

UPPSALA UNIVERSITET

LECTURE NOTES

Complex Analysis

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1. INTRO

In this course, we shall study functions $f : \mathbb{C} \rightarrow \mathbb{C}$ (or more generally, $f : D \rightarrow \mathbb{C}$ where $D \subseteq \mathbb{C}$)

Definition/Sats 1.1: Complex Number

A *complex number* is a number of the form $x + iy$, where $x, y \in \mathbb{R}$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$

Anmärkning:

The number x is called the *real part* ($\operatorname{Re}(z) = x$) of the complex number, and y is called the *imaginary part* ($\operatorname{Im}(z) = y$) of the complex number

Anmärkning:

The set of all complex numbers is denoted by \mathbb{C}

Anmärkning:

\mathbb{C} is the *smallest* field extension to \mathbb{R} that is algebraically closed.

Anmärkning:

$i^2 = -1$

1.1. Operations over \mathbb{C} .

We define the operations *addition* and *multiplication* of two complex numbers as follows:

Definition/Sats 1.2: Addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Definition/Sats 1.3: Multiplication of complex numbers

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

With respect to these two operations, \mathbb{C} forms a commutative field.

This means that the following holds for addition:

- $z_1 + z_2 = z_2 + z_1$
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And for multiplication:

- $z_1z_2 = z_2z_1$
- $z_1(z_2z_3) = (z_1z_2)z_3$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Definition/Sats 1.4: Complex conjugate

The *complex conjugate* of a complex number $z = x + iy$, denoted by \bar{z} , is defined by $\bar{z} = x - iy$

The following holds for the complex conjugate:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$
- $\overline{\bar{z}} = z$
- $z \cdot \bar{z} = |z|^2$
- $z^{-1} = \frac{\bar{z}}{|z|^2}$
- $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$

Anmärkning:

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

Anmärkning:

Multiplication by i is simply rotation by $\frac{\pi}{2}$ counterclockwise.

Definition/Sats 1.5

Let $z \in \mathbb{C}$. Then there exists a $w \in \mathbb{C}$ such that $w^2 = z$ (where $-w$ also satisfies this equation)

Bevis 1.1

Let $z = a + bi$ and $w = x + iy$ such that $a + bi = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

Then $a = x^2 - y^2$ and $b = 2xy$

We also know that $|z| = a^2 + b^2 = |x^2 + y^2|^2 = (x^2 - y^2)^2 + 4x^2y^2$

Therefore, $x^2 + y^2 = \sqrt{a^2 + b^2}$ and:

$$\left. \begin{array}{l} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\left. \begin{array}{l} -x^2 + y^2 = -a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{array} \right\} \Rightarrow y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}$$

Now let $\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$ and $\beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$ and let $\sqrt{}$ denote the positive square root of positive real numbers.

If b is positive, then either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$

If b is negative, then either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$

Therefore, the equation has solutions $\pm(\alpha + \mu\beta i)$ where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$

□

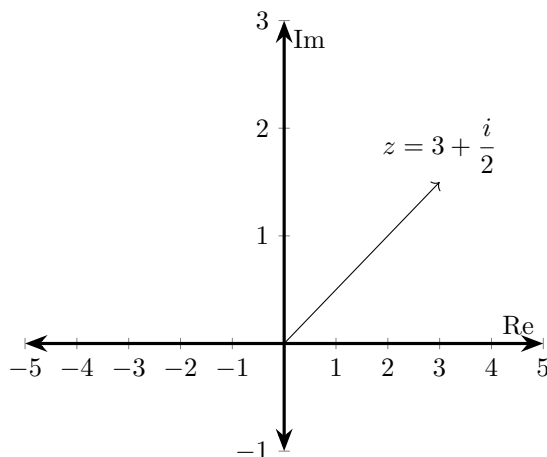
Anmärkning:

From the proof above, we can conclude the following:

- The square roots of a complex number are real \Leftrightarrow the complex number is real and positive
- The square roots of a complex number are purely imaginary \Leftrightarrow the complex number is real and negative
- The two square roots of a number coincide \Leftrightarrow the complex number is zero

1.2. Cartesian representation.

It is natural to represent a complex number $z = x + iy$ as a tuple (x, y) , and we can therefore represent it in the standard cartesian plane:



Anmärkning:

This is sometimes called the *complex plane*

Definition/Sats 1.6: Absolute value/Modulus

The absolute value of a complex number $z = x + iy$ (geometrically the length of the vector), denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2}$$

