

UPPSALA UNIVERSITET

LECTURE NOTES

# Complex Analysis

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## 1. INTRO

In this course, we shall study functions  $f : \mathbb{C} \rightarrow \mathbb{C}$  (or more generally,  $f : D \rightarrow \mathbb{C}$  where  $D \subseteq \mathbb{C}$ )

**Definition/Sats 1.1: Complex Number**

A *complex number* is a number of the form  $x + iy$ , where  $x, y \in \mathbb{R}$

Two complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  are said to be equal iff  $x_1 = x_2$  and  $y_1 = y_2$

**Anmärkning:**

The number  $x$  is called the *real part* ( $\operatorname{Re}(z) = x$ ) of the complex number, and  $y$  is called the *imaginary part* ( $\operatorname{Im}(z) = y$ ) of the complex number

**Anmärkning:**

The set of all complex numbers is denoted by  $\mathbb{C}$

**Anmärkning:**

$\mathbb{C}$  is the *smallest* field extension to  $\mathbb{R}$  that is algebraically closed.

**Anmärkning:**

$i^2 = -1$

1.1. Operations over  $\mathbb{C}$ .

We define the operations *addition* and *multiplication* of two complex numbers as follows:

**Definition/Sats 1.2: Addition of complex numbers**

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

**Definition/Sats 1.3: Multiplication of complex numbers**

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations,  $\mathbb{C}$  forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

**Definition/Sats 1.4: Complex conjugate**

The *complex conjugate* of a complex number  $z = x + iy$ , denoted by  $\bar{z}$ , is defined by  $\bar{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$
- $\overline{\bar{z}} = z$
- $z \cdot \bar{z} = |z|^2$
- $z^{-1} = \frac{\bar{z}}{|z|^2}$
- $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$

**Anmärkning:**

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

**Anmärkning:**

Multiplication by  $i$  is simply rotation by  $\frac{\pi}{2}$  counterclockwise.

**Definition/Sats 1.5**

Let  $z \in \mathbb{C}$ . Then there exists a  $w \in \mathbb{C}$  such that  $w^2 = z$  (where  $-w$  also satisfies this equation)

**Bevis 1.1**

Let  $z = a + bi$  and  $w = x + iy$  such that  $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then  $a = x^2 - y^2$  and  $b = 2xy$

We also know that  $|z| = a^2 + b^2 = |x^2 + y^2|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore,  $x^2 + y^2 = \sqrt{a^2 + b^2}$  and:

$$\left. \begin{aligned} x^2 - y^2 &= a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned} \right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left. \begin{aligned} -x^2 + y^2 &= -a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{aligned} \right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let  $\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$  and  $\beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$  and let  $\sqrt{\phantom{x}}$  denote the positive square root of positive real numbers.

If  $b$  is positive, then either  $x = \alpha, y = \beta$  or  $x = -\alpha, y = -\beta$

If  $b$  is negative, then either  $x = \alpha, y = -\beta$  or  $x = -\alpha, y = \beta$

Therefore, the equation has solutions  $\pm(\alpha + \mu\beta i)$  where  $\mu = 1$  if  $b \geq 0$  and  $\mu = -1$  if  $b < 0$

□

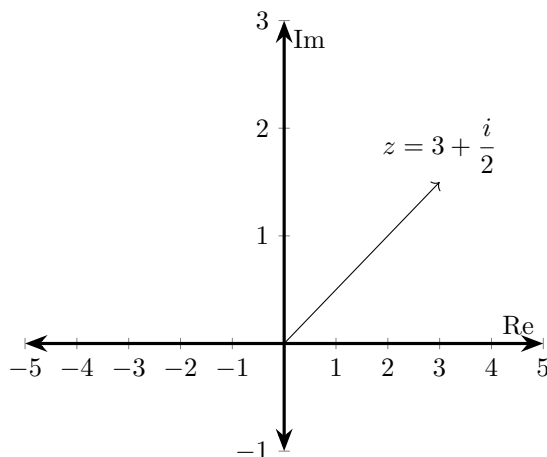
**Anmärkning:**

From the proof above, we can conclude the following:

- The square roots of a complex number are real  $\Leftrightarrow$  the complex number is real and positive
- The square roots of a complex number are purely imaginary  $\Leftrightarrow$  the complex number is real and negative
- The two square roots of a number coincide  $\Leftrightarrow$  the complex number is zero

## 1.2. Cartesian representation.

It is natural to represent a complex number  $z = x + iy$  as a tuple  $(x, y)$ , and we can therefore represent it in the standard cartesian plane:



### Anmärkning:

This is sometimes called the *complex plane*

#### Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number  $z = x + iy$  (geometrically the length of the vector), denoted by  $|z|$ , is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:

- $|z|^2 = z \cdot \bar{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

### Anmärkning:

Every  $z \in \mathbb{C}$  such that  $z \neq 0$  (that is,  $x \neq 0$  or  $y \neq 0$ ) has a multiplicative inverse  $\frac{1}{z}$  given by:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

#### Definition/Sats 1.7: Triangle inequality

For  $z_1, z_2 \in \mathbb{C}$ , it holds that  $|z_1 + z_2| \leq |z_1| + |z_2|$

#### Lemma 1.1: Reversed triangle inequality

For  $z_1, z_2 \in \mathbb{C}$ , it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

**Bevis 1.2**

$$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

$$\text{So that } |z_1| - |z_2| \leq |z_1 - z_2|$$

□

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
- $-\operatorname{Re}(z) \leq \operatorname{Re}(z) \leq |z|$
- $-\operatorname{Im}(z) \leq \operatorname{Im}(z) \leq |z|$
- $|\bar{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \dots + |w_n|^2}$

**1.3. Polar form.**

Let  $z = x + iy \neq 0$ . The point  $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$  lies on the unit circle, and hence there exists  $\theta$  such that:

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta)$$

Therefore  $z = x + iy$  can be written as:

$$z = r(\cos(\theta) + i \sin(\theta))$$

Where  $r = |z|$  is uniquely determined by  $z$ , while  $\theta$  is  $2\pi$ -periodic. This is called the *polar form* of  $z$  and just as the cartesian representation requires a tuple of information  $(|z|, \theta)$

**Definition/Sats 1.8: Argument**

The *argument* of a complex number  $z$ , denoted by  $\arg(z)$ , is the angle  $\theta$  between  $z$  and the real number line in the complex plane

**Anmärkning:**

Since the argument is  $2\pi$  periodic, the angle is usually given as  $\theta + k2\pi$   $k \in \mathbb{Z}$ , but we are only interested in  $\theta$

This  $\theta$  is called the *principal value* of  $\arg(z)$ , denoted by  $\operatorname{Arg}(z)$  and belongs to  $(-\pi, \pi]$

**Anmärkning:**

We are always allowed to change an angle by multiples of  $2\pi$ , the principal value argument is the angle after changing the argument such that it lies between  $(-\pi, \pi]$

**Anmärkning:**

A specification of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that  $\operatorname{Arg}(z)$  is "discontinuous" along the negative real axis. This is called a *branch-cut*

$$\text{Suppose } z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$$

Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 [(\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))] \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

**Anmärkning:**

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

#### 1.4. Exponential form.

##### Definition/Sats 1.9

For  $z = x + iy \in \mathbb{C}$ , let  $e^z = e^x(\cos(y) + i \sin(y))$

##### Anmärkning:

$e^{iy} = \cos(y) + i \sin(y) \quad y \in \mathbb{R}$  (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$\begin{aligned} e^z &= e^{x+iy} = e^x \cdot e^{iy} \\ e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \\ \Rightarrow e^{iy} &= 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos(\theta)} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin(\theta)} \\ \Rightarrow e^z &= e^x(\cos(\theta) + i \sin(\theta)) \end{aligned}$$

##### Anmärkning:

One can through comparing see that  $|e^z| = e^x$ , and that  $|e^{iy}| = 1$

##### Properties of the exponential form:

- $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$
- $e^z \neq 0 \quad \forall z \in \mathbb{C}$
- $x \in \mathbb{R} \Rightarrow e^x > 1$  if  $x > 0$  and  $e^x < 1$  if  $x < 0$
- $|e^{x+iy}| = e^x$
- $e^{i\pi/2} = i \quad e^{i\pi} = -1 \quad e^{3i\pi/2} = -i \quad e^{2i\pi} = 1$
- $e^z$  is  $2\pi$ -periodic
- $e^z = 1 \Leftrightarrow z = 2\pi ki \quad k \in \mathbb{Z}$

##### Definition/Sats 1.10: deMoivre's formula

For  $n \in \mathbb{Z}$ ,  $(r(\cos(\theta) + i \sin(\theta)))^n = r^n(\cos(n\theta) + i \sin(n\theta))$

#### 1.5. Logarithmic form.

In real analysis, we have defined the logarithm as the inverse of  $e^x$ . This has previously worked since for  $x \in \mathbb{R}$ ,  $e^x$  is injective.

The problem is that for  $e^z$  where  $z \in \mathbb{C}$ , it is not injective and should therefore not have an inverse.

Given  $z \in \mathbb{C} \setminus \{0\}$ , we define  $\ln(z)$  as the cut of all  $w \in \mathbb{C}$  whose image under the exponential form is  $z$ , i.e  $w = \ln(z) \Leftrightarrow z = e^w$ .

Here,  $\ln(z)$  is a *multivalued form*

We can use the fact that  $|z| = r = e^x$  to derive some interesting properties of the logarithm:

$$\begin{aligned} z &= r e^{i\theta} & w &= u + iv \\ \text{If } z &= e^w \Leftrightarrow r e^{i\theta} = e^u \cdot e^{iv} \\ \Leftrightarrow u &= \ln(r) = \ln(|z|) & v &= \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z} \end{aligned}$$

**Definition/Sats 1.11: Complex logarithm**

For  $z \neq 0$ , we define the complex logarithm for  $z \in \mathbb{C}$  as:

$$\begin{aligned}\ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\operatorname{Arg}(z) + k2\pi) \quad k \in \mathbb{Z}\end{aligned}$$



## 2. ELEMENTARY COMPLEX FUNCTIONS

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

## 2.1. Branches of the complex logarithm.

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of  $2\pi$ ).

If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

**Definition/Sats 2.1: Principal logarithm**

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\text{Ln}(z) = \ln(|z|) + i \cdot \text{Arg}(z)$$

**Anmärkning:**

We have essentially extended the "normal" logarithm, which is defined on  $(0, \infty)$ , to be defined on  $\mathbb{C} \setminus \{0\}$

**Anmärkning:**

The principal logarithm is discontinuous for negative reals, since their principal value argument is  $-\pi$ , but the principal value argument is discontinuous at  $-\pi$ . This is the so called *branch-cut*

**Anmärkning:**

Even though the principal logarithm is discontinuous for negative reals, it is not undefined. Any negative real number  $z$  will have  $\text{Arg}(z) = \pi$ , which the logarithm very much is defined for.

**Anmärkning:**

When branching, we do not necessarily have to pick  $(-\pi, \pi]$ , we can pick any interval  $(\alpha, \alpha + 2\pi]$ . This is usually denoted by  $\arg_\alpha$ .

## 2.2. Complex mappings.

One can think of a complex mapping  $f : \mathbb{C} \rightarrow \mathbb{C}$  as  $f(z) = f(x + iy) = w = u + iv$

Then it becomes clear which regions map to where by drawing them in their respective  $z$ -plane and  $w$ -plane.

## 2.3. Complex powers.

Given  $z \in \mathbb{C}$ , consider the following equation:

$$(1) \quad w^n = z$$

The set of all solutions  $w$  of (1) is denoted  $z^{1/n}$  and is called the *n-th root of z*.

**Anmärkning:**

If  $z = 0$ , then  $w = 0$

Suppose  $z \neq 0$ , then we may write  $w = |w| e^{i\alpha}$  and  $z = |z| e^{i\theta}$   
 By deMoivre's formula, (1) becomes:

$$|w|^n e^{in\alpha} = |z| e^{i\theta}$$

Then, the following follows:

$$\left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ n\alpha &= \theta + k2\pi \quad k \in \mathbb{Z} \end{aligned} \right\} \Leftrightarrow \left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ \alpha &= \frac{\theta}{n} + k \frac{2\pi}{n} \quad k \in \mathbb{Z} \end{aligned} \right\}$$

Notice now that every  $k \in \mathbb{Z}$  gives a solution to (1)

Since sine and cosine are both  $2\pi$ -periodic, then only  $k = 0, 1, \dots, n-1$  actually give *different* solutions  
 (since  $k = n \Rightarrow \alpha = \frac{\theta}{n} + n \frac{2\pi}{n}$ )

Suppose  $z \neq 0$ . For  $n \in \mathbb{Z}$  it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that  $\ln(z)$  attains.

It is also true, that for  $n = 1, 2, 3, \dots$ :

$$\frac{1}{z^n} = e^{\frac{1}{n} \ln(z)}$$

We can let  $n \in \mathbb{C}$ , and obtain the following definition:

#### Definition/Sats 2.2: Complex power

For  $\alpha \in \mathbb{C}$ , let:

$$z^\alpha = e^{\alpha \ln(z)} \quad z \neq 0$$

#### Anmärkning:

This makes  $z^\alpha$  a multivalued function, but it is possible to have a single-valued output from it.

#### Definition/Sats 2.3

Let  $a, b \in \mathbb{C}$  where  $a \neq 0$ . Then  $a^b$  is single-valued (does not depend on the choice of branch for the logarithm)  $\Leftrightarrow b \in \mathbb{Z}$

If  $b \in \mathbb{Q}$  and is in lowest form (that is,  $b = \frac{p}{q}$  where  $p, q$  have no common factors), then  $a^b$  has exactly  $q$  distinct values (the  $q$ :th roots of  $a^p$ )

If  $b \in \mathbb{C} \setminus \mathbb{Q}$ , then  $a^b$  has infinitely many values.

#### Bevis 2.1

Chose some interval (branch), say  $[0, 2\pi)$ , for the arg function and let  $\ln(z)$  be the corresponding branch of the logarithm. If we chose another branch, we would have  $\ln(a) + 2\pi kbi$  rather than  $\ln(a)$  (where  $k \in \mathbb{Z}$ )

Therefore,  $a^b = e^{b \ln(a) + 2\pi kbi} = e^{b \ln(a)} \cdot e^{2\pi ki}$

Notice that  $e^{2\pi kbi}$  stays the same regardless of  $b \in \mathbb{Z}$ , as long as it is an integer.

In the same way, it can be shown that  $e^{2\pi kip/q}$  has  $q$  distinct values if  $p, q$  have no common factor.

If  $b$  is irrational, and if  $e^{2\pi kbi} = e^{2\pi mbi}$ , then it follows that  $e^{(2\pi bi)(k-m)} = 1$ , and therefore  $b(k-m)$  is an integer.

