Lecture 15



## Ch. 16.1 Discrete Dividends

Consider a stock $S$ that pays dividends at times $T_1, \dots, T_K$ where $0 < T_1 < T_2 < \dots < T_K < T$. In addition to $S$, there is also a bank account $dB_t = rB_t\,dt$.

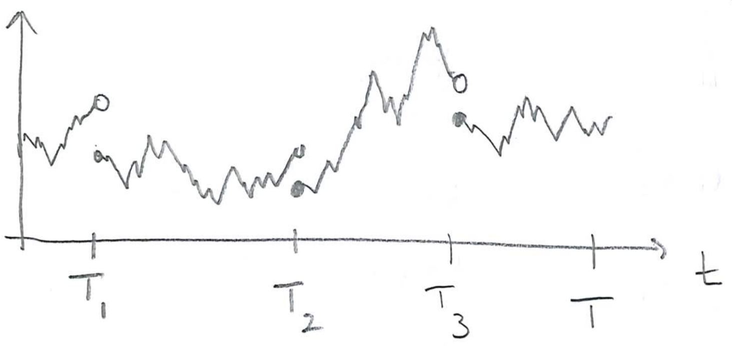
Between dividend dates, $S$ follows

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \qquad (\text{gBM}).$$

At each $t = T_i$, a dividend $\delta(S_{T_i-})$ is paid out.

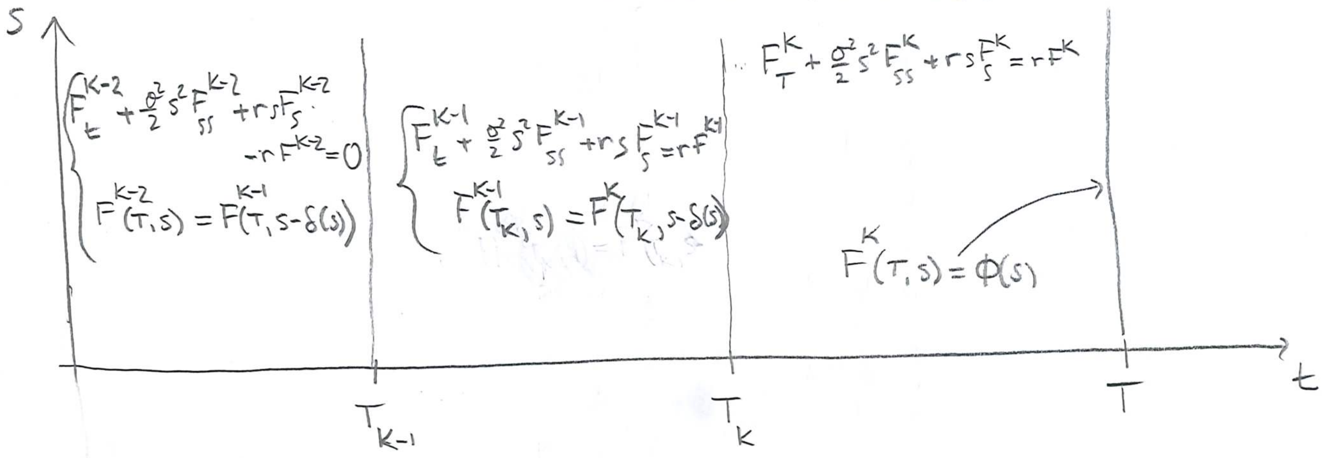
Here $\delta : [0,\infty) \to [0,\infty)$ is a continuous function with $\delta(s) \le s$.

To avoid arbitrage, we must have $S_{T_i} = S_{T_i-} - \delta(S_{T_i-})$.




**Question**: What is the price of a $T$-claim $\mathcal{X} = \Phi(S_T)$?

**Answer**: For $t \in [T_i, T_{i+1}]$ we have $\Pi_t(\mathcal{X}) = F^i(t, S_t)$ where $F^i(t,s)$ is constructed as follows:




(This is Prop. 16.2.)

**Prop 16.3 (Risk-neutral valuation)**

The arbitrage-free price of a simple $T$-claim $\mathcal{X} = \Phi(S_T)$ in the presence of discrete dividends is $F(t, S_t)$, where

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right].$$

Here

$$\begin{cases} dS_u = rS_u\,du + \sigma S_u\,dW^Q_u \quad \text{(between dividend dates)} \\ S_t = s \\ S_{T_i} = S_{T_i-} - \delta(S_{T_i-}) \end{cases}$$

under $Q$.

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**Important special case:** $\delta(s) = \delta \cdot s$

constant, $\delta \in (0,1)$.

Then

$$\begin{aligned} S_T &= S_{T_k} e^{(r-\frac{\sigma^2}{2})(T-T_k) + \sigma(W^Q_T - W^Q_{T_k})} \\ &= (1-\delta) S_{T_k-} e^{(r-\frac{\sigma^2}{2})(T-T_k) + \sigma(W^Q_T - W^Q_{T_k})} \\ &= (1-\delta) S_{T_{k-1}} e^{(r-\frac{\sigma^2}{2})(T-T_{k-1}) + \sigma(W^Q_T - W^Q_{T_{k-1}})} \\ &= (1-\delta)^2 S_{T_{k-1}-} e^{(r-\frac{\sigma^2}{2})(T-T_{k-1}) + \sigma(W^Q_T - W^Q_{T_{k-1}})} \\ &= \ldots = (1-\delta)^n s\, e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma(W^Q_T - W^Q_t)} \end{aligned}$$

where $n$ is the number of dividend times in $[t,T]$.

