

UPPSALA UNIVERSITET

LECTURE NOTES

# Complex Analysis

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## 1. INTRO

In this course, we shall study functions  $f : \mathbb{C} \rightarrow \mathbb{C}$  (or more generally,  $f : D \rightarrow \mathbb{C}$  where  $D \subseteq \mathbb{C}$ )

**Definition/Sats 1.1: Complex Number**

A *complex number* is a number of the form  $x + iy$ , where  $x, y \in \mathbb{R}$

Two complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  are said to be equal iff  $x_1 = x_2$  and  $y_1 = y_2$

**Anmärkning:**

The number  $x$  is called the *real part* ( $\operatorname{Re}(z) = x$ ) of the complex number, and  $y$  is called the *imaginary part* ( $\operatorname{Im}(z) = y$ ) of the complex number

**Anmärkning:**

The set of all complex numbers is denoted by  $\mathbb{C}$

**Anmärkning:**

$\mathbb{C}$  is the *smallest* field extension to  $\mathbb{R}$  that is algebraically closed.

**Anmärkning:**

$i^2 = -1$

1.1. Operations over  $\mathbb{C}$ .

We define the operations *addition* and *multiplication* of two complex numbers as follows:

**Definition/Sats 1.2: Addition of complex numbers**

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

**Definition/Sats 1.3: Multiplication of complex numbers**

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations,  $\mathbb{C}$  forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

**Definition/Sats 1.4: Complex conjugate**

The *complex conjugate* of a complex number  $z = x + iy$ , denoted by  $\bar{z}$ , is defined by  $\bar{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$
- $\overline{\bar{z}} = z$
- $z \cdot \bar{z} = |z|^2$
- $z^{-1} = \frac{\bar{z}}{|z|^2}$
- $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$

**Anmärkning:**

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

**Anmärkning:**

Multiplication by  $i$  is simply rotation by  $\frac{\pi}{2}$  counterclockwise.

**Definition/Sats 1.5**

Let  $z \in \mathbb{C}$ . Then there exists a  $w \in \mathbb{C}$  such that  $w^2 = z$  (where  $-w$  also satisfies this equation)

**Bevis 1.1**

Let  $z = a + bi$  and  $w = x + iy$  such that  $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then  $a = x^2 - y^2$  and  $b = 2xy$

We also know that  $|z| = a^2 + b^2 = |x^2 + y^2|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore,  $x^2 + y^2 = \sqrt{a^2 + b^2}$  and:

$$\left. \begin{array}{l} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left. \begin{array}{l} -x^2 + y^2 = -a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let  $\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$  and  $\beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$  and let  $\sqrt{\phantom{x}}$  denote the positive square root of positive real numbers.

If  $b$  is positive, then either  $x = \alpha, y = \beta$  or  $x = -\alpha, y = -\beta$

If  $b$  is negative, then either  $x = \alpha, y = -\beta$  or  $x = -\alpha, y = \beta$

Therefore, the equation has solutions  $\pm(\alpha + \mu\beta i)$  where  $\mu = 1$  if  $b \geq 0$  and  $\mu = -1$  if  $b < 0$

□

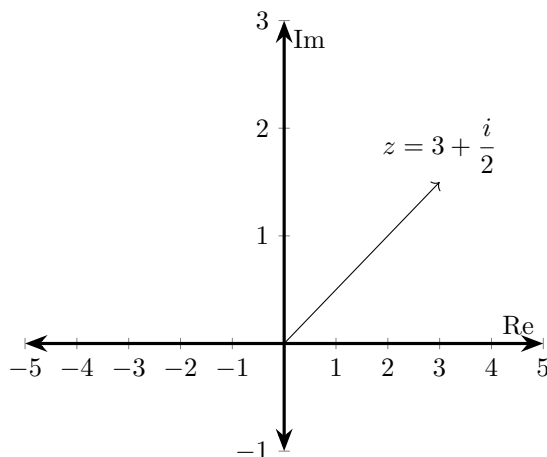
**Anmärkning:**

From the proof above, we can conclude the following:

- The square roots of a complex number are real  $\Leftrightarrow$  the complex number is real and positive
- The square roots of a complex number are purely imaginary  $\Leftrightarrow$  the complex number is real and negative
- The two square roots of a number coincide  $\Leftrightarrow$  the complex number is zero

## 1.2. Cartesian representation.

It is natural to represent a complex number  $z = x + iy$  as a tuple  $(x, y)$ , and we can therefore represent it in the standard cartesian plane:



### Anmärkning:

This is sometimes called the *complex plane*

#### Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number  $z = x + iy$  (geometrically the length of the vector), denoted by  $|z|$ , is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:

- $|z|^2 = z \cdot \bar{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

### Anmärkning:

Every  $z \in \mathbb{C}$  such that  $z \neq 0$  (that is,  $x \neq 0$  or  $y \neq 0$ ) has a multiplicative inverse  $\frac{1}{z}$  given by:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

#### Definition/Sats 1.7: Triangle inequality

For  $z_1, z_2 \in \mathbb{C}$ , it holds that  $|z_1 + z_2| \leq |z_1| + |z_2|$

#### Lemma 1.1: Reversed triangle inequality

For  $z_1, z_2 \in \mathbb{C}$ , it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

**Bevis 1.2**

$$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

$$\text{So that } |z_1| - |z_2| \leq |z_1 - z_2|$$

□

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
- $-\operatorname{Re}(z) \leq \operatorname{Re}(z) \leq |z|$
- $-\operatorname{Im}(z) \leq \operatorname{Im}(z) \leq |z|$
- $|\bar{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \dots + |w_n|^2}$

**1.3. Polar form.**

Let  $z = x + iy \neq 0$ . The point  $\left( \frac{x}{|z|}, \frac{y}{|z|} \right)$  lies on the unit circle, and hence there exists  $\theta$  such that:

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta)$$

Therefore  $z = x + iy$  can be written as:

$$z = r(\cos(\theta) + i \sin(\theta))$$

Where  $r = |z|$  is uniquely determined by  $z$ , while  $\theta$  is  $2\pi$ -periodic. This is called the *polar form* of  $z$  and just as the cartesian representation requires a tuple of information  $(|z|, \theta)$

**Definition/Sats 1.8: Argument**

The *argument* of a complex number  $z$ , denoted by  $\arg(z)$ , is the angle  $\theta$  between  $z$  and the real number line in the complex plane

**Anmärkning:**

Since the argument is  $2\pi$  periodic, the angle is usually given as  $\theta + k2\pi$   $k \in \mathbb{Z}$ , but we are only interested in  $\theta$

This  $\theta$  is called the *principal value* of  $\arg(z)$ , denoted by  $\operatorname{Arg}(z)$  and belongs to  $(-\pi, \pi]$

**Anmärkning:**

We are always allowed to change an angle by multiples of  $2\pi$ , the principal value argument is the angle after changing the argument such that it lies between  $(-\pi, \pi]$

**Anmärkning:**

A specification of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that  $\operatorname{Arg}(z)$  is "discontinuous" along the negative real axis. This is called a *branch-cut*

$$\text{Suppose } z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$$

Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 [(\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))] \\ &\quad r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

**Anmärkning:**

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

#### 1.4. Exponential form.

##### Definition/Sats 1.9

For  $z = x + iy \in \mathbb{C}$ , let  $e^z = e^x(\cos(y) + i \sin(y))$

##### Anmärkning:

$e^{iy} = \cos(y) + i \sin(y) \quad y \in \mathbb{R}$  (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$\begin{aligned} e^z &= e^{x+iy} = e^x \cdot e^{iy} \\ e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \\ \Rightarrow e^{iy} &= 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos(\theta)} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin(\theta)} \\ \Rightarrow e^z &= e^x(\cos(\theta) + i \sin(\theta)) \end{aligned}$$

##### Anmärkning:

One can through comparing see that  $|e^z| = e^x$ , and that  $|e^{iy}| = 1$

##### Properties of the exponential form:

- $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$
- $e^z \neq 0 \quad \forall z \in \mathbb{C}$
- $x \in \mathbb{R} \Rightarrow e^x > 1$  if  $x > 0$  and  $e^x < 1$  if  $x < 0$
- $|e^{x+iy}| = e^x$
- $e^{i\pi/2} = i \quad e^{i\pi} = -1 \quad e^{3i\pi/2} = -i \quad e^{2i\pi} = 1$
- $e^z$  is  $2\pi$ -periodic
- $e^z = 1 \Leftrightarrow z = 2\pi ki \quad k \in \mathbb{Z}$

##### Definition/Sats 1.10: deMoivre's formula

For  $n \in \mathbb{Z}$ ,  $(r(\cos(\theta) + i \sin(\theta)))^n = r^n(\cos(n\theta) + i \sin(n\theta))$

#### 1.5. Logarithmic form.

In real analysis, we have defined the logarithm as the inverse of  $e^x$ . This has previously worked since for  $x \in \mathbb{R}$ ,  $e^x$  is injective.

