Problem Session 5

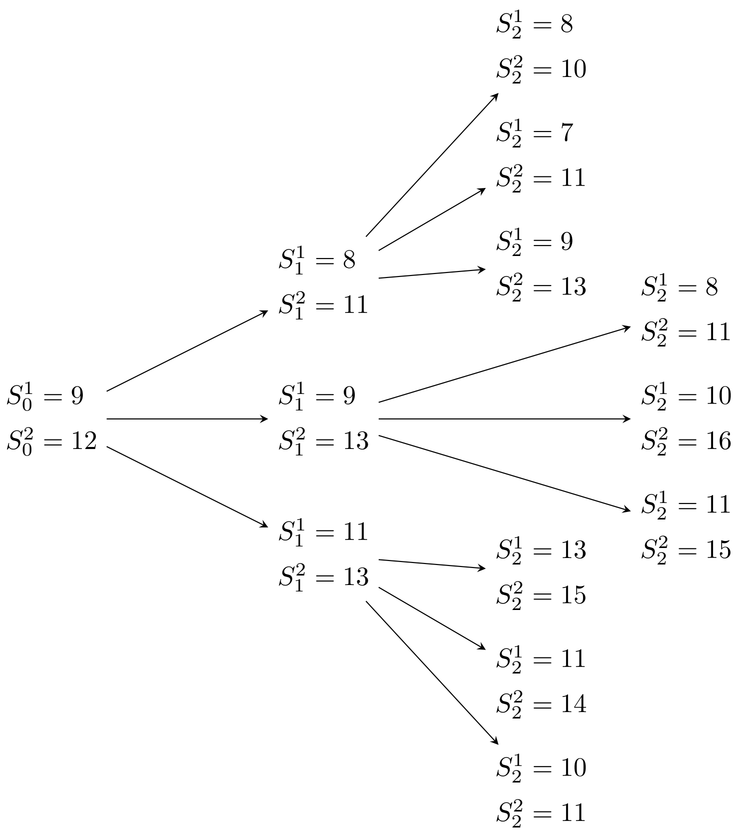
Probability and Martingales, 1MS045

19 December 2024

Note: If not specified otherwise, all random variables are real-valued, with the usual σ-algebra of Borel sets.

Problems

1. Consider a market model with two assets S_t^1 and S_t^2 and two time steps as indicated in the diagram below. The riskless interest rate is assumed to be 0. Determine a martingale measure Q for this model.



2. Consider a market model with a single assets S_t and two time steps as indicated in the diagram below. The riskless bond is assumed to be S_t^0. The Claim C has a payoff of 1

if $S_2 = 24$ at the second time step. Otherwise the payoff is 0. Determine a replicating strategy θ for this claim, as well as its fair price.

$$\begin{tikzcd}
& & S_2^1 = 15 \\
& S_1 = 16 \arrow[ur] \arrow[r] & S_2^1 = 18 \\
S_0 = 20 \arrow[ur] \arrow[dr] & & \\
& S_1 = 22 \arrow[r] \arrow[dr] & S_2^1 = 21 \\
& & S_2^1 = 24
\end{tikzcd}$$

3. Consider a binomial model with $S_0 = 10$, $a = -0.1$, $b = 0.2$, and $r = 0.05$. Determine the fair price of a European call option and a European put option, both with a strike price of $K = 12$ and exercise date $T = 6$.

4. In the lecture, we derived the following formula for the price of a European call option under the Black–Scholes model:

$$\int_{-\infty}^{\infty} \left(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x} - e^{-rT} K \right)^+ \cdot \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

   Show that this is equivalent to

$$S_0 \Phi(d_+) - e^{-rT} K \Phi(d_-),$$

   where Φ is the cumulative distribution function of a standard normal distribution and

$$d_\pm = \frac{\log S_0 - \log K + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

5. Derive the following bounds for the fair price $P_0(E)$ of a European put option with underlying stock S_t, strike price K, exercise date T, and discounting factor β^T using the arbitrage arguments

$$\max\{0, \beta^T K - S_0\} \leq P_0(E) \leq \beta^T K.$$

6. Consider the *trinomial model* with only one asset S_t whose price process is determined by $S_t = R_t \cdot S_{t-1}$, where $R_t \in \{1+a, 1+b, 1+c\}$ with probabilities p_a, p_b, p_c, respectively (with $p_a + p_b + p_c = 1$). The riskless bond is given by $S_t^0 = (1+r)^t$.

   (a) When is the model viable?

   (b) When is the model complete?

7. Consider the binomial model with two assets S_t^1 and S_t^2. Their price processes are determined by

$$S_t^1 = R_t^1 \cdot S_{t-1}^1 \quad \text{and} \quad S_t^2 = R_t^2 \cdot S_{t-1}^2,$$

   where either $R_t^1 = 1+a, R_t^2 = 1+a'$ with probability p_a or $R_t^1 = 1+b, R_t^2 = 1+b'$ with probability $p_b = 1 - p_a$. The riskless bond is given by $S_t^0 = (1+r)^t$.

   (a) When is the model viable?

(b) When is it complete?

8. Prove that the set of all equivalent martingale measures of a market model is always convex, i.e. for any two equivalent martingale measures Q and Q', and every $\lambda \in [0,1]$, the convex combination $\lambda Q + (1-\lambda)Q'$ is also an equivalent martingale measure.