Madison College Textbook for College Mathematics 804-107

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How to use this workbook

Each chapter consists of text plus worked examples. These are followed by **Exercises** labeled as ***Your Turn***. You should work through each chapter checking the answers of the **Example** problems on your calculator. After this you should **work** the exercises. Working through the chapter sample exam will help you review all of the material in the chapter. **Good Luck!**

9/22/2016

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Chapter 1 **Pre-Algebra**

Section 1.1 Calculator Use

Throughout most of human history computation has been a tedious task that was often postponed or avoided entirely. It is only in the last generation that the use of inexpensive handheld calculators has transformed the ways that people deal with quantitative data. Today the use and understanding of electronic computation is nearly indispensable for anyone engaged in technical work. There are a variety of inexpensive calculators available for student use. Some even have graphing and/or symbolic capabilities. Most newer model calculators such as the Casio models *fx-300W*, *fx-300MS*, *fx-115MS*, and the Texas Instruments models TI-30X IIB, TI-30X IIS, TI-34 II enter calculations in standard "algebraic" format. Older calculators such as the Casio *fx-250HC* and the Texas Instruments TI-30Xa and TI-36X enter some calculations in a "reverse" format. Both types of calculators are priced under twenty dollars, yet possess enough computational power to handle the problems faced in most everyday applications. The Casio series is fairly representative of the "newer" format calculators and the TI-30Xa is typical of "older" format ones. While other calculator models have similar or even better features for performing the required computations, the reader will be responsible for learning their detailed use. **Never** throw away the user's manual!

In order to perform a computation, the correct keystrokes must be entered. Although calculators differ in the way keystrokes are entered, this text attempts to provide the reader with a couple of different keystroke options for each example problem in this chapter. The reader should practice the order in which to press the keys on the calculator while reading through the examples. This practice will ensure that the reader knows how to use his/her particular calculator. In order to indicate the sequence of keystrokes the following notation will be used. Digits (0 through 9 plus any decimal point) will be presented in normal typeface. Any additional keystrokes will be enclosed in boxes. For example, to multiply 7 times 8, the command sequence will be written as $7 \boxed{\times}\ 8 \boxed{=}$ and 56 appears on the display.

Section 1.2 Order of Operations

Mathematical expressions which involve more than one operation appear ambiguous. For example, is

$$5+6\times 2=5+12=17 \quad \textbf{or} \quad 5+6\times 2=11\times 2=22?$$

To clarify this question, mathematics has developed the following hierarchy of computations called **order of operations**.

1. Perform all operations that appear in grouping symbols first. If grouping symbols are nested, do the innermost first.

2. Raise all bases to powers in the order encountered moving from left to right.

3. Perform all multiplications/divisions in the order encountered moving from left to right.

4. Perform all additions/subtractions in the order encountered moving from left to right.

Here grouping symbols means parentheses ( ), brackets [ ], braces { }, etc. An example of a nested expression is $(6+2\times(4+1))\div 8$. The innermost grouping symbol is (4+1) so the result is $(6+2\times 5)\div 8=(6+10)\div 8=16\div 8=2$. Raising a base to a power (also known as an **exponent)** means repeated multiplication of the base as in $6^3=6\times 6\times 6=216$.

Newer calculators often use ^ to indicate exponentiation. Older calculators such as the TI-30Xa use a button labeled y^x or x^y. We will cover exponents in greater detail in both **Section 1.4** and **Chapter 2**.

In the original problem posed above the multiplication of 6 with 2 is performed before the addition of 5. The proper answer is therefore 17. The other interpretation could be achieved by using parentheses $(5+6)\times 2=11\times 2=22$.

Order of operations is **built into** all scientific calculators. That is, if you enter the keystrokes in the correct order, the calculator will automatically perform the correct calculation.

In many formulas x occurs as a variable, but then confusion with the times sign can result. To avoid this, alternative symbols for multiplication are used. They are the dot notation and adjacent parentheses as in $7\times 3=7\cdot 3=(7)(3)=21$. Newer calculators recognize that adjacent parentheses means multiplication, but the TI-30Xa does **not**. On the TI-30Xa the times operation $\times$ must be inserted between the parentheses.

