Analysis of Time Series, L15

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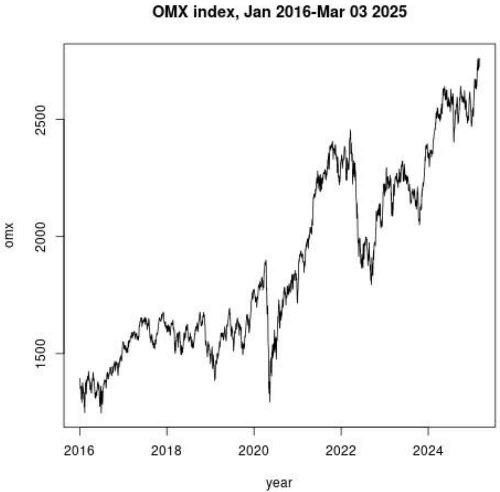
Today

5.3: GARCH models:

- ARCH
- GARCH
- other models

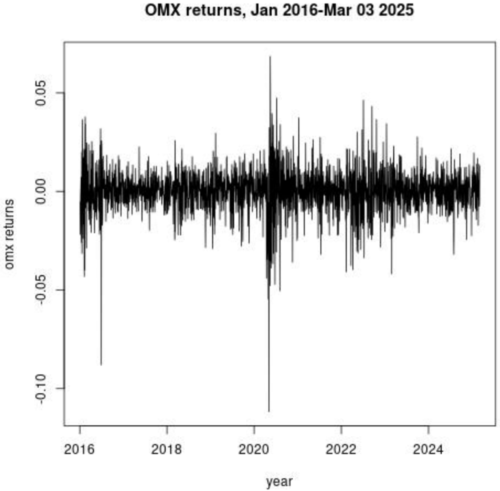
ARCH

OMX series x_t



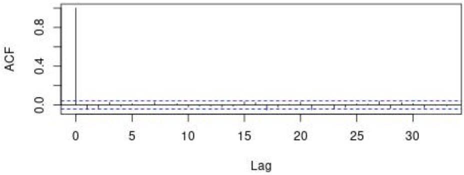
ARCH

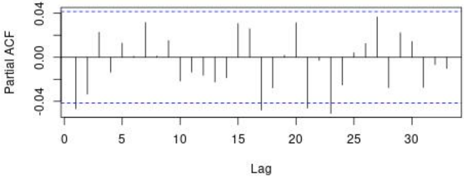
OMX returns $y_t = \ln(x_t) - \ln(x_{t-1})$



ARCH

ACF and PACF for $y_t = \ln(x_t) - \ln(x_{t-1})$. White noise?





Box-Ljung test of H_0: The y_t are uncorrelated. Do not reject for lag length 10.

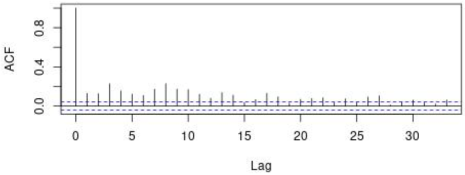
```
> Box.test(y,type="Ljung-Box")

Box-Ljung test

data:  y
X-squared = 4.8895, df = 1, p-value = 0.02702

> Box.test(y,type="Ljung-Box", lag=10)

Box-Ljung test

data:  y
X-squared = 12.755, df = 10, p-value = 0.2377
```

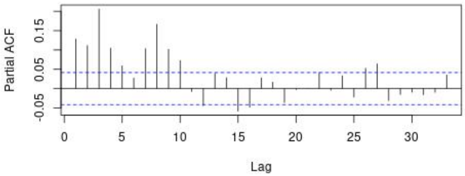
Box-Ljung test of H_0: The y_t^2 are uncorrelated. Reject for both lag lengths.

```
> Box.test(y^2,type="Ljung-Box")

Box-Ljung test

data:  y^2
X-squared = 36.764, df = 1, p-value = 1.333e-09

> Box.test(y^2,type="Ljung-Box",lag=10)

Box-Ljung test

data:  y^2
X-squared = 614.21, df = 10, p-value < 2.2e-16
```

ARCH

ACF and PACF for y_t^2. A lot of structure!





ARCH

Robert Engle, Nobel prize 2003.

https://www.nobelprize.org/nobelprizes/economic-sciences/laureates/2003/engle-photo.html

ARCH

An important formula: $E(X) = E\{E(X|Y)\}$.

In the discrete case, we have

$$E(X|Y = y) = \sum_x xP(X = x|Y = y)$$

which leads to (why?)

$$E\{E(X|Y)\} = \sum_x xP(X = x) = E(X).$$

ARCH

AutoRegressive Conditional Heteroscedasticity (ARCH) model:

- $$y_t = \sigma_t \epsilon_t,$$
  $$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2,$$

  where the ϵ_t are i.i.d. $N(0,1)$ and $\alpha_0 > 0$, $\alpha_1 > 0$.
- It follows that
  $$y_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + v_t,$$
  where $v_t = \sigma_t^2(\epsilon_t^2 - 1)$.
- $\{y_t\}$ is an uncorrelated sequence (why?), but $\{y_t^2\}$ is not.

ARCH

Some further properties:

- $$\mathrm{Var}(y_t) = \alpha_0 + \alpha_1 E(y_{t-1}^2)$$
- If $\alpha_1 < 1$,
$$\mathrm{Var}(y_t) = \frac{\alpha_0}{1-\alpha_1}$$
- and
$$E(y_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)},$$
implying that the kurtosis $\kappa > 3$.
- It is required that $\alpha_1^2 < 1/3$, i.e. $\alpha_1 < 0.577$.

