Exam 14 August 2023

(1) See Lecture 2

(2) Note that

$$\mathbb{E}(|X_a|; |X_a|>K) = \mathbb{E}(X_a; X_a>K)$$
$$= \int_K^\infty x\cdot(a-1)x^{-a} I_{[1,\infty)}(x)\,dx$$
$$= \int_K^\infty (a-1)x^{1-a}\,dx \qquad (\text{for } K\geq 1)$$
$$= \frac{(a-1)x^{2-a}}{2-a}\Big|_K^\infty$$
$$= \frac{a-1}{a-2}K^{2-a} \qquad \text{for } a>2 \text{ (otherwise, it is infinite)}$$

Let $a^* = \inf A$. If $a^*>2$, then we have

$$\mathbb{E}(X_a; X_a>K) = \frac{a-1}{a-2}K^{2-a} = \left(1+\frac{1}{a-2}\right)\frac{1}{K^{a-2}}$$
$$\leq \left(1+\frac{1}{a^*-2}\right)\frac{1}{K^{a^*-2}} \qquad \text{for all } K\geq 1,\ a\in A$$

So if we choose $K > \left(\frac{\varepsilon}{1+\frac{1}{a^*-2}}\right)^{-\frac{1}{a^*-2}}$, then

$$\mathbb{E}(X_a; X_a>K) < \varepsilon \quad \text{for all } a\in A$$

Thus the family is uniformly integrable if $a^* = \inf A > 2$.

On the other hand, if $a^* \leq 2$, then either $a\leq 2$ for some $a\in A$ (so that $\mathbb{E}(X_a)=\infty$), or there is a sequence $a_1 \geq a_2 \geq \dots$ of elements in A such that $a_n \to 2$. For these, we have

$$\lim_{n\to\infty} \mathbb{E}(|X_{a_n}|; |X_{a_n}|>K) = \lim_{n\to\infty}\left(1+\frac{1}{a_n-2}\right)\frac{1}{K^{a_n-2}} = \infty$$

for every choice of K. Thus there is no choice of K such that $\mathbb{E}(|X_a|; |X_a|>K) < \varepsilon$ for all $a\in A$ if ε is given. So the family is not uniformly integrable if $a^* = \inf A \leq 2$.

(3) We can write

$$\int_{-\infty}^{\infty}\big(P(X<x\le Y)-P(Y<x\le X)\big)dx$$

$$=\int_{-\infty}^{\infty}\int_{\Omega}\left(I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}\right)dP\,dx$$

Now note first that

$$\int_{-\infty}^{\infty}\int_{\Omega}\left|I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}\right|dP\,dx$$

$$=\int_{\Omega}\int_{-\infty}^{\infty}\left|I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}\right|dx\,dP$$

$$\le\int_{\Omega}\int_{-\infty}^{\infty}\left(I_{\{X<x\le Y\}}+I_{\{Y<x\le X\}}\right)dx\,dP$$

$$=\int_{\Omega}\left((Y-X)I_{\{X<Y\}}+(X-Y)I_{\{Y<X\}}\right)dP$$

$$=\int_{\Omega}|Y-X|\,dP<\infty \quad \text{since } X,Y \text{ are integrable}$$

So $I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}$ is integrable, and Fubini's theorem applies. Thus

$$\int_{-\infty}^{\infty}\big(P(X<x\le Y)-P(Y<x\le X)\big)dx$$

$$=\int_{-\infty}^{\infty}\int_{\Omega}\left(I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}\right)dP\,dx$$

$$=\int_{\Omega}\int_{-\infty}^{\infty}\left(I_{\{X<x\le Y\}}-I_{\{Y<x\le X\}}\right)dx\,dP$$

$$=\int_{\Omega}\left((Y-X)I_{\{X<Y\}}-(X-Y)I_{\{Y<X\}}\right)dP$$

$$=\int_{\Omega}(Y-X)\,dP=\mathbb{E}Y-\mathbb{E}X$$

(4) See Lecture 4

⑤ (a) A stopping time with respect to a filtration $\mathcal{F}_n$ is a random variable T with values in $\{0, 1, \dots, \infty\}$ such that

$$\{T = n\} \in \mathcal{F}_n$$

τ_1 is not a stopping time:

$\tau_1 = \sup\{n : P_n \le 10\}$ is the last time that $P_n \le 10$. Since the possible values of P_n are $1, 2, 4, 8, \dots$, we can also write

$$\{\tau_1 = n\} = \{P_n = 8\} \cap \{P_{n+1} = 16\} = \{P_n = 8\} \cap \{X_{n+1} = 2\}$$

which is not in $\mathcal{F}_n$, since it involves X_{n+1}.

τ_2 is a stopping time:

$\tau_2 = \inf\{n : P_n > 10\}$ is the first time that $P_n > 10$. We have

$$\{\tau_2 = n\} = \{P_{n-1} = 8\} \cap \{P_n = 16\} = \{P_{n-1} = 8\} \cap \{X_n = 2\} \in \mathcal{F}_n$$

τ_3 is a stopping time:

$\tau_3 = \inf\{n : P_n = P_{n-10}\} = \inf\{n : X_n = X_{n-1} = \dots = X_{n-9} = 1\}$

is the first time that ten consecutive X_i are equal to 1. We have

$$\{\tau_3 = n\} = \{X_n = X_{n-1} = \dots = X_{n-9} = 1\} \cap \{X_j X_{j-1} \cdots X_{j-9} \ne 1 \text{ for all } j < n\}$$

This also lies in $\mathcal{F}_n$.

