Exam, Real analysis, 1MA226, 2020-06-15.
Solution suggestions.

1. **Answer:** Yes, A is closed.

Proof: We know that multiplication is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, i.e. the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$, is continuous (cf. Theorem 4.9). Also $\{1\}$ is a closed subset of $\mathbb{R}$. But note that $A = f^{-1}(\{1\})$; hence by Corollary 4.8, A is a closed set in $\mathbb{R}^2$. $\square$

(Alternatively one may of course give a direct proof, working from Def. 2.18(d).)

2. (a). For n even we get $x_n = -1 + 1 - \cdots + 1 = 0$, while for n odd we get $x_n = -1 + 1 - \cdots - 1 = -1$. It follows that

$$\limsup_{n\to\infty} x_n = 0.$$

and

$$\liminf_{n\to\infty} x_n = -1.$$

(b). We consider $y_n = \log x_n$ instead. By Taylor expansion we have:

$$\begin{aligned} y_n &= n \log\Big(1 + \frac{1}{n^{1/2}}\Big) + n^2 \log\Big(1 + \frac{(-1)^n}{n^{3/2}}\Big) \\ &= n\Big(\frac{1}{n^{1/2}} - \frac{1}{2n} + O\Big(\frac{1}{n^{3/2}}\Big)\Big) + n^2\Big(\frac{(-1)^n}{n^{3/2}} + O\Big(\frac{1}{n^3}\Big)\Big) \\ &= \big(1 + (-1)^n\big)n^{1/2} - \frac{1}{2} + O\Big(\frac{1}{n^{1/2}}\Big). \end{aligned}$$

It follows that for n *even* we have $y_n \to +\infty$ (and thus $x_n \to +\infty$) as $n \to \infty$, while for n *odd* we have $y_n \to -\frac{1}{2}$ (and thus $x_n \to e^{-1/2}$) as $n \to \infty$.

Answer: $\liminf_{n\to\infty} x_n = e^{-1/2}$ and $\limsup_{n\to\infty} x_n = +\infty$.

3. Let a be any positive real number. Note that for all $x \in [-a, a]$ and all $n \in \mathbb{Z}^+$ we have:

$$\left|\frac{x^3+n}{x^2+n^3}\right| \leq \frac{a^3+n}{n^3} \leq \frac{a^3}{n^3} + \frac{1}{n^2}.$$

Furthermore the series $\sum_{n=1}^{\infty}(\frac{a^3}{n^3} + \frac{1}{n^2})$ converges. Hence by Weierstrass' M-test, we conclude that the series defining $F(x)$ is uniformly convergent on $[-a, a]$.

Hence it follows that F is well-defined and continuous in the interval $[-a, a]$. Since this is true for any $a > 0$, it follows that F is well-defined and continuous on the whole real axis.

Next consider the series obtained by formally differentiating the series for $F(x)$ term by term, i.e.:

$$\sum_{n=1}^{\infty} \frac{3x^2(x^2+n^3) - (x^3+n)2x}{(x^2+n^3)^2}. \tag{1}$$

We claim that this series is uniformly convergent on any interval $[-a, a]$ with $a > 0$. Indeed, for all $n \in \mathbb{Z}^+$ and $x \in [-a, a]$ we have

$$\begin{aligned} \left|\frac{3x^2(x^2+n^3) - (x^3+n)2x}{(x^2+n^3)^2}\right| &\leq \frac{\left|3x^4 + 3x^2n^3 - 2x^4 - 2nx\right|}{n^6} \\ &\leq \frac{\left|3x^4\right| + \left|3x^2n^3\right| + \left|2x^4\right| + \left|2nx\right|}{n^6} \\ &\leq \frac{3a^4 + 3a^2n^3 + 2a^4 + 2an}{n^6}. \end{aligned}$$

