Spinors Part 2: Rotations, Möbius Transformations, and SU(2)

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I. INTRODUCTION

Welcome to Part 2 of our special feature on Spinors. In Part 1 “Bloch Sphere and the Hopf Fibration”, we discussed spinors from first principles, and discussed how

1. Spinors represent a body’s rotation by encoding information about the axis and direction of rotation.

2. Spinors that are orthogonal represent antipodal states on the Bloch sphere, and vice versa.

Here, as promised, we will discuss the third key property of spinors:

3. Rotating the physical state space by an angle 2π returns the spinor $-|\psi\rangle$, and rotating again by 2π gets you back to $|\psi\rangle$. In general, rotating a physical state vector by a given angle only rotates the corresponding spinor by half that angle, and vice versa.

We will summarise our discussion of the first two properties, but readers are encouraged to keep Part 1 handy so they can refer back to it as needed. (A copy can be found at https://github.com/RCPN/Spinors.)

II. RECAP

Spinors often seem unintuitive because there isn’t an easy way to plot them or their transformations, since we need four dimensions to plot S^3 and visualise the effect of multiplying a spinor by an SU(2) matrix. But if we connect spinors to the realm of complex functions, we can find a way to reduce the dimensionality enough to plot the effects SU(2) matrices have on them. The Riemann sphere is the most basic object in complex analysis, being the codomain for general complex functions, and it coincides with the Bloch sphere through their shared equivalence to the complex projective line $\mathbb{CP}^1$. (See [1] for a full treatment, or “Spinors Part 1” in Jeremy Issue 3, 2025 for a full discussion of the construction). Algebraically, stereographic projection is a map

$$(x, y, z) \mapsto \zeta = \frac{x + iy}{1 + z}, \tag{1}$$

which has inverse

$$\zeta \mapsto \left(\frac{2\mathrm{Re}(\zeta)}{1 + \zeta\zeta^*}, \frac{2\mathrm{Im}(\zeta)}{1 + \zeta\zeta^*}, \frac{1 - \zeta\zeta^*}{1 + \zeta\zeta^*} \right), \tag{2}$$

where $*$ denotes complex conjugation. Using spherical coordinates on S^2 and polar coordinates on $\mathbb{C}$, we see from Fig. 1 that an equivalent formulation is

$$(\varphi, \theta) \mapsto \zeta = \tan(\theta/2)e^{i\varphi} \tag{3}$$

where $0 \le \varphi < 2\pi$ and $0 \le \theta \le \pi$.

The North Pole is mapped to 0, the equator $x^2 + y^2 = 1$ is mapped to the unit circle $S^1 \equiv \mathrm{U}(1)$, and the entire northern hemisphere is mapped to the interior of the unit disc. Similarly, the southern hemisphere is mapped to the exterior (i.e. complement) of the unit disc towards infinity. We introduce a “number at infinity”, called ∞, and set the projection of the South Pole to be this point. The resulting set

$$\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \tag{4}$$

is called the extended complex plane, and as the stereographic projection is a homeomorphism, we treat it as topologically equivalent to S^2, justifying the name “Riemann sphere”.

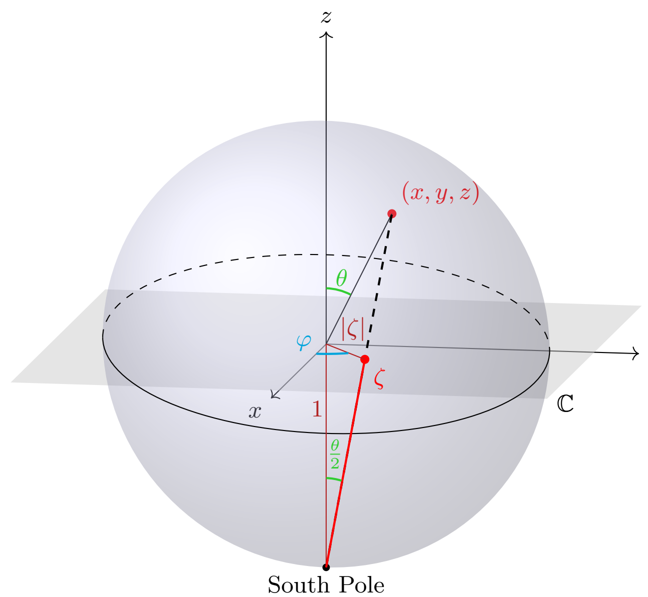

FIG. 1: Stereographic Projection of the point (x, y, z) on the unit sphere to $\zeta \in \mathbb{C}$. By standard results from circle geometry, the angle from the z-axis to (x, y, z) as measured from the South Pole is exactly half the polar angle θ.

