Multiscale methods: non-intrusive implementation, advection-dominated problems and related topics

PhD Defense

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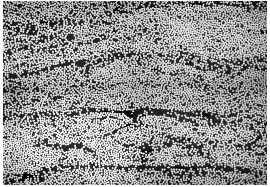
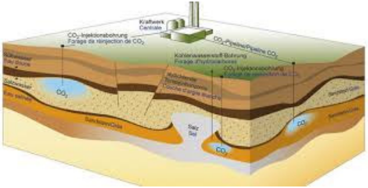
CERMICS, Champs-sur-Marne, France

Multiscale problems

Aircrafts are made of composite materials



Concrete is a multiscale material



Reservoir modelling involves subsurface flow

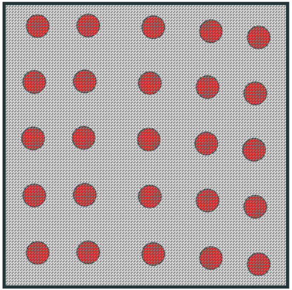
Airflow though a dense city, coolant in nuclear reactors, etc.

Standard finite element methods

We solve

$$\begin{cases} -\operatorname{div}(A^{\varepsilon}\nabla u^{\varepsilon}) = 500 & \text{in } \Omega, \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

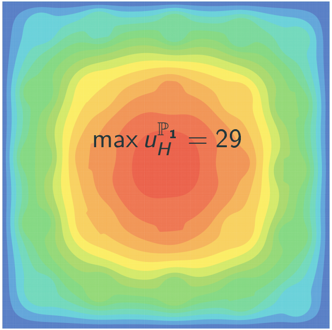
- ○ $A^{\varepsilon} = 1$
- ● $A^{\varepsilon} = 30$



$\mathbb{P}_1$ Lagrange finite element method

Approximation $u_H^{\mathbb{P}_1}$ with 10^6 degrees of freedom

mesh size $\ll$ microstructure



Standard finite element methods

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$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = 500 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

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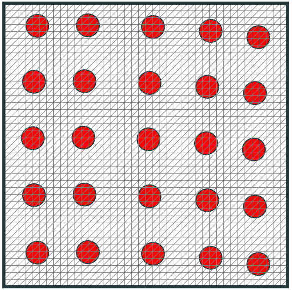
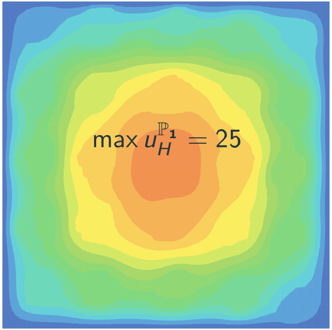


$\mathbb{P}_1$ Lagrange finite element method

Approximation $u_H^{\mathbb{P}_1}$ with 1521 degrees of freedom

mesh size $\sim$ microstructure



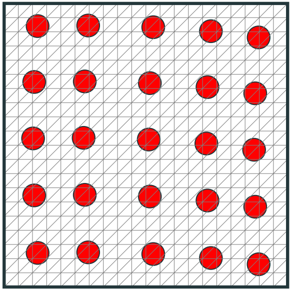
Standard finite element methods

We solve

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = 500 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

○ $A^\varepsilon = 1$

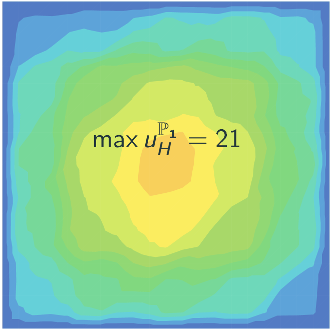
● $A^\varepsilon = 30$



<u>$\mathbb{P}_1$ Lagrange finite element method</u>

Approximation $u_H^{\mathbb{P}_1}$ with 361 degrees of freedom

mesh size $\geq$ microstructure



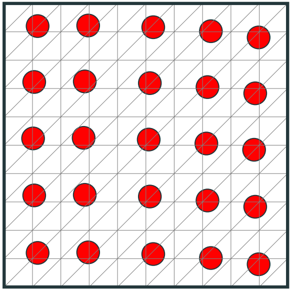
Standard finite element methods

We solve

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = 500 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

○ $A^\varepsilon = 1$

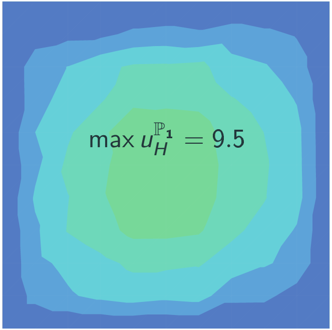
● $A^\varepsilon = 30$



$\mathbb{P}_1$ Lagrange finite element method

Approximation $u_H^{\mathbb{P}_1}$ with 81 degrees of freedom

mesh size $\gg$ microstructure



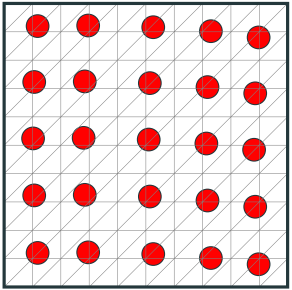
Standard finite element methods

We solve

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = 500 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

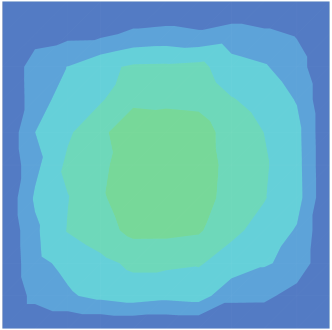
○ $A^\varepsilon = 1$

● $A^\varepsilon = 30$

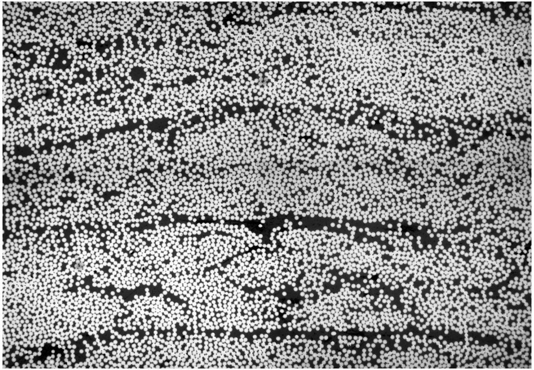


$\mathbb{P}_1$ Lagrange finite element method

Macroscopic properties are lost if the microstructure is not resolved.



