Financial Theory – Lecture 4

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Agenda

- Two-asset portfolios.
- Diversification.
- Some special types of portfolios.
- Portfolio mathematics.

The lecture is based on

- Chapter 4.1-4.2 and 4.4-4.5 in the course book.

Basic definitions in portfolio models

We let $\boldsymbol{r} = (r_1, r_2, \ldots, r_N)^\top$ denote a vector of rates of return of N assets.

A vector of portfolio weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ is a vector satisfying

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 1.$$

The rate of return of a portfolio is denoted r_p, or $r(\boldsymbol{\pi})$ if we want to emphasise the specific portfolio weight vector.

We have previously shown that

$$r_p = r(\boldsymbol{\pi}) = \sum_{i=1}^{N} \pi_i r_i = \boldsymbol{\pi} \cdot \mathbf{r}.$$

Two-asset portfolios

Now assume that there exists only two assets: $N = 2$.

Let w be the portfolio weight in asset 1, i.e. the portfolio weights are

$$(\pi_1, \pi_2)^T = (w, 1 - w)^T.$$

Note that w can be any number. The return on this portfolios is

$$r(w) = wr_1 + (1 - w)r_2.$$

We let for $i = 1, 2$

$$E[r_i] = \mu_i,\ \mathrm{Std}[r_i] = \sigma_i \text{ and } \mathrm{Corr}[r_1, r_2] = \rho,$$

and assume that $\mu_1 \neq \mu_2$.

Two-asset portfolios

The mean return of the portfolio is

$$\begin{aligned}\mu(w) &= E\left[r(w)\right] \\ &= E\left[wr_1 + (1-w)r_2\right] \\ &= w\mu_1 + (1-w)\mu_2,\end{aligned}$$

and the variance of the rate of return of the portfolio is

$$\begin{aligned}\sigma^2(w) &= \mathrm{Var}[r(w)] \\ &= \mathrm{Var}[wr_1 + (1-w)r_2] \\ &= \mathrm{Var}[wr_1] + 2\mathrm{Cov}[wr_1, (1-w)r_2] + \mathrm{Var}[(1-w)r_2] \\ &= w^2\sigma_1^2 + 2w(1-w)\mathrm{Cov}[r_1, r_2] + (1-w)^2\sigma_2^2 \\ &= w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2.\end{aligned}$$

Two-asset portfolios

Using the expression for $\mu(w)$ we can solve for the weight as long as $\mu_1 \neq \mu_2$:

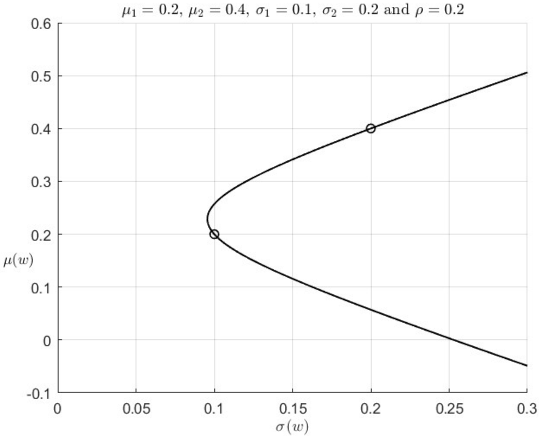
$$\mu(w) = w\mu_1 + (1-w)\mu_2 = \mu_2 + w(\mu_1 - \mu_2) \quad \Leftrightarrow \quad w = \frac{\mu(w) - \mu_2}{\mu_1 - \mu_2}.$$

Now use this expression in the formula for $\sigma(w) = \sqrt{\sigma^2(w)}$:

$$\begin{aligned} \sigma(w) &= \sqrt{w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2} \\ &= \ldots \\ &= \sqrt{K_0 + K_1\mu(w) + K_2\mu(w)^2} \end{aligned}$$

for some constants K_0, K_1 and K_2 depending on μ_1, μ_2, σ_1, σ_2 and ρ (see p. 107 in Munk for details).

