Lecture 10

Theorem (not in the book). Let $F(t,s)$ be the pricing function of a simple T-claim $\mathcal{X} = \Phi(S_T)$ in the standard BS-model. If Φ is convex, then

i) $F(t,s)$ is convex in s

ii) $F(t,s)$ is increasing in σ.

Proof: $F(0,s) = e^{-rT} \int_{\mathbb{R}} \Phi\left(s \exp\{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}x\}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

i) $F_{ss} = e^{-rT} \int_{\mathbb{R}} \Phi''(s \exp\{\dots\}) \exp\{2(\dots)\} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \geq 0$

ii) $\frac{\partial F}{\partial \sigma} = \int_{\mathbb{R}} \Phi'(s \exp\{\dots\})\, s \exp\{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}x\} \sqrt{T}(x - \sigma\sqrt{T}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$= s\sqrt{T} \int_{\mathbb{R}} \Phi'(s \exp\{\ \})\, (x - \sigma\sqrt{T})\, e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} \frac{1}{\sqrt{2\pi}} dx$$

$$\overset{\text{int. by parts}}{=} s\sqrt{T} \int_{\mathbb{R}} \Phi''(\ )\, s \exp\{\dots\} \sigma\sqrt{T}\, e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} \frac{1}{\sqrt{2\pi}} dx \geq 0$$

Drift estimation (not in the book)

Assume $X_t = \mu t + \sigma W_t$ and we want a confidence interval for μ. An estimate for μ is $\hat{\mu} = \frac{X_t}{t} \in N(\mu, \frac{\sigma}{\sqrt{t}})$, and a confidence interval is

$$\left(\hat{\mu} - \frac{\sigma}{\sqrt{t}} 1.96 \;,\; \hat{\mu} + \frac{\sigma}{\sqrt{t}} 1.96\right) \qquad (95\%\text{-confidence})$$

If one wants a certain precision $\Delta\mu$ so that $P(\mu \in (\hat{\mu} - \Delta\mu, \hat{\mu} + \Delta\mu)) = 0.95$, one needs

$$\frac{2\sigma}{\sqrt{t}} = \Delta\mu \;, \quad \text{i.e.} \quad t = \frac{4\sigma^2}{(\Delta\mu)^2}.$$

Plug in reasonable values $\begin{cases} \sigma = 0.3 \\ \Delta\mu = 0.06 \end{cases} \Rightarrow t = 100$ years!

Remark: When pricing options, the drift of the stock needs not be estimated! (since under the pricing measure Q, the drift is r)!

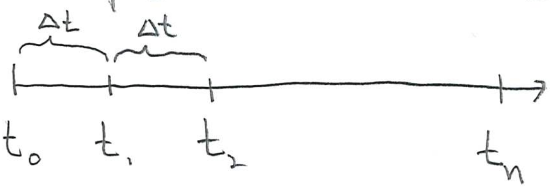
7.8 Volatility

In the BS-formula, $\underbrace{s, r, t}_{\text{observable}}, \underbrace{T, K}_{\text{specified in the contract}}, \sigma$ are needed.

σ: not directly observable

Two approaches:

1. Historic volatility: If $dS_t = \mu S_t dt + \sigma S_t dW_t$, sample S at $n+1$ time points



Let $\xi_i = \ln \frac{S_{t_i}}{S_{t_{i-1}}} = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma(W_{t_i} - W_{t_{i-1}})$

$$\in N\left((\mu - \tfrac{\sigma^2}{2})\Delta t,\ \sigma\sqrt{\Delta t}\right).$$

An estimate of σ^2 is then

$$s^2 := \frac{\sum_{i=1}^{n} (\xi_i - \bar{\xi})^2}{(n-1)\Delta t} \quad \text{where } \bar{\xi} = \frac{1}{n}\sum_{i=1}^{n} \xi_i .$$

2. Implied volatility: Let p be the price in the market of a certain call option (maturity T, strike K).

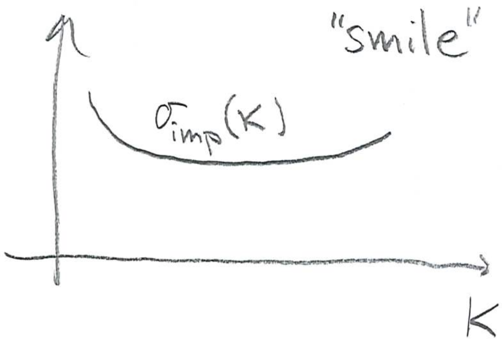
Find σ such that $p = BS(s, t, T, r, \sigma, K)$ (where BS denotes the Black-Scholes formula).

This σ is called implied volatility.

