MSE Hand in 10 - Solution

Part 1

1. $\log\left(14^{71}\right)$ $\boxed{=71\log(14)}$. Using the logarithmic property $\log\left(a^b\right)=b\cdot\log(a)$, the exponent 71 is brought in front as a multiplier.

2. Write the following as a single logarithm:

$$
\begin{aligned}
&3+\frac{1}{2}\log_4(x)+\frac{1}{2}\log_4(y)\\
&=\log_4(4^3)+\frac{1}{2}\log_4(x)+\frac{1}{2}\log_4(y) && \text{(Rewrite 3 as } \log_4(4^3))\\
&=\log_4(64)+\frac{1}{2}\log_4(x)+\frac{1}{2}\log_4(y)\\
&=\log_4\left(64\cdot\sqrt{x}\cdot\sqrt{y}\right) && \text{(Combine using } \log_b(a)+\log_b(c)=\log_b(ac))\\
&=\boxed{\log_4\left(64\sqrt{xy}\right)}.
\end{aligned}
$$

3. Reduce the expression

$$
\begin{aligned}
&\frac{98}{x-1}\div\frac{x}{x-1}\\
&=\frac{98}{x-1}\cdot\frac{x-1}{x} && \text{(Rewrite division as multiplication by the reciprocal)}\\
&=\frac{98\cdot(x-1)}{(x-1)\cdot x}\\
&=\boxed{\frac{98}{x}} && \text{(Cancel the common factor } x-1).
\end{aligned}
$$

4. Consider the following two functions $h(x)=\dfrac{1}{x+5}$ and $g(x)=\dfrac{x}{x-\frac{1}{2}}$. What is the domain of the composite $(h\circ g)(x)$?

   To find the domain of $(h\circ g)(x)$, we follow these steps:

   i. The domain of $g(x)=\dfrac{x}{x-\frac{1}{2}}$ is $x\neq\dfrac{1}{2}$.

   ii. The domain of $h(x)=\dfrac{1}{x+5}$ requires $g(x)+5\neq 0 \implies g(x)\neq -5$.

   iii. Substituting $g(x)=\dfrac{x}{x-\frac{1}{2}}$:

$$
\frac{x}{x-\frac{1}{2}}\neq -5 \implies x=\frac{5}{12} \text{ must be excluded.}
$$

   iv. Combining restrictions, the domain of $(h\circ g)(x)$ is:

$$
\boxed{x\in\mathbb{R}\setminus\left\{\frac{5}{12},\frac{1}{2}\right\}}.
$$

5. Consider the function

$$
f(x)=\frac{102}{3x-24}+76
$$

What is the range of the inverse function f^{-1} ? Write your answer as an integer between 0 and 99.

i. **Find the domain of $f(x)$:**
The denominator $3x - 24$ must not be zero:

$$3x - 24 \neq 0 \implies x \neq 8.$$

Thus, the domain of $f(x)$ is:

$$x \in \mathbb{R} \setminus \{8\}.$$

ii. **Find the range of the inverse function $f^{-1}(x)$:**
The range of the inverse function $f^{-1}(x)$ is the domain of $f(x)$, which is:

$$\boxed{x \in \mathbb{R} \setminus \{8\}}.$$

6. The binary number 101010101_2 can be expanded as a sum of powers of 2. Starting from the rightmost bit (least significant), the binary digits represent coefficients for powers of 2:

$$101010101_2 = \boxed{1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0}$$

Simplify:

$$= 2^8 + 2^6 + 2^4 + 2^2 + 2^0$$

7. To convert 4726_{16} to decimal, expand it as a sum of powers of 16:

$$4726_{16} = 4 \cdot 16^3 + 7 \cdot 16^2 + 2 \cdot 16^1 + 6 \cdot 16^0.$$

In the order stated:

$$\boxed{4726_{16} = 6 \cdot 16^0 + 4 \cdot 16^3 + 2 \cdot 16^1 + 7 \cdot 16^2}.$$