Since  $b$  is irrational, then  $n - m = 0$

□

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up. We can chose to branch  $z^\alpha$ :

$$z^\alpha = e^{\alpha \text{Ln}(z)}$$

## 2.4. Trigonometric and Hyperbolic functions.

We have the following:

$$\left. \begin{aligned} e^{iy} &= \cos(y) + i \sin(y) \\ e^{-iy} &= \cos(y) - i \sin(y) \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \cos(y) &= \frac{e^{iy} + e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy} - e^{-iy}}{2i} \end{aligned} \right\}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

### Definition/Sats 2.4: Complex sine and cosine

For  $z \in \mathbb{C}$ , we define:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals. When defining them for complex numbers, we just extend their domain:

### Definition/Sats 2.5: Complex hyperbolic functions

For  $z \in \mathbb{C}$ , we define:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

- **Sine:**

$$\begin{aligned} \sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &\Rightarrow \frac{e^{-y}(\cos(x) + i \sin(x)) - e^y(\cos(x) - i \sin(x))}{2i} = \frac{(e^{-y} - e^y) \cos(x) + i(e^y - e^{-y}) \sin(x)}{2i} \\ &= \frac{(e^{-y} - e^y) \cos(x)}{2i} + \frac{(e^y - e^{-y}) \sin(x)}{2} \\ &\stackrel{i^{-1} = -i}{\implies} \underbrace{\frac{(e^y - e^{-y})}{2}}_{\sinh(y)} i \cos(x) + \underbrace{\frac{(e^y + e^{-y})}{2}}_{\cosh(y)} \sin(x) \end{aligned}$$

- **Cosine:**

$$\begin{aligned} \cos(x + iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} \\ &= \frac{e^{-y}(\cos(x) + i \sin(x)) + e^y(\cos(x) - i \sin(x))}{2} = \frac{\cos(x)(e^y + e^{-y}) + i(e^{-y} - e^y) \sin(x)}{2} \\ &= \underbrace{\frac{e^y + e^{-y}}{2}}_{\cosh(y)} \cos(x) - \underbrace{\frac{e^y - e^{-y}}{2}}_{\sinh(y)} i \sin(x) \end{aligned}$$

This leads us to the following:

**Definition/Sats 2.6: Addition formulas for complex trigonometric functions**

- $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$
- $\cos(x + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$

**Anmärkning:**

Both sine and cosine can be defined as the unique solution to an ODE, namely:

$$\begin{aligned} f''(x) + f(x) &= 0 & f(0) &= 0, f'(0) = 1 & f(x) &= \sin(x) \\ f''(x) + f(x) &= 0 & f(0) &= 1, f'(0) = 0 & f(x) &= \cos(x) \end{aligned}$$

**2.5. Mapping properties of  $\sin(z)$ .**

Let  $f(z) = \sin(z)$  in  $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$ , let  $A$  be the set of points allowed with respect to the above constraint and let  $B$  be the mapping of those points by  $\sin(A)$

**Claim:**  $f : A \rightarrow B$  is a bijective mapping

**Bevis 2.2**

Take a  $z \in \mathbb{C}$   $z = x + iy$   $-\frac{\pi}{2} < x < \frac{\pi}{2}$ .

Then:

$$f(z) = \sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$f(z) \in \mathbb{R} \Leftrightarrow \cos(x) \sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0$$

If  $y = 0$ , then:

$$f(z) = \sin(x) \cosh(y) = \sin(x) \in (-1, 1)$$

Therefore, if  $z \in A \Rightarrow f(z) \in B$ . Now we need to show that for any  $z \in B$ , there is a  $u$  such that  $f(u) = z$

Let  $u = \sin(x) \cosh(y)$ ,  $v = \cos(x) \sinh(y)$  and pick a vertical line at  $x = a \neq 0$

We will now consider the images of these lines:

$$\cosh(y) = \frac{u}{\sin(a)} \quad \sinh(y) = \frac{v}{\cos(a)}$$

$$(\cosh(y))^2 - (\sinh(y))^2 = 1 \Rightarrow \left( \frac{u}{\sin(a)} \right)^2 - \left( \frac{v}{\cos(a)} \right)^2 = 1$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line  $y = b \neq 0$

$$\sin(x) = \frac{u}{\cosh(b)} \quad \cos(x) = \frac{v}{\sinh(b)}$$

$$\cos^2(x) + \sin^2(x) = 1 \Rightarrow \left( \frac{u}{\cosh(b)} \right)^2 + \left( \frac{v}{\sinh(b)} \right)^2 = 1$$

This is a half-ellipse. Note that  $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

□

3. TOPOLOGY OF  $\mathbb{C}$ **Definition/Sats 3.1: Open disc**

The set  $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$  is called the *open-disc* with center  $z_0$  and radius  $r$

**Anmärkning:**

Since we have a strict inequality, it is open. If we had  $\leq$ , it would be a closed disc.

**Definition/Sats 3.2: Open subset**

A subset  $M$  of  $\mathbb{C}$  is called *open* if for every  $z_0 \in M$  there exists an  $r > 0$  such that  $D_r(z_0) \subseteq M$

**Definition/Sats 3.3: Interior point**

A point  $z_0 \in M$  is called an *interior-point* of  $M$  if there exists an  $r > 0$  such that  $D_r(z_0) \subseteq M$

**Definition/Sats 3.4: Boundary point**

A point  $z_0 \in \mathbb{C}$  is called a *boundary point* of  $M$  if  $\forall r > 0$  it holds that:

$$D_r(z_0) \cap M \neq \emptyset \quad \wedge \quad D_r(z_0) \cap M^c \neq \emptyset$$

**Anmärkning:**

The set of all interior points of  $M$  is denoted by  $\text{int}(M)$  and the set of all boundary points of  $M$  is denoted by  $\partial M$

The following equivalences hold:

- $M$  is closed  $\Leftrightarrow \partial M \subseteq M$
- $M$  is open  $\Leftrightarrow \partial M \subseteq M^c$
- $\mathbb{C}$  is clopen
- $\emptyset$  is clopen
- The union of any collection of open subsets of  $\mathbb{C}$  is open
- The intersection of any finite collection of open subsets of  $\mathbb{C}$  is open

**Definition/Sats 3.5: Closed set**

We say that a set  $X \subseteq \mathbb{C}$  is closed if its complement  $X^c$  is open

**Definition/Sats 3.6: Polygonal path**

A polygonal path  $P$  (sometimes called piecewise linear curve) is a curve specified by a sequence of points  $(A_1, A_2, \dots, A_n)$ .

The curve itself consists of line segments connecting the consecutive points.

**Definition/Sats 3.7: polygonal-path-connected open set**

An open set  $M$  is called *polygonal-path-connected* if every pair of points  $z_1, z_2 \in M$  can be connected by a polygonal path contained in  $M$

**Anmärkning:**

Some would call this just path-connected, or even just connected. This works in  $\mathbb{R}^n$  (recall that  $\mathbb{C} \cong \mathbb{R}^2$ ). Topologically speaking, polygonal-path-connectedness  $\implies$  path-connectedness

**Anmärkning:**

A set  $X$  is connected  $\Leftrightarrow$  the only subsets of  $X$  which are clopen are  $\emptyset$  and  $X$

**Anmärkning:**

One can assume the polygonal paths to have segments parallel to the ordinate ones.

**Anmärkning:**

An open connected set is called a *domain*

**Definition/Sats 3.8**

Suppose that  $u(x, y)$  is a real-valued function defined in a domain  $D \subseteq \mathbb{R}$

Also suppose that:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} =$$

in all of  $D$ . Then  $u$  is contained in  $D$

**Definition/Sats 3.9: Simply connected**

A domain  $D \subseteq \mathbb{C}$  is called *simply connected* if every closed curve in  $D$  can be, within  $D$ , continuously deformed to a point

**Anmärkning:**

Topologically speaking,  $D$  is homotopic to a point.

**Definition/Sats 3.10: Non-connectedness**

A set  $A \subseteq \mathbb{C}$  is *not connected* if there are open sets  $U$  and  $V$  such that:

- $A \subseteq U \cup V$
- $A \cap U \neq \emptyset$  and  $A \cap V \neq \emptyset$

### 3.1. Limits and Continuity.

#### Definition/Sats 3.11: Complex limit

A sequence  $\{z_n\}_{n=1}^{\infty}$  of complex numbers is said to have the limit  $z_0$  (*converges to*  $z_0$ ) if for every given  $\varepsilon > 0$ , there exists an integer  $N \geq 1$  such that

$$|z_n - z_0| < \varepsilon \quad \forall n \geq N$$

We write this as:

$$\lim_{n \rightarrow \infty} z_n = z_0$$

#### Anmärkning:

Every cauchy sequence in  $\mathbb{C}$  converges.

#### Anmärkning:

$z_n \rightarrow z_0 \Leftrightarrow \operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z_0)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z_0)$

This follows from  $|x|, |y| \leq \sqrt{x^2 + y^2} \leq |x| + |y|$

#### Definition/Sats 3.12

Let  $f$  be a function defined in a punctured neighborhood of  $z_0$

We say that  $f$  has the limit  $w_0$  as  $z \rightarrow z_0$ , if for every given  $\varepsilon > 0$  there exists  $\delta > 0$  such that:

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

We write this as:

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

#### Anmärkning:

If a limit exists, it is unique.

#### Definition/Sats 3.13

For  $z = x + iy$ , let:

$$u(x, y) = \operatorname{Re}(f(z)) \quad v(x, y) = \operatorname{Im}(f(z))$$

Let  $z_0 = x_0 + iy_0$  and  $w_0 = u_0 + iv_0$

Then the following holds:

$$\lim_{z \rightarrow z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \end{cases}$$

#### Definition/Sats 3.14: Continuous function

Let  $f$  be a function defined in a neighborhood of  $z_0$ .

$f$  is said to be continuous at  $z_0$  if:

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

A function  $f$  is said to be *continuous on the (open) set*  $M$  if it is continuous at each point of  $M$

**Anmärkning:**

The following statements are equivalent (for  $f : A \rightarrow \mathbb{C}$ ):

- $f$  is continuous
- The inverse image of every closed set is closed relative to  $A$
- The inverse image of every open set is open relative to  $A$
- The image set  $f(A)$  is connected

Assume  $\lim_{z \rightarrow z_0} f(z) = A$  and  $\lim_{z \rightarrow z_0} g(z) = B$

The following properties from the real limit hold for the complex limit:

- $\lim_{z \rightarrow z_0} (f(z) \pm g(z)) = A \pm B$
- $\lim_{z \rightarrow z_0} f(z)g(z) = AB$
- $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad B \neq 0$

**Anmärkning:**

If  $f, g$  are continuous at  $z_0$ , then so are  $f \pm g$  and  $fg$ . The quotient is only continuous if  $g(z_0) \neq 0$

**Anmärkning:**

Constant functions, polynomials, and rational functions (whenever the denominator is non-zero) are all continuous in  $\mathbb{C}$

**3.2. The complex derivative.**

Analogous to the real case, we also have the following:

**Definition/Sats 3.15: Differentiability**

Let  $f$  be a complex-valued function defined in a neighborhood of  $z_0$ .

We say that  $f$  is differentiable at  $z_0$  if the limit:

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

The limit is called the *derivative* of  $f$  at  $z_0$ , and is denoted by  $f'(z_0)$  or  $\frac{df}{dz}(z_0)$

**Anmärkning:**

Since  $\Delta z$  is a complex number, it can approach 0 from different directions. In order for the derivative to exist, the results must be independent of the direction of which  $\Delta z$  approaches 0 (i.e., approaches 0 from all directions)

**Anmärkning:**

If  $X$  is an open connected set and  $a, b \in X$ , then there is a differentiable path  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = a$  and  $\gamma(1) = b$

**Example:**

The function  $f(z) = \bar{z}$  is nowhere differentiable since:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{z_0 + \Delta z} - \bar{z}_0}{\Delta z} = \frac{\bar{\Delta z}}{\Delta z} = \frac{\bar{\Delta x} + i\bar{\Delta y}}{\Delta x + i\Delta y}$$

As  $\Delta z \rightarrow 0$  from the  $x$ -direction (real-line), the limit becomes  $\frac{\bar{x}}{x} = 1$

However, as we approach from the  $y$ -direction (complex axis), the limit becomes  $\frac{\bar{iy}}{iy} = \frac{-y}{y} = -1$

Since  $x, y$  were chosen arbitrarily, this applies to all  $x, y$ . Since the limits did not match, it is not differentiable and at no point.

Of course, all the properties from the real case hold here as well.

Suppose  $f, g$  are differentiable at  $z$ , then:

- $(f \neq g)'(z) = f'(z) \neq g'(z)$



- $(cf)'(z) = cf'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f \circ g)'(z) = f'(g(z))g'(z)$

### 3.3. Analytic functions.

#### Definition/Sats 3.16: Analytic function

A complex-valued function  $f$  is said to be *analytic* in an open set  $G$  if  $f$  is differentiable at every point in  $G$ .

We say that  $f$  is *analytic at  $z_0$*  if  $f$  is differentiable in a neighborhood of  $z_0$ .

#### Anmärkning:

If  $f$  is analytic in all of  $\mathbb{C}$ , then  $f$  is said to be *entire* (or *holomorphic*).

#### Definition/Sats 3.17: wobgowegb

If an entire function  $f(z)$  has a root at  $w$ , then:

$$\lim_{z \rightarrow w} \frac{f(z)}{(z - w)}$$

is an entire function.

## 4. CAUCHY-RIEMANN'S EQUATIONS

Suppose  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  is differentiable at  $z_0 = x_0 + iy_0$

Then:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}$$

1) Let  $\Delta z = \Delta x$  (i.e  $\Delta y = 0$ ):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

2) Let  $\Delta z = i\Delta y$  (i.e  $\Delta x = 0$ ):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

It must therefore hold that:

$$u_x + iv_x = -iu_y + v_y$$

This leads to the Cauchy-Riemann equations:

$$\left. \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \right\}$$

We have therefore arrived at the following:

**Definition/Sats 4.1**

A necessary condition for  $f = u + iv$  to be differentiable at  $z_0 = x_0 + iy_0$  is that the Cauchy-Riemann equations are satisfied at  $(x_0, y_0)$

**Anmärkning:**

We also saw that if  $f$  is differentiable at the point  $z_0$ , then the derivative is given by:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

The following provides a sufficient condition for Differentiability:

**Definition/Sats 4.2**

Suppose that  $f = u + iv$  is defined in a open set  $G$  containing  $z_0 = x_0 + iy_0$ .