For a real multiscale material ...



...far too many degrees of freedom are needed.

Numerical homogenization

Goal: recover the global behaviour of the solution to multiscale elliptic-like PDE on a coarse mesh.

1. Offline stage: resolve the microstructure locally

2. Online stage: one coarse global problem (cheap)

Advantageous in a multi-query context with a fixed microstructure (optimization, inverse problems, uncertainty quantification, etc.), or e.g. when a solution is required in real-time.

Examples:

Heterogeneous Multiscale Method (E and Engquist 2003)
(Super-)Localized Orthogonal Decomposition (Målqvist and Peterseim 2014; Hauck and Peterseim 2023)
Multiscale Finite Element Method (MsFEM) (Hou and Wu 1997)

The multiscale finite element method

Some notation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain.
$u^\varepsilon \in H_0^1(\Omega)$ solving

$$-\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega,$$

for some $f \in L^2(\Omega)$, is also the unique solution to the variational formulation

$$a^\varepsilon(u^\varepsilon, v) = \int_\Omega f\, v \quad \text{for all } v \in H_0^1(\Omega),$$

with

$$\text{for all } u, v \in H^1(\Omega), \quad a^\varepsilon(u, v) = \int_\Omega \nabla v \cdot A^\varepsilon \nabla u.$$

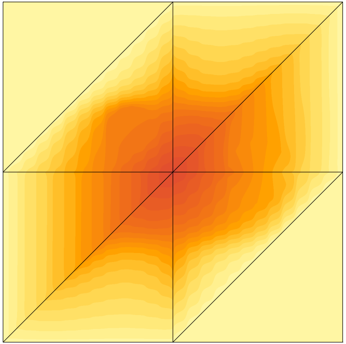
The multiscale finite element method

Adapt the approximation space to the differential operator. For MsFEM-lin:

1. Offline stage: resolve the microstructure locally

Multiscale basis functions:

$$\forall\, K \in \mathscr{T}_H, \quad \begin{cases} -\operatorname{div}(A^\varepsilon \nabla \phi_i^\varepsilon) = 0 & \text{in } K, \\ \phi_i^\varepsilon = \phi_i^{\mathbb{P}_1} & \text{on } \partial K. \end{cases}$$



2. Online stage: one coarse global problem (cheap)

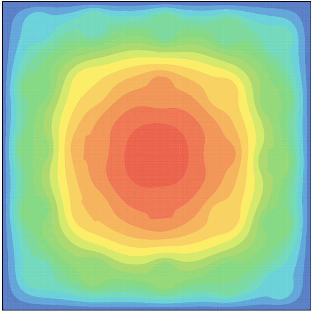
Find $u_H^\varepsilon \in V_H^\varepsilon = \operatorname{span}\{\phi_i^\varepsilon\}$ such that

$$a^\varepsilon(u_H^\varepsilon, v_H^\varepsilon) = \int_\Omega f\, v_H^\varepsilon \quad \text{for all } v_H^\varepsilon \in V_H^\varepsilon.$$

Standard FEM vs MsFEM

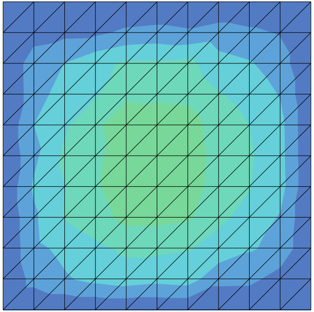
We recall the example shown before:

Fine-scale solution



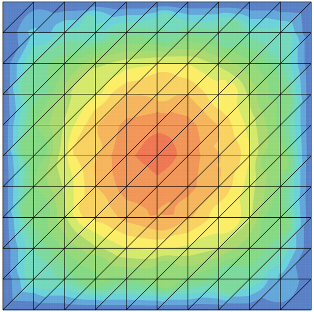
$\max u^{\varepsilon} = 29$

$\mathbb{P}_1$ approximation on coarse mesh



$\max u_H^{\mathbb{P}_1} = 9.5$

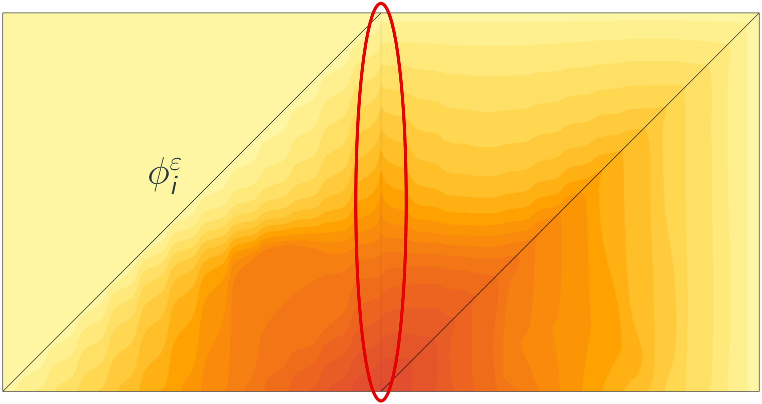
MsFEM approximation
(coarse mesh; with OS)



$\max u_H^{\varepsilon} = 27$

Local boundary conditions

ϕ_i^ε is affine on ∂K, while u^ε is highly oscillatory...



