Lévy processes, cont.

An adapted process $X : (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a <u>Lévy process</u> w.r.t. $(\mathcal{F}_t)$, if

- $X_0 = 0$, $t \to X_t$, $t \geq 0$, cadlag a.s.
- $X_{t+u} - X_t$ independent of $\mathcal{F}_t$, all $t \geq 0$, $u \geq 0$
- $X_{t+u} - X_t \overset{d}{=} X_u$

<u>Thm</u> Let $M(ds, dx)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, with mean $\mu(ds, dx) = \lambda ds\, \nu(dx)$,

where $\int_{\mathbb{R}^d} (|x| \wedge 1)\, \nu(dx) < \infty$.

Then $X_t = \int_{[0,t) \times \mathbb{R}^d} x\, M(ds, dx)$

converges absolutely, is Lévy, and has <u>bounded variation</u>, i.e. $X_t = \sum_{s \leq t} \Delta X_s$, has the property

$$V_t = \sum_{s \leq t} \underbrace{|\Delta X_s|}_{\text{distance in } \mathbb{R}^d} < \infty \text{ a.s. for all } t.$$

Also $(\lambda = 1)$

$$E[e^{i\theta X_t}] = e^{t\int_{\mathbb{R}^d}(e^{i\theta x}-1)\nu(dx)}$$

$$= e^{t\cdot\psi(\theta)},$$

$\psi(\theta)$ is called the characteristic exponent.

Corollary: If ~~$\int_{\mathbb{R}^d}$ ... $dx < \infty$~~ $\int_{\mathbb{R}^d}|x|\,\nu(dx) < \infty$

then $a = \int_{\mathbb{R}^d} x\,\nu(dx) < \infty$

and $X_t = bt + \sigma B_t + \int_0^t\int_{\mathbb{R}^d} x\underbrace{(M(ds,dx) - \mu(ds,dx))}_{\text{compensated Poisson measure}}$

$$= (b-a)t + \sigma B_t + \int_0^t\int_{\mathbb{R}^d} x\,M(ds,dx),$$

is a Lévy process.

Def X is stable with index α

if $X_{ct} \overset{d}{=} c^{1/\alpha} X_t$ for all $c > 0$.

Example $B_{ct} = \sqrt{c}\,B_t$ BM is 2-stable

Example $d = 1$, $\nu(dx)$ supported on $[0,\infty)$, $\int_0^\infty (1\wedge x)\,\nu(dx) < \infty$

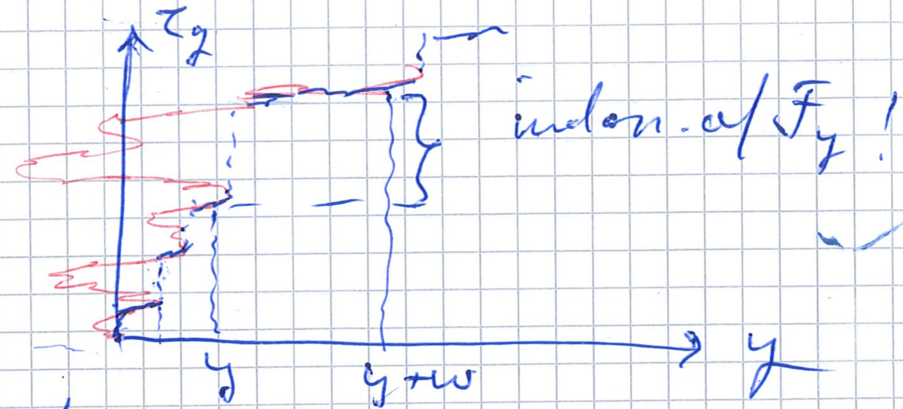
Then (X_t) is non-decreasing Lévy

i.e. subordinator

Example (c

Brownian first passage time

$$\tau_y = \inf\{t \geq 0 : B_t \geq y\}, \quad y \geq 0.$$



$$\tau_{y+w} - \tau_y \overset{d}{=} \tau_w$$

$\{\tau_y\}_{y \geq 0}$ is a Lévy process which is increasing, i.e. a **subordinator**!

Recall $\{B_t\}$ martingale

$\{B_t^2 - t\}$ —"—

$\{e^{B_t - \frac{1}{2}t}\}$ —"—

$\{e^{rB_t - \frac{r^2 t}{2}}\}$ —"— $\qquad M_t = e^{rB_t - r^2 t/2}$

$E[|M_t|] = E[M_t] < \infty, \ \forall t$

$E[M_t | \mathcal{F}_s] = M_s, \quad M_0 = 1,$

τ_y is a stopping time: $\{\tau_y \leq t\} = \{\sup_{s \leq t} B_s \geq y\} \in \mathcal{F}_t, \ t \geq 0$

By optimal stopping theorem,

F13:4

$$1 = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_y}]$$
$$= \mathbb{E}[e^{\lambda B_{\tau_y} - \lambda^2 \cdot \frac{1}{2} \cdot \tau_y}]$$
$$= \mathbb{E}[e^{\lambda y - \frac{\lambda^2}{2}\tau_y}] = e^{\lambda y}\,\mathbb{E}[e^{-\frac{\lambda^2}{2}\tau_y}]$$

and therefore

$$\mathbb{E}[e^{-\frac{\lambda^2}{2}\tau_y}] = e^{-\lambda y}, \quad y > 0, \ \lambda \in \mathbb{R}$$

Let's take $\theta = \frac{\lambda^2}{2} \geq 0$, $\lambda = (\pm)\sqrt{2\theta}$

so that $\boxed{\mathbb{E}[e^{-\theta \tau_y}] = e^{-\sqrt{2\theta}\cdot y}, \ y > 0}$

In particular, $\mathbb{E}[e^{-\theta \tau_{cy}}] = e^{-cy\sqrt{2\theta}}$

$$= e^{-y\cdot\sqrt{2\theta c^2}}$$
$$= \mathbb{E}[e^{-\theta c^2 \tau_y}]$$

Then, $\{\tau_y\}$ is stable with index $\alpha = \frac{1}{2}$, a $\frac{1}{2}$-stable subordinator

Note

Compare: Lalley Prop 5.3, Cor. 5.4

$$\mathbb{E}[\tau_y] = -\frac{d}{d\theta}\mathbb{E}[e^{-\theta \tau_y}]\Big|_{\theta=0}$$
$$= -\frac{d}{d\theta} e^{-y\sqrt{2\theta}}\Big|_{\theta=0}$$
$$= \frac{1}{\sqrt{2\theta}} e^{0}\Big|_{\theta=0} = +\infty$$

Thm

Every subordinator has the form

$$X_t = bt + \int_0^t \int_0^\infty v \, N(du, dv) ,$$

where $N(du, dv)$ is Poisson measure with intensity $\mu(du, dv) = \lambda \, du \, \nu(dv)$, such that $\int_0^\infty (1 \wedge v) \nu(dv) < \infty$

Proof $\{X_t\}$ is infinitely divisible (see F12:5). Then

$$E(e^{i\theta X_t}) = e^{t \cdot \psi(\theta)} ,$$

where the ~~expon~~ characteristic exponent must satisfy the Lévy-Khintchine representation

$$\psi(\theta) = \lambda \int_0^\infty (e^{i\theta v} - 1) \nu(dv)$$