NOTES ON BYPASSES

1. Introduction

After addressing the note you sent in §2, I'll describe how to apply bypasses to convex hypersurfaces Σ described as double positive factorizations (DPFs) in Lemma 4.3. The definition of the DPF bypass from my previous notes was wrong – apparently as the result of sign error in a handle slide. The notes here have a proof which is at least not obviously wrong.

In Definition 4.2 I package the construction of [HH18, §10] as an *obviously overtwisted bypass* (OOB). Applying such a bypass to any $\{0\} \times \Sigma$ gives a contact structure on $[0,1]_t \times \Sigma$ for which $\{1\} \times \Sigma$ has an overtwisted neighborhood.

A specific OOB attachment is described in §5. It is the simplest such example, since it can be applied locally and involves the least possible number of handles. If we apply attach this bypass to the boundary of a standard $\dim = 2n+1$ Darboux disk, we get an *obviously overtwisted disk* (OOD). I describe a DPF for the boundary Σ_{ot} of the OOD around Definition 5.1. The positive and negative regions $R^{\pm}_{ot}$ for Σ_{ot} are "exotic" symplectic manifolds which have been the focus of quite a few articles, cf. [CM19, §3.5].

1.1. **Some speculation.** Any thoughts on if the below is worth attempting or if it's clearly futile? I wouldn't be surprised if you tried something similar (or identical) already.

Question 1.1. *Is every convex hypersurface Σ with an overtwisted t-invariant neighborhood the result of an obviously overtwisted bypass attachment to some Σ'?*

Let's take a fixed Σ which has an overtwisted t-invariant neighborhood $[0,1] \times \Sigma$. We'll attempt to prove that Σ is the result of an OOB attachment.

Although I don't know how to formalize this, the h-principle should allow us to find OODs in a t-invariant $(0,1)_t \times \Sigma$ since this neighborhood is overtwisted. Assuming this is true, there will be OOBs inside $[0,1)_t \times \Sigma$ which can be attached to $\{0\} \times \Sigma$: Attach one end of a contact 1-handle to the dividing set of $\{0\} \times \Sigma$ and the other end to the dividing set of the OOD. This is the same as first connecting $\{0\} \times \Sigma$ to the boundary of a Darboux ball – which gives us back Σ, unmodified – and then applying an OOB attachment.

Say we attach such an OOB, B_1, to $\{0\} \times \Sigma$. Then by [HH19, Theorem 1.2.5] we can get from $\{0\} \times \Sigma$ to $\{1\} \times \Sigma$ by a sequence of bypass attachments $B_1, \cdots, B_k$ so that the union of all of the bypass attachment slices will be isotopic to the t-invariant $[0,1] \times \Sigma$.

If we can perform handle-slides and rearrange the orders of attachment so that there are bypasses $B'_1, \dots, B'_{k'}$ getting us from $\{0\} \times \Sigma$ back to itself and with $B'_{k'}$ obviously overtwisted, then $\Sigma = \{1\} \times \Sigma$ will be the result of an OOB attachment, answering Question 1.1 in the affirmative.

It feels overly optimistic to hope that we could force this $B'_{k'}$ to be of the model form described in §5. On the other hand, the OOB B_1 is attached along a ball in Σ, so maybe it's possible that it could be avoided during any handle-slides and changing of the orders of bypasses, which would guarantee that $B'_{k'} = B_1$ giving. Assuming that the strategy of rearranging the orders of bypasses works, I don't know if it would be any easier to prove using DPFs or contact handle attachments.

2. Notes on Ko's note

The goal of the note is to prove that given bypass attachment data for Legendrians $\Lambda^{\pm} \subset \Gamma$ bounding Lefschetz thimbles $L^{\pm} \subset R^{\pm}$ that we can make a DPF with page W for which $\Lambda^{+} \cup \Lambda^{-}$ sit on a single page.

My understanding is that the tricky part is to realize $\Lambda^{+} \cup \Lambda^{-}$ on a page of a supporting open book for Γ. According to [HH19, Corollary 1.3.3], we can realize a smooth Legendrian link on a page of a supporting open book decomposition, but $\Lambda^{+} \cup \Lambda^{-}$ has a singularity at the point of intersection.

Following the proof at the end of [HH19, §10], we need to find

(1) a neighborhood N of $\Lambda^{+} \cup \Lambda^{-}$ (disjoint union some other smooth Legendrians) and

(2) a contact Morse function f on N for which is constant along ∂N

I will forget the other smooth Legendrian components since the construction is carried out locally on connected components of N.

