Optimized Sampling for Monte Carlo simulations via Dimension Reduction

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Class presentation

Outline

Problem at hand - Optimize Monte Carlo Estimator

Problem at hand

Estimate $E(f(U))$

- U- vector of 'd' independent Random Variables
- f does not depend equally on all its arguments
- Present an algorithm to obtain a lower-variance unbiased estimator than the standard Monte Carlo method.

Outline

Monte Carlo

- Consider a Random variable X with with pdf $\mathbb{f}_X(x)$. What is $E(g(X))$?

$$E(g(X)) = \int_x g(x)\mathbb{f}_X(x)$$

- How to find $E(g(X))$ when the pdf of X, $\mathbb{f}_X(x)$ is not known?
- There comes Monte Carlo !

Definition

Monte Carlo is the art of approximating an expectation by the sample mean of a function of simulated random variables.

- Take n samples of X's $(x_1, x_2, x_3, \ldots, x_n)$ and compute sample mean. That would be the MC estimate of $E(g(X))$

$$\widetilde{g_n}(x) = \frac{1}{n}\sum_{i=1}^{n} g(x_i)$$

Monte Carlo

Recall : Desirable properties in an estimator

- Unbiasedness
- Minimal Variance

- Monte carlo estimator is asymptotically unbiased.

$$E(\widetilde{g_n}(x)) = E\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) = \frac{1}{n}\sum_{i=1}^{n} E(g(x_i)) = E(g(X))$$

- Now improving the efficiency of the estimator means reducing the variance

Note

Monte Carlo relies on repeated Sampling

- Any change in the variance of the estimator can be made only by change in Sampling.

Past Works

- Standard Monte-Carlo : Sampling is random.
  - *Sample Complexity* : $O(d\epsilon^{-2})$ to achieve variance ϵ^2 for a d-dimensional input vector.
- Quasi Monte Carlo method : Sampling is done at predetermined deterministic sequence of points
- Quasi Monte Carlo method is effective for a class of functions where the importance of U_i decreases with i.

Recall

We are estimating $E(f(U))$, where U is a vector of d independent random variables $(U_i, ...U_{i+d-1})$, where i = 1 to d.

- Multilevel Monte Carlo(MLMC) : Low dimensional approximation of $f(U) \Rightarrow$ reduction in computational cost.

Outline

Main Idea

Main Contribution

Presenting an unbiased algorithm to estimate $E[f(U)]$ by optimizing tradeoff between the statistical error and the total running time.
That is $T * \text{Variance}(Estimator)$ is upper bounded by $T * \text{Variance}(MC)$, but in its best case it can be much lesser than that.

- Recollect that we are trying to estimate $E[f(U)]$, where $U \in \mathbb{R}^d$.
- Paper notes that the importance of the component U_i decreases with increasing i.

Key Idea

- As f depends on it's first few components, only those set of arguments of f needs to be simulated at each iteration.
- The remaining arguments are carried over.

Outline

Algorithm

Algorithm

1. First iteration. Simulate a vector $V^{(1)}$ that has the same distribution as U and calculate $f(V(1))$.
2. Loop. In iteration $k+1$, where $1 \leq k \leq n-1$, let $V^{(k+1)}$ be the vector obtained from $V^{(k)}$ by redrawing the first N_k components of $V^{(k)}$, and keeping the remaining components unchanged. Calculate $f(V(k+1))$.
3. Output $f_n \triangleq \frac{f(V(1)),\ldots,f(V(n))}{n}$, where f_n is an unbiased estimator of $E[f(U)]$

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3. Output $f_n \triangleq \frac{f(V(1)),\ldots,f(V(n))}{n}$, where f_n is an unbiased estimator of $E[f(U)]$

- In step 2, N_k is a random integer in $[1, d]$, chosen according a prob. distribution q_i,$0 \leq i \leq d-1$. A decreasing sequence with i with $q_1 = 0$.
- q_i and N_k are related as $\Pr(N_k > i) = q_i$.

Outline

Analysis

- Standard Monte Carlo algorithm:

$$T_{MC}\text{Variance}(f_n) = \tau\text{Variance}(f(U))$$

where τ be the expected time needed to simulate U
$T_{MC} = n\tau$ is the expected total running time of the standard MC scheme in n iterations

- Optimized Sampling via Reduced Dimension:

$$\text{Variance}(f_n)T \le 2\left(\sum_{i=0}^{d-1} \frac{C(i) - C(i+1)}{q_i}\right)\left(\sum_{i=0}^{d-1} q_i(T_{i+1} - T_i)\right)$$

where $C(i) \triangleq \text{Variance}(E(f(U)|(U_{i+1}, ..., U_d))$
$T = (n-1)\sum_{i=0}^{d-1} q_i(T_{i+1} - T_i)$
and q_i decreases with increasing i

- In the special case of Linearly decreasing probabilities, the paper improves upon the standard Monte Carlo algorithm by a factor of $\Theta(d/\ln(d))$