Analysis of Time Series, L12

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Today

- 4.6: Cross Spectra
- 4.7: Linear Filters
- Wavelets (not in the book)

Cross Spectra

Let $\mathbf{x}_t = (x_{t1}, x_{t2}, ..., x_{tp})'$, $\boldsymbol{\mu} = E(\mathbf{x})$ and define the autocovariance matrix

$$\boldsymbol{\Gamma}(h) = E\{(\mathbf{x}_{t+h} - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\}$$

with elements $\gamma_{jk}(h)$, $j, k = 1, ..., p$.

Theorem (Property 4.8)

If, for all j, k,

$$\sum_{h=-\infty}^{\infty} |\gamma_{jk}(h)| < \infty,$$

then the spectral density matrix $\mathbf{f}(\omega)$ satisfies

$$\boldsymbol{\Gamma}(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h}\mathbf{f}(\omega)d\omega, \quad h = 0, \pm 1, \pm 2, ...,$$

$$\mathbf{f}(\omega) = \sum_{h=-\infty}^{\infty} \boldsymbol{\Gamma}(h)e^{-2\pi i\omega h}, \quad -1/2 \leq \omega \leq 1/2.$$

Cross Spectra

Bivariate process (x_t, y_t).

- Autocovariance matrix

$$\mathbf{\Gamma}(h) = \begin{pmatrix} \mathrm{cov}(x_{t+h}, x_t) & \mathrm{cov}(x_{t+h}, y_t) \\ \mathrm{cov}(y_{t+h}, x_t) & \mathrm{cov}(y_{t+h}, y_t) \end{pmatrix} = \begin{pmatrix} \gamma_{xx}(h) & \gamma_{xy}(h) \\ \gamma_{yx}(h) & \gamma_{yy}(h) \end{pmatrix}$$

- Spectral density matrix

$$\mathbf{f}(\omega) = \begin{pmatrix} f_{xx}(\omega) & f_{xy}(\omega) \\ f_{yx}(\omega) & f_{yy}(\omega) \end{pmatrix},$$

$$\gamma_{xy}(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} f_{xy}(\omega) d\omega,\ f_{xy}(\omega) = \sum_{h=-\infty}^{\infty} \gamma_{xy}(h) e^{-2\pi i\omega h}.$$

- *Squared coherence*

$$\rho^2_{y\cdot x}(\omega) = \frac{|f_{yx}(\omega)|^2}{f_{xx}(\omega) f_{yy}(\omega)}.$$

Cross Spectra

- DFT $\mathbf{d}(\omega_j) = (d_1(\omega_j), d_2(\omega_j), ..., d_p(\omega_j))'$, where

$$d_k(\omega_j) = n^{-1/2} \sum_{t=1}^{n} x_{tk} e^{-2\pi i \omega_j t}, \quad j = 0, 1, ..., n-1,$$

  where $\omega_j = j/n$.
- Raw periodogram $\mathbf{I}(\omega_j) = \mathbf{d}(\omega_j)\mathbf{d}^*(\omega_j)'$.
- Smoothed periodogram

$$\begin{pmatrix} \hat{f}_{xx}(\omega_j) & \hat{f}_{xy}(\omega_j) \\ \hat{f}_{yx}(\omega_j) & \hat{f}_{yy}(\omega_j) \end{pmatrix} = \hat{\mathbf{f}}(\omega_j) = \sum_{k=-m}^{m} h_k \mathbf{I}\left(\omega_j + \frac{k}{n}\right),$$

  where $\sum_{k=-m}^{m} h_k = 1$.
- Estimated squared coherence

$$\hat{\rho}^2_{y\cdot x}(\omega) = \frac{|\hat{f}_{yx}(\omega)|^2}{\hat{f}_{xx}(\omega)\hat{f}_{yy}(\omega)}.$$

Cross Spectra

- $$\hat{\rho}^2_{y\cdot x}(\omega) = \frac{|\hat{f}_{yx}(\omega)|^2}{\hat{f}_{xx}(\omega)\hat{f}_{yy}(\omega)}.$$

- For large n,

$$|\hat{\rho}_{y\cdot x}(\omega)| \approx N\left(|\rho_{y\cdot x}(\omega)|, \frac{\{1-\rho^2_{y\cdot x}(\omega)\}^2}{2L_h}\right),$$

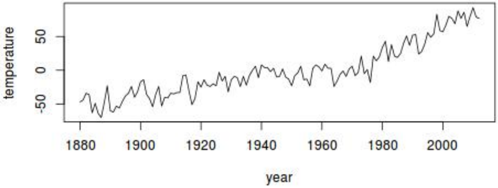
  where

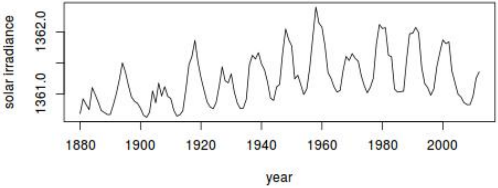
$$L_h = \left(\sum_{k=-m}^{m} h_k^2\right)^{-1}$$

- May be used to construct confidence intervals for $\rho^2_{y\cdot x}(\omega)$.

