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QUADRATIC PROGRAMMING FOR PORTFOLIO OPTIMIZATION

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SUMMARY

Portfolio optimization is a procedure for generating a portfolio composition which yields the highest return for a given level of risk or a minimum risk for given level of return. The problem can be formulated as a quadratic programming problem. We shall present a new and efficient optimization procedure taking advantage of the special structure of the portfolio selection problem. An example of its application to the traditional mean-variance method will be shown. Formulation of the procedure shows that the solution of the problem is vector intensive and fits well with the advanced architecture of recent computers, namely the vector processor.

KEY WORDS Portfolio optimisation Portfolio selection Asset allocation Quadratic programming Mean-variance method

1. INTRODUCTION

Portfolio optimization is a classical problem for which Markowitz[1] has proposed a solution based on the diversification of assets to reduce risks and the optimization to make optimal use of their correlations. The objective of portfolio optimization is to build a portfolio that offers the highest level of return for each level of risk or the lowest level of risk for a desired level of return. Initially, Markowitz proposed to represent risk as the variance of return. The problem can be formulated as a quadratic programming problem. However, the available technology for solving a large quadratic programming problem was not robust enough at that time. A simplified version of this problem was then proposed by Sharpe.[2] The solution to portfolio optimization can be found from successive works on the quadratic optimization solution based on the simplex method (e.g., Wolfe,[3] Beale,[4] Theil and Van de Panne,[5] Dantzig[6]) or based on the dual method for linear programming (Lemke[7]). A good review of quadratic programming can be found in Reference 8 (Dorn 1963) or Reference 9 (Van de Panne and Whinston 1966). Efforts have also been made regarding portfolio selection formulation to make the solution more realistic (see Rudd and Rosenberg,[10] Pogue[11]). However, there is no really new procedure for quadratic programming that would accelerate the process and make it an interactive and thus useful tool for investors. Recently, with the advent of supercomputers and the pressure of competition from the financial market, this problem has again drawn much attention. Speidell *et al.*[12] have even tried to de-obscure the technical jargons of portfolio optimization to shed some light on the subject. The objective of this paper is to introduce a new algorithm for the quadratic programming solution targeting portfolio optimization. In the next section, we will briefly describe the general problem and its formulation. In section 3 we shall present the results, followed by discussions and concluding remarks.

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2. PORTFOLIO OPTIMIZATION

If the risk of a portfolio return can be represented by its variance, the problem can be stated as minimizing the risk associated with a given rate of return in a standard model (see Markowitz[13]):

$$V = X'AX \tag{1}$$

$$R = \Sigma\ x_i r_i \tag{2}$$

$$\Sigma\ x_i = 1 \tag{3}$$

$$x_i \geqslant 0 \text{ (for no short sales)} \tag{4}$$

where V is the variance or risk; A is the variance–covariance matrix; $X' = (x_1 x_2 \ldots x_n)$ is the portfolio composition; R is the portfolio rate of return; and r_i is the rate of return of asset i.

Suppose for the moment we are not interested in the constraint (4). The solution by the lagrangian method can be formulated as

$$\begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} X \\ \lambda_1/2 \\ \lambda_2/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ R \end{bmatrix} \tag{5}$$

where

$$B' = \begin{bmatrix} 1, \ldots, 1 \\ r_1, \ldots, r_n \end{bmatrix}$$

λ_1, λ_2 are lagrangian multipliers; and 0 is the matrix or vector of zero coefficients.

The solution of the portfolio composition is

$$\begin{bmatrix} X \\ \lambda_1/2 \\ \lambda_2/2 \end{bmatrix} = \begin{bmatrix} U & Y \\ Z & W \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ R \end{bmatrix} \tag{6}$$

where

$$W = -(B'A^{-1}B)^{-1}$$
$$Y = -A^{-1}BW$$

The composition of the optimal portfolio is

$$X = [Y] \begin{bmatrix} 1 \\ R \end{bmatrix} \tag{7}$$

The 'efficient frontier' or 'minimum variance set' is

$$V = -\lambda_1/2 - (\lambda_2/2)R \tag{8}$$

It is clear from (6) and (8) that the efficient frontier is a second-order function of the portfolio return, i.e. a piecewise parabola.

