NOTES ON BYPASSES

ABSTRACT. Here we translate some of the key terminology from [HH18] into the language of open books and double positive factorizations (DPFs).

Throughout Σ is a Weinstein convex hypersurface with decomposition $R^{\pm}, \Gamma$. We'll take $\dim \Gamma = 2n-1, \dim \Sigma = 2n$.

Here is our convention for building Lefschetz fibrations: Start with a trivial Lefschetz fibration for $\pi : W \times \mathbb{D} \to \mathbb{D}$ (for which π is the projection onto the $\mathbb{D}$ factor) an open book on the boundary whose page is W and whose monodromy is trivial. Say that the whose mapping torus portion of the open book is $W \times S^1$. If we attach a Weinstein handle at some $\Lambda \subset W \times \{0\}$ then the Lefschetz fibration extends over the surgered Weinstein manifold so that the core of the handle becomes a Lefschetz thimble. The surgery locus in the boundary is $N_W(\Lambda) \times [0, 3\epsilon]$ and the boundary of the cocore is $\Lambda \times \{0\} \simeq \Lambda \times \{3\epsilon\}$. Here $N_W(\Lambda_k)$ is a standard Weinstein neighborhood of Λ in W. The $\simeq$ is a Legendrian isotopy coming from our ability to slide boundary of the cocore through the handle.

To build up to a general Lefschetz fibration, we attach k Weinstein handles to the neighborhoods

$$N_W(\Lambda_k) \times \left[\frac{i-1}{k}, \frac{i-1}{k} + 3\epsilon\right] \subset W \times S^1, \quad S^1 = \mathbb{R}/\mathbb{Z}$$

using the 3ϵ neighborhoods with ϵ small. With this ordering convention the monodromy of the open book on the boundary is

$$\tau_{\Lambda_k} \circ \cdots \circ \tau_{\Lambda_1} : W \to W.$$

We say that the Lefschetz fibration is determined by $\mathbf{\Lambda} = (\Lambda_1 \dots, \Lambda_k)$.

For a double positive factorization (DPF), we have $R^{\pm}$ determined by Lefschetz fibrations with fiber W and ordered collections of (framed) Legendrian spheres $\mathbf{\Lambda}^{\pm} = (\Lambda_i^{\pm})_{i=1}^{k^{\pm}}, \Lambda_i^{\pm} \subset W$. We identify with boundaries when we have an identifications of the fibers of the Lefschetz fibrations on the $R^{\pm}$ with a fixed W and

$$\tau_{\mathbf{\Lambda}^+} := \tau_{\Lambda^+_{k^+}} \circ \cdots \circ \tau_{\Lambda_1^+} = \tau_{\Lambda^-_{k^-}} \circ \cdots \circ \tau_{\Lambda_1^-} =: \tau_{\mathbf{\Lambda}^-}.$$

Specifically the identification of the monodromies allows us identify the mapping tori

$$[0,1] \times W / \sim^{\pm}, \quad (1, x) \sim^{\pm} (0, \tau_{\mathbf{\Lambda}^{\pm}}).$$

From the open book picture, the mapping tori make up all of the $\partial R^{\pm}$ except for the binding. The identification trivially extends over the binding since it is ∂W.

1. LEGENDRIAN SUM AND BYPASSES

1.1. **Legendrian sum.** Given a pair $\Lambda^{\pm}$ of Legendrians in a contact manifold Γ intersecting ξ-transversely at a single point, we'll define the Legendrian sum using the Lagrangian projection formulation from [HH18].

Let $N_{\Gamma}(\Lambda^- \cup \Lambda^+)$ be a neighborhood in Γ inside of a subset of the $[0, 4\epsilon]_t \times W$ where W is a Weinstein manifold containing a plumbing of two copies of $\mathbb{D}^*\mathbb{S}^{n-1}$. The $\Lambda^{\pm}$ are given by the 0-sections of the $\mathbb{D}^*\mathbb{S}^{n-1}$ inside $\{0\} \times W$ and our contact structure is determined by $dt + \beta$ for β a Liouville form on W. Then define

$$\Lambda^- \uplus \Lambda^+ \subset [\epsilon, 2\epsilon] \times W$$

to be the Legendrian lift of the Lagrangian sphere

$$\tau_{\Lambda^+}\Lambda^- = \tau^{-1}_{\Lambda^-}\Lambda^+ \subset W$$

where τ is a Dehn twist. Then

$$F^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+) := \mathrm{Flow}^{\pm\epsilon}_{\partial_t}(\Lambda^- \uplus \Lambda^+) \subset ([0, \epsilon] \cup [2\epsilon, 3\epsilon]) \times W.$$