ARCH

- Observations $y_1, y_2, ..., y_n$.
- The conditional log likelihood of $y_2, ..., y_n$ given y_1 fulfills (why?)

$$l(\alpha_0, \alpha_1) \propto -\frac{1}{2}\sum_{t=2}^{n} \log(\alpha_0 + \alpha_1 y_{t-1}^2) - \frac{1}{2}\sum_{t=2}^{n} \frac{y_t^2}{\alpha_0 + \alpha_1 y_{t-1}^2}.$$

- May be maximized with numerical methods.
- Changing the normality assumption to e.g. Student's t alters the likelihood.
- Still, it can be maximized with numerical methods.

ARCH

Extension: ARCH(m)

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \ldots + \alpha_m y_{t-m}^2. \end{aligned}$$

GARCH

GARCH(1,1):

- $$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

- It follows that

$$y_t^2 = \alpha_0 + (\alpha_1 + \beta_1) y_{t-1}^2 + v_t - \beta_1 v_{t-1},$$

  where $v_t = \sigma_t^2(\epsilon_t^2 - 1)$.
- Not identified if $\alpha_1 = 0$.
- Hence, GARCH(0,1) is not a possible model.

GARCH

Some further properties:

- If $\alpha_1 + \beta_1 < 1$,
$$\mathrm{Var}(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$
- and
$$E(y_t^4) = \frac{3\alpha_0^2(1 + \alpha_1 + \beta_1)}{(1 - \alpha_1 - \beta_1)\{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2\}},$$
implying that the kurtosis $\kappa > 3$.
- It is required that $1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2 > 0$.
- The sequence $\{y_t^2\}$ has autocorrelation function
$$\rho_n = \alpha_1 \frac{1 - \beta_1^2 - \alpha_1\beta_1}{1 - \beta_1^2 - 2\alpha_1\beta_1}(\alpha_1 + \beta_1)^{n-1}.$$

GARCH

- Recall:
$$y_t^2 = \alpha_0 + (\alpha_1 + \beta_1) y_{t-1}^2 + v_t - \beta_1 v_{t-1},$$
where $v_t = \sigma_t^2(\epsilon_t^2 - 1)$.
- May show: y_t^2 is stationary if $\alpha_1 + \beta_1 < 1$.
- The *integrated* GARCH model, IGARCH, assumes that $\alpha_1 + \beta_1 = 1$:

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + (1 - \beta_1) y_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{aligned}$$

GARCH

GARCH(m, r):

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \ldots + \alpha_m y_{t-m}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_r \sigma_{t-r}^2,
\end{aligned}
$$

where $m \geq r$.

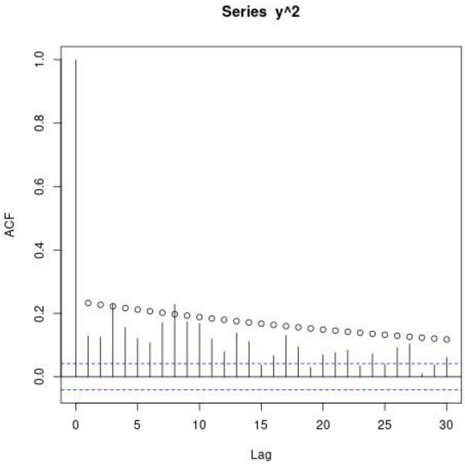
For OMX returns, R library `fGarch`:

```
> garchFit(y~garch(1,1),y,include.mean=FALSE)

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
omega  2.870e-06   8.105e-07    3.541 0.000399 ***
alpha1 9.288e-02   1.419e-02    6.546 5.91e-11 ***
beta1  8.839e-01   1.797e-02   49.185  < 2e-16 ***
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
7123.294    normalized:  3.19717
```

`omega` is α_0.
Observe: The sum of estimated α_1 and β_1 is close to 1.

GARCH

Estimated ACF for y_t^2, compared to ACF from estimated GARCH(1,1) model with rings.



Other models

Some more extensions of GARCH$(1,1)$:

- Quadratic GARCH (QGARCH):

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \phi y_{t-1} + \beta_1 \sigma_{t-1}^2,$$

- Threshold GARCH (TGARCH):

$$\sigma_t = \alpha_0 + \alpha_1^+ y_{t-1}^+ + \alpha_1^- y_{t-1}^- + \beta_1 \sigma_{t-1},$$

  where $y_{t-1}^+ = y_{t-1} I\{y_{t-1} > 0\}$, $y_{t-1}^- = y_{t-1} I\{y_{t-1} \leq 0\}$,
- EGARCH:

$$\log(\sigma_t^2) = \alpha_0 + \alpha_1\{|y_{t-1}| - E(|y_{t-1}|)\} + \gamma y_{t-1} + \beta \log(\sigma_{t-1}^2),$$

- GARCH-M:

$$\begin{cases} y_t = \mu + c\sigma_t^2 + \sigma_t \epsilon_t, \\ \sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{cases}$$

- and many more!

Other models

Further reading:

Tsay, R.S. (2010), Analysis of Financial Time Series, 3rd ed., Wiley.

News of today

- ARCH
- GARCH
- IGARCH
- QGARCH
- TGARCH
- EGARCH
- GARCH-M
- ...