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Counting Rooted Maps by Genus. I

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Using a combinatorial equivalent for maps, we take the first census of maps on orientable surfaces of arbitrary genus. We generalize to higher genus Tutte's recursion formula for counting slicings, and thus obtain an algorithm for counting rooted maps by genus, number of edges, and number of vertices. We then solve a special case of this recursion formula to count slicings with one face by genus. This leads to an explicit formula which counts rooted maps with one face by genus and number of edges.

1. INTRODUCTION

A *map* is a partition of a closed, connected 2-dimensional surface into simply connected polygonal regions (faces) by means of a finite number of simple curves (edges) connecting pairs of points (vertices) in such a way that the edges are disjoint from each other and from the vertices. The enumeration of *planar* maps (maps on the sphere or plane) has been investigated extensively, in particular by W. T. Tutte. Planar maps labeled so as to destroy all possible symmetry have been enumerated for a wide variety of map conditions (cf. [1a–b], [9a–c], and other census papers of W. T. Tutte, W. G. Brown, and R. C. Mullin).

W. G. Brown [1a–b] counted several types of maps on the projective plane and began investigating the torus, but did not obtain any explicit formulae for counting maps on the torus.

In this paper, we present what we believe to be the first census of maps on oriented surfaces of arbitrary genus. Two maps are considered to be equivalent if they are related by an orientation-preserving homeomorphism (cf. [2, p. 54] and [9b, p. 250]. A map was defined in [9b] to be *rooted* if one edge is distinguished, oriented, and assigned a left and a right side. But, since we are working only with oriented surfaces, it suffices to distinguish and orient one edge (or, equivalently, to distinguish one edge-end, the "tail" end of the distinguished edge), because its left and right side are determined by the orientation of the surface. Two rooted maps are considered equivalent if they are related by an orientation-preserving homeomorphism which leaves fixed the distinguished edge-end, called the *root*, and it has been shown [1b, p. 16] that such a homeomorphism leaves fixed every edge-end.

We begin by reviewing the combinatorial equivalent of maps presented in [7] (which allowed us to approach map enumeration combinatorially rather than topologically) and giving a simple application to counting rooted maps without regard to genus.

We then generalize to higher genus Tutte's recursion formula [9b, p. 713, formula 2.1] for counting *slicings*. (In fact, we will consider instead the maps obtained from slicings by contracting each of the bands to a point—we call such a "contracted slicing" a *dicing*. A dicing is thus a map whose vertices are distinguished by labeling each one with a different natural number, and, for each vertex which contains at least one edge-end, one of these edge-ends is distinguished. Two dicings are considered equivalent if they are related by an orientation-preserving homeomorphism which leaves fixed every vertex and the distinguished edge-end in every vertex. Any such homeomorphism must leave at least one edge-end fixed, unless the map consists of a single vertex and no edges, and must therefore leave fixed every edge-end.)

By means of this formula, we may compute the number of dicings given the degree of each vertex and the genus. (In [10, Appendix A, Table 1], these numbers are tabulated for maps with up to 11 edges.) This recursion formula is solved explicitly for maps with one face.

We then obtain a relation between the number of dicings and the number of rooted maps. This relation allows us to compute the number of rooted maps given the genus, the number of vertices, and the number of edges (these numbers are tabulated for maps with up to 14 edges in [10, Appendix A, Table 2], and for maps with up to 11 edges in Table 1 at the end of this paper). From the solution of the recursion formula for maps with one face, we derive an explicit formula for the number of rooted maps with one face, given the genus and the number of vertices.

2. The Combinatorial Equivalent of Maps

The usual method for enumerating maps is to delete an edge (strictly speaking, to merge an edge with its incident faces) and thus find an expression for the number of maps with e edges in terms of the number of maps with fewer than e edges. The success of this method for counting planar maps depends upon the fact that deleting an edge from a planar map either leaves a map or separates the imbedded graph into two components, after which the imbedding surface can be pulled apart, leaving two maps. For non-planar maps, this is no longer true. For example, when an edge is deleted from the map of Figure 1A on the torus (represented as a square with opposite sides identified in the direction of the arrows), the resulting topological graph (1B) is not a map, as the torus minus one loop is not simply connected.

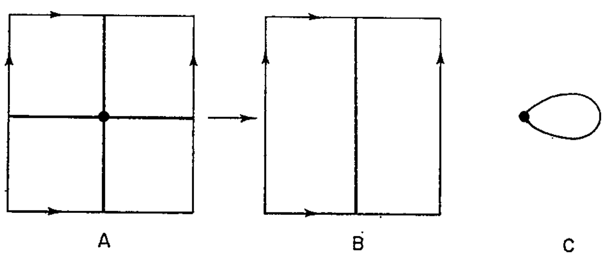

FIGURE 1

Of course, the graph determined by the vertices and edges of Figure 1B can be imbedded as a map in the plane (Fig. 1C). This suggests that the number of maps of genus g with e edges may be expressed in terms of the number of maps of genus *g or less* with fewer than e edges, provided that a method can be found for pulling a graph out of its imbedding surface and, after deleting an edge, imbedding it as a map in whatever surface fits it, in some unique way.

This, in fact, can always be done. A theorem of J. Edmonds [4b] asserts that "for any connected linear graph with an arbitrarily specified cyclic ordering of the edge-ends to each vertex, there exists a topologically unique imbedding in an oriented closed surface so that the clockwise edge-end orderings around each vertex are as specified and so that the complement of the graph in the surface is a set of discs." (Such an imbedding, called a *simple imbedding*, is a map.) Conversely, the vertices and edges of any map determine a graph in the general sense (loops and multiple edges are allowed) which is of necessity connected (otherwise, there exists a region whose boundary is not connected, and this region cannot be simply connected), and the orientation of the surface determines a cyclic ordering of edge-ends to each vertex of any map.

This result appears in Edmonds' Master's thesis [4a], which includes an algorithm for constructing the map from the graph. The algorithm given by Youngs in [12] assumes the graph to have no multiple edges. Edmonds' proof is based on the assumption that any map is homeomorphic to a set of polygons with edges identified in pairs. The same result is assumed by Brown in [1b, p. 2]. We follow tradition and assume the above-mentioned result—and hence Edmonds' theorem—is true. So it suffices to consider connected *ordered graphs*, where an ordered graph is a graph with specified cyclic orderings of the edge-ends at each vertex.

The set of edge-ends of a graph may be considered [cf. 7] as the Cartesian product $2E$ of E (the set of edges) and $\{-1, +1\}$, where for any edge η in E, its two ends are called $(-1, \eta)$ (or $-\eta$) and $(+1, \eta)$ (or $+\eta$). The vertices of a graph (apart from the *isolated* vertices—those incident on no edges) may be considered as sets into which $2E$ is partitioned. If a cyclic ordering of edge-ends is specified at each vertex, the ordering together with the partition is equivalent to a permutation P on $2E$, whose cycles are the vertices. Conversely, a permutation P on the set $2E = \{-1, +1\} \times E$ determines an ordered graph up to the number of isolated vertices. Another permutation, called *minus* and denoted by a minus sign, is a fixed-point-free involution on $2E$ defined by letting $-\beta$ be the other end of the edge containing β, for all β in $2E$. The connected components of the graph are the orbits of the subgroup $\Gamma(P, -)$ of permutations on $2E$ generated by P and $-$.

An imbedding of an ordered graph in an oriented surface is an imbedding of the graph in the surface such that, for any β in $2E$, $P(\beta)$ is the first end reached by counterclockwise rotation from β. That is, P traces each vertex counterclockwise. A traveler tracing a polygonal contour which forms part of the boundary of a region, with the region on his right, will find that the initial ends of successive edges of the contour are successive elements of a single cycle of $P-$ (minus followed by P) [cf. 10, p. 18]. If the imbedding is simple, each region has but one contour, which is traced clockwise by a cycle of $P-$ [cf. also 4a] (see Figure 2, where $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ is a cycle of $P-$ and the arrows indicate the counterclockwise direction).

