

Mini Project - Algebraic Number Theory

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Chapter 1

Noetherian Rings and Dedekind Rings

1.1 Noetherian rings and modules

Lemma 1. Let (T, \leq) be a partially ordered set. The following statements are equivalent :

1. Every non-empty subset of T contains a maximal element.
2. Every increasing sequence $(t_n)_{n \geq 0}$ of elements of T is stationary.

Proof. Let (t_n) be an increasing sequence of elements of T with respect to \leq and t_p be a maximal element of (t_n) . Then for $n \geq p$, $t_p \leq t_n$, so $t_n = t_p$ for all $n \geq p$.

Pick $\emptyset \neq S \subseteq P$. Let $x_1 \in S$ be arbitrary. Given $x_k \in S$, pick $x_{k+1} \in S$ strictly bigger than x_k . By hypothesis, we will eventually run out of bigger elements to pick at say x_n . Then by construction there are no larger elements than x_n , that is, x_n is a maximal element of S . \square

Theorem 1. Let \mathcal{R} be a ring and \mathcal{M} be an \mathcal{R} -module. The following statements are equivalent.

1. Every non-empty collection of submodules of \mathcal{M} contains a maximal element.
2. Every increasing sequence of submodules of \mathcal{M} is stationary.
3. Every submodule of \mathcal{M} is of finite type.

Proof. We will first establish the equivalence of (2) and (3). Assume $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an increasing sequence of submodules of \mathcal{M} . From 3, $N := \cup_{i \geq 0} N_i$ is a finitely generated submodule of \mathcal{M} . Suppose N is generated by $a_1, \dots, a_k \in N$. For all $i \in \{1, \dots, k\}$, there is some $j_i \in \mathbb{N}$ such that $a_i \in N_{j_i}$. For $j := \max\{j_1, \dots, j_k\}$, we have $a_1, \dots, a_k \in N_j$.

Hence $N_j = N$. Therefore, every increasing sequence of submodules of \mathcal{M} is stationary.

Suppose every increasing sequence of submodules of \mathcal{M} is stationary. Let N be a submodule of \mathcal{M} . For the sake of a contradiction, suppose N is not finitely generated. Any finitely generated submodule of N is not equal to N . So we can inductively choose a sequence $a_i \in N \setminus \langle a_1, \dots, a_{i-1} \rangle$. The chain : $\langle a_1 \rangle \subsetneq \langle a_1, a_2 \rangle \subsetneq \dots$ is strictly increasing contradicting 2. Hence, every submodule of \mathcal{M} is finitely generated.

The equivalence of (1) and (2) follows from Lemma 1. \square

Definition 1 (Noetherian Module). An \mathcal{R} -module \mathcal{M} is called Noetherian if it satisfies the equivalent conditions of Theorem 1.

Definition 2 (Noetherian Ring). A ring \mathcal{R} is called Noetherian if, considered as an \mathcal{R} -module, it is a Noetherian module.

Proposition 1. Let $0 \rightarrow \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \rightarrow 0$ be an exact sequence of \mathcal{R} -modules. Then \mathcal{M} is Noetherian if and only if \mathcal{M}' and \mathcal{M}'' are Noetherian.

Proof. Suppose \mathcal{M} is Noetherian. Since \mathcal{M}' is isomorphic to a submodule of \mathcal{M} , \mathcal{M}' is Noetherian. Let N'' be a submodule of \mathcal{M}'' . Then $g^{-1}(N'')$ is a submodule of \mathcal{M} . Therefore there exist $x_1, \dots, x_r \in g^{-1}(N'')$ such that $g^{-1}(N'')$ is generated by x_1, \dots, x_r . Since g is surjective, we have $N'' = g(g^{-1}(N''))$. It follows that N'' is generated by $g(x_1), \dots, g(x_r)$. Thus \mathcal{M}'' is Noetherian.

Conversely, suppose \mathcal{M}' and \mathcal{M}'' are Noetherian. Let N be a submodule of \mathcal{M} . Then $g(N)$ is a submodule of \mathcal{M}'' . Therefore, there exist $x_1, \dots, x_r \in N$ such that $g(x_1), \dots, g(x_r)$ generate $g(N)$. Next, $f^{-1}(N)$ is a submodule of \mathcal{M}' . Therefore there exist $y_1, \dots, y_s \in f^{-1}(N)$ such that $f^{-1}(N)$ is generated by y_1, \dots, y_s . We claim that N is generated by $x_1, \dots, x_r, f(y_1), \dots, f(y_s)$. Let $z \in N$. Then $g(z) = \sum_{i=1}^r a_i g(x_i)$ with $a_1, \dots, a_r \in \mathcal{R}$. Let $z' = z - \sum_{i=1}^r a_i x_i$. Then $z' \in N \cap \ker g = N \cap \text{Im } f$. Therefore $z' = f(x')$ with $x' \in f^{-1}(N)$. There exist $b_1, \dots, b_s \in \mathcal{R}$ such that $x' = \sum_{j=1}^s b_j y_j$. Thus $z = \sum_{i=1}^r a_i x_i + \sum_{j=1}^s b_j f(y_j)$. \square

Proposition 2. Let \mathcal{R} be a ring, \mathcal{M} an \mathcal{R} -module, and \mathcal{M}' a submodule of \mathcal{M} . Then \mathcal{M} is Noetherian if and only if \mathcal{M}' and \mathcal{M}/\mathcal{M}' are Noetherian.

Proof. Consider the short exact sequence :

$$0 \rightarrow \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}/\mathcal{M}' \rightarrow 0$$

where f is the inclusion map and g takes $x \in \mathcal{M}$ to $x + \mathcal{M}'$. Note that f is clearly injective and g is surjective because for all $x + \mathcal{M}' \in \mathcal{M}/\mathcal{M}'$, $g(x) = x + \mathcal{M}'$. We observe that $\ker g = \{x \in \mathcal{M} : x + \mathcal{M}' = \mathcal{M}'\} = \mathcal{M}' = \text{Im } f$. Applying proposition 1 to this sequence completes the proof. \square

Corollary. Let \mathcal{R} be a ring and let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be Noetherian \mathcal{R} -modules. Then the \mathcal{R} -module product $\prod_{i=1}^n \mathcal{M}_i$ is Noetherian.

