Le 20

The argument principle, Rouché's theorem

Recall: Last time we proved the following

Thm (The argument principle)

Let C be a simple closed, positively oriented, contour in $\mathbb{C}$. Suppose that f is analytic and nonzero on C, and meromorphic inside C. Then,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_o(f) - N_p(f),$$

where $N_o(f)$ resp. $N_p(f)$ denotes the number of zeros resp. poles of f inside C counted with multiplicity (or order).

Remark: Note that $\int_C \frac{f'(z)}{f(z)}\,dz = i\Delta_C \arg f$

since, at least locally, we can introduce a branch of log such that

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = \frac{d}{dz}\left(\ln|f(z)| + i\arg f(z)\right)$$

It follows that

$$\int_C \frac{f'(z)}{f(z)}\,dz = \Delta_C \ln|f| + i\Delta_C \arg f = i\Delta_C \arg f$$

The theorem can therefore be written

$$\boxed{\frac{1}{2\pi}\Delta_C \arg f = N_0(f) - N_p(f)}$$

and is therefore called the argument principle. □

<u>Corollary</u> Let C be a simple closed positively oriented contour in $\mathbb{C}$. Suppose that f is analytic inside and on C, and nonzero on C. Then,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_0(f)$$

$$\left(\text{i.e. } \frac{1}{2\pi}\Delta_C \arg f = N_0(f)\right).$$

Let me mention the following:

<u>Thm</u> If f is a nowhere vanishing analytic function in a simply connected domain D, then there exists an analytic function g on D such that $f(z) = e^{g(z)}$.

<u>Remark</u>: The fcn g is usually denoted $\log f$. It is unique up to integer multiples of $2\pi i$.

<u>Proof</u>: Select a point $z_0 \in D$ and choose w_0 such that $e^{w_0} = f(z_0)$. Define

$$s(z) = \int_\Gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + w_0 ,$$

where Γ is any path from z_0 to z.

Since $\frac{f'}{f}$ is analytic in the simply connected domain D, the def. of s does not depend on the choice of path. Arguing as earlier,

$$s'(z) = \frac{f'(z)}{f(z)} .$$

It follows that

$$\frac{d}{dz}\left(f(z)\, e^{-s(z)} \right) = \left(f'(z) - f(z)\, s'(z) \right) e^{-s(z)} = 0 .$$

So $f(z)e^{-s(z)}$ is constant. Evaluating at the point z_0 shows that

$$f(z)\, e^{-s(z)} = f(z_0)\, e^{-s(z_0)} = f(z_0)\, e^{-w_0} = 1$$

Hence, $f(z) = e^{s(z)}$.

$\square$

[If $f = e^{s_1} = e^{s_2}$, then at a fixed z_0 it must hold that $s_1(z_0) = s_2(z_0) + in2\pi$. But $s_1' = s_2' = \frac{f'}{f}$, so $(s_1 - s_2)' = 0 \Rightarrow s_1(z) = s_2(z) + in2\pi$ $\forall z \in D$]

An important consequence of the argument principle is the following:

Thm (Rouché)

Suppose that f and h are analytic inside and on a simple closed contour C, and that

$$|h(z)| < |f(z)| \quad , z \in C. \quad (*)$$

Then f and $f+h$ have the same number of zeros inside C, counting multiplicities.

Remark: The strict inequality $(*)$ implies that f and $f+h$ are both nonzero on C.

Proof: From

$$f(z) + h(z) = f(z)\left(1 + \frac{h(z)}{f(z)}\right) \quad , z \in C,$$

it follows that

$$\arg\left(f(z) + h(z)\right) = \arg f(z) + \arg\left(1 + \frac{h(z)}{f(z)}\right), \ z \in C.$$

Since $|h(z)| < |f(z)|$ clearly $\Delta_C \arg\left(1 + \frac{h}{f}\right) = 0$

$\Rightarrow \Delta_C \arg(f+h) = \Delta_C \arg f$, so by the above corollary $N_0(f+h) = N_0(f)$.

