Exam - Fourier analysis

Department of Mathematics
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Exam in Fourier Analysis, 5 credits
1MA211
KandFy, KandMa, Fristående

Writing time: 08:00–13:00. Allowed equipments: writing materials, table of formulæ. There are 8 problems in this exam. You have to motivate every step in your solution to get the full score from a question.

To pass the exam you need at least one point on exercise 1b, 2 and 3 (or similar exercises). You can obtain the grades 3, 4 and 5 on the exam by the requirements given in the table below.

Grade	Requirements				
3	3 A	7 B	2 C		18 total
4	4 A	10 B	4 C		25 total
5	4 A	10 B	4 C	4*	32 total
Max	8 A	24 B	8 C	10*	40 total

Learning Outcomes:

- Basic concepts and theorems (A points)
- Basic numeracy skill (B points)
- Ordinary or Partial differential equations (C points)

1. (a) State the uniqueness theorem for the Laplace transform. 2 A

   (b) Solve the ODE

$$\begin{cases} y'(t) + y(t) = 3 \\ y(0) = 2 \end{cases}$$

   using some method that has been taught during the course. 3 C

2. Let f be an even, 1 periodic function with $f(x) = x^2$ for $0 \leq x < 1/2$.

   (a) Find the Fourier series of f. 2 B

   (b) Calculate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

   1 B

   (c) Calculate the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

   1 A, 1 B

3. Use the Fourier transform to calculate

$$\int_{\mathbb{R}} \frac{e^{ix}}{(x+1)^2+1}\,\mathrm{d}x$$

5 B

4. Calculate, using separation of variables, the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0,1), t \in (0,\infty) \\ u(0,t) = 1, u(1,t) = 2 & t \in (0,\infty) \\ u(x,0) = 1 & x \in (0,1) \end{cases}$$

5 C

5. (a) Assume that f is a function of at most polynomial growth. Write down the expression of how $f \in \mathscr{S}'(\mathbb{R})$ acts on a test function $\varphi \in \mathscr{S}(\mathbb{R})$. 2 A

   (b) Show that $xg = \mathrm{p.\,v.}\,\frac{1}{x}$ in $\mathscr{S}'(\mathbb{R})$, where

$$g(\varphi) := \lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi(x)-\varphi(0)}{x^2}\,\mathrm{d}x \text{ and } \mathrm{p.\,v.}\,\frac{1}{x}(\varphi) := \lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

3 B

6. (a) State Riemann-Lebesgue Lemma for the Fourier transform. 2 A

   (b) Prove that the Fourier transform of $f \in L^1(\mathbb{R})$ is continuous. Motivate every step in your proof! Hint: It is enough to show that $\lim_{\xi\to\xi_0} \hat{f}(\xi) = \hat{f}(\xi_0)$ using Lebesgue Dominated Convergence Theorem. 3 B

7.* Let $g \in \mathcal{C}^2(\mathbb{T})$ and define T_g as the mapping

$$T_g f(x) := \sum_{n=-\infty}^{\infty} \hat{g}(n)\hat{f}(n)e^{inx}$$

for $f \in L^2(\mathbb{T})$. **Show that** $\|T_g f\|_{L^2(\mathbb{T})} \leq C_g \|f\|_{L^2(\mathbb{T})}$, where C_g only depends on the function g.

1 A, 4 B

8.* Assume that the kernel $K_n(x)$ satisfies the following properties:

   (i) $K_n(x)$ is even and positive for all $n \in \mathbb{N}$.

   (ii) $\displaystyle\int_{\mathbb{T}} K_n(x)\,\mathrm{d}x = 1, \quad \forall n \in \mathbb{N}.$

   (iii) For every $\delta > 0$, $\displaystyle\lim_{n\to\infty} \int_\delta^\pi K_n(y)\,\mathrm{d}y = 0.$

Prove that $\lim_{n\to\infty} \|K_n * f - f\|_{L^1(\mathbb{T})} = 0$

Hint: for all $f \in L^1(\mathbb{T})$, $\lim_{y\to 0} \|f(x-y) - f(x)\|_{L^1_x(\mathbb{T})} = 0$.

5 B