Division is also indicated by a variety of notations. For example, the following all mean 34 divided by 17:

$$34\div 17=34/17=\frac{34}{17}=17\overline{)34}=2 \quad .$$

In addition to parentheses, brackets and braces, certain symbols act as **implied grouping symbols**. The most important of these are the fraction bar and the radical or root symbol. The fraction bar acts to separate the numerator from the denominator. If either or both of the numerator or denominator consist of an expression with operations, these must be performed first before the division indicated by the fraction bar. For example,

$$\frac{7+3}{2+3}=\frac{10}{5}=2$$

To perform this computation on the calculator, parentheses need to be inserted around both the numerator and the denominator $\boxed{(}\boxed{7}\boxed{+}\boxed{3}\boxed{)}\boxed{\div}\boxed{(}\boxed{2}\boxed{+}\boxed{3}\boxed{)}\boxed{=}$. Parentheses are the only grouping symbol the calculator recognizes or uses.

The root or radical symbol also acts as a grouping symbol. Any calculation inside the square root needs to be completed before the root is taken. For example,

$$\sqrt{25+144}=\sqrt{169}\ =13$$

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol. On newer calculators enter the following keystrokes:

$$\boxed{\sqrt{\ }}\boxed{(}25\boxed{+}144\boxed{)}\boxed{=}$$

On the TI-30Xa enter the keystrokes:

$$\boxed{(}25\boxed{+}144\boxed{)}\boxed{\sqrt{\ }}$$

Note: On "newer" calculators like the Casio series one enters expressions the way "they look", i.e., the square root symbol comes first. On older models like the TI-30Xa most functions like $\sqrt{\ }$ come after the expressions they are to evaluate. In any case, using parentheses keys when necessary is a good habit to acquire. Failure to do so usually results in wrong answers!

Your Turn!!

Perform the following arithmetical operations

1.	$47 + 271$	=__________	2.	$19 + 67 + 43$	=__________
3.	$269 + 151 – 237$	=__________	4.	$135+932–612$	=__________
5.	45×125	=__________	6.	$167\times134 - 21679$	=__________
7.	$68\times42\div136$	=__________	8.	$72498\div129$	=__________
9.	$12\times9-8\times7\div(4\times2)$	=__________	10.	$12\times9-8\times7\div4\times2$	=__________
11.	$32-3\times9+16\div4\times2$	=__________	12.	$4(19-12)\div2-11$	=__________
13.	$12+\frac{18}{3}-5$	=__________	14.	$\frac{561}{51}$	=__________
15.	$\frac{12\times5\times4}{6\times2+3}$	=__________	16.	$\frac{5+13+2}{1+3}$	=__________

From a board 10 feet long a piece 29 inches was cut off. How long is the piece remaining? Ignore the width of the cut.

17) ________________

In 1993 General Motors had total sales of \$133,621,900,000 , while Ford Motor Company had sales of \$108,521,000,000 . How much more were GM's sales than Ford's?

18) _______________

If 128 copies of a software package cost \$5760, what is the cost per copy?

19) _______________

A carpenter earns \$17 per hour plus time and a half for overtime (more than 40 hours in one week). If she works 58 hours one week, what is her gross pay for that week?

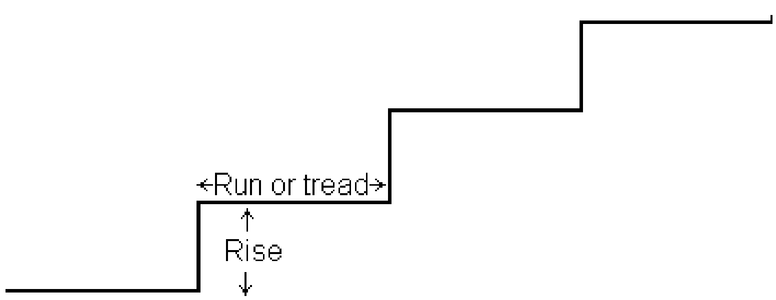
20) _______________

You order 15 DVD's at \$8 a DVD and 24 books at \$5 a book. What is the cost of the order?

21) _______________

A truck averages 16 miles per gallon and has a 25 gallon gas tank. What is the furthest distance the truck can travel without stopping for gas?

22) _______________



A stairway consists of rises of 6 in and must reach a height of 10 feet. How many rises are needed?

23) _______________

A stairway consists of 4 in rises and treads of 18 in. If the height of the stairs is 4 feet, what is the distance taken up by the stairway on the lower floor?