Proof. For $n = 2$, we want to show that if \mathcal{M}_1 and \mathcal{M}_2 are Noetherian, then the product $\mathcal{M}_1 \times \mathcal{M}_2$ is Noetherian. Consider the sequence :

$$0 \rightarrow \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_1 \times \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_2 \rightarrow 0,$$

where $f(x) = (x, 0)$ for all $x \in \mathcal{M}_1$ and $g(x, y) = y$ for all $x, y \in \mathcal{M}_1 \times \mathcal{M}_2$. Note that f is injective and g is surjective. We observe that $\text{Im } f = \{(x, 0) : x \in \mathcal{M}_1\} \cong \mathcal{M}_1 = \ker g$. Therefore, this is a short exact sequence. Applying Proposition 1 to this sequence proves the result for $n = 2$. Inductively, it follows that $\prod_{i=1}^n \mathcal{M}_i$ is Noetherian. \square

Corollary. Let \mathcal{R} be a Noetherian ring and let \mathcal{M} be an \mathcal{R} -module of finite type. then \mathcal{M} is a Noetherian module.

Proof. Suppose \mathcal{R} is generated by x_1, \dots, x_r . We prove the assertion by induction on r . First suppose $r = 1$. Let $g : \mathcal{R} \rightarrow \mathcal{M}$ be the map defined by $g(a) = ax_1$. Then g is a surjective homomorphism and it follows that \mathcal{M} is Noetherian from Proposition 1.

Now, suppose $r \geq 2$. Let $\mathcal{M}' = Ax_r$. Let $g : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{M}'$ be the natural surjection. Then \mathcal{M}/\mathcal{M}' is generated by $g(x_1), \dots, g(x_{r-1})$. Therefore by induction both \mathcal{M}' and \mathcal{M}/\mathcal{M}' are Noetherian. Therefore by Proposition 1, \mathcal{M} is Noetherian. \square

1.2 An application concerning integral elements

Lemma 2. Let \mathcal{R} be an integrally closed ring. Let \mathcal{K} be its field of fractions, \mathcal{L} be an extension of finite degree n of \mathcal{K} , and \mathcal{R}' is the integral closure of \mathcal{R} in \mathcal{L} . Suppose \mathcal{K} is of characteristic 0. Then \mathcal{R}' is an \mathcal{R}' -submodule of a free \mathcal{R} -module of rank n .

Proof. Let (x_1, \dots, x_n) be a base of \mathcal{L} over \mathcal{K} . Each x_i is algebraic over \mathcal{K} , so for any i , we have an equation of the form $a_n x_i^n + a_{n-1} x_i^{n-1} + \dots + a_0 = 0$ ($a_j \in \mathcal{R} \forall j$). We may assume $a_n \neq 0$. Multiplying through by a_n^{n-1} , we see that $a_n x_i$ is integral over \mathcal{R} . Put $x'_i = a_n x_i$. Then (x'_1, \dots, x'_n) is a base for \mathcal{L} over \mathcal{K} contained in \mathcal{R}' . Hence, there exists another base (y_1, \dots, y_n) of \mathcal{L} over \mathcal{K} such that $\text{Tr}(x'_i y_j) = \delta_{ij}$. Let $z \in \mathcal{R}'$. Since (y_1, \dots, y_n) is a base for \mathcal{L} over \mathcal{K} , we may write $z = \sum_{j=1}^n b_j y_j$ with $b_j \in \mathcal{K}$. For any i , we have $x'_i z \in \mathcal{R}'$. Therefore, $\text{Tr}(x'_i z) \in \mathcal{R}$. Thus, $\text{Tr}(x'_i z) = \text{Tr}(\sum_j b_j x'_i y_j) = \sum_j b_j \text{Tr}(x'_i y_j) = \sum_j b_j \delta_{ij} = b_i$. Hence, it follows that $b_i \in \mathcal{R}$ for all i , which implies that \mathcal{R}' is a submodule of the free \mathcal{R} -module $\sum_{j=1}^n \mathcal{R} y_j$. \square

Proposition 3. Let \mathcal{R} be a Noetherian integrally closed ring. Let \mathcal{K} be its field of fractions, \mathcal{L} a finite extension of \mathcal{K} , and \mathcal{R}' the integral closure of \mathcal{R} in \mathcal{L} . Suppose that \mathcal{K} is of characteristic 0. Then \mathcal{R}' is a \mathcal{R} -module of finite type and a Noetherian ring.

Proof. From previous lemma we know that \mathcal{R}' is a submodule of a free \mathcal{R} -module of rank n . Thus \mathcal{R}' is a \mathcal{R} -module of finite type, and therefore, a Noetherian module. On the other hand, the ideals of \mathcal{R}' are special cases of \mathcal{R} -submodules of \mathcal{R}' . They satisfy the maximal condition, so \mathcal{R}' is a Noetherian ring. \square

1.3 Some preliminaries concerning ideals

Definition 3 (Prime and Maximal Ideals). Let \mathcal{R} be a non-zero ring and \mathcal{P} be an ideal of \mathcal{R} . We say that \mathcal{P} is a **prime ideal** of \mathcal{R} if the following hold :

1. $\mathcal{P} \neq \mathcal{R}$,
2. if there exist ideals \mathcal{I}, \mathcal{J} of \mathcal{R} such that $\mathcal{I}\mathcal{J} \subseteq \mathcal{P}$, then $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$.