The complex projective line $\mathbb{CP}^1$ is constructed by starting with $\mathbb{C}^2 \setminus [0, 0]$ and quotienting out the equivalence relation $[a, b] \sim \lambda[a, b] \quad \forall\lambda \in \mathbb{C} \setminus \{0\}$, so that all scalar multiples of vectors are made equivalent. Denoting the equivalence class of $[a, b]$ as $[a : b]$, the easiest choice of representative when $b \ne 0$ is $[a/b, 1]$, while for $b = 0$ there is only one equivalence class $[1 : 0]$. Thus the complex projective line is

$$\mathbb{CP}^1 = \{[\zeta : 1] : \zeta \in \mathbb{C}\} \cup \{[1 : 0]\}, \tag{5}$$

and we can see by comparison with Eq. (4) that the set on the left is isomorphic to $\mathbb{C}$ while $[1 : 0]$ is equivalent to the point at infinity. Essentially, we’ve constructed the Bloch sphere in one step: usually we normalise our $\mathbb{C}^2$ vectors first (quotienting out the real modulus), and then quotient out the global phases $e^{i\theta}$, but this is just equivalent to quotienting out multiplication by a complex number in modulus-argument form. This demonstrates the equivalence of $\mathbb{CP}^1$ and the Bloch sphere. Since we have also shown equivalence to the Riemann sphere, which is topologically equivalent to S^2, we have shown that $S^2 \cong \mathbb{CP}^1$.

But in truth, the Riemann sphere is not just topologically a sphere; it can also be given the geometry of a sphere using tools from differential geometry. This means we can rotate the Riemann sphere as if it is literally S^2 with numbers stuck to points on its surface. Rotations of S^2 are given by the Special Orthogonal group

$$\mathrm{SO}(3) = \{O \in \mathrm{Mat}_{3\times3}(\mathbb{R}) : OO^T = I, \det O = 1\}, \quad (6)$$

but the rotations of the Riemann sphere are a group of functions mapping $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Our idea will be to show that these functions are naturally represented by SU(2) matrices when we move from the Riemann sphere to our spinor representation. The reason we want to do this is that plotting complex functions is very easy to do, so unlike with spinors and SU(2) matrices, we can actually visualise what these rotations look like.

Having summarised the connection between the Riemann sphere, $\mathbb{CP}^1$, and the Bloch sphere, we can now recap spinor construction. Let

$$\chi \in \mathbb{C}^2, \quad \chi = [\chi_1, \chi_2] \quad (7)$$

denote what will become a spinor, and since this will represent a quantum state, assume it is normalised such that $|\chi_1|^2 + |\chi_2|^2 = 1$. (Normally we would use column vectors here, but we'll stick to row vectors for now.) For a vector $(x, y, z) \in S^2$, we can use Eq. (1) to get the corresponding point on the Riemann sphere:

$$(x, y, z) \mapsto \frac{x + iy}{1 + z} = \zeta, \quad (8)$$

which we can suggestively write as the ratio of two complex numbers

$$\zeta = \frac{\chi_1}{\chi_2}, \quad \chi_1 = \alpha(x + iy), \quad \chi_2 = \alpha(1 + z). \quad (9)$$

where $\alpha \in \mathbb{C}$ is a normalisation constant. It turns out that

$$[\chi_1, \chi_2] = [\alpha(x + iy), \alpha(1 + z)] \quad (10)$$

is a spinor representation of (x, y, z) [2], and we can use the normalisation requirement to show that

$$|\alpha| = \frac{1}{\sqrt{2(1 + z)}}. \quad (11)$$

However, $\theta := \arg\alpha$ is a free parameter in this representation, so there is an infinite family of spinors that represent this S^2 vector, given by the general spinor representation

$$\chi = \left[\frac{x + iy}{\sqrt{2(1 + z)}}\mathrm{e}^{i\theta}, \sqrt{\frac{1 + z}{2}}\mathrm{e}^{i\theta}\right]. \quad (12)$$

This form can be immediately connected to the Bloch sphere, as the immeasurable $e^{i\theta}$ global phase has been extracted from the rest of the expression. Thus we can quotient it out to get the physical state

$$\chi \sim \left[\frac{x + iy}{\sqrt{2(1 + z)}}, \sqrt{\frac{1 + z}{2}}\right]. \quad (13)$$

