A1

IBVP 1: The PDE is $u_t + u_x = 0$, which is the advection equation with positive wave speed. This means that waves travel *to the right*. We need a boundary condition on the inflow, which is the left boundary. We know that we need exactly one BC because the PDE is scalar and first order in space. However, IBVP one includes the condition $u(1,t) = 1$, which is a BC on the right boundary. This IBVP is **ill-posed**.

(If you are unsure which boundary the BC should be imposed on, you can always use the energy method to determine which boundary could contribute to energy growth and therefore requires a BC.)

IBVP 2: This is the advection equation with forcing function $F(x,t) = \sin(x)\cos(t)$. We know that the forcing function does not affect well-posedness, so we may pretend like $F(x,t) = 0$. We should have one BC at the left boundary (see discussion for IBVP 1) and one IC, which is precisely what we have. This problem is **well-posed**.

IBVP 3: This is the **backwards** heat equation (wrong sign in front of u_{xx}), which we know is **ill-posed**. We do not even need to think about the BCs or the IC.

In case you do not remember that this is the backward heat equation, you can also prove this using the energy method or Fourier analysis. The energy method leads to (assuming that u is real)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -uu_x|_0^1 + \|u_x\|^2.$$

There is no way to bound the growth rate. The term $\|u_x\|^2$ is the dissipation term that we are used to from the heat equation, but with the wrong sign.

Fourier analysis (assuming periodic BCs) yields

$$\frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} = k^2\hat{u}_k \quad \Longleftrightarrow \quad \hat{u}_k(t) = \hat{u}_k(0)e^{k^2t}.$$

For well-posedness we can tolerate growth of the form

$$\hat{u}_k(t) = Ke^{\alpha t}\hat{u}_k(0)$$

for some *constants* α and K, but since the wavenumber k is unbounded, our growth rate can not be bounded in this way.

A2

This is the wave equation with constant wave speed, a Dirichlet condition on the left boundary, and the condition $c^2u_x = -\alpha u_t$ on the right boundary. The energy method

yields:

$$\underbrace{(u_t, u_{tt})}_{=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2} = c^2(u_t, u_{xx}) = [IBP] = c^2 u_t u_x|_0^L - c^2 \underbrace{(u_{xt}, u_x)}_{=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|^2}.$$

We get the energy rate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2 + c^2\|u_x\|^2\right) = c^2 u_t(L,t) \underbrace{u_x(L,t)}_{=-\alpha c^{-2} u_t(L,t)} - c^2 \underbrace{u_t(0,t)}_{=0} u_x(0,t).$$

It follows from the Dirichlet condition $u(0,t) = 0$ that $u_t(0,t) = 0$ (since $u(0,t)$ is constant in time). We arrive at:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2 + c^2\|u_x\|^2\right) = -\alpha u_t(L,t)^2.$$

The question is which of the following relations that u satisfies for any smooth initial data.

1. $\|u\|^2 = 0$.
   False. Any nonzero initial data contradict this.

2. $\|u\|^2 = -\alpha u_t(L,t)^2$.
   False. One can easily select initial data that contradict this.

3. $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2\right) = 0$.
   False. The easiest way to contradict this is probably to set initial data $u = 0, u_t = 1$.

4. $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2\right) = -\alpha u_t(L,t)^2$.
   False. The easiest way to contradict this is probably to set initial data $u = 0, u_t = 1$.

5. $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2 + c^2\|u_x\|^2\right) = 0$.
   False. Correct energy, but we know that there is dissipation from the right boundary.

6. $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2 + c^2\|u_x\|^2\right) = -\alpha u_t(L,t)^2$.
   True! This is the energy rate that we derived.

7. $\|u_t\|^2 = 0$.
   False. Can choose initial data for u_t that does not satisfy this.

8. $\|u_t\|^2 = -\alpha u_t(L,t)^2$.
   False. Can choose initial data for u_t that does not satisfy this.

A3

1. LU
2. Gaussian elimination
3. CG

A4

The total matrix $B = cD + \epsilon A$ is *not* symmetric. CG requires the matrix to be SPD. CG is therefore the worst choice. LU factorization will be more efficient than Gaussian elimination since we are solving many systems with *the same matrix*.

A5

A correct statement of the weak form is:

Find $u \in V_g$ such that

$$(v, u_t) = -a(v_x, u_x)$$

for all $v \in V_0$.

That is, the correct answers are:

- X1 = Alt. 1
- X2 = Alt. 6
- X3 = Alt. 4

A6

Three important things to consider:

- A small relative error tolerance ($<< 10^{-3}$) favors a high-order method.
- Complex geometry favors FEM over FD.
- Wave propagation problems (slightly) favor FD over FEM, since there is no mass matrix to invert.