Suppose also that  $u_x, u_y, v_x, v_y$  exists in  $G$  and are continous at  $(x_0, y_0)$ , and satisfy the Cauchy-Riemann equations at  $(x_0, y_0)$

Then  $f$  is differentiable at  $z_0$

**Anmärkning:**

Cauchy-Riemann equations +  $u, v \in C^1 \Rightarrow f$  is differentiable

### Bevis 4.1

In view of the continuity of the first partial derivatives at  $(x_0, y_0)$ , it holds that:

$$\begin{aligned} u(x_0 + \Delta x, y_0 + \Delta y) &= u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_1(\Delta x, \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) &= v(x_0, y_0) + v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_2(\Delta x, \Delta y) \end{aligned}$$

Where  $\rho_1, \rho_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$

Then:

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= u_x(x_0, y_0)\Delta x + \underbrace{u_y(x_0, y_0)}_{= -v_x(x_0, y_0)}\Delta y + i(v_x(x_0, y_0)\Delta x + \underbrace{v_y(x_0, y_0)}_{= u_x(x_0, y_0)}\Delta y) \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \\ &\stackrel{\text{CR-eq.}}{=} u_x(x_0, y_0)\Delta z + iv_x(x_0, y_0)\Delta z + |\Delta z|(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \end{aligned}$$

Since  $\rho_1, \rho_2 \rightarrow 0$  as  $\Delta z \rightarrow 0$ , it follows that:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is equal to  $u_x(x_0, y_0) + iv_x(x_0, y_0)$  □

#### 4.1. Inverse mappings.

Suppose  $f = u + iv$  is analytic in a domain  $D$  (with  $f'$  continuous).

Consider the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

As a mapping of  $D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Its Jacobian matrix:

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

has determinant:

$$\det(J_f) = u_x v_y - u_y v_x \stackrel{\text{CR-eq.}}{=} u_x^2 + v_x^2 = |f'(z)|^2$$

The inverse function then leads to the following:

#### Definition/Sats 4.3: Inverse function theorem

Suppose  $f(z)$  is analytic on a domain  $D$  with  $f'(z) \neq 0$  continuous.

Then there is a neighborhood  $U$  of  $z_0$  and a neighborhood  $V$  of  $f(z_0)$  such that  $f : U \rightarrow V$  is bijective, and the inverse function  $f^{-1} : V \rightarrow U$  is analytic with derivative:

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \quad w = f(z)$$

## 5. HARMONIC FUNCTIONS

**Definition/Sats 5.1: Harmonic function**

A real-valued function  $\phi(x, y)$  is said to be *harmonic* in a domain  $D$  if  $\phi \in C^2(D)$  and  $\phi$  satisfies Laplace's equations:

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

in  $D$

**Definition/Sats 5.2**

Suppose  $f = u + iv$  is analytic in a domain  $D$ . Then  $u, v$  are harmonic in  $D$

**Bevis 5.1**

One can show that  $u, v \in C^\infty$ :

$$\begin{aligned} u_x = v_y &\Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x &\Rightarrow u_{yy} = -v_{xy} \end{aligned}$$

As  $v_{yx} = v_{xy}$ , we have  $u_{xx} + u_{yy} = 0$

Similarly,  $v_{xx} + v_{yy} = 0$

□

**Definition/Sats 5.3: Harmonic Conjugacy**

If  $u$  is harmonic in a domain  $D$  and  $v$  is a harmonic function in  $D$  such that  $u + iv$  is analytic in  $D$ , then we say that  $v$  is a *harmonic conjugate* of  $u$  in  $D$

**Definition/Sats 5.4**

If  $u$  is harmonic in a simply connected domain  $D \subseteq \mathbb{C}$ , then there exists a harmonic conjugate  $v$  of  $u$  in  $D$ , and  $v$  is unique up to addition of a real constant

**Bevis 5.2**

Suppose  $u$  is harmonic in  $D \subseteq \mathbb{C}$

Consider the vector-field  $\overline{F} = (-u_y, u_x) \in C^1(0)$ .

Note that:

$$\frac{\partial F_1}{\partial y} = -u_{yy} \stackrel{u \text{ harm.}}{=} u_{xx} = \frac{\partial F_2}{\partial x}$$

Since  $D$  is simply connected  $\Rightarrow \overline{F}$  is conservative  $\Rightarrow \exists v : \nabla v = \overline{F}$ , i.e.  $(v_x, v_y) = (-u_y, u_x)$

$$\Rightarrow \left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \Rightarrow f = u + iv \text{ is analytic in } D$$

If  $\bar{v}$  is another harmonic conjugate, then:

$$\begin{aligned} \bar{v}_x &= -u_y = v_x \\ \bar{v}_y &= u_x = v_y \\ \Rightarrow \nabla(v - \bar{v}) &= \bar{0} \Rightarrow v - \bar{v} = c \in \mathbb{C} \end{aligned}$$

□

**Anmärkning:**

A vector field is conservative if it is the gradient of some function.

It has the property that its line integral is path independent.

## 6. CONFORMAL MAPPINGS

Let  $D$  be a domain in  $\mathbb{C}$ ,  $z_0 \in D$ .

Suppose  $f : D \rightarrow \mathbb{C}$  is analytic with  $f'(z_0) \neq 0$ . Let  $\gamma(t) = x(t) + iy(t)$  be a  $C^1$ -curve in  $D$  through  $z_0 = \gamma(0)$  with  $\gamma'(0) \neq 0$ . Then  $(f \circ \gamma)(t) = f(\gamma(t))$  is a  $C^1$ -curve through  $(f \circ \gamma)(0) = f(z_0)$ .

Moreover,

$$\begin{aligned} (f \circ \gamma)'(0) &= \frac{d}{dt} f(\gamma(t))|_{t=0} = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \cdot \frac{\gamma(t) - \gamma(0)}{t} = f'(z_0) \gamma'(0) \end{aligned}$$

From this, we can conclude  $(f \circ \gamma)'(0) = f'(z_0) \gamma'(0)$  is a tangent vector to  $f \circ \gamma$  at  $f(z_0)$

Note that  $\arg(f \circ \gamma)'(0) = \arg(f'(z_0) \gamma'(0))$

If  $\gamma_1$  and  $\gamma_2$  are two  $C^1$ -curves which intersect at  $z_0$ , then the angle from  $(f \circ \gamma_1)'(0)$  to  $(f \circ \gamma_2)'(0)$  is the same as the angle from  $\gamma_1'(0)$  to  $\gamma_2'(0)$

**Definition/Sats 6.1: Conformal  $C^1$ -mapping**

A  $C^1$ -mapping  $f : D \rightarrow \mathbb{C}$  is said to be *conformal* at  $z_0$  if it satisfies the above paragraph.

If  $f$  maps  $D$  bijectively onto  $V$ , and if  $f$  is conformal at one point  $z_0 \in D$ , we call  $f : D \rightarrow V$  a *conformal mapping*

**Definition/Sats 6.2**

If  $f$  is analytic at  $z_0$  and  $f'(z_0) \neq 0$ , then  $f$  is conformal at  $z_0$

**Anmärkning:**

One can in fact prove the converse of this theorem.

## 7. STEREOGRAPHIC PROJECTION

Consider the unit sphere  $S \in \mathbb{R}^3$ .

Given any point  $P = (x_1, x_2, x_3) \in S$  other than the north pole  $N = (0, 0, 1)$ , we draw the line through  $N$  and  $P$ .

We define the *stereographic projection* of  $P$  to be the point  $z = x + iy \in \mathbb{C} \sim (x, y, 0)$ , where the line intersects the plane  $x_3 = 0$ . Then the following holds:

$$(x, y, 0) = (0, 0, 1) + t[(x_1, x_2, x_3) - (0, 0, 1)]$$

Where  $t$  is given by  $1 + t(x_3 - 1) = 0 \Leftrightarrow t = \frac{1}{1 - x_3}$ . We arrive at the following:

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$$

Conversely, given  $z = x + iy \in \mathbb{C} \sim (x, y, 0)$  the line through  $N$  and  $z$  is given by:

$$(x_1, x_2, x_3) = (0, 0, 1) + t[(x, y, 0) - (0, 0, 1)] \quad t \in \mathbb{R}$$

**Anmärkning:**

The line intersects  $S$  when:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1 \\ \Leftrightarrow (tx)^2 + (ty)^2 + (1 - t)^2 &= 1 \\ \Leftrightarrow t^2(x^2 + y^2 + 1) - 2t &= 0 \\ \Leftrightarrow t = 0 \vee t = \frac{2}{x^2 + y^2 + 1} &= \frac{2}{|z|^2 + 1} \end{aligned}$$

This corresponds to  $P = N$  or:

$$P = \left( \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

Thus, stereographic projections  $s : S \setminus N \rightarrow \mathbb{C}$  define a bijection.

Letting  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  denote the *extended complex plane* and define  $s(N) = \infty$ , then  $s$  becomes a bijective map from  $S$  onto  $\hat{\mathbb{C}}$

**Definition/Sats 7.1**

Under stereographic projections, circles on  $S$  correspond to circles and lines in  $\mathbb{C}$

**Anmärkning:**

We therefore call circles and lines in  $\mathbb{C}$  "circles" in  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , where lines are considered as "circles through  $\infty$ "



**Bevis 7.1**

The general equation for a circle or line in the  $z = x + iy$  plane is:

$$A(x^2 + y^2) + Cx + Dy + E = 0$$

Using  $z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$ , we get:

$$\begin{aligned} A \left( \left( \frac{x_1}{1 - x_3} \right)^2 + \left( \frac{x_2}{1 - x_3} \right)^2 \right) + \frac{Cx_1}{1 - x_3} + \frac{Dx_2}{1 - x_3} + E &= 0 \\ \Leftrightarrow A(x_1^2 + x_2^2) + Cx_1(1 - x_3) + Dx_2(1 - x_3) + E(1 - x_3)^2 &= 0 \end{aligned}$$

Using  $x_1^2 + x_2^2 + x_3^2 = 1$ , we get:

$$A(1 - x_3^2) + Cx_1(1 - x_3) + Dx_2(1 - x_3) + E(1 - x_3)^2 = 0$$

Dividing by  $1 - x_3$  yields:

$$\begin{aligned} A(1 + x_3) + Cx_1 + Dx_2 + E(1 - x_3) &= 0 \\ \Leftrightarrow Cx_1 + Dx_2 + (A - E)x_3 + A + E &= 0 \end{aligned}$$

This is the equation for a plane in  $\mathbb{R}^3$ , which intersects  $S$  in a circle □

## 8. MÖBIUS TRANSFORMATIONS

**Definition/Sats 8.1: Moebius transformation**

A *Möbius transformation* (sometimes called *fractional linear transformation*) is a conformal mapping of the form:

$$T(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C}$$

Where  $ad - bc \neq 0$  (we do not want  $T$  as constant)

**Anmärkning:**

If  $c = 0$ , we let  $T(\infty) = \infty$ . Then  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is bijective

If  $c \neq 0$ , then:

$$T : \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\} \rightarrow \mathbb{C} \setminus \left\{ \frac{a}{c} \right\}$$

is a bijection. Letting  $T\left(-\frac{d}{c}\right) = \infty$ , and  $T(\infty) = \frac{a}{c}$ , we extend  $T$  to a bijective map  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$

The inverse is found by solving:

$$w = T(z)$$

which gives:

$$z = T^{-1}(w) = \begin{cases} \frac{-dw + b}{cw - a}, & \text{if } w \neq \frac{a}{c} \\ \infty & \text{if } w = \frac{a}{c} \\ \frac{-d}{c} & \text{if } w = \infty \end{cases}$$

**Anmärkning:**

If we interpret  $\frac{a}{c}$  and  $-\frac{d}{c}$  as  $\infty$ , it also holds for  $c = 0$

**Definition/Sats 8.2**

Any conformal map from  $D = \{z : \text{such that } |z| < 1\}$  to  $D$  is a Möbius transformation of the form:

$$T(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}$$

For some fixed  $z_0 \in D$  and  $\theta \in [0, 2\pi)$ , and any  $T$  of this form is a conformal map of  $D$  onto  $D$

**Anmärkning:**

$$\begin{aligned} T'(z) &= \frac{d}{dz} \left( \frac{az + b}{cz + d} \right) = \frac{a(cz + d) - (az + b) \cdot c}{(cz + d)^2} \\ &= \frac{ad - bc}{(cz + d)^2} \neq 0 \end{aligned}$$

Thus  $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is conformal

**Anmärkning:**

If:

$$\begin{aligned} T(z) &= \frac{az + b}{cz + d} & S(z) &= \frac{\alpha z + \beta}{\gamma z + \delta} \\ \Rightarrow (S \circ T)(z) &= \frac{\alpha T(z) + \beta}{\gamma T(z) + \delta} \\ &= \frac{\alpha \left( \frac{az + b}{cz + d} \right) + \beta}{\gamma \left( \frac{az + b}{cz + d} \right) + \delta} = \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)} \end{aligned}$$

This shows that compositions of Moebius transformations are Möbius transformations.

**Anmärkning:**

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

**Lemma 8.1**

If a Moebius transformation  $T$  has more than two fixed points in  $\hat{\mathbb{C}}$  ( $z_0$  is a fixpoint if  $T(z_0) = z_0$ ), then  $T(z) = z \forall z \in \hat{\mathbb{C}}$

**Bevis 8.1**

If  $c = 0$ , then  $T(z) = \frac{az + b}{d}$ , so:

$$T(z) = z \Leftrightarrow \frac{az + b}{d} = z \Leftrightarrow (a - d)z + b = 0$$

So  $T$  has at most one fixed point in  $\mathbb{C}$  unless  $a = d$  and  $b = 0 \Leftrightarrow T(z) = z \forall z \in \mathbb{C}$

So if  $c = 0$ ,  $T$  has at most 2 fixed points in  $\widehat{\mathbb{C}}$  ( $T(\infty) = \infty$ ) unless  $T(z) = z \forall z \in \mathbb{C}$

If  $c \neq 0$ , then:

$$\begin{aligned} T(z) = z &\Leftrightarrow \frac{az + b}{cz + d} = z \\ &\Leftrightarrow cz^2 + (d - c)z - b = 0 \end{aligned}$$

So  $T$  has at most 2 fixed points in  $\mathbb{C}$  (and  $T(\infty) = \frac{a}{c} \neq \infty$ ) unless  $c = 0, a = d, b = 0$

This contradicts  $c \neq 0$  □

**Definition/Sats 8.3**

If  $S, T$  are Möbius transformations such that  $S(z_i) = T(z_i)$  at three different points  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ , then  $S = T$

**Bevis 8.2**

If  $S(z_i) = T(z_i)$  for  $i = 1, 2, 3$ , then the Möbius transformation  $T^{-1} \circ S$  has at least 3 fixed points. By the previous lemma:

$$T^{-1} \circ S(z) = z \quad \forall z \in \widehat{\mathbb{C}}$$

i.e  $S(z) = T(z) \forall z \in \widehat{\mathbb{C}}$  □

**Anmärkning:**

Particular cases of the Möbius transformation are:

- $T(z) = z + b$  (*translation*)
- $T(z) = az = |a|e^{i\arg(a)z}$  (*rotation & magnification*)
- $T(z) = \frac{1}{z}$  (*inversion*)

**Anmärkning:**

If  $c \neq 0$ :

$$T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2} \frac{1}{z + \frac{d}{c}}$$

This means that every Möbius transformation is a composition of translations, rotations, magnifications, and inversions.