8. Convert the binary number 10100011010_2 to a hexadecimal number.

To convert the binary number 10100011010_2 to a hexadecimal number, group the binary digits into groups of four from right to left (adding leading zeros if necessary):

$$10100011010_2 = 0001\ 0100\ 0110\ 1010_2$$

Now, convert each group of four binary digits into its hexadecimal equivalent:

$$0001_2 = 1, \quad 0100_2 = 4, \quad 0110_2 = 6, \quad 1010_2 = A$$

Thus, the hexadecimal representation of 10100011010_2 is:

$$\boxed{146A_{16}}$$

9. Calculate $1000101001_2 + 100001110_2$. State the answer as a binary number.

   To calculate $1000101001_2 + 100001110_2$, align the binary numbers for addition:

$$\begin{array}{r} 1000101001 \\ +\ 100001110 \\ \hline 1010010001 \end{array}$$

   Thus, the result is:

$$\boxed{1010010001_2}$$

10. What is $110_2 \times B_{16}$ in decimal?

    To compute $110_2 \times B_{16}$ in decimal, we first convert both numbers to decimal and then perform the multiplication.

    i. Convert 110_2 to decimal:

$$110_2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 4 + 2 + 0 = 6.$$

    ii. Convert B_{16} to decimal:

$$B_{16} = 11 \text{ (since } B \text{ represents 11 in hexadecimal).}$$

    iii. Multiply the decimal equivalents:

$$6 \times 11 = 66.$$

    Thus, $110_2 \times B_{16}$ in decimal is:

$$\boxed{66}$$

11. Give the prime factorisation of 224

    To find the prime factorisation of 224, we proceed as follows:

    i. Divide by the smallest prime number (2):

$$224 \div 2 = 112$$

    ii. Continue dividing by 2:

$$112 \div 2 = 56, \quad 56 \div 2 = 28, \quad 28 \div 2 = 14, \quad 14 \div 2 = 7$$

    iii. The final result is 7, which is a prime number.

    Thus, the prime factorisation of 224 is:

$$\boxed{224 = 2^5 \cdot 7}$$

12. Find the greatest common divisor (gcd) and least common multiple (lcm) of the integers 48 and 120.

    To find the greatest common divisor (gcd) and least common multiple (lcm) of 48 and 120, we proceed as follows:

    i. Find the prime factorisation of each number:

$$48 = 2^4 \cdot 3, \quad 120 = 2^3 \cdot 3 \cdot 5$$

    ii. For the gcd, take the lowest powers of all common prime factors:

Common prime factors: 2 and 3

$$\gcd(48, 120) = 2^3 \cdot 3 = 24$$

iii. For the lcm, take the highest powers of all prime factors:
Prime factors: 2, 3, 5

$$\operatorname{lcm}(48, 120) = 2^4 \cdot 3 \cdot 5 = 240$$

Thus, the greatest common divisor is:

$$\boxed{\gcd(48, 120) = 24}$$

and the least common multiple is:

$$\boxed{\operatorname{lcm}(48, 120) = 240}$$

13. Let a be positive integer. Find the smallest possible remainder in the expression below.
Rewrite the equation to isolate r:

$$r = 88 - a \times 7$$

Since r must satisfy $0 \leq r < 7$, calculate $88 \mod 7$:

$$88 \div 7 = 12 \quad \text{remainder } 4.$$

$$\boxed{r = 4}$$

14. Order the following remainders from smallest to largest.
    1. 12 mod 3
    2. 23 mod 11
    3. 7 mod 5
    4. 100 mod 12
    5. 45 mod 8
15. Consider the Boolean function $F(x, y, z) = xy + y(z + x)$.
To verify the decimal equivalent of the truth table for $F(x, y, z) = xy + y(z + x)$, we proceed step by step.
i. Expand the Boolean function:

$$F(x, y, z) = xy + y(z + x)$$

Simplify using Boolean algebra:

$$F(x, y, z) = xy + yz + yx = xy + yz \quad (\text{since } yx = xy).$$

ii. Generate the truth table:

x	y	z	$F(x, y, z) = xy + yz$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

iii. Read the last column to form the binary number: The last column is $0, 0, 0, 1, 0, 0, 1, 1$. This gives the binary number:

$$00010011_2$$

iv. Convert 00010011_2 to decimal:

$$00010011_2 = 1 \cdot 2^4 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 2 + 1 = 19$$