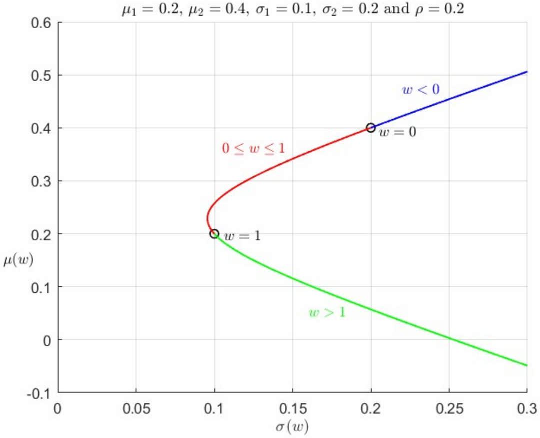
Two-asset portfolios



By letting w vary we can get any expected return we want – given that we accept the standard deviation of that portfolio.

Two-asset portfolios

The weight w can be any real number.



When $w \in [0, 1]$ then also $1 - w \in [0, 1]$, and there is no short-selling in any of the assets (a "long-only portfolio").

Two-asset portfolios

Which portfolios has the lowest possible variance and how large is this variance?

We use

$$\sigma^2(w) = w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2$$

and look for a portfolio with

$$\frac{\partial \sigma^2(w)}{\partial w} = 2w\sigma_1^2 + 2\big(1-w-w\big)\rho\sigma_1\sigma_2 - 2(1-w)\sigma_2^2 = 0.$$

Solving this equation yields the portfolio weights

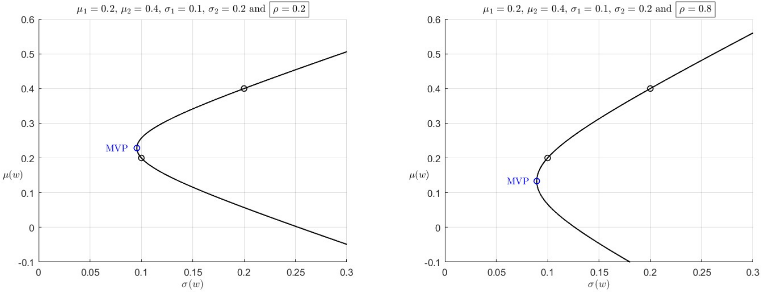
$$w_{\text{min}} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad \text{and} \quad 1 - w_{\text{min}} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Two-asset portfolios

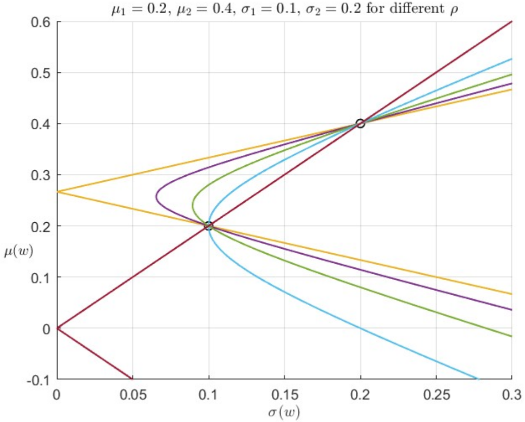
The minimum variance is given by

$$\sigma^2(w_{\min}) = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}.$$

Check this by doing Exercise 4.1 in the course book, and there you can also find an expression for $\mu(w_{\min})$.



Two-asset portfolios



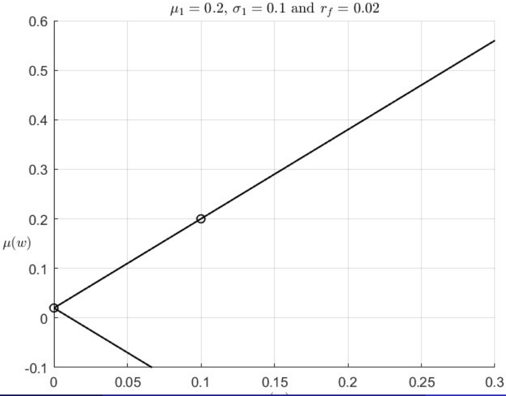
Colour	Yellow	Purple	Green	Blue	Red
ρ	-1	-0.5	0	0.5	1

Two-asset portfolios

Now assume that asset 2 is a risk-free asset with rate of return r_f.