**Definition/Sats 8.4**

Every Möbius transformation maps "circles" onto "circles"

**Anmärkning:**

Recall that a "circle" in  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is a circle or line in  $\mathbb{C}$ . A line in  $\mathbb{C}$  is a "circle" through  $\infty$  in  $\widehat{\mathbb{C}}$

**Bevis 8.3**

It is easy to see that translations and rotations/magnifications map circles onto circles and line onto lines. This gives enough to prove that inversion:

$$T(z) = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} = u + iv$$

maps circles onto circles

A circle in  $\hat{\mathbb{C}}$  has the equation:

$$\begin{aligned} A(x^2 + y^2) + Cx + Dy + E &= 0 \\ \Leftrightarrow A + C \frac{x}{x^2 + y^2} + D \frac{y}{x^2 + y^2} + E \frac{1}{x^2 + y^2} &= 0 \\ \Leftrightarrow E(u^2 + v^2) + Cu - Dv + A &= 0 \end{aligned}$$

□

Given a "circle"  $C_z$  in the  $z$ -plane and a "circle"  $C_w$  in the  $w$ -plane, can one find a Moebius transformation  $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  such that  $T(C_z) = C_w$ ? Yes!

## 8.1. The cross-ratio.

**Definition/Sats 8.5: Cross-ratio**

Let  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$  be distinct and put:

$$(z, z_1, z_2, z_3) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \in \widehat{\mathbb{C}}$$

If some of the  $z_i$  is  $\infty$ , the right hand side should be interpret as:

$$(z, z_1, z_2, z_3) = \begin{cases} \frac{z_2 - z_3}{z - z_3} & \text{if } z_1 = \infty \\ \frac{z - z_3}{z - z_1} & \text{if } z_2 = \infty \\ \frac{z - z_3}{z_2 - z_1} & \text{if } z_3 = \infty \end{cases}$$

$(z, z_1, z_2, z_3)$  is called the *cross-ratio* of the four points

**Anmärkning:**

$S(z) = (z, z_1, z_2, z_3)$  is a Möbius transformation such that:

$$S(z_1) = 0 \quad S(z_2) = 1 \quad S(z_3) = \infty$$

By an earlier remark, this is the unique Möbius transformation mapping  $z_1, z_2, z_3$  to  $0, 1, \infty$

**Definition/Sats 8.6**

Given a tripple  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$  of distinct points, and another tripple  $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$  of distinct points, then there is a unique Möbius transformation  $T$  such that  $T(z_i) = w_i$

The mappings  $w = T(z)$  is found by solving the cross-ratio equation:

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

**Bevis 8.4**

By an earlier remark, there is at most one such mapping. We now prove that there is exactly one by contradicting it.

Put  $S(z) = (z, z_1, z_2, z_3)$ ,  $U(w) = (w, w_1, w_2, w_3)$ :

$$\Rightarrow T(z) = (U^{-1} \circ S)(z) = U^{-1}(S(z))$$

$U^{-1}(S(z))$  is a Möbius transformation such that:

$$T(z_1) = U^{-1}(S(z_1)) = U^{-1}(0) = w_1$$

$$T(z_2) = \dots$$

$$\vdots$$

Then:

$$w = T(z) \Leftrightarrow w = U^{-1}(S(z)) \Leftrightarrow U(w) = S(z)$$

$$\Leftrightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

□

This theorem can be used to construct a  $T$  as above, mapping  $C_z$  to  $C_w$

Let  $z_1, z_2, z_3$  be three distinct points on a circle  $C_z$  in  $\widehat{\mathbb{C}}$ . Note that  $C_z$  is *oriented* by the order of these points, that is  $C_z$  acquires an orientation by proceeding through  $z_1, z_2, z_3$  in succession

Since a Möbius transformation is conformal, it maps the region to the left of  $C_z$ , oriented by  $z_1, z_2, z_3$ , to the region left of  $C_w = T(C_z)$  oriented by  $w_1, w_2, w_3$

## 8.2. Symmetry-preserving property.

Two points  $z_1$  and  $z_2$  are said to be *symmetric* with respect to a line  $L$  if  $L$  is the perpendicular bisector of the line-segment joining  $z_1$  and  $z_2$

This means that every circle or line through  $z_1$  and  $z_2$  intersects  $L$  orthogonally

### Definition/Sats 8.7

Two points  $z_1$  and  $z_2$  are said to be *symmetric* with respect to a circle  $C$  if every circle or line through  $z_1$  and  $z_2$  intersects  $C$  orthogonally

In particular, the center  $a$  of  $C$  and  $\infty$  are symmetric with respect to  $C$

### Definition/Sats 8.8: Symmetry principle

Let  $C_z$  be a circle or line in the  $z$ -plane and  $w = T(z)$  be any Möbius transformation. Then two points  $z_1$  and  $z_2$  are symmetric with respect to  $C_z$  if and only if their images  $w_1 = T(z_1)$  and  $w_2 = T(z_2)$  are symmetric with respect to the image  $C_w = T(C_z)$  under  $T$ .

### Bevis 8.5

"Two points are symmetric with respect to a given circle if and only if every circle containing the points intersects the given circle orthogonally" is a re-formulation of the theorem.

Möbius transformations preserve the class of circles, and they also preserve orthogonality. Hence, they preserve the symmetric condition.  $\square$

## 9. DIRICHLET PROBLEMS

We have previously discussed harmonic function over a domain  $D$ . These have many applications in solving Dirichlets problem.

The formulation of Dirichlets problems are as follows:

**Definition/Sats 9.1: Dirichlet problems**

Let  $A$  be an open and bounded region, and let  $u_0$  be a given continuous function on  $\partial A$ , find a real-valued function on  $\bar{A}$  that is continuous on  $\bar{A}$  and harmonic on the entire region  $A$ .

Of course, on  $\partial A$ ,  $u = u_0$

Next step is to start pondering whether or not there exists solutions, and whether or not these solutions are unique.

There are theorems that state if  $\partial A$  is "sufficiently smooth" then there is always a solution. These theorems are difficult, but we can show that when there is a solution it is unique.

**Definition/Sats 9.2: Uniqueness of solutions**

Solution to the Dirichlet problems are unique.

**Bevis 9.1**

We go about using the standard proof method, assume we have two solutions  $u, \bar{u}$ .

Since they are solutions, they are both harmonic and equal to  $u_0$  on  $\partial A$ , therefore let  $\phi = u - \bar{u}$ .

Then,  $\phi = 0$  on  $\partial A$ . We now need to show that  $\phi = 0$  on  $A$ .

By the maximum-principle for harmonic functions,  $\phi(x, y) \leq 0$ , and by the minimum-principle for harmonic functions  $\phi(x, y) \geq 0$ . This implies  $\phi = 0 = u - \bar{u} \Leftrightarrow u = \bar{u}$   $\square$

**9.1. Standard cases.**

This can be easily solved in some standard cases:

$$\bullet \begin{cases} \nabla^2 \phi = 0 \text{ in } D \\ \phi(a, y) = A \\ \phi(b, y) = B \end{cases} \quad \text{Let } \phi(x, y) = \alpha x + \beta, \text{ choose } \alpha, \beta \text{ such that}$$

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{B - A}{b - a} \\ \beta = A - \frac{a(B - A)}{b - a} = \frac{AB - aB}{b - a} \end{cases}$$

$$\Rightarrow \phi(x, y) = \frac{(B - A)x + Ab - aB}{b - a}$$

$$\bullet \phi(x, y) = \frac{2}{\pi} \text{Arg}(z) = \frac{2}{\pi} \arctan\left(\frac{y}{x}\right)$$

$$\bullet \phi(x, y) = \alpha \text{Arg}(z) + \beta \text{ leads to:}$$

$$\begin{cases} \alpha \frac{\pi}{2} + \beta = A \\ -\alpha \frac{\pi}{2} + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{A - B}{\pi} \\ \beta = \frac{A + B}{2} \end{cases}$$

$$\text{i.e } \phi(x, y) = \frac{A - B}{\pi} \text{Arg}(z) + \frac{A + B}{2}$$

- $\phi(x, y) = \frac{1}{\alpha} \text{Arg}(z)$
- $\phi(x, y) = \frac{1}{\pi} \text{Arg}(z - z_0)$
- $\phi(x, y) = a_n + \frac{1}{\pi} \sum_{b=1}^n (a_{k-1} - a_k) \text{Arg}(z - x_k)$ :

$$\text{Arg}(z - x_k) = \begin{cases} \pi & x < x_k \\ 0 & x > x_k \end{cases}$$

$\Rightarrow$  if  $x_j < x < x_{j+1}$  then:

$$\phi(x, 0) = a_n + \sum_{k=j+1}^n (a_{k-1} - a_k) = a_j$$

- $\phi(x, y) = \alpha \ln(|z|) + \beta$  leads to:

$$\begin{cases} \alpha \ln(r_1) + \beta = A \\ \alpha \ln(r_2) + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{B - A}{1 - r_2 - \ln(r_1)} \\ \beta = \frac{A \ln(r_2) - B \ln(r_1)}{\ln(r_2) - \ln(r_1)} \end{cases}$$

$$\Rightarrow \phi(x, y) = \frac{B - A}{1 - r_2 - \ln(r_1)} \ln(|z|) + \frac{A \ln(r_2) - B \ln(r_1)}{\ln(r_2) - \ln(r_1)}$$

How about more complicated Dirichlet problems?

The idea is to simplify the complicated problems to an easier one using a conformal mapping.

### Definition/Sats 9.3

Suppose  $f : D \rightarrow D'$  is analytic,  $f = u + iv$ .

If  $\psi(u, v)$  is harmonic in  $D'$ , then:

$$\phi(x, y) := \psi(u(x, y), v(x, y))$$

is harmonic in  $D$

### Bevis 9.2

Take  $z_0 \in D$ . Then  $w_0 = f(z_0) \in D'$  and since  $D'$  is open, there is a disk  $w_0 \in V$  contained in  $D'$ .

Since  $f$  is continuous, there is a disk  $z_0 \in U$  in  $D$  such that  $f(U) \subseteq V$ . Since  $\psi$  is harmonic in  $V$ , which is simply connected, there is an analytic function  $g$  in  $V$  such that  $\text{Re}(g) = \psi$

But then  $g \circ f$  is an analytic function in  $U$  such that:

$$\text{Re}(g \circ f)(z) = \psi(u(x, y), v(x, y)) = \phi(x, y)$$

Hence,  $\phi$  is harmonic in  $U$ . Since  $z_0$  was arbitrarily chosen,  $\phi$  is harmonic in  $D$  □

Suppose now that the analytic function  $f : D \rightarrow D'$  maps  $D$  bijectively onto  $D'$  and extends to a continuous bijection  $f : \bar{D} \rightarrow \bar{D}'$ .

Suppose also that the boundary conditions for  $\psi$  in  $D'$  corresponds to the boundary conditions for  $\phi$  in  $D$ .

Then, if we can solve the Dirichlet problem for  $\psi$ , we can also solve it for  $\phi$



## 10. COMPLEX INTEGRATION

We shall now study so-called *contour integrals* (or the line integrals) of complex-valued functions.

This theory will teach us more about the properties of analytic functions

## 10.1. Contours.

**Definition/Sats 10.1: Smooth arc**

A point set  $\gamma \in \mathbb{C}$  is said to be a *smooth arc* if it is the image of some continuous complex-valued function  $z = z(t)$ ,  $a \leq t \leq b$  such that:

- $z(t)$  has a continuous derivative on  $[a, b]$
- $z'(t) := x'(t) + iy'(t) \neq 0$  on  $[a, b]$
- $z(t)$  is bijective on  $[a, b]$

**Definition/Sats 10.2: Smooth closed curve**

A point set  $\gamma$  in  $\mathbb{C}$  is said to be a *smooth closed curve* if it is the range of some continuous function  $z = z(t)$ ,  $a \leq t \leq b$  satisfying the requirements for a smooth arc and:

- $z(t)$  is bijective on  $[a, b)$  but:

$$z(a) = z(b) \quad z'(a) = z'(b)$$

**Anmärkning:**

The phrase "γ is a smooth curve" means that γ is either a smooth arc or a smooth closed curve.

There are infinitely many choices of "admissible" parametrizations of a curve. One can, for example, change the direction of a parametrization yet still represent the same fundamental curve.

There is of course, a natural choice of the orientation of curves.

**Definition/Sats 10.3: Directed smooth curve**

A smooth curve with a specified orientation is called a *directed smooth curve*

**Definition/Sats 10.4: Contour**

A *contour*  $\Gamma$  is either a single point or a finite sequence  $(\gamma_1, \dots, \gamma_n)$  of directed smooth curves such that the terminal point on  $\gamma_k$  coincides with the initial point on  $\gamma_{k+1}$

We write  $\Gamma = \gamma_1 + \dots + \gamma_n$

**Definition/Sats 10.5: Closed contour**

$\Gamma$  is said to be a *closed contour* if the initial point on  $\gamma_1$  coincides with the terminal point on  $\gamma_n$ .

**Anmärkning:**

If the only two coinciding points on a closed contour is the initial and the terminal point, then we say it is a *simple closed contour*.

**Definition/Sats 10.6: Length of curve**

If  $\gamma$  is a smooth curve, then the length of the curve is given by the multivariate-curve length:

$$L(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of  $\Gamma = \gamma_1 + \dots + \gamma_n$  is given by:

$$L(\Gamma) = L(\gamma_1) + \dots + L(\gamma_n)$$

**10.2. Contour integrals.**

Lets see how to define  $\int_{\Gamma} f(z)dz$ , aka the contour integral of a complex-valued function  $f$  over the contour  $\Gamma$ .

We start by defining:

$$\int_{\Gamma} f(z)dz$$

where  $\gamma$  is a directed smooth curve.

For each  $i \in \{0, \dots, n\}$ , we form a partition  $\mathcal{P}_n = \{z_0, \dots, z_n\}$  of  $\gamma_i$

Let  $L(\gamma; z_{k-1}, z_k)$  denote the length of  $\gamma$  from  $z_{k-1}$  to  $z_k$ , then:

$$\mu(\mathcal{P}_n) = \max_{1 \leq k \leq n} L(\gamma; z_{k-1}, z_k)$$

is a measure of the "firmness" (**CHECK**) of the partition.

Take, for  $k = 1, \dots, n$  an arbitrary point  $c_k$  on  $\gamma$  between  $z_{k-1}$  and  $z_k$ .