Cross Spectra

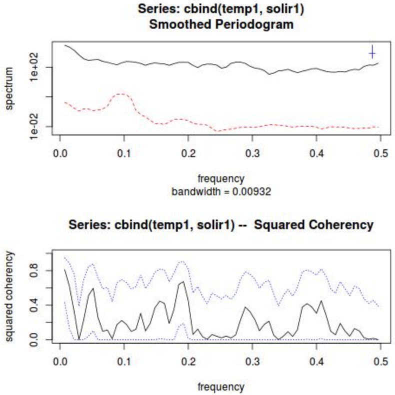
Yearly mean temperature and solar irradiance, 1880-2012.





Cross Spectra

Smoothing with spans=4



Significance (lower dotted curve above 0) basically only at 0 (common trend).

R code:

```
> par(mfrow=c(2,1))
> s=spec.pgram(cbind(temp1,solir1),spans=4)
> plot(s,plot.type="coh")
```

Linear Filters

- Input series $\{x_t\}$.
- A *linear filter* $\{a_t\}$ such that $\sum_{j=-\infty}^{\infty} |a_j| < \infty$.
- Output series $\{y_t\}$ such that we have the convolution

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}.$$

- Examples:
  - Moving average, e.g. $a_0 = a_1 = a_2 = a_3 = \frac{1}{4}$ (all the other $a_j = 0$).
  - Difference, e.g. $a_0 = 1$, $a_1 = -1$ (all the other $a_j = 0$).

Linear Filters

Frequency response function

$$A_{yx}(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i \omega j}.$$

Theorem (Property 4.9)

The spectral density $f_{yy}(\omega)$ of the filtered output y_t is related to the spectral density $f_{xx}(\omega)$ of the input x_t through

$$f_{yy}(\omega) = |A_{yx}(\omega)|^2 f_{xx}(\omega).$$

Prove that a causal ARMA process $\phi(B)x_t = \theta(B)w_t$ has spectral density

$$f(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i \omega})|^2}{|\phi(e^{-2\pi i \omega})|^2}.$$

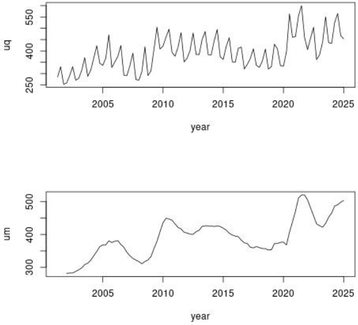
Linear Filters

Calculate $A_{yx}(\omega)$ and $|A_{yx}(\omega)|^2$ for

1. the moving average filter $a_0 = a_1 = a_2 = a_3 = \frac{1}{4}$ (all the other $a_j = 0$) and show that $|A_{yx}(1/4)|^2 = 0$.
2. the difference filter $a_0 = 1$, $a_1 = -1$ (all the other $a_j = 0$) and show that $|A_{yx}(0)|^2 = 0$.

Linear Filters

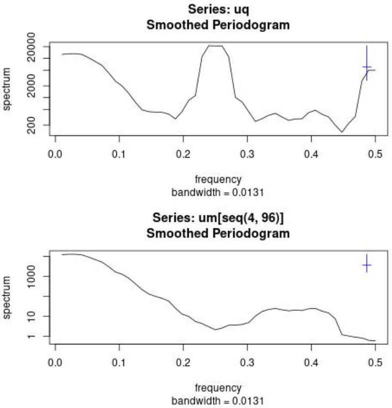
The quarterly unemployment series and its yearly moving average:
In R: `um=filter(uq,c(rep(1/4,4)),sides=1)`



The season is smoothed out!

Linear Filters

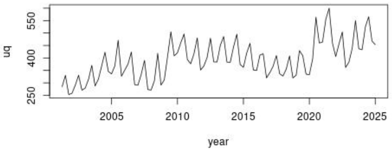
Corresponding non parametric spectral estimates, spans=4:

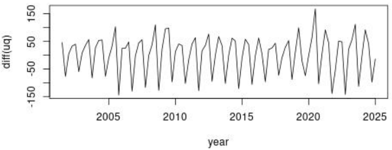


The peak at 0.25 has disappeared!

