Conformal mappings

Let D be a domain in $\mathbb{C}$, $z_0 \in D$.

Suppose $f : D \to \mathbb{C}$ analytic with $f'(z_0) \neq 0$

Let $\gamma(t) = x(t) + i y(t)$ be a C^1-curve in D through $z_0 = \gamma(0)$ with $\gamma'(0) \neq 0$.

Then $(f \circ \gamma)(t) = f(\gamma(t))$ is a C^1-curve through $(f\circ\gamma)(0) = f(z_0)$. Moreover,

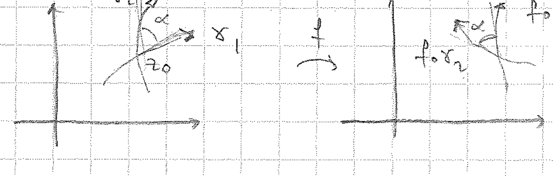
$$(f\circ\gamma)'(0) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}$$

$$= \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\underbrace{\gamma(t) - \gamma(0)}_{\neq 0 \text{ for small } t}} \cdot \frac{\gamma(t) - \gamma(0)}{t} = f'(z_0)\gamma'(0)$$

i.e. $(f\circ\gamma)'(0) = f'(z_0)\gamma'(0)$ is a tangent vector to $f\circ\gamma$ at $f(z_0)$. Note that

$$\arg (f\circ\gamma)'(0) = \underset{\uparrow \text{ same for any } \gamma}{\arg f'(z_0)} + \arg \gamma'(0)$$

(1) So if γ_1 and γ_2 are two C^1-curves (as above) which intersect at z_0, then the angle from $(f\circ\gamma_1)'(0)$ to $(f\circ\gamma_2)'(0)$ is the same as the angle from $\gamma_1'(0)$ to $\gamma_2'(0)$.



Def A C^1-mapping $f: D \to \mathbb{C}$ is said to be conformal at z_0 if it satisfies (1).

If f maps the domain D bijectively onto V and if f is conformal at each point $z_0 \in D$, we shall call $f: D \to V$ a conformal mapping.

We have proven:

Thm If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.

Remark: One can in fact prove a converse of this theorem. See the theorem on page 126 in the book.

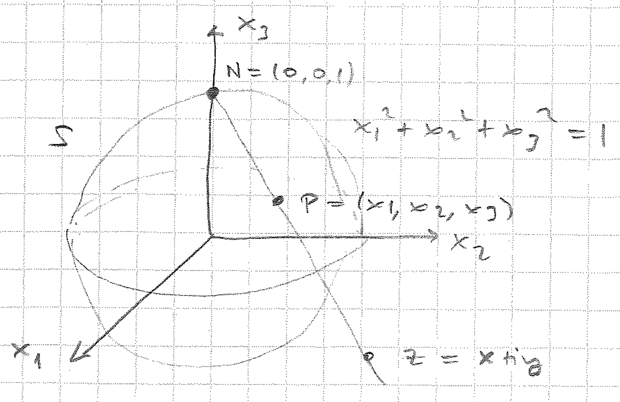
See also Simon book on page 36

Ex 1) $f(z) = e^z$ is conformal at every point $z \in \mathbb{C}$, since $f'(z) = e^z \neq 0$.

2) $f(z) = z^2$ is conformal at every point $z \in \mathbb{C} \setminus \{0\}$, since $f'(z) = 2z \neq 0$ if $z \neq 0$.

Stereographic projection

Consider the unit sphere S in $\mathbb{R}^3$ (the Riemann sphere)



Given any point $P=(x_1,x_2,x_3)\in S$ other than the north pole $N=(0,0,1)$ we draw the line through N and P. We define the stereographic projection of P to be the point $z=x+iy\in\mathbb{C}\sim(x,y,0)$ where the line intersects the plane $x_3=0$. Clearly

$$(x,y,0)=(0,0,1)+t\left[(x_1,x_2,x_3)-(0,0,1)\right]$$

where t is given by $1+t(x_3-1)=0 \Leftrightarrow t=\dfrac{1}{1-x_3}$

I.e.

$$z=x+iy=\frac{x_1+ix_2}{1-x_3} \qquad (2)$$

Conversely, given $z=x+iy\in\mathbb{C}\sim(x,y,0)$ the line through N and z is given by

$$(x_1,x_2,x_3)=(0,0,1)+t\left[(x,y,0)-(0,0,1)\right],\ t\in\mathbb{R},$$

It intersect S when $x_1^2+x_2^2+x_3^2=1$

$$\Leftrightarrow \quad (tx)^2 + (ty)^2 + (1-t)^2 = 1$$

$$\Leftrightarrow \quad t^2(x^2+y^2+1) - 2t = 0 \Leftrightarrow$$

$$\Leftrightarrow \quad t = 0 \quad \text{or} \quad t = \frac{2}{x^2+y^2+1} = \frac{2}{|z|^2+1}$$

This corresponds to $P = N$ or

$1 - \frac{2}{|z|^2+1}$

$$P = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

Thus, stereographic projection $s : S \setminus N \to \mathbb{C}$ defines a bijection. Letting $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the <u>extended complex plane</u> and defining $s(N) = \infty$, s becomes a bijective map from S onto $\hat{\mathbb{C}}$.