Then form the Riemann-sum:

$$S(\mathcal{P}_n) = \sum_{k=1}^n f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^n f(c_k)\delta z_k$$

**Definition/Sats 10.7**

We say that  $f$  is integrable along the directed smooth curve  $\gamma$  if there exists a complex number  $L \in \mathbb{C}$  such that:

$$\lim_{n \rightarrow \infty} \mu(\mathcal{P}_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} S(\mathcal{P}_n) = L$$

(independent of the choice of partition and Riemann-sum)

**Anmärkning:**

The number  $L$  is called the *integral of  $f$  along  $\gamma$* , and is denoted:

$$\int_{\gamma} f(z)dz$$

**Anmärkning:**

The integral has the following properties:

- $\int_{\gamma} (f(z) \pm g(z))dz = \int_{\gamma} f(z)dz \pm \int_{\gamma} g(z)dz$
- $\int_{\gamma} c \cdot f(z)dz = c \int_{\gamma} f(z)dz$
- $\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$

**Definition/Sats 10.8**

If  $f$  is continuous along  $\gamma$ , then  $f$  is integrable along  $\gamma$

**10.3. How to compute the contour integral.**

First, consider  $\int_a^b f(t)dt$ , where  $f(t) = u(t) + iv(t)$ , and  $u, v$  is continuous on  $[a, b]$

Let  $F(t)$  be an antiderivative of  $f(t)$ , i.e:

$$\begin{aligned} F(t) &= U(t) + iV(t) & U' &= u & V' &= v \\ \Rightarrow \int_a^b f(t)dt &= \int_a^b (u(t) + iv(t))dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \\ U(t)|_a^b + iV(t)|_a^b &= F(b) - F(a) \end{aligned}$$

**Definition/Sats 10.9**

If  $f$  is continuous on  $[a, b]$ , and  $F'(t) = f(t)$  for  $t \in [a, b]$ , then:

$$\int_a^b f(t)dt = F(b) - F(a)$$

The integral of  $f$  along an arbitrary directed smooth curve can be reduced to integrals as above, by the parametrization  $z(t)$  ( $a \leq t \leq b$ )

Let:

$$z_0 = z(t_0) \quad z_1 = z(t_1) \quad \cdots \quad z_n = z(t_n)$$

where  $a = t_0 < t_1 < \cdots < t_n = b$ :

$$\begin{aligned} \xrightarrow{c_k \approx z_k} \sum_{k=1}^n f(z_k) \Delta z_k &= \sum_{k=1}^n f(z(t_0)) \Delta z_k \\ \Delta z_k = z_k - z_{k-1} &= z(t_k) - z(t_{k-1}) \approx z'(t_k)(t_k - t_{k-1}) = z'(t_k) \Delta t_k \\ \Rightarrow \sum_{k=1}^n f(z(t_k)) z'(t_k) \Delta t_k \end{aligned}$$

This is a Riemann-sum for the continuous function  $f(z(t))z'(t)$  on  $[a, b]$

This suggests the following:

**Definition/Sats 10.10: Curve integral**

Let  $f$  be a continuous function on a directed smooth curve having admissible parametrization  $z(t)$  ( $a \leq t \leq b$ ), then:

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt$$

**Definition/Sats 10.11**

Suppose  $\Gamma = \gamma_1 + \cdots + \gamma_n$  and let  $f$  be continuous on  $\Gamma$ , then we let:

$$\int_{\Gamma} f(z)dz := \sum_{k=1}^n \int_{\gamma_k} f(z)dz$$

(if  $\Gamma = \{z_0\}$ , we let  $\int_{\Gamma} f(z)dz = 0$ )

**Definition/Sats 10.12: ML-inequality**

Suppose  $|f(z)| \leq M \forall z \in \gamma$ :

$$\Rightarrow \left| \sum_{k=1}^n f(c_k) \Delta z_k \right| \leq \sum_{k=1}^n |f(c_k)| |\Delta z_k| \leq M \sum_{k=1}^n |\Delta z_k| \leq ML(\gamma)$$

As  $\mu(\mathcal{P}_n) \rightarrow 0$ , this implies:

$$\left| \int_{\gamma} f(z) dz \right| \leq ML(\gamma)$$

**Anmärkning:**

Here  $L(\gamma)$  (more commonly referred to as  $l(\gamma)$ ) is the *arc-length* of  $\gamma$ .

**Definition/Sats 10.13**

Suppose  $f$  is continuous on  $\Gamma$  and that  $|f(z)| \leq M, z \in \Gamma$

Then:

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML \quad \text{where } L = L(\Gamma)$$

## 11. INDEPENDENCE OF PATHS &amp; CAUCHY'S INTEGRAL THEOREM

## 11.1. Independence of paths.

**Definition/Sats 11.1**

Suppose that  $f(z)$  is continuous in a domain  $D$  and that  $f(z)$  has an antiderivative  $F(z)$  in  $D$ , i.e  $F'(z) = f(z) \forall z \in D$

Let  $\Gamma$  be a contour in  $D$  with initial point  $z_I$  and terminal point  $z_T$ , then:

$$\int_{\Gamma} f(z)dz = F(z_T) - F(z_I)$$

**Bevis 11.1**

$$\int_{\Gamma} f(z)dz = \sum_k \int_{\gamma_k} f(z)dz = \sum_k \int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt$$

where  $z(t)$ ,  $\tau_{k-1} \leq t \leq \tau_k$  is a parametrization of  $\gamma_k$

Now:

$$\frac{d}{dt}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t)$$

So by Sats 11.66, we have:

$$\int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt = F(z(\tau_k)) - F(z(\tau_{k-1}))$$

Sum over  $k$

□

**Anmärkning:**

If  $f$  is continuous in a domain  $D$  and has an antiderivative in  $D$ , then:

$$\int_{\Gamma} f(z)dz = 0$$

for every closed contour in  $D$

**Definition/Sats 11.2**

Let  $f$  be continuous in a domain  $D$ , then the following are equivalent:

- $f$  has an antiderivative in  $D$
- $\int_{\Gamma} f(z)dz = 0$  for every closed contour  $\Gamma$  in  $D$
- Contour integrals are independent of path in  $D$ 
  - If  $\Gamma_1$  and  $\Gamma_2$  are 2 contours with the same initial and terminal point, then  $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$

**Bevis 11.2**

(I)  $\Rightarrow$  (II): see anmärkning above.

(II)  $\Rightarrow$  (III): Given  $\Gamma_1$  and  $\Gamma_2$ , let:

$$\begin{aligned} \Gamma = \Gamma_1 + (-\Gamma_2) &\Rightarrow 0 = \int_{\Gamma} = \int_{\Gamma_1} + \int_{-\Gamma_2} = \int_{\Gamma_1} - \int_{\Gamma_2} \\ &\Rightarrow \int_{\Gamma_1} = \int_{\Gamma_2} \end{aligned}$$

(III)  $\Rightarrow$  (I): Fix  $z_0 \in D$ . Since  $D$  is a domain,  $\Rightarrow$  for any  $z \in D$  there is a polygonal path  $\Gamma$  from  $z_0$  to  $z$

Define  $F(z) = \int_{\Gamma} f(s)ds$

$F(z)$  is well defined, i.e. independent of the choice of  $\Gamma$ . By (III).

We now show that  $F'(z) = f(z) \forall z \in D$  by looking at a line segment  $L$ :

$$\begin{aligned} \Rightarrow F(z + \Delta z) - F(z) &= \int_L f(s)ds = \int_L f(z)ds + \int_L (f(s) - f(z))ds \\ &= f(z)\Delta z + \int_L (f(s) - f(z))ds \end{aligned}$$

i.e:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z} \int_L (f(s) - f(z))ds$$

By the ML-inequality:

$$\left| \frac{1}{\Delta z} \int_L (f(s) - f(z))ds \right| \leq \frac{1}{|\Delta z|} \cdot \max_{s \in L} |f(s) - f(z)| \cdot |\Delta z| \xrightarrow{\Delta z \rightarrow 0} 0$$

Thus:

$$F'(z) = \lim_{\Delta z \rightarrow 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z)$$

□

## 11.2. Cauchy's integral theorem.

Let  $\Gamma$  be a simple closed contour in  $\mathbb{C}$  parametrized by  $z = z(t)$   $a \leq t \leq b$ :

$$\begin{aligned} \int_{\Gamma} f(z)dz &= \int_a^b f(z(t)) \frac{dz}{dt} dt \\ &= \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))) \left( \frac{dx}{dt} + i \frac{dy}{dt} \right) dt \\ &= \int_a^b \left( u(x(t), y(t)) \frac{dx}{dt} - v(x(t), y(t)) \frac{dy}{dt} \right) dt + i \int_a^b (v(x(t), y(t)) + u(x(t), y(t)) \frac{dy}{dt}) dt \end{aligned}$$

This leads us to:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (udx - vdy) + i \int_{\Gamma} (vdx + udy)$$

We can use Greens theorem:

### Definition/Sats 11.3: Greens theorem

Let  $\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$  be a  $C^1$ -vector field defined on a simply connected domain  $D$ , and let  $\Gamma$  be a positively oriented simple closed contour in  $D$

Then:

$$\int_{\Gamma} (F_1 dz + F_2 dy) = \iint_{\Omega} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Where  $\Omega$  denotes the region interior to  $\Gamma$

Let's see if we can use this on the expression for  $\int_{\Gamma} f(z)dz$ :

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (udx - vdy) + i \int_{\Gamma} (vdx + udy) = \iint_{\Omega} \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_{\Omega} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Now, if we suppose that  $u, v \in C^1$  and assume that  $f$  is analytic in  $D$ , then:

$$\int_{\Gamma} f(z)dz = 0$$

In view of the Cauchy-Riemann equations

The following holds:

**Definition/Sats 11.4: Cauchy's integral theorem**

Suppose that  $f$  is analytic in a simply connected domain  $D$ , and let  $\Gamma$  be any closed contour in  $D$ , then:

$$\int_{\Gamma} f(z) dz = 0$$

**Anmärkning:**

The theorem generalizes our discussion in two ways.

First,  $\Gamma$  can be any closed contour, it does not need to be simple.

Second, the assumption that  $u, v \in C^2$  has been dropped. The fact that the second assumption is not necessary was first demonstrated by Edouard Goursat

**Definition/Sats 11.5: Cauchy-Goursat theorem**

If  $f$  is analytic inside and on a simple closed contour, then:

$$\int_{\Gamma} f(z) dz = 0$$

Combined with the theorem of path Independence, we have the following:

**Definition/Sats 11.6**

Suppose that  $f$  is analytic in a simply connected domain. Then  $f$  has an antiderivative, contour integrals are independent of path, and integrals over closed contours are 0

## 12. GOURSAT'S ARGUMENT

**Definition/Sats 12.1: Goursat**

Let  $R$  be a rectangle, and let  $f$  be analytic on  $R$ . Then:

$$\int_{\partial R} f(z) dz = 0$$

**Bevis 12.1**

Decompose  $R$  into four sub-rectangles by bisecting the sides.

Then:

$$\begin{aligned} \int_{\partial R} f(z) dz &= \sum_{j=1}^4 \int_{\partial R_j} f(z) dz \\ \Rightarrow \left| \int_{\partial R} f(z) dz \right| &\leq \sum_{j=1}^4 \left| \int_{\partial R_j} f(z) dz \right| \end{aligned}$$

Then there is some rectangle  $R^{(1)}$  among  $R_1, R_2, R_3, R_4$  such that:

$$\int_{\partial R^{(1)}} f(z) dz \geq \frac{1}{4} \left| \int_{\partial R} f(z) dz \right|$$

Next, decompose  $R^{(1)}$  into four sub-rectangles by bisecting the sides. Similarly, one of these, say  $R^{(2)}$  will satisfy:

$$\left| \int_{\partial R^{(2)}} f(z) dz \right| \geq \frac{1}{4} \left| \int_{\partial R^{(1)}} f(z) dz \right| \geq \frac{1}{4^2} \left| \int_{\partial R} f(z) dz \right|$$

We continue to obtain a sequence of rectangles  $R^{(1)} \supseteq R^{(2)} \supseteq \dots$

Following this, we have:

$$\left| \int_{\partial R^{(n)}} f(z) dz \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f(z) dz \right|$$

Let  $L$  be the length of  $\partial R$  and  $L_n$  the length of  $\partial R^{(n)}$ . Then:

$$L_n = \frac{1}{2^n} L$$

It can be shown that  $\bigcap_{n=1}^{\infty} R^{(n)}$  consists of a single point  $z_0$ .

Since  $f$  is differentiable at  $z_0$ :

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \xrightarrow{z \rightarrow z_0} 0$$

Let  $\varepsilon > 0$  be given. Then  $\exists \delta > 0$  such that:

$$0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

I.e:

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0| \quad |z - z_0| < \delta$$



Choose  $n$  so large that  $R^{(n)}$  belongs to the disc  $|z - z_0| < \delta$ . Then:

$$\begin{aligned} \left| \int_{\partial R} f(z) dz \right| &\leq 4^n \left| \int_{\partial R^{(n)}} f(z) dz \right| \\ &= 4^n \left| \int_{\partial R^{(n)}} \underbrace{\left( f(z) - \overbrace{f(z_0)}^0 - \overbrace{f'(z_0)}^0 (z - z_0) \right)}_{\substack{1 \leq \varepsilon |z - z_0| \leq \varepsilon \cdot \text{diam}(R^{(n)}) = \varepsilon 2^{-n} \cdot \text{diam}(R)}} dz \right| \\ &\stackrel{\text{ML-ineq}}{\leq} 4^n \cdot \frac{\varepsilon}{2^n} \cdot \text{diam}(R) \cdot \frac{1}{2^n} L = L \cdot \text{diam}(R) \varepsilon \end{aligned}$$

Where  $\text{diam}(R)$  is the length of the diagonal of  $R$

This is true for all  $\varepsilon > 0 \implies \int_{\partial R} f(z) dz = 0$  □

### Definition/Sats 12.2

Let  $D$  be an open disc centered at  $z_0$ . let  $f$  be continue in  $D$ , and assume that for each rectangle  $R$  contained in  $D$ , we have:

$$\int_{\partial R} f(z) dz = 0$$

For any point  $z \in D$ , define:

$$F(z) = \int_{\Gamma_z} f(s) ds$$

Where  $\Gamma_z$  is the contour along the perimeter along the catheters of the triangle formed from  $z_0$  to  $z$

Then  $F'(z) = f(z)$

### Bevis 12.2

This proof uses some similar techniques found in the proof of the path independence theorem, we have that:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z} \int_{\Gamma} (f(s) - f(z)) ds$$

Where  $\Gamma$  is the contour constructed in the same fashion as  $\Gamma_z$

Then, by the ML-inequality:

$$\begin{aligned} \left| \frac{1}{\Delta z} \int_{\Gamma} (f(s) - f(z)) ds \right| &\leq \frac{1}{|\Delta z|} \max_{s \in \Gamma} |f(s) - f(z)| \cdot \underbrace{(|\Delta x| + |\Delta y|)}_{\leq 2|\Delta z|} \\ &\leq 2 \max_{s \in \Gamma} |f(s) - f(z)| \rightarrow 0 \end{aligned}$$

So  $F'(z) = f(z)$  □

Let's see what happens if we combine the two theorems above with the independence of point theorem. We get the following local result:

### Definition/Sats 12.3

Let  $D$  be an open disc and uspose that  $f$  is analytic in  $D$ . Then  $f$  has an antiderivative in  $D$ , contour integrals are independent of path, and integrals over closed contours are 0

## 13. HOMOTOPY

We operate under the assumption that  $D$  is a domain,  $I = [0, 1]$

**Definition/Sats 13.1: Homotopic relationship**

Suppose  $\gamma_0, \gamma_1 : I \rightarrow D$  is continuous and that  $\gamma_0(0) = \gamma_1(0) = z_0$  and  $\gamma_0(1) = \gamma_1(1) = z_1$

We say that  $\gamma_0$  is *homotopic to  $\gamma_1$  with endpoints fixed in  $D$*  if there is a continuous mapping  $H : I \times I \rightarrow D$  such that:

- $H(0, t) = \gamma_0(t) \quad \forall t \in I$
- $H(1, t) = \gamma_1(t) \quad \forall t \in I$
- $H(s, 0) = z_0 \quad H(s, 1) = z_1 \quad \forall s \in I$

This is sometimes denoted  $\gamma_0 \simeq \gamma_1$

**Anmärkning:**

"Homotopic" is an equivalence relation.

