Financial Theory – Lecture 10

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Agenda

- Valuation of stocks.

The lecture is based on

- Chapter 6 in the course book.

Prices and returns over time

Recall:

$$r_{t,t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}$$

is the rate of return for an asset over $(t, t+1]$.

To simplify notation, we write this equation as

$$r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t}.$$

This is our "master equation".

It is an accounting identity, i.e. it holds by definition.

Prices and returns over time

$$r_{t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} \quad \Leftrightarrow \quad 1 + r_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t},$$

or

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}}.$$

Now use

$$P_{t+1} = \frac{D_{t+2} + P_{t+2}}{1 + r_{t+2}}.$$

Then

$$\begin{aligned} P_t &= \frac{D_{t+1} + \frac{D_{t+2}+P_{t+2}}{1+r_{t+2}}}{1 + r_{t+1}} \\ &= \frac{D_{t+1}}{1 + r_{t+1}} + \frac{D_{t+2}}{(1 + r_{t+1})(1 + r_{t+2})} + \frac{P_{t+2}}{(1 + r_{t+1})(1 + r_{t+2})}. \end{aligned}$$

Prices and returns over time

We iterate this in total $T - t$ times to arrive at

$$
\begin{aligned}
P_t \quad = \quad & \frac{D_{t+1}}{1+r_{t+1}} + \frac{D_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \cdots \\
& + \frac{D_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)} \\
& + \frac{P_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)}.
\end{aligned}
$$

This equation is written ex post, i.e. after the randomness has been resolved.

To be useful, we need the ex ante version, i.e. before the randomness is resolved.

Prices and returns over time

We achieve this by taking the expectation with respect to the information up to and including time t, i.e. we take E_t on both sides of the equation on the previous slide.

Since P_t is known at time t, we have $P_t = E_t[P_t]$:

$$\begin{aligned} P_t = E_t\left[P_t\right] &= E_t\Bigg[\frac{D_{t+1}}{1+r_{t+1}} + \frac{D_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \dots \\ &\quad + \frac{D_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)} \\ &\quad + \frac{P_T}{(1+r_{t+1})(1+r_{t+2})\cdots(1+r_T)}\Bigg]. \end{aligned}$$

Now we want to let $T \to \infty$. If the last term goes to zero, then the price is given by an infinite sum of expected discounted dividends.

We'll soon get back to this equation.

Prices and returns over time

Now we divide the equation

$$P_t = \frac{D_{t+1} + P_{t+1}}{1 + r_{t+1}}$$

with D_t:

$$\frac{P_t}{D_t} = \frac{\frac{D_{t+1}}{D_t} + \frac{D_{t+1}}{D_t} \cdot \frac{P_{t+1}}{D_{t+1}}}{1 + r_{t+1}}.$$

This is a recursion for the price-dividend ratio P_t/D_t.

It depends on the return and the dividend growth D_{t+1}/D_t.

It turns out that the price-dividend ratio has better statistical properties than the price. Furthermore, it has been shown that the price-dividend ratio can predict returns on several assets for longer horizons (1+ years).

Constant discount rate

We now make the following assumption:

$$E_t[r_{t+1}] = r = \text{a constant}.$$

What does this mean? Write

$$\begin{aligned} r_{t+1} &= E_t[r_{t+1}] + \big(r_{t+1} - E_t[r_{t+1}]\big) \\ &= r + \varepsilon_{t+1}. \end{aligned}$$

We see that

$$E_t\left[\varepsilon_{t+1}\right] = E_t\left[r_{t+1} - E_t\left[r_{t+1}\right]\right] = E_t\left[r_{t+1}\right] - E_t\left[r_{t+1}\right] = 0.$$

The rate of return r_{t+1} is the constant r plus zero-mean noise.

We can use the CAPM or APT equation to determine the discount rate r.

Constant discount rate

Recall that the "master equation" can be written

$$1 + r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}.$$

Take E_t on both sides:

$$\underbrace{E_t\left[1 + r_{t+1}\right]}_{=1+E_t[r_{t+1}]=1+r} = E_t\left[\frac{P_{t+1} + D_{t+1}}{P_t}\right] = \frac{1}{P_t} E_t\left[P_{t+1} + D_{t+1}\right]$$

$$\Leftrightarrow$$

$$P_t = \frac{E_t\left[P_{t+1} + D_{t+1}\right]}{1 + r} = E_t\left[\frac{P_{t+1} + D_{t+1}}{1 + r}\right].$$