Thus, the decimal equivalent is:

$$\boxed{19}$$

16. Consider the following Boolean expression $x(yz + z\bar{x}) + \bar{x}$.

i. Distribute x across $(yz + z\bar{x})$:

$$x(yz + z\bar{x}) + \bar{x} = xyz + xz\bar{x} + \bar{x}.$$

ii. Simplify $xz\bar{x}$ using the complement rule ($x\bar{x} = 0$):

$$xyz + xz\bar{x} + \bar{x} = xyz + 0 + \bar{x} = xyz + \bar{x}.$$

iii. Factor z from xyz:

$$xyz + \bar{x} = z(yx) + \bar{x}.$$

iv. Recognise that yx is just y when multiplied by z, because z is independent of x. So:

$$z(yx) + \bar{x} = zy + \bar{x}.$$

Thus, the simplified expression is:

$$\boxed{zy + \bar{x}}$$

17. Which boolean function $F(X, Y, Z)$ returns 1 if you are able to study efficiently, 0 if you are not.

$X = 1$: You must decide to study.

$Y = 1$: You must have access to the course material.

$Z = 0$: You must not be distracted by social media.

These conditions must all be true simultaneously. Therefore, the Boolean function $F(X, Y, Z)$ is:

$$\boxed{F(X, Y, Z) = X \cdot Y \cdot \bar{Z}}.$$

18. One of the five circuits below is equivalent to the Boolean expression: $F = x(\bar{x} + yz)$

i. $\boxed{CircuitA}$ computes the sub-expression $\bar{x}$ first. ii. It then combines $\bar{x}$ with yz using an OR gate to form $(\bar{x} + yz)$. iii. Finally, the output of the OR gate is ANDed with x.

This matches the original, unsimplified expression: $F = x(\bar{x} + yz)$

19. Determine this binary number, and convert it to decimal.

Step 1: Evaluate $F(x, y, z)$ for the given inputs

The circuit implements the Boolean function $F(x, y, z)$. We compute $F(x, y, z)$ step by step for the specified inputs:

i. For $F(1, 1, 0)$:

- $y = 1 \to \bar{y} = 0,$
- $z = 0 \to y \wedge z = 1 \wedge 0 = 0,$

- Combine $x \wedge (\bar{y} + (y \wedge z)) = 1 \wedge (0 + 0) = 1 \wedge 0 = 0$.

Thus, $F(1, 1, 0) = 0$.

ii. For $F(0, 0, 0)$:

- $y = 0 \to \bar{y} = 1$,
- $z = 0 \to y \wedge z = 0 \wedge 0 = 0$,
- Combine $x \wedge (\bar{y} + (y \wedge z)) = 0 \wedge (1 + 0) = 0 \wedge 1 = 0$.

Thus, $F(0, 0, 0) = 0$.

iii. For $F(0, 1, 0)$:

- $y = 1 \to \bar{y} = 0$,
- $z = 0 \to y \wedge z = 1 \wedge 0 = 0$,
- Combine $x \wedge (\bar{y} + (y \wedge z)) = 0 \wedge (0 + 0) = 0 \wedge 0 = 0$.

Thus, $F(0, 1, 0) = 0$.

iv. For $F(0, 0, 1)$:

- $y = 0 \to \bar{y} = 1$,
- $z = 1 \to y \wedge z = 0 \wedge 1 = 0$,
- Combine $x \wedge (\bar{y} + (y \wedge z)) = 0 \wedge (1 + 0) = 0 \wedge 1 = 0$.

Thus, $F(0, 0, 1) = 0$.

v. For $F(1, 0, 1)$:

- $y = 0 \to \bar{y} = 1$,
- $z = 1 \to y \wedge z = 0 \wedge 1 = 0$,
- Combine $x \wedge (\bar{y} + (y \wedge z)) = 1 \wedge (1 + 0) = 1 \wedge 1 = 1$.

Thus, $F(1, 0, 1) = 1$.

