On fair cooperation among competing fog providers

APPENDIX

Lemma 1. *For any internal point* $\mathbf{p} \in [0,1]^2$*, a Pareto-improvement* $\mathbf{p}^*$ *of* $\mathbf{b}$*, i.e., such that* $b_1(\mathbf{p}^*) < b_1(\mathbf{p})$ *and* $b_2(\mathbf{p}^*) < b_2(\mathbf{p})$ *exists along directions* $\hat{n} = (n_1, n_2)$*, such that* $n_1, n_2 > 0$.

Proof. Step 1. Analytical expression of b_i. Let $\pi = [\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}]$. With this notation the infinitesimal generator matrix of the chain for $K = 1$:

$$\mathbf{Q} = \begin{bmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 1 & -1 - p_2\lambda_1 - \lambda_2 & 0 & \lambda_2 + p_2\lambda_1 \\ 1 & 0 & -1 - p_1\lambda_2 - \lambda_1 & \lambda_1 + p_1\lambda_2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

The steady state probability vector is the solution of a system of four linear equations in four unknowns, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = \pi^T$, $\mathbf{A} = \mathbf{Q}^T$, excepts the last row replaced with all all 1s:

$$\begin{bmatrix} -\lambda_1 - \lambda_2 & 1 & 1 & 0 \\ \lambda_1 & -1 - p_2\lambda_1 - \lambda_2 & 0 & 1 \\ \lambda_2 & 0 & -1 - p_1\lambda_2 - \lambda_1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_{00} \\ \pi_{01} \\ \pi_{10} \\ \pi_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We get:

$$b_1 = \pi_{11} + \pi_{01}(1 - p_2) \quad b_2 = \pi_{11} + \pi_{10}(1 - p_1)$$

that means, a job from node one is blocked if the both nodes are busy, or node one is busy and node two is idle but it doesn't accept to serve the job. Due to the symmetry of the expressions, without loss of generality we will focus on b_1. Using the Kramer rule:

$$b_1 = \frac{det(Q_{11}) + (1 - p_2)det(Q_{01})}{det(Q)}$$

In particular, after some manipulation, we get the following expression:

$$\begin{aligned} b_1(p_1, p_2) &= \frac{(\kappa_1 + \beta_1 p_2) + (\alpha_1 + \gamma_1 p_2)p_1}{(\kappa + \beta p_2) + (\alpha + \gamma p_2)p_1} \\ &= \frac{(\kappa_1 + \alpha_1 p_1) + (\beta_1 + \gamma_1 p_1)p_2}{(\kappa + \alpha p_1) + (\beta + \gamma p_1)p_2} \end{aligned}$$

where:

$$\begin{aligned} \kappa &= (1 + \lambda_1)(1 + \lambda_2)(2 + \lambda_1 + \lambda_2) \\ \alpha &= \lambda_2(1 + \lambda_2)(1 + \lambda_1 + \lambda_2) \\ \beta &= \lambda_1(1 + \lambda_1)(1 + \lambda_1 + \lambda_2) \\ \gamma &= \lambda_1\lambda_2(\lambda_1 + \lambda_2) \\ \alpha_1 &= \lambda_2(\lambda_1 + 2\lambda_2 + (1 + \lambda_1)\lambda_2^2) \\ \kappa_1 &= \lambda_1(1 + \lambda_2)(2 + \lambda_1 + \lambda_2) \\ \beta_1 &= \lambda_1^3 + \lambda_1^2\lambda_2 + \lambda_2^2 - \lambda_1(2 + \lambda_2) \\ \gamma_1 &= (\lambda_1 - 1)\lambda_2(\lambda_1 + \lambda_2) \end{aligned}$$

Note that the symbols at the denominators have not indices, i.e., the denominator is the same for both b_1, b_2.
Step 2. Compute the derivatives of $\mathbf{b}$.