**Definition/Sats 13.2**

Suppose that  $\gamma_0, \gamma_1 : I \rightarrow D$  are continuous and that  $\gamma_0(0) = \gamma_0(1) = z_0$  and  $\gamma_1(0) = \gamma_1(1) = z_1$  (this is a closed curve)

We say that  $\gamma_0$  and  $\gamma_1$  are *homotopic as closed curves in  $D$*  if there is a continuous mapping  $H : I \times I \rightarrow D$  such that

- $H(0, t) = \gamma_0(t) \quad \forall t \in I$
- $H(1, t) = \gamma_1(t) \quad \forall t \in I$
- $H(s, 0) = H(s, 1) \quad \forall s \in I$

**Definition/Sats 13.3: Simply connected domain**

A domain  $D$  is called *simply connected* if every closed curve in  $D$  is homotopic to a point (= constant closed curve) in  $D$

**Definition/Sats 13.4: Deformation theorem**

Suppose that  $f$  is analytic in a domain  $D$

- If  $\Gamma_0$  and  $\Gamma_1$  are contours from  $z_0$  to  $z_1$  which are homotopic with endpoints fixed in  $D$ , then:

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz$$

- If  $\Gamma_0$  and  $\Gamma_1$  are closed contours which are homotopic as closed curves in  $D$ , then:

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz$$

**Bevis 13.1**

Let  $H : I \times I \rightarrow D$  be a homotopy from  $\Gamma_0$  to  $\Gamma_1$ .

If  $H(I \times I)$  is a compact subset of  $D$  and  $D$  is open, then  $\exists \varepsilon > 0$  such that  $\forall z \in H(I \times I)$  the disk  $D_\varepsilon(z) \subset D$  (**CHECK**)

$H$  is uniformly continuous  $\implies \exists \delta > 0$  such that:

$$|(s, t) - (\tilde{s}, \tilde{t})| < \delta \implies |H(s, t) - H(\tilde{s}, \tilde{t})| < \varepsilon$$

Subdivide  $I \times I$  into subsquares  $Q$  with  $\text{diam } Q < \delta$ . For every subsquare  $Q$ ,  $H(Q)$  belongs to a disk contained in  $D$ . Orient each  $\partial Q$  counterclockwise, and  $H(\partial Q)$  accordingly. By Cauchy's integral theorem for a disk:

$$\begin{aligned} &\implies \int_{H(\partial Q)} f(z) dz = 0 \quad \forall Q \\ \implies 0 &= \sum_Q \int_{H(\partial Q)} f(z) dz = \int_{H(\partial(I \times I))} f(z) dz \\ &= \int_{\lambda_0} + \int_{\Gamma_1} - \int_{\lambda_1} - \int_{\Gamma_0} \end{aligned}$$

It follows that if  $\lambda_0, \lambda_1$  are constant, then:

$$\int_{\lambda_0} = \int_{\lambda_1} = 0 \implies \int_{\Gamma_0} = \int_{\Gamma_1}$$

If  $\lambda_0 = \lambda_1$  then:

$$\int_{\lambda_0} = \int_{\lambda_1} \implies \int_{\Gamma_0} = \int_{\Gamma_1}$$

□

**Anmärkning:**

One can argue that the above proof only works for a smooth homotopy.

Now, if  $\gamma : I \rightarrow D$  is a contour, then we can find a partition  $D = \{t_0, \dots, t_n\}$  of  $I$ :

$$0 = t_0 < t_1 < \dots < t_n = 1$$

Such that  $\gamma([t_i, t_{i+1}])$  belongs to a disk  $D_i$  contained in  $D$ . If  $F_i$  is an antiderivative for  $f$  in  $D_i$ , then letting  $\gamma_i$  denote the restriction of  $\gamma$  to  $[t_i, t_{i+1}]$  it follows that:

$$\begin{aligned} \int_{\gamma} f(z) dz &= \sum_{i=1}^{n-1} \int \gamma_i f(z) dz \\ (2) \quad &= \sum_{i=1}^{n-1} (F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i))) \end{aligned}$$

For any continuous curve  $\gamma : [0, 1] \rightarrow D$  one can define  $\int_{\gamma} f(z) dz$ , if  $f$  is analytic in  $D$  then by (2) for any partition  $P$  of  $[0, 1]$  such that  $\gamma([t_i, t_{i+1}])$  belongs to a disk  $D_i \subset D$  and  $F_i$  is an antiderivative of  $f$  in  $D_i$  independent of the choice of  $P, D_i, F_i$

**Definition/Sats 13.5: Corollary**

Suppose that  $f$  is analytic in a simply connected domain, and let  $\Gamma$  be any closed contour (**CHECK**) in  $D$ , then:

$$\int_{\Gamma} f(z) dz = 0$$

## 14. CAUCHYS INTEGRAL FORMULA &amp; APPLICATION

Using Cauchy's integral theorem, one can prove the following important result:

**Definition/Sats 14.1: Cauchy's integral formula**

Suppose  $f$  is analytic in a simply connected domain  $D$ . Let  $\Gamma$  be a simple closed positively oriented contour in  $D$  and  $z_0$  any point inside  $\Gamma$ . Then:

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz$$

**Bevis 14.1**

$\frac{f(z)}{z - z_0}$  is analytic in  $D \setminus \{z_0\}$ .

As in the last **example of Lecture 9**, we see that:

$$\begin{aligned} \int_{\Gamma} f(z) dz &= \int_{C_r} \frac{f(z)}{z - z_0} dz \\ &= \int_{C_r} \frac{f(z_0)}{z - z_0} dz + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \end{aligned}$$

Note that  $\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz$  is independent of  $r$ , it is therefore enough to show that:

$$\lim_{r \rightarrow 0^+} \int_{C_r}$$

Let  $M_r = \max_{z \in C_r} |f(z) - f(z_0)|$ . If  $f$  is continuous, then  $M_r \rightarrow 0$  as  $r \rightarrow 0^+$ , so by the ML-inequality:

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{M_r}{r} \cdot 2\pi \xrightarrow{r \rightarrow 0^+} 0$$

□

**Anmärkning:**

$f(z_0)$  is determined by  $f(z)$   $z \in \Gamma$

What we have seen is that Cauchy's integral formula says that:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi \quad z \in \Gamma$$

By differentiation under the integral sign, it seems plausible that the following holds:

**Definition/Sats 14.2: Cauchy's generalised integral formula**

If  $f$  is analytic inside and on a simple closed positively oriented contour, and  $z \in \Gamma$ , then:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z}^{n+1} d\xi \quad n \in \mathbb{N}$$

The proof is by induction using the definition of the derivative. **This is left in the old notes**

**Example:**

Compute  $\int_{\Gamma} \frac{2z + 1}{z(z - 1)^2} dz$  where  $\Gamma$  is a one-dimensional Moebius strip.

By symmetry, we can divide  $\Gamma = \Gamma_1 + \Gamma_2$  in a way such that they are both simple and closed, then:

$$\begin{aligned} \int_{\Gamma} \frac{2z+1}{z(z-1)^2} dz &= \int_{\Gamma} \frac{(2z+1)/z^2}{z-1} dz + \int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z} dz \\ &= 2\pi i \frac{d}{dz} \left( \frac{2z+1}{z} \right) \Big|_{z=1} = 2\pi i \frac{2z+1}{(z-1)^2} \Big|_{z=0} = -4\pi i \end{aligned}$$

Note that Cauchy's generalised integral formula implies the following:

**Definition/Sats 14.3**

If  $f$  is analytic in a domain  $D$ , then all derivatives exist and are analytic in  $D$

**Bevis 14.2**

Apply Cauchy's generalised integral formula on a positively oriented circle around an arbitrary  $z_0 \in D$  □

In particular, the derivative of an analytic function is analytic.

Suppose  $f$  is analytic. Recall that:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Since  $f'$  is analytic, and in particular continuous, it follows that  $u, v \in C^1$ . Since  $f''$  exists and is continuous, and:

- $f''(z) = u_{xx} + iv_{xx} = v_{xy} - iuv_{yy}$
- $f''(z) = v_{yx} - iu_{yx} = -u_{yy} - iv_{yy}$

It follows that also all second derivatives are continuous (etc.)

**Definition/Sats 14.4**

If  $f = u + iv$  is analytic in a domain  $D$ , then  $u, v \in C^\infty(D)$

**Anmärkning:**

This completes the proof that  $u, v$  are harmonic.

Suppose that  $f$  is continuous in a domain  $D$  and that:

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 0 \quad \forall \text{ closed contours } \Gamma \in D \\ \implies f &\text{ has antiderivative } F \in D \Leftrightarrow F' = f \in D \end{aligned}$$

Since  $F$  is analytic, so if  $F' = f$  according to the theorem above. We have therefore proven the following:

**Definition/Sats 14.5: Morera**

If  $f$  is continuous in a domain  $D$ , and:

$$\int_{\Gamma} f(z) dz = 0$$

For all closed contours  $\Gamma \in D$ , then  $f$  is analytic in  $D$

### 14.1. Consequences of Cauchy's generalised integral formula.

#### Definition/Sats 14.6: Cauchy estimate

Let  $f$  be analytic inside and on a circle  $C_R$  of radius  $R$  centered at  $z_0$

Suppose  $|f(z)| \leq M$  for all  $z \in C_R$ , then it holds that:

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n} \quad n \in \mathbb{N}$$

#### Bevis 14.3

Give  $C_R$  a positive orientation. Then, by Cauchy's generalised integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z_0}^{n+1} d\xi$$

For  $\xi \in C_R$ , it holds that:

$$\left| \frac{f(\xi)}{\xi - z_0}^{n+1} \right| \leq \frac{M}{R^{n+1}}$$

□

The length of  $C_R$ ,  $l(C_R)$  is  $2\pi R$ , so by the ML-inequality:

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n!M}{R^n}$$

Suppose  $f$  is entire,  $|f(z)| \leq M \forall z \in \mathbb{C}$ .

Then the Cauchy estimate implies that:

$$|f'(z_0)| \leq \frac{M}{R}$$

- This is true  $\forall R > 0 \Rightarrow |f'(z_0)| = 0$ , i.e.  $f'(z_0) = 0$
- This is true  $\forall z_0 \in \mathbb{C} \Rightarrow f'(z) = 0 \Rightarrow f(z) = \text{constant}$

We have proven the following:

#### Definition/Sats 14.7: Liouville

The only bounded entire functions are the constant functions.

Liouville's theorem can be used to prove the following well-known result:

#### Definition/Sats 14.8: Fundamental theorem of algebra

Every non-constant polynomial with complex coefficients has at least one zero.

**Bevis 14.4**

Let  $P(z) = a_n z^n + \cdots + a_0$  where  $a_n \neq 0$ . Also, suppose that  $P(z)$  has no zeroes.

Put  $f(z) = \frac{1}{P(z)}$ . Then  $f$  is entire.

We now show that  $f(z)$  is bounded:

•

$$P(z) = z^n \left( a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) \text{ so } \frac{P(z)}{z^n} \rightarrow a_n \text{ as } z \rightarrow \infty$$

$$\Rightarrow \exists \rho \text{ s.t. } |z| \geq \rho \Rightarrow \left| \frac{P(z)}{z^n} \right| \geq \frac{|a_n|}{2}$$

$$\Rightarrow |f(z)| = \frac{1}{|P(z)|} \leq \frac{2}{|z|^n |a_n|} \leq \frac{2}{\rho^n |a_n|} \quad |z| \geq \rho$$

- For  $|z| \leq \rho$ , the function  $|f(z)|$  is a continuous function on a compact set  $\Rightarrow |f(z)|$  has a maximum and in particular is bounded.

Thus,  $\frac{1}{P(z)}$  is a bounded entire function and must therefore be constant according to Liouville's theorem.

But then  $P(z)$  must be constant, in other words, the only polynomials without zeroes are the constant ones.

□

## 15. APPLICATIONS TO HARMONIC FUNCTIONS

## 15.1. The mean-value property.

Suppose that  $f$  is analytic inside and on a circle  $C_R$  of radius  $R$  centered at  $z_0$ .

By Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0} dz$$

Parametrise  $C_R$ :  $z(t) = z_0 + Re^{it}$  where  $0 \leq t \leq 2\pi$ :

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} \cdot iRe^{it} dt$$

I.e:

$$(3) \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) dt$$

(3) is called the *mean-value* property.

A direct consequence of (3) is the following:

**Lemma 15.1**

Suppose that  $f$  is analytic in a disk centered at  $z_0$ , say  $D(z_0)$ .

Suppose also that:

$$\max_{z \in D(z_0)} |f(z)| = |f(z_0)|$$

Then:

$$|f(z)| = |f(z_0)| \quad \forall z \in D(z_0)$$

**Bevis 15.1**

Suppose that  $|f(z)|$  is *not* constant.

Then  $\exists z_1 \in D(z_0)$  such that  $|f(z_1)| < |f(z_0)|$

Let  $C_R$  be the circle with center at  $z_0$  passing through  $z_1$ .

By assumption,  $|f(z)| \leq |f(z_0)| \quad \forall z \in C_R$

Since  $f$  is continuous,  $\exists$  a segment of  $C_R$  containing  $z_1$  in which  $|f(z)| < |f(z_0)|$

Say that  $|f(z)| < |f(z_0)| - 2\pi\varepsilon$  on a segment of opening angle (**CHECK**)  $\delta$ :

$$\begin{aligned} \Rightarrow |f(z_0)| &\stackrel{(3)}{=} \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})| dt \\ &< \frac{1}{2\pi} [(2\pi - \delta) |f(z_0)| + \delta(|f(z_0)| - 2\pi\varepsilon)] \\ &= |f(z_0)| < \delta\varepsilon \end{aligned}$$

Contradiction!