Constant discount rate

Using the same iteration technique as previously we get

$$
\begin{aligned}
P_t &= E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} + \ldots + \frac{D_T}{(1+r)^{T-t}} + \frac{P_T}{(1+r)^{T-t}}\right] \\
&= \frac{E_t\left[D_{t+1}\right]}{1+r} + \frac{E_t\left[D_{t+2}\right]}{(1+r)^2} + \ldots + \frac{E_t\left[D_T\right]}{(1+r)^{T-t}} + \frac{E_t\left[P_T\right]}{(1+r)^{T-t}} \\
&= \sum_{j=1}^{T-t} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j} + \frac{E_t\left[P_T\right]}{(1+r)^{T-t}}.
\end{aligned}
$$

Assuming that

$$\frac{E_t\left[P_T\right]}{(1+r)^{T-t}} \to 0 \quad \text{as} \quad T \to \infty,$$

we get

$$P_t = \sum_{j=1}^{\infty} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j}.$$

Constant growth models

Now assume that the expected growth rate in dividends is a constant g:

$$E_t\left[\frac{D_{t+1}-D_t}{D_t}\right]=g \quad \Leftrightarrow \quad E_t\left[D_{t+1}\right]=(1+g)D_t.$$

Iterating this we get

$$\begin{aligned} E_t\left[D_{t+2}\right] &= E_t\left[(1+g)D_{t+1}\right] \\ &= (1+g)\cdot E_t\left[D_{t+1}\right] \\ &= (1+g)\cdot(1+g)D_t \\ &= (1+g)^2 D_t. \end{aligned}$$

More generally,

$$E_t\left[D_{t+j}\right]=(1+g)^j D_t \text{ for } j=1,2,\ldots$$

Constant growth models

Now,

$$\begin{aligned} P_t &= \sum_{j=1}^{\infty} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j} \\ &= \sum_{j=1}^{\infty} \frac{(1+g)^j D_t}{(1+r)^j} \\ &= D_t \cdot \underbrace{\sum_{j=1}^{\infty} \left(\frac{1+g}{1+r}\right)^j}_{=S} \end{aligned}$$

Constant growth models

Recall: If $|\alpha| < 1$, then

$$\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}.$$

This implies

$$\sum_{j=1}^{\infty} \alpha^j = \sum_{j=0}^{\infty} \alpha^j - 1 = \frac{1}{1-\alpha} - 1 = \frac{\alpha}{1-\alpha}.$$

We get

$$S = \sum_{j=1}^{\infty} \left(\frac{1+g}{1+r}\right)^j = \frac{\frac{1+g}{1+r}}{1 - \frac{1+g}{1+r}} = \frac{1+g}{1+r-(1+g)} = \frac{1+g}{r-g}.$$

For this to be well defined, we need

$$\frac{1+g}{1+r} < 1 \quad \Leftrightarrow \quad 1+g < 1+r \quad \Leftrightarrow \quad g < r.$$

Constant growth models

Using

$$P_t = S \cdot D_t \quad \text{and} \quad S = \frac{1+g}{r-g}$$

we arrive at

$$\begin{aligned} P_t &= \frac{(1+g)D_t}{r-g} \\ &= \frac{E_t\left[D_{t+1}\right]}{r-g}. \end{aligned}$$

This is known as Gordon's formula.

Constant growth models

Using Gordon's formula we get the following in our model.

- The price-dividend ratio is

$$\frac{P_t}{D_t} = \frac{1+g}{r-g}.$$

- The discount rate can be written

$$\begin{aligned} r &= g + \frac{(1+g)D_t}{P_t} \\ &= g + \frac{E_t\left[D_{t+1}\right]}{P_t} \\ &= g + E_t\left[\frac{D_{t+1}}{P_t}\right] \\ &= \text{Growth rate of dividends} + \text{Expected dividend yield.} \end{aligned}$$

Rational bubbles

Let us return to the equation

$$P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right]. \qquad (*)$$

If

$$E_t\left[\frac{P_T}{(1+r)^{T-t}}\right] \to 0 \quad \text{as} \quad T \to \infty,$$

then we have seen that we get the solution

$$P_t = \frac{(1+g)D_t}{r-g}.$$

But without this condition there are other solutions.

The solution P_t above is known as the bubble-free solution.

Now let $\hat{P}_t$ be any solution to Equation $(*)$.

Rational bubbles

We have

$$\hat{P}_t = E_t\left[\frac{\hat{P}_{t+1} + D_{t+1}}{1+r}\right] \quad \text{and} \quad P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right].$$

Subtract these two equations:

$$\begin{aligned}
\hat{P}_t - P_t &= E_t\left[\frac{\hat{P}_{t+1} + D_{t+1}}{1+r}\right] - E_t\left[\frac{P_{t+1} + D_{t+1}}{1+r}\right] \\
&= E_t\left[\frac{\hat{P}_{t+1}}{1+r}\right] - E_t\left[\frac{P_{t+1}}{1+r}\right] \\
&= E_t\left[\frac{\hat{P}_{t+1} - P_{t+1}}{1+r}\right].
\end{aligned}$$

Rational bubbles

Now let

$$M_t = \frac{\hat{P}_t - P_t}{(1+r)^t}$$

$$\Leftrightarrow$$

$$\hat{P}_t - P_t = (1+r)^t M_t.$$

One can show that $E_t[M_{t+1}] = M_t$.

