Proposition: Arbitrage, i.e. the existence of a strategy with $V_0(\theta)=0$, $V_t(\theta)\geq 0$ for all t, and $\mathbb{E}(V_T(\theta))>0$; is equivalent to weak arbitrage, i.e. the existence of a strategy with $V_0(\theta)=0$, $V_T(\theta)\geq 0$ at time T, and $\mathbb{E}(V_T(\theta))>0$.

Proof: Arbitrage $\Rightarrow$ weak arbitrage is trivial and we only have to prove the converse.

Suppose θ is a portfolio that offers weak arbitrage but $V_t(\theta)$ is not (a.s.) non-negative for all t. Then there exists a time $t<T$ and an event $A\in\mathcal{F}_t$ with $\mathbb{P}(A)>0$ s.t.

$$V_t(\theta)(\omega)=(\theta_t\cdot S_t)(\omega)<0 \quad \text{for } \omega\in A.$$

We may assume t is the latest such time (discrete model!) and $V_u(\theta)(\omega)\geq 0$ a.s. for all $t<u\leq T$. We now construct a portfolio that offers (strong) arbitrage:

$$\varphi_u(\omega) = 0 \quad \text{for } \omega \notin A \text{ and all } u.$$

$$\varphi_u(\omega) = 0 \quad \text{for } \omega \in A \text{ and } u \le t$$

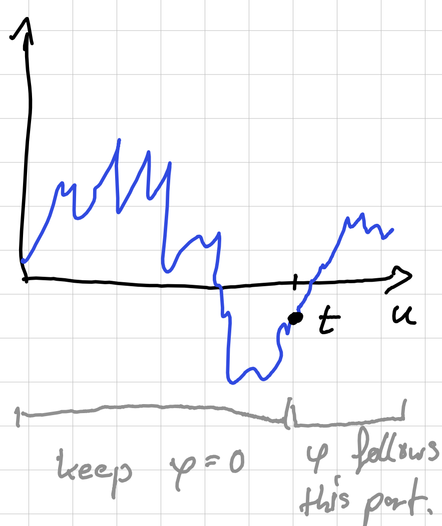
$$\left.\begin{aligned} \varphi_u^0(\omega) &= \theta_u^0(\omega) - \frac{\theta_t(\omega)\cdot S_t(\omega)}{S_t^0(\omega)} \\ \varphi_u^i(\omega) &= \theta_u^i(\omega) \end{aligned}\right\} \text{for } \omega \in A,\ u > t$$

Note that $\theta_t \cdot S_t < 0$ by assumption and so this term is positive.

This is pre-visible/predictible self-financing strategy.

We also have $V_t(\varphi) \ge 0$ for all t, $V_0(\varphi) = 0$ and $\mathbb{E}(V_T(\varphi)) > 0$. Thus we have arbitrage. $\square$

Pictorially,

A does not occur	A occurs
Do nothing, $V_t(\theta) \ge 0$ throughout	

Arbitrage Prices

H ... claim with maturity T

If the claim has a generating/replicating strategy θ (i.e. a self-financing strategy with $V_T(\theta)=H$), then we say H is attainable. In this case, $V_0(\theta)$ can be taken as a fair price for H.

We want to know if this price is unique.

Lemma: For all generating strategies of H, the associated value process is the same at all times (a.s.), provided the model is free of arbitrage.

Proof: Assume to the contrary and we have two strategies θ, φ s.t. $V_T(\theta) = H = V_T(\varphi)$ a.s. but $V_t(\theta) \neq V_t(\varphi)$ for some t with positive probability. WLOG we may assume $A = \{V_t(\theta) > V_t(\varphi)\} \in \mathcal{F}_t$ has positive probability. Now consider the strategy ψ with $\psi_u(\omega) = \theta_u(\omega) - \varphi_u(\omega)$ for all u if $\omega \notin A$ and for $\omega \in A$ let

$\psi_u(\omega) = \theta_u(\omega) - \varphi_u(\omega)$ for $u \le t$

$$\left.\begin{aligned} \psi_u^0(\omega) &= \frac{V_t(\theta) - V_t(\varphi)}{S_t^0} \\ \psi_u^i(\omega) &= 0 \quad ; i > 0 \end{aligned}\right\} \text{ for } u > t.$$

If A does not occur $V_T(\psi) = V_T(\theta) - V_T(\varphi) = 0$

If A occurs, positive difference is converted into cash and $V_T(\psi) > 0$. Hence there is weak arbitrage, contradicting the arbitrage-free assumption. $\square$

Thus the fair price can be uniquely defined by $V_0(\theta)$ (a.s.).

Martingales and Pricing

Recall: if M_t is a martingale, then

$$\mathbb{E}(M_t \mid \tilde{\mathcal{F}}_{t-1}) = M_{t-1} \iff \mathbb{E}(\underbrace{M_t - M_{t-1}}_{\Delta M_t} \mid \tilde{\mathcal{F}}_{t-1}) = 0$$

For predictable/previsible φ, we define the martingale transform $X = \varphi \bullet M$

$$X_t = \varphi_1 \Delta M_1 + \varphi_2 \Delta M_2 + \dots + \varphi_t \Delta M_t$$
$$= \sum_{k=1}^{t} \varphi_k (M_k - M_{k-1})$$

M_k martingale

Note that $\mathbb{E}(\varphi_k \Delta M_k \mid \tilde{\mathcal{F}}_{k-1}) = \varphi_k \mathbb{E}(\Delta M_k \mid \tilde{\mathcal{F}}_{k-1}) = 0$

φ predictable

Thus $\mathbb{E}(\varphi_k \Delta M_k) = 0 \Rightarrow \mathbb{E}(X_t) = 0$ for all $t \leq k$.