Furthermore, the series $\sum_{n=1}^{\infty} \frac{3a^4+3a^2n^3+2a^4+2an}{n^6}$ converges. Hence by Weierstrass' M-test, we conclude that the series in (1) is indeed uniformly convergent on $[-a, a]$. Hence by Rudin's Thm. 7.17, we have that $F'(x)$ exists for all $x \in [-a, a]$, and

$$F'(x) = \sum_{n=1}^{\infty} \frac{3x^2(x^2+n^3) - (x^3+n)2x}{(x^2+n^3)^2}.$$

The uniform convergence pointed out above (together with the fact that each term is a continuous function of x) implies that this function is continuous in $[-a, a]$. Hence F is C^1 in $[-a, a]$. Since this is true for any $a > 0$, we conclude that F is C^1 on the whole real line. □

4. Let $\phi : [1, e] \to [1, e]$ be the map $\phi(x) = e^{f(x)}$. (Note that we have $f(x) \in [0, 1]$ for every $x \in [1, e]$, and hence indeed $\phi(x) \in [1, e]$.)

By the Mean Value Theorem applied to the function $x \mapsto e^x$, for any two real numbers $\alpha < \beta$ there exists some $\xi \in (\alpha, \beta)$ such that $e^\beta - e^\alpha = e^\xi(\beta - \alpha)$. Hence if $0 \le \alpha < \beta \le 1$ then $0 \le e^\beta - e^\alpha \le e(\beta - \alpha)$. Of course this also holds if $\alpha = \beta$. Hence, allowing also the symmetric case $\beta < \alpha$, we conclude that:

$$\left|e^\beta - e^\alpha\right| \le e\left|\beta - \alpha\right|, \qquad \forall \alpha, \beta \in [0, 1].$$

Now for any $x, y \in [1, e]$ we have $f(x), f(y) \in [0, 1]$, and hence by the above:

$$\left|\phi(x) - \phi(y)\right| = \left|e^{f(x)} - e^{f(y)}\right| \le e\left|f(x) - f(y)\right|.$$

The above is $\le e \cdot \frac{1}{3}\left|x - y\right|$, by the assumption in the problem formulation. Hence we have proved:

$$\left|\phi(x) - \phi(y)\right| \le \frac{e}{3}\left|x - y\right|, \qquad \forall x, y \in [1, e].$$

Since $\frac{e}{3} < 1$, this proves that ϕ *is a contraction of* $[1, e]$ *into* $[1, e]$.

We also know that $[1, e]$ as a metric space is complete (since $[1, e]$ is a closed subset of $\mathbb{R}$ and $\mathbb{R}$ with its standard metric is complete). Hence by the Banach Contraction Principle, there is a unique $x \in [1, e]$ such that $\phi(x) = x$. But note that for any $x \in [1, e]$ we have the following chain of equivalences: $\phi(x) = x \Leftrightarrow e^{f(x)} = x \Leftrightarrow f(x) = \log x$. Hence we have proved that there is a unique $x \in [1, e]$ such that $f(x) = \log x$. $\square$

5. Let $\varepsilon > 0$ be given. The set C is a closed and bounded subset of $\mathbb{R}^2$, hence compact. Hence by Theorem 4.19, f is uniformly continuous. Hence there exists $\delta > 0$ such that $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$ whenever (x_1, y_1) and (x_2, y_2) are points in C with $|(x_1, y_1) - (x_2, y_2)| < \delta$. Now for any $n \in \mathbb{Z}^+$ and any points $x, x' \in [0, 1]$ with $|x - x'| < \delta$, we have $|(x, a_n) - (x', a_n)| < \delta$ and hence

$$|f_n(x) - f_n(x')| = |f(x, a_n) - f(x', a_n)| < \varepsilon.$$

Hence the sequence (f_n) is equicontinuous. $\square$

6. Using the fact that the function $\alpha \mapsto \alpha^3$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$, with inverse $\beta \mapsto \sqrt[3]{\beta}$, it follows that for any $x, y, t, s \in \mathbb{R}$, the following chain of equivalences holds:

$$f(x,y) = (t,s) \Leftrightarrow \begin{cases} x = t \\ (x+y)^3 = s \end{cases} \Leftrightarrow \begin{cases} x = t \\ x+y = \sqrt[3]{s} \end{cases}$$
$$\Leftrightarrow (x,y) = (t, \sqrt[3]{s} - t).$$