$$\frac{\partial b_1}{\partial p_1} = \frac{(\alpha_1 + \gamma_1 p_2)(\kappa + \beta p_2) - (\alpha + \gamma p_2)(\kappa_1 + \beta_1 p_2)}{(\kappa + \alpha p_1 + \beta p_2 + \gamma p_1 p_2)^2} > 0$$

because the denominator is positive as it is a square, and the is also positive since after some manipulation it can be written as:

$$\lambda_2(2\lambda_2 + ...)(2 + ... - p_2 - \lambda_2 p_2) > 0$$

similarly

$$\frac{\partial b_1}{\partial p_2} = \frac{(\beta_1 + \gamma_1 p_1)(\kappa + \alpha p_1) - (\beta + \gamma p_1)(\kappa_1 + \alpha_1 p_1)}{(\kappa + \alpha p_1 + \beta p_2 + \gamma p_1 p_2)^2}$$

since the numerator is:

$$-(2\lambda_1 + \lambda_1^2 + ..)(2 + 3\lambda_1 + ..) < 0$$

Figure 1 shows the gradient field of b_1 for $\lambda_1 = 0.9, \lambda_2 = 0.6$. Note that the above two derivatives have the expected intuitive

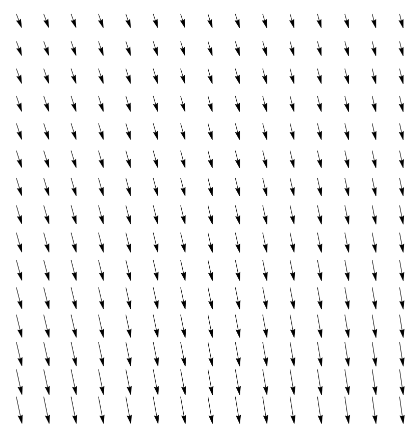

Figure 1: An example of gradient field of $b_1(0.9, 0.6, p_1, p_2)$ revealing the directions where the blocking probably *increases*

signs: the blocking probability b_1 increases when node 1 increases its cooperation probability, p_1 as this means to serve, beside its own tasks, more often tasks coming from the node 2. Similarly, this probability decreases if node 2 cooperates more often, by increases p_2.
Step 3. Find the directions along which both b_i *decrease.* The direction $\hat{n} = (n_1, n_2)$ along which both components b_i *decrease*, points towards the positive- positive quadrant, $n_1, n_2 > 0$. In fact, the two gradients $\nabla b_1 = (\frac{\partial b_1}{\partial p_1}, \frac{\partial b_1}{\partial p_2}), \nabla b_2 = (\frac{\partial b_2}{\partial p_1}, \frac{\partial b_2}{\partial p_2})$ are not orthogonal since $\nabla b_1 \cdot \nabla b_2 < 0$, see Figure 2.

Step 4. The only point of the domain $D = [0,1]^2$ *where* b *cannot Pareto improved are the border of* D. In fact, from the previous step, b_i computed at any internal point can be improved along $\hat{n}$. On the contrary, for a point $\in D$ with a $p_i = 1$ the admissible directions $\hat{n}$ must have $n_i = 0$. However, b_i increases along this direction since $\nabla b_i \cdot \hat{n} = \frac{\partial b_i}{\partial p_i} > 0$. This

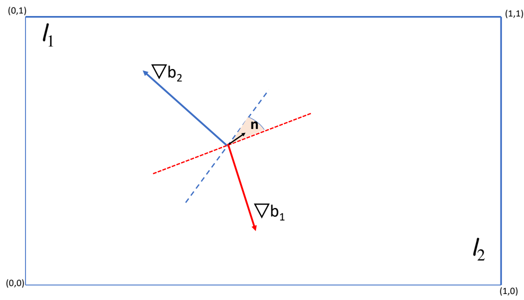

Figure 2: A sketch of the position of the two gradients.

are the only points of D for which a Pareto improvement doesn't exit. This conclude the proof. $\square$

Lemma 2. *Let $A = (1,0), B = (1,1), C = (0,1)$. The Pareto frontier of the problem $\mathcal{P}_{21}$ is $\partial P \subset \overline{AB} \cup \overline{BC}$.*

Proof. Step1. From Lemma 1, the only blocking probabilities that cannot be Pareto improved *and* may be less than those without cooperation, are the ones computed at points in $\overline{AB} \cup \overline{BC}$.