Step 2: Construct the binary number The outputs are $F(1, 1, 0) = 1$, $F(0, 0, 0) = 0$, $F(0, 1, 0) = 0$, $F(0, 0, 1) = 1$, and $F(1, 0, 1) = 1$. Reading from left to right:

$$\text{Binary number: } 10011_2$$

Step 3: Convert the binary number to decimal

$$10011_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 0 + 0 + 2 + 1 = 19.$$

Final Answer: The decimal equivalent is:

$$\boxed{19}$$

20. In the following, $\log n$ refers to log base 2, i.e. the binary logarithm. Mark each statement as true or false.

   i. $n^2 \log n = O(3^3)$:
      - $3^3 = 27$ is a constant, while $n^2 \log n$ grows unbounded as $n \to \infty$.
      - $\boxed{False}$

   ii. $\log n^2 = O(1)$:
      - $\log n^2 = 2 \log n$, which grows unbounded as $n \to \infty$. It is not $O(1)$, which denotes a constant bound.
      - $\boxed{False}$

   iii. $6\sqrt{n} = O(n\sqrt{n})$:
      - Dividing both sides by $\sqrt{n}$, we get $6 = O(n)$, which is true since $n \to \infty$ bounds the constant 6.
      - $\boxed{True}$

iv. $\sqrt{n} + \sqrt{n} = O(n \cdot \log n)$:
  - $\sqrt{n} + \sqrt{n} = 2\sqrt{n}$, which grows slower than $n \cdot \log n$ as $n \to \infty$.
  - $\boxed{True}$

v. $n \cdot \log n = O((\log n)^3)$:
  - $n \cdot \log n$ grows faster than $(\log n)^3$ as $n \to \infty$.
  - $\boxed{False}$

vi. $2^n = O(\sqrt{n} \cdot \log n)$:
  - Exponential growth 2^n far outpaces $\sqrt{n} \cdot \log n$ as $n \to \infty$.
  - $\boxed{False}$

vii. $n\sqrt{n} = O(n^{3/2})$:
  - $n\sqrt{n} = n^{3/2}$, so this is true.
  - $\boxed{True}$

viii. $8^{\log n} = O(n^{2/3})$:
  - $8^{\log n} = (2^3)^{\log n} = 2^{3 \log n} = n^3$, which grows much faster than $n^{2/3}$.
  - $\boxed{False}$

ix. $n \cdot \log n = O((\log n)^2)$:
  - $n \cdot \log n$ grows faster than $(\log n)^2$ as $n \to \infty$.
  - $\boxed{False}$

x. $n^n = O(3^n)$:
  - n^n grows far faster than 3^n as $n \to \infty$.
  - $\boxed{False}$

xi. $7n \cdot \log n = \Theta(\log(n!))$:
  - Using Stirling's approximation, $\log(n!) \sim n \log n - n$, so they are equivalent.
  - $\boxed{True}$

xii. $\sum_{i=1}^{n} i = O(\sqrt{n} \cdot \log n)$:
  - The sum $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ grows quadratically, which is much faster than $\sqrt{n} \cdot \log n$.
  - $\boxed{False}$

21. Order the following functions by their growth rate, from slowest to fastest.

$\log \log n$

$\log(n\sqrt{n})$

$\sqrt{n}$

$10n$

$n \log n$

$\frac{n^2}{\sqrt{n}}$

5^n

$\frac{1}{2}n!$

22. Analyse the two code snippets in terms of their time complexity.

Code Snippet 1: `findGCD1`

- The code iterates through all integers from 1 to a, checking if i divides both a and b.
- The loop runs from $i = 1$ to $i = a$, which makes the number of iterations proportional to a.
- For each iteration, the modulo operation $a\%i$ and $b\%i$ are performed, each taking $O(1)$ time.

- Therefore, the overall time complexity of `findGCD1` is:

  $$\boxed{O(a)}.$$

Code Snippet 2: `findGCD2`

- In each iteration of the `while` loop, the remainder $b = a\%b$ is computed, and a is updated to the value of b.
- The size of b decreases significantly with each iteration, as the Euclidean algorithm ensures that $b < a$ after every step.
- The number of iterations is bounded by $O(\log(\min(a, b)))$, since the size of b reduces approximately by half at each step.
- Each iteration involves a single modulo operation, which takes $O(1)$ time.
- Thus, the overall time complexity of `findGCD2` is:

  $$O(\log(\min(a, b)))$$

  and since we know $a < b$ we can simplify to $\boxed{O(\log(a))}$

Part 2

Item 1

a. Solve the equation $AX = B$. State your inputs as four integers between 0 and 99. **Step 1: Solve $AX = B$.** The matrix equation is:

$$\begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 0 \\ 2 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}.$$

We solve for X using the formula:

$$X = A^{-1} \cdot B.$$

Step 2: Compute A^{-1}.

