Independence of path, Cauchy's Integral thm

Independence of path.

Thm Suppose that $f(z)$ is continuous in a domain D and that $f(z)$ has an antiderivative $F(z)$ in D, i.e. $F'(z) = f(z)$ $\forall z \in D$. Let Γ be a contour in D with initial point z_I and terminal point z_T. Then,

$$\int_\Gamma f(z)\,dz = F(z_T) - F(z_I)$$

Proof: $\displaystyle\int_\Gamma f(z) = \sum_k \int_{\gamma_k} f(z)\,dz = \sum_k \int_{\tau_{k-1}}^{\tau_k} f(z(t))\,z'(t)\,dt,$

where $z(t)$, $\tau_{k-1} \le t \le \tau_k$, is a parametrization of γ_k. Now,

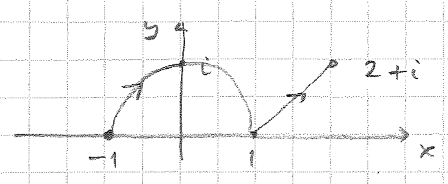
$$\frac{d}{dt} F(z(t)) = F'(z(t))\,z'(t) = f(z(t))\,z'(t)$$

So by theorem from last lecture

$$\int_{\tau_{k-1}}^{\tau_k} f(z(t))\,z'(t)\,dt = F(z(\tau_k)) - F(z(\tau_{k-1}))$$

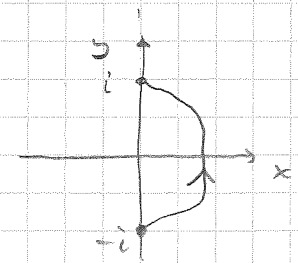
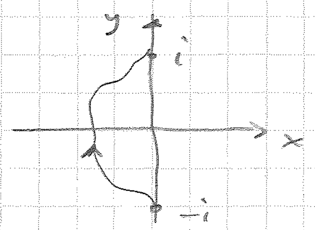
Sum over k. ∎

Ex. Compute $\int_\Gamma \cos z \, dz$, with Γ as in the figure:



Sol: $\int_\Gamma \cos z \, dz = \left[\sin z\right]_{-1}^{2+i} = \sin(2+i) - \sin(-1)$ $\square$

Ex. Compute $\int_\Gamma \frac{1}{z} \, dz$, when Γ is the contour

(a)  (b) 

Sol. (a) $\int_\Gamma \frac{1}{z} \, dz = \left[\operatorname{Log} z\right]_{-i}^{i} = i\frac{\pi}{2} - i\left(-\frac{\pi}{2}\right) = i\pi$

(b) Let $\log z = \ln|z| + i \arg z$; $0 < \arg z < 2\pi$

$\Rightarrow \int_\Gamma \frac{1}{z} \, dz = \left[\log z\right]_{-i}^{i} = i\frac{\pi}{2} - i\left(\frac{3\pi}{2}\right) = -i\pi$ $\square$

Corollary: If f is continuous in a domain D and has an antiderivative in D, then $\int_\Gamma f(z) \, dz = 0$ for every closed contour in D.

Ex. Let $f(z) = z^n$, $n \neq -1$, and let Γ be any closed contour not passing through 0.

Then, $\int_\Gamma f(z) \, dz = 0$,

because z^n has an antiderivative $\frac{z^{n+1}}{n+1}$ in $\mathbb{C} \setminus \{0\}$. $\square$

Thm Let f be continuous in a domain D.

Then the following are equivalent:

(i) f has an antiderivative in D

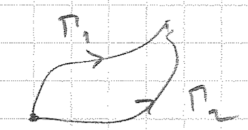
(ii) $\int_\Gamma f(z)\,dz = 0$ for every closed contour Γ in D.

(iii) Contour integrals are independent of path in D.

(i.e. if Γ_1 and Γ_2 are two contours with the same initial and terminal pts $\Rightarrow \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$)

Proof: (i) $\Rightarrow$ (ii): shown above (as well as (i) $\Rightarrow$ (iii))

(ii) $\Rightarrow$ (iii) Given Γ_1 and Γ_2 let $\Gamma := \Gamma_1 + (-\Gamma_2)$



$$\Rightarrow 0 = \int_\Gamma = \int_{\Gamma_1} + \int_{-\Gamma_2} = \int_{\Gamma_1} - \int_{\Gamma_2}, \text{ i.e. } \int_{\Gamma_1} = \int_{\Gamma_2}.$$

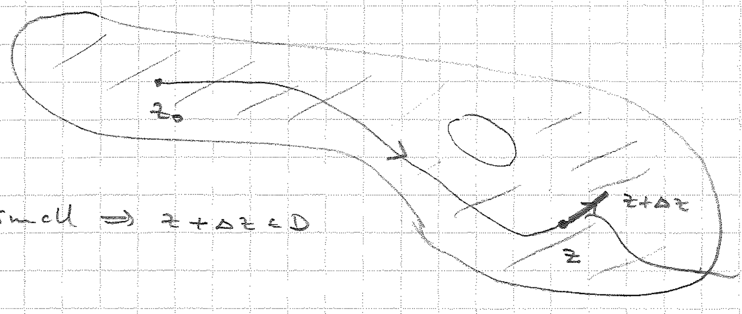
(iii) $\Rightarrow$ (i): Fix $z_0 \in D$. D domain $\Rightarrow$ For any $z \in D$

$\exists$ polygonal path Γ from z_0 to z.

