Remember to justify your answers clearly. You may answer in Swedish or in English. There are 8 problems, each worth of 0–5 points. Grade limits are 16-24 (grade 3), 25-31 (Grade 4), and 32-40 (grade 5).

1. Is the function $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$, $\quad (x,y) \neq (0,0)$, $f(0,0) = 1$ continuous? (5p)

   Solution sketch: As $f(x,y) = \frac{P(x,y)}{Q(x,y)}$, where P and Q are continuous as polynomials, it suffices to study the continuity at the points where $Q(x,y) = 0$. Now $Q(x,y) = x^2y^2 + (x-y)^2 = 0$ only if $(x,y) = (0,0)$, so hence the function is continuous if we have

   $$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 1.$$

   Remember that direction matters! So we move to polar coordinates and get

   $$f(r,\varphi) = \frac{r^4\cos^2\varphi\sin^2\varphi}{r^4\cos^2\varphi\sin^2\varphi + r^2(\cos\varphi - \sin\varphi)^2} = \frac{\cos^2\varphi\sin^2\varphi}{\cos^2\varphi\sin^2\varphi + \frac{(\cos\varphi-\sin\varphi)^2}{r^2}}.$$

   Here

   $$\frac{(\cos\varphi - \sin\varphi)^2}{r^2} \to \pm\infty$$

   as $r \to 0$ unless $\cos\varphi = \sin\varphi$ (along which one gets the limit 1 as wanted!), so for all such directions we get

   $$\lim_{r\to 0} f(r,\varphi) = \frac{\cos^2\varphi\sin^2\varphi}{\cos^2\varphi\sin^2\varphi \pm \infty} = 0.$$

   Hence the function is not continuous (and the limit at $(0,0)$ does not even exists).

2. Find the largest and smallest value of $f(x,y,z) = x + y$ on $x^2 + y^2 + z^2 \leq 1$. (5p)

   Solution sketch: Let us recall that a continuous function attains largest and smallest value on compact sets (now a ball), giving us the sanity check. Let us also recall that largest and smallest values are attained at extreme points where $\nabla f = 0$ or at the boundary $x^2 + y^2 + z^2 = 1$. Now $f'_x = f'_y = 1$ so the gradient is never zero, and hence it suffices to study the boundary. Next let us obtain that due to symmetry, we obtain maximum value for some (x_0, y_0), $x_0, y_0 \geq 0$ (if they have different signs, then we obtain cancellation), and the minimum value is obtained at $(-x_0, -y_0)$. Also, the boundary can be written as $x^2 + y^2 = 1 - z^2$ so we obtain larger values of both x and y when z is as small as possible, so we can reduce the problem by setting $z = 0$. Then to obtain the largest value we have $x^2 + y^2 = 1$ leading to $y = \sqrt{1-x^2}$, and we get a one variable problem

   $$\max g(x) = x + \sqrt{1-x^2}, \quad 0 \leq x \leq 1.$$

   From condition $g'(x) = 0$ we get

   $$1 = \frac{x}{\sqrt{1-x^2}}$$

   that gives $x = \frac{1}{\sqrt{2}}$. Then $y = \frac{1}{\sqrt{2}}$ and we get maximum value

   $$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

   and the minimum value

   $$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}.$$

3. Find all $u(x,y) = f(xy)$, where f is a function of one-variable, that satisfy $xu'_x + yu'_y = x^2y^2$. (5p)

Solution sketch: Taking partial derivatives we get

$$u'_x(x,y) = yf'(xy), \quad u'_y(x,y) = xf'(xy)$$

from which plugging into the equation gives

$$xyf'(xy) + xyf'(xy) = x^2y^2.$$

Denoting $u = xy$ and dividing with u leads to

$$2f'(u) = u$$

so integrating gives

$$f(u) = \frac{u^2}{4} + C.$$

4. Let $f(x,y,z) = \cos(xyz) + xz^2 - y^2x$. (5p)

   (a) Find the direction of most decrease at point $(x_0, y_0, z_0) = (0,0,1)$.

   (b) Compute the directional derivative into the direction $v = (1,1,1)$ at the point $(x_0, y_0, z_0) = (0,0,1)$.