PDE 1: Wave propagation (advection), trivial geometry, small error tolerance. Fourth-order FD should be better than second-order FEM.

PDE 2: No wave propagation (stationary heat equation), complex geometry, large error tolerance. Perfect for second-order FEM.

PDE 3: Wave propagation (Schrödinger), simple geometry (a square), small error tolerance. Perfect for high-order FD.

A7

BC1: $u_x(L,t) = 0$. This is a Neumann condition. The usual SAT ansatz is:

$$SAT_1 = \tau_r H^{-1} \mathbf{e}_r (\mathbf{d}_r^T \mathbf{u} - 0).$$

There is only one alternative that is of this form, with $\tau_r = -1$. Answer:

$$SAT_1 = -H^{-1} \mathbf{e}_r (\mathbf{d}_r^T \mathbf{u} - 0).$$

BC2: $u(L,t) = 0$. We have not practiced SAT for Dirichlet BC very much, but we know that SAT_2 **must** be consistent with the BC, i.e.,

$$SAT_2 \sim (\mathbf{e}_r^T \mathbf{u} - 0).$$

There is only one alternative of this form, so that must be the correct one. Answer:

$$SAT_2 = -H^{-1} \mathbf{d}_r (\mathbf{e}_r^T \mathbf{u} - 0)$$

This is also similar to the stated SAT for the left boundary, which is a good sign.

BC3: $u_x(L,t) + \alpha u(L,t) = 0$. The SBP discretization of this BC is $\mathbf{d}_r^T \mathbf{u} + \alpha \mathbf{e}_r^T \mathbf{u} = 0$. There is only one alternative that matches this. Answer:

$$SAT_3 = -H^{-1} \mathbf{e}_r^T (\mathbf{d}_r^T \mathbf{u} + \alpha \mathbf{e}_r^T \mathbf{u} - 0).$$

A8

One might recognize these finite-difference approximations as:

1. The forward difference operator (D_+), which is a first-order approximation of the first derivative.
2. The centered second-order difference operator (D_0), which is a second-order approximation of the first derivative.
3. The centered second-derivative ($D_+ D_-$), which is a second-order approximation.

There are many clues. For example, if the coefficients are of order h^{-1}, it must be a first-derivative approximation (or not consistent with any derivative). If the coefficients are of order h^{-2}, it must be a second-derivative approximation (or not consistent with any derivative), and so on. Also, the stencils are too narrow to be high-order accurate, so they are at most first- or second-order accurate.

If one does not recognize the approximation, they can always be analyzed by Taylor expansion. We have

$$u(x+h) = u + hu' + \frac{h^2}{2!}u'' + \frac{h^3}{3!}u''' + \frac{h^4}{4!}u'''' + ...$$

$$u(x-h) = u - hu' + \frac{h^2}{2!}u'' - \frac{h^3}{3!}u''' + \frac{h^4}{4!}u'''' + ...$$

where we use shorthand notation $u' = u'(x)$, etc. It follows that

$$\frac{u(x+h)-u(x)}{h} = \frac{1}{h}\left(u + hu' + \frac{h^2}{2!}u'' + \mathcal{O}(h^3) - u\right) = u' + \frac{h}{2!}u'' + \mathcal{O}(h^2) = \underbrace{u' + \mathcal{O}(h)}_{\text{1st order approx. of } u'},$$

$$\begin{aligned}\frac{u(x+h)-u(x-h)}{2h} &= \frac{1}{2h}\left(u + hu' + \frac{h^2}{2!}u'' + \frac{h^3}{3!}u''' - u + hu' - \frac{h^2}{2!}u'' + \frac{h^3}{3!}u''' + \mathcal{O}(h^4)\right) \\ &= u' + \frac{h^2}{3!}u''' + \mathcal{O}(h^3) = \underbrace{u' + \mathcal{O}(h^2)}_{\text{2nd order approx. of } u'},\end{aligned}$$

$$\begin{aligned}\frac{u(x+h)-2u(x)+u(x-h)}{h^2} &= \frac{1}{h^2}\left((1-2+1)u + (h-h)u' + h^2u'' + \frac{2h^4}{4!}u'''' + \mathcal{O}(h^5)\right) \\ &= \underbrace{u'' + \mathcal{O}(h^2)}_{\text{2nd order approx. of } u''}.\end{aligned}$$

B1

The PDE is

$$u_t = u_{xx} - u_x, \quad x \in (0,1), \quad t > 0,$$

which is the advection-diffusion equation with advection to the right. The energy method yields

$$\underbrace{(u, u_t)}_{=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2} = \underbrace{(u, u_{xx})}_{IBP} - \underbrace{(u, u_x)}_{=\frac{1}{2}u^2|_0^1} = [IBP] = \left[uu_x - \frac{1}{2}u^2\right]_0^1 - \|u_x\|^2.$$

We have the energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \leq \left[uu_x - \frac{1}{2}u^2\right]_0^1 = \underbrace{\left[\frac{1}{2}u\left(2u_x - u\right)\right]_0^1}_{:=BT}.$$

For well-posedness, we need $BT \leq 0$. Let us try the sets of BCs one by one.

