Inference 2, 2023, lecture 12

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Today

Chap. 5. Testing hypotheses (continued): Unbiased tests

Unbiased tests

Example 1:

- Let $\mathbf{X} = (X_1, ..., X_n)$ be a i.i.d. sample from $N(\mu, \sigma^2)$ with known σ^2.
- Test H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$.
- Is there any UMP test for this situation?

Unbiased tests

- Suppose we want to test H_0: $\theta \in \Theta_0$ vs H_1: $\theta \in \Theta_1$ at significance level α.
- It is reasonable to require that the power function is at least equal to α for all $\theta \in \Theta_1$.

Definition (5.11)

A test is called an **unbiased α-test** if.f.
$\alpha = \sup_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\mathbf{X})$ and $\inf_{\theta \in \Theta_1} \mathrm{E}_\theta \varphi(\mathbf{X}) \geq \alpha$.

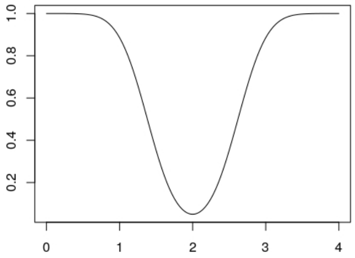
Unbiased tests

Example 1':

- Suggest an unbiased test for example 1:
- Reject if $|\bar{x} - \mu_0| > \lambda_{\alpha/2}\sigma/\sqrt{n}$.
- The power function is (why?)

$$\pi(\mu) = \Phi\left(-\lambda_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right) + \Phi\left(-\lambda_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right).$$

Power function, for $\mu_0 = 2$, $\sigma = 1$, $n = 10$, $\alpha = 0.05$:



Unbiased tests

Definition (5.12)

A test φ^* is called
a **uniformly most powerful unbiased (UMPU)** α**-test** if.f.
$\alpha = \sup_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi^*(\mathbf{X})$ and $\mathrm{E}_\theta \varphi^*(\mathbf{X}) \geq \mathrm{E}_\theta \varphi(\mathbf{X})$ for all $\theta \in \Theta_1$
and **for all unbiased** α-tests φ.

Unbiased tests

Theorem (5.5)

Let $\mathbf{X} = (X_1, ..., X_n)$ *be a sample from a distribution* P_θ *belonging to a one-parameter exponential family with*

$$p(\mathbf{x}; \theta) = C(\theta) \exp\{\theta T(\mathbf{x})\} h(\mathbf{x}),$$

where $\theta \in \mathcal{R}$. *For* H_0: $\theta = \theta_0$ *vs* H_1: $\theta \neq \theta_0$, *the UMPU* α*-test is given by*

$$\varphi\{T(\mathbf{x})\} = \begin{cases} 1 & \textit{if} \quad T(\mathbf{x}) > k_u; \; T(\mathbf{x}) < k_l, \\ \gamma_1 & \textit{if} \quad T(\mathbf{x}) = k_u, \\ \gamma_2 & \textit{if} \quad T(\mathbf{x}) = k_l, \\ 0 & \textit{if} \quad k_l < T(\mathbf{x}) < k_u, \end{cases}$$

with γ_1, γ_2, k_u, k_l *such that*

$$\mathrm{E}_{\theta_0} \varphi(\mathbf{X}) = \alpha$$

and

$$\mathrm{E}_{\theta_0}\{\varphi(\mathbf{X}) T(\mathbf{X})\} = \alpha \mathrm{E}_{\theta_0}\{T(\mathbf{X})\}.$$

Unbiased tests

Corollary (5.3)

Under the conditions of theorem 5.5 and under the additional assumption that T has a symmetrical distribution around some point a under the null hypothesis, the UMPU α-test has the form

$$\varphi\{T(\mathbf{x})\} = \begin{cases} 1 & \text{if} \quad T(\mathbf{x}) - a > a - C; \;\; T(\mathbf{x}) - a < C - a, \\ \gamma & \text{if} \quad T(\mathbf{x}) - a = a - C, \\ \gamma & \text{if} \quad T(\mathbf{x}) - a = C - a, \\ 0 & \text{if} \quad C - a < T(\mathbf{x}) - a < a - C, \end{cases}$$

with γ, C such that

$$\mathrm{P}_{\theta_0}^{T}(T - a < C - a) + \gamma \mathrm{P}_{\theta_0}^{T}(T - a = C - a) = \frac{\alpha}{2}.$$

Unbiased tests

Example 2: Derive UMPU tests on level $\alpha = 0.05$ as explicitly as possible for the following situations.

1. Let $\mathbf{X} = (X_1, ..., X_n)$ be an i.i.d. sample from $N(\mu, \sigma^2)$ with known σ^2. Test H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$.
2. $X \sim \mathrm{Bin}(5, p)$. Test $H_0 : p = 1/2$ vs $H_1 : p \neq 1/2$.
3. Let $\mathbf{X} = (X_1, ..., X_n)$ be an i.i.d. sample from $N(\mu, \theta)$ with known μ. Test H_0: $\theta = \theta_0$ vs H_1: $\theta \neq \theta_0$.

News of today

Generalization to two-sided tests:

- Unbiased test
- UMPU test
- The exponential family: A (certain) test is UMPU if it has the correct size and if the test function is uncorrelated with the test statistic.