If we are given a spinor $[\chi_1, \chi_2]$ with $\chi_2 \neq 0$, we can always divide by χ_2 to get

$$\chi \sim \left[\frac{x + iy}{1 + z}, 1\right] \in \left[\frac{x + iy}{1 + z} : 1\right] \sim \frac{x + iy}{1 + z} \in \overline{\mathbb{C}}, \quad (14)$$

so such a spinor can always be associated with a unique equivalence class $[\zeta : 1] \in \mathbb{CP}^1$, and hence with a unique complex number ζ on the Riemann sphere. Similarly, when $\chi_2 = 0$, the corresponding equivalence class is $[1 : 0]$, and hence the corresponding point on the Riemann sphere is ∞. This is the mathematical realisation of the equivalence of the Bloch sphere, Riemann sphere, and $\mathbb{CP}^1$. Furthermore, orthogonal spinors correspond to antipodal points on the Bloch sphere because the quotienting process makes points separated by a half turn about the origin (like $[x, y]$ and $[-x, -y]$) equivalent. Thus half turns in $\mathbb{C}^2$ become full turns in $\mathbb{CP}^1$, and quarter turns become half turns.

III. MÖBIUS TRANSFORMATIONS

Recall rotations of S^2 are elements of SO(3). The properties of this group can help us identify what kinds of functions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ represent rotations of the Riemann sphere. First, the functions must be smooth in the complex variable ζ since rotation is a smooth operation; this is codified by the fact that SO(3) is a Lie group, which is by definition a smooth manifold. (Don't worry if you don't know what either of those are, just accept that things rotate smoothly.) Second, the functions must be invertible and preserve the angles between any curves we draw on the sphere; this is because rotations are invertible and "rigid", as codifed by the conditions that $OO^T = I$ and $\det O = 1$. Such functions are called Möbius transformations, and they are the second-simplest type of complex function after linear functions. In this section, we will explore the basic properties of Möbius transformations to build intuition, before using them to explain how SU(2) relates to rotations in the next section.

Möbius transformations are always of the form

$$\varphi(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0. \quad (15)$$

We will denote the class of such functions as $\mathcal{M}$. The condition $ad - bc \neq 0$ is very important, as it ensures the numerator is not a multiple of the denominator. To see why this would be bad, consider the function

$$\phi(\zeta) = \frac{k(c\zeta + d)}{(c\zeta + d)}, \quad c, d, k \in \mathbb{C}. \quad (16)$$

Clearly, $\phi(\zeta) = k$ everywhere (except at $\zeta = -d/c$, where it is ∞), so neither is this function particularly interesting, nor is it invertible, meaning it could never represent a rotation (which by their nature are invertible). So we want to throw away functions like it. Why does $ad - bc \neq 0$ guarantee this? Basically, if there is some $k \in \mathbb{C}$ such that $(a\zeta + b) = k(c\zeta + d)$, then since this needs to happen for all $\zeta \in \mathbb{C}$, we're forced to have $a = kc$ and $b = kd$, and this means $ad - bc$ must be 0. But we can reverse this logic a bit: if $ad - bc \neq 0$, we can be sure there is no such $k \in \mathbb{C}$, and so the function is fine. The astute reader will notice a resemblance to the condition required for a 2×2 matrix to be invertible; this is not a coincidence (as we will discuss later), so for the time being we will let this similarity inspire us to abuse terminology and call $ad - bc$ the determinant of the Möbius transformation $\varphi(\zeta)$.

Suppose we take two Möbius transformations:

$$\varphi(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \quad \psi(\zeta) = \frac{p\zeta + q}{r\zeta + s}. \tag{17}$$

Recall that two rotations can be "added" together by performing one and then performing the other; mathematically, this means we are *composing* the rotations. Since we want to eventually link Möbius transformations to rotations, let's see what happens when we compose Möbius transformations:

$$(\varphi \circ \psi)(\zeta) = \frac{(ap + br)\zeta + (aq + bs)}{(cp + dr)\zeta + (cq + ds)}. \tag{18}$$

