Magic Squares

Thursday, July 6, 2023 21:14

A magic square M is an $n \times n$ matrix whose rows, columns, and diagonals sum to the same constant k.

A common exercise for younger math students is, given a partially complete magic square (where α entries are unknown and $n^2 - \alpha$ entries are known), and given the constant k, find M.

For example:

$k =$ Magic Sum = 34

$M =$

16	a	3	b
c	11	d	8
e	7	6	f
4	g	h	1

Here, the matrix M is 4×4.

To solve a magic square of size $n \times n$ in the general case, construct a matrix A and a vector $\underline{b}$ as follows:

A is a matrix representing the left side of a system of equations. Since all rows, columns, and diagonals sum to k, and there are n rows, n cols, and 2 diagonals, the # of equations is $2n+2$, and thus: $\boxed{A \text{ has } 2n+2 \text{ rows}}$

The number of variables is α, and so: $\boxed{A \text{ has } \alpha \text{ columns}}$

We then fill in the matrix A row by row, such that:

1. rows 1 to n contain the equations for summing the rows of M
2. rows $n+1$ to $2n$ contain the equations for summing the columns of M
3. row $2n+1$ contains the equation for summing the top left – bottom right diagonal
4. row $2n+2$ contains the equation for summing the top right – bottom left diagonal

To construct $\underline{b}$:

First, define a vector $\underline{k}$, such that

$$\underline{k} = k \cdot \underline{1}, \quad \text{where } \underline{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{2n+2}$$

Then, define a vector $\underline{e}$ such that $\underline{e} \in \mathbb{R}^{2n+2}$, and:

1. The 1st to nth entries of $\underline{e}$ contain the sums of the already given values in the 1st to nth rows of M.
2. The $(n+1)$th to $(2n)$th entries of $\underline{e}$ contain the sums of the already given values in the 1st to nth columns of M.
3. The $(2n+1)$th entry of $\underline{e}$ is the sum of the alr. giv. val. in the TL – BR diagonal of M.
4. The $(2n+2)$th entry of $\underline{e}$ is the sum of the a.g.v. in the TR – BL diag. of M.

Then $\boxed{\underline{b} = \underline{k} - \underline{e}}$

To find M, solve the system $A\underline{x} = \underline{b}$, and $\underline{x} \in \mathbb{R}^{\alpha}$ contains the missing entries of M.

Example: See the problem presented earlier:

$k =$ Magic Sum = 34

$M =$

16	a	3	b
c	11	d	8
e	7	6	f
4	g	h	1

A will be a $(2n+2)$ by (α)

$= (2 \cdot 4 + 2)$ by (8)

$= 10 \times 8$ matrix.

$$A = \begin{array}{c} (a \quad b \quad c \quad d \quad e \quad f \quad g \quad h) \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ (\text{row } 1) \\ (\text{row } 2) \\ (\text{row } 3) \\ (\text{row } 4) \\ (\text{col } 1) \\ (\text{col } 2) \\ (\text{col } 3) \\ (\text{col } 4) \\ (\text{diag } 1) \\ (\text{diag } 2) \end{array}$$

$\underline{b}$ will be in $\mathbb{R}^{10}$, and $\underline{b} = \underline{k} - \underline{e}$

$$= \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \\ 34 \end{bmatrix} - \begin{bmatrix} 19 \\ 19 \\ 13 \\ 5 \\ 20 \\ 18 \\ 9 \\ 9 \\ 34 \\ 11 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 21 \\ 29 \\ 14 \\ 16 \\ 25 \\ 25 \\ 0 \\ 23 \end{bmatrix}$$

To solve $A\underline{x} = \underline{b}$, we can instead solve $A^T A \underline{x} = A^T \underline{b}$, because if $A\underline{x} = \underline{b}$ has a solution $\underline{x}$ then $A^T A \underline{x} = A^T \underline{b}$. Python can easily solve invertible square matrices but not rectangular ones, which is why changing the equation to $A^T A \underline{x} = A^T \underline{b}$ is useful.

Using Python, $\underline{x} = \begin{bmatrix} 2 \\ 13 \\ 5 \\ 10 \\ 9 \\ 12 \\ 14 \\ 15 \end{bmatrix}$

```
[3]: A = np.array([
         [1, 1, 0, 0, 0, 0, 0, 0],
         [0, 0, 1, 1, 0, 0, 0, 0],
         [0, 0, 0, 0, 1, 1, 0, 0],
         [0, 0, 0, 0, 0, 0, 1, 1],
         [0, 0, 1, 0, 1, 0, 0, 0],
         [1, 0, 0, 0, 0, 0, 1, 0],
         [0, 0, 0, 1, 0, 0, 0, 1],
         [0, 1, 0, 0, 0, 1, 0, 0],
         [0, 0, 0, 0, 0, 0, 0, 0],
         [0, 1, 0, 1, 0, 0, 0, 0]
     ])

     b = np.array([15, 15, 21, 29, 14, 16, 25, 25, 0, 23])

     la.solve(A.T @ A, A.T @ b)

[3]: array([ 2., 13.,  5., 10.,  9., 12., 14., 15.])
```

and so our solution M is:

$k =$ Magic Sum = 34

$M =$

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1