The inverse of A is computed in Python:

$$A^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{9}{13} & \frac{3}{13} \\ \frac{21}{13} & \frac{13}{13} & -\frac{3}{13} \\ \frac{9}{13} & \frac{15}{13} & \frac{7}{13} \end{bmatrix}.$$

Step 3: Compute $X = A^{-1} \cdot B$. You have to use Python for this part, but it is possible to find it using the formula for the inverse of a 3×3 matrix A: Multiply A^{-1} by B:

$$X = \begin{bmatrix} \frac{3}{13} & \frac{9}{13} & \frac{3}{13} \\ \frac{21}{13} & \frac{13}{13} & -\frac{3}{13} \\ \frac{9}{13} & \frac{15}{13} & \frac{7}{13} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}.$$

Perform the multiplication (detailed steps omitted for brevity):

Final Answer:

$$\boxed{X = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & -2 \end{bmatrix}.}$$

b. $A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & k & 0 \\ 2 & 3 & -3 \end{bmatrix}$

Find the value of k such that the equation $AX = B$ has no solution.

Reduce to echelon form and check if the last row is all zeros except for the last element.

$$\begin{bmatrix} 1 & 2 & -4 \\ 3 & k & 0 \\ 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 5 \\ 0 & 0 & 18-5k \end{bmatrix}$$

For the last row to be all zeros except for the last element, we must have $18-5k = 0$ for it to be consistent. So $\boxed{\text{k} = 18/5}$.

Item 2

a. State in augmented form

$$\boxed{\begin{bmatrix} 1 & 0 & 8 & 6 & 0 \\ 2 & 3 & -1 & 4 & 0 \\ 4 & 5 & 0 & 7 & 0 \end{bmatrix}}$$

b. Solve the systems and state in parametric form.

Reduce to reduced echelon form using Python:

$$\begin{bmatrix} 1 & 0 & 8 & 6 & 0 \\ 2 & 3 & -1 & 4 & 0 \\ 4 & 5 & 0 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The free variable is x_4 and the system can be written in parametric form as:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ -3 \\ -1 \\ 1 \end{bmatrix}}$$

Item 3

A: RREF

B: RREF

C: Not since row of zeros must appear at the bottom.

D: Not since the leading 1 in the first row is to the right of the leading 1 in the second row, violating the rule that leading 1s must move strictly to the right as you go down rows.

E: RREF

F: RREF

G: RREF

H: Not since the second column contains a leading 1, but there is a nonzero entry (2) above it.

Item 4

How many card hands of 4 cards, where order does not matter, can be dealt from a standard deck such that the hand contains at least one spade?

To solve the problem, we calculate the total number of hands with at least one spade.

i. Total number of 4-card hands:

$$\binom{52}{4}$$

ii. Number of 4-card hands with no spades:

$$\binom{39}{4}$$

iii. Number of hands with at least one spade:

$$\binom{52}{4} - \binom{39}{4}$$

Substitute the values:

$$\binom{52}{4} = \frac{52 \cdot 51 \cdot 50 \cdot 49}{4!} = 270725,$$

$$\binom{39}{4} = \frac{39 \cdot 38 \cdot 37 \cdot 36}{4!} = 82251.$$

Thus:

$$\binom{52}{4} - \binom{39}{4} = 270725 - 82251 = 188474.$$

Final answer:

$$\boxed{188474}$$

Item 5

80% of Danes go to the doctor regularly; 35% of those have no health issues during the following year. Of the remaining Danes, only 5% have no health issues during the following year.
What is the probability a randomly chosen Dane will have health issues in the following year?
To find the probability that a randomly chosen Dane will have health issues in the following year, we proceed as follows:
i. Define the events:

