Uniform Integrability

Problem: Given $X_n \to X_\infty$, when can we say that $\mathbb{E}(X_n) \to \mathbb{E}(X_\infty)$?

Example: $X_n = \begin{cases} n^2 & \text{with prob } 1/n^2 \\ 0 & \text{otherwise.} \end{cases}$

Then $\mathbb{E}(X_n) = 1$. Since $\sum_n P(X_n \neq 0) = \sum_n 1/n^2 < \infty$

we have $X_n \to X_\infty = 0$ a.s.

But $\mathbb{E}(X_\infty) = 0 \neq 1 = \lim_n \mathbb{E}(X_n)$!

Uniform integrability is a key condition that allows exchange of $\mathbb{E}$ and lim.

Lemma: Let X be an integrable random variable. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all events E with $P(E) < \delta$, we have $\mathbb{E}(|X|; E) = \mathbb{E}(|X| \cdot I_E) < \varepsilon$.

(This is a special case of Egorov's theorem.)

Proof: Suppose this was not the case:

For some $\varepsilon_0 > 0$ there exists a sequence of events E_n s.t. $\mathbb{P}(E_n) < 2^{-n}$ but $\mathbb{E}(|X| \cdot I_{E_n}) \geq \varepsilon_0$. Since $\sum_n \mathbb{P}(E_n) < \infty$, the B.C. lemma implies that only finitely many E_n occur. Let $F = \limsup_{n \to \infty} E_n$.

Then $\mathbb{P}(F) = 0$.

Hence $\mathbb{E}(|X| \cdot I_F) = 0$.

But by the reverse Fatou lemma:

$$\limsup_{n \to \infty} \mathbb{E}(|X| \, I_{E_n}) \leq \mathbb{E}(|X| \limsup_n I_{E_n})$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\mathbb{E}(|X| \cdot I_F) = 0$$

But the LHS is bounded below by $\varepsilon_0 > 0$, a contradiction ↯

□

In particular, there exists $K > 0$ s.t.

$$\mathbb{E}(|X| \,;\, |X| > K) < \varepsilon .$$

This holds because $\mathbb{P}(|X| > K) \leq \dfrac{\mathbb{E}(|X|)}{K}$

by Markov's inequality so we can take

$$K > \frac{\mathbb{E}(|X|)}{\delta} .$$

<u>Note</u>: K generally depends on ε <u>and</u> X !

Def^n Let $\mathcal{C}$ be a family of random variables. We say $\mathcal{C}$ is uniformly integrable if, for every $\varepsilon > 0$, there exists $K > 0$ s.t.

$$\mathbb{E}(|X| \,;\, |X| > K) < \varepsilon \quad \text{for all } X \in \mathcal{C}$$

Note: K does not depend on X (just $\varepsilon, \mathcal{C}$).

Example:
$$X_n = \begin{cases} n^2 & \text{with probability } 1/n^2 \\ 0 & \text{otherwise} \end{cases}$$

is <u>not</u> uniformly integrable. No matter $K > 0$, for large enough n, $\mathbb{E}(|X| \,;\, |X| \geq K) = n^2 \cdot \frac{1}{n^2} = 1.$

Uniform integrability and whether $\lim_n \mathbb{E}(X_n) = \mathbb{E}(\lim_n X_n)$ are closely connected.

We start with a sufficient condition:

Proposition: Assume there exists $p > 1$ and $C > 0$ s.t. $\mathbb{E}(|X|^p) \leq C$ for all $X \in \mathcal{C}$. Then $(X)_{X \in \mathcal{C}}$ is uniformly integrable.

Proof: We have, for all $K > 0$.

$$\mathbb{E}(|X|; |X| > K) \leq \mathbb{E}\left(|X| \cdot \left(\frac{|X|}{K}\right)^{p-1}; |X| > K\right)$$

$$= \mathbb{E}\left(|X|^p K^{1-p}; |X| > K\right)$$

$$\leq K^{1-p} \mathbb{E}(|X|^p) \leq C K^{1-p}.$$

Hence, choosing $K = \left(\varepsilon / C\right)^{\frac{1}{1-p}} = \left(\frac{C}{\varepsilon}\right)^{\frac{1}{p-1}}$ suffices. $\square$

Another sufficient condition:

<u>Proposition</u>: If $|X| \leq Y$ for all $X \in \mathcal{C}$ where Y is an integrable random variable, then $\mathcal{C}$ is uniformly integrable.

<u>Proof</u>: [Exercise!]

<u>Theorem</u>: Let X be an integrable random variable. The family

$$\mathcal{C} = \{ \mathbb{E}(X \mid \mathcal{G}) : \mathcal{G} \text{ is a sub } \sigma\text{-algebra of } \mathcal{F} \}$$

is uniformly integrable.

<u>Proof</u>: For given $\varepsilon > 0$ choose δ such that $\mathbb{P}(F) < \delta$ implies $\mathbb{E}(X; F) < \varepsilon$ for all $F \in \mathcal{F}$.

