Middle-schoolers' misconceptions in discretized nonsymbolic proportional reasoning explain fraction biases better than their continuous reasoning: Evidence from correlation and cluster analyses

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ABSTRACT

Early emerging nonsymbolic proportional skills have been posited as a foundational ability for later fraction learning. A positive relation between nonsymbolic and symbolic proportional reasoning has been reported, as well as successful nonsymbolic training and intervention programs enhancing fraction magnitude skills. However, little is known about the mechanisms underlying this relationship. Of particular interest are nonsymbolic representations, which can be in continuous formats that may emphasize proportional relations and in discretized formats that may prompt erroneous whole-number strategies and hamper access to fraction magnitudes. We assessed the proportional comparison skills of 159 middle-school students (mean age = 12.54 years, 43% females, 55% males, 2% other or prefer not to say) across three types of representations: (a) continuous, unsegmented bars, (b) discretized, segmented bars that allowed counting strategies, and (c) symbolic fractions. Using both correlational and cluster approaches, we also examined their relations to symbolic fraction comparison ability. Within each stimulus type, we varied proportional distance, and in the discretized and symbolic stimuli, we also manipulated whole-number congruency. We found that fraction distance across all formats modulated middle-schoolers' performance; however, whole-number information affected discretized and symbolic comparison performance. Further, continuous and discretized nonsymbolic performance was related to fraction comparison ability; however, discretized skills explained variance above and beyond the contributions of continuous skills. Finally, our cluster analyses revealed three nonsymbolic comparison profiles: students who chose the bars with the largest number of segments (whole-number bias), chance-level performers, and high performers. Crucially, students with a whole-number bias profile showed this bias in their fraction skills and failed to show any symbolic distance modulation. Together, our results indicate that the relation between nonsymbolic and symbolic proportional skills may be determined by the (mis)conceptions based

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on discretized representations, rather than understandings of proportional magnitudes, suggesting that interventions focusing on competence with discretized representations may show dividends for fraction understanding.

1. Introduction

A milestone in numerical development is understanding that all numerical symbols denote numerical magnitudes (Siegler, 2016). This understanding affords individuals to realize the critical commonality across all real numbers; that is, they can be placed in the same numberline whether they share other superficial features, for example notation format. Fraction learning requires students to broaden their knowledge of numerical quantities (Siegler et al., 2011) to understand that, akin to whole numbers, fractions, as well as other rational numbers possess magnitudes. Additionally, students may need to refine rules about number magnitudes and avoid those that apply only to whole numbers, for example, larger numerals do not always denote larger numerical quantities for fractions (Rosenberg-Lee, 2021). Unfortunately, across countries, many students fail to overcome the difficulties that fractions pose (Abreu-Mendoza et al., 2019; Bailey et al., 2015; Carpenter et al., 1980; DeWolf & Vosniadou, 2015; Gómez & Dartnell, 2019). This failure to understand fractions sharply contrasts with children's remarkable abilities to work with nonsymbolic representations of proportions (e.g., bicolored bars and pie charts). For instance, before formal instruction, children can compare and match objects based on their proportional magnitudes (Boyer et al., 2008; Hurst & Cordes, 2018; Jeong et al., 2007). Combined with correlational evidence for the positive relationship between nonsymbolic and symbolic proportional reasoning (Begolli et al., 2020; Matthews et al., 2016; Möhring et al., 2016), these findings have led researchers to develop fraction instructions that scaffold on nonsymbolic representations (Abreu-Mendoza et al., 2021; Braithwaite & Siegler, 2021; Gouet et al., 2020; Hamdan & Gunderson, 2017); for a review of those instructional interventions, see Abreu-Mendoza and Rosenberg-Lee, (2023). However, the mechanisms underlying the relationship between
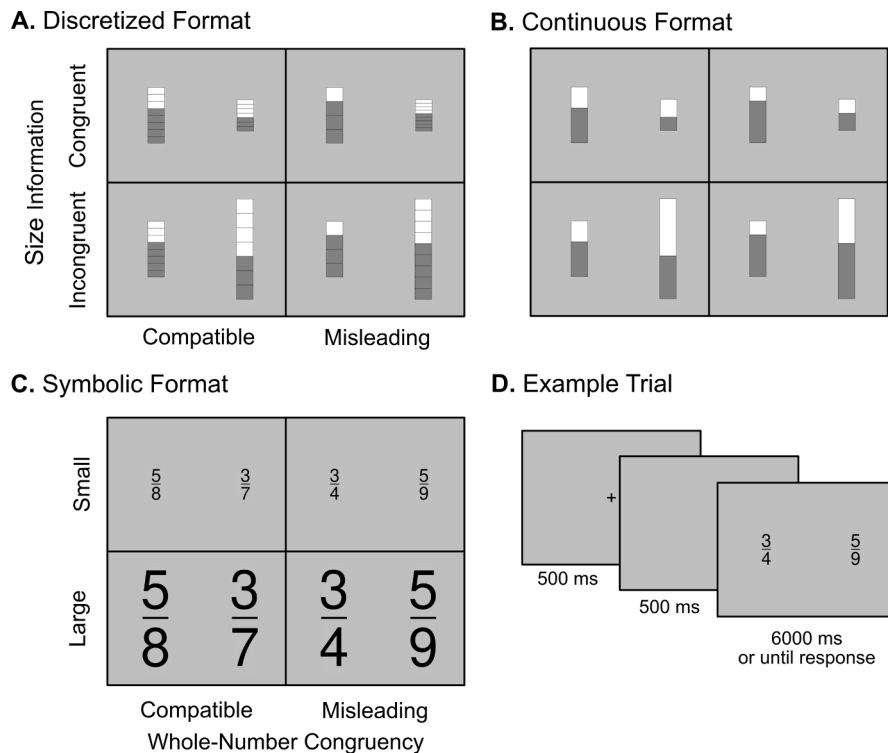

Fig. 1. Examples of the stimuli used in the nonsymbolic and symbolic proportional comparison task and trial presentation timing. **A to C.** Participants had to indicate which of two proportions was the largest. Proportions were presented in two nonsymbolic formats, discretized and continuous, as well as a symbolic format, each presented in a separate block. To prevent participants from using non-numerical cues, in the two nonsymbolic formats, the bar with the proportionally largest grey area could have either the grey segment with the largest absolute size (size congruent trials) or the one with the smallest absolute size (size incongruent). In the nonsymbolic discretized format, the bar with the proportionally largest grey area could have the largest number of segments (whole-number compatible trials) or the smallest number of segments (whole-number misleading trials). Similarly, in the symbolic format, the largest fraction could have the largest components (whole-number compatible trials) or the smallest components (whole-number misleading trials). To mirror the size manipulation in the nonsymbolic formats without introducing size interference (i.e., the numerical Stroop effect), fractions could appear in either a small or a large font. **D.** Stimuli were presented until participants responded (up to 6000 ms).

nonsymbolic and symbolic proportional reasoning remain unknown. Notably, not all nonsymbolic formats may promote proportional reasoning equally well. In fact, some formats with countable items may encourage students to rely on inappropriate whole-number rules instead of drawing on their proportional skills. To improve children's fraction knowledge through nonsymbolic instruction of fractions, we must first determine which formats more strongly engage students with proportional reasoning and provide a more comprehensive explanation for the relationships observed between nonsymbolic and symbolic proportional skills.

1.1. Continuous and discretized nonsymbolic proportional reasoning

Long before any formal instruction, children can reason about proportional quantities if presented nonsymbolically (e.g., collections of objects or bicolored bars). Infants as young as six months of age can discriminate the changes in the proportion of dots of different colors (McCrink & Wynn, 2007). Preschool children can successfully indicate which figure has the proportionally larger quantity (Hurst & Cordes, 2018; Jeong et al., 2007) and match objects of different sizes but same proportions (Boyer et al., 2008; Boyer & Levine, 2015; Sophian, 2000). From age four, children can also perform addition and subtraction of nonsymbolic proportions (Mix et al., 1999), and, by age six, they can solve a:b::c:d analogies using proportional information (Goswami, 1989).

Critically, children's nonsymbolic proportional reasoning reflects an understanding of proportional magnitudes (Bhatia et al., 2020; Kalra et al., 2020; Meert et al., 2013; Park et al., 2021; Sophian, 2000). A key signature of magnitude understanding is distance effects (Moyer & Landauer, 1967), that is, faster and more accurate performance in comparing far numerical distances (e.g., 3 vs. 9) than near distances (e.g., 3 vs. 4). Akin to whole-number processing (Holloway & Ansari, 2009, 2010), children's performance in nonsymbolic proportional comparison tasks show distance effects (Kalra et al., 2020): elementary-school children are more accurate when comparing proportions with far distances (e.g., 7/9 vs. 2/5) than near ones (e.g., 7/9 vs. 6/7). Proportional magnitude processing is not only observed in children's comparison skills but also in their ability to estimate (Meert et al., 2013) and match (Bhatia et al., 2020) nonsymbolic ratios, and it continues to develop beyond elementary school (Park et al., 2021).

This body of research suggests that children are well equipped to understand proportional magnitudes and puts forward nonsymbolic representations as a tool to engage students with proportional reasoning; however, not all nonsymbolic formats may draw on this capacity. Children's proportional reasoning goes awry when representations of proportions are segmented into pieces (Abreu-Mendoza et al., 2020; Begolli et al., 2020; Boyer et al., 2008; Hurst & Cordes, 2018; Jeong et al., 2007). These segments allow for counting, which in turn introduces whole-number information (see Fig. 1A). This segmentation results in cases where the whole-number information is *compatible* with the proportional information (e.g., 5/8 vs. 3/7 as the proportion with the largest number of segments is also the one with the largest proportional magnitude) and cases where it is *misleading* (e.g., 3/4 vs. 5/9 as the proportion with the largest number of segments is actually the one with the smaller proportional magnitude). As a result, children, and even adults, may employ erroneous counting strategies instead of proportional reasoning (Plummer et al., 2017). For example, even though six-year-olds successfully compare nonsymbolic proportions presented as bicolored pie-chart-like shapes (continuous format), they fail to do so when the same proportions are presented with these shapes but further segmented into smaller pieces (discretized format) and the whole number information is misleading (Jeong et al., 2007). Only after age nine do children perform above chance in both compatible and misleading discretized proportions, yet they still perform worse on misleading relative to compatible trials (Begolli et al., 2020; Jeong et al., 2007). Note, whole-number information can be introduced by either non-contiguous objects (e.g., dots or unattached elements) or segmented stimuli leading to adjacent items (e.g., segmented bars). Here, following Begolli et al. (2020), we refer to the non-contiguous objects as *discrete* and segmented stimuli as *discretized* and group them both as *countable* formats.

Within the nonsymbolic proportional reasoning literature, these two lines of research — distance-based effects and whole-number interference — have been pursued largely independently. No study, to our knowledge, has directly manipulated both in the same
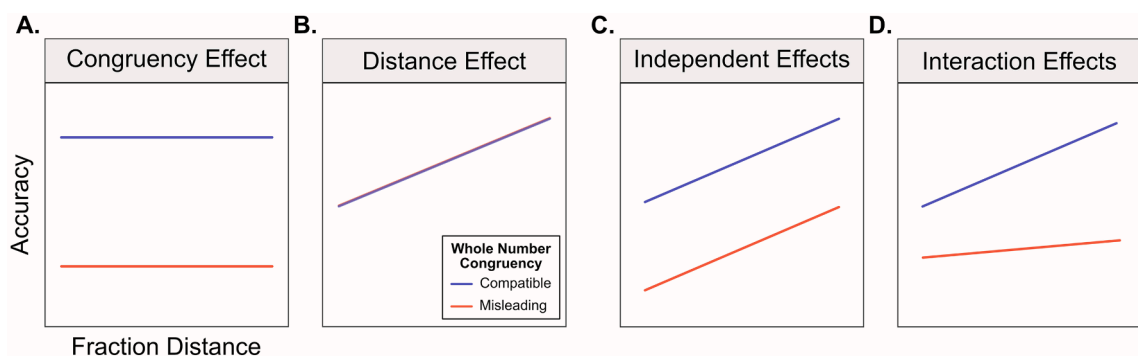

Fig. 2. Schematics representations of theoretical outcomes for the interplay between fraction distance and whole-number congruency effects. **A.** Only whole-number congruency effects: whole-number information may impede access to proportional magnitudes, resulting in much better performance on compatible than misleading trials and no effect of distance. **B.** Only distance effects: once proportional magnitudes are accessed, they may override whole number congruency effects. **C.** Independent main effects: proportional magnitude processing and whole-number interference may have independent effects, leading to similar distance effects but lower accuracy in misleading trials. **D.** Interaction effects: proportional magnitude processing and whole-number information interact; particularly, average performance in misleading trials may remain as in Panel C, but magnitude processing is impaired by whole-number interference.

experimental design, as we do here. At least four outcomes are theoretically possible to describe the interplay between these two factors (see Fig. 2). Participants' performance could display: 1) only whole-number congruency effects, 2) only distance effects 3) independent main effects of congruency and distance and 4) interactions between congruency and distance effects. First, whole-number information may impede access to proportional magnitudes, resulting in much better performance on compatible than misleading trials and no effect of distance (Fig. 2A). Consistent with this possibility, studies that manipulate congruency, but not distance report that preschooler and second graders show chance level performance or worse on misleading discretized proportions (Abreu-Mendoza et al., 2020; Begolli et al., 2020; Hurst & Cordes, 2018; Jeong et al., 2007). Second, proportional magnitudes, once they are accessed, may override whole number congruency effects (Fig. 2B). Consistently, children can appropriately work with discrete proportions when paired with continuous proportions, which do not introduce whole-number interference (Boyer et al., 2008), or when discretized proportional reasoning is primed by continuous representations (Abreu-Mendoza et al., 2020; Boyer & Levine, 2015; Hurst & Cordes, 2018). Third, proportional magnitude processing and whole-number congruency may have independent effects on students' proportional comparison skills (Fig. 2C). This view aligns with current perspectives of conceptual change, suggesting that naive initial models (whole-number based number knowledge) and scientific models (rational number knowledge) co-exist (Carey, 2009; Vosniadou et al., 2008). Fourth, proportional magnitude processing and whole-number information may interact with each other (Fig. 2D). Specifically, magnitude processing might be impaired by the interference of whole-number information. This pattern was evident in a symbolic fraction comparison task conducted by Ischebeck et al. (2009), which found an interaction between fraction distance and whole-number condition in adults' reaction times when comparing proper and improper fractions, although they implement whole number interference in a nonstandard way (Rosenberg-Lee, 2021).

