NOTES ON BYPASSES

ABSTRACT. Here we translate some of the key terminology from [HH18] into the language of open books and double positive factorizations (DPFs).

Throughout Σ is a Weinstein convex hypersurface with decomposition $R^{\pm}, \Gamma$. We'll take $\dim \Gamma = 2n-1, \dim \Sigma = 2n$.

Here is our convention for building Lefschetz fibrations: Start with a trivial Lefschetz fibration for $\pi : W \times \mathbb{D} \to \mathbb{D}$ (for which π is the projection onto the $\mathbb{D}$ factor) an open book on the boundary whose page is W and whose monodromy is trivial. Say that the whose mapping torus portion of the open book is $W \times S^1$. If we attach a Weinstein handle at some $\Lambda \subset W \times \{0\}$ then the Lefschetz fibration extends over the surgered Weinstein manifold so that the core of the handle becomes a Lefschetz thimble. The surgery locus in the boundary is $N_W(\Lambda) \times [0, 3\epsilon]$ and the boundary of the cocore is $\Lambda \times \{0\} \simeq \Lambda \times \{3\epsilon\}$. Here $N_W(\Lambda_k)$ is a standard Weinstein neighborhood of Λ in W. The $\simeq$ is a Legendrian isotopy coming from our ability to slide the boundary of the cocore through the handle.

To build up to a general Lefschetz fibration, we attach k Weinstein handles to the neighborhoods

$$N_W(\Lambda_k) \times \left[\frac{i-1}{k}, \frac{i-1}{k} + 3\epsilon\right] \subset W \times S^1, \quad S^1 = \mathbb{R}/\mathbb{Z}$$

using the 3ϵ neighborhoods with ϵ small. With this ordering convention the monodromy of the open book on the boundary is

$$\tau_{\Lambda_k} \circ \cdots \circ \tau_{\Lambda_1} : W \to W.$$

We say that the Lefschetz fibration is determined by $\mathbf{\Lambda} = (\Lambda_1 \dots, \Lambda_k)$.

For a double positive factorization (DPF), we have $R^{\pm}$ determined by Lefschetz fibrations with fiber W and ordered collections of (framed) Legendrian spheres $\mathbf{\Lambda}^{\pm} = (\Lambda_i^{\pm})_{i=1}^{k\pm}, \Lambda_i^{\pm} \subset W$. We identify with boundaries when we have an identifications of the fibers of the Lefschetz fibrations on the $R^{\pm}$ with a fixed W and

$$\tau_{\mathbf{\Lambda}^+} := \tau_{\Lambda^+_{k+}} \circ \cdots \circ \tau_{\Lambda^+_1} = \tau_{\Lambda^-_{k-}} \circ \cdots \circ \tau_{\Lambda^-_1} =: \tau_{\mathbf{\Lambda}^-}.$$

Specifically the identification of the monodromies allows us identify the mapping tori

$$[0,1] \times W / \sim^{\pm}, \quad (1, x) \sim^{\pm} (0, \tau_{\mathbf{\Lambda}^{\pm}}).$$

From the open book picture, the mapping tori make up all of the $\partial R^{\pm}$ except for the binding. The identification trivially extends over the binding since it is ∂W.

1. LEGENDRIAN SUM

Given a pair $\Lambda^{\pm}$ of Legendrians in a contact manifold Γ intersecting ξ-transversely at a single point, we'll define the Legendrian sum using the Lagrangian projection formulation from [HH18].

Let $N_{\Gamma}(\Lambda^- \cup \Lambda^+)$ be a neighborhood in Γ inside of a subset of the $[0, 4\epsilon]_t \times W$ where W is a Weinstein manifold containing a plumbing of two copies of $\mathbb{D}^*\mathbb{S}^{n-1}$. The $\Lambda^{\pm}$ are given by the 0-sections of the $\mathbb{D}^*\mathbb{S}^{n-1}$ inside $\{0\} \times W$ and our contact structure is determined by $dt + \beta$ for β a Liouville form on W. Then define

$$\Lambda^- \uplus \Lambda^+ \subset [\epsilon, 2\epsilon] \times W$$

to be the Legendrian lift of the Lagrangian sphere

$$\tau_{\Lambda^+}\Lambda^- = \tau^{-1}_{\Lambda^-}\Lambda^+ \subset W$$

where τ is a Dehn twist. Then

$$F^{\pm\epsilon}(\Lambda^- \uplus \Lambda^+) := \mathrm{Flow}^{\pm\epsilon}_{\partial_t}(\Lambda^- \uplus \Lambda^+) \subset ([0, \epsilon] \cup [2\epsilon, 3\epsilon]) \times W.$$