An ideal \mathcal{M} of \mathcal{R} is called a **maximal ideal** if the following hold :

1. $\mathcal{M} \neq \mathcal{R}$,
2. if there exists any ideal \mathcal{I} of \mathcal{R} such that $\mathcal{M} \subseteq \mathcal{I}$, then either $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{R}$.

Proposition 4. Let \mathcal{R} be a non-zero commutative ring with unity. Then an ideal \mathcal{P} is prime ideal if and only if for any $a, b \in \mathcal{R}$ whenever $a \cdot b \in \mathcal{P}$, then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Proof. Let $x, y \in \mathcal{R}$ such that $xy \in \mathcal{P}$. So, $(xy) \subseteq \mathcal{P}$. Since \mathcal{R} is commutative $(xy) = (x)(y) \subseteq \mathcal{R}$. By definition, either $(x) \subseteq \mathcal{P}$ or $(y) \subseteq \mathcal{P}$. Hence, either $x \in \mathcal{P}$ or $y \in \mathcal{P}$, proving the forward direction.

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}$. Without loss of generality, let $\mathcal{I} \subsetneq \mathcal{P}$. Then there exists $x \in \mathcal{I}$ such that $x \notin \mathcal{P}$. Let $y \in \mathcal{J}$. Since \mathcal{J} is an ideal, $xy \in \mathcal{J}$. But $xy \in \mathcal{I}\mathcal{J} \subseteq \mathcal{P}$, so $xy \in \mathcal{P}$. Hence, either $x \in \mathcal{P}$ or $y \in \mathcal{P}$. Since we assumed $x \notin \mathcal{P}$, $y \in \mathcal{P}$. Since choice of y was arbitrary, $\mathcal{J} \subseteq \mathcal{P}$, proving the reverse direction. \square

Proposition 5. Let \mathcal{R} be a commutative ring with unity. Then an ideal \mathcal{P} of \mathcal{R} is a prime ideal if and only if the quotient ring \mathcal{R}/\mathcal{P} is an integral domain.

Proof. Let \mathcal{P} be a prime ideal, $a + \mathcal{P}, b + \mathcal{P} \in \mathcal{R}/\mathcal{P}$. If $(a + \mathcal{P})(b + \mathcal{P}) = \mathcal{P}$, then we get $ab \in \mathcal{P}$. Hence, $a \in \mathcal{P}$ or $b \in \mathcal{P}$. So $a + \mathcal{P} = \mathcal{P}$ or $b + \mathcal{P} = \mathcal{P}$.

Hence, \mathcal{R}/\mathcal{P} is an integral domain.

Now let \mathcal{R}/\mathcal{P} be an integral domain. Let $a \notin \mathcal{P}$ and $b \notin \mathcal{P}$ then $a + \mathcal{P} \neq \mathcal{P}$ and $b + \mathcal{P} \neq \mathcal{P}$. Therefore $(a + \mathcal{P})(b + \mathcal{P}) \neq \mathcal{P}$. Hence, $ab + \mathcal{P} \neq \mathcal{P}$ implies $ab \notin \mathcal{P}$. Hence, \mathcal{P} is prime. \square

Proposition 6. Let \mathcal{R} be a commutative ring with unity. Then an ideal \mathcal{M} of \mathcal{R} is a maximal ideal if and only if the quotient ring \mathcal{R}/\mathcal{M} is a field.

Proof. Let \mathcal{M} be a maximal ideal. Let $a + \mathcal{M}$ be a non zero element of \mathcal{R}/\mathcal{M} . Hence $a + \mathcal{M} \neq \mathcal{M}$, or $a \notin \mathcal{M}$. Consider the ideal $\mathcal{M} + (a)$. Observe that $\mathcal{M} \subsetneq \mathcal{M} + (a) \subseteq \mathcal{R}$. Since \mathcal{M} is maximal $\mathcal{M} + (a) = \mathcal{R}$. Hence, there exist $r \in \mathcal{R}$, $m_0 \in \mathcal{M}$ such that $m_0 + ra = 1$. It follows that $(a + \mathcal{M})(r + \mathcal{M}) = ar + \mathcal{M} = 1 - m_0 + \mathcal{M} = 1 + \mathcal{M}$. Hence, $a + \mathcal{M}$ is a unit. Since choice of a was arbitrary, it follows every non-zero element in \mathcal{R}/\mathcal{M} is a unit. Hence, \mathcal{R}/\mathcal{M} is a field.

Now let \mathcal{I} be an ideal such that $\mathcal{M} \subsetneq \mathcal{I} \subseteq \mathcal{R}$. Then there exists an $r \in \mathcal{I} \setminus \mathcal{M}$. Since \mathcal{R}/\mathcal{M} is a field, since $r \notin \mathcal{M}$, $r + \mathcal{M}$ has an inverse, say $r_1 + \mathcal{M}$. Now $(r + \mathcal{M})(r_1 + \mathcal{M}) = 1 + \mathcal{M} \Rightarrow rr_1 + \mathcal{M} = 1 + \mathcal{M}$, or $rr_1 - 1 \in \mathcal{M} \subset \mathcal{I}$. Since $r \in \mathcal{I}$, $r_1 \in \mathcal{R}$, we get $rr_1 \in \mathcal{I}$, so $rr_1 - (rr_1 - 1) \in \mathcal{I}$ or $1 \in \mathcal{I}$. Hence, $\mathcal{I} = \mathcal{R}$. Therefore, \mathcal{M} is maximal. \square

Lemma 3. Let \mathcal{R} be a ring, \mathcal{P} be a prime ideal of \mathcal{R} , and let \mathcal{R}' be a subring of \mathcal{R} . Then $\mathcal{P} \cap \mathcal{R}'$ is a prime ideal of \mathcal{R}' .

