MORE NOTES ON BYPASSES: August 7, 2023

1. Introduction

After addressing the note you sent in §2, I'll describe how to apply bypasses to convex hypersurfaces Σ described as double positive factorizations (DPFs) in Lemma 4.2. The definition of the DPF bypass from my previous notes was wrong – the result of sign error in a handle slide. The notes here have a proof which is at least not obviously wrong.

In Definition 5.1 I package the construction of [HH18, §10] as an *obviously overtwisted bypass* (OOB). Applying such a bypass to any $\{0\} \times \Sigma$ gives a contact structure on $[0,1]_t \times \Sigma$ for which $\{1\} \times \Sigma$ has an overtwisted neighborhood.

A specific OOB attachment is described in §5.1. It is the simplest such example, since it can be applied locally and involves the least possible number of handles. If we apply attach this bypass to the boundary of a standard $\dim = 2n+1$ Darboux disk, we get an *obviously overtwisted disk* (OOD).

I describe a DPF for the boundary Σ_{ot} of the OOD in §5.2. If everything is correct, the $R^{\pm}_{ot}$ for Σ_{ot} are "exotic" symplectic manifolds which have been the focus of quite a few articles. I've double checked the calculations here, but am still a little skeptical: I don't know if the $R^{\pm}_{ot}$ have "enough" holomorphic planes for a neighborhood of Σ_{ot} to have vanishing contact homology as in [Av23].

1.1. **Some speculation.** Any thoughts on if the below is worth attempting or if it's clearly futile? You've likely considered something similar. It's independent of the $R^{\pm}_{ot}$ calculation mentioned above.

Question 1.1. *Is every convex hypersurface Σ with an overtwisted t-invariant neighborhood the result of an obviously overtwisted bypass attachment to some Σ'?*

Let's take a fixed Σ which has an overtwisted t-invariant neighborhood $[0,1] \times \Sigma$. We'll attempt to prove that Σ is the result of an OOB attachment.

The h-principle should allow us to find OODs in a t-invariant $(0,1)_t \times \Sigma$ since this neighborhood is overtwisted. Assuming this is true, there will be OOBs inside $[0,1)_t \times \Sigma$ which can be attached to $\{0\} \times \Sigma$: Attach one end of a contact 1-handle to the dividing set of $\{0\} \times \Sigma$ and the other end to the dividing set of the OOD. This is the same as first connecting $\{0\} \times \Sigma$ to the boundary of a Darboux ball – which gives us back Σ, unmodified – and then applying an OOB attachment.

Say we attach such an OOB, B_1, to $\{0\} \times \Sigma$. Then by [HH19, Theorem 1.2.5] we can get from $\{0\} \times \Sigma$ to $\{1\} \times \Sigma$ by a sequence of bypass attachments $B_1, \cdots, B_k$ so that the union of all of the bypass attachment slices will be isotopic to the t-invariant $[0,1] \times \Sigma$.

If we can perform handle-slides and rearrange the orders of attachment so that there are bypasses $B'_1, \dots, B'_{k'}$ getting us from $\{0\} \times \Sigma$ back to itself and with $B'_{k'}$ obviously overtwisted, then $\Sigma = \{1\} \times \Sigma$ will be the result of an OOB attachment to Σ', which is given by attaching the bypasses $B'_1, \dots, B'_{k'-1}$, answering Question 1.1 in the affirmative.

The OOB B_1 is attached along a ball in Σ, so maybe it's possible that it could be avoided during any handle-slides and changing of the orders of bypasses, guaranteeing $B'_{k'} = B_1$. If true, then Σ would be determined by the model OOB attachment described in §5. Perhaps this is overly optimistic. Assuming that the strategy of rearranging the orders of bypasses works, I don't know if it would be any easier to prove using DPFs or contact handle attachments.

2. Notes on Ko's note

The goal of Ko's note is to prove that provided bypass attachment data for Legendrians $\Lambda^{\pm} \subset \Gamma$ bounding thimbles $L^{\pm} \subset R^{\pm}$, we can make a DPF with page W for which $\Lambda^+ \cup \Lambda^-$ sit on a single page. My understanding is that the tricky part is to realize $\Lambda^+ \cup \Lambda^-$ on a page of a supporting open book for Γ. By [HH19, Corollary 1.3.3], we can realize a smooth Legendrian on a page of a supporting open book decomposition, but $\Lambda^+ \cup \Lambda^-$ has a singularity at the point of intersection.