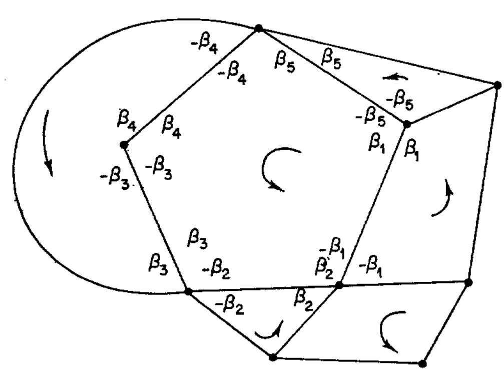

FIGURE 2

Thus *every map with e edges, v vertices, and f faces is equivalent to a permutation P on a set* $2E = \{-1, +1\} \times E$, *where* $\#(E)$, *the cardinality of E, is e, P has v cycles, P— has f cycles, and* $\Gamma(P, -)$ *is transitive on 2E.* For example, the map in Figure 1A is represented by the permutation $P = (-1, -2, +1, +2)$, which has one cycle, as does $P-$; while the map in 1C is represented by the permutation $P = (-1, +1)$, which has one cycle, while $P- = (-1)(+1)$, which has two cycles. P, or rather the isomorphism class containing P (see the beginning of Section 7) is a combinatorial equivalent of a map, and is thus called a *combinatorial map*.

The genus of a combinatorial map is given by the genus of the surface in which it may be imbedded as a map: $1 - (f - e + v)/2$ by Euler's formula [cf. 2, p. 118], or

$$g = 1 - \tfrac{1}{2}(z(P-) - e + z(P)), \tag{1}$$

where $z(a_1, a_2, \ldots)$ is the number of orbits of the permutation group on $2E$ generated by the permutations $a_1, a_2, \ldots$. This is also the minimal genus of an orientable surface in which the combinatorial map can be imbedded as a topological graph [cf. 10, pp. 20–27]. Formula 1 holds for the map with one vertex and no edges provided the isolated vertex is regarded as an empty cycle of P and an empty cycle of $P-$.

Combinatorial maps are very similar to the *constellations* studied by A. Jacques [cf. 3, 6a], who generalized them in [6b] by replacing "minus" by an arbitrary permutation. (The work of [6a] and of [7], although containing some overlapping material, each proceeded independently of the other.)

Since a map on an oriented surface is rooted by distinguishing an edge-end, a (connected) combinatorial map is defined to be rooted if an end (an element of $2E$) is distinguished.

3. Counting Rooted Maps without Regard to Genus

As a simple example of the application of combinatorial maps to enumeration, we derive a recursion relation for $F_{b,p}$, the number (without regard to genus) of rooted maps with $b + p$ edges and $p + 1$ vertices—that is, the sum over all $g \leqslant (p + b)/2$ of the number of rooted maps with $b + p$ edges and $p + 1$ vertices on an orientable surface of genus g.

Since there is one map with one vertex and no edges, $F_{0,0} = 1$. Now take any rooted map with $b + p > 0$ edges and $p + 1$ vertices and delete the root-edge (see Figure 3, where the arrow points from the root β to the other end $-\beta$ of the root-edge). If this reduction separates the graph into two connected components, consider the two components as connected ordered graphs with (respectively) $j+1$ vertices and $k+j$ edges, and $(p-j-1)+1$ vertices and $(b-k)+(p-j-1)$ edges, for some j and k with $0 \leqslant j \leqslant p-1$ and $0 \leqslant k \leqslant b$. Then $P(\beta)$ and $P-(\beta)$ will be in separate components, and so may be regarded as the roots of their respective ordered graphs (unless one or both of the vertices incident with the deleted edge have become isolated). For any choice of j and k, this reduction may be reversed uniquely.

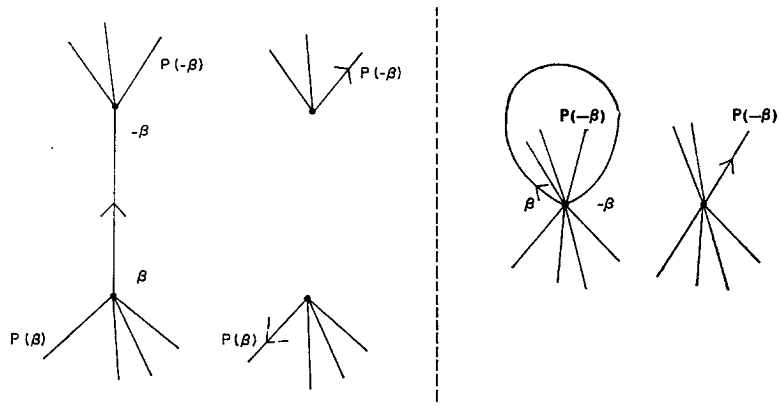

FIGURE 3

If the reduction does not disconnect the graph, it leaves an ordered graph with $(b-1)+p$ edges and $p+1$ vertices. Unless there are no longer any edges, the root-vertex has not been isolated; so shift the root to $P-(\beta)$. This reduction may be reversed in $2(b+p)-1$ ways, because the non-root end $-\beta$ of the root-edge must be the next end clockwise after the root of the reduced ordered graph, and the root β may now be inserted into one of $2[(b-1)+p]+1 = 2(b+p)-1$ slots. So we have

$$F_{0,0} = 1, \qquad \text{and unless } b \text{ and } p \text{ are both zero}$$

$$F_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} F_{k,j} F_{b-k,p-j-1} + [2(b+p)-1]\, F_{b-1,p}. \tag{2}$$

(See [10, Appendix A, Table 5] for the number of rooted maps with e edges and v vertices, for $1 \leqslant v \leqslant e+1$ and $0 \leqslant e \leqslant 20$, computed from formula 2.)

Now let $F_b(x) = \sum_{p=0}^{\infty} F_{b,p} x^p$. Then by substitution in (2) and a routine calculation [cf. 10, p. 120], we have

$$F_b(x) - F_b(0) = x \sum_{k=0}^{b} F_k(x) F_{b-k}(x) + (2b-1)[F_{b-1}(x) - F_{b-1}(0)] + 2x \frac{d}{dx} F_{b-1}(x).$$

With $b = 0$, we have $F_0(x) - 1 = x[F_0(x)]^2$; so $F_0(x) = (1 - \sqrt{1-4x})/2$, and

$$F_{0,p} = \frac{(2p)!}{p!\,(p+1)!}. \tag{3}$$

This is the well-known formula for the number of rooted trees with p edges.

With $b > 0$ and $p = 0$, (2) becomes $F_{b,0} = (2b-1) F_{b-1,0}$. Since $F_{0,0} = 1$, it follows by induction on b that

$$F_{b,0} = 1 \times 3 \times 5 \times \cdots \times (2b-1) = \frac{(2b)!}{b! \times 2^b}. \tag{4}$$

This is the number of rooted maps with one vertex and b edges.

Then, since $F_b(0) = (2b-1) F_{b-1}(0)$,

$$F_b(x) = x \sum_{k=0}^{b} F_k(x) F_{b-k}(x) + (2b-1) F_{b-1}(x) + 2x \frac{d}{dx} F_{b-1}(x).$$

Let

$$F_b(x) = \frac{1 - \sqrt{1-4x}}{2} (1-4x)^{-b} g_b(y),$$

where

$$y = \frac{(1-4x)^{-1/2} - 1}{2}. \tag{5}$$

Then $g_0(y) = 1$, and, if $b > 0$,

$$g_b(y) = y \sum_{k=1}^{b-1} g_k(y) g_{b-k}(y) + 2y(y+1) \frac{d}{dy} g_{b-1}(y) + [2(b-1)(1+2y) + 1] g_{b-1}(y). \tag{5a}$$

The first few functions are:

$g_1(y) = g_0(y) = 1,$
$g_2(y) = 5y + 3,$
$g_3(y) = 60y^2 + 65y + 15,$
$g_4(y) = 1105y^3 + 1730y^2 + 794y + 105.$

It follows from (5a) by induction on b that, if $b > 0$, $g_b(y)$ is a polynomial of degree $b - 1$ with positive integral coefficients. The problem of finding an explicit formula for the coefficients is still unsolved.