- $P(D)$: Probability a Dane goes to the doctor regularly, $P(D) = 0.80$.
- $P(D^c)$: Probability a Dane does not go to the doctor regularly, $P(D^c) = 1 - P(D) = 0.20$.
- $P(H^c|D)$: Probability a Dane who goes to the doctor has no health issues, $P(H^c|D) = 0.35$.
- $P(H^c|D^c)$: Probability a Dane who does not go to the doctor has no health issues, $P(H^c|D^c) = 0.05$.
- $P(H|D)$: Probability a Dane who goes to the doctor has health issues, $P(H|D) = 1 - P(H^c|D) = 0.65$.
- $P(H|D^c)$: Probability a Dane who does not go to the doctor has health issues, $P(H|D^c) = 1 - P(H^c|D^c) = 0.95$.

ii. Use the law of total probability to calculate $P(H)$:

$$P(H) = P(H|D)P(D) + P(H|D^c)P(D^c).$$

iii. Substitute the values:

$$P(H) = (0.65)(0.80) + (0.95)(0.20).$$

iv. Perform the calculations:

$$P(H) = 0.52 + 0.19 = 0.71.$$

Thus, the probability that a randomly chosen Dane will have health issues in the following year is:

$$\boxed{0.71}.$$

Item 6

Let A be the set of 7-bit binary numbers and let B be the set of binary numbers ending with 11 .

a. What is the number of elements in the set $A \cap B$?

i. A 7-bit binary number has the form:

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7,$$

where each $b_i \in \{0, 1\}$.

ii. For numbers in B, the last two bits are fixed as $b_6 = 1$ and $b_7 = 1$. This leaves b_1, b_2, b_3, b_4, b_5 free to vary.

iii. Each of the 5 remaining bits can independently take a value of either 0 or 1, resulting in:

$$2^5 = 32 \text{ numbers.}$$

Thus, the number of elements in the set $A \cap B$ is:

$$\boxed{32}.$$

b. b. Let x be a number in the set $A \cap B$. What is the probability that $x = 23$ or $x = 27$?

To compute the probability that $x = 23$ or $x = 27$ for a number x in the set $A \cap B$, follow these steps:

i. The total number of elements in $A \cap B$ is:

$$|A \cap B| = 32.$$

ii. The set $A \cap B$ contains all 7-bit binary numbers ending with "11".

iii. Convert 23 and 27 to their 7-bit binary representations:

$$23_{10} = 0010111_2 \quad \text{and} \quad 27_{10} = 0011011_2.$$

iv. Both 23 and 27 belong to $A \cap B$, as they are 7-bit binary numbers ending with "11".

v. The probability of a specific number x in $A \cap B$ is:

$$P(x) = \frac{1}{|A \cap B|} = \frac{1}{32}.$$

vi. The probability that $x = 23$ or $x = 27$ is the sum of their individual probabilities:

$$P(x = 23 \text{ or } x = 27) = P(x = 23) + P(x = 27) = \frac{1}{32} + \frac{1}{32} = \frac{2}{32}.$$

vii. Simplify the fraction:

$$P(x = 23 \text{ or } x = 27) = \frac{1}{16}.$$

Final Answer:

$$\boxed{\frac{1}{16}}$$

Item 7

What is the probability that a fish has a defect given that environmental toxin was found?
To calculate the probability that a fish has a defect given that an environmental toxin was found, we use the formula for conditional probability:

$$P(\text{Defect}|\text{Toxin}) = \frac{P(\text{Defect and Toxin})}{P(\text{Toxin})}.$$

1. From the table: - The number of fish with a defect and an environmental toxin found is 156. - The total number of fish where an environmental toxin was found is 161.
2. Substitute these values into the formula:

$$P(\text{Defect}|\text{Toxin}) = \frac{156}{161}.$$

3. Simplify the fraction:

$$P(\text{Defect}|\text{Toxin}) \approx 0.9696.$$

Final Answer:

$$\boxed{P(\text{Defect}|\text{Toxin}) = \frac{156}{161} \approx 0.97.}$$

Item 8

For each algorithm, we analyse its execution time in Θ-notation.
Algorithm loop1(n):

For loop: Iterates from $i = n$ to $i = 1$, decrementing by 1.The loop executes n iterations, and $s = s+1$ is performed each ti

Total operations:

$$\Theta(1) + \Theta(n) = \Theta(n).$$

Algorithm loop2(n):

Outer loop: Iterates $i = 1$ to n (n iterations).Inner loop: For each iteration of the outer loop, $j = i$ and decrements to 0.