1.0.1. *Translation.* We'll always think of $\Lambda^{\pm}$ as being Legendrian spheres sitting on a single page W of an open book for Γ. We'll always work far from the binding so that we can assume that each $t = t_0$ slice of $[-2\epsilon, 2\epsilon] \times W$ also sits in a page. After a slight perturbation of the open book the $(\Lambda^- \uplus \Lambda^+)^{\pm\epsilon}$ can be assumed to sit on pages as well.

2. Bypasses for DPFs

Definition 2.1. *Bypass attachment data consists of a tuple $(\Lambda^{\pm}, L^{\pm})$ for which $\Lambda^{\pm}$ is a pair of Legendrians intersecting in a single point and $L^{\pm}$ are Lagrangian slice disks in the $R^{\pm}$ for the $\Lambda^{\pm}$.*

If we attach a bypass to $[-\epsilon, 0] \times \Sigma$ at $\{0\} \times \Sigma$ using the data of $(\Lambda^{\pm}, L^{\pm})$ then we get a contact structure on $[-\epsilon, 1] \times \Sigma$ with $\Sigma_1 = \{1\} \times \Sigma$ described as in [HH18, Theorem 5.1.3]:

(1) $R_1^{\pm}$ is obtained by attaching a Weinstein handle to $\Lambda^- \uplus \Lambda^+$ and cutting out standard neighborhoods of the $F^{\mp\epsilon}L^{\pm}$.
(2) Γ_1 is obtained by performing a contact $+1$ surgery on $F^{\mp\epsilon}\Lambda^{\pm}$ and a contact -1 surgery along $(\Lambda^- \uplus \Lambda^+)$.
(3) The boundaries of the $R_1^{\pm}$ are identified by the contactomorphism $\partial R_1^+ \to \partial R_1^-$ obtained by sliding $F^{-\epsilon}\Lambda^+$ up over $(\Lambda^- \uplus \Lambda^+)$ to $F^{\epsilon}\Lambda^-$.

To perform a *trivial bypass*, we require that both $\Lambda^{\pm}$ are unknots with the $L^{\pm}$ being the standard disk fillings with Λ^+ below Λ^-.

We'll say that a bypass as described in [HH18, §10] is an *obviously overtwisted bypass*. This means that Λ^+ is the standard unknot, L^+ is the standard Lagrangian disk, and Λ^+ is above Λ^-. The definition ensures that the convex hypersurface resulting from the bypass will have an overtwisted neighborhood. It is a priori stronger than the *overtwisted bypass* of [HH18].

Question 2.2. *Is every convex hypersurface with an overtwisted neighborhood the result of an obviously overtwisted bypass?*

2.0.1. *Translation.*

Lemma 2.3. *Suppose that the convex hypersurface is described by a DPF $\boldsymbol{\Lambda}^{\pm}$ with the $\Lambda_1^{\pm}$ intersecting transversely at a single point in W. Then the $\Lambda_1^{\pm}$ together with their thimbles $L_1^{\pm} \subset R^{\pm}$ give bypass attachment data. The result of the bypass can be described by a DPF $\boldsymbol{\Lambda}_b^{\pm}$ given by*

$$\boldsymbol{\Lambda}_b^+ = (\tau^2_{\Lambda_1^+}\Lambda_1^-, \Lambda_2^+, \dots, \Lambda_{k^+}^+), \quad \boldsymbol{\Lambda}_b^- = (\tau_{\Lambda_1^+}\Lambda_1^-, \Lambda_2^-, \dots, \Lambda_{k^-}^-)$$

