Formulas for Fourier Analysis course

Triangle inequalities
Let $x, y \in \mathbb{R}$ and f, g be functions. Then

- $\big| |x| - |y| \big| \leq |x \pm y| \leq |x| + |y|$
- $\left| \int_\Omega f(x)\,\mathrm{d}x \right| \leq \int_\Omega |f(x)|\,\mathrm{d}x$, for a subset $\Omega \subset \mathbb{R}$.

Some useful identities

- $e^{a+ib} = e^a(\cos(b) + i\sin(b))$
- $\int_\mathbb{R} x^n e^{-x^2/2}\,\mathrm{d}x = \begin{cases} \sqrt{2\pi}(n-1)(n-3)\dots 5\cdot 3\cdot 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

Gram–Schmidt orthogonalisation
Let V be an inner product space and $\{v_1, \dots, v_k\} \subset V$ be a linearly independent set of vectors. Then the Gram–Schmidt orthogonalisation is given by

$$\begin{aligned}
u_1 &= v_1, & e_1 &= \frac{u_1}{\|u_1\|} \\
u_2 &= v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle}u_1, & e_2 &= \frac{u_2}{\|u_2\|} \\
u_3 &= v_3 - \frac{\langle u_1, v_3\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle u_2, v_3\rangle}{\langle u_2, u_2\rangle}u_2, & e_3 &= \frac{u_3}{\|u_3\|} \\
&\vdots & &\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} \frac{\langle u_j, v_k\rangle}{\langle u_j, u_j\rangle}u_j, & e_k &= \frac{u_k}{\|u_k\|}.
\end{aligned}$$

Laplace transform

$f(t)$	$\tilde{f}(s) = F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
General formulas	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$e^{at} f(t)$	$F(s-a)$
$f(at), \quad a > 0$	$\frac{1}{a} F(\frac{s}{a})$
$f(t-a)H(t-a), \quad a > 0$	$e^{-as} F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0)$ $\quad - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(u)\,du$	$s^{-1} F(s)$
$f * g(t) = \int_0^t f(u) g(t-u)\,du$	$F(s)\,G(s)$
Particular cases	
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$

Fourier Series

Functions of period 2π

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n})$$

Parseval's formula:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Convolution:

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) g(t-u)\, du.$$

Functions of period T: Let $\Omega = 2\pi/T$.

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t),$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\Omega t}\, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\Omega t\, dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\Omega t\, dt.$$

Parseval's formula:

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Convolution:

$$(f * g)(t) = \frac{1}{T} \int_{-T/2}^{T/2} f(u) g(t-u)\, du.$$

Some trigonometric identities

$$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$$

$$2 \sin a \cos b = \sin(a-b) + \sin(a+b)$$

$$2 \cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2 \sin^2 t = 1 - \cos 2t, \qquad 2 \cos^2 t = 1 + \cos 2t$$

Fourier transform

$f(t)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$
General formulas	
$\alpha f(t) + \beta g(t)$	$\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$
$e^{i\alpha t} f(t)$	$\hat{f}(\omega - \alpha)$
$f(t - t_0)$	$e^{-it_0\omega} \hat{f}(\omega)$
$f(-t)$	$\hat{f}(-\omega)$
$f(at) \quad (a \neq 0)$	$\frac{1}{\lvert a\rvert} \hat{f}(\frac{\omega}{a})$
$t f(t)$	$i \frac{d\hat{f}}{d\omega}$
$f'(t)$	$i\omega \hat{f}(\omega)$
$\hat{f}(t)$	$2\pi f(-\omega)$
$f * g(t) = \int_{-\infty}^{\infty} f(u) g(t-u)\, du$	$\hat{f}(\omega)\hat{g}(\omega)$
Particular cases	
$\chi_{[-a,a]}$	$\frac{2 \sin a\omega}{\omega}$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert\omega\rvert}$
$e^{-t^2/2}$	$\sqrt{2\pi} e^{-\omega^2/2}$
δ	1
1	$2\pi\delta$

Plancherel's formulas:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, d\omega$$