Some MsFEM developments

Basis functions that oscillate on local boundaries: oversampling (Hou and Wu 1997) and MsFEM à la Crouzeix-Raviart (Le Bris et al. 2013).

Crouzeix-Raviart MsFEM with bubble functions for the case of perforated domains (Le Bris et al. 2014). Extension to Stokes flow in (Muljadi et al. 2015), with an enriched approximation space in (Feng et al. 2022).

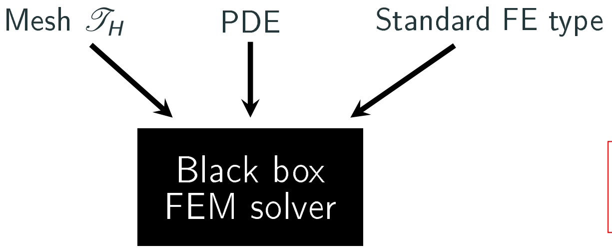
Enrichment of the multiscale space through spectral problems: Generalized MsFEM (Efendiev et al. 2013; Calo et al. 2016), also (Hetmaniuk and Lehoucq 2010), (Hou and Liu 2016).

Polynomial enrichment (Legoll et al. 2022), $\mathbb{P}_k$ MsFEM (Allaire and Brizzi 2005; Hesthaven 2014), multiscale hybrid high-order (Cicuttin et al. 2019).

Combination with localized model order reduction in (Diercks et al. 2023).

Various applications to flow problems and reservoir modeling, nonlinear MsFEM (Efendiev and Hou 2009).

Non-intrusive implementation



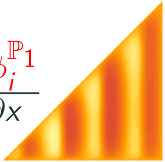
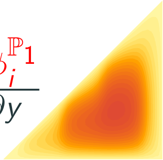
$\boxed{\phi_i^\varepsilon\ ??}$

New solver? $\rightarrow u_H^\varepsilon = \sum_i U_i^\varepsilon \phi_i^\varepsilon$ where $\mathbb{A}^\varepsilon U^\varepsilon = \mathbb{F}^\varepsilon$,

$$\mathbb{A}_{j,i}^\varepsilon = a^\varepsilon\left(\phi_i^\varepsilon, \phi_j^\varepsilon\right) = \sum_{K \in \mathscr{T}_H} \int_K \nabla \phi_j^\varepsilon \cdot A^\varepsilon \nabla \phi_i^\varepsilon, \quad \mathbb{F}_j^\varepsilon = \sum_{K \in \mathscr{T}_H} \int_K f\, \phi_j^\varepsilon.$$

Building $\mathbb{A}^\varepsilon$ and $\mathbb{F}^\varepsilon$ efficiently is not trivial.

Decoupling the microscale from the macroscopic scale

 $=$  $+\ \dfrac{\partial \phi_i^{\mathbb{P}1}}{\partial x}$  $+\ \dfrac{\partial \phi_i^{\mathbb{P}1}}{\partial y}$ 

ϕ_i^{ε} $\qquad$ $\phi_i^{\mathbb{P}1}$

Decoupling the microscale from the macroscopic scale

$$\forall K \in \mathscr{T}_H, \quad \begin{cases} -\operatorname{div}\left(A^\varepsilon \nabla\left(\phi_i^\varepsilon - \phi_i^{\mathbb{P}_1}\right)\right) = \operatorname{div}\left(A^\varepsilon \nabla \phi_i^{\mathbb{P}_1}\right) & \text{in } K, \\ \phi_i^\varepsilon - \phi_i^{\mathbb{P}_1} = 0 & \text{on } \partial K. \end{cases}$$

Since $\nabla \phi_i^{\mathbb{P}_1}$ is piecewise constant, we can solve for each dimension separately.
Define numerical correctors ($\alpha = 1, \dots, d$)

$$\forall K \in \mathscr{T}_H, \quad \begin{cases} -\operatorname{div}\left(A^\varepsilon \nabla \chi_K^{\varepsilon,\alpha}\right) = \operatorname{div}\left(A^\varepsilon e_\alpha\right) & \text{in } K, \\ \chi_K^{\varepsilon,\alpha} = 0 & \text{on } \partial K. \end{cases}$$

Then

$$\phi_i^\varepsilon - \phi_i^{\mathbb{P}_1} = \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} \left(\partial_\alpha \phi_i^{\mathbb{P}_1}\right)\Big|_K \chi_K^{\varepsilon,\alpha}$$

$$\text{or} \quad \phi_i^\varepsilon = \phi_i^{\mathbb{P}_1} + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} \left(\partial_\alpha \phi_i^{\mathbb{P}_1}\right)\Big|_K \chi_K^{\varepsilon,\alpha}.$$

Effective PDE

$$\boxed{\text{Decoupling:} \qquad \nabla \phi_i^\varepsilon = \sum_{K \in \mathcal{T}_H} \sum_{\alpha=1}^{d} \left(\partial_\alpha \phi_i^{\mathbb{P}_1} \right)\Big|_K \left(e_\alpha + \nabla \chi_K^{\varepsilon,\alpha} \right)}$$