□

The lemma implies the following theorem:



**Definition/Sats 15.1: Maximum modulus theorem**

If  $f$  is analytic in a domain  $D$ , and  $|f(z)|$  attains its maximum value at a point  $z_0 \in D$ , then  $f$  is constant in  $D$

**Bevis 15.2**

We show that  $|f|$  is constant in  $D$

The result then follows from an exercise (**DO IT AND TEX**)

Suppose  $|f(z)|$  is *not* constant. Then  $\exists z_1 \in D$  such that  $|f(z_1)| < |f(z_0)|$ . Let  $\gamma$  be a polygonal path from  $z_0$  to  $z_1$ . We now consider the values of  $|f(z)|$  for  $z$  on  $\gamma$  starting at  $z_0$ .

There exists a  $w \in \gamma$  such that:

- $|f(z)| = |f(z_0)| \quad \forall z$  preceeding  $w$  on  $\gamma$
- $\exists$  points  $z$  on  $\gamma$  orbiting close to  $w$  such that  $|f(z)| < |f(z_0)|$

**Anmärkning:**

Follows from the supremum property of  $\mathbb{R}$ . Parametrise  $\gamma$  with  $z = z(t)$ ,  $0 \leq t \leq 1$ ,  $z(0) = z_0$ ,  $z(1) = z_1$

Let  $M = \{t \in [0, 1] \text{ s.t. } |f(z(t))| < |f(z_0)|\}$

$M \neq \emptyset$  ( $1 \in M$ ) and  $M$  is bounded below (by 0).

Let  $\alpha = \inf M$ ,  $w = f(z(\alpha))$ . Then:

- $\alpha$  is a lower bound for  $M$ :

$$\Rightarrow \forall t \in M \text{ it holds that } \alpha \leq t$$

$$\Rightarrow \text{If } t < \alpha \Rightarrow t \notin M, \quad |f(z(t))| = |f(z_0)|, \quad \text{proving the first point}$$

- $\alpha$  is the greatest lower bound:

$$\Rightarrow \forall \beta > \alpha \exists t \in M \text{ s.t. } t < \beta$$

i.e there are points  $z(t)$  with  $|f(z(t))| < |f(z_0)|$  orbiting close to  $w$ , proving the second point.

Since  $f$  is continuous,  $|f(z)| = |f(z_0)| \quad \forall z$  preceeding  $w$  on  $\gamma$  implies that  $|f(w)| = |f(z_0)|$ . There  $\exists$  disk  $D(w)$  centered at  $w$  contained in  $D$ . By the lemma above,  $|f|$  is constant in  $D(w)$ , but this contradicts the fact that  $\exists$  points  $z$  on  $\gamma$  orbiting close to  $w$  such that  $|f(z)| < |f(z_0)|$

Thus, the assumption that  $|f|$  is *not* constant must be false.

□

Now, suppose  $f$  is analytic in a bounded domain  $D$  and that  $f$  is continuous up to the boundary of  $D \Rightarrow |f(z)|$  attains a maximum in  $\overline{D}$ .

According to the above theorem, the maximum can only be attained in  $D$  if  $f$  is constant. In any case we have the following variant of the maximum modulus principle:

**Definition/Sats 15.2**

A function which is analytic in a bounded domain and continuous up to the boundary attains its maximum modulus on the boundary.

**15.2. Applications to harmonic functions.**

Lets first give a new proof of the following:

**Definition/Sats 15.3**

Suppose  $\phi$  is harmonic in a simply connected domain  $D$ . Then there exists an analytic function  $f$  such that  $\phi = \operatorname{Re} f$

**Bevis 15.3**

If such a function  $f$  exists, say that  $f = \phi + i\psi$ , then  $f' = \phi_x + \psi_x = \phi_x - i\phi_y$ , so  $f$  would be a primitive of  $\phi_x - i\phi_y$ .

So, we expect that  $\phi_x - i\phi_y$  has an antiderivative being the sought function  $f$ .

Therefore, define  $g(z) = \phi_x - i\phi_y$ . Note that:

- $(\phi_x)_x = (-\phi_y)_y$  since  $\Delta\phi = 0$
- $(\phi_x)_y = -(\phi_y)_x$  since  $\phi \in C^2$

$\Rightarrow$  Cauchy-riemanns equations are satisfied and  $g$  real and imaginary parts are  $C^1$  since  $\phi \in C^2$

$\Rightarrow g$  is analytic in a simply connected domain  $D$

$\Rightarrow g$  has an antiderivative  $G = u + iv$  in  $D$

Since  $G' = g$ , we have:

$$\begin{aligned} u_x - iu_y &= \phi_x - i\phi_y \\ \Rightarrow (\phi - u)_x &= (\phi - u)_y = 0 \\ \Rightarrow \phi - u &= \text{constant, say } \phi = u + c \quad c \in \mathbb{R} \end{aligned}$$

Put  $f(z) = G(z) + c$ , this  $f$  has the sought properties. □

Now, let  $\phi$  be a harmonic function in a simply connected domain  $D$

Let  $f = \phi + i\psi$  be analytic in  $D$  (exists by previous theorem (**CHECK**)):

$$\Rightarrow \underbrace{|e^f|}_{\text{analytic}} = |e^{\phi+i\psi}| = e^{\phi}$$

The exponential function is monotonically increasing. If  $\phi$  (and therefore  $|e^f|$ ) attains a maximum in  $D$ , then  $e^f$  is constant.

Hence,  $\phi$  is constant.

Since  $\phi$  attains its maximum precisely when  $-\phi$  attains its maximum, we also have a corresponding minimum principle.

We have proven the following:

**Definition/Sats 15.4**

If  $\phi$  is harmonic in a simply connected domain  $D$  and if  $\phi$  attains its maximum or minimum at some point  $z_0 \in D$ , then  $\phi$  is constant

**Definition/Sats 15.5**

A function which is harmonic in a bounded simply connected domain, and continuous up to the boundary, attains its maximum and minimum on the boundary.

**Anmärkning:**

The assumption of *simple* connectedness is not necessary in these two theorems.

*Recall Dirichlets problem:*

Find a function  $\phi(x, y)$  which is harmonic in a domain  $D$  and continuous in  $\overline{D}$  with given values on the boundary of  $D$ .

*You ask:*

Does a solution of Dirichlets problem exist? If so, is it unique?  
The above theorem implies uniqueness for *bounded*  $D$

#### Definition/Sats 15.6

Let  $\phi_1(x, y)$  and  $\phi_2(x, y)$  be harmonic in a bounded domain  $D$ , and continuous on  $\overline{D}$  such that  $\phi_1(x, y) = \phi_2(x, y)$  on  $\partial D$   
Then  $\phi_1 = \phi_2$  in  $D$

#### Bevis 15.4

Let  $\phi = \phi_1 - \phi_2$ .

By the geometrisation of the second theorem,  $\phi$  attains both its maximum and minimum value on  $\partial D$ . But  $\phi = 0$  on  $\partial D \Rightarrow \phi = 0$

An explicit solution can be found if for example  $D$  is a disk. □

## 16. SEQUENCES AND SERIES OF FUNCTIONS &amp; UNIFORM CONVERGENCE

Chapter 3 covers what is meant by convergence of a sequence of complex numbers. Lets recall:

**Definition/Sats 3.11: Complex limit**

A sequence  $\{z_n\}_{n=1}^{\infty}$  of complex numbers is said to have the limit  $z_0$  (we say *converges to*  $z_0$ ) if for every given  $\varepsilon > 0$  there exists an integer  $N \geq 1$  such that:

$$|z_n - z_0| < \varepsilon \quad \forall n \geq N$$

We write this as:

$$\lim_{n \rightarrow \infty} z_n = z_0$$

**Definition/Sats 16.1: Complex series**

A *series* is a formal expression of the form  $c_0 + c_1 + c_2 + \cdots$ , or equivalently:

$$\sum_{j=1}^{\infty} c_j$$

Where  $c_j \in \mathbb{C}$ .

The  $n$ :th partial sum of the series, denoted by  $S_n$ , is the sum of the first  $n + 1$  terms, that is:

$$S_n = \sum_{j=0}^n c_j$$

If  $\{S_n\}_{n=0}^{\infty}$  has limit  $S$ , the series is said to *converge* to  $S$  and we write:

$$S = \sum_{j=0}^{\infty} c_j$$

If the series does not converge, it is said to *diverge*

**Example:**

It holds that:

$$\sum_{j=0}^{\infty} c^j = \frac{1}{1-c} \quad |c| < 1$$

**Bevis 16.1**

$$\begin{aligned} (1-c)(1+c+\cdots+c^{n-1}+c^n) &= 1+c+\cdots+c^{n-1}+c^n - (c+c^2+\cdots+c^n+c^{n+1}) \\ &= 1-c^{n+1} \Leftrightarrow \frac{1}{1-c} - (1+c+\cdots+c^n) = \frac{c^{n+1}}{1-c} \end{aligned}$$

The result follows since  $\frac{c^{n+1}}{1-c} \xrightarrow{n \rightarrow \infty} 0$  when  $|c| < 1$

□

You can often show that a series converges by comparing it with another series whose convergence is known.

**Definition/Sats 16.2: Comparison test**

Suppose that:

$$|c_j| \leq M_j \quad \forall j \geq J$$

Then:

$$\sum_{j=0}^{\infty} M_j \text{ converges} \Rightarrow \sum_{j=0}^{\infty} c_j \text{ converges}$$

The proof is rather straightforward if one is familiar with the Cauchy convergence criterion:

$$\{A_n\}_{n=1}^{\infty} \text{ converges} \Leftrightarrow \{A_n\}_{n=0}^{\infty} \text{ is Cauchy sequence}$$

Also follows if you combine **Thm 1 and Thm 2** in 5.1 ?? (**CHECK**)

### Definition/Sats 16.3: Absolute convergence

The series

$$\sum_{j=0}^{\infty} c_j$$

is said to be *absolutely convergent* if

$$\sum_{j=0}^{\infty} |c_j|$$

converges.

*An absolutely convergent series is convergent as seen by a trivial use of the comparison* (**EVAL NEED**)

The following result can often be used to prove convergence of a series:

### Definition/Sats 16.4: Ratio test

Suppose that the terms of the series:

$$\sum_{j=0}^{\infty} c_j$$

have the property that the ratios  $\left| \frac{c_{j+1}}{c_j} \right| \xrightarrow{j \rightarrow \infty} L$

Then the series converges if  $L < 1$ , diverges if  $L > 1$ , and provides no information if  $L = 1$

**(PROOF)**

**Example:**

Show that the series

$$\sum_{j=0}^{\infty} \frac{4^j}{j!}$$

Converges.

To solve this, let  $c_j = \frac{4^j}{j!}$ . Then, by the ratio test:

$$\Rightarrow \left| \frac{c_{j+1}}{c_j} \right| = \frac{4^{j+1}}{(j+1)!} \cdot \frac{j!}{4^j} = \frac{4}{j+1} \xrightarrow{j \rightarrow \infty} 0$$

Convergence therefore follows.

**Definition/Sats 16.5: Pointwise convergence**

Let  $\{f_n\}_{n=0}^{\infty}$  be a sequence of functions defined on some set  $E \subseteq \mathbb{C}$ .

We say that the sequence  $\{f_n\}_{n=0}^{\infty}$  *converges pointwise to  $f$  on  $E$*  if for each  $z \in E$ , the sequence of complex numbers  $\{f_n(z)\}_{n=0}^{\infty}$  converges to  $f(z)$

I.e, for every  $z \in E$  and every  $\varepsilon > 0$ , there exists  $N$  (dependent on  $z$  and  $\varepsilon$ ) such that:

$$n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$$

**Definition/Sats 16.6: Uniform convergence**

Let  $\{f_n\}_{n=0}^{\infty}$  be a sequence of functions defined on  $E \subseteq \mathbb{C}$ . We say that the sequence  $\{f_n\}_{n=0}^{\infty}$  of functions *converges uniformly to  $f$  on  $E$*  if for every  $\varepsilon > 0$  there exists  $N$  (dependent on  $\varepsilon$ ) such that:

$$n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon \quad \forall z \in E$$

$$\Leftrightarrow \sup_{z \in E} |f_n(z) - f(z)| \xrightarrow{n \rightarrow \infty} 0$$

Why do we care about uniform convergence?

Well, because uniform convergence preserves the properties of the functions under limit operators!

**Definition/Sats 16.7**

Let  $\{f_n\}_{n=0}^{\infty}$  be a sequence of functions continuous on a set  $E \subseteq \mathbb{C}$  and converging uniformly to  $f$  on  $E$ .

Then  $f$  is continuous on  $E$

**Bevis 16.2**

Take  $z_0 \in E$  and let  $\varepsilon > 0$  be given.

We want to show that there is a  $\delta > 0$  such that  $z \in E$ :

$$|z - z_0| < \delta \Rightarrow |f(z_0) - f(z)| < \varepsilon$$

First, choose  $N$  so big that

$$|f(z) - f_N(z)| < \frac{\varepsilon}{3} \quad \forall z \in E$$

This is possible thanks to uniform convergence.

Since  $f_N$  is continuous at  $z_0$ , there  $\exists \delta > 0$  such that:

$$|f_N(z_0) - f_N(z)| < \frac{\varepsilon}{3} \quad \forall z \in E$$

s.t  $|z - z_0| < \delta$

For small  $z$  it follows that:

$$\begin{aligned} |f(z_0) - f(z)| &= |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\ &\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

□

We can therefore integrate the limit  $f(z)$  over contours  $\Gamma$  in  $E$

The following holds:

**Definition/Sats 16.8**

Let  $\{f_n\}_{n=0}^{\infty}$  be a sequence of functions continuous on  $E \subseteq \mathbb{C}$  and converging uniformly to  $f$  on  $E$ . Suppose that the contour  $\Gamma \subset E$

Then it holds that:

$$\int_{\Gamma} f_n(z) dz \rightarrow \int_{\Gamma} f(z) dz \quad n \rightarrow \infty$$

**Bevis 16.3**

Let  $L = L(\Gamma)$

Choose  $N$  such that:

$$|f(z) - f_n(z)| < \frac{\varepsilon}{L} \quad \forall n \geq N \quad \forall z \in E$$

Then, for  $n \geq N$ :

$$\begin{aligned} \left| \int_{\Gamma} f(z) dz - \int_{\Gamma} f_n(z) dz \right| &= \left| \int_{\Gamma} (f(z) - f_n(z)) dz \right| \\ &\stackrel{\text{ML}}{<} \frac{\varepsilon}{L} \cdot L = \varepsilon \end{aligned}$$

□

We now turn to series of functions

$$\sum_{j=0}^{\infty} f_j(z)$$

**Definition/Sats 16.9: Pointwise resp. uniform convergence, series of functions**

The series

$$\sum_{j=0}^{\infty} f_j(z)$$

is said to *converge pointwise resp. uniformly to  $f(z)$  on  $E$*  if the sequence  $\{S_n\}_{n=0}^{\infty}$  of partial sums:

$$S_n(z) = \sum_{j=0}^n f_j(z)$$

converges pointwise resp. uniformly to  $f(z)$  on  $E$

**Example:**

We saw that  $\sum_{j=0}^{\infty} z^j$  converges pointwise to  $\frac{1}{1-z}$  on  $|z| < 1$

From the identity:

$$\left| \frac{1}{1-z} - \sum_{j=0}^n z^j \right| = \left| \frac{z^{n+1}}{1-z} \right|$$

we see that the convergence is uniform on any disk  $|z| \leq r$  for  $r < 1$  (but not on  $|z| < 1$ )

**Definition/Sats 16.10: Weierstrass  $M$ -test**

Suppose that:

$$\sum_{j=0}^{\infty} M_j$$

is a convergent series with non-negative terms, and that

$$|f_j(z)| \leq M_j \quad \forall z \in E \quad \text{and } j \geq J$$

Then the series  $\sum_{j=0}^{\infty} f_j(z)$  converges uniformly on  $E$

Also easy if you know the Cauchy criterion (**TEX**)

We now turn to analytic functions

**Definition/Sats 16.11**

Let  $\{f_n\}_{n=0}^{\infty}$  be a sequence of analytic functions in a domain  $D$ , which converges uniformly to  $f$  on  $D$

Then  $f$  is analytic in  $D$

**Bevis 16.4**

Let  $\tilde{D}$  be any disk in  $D$ . Then:

$$\int_{\Gamma} f(z) dz \stackrel{\text{Thm.}}{=} \lim_{n \rightarrow \infty} \int_{\Gamma} f_n(z) dz$$

Cauchy  $\stackrel{\text{int. thm.}}{=} 0$

for all closed contours  $\Gamma$  in  $D$

By Morera's theorem it follows that  $f$  is analytic in  $\tilde{D}$ . But since  $D$  is a union of disks  $\tilde{D}$ , the result follows.  $\square$

We finally mention the following:

**Definition/Sats 16.12**

Suppose that  $\{f_n\}_{n=0}^{\infty}$  is a sequence of functions analytic in  $|z - z_0| \leq R$  which converges uniformly to  $f(z)$  on  $|z - z_0| \leq R$ .

