Retake exam
Graph Theory, 1MA170, Period 2 2017

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The exam consists of 5 questions worth 10 points each. Choose any 4 questions to answer. If you attempt all 5, the best 4 scores will count for your final grade. Answer each question carefully and with attention to details, naming any results from the lecture notes that you use. You may write your solutions in Swedish or English. Calculators are not allowed.

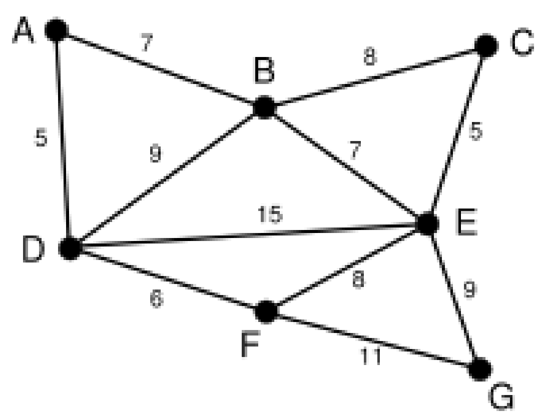
Good luck!

1. (Trees)

   (a) Define *tree*. (1)

   (b) Prove that any tree on $n > 1$ vertices has at least 2 leaves. (2)

   (c) State either Prim's or Kruskal's algorithm for finding a minimum weight spanning tree in a weighted graph. Using the algorithm you stated, find a minimum weight spanning tree in the graph below, clearly showing each step. (3)



   (d) Let G be a graph containing k edge-disjoint spanning trees. Show that for each partition $V_1, V_2, \ldots, V_n$ of the vertex set, the number of edges of G which have endpoints in different parts of the partition is at least $k(n-1)$. (4)

2. (Planarity)

   (a) Define the *dual* of a planar graph G. (1)

   (b) Using Euler's formula, or otherwise, show that for a planar graph G on n vertices

   $$\sum_{v \in V(G)} d(v) \leq 6n - 12$$

   where $d(v)$ is the degree of vertex v. (3)

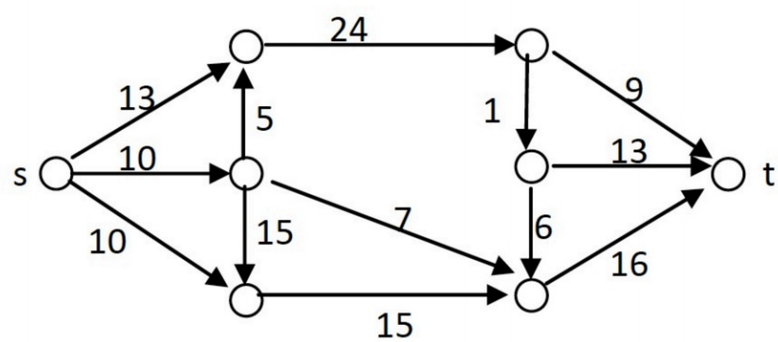
   (c) Show that every n-vertex graph isomorphic to its dual has $2n - 2$ edges. For all $n \geq 4$, construct a graph on n vertices isomorphic to its dual. (4)

   (d) Show that K_5 can be drawn on a torus without any edges crossing. (2)

3. (Flows in networks)

   (a) Define *flow* and *source/sink cut.* (2)

   (b) State the Ford-Fulkerson algorithm. Use it to determine the maximum flow in the network below, clearly indicating each step in the algorithm. (4)



   (c) State the max-flow min-cut theorem, and explain how the Ford-Fulkerson algorithm can be used to prove it for rational edge weights. (A detailed proof is not necessary, only the main ideas of the proof). (4)

4. (Random graphs)

   (a) Explain briefly how to construct the Erdös-Renyi random graphs $G_{n,p}$ and $G_{n,m}$. (2)

   (b) What is the probability of $G_{n,p}$ taking the value of a given labelled graph G with m edges? (2)

   (c) Explain briefly how to construct a Barabasi-Albert preferential attachment graph. (2)

   (d) Give an example of a real world network for which either the Erdös-Renyi model or the preferential attachment model is a good simulation. Your answer should specify which properties of the real world network make one model better than the other. (4)

5. (Chromatic numbers)

(a) Define the *chromatic number*, $\chi(G)$ of a graph G. (1)

(b) Recall that the *clique number*, $\omega(G)$, is the size of the largest complete subgraph of G, and the *independence number*, $\alpha(G)$, is the size of the largest set of pairwise nonadjacent vertices of G. Prove that

$$\chi(G) \geq \omega(G)$$

and

$$\chi(G) \geq \frac{v(G)}{\alpha(G)}$$

(3)

(c) Prove that the chromatic number of the Petersen graph is 3. (3)

(d) Prove that the chromatic number of a planar graph is at most 6. (3)