Hence f is a *bijection* from $\mathbb{R}^2$ onto $\mathbb{R}^2$, with inverse function

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad g(s,t) = (t, \sqrt[3]{s} - t).$$

Note that the function g is continuous; hence for *every* open set $U \subset \mathbb{R}^2$ we have that $V := f(U) = g^{-1}(U)$ is open, by Theorem 4.8; also $f|_U$ is a bijection from U onto V. Hence we have solved the first part of the problem by proving that *every* open set $U \subset \mathbb{R}^2$ with $(0,0) \in U$ satisfies the desired conclusion!

[Alternative solution to the first part of the problem, perhaps easier to come up with: Simply choose $U = \mathbb{R}^2$ and $V = \mathbb{R}^2$; then V is open and f is a bijection from U onto V.]

We now turn to the last part of the problem. Thus assume that $U \subset \mathbb{R}^2$ is *any* open set containing $(0,0)$, and that $f|_U$ is a bijection from U onto the open set $V \subset \mathbb{R}^2$. By the above discussion we have $(f|_U)^{-1} = g$ on all V. Also $(0,0) = f(0,0) \in V$. Hence it suffices to prove that g is not differentiable at $(0,0)$. However, by Theorem 9.17, if g were differentiable at $(0,0)$ then the partial derivatives of the functions $g_1(t,s) = t$ and $g_2(t,s) = \sqrt[3]{s} - t$ with respect to t and s would exist at $(t,s) = (0,0)$; but we know that $\frac{\partial}{\partial s}(\sqrt[3]{s} - t)\big|_{(t,s)=(0,0)}$ does *not* exist. [Proof: By definition,

$$\frac{\partial}{\partial s}(\sqrt[3]{s} - t)\bigg|_{(t,s)=(0,0)} = \lim_{h \to 0} \frac{(\sqrt[3]{h} - 0) - (\sqrt[3]{0} - 0)}{h} = \lim_{h \to 0} h^{-2/3},$$

and this limit does not exist.] Hence g is not differentiable at the point $(0,0) \in V$. $\square$

7. For any $m \geq 2$ we let P_m be the partition of $[0,1]$ determined by the following numbers:

$$0 < 2^{-m} - 2^{-2m} < 2^{-m} + 2^{-2m} < 2^{1-m} - 2^{-2m} < 2^{1-m} + 2^{-2m} < \cdots$$
$$< 2^{-1} - 2^{-2m} < 2^{-1} + 2^{-2m} < 1.$$

[Verification that all the above inequalities really hold: This is obvious except for the inequalities of the form $2^{j-m} + 2^{-2m} < 2^{j+1-m} - 2^{-2m}$ for $j \in \{0, 1, \ldots, m-2\}$. However we have $2^{j-m} + 2^{-2m} < 2^{j+1-m} - 2^{-2m}$ $\Leftrightarrow 2 \cdot 2^{-2m} < 2^{j-m} \Leftrightarrow 1 - 2m < j - m \Leftrightarrow 1 < j + m$, and the last inequality is clearly true for every $j \in \{0, 1, \ldots, m-2\}$. Done!]

Note that the function f is *identically equal to* 3 on every interval in the partition P_m except for the intervals

$$(*) \qquad \big[0, 2^{-m} - 2^{-2m}\big]$$

and

$$(**) \qquad \Big[2^{-j} - 2^{-2m}, 2^{-j} + 2^{-2m}\Big] \qquad (j = 1, 2, \ldots, m).$$