Now take $K > \mathbb{E}(|X|)/\delta$. For $Y = \mathbb{E}(X \mid \mathcal{G})$ we get $|Y| = |\mathbb{E}(X \mid \mathcal{G})| \leq \mathbb{E}(|X| \mid \mathcal{G})$ (Jensen) and so $\mathbb{E}|Y| \leq \mathbb{E}(\mathbb{E}(|X| \mid \mathcal{G})) = \mathbb{E}|X|$ and

$$K\,\mathbb{P}(|Y| > K) \underset{\uparrow \text{Markov}}{\leq} \mathbb{E}(|Y|) \leq \mathbb{E}|X| \underset{\uparrow \text{choice of } K.}{<} K\delta$$

and so $\mathbb{P}(|Y| > K) < \delta$.

And we get

$$\mathbb{E}(|Y|\,;\,|Y|>K) \le \mathbb{E}(|X|\,;\,\underbrace{|Y|>K}_{\text{event } F \text{ with prob } < \delta}) < \varepsilon. \qquad \square$$

Definition: A sequence X_n of random variables is said to converge in probability ($X_n \xrightarrow[p]{} X$) if, for all $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Lemma If $X_n \xrightarrow[a.s.]{} X$, then also $X_n \xrightarrow[p]{} X$.

If $X_n \xrightarrow[L^p]{} X$ for some $p > 1$ (i.e. $\|X_n - X\|_p \to 0$) then also $X_n \xrightarrow[p]{} X$.

Proof: For the first part, assume $X_n \to X$ a.s. and apply reverse Fatou lemma:

$$\limsup_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{P}(\limsup\{|X_n - X| > \varepsilon\})$$

$$= \mathbb{P}(|X_n - X| > \varepsilon \text{ infinitely often})$$

$$\le \mathbb{P}(X_n \not\to X) = 0 \text{ by a.s. convergence.}$$

So $X_n \xrightarrow[p]{} X$.

For the second part, suppose $X_n \xrightarrow[L^p]{} X$.

That is $\|X_n - X\|_p = \mathbb{E}(|X_n - X|^p)^{1/p} \to 0$.

We use Markov's inequality,

$$\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(|X_n - X|^p > \varepsilon^p)$$
$$\leq \varepsilon^{-p}\, \mathbb{E}(|X_n - X|^p) \xrightarrow[n\to\infty]{} 0$$

From which we again have $X_n \xrightarrow[P]{} X$. $\square$

<u>Theorem</u>: Suppose that $X_n \xrightarrow[P]{} X$ and $|X_n| \leq K$ for some $K > 0$ for all $n \in \mathbb{N}$.

Then we have $\mathbb{E}(|X_n - X|) \to 0$ and thus $X_n \xrightarrow[L^1]{} X$.

<u>Proof</u>: For every $k \in \mathbb{N}$ we have

$$\mathbb{P}(|X| > K + \tfrac{1}{k}) \leq \mathbb{P}(|X_n - X| > \tfrac{1}{k}) \xrightarrow[n\to\infty]{} 0$$

so $\mathbb{P}(|X| > K + 1/k) = 0$ and $|X| \leq K$ a.s.

Let $\varepsilon > 0$ and pick n_0 large enough s.t.

$$\mathbb{P}(|X_n - X| > \varepsilon/3) < \varepsilon/3K \quad \text{for all } n \geq n_0.$$

Then,

$$\mathbb{E}(|X_n - X|) = \mathbb{E}(\overbrace{|X_n - X|}^{\leq \varepsilon/3} \,;\, |X_n - X| \leq \tfrac{\varepsilon}{3})$$
$$+ \mathbb{E}(\underbrace{|X_n - X|}_{\leq |X_n| + |X| \leq 2K} \,;\, |X_n - X| > \tfrac{\varepsilon}{3})$$
$$\leq \frac{\varepsilon}{3} + \mathbb{P}(|X_n - X| > \varepsilon/3)\, 2K$$
$$< \varepsilon/3 + 2K \frac{\varepsilon}{3K} = \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, $\mathbb{E}(|X_n - X|) \to 0$

And $X_n \xrightarrow[L^1]{} X$. $\square$

<u>Theorem</u>: Suppose that X_n is a sequence of integrable random variables. The following are equivalent:

1) $\mathbb{E}(|X_n - X|) \to 0$

2) $X_n \xrightarrow[P]{} X$ and $\{X_n\}$ is uniformly integrable.

