Proposed Problems

Seminar 1

1. A home computer is tied to a mainframe computer via a telephone modem. The home computer will dial repeatedly until contact is made. Let c denote the fact that contact is made on a particular attempt, and n that contact is not made.
a) List the sample space of this experiment.
b) List the elementary events that constitute the following events:
A: contact is made in at most five attempts;
B: contact is made after at least 10 tries.

2. Consider a game of darts consisting of 7 concentric circles of radii $r_1 < r_2 < ... < r_7$. Let A_i be the event: the circle C_i of radius r_i is hit. Describe (in words) the following events:
a) $A = A_1 \cup ... \cup A_5$;
b) $B = A_4 \cap ... \cap A_7$;
c) $C = \overline{A_2} \cap A_3$;
d) $D = A_5 \setminus A_4$;
e) $E = A_4 \triangle A_5$ $(A \triangle B = (A \cup B) \setminus (A \cap B))$.

3. A computer system uses passwords that consist of 5 letters followed by 2 digits. There is no difference between lowercase and capital letters.
a) How many passwords are possible? (Ans: $(26)^5(10)^2$)
b) How many passwords are possible, without repetition of characters?
(Ans: $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 10 \cdot 9$)
c) How many passwords begin with letter A and end in 5? (Ans: $(26)^4 \cdot 10$)
d) How many passwords contain the letters A, B, C (in that order) and end in an even digit? (Ans: $3 \cdot (26)^2 \cdot 10 \cdot 5$)

4. There are 6 boys and 6 girls in a study group. For a certain project they have to work in teams of two people.
a) In how many ways can they be teamed up? (Ans: $C_{12}^2 \cdot C_{10}^2 \cdot C_8^2 \cdot C_6^2 \cdot C_4^2 \cdot C_2^2 = \dfrac{12!}{2^6}$)
b) In how many of those, each team consists of a boy and a girl? (Ans: $6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2 = (6!)^2$)

Seminar 2

1. The natural numbers $1, 2, ..., n$ (where n is a fixed natural number) are placed randomly in a sequence. Find the probability of the following events:
a) A: the numbers 1 and 2 are placed consecutively, in increasing order; (Ans: $\dfrac{1}{n}$)
b) B: the numbers 1 and 2 are placed consecutively; (Ans: $\dfrac{2}{n}$)
c) C: the numbers 1 and 2 are placed in increasing order; (Ans: $\dfrac{1}{2}$)
d) D: the numbers i, j, k are placed consecutively, in increasing order. (Ans: $\dfrac{1}{n(n-1)}$)

2. A computer program is tested by 3 independent tests. When there is an error, these tests will discover it with probabilities 0.2, 0.3 and 0.5, respectively. Suppose that the program contains an error. What is the probability that it will be found by at least one test? (Ans: 0.72)

3. In a study of waters near industrial plants it was found that 30% showed signs of chemical pollution (event C), 25% showed evidence of thermal pollution (event T) and 10% showed signs of chemical and thermal pollution.
a) From the data above, is chemical pollution independent of thermal pollution in that area? (Ans: No)
b) What is the probability that a stream that shows some thermal pollution will also show signs of chemical pollution (event A)? (Ans: $\frac{2}{5}$)
c) What is the probability that a stream showing chemical pollution will not show signs of thermal pollution (event B)? (Ans: $\frac{2}{3}$)

4. Three highways Connect city A from city B and two highways connect city B to city C. During a rush hour, the traffic on each highway is blocked with probability 0.2, independently of other highways.
a) Compute the probability that there is at least one open route from A to C. (Ans: 0.9523)
b) How will a new highway, also blocked with probability 0.2 independently of other highways, change that probability if it is built
b1) between B and C? (Ans: 0.9841)
b2) between A and C? (Ans: 0.991)

5. A system may become infected by some spyware through the internet or e-mail. Seventy percent of the time the spyware arrives via the internet, the rest of the time via e-mail. If it enters via the internet, the system detects it with probability 0.6, whereas if via e-mail, it is detected with probability 0.8. What percentage of times is the spyware detected? (Ans: 66%)

Seminar 3

1. A person has 40 homing (messenger) pigeons. When released, the probability that a pigeon will come back is 0.7. Find the probability of the events:
a) A: 10 pigeons do not come back; (Ans: $C_{40}^{10}(0.3)^{10}(0.7)^{30} = 0.1128$)
b) B: all of them come back; (Ans: $(0.7)^{40} = 6.3668e - 07$)
c) C: at least 38 of them come back. (Ans: $\sum_{k=38}^{40} C_{40}^{k}(0.7)^k(0.3)^{40-k} = 1.0277e - 04$)