Let's first verify that the monodromies of the $\boldsymbol{\Lambda}_b^{\pm}$ agree as a sanity check:

$$\begin{aligned}
\tau_{\boldsymbol{\Lambda}_b^+} &= \tau_{\Lambda_{k^+}^+} \circ \cdots \circ \tau_{\Lambda_2^+} \circ \tau_{\tau^2_{\Lambda_1^+}\Lambda_1^-} = \tau_{\Lambda_k^+} \circ \cdots \circ \tau_{\Lambda_2^+} \circ \tau^2_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-2}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^+} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-2}_{\Lambda_1^+} = \tau_{\boldsymbol{\Lambda}^+} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} = \tau_{\boldsymbol{\Lambda}^-} \circ \tau_{\tau^{-1}_{\Lambda_1^-}\Lambda_1^+} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\boldsymbol{\Lambda}^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k^-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k^-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\Lambda_1^+} \circ \tau_{\Lambda_1^-} \circ \tau^{-1}_{\Lambda_1^+} \\
&= \tau_{\Lambda_{k^-}^-} \circ \cdots \circ \tau_{\Lambda_2^-} \circ \tau_{\tau_{\Lambda_1^+}\Lambda_1^-} = \tau_{\boldsymbol{\Lambda}_b^-}.
\end{aligned}$$

Now to prove the lemma we apply the definition of the bypass. Suppose that the $\Lambda_1^{\pm}$ lie on $\{3\epsilon\} \times W$. The cocores of the surgery handles can then be pushed down to $\{2\epsilon\} \times W$ as shown in the left-hand side of Figure 1 so that they share a single transverse intersection.

Working out $\boldsymbol{\Lambda}_b^-$ is very easy. According to the definition of the bypass, we must perform a contact $+1$ surgery on a copy of Λ_1^- placed slightly above the new surgery locus $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ in the boundary of ∂R_1^-. This is as shown in the left-hand side of Figure 1. The ± 1 surgeries along the Λ_1^- at heights 2ϵ and 3ϵ cancel, so that we are only left with the new -1 surgery locus.

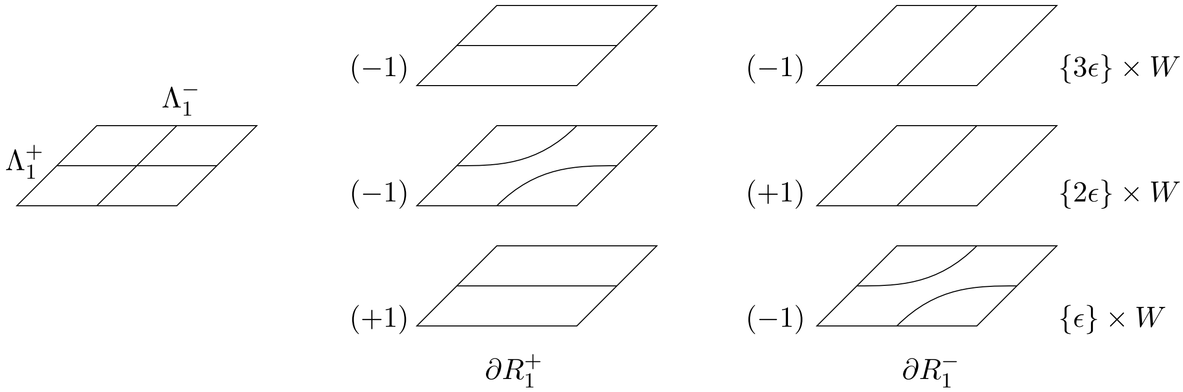

FIGURE 1. Depiction of bypass surgeries in an open book.

To describe the boundary of the new positive region R_1^+ we must perform a contact -1 surgery along $\tau_{\Lambda_1^+}\Lambda_1^-$ (at height 2ϵ) along with a contact $+1$ surgery slightly below it, say at $\{\epsilon\} \times W$. This is as depicted in the center row of Figure 1. We want to make the ± 1 surgeries on Λ_1^+ cancel. To that end, we handle-slide $\Lambda_1^- \uplus \Lambda_1^+ = \tau_{\Lambda_1^+}\Lambda_1^-$ (currently at height 2ϵ) down through the surgery locus at $\{\epsilon\} \times \Lambda_1^+$. The before and after pictures are as shown in the left and right columns of Figure 2, respectively. Because we are flowing down through a $+1$ surgery handle, the effect is a positive Dehn twist: We obtain $\tau^2_{\Lambda_1^+}\Lambda_1^-$ on the bottom (say, at height ϵ) and canceling ± 1 surgeries (say at heights 2ϵ, 3ϵ). Deleting the canceling ± 1 surgeries, we are left with $\mathbf{\Lambda}_b^+$ as claimed.