This suggests the possibility that the function $A_{g,b}(x) = \sum_{p=0}^{\infty} a_{g,b,p}x^p$, where $a_{g,b,p}$ is the number of rooted maps of genus g with $b + p$ edges and $p + 1$ vertices, is also of the form of (5), with the same restriction on the polynomial factor. The proof of this conjecture appears in [10, pp. 123–140] and is contained in "Counting Rooted Maps by Genus. II."

4. The Recursion Formula for Counting Slicings and Dicings by Genus

The combinatorial equivalent of a dicing is a permutation P on $2E$ with $z(P, -) = 1$ and with the v cycles of P (the vertices) distinguished by labeling them with the numbers $1,..., v$, and with each cycle of P containing a distinguished end. Consider a genus-g dicing whose i-th vertex is of degree $d_i \geqslant 1$ $(i = 1,..., v)$ and let β be the distinguished end of the v-th vertex.

If the edge containing β is a link (so that $-\beta$ belongs to a different vertex, say the i-th), contract the link to a point as in Figure 4. This

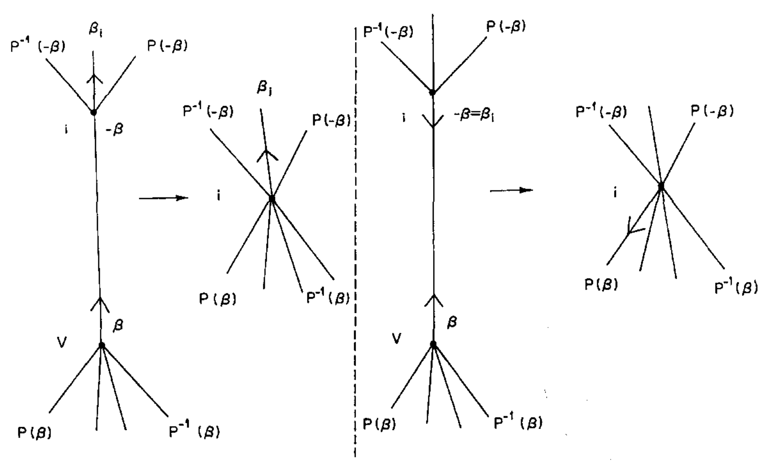

FIGURE 4

eliminates one edge and the v-th vertex. The degree of the i-th vertex changes from d_i to $d_i + d_v - 2$, and its distinguished end β_i remains where it was, unless it was $-\beta$, in which case shift it to $P(\beta)$.

The cycle of $P-$ containing β changes from

$$(\ldots, -P^{-1}(\beta), \beta, P(-\beta), \ldots) \quad \text{to} \quad (\ldots, -P^{-1}(\beta), P(-\beta), \ldots)$$

and the cycle of $P-$ containing $-\beta$ changes from

$$(\ldots, -P^{-1}(-\beta), -\beta, P(\beta), \ldots) \quad \text{to} \quad (\ldots, -P^{-1}(-\beta), P(\beta), \ldots),$$

while the other cycles of $P-$ remain unchanged. So β and $-\beta$ merely get dropped from the cycle(s) of $P-$ which contain them. The number of cycles (including empty cycles) of $P-$ remains unchanged; hence, by formula 1, so does the genus.

Reversing this reduction means selecting vertex i and breaking off $d_v - 1$ consecutive ends not containing β_i (and joining the pieces with a link), which can be done in $d_i - 1$ ways, or breaking off $d_v - 1$ consecutive ends starting with β_i, which can be done in one way. This can be done to any of the vertices from the first to the $v - 1$-st; so the contribution of this reduction to $C_g(d_1, \ldots, d_v)$, which is the number of genus-g dicings whose vertices are of degrees $d_1, \ldots, d_v$, is

$$\sum_{i=1}^{v-1} d_i C_g(d_1, \ldots, d_{i-1}, d_i + d_v - 2, d_{i+1}, \ldots, d_{v-1}).$$

If the edge containing β is a loop, delete it and split its incident vertex into two, one labeled v and containing the ends from $P(\beta)$ to $P^{-1}(-\beta)$ with distinguished end $P(\beta)$, and the other labeled $v + 1$ and containing the ends from $P(-\beta)$ to $P^{-1}(\beta)$ with distinguished end $P(-\beta)$ (see Fig. 5). The sum of the degrees of the two new vertices is $d_v - 2$.

This reduction also does not change the number of cycles of $P-$: just as in the previous reduction, β and $-\beta$ merely get dropped from the cycle(s) of $P-$ which contain them, while the other cycles of $P-$

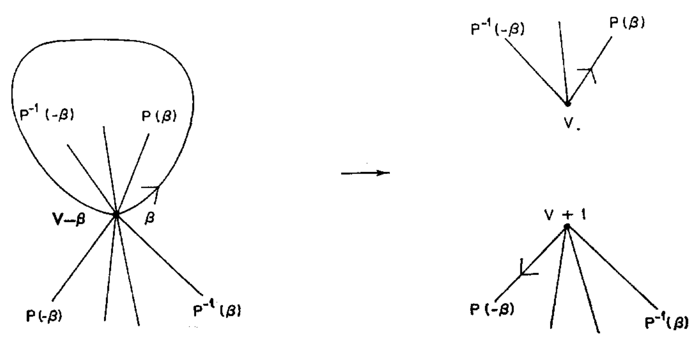

FIGURE 5

remain unchanged. One edge is eliminated and one new vertex is created. So (by formula 1) if the reduction disconnects the graph, the sum of the genera of the two components is g; otherwise, the genus of the reduced combinatorial map is $g - 1$. (This reduction *must*, therefore, disconnect a *planar* map.)

This reduction is uniquely reversible: so the contribution from the case in which the graph is not disconnected is

$$\sum_{k+m=d_v-2} C_{g-1}(d_1, \ldots, d_{v-1}, k, m),$$

where k and m are non-negative. If the graph is disconnected, some of the first $v - 1$ vertices will go into the connected component containing the new v-th vertex, and the remainder into the component containing the new $v + 1$-st vertex. Let D_1 be the subsequence of $D = d_1, \ldots, d_{v-1}$ consisting of every term d_j such that the j-th vertex goes into the component containing the v-th vertex, and let D_2 be the complement of D_1 in D (borrowing notation from set theory, we say that $D_1 \cap D_2 = \phi$ and $D_1 \cup D_2 = D$). There will be a contribution from every subsequence D_1 of D, and the total contribution from this case is

$$\sum_{\substack{D_1 \cup D_2 = D \\ D_1 \cap D_2 = \phi}} \sum_{h+f=g} \sum_{k+m=d_v-2} C_h(D_1, k)\, C_f(D_2, m),$$

where k and m are non-negative.

Summing all the contributions, we find that, for every sequence of positive integers $d_1, \ldots, d_v$ *and for every integer* g,

$$\begin{aligned} C_g(d_1, \ldots, d_v) = {} & \sum_{i=1}^{v-1} d_i C_g(d_1, \ldots, d_{i-1}, d_i + d_v - 2, d_{i+1}, \ldots, d_{v-1}) \\ & + \sum_{\substack{k+m=d_v-2 \\ k \geqslant 0, m \geqslant 0}} C_{g-1}(d_1, \ldots, d_{v-1}, k, m) \\ & + \sum_{\substack{D_1 \cup D_2 = d_1, \ldots, d_{v-1} \\ D_1 \cap D_2 = \phi}} \sum_{h+f=g} \sum_{\substack{k+m=d_v-2 \\ k \geqslant 0, m \geqslant 0}} C_h(D_1, k)\, C_f(D_2, m), \end{aligned} \tag{6}$$

which reduces when $g = 0$ to formula 2.1, page 713 of [9a]. To anchor formula 6, define $C_g(d_1, \ldots, d_v)$ to be 0 if any $d_i = 0$ unless $v = 1$ and $g = 0$ (because the graph is disconnected), and define $C_0(0)$ to be 1 (to represent the map with one vertex and no edges).