The problem is that for  $e^z$  where  $z \in \mathbb{C}$ , it is not injective and should therefore not have an inverse.

Given  $z \in \mathbb{C} \setminus \{0\}$ , we define  $\ln(z)$  as the cut of all  $w \in \mathbb{C}$  whose image under the exponential form is  $z$ , i.e  $w = \ln(z) \Leftrightarrow z = e^w$ .

Here,  $\ln(z)$  is a *multivalued form*

We can use the fact that  $|z| = r = e^x$  to derive some interesting properties of the logarithm:

$$\begin{aligned} z &= r e^{i\theta} & w &= u + iv \\ \text{If } z &= e^w \Leftrightarrow r e^{i\theta} = e^u \cdot e^{iv} \\ \Leftrightarrow u &= \ln(r) = \ln(|z|) & v &= \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z} \end{aligned}$$

**Definition/Sats 1.11: Complex logarithm**

For  $z \neq 0$ , we define the complex logarithm for  $z \in \mathbb{C}$  as:

$$\begin{aligned}\ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\operatorname{Arg}(z) + k2\pi) \quad k \in \mathbb{Z}\end{aligned}$$



## 2. ELEMENTARY COMPLEX FUNCTIONS

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

## 2.1. Branches of the complex logarithm.

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of  $2\pi$ ).

If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

**Definition/Sats 2.12: Principal logarithm**

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\text{Ln}(z) = \ln(|z|) + i \cdot \text{Arg}(z)$$

**Anmärkning:**

We have essentially extended the "normal" logarithm, which is defined on  $(0, \infty)$ , to be defined on  $\mathbb{C} \setminus \{0\}$

**Anmärkning:**

The principal logarithm is discontinuous for negative reals, since their principal value argument is  $-\pi$ , but the principal value argument is discontinuous at  $-\pi$ . This is the so called *branch-cut*

**Anmärkning:**

Even though the principal logarithm is discontinuous for negative reals, it is not undefined. Any negative real number  $z$  will have  $\text{Arg}(z) = \pi$ , which the logarithm very much is defined for.

**Anmärkning:**

When branching, we do not necessarily have to pick  $(-\pi, \pi]$ , we can pick any interval  $(\alpha, \alpha + 2\pi]$ . This is usually denoted by  $\arg_\alpha$ .

## 2.2. Complex mappings.

One can think of a complex mapping  $f: \mathbb{C} \rightarrow \mathbb{C}$  as  $f(z) = f(x + iy) = w = u + iv$

Then it becomes clear which regions map to where by drawing them in their respective  $z$ -plane and  $w$ -plane.

## 2.3. Complex powers.

Given  $z \in \mathbb{C}$ , consider the following equation:

$$(1) \quad w^n = z$$

The set of all solutions  $w$  of (1) is denoted  $z^{1/n}$  and is called the *n-th root of z*.

**Anmärkning:**

If  $z = 0$ , then  $w = 0$

Suppose  $z \neq 0$ , then we may write  $w = |w| e^{i\alpha}$  and  $z = |z| e^{i\theta}$   
 By deMoivre's formula, (1) becomes:

$$|w|^n e^{in\alpha} = |z| e^{i\theta}$$

Then, the following follows:

$$\left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ n\alpha &= \theta + k2\pi \quad k \in \mathbb{Z} \end{aligned} \right\} \Leftrightarrow \left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ \alpha &= \frac{\theta}{n} + k \frac{2\pi}{n} \quad k \in \mathbb{Z} \end{aligned} \right\}$$

Notice now that every  $k \in \mathbb{Z}$  gives a solution to (1)

Since sine and cosine are both  $2\pi$ -periodic, then only  $k = 0, 1, \dots, n-1$  actually give *different* solutions  
 (since  $k = n \Rightarrow \alpha = \frac{\theta}{n} + n \frac{2\pi}{n}$ )

Suppose  $z \neq 0$ . For  $n \in \mathbb{Z}$  it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that  $\ln(z)$  attains.

It is also true, that for  $n = 1, 2, 3, \dots$ :

$$\frac{1}{z^n} = e^{\frac{1}{n} \ln(z)}$$

We can let  $n \in \mathbb{C}$ , and obtain the following definition:

#### Definition/Sats 2.13: Complex power

For  $\alpha \in \mathbb{C}$ , let:

$$z^\alpha = e^{\alpha \ln(z)} \quad z \neq 0$$

#### Anmärkning:

This makes  $z^\alpha$  a multivalued function, but it is possible to have a single-valued output from it.

#### Definition/Sats 2.14

Let  $a, b \in \mathbb{C}$  where  $a \neq 0$ . Then  $a^b$  is single-valued (does not depend on the choice of branch for the logarithm)  $\Leftrightarrow b \in \mathbb{Z}$

If  $b \in \mathbb{Q}$  and is in lowest form (that is,  $b = \frac{p}{q}$  where  $p, q$  have no common factors), then  $a^b$  has exactly  $q$  distinct values (the  $q$ :th roots of  $a^p$ )

If  $b \in \mathbb{C} \setminus \mathbb{Q}$ , then  $a^b$  has infinitely many values.

#### Bevis 2.1

Chose some interval (branch), say  $[0, 2\pi)$ , for the arg function and let  $\ln(z)$  be the corresponding branch of the logarithm. If we chose another branch, we would have  $\ln(a) + 2\pi kbi$  rather than  $\ln(a)$  (where  $k \in \mathbb{Z}$ )

Therefore,  $a^b = e^{b \ln(a) + 2\pi kbi} = e^{b \ln(a)} \cdot e^{2\pi ki}$

Notice that  $e^{2\pi kbi}$  stays the same regardless of  $b \in \mathbb{Z}$ , as long as it is an integer.

In the same way, it can be shown that  $e^{2\pi kip/q}$  has  $q$  distinct values if  $p, q$  have no common factor.

If  $b$  is irrational, and if  $e^{2\pi kbi} = e^{2\pi mbi}$ , then it follows that  $e^{(2\pi bi)(k-m)} = 1$ , and therefore  $b(k-m)$  is an integer.