Here, our first goal is to provide, for the first time, an empirical examination of how whole-number information interacts with magnitude processing as measured by distance effects, during nonsymbolic proportional reasoning. In particular, we assessed children's magnitude processing in contexts where discretized proportional information could be compatible or misleading with the whole-number information (Fig. 1A), as well as in a context where there was no whole-number information (continuous format, Fig. 1B), while manipulating distance effects comparably in the two nonsymbolic conditions. In particular, we focused on discretized representations of proportions instead of discrete ones, as they are the ones that most visually resemble the continuous format.

1.2. Symbolic proportional reasoning

In contrast to the remarkable early ability to work with nonsymbolic proportions, many students struggle with symbolic fractions (Carpenter et al., 1980). In the United States, students are introduced to fractions around third grade (Common Core State Standards Initiative, 2020); yet, even community college students continue to show great difficulty with fractions (Ngo, 2019). Corresponding to the equivalent nonsymbolic discretized proportions, symbolic fractions can also be categorized as compatible (e.g., 5/8 vs. 3/7) or misleading (e.g., 3/4 vs. 5/9) with whole number knowledge. While learning new knowledge that may appear to contradict previous knowledge, students may develop misconceptions; that is, erroneous interpretations about this new knowledge (Stafylidou & Vos-niadou, 2004; Van Dooren & Inglis, 2015). One of the most common misconceptions about fractions held by students, referred to as the whole-number bias (Ni & Zhou, 2005), is thinking that the same rules that apply to whole numbers apply to fractions. Many choose the fraction with the largest components (numerator and denominator) as the one with the largest magnitude (Rinne et al., 2017) regardless of its actual proportional magnitude. Only by sixth grade have more than half of the students overcome this misconception and achieved a normative understanding of fractions (Rinne et al., 2017). Recently, research has identified an intermediate stage between a whole-number bias and a normative understanding of fraction magnitudes (Gómez & Dartnell, 2019; Leib et al., 2023; Reinhold et al., 2020; Rinne et al., 2017). Students in this stage tend to choose the fraction with the smallest components as the largest fraction, suggesting that they have understood that larger whole numbers, when placed as denominators, can potentially refer to smaller quantities, but have overgeneralized this feature to all fractions.

Largely independent from studies of congruency effects is the debate about whether people access fraction magnitudes automatically, only via calculations, or not at all because they are focused on the magnitude of the fraction components alone (Binzak & Hubbard, 2020; Bonato et al., 2007). In particular, adults may use more efficient but restricted strategies to compare symbolic fractions instead of accessing their magnitudes. For example, when participants compare fractions that share either the same numerators or denominators, they rely on componential strategies instead of focusing on the fraction magnitude (Bonato et al., 2007; Schneider & Siegler, 2010). Yet, under some conditions (e.g., when fractions do not share components), adults access fraction magnitudes and distance effects are observed (Binzak & Hubbard, 2020). Surprisingly, symbolic distance effects have been reported in children even before they encounter formal instruction. When six-to-eight-year-old children were asked to compare symbolic proportions in a context of gumball probability game, Szkudlarek and Brannon (2021) found children's judgments of proportions were modulated by the distance between the two symbolic proportions to be compared. Similarly, after a brief instruction of fraction symbols, Kalra et al. (2020) found that the proportional distance modulated second-graders' performance in symbolic comparisons, during a proportional comparison task that was comprised of nonsymbolic, symbolic, and mixed stimuli.

In contrast to the dearth of studies in the nonsymbolic literature, there are a handful of fraction comparison studies examining both fraction distance and whole-number congruency. For example, Reinhold et al. (2020) showed that performance of a group of low-performing sixth-grade German students was consistent with the congruency effect outcome (Fig. 2A) while another was consistent with the distance effect outcome (Fig. 2B). In contrast, Meert et al. (2010) found that although young adults were overall slower for misleading trials in comparison to compatible ones, both type of trials showed equivalent distance effects, consistent with independent effect relation (Fig. 2C). These results suggest that there are not only developmental differences but also differences within the same school grade, as students show large individual differences in their understanding of fractions. Our second goal was to examine the

interplay between fraction distance and whole-number congruency during fraction magnitude processing, allowing us to contrast it with that of nonsymbolic proportional reasoning.

1.3. Relationship between nonsymbolic and symbolic proportions

Emerging evidence supports the hypothesis that nonsymbolic proportional reasoning is a critical foundation for symbolic fraction learning. Early work established a relation between children's nonsymbolic proportional reasoning and general fraction knowledge, such as fraction arithmetic, fraction concepts and fraction numberline estimation (Hansen et al., 2015; Möhring et al., 2016). Critically, Matthews et al. (2016) were the first to examine the relation between nonsymbolic and symbolic comparison abilities, showing that college students' performance on a comparison task comprising continuous (e.g., unsegmented bicolored bars) and discrete (e.g., collection of dots) nonsymbolic proportions was related to students' performance when comparing fractions, as well as their performance on a fraction knowledge assessment, and even college entrance exams. These results led the authors to propose the Ratio Processing System, a cognitive architecture that serves as a foundation for understanding symbolic proportions (Lewis et al., 2016). However, later findings regarding this relation have been mixed. While one study found this relation even in younger populations (e.g., Szkudlarek & Brannon, 2021), another study only partially replicated it in another sample of the same population (e.g., Park & Matthews, 2021).

These divergent patterns point to two features that may affect the strength of this relation. The first is the outcome measure for symbolic fraction knowledge (e.g., performance on a fraction comparison task vs. general fraction assessments). A consistently replicated finding is a relationship between performance on nonsymbolic proportional tasks (comparison or match-to-sample tasks) and measures of general fraction knowledge (Begolli et al., 2020; Hansen et al., 2015; Matthews et al., 2016; Möhring et al., 2016; Park & Matthews, 2021). This robust result, however, may reflect a general association between mathematical abilities and not a specific relation between symbolic and nonsymbolic proportional reasoning. If nonsymbolic proportional and symbolic fraction magnitude processing are directly related, we should observe a relation between nonsymbolic and symbolic proportional comparison tasks. However, this relation has not shown the same robustness, as one direct replication study (Park & Matthews, 2021) failed to find the relation between nonsymbolic and symbolic comparison skills (Matthews et al., 2016), and instead reported a relation between nonsymbolic skills and general fraction knowledge.

The second feature affecting the relation between nonsymbolic and symbolic proportional skills is the type of nonsymbolic format: continuous vs. discretized vs. discrete. This feature is critical, as the formats introduce whole-number information differently: while noncountable continuous formats do not carry whole-number information, countable discretized and discrete representations can be compatible or misleading regarding whole-number quantities. Math education research has argued against the use of nonsymbolic countable representations of fractions, sometimes referred as area models, as they tend to highlight part-whole interpretations (Charalambous et al., 2010; Powell, 2023). In their original study, Matthews et al. (2016) used a composite score that consisted of performance on continuous and discrete representations, leaving unanswered whether each format relates to symbolic skills. In their later replication study, Park and Matthews (2021) found that only the composite score and accuracy on the continuous proportions were related to general fraction knowledge. Similarly, past studies using match-to-sample tasks combining continuous and discretized formats (Hansen et al., 2015) and numberline tasks with continuous stimuli (Möhring et al., 2016) have also shown a relation between performance on these tasks and general fraction knowledge. Other studies, in contrast, have used countable representations of proportions. Szkudlarek and Brannon (2021) found that six-to-eight-year-olds who have yet to receive fraction instructions showed distance effects when comparing discrete (i.e., sets of dots) and symbolic (i.e. fractions) proportions and a strong relationship between the two abilities. Wong (2019) also reported small-to-medium strength correlations between fourth-graders' performance across different proportional reasoning tasks (numberline and comparison tasks) involving discrete proportions (dots) and fractions.

To the best of our knowledge, only one study has examined the distinct relations for the different nonsymbolic representations, but the outcome measure was general fraction knowledge, not specific symbolic comparison skills. Begolli et al. (2020) assessed the nonsymbolic proportional reasoning of seven-to-twelve-year-old children across continuous, discretized, and discrete using a match-to-sample task. They found that performance in each nonsymbolic format correlated with fraction knowledge; however, it was discretized performance that showed the strongest relationship, followed by discrete and then continuous. These results suggest that the importance of nonsymbolic proportional reasoning for symbolic fraction understanding may be primarily through discretized skills, highlighting the ability to overcome whole number interference in both formats as crucial.

Neuroimaging findings have also suggested a relation between nonsymbolic and symbolic proportional magnitude processing by showing overlapping brain areas for both types of proportions (Jacob & Nieder, 2009b, 2009a; Mock et al., 2018; for a summary, see Rosenberg-Lee, 2021). However, these results stem from univariate analyses which capture changes in brain activity within a voxel. In contrast, multivariate approaches evince a more complex picture. Using representational similarity analyses, Mock et al. (2019) found that adults' brain activity in the bilateral inferior parietal lobule, a key area for quantity processing, showed stronger similarity for fractions with a discrete format (dots) than for fractions with a continuous one (pie charts). More recently, Bhatia et al. (2022) examined the similarity in brain activity patterns of adults while passively observing continuous nonsymbolic (either single line lengths, or pairs of lines arranged to suggest a proportion) and symbolic representations (either whole numbers or fractions). Their results indicated that greater similarity within representations (e.g., line lengths, or numerals) than between number systems (e.g., whole numbers, or proportions). These findings suggest that at the neural level continuous nonsymbolic and symbolic proportional quantities may be coded independently. Together, these imaging results suggest that neural representations of nonsymbolic proportions and symbolic fractions may be represented differently, but further research is warranted on the relation of countable proportions (i.e., discrete/discretized) with symbolic fractions.

In summary, these correlational and imaging studies suggest that while performance on tasks using continuous, discrete, and discretized formats may be related to symbolic fraction knowledge, findings for the countable formats are stronger. Moreover, whole-number information may play an integral role in this relation, as both countable formats and symbolic fractions can be misleading from a whole-number perspective, while continuous cannot. Yet, no study to date has examined the effects of nonsymbolic formats and whole-number congruency on the relation between nonsymbolic proportional skills and symbolic fraction comparison ability. Here, we first aim to replicate previously reported correlations between nonsymbolic and symbolic fraction comparison performance. Further, we decompose our nonsymbolic measure to examine the individual relations of continuous and discretized skills with symbolic fractions. Finally, as discretized and symbolic proportions can be compatible or misleading with respect to whole-number information, we further decomposed our measures to assess whether this relationship depends on the congruency of whole-number information. This methodological approach affords testing whether proportional magnitude understanding is the driving feature for the relation between nonsymbolic and symbolic skills, as proposed by the dominant theory, or whether whole-number information also plays a role. Critically, we used an experimental design where the same task and, more importantly, the same proportions were used in the nonsymbolic formats as presented in the symbolic format. This approach, similar to Bhatia et al. (2020) and Szkudlarek and Brannon (2021), ensures that task difficulty is similar across formats and provides stronger evidence for the relation between nonsymbolic and symbolic comparison abilities.