Rational bubbles

We can thus write

$$\hat{P}_t = P_t + (1+r)^t M_t,$$

or

Any price satisfying Equation $(*)$ = Bubble-free solution + Bubble.

We call

$$(1+r)^t M_t$$

a rational bubble since it satisfies Equation $(*)$.

A firm's dividend policy

Let e_t denote the earnings of a firm during year t. The earnings are used to reinvest the amount I_t in the firm, and to pay the amount D_t as dividends to the share holders:

$$e_t = I_t + D_t.$$

The earnings grow due to the return r_e on the investments or on the equity:

$$e_{t+1} = e_t + r_e I_t.$$

It is common to refer to r_e as the return on equity (ROE).

We let b denote the plowback ratio, i.e. the fraction of the earnings that are invested:

$$I_t = b e_t.$$

Throughout, we assume that both r_e and b are constants.

A firm's dividend policy

Remark

We can talk about earnings, dividends and price per share, in which case we call earnings earnings per share (EPS).

Or, we can talk about the total amount of these three quantities.

They only differ by a multiple of the number of shares.

A firm's dividend policy

The choice of how much of the earnings that should be paid out to the share holders is the firm's dividend policy.

Now,

$$e_{t+1} = e_t + r_e \underbrace{I_t}_{=be_t} = (1 + br_e)e_t \quad \Rightarrow \quad \frac{e_{t+1} - e_t}{e_t} = br_e.$$

The growth rate in earnings is equal to br_e. For the dividends:

$$D_t = e_t - I_t = e_t - be_t = (1 - b)e_t.$$

A firm's dividend policy

It follows that

$$\frac{D_{t+1} - D_t}{D_t} = \frac{(1-b)e_{t+1} - (1-b)e_t}{(1-b)e_t} = \frac{e_{t+1} - e_t}{e_t} = br_e.$$

Conclusion: If the plowback ratio and return on new investements are constant, then the growth rate of dividends is constant and equal to

$$g = br_e.$$

Exercise: Show that also

$$\frac{P_{t+1} - P_t}{P_t} = r_e b$$

in this model.

A firm's dividend policy

An important "multiple" is the price-earnings ratio (P/E).

In this model

$$P_t = \frac{(1+g)D_t}{r-g} = \frac{(1+br_e)(1-b)e_t}{r-br_e} \quad \Rightarrow \quad \frac{P_t}{e_t} = \frac{(1+br_e)(1-b)}{r-br_e}.$$

We also have

$$e_{t+1} = (1+br_e)e_t \ \ \text{(ex post)} \quad \Rightarrow \quad E_t\left[e_{t+1}\right] = (1+br_e)e_t \ \ \text{(ex ante)}.$$

The forward price-earnings ratio is in this model given by

$$\frac{P_t}{E_t\left[e_{t+1}\right]} = \frac{1-b}{r-br_e}.$$

A firm's dividend policy

Why do firm's pay dividends?

- It is a way of share holders to get cash flows without having to sell stocks.
- Some funds are only allowed to use dividends and other cash payments in their distribution of funds; they are not allowed to sell anything of their capital.

Investment opportunities

Consider a firm whose dividend policy is

$$e_t = D_t \quad \Leftrightarrow \quad b = 0,$$

i.e. all earnings are paid out as dividends.

The value of this firm is

$$\frac{e_t}{r}.$$

To see this we use Gordon's formula:

$$\begin{aligned}\frac{(1+g)D_t}{r-g} &= \left\{g = br_e = 0 \text{ and } D_t = e_t\right\} \\ &= \frac{e_t}{r}.\end{aligned}$$

Investment opportunities

Now let

$$O_t = P_t - \frac{e_t}{r},$$

i.e. we can write

$$P_t = \frac{e_t}{r} + O_t.$$

Here O is the present value of growth opportunities (PVGO).

$$\text{Price today} = \text{Value of assets in place} + \text{PVGO}.$$

We can write the P/E ratio as

$$\frac{P_t}{e_t} = \frac{1}{r} + \frac{O_t}{e_t} = \frac{1}{r}\left(1 + \frac{O_t}{e_t/r}\right).$$

Two-period growth models

Let us consider the price P_t decomposed into two sums:

$$\begin{aligned} P_t &= E_t\left[\sum_{j=1}^{T-t}\frac{D_{t+j}}{(1+r)^j} + \sum_{j=T-t+1}^{\infty}\frac{D_{t+j}}{(1+r)^j}\right] \\ &= \underbrace{E_t\left[\sum_{j=1}^{T-t}\frac{D_{t+j}}{(1+r)^j}\right]}_{=S_1} + \underbrace{E_t\left[\sum_{j=T-t+1}^{\infty}\frac{D_{t+j}}{(1+r)^j}\right]}_{=S_2} \end{aligned}$$

Now assume:

- The dividends grow with constant rate $G \neq r$ from $t+1$ to T (we assume that $t < T$).
- The dividends grow with constant rate $g < r$ from $T+1$ onwards.