Since  $b$  is irrational, then  $n - m = 0$

□

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up. We can chose to branch  $z^\alpha$ :

$$z^\alpha = e^{\alpha \text{Ln}(z)}$$

## 2.4. Trigonometric and Hyperbolic functions.

We have the following:

$$\left. \begin{aligned} e^{iy} &= \cos(y) + i \sin(y) \\ e^{-iy} &= \cos(y) - i \sin(y) \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \cos(y) &= \frac{e^{iy} + e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy} - e^{-iy}}{2i} \end{aligned} \right\}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

### Definition/Sats 2.15: Complex sine and cosine

For  $z \in \mathbb{C}$ , we define:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals. When defining them for complex numbers, we just extend their domain:

### Definition/Sats 2.16: Complex hyperbolic functions

For  $z \in \mathbb{C}$ , we define:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

- **Sine:**

$$\begin{aligned} \sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &\Rightarrow \frac{e^{-y}(\cos(x) + i \sin(x)) - e^y(\cos(x) - i \sin(x))}{2i} = \frac{(e^{-y} - e^y) \cos(x) + i(e^y - e^{-y}) \sin(x)}{2i} \\ &= \frac{(e^{-y} - e^y) \cos(x)}{2i} + \frac{(e^y - e^{-y}) \sin(x)}{2} \\ &\stackrel{i^{-1} = -i}{\Rightarrow} \underbrace{\frac{(e^y - e^{-y})}{2}}_{\sinh(y)} i \cos(x) + \underbrace{\frac{(e^y + e^{-y})}{2}}_{\cosh(y)} \sin(x) \end{aligned}$$

- **Cosine:**

$$\begin{aligned} \cos(x + iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} \\ &= \frac{e^{-y}(\cos(x) + i \sin(x)) + e^y(\cos(x) - i \sin(x))}{2} = \frac{\cos(x)(e^y + e^{-y}) + i(e^{-y} - e^y) \sin(x)}{2} \\ &= \underbrace{\frac{e^y + e^{-y}}{2}}_{\cosh(y)} \cos(x) - \underbrace{\frac{e^y - e^{-y}}{2}}_{\sinh(y)} i \sin(x) \end{aligned}$$

This leads us to the following:

**Definition/Sats 2.17: Addition formulas for complex trigonometric functions**

- $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$
- $\cos(x + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$

**Anmärkning:**

Both sine and cosine can be defined as the unique solution to an ODE, namely:

$$\begin{aligned} f''(x) + f(x) &= 0 & f(0) &= 0, f'(0) = 1 & f(x) &= \sin(x) \\ f''(x) + f(x) &= 0 & f(0) &= 1, f'(0) = 0 & f(x) &= \cos(x) \end{aligned}$$

**2.5. Mapping properties of  $\sin(z)$ .**

Let  $f(z) = \sin(z)$  in  $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$ , let  $A$  be the set of points allowed with respect to the above constraint and let  $B$  be the mapping of those points by  $\sin(A)$

**Claim:**  $f : A \rightarrow B$  is a bijective mapping

**Bevis 2.2**

Take a  $z \in \mathbb{C}$   $z = x + iy$   $-\frac{\pi}{2} < x < \frac{\pi}{2}$ .

Then:

$$\begin{aligned} f(z) &= \sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y) \\ f(z) \in \mathbb{R} &\Leftrightarrow \cos(x) \sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0 \end{aligned}$$

If  $y = 0$ , then:

$$f(z) = \sin(x) \cosh(y) = \sin(x) \in (-1, 1)$$

Therefore, if  $z \in A \Rightarrow f(z) \in B$ . Now we need to show that for any  $z \in B$ , there is a  $u$  such that  $f(u) = z$

Let  $u = \sin(x) \cosh(y)$ ,  $v = \cos(x) \sinh(y)$  and pick a vertical line at  $x = a \neq 0$

We will now consider the images of these lines:

$$\begin{aligned} \cosh(y) &= \frac{u}{\sin(a)} & \sinh(y) &= \frac{v}{\cos(a)} \\ (\cosh(y))^2 - (\sinh(y))^2 &= 1 \Rightarrow \left( \frac{u}{\sin(a)} \right)^2 - \left( \frac{v}{\cos(a)} \right)^2 = 1 \end{aligned}$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line  $y = b \neq 0$

$$\begin{aligned} \sin(x) &= \frac{u}{\cosh(b)} & \cos(x) &= \frac{v}{\sinh(b)} \\ \cos^2(x) + \sin^2(x) &= 1 \Rightarrow \left( \frac{u}{\cosh(b)} \right)^2 + \left( \frac{v}{\sinh(b)} \right)^2 = 1 \end{aligned}$$

This is a half-ellipse. Note that  $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

□

3. TOPOLOGY OF  $\mathbb{C}$ **Definition/Sats 3.18: Open disc**

The set  $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$  is called the *open-disc* with center  $z_0$  and radius  $r$

**Anmärkning:**

Since we have a strict inequality, it is open. If we had  $\leq$ , it would be a closed disc.

**Definition/Sats 3.19: Open subset**

A subset  $M$  of  $\mathbb{C}$  is called *open* if for every  $z_0 \in M$  there exists an  $r > 0$  such that  $D_r(z_0) \subseteq M$

**Definition/Sats 3.20: Interior point**

A point  $z_0 \in M$  is called an *interior-point* of  $M$  if there exists an  $r > 0$  such that  $D_r(z_0) \subseteq M$

**Definition/Sats 3.21: Boundary point**

A point  $z_0 \in \mathbb{C}$  is called a *boundary point* of  $M$  if  $\forall r > 0$  it holds that:

$$D_r(z_0) \cap M \neq \emptyset \quad \wedge \quad D_r(z_0) \cap M^c \neq \emptyset$$

**Anmärkning:**

The set of all interior points of  $M$  is denoted by  $\text{int}(M)$  and the set of all boundary points of  $M$  is denoted by  $\partial M$

The following equivalences hold:

- $M$  is closed  $\Leftrightarrow \partial M \subseteq M$
- $M$  is open  $\Leftrightarrow \partial M \subseteq M^c$
- $\mathbb{C}$  is clopen
- $\emptyset$  is clopen
- The union of any collection of open subsets of  $\mathbb{C}$  is open
- The intersection of any finite collection of open subsets of  $\mathbb{C}$  is open

**Definition/Sats 3.22: Closed set**

We say that a set  $X \subseteq \mathbb{C}$  is closed if its complement  $X^c$  is open

**Definition/Sats 3.23: Polygonal path**

A polygonal path  $P$  (sometimes called piecewise linear curve) is a curve specified by a sequence of points  $(A_1, A_2, \dots, A_n)$ .

The curve itself consists of line segments connecting the consecutive points.

**Definition/Sats 3.24: polygonal-path-connected open set**

An open set  $M$  is called *polygonal-path-connected* if every pair of points  $z_1, z_2 \in M$  can be connected by a polygonal path contained in  $M$

**Anmärkning:**

Some would call this just path-connected, or even just connected. This works in  $\mathbb{R}^n$  (recall that  $\mathbb{C} \cong \mathbb{R}^2$ ). Topologically speaking, polygonal-path-connectedness  $\implies$  path-connectedness

**Anmärkning:**

A set  $X$  is connected  $\Leftrightarrow$  the only subsets of  $X$  which are clopen are  $\emptyset$  and  $X$

**Anmärkning:**

One can assume the polygonal paths to have segments parallel to the ordinate ones.

**Anmärkning:**

An open connected set is called a *domain*

**Definition/Sats 3.25**

Suppose that  $u(x, y)$  is a real-valued function defined in a domain  $D \subseteq \mathbb{R}$

Also suppose that:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} =$$

in all of  $D$ . Then  $u$  is contained in  $D$

**Definition/Sats 3.26: Simply connected**

A domain  $D \subseteq \mathbb{C}$  is called *simply connected* if every closed curve in  $D$  can be, within  $D$ , continuously deformed to a point

**Anmärkning:**

Topologically speaking,  $D$  is homeomorphic to a point.

**Definition/Sats 3.27: Non-connectedness**

A set  $A \subseteq \mathbb{C}$  is *not connected* if there are open sets  $U$  and  $V$  such that:

- $A \subseteq U \cup V$
- $A \cap U \neq \emptyset$  and  $A \cap V \neq \emptyset$

### 3.1. Limits and Continuity.

#### Definition/Sats 3.28: Complex limit

A sequence  $\{z_n\}_{n=1}^{\infty}$  of complex numbers is said to have the limit  $z_0$  (*converges to*  $z_0$ ) if for every given  $\varepsilon > 0$ , there exists an integer  $N \geq 1$  such that

$$|z_n - z_0| < \varepsilon \quad \forall n \geq N$$

We write this as:

$$\lim_{n \rightarrow \infty} z_n = z_0$$

#### Anmärkning:

Every cauchy sequence in  $\mathbb{C}$  converges.