1.4. Profiles of proportional reasoning

Correlational analyses are one approach to dealing with the considerable heterogeneity among students' fraction knowledge at every stage of development; however, they fail to capture non-linear relations between symbolic and nonsymbolic representations and possible strategies shared between formats. Recently, person-oriented approaches (e.g., latent class growth, latent transition, and cluster analyses) have uncovered patterns of performance that were obscured when averaging performance from participants with different levels of fraction knowledge (e.g., Braithwaite & Siegler, 2018). Specifically, person-oriented approaches have allowed characterizing participants' performance based on their fraction comparison strategies. Consistently across studies (Gómez & Dartnell, 2019; Miller Singley et al., 2020; Reinhold et al., 2020; Rinne et al., 2017), three profiles of fraction comparison skills have emerged: (1) Students with a *whole-number bias* or *larger-number bias* profile are prone to choose fractions with the largest components, resulting in near-to-ceiling performance where whole-number information is compatible with the fraction magnitude (e.g., 5/8 vs. 3/7), and near-to-floor performance when the whole-number information is misleading (e.g., 3/4 vs. 5/9). (2) Students with a *reverse bias* or *smaller-number-bias* profile generally choose fractions with the smallest components, resulting in the opposite performance pattern to those with a whole-number bias profile. (3) Finally, students with a high-performing or normative profile show strong proficiency in comparison tasks regardless of the stimuli properties of fraction magnitudes and whole-number information.

Those studies have reported that students with these different profiles also exhibit differences in general fraction knowledge, math achievement, and the growth of their fraction knowledge. For instance, sixth-grade students with normative performance show distance effects when comparing fractions, indicating appropriate magnitude processing, and are better at placing nonsymbolic proportional magnitudes on a numberline than students who use either biased strategy (Reinhold et al., 2020). In addition, middle-school students with reverse and normative profiles have greater mathematical achievement than students with a whole-number bias profile (Gómez & Dartnell, 2019). Finally, elementary-school students with a whole-number bias profile are less likely to transition to a high-performing profile than those with a reverse bias profile (Rinne et al., 2017).Together, these studies highlight the importance of person-oriented approaches to describe fraction knowledge. Here, we extend these approaches to characterize, for the first time, students' strategies to compare continuous and discretized nonsymbolic proportions, using cluster analyses. Notably, this approach will allow us to examine whether a majority of students show a whole-number bias on the discretized trials or whether it is only a small number of students. Next, we examine whether these subgroups show distance effects in their nonsymbolic skills and which of the four theoretical outcomes they display (Fig. 2). Lastly, to complement our correlational analyses, our final goal was to investigate whether the strategies observed in the nonsymbolic formats relate to symbolic fractions performance, especially regarding strategies and distance effects.

1.5. The current study

Here, we assessed middle-school students' ability to compare proportions in three different formats — nonsymbolic continuous, nonsymbolic discretized, and symbolic (fractions) — to achieve four overarching goals: Our first two goals were to determine how compatible and misleading whole-number information interacts with magnitude processing in discretized nonsymbolic proportional reasoning and in symbolic fractions. Our third goal was to examine relations between nonsymbolic proportional comparison skills and symbolic fraction comparison ability. We first replicated prior work establishing this relationship. Then we examined independent contributions of continuous and discretized nonsymbolic skills to fraction ability to determine if a specific format drives the nonsymbolic and symbolic relation. Our fourth goal was to characterize students' nonsymbolic comparison strategies using a person-oriented approach, cluster analyses. We then examined the performance of these profiles with respect to modulation of nonsymbolic and symbolic magnitudes. Together, these results will contribute to unveiling the reasons why some nonsymbolic formats promote a stronger understanding of proportions. By contrast, other formats introduce misconceptions, which in turn will help to identify targets for evidence-based educational interventions to improve rational number teaching.

2. Methods

2.1. Mathematical thinkers like Me (MLM) project

Two-hundred forty-four 6th-to 8th-grade students participated in the second phase of the Mathematical Thinkers Like Me (MLM) project. The MLM project aims to develop and study a collaborative educational environment for learning mathematics among middle-school students from historically, economically, and socially disadvantaged ethnic-racial groups, to improve students' mathematical conceptual understanding and sense of belonging in mathematics. Participants were recruited from four schools, two schools each from two districts located on the east and west coasts of the United States. However, as recruitment in one school was low (n = 7), we excluded these participants, as well as six sixth graders from one of the three remaining schools (n = 6).

The data presented here were collected between October 2021 and March 2022. We obtained either paper-based or online consents and assents from legal guardians and students, respectively. Data collection took place at students' schools and was conducted by their classroom teachers, who received a protocol from the researchers. Students completed paper-and-pencil math assessments and computerized online math and executive function tasks divided into three sessions. Within each session, tasks were completed in a fixed order for ease of administration, but it was up to the classroom teacher to decide the order of the sessions. For the computerized tasks, teachers provided links to the online assessment tools that students followed to complete the tasks.

In Session 1, students completed three pen-and-paper assessments: the Calculation subtest of the Woodcock-Johnson III (Woodcock et al., 2001), a fraction arithmetic task (Siegler & Pyke, 2013), and an abbreviated version of the Rational Number Knowledge test (Van Hoof et al., 2018). In Session 2, students completed two online math tasks: a decimal comparison task and, the focus of this study, a nonsymbolic and symbolic proportional comparison task. Then, participants completed a series of online surveys regarding math anxiety, math-related social psychological variables (e.g., attitudes towards math, experiences with gender bias, among others), and demographic questions. In Session 3, participants completed six online executive function tasks: the Alternate Uses task (Guilford et al., 1960), Corsi-block tapping task (Corsi, 1972), a digit span task, the Hearts and Flowers task (Davidson et al., 2006), the Wisconsin Card Sorting task (Berg, 1948), and the Tower of London task (Culbertson & Zillmer, 1998). After the executive function tasks, students completed an online version of the Implicit Response Test. All computerized tasks were implemented in Psychopy v2020.2.8 (Peirce et al., 2019), an open source, Python-based stimulus presentation software, and hosted online via Pavlovia.org while the Alternate Uses task and the online surveys were hosted on Qualtrics. All protocols were in accordance with the Rutgers University Institutional Review Board.

2.2. Participants

A sample comprising 175 students who completed Session 2, which included the task of interest, the nonsymbolic and symbolic proportional comparison task, was identified. However, after removing students who had<70% of valid trials of each of the three experimental blocks ($n = 16$), the final sample was comprised of 159 students. Table 1 shows the full description of the demographic characteristics of the 151 students (mean age = 12.54 years, $SD = 0.88$) who reported this information.

The sample size was a convenience sample determined by the number of students in the Mathematical Thinkers Like Me participating schools and the number of legal guardians who consented for their children to participate. However, we ensured we had sufficient power to detect the effects of interest: whole-number congruency and distance effects. Past studies reporting whole-number congruency effects in fraction comparison tasks have found effect sizes of larger magnitudes (e.g., Cohen's f = 0.40, Avgerinou & Tolmie, 2019). For the sake of comparison with Avgerinou and Tolmie's study, we performed a sensitivity power analysis for a repeated measures ANOVA with our sample size (n = 159), power of 80%, alpha of 0.05, and the correlation coefficient for the relation between misleading and compatible symbolic trials ($r = -.73$), which indicated that our sample size allowed detection of effect sizes as small as Cohen's $f = 0.21$, suggesting that our sample size was sufficiently powered to detect this effect. For fraction distance effects, we used the multilevel power calculator tool Murayama et al. (2022), which requiresthe sample size and t-scores from linear mixed

Table 1
Sample demographics.

Variable	
Age (mean years)	12.54 (0.88)
Gender	
Male	65 (43.05%)
Female	83 (54.97%)
Other	1 (0.66%)
Prefer not to say	2 (1.32%)
Ethnic-Racial Group	
Black/African American, Caribbean	6 (3.97%)
Latino, Hispanic, Chicano or Puerto Rican	105 (69.54%)
White/Anglo or European American	14 (9.27%)
Asian, Asian American, or Pacific Islander	4 (2.65%)
Native American	2 (1.32%)
Multiracial	7 (4.64%)
Other Identity	13 (8.61%)

models. Although this tool was not created for generalized linear mixed effect models, we used the *z*-scored reported in Kalra et al. (2020) as a proxy of the required *t*-scores, as these scores converge as sample sizes approach 30. Thus, we used the sample size (n = 306) and stats (*z*-score = 15.84) reported for the distance effect, a target power of 80% and alpha of 0.05. This analysis indicated that 15 participants were enough to detect a distance effect. Overall, our sample size of 159 is sufficiently powered to detect our effects of interest.

2.3. Materials

2.3.1. Nonsymbolic and symbolic proportional comparison task

Design and procedure. To measure nonsymbolic and symbolic proportional reasoning, participants completed a task in which they had to compare proportions represented in two nonsymbolic formats (continuous and discretized) and one symbolic format (fractions), each format presented in a separate experimental block. The study always started with one of the two nonsymbolic formats, followed by the other format. The order of these blocks was counterbalanced across participants. For all participants, the symbolic format was completed last. In the nonsymbolic continuous block (Fig. 1B), participants compared bicolored bars with a grey and a white segment. In the nonsymbolic discretized block (Fig. 1A), the grey and white segments were further divided into sub-segments, introducing whole-number information. Finally, in the symbolic block (Fig. 1C), participants compared symbolic fractions instead of bars. Blocks comprised 38 trials for a total of 114 trials in the experiment.

Regarding whole-number information, discretized and symbolic trials could be either *compatible* or *misleading*. For discretized compatible trials, the bar with the larger number of grey segments and the larger number of total segments was the one with the proportionally larger grey area (e.g., 5/8 vs. 3/7). Correspondingly, for the symbolic compatible trials, the fraction with the largest components (numerator and denominator) was the fraction with the larger magnitude (again, 5/8 vs. 3/7). By contrast, discretized and symbolic misleading trials showed the opposite pattern (e.g., 3/4 vs. 5/9); that is, the figure with the smaller number of grey and total segments was the one with the proportionally larger grey area, and accordingly, the fraction with the smaller components was the one with the larger magnitude (again, 3/4 vs. 5/9). For consistency, continuous trials showing the same proportions were labeled as compatible or misleading. For example, the continuous trial showing the 3/4 vs. 5/9 pair was labeled as a misleading trial; however, this label was meaningless, as there was no whole-number information in continuous trials. Half of the comparisons were whole-number compatible, and the other was misleading.

At the beginning of each nonsymbolic block, participants saw these on-screen instructions: " Your job is to decide which of the two figures has the proportionally larger grey area." Furthermore, they were instructed to press 'z' if the stimulus on the left was the one with the proportionally larger quantity or 'm' if the one on the right was a larger proportional quantity (Fig. 1D). Next, they viewed an example trial where they had to press the space bar to continue. After this example trial, participants completed four practice trials with performance feedback (i.e., correct, incorrect, or no response messages). All practice trials were neutral (i.e., pairs in which the larger fraction had the larger numerator but the smaller denominator), had large distance (>0.42) and were not presented during the main task (Appendix A). Participants had to answer three of the four trials correctly to advance to the experimental trials. Otherwise, participants saw the instructions for a second time and completed the same four practice trials with feedback but in a different order. After the second set of practice trials, if participants failed to answer at least three trials correctly, they saw the instructions a third time before starting the experimental task; otherwise, they moved to the experimental trials. Example and practice trials for the symbolic block were identical to those of the nonsymbolic blocks, but participants compared symbolic fractions instead of bicolored bars.

For all blocks, practice and experimental trials started with a fixation cross that remained on the screen for 500 ms (ms), followed by a blank screen for another 500 ms. Then, the stimuli appeared and remained on the screen for 6000 ms or until the participants responded (Fig. 1D).

As the primary outcome variable for this study was the proportion of correct responses, we excluded anticipatory responses (reaction times [RT] shorter than 250 ms) and outlier responses (RTs at least 3 standard deviations above the individual's mean). After applying these criteria, 19 participants from the full sample did not have at least 70% of trials from each experimental block and were excluded from the final sample. Among the remaining 159 students of the final sample, we analyzed 18,785 (97.50%) of 19,266 trials.

Stimuli. Fraction pairs were chosen from the 98 pairs reported in Binzak and Hubbard's (2020), which had the following properties: all fractions were single-digit, irreducible proper fractions, and pairs did not include unit fractions, common components, or pairs where denominators or numerators were multiples of each other. For this study, we also removed neutral trials, resulting in 71 pairs. Then, we removed 12 pairs for which our size inconsistent manipulation (more details below) could not be applied. In these trials, the smaller bar, with the proportionally larger grey area, had an absolute larger grey area instead of a smaller one. For example, for the 7/8 vs. 2/7 pair, the smaller bar (of size 250 px) would have a grey area of 219 px, while the larger bar would have a smaller

Table 2
Fraction distance matching between whole-number compatible and misleading trials.

