NOTES ON BYPASSES

ABSTRACT. Here we take notes on [HH18] and translate some of the key terminology into the language of double positive factorizations (DPFs).

Throughout Σ is a Weinstein convex hypersurface with decomposition $R^{\pm}, \Gamma$. We'll take $\dim \Gamma = 2n-1, \dim \Sigma = 2n$.

Here is our convention for building Lefschetz fibrations: Start with a Lefschetz fibration for $W \times \mathbb{D} \to \mathbb{D}$ inducing an open book on the boundary whose page is W and whose monodromy is trivial. Say that the whose mapping torus portion of the open book is $W \times S^1$. If we attach a Weinstein handle at some $\Lambda \subset W \times \{0\}$ then the Lefschetz fibration extends over the surgered Weinstein manifold so that the core of the handle becomes a Lefschetz thimble. The surgery locus in the boundary is $N_W(\Lambda) \times [0, 3\epsilon]$ and the boundary of the cocore is $\Lambda \times \{0\} \simeq \Lambda \times \{3\epsilon\}$. Here $N_W(\Lambda_k)$ is a standard Weinstein neighborhood of Λ in W. The $\simeq$ is a Legendrian isotopy coming from our ability to slide boundary of the cocore through the handle.

To build up to a general Lefschetz fibration, we attach k Weinstein handles to the neighborhoods

$$N_W(\Lambda_k) \times \left[\frac{i-1}{k}, \frac{i-1}{k} + 3\epsilon\right] \subset W \times S^1, \quad S^1 = \mathbb{R}/\mathbb{Z}$$

using the 4ϵ neighborhoods with ϵ small. With this ordering convention the monodromy of the open book on the boundary is

$$\tau_{\Lambda_k} \circ \cdots \circ \tau_{\Lambda_1} : W \to W.$$

For a double positive factorization (DPF), we have $R^{\pm}$ determined by Lefschetz fibrations with fiber W and ordered collections of (framed) Legendrian spheres $\mathbf{\Lambda}^{\pm} = (\Lambda_i^{\pm})_{i=1}^{k\pm}, \Lambda_i^{\pm} \subset W$. We identify with boundaries when we have an identifications of the fibers of the Lefschetz fibrations on the $R^{\pm}$ with a fixed W and

$$\tau_{\mathbf{\Lambda}^+} := \tau_{\Lambda_{k+}^+} \circ \cdots \circ \tau_{\Lambda_1^+} = \tau_{\Lambda_{k-}^-} \circ \cdots \circ \tau_{\Lambda_1^-} =: \tau_{\mathbf{\Lambda}^-}.$$

Specifically the identification of the monodromies allows us identify the mapping tori

$$[0,1] \times W / \sim^{\pm}, \quad (1, x) \sim^{\pm} (0, \tau_{\mathbf{\Lambda}^{\pm}}).$$

From the open book picture, the mapping tori make up all of the $\partial R^{\pm}$ except for the binding. The identification trivially extends over the binding since it is ∂W.

1. LEGENDRIAN SUM AND BYPASSES

1.1. Legendrian sum. Given a pair $\Lambda^{\pm}$ of Legendrians in a contact manifold Γ intersecting ξ-transversely at a single point, we'll define the Legendrian sum using the Lagrangian projection formulation from [HH18].

Let $N_\Gamma(\Lambda^- \cup \Lambda^+)$ be a neighborhood in Γ inside of a subset of the $[0, 4\epsilon]_t \times W$ where W is a Weinstein manifold containing a plumbing of two copies of $\mathbb{D}^*\mathbb{S}^{n-1}$. The $\Lambda^{\pm}$ are given by the 0-sections of the $\mathbb{D}^*\mathbb{S}^{n-1}$ inside $\{0\} \times W$ and our contact structure is determined by $dt + \beta$ for β a Liouville form on W. Then define

$$\Lambda^- \uplus \Lambda^+ \subset [\epsilon, 2\epsilon] \times W$$

to be the Legendrian lift of the Lagrangian sphere

$$\tau_{\Lambda^+}\Lambda^- = \tau_{\Lambda^-}^{-1}\Lambda^+ \subset W$$

where τ is a Dehn twist. Then

$$F^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+) := \text{Flow}_{\partial_t}^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+) \subset ([0, \epsilon] \cup [2\epsilon, 3\epsilon]) \times W.$$

1.1.1. *Translation.* We'll always think of $\Lambda^{\pm}$ as being Legendrian spheres sitting on a single page W of an open book for Γ. We'll always work far from the binding so that we can assume that each $t = t_0$ slice of $[-2\epsilon, 2\epsilon] \times W$ also sits in a page. After a slight perturbation of the open book the $(\Lambda^- \uplus \Lambda^+)^{\pm\epsilon}$ can be assumed to sit on pages as well.

One key observation is that we have conjugacy relations and a braid relation when $\Lambda^{\pm}$ intersect at a single point in W,

$$\tau_{\tau_{\Lambda^+}\Lambda^-} = \tau_{\Lambda^+} \circ \tau_{\Lambda^-} \circ \tau_{\Lambda^+}^{-1}, \quad \tau_{\Lambda^+}\tau_{\Lambda^-}\tau_{\Lambda^+} \simeq \tau_{\Lambda^-}\tau_{\Lambda^+}\tau_{\Lambda^-}.$$

Another observation is that if Λ^+ is an unknot, then $\Lambda^- \uplus \Lambda^+$ will be loose in the ambient contact manifold. In the low dimensional case, $\dim \Gamma = 3$, this will be Λ^- both positively and negatively stabilized.