Linear Filters

The quarterly unemployment series and its difference:

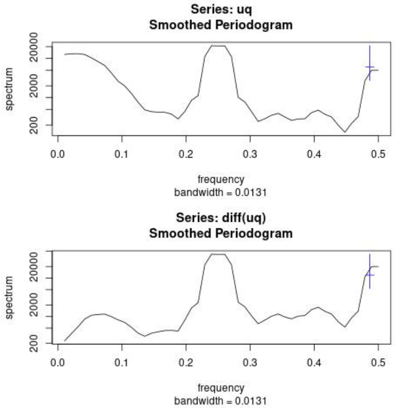




The trend is differenced out!

Linear Filters

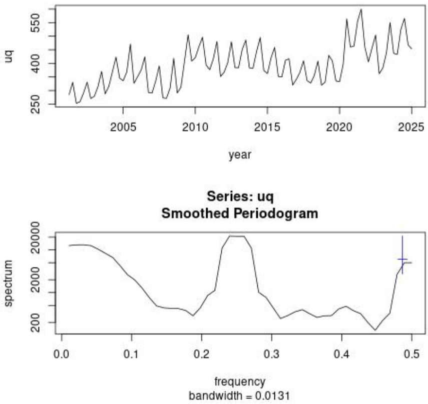
Corresponding non parametric spectral estimates, spans=4:



The peak at 0 has disappeared!

Wavelets

The unemployment series and its estimated periodogram (spans=4):



Different trend behavior for different years.

Wavelets

Chap. 2:

- "Exact" regression of x_t on periodic functions:

$$x_t = \sum_{j=1}^{n/2} \left\{ \hat{\beta}_{1j} \cos(2\pi\omega_j t) + \hat{\beta}_{2j} \sin(2\pi\omega_j t) \right\}$$

  where $\omega_j = j/n$.

- *The periodogram* consists of the estimated weights

$$P(\omega_j) = \hat{\beta}_{1j}^2 + \hat{\beta}_{2j}^2.$$

- The *basis functions* $\{cos(2\pi\omega_j t), \sin(2\pi\omega_j t)\}$ are periodic and *orthogonal*.
- $f(t)$ and $g(t)$ are orthogonal if $\int f(t)g(t)dt = 0$.

Wavelets

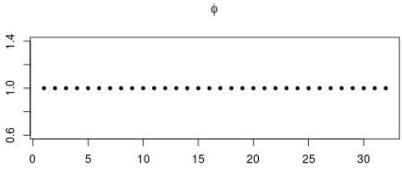
- A linear combination of orthogonal, *periodic* functions is not a good description of data if the trends or periodicities are not the same during the sampling interval.
- If so, it is better to use a linear combination of orthogonal, *non periodic functions*!
- *The wavelet transform* is

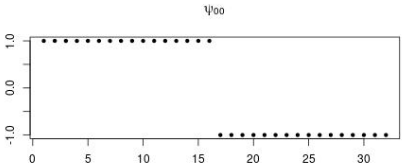
$$x_t = s\phi(t) + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(t),$$

  where $n = 2^m$ and $\phi(t)$ (father wavelet) and the $\psi_{jk}(t)$ (mother wavelets) together form an orthonormal basis.
- Truncating the sum gives a smoothed version of the series.
- One simple set of wavelets are the *Haar wavelets*, as illustrated below.

Wavelets

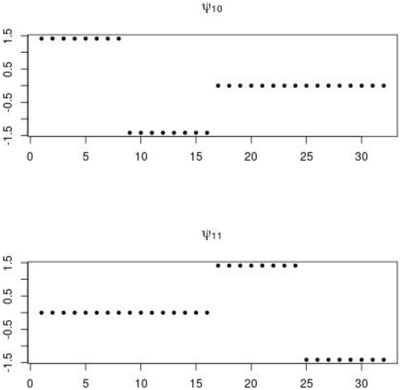
$n = 32$, Haar wavelets $\phi(t)$ and $\psi_{00}(t)$





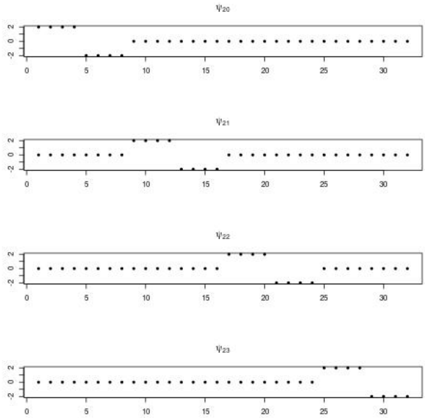
Wavelets

$n = 32$, Haar wavelets $\psi_{10}(t)$ and $\psi_{11}(t)$



Wavelets

$n = 32$, Haar wavelets $\psi_{20}(t)$, $\psi_{21}(t)$, $\psi_{22}(t)$ and $\psi_{23}(t)$



Wavelets

- $$x_t = s\phi(t) + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(t).$$

- Example with $n = 8 = 2^3$, Haar wavelet:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} s \\ d_{00} \\ d_{10} \\ d_{11} \\ d_{20} \\ d_{21} \\ d_{22} \\ d_{23} \end{pmatrix}$$