5. Proof of Uniqueness of the Solution of the Recursion Formula

We now show that *the anchoring condition together with* (6) *determine* $C_g(d_1, ..., d_v)$ *for all* g *and for all sequences* $(d_1, ..., d_v)$ *of non-negative integers, and that* $C_g = 0$ *if the sum of the* d_i *is odd or if* $g < 0$.

We proceed by induction on the sum of the degrees (which is, of course, twice the number of edges). Suppose that, for all sequences of non-negative integers whose sum is less than n (clearly $n \geqslant 0$), C_g is known for all g, and $C_g = 0$ if the sum is odd or if $g < 0$. Let $d_1, ..., d_{v-1}, d_v$ be a sequence of non-negative integers whose sum is n, and let $D = d_1, ..., d_{v-1}$.

In the first term on the RHS of (6), the sum of the arguments of C_g is $n - 2$. All the arguments are non-negative unless either $d_i = 0$ and $d_v = 1$ or $d_v = 0$ and $d_i = 1$. If $d_v = 0$, the LHS is already known by the anchoring conditions. If $d_i = 0$, this term may be ignored, since C_g is multiplied by d_i. In any case, the contribution is known, and, if n is odd, so is $n - 2$ and the contribution is 0, as it is if $g < 0$.

In the second term in the RHS of (6), the sum of the arguments of C_{g-1} is $n - 2$. Since k and m are non-negative, so are all the arguments; so the contribution of this term is known, and, if n is odd or if $g < 0$, the contribution is 0.

In the last term on the RHS, all the arguments of both C_h and C_f are non-negative. If the sum of the arguments of C_h is n_1, the sum of the arguments of C_f must be $n_2 = n - 2 - n_1$. Since n_1 and n_2 are both non-negative, they are both at most $n - 2$, and the contributions of both C_h and C_f are known. If n is odd, one of n_1 and n_2 is odd, and, if g is negative, one of f and h is negative. Under either condition, one of C_h and C_f is 0; so their product is 0. Thus $C_g(D, d_v)$ is determined, and is 0 if n is odd or if g is negative.

Since $n \geqslant 0$, it remains only to prove the statement for $n = 0$ and $n = 1$. If $n = 0$, all the arguments must be 0. So $C_g(D, d_v)$ is known by the anchoring conditions to be 0 unless $g = 0$ and D is empty, in which case $C_0(0) = 1$. In particular, if g is negative, $C_g = 0$. If $n = 1$, one of the arguments is 1 and the rest are 0, and, by the anchoring conditions, $C_g = 0$ except perhaps for $C_g(1)$. In this last case, $v = 1$; so the first term in the RHS of (6) is 0, and, since $d_v - 2 = -1$ and k and m are non-negative, the rest of the RHS is 0. Therefore, $C_g(1) = 0$ for all g. This completes the induction.

C_g is clearly symmetric in its arguments, since the definition of dicing is symmetric in the vertex labels. It suffices, therefore, to compute C_g for *descending* vertex-degree sequences. This has been done for all sequences corresponding to maps with at most 11 edges (cf. [10, Appendix A, Table 1]).

6. A Census by Genus of Dicings with One Face

An explicit solution to (6) has been found for maps with one face. The number of faces in the dicings counted by the LHS of (6) is

$$f = \frac{1}{2}\left(\sum_{i=1}^{v} d_i\right) - v + 2(1 - g)$$

by formula 1. The first two terms in the RHS of (6) count dicings with f faces. The last term counts *pairs* of dicings, and for each pair the sum of the number of faces in each member of the pair is f. Since every map has at least one face, this last term disappears when f is set equal to 1. So if we let $F(d_1, ..., d_v)$ be the number of dicings with one face and v vertices of degrees $d_1, ..., d_v$, then, by (6) and the anchoring conditions,

$$\begin{aligned} &F(0) = 1,\ F(D) = 0,\ \text{if } D \text{ contains } 0 \text{ unless } D = (0), \text{ and,}\\ &\quad \text{if } d_1, ..., d_v \text{ are all positive,}\\ &F(d_1, ..., d_v) = \sum_{i=1}^{v-1} d_i F(d_1, ..., d_{i-1}, d_i + d_v - 2, d_{i+1}, ..., d_{v-1})\\ &\qquad\qquad + \sum_{\substack{k+m=d_v-2\\ k\geq 0, m\geq 0}} F(d_1, ..., d_{v-1}, k, m). \end{aligned} \tag{7}$$

As a special case of (6), (7) has a unique solution for sequences of non-negative integers, which is 0 if the sum is odd. We may therefore proceed by quoting a formula and proving that it satisfies (7) (cf. [10, Appendix B] for a summary of the original derivation of the formula). Suppose that

$$\begin{aligned} &F(0) = 1,\ F(0, ..., 0) = 0 \text{ unless there is only one } 0,\\ &\quad \text{and if } d_1, ..., d_v \text{ are all positive, then}\\ &F(d_1, ..., d_v) = \frac{(e-1)!}{2^{e+1}} \times \text{the coefficient of } x^{e+1} \text{ in}\\ &\quad \prod_{j=1}^{v} [(1 + x)^{d_j} - (1 - x)^{d_j}], \text{ where } e = \frac{1}{2}\sum_{j=1}^{v} d_j. \end{aligned} \tag{8}$$

Formula 8 is a symmetric function of the d_j, and is 0 if any one of the d_j is 0 unless $v = 1$; so it satisfies the anchoring conditions. With $e = 1$, the only numbers not covered by the anchoring conditions are $F(1, 1)$ and $F(2)$. From (7),

$$F(1, 1) = 1F(0) = 1 \quad \text{and} \quad F(2) = \sum_{k+m=0} F(k, m) = F(0, 0) = 0.$$

From (8),

$$F(1, 1) = \frac{0!}{4} \times \{\text{the coefficient of } x^2 \text{ in } [(1 + x) - (1 - x)]^2\} = 1,$$

and

$$F(2) = \frac{0!}{4} \times \{\text{the coefficient of } x^2 \text{ in } (1 + x)^2 - (1 - x)^2\} = 0.$$

So (8) satisfies (7) when $e = 1$.

We now show that (8) satisfies (7) for $e \geqslant 2$. The sum of the arguments of F in all the terms in the RHS of (7) is $2(e - 1)$; so the RHS is

$$\frac{(e - 2)!}{2^e} \times \text{the coefficient of } x^e$$

in

$$\sum_{i=1}^{v-1} d_i \prod_{j=1}^{v-1} [(1 + x)^{d_j} - (1 - x)^{d_j}] \text{ with } d_i + d_v - 2 \text{ replacing } d_i$$

$$+ \sum_{k+m=d_v-2} \left\{ \prod_{j=1}^{v-1} [(1 + x)^{d_j} - (1 - x)^{d_j}] \right\}$$

$$\times [(1 + x)^k - (1 - x)^k][(1 + x)^m - (1 - x)^m].$$

In the second sum, take out the common factor

$$\prod_{j=1}^{v-1} [(1 + x)^{d_j} - (1 - x)^{d_j}].$$

The remaining factor reduces (by summing geometric series—cf. [10, p. 50]) to

$$(d_v - 1)[(1 + x)^{d_v-2} + (1 - x)^{d_v-2}] - [(1 + x)^{d_v-1} - (1 - x)^{d_v-1}]/x.$$