It holds that:

- $|z|^2 = z \cdot \bar{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Anmärkning:

Every $z \in \mathbb{C}$ such that $z \neq 0$ (that is, $x \neq 0$ or $y \neq 0$) has a multiplicative inverse $\frac{1}{z}$ given by:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Definition/Sats 1.7: Triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that $|z_1 + z_2| \leq |z_1| + |z_2|$

Lemma 1.1: Reversed triangle inequality

For $z_1, z_2 \in \mathbb{C}$, it holds that:

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Bevis 1.2

$$z_1 = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

$$\text{So that } |z_1| - |z_2| \leq |z_1 - z_2|$$

□

The following properties holds:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
- $-\operatorname{Re}(z) \leq \operatorname{Re}(z) \leq |z|$
- $-\operatorname{Im}(z) \leq \operatorname{Im}(z) \leq |z|$
- $|\bar{z}| = |z|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$
- $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \dots + |w_n|^2}$

1.3. Polar form.

Let $z = x + iy \neq 0$. The point $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ lies on the unit circle, and hence there exists θ such that:

$$\frac{x}{|z|} = \cos(\theta) \quad \frac{y}{|z|} = \sin(\theta)$$

Therefore $z = x + iy$ can be written as:

$$z = r(\cos(\theta) + i \sin(\theta))$$

Where $r = |z|$ is uniquely determined by z , while θ is 2π -periodic. This is called the *polar form* of z and just as the cartesian representation requires a tuple of information $(|z|, \theta)$

Definition/Sats 1.8: Argument

The *argument* of a complex number z , denoted by $\arg(z)$, is the angle θ between z and the real number line in the complex plane

Anmärkning:

Since the argument is 2π periodic, the angle is usually given as $\theta + k2\pi$ $k \in \mathbb{Z}$, but we are only interested in θ

This θ is called the *principal value* of $\arg(z)$, denoted by $\operatorname{Arg}(z)$ and belongs to $(-\pi, \pi]$

Anmärkning:

We are always allowed to change an angle by multiples of 2π , the principal value argument is the angle after changing the argument such that it lies between $(-\pi, \pi]$

Anmärkning:

A specification of choosing a particular range for the angles is called choosing a *branch* of the argument. Also, note that $\operatorname{Arg}(z)$ is "discontinuous" along the negative real axis. This is called a *branch-cut*

$$\text{Suppose } z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1)), z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$$

Then:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 [(\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2))] \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

Anmärkning:

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

1.4. Exponential form.

Definition/Sats 1.9

For $z = x + iy \in \mathbb{C}$, let $e^z = e^x(\cos(y) + i \sin(y))$

Anmärkning:

$e^{iy} = \cos(y) + i \sin(y) \quad y \in \mathbb{R}$ (Eulers formula)

We can see that the definition holds through some Taylor expansions:

$$\begin{aligned} e^z &= e^{x+iy} = e^x \cdot e^{iy} \\ e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \\ \Rightarrow e^{iy} &= 1 + iy - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos(\theta)} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin(\theta)} \\ \Rightarrow e^z &= e^x(\cos(\theta) + i \sin(\theta)) \end{aligned}$$

Anmärkning:

One can through comparing see that $|e^z| = e^x$, and that $|e^{iy}| = 1$

Definition/Sats 1.10: deMoivre's formula

For $n \in \mathbb{Z}$, $(r(\cos(\theta) + i \sin(\theta)))^n = r^n(\cos(n\theta) + i \sin(n\theta))$

1.5. Logarithmic form.

In real analysis, we have defined the logarithm as the inverse of e^x . This has previously worked since for $x \in \mathbb{R}$, e^x is injective.

The problem is that for e^z where $z \in \mathbb{C}$, it is not injective and should therefore not have an inverse.

Given $z \in \mathbb{C} \setminus \{0\}$, we define $\ln(z)$ as the cut of all $w \in \mathbb{C}$ whose image under the exponential form is z , i.e $w = \ln(z) \Leftrightarrow z = e^w$.