2. Successful implementation of a new system is based on three independent modules. Module 1 works properly with probability 0.96. For modules 2 and 3, these probabilities are 0.95 and 0.9, respectively. Find the probability that
a) exactly one of these modules fails to work properly; (Ans: 0.0266)
b) the system does not work properly. (Ans: 0.1792)

3. An internet search engine looks for a keyword in 9 databases, searching them in random order. Only 5 of these databases contain the given keyword. Find the probability that it will be found in at least 2 of the first 4 searched databases. (Ans: 0.8333)

4. A flu vaccine meets specifications with probability 0.9. Would it be unusual if 7 or more vaccines have to be tested to find three that meet specifications (event A)? Explain. (Ans: Yes, the probability of that event is 0.00127, very small)

5. Students from 3 departments participate in a debate. The boys/girls ratios for the 3 groups are $(6, 4)$, $(4, 5)$ and $(5, 5)$, respectively. If one student is chosen from each department as spokesperson, what is the probability that the spokespersons are 2 boys and 1 girl (ev. A)? (Ans: $\frac{7}{18}$)

6. An internet search engine looks for a keyword in a sequence of independent web sites. It is believed that 20% of the sites contain this keyword. Find the probability that
a) at least 5 of the first 10 sites contain the given keyword; (Ans: 0.0328)
b) the search engine had to visit at least 5 sites in order to find the first occurrence of the keyword (Ans: 0.4096)

Seminar 4

1. A computer virus attacks a folder consisting of 250 files. Files are affected by the virus independently of one another, each with probability 0.032. Let X denote the number of infected files.
a) Find the pdf of X. What type of distribution does X have? (Ans: $B(250, 0.032)$)
b) What is the probability that more than 7 files are affected by this virus? (Ans: 0.5493)

2. Messages arrive at an electronic message center at random times, with an average of 9 messages per hour. What is the probability of receiving
a) exactly 5 messages during the next hour? (Ans: 0.061)
b) at least 5 messages during the next hour? (Ans: 0.945)

3. Two out of six computers in a lab have problems with hard drives. Three computers are selected at random for inspection. Let X denote the number of computers that are found to have hard drive problems.
a) Find the pdf of X. What type of distribution does X have? (Ans: $H(6, 2, 3)$)
b) What is the probability that at most 1 of them has hard drive problems? (Ans: 0.8)

4. Eight letters are randomly distributed into 3 mailboxes. Let X be the number of letters in the 1^{st} mailbox. Find the pdf of X.

5. On any day, in a small computer lab, the number of hardware failures, X, and the number of software failures, Y, have the joint distribution $P(x, y)$, with $P(0, 0) = 0.6, P(0, 1) = 0.1, P(1, 0) = 0.1, P(1, 1) = 0.2$.
a) Write the joint pdf of the vector (X, Y);
b) Find the pdf's of X and Y, respectively;
c) Are hardware and software failures independent in this lab? (Ans: No)
d) Find the probability that no software failures occur on a given day; (Ans: 0.7)
e) Find the probability that 1 hardware failure and at most 1 software failure occur on a given day. (Ans: 0.3)

6. Network breakdowns are unexpected rare events that occur every 3 weeks, on average. Find the probability of more than 4 breakdowns occurring during a 21-week period. (Ans: 0.827)

Seminar 5

1. The time, in minutes, it takes to reboot a certain system is random variable with density

$$f(x) = \begin{cases} C(10-x)^2, & 0 < x < 10 \\ 0, & \text{otherwise} \end{cases}$$

Find
a) the constant C; (Ans: $C = 0.003$)
b) the probability that it takes between 1 and 2 minutes to reboot the system. (Ans: 0.217)

2. The random vector (X, Y) has the joint pdf given by

$$f(x,y) = k(x^2 + y), \quad \text{for } -1 \le x \le 1,\ 0 \le y \le 1.$$

Find
a) the constant k; (Ans: $k = 3/5$)
b) the marginal densities of X and Y; are X and Y independent? (Ans: No)
c) the probabilities $P(Y < 0.6)$ and $P(Y < 0.6 \mid X < 0.5)$ (Ans: 0.456 and 0.44)

3. Consider a satellite whose work is based on a certain block A. This block has an independent backup B. The satellite performs its task until both A and B fail. The lifetime of A and B are exponentially distributed random variables with average lifetime of 10 years. What is the probability that the satellite will work for more than 10 years? (Ans: 0.6004)