Now let

$$A(x) = \prod_{j=1}^{v} [(1+x)^{d_j} - (1-x)^{d_j}],$$

$$B(x) = \sum_{i=1}^{v-1} d_i \prod_{j=1}^{v-1} [(1+x)^{d_j} - (1-x)^{d_j}] \text{ with } d_i \text{ replaced by } d_i + d_v - 2,$$

$$C(x) = \left[\prod_{j=1}^{v-1} [(1+x)^{d_j} - (1-x)^{d_j}]\right] (d_v - 1)[(1+x)^{d_v-2} + (1-x)^{d_v-2}],$$

and

$$D(x) = \left[\prod_{j=1}^{v-1} [(1+x)^{d_j} - (1-x)^{d_j}]\right] [(1+x)^{d_v-1} - (1-x)^{d_v-1}].$$

We must prove that the coefficient of x^{e+1} in

$$E(x) = (e-1)\, A(x) - 2xB(x) - 2xC(x) + 2D(x) \text{ is } 0.$$

We expand each of these functions as polynomials in $1 + x$ and $1 - x$. Let $D = d_1, \ldots, d_v$. If δ is any subsequence of D, we choose all the elements of δ to be exponents of $1 - x$, and all the elements of $D - \delta$ to be exponents of $1 + x$; then sum over all $\delta \subseteq D$.

Now group each δ with its complement $D - \delta$. Replacing δ by $D - \delta$ means reversing the roles of $1 + x$ and $1 - x$, or changing x to $-x$.

$A_\delta(-x) = (-1)^v A_\delta(x)$, where $A_\delta(x)$ is that part of $A(x)$ using δ as above;
$B_\delta(-x) = (-1)^{v-1} B_\delta(x)$; so $(-x)\, B_\delta(-x) = (-1)^v\, [xB_\delta(x)]$;
$C_\delta(-x) = (-1)^{v-1} C_\delta(x)$; so $(-x)\, C_\delta(-x) = (-1)^v\, [xC_\delta(x)]$; and
$D_\delta(-x) = (-1)^v D_\delta(x)$.

Then $E_\delta(-x) = (-1)^v E_\delta(x)$; so the coefficient of $(-x)^{e+1}$ in $E_\delta(-x)$ is just $(-1)^{e-v+1}$ times the coefficient of x^{e+1} in $E_\delta(x)$.

If $e - v + 1$ is odd, these coefficients cancel out in pairs as δ is grouped with $D - \delta$. This means (since $A_\delta(-x) = (-1)^v A_\delta(x)$) that the number of dicings is 0 if $e - v + 1$ is odd, which is hardly surprising since with $f = 1$, the genus is $\frac{1}{2}(e - v + 1)$. If $e - v + 1$ is even, add over all δ not containing d_v, and then multiply by 2 to account for the complementary subsequences—those that contain d_v.

Let δ be any subsequence of D not containing d_v. Let $t = \sum_{d \in \delta} d$ and $u = \sum_{d \notin \delta} d$. Then $t + u = 2e$.

$$A_\delta(x) = (-1)^{\#(\delta)}\, (1+x)^u\, (1-x)^t.$$

If $u \geqslant 2$,

$$B_\delta(x) = (-1)^{\#(\delta)}\,(u - d_v)(1 + x)^{u-2}\,(1 - x)^t,$$

since d_v combines with each $d_i \notin \delta$ with a multiplicity of d_i. If $D - \delta = \{d_v\}$, $u - d_v = 0$; so this case need not be considered separately. If $u \geqslant 2$,

$$C_\delta(x) = (-1)^{\#(\delta)}\,(d_v - 1)(1 + x)^{u-2}\,(1 - x)^t,$$

and

$$D_\delta(x) = (-1)^{\#(\delta)}\,(1 + x)^{u-1}\,(1 - x)^t.$$

It will suffice if we can show that the coefficient of x^{e+1} in

$$\begin{aligned} E_\delta(x) &= (e - 1)(1 + x)^u\,(1 - x)^t - 2x(u - d_v)(1 + x)^{u-2}\,(1 - x)^t \\ &\quad - 2x(d_v - 1)(1 + x)^{u-2}\,(1 - x)^t + 2(1 + x)^{u-1}\,(1 - x)^t \end{aligned}$$

is 0; for then the coefficient of x^{e+1} in

$$E(x) = 2 \sum_{d_v \notin \delta} (-1)^{\#(\delta)}\, E_\delta(x)$$

will surely be 0. If $u \geqslant 2$, this coefficient is [cf. 10, p. 52]

$$\begin{aligned} &\sum_{i+j=e+1} \frac{(u-1)!\,t!\,(-1)^j}{i!\,(u-i)!\,j!\,(t-j)!}\,[u(e - 1) - 2i(u - i) + 2(u - i)] \\ &= \sum_{i+j=e+1} \frac{(u-1)!\,t!\,(-1)^j}{i!\,(u-i!)\,j!\,(t-j)!}\,[i(t - j) + j(u - i)], \end{aligned}$$

since

$$\begin{aligned} &i + j = e + 1 \quad \text{and} \quad t + u = 2e \\ &= \sum_{i+j=e+1} \frac{(u-1)!\,t!\,(-1)^j}{(i-1)!\,(u-i)!\,j!\,(t-j-1)!} \\ &\quad + \sum_{i+j=e+1} \frac{(u-1)!\,t!\,(-1)^j}{i!\,(u-i-1)!\,(j-1)!\,(t-j)!}. \end{aligned}$$

In the second sum of the previous line, add 1 to j and subtract 1 from i. It then differs from the first sum by a factor of -1; so, if $u \geqslant 2$, the coefficient is 0. If $u = 1$, $D - \delta$ has only the term d_v which is 1; so

$$B_\delta(x) = C_\delta(x) = 0$$

and

$$E_\delta(x) = (e - 1)\,A_\delta(x) + 2D_\delta(x) = (1 - x)^{2e-1} \times [(e - 1)\,x + (e + 1)],$$

in which the coefficient of x^{e+1} is 0 (cf. [10, p. 54]).

This completes the proof that (8) satisfies (7). Together with the uniqueness of the solution of (7), it establishes that *the number of dicings with e edges, one face, and v vertices of degrees $d_1, \ldots, d_v$ is given by formula* 8.

Expanding the RHS of (8) by the binomial theorem with e replaced by $v + 2g - 1$ (as required by formula 1 with $f = 1$), we obtain

$$F(d_1, \ldots, d_v) = \frac{(v + 2g - 2)!}{2^{2g}} \left(\prod_{j=1}^{v} d_j\right) \sum_{\substack{i_1+\cdots+i_v=g \\ i_1,\ldots,i_v \geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j + 1} \binom{d_j - 1}{2i_j}, \tag{9}$$

where g is the genus.

Letting $g = 0$, we find that *the number of trees with $v \geqslant 2$ labeled rooted vertices of degrees $d_1, \ldots, d_v$ is*

$$(v - 2)! \left(\prod_{j=1}^{v} d_j\right). \tag{10}$$

Letting $g = 1$, we find that

$$\begin{aligned} F(d_1, \ldots, d_v) &= \frac{v!}{4} \left(\prod_{j=1}^{v} d_j\right) \sum_{k=1}^{v} \frac{1}{3} \binom{d_k - 1}{2} \\ &= \frac{v!}{24} \left(\prod_{j=1}^{v} d_j\right) \left[-(4v + 6) + \sum_{k=1}^{v} (d_k)^2\right] \end{aligned} \tag{11}$$

because, with $g = 1$, $\sum_{k=1}^{v} d_k = 2e = 2(v + 1)$.

Similarly, for any given g, $F(d_1, \ldots, d_v)$ can be expressed as $(v + 2g - 2)!$ $(\prod_{j=1}^{v} d_j)$ multiplied by a polynomial of degree $2g$ in the power symmetric functions of the degrees (cf. [10, Appendix B, formulae 8–11]).