The first n equations in (5) are, in fact, the marginal utility functions of n assets (see Sharpe[14]). The physical meanings of the lagrangian multipliers are clear here. $-\lambda_2$ is the risk tolerance, and can be considered as the trade-off between risk and return or the price of unit return per unit risk. In other words, it is the rate of change of variance with respect to the portfolio return. The ratio $-\lambda_1/\lambda_2$ is the portfolio marginal utility. It can be proved that this marginal utility is the riskless rate associated with an efficient portfolio having a return R. Note that this riskless rate is the interception of the portfolio capital market line and the return axis (zero risk).

In the above solution we may have a portfolio with negative x_i because we have not taken into account the constraint (4). If there exists a subset of m assets with $m \leqslant n$ whose solution has x_i all non-negative, then that solution is the optimal solution for that subset and a feasible solution to the original problem. It can also be the optimal solution to the original problem if no marginal utility function with respect to any given asset x_i yields a higher value than the riskless rate associated with the efficient portfolio of the above subset. If there exists an asset whose marginal utility is higher than the subset portfolio's riskless rate, a swap of portfolio assets with this asset is necessary to improve the portfolio utility or the objective function. The optimal solution is found when no more swapping is necessary (see also Sharpe[14]). In our procedure, however, no actual swapping is done because it is too time consuming. From observation, the final optimal portfolio usually consists of only a small number of assets, although the population of the market can be considerable. This means a large number of assets will not participate in the final solution. The first step is to use the lagrangian method where, ignoring constraint (4), we eliminate all assets believed to contribute to negative portfolio return. This is the simplification process to reduce the size of the original variance–covariance matrix. We now work on this downscaled matrix. The next step is to look for a feasible solution to the original problem. Instead of pivoting in and out of the simplex table of individual variables one by one, we massively eliminate all negative x_i. A new reduced matrix is generated. The process continues until a feasible solution is reached. By comparing the riskless rate of this feasible portfolio to the marginal utilities of the original problem, we verify whether there exist any other assets which can still contribute to the improvement of the solution. If not, this is also the the optimal solution to the original problem. If there are, we re-insert all of these assets into the previous solution matrix to generate a new matrix, and the process continues until the optimal solution is reached. This re-verification process is designed to make sure that we do not overlook or miss out any asset during the simplification and massive elimination processes. Each re-verification iteration will, of course, improve the solution. The optimal solution exists, thus the convergence is assured. The whole process can be summarized as follows.

BEGIN
1. Simplification of the original matrix
2. Massive elimination to obtain a feasible solution
3. Re-verification of the riskless rate with respect to the marginal utilities
   (i) if no improvement is possible, EXIT
   (ii) else, re-insert into the solution matrix all assets whose marginal utilities are higher than the riskless rate of the feasible portfolio, return to 2

END

3. RESULTS

We have tested the above procedure with a sample of 100 French stocks. The variance–covariance matrix and the stock expected return vector were kindly provided to us by Credit Commercial de France. Figure 1 shows the efficient frontier of the solution in a solid line. The cluster of points on this figure is the expected return of these 100 stocks and its variances. The above efficient frontier was obtained and plotted on an IBM RS/6000-320

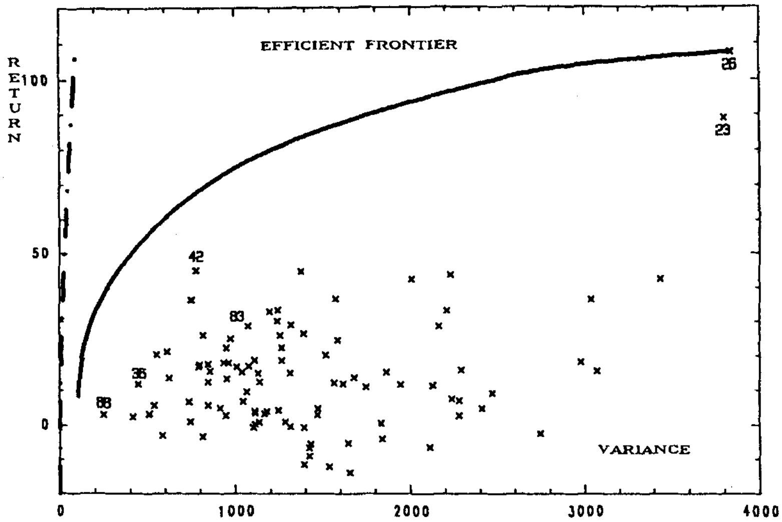

Figure 1. The efficient frontier is a piecewise parabola, shown as the solid line. The broken parabola near the vertical axis is the efficient frontier when short selling is allowed. The cluster of crosses is the population of stocks used to calculate the efficient frontier