Following the proof at the end of [HH19, §10], we need to find

(1) a neighborhood N of $\Lambda^+ \cup \Lambda^-$ (disjoint union some other smooth Legendrians) and
(2) a contact Morse function f on N for which is constant along ∂N

I'll neglect the smooth Legendrian components since the construction applies to components of N.

Identify N with a contactization $[-1,1]_t \times V$ where V a plumbing of two copies of $\mathbb{D}^*\mathbb{S}^{n-1}$, or equivalently an A_2 Milnor fiber. This is a Stein manifold and so the Liouville vector field X_V on V is gradient like for a Morse function f_V on V, $\langle \nabla f_V, X_V\rangle \geq \mathrm{const}\, \|\nabla f_V\|^2$. By rescaling f_V and possibly modifying it near ∂V (without creating critical points), we can assume that $\mathrm{const} = f_V|_{\partial V} = 1$. On this contactization neighborhood, we can take our contact structure to be determined by $\alpha = dt + \beta$ where β is the Liouville form for V, set $f = \frac{1}{2}t^2 + f_V$, and define $X = t\partial_t + X_V$. Then we compute

$$\mathcal{L}_X \alpha = \alpha, \quad \nabla f = t\partial_t + \nabla f_V, \quad \langle \nabla f, X\rangle = t^2 + \langle \nabla f_V, X_V\rangle \geq \|\nabla f\|^2.$$

So X is a contact vector field which is gradient-like for f. Moreover $f^{-1}(1)$ gives us N with its corners rounded and so contains the singular Legendrian $\Lambda^+ \cup \Lambda^-$ in its interior as desired. Moreover, the Morse critical points of f are in one-to-one correspondence with those of f_V in a way which preserves indices. So if we extend f to the exterior $\Gamma \setminus N$ of N arbitrarily, then N will be contained in a neighborhood of the n-skeleton of Γ since n is the maximum index of a critical point of f_V.

3. Conventions

Throughout Σ is a Weinstein convex hypersurface with decomposition $R^\pm, \Gamma$. We'll take $\dim \Gamma = 2n-1, \dim \Sigma = 2n$.

For Lefschetz fibrations we assume that neighborhoods of the boundaries of thimbles (AKA cocores) lie with a neighborhood of a page W. We label these Legendrians Λ_j so that they sit on $W \times \{j\epsilon\}$ given by the time $j\epsilon$ Reeb flow applied to $W = W \times \{0\}$. With this ordering convention the monodromy of the open book on the boundary is

$$\tau_{\Lambda_k} \circ \cdots \circ \tau_{\Lambda_1} : W \to W.$$

We say that the Lefschetz fibration is determined by $\mathbf{\Lambda} = (\Lambda_1 \ldots, \Lambda_k)$.

Editor notes 3.1. *This is opposite the convention from Ko's note! Sorry, I started this note a while ago.*

Legendrian surgeries can be thought of as taking out a neighborhood $N = [-\epsilon, \epsilon] \times \mathbb{D}^*\Lambda$ of a Legendrian and gluing it back with a Dehn twist on the top $\{\epsilon\} \times \mathbb{D}^*\Lambda$. In this picture, the cocore of the handle is the zero section of N, cf. [Av20, §10]. So we can flow the cocore to the handle exterior using the $\pm$ Reeb flow to $\{\pm\epsilon\} \times \Lambda$.

For a DPF, we have $R^\pm$ determined by Lefschetz fibrations with fiber W and ordered collections of (framed) Legendrian spheres $\mathbf{\Lambda}^\pm = (\Lambda_i^\pm)_{i=1}^{k\pm}, \Lambda_i^\pm \subset W$. We identify their boundaries when we have an identifications of the fibers of the Lefschetz fibrations on the $R^\pm$ with a fixed W and

$$\tau_{\mathbf{\Lambda}^+} := \tau_{\Lambda^+_{k+}} \circ \cdots \circ \tau_{\Lambda^+_1} = \tau_{\Lambda^-_{k-}} \circ \cdots \circ \tau_{\Lambda^-_1} =: \tau_{\mathbf{\Lambda}^-}.$$

Specifically the identification of the monodromies allows us identify the mapping tori

$$[0,1] \times W/ \sim^\pm, \quad (1,x) \sim^\pm (0, \tau_{\mathbf{\Lambda}^\pm}).$$

From the open book picture, the mapping tori make up all of the $\partial R^\pm$ except for the binding. The identification trivially extends over the binding since it is ∂W.