Define $F(z) = \int_\Gamma f(s)\,ds$.

$F(z)$ is well-defined, i.e. independent of the choice of Γ, by (iii). We now show that

$F'(z) = f(z) \quad \forall z \in D$. See figure



Δz small $\Rightarrow$ $z+\Delta z \in D$

$$\Rightarrow F(z+\Delta z) - F(z) = \int_L f(s)\,ds = \int_L f(z)\,ds + \int_L (f(s)-f(z))\,ds$$

$$= f(z)\Delta z + \int_L (f(s)-f(z))\,ds,$$

i.e. $$\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z}\int_L (f(s)-f(z))\,ds$$

But by the ML-ineq.

$$\left|\frac{1}{\Delta z}\int_L (f(s)-f(z))\,ds\right| \le \frac{1}{|\Delta z|}\cdot \max_{s\in L}|f(s)-f(z)|\,|\Delta z|$$

$\to 0$ as $\Delta z \to 0$ by cont of f.

Thus, $$F'(z) = \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

∎

Cauchy's integral thm

Let Γ be a simple closed contour in $\mathbb{C}$

parametrized by $z = z(t)$, $a \le t \le b$.

$$\Rightarrow \int_\Gamma f(z)\,dz = \int_a^b f(z(t))\frac{dz}{dt}\,dt =$$

$$= \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right]\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt =$$

$$= \int_a^b \left[u(x(t),y(t))\frac{dx}{dt} - v(x(t),y(t))\frac{dy}{dt}\right]dt$$

$$+ i\int_a^b \left[v(x(t),y(t))\frac{dx}{dt} + u(x(t),y(t))\frac{dy}{dt}\right]dt$$

i.e. $\int_\Gamma f(z)\,dz = \int_\Gamma (u\,dx - v\,dy) + i \int_\Gamma (v\,dx + u\,dy)$.

Recall the following theorem from vector calculus.

<u>Thm</u> (Green's thm)

Let $F(x,y) = (F_1(x,y), F_2(x,y))$ be a C^1-vector field defined on a simply connected domain D, and let Γ be a positively oriented simple closed contour in D.

Then,

$$\int_\Gamma (F_1\,dx + F_2\,dy) = \iint_\Omega \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where Ω denotes the region interior to Γ.

Let us use this on the expression for $\int_\Gamma f(z)dz$.

$$\Rightarrow \int_\Gamma f(z)\,dz = \int_\Gamma (u\,dx - v\,dy) + i \int_\Gamma (v\,dx + u\,dy) =$$

$$= \iint_\Omega \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

if we suppose that $u, v \in C^1$.

If we moreover assume that f is analytic in D

$$\Rightarrow \int_\Gamma f(z)\,dz = 0$$

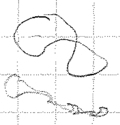
in view of the Cauchy-Riemann equations.

The following holds:

Thm (Cauchy's Integral theorem)

Suppose that f is analytic in a simply connected domain D, and let Γ be any closed contour in D. Then,

$$\int_\Gamma f(z)\,dz = 0 .$$



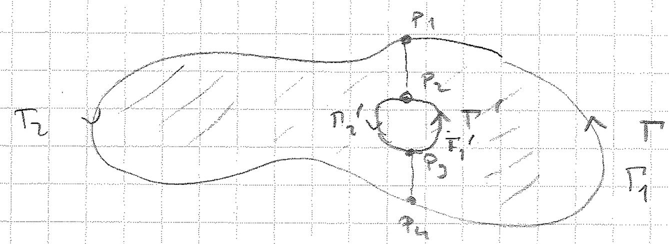
Remark: 1) The theorem generalizes our discussion in two ways. First, Γ can be any closed contour, i.e. need not be simple. Second, the assumption that $u, v \in C^1$ has been dropped. The fact that the second assumption is not necessary was first demonstrated by Edouard Goursat. The theorem is therefore often called the Cauchy–Goursat thm.

2) The theorem implies that: "If f is analytic inside and on a simple closed contour, then $\int_\Gamma f(z)\,dz = 0$."

Combined with the thm of path independence, we have the following:

<u>Thm</u> Suppose that f is analytic in a simply conn. domain. Then f has an antiderivative, contour integrals are indep. of path, and integrals over closed contours are 0.

<u>Ex.</u> Consider the contours Γ and Γ' below.



Suppose f analytic is analytic in a domain containing Γ, Γ' and the region between Γ and Γ'.

Let P_1, P_2, P_3, P_4 be as in the figure and decompose Γ into Γ_1, Γ_2 and Γ' into Γ_1', Γ_2'; as in figure.

By Cauchy's integral thm

$$\int_{\Gamma_1} + \int_{P_1P_2} + \int_{-\Gamma_1'} + \int_{P_3P_4} = 0 \qquad (1)$$

$$\int_{\Gamma_2} + \int_{P_4P_3} + \int_{-\Gamma_2'} + \int_{P_2P_1} = 0 \qquad (2)$$

Add $\Rightarrow \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{-\Gamma_1'} + \int_{-\Gamma_2'} = 0 \Leftrightarrow \int_{\Gamma} = \int_{\Gamma'}$

So we can "deform" Γ into Γ' without affecting the integral. This illustrates the <u>deformation thm</u>.