	Compatible	Misleading
Distance group	*M (SD)*	*M (SD)*
Group 1	0.046 (0.02)	0.046 (0.02)
Group 2	0.176 (0.02)	0.176 (0.02)
Group 3	0.270 (0.04)	0.270 (0.03)
Full set	0.160 (0.10)	0.160 (0.10)

grey area of 129 px out of 450 px. Finally, we wanted to ensure that the compatible and misleading trials were matched in distance between the pairs, and that they sampled equivalently across the distance space. Thus, from the remaining 59 trials, we categorized them into three groups based on their fraction distance: Group 1 had pairs with distances smaller than 0.10, Group 2 had distances between 0.10 and smaller than 0.20, and Group 3 had distances from 0.20 to 0.32. Then, we chose all the misleading trials of each group: eleven trials from Group 1, four from Group 2, and four from Group 3, for a total of 19 misleading trials. For each group, the same number of compatible trials was randomly chosen to result in a set of compatible trials with the same average fraction distance and standard deviation (Table 2). Thus, there were 38 distinct pairs in total, 19 compatible and 19 misleading (Appendix B).

To avoid participants relying on the absolute size of the grey area, continuous and discretized trials could be size *congruent* or size *incongruent*. For congruent trials, the figure with the larger grey area in the longer bar was the one with the proportionally larger grey area. By contrast, for incongruent trials, the figure with the smaller grey area in the shorter bar was the one with the proportionally larger grey area. All bars had a width of 75 pixels (px) but varied in height. Bars' height could be 140, 250, or 450 px. For size congruent trials, the bar with the proportionally larger grey area was always the 450-px bar, and the other bar was the 250-px bar. For size incongruent trials, the bar with the proportionally larger area was the 140-px bar and the other bar was the 250-px bar. Notably, during online data collection, bars were scaled proportionally, based on the resolution of the participant's monitor. Concerning the symbolic trails, to include size variation, but not introduce numerical Stroop effects (Henik & Tzelgov, 1982), we varied the height of the fractions: within each trial both fractions had the same size, either large (450 px) or small (140 px).

The manipulations of size congruency (congruent and incongruent) and side of correct response (left and right) results in four different presentations for each of the 38 stimulus pairs. To present only one version of each of the 38 trials per condition per participant, while counterbalancing size congruency and response side, we created four stimulus schedules (A to D) by dividing the four potential presentations of each stimulus pair across the four schedules. Within each stimulus schedule, half of the trials were size congruent, while the other half were size incongruent, and half of the trials presented the correct answer on the right and the other half on the left. Participants were randomly assigned to one of these stimulus schedules and saw the corresponding stimulus schedule for all formats. For example, if participants saw schedule A for the continuous format, they also saw this schedule for the discretized and symbolic formats.

2.4. Analysis methods

All statistical analyses were performed using R 3.5.3 (R Core Team, 2019). Generalized linear mixed-effects and linear mixed-effects models were conducted using the *glmer* and *lmer* functions, respectively, from the *lme4* package (Bates et al., 2015). For all fixed effects, we used treatment coding (i.e., one of the categorical variable levels was assigned 0, and the other levels represent differences concerning that level). The reference level was assigned alphabetically when there were only two levels; when there were more than two, the reference level is specified in the text. Lastly, the comparison levels are presented in square brackets (e.g. [Chance-Level Group]) both in the text and in the tables. Post-hoc comparisons and simple slope analyses were performed using the *emmeans* and *emtrends* functions from the *emmeans* package (Lenth et al., 2018). Interactions that involved continuous variables (e.g., fraction distance) were plotted using the *ggpredict* function from the *ggeffects* package (Lüdecke et al., 2018). To account for the nested nature of our data (participants attended one of three schools and were in either 6, 7, or 8 grade), we included School as a fixed factor in all our models, as School had a more consistent effect than grade (see Supplementary Analyses 1). Further, for simplicity, we did not include the Size (congruent and incongruent) in our models, as Size was only a control variable and did not interact with any of the variables of interest (see Supplementary Analyses 2). The cluster analyses were carried out using the *kmeans* function from the *stats* package. Then, the *n_clusters* function from the *parameters* package (Lüdecke et al., 2020) was used to determine the optimal number of clusters.

To achieve our first goal of examining the effects of Whole-Number Congruency (hereafter Congruency), Format, and Fraction Distance within the nonsymbolic formats, we used a generalized linear mixed-effect models, with a binomial distribution with a logit link function and students' accuracy in the proportional comparison task as the dependent variable. In the Nonsymbolic Distance Model, Congruency (compatible and misleading), Format (continuous and discretized), and Fraction Distance (z-scored to improve convergence) were fixed effects as well as their interactions. A random intercept was also included for participants and a random slope for Congruency by participant. To control for the effects of school, we introduced School as a categorical covariate using two dummy variables: School 1 vs. School 2 and School 1 vs. School 3.

To achieve our second goal of examining the effects of Congruency and Fraction Distance on students' performance within the symbolic format, again, we used generalized linear mixed-effects models. In the Symbolic Distance Model, we introduced Congruency (compatible and misleading) and the *z*-scored Fraction Distance as fixed effects as well as their interactions. Similar to the nonsymbolic model, we also included a random intercept for participants, a random slope for Whole-Number Congruency by participant, and School (two dummy variables: school 1 vs. school 2, school 1 vs. school 3) as a categorical covariate.

Third, we examined the relationship between performance on the nonsymbolic continuous and discretized formats and symbolic format (third goal). First, to compare our results with previous findings, we conducted a Pearson's correlation between overall performance in the nonsymbolic conditions (averaged across all four types of nonsymbolic trials) and overall performance in symbolic conditions (averaged across the two symbolic trial types). Then, to determine the contributions of each nonsymbolic format to symbolic proportional reasoning, we performed Pearson's correlations between the two nonsymbolic formats (continuous and discrete) and symbolic fraction performance. Further, to compare the nonsymbolic contributions to symbolic fractions, we included performance on each nonsymbolic format as predictors of symbolic fraction performance in the same model. Lastly, to examine whether these relations may vary depending on whole-number information compatibility, we conducted a series of Pearson's correlations with averaged accuracy scores for the six types of trials (three formats by two whole-number information trials).

To achieve our final goal, to characterize students' nonsymbolic comparison strategies, we performed cluster analyses, using a four-dimensional space that only included performance on the four types of nonsymbolic trials. To select the optimal number of clusters for the k-means algorithm, we used three indices: the percentage of explained variance (R^2), the Akaike Information Criterion (AIC), and the maximum number of converging methods according to the $n\_clusters$ function. This function conducts 29 different methods (e.g., elbow, silhouette, gap, Dunn, among others) to determine the optimal number of clusters and selects the number based on the larger consensus across methods.

Finally, we then returned to the nonsymbolic and symbolic generalized linear-mixed models and examined the effects of Cluster as a fixed effect and its interactions with Format, Congruency, and Fraction Distance.

3. Results

3.1. Middle-school students' continuous and discretized nonsymbolic proportional reasoning

To examine the effects of Format (continuous and discretized), Congruency (compatible and misleading), and Fraction Distance (z-scored) among the nonsymbolic trials, we performed a generalized linear mixed model (Nonsymbolic Distance Model, Table 3) with accuracy as the dependent variable and School as a covariate (two dummy variables: school 2 vs. school 1, school 3 vs. school 1). This analysis yielded a main effect of Format (estimate = 0.144, SE = 0.057, z-value = 2.154, p =.012), which was qualified by an interaction between Format and Congruency (estimate = -0.590, SE = 0.080, z-value = -7.355, p <.001, Fig. 3A). To further understand this interaction, we compared the marginal means using *emmeans*. First, participants performed above chance level (50%) across all types of trials (all p-values <.007), suggesting that they were engaged with the task. Second, as expected, there were no differences between the compatible and misleading trials among the continuous format (z-ratio = -0.487, p =.626) as there was no whole number information in these conditions, and difficulty (i.e., fraction distance) was matched between them. Finally, students had lower performance in the discretized misleading trials in comparison to the discretized compatible trials (z-ratio = 7.417, p <.001), suggesting that even 6th-to-8th-graders show interference of whole number information in their nonsymbolic discretized proportional reasoning (i.e., a nonsymbolic whole-number bias).

This analysis also yielded a main effect of Fraction Distance (estimate = 0.488, SE = 0.042, z-value = 11.49, p <.001) and a three-way interaction between Format, Congruency and Fraction Distance (estimate = -0.202, SE = 0.084, z-value = -2.402, p =.016, Fig. 4A). In addition, simple slope analyses indicated that the slopes for all types of trials were different from zero (p <.001), suggesting that students' performance was always modulated by the fraction distance between the two stimuli. Furthermore, slope comparisons using *emtrends* show no differences between the slopes of compatible and misleading trials for the continuous stimuli (estimate = 0.034, SE = 0.060, z-ratio = 0.566, p =.571). In contrast, among the discretized stimuli, students' performance in the compatible trials were more strongly modulated by fraction distance than misleading trials (estimate = 0.236, SE = 0.60, z-ratio = 3.986, p <.001).

Together, these results indicate that middle-school students struggle with discretized trials when the whole-number information is at odds with the proportional information. Notably, whole number congruency not only impacts general performance of these trials but also impairs fraction magnitude processing. Consistent with the interaction effects theoretical outcome shown in Fig. 2D, performance was around the chance level for both discretized compatible and misleading trials with near distances. Still, students were more accurate for compatible trials with far distances than misleading trials with similar distances, suggesting that whole-number information may have suppressed fraction magnitude processing of the latter trials.

3.2. Middle-School students' fraction magnitude processing

To determine the effects of Congruency (compatible and misleading) and Fraction distance (z-scored) among the symbolic trials, we also performed a generalized linear mixed model with accuracy as the dependent variable and school as a covariate (Symbolic Distance

Table 3
Nonsymbolic model.

| Predictors	Nonsymbolic Distance Model			
	Estimate	SE	z-value	p-value
Intercept	**0.642**	**0.062**	**10.367**	**<.001**
Congruency [misleading]	0.036	0.075	0.487	.626
Format [discretized]	**0.144**	**0.057**	**2.154**	**.012**
School [school 2]	−0.016	0.084	−0.187	.852
School [school 3]	**−0.183**	**0.079**	**−2.308**	**.021**
Congruency [misleading] × Format [discretized]	**−0.590**	**0.080**	**−7.355**	**<.001**
Fraction Distance	**0.488**	**0.042**	**11.495**	**<.001**
Congruency [misleading] × Fraction Distance	−0.034	0.060	−0.566	.571
Format [discretized] × Fraction Distance	0.039	0.061	0.632	.527
Congruency [misleading] × Format [discretized] × Fraction Distance	**−0.202**	**0.084**	**−2.402**	**.016**
N	159			
Observations	11,843			
Marginal R^2/ Conditional R^2	0.067 / 0.127			
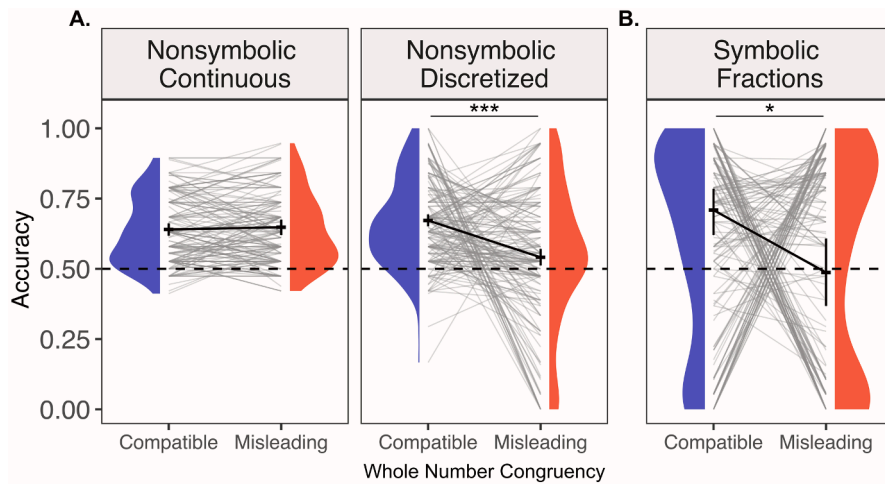
Fig. 3. Overall performance on the nonsymbolic and symbolic proportional comparison tasks. **A.** Middle schoolers' performance in the nonsymbolic formats was differentially affected by the whole number congruency. As expected, there were no differences in students' accuracy in comparing nonsymbolic compatible and misleading continuous trials, as there was no whole-number information in these trials. By contrast, students in the nonsymbolic discretized format had a lower performance on trials with misleading whole-number information (e.g., 3/4 vs. 5/9) as opposed to trials with compatible information (e.g., 5/8 vs. 3/7). **B.** Similarly, whole-number congruency also affected participants' performance in the symbolic format. Participants had lower accuracy in symbolic misleading trials in comparison to compatible trials. Notably, students' performance was no better than guessing on the misleading trials. *Notes.* Grey lines represent individual participants. Horizontal black bars represent the marginal means from the corresponding generalized linear mixed effect models (nonsymbolic Table 3, symbolic Table 4), and error bars represent 95% confidence intervals. Density clouds show the probability density of the observed accuracy scores. *$p <$ .05, **$p <$ .01, ***$p <$ .001.