One can also argue as follows:

Let $F(z) = \dfrac{f+h}{f} = 1 + \dfrac{h}{f}$.

As above $\Delta_C \arg F = 0$.

Now,

$$F' = \frac{f'+h'}{f} - \frac{(f+h)f'}{f^2}$$

$$\Rightarrow \quad \frac{F'}{F} = \frac{f'+h'}{f+h} - \frac{f'}{f}$$

$$\Rightarrow \quad 0 = \frac{1}{2\pi i}\int_C \frac{F'}{F}\,dz = \frac{1}{2\pi i}\int \frac{f'+h'}{f+h}\,dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$$

Since $\Delta_C \arg F = 0$, according to above

$$= N_0(f+h) - N_0(f)$$

Corollary

∎

<u>Ex.</u> Determine the number of zeros of

$$p(z) = z^6 + 9z^4 + z^3 + 2z + 4 \quad \text{in} \quad |z|<1.$$

<u>Sol.</u> Put $f(z) = 9z^4$, $h(z) = z^6 + z^3 + 2z + 4$

Clearly f has 4 zeros in the unit disk.

Now, $|f(z)| = 9|z|^4 = 9$, $|z|=1$.

Moreover,

$$|h(z)| \leq |z|^6 + |z|^3 + 2|z| + 4 = 8, \quad |z|=1$$

$$\Rightarrow \quad |h(z)| < |f(z)|, \quad |z|=1.$$

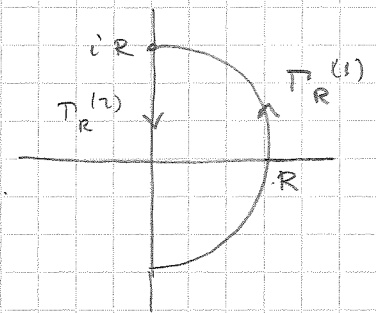
By Rouché's thm $p(z) = f(z)+h(z)$ also has 4 zeros in $|z|<1$.

∎

Also the argument principle can be used to study zeros:

<u>Ex.</u> Show that $p(z) = z^4 + 2z^3 + 3z^2 + z + 2$ has exactly two zeros in the right half-plane.

<u>Sol</u> Use the corollary. Let Γ_R be the simple closed positively oriented contour below:



Since $p(z) = z^4 \left(1 + \frac{2}{z} + \frac{3}{z^2} + \frac{1}{z^3} + \frac{2}{z^4}\right)$ it is clear that $\Delta_{\Gamma_R^{(1)}} \arg p \simeq 4\pi$ for large R.