BC set 1:

$$u = 0, \quad x = 0 \quad \text{and} \quad u = 0, \quad x = 1.$$

This set of BCs yields

$$BT = 0.$$

Well-posed!

BC set 2:

$$u = 0, \quad x = 0 \quad \text{and} \quad u_x - u = 0, \quad x = 1.$$

This set of BCs yields

$$BT = \frac{1}{2}u\left(2u - u\right)|_1 - 0 = \frac{1}{2}u(1,t)^2 \geq 0.$$

Not well-posed!

BC set 3:

$$2u_x - u = 0, \quad x = 0 \quad \text{and} \quad u_x = 0, \quad x = 1.$$

This set of BCs yields

$$BT = \frac{1}{2}u\left(2 \cdot 0 - u\right)|_1 - 0 = -\frac{1}{2}u(1,t)^2 \leq 0.$$

Well-posed!

BC set 4:

$$u_x = 0, \quad x = 0 \quad \text{and} \quad u_x = 0, \quad x = 1.$$

This set of BCs yields

$$BT = \frac{1}{2}u\left(0 - u\right)|_1 - \frac{1}{2}u\left(0 - u\right)|_0 = -\frac{1}{2}u(1,t)^2 + \frac{1}{2}u(0,t)^2.$$

Not well-posed! (Possible growth at the left boundary)

BC set 5:

$$u_x - u = 0, \quad x = 0 \quad \text{and} \quad u_x + u = 0, \quad x = 1.$$

This set of BCs yields

$$BT = \frac{1}{2}u\left(-2u - u\right)|_1 - \frac{1}{2}u\left(2u - u\right)|_0 = -\frac{3}{2}u(1,t)^2 - \frac{1}{2}u(0,t)^2 \leq 0.$$

Well-posed!

B2

The PDE is the advection equation:

$$u_t + cu_x = 0, \quad 0 \leq x \leq L,$$

with periodic boundary conditions.

Semi-discrete approximation 1

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + c \underbrace{\frac{u_{j+1} - u_{j-1}}{2h}}_{=(D_0 u)_j \approx u_x} = 0$$

This is a standard, centered, second-order accurate finite difference approximation of u_x. We expect second-order convergence, i.e., the error behaves as Ch^2, for some constant C, as $h \to 0$.

There is no numerical dissipation. Recall, for example, that the standard SBP operator D_1 is based on centered finite differences. The discrete energy method then shows that the only possible sources of dissipation are the boundaries, but since we are here dealing with periodic BC, we get perfect energy conservation.

Semi-discrete approximation 2

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + c \underbrace{\frac{u_{j+1} - u_{j-1}}{2h}}_{=(D_0 u)_j \approx u_x} = \varepsilon h \underbrace{\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}}_{=(D_+ D_- u)_j \approx u_{xx}}.$$

Notice that the right-hand side contains a second-order accurate approximation of u_{xx}. This is actually a semi-discrete approximation of the advection-diffusion equation:

$$u_t + cu_x = \varepsilon h u_{xx}.$$

Notice that the diffusion constant here is εh, where h is the grid spacing!

Let us first discuss the sign of ε. For $\varepsilon < 0$, the diffusion term has the wrong sign and causes growth (like in the backward heat equation). It is reasonable to assume that the approximation is **unstable** for $\varepsilon < 0$.

For $\varepsilon > 0$, this term causes dissipation, so the approximation is **stable**. Since the diffusion constant is proportional to h, this term is a first-order modification of the original PDE. On coarse grids (large h), there will be a lot of diffusion and the numerical solution will be much smoother/flatter than the true solution. As we refine the grid, h decreases, and the diffusion term becomes smaller. We will obtain first-order convergence to the true solution.

In matrix-vector form, both approximation 1 and approximation 2 can be written as

$$\mathbf{u}_t = D\mathbf{u}.$$

In both cases, the nonzero elements of the matrix D are proportional to h^{-1}. We therefore expect the spectral radius of D to be proportional to h^{-1}, which means that with an explicit time-integrator, we can select the time step according to the usual CFL condition:

$$\Delta t = C\frac{h}{c},$$

where C is some constant that depends on c and ε.