Model, Table 4). The analysis showed a main effect of Congruency (estimate = -0.943, SE = 0.432, z-value = -2.180, p =.029, Fig. 3B). Comparing marginal means against chance level (50%) revealed that while students were able to successfully compare compatible pairs of fractions ($p <$.001), they were not better than chance in the misleading trials (p =.837). Fraction Distance was also significant (estimate = 0.466, SE = 0.054, z-value = 8.532, p = .004). However, these effects were qualified by a marginal interaction between them (estimate = -0.143, SE = 0.079, z-value = -1.816, p = .069, Fig. 4B). Follow-up simple slope analyses showed both slopes were different from zeros ($p <$.001), suggesting that although participants' performance was modulated by fraction distance in both types of trials, distance effects were stronger for compatible trials than misleading ones, consistent with the interaction effects outcome (Fig. 2D).

Overall, these results suggest that whole-number information affects middle-school students' symbolic fraction comparison skills since performance plummeted when fraction components were at odds with the proportional information (e.g., 3/4 vs. 5/9). Yet, performance was still modulated by the fraction distance, marginally more strongly for compatible trials than misleading ones.

3.3. Relations between nonsymbolic and symbolic proportional reasoning

To compare our results with previous findings, we first examined the relationship between overall performance in the nonsymbolic conditions (averaged across all four types of nonsymbolic trials) and overall performance in the symbolic conditions (averaged across the two symbolic trial types). This correlation showed a moderate relationship between nonsymbolic and symbolic performance (r (1 5 7) = .386, $p <$.001, Fig. 5A). Then, to determine the contributions of each nonsymbolic format to symbolic proportional reasoning,

Table 4
Symbolic model.

| Predictors	Symbolic Distance Model			
	Estimate	SE	z-value	p-value
Intercept	**0.919**	**0.220**	**4.184**	**<.001**
Congruency [misleading]	**−0.943**	**0.432**	**−2.180**	**.029**
School [school 2]	**0.383**	**0.166**	**2.309**	**<.001**
School [school 3]	**−0.467**	**0.162**	**−2.881**	**.021**
Fraction Distance	**0.466**	**0.054**	**8.532**	**.004**
Congruency [misleading] × Fraction Distance	−0.143	0.079	−1.816	.069
N	159			
Observations	5884			
Marginal R^2/ Conditional R^2	0.043 / 0.708			
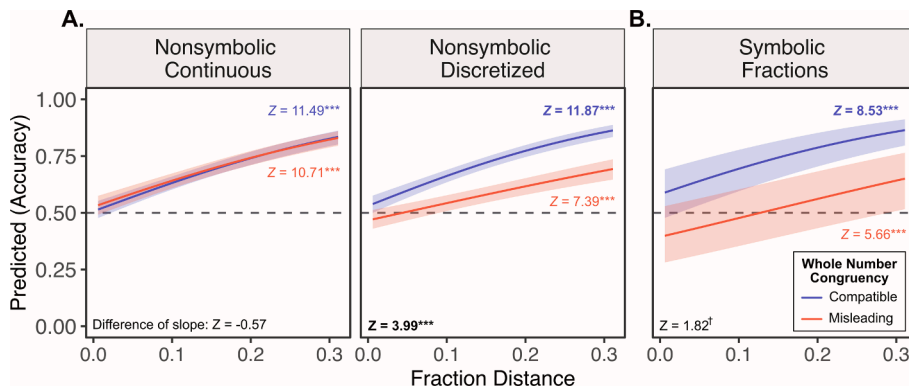
Fig. 4. Distance effects across nonsymbolic and symbolic formats and whole-number congruency conditions. **A.** Middle-schoolers modulated their responses by the fraction distance in both continuous and discretized formats. However, whole-number congruency hindered magnitude processing for the discretized format. Particularly, students showed a weaker distance modulation (flatter slope) for misleading trials than compatible trials. **B.** Students' symbolic fraction performance was also modulated by the fraction distance, but whole-number congruency had a marginal effect on this modulation, where, akin to performance on the discretized format, students had a weaker modulation for misleading than compatible trials, although differences was only marginal. *Notes.* Lines represent the fitted lines from the corresponding generalized linear mixed effect models (nonsymbolic Table 3, symbolic Table 4), and shaded areas represent 95% confidence intervals. When the difference of slopes is significant, the *z*-scores in black appear in bold, while the *z*-score for the corresponding line also appears in bold. *$p < .05$, **$p < .01$, ***$p < .001$.

we looked at the relationship between performance in the two nonsymbolic formats (continuous and discrete) and symbolic fraction performance. These analyses revealed a weaker but significant relationship between performance on the continuous trials and the symbolic trials ($r(157) = .268, p < .001$, Fig. 5B) and a moderate relationship between discretized trials and symbolic trials ($r(157) = .388, p < .001$, Fig. 4C). Notably, the two nonsymbolic formats showed a robust relationship with each other ($r(157) = .457, p < .001$). Further, to determine the independent contributions of each nonsymbolic format to symbolic skills, we performed a linear regression model with both nonsymbolic continuous and nonsymbolic discretized skills in the same model. This model was significant ($F(2,156) = 14.95, p < .001$) and, critically, revealed that only discretized skills ($p < .001$), not continuous ones ($p = .165$), have independent contributions to symbolic fraction ability.

These results converge with past findings indicating that nonsymbolic performance relates to symbolic proportional skills when combining different nonsymbolic formats, like dots and continuous bars (Matthews et al., 2016), and that discretized proportional skills have a stronger relation to symbolic skills than continuous ones (Begolli et al., 2020). They are also aligned with recent findings showing that across formats, nonsymbolic proportional skills show moderate-to-strong relations to each other (Park et al., 2021). Nevertheless, as some past work has collapsed performance across nonsymbolic continuous and discretized formats, it is unknown whether these relations may vary depending on whether trials are compatible or not with whole number information, particularly, as performance in the discretized and symbolic formats differed depending on their whole number congruency (i.e., higher performance on whole number compatible than misleading trials). Thus, we performed pairwise correlations between the six types of nonsymbolic and symbolic trials (Fig. 5D). These analyses revealed the following findings: A) Performance in the two types of continuous trials was strongly correlated ($r(157) = .64, p < .001$), indicating excellent internal reliability of our measure. B) For nonsymbolic discretized and symbolic trials, performance between the compatible and misleading trials was negatively related, suggesting that a large number of participants used a strategy that was adequate for one type of trial but unsuccessful for the other. C) Performance in the continuous trials did not contribute to either of the two symbolic trials (*p*-values ranged from .061 to .726). D) Finally, performance in the compatible and misleading trials of the discretized format was related to their corresponding symbolic trials. In fact, the strongest cross-format relationship was between misleading discretized and misleading symbolic. Given that nonsymbolic discretized and symbolic fractions both introduce whole-number information, these results suggest that a driving feature for the relation between nonsymbolic and symbolic skills is the use of similarly inappropriate whole number strategies for both trial types.

3.4. Cluster analyses

3.4.1. Nonsymbolic strategy profiles

To further explore the pattern of responses across the nonsymbolic formats and their relation to symbolic proportional reasoning, we categorized participants based on their performance in the four nonsymbolic conditions using a *k*-means clustering algorithm. According to the percentage of explained variance (R^2), the AIC, and the number for converging methods, the optimal number of clusters was three with an R^2 of 0.60, an AIC of 31.20, and 14 of 29 (48%) methods selecting this number of clusters (Table 5).

Fig. 6A shows the accuracy in the different types of nonsymbolic trials of the three profiles, according to the cluster analysis. We labeled the profiles (Columns in Fig. 6) based on the pattern of performance in the discretized trials, where accuracy between clusters differed the most. The first profile comprised 28 participants (17.6%) who showed a close-to-ceiling effect on discretized compatible trials but a near-to-floor effect on the misleading ones (*Whole-Number Biased* profile). The second profile included 68 students (42.8%) who had a smaller advantage in the compatible trials in comparison to the misleading ones of the discretized format but were around
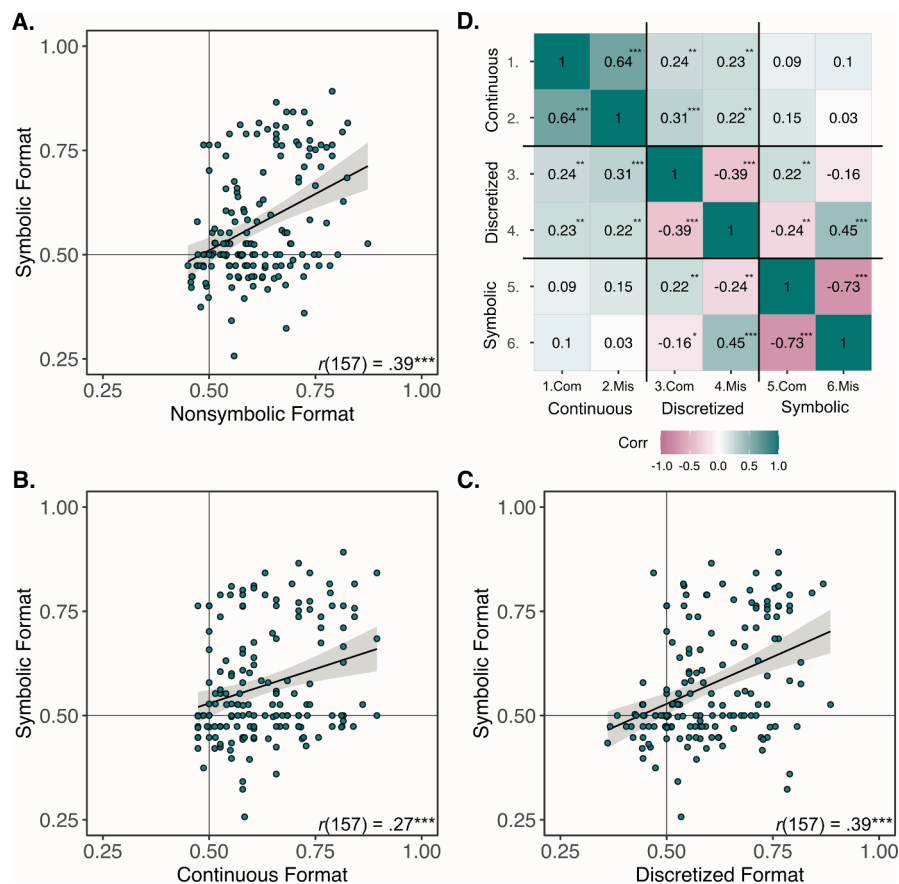
Fig. 5. Correlations between nonsymbolic and symbolic proportional comparison skills. **A.** There was a moderate relation between performance in the nonsymbolic formats (averaged across all four types of nonsymbolic trials) and overall performance in the symbolic conditions (averaged across the two symbolic trial types). **B, C.** When assessing the contributions of each nonsymbolic format to symbolic proportional reasoning, there was a weaker but significant relationship between performance on the continuous trials and the symbolic trials and a moderate relationship between discretized trials and symbolic trials. **D.** Pairwise correlations between the six types of nonsymbolic and symbolic trials were used to explore whether relations varied depending on whether trials were compatible or not with whole-number information. Notably, the strongest cross-format correlations were between the compatible and misleading trials of the discretized format with the corresponding symbolic trials. *Notes.* $*p < .05$, $**p < .01$, $***p < .001$.

Table 5
Statistics for the selection of the number of clusters.

