Scriba Robot – Kinematics

This document describes the kinematic model & equations needed for the Scriba robot.

Summary

1 General equations

The kinematics equations of the robot are needed to implement move command and odometry.

ROS move command is sent as a velocity (kinematic center speed) and twist (angular speed) command.

There are two main configurations to consider:

- Steering wheel with no angle
- Steering wheel with angle

In both cases we want to express the angle of rotation and the kinematic center displacement relative to the front wheel angle and front wheel displacement.

1.1 Steering wheel with no angle

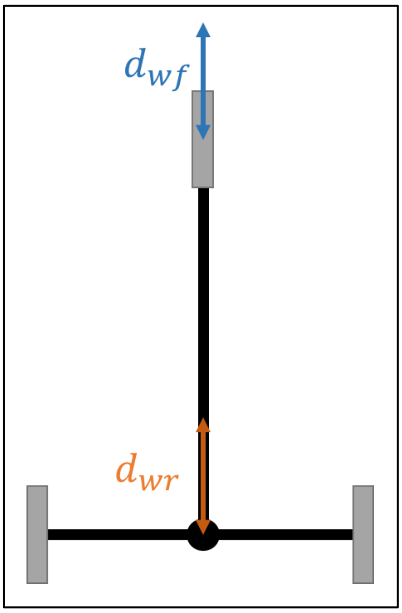


Figure 1: Kinematics with straight front wheel

It's easy to get the angle of rotation and the kinematic center displacement relative to the front wheel angle and front wheel displacement:

- Angle of rotation: $\omega = 0$
- Kinematic center displacement: $d_{wr} = d_{wf}$

1.2 Steering wheel with angle

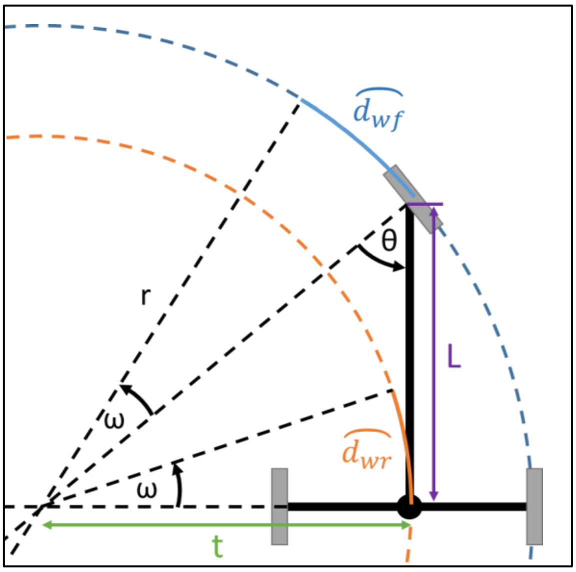


Figure 2: Kinematics with steered front wheel

$$\tan(\theta) = \frac{t}{L} \quad \Leftrightarrow \quad t = L * \tan(\theta)$$

$$\cos(\theta) = \frac{L}{r} \quad \Leftrightarrow \quad r = \frac{L}{\cos(\theta)}$$

$$\widehat{d_{wf}} = \omega * r = \frac{\omega * L}{\cos(\theta)} \quad \Leftrightarrow \quad \omega = \widehat{d_{wf}} * \frac{\cos(\theta)}{L}$$

$$\widehat{d_{wr}} = \omega * t = \omega * L * \tan(\theta) = \widehat{d_{wf}} * \frac{\cos(\theta)}{L} * L * \tan(\theta) = \widehat{d_{wf}} * \cos(\theta) * \tan(\theta)$$

From those equations we obtain the angle of rotation and the kinematic center displacement relative to the front wheel angle and front wheel displacement:

- Angle of rotation: $\omega = \widehat{d_{wf}} * \frac{\cos(\theta)}{L}$
- Kinematic center displacement: $\widehat{d_{wr}} = \widehat{d_{wf}} * \cos(\theta) * \tan(\theta)$

