum_{i,j}\{P[A=(i,j)]\,P[B=(h,k),\ h>i,\ k>j \mid A=(i,j)]\} \\ &\quad +\sum_{i,j}\{P[A=(i,j)]\,P[B=(h,k),\ h<i,\ k<j \mid A=(i,j)]\} \\ &= 2\sum_{i,j}\left\{\pi_{ij}\sum_{h>i}\sum_{k>j}\pi_{hk}\right\}.\end{aligned}$$

2. The probability of a discordant pair is

$$\Pi_d = 2\sum_{i,j}\left\{\pi_{ij}\sum_{h>i}\sum_{k<j}\pi_{hk}\right\}.$$

Gamma Coefficient

We define the Goodman-Kruskal's gamma as

$$\gamma = \frac{\Pi_c - \Pi_d}{\Pi_c + \Pi_d}.$$

1. γ has the range $-1 \leq r \leq 1$. It work in a similar way as the Pearson correlation coefficient.
2. If $\gamma > 0$ $(\Pi_c > \Pi_d)$, then it is more likely to have concordant pairs than discordant pairs (positive trend).
3. If $\gamma < 0$ $(\Pi_c < \Pi_d)$, then it is less likely to have concordant pairs than discordant pairs (negative trend).
4. If $\gamma = 0$, then no trend.
5. If X and Y are independent, then $\gamma = 0$. But $\gamma = 0$ does not mean independence.

Alternative Method

For ordinal data, we use the sample Goodman-Kruskal's gamma is

$$\hat{\gamma} = \frac{C - D}{C + D}$$

to check whether they have a monotone trend, where C is the total number of concordant pairs of observations, and D is the total number of discordant pairs of observations.

- If $\gamma = 0$, there is no trend between X and Y.
- For a large sample size, $\hat{\gamma}$ is approximately normal.

Example Sample Gamma Coefficient

The sample version of γ is

$$\hat{\gamma} = \frac{C - D}{C + D},$$

where C is the total number of concordant pairs and D is the total number of discordant pairs.

Compute $\hat{\gamma}$

	Job satisfaction		
Age	1: Not satisfied	2: Satisfied	3: Very satisfied
1: < 30	34	53	88
2: $30 - 50$	80	174	304
3: > 50	29	75	172

Wilcoxon Test or Mann-Whitney Test

Suppose that we have two random variables Y_0 and Y_1. The Wilcoxon test or the Mann-Whitney test tests whether

$$P(Y_0 > Y_1) = P(Y_0 < Y_1).$$

In the special case where we have a $2 \times J$ table $(I = 2)$ and the scores for X are $\{0, 1\}$. We have two groups, one group with $X = 0$ and another group with $X = 1$. Then the general idea is that

1. Assign ranks to the whole sample of size $n_{0+} + n_{1+}$.
2. Compute the sum of ranks assigned to the group $X = 0$.
3. If H_0 is not true, the sum of ranks tends to be either small or large.

The Kruskal-Wallis test generalizes the Mann-Whitney test to more than 2 groups. The Kruskal-Wallis test can be viewed as a non-parametric version of one-way ANOVA.

Be Careful With Their Hypotheses

	Y			
X	1	2	3	4
0	0.05	0.5	0.35302019	0.09697981
1	0.1666553	0.2833447	0.5000000	0.0500000