#### Anmärkning:

$z_n \rightarrow z_0 \Leftrightarrow \operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z_0)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z_0)$

This follows from  $|x|, |y| \leq \sqrt{x^2 + y^2} \leq |x| + |y|$

#### Definition/Sats 3.29

Let  $f$  be a function defined in a punctured neighborhood of  $z_0$

We say that  $f$  has the limit  $w_0$  as  $z \rightarrow z_0$ , if for every given  $\varepsilon > 0$  there exists  $\delta > 0$  such that:

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

We write this as:

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

#### Anmärkning:

If a limit exists, it is unique.

#### Definition/Sats 3.30

For  $z = x + iy$ , let:

$$u(x, y) = \operatorname{Re}(f(z)) \quad v(x, y) = \operatorname{Im}(f(z))$$

Let  $z_0 = x_0 + iy_0$  and  $w_0 = u_0 + iv_0$

Then the following holds:

$$\lim_{z \rightarrow z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \end{cases}$$

#### Definition/Sats 3.31: Continuous function

Let  $f$  be a function defined in a neighborhood of  $z_0$ .

$f$  is said to be continuous at  $z_0$  if:

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

A function  $f$  is said to be *continuous on the (open) set*  $M$  if it is continuous at each point of  $M$

**Anmärkning:**

The following statements are equivalent (for  $f : A \rightarrow \mathbb{C}$ ):

- $f$  is continuous
- The inverse image of every closed set is closed relative to  $A$
- The inverse image of every open set is open relative to  $A$
- The image set  $f(A)$  is connected

Assume  $\lim_{z \rightarrow z_0} f(z) = A$  and  $\lim_{z \rightarrow z_0} g(z) = B$

The following properties from the real limit hold for the complex limit:

- $\lim_{z \rightarrow z_0} (f(z) \pm g(z)) = A \pm B$
- $\lim_{z \rightarrow z_0} f(z)g(z) = AB$
- $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad B \neq 0$

**Anmärkning:**

If  $f, g$  are continuous at  $z_0$ , then so are  $f \pm g$  and  $fg$ . The quotient is only continuous if  $g(z_0) \neq 0$

**Anmärkning:**

Constant functions, polynomials, and rational functions (whenever the denominator is non-zero) are all continuous in  $\mathbb{C}$

**3.2. The complex derivative.**

Analogous to the real case, we also have the following:

**Definition/Sats 3.32: Differentiability**

Let  $f$  be a complex-valued function defined in a neighborhood of  $z_0$ .

We say that  $f$  is differentiable at  $z_0$  if the limit:

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

The limit is called the *derivative* of  $f$  at  $z_0$ , and is denoted by  $f'(z_0)$  or  $\frac{df}{dz}(z_0)$

**Anmärkning:**

Since  $\Delta z$  is a complex number, it can approach 0 from different directions. In order for the derivative to exist, the results must be independent of the direction of which  $\Delta z$  approaches 0 (i.e., approaches 0 from all directions)

**Anmärkning:**

If  $X$  is an open connected set and  $a, b \in X$ , then there is a differentiable path  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = a$  and  $\gamma(1) = b$

**Example:**

The function  $f(z) = \bar{z}$  is nowhere differentiable since:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{z_0 + \Delta z} - \bar{z}_0}{\Delta z} = \frac{\bar{\Delta z}}{\Delta z} = \frac{\bar{\Delta x} + i\bar{\Delta y}}{\Delta x + i\Delta y}$$

As  $\Delta z \rightarrow 0$  from the  $x$ -direction (real-line), the limit becomes  $\frac{\bar{x}}{x} = 1$

However, as we approach from the  $y$ -direction (complex axis), the limit becomes  $\frac{\bar{iy}}{iy} = \frac{-y}{y} = -1$

Since  $x, y$  were chosen arbitrarily, this applies to all  $x, y$ . Since the limits did not match, it is not differentiable and at no point.

Of course, all the properties from the real case hold here as well.

Suppose  $f, g$  are differentiable at  $z$ , then:

- $(f \neq g)'(z) = f'(z) \neq g'(z)$



- $(cf)'(z) = cf'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f \circ g)'(z) = f'(g(z))g'(z)$

### 3.3. Analytic functions.

#### Definition/Sats 3.33: Analytic function

A complex-valued function  $f$  is said to be *analytic* in an open set  $G$  if  $f$  is differentiable at every point in  $G$ .

We say that  $f$  is *analytic at  $z_0$*  if  $f$  is differentiable in a neighborhood of  $z_0$ .

#### Anmärkning:

If  $f$  is analytic in all of  $\mathbb{C}$ , then  $f$  is said to be *entire* (or *holomorphic*).

#### Definition/Sats 3.34

If an entire function  $f(z)$  has a root at  $w$ , then:

$$\lim_{z \rightarrow w} \frac{f(z)}{(z - w)}$$

is an entire function.

## 4. CAUCHY-RIEMANN'S EQUATIONS

Suppose  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  is differentiable at  $z_0 = x_0 + iy_0$

Then:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}$$

1) Let  $\Delta z = \Delta x$  (i.e  $\Delta y = 0$ ):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{(u(x_0 + \Delta x, y_0) - u(x_0, y_0)) + i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

2) Let  $\Delta z = i\Delta y$  (i.e  $\Delta x = 0$ ):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{(u(x_0, y_0 + \Delta y) - u(x_0, y_0)) + i(v(x_0, y_0 + \Delta y) - v(x_0, y_0))}{i\Delta y} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

It must therefore hold that:

$$u_x + iv_x = -iu_y + v_y$$

This leads to the Cauchy-Riemann equations:

$$\left. \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \right\}$$

We have therefore arrived at the following:

**Definition/Sats 4.35**

A necessary condition for  $f = u + iv$  to be differentiable at  $z_0 = x_0 + iy_0$  is that the Cauchy-Riemann equations are satisfied at  $(x_0, y_0)$

**Anmärkning:**

We also saw that if  $f$  is differentiable at the point  $z_0$ , then the derivative is given by:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

The following provides a sufficient condition for Differentiability:

**Definition/Sats 4.36**

Suppose that  $f = u + iv$  is defined in a open set  $G$  containing  $z_0 = x_0 + iy_0$ .

Suppose also that  $u_x, u_y, v_x, v_y$  exists in  $G$  and are continous at  $(x_0, y_0)$ , and satisfy the Cauchy-Riemann equations at  $(x_0, y_0)$

Then  $f$  is differentiable at  $z_0$

**Anmärkning:**

Cauchy-Riemann equations +  $u, v \in C^1 \Rightarrow f$  is differentiable

### Bevis 4.1

In view of the continuity of the first partial derivatives at  $(x_0, y_0)$ , it holds that:

$$\begin{aligned} u(x_0 + \Delta x, y_0 + \Delta y) &= u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_1(\Delta x, \Delta y) \\ v(x_0 + \Delta x, y_0 + \Delta y) &= v(x_0, y_0) + v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\rho_2(\Delta x, \Delta y) \end{aligned}$$

Where  $\rho_1, \rho_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$

Then:

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= u_x(x_0, y_0)\Delta x + \underbrace{u_y(x_0, y_0)}_{= -v_x(x_0, y_0)}\Delta y + i(v_x(x_0, y_0)\Delta x + \underbrace{v_y(x_0, y_0)}_{= u_x(x_0, y_0)}\Delta y) \\ &\quad + \sqrt{(\Delta x)^2 + (\Delta y)^2}(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \\ &\stackrel{\text{CR-eq.}}{=} u_x(x_0, y_0)\Delta z + i v_x(x_0, y_0)\Delta z + |\Delta z|(\rho_1(\Delta x, \Delta y) + i\rho_2(\Delta x, \Delta y)) \end{aligned}$$

Since  $\rho_1, \rho_2 \rightarrow 0$  as  $\Delta z \rightarrow 0$ , it follows that:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is equal to  $u_x(x_0, y_0) + i v_x(x_0, y_0)$  □

#### 4.1. Inverse mappings.

Suppose  $f = u + iv$  is analytic in a domain  $D$  (with  $f'$  continuous).