1.1.1. *Translation.* We'll always think of $\Lambda^\pm$ as being Legendrian spheres sitting on a single page W of an open book for Γ. We'll always work far from the binding so that we can assume that each $t = t_0$ slice of $[-2\epsilon, 2\epsilon] \times W$ also sits in a page. After a slight perturbation of the open book the $(\Lambda^- \uplus \Lambda^+)^{\pm\epsilon}$ can be assumed to sit on pages as well.

One key observation is that we have conjugacy relations and a braid relation when $\Lambda^\pm$ intersect at a single point in W,

$$\tau_{\tau_{\Lambda^+}\Lambda^-} = \tau_{\Lambda^+} \circ \tau_{\Lambda^-} \circ \tau_{\Lambda^+}^{-1}, \quad \tau_{\Lambda^+} \circ \tau_{\Lambda^-} \circ \tau_{\Lambda^+} \simeq \tau_{\Lambda^-} \circ \tau_{\Lambda^+} \circ \tau_{\Lambda^-}.$$

Another observation is that if Λ^+ is an unknot, then $\Lambda^- \uplus \Lambda^+$ will be loose in the ambient contact manifold. In the low dimensional case, $\dim \Gamma = 3$, this will be Λ^- both positively and negatively stabilized.

1.2. **Bypass.**

Definition 1.1. *Bypass attachment data consists of a tuple $(\Lambda^\pm, L^\pm)$ for which $\Lambda^\pm$ is a pair of Legendrians intersecting in a single point and $L^\pm$ are Lagrangian slice disks in the $R^\pm$ for the $\Lambda^\pm$.*

If we attach a bypass to $[-\epsilon, 0] \times \Sigma$ at $\{0\} \times \Sigma$ using the data of $(\Lambda^\pm, L^\pm)$ then we get a contact structure on $[-\epsilon, 1] \times \Sigma$ with $\Sigma_1 = \{1\} \times \Sigma$ described as in [HH18, Theorem 5.1.3]:

(1) $R_1^\pm$ is obtained by attaching a Weinstein handle to $\Lambda^- \uplus \Lambda^+$ and cutting out standard neighborhoods of the $F^{\mp\epsilon}L^\pm$.
(2) Γ_1 is obtained by performing a contact $+1$ surgery on $F^{\mp\epsilon}\Lambda^\pm$ and a contact -1 surgery along $(\Lambda^- \uplus \Lambda^+)$.
(3) The boundaries of the $R_1^\pm$ are identified by the contactomorphism $\partial R_1^+ \to \partial R_1^-$ obtained by sliding $F^{-\epsilon}\Lambda^+$ up over $(\Lambda^- \uplus \Lambda^+)$ to $F^\epsilon \Lambda^-$.

1.2.1. *Translation.*

Lemma 1.2. *Suppose that the convex hypersurface is described by a DPF $\mathbf{\Lambda}^\pm$ with the $\Lambda_1^\pm$ intersecting transversely at a single point in W. Then the $\Lambda_1^\pm$ together with their thimbles $L_1^\pm \subset R^\pm$ give bypass attachment data. The result of the bypass can be described by a DPF $\mathbf{\Lambda}_b^\pm$ given by*

$$\mathbf{\Lambda}_b^+ = (\tau_{\Lambda_1^+}^2 \Lambda_1^-, \Lambda_2^+, \dots, \Lambda_{k^+}^+), \quad \mathbf{\Lambda}_b^- = (\tau_{\Lambda_1^+}\Lambda_1^-, \Lambda_2^-, \dots, \Lambda_{k^-}^-)$$

