Esercizi tutorato in preparazione del primo parziale

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1

1.1

$$\pi: \quad x + 3y - z = 4 \qquad P = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$r: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\pi_1 \perp \pi \quad \wedge \quad \pi_1 \parallel s_1$$

$$s_1: \begin{cases} x + 2y = 2 \\ y - z = 2 \end{cases} \qquad P = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$n_{\pi_1} = v_1 \times n_\pi$$

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$n_{\pi_1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -7 \end{bmatrix}$$

$$\pi_1: \quad 4x + y + 7z = 11$$

Punto c: s_2 incidente a $\pi : x + 3y - z = 4$, passante per $P = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$r: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + tV \quad \text{tale che} \quad \langle V, n \rangle \neq 0$$

2

2.1

$$r: \begin{cases} x = 0 \\ y + z = 5 \end{cases} \qquad P = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\pi : 4x + y + z = 5$$

$$r_1 : \begin{cases} x - y = 0 \\ x + y - z = 2 \end{cases}$$

Mutua posizione fra r ed r_1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

3

3.1

$$\begin{bmatrix} \alpha^2 - 4 & \alpha^2 + 1 & \alpha + 1 \\ -5 & 0 & -\alpha + 5 \\ -\alpha - 1 & -3 & -\alpha^2 + 4 \end{bmatrix}$$

$$\begin{cases} \alpha^2 - 4 = 0 \\ \alpha^2 + 1 = -(-5) \\ \alpha + 1 = -(-\alpha - 1) \\ -\alpha + 5 = -(-3) \end{cases}$$

$$\begin{cases} \alpha = \pm 2 \\ \alpha = \pm 2 \\ \alpha = 2 \end{cases}$$

$$\alpha = 2$$

3.2

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R}) \qquad B = \begin{bmatrix} -\alpha & 2 \\ \alpha & -\alpha \\ \alpha^2 & -\alpha \end{bmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{R})$$

$$\mathrm{Tr}(AB) = \mathrm{Tr} \begin{bmatrix} -\alpha + \alpha^2 & 2 - \alpha \\ \alpha^2 & 2 - 2\alpha \end{bmatrix} = \alpha^2 - 3\alpha + 2$$

$$\alpha^2 - 3\alpha + 2 = 0 \qquad \alpha \in 1; 2$$

$$\mathrm{Tr}(AB) = 0 \quad \Leftrightarrow \quad \alpha \in 1; 2$$

3.3

$$D = \begin{bmatrix} i+2 & \beta & i \\ 0 & 1 & \beta \\ i+2 & i-2 & i \end{bmatrix}$$

$$\begin{aligned}
\det D &= \det \begin{bmatrix} i+2 & \beta & i \\ 0 & 1 & \beta \\ 0 & i-2-\beta & 0 \end{bmatrix} \\
&= (i+2) \det \begin{bmatrix} 1 & \beta \\ i-2-\beta & 0 \end{bmatrix} \\
&= (i+2)\beta(\beta+2-i) \\
\det D \neq 0 \quad &\Leftrightarrow \quad i \notin \{0, -2+i\}
\end{aligned}$$

4

4.1

Sia $\beta \in \mathbb{R}$. Determinare se il sistema è compatibile:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \beta+1 \\ 1 & 0 & 1 & 2\beta \\ 2 & 1 & 1 & \beta-3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & \beta+1 \\ 0 & -1 & 1 & \beta-1 \\ 0 & -1 & 1 & -\beta-5 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & \beta+1 \\ 0 & -1 & 1 & \beta-1 \\ 0 & 0 & 0 & -2\beta-4 \end{array}\right]$$

Il sistema è compatibile se e solo se $-2\beta - 4 = 0$, ovvero $\beta = -2$.

4.2

Determinare il rango della matrice al variare di $\alpha \in \mathbb{R}$:

$$A = \begin{bmatrix} 1 & -1 & 1 & \alpha^2-\alpha-1 \\ 0 & 1 & \alpha & \alpha^2+2\alpha+1 \\ 1 & 2 & 1 & \alpha^2+2\alpha+2 \\ -1 & 1 & \alpha-1 & \alpha^2+3\alpha+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & \alpha^2-\alpha-1 \\ 0 & 1 & \alpha & \alpha^2+2\alpha+1 \\ 0 & 3 & 0 & 3\alpha+3 \\ 0 & 0 & \alpha & 2\alpha^2+2\alpha \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & \alpha^2-\alpha-1 \\ 0 & 1 & \alpha & \alpha^2+2\alpha+1 \\ 0 & 0 & -3\alpha & -3\alpha^2-3\alpha \\ 0 & 0 & \alpha & 2\alpha^2+2\alpha \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & \alpha^2-\alpha-1 \\ 0 & 1 & \alpha & \alpha^2+2\alpha+1 \\ 0 & 0 & \alpha & \alpha^2+\alpha \\ 0 & 0 & 0 & \alpha^2+\alpha \end{bmatrix}$$

$$\text{rg}\, A = \begin{cases} 2 & \alpha = 0 \\ 3 & \alpha = -1 \\ 4 & \text{altrimenti} \end{cases}$$

5 Altri esercizi

5.1

Determinare la retta s_2 tale che:

$$s_2 \parallel \pi_4 \wedge P \in s_2$$

con:

$$s_2 \parallel \pi_4 \qquad \pi_4 : x - y = 3 \qquad r : \begin{cases} y = 0 \\ y + z = 4 \end{cases} \qquad P = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Allora, la soluzione sarà:

$$s_2 : \begin{cases} 2y + z = 4 \\ x - y = 1 \end{cases}$$

5.2

17.3

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad r : \begin{cases} 3x - 2z = 3 \\ 3y - z = 1 \end{cases}$$

$$V_{12} = P_1 - P_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$n = V \times V_{12} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\pi : 3x - 2z = 1$$

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$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & t+1 & 2 \\ 0 & 3 & t & 3+t \\ 1 & 1 & t & 1+t \\ 1 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & t+1 & 2 \\ 0 & 1 & t1 & 1 \\ 0 & 3 & t & 3+t \\ 0 & -1 & -1 & t-1 \\ 0 & -2 & -t & -1 \end{bmatrix}$$