Proof. Let $x \in \mathcal{R}'$ and $\alpha \in \mathcal{P} \cap \mathcal{R}'$, then $\alpha \in \mathcal{P} \Rightarrow x\alpha \in \mathcal{P}$. Since, $x \in \mathcal{R}'$ and $\alpha \in \mathcal{R}$, we get $x\alpha \in \mathcal{P} \cap \mathcal{R}'$. Therefore, $\mathcal{P} \cap \mathcal{R}'$ is an ideal of \mathcal{R}' . Consider the map ψ defined as $\psi : \mathcal{R}'/\mathcal{P} \cap \mathcal{R}' \rightarrow \mathcal{R}/\mathcal{P}$, $x + \mathcal{P} \cap \mathcal{R}' \mapsto x + \mathcal{P}$. ψ is clearly a homomorphism. The kernel of $\psi = \{x + \mathcal{P} \cap \mathcal{R}' : x + \mathcal{P} = \mathcal{P}\} = \{x + \mathcal{P} \cap \mathcal{R}' : x \in \mathcal{P}\} = \{\mathcal{P} \cap \mathcal{R}'\}$. Hence, we have $\mathcal{R}'/\mathcal{P} \cap \mathcal{R}'$ is a subring of \mathcal{R}/\mathcal{P} , so it must be an integral domain. \square

Definition 4 (Sum and product of ideals). Let \mathcal{R} be a ring and \mathcal{I}, \mathcal{J} be two ideals of \mathcal{R} . We define the sum of two ideals \mathcal{I}, \mathcal{J} as follows :

$$\mathcal{I} + \mathcal{J} := \{x + y : x \in \mathcal{I}, y \in \mathcal{J}\}.$$

We define the product of two ideals \mathcal{I}, \mathcal{J} of \mathcal{R} as follows :

$$\mathcal{I}\mathcal{J} := \left\{ \sum_{i=1}^n x_i y_i : n \in \mathbb{N}, x_i \in \mathcal{I}, y_i \in \mathcal{J} \right\}.$$

Lemma 4. If a prime ideal \mathcal{P} of \mathcal{R} contains a product $\mathcal{I}_1 \cdots \mathcal{I}_n$ of ideals. Then \mathcal{P} contains at least one of the ideals \mathcal{I}_i .

Proof. If $\mathcal{I}_i \not\subseteq \mathcal{P}$ for any i , then there exist $a_i \in \mathcal{I}_i \setminus \mathcal{P}$ for all i . Therefore, $a_i \cdots a_n \notin \mathcal{P}$, since \mathcal{P} is prime. But $a_i \cdots a_n \in \mathcal{I}_1 \cdots \mathcal{I}_n$ which contradicts the hypothesis of the lemma. \square

Lemma 5. In a Noetherian ring every ideal contain a product of prime ideals. In a Noetherian integral domain \mathcal{R} , every non-zero ideal contains a product of prime ideals.

Proof. Let Φ be the set of non zero ideals of \mathcal{R} which don't contain product of non-zero prime ideals. We want to show that Φ is non-empty. For the sake of a contradiction, let $|\Phi| > 0$. Since \mathcal{R} is Noetherian, Φ contains a maximal element \mathcal{B} . The ideal \mathcal{B} cannot be prime; otherwise $\mathcal{B} \in \Phi$. Thus, there exist $x, y \in \mathcal{R} \setminus \mathcal{B}$ such that $xy \in \mathcal{B}$. The ideals $\mathcal{B} + (x)$ and $\mathcal{B} + (y)$ contain \mathcal{B} as a proper subset. Therefore, since \mathcal{B} is maximal, they do not belong to Φ . It follows that they both contain products of non zero prime ideals.

$$\mathcal{B} + (x) \supset p_1 \cdots p_n, \quad \mathcal{B} + (y) \supset q_1 \cdots q_r$$

Since $xy \in \mathcal{B}$,

$$(\mathcal{B} + (x))(\mathcal{B} + (y)) \subset \mathcal{B}.$$

Hence, $p_1 \cdots p_n \cdot q_1 \cdots q_r \subset \mathcal{B}$, a contradiction. Hence, $|\Phi| = 0$. \square

Definition 5 (Fractional ideals). Let \mathcal{R} be an integral domain and \mathcal{K} be its field of fractions. Let \mathcal{I} be an \mathcal{R} -submodule. We call \mathcal{I} , a fractional ideal of \mathcal{K} if there exists a $d \in \mathcal{R} \setminus \{0\}$ such that $d \cdot \mathcal{I} \subseteq \mathcal{R}$.

The ordinary ideals of \mathcal{R} are fractional ideals with $d = 1$. They are also called **integral ideals** to distinguish them from fractional ideals.

Proposition 7. The following are true :

1. Any \mathcal{R} -submodule \mathcal{I} of finite type contained in \mathcal{K} is a fractional ideal.
2. If \mathcal{R} is Noetherian, every fractional ideal \mathcal{I} is an \mathcal{R} -module of finite type.
3. If \mathcal{I} and \mathcal{I}' are fractional ideals, then the sets $\mathcal{I} \cap \mathcal{I}'$, $\mathcal{I} + \mathcal{I}'$, and $\mathcal{I}\mathcal{I}'$ are all fractional ideals.

Proof. 1. Since \mathcal{I} is an \mathcal{R} -submodule of finite type it must be generated by a finite set of generators $\langle a_1, \dots, a_n \rangle$. If $a_i = p_i/q_i$ for all i , then the product $d = \prod_{i=1}^n q_i$ is a common denominator for \mathcal{I} .

2. Since $d \cdot \mathcal{I} \subseteq \mathcal{R}$, we get $\mathcal{I} \subseteq d^{-1}\mathcal{R}$. Since $d^{-1}\mathcal{R}$ is isomorphic to \mathcal{R} , \mathcal{I} is a Noetherian module.

3. If d and d' are the common denominators for \mathcal{I} and \mathcal{I}' respectively then dd' is a common denominator for $\mathcal{I} \cap \mathcal{I}'$, $\mathcal{I} + \mathcal{I}'$, and $\mathcal{I}\mathcal{I}'$. \square

1.4 Dedekind Domains

Definition 6 (Dedekind domain). An integral domain \mathcal{R} is called a Dedekind domain if it is Noetherian and integrally closed, and if every non-zero prime ideal of \mathcal{R} is maximal.

