Non-Convex Shredded Signal Reconstruction via Sparsity Enhancement

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CS531, Detection and Estimation Theory
Class presentation

Outline

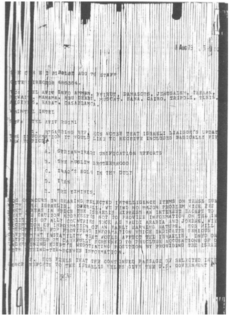
Introduction

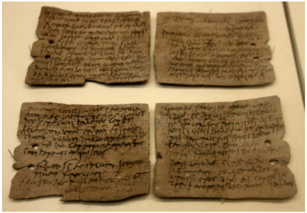
Preliminaries
- Optimization
- Sparsity

Approach

Results

Problem at hand - Shredded Signal(document) reconstruction





- The documents are shredded.
- No piece is lost or missing.
- The pieces are of same size and identical in shape.

Past work

- Optimum matching pieces with respect to their contents, color or texture appearance
- Apictorial jigsaw puzzle

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How to frame this as a mathematical problem

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Optimization Problem

- Maximizing or minimizing some function $f(x)$ over the variable x, when the function is subject to some constraints.

Formal definition of an optimization problem

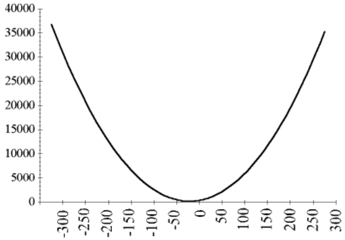
$$\min_x f(x)$$

$$\text{subject to } g_i(x) \leq c$$

$$h_j(x) = d \text{ , where } i = 1, ..., m; j = 1, ..., n$$

- $f(x)$ is called the *Objective* - The main problem specification
- g, h are called constraints - Additional details that define the problem
- x is called the optimization variable

Convexity



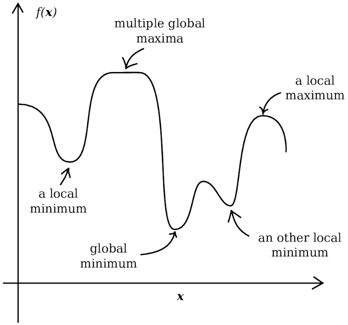


Figure: Figure on the left *Convex* and figure on the right *Non-convex*.

- **Convexity** : $f(x), g(x) \& h(x)$ are Convex. Else Non-convex
- Convex functions guarantee *Global minima*

Sparsity

- A signal is sparse if that can be compactly expressed as a linear combination of a few small number of basis vectors.
- In other words , it is a matrix in which most of the elements are zero.
- Why is sparsity interesting ?

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

  can be simply stored as value = [5,8,3,6]; row = [2,2,3,4]; column = [1,2,2,3]
- Storing sparse matrix and computing are easier
- Natural signals are sparse !

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Key Ideas

- Formulate the problem as a non-convex optimization problem
- Solve the optimization problem using alternate minimization technique.
- Simulataneously force sparsity of the recovered signal

Formulate the problem as optimization problem

- What do we observe?
  Shredded signals, each signal of length N and M shreds.

$$\mathbf{y}_{MN\times 1} = [\mathbf{y_1}^T \mathbf{y_2}^T ......\mathbf{y_M}^T]^T$$

- The original signal(document) to be recovered is $\mathbf{x}_{MN\times 1}$
- If $\mathbf{y}$ is arranged in right order, we can retrieve $\mathbf{x}$. This ordering is given by permutation matrix $\mathbf{P}_{M\times M}$.

$$\mathbf{x}_{MN\times 1} = (\mathbf{P}_{M\times M} \otimes \mathbf{I}_{N\times N})_{MN\times MN}\mathbf{y}_{MN\times 1} \tag{1}$$

- We also know that the signals are sparse in frequency domain. How do I access that?

$$\mathbf{x}_{MN\times 1} = \psi^H_{MN\times MN}\mathbf{v}_{MN\times 1} \tag{2}$$

Formulate the problem as optimization problem

- Given optimal $\mathbf{P}$ and $\mathbf{v}$ the error between the functions 1 and 2 should be minimum. That is exactly our objective function.

$$\min_{\mathbf{P},\mathbf{v}} \quad ||(\mathbf{P} \otimes \mathbf{I})\mathbf{y} - (\psi^H \mathbf{v})||_2$$
$$\text{subject to } ||\mathbf{v}||_2 = ||\mathbf{y}||_2$$

Specifying sparsity

- Let s be the sparsity order of $\mathbf{v}$. The number of non-zero values in $\mathbf{v}$ is s. How do I represent this mathematically?