Consider the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

As a mapping of  $D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Its Jacobian matrix:

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

has determinant:

$$\det(J_f) = u_x v_y - u_y v_x \stackrel{\text{CR-eq.}}{=} u_x^2 + v_x^2 = |f'(z)|^2$$

The inverse function then leads to the following:

#### Definition/Sats 4.37: Inverse function theorem

Suppose  $f(z)$  is analytic on a domain  $D$  with  $f'(z) \neq 0$  continuous.

Then there is a neighborhood  $U$  of  $z_0$  and a neighborhood  $V$  of  $f(z_0)$  such that  $f : U \rightarrow V$  is bijective, and the inverse function  $f^{-1} : V \rightarrow U$  is analytic with derivative:

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \quad w = f(z)$$

## 5. HARMONIC FUNCTIONS

**Definition/Sats 5.38: Harmonic function**

A real-valued function  $\phi(x, y)$  is said to be *harmonic* in a domain  $D$  if  $\phi \in C^2(D)$  and  $\phi$  satisfies Laplace's equations:

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

in  $D$

**Definition/Sats 5.39**

Suppose  $f = u + iv$  is analytic in a domain  $D$ . Then  $u, v$  are harmonic in  $D$

**Bevis 5.1**

One can show that  $u, v \in C^\infty$ :

$$\begin{aligned} u_x = v_y &\Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x &\Rightarrow u_{yy} = -v_{xy} \end{aligned}$$

As  $v_{yx} = v_{xy}$ , we have  $u_{xx} + u_{yy} = 0$

Similarly,  $v_{xx} + v_{yy} = 0$

□

**Definition/Sats 5.40: Harmonic Conjugacy**

If  $u$  is harmonic in a domain  $D$  and  $v$  is a harmonic function in  $D$  such that  $u + iv$  is analytic in  $D$ , then we say that  $v$  is a *harmonic conjugate* of  $u$  in  $D$

**Definition/Sats 5.41**

If  $u$  is harmonic in a simply connected domain  $D \subseteq \mathbb{C}$ , then there exists a harmonic conjugate  $v$  of  $u$  in  $D$ , and  $v$  is unique up to addition of a real constant

### Bevis 5.2

Suppose  $u$  is harmonic in  $D \subseteq \mathbb{C}$

Consider the vector-field  $\overline{F} = (-u_y, u_x) \in C^1(0)$ .

Note that:

$$\frac{\partial F_1}{\partial y} = -u_{yy} \stackrel{u \text{ harm.}}{=} u_{xx} = \frac{\partial F_2}{\partial x}$$

Since  $D$  is simply connected  $\Rightarrow \overline{F}$  is conservative  $\Rightarrow \exists v : \nabla v = \overline{F}$ , i.e.  $(v_x, v_y) = (-u_y, u_x)$

$$\Rightarrow \left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \Rightarrow f = u + iv \text{ is analytic in } D$$

If  $\bar{v}$  is another harmonic conjugate, then:

$$\begin{aligned} \bar{v}_x &= -u_y = v_x \\ \bar{v}_y &= u_x = v_y \\ \Rightarrow \nabla(v - \bar{v}) &= \bar{0} \Rightarrow v - \bar{v} = c \in \mathbb{C} \end{aligned}$$

□

### Anmärkning:

A vector field is conservative if it is the gradient of some function.  
It has the property that its line integral is path independent.

## 6. CONFORMAL MAPPINGS

Let  $D$  be a domain in  $\mathbb{C}$ ,  $z_0 \in D$ .

Suppose  $f : D \rightarrow \mathbb{C}$  is analytic with  $f'(z_0) \neq 0$ . Let  $\gamma(t) = x(t) + iy(t)$  be a  $C^1$ -curve in  $D$  through  $z_0 = \gamma(0)$  with  $\gamma'(0) \neq 0$ . Then  $(f \circ \gamma)(t) = f(\gamma(t))$  is a  $C^1$ -curve through  $(f \circ \gamma)(0) = f(z_0)$ .

Moreover,

$$\begin{aligned} (f \circ \gamma)'(0) &= \frac{d}{dt} f(\gamma(t))|_{t=0} = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \cdot \frac{\gamma(t) - \gamma(0)}{t} = f'(z_0) \gamma'(0) \end{aligned}$$

From this, we can conclude  $(f \circ \gamma)'(0) = f'(z_0) \gamma'(0)$  is a tangent vector to  $f \circ \gamma$  at  $f(z_0)$

Note that  $\arg(f \circ \gamma)'(0) = \arg(f'(z_0) \gamma'(0))$

If  $\gamma_1$  and  $\gamma_2$  are two  $C^1$ -curves which intersect at  $z_0$ , then the angle from  $(f \circ \gamma_1)'(0)$  to  $(f \circ \gamma_2)'(0)$  is the same as the angle from  $\gamma_1'(0)$  to  $\gamma_2'(0)$

**Definition/Sats 6.42: Conformal  $C^1$ -mapping**

A  $C^1$ -mapping  $f : D \rightarrow \mathbb{C}$  is said to be *conformal* at  $z_0$  if it satisfies the above paragraph.

If  $f$  maps  $D$  bijectively onto  $V$ , and if  $f$  is conformal at one point  $z_0 \in D$ , we call  $f : D \rightarrow V$  a *conformal mapping*

**Definition/Sats 6.43**

If  $f$  is analytic at  $z_0$  and  $f'(z_0) \neq 0$ , then  $f$  is conformal at  $z_0$

**Anmärkning:**

One can in fact prove the converse of this theorem.

## 7. STEREOGRAPHIC PROJECTION

Consider the unit sphere  $S \in \mathbb{R}^3$ .

Given any point  $P = (x_1, x_2, x_3) \in S$  other than the north pole  $N = (0, 0, 1)$ , we draw the line through  $N$  and  $P$ .

We define the *stereographic projection* of  $P$  to be the point  $z = x + iy \in \mathbb{C} \sim (x, y, 0)$ , where the line intersects the plane  $x_3 = 0$ . Then the following holds:

$$(x, y, 0) = (0, 0, 1) + t[(x_1, x_2, x_3) - (0, 0, 1)]$$

Where  $t$  is given by  $1 + t(x_3 - 1) = 0 \Leftrightarrow t = \frac{1}{1 - x_3}$ . We arrive at the following:

$$z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$$

Conversely, given  $z = x + iy \in \mathbb{C} \sim (x, y, 0)$  the line through  $N$  and  $z$  is given by:

$$(x_1, x_2, x_3) = (0, 0, 1) + t[(x, y, 0) - (0, 0, 1)] \quad t \in \mathbb{R}$$

**Anmärkning:**

The line intersects  $S$  when:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1 \\ \Leftrightarrow (tx)^2 + (ty)^2 + (1 - t)^2 &= 1 \\ \Leftrightarrow t^2(x^2 + y^2 + 1) - 2t &= 0 \\ \Leftrightarrow t = 0 \vee t = \frac{2}{x^2 + y^2 + 1} &= \frac{2}{|z|^2 + 1} \end{aligned}$$

This corresponds to  $P = N$  or:

$$P = \left( \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

Thus, stereographic projections  $s : S \setminus N \rightarrow \mathbb{C}$  define a bijection.