workstation in less than nine seconds. The broken line to the left is the efficient frontier when short sales are allowed. Figure 2 shows the composition of all optimal portfolios on the frontier. The vertical axis is the return and the horizontal axis is the percentage of each stock in the portfolio. For example at 50 per cent return, the optimal portfolio consists of six stocks of 9 per cent of the portfolio to be invested in stock number 23, 17·2 per cent for stock number 26, 12·8 per cent for stock number 36, 40·5 per cent for stock number 42, 12 per cent for stock number 83, and 8·4 per cent for stock number 88. We can also see the in and out of each

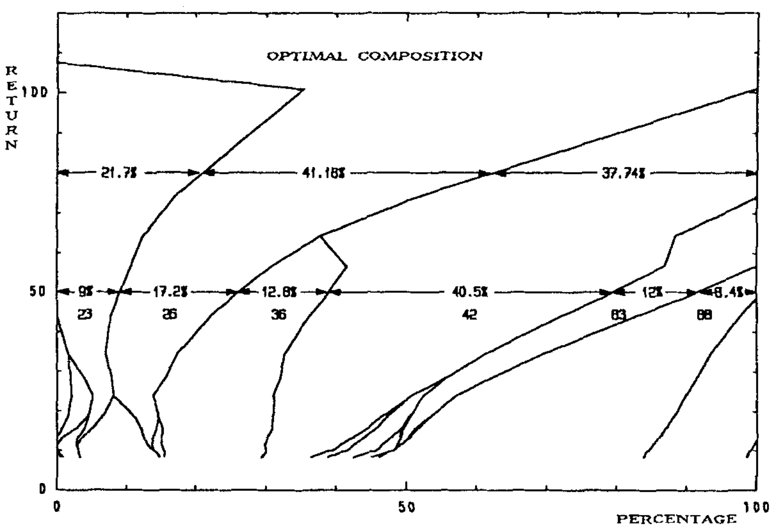

Figure 2. The portfolio composition corresponding to the above efficient frontier. The horizontal axis represents the percentage of the stocks in the portfolio for different levels of return

individual stock as we move the portfolio return to different levels. At the 80 per cent level of return, the efficient portfolio only consists of three stocks: 21·7 per cent for number 23, 41·18 per cent for number 26 and 37·74 per cent for number 42. The other stocks are dropped out from the previous portfolio. It is also noted that as the portfolio return decreases, the portfolio is more diversified as expected. The numbers labelling the stocks in the above example are shown on both figures for easy reference. For the whole frontier plotted above, we never have to re-verify more than twice to arrive at the optimal solution. The simplification process reduces the 100×100 matrix to a 33×33 matrix.

As seen above, our solution involves many vector and matrix manipulations because it operates on a system of linear equations and not a simplex table. This large scale problem can be easily vectorized to take advantage of the parallel processing of the new hardware technology vector processor.

4. DISCUSSIONS AND CONCLUDING REMARKS

The differences between our approach and the simplex type of solution are: (i) our initial feasible solution is in the basis; (ii) we use massive elimination and insertion instead of single pivoting; (iii) the simple verification rule is based on the problem structure to accelerate the procedure.

In the above formulation, we have used the standard model of Markowitz.[13] In fact other constraints, such as the level of asset holdings, cash yields, index tracking, etc., should be included to make the problem more realistic. Additional linear constraints can be added to the above solution. Each constraint will increase the matrix by one more row and column. And the processing is still identical. The above procedure has taken into account the particularity of the constraints (2) and (3) of the portfolio to solve the quadratic programming problem as a system of linear equations. Other quadratic programming problems that have this structure can apply our method. However, when the constraint of type (2) is used, normalization is necessary. The above description looks easy, but the implementation is more complex due to the fact that care has to be taken to assure that all the matrices are invertible.

In the above example we have shown the mean–variance approach. This approach has been challenged. Other alternatives (in particular the semi-variance approach) have been proposed (see, for example, Harlow,[15] King and Jensen,[16] Dueck and Winker[17]). However, the correct and realistic implementation of different risk measures in lieu of the variance is not evident or easily done. Our algorithm should work with any formulation provided that the marginal utility functions can be derived.

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