210614 - Solutions

(1) $\begin{cases} dX = 2\,dt + X\,dW \\ X(0) = x \end{cases}$

Integrate + take expected value:

$$E[X(t)] = x + E\left[\int_0^t 2\,ds\right] + \underbrace{E\left[\int_0^t X(s)\,dW(s)\right]}_{=0}$$

$$= \underline{x + 2t}$$

Let $Y = X^2$. Then, by Ito,

$$dY = 2X\,dX + (dX)^2$$

$$= (4X + X^2)\,dt + 2X^2\,dW$$

Integrate + take expected value:

$$E[Y(t)] = x^2 + \int_0^t \left(4E[X(s)] + E[X^2(s)]\right) ds + \underbrace{E\left[\int_0^t 2X^2(s)\,dW(s)\right]}_{0}$$

$$= x^2 + \int_0^t \left(4(x+2s) + E[Y(s)]\right) ds$$

Let $m(t) = E[Y(t)]$. Then

$$\begin{cases} \dot{m}(t) = 4(x+2t) + m(t) \\ m(0) = x^2 \end{cases}$$

so $m(t) = Ce^t - 8t - 4x - 8$

$m(0) = x^2$

$= (x^2+4x+8)e^t - (8t+4x+8)$

$$\text{Var}\, X(t) = E[X^2(t)] - (E[X(t)])^2$$

$$= (x^2+4x+8)(e^t - 1) - 4t(2+x+t)$$

<u>Answer</u> $E[X(t)] = x + 2t$ $\quad \text{Var}(X(t)) = (x^2+4x+8)(e^t-1) - 4t(2+x+t)$

② $\begin{cases} u_t + \frac{1}{2} u_{xx} + \frac{y^2}{2} u_{yy} = 0 \\ u(T,x,y) = (x+y)^2 \end{cases}$

Let $\begin{cases} dX = dW \\ X(t) = x \end{cases}$ and $\begin{cases} dY = Y\,dV \\ Y(t) = y \end{cases}$ (W,V indep.)

By Feynman-Kac,

$$u(t,x,y) = E_{t,x,y}\left[(X(T)+Y(T))^2\right] = E_{t,x,y}\left[X^2(T) + 2X(T)Y(T) + Y^2(T)\right]$$

$$\underset{\text{indep.}}{=} E_{t,x,y}\left[X^2(T)\right] + 2\,E_{t,x,y}\left[X(T)\right] E_{t,x,y}\left[Y(T)\right] + E_{t,x,y}\left[Y^2(T)\right]$$

$$= x^2 + T - t + 2xy + E_{t,x,y}\left[Y^2(T)\right]$$

Let $Z(s) = Y^2(s)$. Ito $\Rightarrow \begin{cases} dZ(s) = Z\,ds + 2Z\,dV \\ Z(t) = y^2 \end{cases}$

so $E\left[Y^2(T)\right] = y^2 e^{T-t}$.

<u>Answer</u> $u(t,x,y) = x^2 + T - t + 2xy + y^2 e^{T-t}$.

(3) $X = \begin{cases} 2S(T) & \text{if } S(T) < b \\ 0 & \text{if } S(T) \geq b \end{cases} = 2 \cdot Y$

where $Y = \begin{cases} S(T) & \text{if } S(T) < b \\ 0 & \text{if } S(T) \geq b \end{cases}$.

In the BS-model, $S(T) = s\, e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} N(0,1)}$, so

$$E\left[e^{-rT} Y\right] = E\left[e^{-rT} S(T) 1_{\{S(T) < b\}}\right] = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} s\, e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}x} \, dx$$

d is such that $s\, e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T} d} = b$

i.e $d = \dfrac{\ln \frac{b}{s} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$$= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma\sqrt{T})^2}{2}} s \, dx$$

$$= \int_{-\infty}^{d - \sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} s \, dx = s N(d - \sigma\sqrt{T})$$

<u>Answer</u>: $2sN\left(\dfrac{\ln \frac{b}{s} - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$

④ i) The variance of $W(t)$ is

$$E\left[W^2(t)\right] = a^2\, Var\left(2W_1(t) + 5W_2(t)\right)$$

$$\underset{\text{indep.}}{=} a^2\left(Var\left(2W_1(t)\right) + Var\left(5W_2(t)\right)\right)$$

$$= a^2\left(4t + 25t\right) = 29a^2 t = t$$

if $a = \frac{1}{\sqrt{29}}$

Answer: $a = \frac{1}{\sqrt{29}}$

ii) Let $M = e^{-ct}X^3(t)$ Ito $\Rightarrow$

$$dM = \left(3\mu + 3\sigma^2 - c\right)M\,dt + 3\sigma M\,dW$$

so M is a martingale if $c = 3\mu + 3\sigma^2$

Answer: $c = 3\mu + 3\sigma^2$

iii) Price $= E^Q\left[e^{-rT}\left(S(T) - K\right)^+\right] \geq e^{-rT}E^Q\left[S(T)\right] - Ke^{-rT}$

$$= s - Ke^{-rT} \geq s - K$$

Answer: Yes, true!