3.1. **Legendrian sum.** Given a pair $\Lambda^\pm$ of Legendrians in a contact manifold Γ intersecting ξ-transversely at a single point, we define the Legendrian sum using the Lagrangian projection formulation from [HH18]. If the $\Lambda^\pm$ sit on $\{\epsilon\} \times W$, then we'll get $\Lambda^- \uplus \Lambda^+$ to be the Legendrian lift of the Lagrangian sphere

$$\tau_{\Lambda^+}\Lambda^- = \tau^{-1}_{\Lambda^-}\Lambda^+ \subset W$$

where τ is a Dehn twist. Then

$$F^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+) := \mathrm{Flow}^{\pm\epsilon}_{\partial_t}(\Lambda^- \uplus \Lambda^+) \subset ([0,\epsilon] \cup [2\epsilon, 3\epsilon]) \times W.$$

After a slight perturbation of the open book the $F^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+)$ can be assumed to sit on pages as well.

When Λ^+ is a standard unknot, we say that Λ^+ is *θ-above* Λ^- if it has a θ disk which is disjoint from $F^{-\epsilon}\Lambda^-$ for ϵ small.

4. Bypasses

Definition 4.1. *Bypass attachment data consists of a tuple $(\Lambda^\pm, L^\pm)$ for which $\Lambda^\pm$ is a pair of Legendrians intersecting in a single point and $L^\pm$ are Lagrangian slice disks in the $R^\pm$ for the $\Lambda^\pm$.*

If we attach a bypass to $[-\epsilon, 0] \times \Sigma$ at $\{0\} \times \Sigma$ using the data of $(\Lambda^\pm, L^\pm)$ then we get a contact structure on $[-\epsilon, 1] \times \Sigma$ with $\Sigma_1 = \{1\} \times \Sigma$ described as in [HH18, Theorem 5.1.3]:

(1) $R_1^\pm$ is obtained by attaching a Weinstein handle to $\Lambda^- \uplus \Lambda^+$ and cutting out standard neighborhoods of the $F^{\mp\epsilon} L^\pm$.
(2) Γ_1 is obtained by performing a contact $+1$ surgery on $F^{\mp\epsilon}\Lambda^\pm$ and a contact -1 surgery along $(\Lambda^- \uplus \Lambda^+)$.
(3) The boundaries of the $R_1^\pm$ are identified by the contactomorphism $\partial R_1^+ \to \partial R_1^-$ obtained by sliding $F^{-\epsilon}\Lambda^+$ up over $(\Lambda^- \uplus \Lambda^+)$ to $F^\epsilon \Lambda^-$.

4.1. Bypasses for DPFs.

Lemma 4.2. *Suppose that the convex hypersurface $\Sigma = \Sigma_0$ is described by a DPF $\mathbf{\Lambda}^\pm$ with the $\Lambda_1^\pm$ intersecting transversely at a single point in W. Then the $\Lambda_1^\pm$ together with their thimbles $L_1^\pm \subset R^\pm$ give bypass attachment data. The result of the bypass, Σ_1, can be described by a DPF $\mathbf{\Lambda}_b^\pm$ given by*

$$\mathbf{\Lambda}_b^+ = (\tau^2_{\Lambda_1^+}\Lambda_1^-, \Lambda_2^+, \dots, \Lambda_{k+}^+), \quad \mathbf{\Lambda}_b^- = (\tau_{\Lambda_1^+}\Lambda_1^-, \Lambda_2^-, \dots, \Lambda_{k-}^-)$$

Let's first verify that the monodromies of the $\mathbf{\Lambda}_b^\pm$ agree as a sanity check:

$$\begin{aligned}
\tau_{\mathbf{\Lambda}_b^+} &= \tau_{\Lambda_{k+}^+} \circ \dots \circ \tau_{\Lambda_2^+} \circ \tau_{\tau^2_{\Lambda_1^+}\Lambda_1^-} = \tau_{\Lambda_k^+} \circ \dots \circ \tau_{\Lambda_2^+} \circ \tau_{\Lambda_1^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\mathbf{\Lambda}^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\mathbf{\Lambda}^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\mathbf{\Lambda}^-} \circ \tau_{\tau^{-1}_{\Lambda_1^-}\Lambda_1^+} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\mathbf{\Lambda}^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\Lambda_{k-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\Lambda_{k-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} = \tau_{\mathbf{\Lambda}_b^-}.
\end{aligned}$$

