Example. Let R be the field of real numbers. Is the following a subspaces of V_3 (R)?

(i) $W_1 = \{(x, 2y, 3z): x, y, z \in \mathbf{R}\}$

Solution (i) Here

$$W_1 = \{(x, 2y, 3z): x, y, z \in \mathbf{R}\}.$$

Let $\alpha = (x_1, 2y_1, 3z_1)$ and $\beta = (x_2, 2y_2, 3z_2)$ be any two arbitrary elements of W_1, then $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbf{R}$ be any two real numbers, then we have

$$a\alpha + b\beta = a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2)$$

$$= (ax + bx_2, 2ay_1 + 2by_2, 3az_1 + 3az_2)$$

$$= (ax_1 + bx_2, 2[ay_1 + by_2], 3[az_1 + bz_2]) \in W_1$$

$$[\because ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2 \in \mathbf{R}]$$

$\therefore$ $a, b \in \mathbf{R}$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1.$

Hence W_1 is a subspace of V_3**(R).**

ALGEBRA OF SUBSPACES

Theorem 1. The intersection of any two subspaces of a vector space V (F) is also a subspaces of V(F).

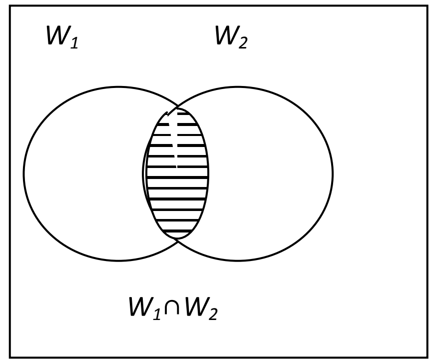
Or

If W_1 and W_2 are two vector subspaces of a vector space V(F), then $W_1 \cap W_2$ is also a vector subspace of V(F).

Proof Let W_1 and W_2 be any two subspaces of a vector space V(F). Now additive identity **0** of V belongs to every subspace of V, so $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$

Thus $\mathbf{0} \in W_1 \cap W_2$ and $W_1 \cap W_2$ is non-empty.

Let $\alpha, \beta \in W_1 \cap W_2$ be any two elements. Also let a, b ϵ F.



Now $\alpha \in W_1 \cap W_2 \Longrightarrow \alpha \in W_1$ and $\alpha \in W_2$

and $\beta \in W_1 \cap W_2 \Longrightarrow \beta \in W_1$ and $b\beta \in W_2$.

Since W_1 and W_2 are subspaces, so

a ,b $\in$ F and $\alpha, \beta \in W_1 \Longrightarrow a\alpha + b\beta \in W_1$ ...... (1)

and a,b $\in$ F and $\alpha, \beta \in W_2 \Longrightarrow a\alpha + b\beta \in W_2$ .... (2)

From (1) and (2), we have

$$a\alpha + b\beta \in W_1 \text{ and } a\alpha + b\beta \in W_2 \Longrightarrow a\alpha + b\beta \in W_1 \cap W_2.$$

$$\therefore \quad a\alpha + b\beta \in W_1 \text{ and } a\alpha + b\beta \in W_2 \Longrightarrow a\alpha + b\beta \in W_1 \cap W_2.$$

Hence $W_1 \cap W_2$ is a subspace of V (F)

Note. Smallest subspace containing any subset of V (F) Let S be any subset of vector space V (F). Let T be a subspace of V(F) containing S, and itself is contained in every subspace of V containing S, then T is said to be the **smallest subspace of V** containing S. T is also called the subspace of V **spanned** or **generated** by S and is denoted by T = [S].

Thus $[S] = \{W_n \mid S \subseteq W_n, W_n \text{ (F) is a subspace of V(F)}\}$.

We know that there is at least one subspace *i.e.*, the vector space V (F) containing S. Therefore [S] definitely exists and is unique. From theorem 2 above it follows that [S] (F) is a subspace of V (F) and is called the subspace generated or spanned by *S*. If [S] = V then it is said that V is **spanned** by S.

Important

We have seen above that the intersection of two (or more) subspaces is always a subspace, but the union of two subspaces of *V*(*F*) is not necessarily a subspace of V(F). Since if a, b $\in$ Fand $\alpha, \beta \in W_1 \cup W_2$ then $a\alpha + b\beta$ may or may not belong to $W_1 \cup W_2$. For example, if **R** is the field of real numbers and

$$W_1 \{(x_1, 0, 0): x \in \mathbf{R}\}, W_2 = \{(0, y_1, 0,): y \in \mathbf{R}\}$$

are two subspaces of V_3 (**R**). Let

$$\alpha = (x_1, 0, 0) \in W_1, \beta = (0, y_1, 0,) \in W_2$$

Where $x_1, y_1 \in \mathbf{R}$. Now α and β are both elements of $W_1 \cup W_2$, then for a, b $\in$ **R,** we have

$$a\alpha + b\beta = a(x_1, 0, 0) + b(0, y_1, 0)$$

$$= (ax_1, by_1, 0) \notin W_1 \cup W_2.$$

Since $(ax_1, by_1, 0)$ neither belongs to W_1 nor W_2. Thus $W_1 \cup W_2$ is not a subspace of V (F).

Now we have an important theorem.

Theorem 3. The union of two subspaces is a subspace if and only if one is contained in the other.

Proof: Let V (F) be a vector space and let W_1 and W_2 be two subspaces of V(F).

If part: suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_2$ or W_1 Therefore, $W_1 \cup W_2$ is also a subspace of V(F) since W_1, W_2 are subspaces.

Conversely (only if part), let $W_1 \cup W_2$ be a subspace of V (F), then we are to prove $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. We shall prove it by contradiction.

Suppose that W_1 is not a subset of W_2 and W_2 is not a subset of W_1

Since $W_1 \not\subseteq W_2 \Rightarrow \exists \alpha \in W_1, \alpha \notin W_2$ .... (1)

$W_2 \not\subseteq W_1 \Rightarrow \exists \beta \in W_2, \beta \notin W_1$ .... (2)

Now from (1) and (2), we get

$$\alpha \in W_1 \Rightarrow \alpha \in W_1 \cup W_2$$

$$\beta \in W_2 \Rightarrow \beta \in W_1 \cup W_2.$$

Again since $W_1 \cup W_2$ is a subspace and so we have

$$\alpha, \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1 \cup W_2$$

$$\Rightarrow \alpha + \beta \in W_1 \text{ or } \alpha + \beta \in W_2.$$

If $\alpha + \beta \in W_1$, then

$(\alpha + \beta) - \alpha = \beta \in W_1$ [since W_1 is a subspace and $\alpha \in W_1$]

But from (2), we see that $\beta \notin W_1$, which is a contradiction.
Hence either W_1 is a subset of W_2 or W_2 is a subset of W_1.