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Matematiska Institutionen
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Prov i matematik
Reell analys, 1MA226
2020-08-19

Duration: 8.00 – 13.00. The exam consists of 8 problems, each worth 5 points. Solutions may be written in Swedish or English, and should contain detailed arguments. Permitted aids: Course material, lecture notes, old problems and solutions.

1. Give an example of a sequence of nonempty compact subsets $C_1, C_2, \ldots$ of $\mathbb{R}^2$ (equipped with its standard metric) such that the union $\cup_{n=1}^{\infty} C_n$ is an open set. Prove your claim.

2. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences:

(a). $x_n = \big(1 + \cos\big(\frac{\pi n}{2}\big)\big)2^n + (-1)^n$.

(b). $x_n = \Big(1 + \dfrac{(-1)^n}{n}\Big)^{n^2} - e^n$.

3. Prove that the series $F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^3 + x^4}$ converges for all $x \in \mathbb{R}$, and that the function $F : \mathbb{R} \to \mathbb{R}$ is C^1.

4. For the following two sequences (f_n) in $C([0,1])$, determine if (f_n) converges uniformly on $[0,1]$:

(a) $f_n(x) = \dfrac{xn+1}{x+n}$.

(b) $f_n(x) = \sin(\pi x^n)$.

5. Prove that there exists a unique function $f \in C([0,1])$ satisfying

$$f(x) = 2x + \frac{1}{3} f(x^2) + \int_0^1 (y-x) f(y)\, dy.$$

6. The system

$$\begin{cases} uvw = 1 \\ e^{u-v} + e^{v-w} = 2 \end{cases}$$

is satisfied at the point $(u,v,w) = (1,1,1)$. Show that u and v can be solved in a neighbourhood of $(1,1,1)$ as a function of w. Calculate also $u'(1)$ and $v'(1)$, where u and v are regarded as functions of w.

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7. Give an example of an *unbounded*, continuous, real-valued function f on $[0, \infty)$ such that the limit

$$\lim_{b \to +\infty} \int_0^b f(x)\, dx$$

exists. Prove your claim.

8. Let F be an equicontinuous family of functions from $[0, 1]$ to $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists some $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ and all $f \in F$, we have

$$\left| \frac{1}{2^n} \sum_{k=1}^{2^n} f\left(\frac{k}{2^n}\right) - \frac{1}{2^m} \sum_{j=1}^{2^m} f\left(\frac{j}{2^m}\right) \right| < \varepsilon.$$

LYCKA TILL / GOOD LUCK!