(b) We have

$$\mathbb{E}(P_n \mid \mathcal{F}_{n-1}) = \mathbb{E}(P_{n-1} X_n \mid \mathcal{F}_{n-1})$$
$$= P_{n-1}\,\mathbb{E}(X_n) = \tfrac{3}{2} P_{n-1}$$

Thus for $c = \frac{3}{2}$,

$$\mathbb{E}(c^{-n} P_n \mid \mathcal{F}_{n-1}) = c^{-n}\,\mathbb{E}(P_n \mid \mathcal{F}_{n-1}) = c^{-n} \cdot \tfrac{3}{2} P_{n-1} = c^{-(n-1)} P_{n-1},$$

meaning that $c^{-n} P_n$ is a martingale.

Note finally that

$$\ln(c^{-n} P_n) = -n \ln c + \ln P_n$$
$$= -n \ln c + \ln(X_1 X_2 \cdots X_n)$$
$$= (\ln X_1 - \ln c) + (\ln X_2 - \ln c) + \dots + (\ln X_n - \ln c)$$

Since $\mathbb{E}(\ln X_i - \ln c) = \left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln 1\right) - \ln c$
$$= \frac{1}{2}\ln 2 - \ln\frac{3}{2} = \frac{1}{2}\left(\ln 2 - \ln\frac{9}{4}\right)$$
$$= \frac{1}{2}\ln\frac{8}{9} < 0$$
it follows from the strong law of large numbers that
$$\frac{1}{n}\ln(c^{-n}P_n) \xrightarrow{a.s.} \frac{1}{2}\ln\frac{8}{9} < 0$$
Thus $\ln(c^{-n}P_n) \xrightarrow{a.s.} -\infty$ and $c^{-n}P_n \xrightarrow{a.s.} 0$.

(6) Note first that X_n is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$:
$$\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = \frac{1}{2}\cdot\frac{X_{n-1}^2 + 2X_{n-1} - 1}{2} + \frac{1}{2}\cdot\frac{-X_{n-1}^2 + 2X_{n-1} + 1}{2} = X_{n-1}.$$
Moreover, $X_n \in (-1,1)$ for all n: this follows by induction since $X_0 \in (-1,1)$, and
$$\frac{X_{n-1}^2 + 2X_{n-1} - 1}{2} = \frac{(X_{n-1}+1)^2 - 2}{2} \in \left(\frac{0^2-2}{2}, \frac{2^2-2}{2}\right) = (-1,1)$$
if $X_{n-1} \in (-1,1)$ as well as
$$\frac{-X_{n-1}^2 + 2X_{n-1} + 1}{2} = \frac{-(X_{n-1}-1)^2 + 2}{2} \in \left(\frac{-2^2+2}{2}, \frac{-0^2+2}{2}\right) = (-1,1)$$
if $X_{n-1} \in (-1,1)$.
So X_n is a bounded martingale, thus convergent (almost surely) by the martingale convergence theorem.
A possible limit L needs to satisfy $L = \lim_{n\to\infty} X_n = \lim_{n\to\infty} X_{n-1}$, thus
$$L = \frac{L^2 + 2L - 1}{2} \quad \text{and} \quad L = \frac{-L^2 + 2L + 1}{2}$$
$$\Leftrightarrow L^2 = 1 \qquad\qquad \Leftrightarrow L^2 = 1$$
$$\Leftrightarrow L = \pm 1 \qquad\qquad \Leftrightarrow L = \pm 1$$
So the possible limits are ± 1. Finally, $\mathbb{E}(X_n) = \mathbb{E}(X_0) = a$ for all n, thus
$$\mathbb{E}\left(\lim_{n\to\infty} X_n\right) = \lim_{n\to\infty}\mathbb{E}(X_n) = a$$

and at the same time

$$\mathbb{E}\left(\lim_{n\to\infty} X_n\right) = P\left(\lim_{n\to\infty} X_n = 1\right) - P\left(\lim_{n\to\infty} X_n = -1\right)$$
$$= P\left(\lim_{n\to\infty} X_n = 1\right) - \left(1 - P\left(\lim_{n\to\infty} X_n = 1\right)\right)$$
$$= 2P\left(\lim_{n\to\infty} X_n = 1\right) - 1,$$

thus $P\left(\lim_{n\to\infty} X_n = 1\right) = \frac{a+1}{2}$ and $P\left(\lim_{n\to\infty} X_n = -1\right) = \frac{1-a}{2}$.

(7) See Lectures 15 and 16

(8) (a) See Lectures 16 and 17

(b) See Lectures 19 and 20

(c) For example, consider the following simple model with a single time step:

$$S_0 = 2 \quad \begin{array}{l} \xrightarrow{q_1} S_1 = 1 \\ \xrightarrow{q_2} S_1 = 2 \\ \xrightarrow{q_3} S_1 = 3 \end{array}$$

Assume also that the risk-free rate is 0. Now both $q_1 = q_2 = q_3 = \frac{1}{3}$ and $q_1 = q_3 = \frac{1}{4}$, $q_2 = \frac{1}{2}$ (and infinitely many others) define equivalent martingale measures.

Since there is a martingale measure, but not a unique one, the model is viable, but not complete.