Integral calculations using residue calculus

Recall the following:

Thm (Residue thm)

Let Γ be a simple closed positively oriented contour, and let f be analytic inside and on Γ with the exception of a finite number of isolated sing. $z_1, \dots, z_n$ inside Γ. Then,

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

Ex. Compute $\oint_{|z|=4} z e^{3/z}\,dz$.

Sol. $e^w = \sum_{j=0}^{\infty} \frac{w^j}{j!}, \quad w \in \mathbb{C}$

$$\Rightarrow e^{\frac{3}{z}} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{3}{z}\right)^j = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$$

for $z \neq 0$.

$$\Rightarrow z e^{\frac{3}{z}} = z + 3 + \frac{3^2}{2!}\frac{1}{z} + \frac{3^3}{3!}\frac{1}{z^2} + \cdots, \quad z \neq 0$$

$$\Rightarrow \operatorname{Res}(z e^{3/z}, 0) = \frac{3^2}{2!}$$

$$\Rightarrow \oint_{|z|=4} z e^{3/z}\,dz = 2\pi i \cdot \frac{3^2}{2!} = \underline{9\pi i}$$

$\square$

Residue calculus

1) If f has a removable sing. at z_0, clearly

$$\mathrm{Res}(f, z_0) = 0$$

2) Suppose that f has a simple pole, i.e. a pole of order 1, at z_0.

Then f has a Laurent series expansion

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

in a punctured disk about z_0.

Hence: $$\boxed{\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

if f has a simple pole at z_0.

Ex. Let $f(z) = \dfrac{e^z}{z(z+1)}$.

f has simple poles at $z = 0$ and $z = -1$, and

$$\mathrm{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^z}{z+1} = 1$$

$$\mathrm{Res}(f, -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{e^z}{z} = -e^{-1}$$

Ex. Suppose that $f(z) = \dfrac{g(z)}{h(z)}$

where g and h are both analytic at z_0, and that h has a simple zero at z_0 while $g(z_0) \neq 0$.

Clearly then f has a simple pole at z_0, so

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} =$$

$$= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - h(z_0)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Thus, in such a case

$$\boxed{\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}}$$

3) Suppose that f has a pole of order m at z_0.

Then f has a Laurent series expansion

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{z - z_0}}_{\text{principal part of } f \text{ at the pole } z_0} + a_0 + a_1(z - z_0) + \dots$$

in a punctured disk about z_0.

It follows that

$$(z - z_0)^m f(z) = a_{-m} + a_{-(m-1)}(z - z_0) + \dots +$$

$$+ a_{-1}(z - z_0)^{m-1} + a_0 (z - z_0)^m + \dots$$

and so

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] = (m-1)!\, a_{-1} + m!\, a_0 (z - z_0) + \dots$$

Taking the limit as $z \to z_0$ we have the following:

<u>Thm</u> If f has a pole of order m at z_0, then

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$

Calculations of Integrals

Trigonometric integrals can sometimes be calculated using the residue calculus.

Ex Compute $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$.

Sol Put $z = e^{i\theta}$. Then $e^{-i\theta} = \frac{1}{z}$, so

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Furthermore, $dz = ie^{i\theta}\,d\theta$, i.e. $d\theta = \frac{dz}{iz}$.

$$\Rightarrow I = \oint_{|z|=1} \frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{5+4\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} =$$

$$= -\frac{1}{4i}\oint_{|z|=1} \frac{(z^2-1)^2}{z^2(2z^2+5z+2)}\, dz$$

Put $f(z) = \frac{(z^2-1)^2}{z^2(2z^2+5z+2)} = \frac{(z^2-1)^2}{2z^2\left(z+\frac{1}{2}\right)(z+2)}$

f has simple poles at $z=-\frac{1}{2}$ and $z=-2$, and a pole of order 2 at $z=0$

Only $z=0$ and $z=-\frac{1}{2}$ lie inside the unit circle.