In this case

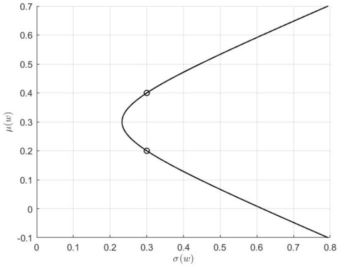
$$\mu(w) = w\mu_1 + (1-w)r_f \quad \text{and} \quad \sigma(w) = \sqrt{w^2\sigma_2^2} = |w|\sigma_2.$$



Two special cases

$\underline{\sigma_1 = \sigma_2}$

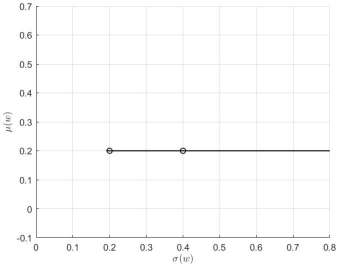
In this case a typical situation is as follows:



This is a situation that is OK.

$\underline{\mu_1 = \mu_2}$

In this case a typical situation is as follows:



This is not a realistic situation from an economic point of view.

Risk reduction through diversification

Now consider a market with N assets.

On this market we form a portfolio with equal weights in each asset:

$$\pi_1 = \pi_2 = \ldots = \pi_N.$$

Since they need to sum to one, we have

$$\pi_i = \frac{1}{N},\ i = 1, 2, \ldots, N.$$

Risk reduction through diversification

How large is the variance of the rate return r_p of this equally weighted portfolio?

$$
\begin{aligned}
\mathrm{Var}[r_p] &= \mathrm{Var}\left[\sum_{i=1}^{N} \pi_i r_i\right] \\
&= \mathrm{Var}\left[\sum_{i=1}^{N} \frac{1}{N} r_i\right] \\
&= \frac{1}{N^2}\mathrm{Var}\left[\sum_{i=1}^{N} r_i\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{Cov}[r_i, r_j] \\
&= \frac{1}{N^2}\left(\sum_{i=1}^{N} \mathrm{Var}[r_i] + \sum_{i \neq j}^{N} \mathrm{Cov}[r_i, r_j]\right)
\end{aligned}
$$

Risk reduction through diversification

There are N variance and $N(N-1)$ covariance terms. Let

$$\overline{\mathrm{Var}}_N = \frac{1}{N}\sum_{i=1}^{N} \mathrm{Var}[r_i]$$

and

$$\overline{\mathrm{Cov}}_N = \frac{1}{N(N-1)}\sum_{i \neq j}^{N} \mathrm{Cov}[r_i, r_j]$$

be their respective averages.

We assume that as the number of assets grow, i.e. when $N \to \infty$, they converge to $\overline{\mathrm{Var}}$ and $\overline{\mathrm{Cov}}$ respectively.

Risk reduction through diversification

Now,

$$
\begin{aligned}
\mathrm{Var}[r_p] &= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}[r_i] + \frac{1}{N^2}\sum_{i\neq j}^{N}\mathrm{Cov}[r_i, r_j] \\
&= \frac{1}{N}\cdot\frac{1}{N}\sum_{i=1}^{N}\mathrm{Var}[r_i] + \frac{N-1}{N}\cdot\frac{1}{N(N-1)}\sum_{i\neq j}^{N}\mathrm{Cov}[r_i, r_j] \\
&= \frac{1}{N}\overline{\mathrm{Var}}_N + \frac{N-1}{N}\overline{\mathrm{Cov}}_N \\
&\xrightarrow{N\to\infty} 0\cdot\overline{\mathrm{Var}} + 1\cdot\overline{\mathrm{Cov}} \\
&= \overline{\mathrm{Cov}}.
\end{aligned}
$$

Conlusion: By investing in more and more asset we can diminish the risk, but the lower limit is given by $\overline{\mathrm{Cov}}$.