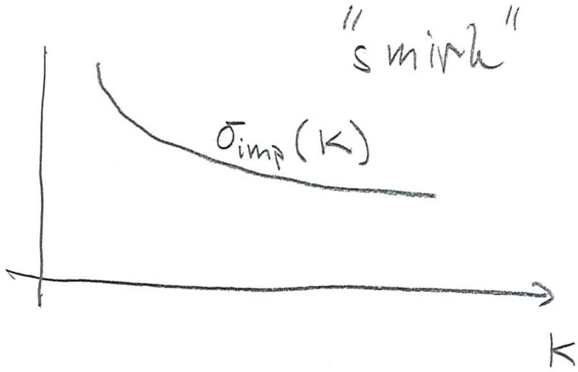
Remark: Recall that the BS-formula is increasing in σ.

If gBm is the correct model (i.e. option prices are calculated using the BS-formula) then the same implied volatility would be obtained for different K and T.

In reality:



or



8 Completeness and Hedging

Def 8.1 A T-claim $\mathcal{X}$ can be replicated if there exists a self-financing portfolio h with $P(V_T^h = \mathcal{X}) = 1$. If every T-claim can be replicated then the market is complete.

Prop 8.2 Assume that a T-claim $\mathcal{X}$ can be replicated using h. Then the only possible arbitrage-free price of $\mathcal{X}$ is $\Pi_t(\mathcal{X}) = V_t^h$.

Proof: If for example $\Pi_t(\mathcal{X}) < V_t^h$ for some t, sell the portfolio and buy the claim $\Rightarrow$ arbitrage.

We now specialise to the model

$$(*) \quad \begin{cases} dB_t = r B_t \, dt \\ dS_t = \mu(t,S_t) S_t \, dt + \sigma(t,S_t) S_t \, dW_t \end{cases} \qquad (\text{with } \sigma(t,s) > 0)$$

Theorem 8.3 The model $(*)$ is complete

We will prove a simpler result, namely that all simple T-claims can be replicated.

Recall: The value $\Pi_t(\mathcal{X})$ of a simple T-claim $\mathcal{X} = \Phi(S_T)$ is $F(t, S_t)$, where $F(t,s)$ is the pricing function. Thus

$$d\Pi_t = F_t \, dt + F_s \, dS_t + \frac{1}{2} F_{ss} (dS_t)^2$$

$$= \left(F_t + \frac{\sigma^2}{2} S_t^2 F_{ss}\right) dt + F_s \, dS_t \, .$$

Moreover, a portfolio $h = (h^B, h^S)$ is self-financing if $dV_t^h = h_t^B \, dB_t + h_t^S \, dS_t$. Choose $h_t^S = F_s(t, S_t)$!

Theorem 8.5 Let $\mathcal{X} = \Phi(S_T)$ and define $F(t,s)$ by

$$\begin{cases} F_t + \frac{\sigma^2 s^2}{2} F_{ss} + r s F_s - r F = 0 \\ F(T,s) = \Phi(s) \, . \end{cases}$$

Define $h = (h^B, h^S)$ by

$$\begin{cases} h_t^B = \dfrac{F(t,S_t) - S_t F_s(t,S_t)}{B_t} \\ h_t^S = F_s(t, S_t) \, . \end{cases}$$

Then h replicates $\mathcal{X}$, and $\Pi_t(\mathcal{X}) = V_t^h = F(t, S_t)$.

Pf: $V_t^h = h_t^B B_t + h_t^S S_t = F(t, S_t)$

so

$$dV_t^h = F_t\,dt + F_s\,dS_t + \tfrac{1}{2} F_{ss} (dS_t)^2$$
$$= \left(F_t + \tfrac{\sigma^2}{2} S_t^2 F_{ss}\right) dt + F_s\,dS_t$$
$$\overset{\text{BS PDE}}{=} r(F - S_t F_s)\,dt + F_s\,dS_t = h_t^B dB_t + h_t^S dS_t$$

Thus h is self-financing. Since $V_T^h = F(T, S_T) = \phi(S_T) = \mathcal{X}$, h replicates $\mathcal{X}$. By no-arbitrage (Prop. 8.2), $\Pi_t(\mathcal{X}) = V_t^h = F(t, S_t)$.

Ex: If $\mathcal{X} = S_T$, then $F(t,s) = s$ so $h_t^S = F_s = 1$.

Ex: For a call option (in the standard BS-model)

$$F(0,s) = sN(d_1) - Ke^{-rT} N(d_2) \quad \text{where} \quad \begin{cases} d_1 = \dfrac{\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T}. \end{cases}$$

Thus

$$F_s(0,s) = N(d_1) + \frac{1}{\sqrt{2\pi}}\left(s e^{-\frac{d_1^2}{2}} - Ke^{-rT} e^{-\frac{d_2^2}{2}}\right)\frac{\partial d_1}{\partial s}.$$

Moreover,

$$s e^{-\frac{d_1^2}{2}} - Ke^{-rT} e^{-\frac{d_2^2}{2}} = e^{-\frac{d_1^2}{2}}\left(s - Ke^{-rT} e^{-\frac{\sigma^2 T}{2}} e^{\sigma\sqrt{T} d_1}\right) = 0$$

so

$$F_s(0,s) = N(d_1).$$

Remark: The derivative $\Delta := F_s$ is called the delta.

In a replicating portfolio one should hold Δ shares of S at each time.