Here, $\ln(z)$ is a *multivalued form*

We can use the fact that $|z| = r = e^x$ to derive some interesting properties of the logarithm:

$$\begin{aligned} z &= re^{i\theta} & w &= u + iv \\ \text{If } z &= e^w \Leftrightarrow re^{i\theta} = e^u \cdot e^{iv} \\ \Leftrightarrow u &= \ln(r) = \ln(|z|) & v &= \theta + k2\pi = \arg(z) \quad k \in \mathbb{Z} \end{aligned}$$

Definition/Sats 1.11: Complex logarithm

For $z \neq 0$, we define the complex logarithm for $z \in \mathbb{C}$ as:

$$\begin{aligned} \ln(z) &= \ln(|z|) + i \cdot \arg(z) \\ &= \ln(|z|) + i(\text{Arg}(z) + k2\pi) \quad k \in \mathbb{Z} \end{aligned}$$

2. ELEMENTARY COMPLEX FUNCTIONS

Branching is not an exclusive phenomenon to the argument, it can be done everywhere

2.1. Branches of the complex logarithm.

In Definition 1.11, we defined the complex logarithm as:

$$\ln(|z|) + i \cdot \arg(z)$$

We also added a line below it, to show that the definition holds for the principal value argument (with multiples of 2π).

If we remove the multiples, we have *branched* the complex logarithm and obtained a single-valued function:

Definition/Sats 2.12: Principal logarithm

By branching the argument of the complex logarithm, we obtain the *principal logarithm*:

$$\text{Ln}(z) = \ln(|z|) + i \cdot \text{Arg}(z)$$

Anmärkning:

We have essentially extended the "normal" logarithm, which is defined on $(0, \infty)$, to be defined on $\mathbb{C} \setminus \{0\}$

Anmärkning:

The principal logarithm is discontinuous for negative reals, since their principal value argument is $= -\pi$, but the principal value argument is discontinuous at $-\pi$. This is the so called *branch-cut*

Anmärkning:

Even though the principal logarithm is discontinuous for negative reals, it is not undefined. Any negative real number z will have $\text{Arg}(z) = \pi$, which the logarithm very much is defined for.

Anmärkning:

When branching, we do not necessarily have to pick $(-\pi, \pi]$, we can pick any interval $(\alpha, \alpha + 2\pi]$. This is usually denoted by \arg_α .

2.2. Complex mappings.

One can think of a complex mapping $f : \mathbb{C} \rightarrow \mathbb{C}$ as $f(z) = f(x + iy) = w = u + iv$

Then it becomes clear which regions map to where by drawing them in their respective z -plane and w -plane.

2.3. Complex powers.

Given $z \in \mathbb{C}$, consider the following equation:

$$(1) \quad w^n = z$$

The set of all solutions w of (1) is denoted $z^{1/n}$ and is called the *n-th root of z*.

Anmärkning:

If $z = 0$, then $w = 0$

Suppose $z \neq 0$, then we may write $w = |w|e^{i\alpha}$ and $z = |z|e^{i\theta}$
 By deMoivre's formula, (1) becomes:

$$|w|^n e^{in\alpha} = |z|e^{i\theta}$$

Then, the following follows:

$$\left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ n\alpha &= \theta + k2\pi \quad k \in \mathbb{Z} \end{aligned} \right\} \Leftrightarrow \left. \begin{aligned} |w| &= \sqrt[n]{|z|} \\ \alpha &= \frac{\theta}{n} + k\frac{2\pi}{n} \quad k \in \mathbb{Z} \end{aligned} \right\}$$

Notice now that every $k \in \mathbb{Z}$ gives a solution to (1)

Since sine and cosine are both 2π -periodic, then only $k = 0, 1, \dots, n-1$ actually give *different* solutions
 (since $k = n \Rightarrow \alpha = \frac{\theta}{n} + n\frac{2\pi}{n}$)

Suppose $z \neq 0$. For $n \in \mathbb{Z}$ it holds that:

$$z^n = e^{n \ln(z)}$$

For every value that $\ln(z)$ attains.

It is also true, that for $n = 1, 2, 3, \dots$:

$$\frac{1}{z^n} = e^{\frac{1}{n} \ln(z)}$$

We can let $n \in \mathbb{C}$, and obtain the following definition:

Definition/Sats 2.13: Complex power

For $\alpha \in \mathbb{C}$, let:

$$z^\alpha = e^{\alpha \ln(z)} \quad z \neq 0$$

Anmärkning:

This makes z^α a multivalued function.