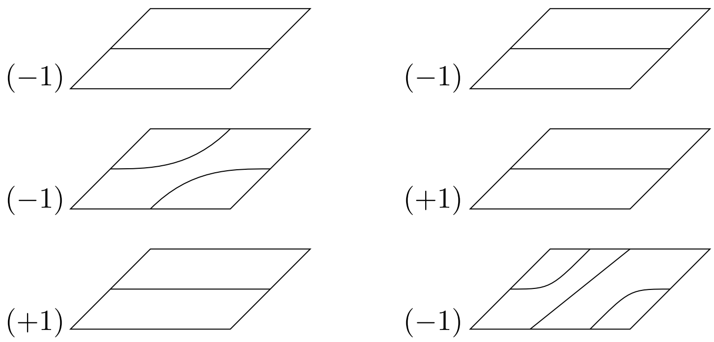

FIGURE 2. Handle-sliding the new -1 surgery locus on ∂R_1^+ down through the new $+1$ surgery locus. On the bottom left, we see $\tau^2_{\Lambda_1^+}\Lambda^-$ sitting on $\{\epsilon\} \times W$.

3. AN OBVIOUSLY OVERTWISTED EXAMPLE

Let's describe an example bypass on $\mathbb{S}^{2n}$ starting with $R^{\pm} = \mathbb{D}^{2n}$. So our convex hypersurface is the boundary of a contact $\dim = 2n+1$ Darboux disk.

For the Lefschetz fibrations on $\mathbb{D}^{2n}$ (on which we take standard complex coordinates z_k) we can use the function $\pi = \delta + z_1^3 + \sum_2^n z_k^2$ with $\delta \in \mathbb{C} \setminus 0$ small. Then a regular fiber $W = W^{2n-2}$ will be a plumbing of two cotangent bundles of spheres x, y and the monodromy is $\tau_y \tau_x$. So we can say that

$$\mathbf{\Lambda}^{\pm} = (\Lambda_1, \Lambda_2) = (x, y)$$

for both $\pm$. Both of the x and y are unknots in $\mathbb{D}^{2n}$. According to our conventions, y sits slightly above x in a thickened neighborhood of W.

In the case $n = 2$ the Legendrians can be seen as a meridian and a longitude on a once-punctured torus. To get the monodromy for the open book in this case, we can follow [Av11, Theorem 4.8] and look at the left-hand side of Figure 3. In that portion of the figure, the 1-handles for the Weinstein decomposition are omitted, so when performing isotopies we have to stay within a neighborhood of a single page, whose skeleton is shown in the figure.

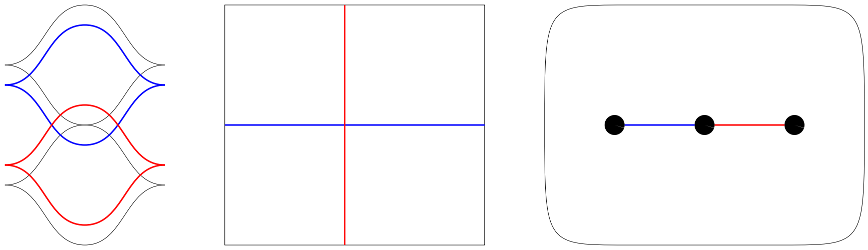

FIGURE 3. In each subfigure we see x in blue and y in red. On the left they are depicted as Legendrians in the front projection with the skeleton of W in black as in [Av11]. Here x and y are shifted off of the 1-skeleton so that x lives on $\{-\epsilon\} \times W$ and y lives on $\{\epsilon\} \times W$. In the center we see the Legendrians as meridians and longitudes on a punctured torus. For these pictures, we identify the sides with each other and the top with the bottom, taking out the puncture determined by the corners. On the right we see matching paths for the Legendrian in the π_W projection.

In general, a linear function (eg. $\pi_W = z_1|_W$) gives a Lefschetz fibration on W with three critical fibers, which after an isotopy we can take to be $1, 0, -1 \in \mathbb{C}$ and with non-singular fiber a $\mathbb{D}^*\mathbb{S}^{n-2}$. Then we can see x as the matching path sphere for the arc $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ and y as the matching path sphere for the arc $[-1,0]$. See the right-hand side of Figure 3. In the $n = 2$ case, we see that x is above y and we'll assume that this is the case as well in higher dimensions (although I'd have to think about how to prove it).