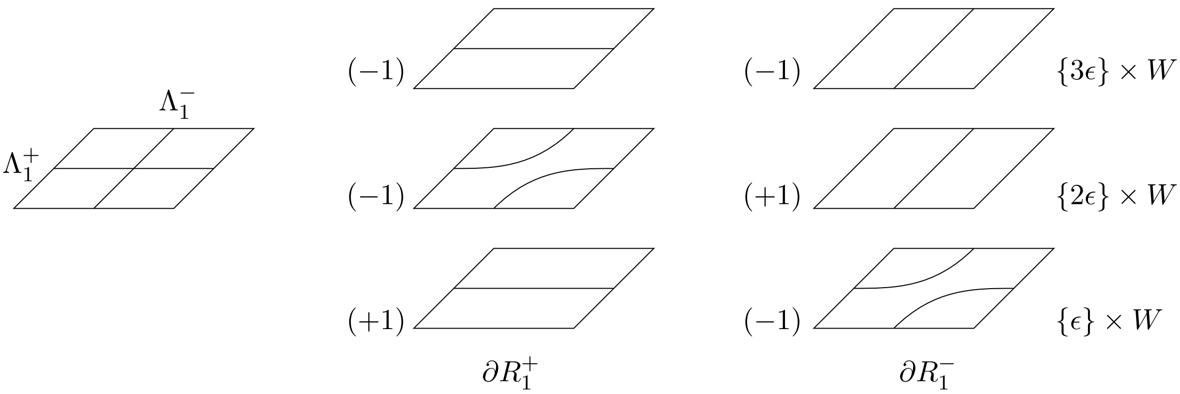

FIGURE 1. Depiction of bypass surgeries in an open book. Each "sheet" is a neighborhood of the point $\Lambda_1^+ \cap \Lambda_1^-$ in W.

To prove the lemma we apply the definition of the bypass. Suppose that the $\Lambda_1^\pm$ lie on $\{3\epsilon\} \times W$. The cocores of the surgery handles can then be pushed down to $\{2\epsilon\} \times W$ by the negative Reeb flow as shown in the left-hand side of Figure 1 so that they share a single transverse intersection.

Working out $\mathbf{\Lambda}_b^-$ is very easy. According to the definition of the bypass, we must perform a contact $+1$ surgery on a copy of Λ_1^- placed slightly above the new surgery locus $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ in the boundary of ∂R_1^-. See the right-hand side of Figure 1. The ± 1 surgeries along the Λ_1^- at heights 2ϵ and 3ϵ cancel, so that we are only left with the new -1 surgery locus, obtaining Λ_b^- as claimed.

To describe the boundary of the new positive region R_1^+ we perform a contact -1 surgery along $\tau_{\Lambda_1^+}\Lambda_1^-$ (at height 2ϵ) along with a contact $+1$ surgery slightly below it, say at $\{\epsilon\} \times W$. See the center row of Figure 1. We want to make the ± 1 surgeries on Λ_1^+ cancel. To that end, we handle-slide

$\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ (currently at height 2ϵ) down through the surgery locus at $\{\epsilon\} \times \Lambda_1^+$. The before and after pictures are as shown in Figure 2. Because we are flowing down through a $+1$ surgery handle, the effect is a positive Dehn twist: We obtain $\tau^2_{\Lambda_1^+}\Lambda_1^-$ on the bottom (say, at height ϵ) and canceling ± 1 surgeries (say at heights 2ϵ, 3ϵ). Deleting the canceling ± 1 surgeries, we are left with $\mathbf{\Lambda}_b^+$ as claimed.

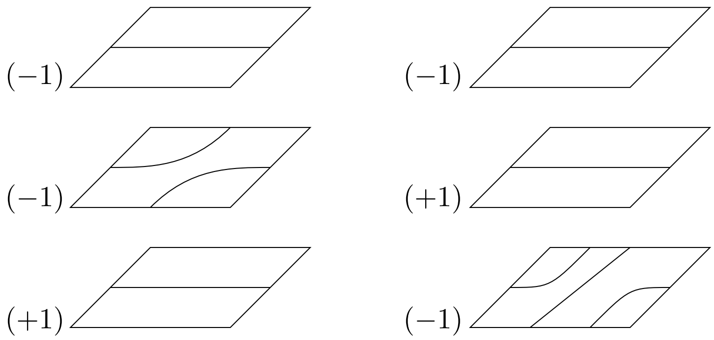

FIGURE 2. Handle-sliding the new -1 surgery locus on ∂R_1^+ down through the new $+1$ surgery locus. The before picture is on the left, and in the after picture on the right $\tau^2_{\Lambda_1^+}\Lambda^-$ sits on the bottom sheet $\{\epsilon\} \times W$.