1.2. **Bypass.**

Definition 1.1. *Bypass attachment data consists of a tuple $(\Lambda^{\pm}, L^{\pm})$ for which $\Lambda^{\pm}$ is a pair of Legendrians intersecting in a single point and $L^{\pm}$ are Lagrangian slice disks in the $R^{\pm}$ for the $\Lambda^{\pm}$.*

If we attach a bypass to $[-\epsilon, 0] \times \Sigma$ at $\{0\} \times \Sigma$ using the data of $(\Lambda^{\pm}, L^{\pm})$ then we get a contact structure on $[-\epsilon, 1] \times \Sigma$ with $\Sigma_1 = \{1\} \times \Sigma$ described as in [HH18, Theorem 5.1.3]:

(1) $R_1^{\pm}$ is obtained by attaching a Weinstein handle to $\Lambda^- \uplus \Lambda^+$ and cutting out standard neighborhoods of the $F^{\mp\epsilon}L^{\pm}$.
(2) Γ_1 is obtained by performing a contact $+1$ surgery on $F^{\mp\epsilon}\Lambda^{\pm}$ and a contact -1 surgery along $(\Lambda^- \uplus \Lambda^+)$.
(3) The boundaries of the $R_1^{\pm}$ are identified by the contactomorphism $\partial R_1^+ \to \partial R_1^-$ obtained by sliding $F^{-\epsilon}\Lambda^+$ up over $(\Lambda^- \uplus \Lambda^+)$ to $F^{\epsilon}\Lambda^-$.

1.2.1. *Translation.* In the DPF picture, we assume that the $\Lambda^{\pm}$ are the $\Lambda_1^{\pm} \in \mathbf{\Lambda}^{\pm}$ so that

$$\Lambda^- \uplus \Lambda^+ = \tau_{\Lambda_1^+}\Lambda_1^-.$$

If we are given a Lefschetz fibration determined by a $\mathbf{\Lambda}$, then cutting out the disk L_1 whose boundary is Λ_1 amounts to replacing $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_k)$ with $(\Lambda_2, \dots, \Lambda_k)$.

Let's assume that the surgery loci for the $\Lambda_1^{\pm}$ are within $[\epsilon, 2\epsilon] \times W$. Since we use the $\{0\} \times W$ to identify the fibers of the fibration, let's say that $\Lambda_1^- \uplus \Lambda_1^+$ sits on $\{\epsilon\} \times W$ and that there are no surgery loci in $[0, \epsilon] \times W$.

From the positive region description of Γ, we perform add a negative Dehn twist for the contact $+1$ surgery at $\{0\} \times W$, and have the Weinstein handle attachment locus at $\{\epsilon\} \times W$. As in step (3) of the definition, we slide the $+1$ surgery locus up through the $\{\epsilon\} \times W$. Now the $+1$ surgery occurs along $\tau_{\Lambda^- \uplus \Lambda^+}\Lambda^+$ and we compute

$$\left(\tau_{\tau_{\Lambda^- \uplus \Lambda^+}\Lambda^+}\right)^{-1} = \left(\tau_{\Lambda^- \uplus \Lambda^+} \circ \tau_{\Lambda^+} \circ \tau_{\Lambda^- \uplus \Lambda^+}^{-1}\right)^{-1} = \left(\tau_{\Lambda^+}\tau_{\Lambda^-}\tau_{\Lambda^+}^{-1}\tau_{\Lambda^+}\tau_{\Lambda^+}^{-1}\tau_{\Lambda^-}^{-1}\tau_{\Lambda^+}\right)^{-1}$$

We use the pushoff directions $(F^{\pm})$ to determine the order in which twists are added to the modnodromies. To follow step (2) in the definition the monodromies become modified as

$$\tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\Lambda_1^+} \mapsto \tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} \circ \tau_{\Lambda_1^+}$$
$$\tau_{\Lambda_k^-} \circ \cdots \circ \tau_{\Lambda_1^-} \mapsto \tau_{\Lambda_k^-} \circ \cdots \circ \tau_{\Lambda_1^-}^{-1} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau_{\Lambda_1^-}.$$

In the first row the $\tau_{\Lambda_1^+}^{-1}$ is added below the new surgery locus and in the second row the $\tau_{\Lambda_1^-}^{-1}$ is added above. These simplify to

$$\tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\Lambda_1^+} \mapsto \tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-}$$
$$\tau_{\Lambda_k^-} \circ \cdots \circ \tau_{\Lambda_1^-} \mapsto \tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\Lambda_1^+}^{-1} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau_{\Lambda_1^+}.$$

REFERENCES

[BEM15] M. S. Borman, Y. Eliashberg, and E. Murphy, *Existence and classification of overtwisted contact structures in all dimensions*, Acta Math. 215, p.281–361, 2015.

[BHH] J. Breen, K. Honda, and Y. Huang, *The Giroux correspondence in arbitrary dimensions*, arXiv:2307.02317, 2023.

[GP17] E. Giroux and J. Pardon, *Existence of Lefschetz fibrations on Stein and Weinstein domains*, Geom. Topol., 21, no. 2, 963–997, 2017.

[HH18] K. Honda and H. Huang, *Bypass attachments in higher-dimensional contact topology*, preprint, arXiv:1803.09142, 2018. 1, 2

[HH19] K. Honda and H. Huang, *Convex hypersurface theory in contact topology*, preprint, arXiv:1907.06025, 2019.

[HT22] Ko Honda and Yin Tian, *Contact categories of disks*, J. Symp. Geom., Vol. 20, 2022.