Number of clusters	Explained variance (R^2)	AIC	Converging methods
1	0.00	26.08	2
2	0.35	27.64	8
3	0.60	31.20	14
4	0.66	38.04	1
5	0.69	45.39	0
6	0.73	52.72	0
7	0.75	60.37	0
8	0.77	67.90	2
9	0.78	75.74	0
10	0.80	83.44	2

chance level (50%) in the latter trials (*Chance-Level Performance* profile). Finally, the third profile comprised 63 participants (39.6%) whose performance was above chance level for the two types of discretized trials (*High-Performance* profile). Regarding their continuous performance, the high-performing profile also had the highest accuracy in this format, followed by the Whole-Number Biased profile and then the Chance-Level Performance profile.
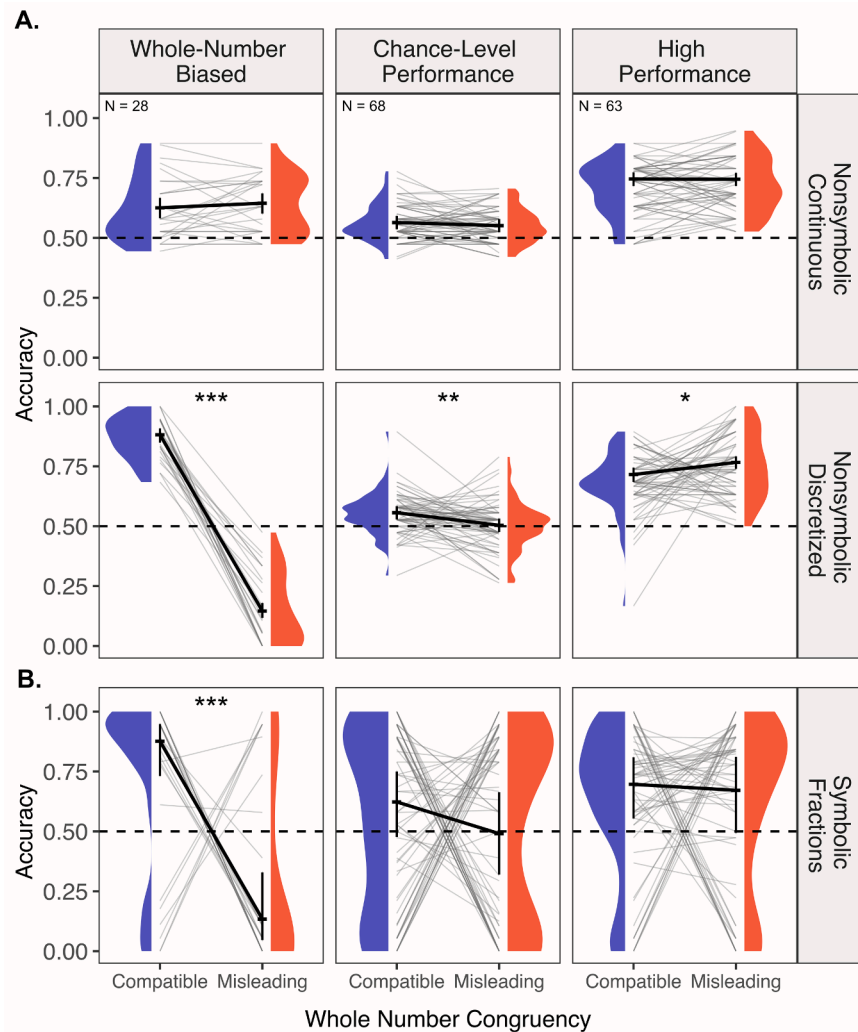
Fig. 6. Nonsymbolic proportional reasoning profiles' performance for the nonsymbolic and symbolic formats. Cluster analyses revealed three nonsymbolic proportional reasoning profiles **A.** The *Whole-Number Biased* profile successfully compared continuous proportions but showed a close-to-ceiling effect on discretized compatible trials but a near-to-floor effect on the misleading ones. The *Chance-Level Performance* profile had the lowest performance in the continuous trials but showed a smaller advantage in the compatible trials in comparison to the misleading ones of the discretized format; however, students with this profile were around chance level (50%) in the latter trials. The *High-Performance* profile showed a performance above chance level for all types of nonsymbolic trials. **B.** Performance of the Whole-Number Biased profile in the symbolic trials resembled that of the discretized trials: close to ceiling-level performance in the compatible trials, but close to floor-level in the misleading ones. Students' performance in the Chance-Level profile was no better than guessing in both types of symbolic trials and there were no differences between them. Similarly, the High-Performance profile also did not show differences between the symbolic trials; however, students with this profile performed above chance in the compatible trials and marginally above chance in the misleading ones. *Notes.* Grey lines represent individual participants. Horizontal black bars represent the marginal means from the corresponding generalized linear mixed effect models (nonsymbolic Table 6, symbolic Table 7), and error bars represent 95% confidence interval. Density clouds show the probability density of the observed accuracy scores. $^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$.

3.4.2. Nonsymbolic models

To quantify differences in the performance between groups across the two nonsymbolic formats, we started with the Nonsymbolic Distance Model—which comprised Format, Congruency, and Fraction Distance as fixed effects—and added Profile (Whole-Number Biased, Chance-Level Performance [reference category], and High-Performance profiles) as a fixed effect, as well as its interactions. However, we did not include the random slope for Congruency by participant, as the model that included it failed to converge. Instead, we only included participant as a random slope.

Table 6 shows the results of the Nonsymbolic Distance Cluster Model. In addition to the main effect of Fraction Distance and the interaction between Congruency and Format already observed in the Nonsymbolic Distance Model, this analysis revealed that students

Table 6
Nonsymbolic Cluster Model.

Predictors	Estimate	SE	z-value	p-value
Intercept	**0.284**	**0.063**	**4.510**	**<.001**
Congruency [misleading]	−0.053	0.081	−0.654	.513
Format [discretized]	−0.032	0.081	−0.396	.692
School [school 2]	0.004	0.051	0.077	.939
School [school 3]	−0.083	0.050	−1.670	.095
Congruency [misleading] × Format [discretized]	**−0.158**	**0.114**	**−1.381**	**.167**
Fraction Distance	**0.268**	**0.059**	**4.574**	**<.001**
Congruency [misleading] × Fraction Distance	−0.008	0.082	−0.097	.923
Format [discretized] × Fraction Distance	0.034	0.083	0.404	.686
Congruency [misleading] × Format [discretized] × Fraction Distance	−0.064	0.116	−0.549	.583
Group [whole-number bias]	**0.256**	**0.109**	**2.344**	**.019**
Group [high performance]	**0.820**	**0.097**	**8.439**	**<.001**
Congruency [misleading] × Group [whole-number bias]	0.135	0.153	0.883	.377
Congruency [misleading] × Group [high performance]	0.047	0.134	0.352	.725
Format [discretized] × Group [whole-number bias]	**1.526**	**0.188**	**8.105**	**<.001**
Format [discretized] × Group [high performance]	−0.122	0.135	−0.907	.365
Fraction Distance × Group [whole-number bias]	0.029	0.111	0.264	.792
Fraction Distance × Group [high performance]	**0.733**	**0.108**	**6.779**	**<.001**
Congruency [misleading] × Format [discretized] × Group [whole-number bias]	**−3.701**	**0.258**	**−14.352**	**<.001**
Congruency [misleading] × Format [discretized] × Group [high performance]	**0.428**	**0.188**	**2.271**	**.023**
Congruency [misleading] × Fraction Distance × Group [whole-number bias]	0.089	0.158	0.563	.574
Congruency [misleading] × Fraction Distance × Group [high performance]	−0.223	0.148	−1.513	.130
Format [discretized] × Fraction Distance × Group [whole-number bias]	0.163	0.203	0.803	.422
Format [discretized] × Fraction Distance × Group [high performance]	−0.079	0.150	−0.529	.597
Congruency [misleading] × Format [discretized] × Fraction Distance × Group [whole-number bias]	**−0.580**	**0.271**	**−2.142**	**.032**
Congruency [misleading] × Format [discretized] × Fraction Distance × Group [high performance]	−0.006	0.206	−0.030	.976
N	159			
Observations	11,834			
Marginal R2/ Conditional R2	0.205 / 0.206			

in the *Whole-Number Biased* profile (58%) and the *High-Performance* profile (74%) had, in general, higher performance in the nonsymbolic trials than those in the *Chance-Level Performance* profile (54%). This main effect of Profile was qualified by both an interaction between Format and Profile [whole-number biased profile], and two three-way interactions between Congruency, Format, and Profile for both the whole-number biased profile and the high-performance profile.

These interactions suggest that profiles' averaged performance differed across the two nonsymbolic formats and the whole-number congruency conditions, which we first examined by comparing the marginal means against chance-level (50%). These analyses showed that while all three profiles performed above chance levels in the two types of continuous trials (all p-values < 0.001, Fig. 6A, upper panel), differences appeared in the discretized format (Fig. 6A, lower panel): students in the *Whole-Number Biased* profile performed above chance in the compatible trials but below chance in the misleading trials. Children in *the Chance-Level* profile were slightly above chance-level in the compatible trials (55%) but were no better than guessing in the misleading trials. Lastly, students in the *High-Performance* profile were well-above chance level in both compatible and misleading trials and were even slightly better on the misleading trials than the compatible ones.

Performance also differed across profiles by Fraction Distance, as suggested by the two-way interaction between Fraction Distance and Profile [high performance] and the four-way interaction between Congruency, Format, Fraction Distance, and Profile [whole-number bias] (Fig. 7A). Further series of follow-up simple-slope and slope-comparison analyses revealed the following results: A) Performance of the *Whole-Number Biased* profile was modulated by Fraction Distance in the Continuous format; however, in the Discretized format, performance in the compatible trials, but not misleading ones, showed distance effects, consistent with the interaction effects theoretical outcome (Fig. 2D). B) For the *Chance-Level* profile, even though performance was close to chance-level in the discretized trials, responses were modulated by fraction distance in both Continuous and Discretized formats, regardless of the whole-number condition, resembling the distance-effect-only outcome (Fig. 2A). C) Similarly, the *High-Performance* profile showed distance effects in both formats and whole-number conditions; however, students with this profile showed a stronger modulation in the discretized compatible trials than in the misleading ones, consistent with the interaction effects outcome (Fig. 2D). This relative reduction in modulation for the misleading trials led to better performance at the more difficult, near distances, resulting in the slightly better overall performance on misleading trials for this group.

Turning to comparisons between profiles, in the Continuous format, the *High-Performance* profile showed the strongest distance effects in both types of continuous trials compared to the *Chance-Level* and *Whole-Number Biased* profiles, which did not differ from each other. In the compatible trials of the Discretized format, the *High-Performance* profile showed the strongest effects, while there were no differences between the two other profiles. Finally, in the misleading discretized trials, the *High-Performance* profile showed the strongest effects, followed by the *Chance-Level* profile, which showed a greater modulation than the *Whole-Number Biased* group. Together, these results suggest whole-number information does not affect students' performance equally. In particular, students in the
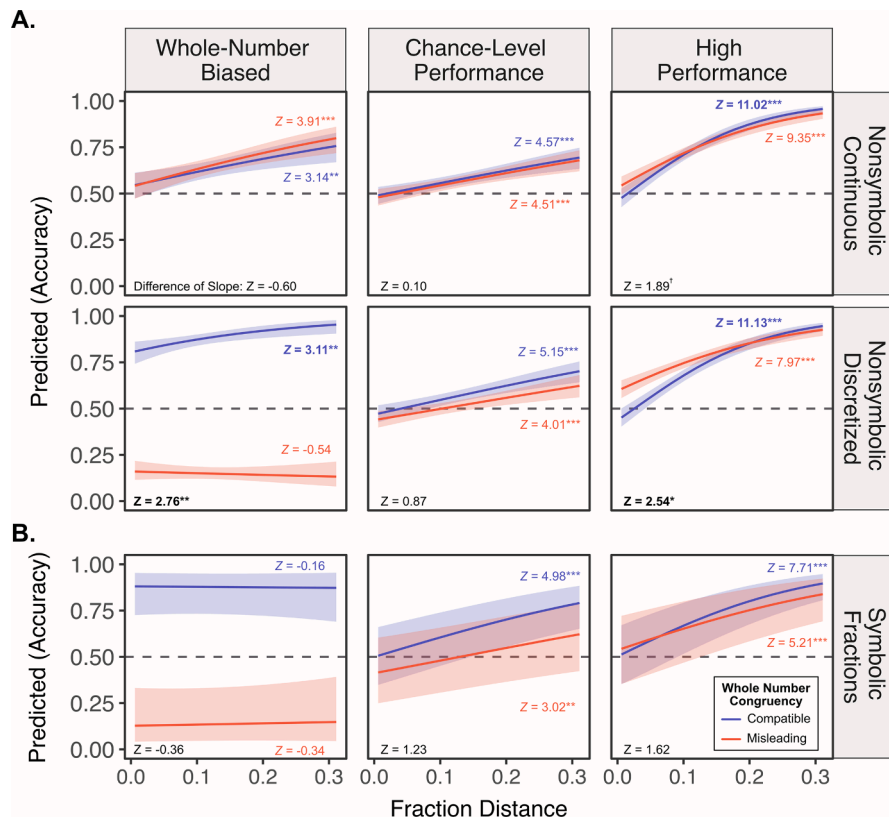
Fig. 7. Distance effects of the nonsymbolic proportional reasoning profiles across the nonsymbolic and symbolic formats and whole-number congruency conditions. **A.** Performance of the *Whole-Number Biased* profile was modulated by Fraction Distance in the Continuous format; however, in the Discretized format, performance in the compatible trials, but not misleading ones, showed distance effects. For the *Chance-Level profile*, responses were modulated by fraction distance in both Continuous and Discretized formats, regardless of the whole-number condition. The *High-Performance* profile showed distance effects in both formats and across the two whole-number conditions. **B.** Students with the Whole-Number Biased profile did not modulate their responses by the fraction distance, while students in the Chance-Level and High-Performance profiles did. *Notes.* Lines represent the fitted lines from the corresponding generalized linear mixed effect models (nonsymbolic Table 6, symbolic Table 7), and shaded areas represent 95% confidence intervals. When the difference of slopes is significant, the *z*-scores in black appear in bold, while the *z*-score for the corresponding line also appears in bold. *$p < .05$, **$p < .01$, ***$p < .001$.