Total iterations of the inner loop:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2).$$

Overall time complexity:

$$\Theta(n^2).$$

Algorithm loop3(n):

Outer loop: While $i > 0$, starts with $i = n$ and decrements $i = i - 1$ per iteration. This executes n iterations.Inner loop: Fo

Total iterations of the inner loop:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2).$$

Overall time complexity:

$$\Theta(n^2).$$

Algorithm loop4(n):

Outer while loop: Iterates while $i \leq n$. If $i < j$, i increments; otherwise, j increments, and i resets to 0.The outer loop exec

Overall time complexity:

$$\Theta(n^2).$$

Summary of Time Complexities:

loop1(n): $\Theta(n)$
loop2(n): $\Theta(n^2)$
loop3(n): $\Theta(n^2)$
loop4(n): $\Theta(n^2)$

Item 9

Consider the word, VIRGINIA. How many unique ways can these letters be arranged?
The word "VIRGINIA" has 8 letters, where certain letters repeat:
- I appears 3 times,
- V, R, G, N, A each appear 1 time.
The number of unique arrangements of the letters in the word is given by the formula for permutations of multiset:

$$\text{Unique arrangements} = \frac{n!}{p_1! \cdot p_2! \cdot \dots \cdot p_k!},$$

where n is the total number of letters, and p_i are the frequencies of the repeated letters.
Substitute the values:

$$n = 8, \quad p_1 = 3 \, (\text{for } I), \quad p_2 = 1, \dots, p_6 = 1 \, (\text{for } V, R, G, N, A).$$

$$\text{Unique arrangements} = \frac{8!}{3!}.$$

Compute factorials:

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320,$$

$$3! = 3 \cdot 2 \cdot 1 = 6.$$

Divide:

$$\frac{8!}{3!} = \frac{40320}{6} = 6720.$$

Final Answer:

$$\boxed{6720}$$

Item 10

a. Fill out the table

i. The company keeps a stock of 2000 circuit boards.

ii. 60% of the boards come from Supplier 2, and the remaining 40% come from Supplier 1:

$$\text{Boards from Supplier } 1 = 0.40 \cdot 2000 = 800,$$

$$\text{Boards from Supplier } 2 = 0.60 \cdot 2000 = 1200.$$

iii. 5% of Supplier 1's boards are defective, and 3% of Supplier 2's boards are defective:

$$\text{Defectives from Supplier } 1 = 0.05 \cdot 800 = 40,$$

$$\text{Defectives from Supplier } 2 = 0.03 \cdot 1200 = 36.$$

iv. The remaining boards in each category are non-defective:

$$\text{Non-Defectives from Supplier 1} = 800 - 40 = 760,$$

$$\text{Non-Defectives from Supplier 2} = 1200 - 36 = 1164.$$

Contingency Table:

	Supplier 1	**Supplier 2**
Defectives	40	36
Non-Defectives	760	1164

b. If a randomly chosen circuit board from the company's stock is chosen and turns out to be defective, what is the probability that the circuit board is from supplier 1.

To calculate the probability that a defective circuit board is from Supplier 1, we use Bayes' Theorem:

$$P(\text{Supplier 1}|\text{Defective}) = \frac{P(\text{Defective}|\text{Supplier 1})P(\text{Supplier 1})}{P(\text{Defective})}.$$

i. $P(\text{Defective}|\text{Supplier 1})$:

$$P(\text{Defective}|\text{Supplier 1}) = \frac{\text{Defective boards from Supplier 1}}{\text{Total boards from Supplier 1}} = \frac{40}{800} = 0.05.$$

ii. $P(\text{Supplier 1})$:

$$P(\text{Supplier 1}) = \frac{\text{Boards from Supplier 1}}{\text{Total boards}} = \frac{800}{2000} = 0.4.$$

iii. $P(\text{Defective})$: Total defective boards:

$$\text{Defective boards from Supplier 1} + \text{Defective boards from Supplier 2} = 40 + 36 = 76.$$