Solution sketch: The direction of most increase is the direction of the gradient ∇f, and the direction of most decrease is opposite, i.e. $-\nabla f$. Also, the directional derivative can be computed by $f'_v = \nabla f \cdot v$, where $|v| = 1$ is the direction vector. Computing the partials now give

$$f'_x = -yz\sin(xyz) + z^2 - y^2,$$

$$f'_y = -xz\sin(xyz) - 2xy,$$

and

$$f'_z = -yz\sin(xyz) + 2xz,$$

which gives the gradient at $(x_0, y_0, z_0) = (0,0,1)$ to be $\nabla f(0,0,1) = (1,0,0)$.

   (a) The direction of most decrease is $-\nabla f(0,0,1) = (-1,0,0)$.

   (b) Now the unit vector v giving the direction is $v = \frac{1}{\sqrt{3}}(1,1,1)$ so

$$f'_v = (1,0,0) \cdot \frac{1}{\sqrt{3}}(1,1,1) = \frac{1}{\sqrt{3}}.$$

5. Show that the equation $yx - \sin x = \cos x + 1$ has a solution $y = y(x)$ that is close to 0 whenever x is close to π. Find the coefficients a_0, a_1, and a_2 in the Taylor approximation

$$y(x) \approx a_0 + a_1x + a_2x^2$$

around $x = \pi$. (5p)

Solution sketch: By the implicit function theorem, $F(x,y) = 0$ has a solution $y = y(x)$ around a point (x_0, y_0) if

$$F'_y(x_0, y_0) \neq 0.$$

Now $(x_0, y_0) = (\pi, 0)$ and $F(x, y) = yx - \sin x - \cos x - 1$, for which $F'_y(\pi, 0) = \pi \neq 0$. For the Taylor approximation, we know that

$$y(x) \approx y(\pi) + y'(\pi)(x - \pi) + \frac{y''(\pi)}{2}(x - \pi)^2.$$

Hence we need to compute $y'(\pi)$ and $y''(\pi)$, from which we get a_0, a_1, a_2 by comparing the constant, the first order, and the second order terms. Now the derivatives can be computed by implicit differentiation. Assuming $y = y(x)$, we get from equation $F(x, y) = 0$ by differentiating both sides that

$$y'x + y - \cos x + \sin x = 0.$$

Plugging in $x = \pi$ and $y = 0$, and then solving for y' gives

$$y'(\pi) = -\frac{1}{\pi}.$$

Differentiating again gives

$$y''x + 2y' + \sin x + \cos x = 0$$

from which we can solve similarly that

$$y''(\pi) = \frac{1}{\pi} - \frac{2y'(\pi)}{\pi}.$$

From these one get the coefficients a_0, a_1, and a_2 by comparing representations

$$y(x) \approx y(\pi) + y'(\pi)(y - \pi) + \frac{y''(\pi)}{2}(y - \pi)^2$$

and

$$y(x) \approx a_0 + a_1 x + a_2 x^2.$$

6. Compute $\int\int_D xy(x^2 + y^2)^3 dxdy$, where $D = \{(x, y) : 0 \leq x, y \leq 1\}$. (5p)

Solution sketch: This is a straightforward integration. To simplify things a bit (not necessary) we take a short cut and observe due to symmetry of the set D (square) and the integrand $xy(x^2+y^2)^3$ that

$$\begin{aligned}
\int\int_D xy(x^2 + y^2)^3 dxdy &= \int_0^1 \int_0^y xy(x^2 + y^2)^3 dxdy + \int_0^1 \int_0^x xy(x^2 + y^2)^3 dydx \\
&= 2\int_0^1 \int_0^y xy(x^2 + y^2)^3 dxdy \\
&= \int_0^1 \frac{y}{4}\left((2y^2)^4 - (y^2)^4\right) dy \\
&= \int_0^1 \frac{15}{4} y^9 dy \\
&= \frac{15}{40} = \frac{3}{8}.
\end{aligned}$$

7. Compute the curve integral $\int_\gamma \left(y + y\sin(e^{xy}) + x^2 + y^2\right) dx + \left(x + x\sin(e^{xy})\right) dy$, where γ is given by the parametrisation $r(t) = (x(t), y(t)) = (2\cos t, 2\sin t), \quad 0 \le t \le \pi$. (5p)