Proof: [Exercise (maybe)]

Uniformly Integrable Martingales

Let M_n be a uniformly integrable martingale. $M_n \to M_\infty$ a.s. by the martingale convergence theorem. By uniform integrability,

$$M_n \xrightarrow[L^1]{} M_\infty .$$

For any fixed n, we have

$$\mathbb{E}(M_r \mid \mathcal{F}_n) = M_n \quad \text{for } r \geq n$$

$$\Rightarrow \quad \mathbb{E}(M_r ; F) = \mathbb{E}(M_n ; F) \text{ for all } F \in \mathcal{F}_n .$$

We get $|\mathbb{E}(M_n ; F) - \mathbb{E}(M_\infty ; F)|$

$$= |\mathbb{E}(M_r ; F) - \mathbb{E}(M_\infty ; F)|$$

$$= |\mathbb{E}(M_r - M_\infty ; F)| \leq \mathbb{E}(|M_r - M_\infty| ; F) \quad \forall r \geq n$$

$$\to 0 \quad \text{as } r \to \infty .$$

So we must have $\mathbb{E}(M_n ; F) = \mathbb{E}(M_\infty ; F)$ for all $F \in \mathcal{F}$. So $M_n = \mathbb{E}(M_\infty \mid \mathcal{F}_n)$ a.s.

We have shown:

<u>Theorem</u>: If M_n is a uniformly integrable martingale with respect to filtration $\mathcal{F}_n$, then
$M_\infty = \lim_n M_n$ exists a.s. and we have
$M_n = \mathbb{E}(M_\infty \mid \mathcal{F}_n)$ a.s. for all $n \in \mathbb{N}$

<u>Remark</u>: Also holds for super-/submartingales with appropriate inequalities.

<u>Doob's submartingale inequality</u>

Th$^{\underline{m}}$ Consider a non-negative submartingale Z_n.
For every $c > 0$, we have

$$c \, \mathbb{P}(\sup_{k \le n} Z_k \ge c) \le \mathbb{E}(Z_n \, ; \sup_{k \le n} Z_k \ge c) \le \mathbb{E}(Z_n)$$

[Note the similarity to Markov's inequality]

Proof: The event $\{\sup_{k \le n} Z_k \ge c\}$ can be decomposed in disjoint events

$$F_0 = \{Z_0 \ge c\}, \quad F_1 = \{Z_0 < c\} \cap \{Z_1 \ge c\}$$
$$F_2 = \{Z_0 < c\} \cap \{Z_1 < c\} \cap \{Z_2 \ge c\}, \quad F_3 = \ldots$$

Note that $F_k \in \tilde{\mathcal{F}}_k \supseteq \sigma(Z_0, \dots, Z_k)$.

So, $\mathbb{E}(Z_n ; F_k) = \int_{F_k} Z_n \, d\mathbb{P} = \int_{F_k} \mathbb{E}(Z_n | \tilde{\mathcal{F}}_k) \, d\mathbb{P}$

$$\underset{\substack{\uparrow \\ \text{submartingale}}}{\geq} \int_{F_k} Z_k \, d\mathbb{P} = \mathbb{E}(Z_k ; F_k).$$

Now since $Z_k \geq c$ on F_k,

$$\mathbb{E}(Z_n ; F_k) \geq \int_{F_k} c \, d\mathbb{P} = c \, \mathbb{P}(F_k).$$

Now summing, gives

$$\sum_{k=0}^{n} \mathbb{E}(Z_n ; F_k) \geq c \sum_{k=0}^{n} \mathbb{P}(F_k)$$

$$= c \, \mathbb{P}\Big(\bigcup_{k=0}^{n} F_k\Big) = c \cdot \mathbb{P}\big(\sup_{k \leq n} Z_k \geq c\big).$$

And LHS gives

$$\sum_{k=0}^{n} \mathbb{E}(Z_n ; F_k) = \sum_{k=0}^{n} \mathbb{E}(Z_n I_{F_k}) = \mathbb{E}\Big(Z_n \sum_{k=0}^{n} I_{F_k}\Big)$$

$$= \mathbb{E}\Big(Z_n \, I_{\bigcup_{k=0}^{n} F_k}\Big) = \mathbb{E}\big(Z_n ; \sup_{k \leq n} Z_k \geq c\big) \leq \mathbb{E}(Z_n).$$

So $\mathbb{E}(Z_n) \geq c \, \mathbb{P}\big(\sup_{k \leq n} Z_k \geq c\big)$ as required $\square$

Jensen's inequality also implies

Lemma If M_n is a martingale and f is a convex function s.t. $f(M_n)$ is integrable for all n, then $f(M_n)$ is a submartingale.

Theorem (Kolmogorov's inequality)
Let X_n be a sequence of independent random variables with $\mathbb{E}(X_n)=0$ and $Var(X_n)=\sigma_n^2<\infty$. Set $S_n = X_1+\dots+X_n$.
Then, for every $c>0$,

$$c^2\, \mathbb{P}\left(\sup_{k\le n} |S_k| \ge c\right) \le V_n = Var(S_n) = \sum_{k=1}^{n} \sigma_k^2$$

Proof: S_n is a martingale and S_n^2 a submartingale as $x \mapsto x^2$ is convex.
By Doob's submartingale inequality, we get

$$c^2\, \mathbb{P}\left(\sup_{k\le n} |S_k| \ge c\right) = c^2\, \mathbb{P}\left(\sup_{k\le n} S_k^2 \ge c^2\right)$$

$$\le \mathbb{E}(S_n^2) = Var(S_n).$$

$\square$