$$\Rightarrow I = -\frac{1}{4i}\cdot 2\pi i\left[\operatorname{Res}\left(f,-\tfrac{1}{2}\right)+\operatorname{Res}(f,0)\right]$$

by the residue theorem

$$\operatorname{Res}(f, -\tfrac{1}{2}) = \lim_{z \to -\frac{1}{2}} (z + \tfrac{1}{2}) f(z) =$$

$$= \lim_{z \to -\frac{1}{2}} \frac{(z^2 - 1)^2}{2z^2(z+2)} = \frac{3}{4}$$

and

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left[z^2 f(z) \right] =$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[\frac{(z^2-1)^2}{2z^2 + 5z + 2} \right] =$$

$$= \left. \frac{2(z^2-1) \cdot 2z (2z^2+5z+2) - (z^2-1)^2 \cdot (4z+5)}{(2z^2+5z+2)^2} \right|_{z=0} =$$

$$= -\frac{5}{4}$$

Thus,

$$I = -\frac{1}{4i} \cdot 2\pi i \left[\frac{3}{4} - \frac{5}{4} \right] = \underline{\underline{\frac{\pi}{4}}}$$

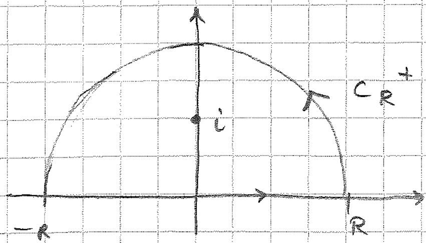
$\boxtimes$

Integrals over $\mathbb{R}$ can often be computed by considering a contour in, say, the upper half-plane.

<u>Ex</u> Compute $I = \displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \, dx$

<u>Sol.</u> Let $f(z) := \dfrac{z^2}{(z^2+1)^2} = \dfrac{z^2}{(z-i)^2(z+i)^2}$

Also, let $\Gamma_R = [-R, R] \cup C_R^+$, where C_R^+ is a half-circle in the upper half-plane of radius $R > 1$; see figure.



By the residue theorem

$$\int_{T_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, i)$$

Since $z = i$ is a pole of order 2, we have

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{1}{1!} \frac{d}{dz}\left[(z-i)^2 f(z)\right] =$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] = \left.\frac{2z(z+i)^{\cancel{2}} - z^2 \cdot 2\cancel{(z+i)}}{(z+i)^{\cancel{4}\,3}}\right|_{z=i}$$

$$= \frac{2i \cdot 2i - (-1)\cdot 2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

Thus,

$$\int_{T_R} f(z)\,dz = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2} \qquad \forall\, R > 1.$$

On the other hand

$$\underbrace{\int_{T_R} f(z)\,dz}_{\frac{\pi}{2}} = \underbrace{\int_{-R}^{R} f(x)\,dx}_{\text{Wanted integral}} + \underbrace{\int_{C_R^+} f(z)\,dz}_{\to 0}$$

But

$$\left|\int_{C_R^+} f(z)\,dz\right| \overset{ML}{\leq} \max_{z \in C_R^+} |f(z)| \cdot L(C_R^+)$$

and

$$|f(z)| = \left| \frac{z^2}{(z^2+1)^2} \right| = \frac{|z|^2}{|z^2+1|^2} \le$$

$$\underset{\text{Δ-ineq.}}{\overset{\text{Rev.}}{\le}} \frac{|z|^2}{\left||z|^2-1\right|^2} = \frac{R^2}{(R^2-1)^2}, \quad z \in C_R^+ \quad (R>1)$$

$$\Rightarrow \left| \int_{C_R^+} f(z)\, dz \right| \le \frac{R^2}{(R^2-1)^2} \pi R \to 0, \quad R \to +\infty.$$

We get that

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = \lim_{R\to+\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)^2}\, dx = \underline{\underline{\frac{\pi}{2}}}$$

E]

Remark: The same method can be used to calculate any integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$$

with $\deg Q \ge \deg P + 2$, $Q \ne 0$ on $\mathbb{R}$.

