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Answers to selected problems in

Shumway and Stoffer(2017): Time Series Analysis and Its Applications, 4th ed.

1.6. (a) x_t is not stationary (the mean is not constant)

1.7.

$$\gamma(h) = \begin{cases} 6\sigma_w^2 & \text{if} \quad h = 0, \\ 4\sigma_w^2 & \text{if} \quad |h| = 1, \\ \sigma_w^2 & \text{if} \quad |h| = 2, \\ 0 & \text{if} \quad |h| \geq 3. \end{cases}$$

$$\rho(h) = \begin{cases} 1 & \text{if} \quad h = 0, \\ 2/3 & \text{if} \quad |h| = 1, \\ 1/6 & \text{if} \quad |h| = 2, \\ 0 & \text{if} \quad |h| \geq 3. \end{cases}$$

1.8. (b) $\mu_t = t\delta$, $\gamma(s,t) = \sigma_w^2 \min(s,t)$

(c) For $h \geq 0$, $\gamma(t+h,t) = \sigma_w^2 t$, which is a function of t (for stationarity, it should be a function only of h).

(d) For large t, x_t is approximately a linear function of x_{t-1}. (In fact, $x_t \approx \delta + x_{t-1}$.)

(e) $y_t = x_t - x_{t-1} = \delta + w_t$ is stationary

1.13. (a)

$$\rho_y(h) = \begin{cases} 1 & \text{if} \quad h = 0, \\ -\frac{\theta\sigma_w^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2} & \text{if} \quad |h| = 1, \\ 0 & \text{if} \quad |h| \geq 2. \end{cases}$$

(b)

$$\rho_{xy}(h) = \begin{cases} \frac{\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}} & \text{if} \quad h = 0, \\ -\frac{\theta\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}} & \text{if} \quad h = -1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) $\rho_{xy}(h)$ is a function of h only.

1.14. (a) $E(y_t) = \exp\left\{\mu_x + \frac{1}{2}\gamma(0)\right\}$

(b) $\gamma(h) = e^{2\mu_x + \gamma(0)}\left\{e^{\gamma(h)} - 1\right\}$

1.15. $\mu_t = 0$, $\rho(h) = \begin{cases} \sigma_w^4 & \text{if} \quad h = 0, \\ 0 & \text{otherwise.} \end{cases}$ Stationary.

1.17. (a) $\phi_w(-\theta\lambda_1)\phi_w(\lambda_1 - \theta\lambda_2)\cdots\phi_w(\lambda_{n-1} - \theta\lambda_n)\phi_w(\lambda_n)$

2.6. (a) $\mu_t = \beta_0 + \beta_1 t$ not constant in t

(b) $\mu_t = \beta_1$, $\gamma(t,t) = 2\sigma_w^2$ finite and constant.

$$\gamma(t+h,t) = \begin{cases} 2\sigma_w^2 & \text{if} \quad h = 0, \\ -\sigma_w^2 & \text{if} \quad |h| = 1, \\ 0 & \text{if} \quad |h| \geq 2, \end{cases}$$

function only of h.

(c) $\mu_t = \beta_1$, $\gamma(t,t) = 2\gamma_y(0) - 2\gamma_y(1)$ finite and constant.
$\gamma(t+h,t) = 2\gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1)$, function only of h.

2.7. $\mu_t = \delta$ constant in t.
$\gamma(t,t) = \sigma_w^2 + 2\gamma_y(0) - 2\gamma_y(1)$, finite and constant.
$\gamma(t+h,t) = \sigma_w^2 I\{h=0\} + 2\gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1)$, function only of h.
($I\{h=0\} = 1$ if $h = 0$ and 0 otherwise)

3.1. $\rho_x(1) = \frac{\theta}{1+\theta^2}$ is maximized at $\theta = 1$ (maximum value 1/2) and minimized at $\theta = -1$.

3.2. (a) Use recursion.

(b) $E(x_t) = 0$.

(c) $\text{var}(x_t) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$.

(e) Not stationary since the autocovariance function (and the variance) is a function of t.

(f) As $t \to \infty$, $\text{var}(x_t) \to \sigma_w^2 \frac{1}{1-\phi^2}$, and so, $\text{corr}(x_t, x_{t-h}) \to \phi^h$.

(g) Use a "burn-in" sample.

(h) Yes, because here $\text{var}(x_t) = \sigma_w^2 \frac{1}{1-\phi^2}$ and $\text{corr}(x_t, x_{t-h}) = \phi^h$.

3.4. (a) AR(1) with $\phi = 0.5$. Causal and invertible.