Example. Every principal ideal ring is a Dedekind domain.

Theorem 2. Let \mathcal{R} be a Dedekind domain, \mathcal{K} be its field of fractions. Let \mathcal{L} be a finite extension of \mathcal{K} and \mathcal{R}' be the integral closure of \mathcal{R} in \mathcal{L} . If \mathcal{K} is of characteristic 0. Then \mathcal{R}' is a Dedekind domain and an \mathcal{R} -module of finite type.

Proof. We need to show three things. That \mathcal{R}' is integrally close, that \mathcal{R}' is Noetherian, and that every non-zero prime ideal of \mathcal{R} is maximal. The first part is done for us by construction. From Proposition 3, we get that \mathcal{R}' is Noetherian and a \mathcal{R} -module of finite type. It remains to show that every prime ideal $\mathcal{P}' \neq (0)$ of \mathcal{R}' is maximal. Let $x \in \mathcal{P}' \not\subseteq (0)$ and the following be its minimal polynomial over \mathcal{R} :

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0. \quad (a_i \in \mathcal{R})$$

Note that $a_0 \neq 0$, because if not then dividing through by x , we get a polynomial of lower degree. Note that since $a_0 = -x(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1)$, we get that $a_0 \in \mathcal{R}'x$. But since $a_0 \in \mathcal{R}$, we have that $x \in \mathcal{R}'x \cap \mathcal{R} \subseteq \mathcal{P}' \cap \mathcal{R}$. Hence, $\mathcal{P}' \cap \mathcal{R} \neq (0)$. Since \mathcal{P}' is a prime ideal, $\mathcal{P}' \cap \mathcal{R}$ is a prime ideal. Since \mathcal{R} is a Dedekind ring $\mathcal{P}' \cap \mathcal{R}$ is a maximal ideal of \mathcal{R} and so $\mathcal{R}/\mathcal{P}' \cap \mathcal{R}$ is a field. Consider the map $\varphi : \mathcal{R} \rightarrow \mathcal{R}'/\mathcal{P}'$ such that $x \mapsto x + \mathcal{P}'$. This is clearly a well defined homomorphism. The kernel of this map is $\ker \varphi := \{x \in \mathcal{R} : x + \mathcal{P}' = \mathcal{P}'\} = \mathcal{R} \cap \mathcal{P}'$. It follows that $\mathcal{R}/\mathcal{R} \cap \mathcal{P}'$ is a subring of $\mathcal{R}'/\mathcal{P}'$.

We note that $\mathcal{R}'/\mathcal{P}'$ is integral over $\mathcal{R}/\mathcal{R} \cap \mathcal{P}'$ ^a. Thus $\mathcal{R}'/\mathcal{P}'$ is a field, so \mathcal{P}' is maximal. \square

^aPick an element $x + \mathcal{P}' \in \mathcal{R}'/\mathcal{P}'$. Since $x \in \mathcal{R}'$, there exist $n \in \mathbb{Z}, a_0, \dots, a_{n-1} \in \mathcal{R}$ such that $a_0 + a_1x + \cdots + x^n = 0$. Hence, $(a_0 + \mathcal{R} \cap \mathcal{P}') + (a_1 + \mathcal{R} \cap \mathcal{P}')(x + \mathcal{P}') + \cdots + (1 + \mathcal{R} \cap \mathcal{P}')(x + \mathcal{P}')^n = \mathcal{R} \cap \mathcal{P}'$ or $x + \mathcal{P}'$ is integral over $\mathcal{R}/\mathcal{R} \cap \mathcal{P}'$.

Theorem 3. Let \mathcal{R} be a Dedekind domain which is not a field. Every maximal ideal of \mathcal{R} is invertible in the monoid of fractional ideals of \mathcal{R} .

Proof. The set of fractional ideals forms a monoid under multiplication. Closure follows Proposition 7.3. Associativity follows from the associativity of \mathcal{R} , and \mathcal{R} acts as the identity.

Let \mathcal{M} be a maximal ideal of \mathcal{R} . Then $\mathcal{M} \neq (0)$, since \mathcal{R} is not a field. Put

$$\mathcal{M}' = \{x \in \mathcal{K} | x\mathcal{M} \subseteq \mathcal{R}\}.$$

Note that \mathcal{M}' is an \mathcal{R} -submodule of \mathcal{K} and is a fractional ideal of \mathcal{R} . We need to show that $\mathcal{M}\mathcal{M}' = \mathcal{R}$. From the definition of \mathcal{M}' , it must be that $\mathcal{M}\mathcal{M}' \subseteq \mathcal{R}$. As \mathcal{M} is a maximal ideal, $\mathcal{M} = \mathcal{R}\mathcal{M} \subseteq \mathcal{M}'\mathcal{M} \subseteq \mathcal{R}$

we get that either $\mathcal{M}\mathcal{M}' = \mathcal{M}$ or $\mathcal{M}\mathcal{M}' = \mathcal{R}$. It suffices to show that $\mathcal{M}\mathcal{M}' \neq \mathcal{M}$.

For the sake of contradiction, suppose $\mathcal{M}'\mathcal{M} = \mathcal{M}$. Then for any $x \in \mathcal{M}'$ we have $x\mathcal{M} \subseteq \mathcal{M}$, $x^2\mathcal{M} \subseteq x\mathcal{M} \subseteq \mathcal{M}$. Inductively, $x^n\mathcal{M} \subseteq \mathcal{M}$ for all $n \in \mathbb{N}$. Hence any non-zero element $d \in \mathcal{M}$ acts as a common denominator for all powers x^n of x , $n \in \mathbb{N}$. It follows that $\mathcal{R}[x]$ is a fractional ideal of \mathcal{M} . Since \mathcal{R} is Noetherian, $\mathcal{R}[x]$ is a \mathcal{R} -module of finite type, so x is integral over \mathcal{R} . But \mathcal{R} is integrally closed, therefore $x \in \mathcal{R}$ and hence $\mathcal{M}'\mathcal{M} = \mathcal{M} \Rightarrow \mathcal{M}' = \mathcal{R}$. It suffices to show that \mathcal{M}' is never equal to \mathcal{R} .