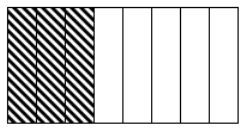
24) _______________

Section 1.3 Fractions

Fractions are ratios of whole numbers, which allow us to express numbers which are between the whole numbers. For example,

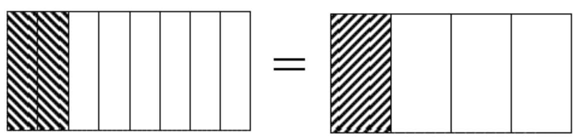
$$2\frac{2}{3} = 2 + \frac{2}{3} \text{ is between 2 and 3 .}$$

Fractions represent "part of a whole". Imagine that we have a freight car with eight equal sized compartments. If three of these compartments are full of grain, we would indicate that we have three eighths of a freight car's worth of grain. This is illustrated below.



$$= \frac{3}{8}$$

Consider a car with eight compartments of which two are full. The fraction of a full car is two eighths. If we look at the same car split into four equal compartments, this same amount of grain fills one fourth of the car. We arrive at the following result.



$$\frac{2}{8} = \frac{1}{4}$$

We say that such equal fractions while they "look different" are **equivalent**. To generate equivalent fractions, we can multiply or divide both numerator (the top number) and denominator (the bottom number) by a common number. So we have the following fractions equivalent to two thirds.

$$\frac{2}{3} = \frac{2\times 3}{3\times 3} = \frac{6}{9}$$
$$= \frac{2\times 17}{3\times 17} = \frac{34}{51}$$

Similarly, $\frac{18}{24} = \frac{18\div 6}{24\div 6} = \frac{3}{4}$. This same result could be stated in terms of "canceling" the common factor of 6 between the numerator and denominator.

$$\frac{18}{24} = \frac{6\times 3}{6\times 4} = \frac{\cancel{6}\times 3}{\cancel{6}\times 4} = \frac{3}{4} .$$

If a fraction has no common factors between its numerator and denominator, the fraction is said to be in lowest terms.

There are three types of fractions.

1. Proper fractions with the numerator less than (symbolized by <) the denominator. All proper fractions are less than 1.

2. Improper fractions with the numerator greater than (symbolized by >) the denominator. All improper fractions are greater than 1 . Improper fractions can be expressed as a **mixed number**, which is a whole number plus a proper fraction. For example,
$$\frac{25}{6} = 6\overline{)25} = 4 + \frac{1}{6} = 4\frac{1}{6} \quad .$$
3. Unit fractions with the numerator equal to the denominator. All unit fractions are equal to one. For example, $\frac{19}{19} = \frac{4}{4} = \frac{25}{25} = \frac{1}{1} = 1$.

Note: when we write $4\frac{1}{6}$, we are using a shorthand notation. There really **is** a + sign between the 4 and the one sixth that's understood but unstated. Working backwards we can convert a mixed number into an improper fraction. For example,
$$7\frac{2}{3} = \frac{7 \times 3 + 2}{3} = \frac{23}{3} \quad .$$

Fractions can be entered on many calculators using the $\boxed{a\frac{b}{c}}$ key. For example, use the following keystrokes to enter the fraction $\frac{14}{24}$: 14 $\boxed{a\frac{b}{c}}$ 24 $\boxed{=}$.

$\boxed{7\lrcorner 12}$ will then appear in the display as the fraction reduced to lowest terms.

For a mixed number such as, $11\frac{5}{6}$ enter the following keystrokes:

11 $\boxed{a\frac{b}{c}}$ 5 $\boxed{a\frac{b}{c}}$ 16 $\boxed{=}$. Some calculators display $\boxed{11\lrcorner 5\lrcorner 6}$, while other calculators display $\boxed{11\_5\lrcorner 6}$. To change this answer to the improper fraction $\boxed{\frac{181}{16}}$, enter $\boxed{\text{shift}}$ $\boxed{a\frac{b}{c}}$ on some calculators, or $\boxed{2nd}$ $\boxed{a\frac{b}{c}}$ on other calculators. Also, on some calculators the $\boxed{a\frac{b}{c}}$ key when pressed after entering a fraction converts it to a decimal and if $\boxed{a\frac{b}{c}}$ is pressed a second

time the decimal is converted back to a fraction. On other calculators fraction – decimal conversions are performed by entering $\boxed{2nd}$ $\boxed{\leftarrow}$. The $\boxed{\leftarrow}$ key is also the back space key, which deletes characters in the display.

