Analysis of Time Series, L4

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Today

- 3.2: Autoregressive moving average (ARMA) models
- 3.3: Difference equations
- Menti

ARMA models

ARMA(1,1):

- Let

$$x_t = \phi x_{t-1} + w_t + \theta w_{t-1}$$

- On operator form:

$$\phi(B) x_t = \theta(B) w_t$$

  where
  $\phi(B) = 1 - \phi B$,
  $\theta(B) = 1 + \theta B$.

- What happens if $\phi = -\theta$?

ARMA models

Definition (3.5)

An *autoregressive moving average model* of order p, q, ARMA(p, q), is given by

$$x_t = \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \dots + \theta_q w_{t-q},$$

where w_t is white noise with mean zero and variance σ^2.

On operator form:

$$\phi(B) x_t = \theta(B) w_t$$

where
$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$,
$\theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q$.

ARMA models

Example 1:

Let

$$x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t - 0.9w_{t-1} + 0.2w_{t-2}.$$

Is this an ARMA(2,2) model?

ARMA models

Potential problems with ARMA models:

- Parameter redundancy
  (cf the previous example)
- Different MA models have the same ACF
  (cf $x_t = w_t + \theta w_{t-1}$, $x_t = w_t + \theta^{-1} w_{t-1}$).
- Stationary AR models that depend on the future
  (cf $x_t = \phi x_{t-1} + w_t$ with $\phi > 1$)

ARMA models

Definition (3.7)

An ARMA model is said to be *causal* if it can be written as a one-sided linear process, i.e.

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j},$$

where $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

Consider the AR(1) process $x_t = \phi x_{t-1} + w_t$.
For which ϕ is it causal?

ARMA models

Definition (3.6)

The *AR and MA polynomials* are defined as

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \quad \phi_p \neq 0,$$

and

$$\theta(z) = 1 + \theta_1 z - \ldots + \theta_q z^q, \quad \theta_q \neq 0,$$

where z is a complex number.

Theorem (Property 3.1)

1. *An ARMA(p, q) model is causal if and only if $\phi(z) \neq 0$ for all $|z| \leq 1$.*
2. *The coefficients of the MA representation may be found through* $\sum_{j=0}^{\infty} \psi_j z^j = \psi(z) = \frac{\theta(z)}{\phi(z)}, \quad |z| \leq 1.$

The proof (Appendix B) builds on a series expansion of $1/\phi(z)$.

ARMA models

Definition (3.8)

An ARMA(p, q) model is said to be *invertible* if it can be written on AR form as

$$\sum_{j=0}^{\infty} \pi_j x_{t-j} = w_t,$$

where $\sum_{j=0}^{\infty} |\pi_j| < \infty$.

Theorem (Property 3.2)

1. *An ARMA(p, q) model is invertible if and only if $\theta(z) \neq 0$ for all $|z| \leq 1$.*
2. *The coefficients of the AR representation may be found through*

$$\sum_{j=0}^{\infty} \pi_j z^j = \pi(z) = \frac{\phi(z)}{\theta(z)}, \quad |z| \leq 1.$$

ARMA models

ARMA model $\phi(B)x_t = \theta(B)w_t$.
Note:

- MA processes are always causal, because $\phi(z) = 1 \neq 0$ for all z (in particular for all $|z| \leq 1$).
- AR processes are always invertible, because $\theta(z) = 1 \neq 0$ for all z (in particular for all $|z| \leq 1$).

ARMA models

Example 1:

Let

$$x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t - 0.9w_{t-1} + 0.2w_{t-2}.$$

1. Is the model causal?
2. Is the model invertible?
3. Find the MA representation.

ARMA models

Example 2: AR(2)

Let

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t.$$

For which choices of ϕ_1 and ϕ_2 is the model causal?

ARMA models

Example 2 cont.:

1. For which (ϕ_1, ϕ_2) do the solutions $z_{1,2}$ of

$$0 = \phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

   fulfill $|z_{1,2}| > 1$?
   Assume $\phi_2 \neq 0$, $z_{1,2} \neq 0$.

2. Solutions (why?):

$$z_{1,2} = -\frac{\phi_1}{2\phi_2} \pm r, \quad r = \sqrt{\frac{\phi_1^2}{4\phi_2^2} + \frac{1}{\phi_2}}.$$

3. It follows that (why?)

$$\frac{1}{z_{1,2}} = \frac{\phi_1 \pm 2\phi_2 r}{2}.$$

ARMA models

Example 2 cont.:

4. This, together with $|z_{1,2}| > 1$, implies (why?)

$$\phi_2 - \phi_1 < 1, \quad \phi_2 + \phi_1 < 1.$$

5. From the factorization theorem,

$$z^2 + \frac{\phi_1}{\phi_2} z - \frac{1}{\phi_2} = (z - z_1)(z - z_2),$$

which implies (why?)

$$1 < |z_1||z_2| = |z_1 z_2| = \frac{1}{|\phi_2|}$$

i.e. $|\phi_2| < 1$.

6. Sketch the region in the (ϕ_1, ϕ_2) plane!

Difference equations

Order one:

- Solve $u_n - \alpha u_{n-1} = 0$, $u_0 = c$.
- Recursion yields $u_n = \alpha^n c$.
- Equivalently, $\alpha(B)u_n = (1 - \alpha B)u_n = 0$, $u_0 = c$
  is solved by $u_n = (z_0^{-1})^n c$
  where $z_0 = \alpha^{-1}$ is a root of $\alpha(z) = 1 - \alpha z$.

Difference equations

Order two:

- Solve
$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0 \tag{1}$$
- Equivalently, write $\alpha(B)u_n = (1 - \alpha_1 B - \alpha_2 B^2)u_n = 0$.
- Denote the roots of $\alpha(z)$ by z_1 and z_2.
- If $z_1 \neq z_2$, the general solution to (1) is
$$u_n = c_1 z_1^{-n} + c_2 z_2^{-n}.$$
- If $z_1 = z_2$, the general solution to (1) is
$$u_n = z_1^{-n}(c_1 + c_2 n).$$

News of today

- ARMA processes
- Parameter redundancy
- Causality
- Invertibility
- Difference equations and their use