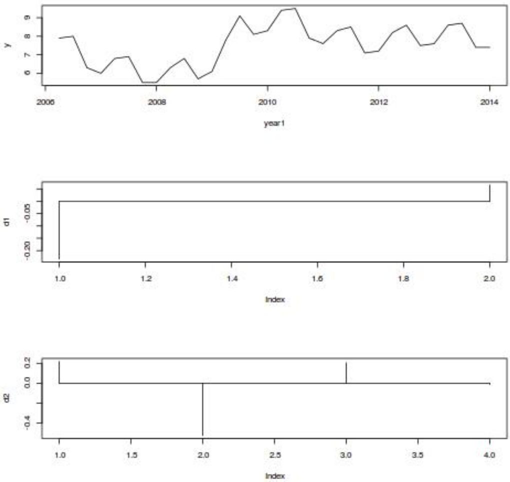
Wavelets

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} s \\ d_{00} \\ d_{10} \\ d_{11} \\ d_{20} \\ d_{21} \\ d_{22} \\ d_{23} \end{pmatrix}$$

$$\mathbf{x}^{(8\times1)} = \begin{pmatrix} \phi^{(8\times1)} & \psi_0^{(8\times1)} & \Psi_1^{(8\times2)} & \Psi_2^{(8\times4)} \end{pmatrix} \begin{pmatrix} s \\ d_{00} \\ \mathbf{d}_1^{(2\times1)} \\ \mathbf{d}_2^{(4\times1)} \end{pmatrix}$$

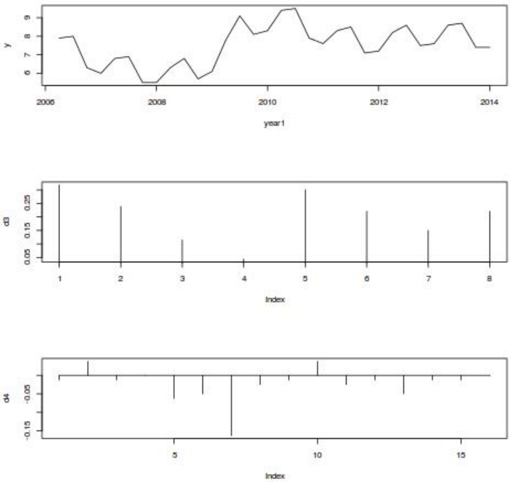
Wavelets

The unemployment series, 2006:1-2013:4, the data, $\mathbf{d_1}$ and $\mathbf{d_2}$:



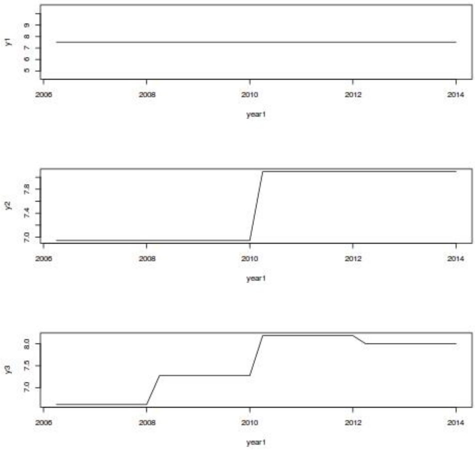
Wavelets

The unemployment series, 2006:1-2013:4, the data, $\mathbf{d_3}$ and $\mathbf{d_4}$:



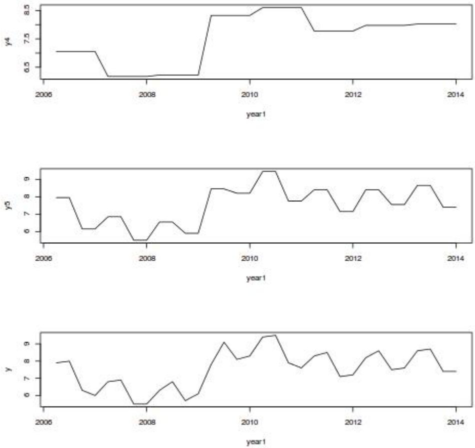
Wavelets

Unemployment, smoothed series with 1-3 components.



Wavelets

Unemployment, smoothed series with 4-6 components.



Wavelets

Some references:

- Shleicher, C. (2002) An Introduction to Wavelets for Economists, working paper.
- Crowley, P.M. (2007) A guide to Wavelets for economists, *Journal of Economic Surveys*, 21, 207-267.
- Zwanzig, S., Mahjani, B. (2020) *Computer Intensive Methods in Statistics*, CRC Press.

News of today

- Cross spectra, coherency
- Linear filters:
  - high-pass (e.g. differences)
  - low-pass (e.g. moving averages)
- Wavelets