Letting $v = 1$, we obtain formula 14 for the number of dicings (and, therefore, of rooted maps) of genus g with one vertex (of degree $4g$) and one face.

7. Labelings of Ordered Graphs: Rooted Maps vs. Dicings

An *isomorphism* from an ordered graph (E, P) onto another one (E', P') is a one-to-one mapping Φ from $2E$ onto $2E'$ such that, for every β in $2E$, $\Phi[P(\beta)] = P'[\Phi(\beta)]$ and $\Phi(-\beta) = -\Phi(\beta)$. Thus the *automorphism group of P* (denoted by aut(P)) is the set of permutations on $2E$ which commute with P and $-$. An ordered graph (E, P) is *connected* if $\Gamma(P, -)$ is transitive on $2E$. It can be shown (cf. [10, pp. 11–20]), using Edmonds' theorem, that two connected ordered graphs are isomorphic if and only if the maps they represent are related by an orientation-preserving homeomorphism. Therefore, an isomorphism class of ordered graphs is called a *combinatorial map.*

A connected ordered graph is *rooted* if one end β is distinguished. A *root-isomorphism* from one rooted (connected) ordered graph (E, P, β) onto another one (E', P', β') is an isomorphism Φ from (E, P) onto (E', P') such that $\Phi(\beta) = \beta'$. Thus a *root-automorphism* on (E, P, β) is a permutation on $2E$ which commutes with P and $-$ and preserves β.

We now show that *the only root-automorphism on a rooted ordered graph is the trivial one.* Let Φ be any root-automorphism on (E, P, β), and let γ be any end in $2E$. Since (E, P) is connected, $\Gamma(P, -)$ is transitive on $2E$; so there exists a word $W(P, -)$ in P and minus such that $\gamma = [W(P, -)](\beta)$. Since Φ commutes with P and $-$, it commutes with $W(P, -)$. Since Φ preserves β,

$$\Phi(\gamma) = \Phi\{[W(P, -)](\beta)\} = [W(P, -)]\,\Phi(\beta) = W[(P, -)](\beta) = \gamma.$$

So Φ is trivial.

This is sufficient to prove, at least for rooted maps on orientable surfaces, that any homeomorphism which preserves the root-edge, its orientation, and its left and right sides, preserves every edge-end (cf. [1b, p. 16] for a topological proof).

In the various census papers on enumeration of rooted maps, what was actually counted was *equivalence classes* of rooted maps, where two rooted maps were said to be equivalent if there was a homeomorphism between them which mapped the root of one onto the root of the other [9b, p. 252]. The combinatorial equivalent of this is counting *root-isomorphism* classes of ordered graphs. We define a *rooted combinatorial map* (E, P, β) as the *root-isomorphism class of rooted ordered graphs containing* (E, P, β).

We now show that *the number of essentially different ways to root the ordered graph* (E, P)—that is, the number of root-isomorphism classes of rooted ordered graphs which are isomorphic as ordered graphs to (E, P)—*is* $2e/\#(\mathrm{aut}(P))$, *where* $e = \#(E)$. (If automorphisms are allowed to include reflections, the number is $4e/\#(\mathrm{aut}(P))$ (cf. [5].)

There are $2e$ ends which may be distinguished; make $2e$ copies of (E, P), each rooted in one of the $2e$ possible ways. Every rooted ordered graph (E', P', β') which is isomorphic as an ordered graph to (E, P) is root-isomorphic to one of these $2e$ copies; so it suffices to count the root-isomorphism classes among these $2e$ copies.

For each $\alpha \in \operatorname{aut}(P)$ and for each i from 1 to $2e$, there exist a unique j such that $\alpha(E, P, \beta_i) = (E, P, \beta_j)$. Since $\alpha^{-1}(E, P, \beta_j) = (E, P, \beta_i)$, α defines a permutation on the set $\Omega = \{(E, P, \beta_i) \mid 1 \leqslant i \leqslant m\}$. The set A of permutations on Ω induced by $\operatorname{aut}(P)$ is a homomorphic image of $\operatorname{aut}(P)$ into the symmetric group on Ω. Since the root-isomorphism group is trivial, Ω has no fixed points under any $\alpha \in A$ except the identity automorphism; so Ω cannot have any fixed points under the whole group A. Thus the kernel of the homomorphism (of $\operatorname{aut}(P)$ into the symmetric group on Ω) is trivial; so A is isomorphic to $\operatorname{aut}(P)$.

The equivalence classes of root-isomorphic (E, P, β_i) are the orbits of A in Ω. For each $(E, P, \beta_i) \in \Omega$, the order of the subgroup of A which fixes (E, P, β_i) is 1; hence (cf. [11, p. 3]), the length of each orbit is $\#(A) = \#(\operatorname{aut}(P))$. And since there are $2e$ objects in Ω, there must be $2e/\#(\operatorname{aut}(P))$ orbits, which completes the proof.

Similarly, to make a dicing, the v vertices of the ordered graph (E, P) may be labeled in $v!$ ways using labels $1, \ldots, v$, and, if the degree of the i-th vertex is d_i, there are d_i ways to distinguish an end in the i-th vertex. Thus $v!\,(\prod_{i=1}^{v} d_i)$ dicings can be made from (E, P). But two dicings are equivalent if there is an isomorphism from one onto the other which preserves the labels on the vertices and the distinguished end of each vertex. An automorphism from a dicing onto itself which preserves the labels on the vertices and the distinguished end of each vertex must preserve at least one end—say the distinguished end of the last vertex; hence, it is a root-automorphism, and is thus trivial. Hence, *the number of equivalence classes of dicings which may be derived from* (E, P) *is* $v!\,(\prod_{i=1}^{v} d_i)/\#(\operatorname{aut}(P))$ (the proof is exactly analogous to the one used to count inequivalent rootings of (E, P); for a more general treatment, see [10, pp. 36–38]).

This is the contribution which (E, P) makes to the number of dicings (of the same genus as (E, P)) whose vertex-degree sequence is some permutation of $(d_1, d_2, \ldots, d_v)$. To find the contribution made by (E, P) to $C_g(d_1, \ldots, d_v)$, the number of genus g dicings whose vertex-degree sequence is $(d_1, \ldots, d_v)$, we must divide the previously obtained number by the number of permutations of the sequence $(d_1, d_2, \ldots, d_v)$. And, of course, (E, P) contributes $2e/\#(\operatorname{aut}(P))$ to the number of genus g rooted maps with e edges and v vertices (cf. [10, Appendix A, Table 3] for the number of rooted maps and dicings contributed by every map with up to 4 edges).

Now the *ratio* of the contributions made by (E, P) to rooted maps and dicings depends upon g and the sequence $d_1, \ldots, d_v$, but *it is independent of the particular combinatorial map* (E, P). The number of rooted maps contributed by all the combinatorial maps of genus g whose vertex-degrees are $d_1, d_2, \ldots, d_v$ is thus

$$\frac{(2e)\, C_g(d_1, \ldots, d_v)}{[\prod_j (m_j)!] \prod_{i=1}^{v} d_i},$$

where m_j is the multiplicity of the j-th largest distinct value of d_i. Summing over all descending sequences $(d_1, \ldots, d_v)$ which add to $2e$ gives the number of rooted maps of genus g with e edges and v vertices.

That is how these numbers were calculated for rooted maps with up to 14 edges (cf. [10, Appendix A, Table 2]). The numbers for maps with up to 11 edges are tabulated at the end of the paper.

On the other hand, if an explicit expression is known for $C_g(d_1, \ldots, d_v)$, it is easier to sum over all sequences $(d_1, \ldots, d_v)$. Thus the number of rooted maps of genus g with v vertices and e edges is

$$\frac{2e}{v!} \sum_{d_1+d_2+\cdots+d_v=2e} \frac{C_g(d_1, d_2, \ldots, d_v)}{[\prod_{i=1}^{v} d_i]}. \tag{12}$$

In the previous section, an explicit expression was found for $C_g(d_1, \ldots, d_v)$ when $d_1 + \cdots + d_v = 4g + 2v - 2$, corresponding to maps with one face (see formula 9, where C_g is denoted by F). In the next section, we apply (12) to (9) and obtain a formula for the number of rooted maps of genus g with one face and v vertices.

