Question 1

The probability mass function, $p_X(x)$, for a discrete variable X is given in the following table.

x	0	1	2
$p_X(x)$	1/3	1/6	1/2

(a) Let $Y = |X - 1|$. Be sure to show your steps for the following questions.

(i) What is the probability mass function of Y?

(ii) What is the moment generating function of Y?

(iii) Compute the mean and variance of Y.

i)

y	$\|0-1\|$	$\|1-1\|$	$\|2-1\|$
$P_y(y)$	$1/3$	$1/6$	$1/2$

$$\mathbb{E}[X] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} = \frac{7}{6}$$

$$\mathbb{E}[X^2] = 1 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{2} = \frac{13}{6}$$

$$Var(X) = \frac{13}{6} - \frac{49}{36} = \frac{29}{36}$$

iii) $$\mathbb{E}[Y] = \mathbb{E}[|X-1|] = |\mathbb{E}[X-1]|$$

$$= |\mathbb{E}[X] - 1| = \left|\frac{7}{6} - 1\right| = \frac{1}{6}$$

$$Var(Y) = Var(|X-1|) = |Var(X-1)|$$

$$= |Var(X) - Var(1)|$$

$$= \left|\frac{29}{36} - 1\right| = \left|-\frac{7}{36}\right| = \frac{7}{36}$$

(b) Suppose we flip a coin $2X$ times, where X follows the pmf given in the table in this question and record the number of heads. Let event *A= more than one head*, and *B=coin is flipped twice*. Consider the following R code.

```
sim <- NULL
N <- 5000
for(i in 1:N){
  x <- sample(0:2, size = 1, prob=c(1/3,1/6,1/2))
  flips<-sample(0:1,size=2*x, replace=TRUE)
  if (sum(flips)>1 & x==1){
      success <- 1
    } else {
      success <- 0
    }
    sim[i] <- success
  }
mean(sim)
```

(i) Which of the following probabilities does the above code estimate? Circle the appropriate option.

$P(A)$	$P(B)$	$P(A \cap B)$
$P(A \cup B)$	$P(A\|B)$	$P(B\|A)$

(ii) Compute this probability and list any probability rules/principles that you apply.

$P(A) = P(\text{more than 1 head}) = 1 - P(0 \text{ heads})$

$= 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$P(B) = P(X=2) = \frac{1}{2}$

$P(A \cap B) = P(A) \cdot P(B) = \frac{3}{8}$

Question 4

You are trying to pull an all-nighter to study the night before an exam (*don't do this for STA237H1!*). There are 10 more hours until your exam. The amount of time, in hours, that you can stay awake to study for can be modelled by a random variable X with PDF $f(x) = \begin{cases} ce^{-0.25(x-2)}, & 2 < x < 10 \\ 0, & otherwise \end{cases}$, where c is a constant.

(a) What is the constant c that makes this a valid PDF?

(b) You know you need 4 hours to go over enough material to pass your exam, and otherwise, you will fail. What is the probability that you pass your exam? Sketch the PDF, mark this probability on your plot, and compute this probability.

(c) What is the probability that you study for exactly 5 hours? Sketch the PDF again and mark this probability on your plot.

(d) You find out that your exam is entirely multiple choice, and if you study for less than 4 hours, there is a 30% probability that you will pass your exam by randomly guessing the answers. If you study for more than 4 hours, you will pass for sure. What is the probability that you will pass the exam now? Compute the probability of this event and list any probability rules/principles that you apply.

a) $\int_0^{10} ce^{-0.25(x-2)} dx = ce^{0.5} \int_0^{10} e^{-0.25x} dx = ce^{0.5}(-4)\left[e^{-0.25x}\Big|_0^{10}\right]$

$= ce^{0.5}(-4)\left[e^{-2.5} - 1\right]$

$= c(6.653544) = 1$

$\Rightarrow c = 0.1651925$

b) $\mathbb{P}[x=4] = c\, e^{-0.25(4-2)} = c e^{-0.5}$

c) $\mathbb{P}[x=5] = c e^{-0.25(5-2)} = c e^{-0.75}$

d) $\mathbb{P}(\text{pass}) = \mathbb{P}(\text{pass})\,\mathbb{P}(\text{study} < 4) + \mathbb{P}[\text{pass}]\,\mathbb{P}[\text{study} > 4]$

$$= 0.3 \int_2^4 c e^{-0.25(x-2)}\, dx + \int_4^\infty c e^{-0.25(x-2)}$$

b) sketch