<u>Thm</u> Under stereographic projection circles on S correspond to circles and lines in $\mathbb{C}$

<u>Remark:</u> We therefore call circles and lines in $\mathbb{C}$ "circles" in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where lines are considered as "circles through ∞".

<u>Proof:</u> The general eq. for a circle or line in the $z = x+iy$ plane is

$$A(x^2+y^2) + Cx + Dy + E = 0$$

Using (2) we get

$$A\left[\left(\frac{x_1}{1-x_3}\right)^2+\left(\frac{x_2}{1-x_3}\right)^2\right]+\frac{Cx_1}{1-x_3}+\frac{Dx_2}{1-x_3}+E=0$$

$$\Leftrightarrow \quad A(x_1^2+x_2^2)+Cx_1(1-x_3)+Dx_2(1-x_3)+E(1-x_3)^2=0$$

Using $x_1^2+x_2^2+x_3^2=1$ we get

$$A(1-x_3^2)+Cx_1(1-x_3)+Dx_2(1-x_3)+E(1-x_3)^2=0$$

Dividing by $1-x_3$ $\Rightarrow$

$$A(1+x_3)+Cx_1+Dx_2+E(1-x_3)=0$$

$$\Leftrightarrow \quad Cx_1+Dx_2+(A-E)x_3+A+E=0$$

This is the eq. for a plane in $\mathbb{R}^3$; which clearly intersects the Riemann-sphere in a circle $\square$

Möbius transformations

Def. A Möbius transformation is a mapping of the form

$$T(z)=\frac{az+b}{cz+d} \qquad (a,b,c,d\in\mathbb{C})$$

where $ad-bc\neq 0$. (so that T is not constant)

If $c=0$ we let $T(\infty)=\infty$.

Then $T:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ is bijective.

If $c \neq 0$, then

$$T : \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\} \to \mathbb{C} \setminus \left\{ \frac{a}{c} \right\}$$

is a bijection. Letting $T\left(-\frac{d}{c}\right) = \infty$ and $T(\infty) = \frac{a}{c}$

we extend T to a bijective map $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

The inverse is found by solving

$$w = T(z)$$

which gives :

$$z = T^{-1}(w) = \begin{cases} \dfrac{-dw + b}{cw - a}, & \text{if } w \neq \frac{a}{c}, w \neq \infty \\ \infty, & \text{if } w = a/c \\ -d/c, & \text{if } w = \infty \end{cases}$$

(this also hold for $c = 0$ if we interpret a/c and $-d/c$ as ∞)

Note that

1) $T'(z) = \dfrac{d}{dz}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{a(cz+d) - (az+b)\cdot c}{(cz+d)^2} =$

$$= \frac{ad - bc}{(cz+d)^2} \neq 0$$

Thus, $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conformal.

2) If $T(z) = \dfrac{az+b}{cz+d}$, $S(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$

$$\Rightarrow \quad (S \circ T)(z) = \frac{\alpha T(z) + \beta}{\gamma T(z) + \delta} =$$

$$= \frac{\alpha \frac{az+b}{cz+d} + \beta}{\gamma \frac{az+b}{cz+d} + \delta} = \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)}$$

$\Rightarrow$ compositions of Möbius transformations are Möbius transformations

Note: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta \delta & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$

<u>Lemma</u> If a Möbius transformation T has more than two fixed points in $\hat{\mathbb{C}}$ (z_0 f.p. if $T(z_0) = z_0$) then $T(z) = z \quad \forall z \in \hat{\mathbb{C}}$.

<u>Proof</u> If $c = 0$ then $T(z) = \frac{az+b}{d}$, so

$$T(z) = z \iff \frac{az+b}{d} = z \iff (a-d)z + b = 0$$

So T has at most one fixed point in $\mathbb{C}$ unless $a = d$ and $b = 0 \iff T(z) = z \quad \forall z \in \mathbb{C}$

So if $c = 0$, T has at most 2 fixed points in $\hat{\mathbb{C}}$ ($T(\infty) = \infty$) unless $T(z) = z \quad \forall z \in \mathbb{C}$.

If $c \neq 0$ then $T(z) = z \iff \frac{az+b}{cz+d} = z$

$\iff cz^2 + (d-a)z - b = 0$

So T has at most 2 fixed points in $\mathbb{C}$

(and $T(\infty) = \frac{a}{c} \neq \infty$) unless $c = 0$, $a = d$, $b = 0$.

But this contradicts $c \neq 0$.

$\square$

<u>Prop</u> If S, T are Möbius transformations s.t. $S(z_i) = T(z_i)$ at three different points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, then $S = T$.

<u>Proof.</u> If $S(z_i) = T(z_i)$, $i = 1,2,3$, then the Möbius transformation $T^{-1} \circ S$ has at least 3 fixed points. So by the lemma $T^{-1} \circ S(z) = z \;\; \forall z \in \hat{\mathbb{C}}$, i.e. $S(z) = T(z) \;\; \forall z \in \hat{\mathbb{C}}$.

$\square$