Problem Session 4

Probability and Martingales, 1MS045

26 November 2024

Note: If not specified otherwise, all random variables are real-valued, with the usual σ-algebra of Borel sets.

Problems

1. Let $X_1, X_2, \dots$ be a sequence of independent uniformly distributed random variables on the interval $[0,1]$. Prove (directly from definition) that $\min(X_1, X_2, \dots, X_n) \to_p 0$ as $n \to \infty$.

2. Let $\{X_1, X_2, \dots\}$ and $\{Y_1, Y_2, \dots\}$ be uniformly integrable sequences of random variables.

   (a) Prove that the sequence $\{X_n + Y_n \mid n \geq 1\}$ is also uniformly integrable.

   (b) Is $\{X_n Y_n \mid n \geq 1\}$ also uniformly integrable?

3. For $n \in \mathbb{N}$, let X_n be normally distributed with mean μ_n and variance σ_n^2. Prove that the family $\{X_n \mid n \geq 1\}$ is uniformly integrable if and only if both μ_n and σ_n^2 are uniformly bounded.

4. Let $X : \Omega \to \{0, 1, 2, \dots\}$ be a random variable with mean $m = \mathbb{E}(X) > 1$ and variance $\sigma^2 = \mathrm{Var}(X) < \infty$. We define the *Galton-Watson* process Z_n associated with X by,

$$Z_0 = 1 \quad \text{and} \quad Z_n = \sum_{j=1}^{Z_{n-1}} X_{j,n} \quad \text{for } n \geq 1,$$

   where $X_{j,n}$ are independent random variables with the same distribution as X.

   (a) Show that $\mathbb{E}(Z_n) = m^n$.

   (b) Prove that $M_n = m^{-n} Z_n$ is a martingale and that it converges to some random variable M_∞ almost surely.

   (c) Show that $\mathbb{E}(M_n) \to \mathbb{E}(M_\infty) = 1$ (Hint show that the martingale is in L^2). Conclude that $\mathbb{P}(M_\infty \neq 0) > 0$.

   (d) Now let $X = 0$ with probability $\frac{1}{2}$ and $X = 2$ with probability $\frac{1}{2}$. Now $m = \mathbb{E}(X) = 1$. What can we say about M_∞?

5. Consider the following sequence of random variables: $X_0 = a$ for some $a \in (0,1)$, and

$$X_n = \begin{cases} X_{n-1}^2 & \text{with probability } \frac{1}{2}, \\ 2X_{n-1} - X_{n-1}^2 & \text{with probability } \frac{1}{2}, \end{cases}$$

for $n > 0$. Prove that the sequence $X_0, X_1, \dots$ converges almost surely. What are the possible limits? For each of the possible limits L, determine

$$\mathbb{P}(\lim_{n\to\infty} X_n = L).$$

6. Prove that if X_n is a non-negative, uniformly integrable submartingale for which $X_n \to 0$ holds almost surely, as $n \to \infty$, then $X_n = 0$ (a.s.) for all $n \in \mathbb{N}$.

7. Let $p \in (0,1)$ be fixed. We have an inexhaustible supply of red and green balls. In a bucket, there is initially one red ball. In each time step, we take a random ball from the bucket. With probability p, we replace it along with another ball of the same colour. With probability $q = 1-p$ we replace it and add a ball of the other colour. Let X_n be the number of red balls in the n-th step. Prove that

$$Y_n = (X_n - n/2) \cdot \binom{n-2q}{n-1}^{-1}$$

is a martingale.

8. In this exercise we will prove *Lévy's Upward Theorem* and give an alternative proof to Kolmogorov's $0-1$ law.
Theorem. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_n$ be a filtration. Define $M_n = \mathbb{E}(X) \mid \mathcal{F}_n)$. Then M_n is a martingale and

$$M_n \to Y := \mathbb{E}(X \mid \mathcal{F}_\infty),$$

almost surely and in L^1, where $\mathcal{F}_\infty = \sigma\left(\bigcup \mathcal{F}_n\right)$.

   (a) Show that M_n is a martingale.
   (b) Show that M_n is UI.
   (c) Define measure μ_1, μ_2 on $(\Omega, \mathcal{F}_\infty)$ by

   $$\mu_1(F) = \mathbb{E}(Y; F) \quad \text{and} \quad \mu_2(F) = \mathbb{E}(M_\infty; F).$$

   Show that $\mu_1 = \mu_2$.
   (d) Show that, almost surely, $Y = M_\infty$. (Hint: consider the expectation of the difference)

Recall Kolmogorov's $0-1$ law:
Theorem. Let $X_1, X_2, \dots$ be a sequence of independent random variables. Define

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \dots) \quad \text{and} \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

Then, for all $E \in \mathcal{T}$, we have $\mathbb{P}(E)$ is either 0 or 1.

   (a) Use Lévy's Upward Theorem with $Y = I_E$ and show that

   $$X = \mathbb{E}(X \mid \mathcal{F}_\infty) = \lim_n \mathbb{E}(X \mid \mathcal{F}_n).$$

   (b) Show that $Y = \mathbb{P}(E)$ and prove the theorem.
   (Hint: Use independence of $\mathcal{F}_n$ and $\mathcal{T}_n$)