Identify N with a contactization $[-1,1]_t \times V$ where V a plumbing of $\mathbb{D}^*\mathbb{S}^n$, or equivalently an A_2 Milnor fiber, say given by the variety for $\epsilon + z_0^3 + \sum_1^n z_k^2$ intersected with $\mathbb{D}^{2n+2} \subset \mathbb{C}^{n+1}$. This is a Stein manifold and so the Liouville vector field X_V on V is gradient like for a Morse function f_V on V, $\langle \nabla f_V, X_V \rangle \geq \text{const} \, \|\nabla f_V\|^2$. By rescaling f_V and possibly modifying it near ∂V (without creating critical points), we can assume that $\text{const} = f_V|_{\partial V} = 1$. On this contactization neighborhood, we can take our contact structure to be determined by $\alpha = dt + \beta$ where β is the Liouville form for V, set $f = \frac{1}{2}t^2 + f_V$, and define $X = t\partial_t + X_V$. Then we compute

$$\mathcal{L}_X \alpha = \alpha, \quad \nabla f = t\partial_t + \nabla f_V, \quad \langle \nabla f, X \rangle = t^2 + \langle \nabla f_V, X_V \rangle \geq \|\nabla f\|^2.$$

So X is a contact vector field which is gradient-like for f. Moreover $f^{-1}(1)$ gives us N with its corners rounded and so contains the singular Legendrian $\Lambda^+ \cup \Lambda^-$ in its interior as desired. Moreover, the Morse critical points of f are in one-to-one correspondence with those of f_V in a way which preserves indices. So if we extend f to the exterior of N arbitrarily, then N will be contained in a neighborhood of the n-skeleton since n is the maximum index of a critical point of f_V.

3. Conventions

Editor notes 3.1. *I need to triple check this.*

Throughout Σ is a Weinstein convex hypersurface with decomposition $R^\pm, \Gamma$. We'll take $\dim \Gamma = 2n-1, \dim \Sigma = 2n$.

Here is our convention for building Lefschetz fibrations: Start with a trivial Lefschetz fibration for $\pi : W \times \mathbb{D} \to \mathbb{D}$ (for which π is the projection onto the $\mathbb{D}$ factor) an open book on the boundary whose page is W and whose monodromy is trivial. Say that the whose mapping torus portion of the open book is $W \times S^1$. If we attach a Weinstein handle at some $\Lambda \subset W \times \{0\}$ then the Lefschetz fibration extends over the surgered Weinstein manifold so that the core of the handle becomes a Lefschetz thimble. The surgery locus in the boundary is $N_W(\Lambda) \times [0, 3\epsilon]$ and the boundary of the cocore is $\Lambda \times \{0\} \simeq \Lambda \times \{3\epsilon\}$. Here $N_W(\Lambda_k)$ is a standard Weinstein neighborhood of Λ in W. The $\simeq$ is a Legendrian isotopy coming from our ability to slide the boundary of the cocore through the handle.

To build up to a general Lefschetz fibration, we attach k Weinstein handles to the neighborhoods

$$N_W(\Lambda_k) \times \left[\frac{i-1}{k}, \frac{i-1}{k} + 3\epsilon\right] \subset W \times S^1, \quad S^1 = \mathbb{R}/\mathbb{Z}$$

using the 3ϵ neighborhoods with ϵ small. With this ordering convention the monodromy of the open book on the boundary is

$$\tau_{\Lambda_k} \circ \cdots \circ \tau_{\Lambda_1} : W \to W.$$

We say that the Lefschetz fibration is determined by $\mathbf{\Lambda} = (\Lambda_1 \ldots, \Lambda_k)$.