Table 7
Symbolic Cluster Model.

Predictors	Estimate	SE	z-value	p-value
Intercept	0.513	0.315	1.630	.103
Congruency [misleading]	−0.539	0.639	−0.844	.399
School [school 2]	**0.348**	**0.161**	**2.161**	**.031**
School [school 3]	**−0.383**	**0.158**	**−2.422**	**.015**
Fraction Distance	**0.408**	**0.082**	**4.985**	**<.001**
Congruency [misleading] × Fraction Distance	−0.146	0.119	−1.229	.219
Group [whole-number bias]	**1.457**	**0.574**	**2.537**	**.011**
Group [high performance]	0.328	0.437	0.750	.453
Congruency [misleading] × Group [whole-number bias]	**−3.286**	**1.199**	**−2.741**	**.006**
Congruency [misleading] × Group [high performance]	0.424	0.920	0.461	.645
Fraction Distance × Group [whole-number bias]	**−0.432**	**0.168**	**−2.576**	**.010**
Fraction Distance × Group [high performance]	**0.250**	**0.118**	**2.126**	**.034**
Congruency [misleading] × Fraction Distance × Group [whole-number bias]	0.223	0.245	0.909	.363
Congruency [misleading] × Fraction Distance × Group [high performance]	−0.052	0.170	−0.307	.758
N	159			
Observations	5884			
Marginal R2/ Conditional R2	0.094 / 0.709			

Whole-Number Biased profile had great difficulty accessing proportional magnitudes when this information is at odds with whole-number information. Further, students in the High-Performance group have strong distance modulation to continuous and discretized trials and do not display any whole number interference effects; instead, they showed an advantage for discretized misleading trials over continuous ones.

3.4.3. Symbolic models

To examine how the different nonsymbolic strategy profiles differed in their symbolic magnitude processing, we included Profile and its interactions as fixed factors to the Symbolic Distance Model, which included Congruency and Fraction Distance, using the Chance-Level Performance profile as the reference level.

Table 7 shows the results of the Symbolic Distance Cluster Model. Aside from the Fraction Distance effect already reported in the Symbolic Distance Model, we found an effect of Profile [whole-number biased], which was qualified by a two-way interaction between Congruency and Profile [whole-number biased] (Fig. 6B). Critically, follow-up marginal mean comparisons indicated that similar to their pattern of responses in the discretized format, children's performance in the *Whole-Number Biased* profile was consistent with this bias: they had a close to ceiling-level performance in the compatible trials, but close to floor-level in the misleading ones, suggesting that they rely on the rule of choosing the fraction with the largest component. Students' performance in the *Chance-Level* profile was no better than guessing in both compatible ($p =$.099) and misleading ($p =$.919) trials and there were no differences between the two types of trials in this profile ($p =$.399). Similarly, the *High-Performance* profile also did not show differences between the two types of trials ($p =$.861); however, students with this profile performed above chance in the compatible trials ($p =$.008) and marginally above chance in the misleading ones ($p =$.059). When comparing performance between profiles, we found that the Whole-Number Bias profile outperformed the High-Performance ($p =$.052) and Chance Level ($p =$.011) profiles in the compatible trials but showed the lowest accuracy in the misleading ones (all p-values $<$ .01). Notably, there were no differences in either type of trials between the High-Performance and Chance-Level groups.

The model also revealed two two-way interactions between Fraction Distance and Profile [whole-number bias] and Fraction Distance and Profile [high-performance] (Fig. 7B). Simple-slope and slope comparisons analyses showed that students with the Whole-Number Biased profile did not modulate their responses by the fraction distance ($p =$.892), matching the congruency effect outcome (Fig. 2A), while students in the High-Performance ($p <$.001) and Chance-Level profiles did ($p <$.001). Finally, group comparisons showed that the High-Performance had the strongest fraction distance effect, followed by the Chance-Level group, and lastly, the Whole-Number Bias group. Together, these results indicate that students who are yet to understand nonsymbolic discretized proportions, particularly misleading cases, do not understand symbolic fractions, and that they might be using the same erroneous whole-number strategies to work with both types of proportions. Finally, differences between students with the other two profiles (Chance-Level and High-Performance) might be due to differences in the precision to compare proportions rather than sensitivity to whole-number interference.

4. Discussion

The mechanisms underlying the relationship between nonsymbolic and symbolic proportional reasoning have remained largely understudied, particularly, when continuous representations may highlight proportional quantities and discretized ones (i.e., segmented stimuli that allow counting strategies) may prompt erroneous whole-number strategies. Most importantly, to our knowledge, no study has examined how these different nonsymbolic representations influence the relation between nonsymbolic and symbolic proportional comparison abilities. In the current study, middle-school students completed a nonsymbolic and symbolic comparison task comprising nonsymbolic continuous and discretized proportional quantities and symbolic fractions. Students were proficient at comparing continuous but struggled in cases where whole-number information contradicted discretized and symbolic proportional information (misleading discretized and symbolic trials). Critically, in aggregate, whole-number information disrupted students' discretized magnitude processing but not continuous or symbolic processing.

Additionally, we replicated the finding that individual differences in fraction comparison ability were related to nonsymbolic proportional reasoning skills. Crucially, however, discretized performance explained symbolic ability above and beyond the contributions of continuous performance when examined separately. Finally, cluster analyses revealed three profiles of nonsymbolic proportional reasoning in middle schoolers: *whole-number biased*, *chance-level performance*, and *high performance*. Students in these profiles differed in their nonsymbolic magnitude processing, as well as in their symbolic fraction skills. Importantly, students with a whole-number biased profile showed this bias in their fraction skills and, remarkably, failed to show any modulation of fraction magnitudes. Together, these results provide insights into the relation between nonsymbolic and symbolic proportional skills, indicating that it may be driven by misconceptions stemming from discretized representations and not based on their understanding of proportional magnitudes as indexed by continuous comparisons. Finally, they call attention to the type of nonsymbolic stimuli used to introduce fractions, as discretized stimuli might be a critical source of students' misconceptions about symbolic fractions.

4.1. Whole-Number information impairs both discretized nonsymbolic and symbolic proportional reasoning in Middle-Schoolers

Consistent with prior findings with younger children (Abreu-Mendoza et al., 2020; Hurst & Cordes, 2018; Jeong et al., 2007), middle-schoolers were proficient at comparing continuous quantities (63%) but struggled with discretized and symbolic proportions, specifically, when whole-number information was at odds with the proportional magnitudes (e.g., 3/4 vs. 5/9). In the discretized format, students were above chance level for trials that were compatible or misleading with whole-number information; however,

performance was significantly lower in this latter type of trial (66% vs. 53% trials). Notably, the visual similarities between the continuous and discretized formats further indicate that segmenting bars, even without separating the segments, is sufficient to disrupt proportional reasoning, consistent with the view that counting may be an automatic process that needs to be inhibited to correctly access proportional information (Abreu-Mendoza et al., 2020). Whole-number information also impacted students' performance in the symbolic format. Middle schoolers had greater accuracy for compatible trials than misleading ones and were not better than chance in these latter trials (62% vs. 52% trials); however, visual inspection suggested that, instead of guessing, students may have used one of two strategies: always choosing the fraction with the largest components or choosing the fraction with the smallest ones. Notably, we cannot attribute these performance effects between formats and whole number information to differences in magnitude distance between conditions, as the same proportions were presented across the three formats and compatible and misleading trials were exactly matched in fraction distance.

Our current design also afforded assessing proportional magnitude processing, as indexed by distance effects (i.e., lower accuracy for near distances than far distances), and probing its interplay with congruency effects as measured by compatibility with whole-number information. While there is a handful of studies examining this interplay in symbolic fractions (e.g., Reinhold et al., 2020), no study has investigated it in nonsymbolic proportions. Here, we considered four possible theoretical outcomes for this interplay (Fig. 2A to 2D): congruency effects only, distance effects only, independent effects, or interaction effects. Students' performance in the continuous format was only modulated by fraction distance (Fig. 4A), as expected, due to the lack of whole-number information in these trials, aligning with the distance effects only outcome. They are also consistent with past studies showing that when children are asked to compare (Kalra et al., 2020) or match-to-sample (Bhatia et al., 2020) nonsymbolic continuous proportional quantities, they modulate their responses by the proportional magnitudes; similarly, adults' estimates of continuous proportions in numberline tasks are also modulated by the proportional magnitude (Meert et al., 2012). By contrast, for discretized trials, students' accuracy was modulated by fraction distance in both compatible and misleading trials, but the distance effects were weaker in the misleading trials compared to the compatible ones (Fig. 4A), consistent with the interaction effect outcome. This pattern of results suggests that even when nonsymbolic proportional magnitudes are accessed, whole-number information impairs magnitude processing. Regarding the symbolic fraction, students' performance was modulated by the fraction magnitude for both compatible and misleading, but overall accuracy was lower for misleading trials than compatible ones (Fig. 4B). These results align with the independent effect outcome and are similar to those reported by Meert et al. (2010), showing that young adults' reaction times are longer in misleading trials than those in compatible ones, but both types of trials showed similar distance effects.

Overall, these results suggest that the interplay between proportional magnitude processing and whole-number information varies depending on the representational format. Specifically, discretized proportional skills may be particularly vulnerable to whole-number information. An outstanding question for future research is whether this same pattern of results can be observed using discrete representations of proportions (e.g., dots), as the one observed here with discretized one. Finally, a note of caution is needed in interpreting these results, as they are observed when averaging students' performance with different levels of proportional skills (group-level analyses). In fact, the cluster analyses (person-oriented approach) provided a more nuanced view for the interplay between proportional reasoning and whole-number information, a point we consider after discussing the results of the correlation analyses.

4.2. The relation between nonsymbolic and symbolic proportional skills is driven by erroneous whole-number strategies and not by continuous proportional reasoning

One of the critical pieces of evidence for the role of nonsymbolic proportional skills as a foundation for later fraction ability is the relation between nonsymbolic and symbolic proportional comparison skills in young adults (Matthews et al., 2016). However, this first study considered a composite nonsymbolic score comprising performance across continuous and discrete formats, leaving unaddressed their specific roles to the relation between nonsymbolic and symbolic skills. Critically, later studies have found mixed evidence for this relation, which point to two critical features that determine the strength of such a relation: the nonsymbolic format and the symbolic outcome measure. Here, we aimed to examine the distinct contributions of the nonsymbolic formats to symbolic fraction magnitude processing.

Our results replicated the original findings by Matthew et al. (2016). Similar to the effect size reported in Matthew et al. study ($r = .33$), here we found a moderate relation between nonsymbolic skills and symbolic comparison ability ($r = .39$) when considering the averaged continuous and discretized nonsymbolic performance (Fig. 5A). However, a closer examination of this relation revealed that discretized skills had the stronger first-order relationship and further explained symbolic performance above and beyond the contributions of continuous performance (Fig. 5B & 5C). This result converges with a previous findings showing that discretized representations have stronger contributions to general fraction knowledge than continuous ones (Begolli et al., 2020). However, our findings partially diverge from those of Park and Matthews' (2021). In their recent replication study, college students' continuous proportional skills showed a stronger relation with fraction knowledge than discrete skills. A possible explanation for these discrepancies is participants' age (middle-schoolers vs. college students). However, some preliminary evidence in college students also shows that discretized misleading trials have the strongest relation with misleading fractions ($r = .32$) in this population (Abreu-Mendoza & Rosenberg-Lee, 2022). Alternatively, the discrepancies may be related to the lack of whole-number interference introduced by Park and Matthews' discrete format, as the number of dots used were outside the counting range. More research is warranted to test the limits of the contributions of whole-number interference, as some formats (discrete vs. discretized) and designs (presenting interspersed nonsymbolic formats) may encourage to rely more or less on counting strategies. Our findings also sharpened our understanding of this relation by showing that it depends on whole-number information, as there were positive relations between trials depicting the same type of whole-number information (e.g., misleading discretized and misleading symbolic) but negative ones

between trials showing the opposite information (e.g., compatible discretized and misleading symbolic).