5 i) At time 0 : Buy $\frac{1+e^{-r(T-T_0)}}{2}$ shares of S.

At time T_0 : Sell $\frac{1}{2}e^{-r(T-T_0)}$ shares, and receive

$\frac{1}{2}e^{-r(T-T_0)}S(T_0)$

Deposit $\frac{1}{2}e^{-r(T-T_0)}S(T_0)$ in the bank.

At time T : Sell $\frac{1}{2}$ share of S and withdraw

$\frac{1}{2}e^{-r(T-T_0)}S(T_0)e^{r(T-T_0)}$ from the bank.

We thus have $\frac{S(T)+S(T_0)}{2}$.

ii) The price at $t=0$ coincides with the value of the replicating strategy in i). Thus the price is $\frac{1+e^{-r(T-T_0)}}{2}s$, where $s=S(0)$.

iii) At $t\in(T_0,T)$, $S(T_0)$ is known. The price of X is then $\frac{S(T_0)e^{-r(T-t)}}{2}+\frac{S(t)}{2}$.

(6) According to put-call-parity, $p = Ke^{-rT} - S + C$.

Here we have $p = 12$ and

$Ke^{-rT} - S + C = 147 - 140 + 6 = 13 > 12$.

An arbitrage strategy is obtained as follows.

At $t=0$: Buy one put option and one share of S.

Sell three zero-coupon bonds and one call option.

In total, we receive $147 + 6 - 12 - 140 = 1$.

At $t=T$ The value of our portfolio is now

$$(K - S(T))^+ + S(T) - K - (S(T) - K)^+ = 0.$$

(7) $dr = \left(\frac{\sigma^2}{4} - a\sqrt{r}\right)dt + \sigma\sqrt{r}\,dW$

<u>Ansatz</u>: $p^T(t,T) = F^T(t, r(t))$, where

$$F^T(t,r) = \exp\left\{A(t,T)\,r + B(t,T)\sqrt{r} - C(t,T)\right\}.$$

Insertion into the term structure equation

$$\begin{cases} F_t + \frac{\sigma^2}{2} r F_{rr} + \left(\frac{\sigma^2}{4} - a\sqrt{r}\right)F_r - rF = 0 \\ F(T,r) = 1 \end{cases}$$

gives

$$A_t r + B_t\sqrt{r} - C_t + \frac{\sigma^2}{2} r\left(\left(A + \frac{B}{2\sqrt{r}}\right)^2 - \frac{B}{4r^{3/2}}\right) + \left(\frac{\sigma^2}{4} - a\sqrt{r}\right)\left(A + \frac{B}{2\sqrt{r}}\right)$$

$$= r.$$

Collecting terms of the same type ($r, \sqrt{r}, 1, \frac{1}{\sqrt{r}}$) yields

$$\text{I}: \begin{cases} A_t + \frac{\sigma^2}{2} A^2 = 1 \\ A(T,T) = 0 \end{cases} \qquad \text{II}: \begin{cases} B_t + \frac{\sigma^2}{2} AB - aA = 0 \\ B(T,T) = 0 \end{cases}$$

$$\text{III}: \begin{cases} -C_t + \frac{\sigma^2}{8} B^2 + \frac{\sigma^2 A}{4} - \frac{aB}{2} = 0 \\ C(T,T) = 0 \end{cases} \qquad \text{IV}: -\frac{\sigma^2 B}{8} + \frac{\sigma^2 B}{8} = 0$$

First solve I to get A, then II to get B and III to get C (IV provides no info).

Thus A, B, C can be found, so the Ansatz works.

8 The price is $F = E^{Q}\left[e^{-rT} g(S(T))\right] =$

$$= e^{-rT} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} g\left(s e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x}\right) dx$$

Differentiation (with respect to δ) gives

$$\frac{\partial F}{\partial \delta} = -e^{-rT} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \underbrace{g'\left(s e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x}\right)}_{\geq 0} \underbrace{s e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x}\, T}_{\geq 0} \, dx$$

≤ 0 so F decreases in δ.

(If g is not differentiable, approximation yields the same result.)