As an application, solve for the following missing numerator: $\frac{\ }{12}=\frac{6}{8}$. As a first step reduce $\frac{6}{8}$ to lowest terms as $\frac{3}{4}$. So $\frac{\ }{12}=\frac{3}{4}$. The first denominator 12 is three times the second denominator 4, so the missing numerator must be three times the second numerator 3. The answer is that the missing numerator is 9.

To compare two fractions and determine which is larger, we can use the following procedure:

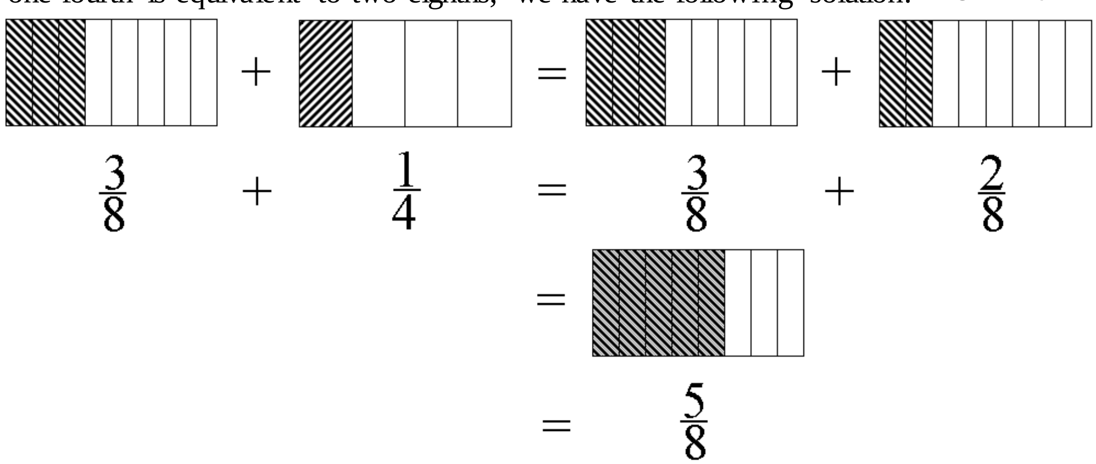
1. If the fractions involve mixed numbers with proper fractions, the number with the larger whole number is the larger number. For example,

$$7\frac{3}{16} > 5\frac{7}{8}\text{, since } 7 > 5 .$$

2. If the fractions are both proper fractions or mixed numbers with equal whole numbers, then convert the fractions into decimals. The number with the larger decimal is the larger number. For example,

$$5\frac{1}{3} < 5\frac{3}{8}\text{, since } \frac{3}{8} = 0.375 > 0.333... = \frac{1}{3} .$$

To add or subtract fractions we need a common denominator. Consider adding $\frac{3}{8}$ to $\frac{1}{4}$. Since one fourth is equivalent to two eighths, we have the following solution:



If mixed numbers are involved, we first deal with the whole numbers, then the fractions. For example,

$$7\frac{1}{2}-5\frac{9}{16}=7-5+\frac{1}{2}-\frac{9}{16}=2+\frac{8}{16}-\frac{9}{16}=1+\frac{16}{16}+\frac{8}{16}-\frac{9}{16}=1+\frac{16+8-9}{16}=1\frac{15}{16} .$$

Note: Since $\frac{9}{16} > \frac{8}{16}$, we had to "borrow" $\frac{16}{16}$ from the 2.

To multiply fractions we form the product of the numerators over the product of the denominators. For example,

$$\frac{5}{8} \times \frac{3}{4} = \frac{5 \times 3}{8 \times 4} = \frac{15}{32}.$$

If the product involves mixed numbers, we first convert them to improper fractions. For example,

$$2\frac{5}{6} \times 4\frac{1}{5} = \frac{2 \times 6 + 5}{6} \times \frac{4 \times 5 + 1}{5} = \frac{17}{6} \times \frac{21}{5} = \frac{17}{\cancel{3} \times 2} \times \frac{\cancel{3} \times 7}{5} = \frac{17 \times 7}{2 \times 5} = \frac{119}{10} = 11\frac{9}{10}.$$

Note: we canceled the common factor of 3 between numerator and denominator in this calculation. In a multiplication problem this can always be done and saves the effort of later having to reduce the final answer. Also note that the answer is "reasonable" in that

$$2\frac{5}{6} \approx 3 \text{ and } 4\frac{1}{5} \approx 4 \text{ , so } 2\frac{5}{6} \times 4\frac{1}{5} \approx 3 \times 4 = 12.$$

A quick estimation like this can often catch silly mistakes even when using a calculator!