$$\begin{aligned}
\mathbb{A}_{j,i}^\varepsilon &= \sum_{K \in \mathcal{T}_H} \int_K \nabla \phi_j^\varepsilon \cdot A^\varepsilon \nabla \phi_i^\varepsilon \\
&= \sum_{K \in \mathcal{T}_H} \sum_{\alpha,\beta=1,2} \partial_\beta \phi_j^{\mathbb{P}_1}\Big|_K \left(\int_K \left(e_\beta + \nabla \chi_K^{\varepsilon,\beta} \right) \cdot A^\varepsilon \left(e_\alpha + \nabla \chi_K^{\varepsilon,\alpha} \right) \right) \partial_\alpha \phi_i^{\mathbb{P}_1}\Big|_K \\
&= \sum_{K \in \mathcal{T}_H} \int_K \nabla \phi_j^{\mathbb{P}_1} \cdot \overline{A} \, \nabla \phi_i^{\mathbb{P}_1} = \overline{\mathbb{A}}_{j,i}.
\end{aligned}$$

$$\text{Effective matrix} \quad \overline{A}_{\beta,\alpha}\Big|_K = \frac{1}{|K|} \int_K \left(e_\beta + \nabla \chi_K^{\varepsilon,\beta} \right) \cdot A^\varepsilon \left(e_\alpha + \nabla \chi_K^{\varepsilon,\alpha} \right)$$

Non-intrusive MsFEM

$$\overline{A} \longrightarrow \boxed{\text{Black box FEM solver}} \longrightarrow$$

Solves $\overline{\mathbb{A}}\,\overline{U} = \overline{\mathbb{F}}$
with
$\overline{\mathbb{A}} = \mathbb{A}^\varepsilon$, $\overline{\mathbb{F}}_j = \displaystyle\int_\Omega f\, \phi_j^{\mathbb{P}_1}$.

Coarse code: $u_H = \displaystyle\sum_i \overline{U}_i\, \phi_i^{\mathbb{P}_1}$.

Set $u_H^{\varepsilon,\text{non-in}} = \displaystyle\sum_i \overline{U}_i\, \phi_i^\varepsilon = u_H + \sum_{K\in\mathscr{T}_H} \sum_{\alpha=1}^{d} (\partial_\alpha u_H)\big|_K\, \chi_K^{\varepsilon,\alpha}$. (Thesis Algorithm 4.2)

Non-intrusive MsFEM

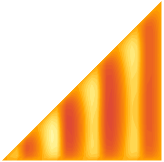
MsFEM:

$$a^\varepsilon(u_H^\varepsilon, \phi_i^\varepsilon) = \int_\Omega f\, \phi_i^\varepsilon$$

Non-intrusive:

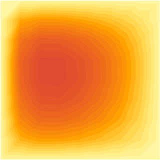
$$a^\varepsilon\left(u_H^{\varepsilon,\text{non-in}}, \phi_i^\varepsilon\right) = \int_\Omega f\, \phi_i^{\mathbb{P}_1}$$

$\chi_K^{\varepsilon,\alpha}$



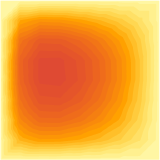
$\overline{A}$

u_H



Legacy code

$u_H^{\varepsilon,\text{non-in}}$



Generalizations

Main ingredients of non-intrusive MsFEM:

$$\boxed{\text{Decoupling: } \phi_i^\varepsilon = \phi_i^{\mathbb{P}_1} + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} \left(\partial_\alpha \phi_i^{\mathbb{P}_1} \right)\Big|_K \chi_K^{\varepsilon,\alpha}}$$

- ϕ_i^ε depends linearly on $\phi_i^{\mathbb{P}_1}$ and $\nabla \phi_i^{\mathbb{P}_1}$ is piecewise constant.

$$\boxed{\text{Effective matrix: } \overline{A}_{\beta,\alpha}\Big|_K = \frac{1}{|K|} \int_K \left(e_\beta + \nabla \chi_K^{\varepsilon,\beta} \right) \cdot A^\varepsilon \left(e_\alpha + \nabla \chi_K^{\varepsilon,\alpha} \right)}$$

- $\nabla \phi_i^{\mathbb{P}_1}$ is piecewise constant and the PDE is linear.

$\rightarrow$ Non-intrusive strategy generalizes to MsFEMs with other local boundary conditions (oversampling, Crouzeix-Raviart) and other linear 2nd order PDEs.

Comparison

Effect of $\int_\Omega f\,\phi_i^\varepsilon \to \int_\Omega f\,\phi_i^{\mathbb{P}_1}$?

Theorem 1 (Thesis Theorem 7.10)

Consider linear or Crouzeix-Raviart type boundary conditions for the ϕ_i^ε (but no oversampling). There exists $C>0$ independent of ε, H and f such that

$$\left\| u_H^\varepsilon - u_H^{\varepsilon,\text{non-in}} \right\|_{H^1(\mathcal{T}_H)} \le CH \left\| f \right\|_{L^2(\Omega)}.$$

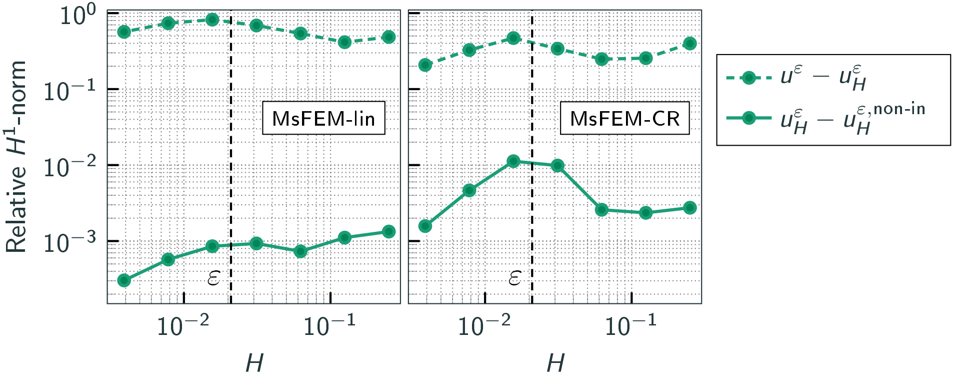
Classical error estimate when $A^\varepsilon(\bullet) = A^{\mathrm{per}}(\bullet/\varepsilon)$, under some technical assumptions (Efendiev and Hou 2009, Le Bris et al. 2013):

$$\left\| u^\varepsilon - u_H^\varepsilon \right\|_{H^1(\mathcal{T}_H)} \le C\left(H + \sqrt{\varepsilon} + \sqrt{\frac{\varepsilon}{H}} \right).$$

$\to$ Same estimate for $\left\| u^\varepsilon - u_H^{\varepsilon,\text{non-in}} \right\|_{H^1(\mathcal{T}_H)}$.