Let $0 \neq a \in \mathcal{M}$. The ideal $\mathcal{R}a$ contains a product $p_1p_2 \cdots p_n$ of non-zero prime ideals. Let n be as small as possible. Note that $\mathcal{M} \supseteq \mathcal{R}a \supseteq p_1p_2 \cdots p_n$, so there exists $i \in \{1, \dots, n\}$ such that $\mathcal{M} \supseteq p_i$. Since p_i is prime, it is also maximal (\mathcal{R} is a Dedekind domain) we get that $\mathcal{M} = p_i$. Therefore, $\mathcal{R}a \supseteq p_i \prod_{j \neq i} p_j$ and $\mathcal{R} \not\supseteq \prod_{j \neq i} p_j$, since our n was minimal. Hence, there exists a $b \in \prod_{j \neq i} p_j$ such that $b \notin \mathcal{R}a$. But $\mathcal{M} \prod_{j \neq i} p_j \subseteq \mathcal{R}a$ so $\mathcal{M}b \subseteq \mathcal{R}a$ or $\mathcal{M}ba^{-1} \subseteq \mathcal{R}$. From the definition of \mathcal{M}' , it follows that $ba^{-1} \in \mathcal{M}'$. Since $b \notin \mathcal{R}a$, we get $ba^{-1} \notin \mathcal{R}$. Therefore $\mathcal{M}' \neq \mathcal{R}$. \square

Theorem 4. Let \mathcal{R} be a Dedekind domain and let $\text{spec}(\mathcal{R})$ be the set of non-zero prime ideals of \mathcal{R} . Then :

1. Every non-zero fractional ideal \mathfrak{b} of \mathcal{R} may be uniquely expressed in the form :

$$\mathfrak{b} = \prod_{\mathfrak{p} \in \text{spec}(\mathcal{R})} \mathfrak{p}^{n_{\mathfrak{p}}(\mathfrak{b})},$$

where, for any $\mathfrak{p} \in \text{spec}(\mathcal{R})$, $n_{\mathfrak{p}}(\mathfrak{b}) \in \mathbb{Z}$ and for almost all $\mathfrak{p} \in \text{spec}(\mathcal{R})$, $n_{\mathfrak{p}}(\mathfrak{b}) = 0$.

2. The monoid of non-zero fractional ideals of \mathcal{R} is a group.

Proof. Let \mathfrak{b} be a non-zero fractional ideal of \mathcal{R} . Then by definition there exists a $d \in \mathcal{R} \setminus \{0\}$ such that $d\mathfrak{b} \subseteq \mathcal{R}$, or $d\mathfrak{b}$ is an integral ideal of \mathcal{R} . Let Γ be the set of non-zero ideals in \mathcal{R} which are not product of prime ideals. For the sake of contradiction, let's assume $|\Gamma| > 0$. By Zorn's Lemma, there exists \mathfrak{a} be a maximal element of Γ . Since \mathcal{R} is the product of the empty collection of prime ideals, so $\mathfrak{a} = \mathcal{R}$.

Every ideal is contained in a maximal ideal, so let $\mathfrak{a} \subseteq \mathfrak{p}$. Let \mathfrak{p}' be the inverse fractional ideal of \mathfrak{p} in the monoid of fractional ideals of \mathcal{R} , the existence of which we proved earlier. Now since $\mathfrak{a} \subseteq \mathfrak{p}$, we get $\mathfrak{a}\mathfrak{p}' \subseteq \mathfrak{p}\mathfrak{p}' = \mathcal{R}$. Since $\mathcal{R} \subseteq \mathfrak{p}'$, $\mathfrak{a} \subseteq \mathfrak{a}\mathfrak{p}'$. \square

Chapter 2

Ideal Classes and the Unit Theorem

2.1 Preliminaries concerning discrete subgroups of \mathbb{R}^n

Definition 7 (Discrete Subgroup of \mathbb{R}^n). A subgroup H of \mathbb{R}^n is discrete if and only if, for any compact subset K of \mathbb{R}^n , the intersection $H \cap K$ is finite.

Theorem 5. Let H be a discrete subgroup of \mathbb{R}^n . Then H is generated over \mathbb{Z} by r vectors e_1, \dots, e_r which are linearly independent over \mathbb{R} .

Proof. Choose $\mathbf{e} = e_1, \dots, e_r$ in H such that they are \mathbb{R} -linearly independent and r is maximal. Define \mathcal{P} as follows :

$$\mathcal{P} := \left\{ \sum_{i=1}^r \alpha_i e_i \mid \alpha_i \in [0, 1] \right\},$$

The set \mathcal{P} is called the fundamental parallelogram of H with respect to the basis e_1, \dots, e_r . We can immediately see that \mathcal{P} is compact because it is homeomorphic to \mathbb{R}^n . Let $x \in H$. Then we can write x in the form :

$$x = \sum_{i=1}^r \lambda_i e_i,$$

for $\lambda_i \in \mathbb{R}$. If x cannot be written in this form, we can add x to $\{e_1, \dots, e_r\}$, contradicting the maximality of r . Now for $j \in \mathbb{Z}$, let :

$$x_j := jx - \sum_{i=1}^r [\lambda_i j] e_i = \sum_{i=1}^r (\lambda_i j - [\lambda_i j]) e_i. \quad (j \in \mathbb{Z})$$