Editor notes 3.2. *This is the opposite of the convention from Ko's note! I started working this out before I received it.*

For a double positive factorization (DPF), we have $R^\pm$ determined by Lefschetz fibrations with fiber W and ordered collections of (framed) Legendrian spheres $\mathbf{\Lambda}^\pm = (\Lambda_i^\pm)_{i=1}^{k^\pm}, \Lambda_i^\pm \subset W$. We identify their boundaries when we have an identifications of the fibers of the Lefschetz fibrations on the $R^\pm$ with a fixed W and

$$\tau_{\mathbf{\Lambda}^+} := \tau_{\Lambda^+_{k^+}} \circ \cdots \circ \tau_{\Lambda^+_1} = \tau_{\Lambda^-_{k^-}} \circ \cdots \circ \tau_{\Lambda^-_1} =: \tau_{\mathbf{\Lambda}^-}.$$

Specifically the identification of the monodromies allows us identify the mapping tori

$$[0,1] \times W / \sim^\pm, \quad (1, x) \sim^\pm (0, \tau_{\mathbf{\Lambda}^\pm}).$$

From the open book picture, the mapping tori make up all of the $\partial R^\pm$ except for the binding. The identification trivially extends over the binding since it is ∂W.

3.1. **Legendrian sum.** Given a pair $\Lambda^{\pm}$ of Legendrians in a contact manifold Γ intersecting ξ-transversely at a single point, we define the Legendrian sum using the Lagrangian projection formulation from [HH18].

Let $N_{\Gamma}(\Lambda^{-} \cup \Lambda^{+})$ be a neighborhood in Γ inside of a subset of the $[0, 4\epsilon]_t \times W$ where W is a Weinstein manifold containing a plumbing of two copies of $\mathbb{D}^*\mathbb{S}^{n-1}$. The $\Lambda^{\pm}$ are given by the 0-sections of the $\mathbb{D}^*\mathbb{S}^{n-1}$ inside $\{0\} \times W$ and our contact structure is determined by $dt + \beta$ for β a Liouville form on W. Then define

$$\Lambda^{-} \uplus \Lambda^{+} \subset [\epsilon, 2\epsilon] \times W$$

to be the Legendrian lift of the Lagrangian sphere

$$\tau_{\Lambda^{+}} \Lambda^{-} = \tau_{\Lambda^{-}}^{-1} \Lambda^{+} \subset W$$

where τ is a Dehn twist. Then

$$F^{\pm\epsilon}(\Lambda^{-} \uplus \Lambda^{+}) := \mathrm{Flow}_{\partial_t}^{\pm\epsilon}(\Lambda^{-} \uplus \Lambda^{+}) \subset ([0, \epsilon] \cup [2\epsilon, 3\epsilon]) \times W.$$

We'll always think of $\Lambda^{\pm}$ as being Legendrian spheres sitting on a single page W of an open book for Γ. We'll always work far from the binding so that we can assume that each $t = t_0$ slice of $[-2\epsilon, 2\epsilon] \times W$ also sits in a page. After a slight perturbation of the open book the $F^{\pm\epsilon}(\Lambda^{-} \uplus \Lambda^{+})$ can be assumed to sit on pages as well.

4. Bypasses

Definition 4.1. *Bypass attachment data consists of a tuple $(\Lambda^{\pm}, L^{\pm})$ for which $\Lambda^{\pm}$ is a pair of Legendrians intersecting in a single point and $L^{\pm}$ are Lagrangian slice disks in the $R^{\pm}$ for the $\Lambda^{\pm}$.*

If we attach a bypass to $[-\epsilon, 0] \times \Sigma$ at $\{0\} \times \Sigma$ using the data of $(\Lambda^{\pm}, L^{\pm})$ then we get a contact structure on $[-\epsilon, 1] \times \Sigma$ with $\Sigma_1 = \{1\} \times \Sigma$ described as in [HH18, Theorem 5.1.3]:

1. $R_1^{\pm}$ is obtained by attaching a Weinstein handle to $\Lambda^{-} \uplus \Lambda^{+}$ and cutting out standard neighborhoods of the $F^{\mp\epsilon} L^{\pm}$.
2. Γ_1 is obtained by performing a contact $+1$ surgery on $F^{\mp\epsilon}\Lambda^{\pm}$ and a contact -1 surgery along $(\Lambda^{-} \uplus \Lambda^{+})$.
3. The boundaries of the $R_1^{\pm}$ are identified by the contactomorphism $\partial R_1^{+} \to \partial R_1^{-}$ obtained by sliding $F^{-\epsilon}\Lambda^{+}$ up over $(\Lambda^{-} \uplus \Lambda^{+})$ to $F^{\epsilon}\Lambda^{-}$.

Definition 4.2. *We'll say that a bypass as described in* [HH18, §10] *is an* obviously overtwisted bypass. *This means that Λ^{+} is the standard unknot, L^{+} is the standard Lagrangian disk, and Λ^{+} is* above *Λ^{-} in the θ-disk sense.*

The definition ensures that the convex hypersurface Σ_1 resulting from the bypass will have an overtwisted neighborhood. It is a priori stronger than the *overtwisted bypass* of [HH18].