$$\mathbf{v} \in \mathcal{X}_s$$

where $\mathcal{X}_s$ is a set of all vectors with atmost s non-zero values. Add this as a constraint and now the problem becomes

$$\min_{\mathbf{P},\mathbf{v}} \quad ||(\mathbf{P} \otimes \mathbf{I})\mathbf{y} - (\psi^H \mathbf{v})||_2 \tag{3}$$
$$\text{subject to } ||\mathbf{v}||_2 = ||\mathbf{y}||_2 \tag{4}$$
$$\mathbf{v} \in \mathcal{X}_s \tag{5}$$

Solving the optimization problem

We have three optimization variables $\mathbf{P}$ and $\mathbf{v}$ and s

Optimization Strategy

- Fix a sparsity order s
- Alternatively optimize $\mathbf{P}$, $\mathbf{v}$
- Called *Cyclic minimisation* or *Alternate minimisation*

Solving the optimization problem

Step 1 : Solving $\mathbf{v}$

- Fix s and $\mathbf{P}$

Optimization problem in $\mathbf{v}$

$$\min_{\mathbf{v}} \quad ||\mathbf{v} - \psi(\mathbf{P} \otimes \mathbf{I})\mathbf{y}||_2 \tag{6}$$

$$\text{subject to } ||\mathbf{v}||_2 = ||\mathbf{y}||_2 \tag{7}$$

$$\mathbf{v} \in \mathcal{X}_s \tag{8}$$

- Let $\widetilde{\mathbf{v}} = \psi(\mathbf{P} \otimes \mathbf{I})\mathbf{y}$
- Observe that

$$||\mathbf{v} - \widetilde{\mathbf{v}}||_2^2 = c - 2\mathbf{v}^{\mathbf{T}}\widetilde{\mathbf{v}}$$

where, $c = ||\mathbf{v}||_2^2 + ||\widetilde{\mathbf{v}}||_2^2 = 2||\mathbf{y}||_2^2$ is constant

- Now the problem becomes maximizing $\mathbf{v}^{\mathbf{T}}\widetilde{\mathbf{v}}$

Solving the optimization problem

Finding the optimal $\mathbf{v}$

- Note that $\mathbf{v}$ has a sparsity constraint(defined by s)
- Idea : Pick the maximal s components of $\widetilde{\mathbf{v}}$
- Define $\mu \in \{0,1\}^{MN \times 1}$ as indices of the chosen components

Optimal $\mathbf{v}$

$$\mathbf{v_{opt}} = ||\mathbf{y}||_2 \left(\frac{\widetilde{\mathbf{v}} \circ \mu}{||\widetilde{\mathbf{v}} \circ \mu||_2} \right) \tag{9}$$

$\mathbf{v_{opt}}$ aligns maximally with $\widetilde{\mathbf{v}}$ and has the same norm as $\mathbf{y}$

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Solving the optimization problem

Step 2 : Solving $\mathbf{P}$

Fix $\mathbf{v}$ and solve for $\mathbf{P}$

- Let $\hat{\mathbf{x}} = \psi^H \mathbf{v}_{opt}$ - a sparse estimate of the $\mathbf{x}$ signals, from Step 1.
- We also have the $\mathbf{y}$ - the observed shreds.
- Now, define $\mathbf{U} \in \mathbb{C}^{M \times M}$; called cost matrix as

$$\mathbf{U}_{k,l} = |\mathbf{y}_k - \widehat{\mathbf{x}}_l|_2^2$$

New objective for solving $\mathbf{P}$

$$\mathbf{P}_{opt} = \arg\min_P [\mathbb{1}_M^T (P \circ U) \mathbb{1}]$$

- The above problem is called the *Assignment problem* or the *Matching problem*
- Can be solved using *Hungarian Algorithm*

Overall algorithm

Final algorithm

Step 0: Set $s = 1$.
Step 1: Monotonically decrease the objective of equation 5 via cyclic minimization until convergence using equations 9 and equation 16.
Step 2: Set $s \leftarrow s + 1$
Step 3: Repeat Step 1 until the decrease in the objective of 5 is negligible.

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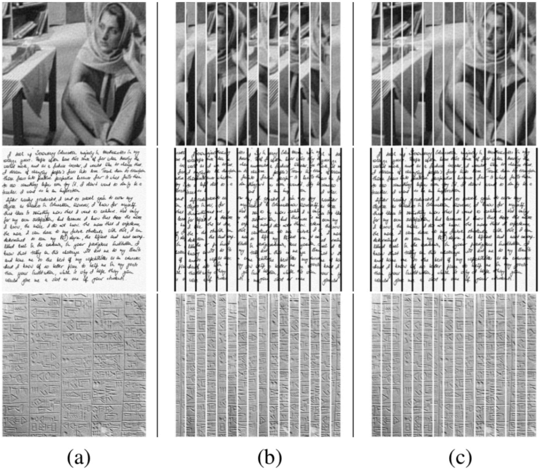

Fig. 2. Reconstruction results: (a) original images, (b) scrambled shredded strips, (c) reconstructed images.

Thank You!