(b) ARMA(2,1), causal but not invertible.

3.6. Roots $\pm i \frac{1}{\sqrt{0.9}}$

3.11. (a) $\tilde{x}_{n+1} = \sum_{j=1}^{\infty} (-\theta)^j x_{n+1-j}$, MSE σ_w^2.

3.15. $x_{t+m}^t = \phi^m x_t$

3.24. (a) $\mu_t = \frac{\alpha}{1-\phi}$,

$$\gamma(h) = \begin{cases} \sigma_w^2 \frac{1+2\theta\phi+\theta^2}{1-\phi^2} & \text{if} \quad h = 0, \\ \sigma^2 \phi^{h-1} \left(\phi \frac{1+2\theta\phi+\theta^2}{1-\phi^2} + \theta \right) & \text{if} \quad |h| \geq 1. \end{cases}$$

$$\rho(h) = \begin{cases} 1 & \text{if} \quad h = 0, \\ \phi^{h-1} \frac{(\theta+\phi)(1+\theta\phi)}{1+2\theta\phi+\theta^2} & \text{if} \quad |h| \geq 1. \end{cases}$$

Weakly stationary. Also strictly stationary if the w_t are normal.

(b) Asymptotically normal with mean $\frac{\alpha}{1-\phi}$ and variance $n^{-1}\sigma_w^2 \left(\frac{1+\theta}{1-\phi}\right)^2$.

3.29. (c) $P_{n+m}^n = \sigma_w^2 \left\{ 1 + \frac{1}{(1-\phi)^2} \left(m - 1 - 2\frac{\phi^2 - \phi^{m+1}}{1-\phi} + \frac{\phi^4 - \phi^{2m+2}}{1-\phi^2} \right) \right\}$

3.38. (a) ARIMA$(0,0,0)\times(0,0,1)_2$

(c) $\tilde{x}_{n+m} = -\sum_{j=1}^{\lfloor m/2\rfloor}(-\Theta)^j \tilde{x}_{n+m-2j} - \sum_{\lfloor m/2\rfloor+1}^{\infty}(-\Theta)^j x_{n+m-2j}$,
where $\lfloor a\rfloor$ is the integer part of a.
$P_{n+m}^n = \begin{cases} \sigma_w^2 & \text{if} \quad m=1,2, \\ \sigma_w^2(1+\Theta^2) & \text{if} \quad m\geq 3. \end{cases}$

4.5. (a) Both have mean zero. $\gamma_w(h)=1$ if $h=0$ and 0 otherwise.

$$\gamma_x(h) = \begin{cases} 1+\theta^2 & \text{if} \quad h=0, \\ -\theta & \text{if} \quad |h|=1, \\ 0 & \text{if} \quad |h|\geq 2. \end{cases}$$

Both are weakly stationary, because the means and variances are constant, the variances are finite and the autocovariance functions only depend on h. In fact, they are also stationary since they are normal processes.

(b) $f_x(\omega) = 1+\theta^2 - 2\theta\cos(2\pi\omega)$

4.18. (a)

$$\gamma_{xy}(h) = \begin{cases} 0 & \text{if} \quad h=0, \\ 1/2 & \text{if} \quad h=1, \\ -1/2 & \text{if} \quad h=-1, \\ 0 & \text{if} \quad |h|\geq 2. \end{cases}$$

The cross autocovariance is a function only of h, hence we have joint stationarity.

(b) $f_x(\omega) = 2-2\cos(2\pi\omega)$, $f_y(\omega) = \frac{1}{2}+\frac{1}{2}\cos(2\pi\omega)$. When one is high, the other one is low.

(c) $a=0.2465$, $b=1.898$.

4.25. (a) $\phi/(1+\phi^2)$

4.28. $f_y(\omega) = \{6+8\cos(2\pi\omega)+2\cos(4\pi\omega)\}^2\sigma_w^2$

4.30. (b) The resulting spectrum is $4\{1-\cos(2\pi\omega)\}\{1-\cos(24\pi\omega)\}f_x(\omega)$.

(c) For $0\leq\omega\leq 1/2$, the filter gives highest weights to frequencies which are odd multiples of $1/24$, increasing in ω.

4.31. (a) $f_y(\omega) = \{1+a^2-2a\cos(2\pi\omega)\}^{-1}f_x(\omega)$

6.1. (b) $\sigma_0^2 = \sigma_1^2 = \sigma_w^2/0.19$

6.2. $\phi^{|s-t|}(P_t^{t-1}+\sigma_v^2)$