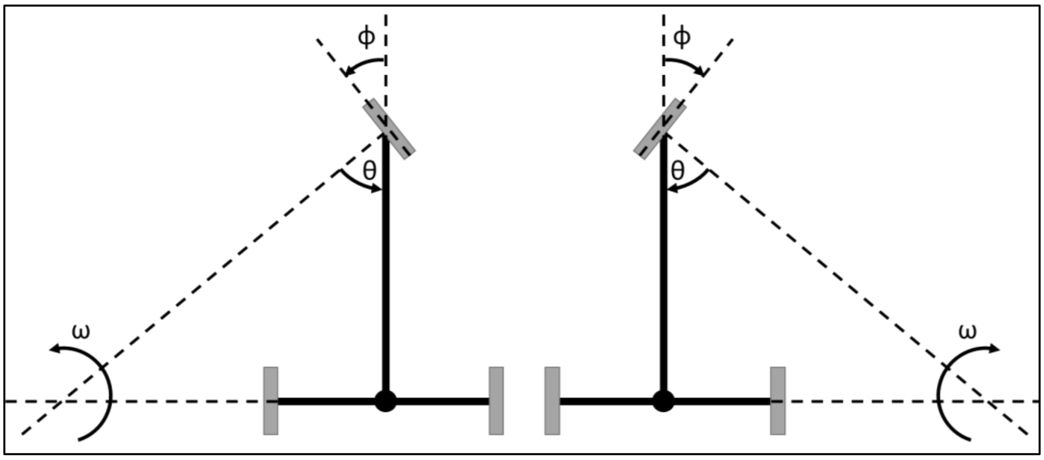
2 Velocity command

ROS command outputs a kinematic center speed and an angular speed. On the other end the motor controls need a front wheel angle and a front wheel speed.

Therefore, command for the robot has to be expressed as a front wheel angle and front wheel speed relative to the angular speed and the kinematic center speed.

The front wheel angle sent as a motor command is ϕ and is expressed as:

$$\varphi = \frac{\pi}{2} - \theta \quad \Leftrightarrow \quad \theta = \frac{\pi}{2} - \varphi$$



2.1 No angular speed command

In the special case of a command with no angular speed.

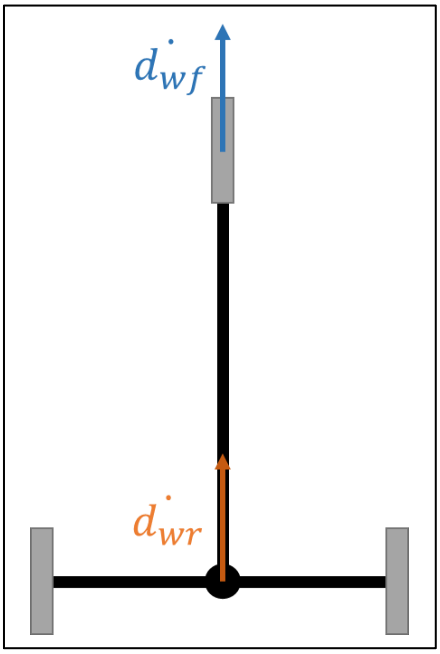


Figure 3: Velocities with straight front wheel

- Front wheel rotation: $\varphi = 0$
- Front wheel speed: $\dot{d_{wf}} = \dot{d_{wr}}$

2.2 With angular speed command

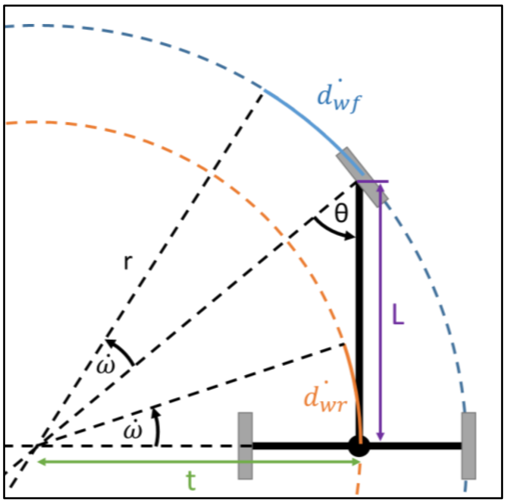


Figure 4: Velocities with steered front wheel

Using the previous equations (1.2 Steering wheel with angle - page 3) the following equations are derived:

$$\overparen{d_{wr}} = \omega L \tan(\theta)$$

$$\Leftrightarrow \dot{d}_{wr} = \dot{\omega} L \tan(\theta) + \dot{\omega} L\, \dot{\theta}(1 + tan^2(\theta))$$