Risk reduction through diversification

What is the intuition behind the previous result?

Let r_i be the rate of return of asset i and let r_m be the return of a market index.

Then we can always write

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

for some α_i, $\beta_i = \mathrm{Cov}[r_i, r_m]/\mathrm{Var}[r_m]$ and $\mathrm{Cov}[r_m, \varepsilon_i] = 0$ (cf. with OLS).

Hence,

$$\mathrm{Var}[r_i] = \mathrm{Var}[\alpha_i + \beta_i r_m + \varepsilon_i] = \beta_i^2 \sigma_m^2 + \sigma_i^2,$$

where

$$\sigma_m^2 = \mathrm{Var}[r_m] \text{ and } \sigma_i^2 = \mathrm{Var}[\varepsilon_i].$$

Risk reduction through diversification

If we make the assumption that $\text{Cov}[\varepsilon_i, \varepsilon_j] = 0$ if $i \neq j$, then we can interpret ε_i as the firm specific variation in the return r_i.

Under this assumption

$$\text{Cov}[r_i, r_j] = \text{Cov}[\alpha_i + \beta_i r_m + \varepsilon_i, \alpha_j + \beta_j r_m + \varepsilon_j] = \beta_i \beta_j \sigma_m^2.$$

It follows that

$$\overline{\text{Cov}}_N = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} \text{Cov}[r_i, r_j] = \frac{\sigma_m^2}{N(N-1)} \sum_{i \neq j}^{N} \beta_i \beta_j.$$

The limit of this as $N \to \infty$ does not depend on the variances σ_i^2.

We can diversify away the firm specific risks, but not the market risk.

Risk reduction through diversification

Now let all the assets have the same standard deviation σ and the same pairwise correlation $\rho \geq 0$.

Remark. One can show that if N random variables have the same pairwise correlation ρ, then this correlation has to satisfy $\rho \geq -1/(N-1)$.

In this case

$$\overline{\mathrm{Var}}_N = \frac{1}{N}\sum_{i=1}^{N}\sigma^2 = \frac{1}{N}\cdot N\sigma^2 = \sigma^2$$

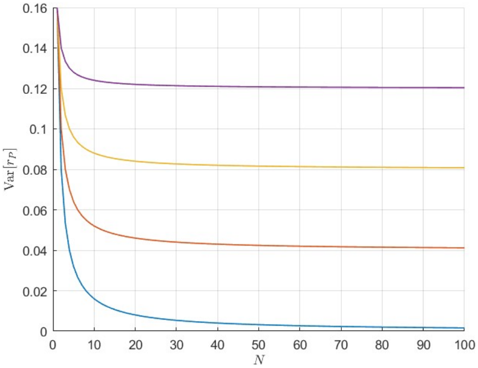
and

$$\overline{\mathrm{Cov}}_N = \frac{1}{N(N-1)}\sum_{i\neq j}^{N}\rho\sigma^2 = \frac{1}{N(N-1)}\cdot N(N-1)\rho\sigma^2 = \rho\sigma^2.$$

Risk reduction through diversification

Now we get

$$\mathrm{Var}[r_p] = \frac{1}{N}\overline{\mathrm{Var}}_N + \frac{N-1}{N}\overline{\mathrm{Cov}}_N = \frac{\sigma^2}{N} + \left(1 - \frac{1}{N}\right)\rho\sigma^2.$$



Corrrelation $\rho = 0, 0.25, 0.5, 0.75$ from bottom and up.

Arbitrage portfolios

An arbitrage is a portfolio with the following properties:

1) It costs zero to buy.
2) It has a payoff that is non-negative, and with strictly positive probability the payoff is strictly positive.

One can think of this as getting a free lottery ticket.

How many units of this portfolio would you like?

Infinitely many!

Arbitrage portfolios

The principle of no arbitrage states that on a market in equilibrium, there can be no arbitrage opportunities, i.e. it is not possible to construct an arbitrage on a market in equilibrium.

This approach is very powerful when determining the price of derivatives such as options and futures.

A model of a market that rules out arbitrage opportunities is said to be free of arbitrage or arbitrage free.