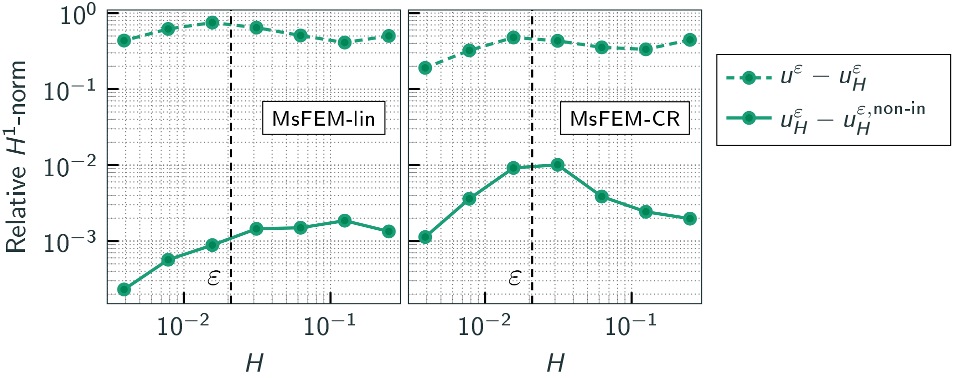
Numerical experiment without oversampling - periodic diffusion

$$A^\varepsilon(x) = \left(1 + 100\,\cos^2(\pi\, x_1/\varepsilon)\sin^2(\pi\, x_2/\varepsilon)\right)\,\mathrm{Id}$$
$$\Omega = (0,1)\times(0,1),\ f(x) = \sin(x_1)\sin(x_2)$$



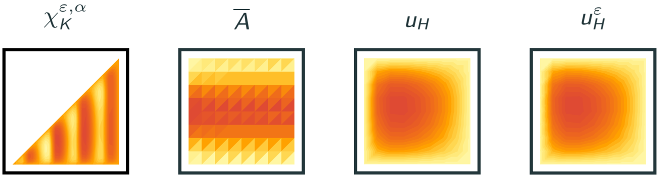
Numerical experiment without oversampling - non-periodic diffusion

$$A^\varepsilon(x) = \left(1 + \left(1 + 100\cos^2(\pi x_1/\varepsilon)\sin^2(\pi x_2/\varepsilon)\right)\cos^2\left(\frac{x_1^2+x_2^2}{\varepsilon}\right)\right) \mathrm{Id}$$
$$\Omega = (0,1)\times(0,1),\ f(x) = \sin(x_1)\sin(x_2)$$



Conclusion

Non-intrusive MsFEM



Publications:

R. A. Biezemans, C. Le Bris, F. Legoll, and A. Lozinski. Non-intrusive implementation of Multiscale Finite Element Methods: An illustrative example. *Journal of Computational Physics*, 477:111914, 2023.

Thesis Chapter 3

R. A. Biezemans, C. Le Bris, F. Legoll, and A. Lozinski. Non-intrusive implementation of a wide variety of Multiscale Finite Element Methods. *Comptes Rendus. Mécanique, Online first*, 2023.

Thesis Chapters 4-7

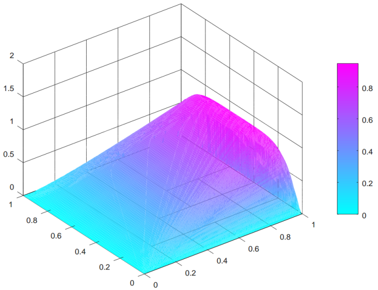
Advection-dominated problems

Single-scale problems with advection

Let $b = (1,0)$. Find $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\alpha\Delta u + b \cdot \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

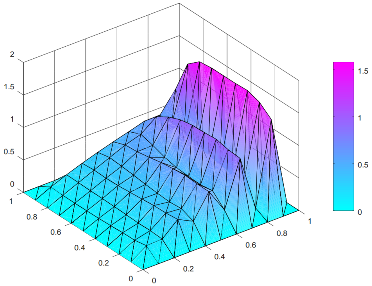
Boundary layers when advection dominates



Fine mesh

$\mathbb{P}_1$ approximation

Spurious oscillations when the boundary layer is not resolved



Coarse mesh

Goal

For the PDE

$$-\operatorname{div}\left(A^\varepsilon \nabla u^\varepsilon\right) + b \cdot \nabla u^\varepsilon = f,$$

recover the macroscopic properties of u^ε with a finite element method on a coarse mesh.

Step towards $b = b(t), b = b^\varepsilon, \ldots$

From here on, $a^\varepsilon(u, v) = \displaystyle\int_\Omega \nabla v \cdot A^\varepsilon \nabla u + v\, b \cdot \nabla u.$

Stabilization methods

Removing the spurious oscillations is a classical problem with many proposed solutions:

- Single scale problems: adding stabilizing terms to the discrete formulation (Streamline-Upwind/Petrov-Galerkin (SUPG), Brooks and Hughes 1982; Galerkin/Least-Squares, Hughes et al. 1985), adding bubble functions to the approximation space (Baiochhi and Hughes 1993, Franca and Russo 1996), Variational Multiscale Method (Hughes et al. 1998).
- Multiscale problems – follow-up on the PhD thesis of F. Madiot (2016):
  - MsFEM-lin SUPG,
  - Basis functions that resolve both A^{ε} and b,
  - Splitting approach.
- Different approaches: MsFEM for transport modeling (Allaire et al. 2012), GMsFEM (Calo et al. 2016), DG-HMM (Abdulle and Huber 2014), LOD (Li et al. 2018; Bonizzoni et al. 2022).