Let's first verify that the monodromies of the $\mathbf{\Lambda}_b^\pm$ agree as a sanity check:

$$\begin{aligned}
\tau_{\mathbf{\Lambda}_b^+} &= \tau_{\Lambda_{k^+}^+} \circ \dots \circ \tau_{\Lambda_2^+} \circ \tau_{\tau^2_{\Lambda_1^+}\Lambda_1^-} = \tau_{\Lambda_k^+} \circ \dots \circ \tau_{\Lambda_2^+} \circ \tau_{\Lambda_1^+}^2 \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-2} \\
&= \tau_{\mathbf{\Lambda}^+} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-2} = \tau_{\mathbf{\Lambda}^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} \\
&= \tau_{\mathbf{\Lambda}^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} = \tau_{\mathbf{\Lambda}^-} \circ \tau_{\tau_{\Lambda_1^-}^{-1}\Lambda_1^+} \circ \tau_{\Lambda_1^+}^{-1} \\
&= \tau_{\mathbf{\Lambda}^-} \circ \tau_{\Lambda_1^-}^{-1} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} \\
&= \tau_{\Lambda_{k^-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^-}^{-1} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} \\
&= \tau_{\Lambda_{k^-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau_{\Lambda_1^+}^{-1} \\
&= \tau_{\Lambda_{k^-}^-} \circ \dots \circ \tau_{\Lambda_2^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} = \tau_{\mathbf{\Lambda}_b^-}
\end{aligned}$$

Now to prove the lemma we apply the definition of the bypass. Suppose that the $\Lambda_1^\pm$ lie on $\{3\epsilon\} \times W$. The cocores of the surgery handles can then be pushed down to $\{2\epsilon\} \times W$ as shown in the left-hand side of Figure 1 so that they share a single transverse intersection.

Working out $\mathbf{\Lambda}_b^-$ is very easy. According to the definition of the bypass, we must perform a contact $+1$ surgery on a copy of Λ_1^- placed slightly above the new surgery locus $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ in the boundary of ∂R_1^-. This is as shown in the left-hand side of Figure 1. The ± 1 surgeries along the Λ_1^- at heights 2ϵ and 3ϵ cancel, so that we are only left with the new -1 surgery locus.

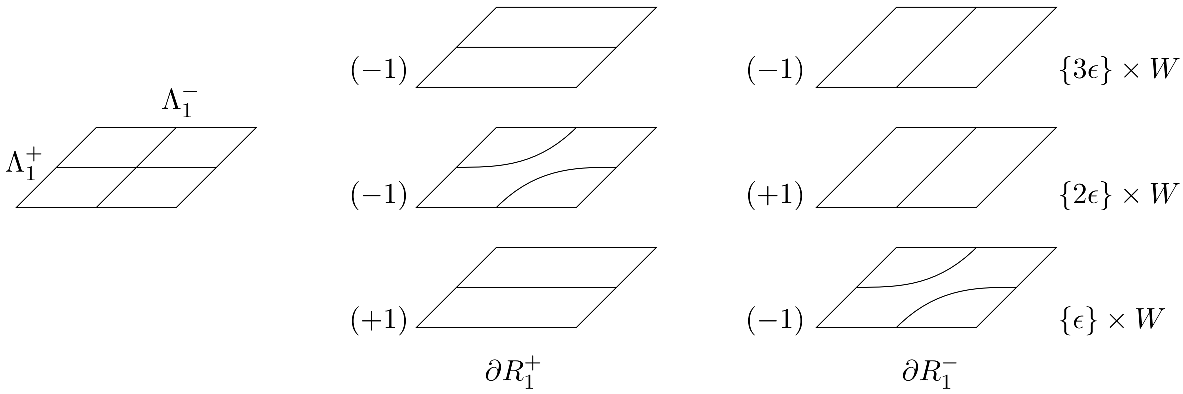

FIGURE 1. Depiction of bypass surgeries in an open book.

To describe the boundary of the new positive region R_1^+ we must perform a contact -1 surgery along $\tau_{\Lambda_1^+}\Lambda_1^-$ (at height 2ϵ) along with a contact $+1$ surgery slightly below it, say at $\{\epsilon\} \times W$. This is as depicted in the center row of Figure 1. We want to make the ± 1 surgeries on Λ_1^+ cancel. To that end, we handle-slide $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ (currently at height 2ϵ) down through the surgery locus at $\{\epsilon\} \times \Lambda_1^+$. The before and after pictures are as shown in the left and right columns of Figure 2, respectively. Because we are flowing down through a $+1$ surgery handle, the effect is a positive Dehn twist: We obtain $\tau^2_{\Lambda_1^+}\Lambda_1^-$ on the bottom (say, at height ϵ) and canceling ± 1 surgeries (say at heights 2ϵ, 3ϵ). Deleting the canceling ± 1 surgeries, we are left with $\mathbf{\Lambda}_b^+$ as claimed.