Consider the division problem $8 \div 2 = 4$. This is the same as $\frac{8}{2} = \frac{8}{1} \times \frac{1}{2} = 8 \times \frac{1}{2} = 4$.

More generally, any division problem can be expressed as follows:

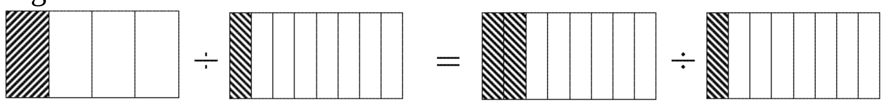
$$a \div b = \frac{a}{b} = \frac{a}{1} \times \frac{1}{b} = a \times \frac{1}{b}$$

This means that division by the number b is equivalent to multiplication by the fraction $\frac{1}{b}$. The fraction $\frac{1}{b}$ is called the **reciprocal** of $b = \frac{b}{1}$. To form the reciprocal of a number we exchange the numerator with the denominator. In summary, division by a **non-zero** number equals multiplication by the reciprocal of that number.

In a division problem 0 is never allowed as the denominator or divisor. The reason for this is as follows. Suppose $20 \div 0 = \frac{20}{0}$ made sense. Then there would be some number, a, which is the answer to this division problem. Restating this as a multiplication problem would give $a \times 0 = 20$. But any number times zero gives zero! So no sensible answer to $20 \div 0 = \frac{20}{0}$ exists.

Another way of explaining this goes to the very meaning of division. $20 \div 4 = \frac{20}{4} = 5$ means that 20 contains five 4's. How many 0's does 20 contain? There's no sensible answer to the question!

Consider now the division $\frac{1}{4} \div \frac{1}{8}$, from the diagram below it is clear that one fourth contains 2 one eighths. So the answer must be 2.



$$\frac{1}{4} \div \frac{1}{8} = 2$$

$$\frac{1}{4} \times 8 = \frac{8}{4}$$

$$= 2$$

The following shows that this result is consistent with the multiplication by the reciprocal definition of division.

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = \frac{8}{4} = 2$$

If the division involves mixed numbers, we first convert them into improper fractions. For example,

$$4\frac{1}{2} \div 1\frac{1}{8} = \frac{9}{2} \div \frac{9}{8} = \frac{\cancel{9}}{2} \times \frac{8}{\cancel{9}} = \frac{8}{2} = 4.$$

In expressions which combine operations the standard order of operations apply as shown in the following:

$$\frac{2}{3} \times 2\frac{1}{2} - 1\frac{1}{4} \div 3 = \frac{\cancel{2}}{3} \times \frac{5}{\cancel{2}} - \frac{5}{4} \times \frac{1}{3} = \frac{5}{3} - \frac{5}{12} = \frac{5 \times 4}{3 \times 4} - \frac{5}{12} = \frac{20-5}{12} = \frac{15}{12} = 1\frac{3}{12} = 1\frac{\cancel{3} \times 1}{\cancel{3} \times 4} = 1\frac{1}{4}.$$

These calculations are all easily performed on the calculator. The keystokes for the previous calculation are as follows:

$2\ \boxed{a\frac{b}{c}}\ 3\ \boxed{\times}\ 2\ \boxed{a\frac{b}{c}}\ 1\ \boxed{a\frac{b}{c}}\ 2\ \boxed{-}\ 1\ \boxed{a\frac{b}{c}}\ 1\ \boxed{a\frac{b}{c}}\ 4\ \boxed{\div}\ 3\ \boxed{=}$.