MsFEM approaches: we investigated the differential operator in the local problems, linear or CR boundary conditions, oversampling, Petrov-Galerkin formulations, bubble functions, etc.

Two types of local problems

$$\textbf{MsFEM:} \qquad -\mathrm{div}(A^\varepsilon \nabla \phi_i^\varepsilon) = 0.$$

$$\textbf{adv-MsFEM:} \qquad -\mathrm{div}(A^\varepsilon \nabla \phi_i^{\varepsilon,\mathrm{adv}}) + b \cdot \nabla \phi_i^{\varepsilon,\mathrm{adv}} = 0.$$

Either can be implemented with our non-intrusive MsFEM strategy.

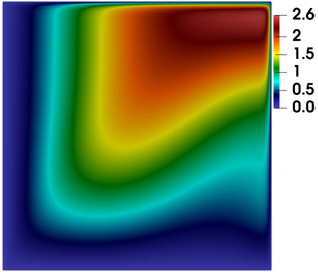
Assess stability by looking at the coarse part:

$$u_H^\varepsilon = u_H + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} (\partial_\alpha u_H)|_K \, \chi_K^{\varepsilon,\alpha},$$

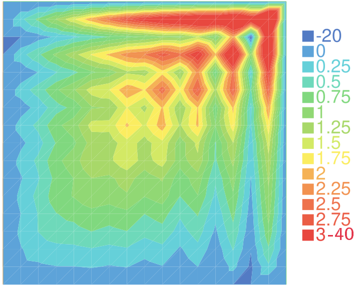
$$u_H^{\varepsilon,\mathrm{adv}} = u_H + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} (\partial_\alpha u_H)|_K \, \chi_K^{\varepsilon,\mathrm{adv},\alpha}.$$

Stability of MsFEMs

Reference solution

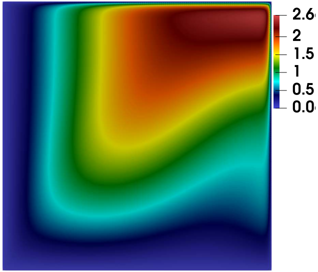


MsFEM-lin

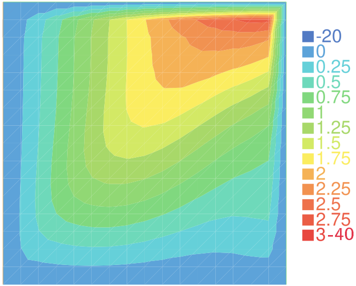


Unstable

Reference solution



MsFEM-lin SUPG

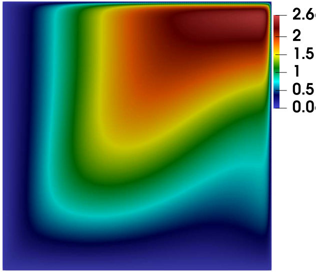


Stable

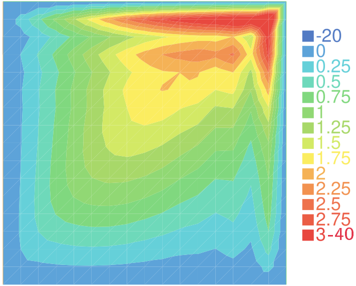
for the correct choice of the stabilization parameter

Stability of MsFEMs

Reference solution



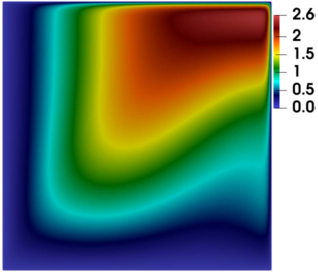
adv-MsFEM-lin



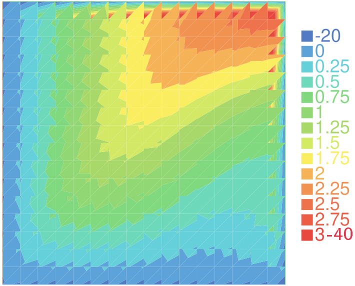
Not fully stabilized

explanation in Thesis Chapter 10

Reference solution



adv-MsFEM-CR



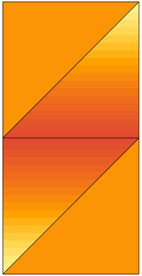
Stable

adv-MsFEM à la Crouzeix-Raviart

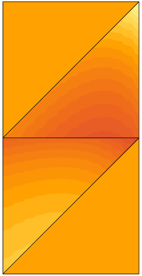
Multiscale function $u_H^{\varepsilon,\mathrm{adv}} = u_H + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} (\partial_\alpha u_H)|_K \, \chi_K^{\varepsilon,\mathrm{adv},\alpha}$.

- u_H is a $\mathbb{P}_1$ Crouzeix-Raviart function, continuous at the midpoints of faces/on average.
- $\chi_K^{\varepsilon,\mathrm{adv},\alpha}$ vanishes in a weak sense: $\int_{face} \chi_K^{\varepsilon,\mathrm{adv},\alpha} = 0$.