Letting  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  denote the *extended complex plane* and define  $s(N) = \infty$ , then  $s$  becomes a bijective map from  $S$  onto  $\hat{\mathbb{C}}$

**Definition/Sats 7.44**

Under stereographic projections, circles on  $S$  correspond to circles and lines in  $\mathbb{C}$

**Anmärkning:**

We therefore call circles and lines in  $\mathbb{C}$  "circles" in  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , where lines are considered as "circles through  $\infty$ "



**Bevis 7.1**

The general equation for a circle or line in the  $z = x + iy$  plane is:

$$A(x^2 + y^2) + Cx + Dy + E = 0$$

Using  $z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$ , we get:

$$\begin{aligned} A \left( \left( \frac{x_1}{1 - x_3} \right)^2 + \left( \frac{x_2}{1 - x_3} \right)^2 \right) + \frac{Cx_1}{1 - x_3} + \frac{Dx_2}{1 - x_3} + E &= 0 \\ \Leftrightarrow A(x_1^2 + x_2^2) + Cx_1(1 - x_3) + Dx_2(1 - x_3) + E(1 - x_3)^2 &= 0 \end{aligned}$$

Using  $x_1^2 + x_2^2 + x_3^2 = 1$ , we get:

$$A(1 - x_3^2) + Cx_1(1 - x_3) + Dx_2(1 - x_3) + E(1 - x_3)^2 = 0$$

Dividing by  $1 - x_3$  yields:

$$\begin{aligned} A(1 + x_3) + Cx_1 + Dx_2 + E(1 - x_3) &= 0 \\ \Leftrightarrow Cx_1 + Dx_2 + (A - E)x_3 + A + E &= 0 \end{aligned}$$

This is the equation for a plane in  $\mathbb{R}^3$ , which intersects  $S$  in a circle □

## 8. MÖBIUS TRANSFORMATIONS

**Definition/Sats 8.45: Moebius transformation**

A *Möbius transformation* is a mapping of the form:

$$T(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C}$$

Where  $ad - bc \neq 0$  ( $T$  is not constant)

**Anmärkning:**

If  $c = 0$ , we let  $T(\infty) = \infty$ . Then  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is bijective

If  $c \neq 0$ , then:

$$T : \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\} \rightarrow \mathbb{C} \setminus \left\{ \frac{a}{c} \right\}$$

is a bijection. Letting  $T\left(-\frac{d}{c}\right) = \infty$ , and  $T(\infty) = \frac{a}{c}$ , we extend  $T$  to a bijective map  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$

The inverse is found by solving:

$$w = T(z)$$

which gives:

$$z = T^{-1}(w) = \begin{cases} \frac{-dw + b}{cw - a}, & \text{if } w \neq \frac{a}{c} \text{ } w \neq \infty \\ \infty & \text{if } w = \frac{a}{c} \\ -\frac{d}{c} & \text{if } w = \infty \end{cases}$$

**Anmärkning:**

If we interpret  $\frac{a}{c}$  and  $-\frac{d}{c}$  as  $\infty$ , it also holds for  $c = 0$

**Anmärkning:**

$$\begin{aligned} T'(z) &= \frac{d}{dt} \left( \frac{ax+b}{cz+d} \right) = \frac{a(cz+d) - (az+b) \cdot c}{(cz+d)^2} \\ &= \frac{ad-bc}{(cz+d)^2} \neq 0 \end{aligned}$$

Thus  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is conformal

**Anmärkning:**

If:

$$\begin{aligned} T(z) &= \frac{az+b}{cz+d} & S(z) &= \frac{\alpha z + \beta}{\gamma z + \delta} \\ \Rightarrow (S \circ T)(z) &= \frac{\alpha T(z) + \beta}{\gamma T(z) + \delta} \\ &= \frac{\alpha \left( \frac{az+b}{cz+d} \right) + \beta}{\gamma \left( \frac{az+b}{cz+d} \right) + \delta} = \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)} \end{aligned}$$

This shows that compositions of Moebius transformations are Möbius transformations.

**Anmärkning:**

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

#### Lemma 8.1

If a Moebius transformation  $T$  has more than two fixed points in  $\widehat{\mathbb{C}}$  ( $z_0$  is a fixpoint if  $T(z_0) = z_0$ ), then  $T(z) = z \forall z \in \widehat{\mathbb{C}}$

#### Bevis 8.1

If  $c = 0$ , then  $T(z) = \frac{az+b}{d}$ , so:

$$T(z) = z \Leftrightarrow \frac{az+b}{d} = z \Leftrightarrow (a-d)z + b = 0$$

So  $T$  has at most one fixed point in  $\mathbb{C}$  unless  $a = d$  and  $b = 0 \Leftrightarrow T(z) = z \forall z \in \mathbb{C}$

So if  $c = 0$ ,  $T$  has at most 2 fixed points in  $\widehat{\mathbb{C}}$  ( $T(\infty) = \infty$ ) unless  $T(z) = z \forall z \in \mathbb{C}$

If  $c \neq 0$ , then:

$$\begin{aligned} T(z) = z &\Leftrightarrow \frac{az+b}{cz+d} = z \\ &\Leftrightarrow cz^2 + (d-c)z - b = 0 \end{aligned}$$

So  $T$  has at most 2 fixed points in  $\mathbb{C}$  (and  $T(\infty) = \frac{a}{c} \neq \infty$ ) unless  $c = 0, a = d, b = 0$

This contradicts  $c \neq 0$

□

#### Definition/Sats 8.46

If  $S, T$  are Möbius transformations such that  $S(z_i) = T(z_i)$  at three different points  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ , then  $S = T$

**Bevis 8.2**

If  $S(z_i) = T(z_i)$  for  $i = 1, 2, 3$ , then the Moebius transformation  $T^{-1} \circ S$  has at least 3 fixed points. By the previous lemma:

$$T^{-1} \circ S(z) = z \quad \forall z \in \widehat{\mathbb{C}}$$

$$\text{i.e. } S(z) = T(z) \quad \forall z \in \widehat{\mathbb{C}}$$

□

**Anmärkning:**

Particular cases of the Möbius transformation are:

- $T(z) = z + b$  (*translation*)
- $T(z) = az = |a| e^{i \arg(a)} z$  (*rotation & magnification*)
- $T(z) = \frac{1}{z}$  (*inversion*)

**Anmärkning:**

If  $c \neq 0$ :

$$T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2} \frac{1}{z + \frac{d}{c}}$$

This means that every Moebius transformation is a composition of translations, rotations, magnifications, and inversions.

**Definition/Sats 8.47**

Every Möbius transformation maps "circles" onto "circles"

**Anmärkning:**

Recall that a "circle" in  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is a circle or line in  $\mathbb{C}$ . A line in  $\mathbb{C}$  is a "circle" through  $\infty$  in  $\widehat{\mathbb{C}}$

**Bevis 8.3**

It is easy to see that translations and rotations/magnifications map circles onto circles and line onto lines. This gives enough to prove that inversion:

$$T(z) = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = u + iv$$

maps circles onto circles

A circle in  $\widehat{\mathbb{C}}$  has the equation:

$$\begin{aligned} A(x^2 + y^2) + Cx + Dy + E &= 0 \\ \Leftrightarrow A + C \frac{x}{x^2 + y^2} + D \frac{y}{x^2 + y^2} + E \frac{1}{x^2 + y^2} &= 0 \\ \Leftrightarrow E(u^2 + v^2) + Cu - Dv + A &= 0 \end{aligned}$$

□

Given a "circle"  $C_z$  in the  $z$ -plane and a "circle"  $C_w$  in the  $w$ -plane, can one find a Moebius transformation  $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $T(C_z) = C_w$ ? Yes!

### 8.1. The cross-ratio.

#### Definition/Sats 8.48: Cross-ratio

Let  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$  be distinct and put:

$$(z, z_1, z_2, z_3) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \in \widehat{\mathbb{C}}$$

If some of the  $z_i$  is  $\infty$ , the right hand side should be interpret as:

$$(z, z_1, z_2, z_3) = \begin{cases} \frac{z_2 - z_3}{z - z_3} & \text{if } z_1 = \infty \\ \frac{z - z_3}{z - z_1} & \text{if } z_2 = \infty \\ \frac{z - z_3}{z_2 - z_1} & \text{if } z_3 = \infty \end{cases}$$

$(z, z_1, z_2, z_3)$  is called the *cross-ratio* of the four points

#### Anmärkning:

$S(z) = (z, z_1, z_2, z_3)$  is a Möbius transformation such that:

$$S(z_1) = 0 \quad S(z_2) = 1 \quad S(z_3) = \infty$$

By an earlier remark, this is the unique Möbius transformation mapping  $z_1, z_2, z_3$  to  $0, 1, \infty$

#### Definition/Sats 8.49

Given a tripple  $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$  of distinct points, and another tripple  $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$  of distinct points, then there is a unique Möbius transformation  $T$  such that  $T(z_i) = w_i$

The mappings  $w = T(z)$  is found by solving the cross-ratio equation:

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

#### Bevis 8.4

By an earlier remark, there is at most one such mapping. We now prove that there is exactly one by contradicting it.