More involved calculations with grouping symbols are also possible. For example,

$$3\frac{3}{4} - \left(5\frac{3}{16} - 3\frac{7}{8}\right) \div 2\frac{1}{2} = 3\frac{3}{4} - \left(2 + \frac{3}{16} - \frac{14}{16}\right) \div \frac{5}{2} = 3\frac{3}{4} - \left(1 + \frac{16+3-14}{16}\right) \times \frac{2}{5} =$$

$$= 3\frac{3}{4} - 1\frac{5}{16} \times \frac{2}{5} = 3\frac{3}{4} - \frac{21}{16} \times \frac{2}{5} = 3\frac{3}{4} - \frac{21}{40} = 3 + \frac{30\text{-}21}{40}$$

$$= 3\frac{9}{40}$$

This is keystroked as follows:

$3\ \boxed{a\frac{b}{c}}\ 3\ \boxed{a\frac{b}{c}}\ 4\ \boxed{-}\ \boxed{(}\ 5\ \boxed{a\frac{b}{c}}\ 3\ \boxed{a\frac{b}{c}}\ 16\ \boxed{-}\ 3\ \boxed{a\frac{b}{c}}\ 7\ \boxed{a\frac{b}{c}}\ 8\ \boxed{)}\ \boxed{\div}\ 2\ \boxed{a\frac{b}{c}}\ 1\ \boxed{a\frac{b}{c}}\ 2\ \boxed{=}$.

Your Turn!!

Write as an improper fraction.

1. $6\frac{11}{16}$ = ________

2. $3\frac{7}{10}$ = ________

Write as a mixed number reduced to lowest terms.

3. $\frac{19}{8}$ = ________

4. $\frac{26}{10}$ = ________

Reduce to lowest terms.

5. $\frac{9}{12}$ = ________

6. $7\frac{18}{32}$ = ________

Supply the missing numerators:

7. $\frac{3}{4} = \frac{?}{12}$ = ________

8. $3\frac{3}{4} = \frac{?}{8}$ = ________

Indicate which number is larger.

9. $\frac{5}{8}$ **or** $\frac{19}{32}$ ________

10. $3\frac{7}{16}$ **or** $2\frac{3}{4}$ = ________

Perform the indicated operations and express the answer as a fraction in lowest terms:

11. $\frac{3}{16} \times \frac{4}{9}$ = ________

12. $6 \div \frac{2}{3}$ = ________

13. $\frac{3}{8} + \frac{5}{8}$ = ________

14. $\frac{13}{32} - \frac{9}{32}$ = ________

15. $4\frac{1}{8} \times 2\frac{1}{4}$ = ________

16. $\frac{5}{8} \div \frac{15}{16}$ = ________

17. $5\frac{3}{8}+3\frac{11}{32}$ = ________________

18. $5\frac{3}{8}\div 2\frac{3}{4}$ = ________________

19. $2\frac{1}{4}\times 3\frac{2}{3}$ = ________________

20. $7\frac{19}{32}-5\frac{3}{4}-1\frac{5}{8}$ = ________________

21. $3\frac{1}{2}\times 3\frac{1}{4}-10\frac{5}{8}$ = ________________

22. $1\frac{3}{8}-4\frac{3}{4}\div 4$ = ________________

Solve and state all results as fractions reduced to lowest terms.

How many pieces of $\frac{5}{16}$ inch thick plywood are in a stack 35 inches high?

23) ________________

A lumberyard sells lumber only in even foot lengths. What is the shortest single board of lumber from which a carpenter could cut three $3\frac{1}{4}$ ft long and two $2\frac{3}{4}$ ft long pieces?

24) ________________

A cubic foot contains about $7\frac{1}{2}$ gallons. How many cubic feet are there in 120 gallons?

25) ________________

A nail $3\frac{1}{2}$ inches long, goes through a board $2\frac{3}{8}$ inches thick supporting a joist. How far into the joist does the nail extend?

26) ________________

A part is measured as $2\frac{7}{16}$ inches long on a scale drawing. If the scale is one foot to $\frac{1}{2}$ inch, how long is the actual part?

27) ________________

Section 1.4 Decimals

The difficulty in adding or subtracting fractions "by hand" compared to adding or subtracting whole numbers is obvious to anyone who has done such calculations. This difficulty motivated the development of representing fractions as decimal numbers. Using decimal fractions all arithmetical operations are similar to computations with whole numbers. The only complication is keeping track of the position of the decimal point.

The basis of the decimal representation of numbers is the use of **place value**. This allows us to represent an infinite range of numbers with only ten symbols (the digits 0 through 9). Contrast this with Roman Numerals or other early number systems where new symbols are constantly added to represent larger values. Place value uses the powers of 10.