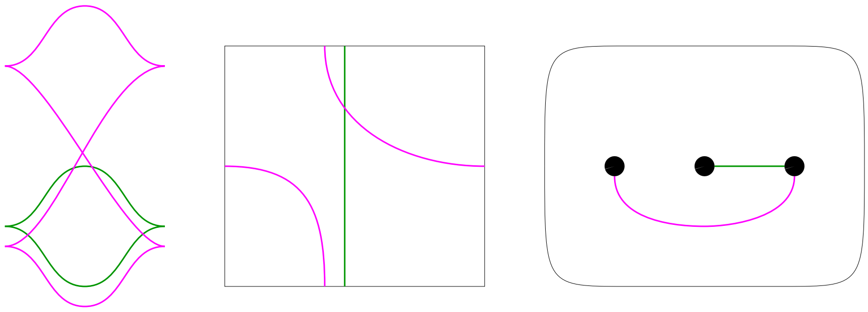

FIGURE 4. The Legendrians $(\Lambda_1'', \Lambda_2'')$ shown in purple and green, respectively.

We can handle-slide x up through y to obtain another positive factorization for $\mathbb{D}^{2n}$ given by $(\Lambda_1', \Lambda_2') = (y, \tau_y x)$ as seen in Figure 4. We see that $\Lambda_1 = x$ is above $\Lambda_1' = y$ so that

$$\mathbf{\Lambda}^+ = (\Lambda_1, \Lambda_2) = (x, y), \quad \mathbf{\Lambda}^- = (\Lambda_1', \Lambda_2') = (y, \tau_y x)$$

is a double positive factorization for the standard $\mathbb{S}^{2n}$ for which the first Legendrians give obviously overtwisted bypass data. After the bypass is performed, we apply Lemma 2.3 to obtain

$$\mathbf{\Lambda}_b^+ = (\tau_x^2 y, y), \quad \mathbf{\Lambda}_b^- = (\tau_x y, \tau_y x)$$

The bypass data determines a contact structure on $[0,1] \times \mathbb{S}^{2n}$ with $\{0\} \times \mathbb{S}^{2n}$ the original hypersurface and $\Sigma_{ot} = \{1\} \times \mathbb{S}^{2n}$ the result of the bypass. The positive and negative regions for Σ_{ot} are denoted $R_{ot}^{\pm}$.

A Lefschetz fibrations for R_{ot}^+ appears in the top row of Figure 5 with $\tau_x^2 y$ in blue and y in red. On the bottom row of the figure we have a Lefschetz fibration for R_{ot}^- with $\tau_x y$ in green and $\tau_y x$ in purple. From the matching path picture we see that a diffeomorphism of the plane interchanging the middle and left marked points induces a symplectomorphism $R_{ot}^+ \to R_{ot}^-$.

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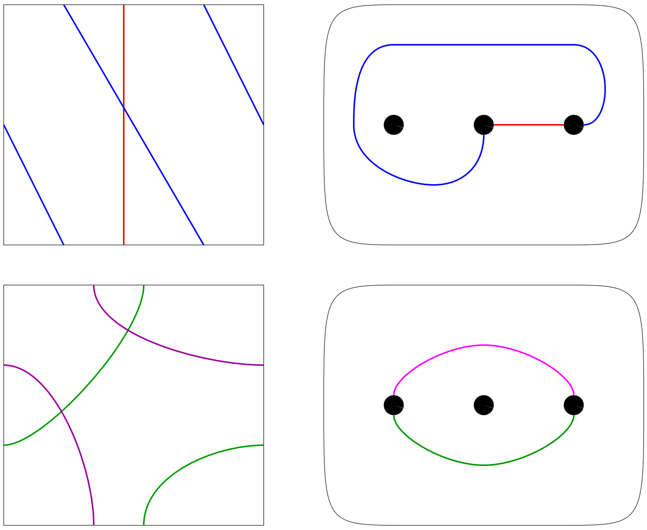

FIGURE 5. Lefschetz fibrations for the positive region (top) and negative region (bottom) after the obviously overtwisted bypass has been performed.

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