Put  $S(z) = (z, z_1, z_2, z_3)$ ,  $U(w) = (w, w_1, w_2, w_3)$ :

$$\Rightarrow T(z) = (U^{-1} \circ S)(z) = U^{-1}(S(z))$$

$U^{-1}(S(z))$  is a Möbius transformation such that:

$$T(z_1) = U^{-1}(S(z_1)) = U^{-1}(0) = w_1$$

$$T(z_2) = \dots$$

$$\vdots$$

Then:

$$w = T(z) \Leftrightarrow w = U^{-1}(S(z)) \Leftrightarrow U(w) = S(z)$$

$$\Leftrightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

□

This theorem can be used to construct a  $T$  as above, mapping  $C_z$  to  $C_w$

Let  $z_1, z_2, z_3$  be three distinct points on a circle  $C_z$  in  $\widehat{\mathbb{C}}$ . Note that  $C_z$  is *oriented* by the order of these points, that is  $C_z$  acquires an orientation by proceeding through  $z_1, z_2, z_3$  in succession

Since a Möbius transformation is conformal, it maps the region to the left of  $C_z$ , oriented by  $z_1, z_2, z_3$ , to the region left of  $C_w = T(C_z)$  oriented by  $w_1, w_2, w_3$

## 8.2. Symmetry-preserving property.

Two points  $z_1$  and  $z_2$  are said to be *symmetric* with respect to a line  $L$  if  $L$  is the perpendicular bisector of the line-segment joining  $z_1$  and  $z_2$

This means that every circle or line through  $z_1$  and  $z_2$  intersects  $L$  orthogonally

### Definition/Sats 8.50

Two points  $z_1$  and  $z_2$  are said to be *symmetric* with respect to a circle  $C$  if every circle or line through  $z_1$  and  $z_2$  intersects  $C$  orthogonally

In particular, the center  $a$  of  $C$  and  $\infty$  are symmetric with respect to  $C$

### Definition/Sats 8.51: Symmetry principle

Let  $C_z$  be a circle or line in the  $z$ -plane and  $w = T(z)$  be any Möbius transformation. Then two points  $z_1$  and  $z_2$  are symmetric with respect to  $C_z$  if and only if their images  $w_1 = T(z_1)$  and  $w_2 = T(z_2)$  are symmetric with respect to the image  $C_w = T(C_z)$  under  $T$ .

### Bevis 8.5

"Two points are symmetric with respect to a given circle if and only if every circle containing the points intersects the given circle orthogonally" is a re-formulation of the theorem.

Möbius transformations preserve the class of circles, and they also preserve orthogonality. Hence, they preserve the symmetric condition.  $\square$

## 9. DIRICHLET PROBLEMS

We have previously discussed harmonic function over a domain  $D$ . These have many applications in solving Dirichlets problem:

Find a function  $\phi(x, y)$  continuous on  $D \cup \partial D$  of class  $C^2$  in  $D$  such that

- $\nabla^2 \phi = \phi_{xx} + \phi_{yy} = 0$  in  $D$  (second derivatives are 0)
- $\phi =$  some given function on  $\partial D$

## 9.1. Standard cases.

This can be easily solved in some standard cases:

$$\bullet \begin{cases} \nabla^2 \phi = 0 \text{ in } D \\ \phi(a, y) = A \\ \phi(b, y) = B \end{cases} \quad \text{Let } \phi(x, y) = \alpha x + \beta, \text{ choose } \alpha, \beta \text{ such that}$$

$$\begin{cases} \alpha a + \beta = A \\ \alpha b + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{B - A}{b - a} \\ \beta = A - \frac{a(B - A)}{b - a} = \frac{AB - aB}{b - a} \end{cases}$$

$$\Rightarrow \phi(x, y) = \frac{(B - A)x + Ab - aB}{b - a}$$

$$\bullet \phi(x, y) = \frac{2}{\pi} \text{Arg}(z) = \frac{2}{\pi} \arctan\left(\frac{y}{x}\right)$$

- $\phi(x, y) = \alpha \text{Arg}(z) + \beta$  leads to:

$$\begin{cases} \alpha \frac{\pi}{2} + \beta = A \\ -\alpha \frac{\pi}{2} + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{A - B}{\pi} \\ \beta = \frac{A + B}{2} \end{cases}$$

$$\text{i.e } \phi(x, y) = \frac{A - B}{\pi} \text{Arg}(z) + \frac{A + B}{2}$$

$$\bullet \phi(x, y) = \frac{1}{\alpha} \text{Arg}(z)$$

$$\bullet \phi(x, y) = \frac{1}{\pi} \text{Arg}(z - z_0)$$

$$\bullet \phi(x, y) = a_n + \frac{1}{\pi} \sum_{k=1}^n (a_{k-1} - a_k) \text{Arg}(z - x_k):$$

$$\text{Arg}(z - x_k) = \begin{cases} \pi & x < x_k \\ 0 & x > x_k \end{cases}$$

$\Rightarrow$  if  $x_j < x < x_{j+1}$  then:

$$\phi(x, 0) = a_n + \sum_{k=j+1}^n (a_{k-1} - a_k) = a_j$$

- $\phi(x, y) = \alpha \ln(|z|) + \beta$  leads to:

$$\begin{cases} \alpha \ln(r_1) + \beta = A \\ \alpha \ln(r_2) + \beta = B \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{B - A}{1 - r_2 - \ln(r_1)} \\ \beta = \frac{A \ln(r_2) - B \ln(r_1)}{\ln(r_2) - \ln(r_1)} \end{cases}$$

$$\Rightarrow \phi(x, y) = \frac{B - A}{1 - r_2 - \ln(r_1)} \ln(|z|) + \frac{A \ln(r_2) - B \ln(r_1)}{\ln(r_2) - \ln(r_1)}$$

How about more complicated Dirichlet problems?

The idea is to simplify the complicated problems to an easier one using a conformal mapping.

**Definition/Sats 9.52**

Suppose  $f : D \rightarrow D'$  is analytic,  $f = u + iv$ .

If  $\psi(u, v)$  is harmonic in  $D'$ , then:

$$\phi(x, y) := \psi(u(x, y), v(x, y))$$

is harmonic in  $D$

**Bevis 9.1**

Take  $z_0 \in D$ . Then  $w_0 = f(z_0) \in D'$  and since  $D'$  is open, there is a disk  $w_0 \in V$  contained in  $D'$ .

Since  $f$  is continuous, there is a disk  $z_0 \in U$  in  $D$  such that  $f(U) \subseteq V$ . Since  $\psi$  is harmonic in  $V$ , which is simply connected, there is an analytic function  $g$  in  $V$  such that  $\operatorname{Re}(g) = \psi$

But then  $g \circ f$  is an analytic function in  $U$  such that:

$$\operatorname{Re}(g \circ f)(z) = \psi(u(x, y), v(x, y)) = \phi(x, y)$$

Hence,  $\phi$  is harmonic in  $U$ . Since  $z_0$  was arbitrarily chosen,  $\phi$  is harmonic in  $D$

□

Suppose now that the analytic function  $f : D \rightarrow D'$  maps  $D$  bijectively onto  $D'$  and extends to a continuous bijection  $f : \overline{D} \rightarrow \overline{D}'$ .

Suppose also that the boundary conditions for  $\psi$  in  $D'$  corresponds to the boundary conditions for  $\phi$  in  $D$ .