Probability of a defective board:

$$P(\text{Defective}) = \frac{\text{Total defective boards}}{\text{Total boards}} = \frac{76}{2000} = 0.038.$$

iv. Substitute the values:

$$P(\text{Supplier 1}|\text{Defective}) = \frac{0.05 \cdot 0.4}{0.038}.$$

v. Perform the calculation:

$$P(\text{Supplier 1}|\text{Defective}) = \frac{0.02}{0.038} \approx 0.5263.$$

Final Answer:

$$\boxed{P(\text{Supplier 1}|\text{Defective}) \approx 0.5263 \text{ or } 52.63\%}.$$

Item 11

Disease A occurs with probability 0.1 , and disease B occurs with probability 0.2 . It is not possible to have both diseases. You have a single test. This test reports positive with probability 0.8 for a patient with disease A, with probability 0.5 for a patient with disease B, and with probability 0.01 for a patient with no disease - call the latter event W.

If the test comes back positive, what is the probability you have either:

1. disease A
2. disease B, or
3. neither

These probabilities can be computed using Bayes' Theorem.

Step 1: Define events and probabilities.

- $P(A) = 0.1$: Probability of disease A
- $P(B) = 0.2$: Probability of disease B
- $P(W) = 1 - P(A) - P(B) = 0.7$: Probability of no disease (event W)
- $P(P|A) = 0.8$: Probability of a positive test given disease A
- $P(P|B) = 0.5$: Probability of a positive test given disease B
- $P(P|W) = 0.01$: Probability of a positive test given no disease

Step 2: Compute $P(P)$ (total probability of a positive test).
Using the law of total probability:

$$P(P) = P(P|A)P(A) + P(P|B)P(B) + P(P|W)P(W).$$

Substitute the values:

$$P(P) = (0.8)(0.1) + (0.5)(0.2) + (0.01)(0.7).$$

$$P(P) = 0.08 + 0.1 + 0.007 = 0.187.$$

Step 3: Compute conditional probabilities.
(1) Probability of having disease A given a positive test:

$$P(A|P) = \frac{P(P|A)P(A)}{P(P)}.$$

$$P(A|P) = \frac{(0.8)(0.1)}{0.187} = \frac{0.08}{0.187} \approx 0.4278.$$

(2) Probability of having disease B given a positive test:

$$P(B|P) = \frac{P(P|B)P(B)}{P(P)}.$$

$$P(B|P) = \frac{(0.5)(0.2)}{0.187} = \frac{0.1}{0.187} \approx 0.5348.$$

(3) Probability of having no disease (event W) given a positive test:

$$P(W|P) = \frac{P(P|W)P(W)}{P(P)}.$$

$$P(W|P) = \frac{(0.01)(0.7)}{0.187} = \frac{0.007}{0.187} \approx 0.0374.$$

Final Answers:

$$\boxed{P(A|P) \approx 0.4278, \quad P(B|P) \approx 0.5348, \quad P(W|P) \approx 0.0374.}$$

Item 12

Determine the homogenuous solution for the coefficient matrix A by setting up the solution in parametric form.

Reduce to reduced echelon form using Python:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 2 & 4 & 3 & 10 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 3/4 & 5/2 \end{bmatrix}$$

The free variables are x_3 and x_4. The parametric solution is:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ -3/4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}}$$

Item 13

A FALSE: The columns of the matrix are linearly **dependent** when $k = 3$ because the last entry, $2k - 6$, becomes 0. Therefore, $k = 3$ does not make the columns linearly independent - quite the opposite.

B TRUE: The matrix A has a solution for every $\mathbf{b}$ because the system $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$ since the last row of A is entirely zero.

C TRUE: The vectors are linearly independent and have 4 entries. Therefore A would be a 4×4 matrix with linearly independent columns and hence is invertible.

D TRUE: As mentioned in the previous item, the vectors are linearly independent.

E FALSE: u_4 is not a linear combination of u_1, u_2, u_3 and does therefore not belong to the span of the set.