Also integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \omega x\, dx$

and $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \omega x\, dx$ $(\omega \in \mathbb{R})$ can be computed

using the method above.

Ex. Compute $I = \int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\, dx$ (Abs. integrable!)

Sol It is tempting to consider $\frac{\cos 3z}{z^2+4}$

It holds that $\cos 3z = \frac{e^{i3z}+e^{-i3z}}{2}$.

Since $|e^{i3z}| = e^{-3y}$ whereas $|e^{-i3z}| = e^{+3y}$,

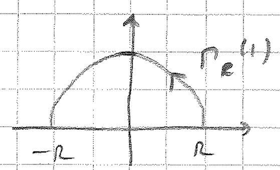
we cannot consider $\displaystyle\int_{\Gamma_R} \frac{\cos 3z}{z^2+4}\, dz$ over a contour

in the upper half-plane. We could use that

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i3x}}{x^2+4}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-i3x}}{x^2+4}\, dx =: I_1 + I_2$$

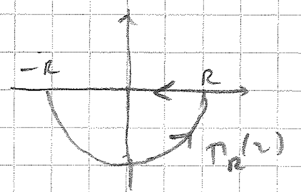
The integral I_1 can be computed by considering

$\displaystyle\int_{\Gamma_R^{(1)}} \frac{e^{i3z}}{z^2+4}\, dz$ with $\Gamma_R^{(1)}$ as follows:



and I_2 by considering $\displaystyle\int_{\Gamma_R^{(2)}} \frac{e^{-i3z}}{z^2+4}\, dz$ with $\Gamma_R^{(2)}$

as follows:



This works fine, and is the way to proceed if

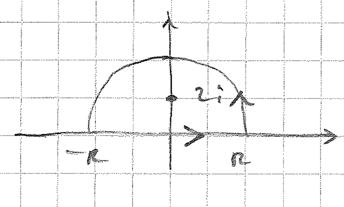
say P or Q has non-real coefficients.

Now, it is easier to note that

$$I = \mathrm{Re} \int_{-\infty}^{\infty} \frac{\overbrace{e^{i3x}}^{=\cos 3x + i\sin 3x}}{x^2+4}\, dx$$

Therefore, let $f(z) = \dfrac{e^{i3z}}{z^2+4}$. Consider $\displaystyle\int_{\Gamma_R} f(z)\, dz$

with $\Gamma_R = [-R, R] \cup C_R^+$ as below



$(R > 2)$

Clearly,

$$\int_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz \qquad (*)$$

and by the residue thm

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \ \operatorname{Res}(f, 2i) = \text{/ see ex. in 2) /}$$

$$= 2\pi i \cdot \frac{e^{i3\cdot 2i}}{2\cdot 2i} = \frac{\pi e^{-6}}{2}.$$

Now,

$$|f(z)| = \left| \frac{e^{i3z}}{z^2+4} \right| = \frac{e^{-3y}}{|z^2+4|} \le \frac{1}{R^2-4}, \quad z \in C_R^+$$

$$\Rightarrow \left| \int_{C_R^+} f(z)\,dz \right| \le \frac{1}{R^2-4}\pi R \to 0, \quad R \to +\infty$$

So, if we let $R \to +\infty$ in $(*)$, we get

$$\int_{-\infty}^{\infty} \frac{e^{i3x}}{x^2+4}\,dx = \frac{\pi e^{-6}}{2}$$

Thus, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx = \operatorname{Re}\left(\frac{\pi e^{-6}}{2}\right) = \frac{\pi e^{-6}}{2}$

(we also see that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{x^2+4}\,dx = \operatorname{Im}\left(\frac{\pi e^{-6}}{2}\right) = 0$,

which is clear since $\dfrac{\sin 3x}{x^2+4}$ is odd) $\square$

Remark: 1) The same method can be used to compute $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\cos wx\,dx$ and $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\sin wx\,dx$

as long as $\deg Q \geq \deg P + 2$.

2) Integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$

are important in fourier analysis. $\square$