Hence, $x_j \in \mathcal{P}$. It follows that $x_j \in \mathcal{P} \cap H$. Since H is discrete, $\mathcal{P} \cap H$

is finite since \mathcal{P} is compact. For $j = 1$, we have :

$$x = x_1 + \sum_{i=1}^r \lfloor \lambda_i \rfloor e_i.$$

So x is in the \mathbb{Z} span of a finite set. Hence, H is finitely generated over \mathbb{Z} . Since $\mathcal{P} \cap H$ is finite, there exist $j \neq k$ such that $x_j = x_k$. It follows that :

$$\sum_{i=1}^r \lambda_i(j - k)e_i = \sum_{i=1}^r (\lfloor j\lambda_i \rfloor - \lfloor k\lambda_i \rfloor)e_i.$$

Linear independence of $\{e_i\}$ gives us :

$$\lambda_i(j - k) = \lfloor j\lambda_i \rfloor - \lfloor k\lambda_i \rfloor.$$

Hence $\lambda_i \in \mathbb{Q}$ for all i . So far we've shown that for any $x \in \mathcal{P} \cap H$, $x = \sum_{i=1}^r \lambda_i e_i$ where $\lambda_i \in \mathbb{Q}$ for all i . Let d be the least common multiple of the denominators of λ_i 's. Then for any $x \in H$, we know :

$$x = x_1 + \sum_{i=1}^r \lfloor \lambda_i \rfloor e_i \in \frac{1}{d} \sum_{i=1}^r \mathbb{Z} e_i.$$

We conclude that : $H \subseteq \frac{1}{d} \sum_{i=1}^r \mathbb{Z} e_i \Rightarrow \sum_{i=1}^r \mathbb{Z} e_i \subseteq H \subseteq \frac{1}{d} \mathbb{Z} e_i$. Therefore we must have that H is finitely generated of rank r over \mathbb{Z} on some linear combination of the vectors $\{\frac{1}{d}e_i\}$. This basis is linearly independent over \mathbb{R} as desired. \square

Definition 8 (Lattice). A discrete subgroup of rank n of \mathbb{R}^n is called a lattice in \mathbb{R}^n .

By Theorem 1, a lattice is generated over \mathbb{Z} by a base of \mathbb{R}^n , which is then a \mathbb{Z} -base for the given lattice. For each \mathbb{Z} -base $e = (e_1, \dots, e_n)$ of a lattice H we shall write \mathcal{P}_e for the half open parallelotope :

$$\mathcal{P}_e = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^n \alpha_i e_i, \alpha_i \in [0, 1) \right\}$$

Thus every point of \mathbb{R}^n is congruent modulo H to one and only one point of \mathcal{P}_e for any fixed e (we say, in this case, that \mathcal{P}_e is a fundamental domain for H). We shall write μ to denote the Lebesgue measure in \mathbb{R}^n , i.e. if S is a measurable subset of \mathbb{R}^n , $\mu(S)$ will stand for its measure (which we will also call its volume).

Lemma 6. The volume $\mu(\mathcal{P}_e)$ is independent of the base e chosen for H .

Proof. Let $f = (f_1, \dots, f_n)$ be another base for H . Then :

$$f_i = \sum_{j=1}^n \alpha_{ij} e_j. \quad (\alpha_{ij} \in \mathbb{Z})$$

By calculus we know that $\mu(\mathcal{P}_f) = |\det(\alpha_{ij})| \mu(\mathcal{P}_e)$. The change of bases matrix $(\alpha_{ij}) \in GL_n(\mathbb{Z})$, so $\det(\alpha_{ij}) = \pm 1$. Hence, $\mu(\mathcal{P}_f) = \mu(\mathcal{P}_e)$. \square

Definition 9 (Volume of a Lattice). The volume of the parallelotope \mathcal{P}_e associated with any base e of H is called the volume of the lattice H and is denoted by $\text{vol}(H)$.

Theorem 6 (Minkowski). Let H be a lattice in \mathbb{R}^n and let S be a measurable subset of \mathbb{R}^n such that $\mu(S) > \text{vol}(H)$. Then there exist two distinct points $x, y \in S$ such that $x - y \in H$.

Proof. Consider the sets $S_x = S \cap (x + \mathcal{P}_e)$, where $x \in H$. Notice that these sets form a partition of S , i.e. they are pairwise disjoint and :

$$S = \cup_{x \in H} S_x.$$

In particular we have :

$$\text{vol}(S) = \sum_{x \in H} \text{vol}(S_x).$$

Notice that the translated sets $S_x - x = (S - x) \cap \mathcal{P}_e$ are all contained in \mathcal{P}_e . We want to prove that the S_x cannot be all mutually disjoint. Since $\text{vol}(S_x) = \text{vol}(S_x - x)$, we have :

$$\text{vol}(H) < \text{vol}(S) = \sum_{x \in H} \text{vol}(S_x) = \sum_{x \in H} \text{vol}(S_x - x).$$

The facts that $S_x - x \subseteq \mathcal{P}_e$ and $\sum_{x \in H} \text{vol}(S_x - x) > \text{vol}(H)$ imply that these sets cannot be disjoint, i.e. there exist two distinct vectors $x \neq y \in H$ such that $(S_x - x) \cap (S_y - y) \neq \emptyset$. Let z be any vector in the (non-empty) intersection $(S_x - x) \cap (S_y - y)$ and define :

$$\begin{aligned} z_1 &= z + x \in S_x \subseteq S \\ z_2 &= z + y \in S_y \subseteq S. \end{aligned}$$

These two vectors satisfy $z_1 - z_2 = x - y \in H$. \square

Theorem 7 (Minkowski's convex body theorem). Let H be a full-dimensional lattice in \mathbb{R}^n and let $C \subseteq \mathbb{R}^n$ be a convex set symmetric about the origin (i.e. $x \in C \Rightarrow -x \in C$). Suppose that either :

1. $\text{vol}(C) > 2^n \cdot \text{vol}(H)$, or

2. $\text{vol}(C) \geq \cdot 2^n \cdot \text{vol}(H)$ and C is compact.