Solution sketch: We make the curve closed by adding γ_1 given by the parametrisation $r(t) = (t, 0), \quad -2 \le t \le 2$. Now $\gamma + \gamma_1 = \partial D$ (with positive orientation) for D that is an upper half-circle of the ball of radius 2. By Green formula we have

$$\int_{\gamma+\gamma_1} Pdx + Qdy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy.$$

Here

$$P(x,y) = y + y\sin(e^{xy}) + x^2 + y^2$$

and

$$Q(x,y) = x + x\sin(e^{xy}).$$

Straightforward differentiation gives

$$Q'_x - P'_y = -2y$$

so the right hand side is

$$\begin{aligned}
&\int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= -\int\int_D 2y dxdy \\
&= -\int_0^\pi \int_0^2 2r\sin\varphi r dr d\varphi \\
&= -4\int_0^2 r^2 dr \\
&= -\frac{32}{3}.
\end{aligned}$$

On the other hand, we have

$$\int_{\gamma+\gamma_1} Pdx + Qdy = \int_\gamma Pdx + Qdy + \int_{\gamma_1} Pdx + Qdy$$

where, by the very definition and since $y = 0$ and $dy = 0$,

$$\begin{aligned}
&\int_{\gamma_1} Pdx + Qdy \\
&= \int_{\gamma_1} \left(y + y\sin(e^{xy}) + x^2 + y^2\right) dx + \left(x + x\sin(e^{xy})\right) dy \\
&= \int_{-2}^2 \left(0 + 0 + t^2 + 0\right) dt + 0 \\
&= \int_{-2}^2 t^2 dt \\
&= \frac{16}{3}.
\end{aligned}$$

Combining everything, we get

$$\int_\gamma Pdx + Qdy = \int\int_D Q'_x - P'_y dxdy - \int_{\gamma_1} Pdx + Qdy = -\frac{32}{3} - \frac{16}{3} = -16.$$

8. Find the values of α such that the integral $\int\int\int_{\mathbb{R}^3} \frac{\min(1,x^2+y^2+z^2)}{(x^2+y^2+z^2)^\alpha} dxdydz$ converges. (5p)

Solution sketch: This is essentially a 3-dimensional version of the last problem in home assignment 2. Indeed, the integral is an indefinite integral, where we have to consider the singularity at the origin and the infinities. Since now the integrand

$$f(x,y,z) = \frac{\min(1, x^2+y^2+z^2)}{(x^2+y^2+z^2)^\alpha} \geq 0,$$

we know that the value of the indefinite integral is independent of the approximation. Let B_R be a ball-of radius R centered at the origin. Then we consider

$$\int\int_{\mathbb{R}^3} f(x,y,z)dxdydz = \lim_{R\to\infty} \int\int_{B_R \setminus B_{R^{-1}}} f(x,y,z)dxdydz.$$

Moving to the spherical coordinates, here (Jacobian determinant equals $r^2 \sin\theta$

$$\begin{aligned}
&\int_{B_R\setminus B_{R^{-1}}} \frac{\min(1,x^2+y^2+z^2)}{(x^2+y^2+z^2)^\alpha} dxdydz \\
&= \int_0^\pi \int_0^{2\pi} \int_{R^{-1}}^R \frac{\min(1,r^2)}{r^{2\alpha}} r^2 \sin\theta dr d\varphi d\theta \\
&= 4\pi \left(\int_{R^{-1}}^1 r^{4-2\alpha} dr + \int_1^R r^{2-2\alpha} dr \right) \\
&= 4\pi \left(\frac{1}{5-2\alpha}(1 - R^{2\alpha-5}) + \frac{1}{3-2\alpha}(R^{3-2\alpha} - 1) \right).
\end{aligned}$$

The first term converges to a finite number $\frac{4\pi}{5-2\alpha}$ as $R\to\infty$ exactly when $5-2\alpha > 0$ while the second term converges to a finite number $\frac{4\pi}{3-2\alpha}$ as $R\to\infty$ exactly when $3-2\alpha < 0$. Combining the two gives us condition $\frac{3}{2} < \alpha < \frac{5}{2}$.