$$
\begin{aligned}
\sum_{k=0}^{+\infty} \frac{k}{3^k} &= \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{k}{3^k} \\
&= \lim_{n \to +\infty} \left(\frac{0}{3^0} + \sum_{k=1}^{n} \frac{k}{3^k} \right) \\
&= \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{k}{3^k} \\
&\geq \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{3^k} \\
&= \lim_{n \to +\infty} \left(-1 + \frac{1}{3^0} + \sum_{k=1}^{n} \frac{1}{3^k} \right) \\
&= \lim_{n \to +\infty} \left(-1 + \sum_{k=0}^{n} \frac{1}{3^k} \right) \\
&= \lim_{n \to +\infty} \left(-1 + \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} \right) \\
&= \lim_{n \to +\infty} \left(-1 + \frac{3}{2} \left(1 - \left(\frac{1}{3} \right)^{n+1} \right) \right) \\
&= \lim_{n \to +\infty} \left(-1 + \frac{3}{2} - \frac{3}{2} \left(\frac{1}{3} \right)^{n+1} \right) \\
&= \lim_{n \to +\infty} \left(\frac{1}{2} - \frac{1}{2} \left(\frac{1}{3} \right)^{n} \right) \\
&= \frac{1}{2} \lim_{n \to +\infty} \left(1 - \left(\frac{1}{3} \right)^{n} \right) \\
&= \frac{1}{2} (1 - 0) \\
&= \frac{1}{2}
\end{aligned}
$$

$$\begin{aligned}\sum_{k=0}^{n}\frac{k}{3^k} &= \sum_{k=1}^{n}\frac{k}{3^k}\\ &= \sum_{k=1}^{n}\frac{k-1+1}{3^k}\\ &= \sum_{k=1}^{n}\frac{k-1}{3^k}+\sum_{k=1}^{n}\frac{1}{3^k}\end{aligned}$$

Ma:

$$\begin{aligned}\sum_{k=1}^{n}\frac{k-1}{3^k} &= \sum_{k=0}^{n-1}\frac{k}{3^{k+1}}\\ &= \frac{1}{3}\sum_{k=0}^{n-1}\frac{k}{3^k}\\ &= \frac{1}{3}\left(\sum_{k=0}^{n}\frac{k}{3^k}-\frac{n}{3^n}\right)\\ &= \frac{1}{3}\sum_{k=0}^{n}\frac{k}{3^k}-\frac{n}{3^{n+1}}\end{aligned}$$

Allora:

$$\begin{aligned}\sum_{k=0}^{n}\frac{k}{3^k} &= \sum_{k=1}^{n}\frac{k-1}{3^k}+\sum_{k=1}^{n}\frac{1}{3^k}\\ &= \frac{1}{3}\sum_{k=0}^{n}\frac{k}{3^k}-\frac{n}{3^{n+1}}+\sum_{k=1}^{n}\frac{1}{3^k}\end{aligned}$$

Ovvero:

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{k}{3^k} &= \frac{1}{3}\sum_{k=0}^{n} \frac{k}{3^k} - \frac{n}{3^{n+1}} + \sum_{k=1}^{n} \frac{1}{3^k} \\
-\frac{1}{3}\sum_{k=0}^{n} \frac{k}{3^k} + \sum_{k=0}^{n} \frac{k}{3^k} &= \sum_{k=1}^{n} \frac{1}{3^k} - \frac{n}{3^{n+1}} \\
\frac{2}{3}\sum_{k=0}^{n} \frac{k}{3^k} &= \sum_{k=1}^{n} \frac{1}{3^k} - \frac{n}{3^{n+1}} \\
\sum_{k=0}^{n} \frac{k}{3^k} &= \frac{3}{2}\sum_{k=1}^{n} \frac{1}{3^k} - \frac{n}{2 \cdot 3^n} \\
&= \frac{3}{2}\left(\sum_{k=0}^{n} \frac{1}{3^k} - 1\right) - \frac{n}{2 \cdot 3^n} \\
&= \frac{3}{2}\sum_{k=0}^{n} \frac{1}{3^k} - \frac{3}{2} - \frac{n}{2 \cdot 3^n} \\
&= \frac{3}{2} \cdot \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} - \frac{3}{2} - \frac{n}{2 \cdot 3^n} \\
&= \frac{9}{4}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right) - \frac{3}{2} - \frac{n}{2 \cdot 3^n} \\
&= \frac{9}{4} - \frac{3}{2} - \frac{9}{4 \cdot 3^{n+1}} - \frac{n}{2 \cdot 3^n} \\
&= \frac{9-6}{4} - \frac{3}{4 \cdot 3^n} - \frac{n}{2 \cdot 3^n} \\
&= \frac{3}{4} - \frac{2n+3}{4 \cdot 3^n}
\end{aligned}
$$

Passando al limite:

$$
\lim_{n \to +\infty} \sum_{k=0}^{n} \frac{k}{3^k} = \lim_{n \to +\infty} \left(\frac{3}{4} - \frac{2n+3}{4 \cdot 3^n}\right) = \frac{3}{4} - 0 = \frac{3}{4}
$$

Fissiamo $\alpha \in \mathbb{R} \setminus \{1\}$.

$$\begin{aligned}
S &= \sum_{k=0}^{n} \alpha^k \\
\alpha S &= \sum_{k=0}^{n} \alpha^{k+1} \\
&= \sum_{k=1}^{n+1} \alpha^k \\
&= \sum_{k=1}^{n} \alpha^k + \alpha^{n+1} \\
&= \sum_{k=0}^{n} \alpha^k + \alpha^{n+1} - \alpha^0 \\
&= S + \alpha^{n+1} - \alpha^0 \\
\alpha S - S &= \alpha^{n+1} - 1 \\
(\alpha - 1)\, S &= \alpha^{n+1} - 1 \\
S &= \frac{\alpha^{n+1} - 1}{\alpha - 1} \\
S &= \frac{1 - \alpha^{n+1}}{1 - \alpha} \\
\sum_{k=0}^{n} \alpha^k &= \frac{1 - \alpha^{n+1}}{1 - \alpha}
\end{aligned}$$