Two-period growth models

With this model

$$
\begin{aligned}
S_1 &= E_t\left[\sum_{j=1}^{T-t} \frac{D_{t+j}}{(1+r)^j}\right] = \sum_{j=1}^{T-t} \frac{E_t\left[D_{t+j}\right]}{(1+r)^j} \\
&= \sum_{j=1}^{T-t} \frac{D_t(1+G)^j}{(1+r)^j} = D_t \sum_{j=1}^{T-t} \left(\frac{1+G}{1+r}\right)^j .
\end{aligned}
$$

Now we use that when $\alpha \neq 1$

$$\sum_{j=0}^{N} \alpha^s = \frac{1-\alpha^{N+1}}{1-\alpha},$$

with $N = T - t$:

$$\sum_{j=1}^{T-t} \left(\frac{1+G}{1+r}\right)^j = \frac{1-\left(\frac{1+G}{1+r}\right)^{T-t+1}}{1-\frac{1+G}{1+r}} - 1 = \cdots$$

Two-period growth models

$$\cdots = (1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r-G}.$$

Hence,

$$S_1 = D_t(1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r-G}.$$

One can show that

$$S_2 = D_t \frac{1+g}{r-g} \left(\frac{1+G}{1+r}\right)^{T-t}$$

Two-period growth models

To summarise:

$$
\begin{aligned}
P_t &= S_1 + S_2 \\
&= D_t \left[(1+G) \cdot \frac{1 - \left(\frac{1+G}{1+r}\right)^{T-t}}{r-G} + \frac{1+g}{r-g} \cdot \left(\frac{1+G}{1+r}\right)^{T-t} \right].
\end{aligned}
$$

Two-period growth models

For a finite number of cash flows one can have

$$\text{Growth rate} \geq \text{Discount rate}.$$

But this can not be true for an infinite number of cash flows, since this would imply

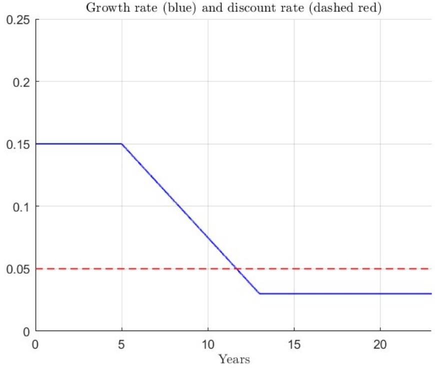
$$P_t = \infty.$$

This is one reason for having more than one time period when modelling dividend growth rate: You can have exceptional growth in the dividends for the first period (which includes a finite number of times).

The final growth rate is the long run steady state growth rate.

Three-period growth models

We can extend to three (or, of course, any number of periods). One way of using a three-period model is the following.



- Assume an initial high constant growth rate.
- Assume final (lower) steady state growth rate.
- Assume that the growth rate diminishes linearly from the initial high to the steady state one.

Three-period growth models

Another version is the following:

- Estimate explicitly the dividends in the first period.
- Assume a high growth rate in the second period.
- Assume (lower) steady state growth in the last premium.

This model could e.g. be used to value growth stocks.

Free cash flows

The "discounted dividend" approach to valuing firms is the typical way we use it in (financial) economics.

A practitioner valuing a firm may find it hard to estimate dividends, and there are firms who do not (yet) have paid any dividends.

An alternative is to take a more "business" or "accounting" approach.

The free cash flow (FCF) of a firm is defined as

> *... the after-tax cash flow generated by the firms operations which is available for distribution among shareholders and creditors.*

(Munk p. 221.)

Free cash flows

The value of the firm is then calculated as

$$V_t = \sum_{j=1}^{\infty} \frac{E_t \left[\mathrm{FCF}_{t+j}\right]}{\left(1 + r_{\text{firm}}\right)^j}$$

Here the discount rate r_{firm} is the weighted average cost of capital (WACC).

Since we are valuing the whole form, not just its equity, we need to discount using the WACC.

In order to get the value of the equity, we need to subtract the value of the firm's debt.