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Financial Derivatives
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Each problem gives at most 5 points. To pass the course (grade 3), a total of 18 points are needed. The limits for higher grades (4 and 5) are 25 and 32 points. No means of assistance other than pencil and paper are allowed. Motivate your answers carefully!

1. Solve the stochastic differential equation

$$\begin{cases} dX(t) = 3t^2 X(t)\,dt + tX(t)\,dW(t) \\ X(0) = x, \end{cases}$$

where $x > 0$, and determine $\mathbb{E}[X(t)]$.

2. Solve the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x,y) + 2\frac{\partial^2 u}{\partial x^2}(t,x,y) + 2\frac{\partial^2 u}{\partial y^2}(t,x,y) + \frac{\partial^2 u}{\partial x \partial y}(t,x,y) + \frac{\partial u}{\partial x}(t,x,y) = 0 \\ u(T,x,y) = (x+y)^2 \end{cases}$$

on $[0,T] \times \mathbb{R}^2$.

3.

(i) If W_1 and W_2 are Brownian motions with instantaneous correlation ρ (i.e. $dW_1(t)dW_2(t) = \rho\,dt$), can one find a constant $b > 0$ such that the process $W(t) := b(W_1(t) - 3W_2(t))$ is a Brownian motion? Motivate your answer.

(ii) Explain briefly the following notions:
  (a) completeness of a model;
  (b) implied volatility;
  (c) inversion of the yield curve.

4. In the standard Black-Scholes model with volatility σ and interest rate r, determine the arbitrage-free price at time $t = 0$ of the "all-or-nothing" T-claim

$$\mathcal{X} = \begin{cases} K & \text{if } S(T) \leq A \\ 0 & \text{if } S(T) > A. \end{cases}$$

5. Consider a market consisting of a bank account with a constant interest rate r and a stock S with initial stock price $S(0) = s$ and constant volatility $\sigma > 0$. Moreover, the stock pays a continuous dividend $\delta S(t)dt$ during each interval $[t, t + dt]$. Consider a T-claim that pays $\mathcal{X} = S(T)$ at time T.

a) What is the arbitrage-free price of $\mathcal{X}$ at time 0?

b) Find a replicating strategy for $\mathcal{X}$.

6. Your broker quotes the price 4 for a call option on a (non-dividend paying) asset S, with strike price $K = 100$ and maturity T. Consider a so-called *straddle* which at time T pays the amount

$$\mathcal{X} = \begin{cases} 100 - S(T) & \text{if } S(T) \leq 100 \\ S(T) - 100 & \text{if } S(T) > 100. \end{cases}$$

Assume that a zero-coupon T-bond with face value 50 trades at 49, and that the current stock price is $S(0) = 95$. Show how to construct an arbitrage if the price of the straddle is 10.

7. An agent models the short rate by

$$dr(t) = \sigma(t)\, dW(t)$$

under the pricing measure, where $\sigma(t)$ is a deterministic function. Determine the term structure $\{p(0,T), T \geq 0\}$, i.e. bond prices $p(0,T)$ for all maturities $T \geq 0$.

8. Consider a currency derivative that gives the right (but not the obligation) to buy 100 USD at a given price K SEK at a given future date T. Assume that the US interest rate r_f, the Swedish (domestic) interest rate r_d and the volatility σ of the exchange rate are positive constants. Which of the following statements are correct? Motivate your answers.

The value of the currency derivative is

(i) increasing in K;
(ii) increasing in r_f;
(iii) increasing in σ.

GOOD LUCK!