And

$$\overparen{d_{wr}} = \overparen{d_{wf}} \cos(\theta) \tan(\theta)$$

$$\dot{d}_{wr} = \dot{d}_{wf} \cos(\theta) \tan(\theta) + \overparen{d_{wf}}\, \dot{\theta}\big(\cos(\theta)(1 + tan^2(\theta)) - \sin(\theta) \tan(\theta)\big)$$

$$\dot{d}_{wr} = \dot{d}_{wf} \sin(\theta) + \overparen{d_{wf}}\, \dot{\theta}(\cos(\theta) + \sin(\theta) \tan(\theta) - \sin(\theta) \tan(\theta))$$

$$\dot{d}_{wr} = \dot{d}_{wf} \sin(\theta) + \overparen{d_{wf}}\, \dot{\theta} \cos(\theta)$$

Θ is considered constant for several reasons. Controlling both the front wheel angle and the front wheel angular speed is unpractical. Indeed the base controller would need the exact front wheel angle and the communication time with the Arduino sending that data is too long. Moreover, the front wheel angular speed will be very small in practice.

If θ constant then $\dot{\theta} = 0$, the previous equations can be rewritten:

$$\dot{d}_{wr} = \dot{\omega} L \tan(\theta) = \dot{\omega} L tan\left(\frac{\pi}{2} - \varphi\right) = \frac{\dot{\omega} L}{\tan(\varphi)} \Leftrightarrow \tan(\varphi) = \frac{\dot{\omega} L}{\dot{d}_{wr}} \Leftrightarrow \varphi = atan\left(\frac{\dot{\omega} L}{\dot{d}_{wr}}\right)$$

$$\dot{d}_{wr} = \dot{d}_{wf} \cos(\theta) \tan(\theta) = \dot{d}_{wf} \cos\left(\frac{\pi}{2} - \varphi\right) \tan\left(\frac{\pi}{2} - \varphi\right) = \dot{d}_{wf} \sin(\varphi) \frac{\cos(\varphi)}{\sin(\varphi)}$$

$$= \dot{d}_{wf} \cos(\varphi) \Leftrightarrow \dot{d}_{wf} = \frac{\dot{d}_{wr}}{\cos(\varphi)}$$

- Front wheel rotation: $\varphi = atan\left(\frac{\dot{\omega}L}{\dot{d_{wr}}}\right)$
- Front wheel speed: $\dot{d_{wf}} = \frac{\dot{d_{wr}}}{\cos(\varphi)}$

2.3 Only angular speed command

Another special case is needed if only angular speed is required. Indeed, for an angle of $\varphi = \frac{\pi}{2} \longrightarrow$ $\cos(\varphi) = 0$ and $\dot{d_{wf}} = \frac{\dot{d_{wr}}}{\cos(\varphi)}$ would go to infinite.

Using the previous equations (1.2 Steering wheel with angle - page 3) the following equations are derived:

$$\omega = \widehat{d_{wf}} * \frac{\cos(\theta)}{L}$$

$$\dot{\omega} = \dot{d_{wf}} * \frac{\cos(\theta)}{L} + \frac{\widehat{d_{wf}}}{L} * \left(-\dot{\theta}\sin(\theta)\right)$$

If θ constant then $\dot{\theta} = 0$, the previous equations can be rewritten:

$$\dot{\omega} = \dot{d_{wf}} * \frac{\cos(\theta)}{L} = \dot{d_{wf}} * \frac{\sin\left(\frac{\pi}{2} - \varphi\right)}{L} = \dot{d_{wf}} * \frac{\sin(\varphi)}{L}$$

$$\Leftrightarrow \quad \dot{d_{wf}} = \frac{\dot{\omega}L}{\sin(\varphi)}$$

- Front wheel rotation: $\varphi = \frac{\pi}{2}\ or - \frac{\pi}{2}$
- Front wheel speed: $\dot{d_{wf}} = \frac{\dot{\omega}L}{\sin(\varphi)}$

3 Odometry

In ROS, a 2D odometry is expressed as a displacement on the x-axis, a displacement on the y-axis and a rotation around the z-axis, all expressed in a global odometry frame.