Therefore

$$F^\delta(t,s) = F^0(t, s(1-\delta)^n)$$

$F^\delta$: pricing function in the presence of dividends

$F^0$: pricing function with no dividends.

Ex: Assume $\delta(s) = \delta s$. What is the price of a call option $\mathcal{X} = (S_T - K)^+$ ?

Answer: $F^{\delta}(t,s) = F^{0}(t, s(1-\delta)^n) = (1-\delta)^n s N(d_1) - Ke^{-r(T-t)} N(d_2)$

where
$$\begin{cases} d_1 = \dfrac{\ln \frac{s(1-\delta)^n}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases}$$

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Ex: Find a replicating strategy for $\mathcal{X} = S_T$. (assume $n$ remaining dividends).

Solution: The value of $\mathcal{X}$ is $F^{\delta}(0,s) = F^{0}(0, s(1-\delta)^n) = s(1-\delta)^n$.

At $t=0$, buy $(1-\delta)^n$ shares of $S$.

At $t=T_1$, receive $(1-\delta)^n \delta S_{T_1-}$ in dividends. New stock price is $S_{T_1} = (1-\delta) S_{T_1-}$, so we can buy $\dfrac{(1-\delta)^n \delta S_{T_1-}}{(1-\delta) S_{T_1-}} = \delta (1-\delta)^{n-1}$ new shares. Total holdings: $(1-\delta)^n + \delta(1-\delta)^{n-1} = (1-\delta)^{n-1}$

Continue similarly at $T_2, \dots, T_n$. After $T_k$ we have $(1-\delta)^{n-k}$ shares, so at $t=T$ we have $(1-\delta)^{n-n} = 1$ share of $S$.

Thus $\mathcal{X}$ is replicated!

## 16.2 Continuous dividends

Market: $\begin{cases} dB_t = rB_t\,dt \\ dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \end{cases}$

Dividend structure: $dD_t = \delta(S_t)S_t\,dt$,

where $\delta(\cdot)$ is a continuous function.

Interpretation: During an interval $[t_1, t_2]$, the holder of one share of $S$ receives the amount $\int_{t_1}^{t_2} \delta(S_u)S_u\,du$.

To price a T-claim $\mathcal{X} = \Phi(S_T)$, we follow our usual approach. Assume $\Pi_t(\mathcal{X}) = F(t, S_t)$ and let $(w^S, w^F)$ be a self-financing relative portfolio of $S$ and $F$.

$$dV_t^w \underset{\text{self-fin.}}{=} V_t^w w^S \frac{dS_t + dD_t}{S_t} + V_t^w w^F \frac{dF_t}{F_t}$$

$$= V_t^w\left(w^S(\mu+\delta) + w^F\mu_F\right)dt + V_t^w\left(w^S\sigma + w^F\sigma_F\right)dW_t$$

where $\begin{cases} \mu_F = \dfrac{F_t + \mu S F_s + \frac{\sigma^2 S^2}{2}F_{ss}}{F} \\ \sigma_F = \dfrac{\sigma S F_s}{F}. \end{cases}$

Choose $(w^S, w^F)$ so that

$$\begin{cases} w^S + w^F = 1 \\ \sigma w^S + \sigma_F w^F = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} w^S = \dfrac{-\sigma_F}{\sigma - \sigma_F} \\ w^F = \dfrac{\sigma}{\sigma - \sigma_F}. \end{cases}$$

Comparing with the bank account, to avoid arbitrage we must have

$$w^S(\mu+\delta) + w^F\mu_F = r.$$

Thus

$$-\sigma_F(\mu+\delta) + \mu_F\sigma = r(\sigma-\sigma_F)$$

$$-SF_s(\mu+\delta) + F_t + \mu SF_s + \frac{\sigma^2 S^2}{2}F_{ss} = rF - rSF_s \qquad \text{(Magic! } \mu \text{ disappears!)}$$

$$F_t + \frac{\sigma^2 S_t^2}{2}F_{ss} + (r-\delta)S_t F_s - rF = 0$$

Since $S_t$ can take any value, the PDE must hold at all points $(t,s)$.

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Propositions 16.6+16.7

The pricing function $F(t,s)$ of $\mathcal{X} = \Phi(S_T)$ solves

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 s^2 F_{ss} + (r-\delta)sF_s - rF = 0 \\ F(T,s) = \Phi(s) \end{cases}.$$

Moreover, $F(t,s) = E^Q_{t,s}\left[e^{-r(T-t)}\Phi(S_T)\right]$

where $\begin{cases} dS_u = (r-\delta)S_u\,dt + \sigma S_u\,dW_u^Q \\ S_t = s \end{cases}$ under $Q$.

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Remark: If $\delta(s) = \delta$ (constant), then

$$S_T = s\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right\} =$$

$$= se^{-\delta(T-t)}\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T-W_t)\right\}$$

Thus $F^\delta(t,s) = F^0(t, se^{-\delta(T-t)})$

Pricing function with cont. dividends

Pricing function with no dividends.

Ex: What is the price of $X = S_T$ if continuous dividends are paid (at a constant proportional rate $\delta$)?

Answer: $F^{\delta}(0,s) = F^{0}(0, se^{-\delta T}) = se^{-\delta T}$.

Can we find a replicating strategy?

At $t=0$, buy $e^{-\delta T}$ shares of $S$. Use all dividends to buy new shares. If $f(t)$ shares are held at time $t$, then $\delta f(t)\,dt$ new shares can be bought during $(t, t+dt)$. Thus
$$\begin{cases} \dot{f}(t) = \delta f(t) \\ f(0) = e^{-\delta T} \end{cases}$$
so $f(t) = e^{-\delta(T-t)}$. In particular, $f(T) = 1$, so $X$ is replicated!