We conclude this section with a simpler example of an application of (12), and find the number of planar rooted Eulerian maps with e edges and v vertices. An Eulerian map is a map all of whose vertices are of even degree. From [9a, p. 709], we have

$$\frac{(e-1)!}{(e-v+2)!} \prod_{i=1}^{v} \frac{(2n_i)!}{(n_i)!\,(n_i-1)!}$$

dicings with vertex-degree sequence $(2n_1, \ldots, 2n_v)$. Now, for each map, the number of rootings divided by the number of dicings it contributes is

$$\frac{2e}{v! \prod_{i=1}^{v} (2n_i)}.$$

So the number of planar rooted Eulerian maps with e edges and v vertices is

$$\frac{2(e!)}{v!\,(e-v+2)!} \sum_{n_1+\cdots+n_v=e} \prod_{i=1}^{v} \frac{(2n_i-1)!}{n_i!\,(n_i-1)!}.$$

It is easily verified that

$$\sum_{j=0}^{\infty} \frac{(2j-1)!}{j!\,(j-1)!}\, x^j = \frac{(1-4x)^{-1/2}-1}{2},$$

so *the number of rooted Eulerian maps with e edges and v vertices is*

$$\frac{2(e!)}{v!\,(e-v+2)!} \times \text{the coefficient of } x^e \text{ in } \left[\frac{(1-4x)^{1/2}-1}{2}\right]^v.$$

If instead 2 of the vertices are of odd degree, from [9a, p. 722] the formula for the number of dicings is modified by replacing

$$\frac{(2n_i)!}{n_i\,!\,(n_i-1)!} \quad \text{by} \quad \frac{(2n_i+1)!}{(n_i\,!)^2}$$

whenever the degree of the vertex is an odd number $2n+1$. The number of rooted maps in this case is

$$\frac{2(e!)}{v!\,(e-v+2)!} \sum_{n_1+\cdots+n_v=e-1} \left\{\frac{(2n_1)!}{n_1\,!\,n_1\,!}\,\frac{(2n_2)!}{n_2\,!\,n_2\,!}\left(\prod_{i=1}^{v}\frac{(2n_i-1)!}{n_i\,!\,(n_i-1)!}\right)\right\},$$

since the number of edges is $n_1+\cdots+n_v+1$. Since

$$(1-4x)^{-1/2} = \sum_{j=0}^{\infty} \frac{(2j)!}{j!\,j!}\, x^j,$$

the number of rooted maps is

$$\frac{2(e!)}{v!\,(e-v+2)!} \times \text{the coefficient of } x^{e-1} \text{ in}$$

$$(1-4x)^{-1}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{(v-2)}.$$

Similarly, it may be shown that the number of maps of the above type with the root belonging to a vertex of odd degree is

$$\frac{2(e-1)!}{v!\,(e-v+2)!} \times \text{the coefficient of } x^{e-1} \text{ in}$$

$$(1-4x)^{-2}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2},$$

and the number with the root belonging to a vertex of even degree is

$$\frac{(v-2)(e-1)!}{v!\,(e-v+2)!} \times \text{the coefficient of } x^{e-1} \text{ in}$$

$$2x(1-4x)^{-5/2}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-3}.$$

8. A Census by Genus of Rooted Maps with One Face

Applying (12) to (9), we find that the number of maps of genus g with one face and v vertices is

$$\frac{(v+2g-1)!}{2^{2g-1}v!} \sum_{d_1+\cdots+d_v=2(v+2g-1)} \sum_{\substack{i_1+\cdots+i_v=g \\ i_1,\ldots,i_v\geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j+1}\binom{d_j-1}{2i_j}$$

$$= \frac{(v+2g-1)!}{2^{2g-1}v!} \sum_{\substack{i_1+\cdots+i_v=g \\ i_1,\ldots,i_v\geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j+1} \sum_{d_1+\cdots+d_v=2(v+2g-1)} \binom{d_j-1}{2i_j}.$$

Now $\binom{d_j-1}{2i_j}$ is the coefficient of $x^{d_j-2i_j-1}$ in $(1-x)^{-(2i_j+1)}$. So $\sum_{d_1+\cdots+d_v=2(v+2g-1)} \binom{d_j-1}{2i_j}$ is the coefficient of $x^{\sum_{j=1}^{v}(d_j-2i_j-1)}$ in $(1-x)^{-\sum_{j=1}^{v}(2i_j+1)}$ or of x^{v+2g-2} in $(1-x)^{-(2g+v)}$, which is $\binom{2v+4g-3}{v+2g-2}$.

So the number is

$$\frac{(2v+4g-3)!}{2^{2g-1}v!\,(v+2g-2)!} \sum_{\substack{i_1+\cdots+i_v=g \\ i_1,\ldots,i_v\geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j+1}$$

$$= \frac{(2v+4g-2)!}{2^{2g}v!\,(v+2g-1)!} \sum_{\substack{i_1+\cdots+i_v=g \\ i_1,\ldots,i_v\geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j+1}. \tag{13}$$

Formula 13 *is the number of rooted maps of genus g with* 1 *face, v vertices, and* $v+2g-1$ *edges, and, by duality between vertices and faces, the number of rooted maps of genus g with* 1 *vertex, v faces, and* $v+2g-1$ *edges.*

If v is set equal to 1, (13) becomes

$$\frac{(4g)!}{2^{2g}(2g+1)!}. \tag{14}$$

This is the number of rooted maps of genus g with 1 *vertex*, 1 *face*, *and* 2*g edges*. From (4), the number of rooted maps with one vertex and $2g$ edges is

$$\frac{(4g)!}{2^{2g}(2g)!},$$

which is just $2g+1$ times as many as the number with one vertex, $2g$ edges, *and one face*. Can a natural $2g+1$ to 1 correspondence between these two sets of maps be found?

We consider several alternative forms for (13). Let $a_{g,b,p}$ be the number of genus g rooted maps with $b+p$ edges, $p+1$ vertices, and $b-2g+1$ faces. By duality, $a_{g,b,p} = a_{g,p+2g,b-2g}$. Then, from (13),

$$a_{g,2g,p} = a_{g,p+2g,0} = \frac{(2p+4g)!}{2^{2g}(p+1)!\,(p+2g)!} \sum_{\substack{i_1+\cdots+i_{p+1}=g \\ i_1,\ldots,i_{p+1}\geqslant 0}} \prod_{j=1}^{p+1} \frac{1}{2i_j+1}. \tag{15}$$

When $g=0$, (15) becomes

$$\frac{(2p)!}{p!\,(p+1)!},$$

which is the number of rooted trees with p edges, in agreement with (3).