$\mathbb{P}_1$-CR:

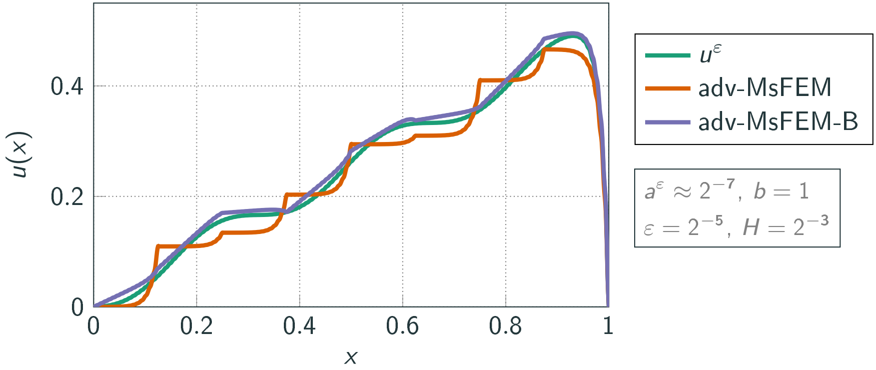


adv-MsFEM-CR:



Inserting the multiscale featuresThe adv-MsFEM-B

A typical example in 1D. showing $u_H^{\varepsilon,\mathrm{adv}} = u_H + \sum_{K \in \mathscr{T}_H} \sum_{\alpha=1}^{d} (\partial_\alpha u_H)|_K \, \chi_K^{\varepsilon,\mathrm{adv},\alpha}$.

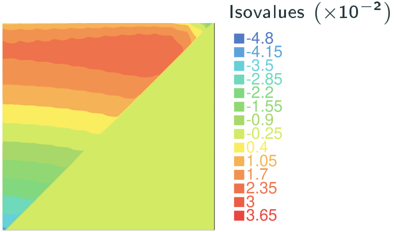


Stability is captured by the multiscale space, but not the general shape of the solution.

Adding bubble functions to the adv-MsFEM-CR

For all $K \in \mathscr{T}_H$, we introduce $B_K^{\varepsilon,\mathrm{adv}} \in H^1(K)$ satisfying

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla B_K^{\varepsilon,\mathrm{adv}}) + b \cdot \nabla B_K^{\varepsilon,\mathrm{adv}} = 1 & \text{in } K, \\ \displaystyle\int_{\mathit{face}} B_K^{\varepsilon,\mathrm{adv}} = 0 & \text{for each face,} \\ \text{constant flux} & \text{on each face.} \end{cases}$$



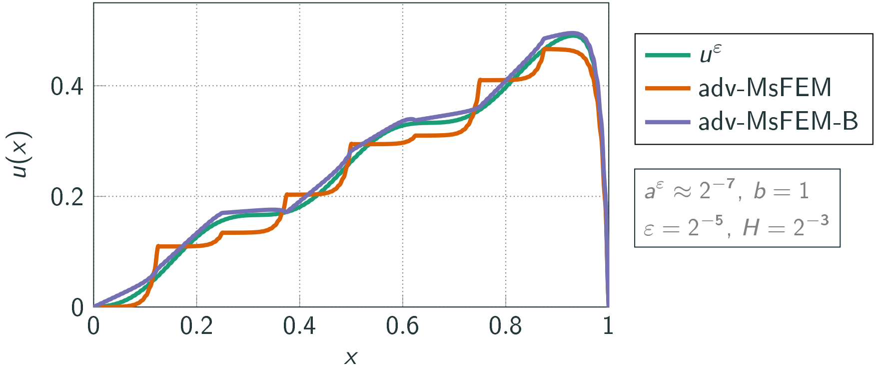
The adv-MsFEM-CR-B is: find $u_{H,B}^{\varepsilon,\mathrm{adv}} \in V_{H,B}^{\varepsilon,\mathrm{adv,CR}} = V_H^{\varepsilon,\mathrm{adv,CR}} \bigoplus_{K \in \mathscr{T}_H} \mathbb{R}\, B_K^{\varepsilon,\mathrm{adv}}$ such that

$$a^\varepsilon\left(u_{H,B}^{\varepsilon}, v_{H,B}^{\varepsilon,\mathrm{adv}}\right) = \int_\Omega f\, v_{H,B}^{\varepsilon,\mathrm{adv}} \quad \text{for all } v_{H,B}^{\varepsilon,\mathrm{adv}} \in V_{H,B}^{\varepsilon,\mathrm{adv,CR}}.$$

MsEFM with bubble functions was also introduced by (Le Bris et al. 2014) for diffusion problems in *perforated domains* and by (Le Bris et al. 2019) for advection-diffusion problems in perforated domains.

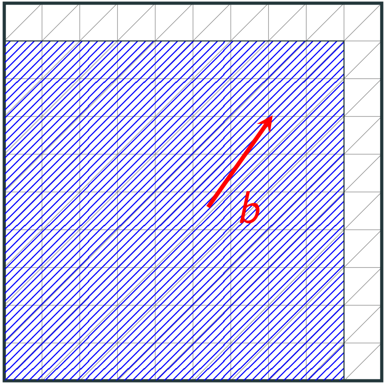
The adv-MsFEM-B

A typical example in 1D.