Then $\mathbb{C} \cap (H \setminus \{0\}) \neq \emptyset$.

Proof. It is easy to see that the volume of the set $\frac{1}{2}C = \{x/2 : x \in C\}$ is $2^{-m} \text{vol}(C)$, and therefore, we can apply previous theorem to find $\frac{1}{2}x_0, \frac{1}{2}x_1 \in \frac{1}{2}C$ such that $z = \frac{1}{2}x_1 - \frac{1}{2}x_0 \in H$. Clearly $z = \frac{1}{2}x_1 + \frac{1}{2}(-x_0) \in C$, since C is convex and symmetric. \square

2.2 The canonical imbedding of a number field

Definition 10 (Canonical imbedding of a number field). Let K be a number field and let n be its degree. There are n distinct isomorphisms $\sigma_i : K \rightarrow \mathbb{C}$. There are exactly n , because the minimal polynomial for a primitive element of K over \mathbb{Q} has only n roots in \mathbb{C} . Let $\alpha : \mathbb{C} \rightarrow \mathbb{C}$ be complex conjugation. Then, for any $i = 1, \dots, n$, we have $\alpha \sigma_i = \sigma_j$ if and only if $\sigma_i(K) \subseteq \mathbb{R}$. We write r_1 for the number of indices such that $\sigma_i(K) \subseteq \mathbb{R}$. Then $n - r_1$ is an even number, so we may write :

$$r_1 + 2r_2 = n$$

Let us renumber the σ_i 's so that $\sigma_i(K) \subseteq \mathbb{R}$ for $1 \leq i \leq r_1$ and so that $\sigma_{i+r_2}(x) = \overline{\sigma_j(x)}$ for $r_1+1 \leq j \leq r_1+r_2$. Then the first r_1+r_2 isomorphisms determine the last r_2 . For $x \in K$, we define :

$$\sigma(x) = (\sigma_1(x), \dots, \sigma_{r_1+r_2}(x)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

We call σ the canonical imbedding of K in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$; it is an injective ring homomorphism. We shall frequently identify $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ with \mathbb{R}^n .

Proposition 8. If M is a free \mathbb{Z} -submodule of K of rank n and if $(x_i)_{1 \leq i \leq n}$ is a \mathbb{Z} -base for M then $\sigma(M)$ is a lattice in \mathbb{R}^n , whose volume is :

$$\text{vol}(\sigma(M)) = 2^{-r_2} \left| \det_{1 \leq i, j \leq n} (\sigma_i(x_j)) \right|.$$

Proof. For fixed i the coordinates of $\sigma(x_i)$ with respect to the canonical base of \mathbb{R}^n are :

$$\langle \sigma_1(x_i), \dots, \sigma_{r_1}(x_i), \Re(\sigma_{r_1+1}(x_i)), \Im(\sigma_{r_1+1}(x_i)), \dots, \Re(\sigma_{r_1+r_2}(x_i)), \Im(\sigma_{r_1+r_2}(x_i)) \rangle$$

We calculate the determinant D of the matrix whose i th column is given as above. We know that $\Re(z) = \frac{1}{2}(z + \bar{z})$ and $\Im(z) = \frac{1}{2i}(z - \bar{z})$ for $z \in \mathbb{C}$. We obtain $D = (2i)^{-r_2} \det(\sigma_j(x_i))$. We apply the transformation $R_i \mapsto iR_{i+1}$ for $i = r_1, r_1+2, \dots, r_1+2r_2$. So we end up with the determinant $D = (2i)^{-r_2} \det_{1 \leq i, j \leq n} (\sigma_j(x_i))$. Since x_i 's form a base for K over \mathbb{Q} , $\det_{1 \leq i, j \leq n} (\sigma_j(x_i)) \neq 0$ and therefore $D \neq 0$. Thus the vectors $\sigma(x_i)$ are linearly independent in \mathbb{R}^n , so that the \mathbb{Z} -module they generate (call it $\sigma(M)$) is a lattice in \mathbb{R}^n . So we get

$$\text{vol}(\sigma(M)) = |(2i)^{-r_2} \det(\sigma_j(x_i))| = 2^{-r_2} |\det(\sigma_j(x_i))|$$

as required. \square

Proposition 9. Let d be the absolute discriminant of K , let A be the ring of integers in K , and let \mathfrak{a} be a non-zero integral ideal of A . Then $\sigma(A)$ and $\sigma(\mathfrak{a})$ are lattices. Moreover,

$$\text{vol}(\sigma(A)) = 2^{-r_2} |d|^{\frac{1}{2}} \quad \text{and} \quad \text{vol}(\sigma(\mathfrak{a})) = 2^{-r_2} |d|^{\frac{1}{2}} N(\mathfrak{a}).$$

Proof. We know that A and \mathfrak{a} are free \mathbb{Z} -modules of rank n , so we may apply the previous proposition. On the other hand, if (x_i) is a \mathbb{Z} -base for A , then $d = \det_{1 \leq i, j \leq n} (\sigma_i(x_j))^2$. This proves the first result. The second formula follows from the first and the observation that $\sigma(\mathfrak{a})$ is a subgroup of $\sigma(A)$ of index $N(\mathfrak{a})$. A fundamental domain for $\sigma(\mathfrak{a})$ may obviously be constructed as the disjoint union of $N(\mathfrak{a})$ copies of a fundamental domain for $\sigma(A)$. \square

2.3 Finiteness of the ideal class group

Proposition 10. Let K be a number field, n its degree, r_1 and r_2 are integers defined earlier, d the absolute discriminant of K , and \mathfrak{a} a non-zero integral ideal of K . Then \mathfrak{a} contains a non-zero element x such that :

$$|N_{K/\mathbb{Q}}(x)| \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} |d|^{\frac{1}{2}} N(\mathfrak{a}).$$