PROBLEMS ON TENSOR PRODUCTS (SOLUTIONS TO HOMEWORK ASSIGNMENT #9)

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1. Let V, W, and U be finite-dimensional vector spaces over a field F. (All Homs and tensor and exterior products in this problem are over F.) Prove that there are natural isomorphisms

$$\mathrm{Hom}\,(V \otimes W,\ U) \cong \mathrm{Hom}\,(V,\ W^* \otimes U)$$

and

$$\left(\bigwedge^n V\right)^* \cong \bigwedge^n V^*$$

(for any n).

Solution. For the first part, first observe that we have a natural isomorphism

$$\mathrm{Hom}\,(V \otimes W,\ U) \cong \mathrm{Bilin}\,(V \times W,\ U)$$

to the bilinear maps $V \times W \to U$ (by the universal property of the tensor product). Then we get a natural isomorphism

$$\mathrm{Bilin}\,(V \times W,\ U) \cong \mathrm{Hom}\,(V,\ \mathrm{Hom}(W, U))$$

via restriction to the first factor. (I.e., a bilinear map $V \times W \to U$ is the same thing as a family of linear maps $W \to U$ varying linearly in V.)

So to finish the first part, we have to show there is a natural isomorphism $\mathrm{Hom}(W, U) \cong W^* \otimes U$ and plug this in. Now $\mathrm{Hom}(W, F) = W^*$ (by definition) and we can tensor both sides by U. But there is a natural isomorphism $\mathrm{Hom}(W, F) \otimes_F U \cong \mathrm{Hom}(W, U)$ via $f \otimes u \mapsto (w \mapsto f(w)u)$. (To check this is well-defined, start with the bilinear map $(f, u) \mapsto (w \mapsto f(w)u)$ and observe that it has to factor through the tensor product.)

For the second part, let $v_1, \cdots, v_m$, $m = \dim V$, be a basis for V and let $v_1^*, \cdots, v_m^*$ be the dual basis for V^* (so $v_j^*(v_k) = \delta_{jk}$). Recall that a basis for $\bigwedge^n V$, $n \le m$, consists of $v_I = v_{i_1} \wedge \cdots \wedge v_{i_n}$ with $I = (i_1, \cdots, i_n)$, where $1 \le i_1 < \cdots < i_n \le m$. Similarly, the $v_I^* = v_{i_1}^* \wedge \cdots \wedge v_{i_n}^*$ will be a basis for $\bigwedge^n V^*$. We simply identify v_I^* with the element of $(\bigwedge^n V)^*$ that sends v_I to 1 and v_J to 0 for $J \ne I$; this gives our isomorphism $\Phi\colon \bigwedge^n V^* \to (\bigwedge^n V)^*$.

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We have to show this isomorphism is independent of the choice of basis. Let $w_1, \cdots, w_m$ be another basis for V, and let $w_j = \sum_k a_{jk} v_k$. Here $A = (a_{jk})$ is an invertible "change of basis" matrix. The change of basis matrix $B = (b_{jk})$ for the dual bases satisfies $w_j^* = \sum_k b_{jk} v_k^*$, and since

$$\delta_{j\ell} = w_j^*(w_\ell) = \sum_{k,r} b_{jk} v_k^*\big(a_{\ell r} v_r\big) = \sum_{k,r} b_{jk} a_{\ell r} \delta_{kr} \sum_k b_{jk} a_{\ell k},$$

we see that $BA^t = I$, or $B = (A^t)^{-1}$.

The new basis defines an isomorphism $\Phi' \colon \bigwedge^n V^* \to \left(\bigwedge^n V\right)^*$ sending w_I^* to the linear functional with $w_I \mapsto 1$, $w_J \mapsto 0$ for $J \neq I$. Let's compare it with Φ. First suppose $n = m$ (perhaps the most interesting case). Then the is only one applicable multi-index $I = (1, \cdots, n)$, and $w_I = (\det A) v_I$, while $w_I^* = (\det B) v_I^*$. So

$$\begin{aligned}
\Phi(w_I^*)(w_I) &= \Phi\big((\det B) v_I^*\big)\big((\det A) v_I\big) \\
&= (\det B)(\det A)\Phi(v_I^*)(v_I) = (\det B)(\det A) \\
&= 1 = \Phi'(w_I^*)(w_I),
\end{aligned}$$

since $\det B = (\det A^t)^{-1} = (\det A)^{-1}$, and so $\Phi' = \Phi$.