Now,

$$p(iy) = y^4 - i2y^3 - 3y^2 + iy + 2 =$$

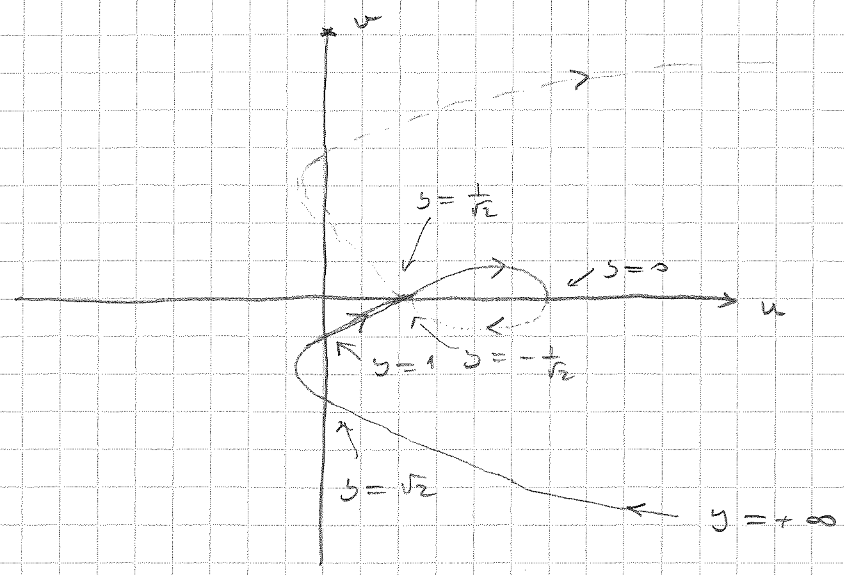
$$= y^4 - 3y^2 + 2 + i(y - 2y^3) =$$

$$= (y^2-1)(y^2-2) + iy(1-2y^2) =: u(y) + iv(y).$$

Note that

$$u(y) = 0 \iff y = \pm 1 \quad \text{or} \quad y = \pm\sqrt{2}$$

$$v(y) = 0 \iff y = 0 \quad \text{or} \quad y = \pm \frac{1}{\sqrt{2}},$$



y		$-\sqrt{2}$		-1		$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$		1		$\sqrt{2}$	
u	+	0	−	0				+				0	−	0	+
v				+		0	−	0	+	0			−		

symmetric

We see that $\Delta_{T_R(2)} \arg p \simeq 0$ for large R.

So clearly $\Delta_{T_R} \arg p = 4\pi$, R large

Thus, $\frac{1}{2\pi} \Delta_{T_R} \arg p = 2 = N_0(p)$

∎

<u>Def.</u> We say that a sequence $\{f_k\}$ of functions analytic on a domain D <u>converges normally</u> to the analytic fcn f on D if it converges uniformly to f on every closed disk contained in D.
(or compact subset of D)

<u>Thm</u> (Hurwitz's thm)
Suppose $\{f_k\}$ is a seq. of analytic fcns on a domain D that converges normally to f on D, and suppose that f has a zero of order N at z_0. Then there exists a $\rho > 0$ such that for k large, f_k has exactly N zeros in the disk $|z - z_0| < \rho$, counting multiplicity, and these zeros converge to z_0 as $k \to \infty$.

<u>Proof</u>: Choose $\rho > 0$ so small that the disk $|z - z_0| < \rho$ is contained in D and so that $f(z) \neq 0$ for $0 < |z - z_0| \leq \rho$. Choose $\delta > 0$ s.t. $|f(z)| \geq \delta$ on $|z - z_0| = \rho$. Since f_k conv. unif. to f on $|z - z_0| \leq \rho$, for k large we have $|f_k(z)| \geq \frac{\delta}{2}$ for $|z - z_0| = \rho$.

It follows that $\frac{f_k'}{f_k}$ conv. unif. to $\frac{f'}{f}$ on $|z - z_0| = \rho$, so that

$$\frac{1}{2\pi i} \int_{|z - z_0| = \rho} \frac{f_k'}{f_k}\, dz \longrightarrow \frac{1}{2\pi i} \int_{|z - z_0| = \rho} \frac{f'}{f}\, dz.$$

The left-hand side is the number of zeros N_k of f_k in $|z-z_0|<\rho$, while the right-hand side is the number N of zeros of f in $|z-z_0|<\rho$. Since $N_k \to N$, for k large f_k has N zeros in the disk $|z-z_0|<\rho$. The same argument works for any smaller $\rho>0$, so these zeros converge to z_0 as $k \to \infty$. ∎

__Def.__ We say that a fcn is __univalent__ on a domain D if it is analytic and one-to-one on D.

__Thm__ Suppose $\{f_k\}$ is a seq. of univalent fcns on a domain D that converges normally to f on D. Then either f is univalent or f is constant.

__Proof.__ Suppose f is not constant.

Suppose z_0 and ζ_0 satisfy $f(z_0)=f(\zeta_0)=: w_0$. Then z_0 and ζ_0 are zeros of finite order for $f(z)-w_0$. By the preceding thm there are sequences $z_k \to z_0$ and $\zeta_k \to \zeta_0$ s.t. $f_k(z_k)-w_0=0$ and $f_k(\zeta_k)-w_0=0$. Since f_k is univalent $z_k=\zeta_k$ and taking limits $z_0=\zeta_0$ ∎