Then, if we can solve the Dirichlet problem for  $\psi$ , we can also solve it for  $\phi$

## 10. COMPLEX INTEGRATION

We shall now study so-called *contour integrals* (or the line integrals) of complex-valued functions.

This theory will teach us more about the properties of analytic functions

## 10.1. Contours.

**Definition/Sats 10.53: Smooth arc**

A point set  $\gamma \in \mathbb{C}$  is said to be a *smooth arc* if it is the image of some continuous complex-valued function  $z = z(t)$ ,  $a \leq t \leq b$  such that:

- $z(t)$  has a continuous derivative on  $[a, b]$
- $z'(t) := x'(t) + iy'(t) \neq 0$  on  $[a, b]$
- $z(t)$  is bijective on  $[a, b]$

**Definition/Sats 10.54: Smooth closed curve**

A point set  $\gamma$  in  $\mathbb{C}$  is said to be a *smooth closed curve* if it is the range of some continuous function  $z = z(t)$ ,  $a \leq t \leq b$  satisfying the requirements for a smooth arc and:

- $z(t)$  is bijective on  $[a, b)$  but:

$$z(a) = z(b) \quad z'(a) = z'(b)$$

**Anmärkning:**

The phrase "γ is a smooth curve" means that γ is either a smooth arc or a smooth closed curve.

There are infinitely many choices of "admissible" parametrizations of a curve. One can, for example, change the direction of a parametrization yet still represent the same fundamental curve.

There is of course, a natural choice of the orientation of curves.

**Definition/Sats 10.55: Directed smooth curve**

A smooth curve with a specified orientation is called a *directed smooth curve*

**Definition/Sats 10.56: Contour**

A *contour*  $\Gamma$  is either a single point or a finite sequence  $(\gamma_1, \dots, \gamma_n)$  of directed smooth curves such that the terminal point on  $\gamma_k$  coincides with the initial point on  $\gamma_{k+1}$

We write  $\Gamma = \gamma_1 + \dots + \gamma_n$

**Definition/Sats 10.57: Closed contour**

$\Gamma$  is said to be a *closed contour* if the initial point on  $\gamma_1$  coincides with the terminal point on  $\gamma_n$ .

**Anmärkning:**

If the only two coinciding points on a closed contour is the initial and the terminal point, then we say it is a *simple closed contour*.



**Definition/Sats 10.58: Length of curve**

If  $\gamma$  is a smooth curve, then the length of the curve is given by the multivariate-curve length:

$$L(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of  $\Gamma = \gamma_1 + \dots + \gamma_n$  is given by:

$$L(\Gamma) = L(\gamma_1) + \dots + L(\gamma_n)$$

**10.2. Contour integrals.**

Lets see how to define  $\int_{\Gamma} f(z)dz$ , aka the contour integral of a complex-valued function  $f$  over the contour  $\Gamma$ .

We start by defining:

$$\int_{\Gamma} f(z)dz$$

where  $\gamma$  is a directed smooth curve.

For each  $i \in \{0, \dots, n\}$ , we form a partition  $\mathcal{P}_n = \{z_0, \dots, z_n\}$  of  $\gamma_i$

Let  $L(\gamma; z_{k-1}, z_k)$  denote the length of  $\gamma$  from  $z_{k-1}$  to  $z_k$ , then:

$$\mu(\mathcal{P}_n) = \max_{1 \leq k \leq n} L(\gamma; z_{k-1}, z_k)$$

is a measure of the "firmness" (**CHECK**) of the partition.

Take, for  $k = 1, \dots, n$  an arbitrary point  $c_k$  on  $\gamma$  between  $z_{k-1}$  and  $z_k$ .

Then form the Riemann-sum:

$$S(\mathcal{P}_n) = \sum_{k=1}^n f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^n f(c_k)\delta z_k$$

**Definition/Sats 10.59**

We say that  $f$  is integrable along the directed smooth curve  $\gamma$  if there exists a complex number  $L \in \mathbb{C}$  such that:

$$\lim_{n \rightarrow \infty} \mu(\mathcal{P}_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} S(\mathcal{P}_n) = L$$

(independent of the choice of partition and Riemann-sum)

**Anmärkning:**

The number  $L$  is called the *integral of  $f$  along  $\gamma$* , and is denoted:

$$\int_{\gamma} f(z)dz$$

**Anmärkning:**

The integral has the following properties:

- $\int_{\gamma} (f(z) \pm g(z))dz = \int_{\gamma} f(z)dz \pm \int_{\gamma} g(z)dz$
- $\int_{\gamma} c \cdot f(z)dz = c \int_{\gamma} f(z)dz$
- $\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$

**Definition/Sats 10.60**

If  $f$  is continuous along  $\gamma$ , then  $f$  is integrable along  $\gamma$

**10.3. How to compute the contour integral.**

First, consider  $\int_a^b f(t)dt$ , where  $f(t) = u(t) + iv(t)$ , and  $u, v$  is continuous on  $[a, b]$

Let  $F(t)$  be an antiderivative of  $f(t)$ , i.e:

$$\begin{aligned} F(t) &= U(t) + iV(t) & U' &= u & V' &= v \\ \Rightarrow \int_a^b f(t)dt &= \int_a^b (u(t) + iv(t))dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \\ U(t)|_a^b + iV(t)|_a^b &= F(b) - F(a) \end{aligned}$$

**Definition/Sats 10.61**

If  $f$  is continuous on  $[a, b]$ , and  $F'(t) = f(t)$  for  $t \in [a, b]$ , then:

$$\int_a^b f(t)dt = F(b) - F(a)$$

The integral of  $f$  along an arbitrary directed smooth curve can be reduced to integrals as above, by the parametrization  $z(t)$  ( $a \leq t \leq b$ )

Let:

$$z_0 = z(t_0) \quad z_1 = z(t_1) \quad \cdots \quad z_n = z(t_n)$$

where  $a = t_0 < t_1 < \cdots < t_n = b$ :

$$\begin{aligned} \xrightarrow{c_k \approx z_k} \sum_{k=1}^n f(z_k) \Delta z_k &= \sum_{k=1}^n f(z(t_0)) \Delta z_k \\ \Delta z_k = z_k - z_{k-1} &= z(t_k) - z(t_{k-1}) \approx z'(t_k)(t_k - t_{k-1}) = z'(t_k) \Delta t_k \\ \Rightarrow \sum_{k=1}^n f(z(t_k)) z'(t_k) \Delta t_k \end{aligned}$$

This is a Riemann-sum for the continuous function  $f(z(t))z'(t)$  on  $[a, b]$

This suggests the following:

**Definition/Sats 10.62: Curve integral**

Let  $f$  be a continuous function on a directed smooth curve having admissible parametrization  $z(t)$  ( $a \leq t \leq b$ ), then:

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt$$

**Definition/Sats 10.63**

Suppose  $\Gamma = \gamma_1 + \cdots + \gamma_n$  and let  $f$  be continuous on  $\Gamma$ , then we let:

$$\int_{\Gamma} f(z)dz := \sum_{k=1}^n \int_{\gamma_k} f(z)dz$$

(if  $\Gamma = \{z_0\}$ , we let  $\int_{\Gamma} f(z)dz = 0$ )

**Definition/Sats 10.64: ML-inequality**

Suppose  $|f(z)| \leq M \ \forall z \in \gamma$ :

$$\Rightarrow \left| \sum_{k=1}^n f(c_k) \Delta z_k \right| \leq \sum_{k=1}^n |f(c_k)| |\Delta z_k| \leq M \sum_{k=1}^n |\Delta z_k| \leq ML(\gamma)$$

As  $\mu(\mathcal{P}_n) \rightarrow 0$ , this implies:

$$\left| \int_{\gamma} f(z) dz \right| \leq ML(\gamma)$$

**Definition/Sats 10.65**

Suppose  $f$  is continuous on  $\Gamma$  and that  $|f(z)| \leq M, z \in \Gamma$

Then:

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML \quad \text{where } L = L(\Gamma)$$