Numerical experiment

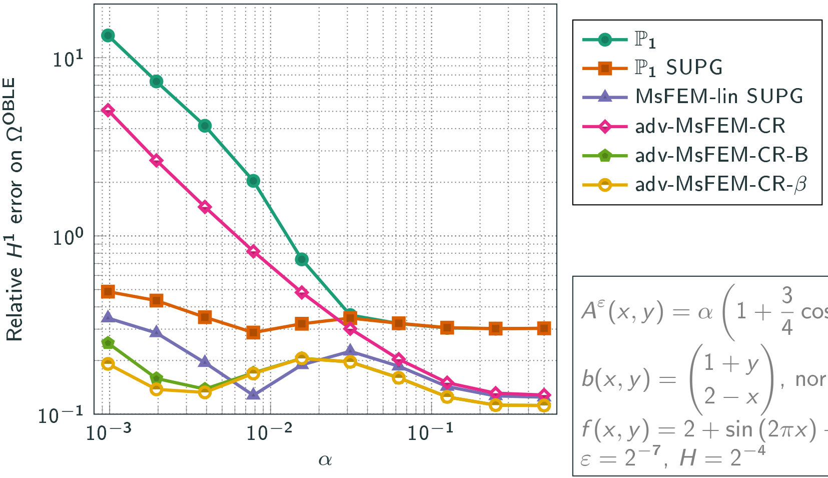
Error measurement



We don't focus on the boundary layer, but on bulk phenomena in Ω^{OBLE} (outside boundary layer elements).

$$\|u_h^\varepsilon - u_H^\varepsilon\|^2_{H^1(\mathscr{T}_H^{\mathbf{OLME}})} = \sum_{K\in\mathscr{T}_H, K\subset\Omega^{\mathbf{OLME}}} \|u_h^\varepsilon - u_H^\varepsilon\|^2_{H^1(K)}$$

Numerical experiment



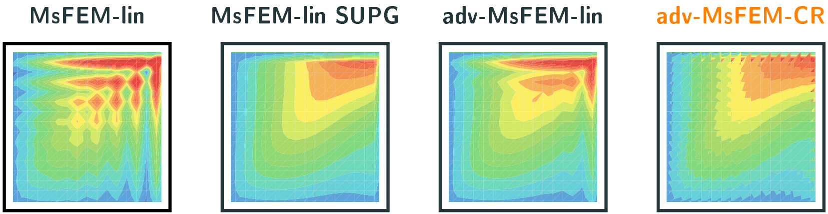
$$A^\varepsilon(x,y) = \alpha\left(1 + \frac{3}{4}\cos\frac{2\pi x}{\varepsilon}\sin\frac{2\pi y}{\varepsilon}\right)\mathrm{Id}$$

$$b(x,y) = \begin{pmatrix} 1+y \\ 2-x \end{pmatrix}, \text{ normalized}$$

$$f(x,y) = 2 + \sin(2\pi x) + x\cos(2\pi y)$$

$\varepsilon = 2^{-7}$, $H = 2^{-4}$

Conclusion



- Stability of the adv-MsFEM-CR without an additional stabilization parameter.
- Bubble functions for accuracy in advection-dominated regime.
- Case of varying $b \to$ reduced basis techniques?

Convergence analysis under minimal regularity hypotheses

Convergence analysis for MsFEM

Consider $-\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f$ with $A^\varepsilon(\bullet) = A^{\mathrm{per}}\left(\frac{\bullet}{\varepsilon}\right)$ and let $u^\star$ be the homogenized limit of u^ε (i.e. $u^\varepsilon \xrightarrow{\varepsilon \to 0} u^\star$ (Bensoussan et al. 1978, Allaire 2002)):

Theorem 2

Let u_H^ε be the MsFEM-lin or MsFEM-CR approximation of u^ε, without oversampling. Then

$$\|u^\varepsilon - u_H^\varepsilon\|_{H^1(\mathscr{T}_H)} \leq CH\|u^\star\|_{H^2(\Omega)} + C\left(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\right)\|\nabla u^\star\|_{W^{1,\infty}(\Omega)}.$$

(Hou et al. 1999 for MsFEM-lin; Le Bris et al. 2013 for MsFEM-CR)

Relies on a result from homogenization theory using A^{per} Hölder continuous, $u^\star \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$.

Our convergence result

Theorem 3 (Thesis Theorem 13.1)

Suppose that $A^\varepsilon(\bullet) = A^{\mathrm{per}}\left(\frac{\bullet}{\varepsilon}\right)$ with A^{per} bounded. Suppose that the family of meshes $(\mathscr{T}_H)_H$ is regular and quasi-uniform and that $u^\star \in H^2(\Omega)$. Let u^ε_H be the MsFEM-lin or the MsFEM-CR approximation of u^ε. Then

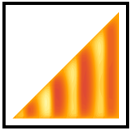
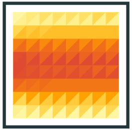
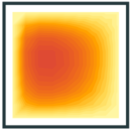
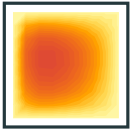
$$|u^\varepsilon - u^\varepsilon_H|_{H^1(\mathscr{T}_H)} \leq C \left(H|u^\star|_{2,\Omega} + \sqrt{\frac{\varepsilon}{H}}\,|u^\star|_{1,\Omega} \right).$$

Summary & outlook

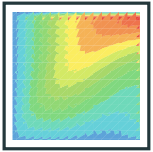
Summary

Non-intrusive MsFEM

$\chi_K^{\varepsilon,\alpha}$ | $\overline{A}$ | u_H | u_H^{ε}

MsFEM for advection-dominated problems



Stability of the adv-MsFEM-CR(-B)

Convergence analysis

$$\boxed{|u^\varepsilon - u_H^\varepsilon|_{H^1(\mathscr{T}_H)} \leq C \left(H |u^\star|_{2,\Omega} + \sqrt{\frac{\varepsilon}{H}} |u^\star|_{1,\Omega} \right)}$$

Directions for future research

- A study of 'residual-free bubble' type for CR methods in a single scale context to understand the stabilizing properties of the adv-MsFEM-CR.
- Design of improved boundary conditions/enrichment of the multiscale space to delay the resonance effect. Links with localized model order reduction and domain decomposition methods may be further explored.
- Extension of the convergence analysis to perturbations of the periodic setting, to locally periodic coefficients, the ultimate goal being to avoid structural hypotheses on the coefficients.