4.1. Bypasses for DPFs.

Lemma 4.3. *Suppose that the convex hypersurface $\Sigma = \Sigma_0$ is described by a DPF $\mathbf{\Lambda}^{\pm}$ with the $\Lambda_1^{\pm}$ intersecting transversely at a single point in W. Then the $\Lambda_1^{\pm}$ together with their thimbles $L_1^{\pm} \subset R^{\pm}$ give bypass attachment data. The result of the bypass, Σ_1, can be described by a DPF $\mathbf{\Lambda}_b^{\pm}$ given by*

$$\mathbf{\Lambda}_b^{+} = (\tau_{\Lambda_1^{+}}^{2} \Lambda_1^{-}, \Lambda_2^{+}, \dots, \Lambda_{k^{+}}^{+}), \quad \mathbf{\Lambda}_b^{-} = (\tau_{\Lambda_1^{+}} \Lambda_1^{-}, \Lambda_2^{-}, \dots, \Lambda_{k^{-}}^{-})$$

Let's first verify that the monodromies of the $\boldsymbol{\Lambda}_b^{\pm}$ agree as a sanity check:

$$\begin{aligned}
\tau_{\boldsymbol{\Lambda}_b^+} &= \tau_{\Lambda_{k+}^+} \circ \cdots \circ \tau_{\Lambda_2^+} \circ \tau_{\tau^2_{\Lambda_1^+}\Lambda_1^-} \\
&= \tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\Lambda_2^+} \circ \tau_{\Lambda_1^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\boldsymbol{\Lambda}^-} \circ \tau_{\tau^{-1}_{\Lambda_1^-}\Lambda_1^+} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} = \tau_{\boldsymbol{\Lambda}_b^-}.
\end{aligned}$$

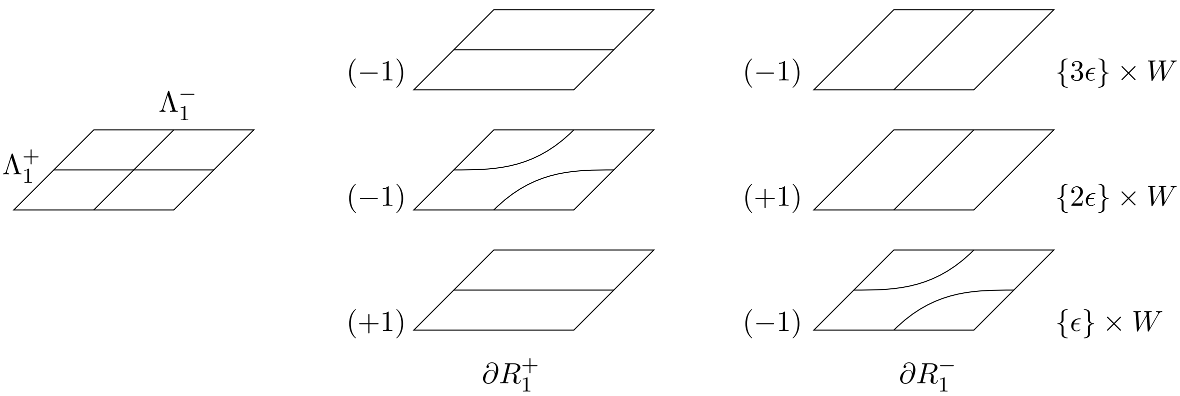

FIGURE 1. Depiction of bypass surgeries in an open book. Each "sheet" is a neighborhood of the point $\Lambda_1^+ \cap \Lambda_1^-$ in W.

Now to prove the lemma we apply the definition of the bypass. Suppose that the $\Lambda_1^{\pm}$ lie on $\{3\epsilon\} \times W$. The cocores of the surgery handles can then be pushed down to $\{2\epsilon\} \times W$ by the negative Reeb flow as shown in the left-hand side of Figure 1 so that they share a single transverse intersection.

Working out $\boldsymbol{\Lambda}_b^-$ is very easy. According to the definition of the bypass, we must perform a contact $+1$ surgery on a copy of Λ_1^- placed slightly above the new surgery locus $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ in the boundary of ∂R_1^-. This is as shown in the right-hand side of Figure 1. The ± 1 surgeries along the Λ_1^- at heights 2ϵ and 3ϵ cancel, so that we are only left with the new -1 surgery locus.