4. Let $X \in N(0,1)$. Find the probability density function of $Z = X^2 - 1$.

5. Let X be a random variable with density $f_X(x) = xe^{-x}$, $x \ge 0$ and let $Y = e^X$. Find f_Y.

Seminar 6

1. (Refer to proposed problem 5, for Seminar 4) On any day, in a small computer lab, the number of hardware failures, X, and the number of software failures, Y, have the joint distribution $P(x,y)$, with $P(0,0) = 0.6, P(0,1) = 0.1, P(1,0) = 0.1, P(1,1) = 0.2$.
a) Are hardware and software failures independent in this lab? (Ans: No)
b) What is the expected total number of failures during one day? (Ans: 0.6)

2. (Optimal portfolio) Shares of company A are sold at \$10 per share and shares of company B are sold at \$50 per share. According to a market analyst, 1 share of each company can either gain \$1, with probability 0.5, or lose \$1, with the same probability, independently of the other company. Which of the following portfolios has the lowest risk:
a) 100 shares of company A; (Ans: Var = 10000)
b) 50 shares of company A + 10 shares of company B; (Ans: Var = 2600)
c) 40 shares of company A + 12 shares of company B? (Ans: Var = 1744.)

3. Lifetime (in years) of a certain hardware is a continuous random variable with pdf

$$f(x) = \begin{cases} C - x/50, & 0 < x < 10 \\ 0, & \text{otherwise} \end{cases}$$

Find

a) the constant C; (Ans: $C = 0.2$)

b) how long is this hardware expected to last. (Ans: $\dfrac{10}{3}$ years)

4. An internet search engine looks for a certain keyword in a sequence of independent web sites. It is believed that 20% of the sites contain this keyword. On the average, how many of the first 15 sites will contain that keyword?

5. (Refer to proposed problem 2, for Seminar 5) The random vector (X, Y) has the joint pdf given by

$$f(x, y) = \frac{3}{5}(x^2 + y), \quad \text{for } -1 \le x \le 1,\ 0 \le y \le 1.$$

Find

a) the means and variances of X and Y; $\left(\text{Ans} : E(X) = 0, E(Y) = \dfrac{3}{5}, V(X) = \dfrac{11}{25}, V(Y) = \dfrac{11}{150}\right)$

b) the correlation coefficient $\rho(X, Y)$. (Ans: $\rho(X, Y) = 0$, X and Y **uncorrelated**, but, as seen in proposed problem 2 (Sem 5), they are **NOT independent**)

Seminar 7

1. Let X be a r. v. with pdf $f(x) = \dfrac{x^m e^{-x}}{m!}$, $x \ge 0$. Show that $P(0 < X < 2(m+1)) \ge \dfrac{m}{m+1}$.

2. An average scanned image occupies 0.6 megabytes of memory with a standard deviation of 0.4 megabytes. If you plan to publish 80 images on your website, what is the probability that their total size is between 47 and 50 megabytes? (Ans: 0.322)

3. Every day, George takes the same street from his home to the university. There are 4 street lights along his way, and he noticed the following Markov dependence: if he sees a green light at an intersection, then 60% of the time the next light is also green, whereas if he sees a red light, then 70% of the time the next light is also red.

a) Find the transition probability matrix for the street lights;

b) If the first light is green, what is the probability that the third light is red? (Ans: 0.52)

c) George's classmate John has many street lights between his home and the university, all following the same pattern that George noticed. If the *first* street light is green, what is the probability that the *last* street light is red? (Ans: $\dfrac{4}{7} \approx 0.5714$)

4. The probability that a certain computer code runs without errors is a parameter $p \in (0, 1)$, unknown. Let X denote the number of times the computer code has to be debugged before it runs without errors. For 5 independent computer projects, a student records the number of times they each had to be corrected as: $3, 7, 5, 3, 2$. Estimate p

a) by the method of moments; (Ans: $\overline{p} = 0.2$)

b) by the method of maximum likelihood. (Ans: $\hat{p} = 0.2$)

5. Let $X_1, \ldots, X_n$ be a sample drawn from a distribution with pdf

$$f(x;\lambda) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}, \quad x > 0$$

($\lambda > 0$), which has mean $\mu = E(X) = \lambda$ and variance $\sigma^2 = V(X) = \lambda^2$. Find
a) the method of moments estimator, $\hat{\lambda}$, for λ; (Ans: $\hat{\lambda} = \overline{X}$)
b) the efficiency of $\hat{\lambda}$, $e(\hat{\lambda})$; (Ans: $e(\hat{\lambda}) = 1$)
c) an approximation for the standard error of the estimate in a), $\sigma_{\hat{\lambda}}$, if the sum of 100 observations is 150. (Ans: $\sigma_{\hat{\lambda}} \approx 0.15$)