Just as before, whenever we are dealing with the argument, the argument (heh) of branching comes up.
 We can chose to branch z^α :

$$z^\alpha = e^{\alpha \text{Ln}(z)}$$

2.4. Trigonometric and Hyperbolic functions.

We have the following:

$$\left. \begin{aligned} e^{iy} &= \cos(y) + i \sin(y) \\ e^{-iy} &= \cos(y) - i \sin(y) \end{aligned} \right\} \Rightarrow \begin{aligned} \cos(y) &= \frac{e^{iy} + e^{-iy}}{2} \\ \sin(y) &= \frac{e^{iy} - e^{-iy}}{2i} \end{aligned}$$

In fact, this will be used in the definition of the complex valued trigonometric functions:

Definition/Sats 2.14: Complex sine and cosine

For $z \in \mathbb{C}$, we define:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Recall that the definition of the hyperbolic trigonometric functions are defined using reals.
 When defining them for complex numbers, we just extend their domain:

Definition/Sats 2.15: Complex hyperbolic functions

For $z \in \mathbb{C}$, we define:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

Now we can look at how the addition formulas for sine and cosine change when the input is complex:

• **Sine:**

$$\begin{aligned} \sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &\Rightarrow \frac{e^{-y}(\cos(x) + i \sin(x)) - e^y(\cos(x) - i \sin(x))}{2i} = \frac{(e^{-y} - e^y) \cos(x) + i(e^y - e^{-y}) \sin(x)}{2i} \\ &= \frac{(e^{-y} - e^y) \cos(x)}{2i} + \frac{(e^y - e^{-y}) \sin(x)}{2} \\ &\stackrel{i^{-1} = -i}{\Rightarrow} \underbrace{\frac{(e^y - e^{-y})}{2}}_{\sinh(y)} i \cos(x) + \underbrace{\frac{(e^y + e^{-y})}{2}}_{\cosh(y)} \sin(x) \end{aligned}$$

• **Cosine:**

$$\begin{aligned} \cos(x + iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} \\ &= \frac{e^{-y}(\cos(x) + i \sin(x)) + e^y(\cos(x) - i \sin(x))}{2} = \frac{\cos(x)(e^y + e^{-y}) + i(e^{-y} - e^y) \sin(x)}{2} \\ &= \underbrace{\frac{e^y + e^{-y}}{2}}_{\cosh(y)} \cos(x) - \underbrace{\frac{e^y - e^{-y}}{2}}_{\sinh(y)} i \sin(x) \end{aligned}$$

This leads us to the following:

Definition/Sats 2.16: Addition formulas for complex trigonometric functions

- $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$
- $\cos(x + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$

2.5. Mapping properties of $\sin(z)$.

Let $f(z) = \sin(z)$ in $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$, let A be the set of points allowed with respect to the above constraint and let B be the mapping of those points by $\sin(A)$

Claim: $f : A \rightarrow B$ is a bijective mapping

Bevis 2.1

Take a $z \in \mathbb{C}$ $z = x + iy$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Then:

$$f(z) = \sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$f(z) \in \mathbb{R} \Leftrightarrow \cos(x) \sinh(y) = 0 \Leftrightarrow \sinh(y) = 0 \Leftrightarrow y = 0$$

If $y = 0$, then:

$$f(z) = \sin(x) \cosh(y) = \sin(x) \in (-1, 1)$$

Therefore, if $z \in A \Rightarrow f(z) \in B$. Now we need to show that for any $z \in B$, there is a u such that $f(u) = z$

Let $u = \sin(x) \cosh(y)$, $v = \cos(x) \sinh(y)$ and pick a vertical line at $x = a \neq 0$

We will now consider the images of these lines:

$$\cosh(y) = \frac{u}{\sin(a)} \quad \sinh(y) = \frac{v}{\cos(a)}$$

$$(\cosh(y))^2 - (\sinh(y))^2 = 1 \Rightarrow \left(\frac{u}{\sin(a)} \right)^2 - \left(\frac{v}{\cos(a)} \right)^2 = 1$$

In the plane, this represents a hyperbolic function. Now pick a horizontal line $y = b \neq 0$

$$\sin(x) = \frac{u}{\cosh(b)} \quad \cos(x) = \frac{v}{\sinh(b)}$$

$$\cos^2(x) + \sin^2(x) = 1 \Rightarrow \left(\frac{u}{\cosh(b)} \right)^2 + \left(\frac{v}{\sinh(b)} \right)^2 = 1$$

This is a half-ellipse. Note that $v > 0 \Leftrightarrow \sinh(b) > 0 \Leftrightarrow b > 0$

□