To describe the boundary of the new positive region R_1^+ we must perform a contact -1 surgery along $\tau_{\Lambda_1^+}\Lambda_1^-$ (at height 2ϵ) along with a contact $+1$ surgery slightly below it, say at $\{\epsilon\} \times W$. This is as depicted in the center row of Figure 1. We want to make the ± 1 surgeries on Λ_1^+ cancel. To that end, we handle-slide $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ (currently at height 2ϵ) down through the surgery locus at $\{\epsilon\} \times \Lambda_1^+$. The before and after pictures are as shown in the left and right columns of Figure 2, respectively. Because we are flowing down through a $+1$ surgery handle, the effect is a positive Dehn twist: We obtain $\tau^2_{\Lambda_1^+}\Lambda_1^-$ on the bottom (say, at height ϵ) and canceling ± 1 surgeries (say at heights 2ϵ, 3ϵ). Deleting the canceling ± 1 surgeries, we are left with $\boldsymbol{\Lambda}_b^+$ as claimed.

4.2. **Antibypasses for DPFs.** To perform an anti-bypass, we change the orientation on our convex hypersurface, apply a bypass, and then change the orientation again. Hence the analogue of Lemma 4.3 for anti-bypasses is as follows:

Lemma 4.4. *Suppose that the convex hypersurface is described by a DPF $\boldsymbol{\Lambda}^{\pm}$ with the $\Lambda_1^{\pm}$ intersecting transversely at a single point in W. Then the $\Lambda_1^{\pm}$ together with their thimbles $L_1^{\pm} \subset R^{\pm}$ give anti-bypass*

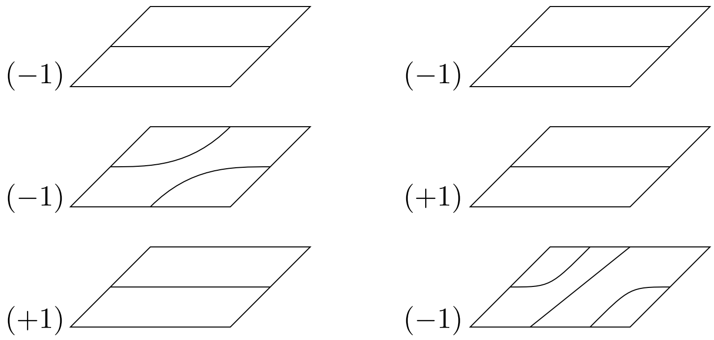

FIGURE 2. Handle-sliding the new -1 surgery locus on ∂R_1^+ down through the new $+1$ surgery locus. The before picture is on the left, and in the after picture on the right $\tau^2_{\Lambda_1^+}\Lambda^-$ sits on the bottom sheet $\{\epsilon\} \times W$.

attachment data. The result of the anti-bypass can be described by a DPF $\mathbf{\Lambda}_a^\pm$ *given by*

$$\mathbf{\Lambda}_a^+ = (\tau_{\Lambda_1^-}\Lambda_1^+, \Lambda_2^+, \dots, \Lambda_{k+}^+), \quad \mathbf{\Lambda}_a^- = (\tau^2_{\Lambda_1^-}\Lambda_1^+, \Lambda_2^-, \dots, \Lambda_{k-}^-)$$

5. AN OBVIOUSLY OVERTWISTED EXAMPLE

Let's describe an example bypass on $\mathbb{S}^{2n}$ starting with $R^\pm = \mathbb{D}^{2n}$. So our convex hypersurface is the boundary of a contact $\dim = 2n+1$ Darboux disk.

For the Lefschetz fibrations on $\mathbb{D}^{2n}$ (on which we take standard complex coordinates z_k) we can use the function $\pi = \delta + z_1^3 + \sum_2^n z_k^2$ with $\delta \in \mathbb{C} \setminus 0$ small. Then a regular fiber $W = W^{2n-2}$ will be a plumbing of two cotangent bundles of spheres x, y and the monodromy is $\tau_y\tau_x$. So we can say that

$$\mathbf{\Lambda}^\pm = (\Lambda_1, \Lambda_2) = (x, y)$$

for both $\pm$. Both of the x and y are unknots in $\mathbb{D}^{2n}$. According to our convention for ordering Dehn twists, $y \subset W \times \{\epsilon\}$ while $x \subset W \times \{-\epsilon\}$.

