(4.1) a) $dZ_t = \alpha Z_t \, dt$

b) $dZ_t = g_t \, dW_t$

c) $dZ_t = \frac{\alpha^2}{2} Z_t \, dt + \alpha Z_t \, dW_t$

d) $dZ_t = (\alpha\mu + \frac{1}{2}\alpha^2\sigma^2) Z_t \, dt + \alpha\sigma Z_t \, dW_t$

e) $dZ_t = (2\alpha + \sigma^2) Z_t \, dt + 2\sigma Z_t \, dW_t$

(4.2) $dZ_t = (\sigma^2 - \alpha) Z_t \, dt - \sigma Z_t \, dW_t$

(4.4) $E[X_t] = x_0 e^{\alpha t}$

(4.5) Let $s < t$.

$$E[X_t | \mathcal{F}_s] = E\Big[X_s + \underbrace{\int_s^t \mu_u \, du}_{\geq 0} + \int_s^t \sigma_u \, dW_u \, \Big| \, \mathcal{F}_s\Big]$$

$$\geq X_s + \underbrace{E\Big[\int_s^t \sigma_u \, dW_u \, \Big| \, \mathcal{F}_s\Big]}_{0} = X_s$$

(4.6) $X_t = h(W_t^1, \dots, W_t^n)$

$$dX_t \overset{\text{Ito}}{=} \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j} dW_t^i dW_t^j$$

$$= \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_t^i + \frac{1}{2} \underbrace{\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}}_{0} dt = \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_t^i ,$$

so X_t is a martingale if h is harmonic

(and a submartingale if h is subharmonic, see (4.5)).

4.7 $E[Q_n] = E\left[\sum_{k=1}^{n} \Delta W^1_{t_k} \cdot \Delta W^2_{t_k}\right] =$

$$= \sum_{k=1}^{n} E\left[\Delta W^1_{t_k} \cdot \Delta W^2_{t_k}\right] \underset{\text{indep.}}{=} \sum_{k=1}^{n} E\left[\Delta W^1_{t_k}\right] E\left[\Delta W^2_{t_k}\right]$$

$$= 0$$

$$\mathrm{Var}[Q_n] = E\left[\left(\sum_{k=1}^{n} \Delta W^1_{t_k} \Delta W^2_{t_k}\right)^2\right]$$

$$= E\left[\sum_{k=1}^{n} (\Delta W^1_{t_k})^2 (\Delta W^2_{t_k})^2\right] + \underbrace{\sum_{j\neq k} E\left[\Delta W^1_{t_j} \Delta W^1_{t_k} \Delta W^2_{t_j} \Delta W^2_{t_k}\right]}_{0}$$

$$\underset{\text{indep.}}{=} \sum_{k=1}^{n} E\left[(\Delta W^1_{t_k})^2\right] E\left[(\Delta W^2_{t_k})^2\right]$$

$$= n \frac{t}{n} \cdot \frac{t}{n} = \frac{t^2}{n} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

4.8 a) $dR_t = 2X_t dX_t + 2Y_t dY_t + (dX_t)^2 + (dY_t)^2$

$$= \dots = (1+2\alpha) R_t \, dt$$

Thus $R_t = (x_0^2 + y_0^2) e^{(1+2\alpha)t}$ which is deterministic.

b) $E[X_t] = x_0 e^{\alpha t}$

5.1 Let $Y_t = x_0 e^{\alpha t}$

$Z_t = \sigma e^{\alpha t}$

$R_t = \int_0^t e^{-\alpha s} dW_s$

$$dX_t = d(Y_t + Z_t R_t) \overset{\text{Ito}}{=} dY_t + Z_t dR_t + R_t dZ_t + \underbrace{dZ_t dR_t}_{0}$$

$$= \alpha Y_t dt + \sigma dW_t + \alpha Z_t R_t dt$$

$$= \alpha X_t dt + \sigma dW_t$$

5.5 $dY_t = \beta\left(\alpha + \frac{\sigma^2}{2}(\beta - 1)\right) Y_t dt + \beta \sigma Y_t dW_t$

5.6 $dZ_t = (\alpha - \gamma + \delta^2) Z_t dt + Z_t(\sigma dW_t - \delta V_t)$

5.7 $dZ_t = (\alpha - \gamma + \delta^2 - \sigma\delta) Z_t dt + Z_t(\sigma - \delta) dW_t$

5.8 $dZ_t = (\alpha + \gamma) Z_t dt + Z_t(\sigma dW_t + \delta dV_t)$

5.9 $F(t,x) = 2 \ln x + (2\mu - \sigma^2)(T-t)$

5.10 We did this in class.

5.11 $F(t,x) = 2 \ln x + (x-1)(T-t)$

5.13 $F(t,x,y) = xy$