In the general case the calculation is a bit messier:

$$\begin{aligned}
\delta_{IJ} = \Phi(w_I^*)(w_J) = \Phi&\Big(\sum_{k_1} b_{i_1 k_1} v_{k_1}^* \wedge \cdots \wedge \sum_{k_n} b_{i_n k_n} v_{k_n}^*\Big) \\
&\Big(\sum_{\ell_1} a_{j_1 \ell_1} v_{\ell_1} \wedge \cdots \wedge \sum_{\ell_n} a_{j_n \ell_n} v_{\ell_n}\Big).
\end{aligned}$$

Only terms where $k_1, \cdots, k_n$ are distinct, $\ell_1, \cdots, \ell_n$ are distinct, and $k_1, \cdots, k_n$ are a permutation of $\ell_1, \cdots, \ell_n$ survive and give ± 1 (multiplied of course by the appropriate product of a's and b's), depending on the sign of the permutation. So the result can be written as

$$\sum_{K,L} \delta_{KL} \det(B_{IK}) \det(A_{JL})$$

for appropriate $n \times n$ minors of A and B, and this gives δ_{IJ} via the relationship between the matrices A and B. $\square$

2. Show that (as rings) there is an isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$, the 4×4 matrices over the reals. Here $\mathbb{H}$ is the division ring of quaternions, which is a 4-dimensional algebra over $\mathbb{R}$. Also show that there is an isomorphism of rings $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

Solution. Let's start with the first isomorphism. Note that $\mathbb{H}$ is an $\mathbb{H}$-$\mathbb{H}$-bimodule (though the left and right actions *do not* coincide). Since both the left and right actions of $\mathbb{H}$ on itself are $\mathbb{R}$-linear, we get a homomorphism of rings

$$\Phi\colon \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{op}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H}) \cong M_4(\mathbb{R}),$$

where $\mathbb{H}^{\mathrm{op}}$ is $\mathbb{H}$ with the multiplication reversed. (The $^{\mathrm{op}}$ is needed because a right $\mathbb{H}$-action is the same as a left action of $\mathbb{H}^{\mathrm{op}}$.) Note that Φ is natural; no choice of basis is involved. Furthermore, $\mathbb{H}^{\mathrm{op}} \cong \mathbb{H}$, since quaternionic conjugation ($1 \mapsto 1$, $i \mapsto -i$, $j \mapsto -j$, $k \mapsto -k$) is an anti-isomorphism from $\mathbb{H}$ to itself (it reverses the order of multiplication). Thus Φ can be viewed as a homomorphism

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to M_4(\mathbb{R}).$$

It is injective since the domain and codomain both have dimension $4^2 = 16$ and there cannot be any kernel, since the $\Phi(a \otimes b)$ for a, b running through $1, i, j, k$ (an $\mathbb{R}$-basis of $\mathbb{H}$) can be seen to be linearly independent.

The second isomorphism is similar. Note that $\mathbb{H}$ is an $\mathbb{H}$-$\mathbb{C}$ bimodule via the left action of $\mathbb{H}$ on itself and the right action of $\mathbb{C} = \mathbb{R} + \mathbb{R}i$. So we get a homomorphism $\Psi\colon \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}^{\mathrm{op}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H})$. Since $\mathbb{C}$ is commutative, this time we can drop the $^{\mathrm{op}}$. We can identify $\mathbb{H}$ with $\mathbb{C}^2$ via the $\mathbb{C}$-basis $\{1, j\}$, and the image of Ψ consists of $\mathbb{C}$-linear maps (for the *right* action of $\mathbb{C}$) since left multiplication commutes with right multiplication and $\mathbb{C}$ commutes with itself. Again, Ψ is an isomorphism by dimension counting. $\square$