On the interval in (*), the function f takes the values 3 and 2^{-m-k} for all $k \geq 1$, but no other values. On each interval in (**), the function f takes the values 3 and 2^{-j}, and no other values. Note that the total length of the intervals in (*) and (**) is $2^{-m} - 2^{-2m} + \sum_{j=1}^{m} 2^{1-2m} = 2^{-m} + (2m-1)2^{-2m}$; hence the total length of the remaining intervals is $1 - 2^{-m} - (2m-1)2^{-2m}$. Hence

$$\begin{aligned} L(P_m, f) &= \sum_i m_i \Delta x_i \\ &= 0 \cdot (2^{-m} - 2^{-2m}) + \sum_{j=1}^{m} 2^{-j} \cdot 2^{1-2m} + 3 \cdot (1 - 2^{-m} - (2m-1)2^{-2m}) \\ &= (1 - 2^{-m})2^{1-2m} + 3\big(1 - 2^{-m} - (2m-1)2^{-2m}\big) \end{aligned}$$

and

$$U(P_m, f) = \sum_i M_i \Delta x_i = 3.$$

From this we conclude:

$$\lim_{m \to \infty} L(P_m, f) = 3 \qquad \text{and} \qquad \lim_{m \to \infty} U(P_m, f) = 3.$$

Hence by Rudin's Theorem 6.6 (and its proof) it follows that f is Riemann integrable on $[0,1]$, and that $\int_0^1 f(x)\,dx = 3$. Hence also $\overline{\int_0^1} f(x)\,dx = \underline{\int_0^1} f(x)\,dx = 3$.

Answer: $\overline{\int_0^1} f(x)\,dx = \underline{\int_0^1} f(x)\,dx = 3.$

$\square$

8. By Theorem 4.19 (and since $[0,1]$ is compact), the function f is uniformly continuous. This implies that there exists some $\delta > 0$ such that $d(f(x), f(y)) < \frac{1}{100}$ for all $x, y \in [0,1]$ with $|x-y| < \delta$. In view of the assumption in the problem formulation, this means that

$$\forall x, y \in [0,1]: \quad |x-y| < \delta \Rightarrow f(x) = f(y).$$

Now let n be a positive integer which is so large that $\frac{1}{n} < \delta$. Then $\frac{k+1}{n} - \frac{k}{n} = \frac{1}{n} < \delta$ for all $k \in \{0, 1, \ldots, n-1\}$, and hence

$$f(0) = f\Big(\frac{1}{n}\Big) = f\Big(\frac{2}{n}\Big) = \cdots = f(1).$$

$\square$

Alternative: Assume that $f(0) \neq f(1)$. Let $E = \{x \in [0,1] : f(x) \neq f(0)\}$. Then $1 \in E$; in particular $E \neq \emptyset$. Also 0 is a lower bound of E. Hence $x_0 := \inf E$ exists, and $x_0 \in [0,1]$. Since f is continuous there exists $r > 0$ such that $d(f(x), f(x_0)) < \frac{1}{10}$ for all $x \in N_r(x_0) \cap [0,1]$. Using the assumption in the problem formulation, this implies that

$$f(x) = f(x_0), \qquad \forall x \in N_r(x_0) \cap [0,1]. \tag{2}$$

By the definition of infimum, $x_0 + r$ is not a lower bound of E; hence there exists some $x_1 \in E$ with $x_1 < x_0 + r$. This x_1 must satisfy $x_0 \leq x_1$, since x_0 is a lower bound of E. Hence $x_1 \in N_r(x_0)$; also $x_1 \in [0,1]$ since $x_1 \in E$; hence by (2) we have $f(x_0) = f(x_1)$. But we have $f(x_1) \neq f(0)$ since $x_1 \in E$; hence

$$f(x_0) \neq f(0). \tag{3}$$

This implies in particular $x_0 \neq 0$, i.e. $x_0 \in (0,1]$. Now since x_0 is a lower bound of E, every $x \in [0, x_0)$ satisfies $x \notin E$, i.e. $f(x) = f(0)$. However there exists some $x \in [0, x_0) \cap N_r(x_0)$, and by (2) this x satisfies $f(x_0) = f(x) = f(0)$, contradicting (3). $\square$