This looks like another Möbius transformation, and in fact we can simplify its determinant as

$$(ap+br)(cq+ds) - (aq+bs)(cp+dr) = (ad-bc)(ps-qr),$$

which we know is non-zero since the determinants of φ and ψ are non-zero. So composing Möbius transformations always yields another Möbius transformation. However, we can't flip the order of composition around:

$$\left(\frac{1}{\zeta}\right) \circ (\zeta + 1) = \frac{1}{\zeta + 1} \neq \frac{1 + \zeta}{\zeta} = (\zeta + 1) \circ \left(\frac{1}{\zeta}\right). \tag{19}$$

Luckily this isn't an issue for us, since rotations don't usually commute either in three dimensions. But what about that requirement of invertibility? It's easy to show that

$$\varphi^{-1}(\zeta) = \frac{d\zeta - b}{-c\zeta + a}, \tag{20}$$

and since its determinant is also $ad - bc$, it is a Möbius transformation as well (as we would hope). Finally, the identity function (which we would expect to represent rotation by angle zero) is also in $\mathcal{M}$, as

$$\mathrm{Id}(\zeta) = \zeta = \frac{\zeta}{1} = \frac{1\zeta + 0}{0\zeta + 1}. \tag{21}$$

These properties mean that $\mathcal{M}$ forms a group under function composition, so Möbius transformations have enough algebraic structure that they could conceivably (though not yet confirmedly) be used to represent rotations.

Another important property of Möbius transformations that we will need is "3-transitivity". Given any three inputs $\alpha_1, \beta_1, \gamma_1 \in \overline{\mathbb{C}}$ and any three outputs $\alpha_2, \beta_2, \gamma_2 \in \overline{\mathbb{C}}$, 3-transitivity means there exists a unique Möbius function with

$$\alpha_1 \mapsto \alpha_2, \beta_1 \mapsto \beta_2, \gamma_1 \mapsto \gamma_2.$$

Proving this is a little out of the scope of this article, but the idea is to first show that a Möbius function satisfying the condition

$$(\alpha, \beta, \gamma) \mapsto (1, 0, \infty)$$

is forced to have the unique form

$$f(\zeta) = \frac{\zeta - \beta}{\zeta - \gamma} \cdot \left(\frac{\alpha - \gamma}{\alpha - \beta}\right), \tag{22}$$

except when $\gamma = \infty$, where it has the unique form

$$f(\zeta) = \frac{\zeta - \beta}{\alpha - \beta}. \tag{23}$$

Then if we want $\alpha_1 \mapsto \alpha_2, \beta_1 \mapsto \beta_2, \gamma_1 \mapsto \gamma_2$, we just take the Möbius functions

$$f \in \mathcal{M} \text{ such that } (\alpha_1, \beta_1, \gamma_1) \mapsto (1, 0, \infty) \tag{24}$$

$$g \in \mathcal{M} \text{ such that } (\alpha_2, \beta_2, \gamma_2) \mapsto (1, 0, \infty) \tag{25}$$

and leveraging Eq. (20), we get that

$$g^{-1} \circ f \in \mathcal{M} \tag{26}$$

gives the required mapping. Uniqueness follows from the uniqueness of Eqs. (22) and (23), the uniqueness of inverses in groups (which ensures g^{-1} above is unique), and the cancellation law for groups. In particular, if $\psi, \phi \in \mathcal{M}$ both satisfy $\alpha_1 \mapsto \alpha_2, \beta_1 \mapsto \beta_2, \gamma_1 \mapsto \gamma_2$, then the uniqueness of f implies

$$f = g \circ \psi = g \circ \phi, \tag{27}$$

so that by the cancellation law we must have

$$g^{-1} \circ f = \psi = \phi, \tag{28}$$

and hence $g^{-1} \circ f$ is unique.

We now return to the fact that the $ad - bc \neq 0$ condition resembles the condition for the invertibility of a 2×2 matrix. It turns out that the map

$$\mathcal{M} \to \mathrm{PGL}(2, \mathbb{C}), \quad \frac{a\zeta + b}{c\zeta + d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{29}$$

is a group isomorphism from the Möbius transformations under composition to the Projective General Linear group of degree 2 $\mathrm{PGL}(2, \mathbb{C})$ under matrix multiplication. It's easy to see this is an isomorphism since the map is clearly bijective and taking the product of the matrix representations of φ, ψ in Eq. (17)

$$\varphi(\zeta) \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \psi(\zeta) \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{30}$$

gives

$$\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} \tag{31}$$

which is just the representation of $(\varphi \circ \psi)(\zeta)$ (c.f. Eq. (18)). But these matrices are of the right dimensionality to operate on spinors, and as we'll see, this isomorphism leads to the identification of SU(2) as the natural matrix group for transforming spinors.