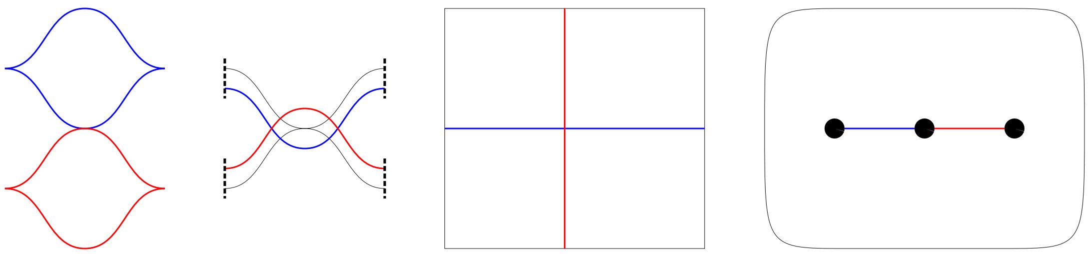

FIGURE 3. In each subfigure we see x in blue and y in red. On the left they are boundaries of Lefschetz thimbles in (S^3, ξ_{std}). In the middle left, a Gompf diagram [Go98], they are Legendrians in the front projection with the skeleton of W in black as in [Av11]. Here x and y are shifted off of the 1-skeleton (thin black graph) so that x lives on $\{-\epsilon\} \times W$ and y lives on $\{\epsilon\} \times W$ and the dashed vertical lines represent 1-handles. In the center-right the Legendrians give a meridian and longitude on a punctured torus. We identify the sides of the square with each other and the top with the bottom, taking the puncture to be the corner. On the right are matching paths for x, y in the π_W projection.

When $n = 2$ the Legendrians can be seen as a meridian and a longitude on a once-punctured torus. To get the monodromy for the open book in this case, we can follow [Av11, Theorem 4.8] and look at the left-hand side of Figure 3.

In general, a linear function (eg. $\pi_W = z_1|_W$) gives a Lefschetz fibration on W with three singular fibers, which after an isotopy we can take to be $1, 0, -1 \in \mathbb{C}$ and with non-singular fiber a $\mathbb{D}^*\mathbb{S}^{n-2}$. Then we can see x as the matching path sphere for the arc $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ and y as the matching path sphere for the arc $[-1, 0]$. See the right-hand side of Figure 3. In the $n = 2$ case, we see that x is above

y – in the θ-disk sense of [HH18] – and we'll assume that this is the case as well in higher dimensions (although I'd have to think about how to prove it).

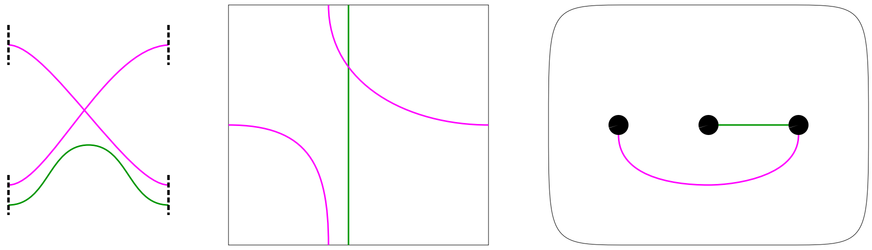

FIGURE 4. The Legendrians $(y, \tau_y x)$ shown in green and purple, respectively.

5.1. **Applying the bypass.** We can handle-slide x up through y to obtain another positive factorization for $\mathbb{D}^{2n}$ given by $(y, \tau_y x)$ as seen in Figure 4. So

$$\mathbf{\Lambda}^+ = (x, y), \quad \mathbf{\Lambda}^- = (y, \tau_y x) \tag{1}$$

is a double positive factorization for the standard $\mathbb{S}^{2n}$ for which the first Legendrians give obviously overtwisted bypass data. A bypass for the pair x, y is obviously overtwisted, since x is above y – in the sense of θ-disks. After the bypass is performed, we apply Lemma 4.3 to obtain

$$\mathbf{\Lambda}_b^+ = (\tau_x^2 y, y), \quad \mathbf{\Lambda}_b^- = (\tau_x y, \tau_y x)$$