Step 2 [sketch]. For any pair of loads λ_1, λ_2, there is a nonempty subset of points $S \subset \overline{AB} \cup \overline{BC}$ such that the blocking probabilities evaluated for these points are lower than the probabilities under no cooperation. First of all, it is not possible that $b_1(1,1) > b_1(0,0)$ *and* $b_2(1,1) > b_2(0,0)$. In fact, for $p_1 = p_2 = 1$ we have:

$$b_1(1,1) = b_2(1,1) =$$

$$\frac{(\lambda_1+\lambda_2)^2}{2+2(\lambda_1+\lambda_2)+(\lambda_1+\lambda_2)^2} > \frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}$$

which is unfeasible [1]. There are then two cases:

(a) $b_1(1,1) \leq b_1(0,0), b_2(1,1) \leq b_2(0,0)$: the point $(1,1) \in S$

(b) $b_1(1,1) > b_1(0,0), b_2(1,1) \leq b_2(0,0)$

(c) $b_2(1,1) > b_2(0,0), b_1(1,1) \leq b_1(0,0)$

As the solutions are symmetric (b) and (c) are indeed the same case. Let's focus on case (b). Since b_1 increases if p_2 decreases, i.e., $b_1(1,p_2) > b_1(1,1) > b_1(0,0)$, the points in $\overline{BC}$ can't be a solution of the problem $\mathcal{P}_{21}$. Consider now the segment $\overline{AB}$. As we move from point B to A, b_1 decreases while b_2 increases. We are also sure that there is a value p_1^* such that $b_1(p_1^*,1) = b_1(0,0)$ because $b_1(0,1) < b_1(0,0)$ and a value p_1^{**} such that $b_2(p_1^{**},1) = b_2(0,0)$, because $b_2(0,1) > b_2(0,0)$. We can show that $p_1^* > p_1^{**}$, i.e., b_1 starts to be lower than $b_1(0,0)$ before b_2 becomes to be worst than $b_2(0,0)$ by solving

$$b_1(\lambda_1,\lambda_2,p_1^*,1) = \frac{\lambda_1}{1+\lambda_1} \quad b_2(\lambda_1,\lambda_2,1,p_1^{**}) = \frac{\lambda_2}{1+\lambda_2}$$

[1]This can be tested with any symbolic solver.

$\square$

Theorem 1. *A Pareto-optimal solution $\mathbf{p}$ of the problem $\mathcal{P}_{21}$ has at least one component $p_i = 1$.*

Proof. For any pair of loads λ_1, λ_2, lemma 2 applies. In fact, for $p_1 = p_2 = 1$, our model is equivalent to an $M/M/2/2$ system queue with a flow $\lambda_T = \lambda_1 + \lambda_2$ (this can be seen by grouping states $(1,0)$ and $(0,1)$ into a single state). Hence:

$$b_1(1,1) = \frac{\frac{\lambda_T^2}{2}}{\frac{\lambda_T^2}{2}+\lambda_T+1} = \frac{(\lambda_1+\lambda_1)^2}{2+2(\lambda_1+\lambda_1)+(\lambda_1+\lambda_1)^2}$$

Case (a) of the Lemma holds if

$$b_1(1,1) \leq \frac{\lambda_1}{1+\lambda_1}, b_1(1,1) \leq \frac{\lambda_2}{1+\lambda_2}$$

that is

$$\sqrt{\lambda_1^2+1}-1 \leq \lambda_2 \leq \sqrt{\lambda_1^2+2\lambda_1}$$

Otherwise the case (b) or (c) hold. $\square$