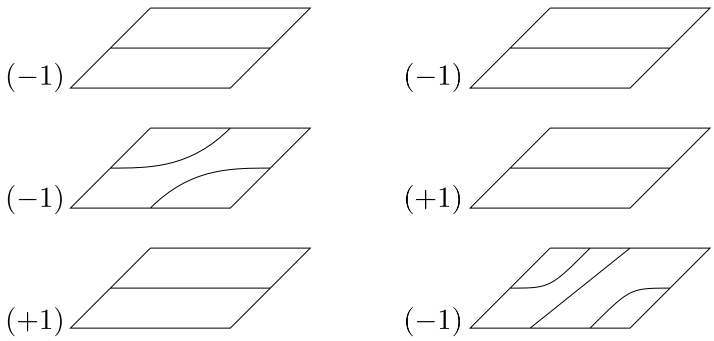

FIGURE 2. Handle-sliding the new -1 surgery locus on ∂R_1^+ down through the new $+1$ surgery locus. On the bottom left, we see $\tau^2_{\Lambda_1^+}\Lambda^-$ sitting on $\{\epsilon\} \times W$.

2. STABILIZATION, UNKNOTS, AND OBVIOUSLY OVERTWISTED BYPASSES

Suppose that we have a Lefschetz fibration π (whose total space has dimension $2n$) determined by some $\mathbf{\Lambda}$ with non-singular fiber W (of dimension $2n-2>0$). Write Γ for the boundary of the total space of π.

Suppose that W admits a Lefschetz fibration $\pi_W : W \to \mathbb{D}$ for which we have a thimble $D \subset W$ with Legendrian boundary $\partial D \subset \partial W$. When $\dim W = 2$ this imposes no restrictions on D, other than it is a Legendrian-realized arc. We may attach a $\mathrm{ind} = n-1, \dim = 2n-2$ Weinstein handle H_D to W along ∂D. The union of the core Lagrangian disk C_D of H_D together with D forms a Lagrangian sphere Λ_D in the surgered Weinstein manifold

$$\Lambda_D := C_D \cup D \subset W_D := W \cup H_D.$$

The $\Lambda_i \subset W$ then determine $\Lambda_i \subset W_D$ via the natural inclusion $W \subset W_D$. It follows that Λ_D intersects to cocore Lagrangian disk $C_D^\perp$ of H_D exactly once,

$$\#(\Lambda_D \cap C_D^\perp) = 1.$$

Adding a $\mathrm{ind} = n, \dim = 2n$ Weinstein handle H_{Λ_D} to the total space of π then yields a *stabilization* of π. It is determined by a tuple of Lagrangians in W_D which we may view as Legendrians in the boundary of the total space of the new Lefschetz fibration,

$$\boldsymbol{\Lambda}_s = (\Lambda_D, \Lambda_1, \ldots, \Lambda_k)$$

The boundary of the new Lefschetz fibration, π_s is then supported by an open book which is a positive stabilization of the open book induced by π. Hence the boundary of the total space is again Γ with the same contact structure. Write L_{Λ_D} for the Lefschetz thimble in the total space of π_s for which

$$\partial L_{\Lambda_D} = \Lambda_D \subset \Gamma$$

is Legendrian.

Lemma 2.1. *Λ_D is a Legendrian unknot in Γ and L_{Λ_D} is a standard bounding disk.*

Proof. In the case that $W = \mathbb{D}^{2n-2}$ with $\pi_W = \sum z_j^2$, we obtain $W_D = \mathbb{D}^*\mathbb{S}^{n-1}$ with $\Lambda_D \subset W_D$ the zero section. The total space of π_s is then $\mathbb{D}^{2n}$ with Λ_D a standard Legendrian unknot bounding a standard Lagrangian disk. $\square$

Editor notes 2.2. *Finish writing this proof. The idea is to build up the total space of π_s from the basic example. The delicate point is that we need to ensure that the disk remains standard.*

References

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