Now suppose we have a probability measure Q such that the discounted price process $\bar{S}_t$ becomes a martingale:

$$\mathbb{E}_Q(\Delta \bar{S}_t^i \mid \tilde{\mathcal{F}}_{t-1}) = 0 \text{ for all } i, t.$$

Equivalently, $\mathbb{E}_Q(\bar{S}_t^i \mid \tilde{\mathcal{F}}_{t-1}) = \bar{S}_{t-1}^i$ for all i, t.

Let θ be an admissible strategy. The discounted value process can be expressed as a martingale transform:

$$\overline{V}_t(\theta) = V_0(\theta) + \sum_{u=1}^{t} \theta_u \cdot \Delta \overline{S}_u$$

$$= \sum_{i=0}^{d} \theta_0^i S_0^i + \sum_{i=1}^{d} \left(\sum_{u=1}^{t} \theta_u^i \Delta \overline{S}_u^i \right)$$

we may ignore $i=0$ as this is the "cash" term which, when discounted is constant & $\Delta \overline{S}_u^0 = 0$.

Since $\mathbb{E}_Q \left(\sum_{u=1}^{t} \theta_u^i \Delta \overline{S}_u^i \right) = 0$ by our observations and the assumption that $\overline{S}_u$ is a martingale, we get $\mathbb{E}_Q(\overline{V}_t(\theta)) = \mathbb{E}_Q(V_0(\theta)) = V_0(\theta)$

Let $\mathbb{P}$ be the probability measure in our model. If $Q, \mathbb{P}$ are equivalent (written $\mathbb{P} \sim Q$, i.e. $\mathbb{P}$ and Q have the same null sets) then this rules out arbitrage:

Assume there is a portfolio θ such that $V_0(\theta) = 0$ but $V_T(\theta) \geq 0$ $\mathbb{P}$ almost surely. Then also $V_T(\theta) \geq 0$ Q-almost surely since Q and $\mathbb{P}$ are equivalent. Note also that $\mathbb{E}_Q(\overline{V}_T(\theta)) = \mathbb{E}_Q(V_0(\theta)) = V_0(\theta) = 0$.

From this it follows that $V_T(\theta) = 0$ Q-a.s. and so $\mathbb{P}$-a.s. In other words, there is no weak (and hence "full") arbitrage.

Q is called the equivalent martingale measure.

Proposition: If H is an attainable claim (i.e., H has a replicating strategy), then for any replicating strategy θ, we have

$$\overline{V}_t(\theta) = \mathbb{E}_Q(\beta H \mid \tilde{\mathcal{F}}_t) \text{ a.s. (wrt } P \text{ and } Q)$$

This follows by taking conditional expectations in

$$\overline{V}_t(\theta) = V_0(\theta) + \sum_{u=1}^{t} \theta_u \cdot \Delta \overline{S}_u$$

and using the martingale property of $\overline{S}$.

We can define the fair price of H by $\overline{V}_0(\theta)$:

$$\pi(H) = \overline{V}_0(\theta) = \mathbb{E}_Q(\beta_T H \mid \tilde{\mathcal{F}}_0) = \mathbb{E}_Q(\beta_T H)$$

Example: In the binomial model, the measure Q was determined by the probability q that turned

$$S_{t-1} \begin{cases} S_t\,(1+b) & \text{prob } 1-q \\ S_t\,(1+a) & \text{prob } q \end{cases}$$

into a martingale up to the discounting factor β.

$$\mathbb{E}\left(\bar{S}_t \mid \tilde{\mathcal{F}}_{t-1}\right) = q\beta\,\bar{S}_{t-1}\cdot(1+a) + (1-q)\beta(1+b)\,\bar{S}_{t-1}$$

q is then determined by the equation

$$1 = q\beta(1+a) + (1-q)\beta(1+b) = \frac{q}{1+r}(1+a) + (1-q)\frac{1+b}{1+r}$$

$$\Updownarrow$$

$$1+r = q(1+a) + (1+b) - q(1+b) = q(a-b) + 1+b$$

$$\Leftrightarrow \quad q = \frac{r-b}{a-b} = \frac{b-r}{b-a}$$

The price of a European call can then be expressed as $\mathbb{E}_Q\left(\beta_T(S_T-K)^+\right)$. Evaluating this expectation gives the Cox-Ross-Rubinstein formula.

Note that the formula requires the existence of a replicating strategy, even if not explicitly referenced. In complete market models, all European contingent claims have a replicating strategy and can be priced this way.

Uniqueness of equivalent martingale measures.

In principle, the martingal measure may not be unique. Suppose Q R are two equivalent measures in a complete model. Then,

$$\mathbb{E}_Q(\beta_T H) = \mathbb{E}_R(\beta_T H) \quad \text{for all } H$$

$$\Rightarrow \quad \mathbb{E}_Q(H) = \mathbb{E}_R(H) \quad \text{as fair price is}$$

unique in absence of arbitrage.

So, $\mathbb{E}_Q(I_A) = \mathbb{E}_R(I_A)$ for all indicators $I_A = H$

and $Q(A) = R(A)$ for all A.