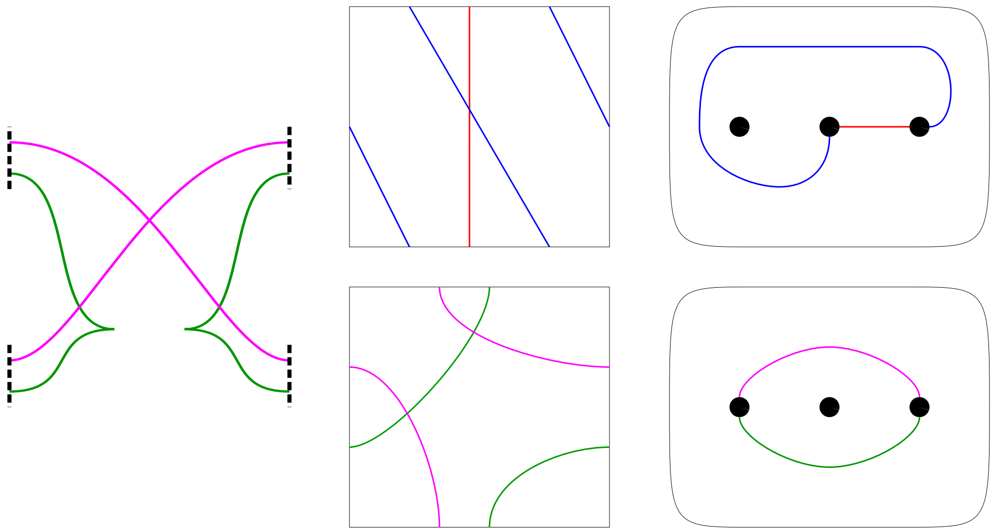

FIGURE 5. Lefschetz fibrations for the positive region (top) and negative region (bottom) after the obviously overtwisted bypass has been performed.

The bypass data determines a contact structure on $[0,1] \times \mathbb{S}^{2n}$ with $\{0\} \times \mathbb{S}^{2n}$ the original hypersurface (given by the boundary of a $\dim = 2n+1$ Darboux disk) and

$$\Sigma_{ot} = \{1\} \times \mathbb{S}^{2n}$$

the result of the bypass. The positive and negative regions for Σ_{ot} are denoted $R^{\pm}_{ot}$.

Definition 5.1. *The* obviously overtwisted disk *(OOD) is the disk with convex boundary Σ_{ot} obtained by attaching an obviously overtwisted bypass to the boundary of a standard Darboux disk.*

A Lefschetz fibrations for R^+_{ot} appears in the top row of Figure 5 with $\tau_x^2 y$ in blue and y in red. On the bottom row of the figure we have a Lefschetz fibration for R^-_{ot} with $\tau_x y$ in green and $\tau_y x$ in purple. From the matching path picture, a diffeomorphism of the disk interchanging the middle and left marked points induces a symplectomorphism $R^+_{ot} \to R^-_{ot}$.

Knowing that [CM19] contained lots of pictures, I took a look and got lucky, finding the Lefschetz fibration described in bottom right of Figure 5 as [CM19, Figure 28]. The $R^{\pm}_{ot}$ are well-studied in the literature. At least in the $2n = 6$ case they can be described as affine algebraic varieties cut out by a single equation

$$R^{\pm}_{ot} = \{x(xy-1) = z_1^2 + z_2^2\} \subset \mathbb{C}^4_{x,y,z_1,z_2}$$

(and I'm sure this could be generalized to other dimensions by adding more z_ks). The following are known facts for the $\dim = 4, 6$ cases: The $R^{\pm}_{ot}$ are not flexible but embed into flexible manifolds of the same dimension. They have symplectic homology $SH = 0$ over $\mathbb{Z}$ but $SH \neq 0$ over a twisted coefficient system.

5.2. **Applying the anti-bypass.** Just to see what the result looks like, let's apply an anti-bypass to the data of Equation (1). Plugging in Lemma 4.4, we get

$$\mathbf{\Lambda}^+ = (x, y), \quad \mathbf{\Lambda}^- = (y, \tau_y x) \quad \mapsto \quad \mathbf{\Lambda}^+_a = (\tau_y x, y), \quad \mathbf{\Lambda}^-_a = (\tau_y^2 x, \tau_y x)$$

In this case, the positive and negative regions are again copies of $\mathbb{D}^{2n}$ as they can both seen to be positive stabilizations.

References

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