For those unfamiliar, the General Linear group $\mathrm{GL}(2,\mathbb{C})$ is the set of invertible 2×2 matrices with complex entries, while the projective version $\mathrm{PGL}(2,\mathbb{C})$ is just this group modulo scalar multiplication of matrices. In other words, for all invertible matrices we let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \forall \lambda \in \mathbb{C}, \lambda \neq 0 \tag{32}$$

and then $\mathrm{PGL}(2,\mathbb{C})$ is just the group of equivalence classes of these. In fact, if $A \in \mathrm{GL}(2,\mathbb{C})$ has determinant m, then by standard properties of the determinant

$$\det\left(m^{-2}A\right) = 1, \tag{33}$$

so we can always choose a representative from the Special Linear Group $\mathrm{SL}(2,\mathbb{C})$ (i.e. invertible matrices with determinant 1), meaning that $\mathrm{PGL}(2,\mathbb{C})$ is the same group as the projective version of $\mathrm{SL}(2,\mathbb{C})$, called $\mathrm{PSL}(2,\mathbb{C})$. This is important because we want to show that Riemann sphere rotations are linked to SU(2), which is a subset of $\mathrm{SL}(2,\mathbb{C})$, and so at the very least we need to know that Möbius transformations can generally be linked to determinant 1 matrices.

More generally, we shouldn't be too surprised that projective groups are appearing here: the elements of linear groups act on vector spaces like $\mathbb{C}^2$, but elements of projective linear groups act on elements of projective spaces like $\mathbb{CP}^1$. So the existence of this isomorphism is a reflection of the fact that the Riemann sphere and $\mathbb{CP}^1$ (and the Bloch sphere) are the same space expressed in different ways, with corresponding ways of expressing transformations of the space.

IV. SPINNING SPINORS (HALFWAY) AROUND

Now, we're not actually interested in all of $\mathcal{M}$, only those Möbius functions that can be identified with rotations of S^2 (since these correspond to rotating our state space), and as SO(3) is a group, we expect these to form a subgroup of $\mathcal{M}$. Finding this subgroup algebraically is tricky, but amounts to using 3-transitivity and the fact that when we rotate the Riemann sphere, the two points sitting at the poles of the rotation axis and the dual great circle (i.e. the equator relative to the axis of rotation) remain unchanged. If we label the rotation poles N and S and a point on the dual great circle E, 3-transitivity says the unique Möbius transformation representing rotation by angle θ about the axis through N and S is the one that sends $N \mapsto N, S \mapsto S$, and $E \mapsto$... well, this is where things get hard. Parameterising the dual great circle in terms of θ and then tracking where E is mapped to is hard to do in general, but can be done for the special cases of rotation about the x, y, and z axes. Fortunately, every rotation can be generated by composing rotations about these axes (i.e. they are generators for SO(3)), so if we find these functions, we can extrapolate their matrix forms and hence find the generators for the matrix group representing rotations for spinors.

For rotation about the x-axis by angle θ, we need to fix the poles (of the rotation, not the Riemann sphere) with $1 \mapsto 1$ and $-1 \mapsto -1$, and then the North Pole ($E = 0$, which lies on the equator of this rotation) should be mapped to a point with spherical coordinates $(\pi/2, \theta)$. So by Eq. (3) we have

$$0 \mapsto \tan(\theta/2)e^{i\pi/2} = i\tan(\theta/2). \tag{34}$$

This is the origin of the $\theta/2$ dependency of spinor rotations. As per the process in Eqs. (22) to (26), we choose the unique

$$f \in \mathcal{M} \text{ such that } (1, 0, -1) \mapsto (1, 0, \infty) \tag{35}$$
$$g \in \mathcal{M} \text{ such that } (1, i\tan(\theta/2), -1) \mapsto (1, 0, \infty), \tag{36}$$

which are given by

$$f(\zeta) = \frac{2\zeta}{\zeta+1} \tag{37}$$
$$g(\zeta) = \frac{2\zeta - 2i\tan(\theta/2)}{(1 - i\tan(\theta/2)\zeta + (1 - i\tan(\theta/2)))}, \tag{38}$$

and then conclude that the Möbius transformation we want is

$$\mathcal{R}_x(\zeta;\theta) = g^{-1} \circ f \tag{39}$$
$$= \frac{2\zeta + 2i\tan(\theta/2)}{(2i\tan(\theta/2))\zeta + 2} \tag{40}$$
$$= \frac{\cos(\theta/2)\zeta + i\sin(\theta/2)}{i\sin(\theta/2)\zeta + \cos(\theta/2)}. \tag{41}$$