5. OBVIOUSLY OVERTWISTED OBJECTS

Definition 5.1. *We'll say that a bypass as described in* [HH18, §10] *is an* obviously overtwisted bypass *(OOB). That is, Λ^+ is the unknot, L^+ is the standard Lagrangian disk, and Λ^+ is θ-above Λ^-.*

The definition ensures that the convex hypersurface Σ_1 resulting from the bypass will have an overtwisted neighborhood. It is a priori stronger than the *overtwisted bypass* of [HH18]. We'll describe an OOB attachment on $\mathbb{S}^{2n}$ the boundary of a contact $\dim = 2n+1$ Darboux disk, whence $R^\pm = \mathbb{D}^{2n}$.

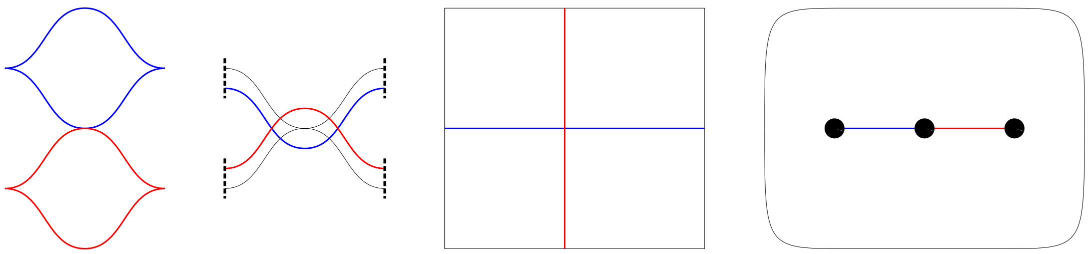

FIGURE 3. In each subfigure we see x in blue and y in red. On the left they are boundaries of Lefschetz thimbles in (S^3, ξ_{std}). In the middle left, a Gompf diagram [Go98], they are Legendrians in the front projection with the skeleton of W in black as in [Av11]. Here x and y are shifted off of the 1-skeleton (thin black graph) so that x lives on $\{-\epsilon\} \times W$ and y lives on $\{\epsilon\} \times W$ and the dashed vertical lines represent 1-handles. In the center-right the Legendrians give a meridian and longitude on a punctured torus. We identify the sides of the square with each other and the top with the bottom, taking the puncture to be the corner. On the right are matching paths for x, y in the π_W projection.

5.1. **The OOD.** For the Lefschetz fibrations on $\mathbb{D}^{2n}$ (on which we take standard complex coordinates z_k) we can use the function $\pi = \delta + z_1^3 + \sum_2^n z_k^2$ with $\delta \in \mathbb{C} \setminus 0$ small. Then a regular fiber $W = W^{2n-2}$ will be a plumbing of two cotangent bundles of spheres x, y and the monodromy is $\tau_y\tau_x$. So we can say that

$$\mathbf{\Lambda}^\pm = (x, y)$$

for both $\pm$. Both of the x and y are unknots in $\mathbb{D}^{2n}$. According to our convention for ordering Dehn twists, $y \subset W \times \{\epsilon\}$ while $x \subset W \times \{-\epsilon\}$.

When $n = 2$ the Legendrians can be seen as a meridian and a longitude on a once-punctured torus. To get the monodromy in this case, we can apply [Av11, Theorem 4.8] and to the left-hand side of Figure 3.

In general, a linear function (eg. $\pi_W = z_1|_W$) gives a Lefschetz fibration on W with three singular fibers, which after an isotopy we can take to be $1, 0, -1 \in \mathbb{C}$ and with non-singular fiber a $\mathbb{D}^*\mathbb{S}^{n-2}$. Then we can see x as the matching path sphere for the arc $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ and y as the matching path sphere for the arc $[-1,0]$. See the right-hand side of Figure 3. In the $n = 2$ case, we see that x is θ-above y and we'll assume that this is the case as well in higher dimensions (although I'd have to think about how to prove it).