Then, for each  $r < R$  and each  $m \geq 1$ , the sequence of  $m$ :th derivatives  $\left\{f_n^{(m)}(z)\right\}_{n=0}^{\infty}$  converges uniformly to  $f^{(m)}(z)$  on  $|z - z_0| \leq r$



**Bevis 16.5**

Let  $\varepsilon > 0$  be given.

Choose  $N$  so large that:

$$|f_n(z) - f(z)| < \varepsilon \quad \forall z : \quad |z - z_0| \leq R \quad \forall n \geq N$$

Fix  $s$  such that  $r < s < R$ . By Cauchy's generalised integral formula:

$$f_n^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\xi - z_0|=s} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{m+1}} d\xi$$

for  $|z - z_0| \leq r$

But if  $|z - z_0| \leq r$  and  $|\xi - z_0| = s$ , then:

$$|\xi - z| \geq s - r$$

so that

$$\left| \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{m+1}} \right| \leq \frac{\varepsilon}{(s - r)^{m+1}}$$

So by the ML-inequality:

$$\left| f_n^{(m)}(z) - f^{(m)}(z) \right| \leq \frac{m!}{2\pi} \cdot \frac{\varepsilon}{(s - r)^{m+1}} \cdot 2\pi s$$

for all  $z$  with  $|z - z_0| \leq r$

This proves uniform convergence of  $\left\{ f_n^{(m)} \right\}_{n=0}^{\infty}$  to  $f^{(m)}$  on  $|z - z_0| \leq r$

□

## 17. POWER SERIES AND TAYLOR SERIES

## 17.1. Power series.

**Definition/Sats 17.1**

A series of the form  $\sum_{j=0}^{\infty} a_j(z - z_0)^j$  is called a *power series*.  
The constants  $a_j$  are called the *coefficients* of the power series.

**Definition/Sats 17.2**

For any power series  $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ , there exists an  $R \in [0, +\infty]$  (dependent on  $\{a_j\}$ ) such that:

- The series converges absolutely for  $|z - z_0| < R$
- The series converges uniformly in any closed subdisk  $|z - z_0| \leq r < R$
- The series diverges for  $|z - z_0| > R$

The number  $R$  is called the *radius of convergence* of the series

The proof depends on the following:

**Lemma 17.1**

Suppose that

$$(4) \quad \sum_{j=0}^{\infty} a_j w^j$$

converges at some point having modulus  $r_0 > 0$ .

Then the series (4) converges absolutely and uniformly in any closed subdisk  $|w| \leq r < r_0$

**Bevis 17.1**

Suppose that the series (4) converges at some  $w_0$  with  $|w_0| = r_0$

$$\Rightarrow |a_j w_0^j| \rightarrow 0 \Rightarrow \exists M \text{ s.t. } |a_j w_0^j| = |a_j| r_0^j \leq M \quad \forall j \geq 0$$

For  $|w| \leq r$  it follows that:

$$|a_j w^j| = |a_j| r_0^j \left( \frac{|w|}{r_0} \right)^j \leq M \left( \frac{r}{r_0} \right)^j$$

Letting  $M_j = M \left( \frac{r}{r_0} \right)^j$  then the result follows by Weierstrass  $M$ -test □

**Bevis 17.2: av Sats 17.2**

Let  $w = z - z_0$

If the series  $\sum_{j=0}^{\infty} a_j w^j$  converges only for  $w = 0$ , or it converges for all  $w \in \mathbb{C}$ , we are done.

Otherwise, let  $R$  be the greatest  $r$  such that (4) converges for some  $w$  with  $|w| = r$ .

More precisely, let  $M = \{r > 0 : (4) \text{ converges for some } w \text{ with } |w| = r\}$ .

Then,  $M$  is non-empty and upper-bounded.

Let  $R = \sup M$ . For any  $r < R$  there exists  $r_1$  with  $r < r_1 \leq R$  such that  $r_1 \in M$ .

By the lemma, (4) converges absolutely and uniformly for  $|w| = |z - z_0| \leq r$

If  $|w| > R$ , then (4) diverges. □

**Example:**

The geometric series  $\sum_{j=0}^{\infty} z^j$  has radius of convergence  $R = 1$ .

The series does not converge if  $|z| = 1$ , since the terms do not tend to zero.

**Example:**

The power series  $\sum_{j=0}^{\infty} \frac{z^j}{j^2}$  converges uniformly for  $|z| \leq 1$ .

This follows from the Weierstrass  $M$ -test with  $M_j = \frac{1}{j^2}$

It diverges if  $|z| > 1$  since  $\frac{r^j}{j^2} \not\rightarrow 0$  if  $r > 1$ , hence  $R = 1$

Are there any formulae for  $R$ ?

In practise, the following is often useful:

**Definition/Sats 17.3**

Consider a power series

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j$$

- *Ratio test:* If

$$L = \lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right|$$

exists, then  $R = \frac{1}{L}$

- *Root test:* If

$$L = \lim_{j \rightarrow \infty} \sqrt[j]{|a_j|}$$

exists, then  $R = \frac{1}{L}$

**Anmärkning:**

- In both cases it is understood that  $R = +\infty$  if  $L = 0$  and  $R = 0$  if  $L = +\infty$
- In fact, the following formula, due to Hadamard, is true for any power series:

$$R = \frac{1}{\lim_{j \rightarrow \infty} \sup \sqrt[j]{|a_j|}}$$

**Bevis 17.3**

Follows directly from the standard ratio and root test.

Let  $c_j(z) = a_j(z - z_0)^j$ :

$$\Rightarrow \left| \frac{c_{j+1}(z)}{c_j(z)} \right| = \left| \frac{a_{j+1}}{a_j} \right| |z - z_0| \rightarrow L |z - z_0| \quad j \rightarrow \infty$$

So by the ratio test, if  $L |z - z_0| < 1$  the series converges, and if  $L |z - z_0| > 1$  the series diverges.

Thus  $R = \frac{1}{L}$  □

**Example:**

- The series  $\sum_{j=0}^{\infty} \frac{a_j}{j!}$  has  $R = +\infty$ , since  $a_j = \frac{1}{j!}$  and so  $\left| \frac{a_{j+1}}{a_j} \right| = \frac{1}{j+1} \xrightarrow{j \rightarrow \infty} 0$
- The series  $\sum_{j=0}^{\infty} j! a^j$  has  $R = 0$ , since  $a_j = j!$  and so  $\left| \frac{a_{j+1}}{a_j} \right| = j+1 \xrightarrow{j \rightarrow \infty} +\infty$

Since the partial sums of a power series are analytic (polynomials!), and converge uniformly in each closed subdisk of the "disks of convergence", the convergence theorems of Chapter 16 imply:

**Definition/Sats 17.4**

Suppose  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  has radius of convergence  $R > 0$ .

Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \quad |z - z_0| < R$$

is analytic. The derivatives of  $f$  are obtained by termwise differentiation of the series:

$$\begin{aligned} f'(z) &= \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \\ f''(z) &= \sum_{k=2}^{\infty} k(k-1) a_k (z - z_0)^{k-2} \\ &\vdots \end{aligned}$$

In particular, the coefficients  $a_k$  are given by

$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

**Anmärkning:**

$$\begin{aligned} f(z_0) &= a_0 \\ f'(z_0) &= \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \Big|_{z=z_0} = 1 \cdot a_1 \\ f''(z_0) &= \sum_{k=2}^{\infty} k(k-1) a_k (z - z_0)^{k-2} \Big|_{z=z_0} = 2 \cdot 1 \cdot a_2 \\ &\vdots \\ f^{(n)}(z_0) &= \sum_{k=n}^{\infty} k(k-1) \cdots (k-n+1) a_k (z - z_0)^{k-n} \Big|_{z=z_0} = n! a_n \end{aligned}$$

**Example:**

We have that:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \quad |z| < 1$$

Since we may differentiate termwise:

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1} \quad |z| < 1$$

Since we may also integrate termwise:

$$\int_0^w \frac{1}{1-z} dz = \sum_{k=0}^{\infty} \int_0^w z^k dz = \sum_{k=0}^{\infty} \frac{w^{k+1}}{k+1} = \sum_{n=1}^{\infty} \frac{w^n}{n} \quad |w| < 1$$

But

$$\begin{aligned} \int_0^w \frac{1}{1-z} dz &= [-\text{Log}(1-z)] = -\text{Log}(1-w) \\ \Rightarrow \text{Log}(1-w) &= -\sum_{n=1}^{\infty} \frac{w^n}{n} \quad |w| < 1 \end{aligned}$$

Let  $w = -z$ :

$$\Rightarrow \text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \quad |z| < 1$$

**Example:**

Let  $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Since  $R = +\infty$ ,  $f$  is entire. One can might quickly notice that

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = f(z) \quad f(0) = 0 \\ \Rightarrow \frac{d}{dz}(f(z)e^{-z}) &= \underbrace{f'(z)}_{f(z)} e^{-z} - f(z)e^{-z} = 0 \\ \Rightarrow f(z)e^{-z} &= c = \text{constant} \end{aligned}$$

Let  $z = 0 \Rightarrow c = 1 \Rightarrow f(z) = e^z$

$$\Rightarrow e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad z \in \mathbb{C}$$

## 17.2. Taylor Series.

We have seen that power series are analytic inside the disk of convergence  $\{|z - z_0| < R\}$

The following is an important converse:

### Definition/Sats 17.5: Taylors theorem

Suppose that  $f(z)$  is analytic for  $|z - z_0| < R$

Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad |z - z_0| < R$$

**Bevis 17.4**

Fix  $z$  and let  $r$  be such that  $|z - z_0| < r < R$

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{|\xi - z_0| = r} \frac{f(\xi)}{\xi - z} d\xi$$

Now,

$$\begin{aligned} \frac{f(\xi)}{\xi - z} &= \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \frac{f(\xi)}{\xi - z_0} \sum_{k=0}^{\infty} \left( \frac{z - z_0}{\xi - z_0} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{f(\xi)(z - z_0)^k}{(\xi - z_0)^{k+1}} \quad |\xi - z_0| = r \end{aligned}$$

Where the convergence is uniform in the variable  $\xi$  on  $|\xi - z_0| = r$

We can therefore interchange integration and summation

$$\begin{aligned} \Rightarrow f(z) &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{|\xi - z_0| = r} \frac{f(\xi)(z - z_0)^k}{(\xi - z_0)^{k+1}} d\xi \\ &= \sum_{k=0}^{\infty} \left( \frac{1}{2\pi i} \oint_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi \right) (z - z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \end{aligned}$$

Where we used the generalised Cauchy's integral formula in the last step □

**Anmärkning:**

The radius of convergence of the Taylor series is the largest number  $R$  such that  $f(z)$  is (or can be extended to be) analytic on the disk  $\{|z - z_0| < R\}$

**Example:**

Let  $f(z) = e^z \Rightarrow f^{(k)}(0) = 1 \quad \forall k \geq 0$

$$\Rightarrow e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad z \in \mathbb{C}$$

**Example:**

Let  $f(z) = \sin(z)$

Then  $f^{(2k)}(0) = 0, \quad f^{(2k+1)}(0) = (-1)^k$

$$\Rightarrow \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \quad z \in \mathbb{C}$$

We finally mention the following:

**Definition/Sats 17.6**

If the power series  $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$  and  $g(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^k$  converge for  $|z - z_0| < R$ , then

$$f(z)g(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$$

Where

$$c_k = \sum_{j=0}^k a_{k-j}b_j$$

**Bevis 17.5**

$f, g$  are analytic in  $|z - z_0| < R \Rightarrow u(z) = f(z)g(z)$  is analytic in  $|z - z_0| < R$ :

$$\Rightarrow u(z) = \sum_{k=0}^{\infty} \frac{u^{(k)}(z_0)}{k!} (z - z_0)^k \quad |z - z_0| < R$$

Now,

$$\begin{aligned} u(z_0) &= f(z_0)g(z_0) \\ u'(z_0) &= f'(z)g(z) + f(z)g'(z) \Big|_{z=z_0} = f'(z_0)g(z_0) + f(z_0)g'(z_0) \\ u''(z_0) &= f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0) \\ u^{(k)}(z_0) &= \sum_{j=0}^k \binom{k}{j} f^{(k-j)}(z_0)g^{(j)}(z_0) \\ \Rightarrow \frac{u^{(k)}(z_0)}{k!} &= \sum_{j=0}^k \frac{f^{(k-j)}(z_0)}{(k-j)!} \frac{g^{(j)}(z_0)}{j!} \\ &\Rightarrow \sum_{j=0}^k a_{k-j}b_j \end{aligned}$$

□

## 18. APPLICATIONS

For now, the theory is covered. In this chapter, we will delve in and try to find some intuition that can help us, as well as figure out ways we can apply our knowledge to solve problems



**18.1. Finding conformal mappings.**

By this part, we know what a conformal mapping is, it just preserves angles (but not necessarily lengths) which is great since we can use this property to solve problems which we already have some ideas of how the answers should behave.

If we know that we want to find some conformal mapping between orthogonal curves, then we at least know that our answers will be orthogonal.