GRAPH THEORY: RETAKE EXAM

10 JUNE 2021

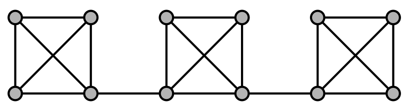
Throughout the entire exam, $G = (V, E)$ denotes a finite simple graph.

Problem 1. Let $V = \{I_1, \ldots, I_n\}$ be a collection of compact intervals. Construct a graph $G = (V, E)$ by drawing an edge between vertices I_i and I_j whenever $I_i \cap I_j \neq \emptyset$, for $1 \leq i < j \leq n$.

(a) Construct an example where G contains a cycle of length at least 4. (1p)

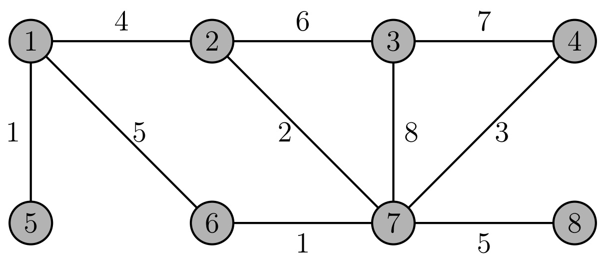
(b) Show that G cannot contain a cyckle C_k with $k \geq 4$ as an *induced* subgraph. (4p)

Problem 2. Let $n, m \geq 1$ be integers. Consider a graph $G_{n,m}$ obtained by "chaining together" m copies of K_n as shown in the following picture (for $n = 4, m = 3$):



How many spanning trees does $G_{n,m}$ have? Does this number change if the edges connecting the different copies of K_n are contracted? (5p)

Problem 3. Consider the weighted graph

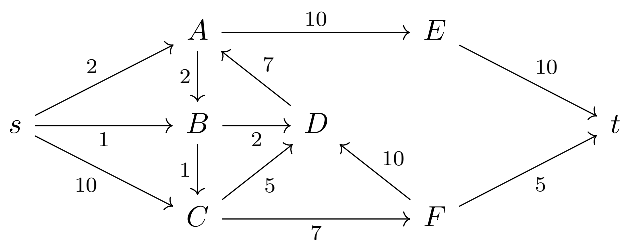


(a) Use Prim's or Kruskal's algorithm to find a minimum weight spanning tree. Your solution must contain all relevant steps in the algorithm. (2p)

(b) What is the Prüfer sequence of the tree obtained in (a)? (1p)

(c) Is the minimum spanning tree obtained in (a) unique? Why (not)? (2p)

Problem 4. Consider the flow network



(a) By applying the Ford-Fulkerson algorithm, find a maximal flow from s to t in the above network. Your solution must contain all relevant steps in the algorithm. (3p)

(b) Construct a minimum s-t-cut, and verify that the capacity of the cut equals the value of the flow in (a). (2p)

Problem 5. Let $n \geq 5$. Consider a graph G_n constructed in the following way: Vertices are the $\binom{n}{2}$ two-element subsets of $\{1, 2, \dots, n\}$, and two vertices are connected by an edge iff the corresponding subsets are disjoint. Show that the graphs G_n are non-planar. (5p)

Problem 6. Consider the complete graph K_n with vertices labelled $\{1, ..., n\}$. Show that for any finite simple graph G, there is a bijection between proper n-colourings of G, and graph homomorphisms $G \to K_n$. (5p)

Problem 7. For $n \geq 3$, construct a graph W_n as follows: To C_n, add a new vertex w, and draw an edge from w to every vertex in C_n. (1p each)

(a) Show that W_n is planar.
(b) What is a planar dual for W_n?
(c) What is the vertex connectivity $\kappa(W_n)$?
(d) What is the independence number $\alpha(W_n)$?
(e) What is the chromatic number $\chi(W_n)$?

Problem 8. The purpose of this problem is to show that the critical function for the existence of a path on three vertices in the Erdős-Renyi graph $G(n, p)$ is $p(n) = n^{-3/2}$.

(a) Show that there are $3\binom{n}{3}$ copies of P_3 in K_n. (1p)
(b) Use the first moment method to show that $G(n, p)$ doesn't contain any copy of P_3 asymptotically almost surely for $p \ll n^{-3/2}$. (2p)
(c) Use the second moment method to show that $G(n, p)$ asymptotically almost surely contains a copy of P_3 for $p \gg n^{-3/2}$. (2p)

Please note: The only aids allowed are course lecture notes/videos and Diestel's book "Graph Theory". Aids not listed here are not permitted.

The problems above are to be solved and written down individually.

Upload your solutions as **one** pdf-file to studium, or send it by mail if uploading causes problems. Please take care that the solution is clearly readable, and that your name is written on it.