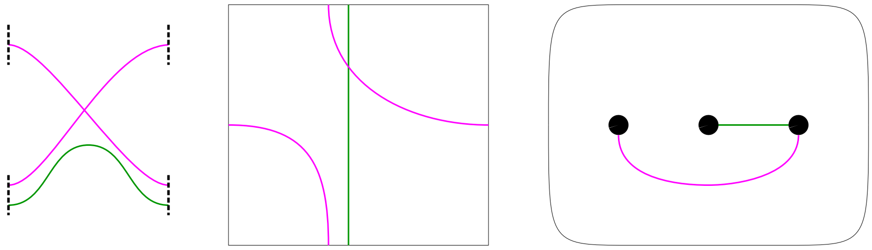

FIGURE 4. The Legendrians $(y, \tau_y x)$ shown in green and purple, respectively.

We can handle-slide x up through y to obtain another positive factorization for $\mathbb{D}^{2n}$ given by $(y, \tau_y x)$ as seen in Figure 4. So

$$\boldsymbol{\Lambda}^+ = (x, y), \quad \boldsymbol{\Lambda}^- = (y, \tau_y x) \tag{1}$$

is a double positive factorization for the standard $\mathbb{S}^{2n}$ for which the first Legendrians $\Lambda^+ = x$ and $\Lambda^- = y$ give OOB data, since x is θ-above y.

Definition 5.2. *The* obviously overtwisted disk *(OOD) is the disk with convex boundary Σ_{ot} obtained by attaching an obviously overtwisted bypass to the boundary of a standard Darboux disk.*

5.2. **Result of the OOB attachment.** After the bypass is performed, we apply Lemma 4.2 to see

$$\boldsymbol{\Lambda}_b^+ = (\tau_x^2 y, y), \quad \boldsymbol{\Lambda}_b^- = (\tau_x y, \tau_y x)$$

giving a DPF on the boundary of the OOD.

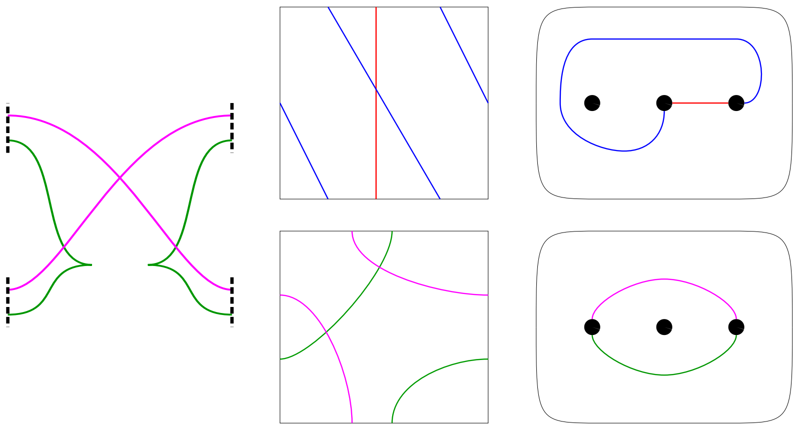

FIGURE 5. Lefschetz fibrations for the positive region (top) and negative region (bottom) after the obviously overtwisted bypass has been performed.

The bypass data determines a contact structure on $[0,1] \times \mathbb{S}^{2n}$ with $\{0\} \times \mathbb{S}^{2n}$ the original hypersurface (given by the boundary of a $\dim = 2n+1$ Darboux disk) and

$$\Sigma_{ot} = \{1\} \times \mathbb{S}^{2n}$$

the result of the bypass. The positive and negative regions for Σ_{ot} are denoted $R_{ot}^{\pm}$.

A Lefschetz fibrations for R_{ot}^+ appears in the top row of Figure 5 with $\tau_x^2 y$ in blue and y in red. On the bottom row of the figure we have a Lefschetz fibration for R_{ot}^- with $\tau_x y$ in green and $\tau_y x$ in purple.

Knowing that [CM19] contained lots of pictures, I took a look and got lucky, finding the Lefschetz fibration described in bottom right of Figure 5 as [CM19, Figure 28]. The $R^{\pm}_{ot}$ are well-studied in the literature. When $2n = 6$ they can be described as affine varieties cut out by a single equation

$$R^{\pm}_{ot} = \{x(xy-1) = z_1^2 + z_2^2\} \subset \mathbb{C}^4_{x,y,z_1,z_2}$$

(and I believe this could be generalized to other dimensions by adding more z_ks). The following are known facts for the $\dim = 4, 6$ cases: The $R^{\pm}_{ot}$ are not flexible but embed into flexible manifolds of the same dimension. They have symplectic homology $SH = 0$ over $\mathbb{Z}$ but $SH \neq 0$ over a twisted coefficient system [MS18], so their holomorphic curves should be quite complicated.

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