By an analogous argument for the y-axis, we have

$$\mathcal{R}_y(\zeta;\theta) = \frac{\cos(\theta/2)\zeta + \sin(\theta/2)}{-\sin(\theta/2)\zeta + \cos(\theta/2)}, \tag{42}$$

and while we could do the same for rotation about the z-axis, there is a simpler way. Notice from Eq. (3) that the stereographic projection has rotational symmetry about the z-axis, meaning that the circle with constant polar angle $\theta = \phi$ on the sphere is mapped to the circle with radius $\tan(\phi/2)$ in $\mathbb{C}$. Thus rotation about the z-axis by an angle θ (pardon the overload of notation) is equivalent to multiplication by $e^{i\theta}$:

$$\mathcal{R}_z(\zeta;\theta) = e^{i\theta}\zeta = \frac{e^{i\theta/2}\zeta}{e^{-i\theta/2}} = \frac{e^{i\theta/2}\zeta + 0}{0\zeta + e^{-i\theta/2}} \in \mathcal{M}. \tag{43}$$

Technically, we could have split $e^{i\theta}$ into any $\frac{e^{i(\theta-\gamma)}}{e^{-i\gamma}}$, but we chose $\gamma = \theta/2$ so that the coefficients also depend on the half angle $\theta/2$, keeping consistency with $\mathcal{R}_x$ and $\mathcal{R}_y$. See https://github.com/RCPN/Spinors for interactive plots of these functions, so you can play around with them and get a feel for what the rotations look like under stereographic projection. (We couldn't include snapshots here because complex plots require colour and this article had to be printed in greyscale.)

Now, the matrix equivalents of these transformations are just

$$R_x(\theta) = \begin{bmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \tag{44}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \tag{45}$$

$$R_z(\theta) = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}, \tag{46}$$

and those of you who have taken QFT are probably very excited right now, as these are the matrices that we usually use to rotate spinors, and in particular taking their derivatives at $\theta = 0$ yields the usual generators of SU(2) (i.e. $i/2$ times the Pauli matrices):

$$R'_x(0) = \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{47}$$

$$R'_y(0) = \frac{i}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{48}$$

$$R'_z(0) = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{49}$$

So we've finally figured out why spinors have the strange tendency to rotate with half the angle that we rotate the state space by. Since converting from states to spinors requires going through the stereographic projection, spinors only see the half-angle measured from the South Pole instead of the full angle measured from the centre of the Bloch sphere when rotating about the x and y axes. Rotation around the z-axis doesn't suffer from this problem, but it is natural to choose a representation of it that transforms consistently with the other two, which is why the half-angle appears there. Then since any rotation can be constructed from these three, we get that general rotations depend on the half angle as well.

As a cherry on top, we can also see the famous result that SU(2) is a double cover of SO(3) (in that it has two matrices that correspond to each rotation). If we multiply every coefficient in a Möbius transformation by -1, then by fraction cancellation this multiple disappears. This means that while the SO(3) rotations are one-to-one with the rotational subgroup of Möbius transformations, that subgroup has two SU(2) matrix representatives per element (differing only by a scalar multiplication of -1), and hence each SO(3) rotation must have exactly two corresponding SU(2) representations.

Thus we see that spinors, SU(2) matrices, and all their seemingly unintuitive properties do arise naturally if we think about rotations of the sphere as being equivalent to rotations of the Riemann sphere — and we didn't need to touch Quaternions, Lie Theory, or Clifford Algebras to do it! A little geometric intuition can go a long way in mathematics, and you can often find new ways of looking at things if you approach them from the perspective of a different field. Regardless of your level of physics or mathematics, we hope these articles have given you a new way of seeing rotations, and an appreciation for why spinors should appear if we try to represent them using a two-dimensional vector space. (Though if you do ever find yourself trying to handle rotations in anything higher than three dimensions, do yourself a favour and do it from the perspective of Clifford Algebra. It's definitely the way to go!)

[1] G. A. Jones and D. Singerman, *Complex functions: an algebraic and geometric viewpoint* (Cambridge university press, 1987).

[2] G. Fano and S. Blinder, *Twenty-first century quantum mechanics: Hilbert space to quantum computers* (Springer, 2017).