If $g>0$, let k be the number of positive numbers in the sequence $i_1,\ldots,i_{p+1}$. Clearly $1\leqslant k\leqslant g$. If $i_j=0$, $1/(2i_j+1)=1$. There are $\binom{p+1}{k}$ ways of choosing which of the i_j are the positive numbers. So

$$a_{g,2g,p} = \frac{(2p+4g)!}{2^{2g}(p+1)!\,(p+2g)!} \sum_{k=1}^{g} \binom{p+1}{k} \sum_{\substack{i_1+\cdots+i_k=g \\ i_1,\ldots,i_k\geqslant 1}} \prod_{j=1}^{k} \frac{1}{2i_j+1} \tag{16}$$

$$= \frac{(2p+4g-1)!}{2^{2g-1}p!\,(p+2g-1)!} \sum_{k=1}^{g} \binom{p}{k-1} \frac{1}{k} \sum_{\substack{i_1+\cdots+i_k=g \\ i_1,\ldots,i_k\geqslant 1}} \prod_{j=1}^{k} \frac{1}{2i_j+1}. \tag{17}$$

From (17), a formula for $a_{g,2g,p}$ as a function of p can be derived for each g. For example,

$$a_{1,2,p} = a_{1,p+2,0} = \frac{(2p+3)!}{p!\,(p+1)!} \times \frac{1}{6};$$

$$a_{2,4,p} = a_{2,p+4,0} = \frac{(2p+7)!}{p!\,(p+3)!} \times \frac{1}{8}\left[\frac{1}{5} + \frac{p}{2} \times \frac{1}{9}\right]$$
$$= \frac{(2p+7)!}{p!\,(p+3)!} \times \frac{(5p+18)}{720};$$

$$a_{3,6,p} = a_{3,p+6,0} = \frac{(2p+11)!}{p!\,(p+5)!} \times \frac{1}{32}\left[\frac{1}{7} + \frac{1}{2} \times \frac{2p}{5.3} + \frac{1}{3} \times \frac{1}{27}\binom{p}{2}\right]$$
$$= \frac{(2p+11)!}{p!\,(p+5)!}\,\frac{\left[1620 + 756p + 140\binom{p}{2}\right]}{9!}.$$

Returning to (13), and using the identity

$$\sum_{j=0}^{\infty} \frac{x^{2j+1}}{2j+1} = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right],$$

we have

$$\sum_{\substack{i_1+\cdots+i_v=g\\ i_1,\ldots,i_v\geqslant 0}} \prod_{j=1}^{v} \frac{1}{2i_j+1}$$
$$= \text{the coefficient of } x^{v+2g} \text{ in } \frac{1}{2^v}\left[\ln(1+x) - \ln(1-x)\right]^v.$$

So (13) can be expressed in terms of v and g as

$$\frac{(2v+4g-2)!}{2^{2g+v}v!\,(v+2g-1)!} \times \text{the coefficient of } x^{v+2g} \text{ in}$$
$$\left[\ln(1+x) - \ln(1-x)\right]^v \tag{18}$$

and in terms of v and e as

$$\frac{(2e)!}{2^{e+1}v!\,e!} \times \text{the coefficient of } x^{e+1} \text{ in } \left[\ln(1+x) - \ln(1-x)\right]^v. \tag{19}$$

Using the identity

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=0}^{\infty} \frac{s(n,k)\,x^n}{n!}$$

TABLE I

The Number of Rooted Maps with e Edges and v Vertices by Genus g

e	v	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
0	1	1					
1	1	1					
1	2	1					
2	1	2	1				
2	2	5					
2	3	2					
3	1	5	10				
3	2	22	10				
3	3	22					
3	4	5					
4	1	14	70	21			
4	2	93	167				
4	3	164	70				
4	4	93					
4	5	14					
5	1	42	420	483			
5	2	386	1720	483			
5	3	1030	1720				
5	4	1030	420				
5	5	386					
5	6	42					
6	1	132	2310	6468	1485		
6	2	1586	14065	15018			
6	3	5868	24164	6468			
6	4	8885	14065				
6	5	5868	2310				
6	6	1586					
6	7	132					
7	1	429	12012	66066	56628		
7	2	6476	100156	258972	56628		
7	3	31388	256116	258972			
7	4	65954	256116	66066			
7	5	65954	100156				
7	6	31388	12012				
7	7	6476					
7	8	429					

Table continued

TABLE I (*continued*)

e	v	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
8	1	1430	60060	570570	1169740	225225	
8	2	26333	649950	3288327	2668750		
8	3	160648	2278660	5554188	1169740		
8	4	442610	3392843	3288327			
8	5	614404	2278660	570570			
8	6	442610	649950				
8	7	160648	60060				
8	8	26333					
8	9	1430					
9	1	4862	291720	4390386	17454580	12317877	
9	2	106762	3944928	34374186	66449432	12317877	
9	3	795846	17970784	85421118	66449432		
9	4	2762412	36703824	85421118	17454580		
9	5	5030004	36703824	34374186			
9	6	5030004	17970784	4390386			
9	7	2762412	3944928				
9	8	795846	291720				
9	9	106762					
9	10	4862					
10	1	16796	1385670	31039008	211083730	351683046	59520825
10	2	431910	22764165	313530000	1171704435	792534015	
10	3	3845020	129726760	1059255456	1955808460	351683046	
10	4	16322085	344468530	1558792200	1171704435		
10	5	37460376	472592916	1059255456	211083730		
10	6	49145460	344468530	313530000			
10	7	37460376	129726760	31039008			
10	8	16322085	22764165				
10	9	3845020	1385670				
10	10	431910					
10	11	16796					
11	1	58786	6466460	205633428	2198596400	7034538511	4304016990
11	2	1744436	126264820	2583699888	16476937840	26225260226	4304016990
11	3	18211380	875029804	11270290416	40121261136	26225260226	
11	4	92400330	2908358552	22555934280	40121261136	7034538511	
11	5	259477218	5188948072	22555934280	16476937840		
11	6	429166584	5188948072	11270290416	2198596400		
11	7	429166584	2908358552	2583699888			
11	8	259477218	875029804	205633428			
11	9	92400330	126264820				
11	10	18211380	6466460				
11	11	1744436					
11	12	58786					

where $s(n, k)$ are Stirling numbers of the first kind—cf. [8, p. 33 and p. 42]), we have

$$
\begin{aligned}
[\ln(1+x) - \ln(1-x)]^v &= \ln\left(1 + \frac{2x}{1-x}\right)^v \\
&= v! \sum_{n=v}^{\infty} \frac{s(n,v)}{n!} \left(\frac{2x}{1-x}\right)^n \\
&= v! \sum_{n=v}^{\infty} \frac{s(n,v)\, 2^n x^n (1-x)^{-n}}{n!} \\
&= v! \sum_{n=v}^{\infty} \frac{s(n,v)\, 2^n x^n}{n!} \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i \\
&= v! \sum_{k=v}^{\infty} x^k \sum_{n=v}^{k} \binom{k-1}{k-n} \frac{s(n,v)\, 2^n}{n!}.
\end{aligned}
$$

So (19) becomes

$$
\begin{aligned}
&\frac{(2e)!}{2^{e+1} e!} \sum_{n=v}^{e+1} \binom{e}{e+1-n} \frac{s(n,v)\, 2^n}{n!} \\
&= \frac{(2e)!}{2^{e-v+1} e!} \sum_{n=0}^{e-v+1} \binom{e}{e+1-v-n} \frac{s(n+v,v)\, 2^n}{(n+v)!}. \qquad (20)
\end{aligned}
$$

If (19) is summed over v with e held constant, it becomes

$$
\frac{(2e)!}{2^{e+1} \times e!} \times \text{the coefficient of } x^{e+1} \text{ in } \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} = \cdots,
$$

where $y = \ln(1+x) - \ln(1-x)$. Now the above power series in y is

$$
\exp(y) - 1 = \frac{1+x}{1-x} - 1 = \frac{2x}{1-x},
$$

in which the coefficient of x^{e+1} is 2. Thus the number of rooted maps with one face and e edges is

$$
\frac{(2e)!}{2^e e!},
$$

in agreement (by duality) with (4).

9. Conclusion

These results demonstrate the utility of the combinatorial approach to non-planar map enumeration. A further demonstration will be given in "Counting Rooted Maps by Genus. II." In that paper, we will use a code for rooted maps presented in [7] to prove the conjecture given at the end of Section 3 and present the generating functions $A_{g,b}(x)$ (for small b) which count rooted maps by genus. We will also derive an explicit formula for the number of *tree-rooted maps* (rooted maps with a distinguished spanning tree) of genus g with e edges and v vertices. Similar results will be given for maps with no vertices of degree 1, and in "Counting Rooted Maps by Genus. III," we will give similar results for *non-separable* (cf. [1c, 9b]) rooted and tree-rooted maps.

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