Replicating portfolios

A replicating portfolio for a given asset is a portfolio that exactly replicates the cash flows of the given asset.

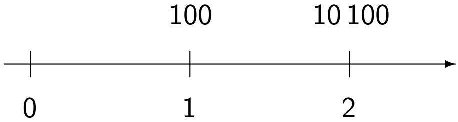
The law of one price states that if two assets have the same cash flows, then they must have the same price.

The law of one price is a weak requirement, and in fact if a market model is free of arbitrage, then the law of one price holds (but the converse is not true).

By using the law of one price, we see that the price of the replicating portfolio and the asset it replicates must be the same.

Replicating portfolios

Example. Asset A is paying out 100 euros one year from now and 10 100 euros in two years from now.



The price of asset B that is paying out 100 euros in one years time is 98.53 euros, and the price of asset C that is paying out 100 euros in two years time is 97.85.

We see that asset A is replicated by the portfolio consisting of 1 unit of asset B and 101 units of asset C. Hence, this is the replicating portfolio.

Using the law of one price we get

$$\text{Price of A} = 1 \cdot 98.53 + 101 \cdot 97.85 = 9\,981.38 \text{ euros.}$$

Tracking portfolios

A tracking portfolio has as its goal to be close to the value of an asset. It is not unusual that the asset being tracked is an index.

The tracking error (TE) measures how much the tracking portfolio deviates from the asset it is tracking and is defined as

$$r_p - r_b,$$

where r_p is the tracking portfolio and r_b the return on the given asset.

One way to quantitatively measure the TE is to calculate $\mathrm{Std}[r_p - r_b]$.

Portfolio mathematics

Given is the vector of rates of return

$$\boldsymbol{r} = (r_1, r_2, \ldots, r_N)^\top.$$

From now on we let

$$\boldsymbol{\mu} = E[\boldsymbol{r}]$$

and

$$\Sigma = \mathrm{Var}[\boldsymbol{r}]$$

denote the mean vector and the variance-covariance matrix of the rate of return vector respectively.

Recall

$$r_p = r(\boldsymbol{\pi}) = \sum_{i=1}^{N} \pi_i r_i = \boldsymbol{\pi} \cdot \boldsymbol{r}.$$

Portfolio mathematics

Then the mean rate of return of a portfolio is

$$\begin{aligned} E\left[r(\boldsymbol{\pi})\right] &= E\left[\sum_{i=1}^{N} \pi_i r_i\right] \\ &= \sum_{i=1}^{N} \pi_i E\left[r_i\right] \\ &= \sum_{i=1}^{N} \pi_i \mu_i \\ &= \boldsymbol{\pi} \cdot \boldsymbol{\mu}. \end{aligned}$$

Portfolio mathematics

The variance of the portfolio rate of return is given by

$$\begin{aligned}
\mathrm{Var}[r(\boldsymbol{\pi})] &= \mathrm{Var}\left[\sum_{i=1}^{N} \pi_i r_i\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \mathrm{Cov}[r_i, r_j] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij} \\
&= \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi}.
\end{aligned}$$

Portfolio optimisation

Using the N assets we can form portfolios with special characteristics.

Typically we want to minimise or maximise some property of a portfolio.

Since we must have

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi} \cdot \mathbf{1} = 1$$

this leads to optimisation with constraints.

Example

1) The portfolio with the smallest variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \end{aligned}$$

2) The long-only portfolio with the smallest variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & \pi_i \geq 0, i = 1, 2, \ldots, N \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \geq \mathbf{0} \end{aligned}$$

3) The portfolio with expected rate of return $\bar{\mu}$ that has the smallest variance:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \mathrm{Var}[r(\boldsymbol{\pi})] \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_i = 1 \\ & E\left[r(\boldsymbol{\pi})\right] = \overline{\mu}. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} \\ \text{s.t.} \quad & \boldsymbol{\pi} \cdot \mathbf{1} = 1 \\ & \boldsymbol{\pi} \cdot \boldsymbol{\mu} = \overline{\mu}. \end{aligned}$$