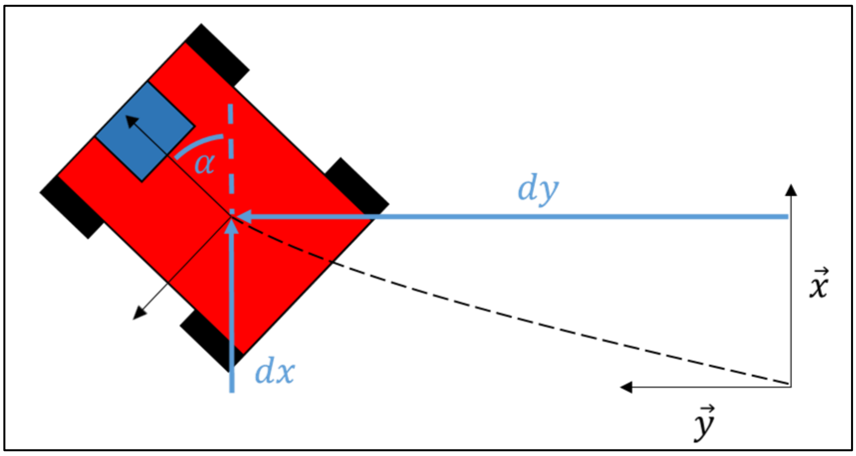


Figure 5: Odometry frame and variables

The displacement will be first expressed in the robot base frame:

- ω
- Δx
- Δy

And then added to the odometry frame using the following equations:

- $\alpha = \alpha + \omega$
- $dx = dx + \Delta x * \cos(\alpha) - \Delta y * \sin(\alpha)$
- $dy = dy + \Delta x * \sin(\alpha) + \Delta y * \cos(\alpha)$

3.1 No angular speed

Without angular speed the odometry is very straight-forward:

- $\omega = 0$
- $\Delta x = d_{wf}$
- $\Delta y = 0$

3.2 With angular speed

Using the previous equations (1.2 Steering wheel with angle - page 3) the following equations are derived:

$$\omega = \widehat{d_{wf}} * \frac{\cos(\theta)}{L} = \widehat{d_{wf}} * \frac{\sin(\varphi)}{L}$$

And

$$\widehat{d_{wr}} = \widehat{d_{wf}} \cos(\theta) \tan(\theta)$$

$$\dot{d_{wr}} = \dot{d_{wf}} \cos(\theta) \tan(\theta) + \widehat{d_{wf}}\, \dot{\theta}\big(\cos(\theta)\big(1 + tan^2(\theta)\big) - \sin(\theta) \tan(\theta)\big)$$

$$\dot{d_{wr}} = \dot{d_{wf}}\sin(\theta) + \widehat{d_{wf}}\,\dot{\theta}(\cos(\theta) + \sin(\theta)\tan(\theta) - \sin(\theta)\tan(\theta))$$

$$\dot{d_{wr}} = \dot{d_{wf}}\sin(\theta) + \widehat{d_{wf}}\,\dot{\theta}\cos(\theta)$$

As before, Θ is considered constant. The odometry is computed from the front wheel displacement ($\widehat{d_{wf}}$) in a defined time period (Δt), the front wheel angle (φ) and displacement in the same defined time period. All that data is received from the motor controller board.

We can express the displacement and angle according to the data:

$$\dot{d_{wr}} = \dot{d_{wf}}\sin(\theta) = \dot{d_{wf}}\cos(\varphi)$$

$$d_{wr} = \dot{d_{wr}} * \Delta\mathrm{t}$$

$$\omega = \widehat{d_{wf}} * \frac{\sin(\varphi)}{L}$$

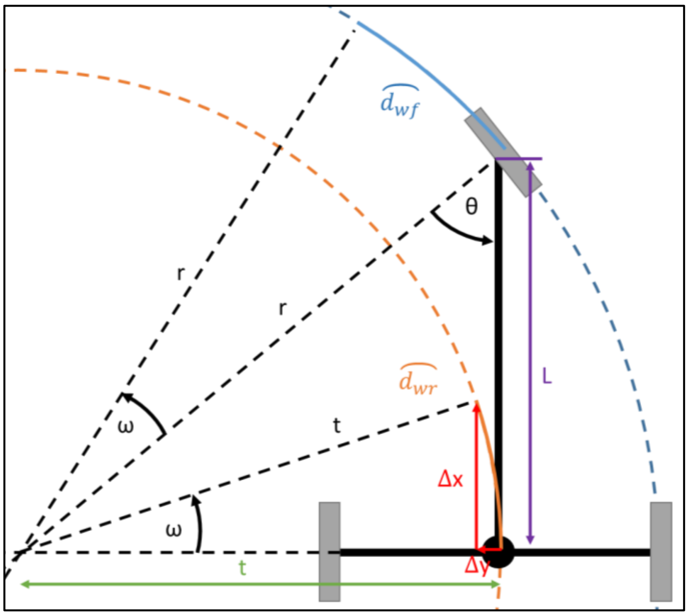


Figure 6: Displacement depending on front wheel angle and travelled distance

The odometry can then be deducted from the previous equations:

- $\omega = \widehat{d_{wf}} * \frac{\sin(\varphi)}{L}$
- $\Delta x = L\,tan(\theta)\sin(\omega) = L\,cotan(\varphi)\sin(\omega) = L\,\frac{\cos(\varphi)}{\sin(\varphi)}\sin(\omega)$
- $\Delta y = t - t\cos(\omega) = t * (1 - \cos(\omega)) = L\,\frac{\cos(\varphi)}{\sin(\varphi)}(1 - \cos(\omega))$

4 Calibration

From odometry, some expressions can be derived to help with calibration.

The three variables to calibrate are:

- Front wheel angle offset
- Front wheel radius
- Robot's length (distance between the kinematic center and the front wheel)

4.1 Front wheel angle calibration

The front wheel angle needs to be expressed relative to the other terms. Using again the equations from 3.2 - With angular speed page 7:

- $\widehat{d_{wr}} = \omega * t$
- $\Delta x = L\, tan(\theta) \sin(\omega) = L\, cotan(\varphi) \sin(\omega) = L\, \frac{\cos(\varphi)}{\sin(\varphi)} \sin(\omega)$
- $\Delta y = t - t\cos(\omega) = t * (1 - \cos(\omega)) = L\, \frac{\cos(\varphi)}{\sin(\varphi)} (1 - \cos(\omega))$

$$\Delta y = L\frac{\cos(\varphi)}{\sin(\varphi)} * \frac{\sin(\omega)}{\sin(\omega)} * (1 - \cos(\omega)) = \frac{\Delta x}{\sin(\omega)} * (1 - \cos(\omega))$$

$$\Leftrightarrow \frac{\Delta y}{\Delta x} = \frac{1 - \cos(\omega)}{\sin(\omega)} = \tan\left(\frac{\omega}{2}\right)$$

$$\Leftrightarrow \omega = 2 * \arctan\left(\frac{\Delta y}{\Delta x}\right)$$

Moreover $\omega = \frac{\widehat{d_{wr}}}{t}$

$$\frac{\widehat{d_{wr}}}{t} = 2 * \arctan\left(\frac{\Delta y}{\Delta x}\right)$$

$$t = \frac{\widehat{d_{wr}}}{2 * \arctan\left(\frac{\Delta y}{\Delta x}\right)} \quad and\ t = \frac{L}{\tan(\varphi)}$$

$$\Leftrightarrow \tan(\varphi) = L * \frac{2 * \arctan\left(\frac{\Delta y}{\Delta x}\right)}{\widehat{d_{wr}}}$$

$$\Leftrightarrow \varphi = \arctan\left(L * \frac{2 * \arctan\left(\frac{\Delta y}{\Delta x}\right)}{\widehat{d_{wr}}}\right)$$

4.2 Front wheel radius

The odometry gives the following equation:

$$traveled\ distance = \frac{steps * \ radius_{front\ wheel} * r_{ms} * \ \pi}{200}$$

The odometry value can be corrected by simply adding a correcting factor:

$$real\ traveled\ distance = \frac{steps * \ factor_{correction} * radius_{front\ wheel} * r_{ms} * \ \pi}{200}$$

This factor can be calculated by comparing the odometry value to the real value:

$$factor_{correction} = \frac{traveled\ distance}{real\ traveled\ distance}$$

4.3 Robot's length

The following equation is given by the robot's model:

$$\widehat{d_{wf}} = \omega * L \Leftrightarrow \omega = \frac{\widehat{d_{wf}}}{L}$$

The odometry value can be corrected by simply adding a correcting factor:

$$\omega_{real} = \frac{\widehat{d_{wf}}}{L * factor_{correction}}$$

This factor can be calculated by comparing the odometry value to the real value:

$$\frac{\omega_{odometry}}{\omega_{real}} = \frac{\widehat{d_{wf}}}{L} * \frac{L * factor_{correction}}{\widehat{d_{wf}}} = factor_{correction}$$

Table of figures