The distinct contributions of the nonsymbolic formats to fraction ability may stem from the information each format highlights. Continuous representations of proportions are ideal for conveying proportional magnitudes, as they do not introduce whole-number interference (Gunderson et al., 2019). According to the "integrated theory of numerical development" (Siegler et al., 2011), a developmental milestone is understanding that different types of numerical representations can be placed in the same numberline; however, even though middle-schoolers of the current study understand continuous nonsymbolic and symbolic fraction magnitudes, they may not yet have integrated them. Remarkably, recent studies have shown that children have yet to integrate different symbolic representations (i.e., decimals, fractions, and percentages) of proportions (Schiller & Siegler, 2022). By contrast, the robust relation between performance on the discretized format and symbolic fraction may be due to non-proportional magnitude features. As both nonsymbolic discretized proportions and symbolic fractions afford comparing stimuli by whole-number strategies (i.e., focusing on the absolute number of segments and the absolute value of the fraction components, respectively), students may be implicitly (or explicitly) using similar strategies while working with these different types of proportions. Notably, our results warrant a further examination of the common neural underpinnings of nonsymbolic and symbolic proportional skills. Previous studies have mostly focused on brain regions responsible for proportional magnitude processing (e.g., Mock et al., 2018), such as the intraparietal sulcus; however, our results suggest that areas responsible for inhibition may also play a critical role in the relation of nonsymbolic and symbolic skills.

4.3. Profiles of nonsymbolic proportional reasoning

Our final goal was to characterize students' profiles of nonsymbolic proportional reasoning, using a person-oriented approach, cluster analysis. Recently, this approach has been used to uncover the distinct strategies that students use when comparing symbolic fractions (Gómez & Dartnell, 2019; Miller Singley et al., 2020; Reinhold et al., 2020) and how these strategies changed throughout the elementary-school years (Rinne et al., 2017). In contrast to those studies, we performed cluster analyses with students' performance across the four different types of nonsymbolic trials. These analyses revealed three different profiles (Fig. 6A): a small group of students (18%) who, even though they were successful at comparing continuous proportions, for the discretized trials, consistently chose the stimuli with the largest absolute number of segments and ignored the proportional information, evidencing strong *whole-number bias* (Ni & Zhou, 2005). Half of the remaining students (43%) were around *chance-level* in the discretized trials showing misleading whole-number information and had the lowest continuous performance. Finally, the third profile comprised *high-performing* students (39%) with strong continuous and discretized nonsymbolic proportional reasoning.

To further map students' understanding of nonsymbolic proportional magnitudes within each profile, we examined their distance effects across the different types of nonsymbolic trials; and probed students' symbolic fraction abilities across the three profiles. Here, for the first time, we showed that when students use whole-number strategies to compare nonsymbolic proportions (whole-number biased profile), they do not show any distance modulation in their judgments of discretized misleading proportions (Fig. 7A). Further, when looking at the symbolic fraction comparison abilities of students with this profile (Fig. 7B), their performance patterns were remarkably similar to those in the nonsymbolic discretized trials: Students had a near-ceiling performance in the symbolic compatible trials but close to floor levels in the misleading trials, suggesting that students not only used whole-number strategies for their discretized judgments but also for their symbolic ones. Critically, the whole-number biased students did not show distance effects in any symbolic trials (Fig. 7B). These results are consistent with prior findings examining students' strategy use in symbolic fraction tasks, showing that when they employ whole-number rules to compare fractions, they do not show symbolic distance effects (Reinhold et al., 2020). Math education research (Powell, 2023) and intervention studies (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Sidney et al., 2019) identify area models, which convey part-whole interpretations of fractions, as potential origins of students' symbolic fraction misconceptions. Our results provide strong evidence to support warning against using these models exclusively in instruction (Gunderson et al., 2019; Hamdan & Gunderson, 2017). However, the correlational nature of our study warrants more research about the directionality of the relation between these shared misconceptions in discretized and symbolic proportional reasoning. Our results also suggest that avoiding whole-number strategies might be a critical step to access correctly both nonsymbolic and symbolic proportions. One possible developmental trajectory is that students first overcome the whole-number bias in nonsymbolic contexts, which, in turn, serves as a foundational skill for later symbolic proportional reasoning. Alternatively, learning about symbolic fractions might refine general quantity processing, leading to improvements in discretized skills (Begolli et al., 2020). To discern between these two alternatives, future longitudinal studies should examine whether students first show distance effects for nonsymbolic discretized or symbolic fractions or whether these effects emerge simultaneously.

In contrast to students with a whole-number biased profile, the other two profiles — chance-level performance and high-performance profiles — modulated their nonsymbolic and symbolic judgments of proportions by the fraction distance. However, students in these two profiles differed in the strength of such modulation. Students with a high-performance profile had the most robust distance modulation, while students with a chance-level profile showed a significant but weaker modulation (Fig. 7B). These results suggest that the critical feature that distinguished students with these two profiles was their sensitivity to nonsymbolic and symbolic proportions instead of overgeneralizing whole-number rules. One interpretation of this pattern is that students in the high-performance profile have a stronger Ratio-Processing System, a cognitive primitive that supports nonsymbolic proportional processing (Lewis et al., 2016; Matthews et al., 2016), which then supports their understanding of symbolic fractions. Alternatively, learning about symbolic fractions may have refined students' nonsymbolic proportional skills, similar to the sharpening of the precision of the approximate number system after learning the meaning of number words (Shusterman et al., 2016). From this interpretation, strong symbolic skills support better discretized performance.

An outstanding question is whether students with these different nonsymbolic proportional reasoning profiles also differed in other domain-specific and domain-general skills. Gomez and Dartnell (2019) showed that students with different symbolic fraction comparison strategies differed in their general math achievement levels. Particularly, fourth-grade students who employ whole-number strategies to compare symbolic fractions have the lowest math achievement levels compared to students who use other strategies. Recently, the role of inhibitory control, the ability to override automatic responses (Diamond, 2013), has been underscored for problems where students need to avoid intuitive responses and access counterintuitive concepts (Van Dooren & Inglis, 2015). In particular to proportional reasoning, emerging evidence suggests that weak inhibitory control relates to difficulties with comparing nonsymbolic (Abreu-Mendoza et al., 2020) and symbolic proportions (Avgerinou & Tolmie, 2019; Coulanges et al., 2021; Leib et al., 2023) in contexts where whole-number information interferes with the proportional information. Together, these studies point to general math achievement and inhibitory control as key candidate abilities that might distinguish between profiles.

4.4. Educational implications

The current results also provide insights into how to conduct tailored interventions for students with different nonsymbolic profiles and which nonsymbolic proportional materials may hinder students' fraction learning. First, our results showed that students' fraction difficulties might stem from at least two different sources: the first one is overgeneralizing whole-number rules (whole-number biased profile), and the second is a weak understanding of nonsymbolic proportional magnitudes (chance-level performance profile). Critically, different intervention approaches would be needed to improve nonsymbolic and symbolic fraction skills of children with these profiles. The first group of students may benefit from interventions comprising 'stop and think' strategies (e.g., Wilkinson et al., 2020) accompanied by explicit instruction warning students of common fraction errors (e.g., Van Hoof et al., 2021). In contrast, children from the second profile may benefit from interventions to improve the understanding of proportional magnitudes (Gouet et al., 2020). The mismatch between students' areas of difficulty and instruction may explain null results in intervention studies. For example, a recent intervention study targeting nonsymbolic continuous proportional skills in second graders using physical manipulatives (Abreu-Mendoza et al., 2021) showed that students improved their continuous proportional skills; however, their discretized skills declined after the intervention. As second graders are more likely to show a strong whole-number bias (Abreu-Mendoza et al., 2020; Jeong et al., 2007), a possibility is that an intervention targeting misconceptions about proportions (e.g., overgeneralization of whole-number rules) might have been more effective. Importantly, our results also proposed a persistent whole-number bias for nonsymbolic discretized proportions as an early predictor for at-risk fraction learning difficulty. However, more research is needed about the direction of the relation between nonsymbolic and symbolic proportional skills.

How can we reconcile the past success of continuous representations (e.g., numberlines) to enhance students' fraction understanding (for a review, see Abreu-Mendoza & Rosenberg-Lee, 2023) with the strong link between discretized and symbolic fraction skills? For example, across two studies, Gunderson and colleagues (Gunderson et al., 2019; Hamdan & Gunderson, 2017) found that unidimensional numberlines enhanced children's fraction understanding. An open question is whether considering fractions as discretized proportions is the default end stage of fraction representation or whether it results from the student's educational history. A possible interpretation for the success of prior interventions using continuous representations of proportions (numberlines) is that instruction linking symbolic fractions to numberlines may prevent the strong link between discretized proportions and symbolic fractions, particularly during early fraction instruction. However, it is worth noting that continuous nonsymbolic interventions have not been sufficient to help children overcome whole-number interference in nonsymbolic discretized skills (Abreu-Mendoza et al., 2021; Hurst et al., 2022). Together, these results call for further research on the causal links between nonsymbolic and symbolic proportional abilities.

Finally, our results also shed light on the origins of the reverse bias. A handful of studies have shown that a group of students persistently choose the fraction with the smallest components as the fraction with the largest magnitude (Gómez & Dartnell, 2019; Reinhold et al., 2020). This strategy has been considered an overgeneralization of an initial understanding that whole-number quantities can refer to smaller magnitudes when working with fractions. Interestingly, our cluster analyses did not capture this strategy when using nonsymbolic performance instead of symbolic fraction ability, suggesting that, in contrast to the whole-number bias, this erroneous strategy may derive from misconceptions of symbolic fractions instead of nonsymbolic proportions.

5. Conclusions

Early nonsymbolic proportional skills have been posited as a foundational ability for later fraction learning. However, little is known about the mechanisms underlying this relationship. Here, we probed the relation of symbolic fraction ability with two nonsymbolic representations of proportions: a continuous format that emphasizes proportional relations and a discretized format that may prompt erroneous whole-number strategies, which, in turn, may hamper access to fraction magnitudes. Our results showed that, contrary to initial proposals, the relation between nonsymbolic and symbolic proportional skills is not based on proportional information but instead may be driven by the misconceptions stemming from discretized representations. However, fraction learning difficulties may stem from either overgeneralizing whole-number rules, which prevent access to proportional magnitudes, or weak nonsymbolic magnitude understanding. These results call attention to the type of nonsymbolic stimuli used to introduce fractions and point to nonsymbolic whole-number bias as a possible marker for later fraction learning difficulties.

Declaration of Competing Interest

Data availability

The authors do not have permission to share data.

Appendix A

Fraction pairs used as practice trials for the proportional comparison task

Neutral Fraction	Distance
4/5 vs. 3/8	0.425
4/9 vs. 7/8	0.430
2/7 vs. 3/4	0.464
7/8 vs. 2/9	0.653

Appendix B

Fraction pairs used in the nonsymbolic and symbolic proportional comparison task.

Compatible				Misleading		
Fraction	Distance	Distance Group		Fraction	Distance	Distance Group
4/9 vs. 3/8	0.069	Group 1		2/3 vs. 3/5	0.067	Group 1
5/6 vs. 4/5	0.033	Group 1		2/3 vs. 5/8	0.042	Group 1
5/8 vs. 4/7	0.054	Group 1		2/5 vs. 3/8	0.025	Group 1
6/7 vs. 4/5	0.057	Group 1		3/4 vs. 5/7	0.036	Group 1
6/7 vs. 5/6	0.024	Group 1		3/5 vs. 4/7	0.029	Group 1
7/8 vs. 4/5	0.075	Group 1		3/5 vs. 5/9	0.044	Group 1
7/9 vs. 3/4	0.028	Group 1		4/5 vs. 5/7	0.086	Group 1
7/9 vs. 5/7	0.063	Group 1		4/5 vs. 7/9	0.022	Group 1
8/9 vs. 5/6	0.056	Group 1		4/7 vs. 5/9	0.016	Group 1
8/9 vs. 6/7	0.032	Group 1		5/6 vs. 7/9	0.056	Group 1
8/9 vs. 7/8	0.014	Group 1		6/7 vs. 7/9	0.079	Group 1
5/8 vs. 3/7	0.196	Group 2		3/4 vs. 4/7	0.179	Group 2
5/9 vs. 3/8	0.181	Group 2		3/4 vs. 5/9	0.194	Group 2
7/9 vs. 5/8	0.153	Group 2		3/5 vs. 4/9	0.156	Group 2
8/9 vs. 5/7	0.175	Group 2		4/5 vs. 5/8	0.175	Group 2
5/7 vs. 2/5	0.314	Group 3		2/3 vs. 3/7	0.238	Group 3
5/8 vs. 2/5	0.225	Group 3		2/3 vs. 3/8	0.292	Group 3
7/8 vs. 3/5	0.275	Group 3		3/4 vs. 4/9	0.306	Group 3
8/9 vs. 5/8	0.264	Group 3		4/5 vs. 5/9	0.244	Group 3

Appendix A. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.cogpsych.2023.101575.

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