10^0	=	1	(The **definition** of an exponent of 0)
10^1	=	10	
10^2	=	100	
10^3	=	1,000	
10^4	=	10,000	
10^5	=	100,000	
10^6	=	1,000,000	
etc.			

Note: 10^n is equal to 1 followed by n zeros.

When we write a number such as 27,483, the digit 2 stands not for 2, but for $2(10^4) = 20{,}000$. The digit 7 represents $7(10^3) = 7000$, etc. The value we associate with each digit comes from its place in the number. The right most digit of a whole number is in the "one's place", the second digit from the right is the "ten's place", etc. To extend the decimal system to fractions, we use the reciprocal powers of 10 and the decimal point to separate the "one's place" from the "tenth's place".

$$10^{-1} = \frac{1}{10^1} = 0.1 \qquad 10^{-2} = \frac{1}{10^2} = 0.01 \qquad 10^{-3} = \frac{1}{10^3} = 0.001 \qquad 10^{-4} = \frac{1}{10^4} = 0.0001$$

$$10^{-5} = \frac{1}{10^5} = 0.00001$$

$$10^{-6} = \frac{1}{10^6} = 0.000001$$

In general $10^{-n} = \frac{1}{10^n}$ is a decimal point followed by $n-1$ zeros.

The leading 0 to left of the decimal point is not required for a number smaller than 1. It is used to emphasize the location of the decimal point.

A decimal fraction such as 0.375 is interpreted as

$$0.375 = 3\times 10^{-1} + 7\times 10^{-2} + 5\times 10^{-3}$$
$$= \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{375}{1000} = \frac{3\times 125}{8\times 125} = \frac{3}{8}$$

Note: adding extra zeros to the right of the rightmost digit to the right of the decimal point does **not** change the value of the decimal fraction. It does, however, imply a greater knowledge of the precision of the value.

A decimal fraction like 0.375 is called a **terminating** decimal because the digits to the right of the decimal point come to an end. The procedure outlined above is how to convert a terminating decimal to a fraction. It is summarized below:

1. Carry along the digits to the left of the decimal point as the whole number part of the resulting mixed number. If there are no non-zero digits to the left of the decimal point, the decimal represents a proper fraction.

2. Put the digits to the right of the decimal point over the power of 10 that goes with the right most decimal place. For example, in converting 0.1145, 1145 is put over 10,000 since the right most digit, 5, is in the ten-thousandth's place.

$$0.1145 = \frac{1145}{10000} = \frac{\cancel{5}\times 229}{\cancel{5}\times 2000} = \frac{229}{2000} \quad .$$

3. Reduce this fraction to lowest terms.

To convert a fraction to a decimal is quite easy. We just translate the fraction bar into a division. Remember that in a mixed number there is an understood but unstated plus sign. So that

$7\frac{11}{16} = 7 + 11 \div 16 = 7.6875$.

This can also be done directly using the calculator as was discussed in **Section 1.2 Fractions**.

If a fraction is in lowest terms and its denominator has a factor besides 2 or 5, then that fraction, when converted to a decimal, will generate a **repeating** decimal. For example,

$\frac{5}{12} = \frac{5}{2\times 2\times 3}$, so 12 has a factor of 3, and $5 \div 12 = 0.416666... = 0.41\overline{6}$.

$$\frac{5}{12} = 0.416666... = 0.41\overline{6} \quad = 0.416\overline{6} \quad = 0.41666\overline{6}.$$

The 6's as indicated either by the ellipsis "…" or 6 with a bar on top repeat "forever".

Note: all of these ways of writing the repeating decimal are the same. Calculators will display 0.416666667 since they work with a fixed number of digits and will round the last digit displayed.

To convert a repeating decimal into a fraction is a little complicated and is rarely encountered in practical problems. As a result no problems requiring such a conversion occur in the unit exercises. However, if you are **curious**, the procedure is summarized and illustrated below:

1. Count and record the number of decimal places from the decimal point to the repeating string.

2. Move the decimal point to the right by this number of places. The result is a decimal number where the repeating pattern of digits begins in the tenth's place immediately to the right of the decimal point.

3. The digits to the left of the decimal point of the result from Step 2 become the whole number part of a mixed number. If there are no non-zero digits to the left of the decimal point, then the original decimal began the repeating pattern with the first digit and the whole number part of the mixed number is zero.

4. Add the whole number from Step 3 to a fraction with the repeating digits as the numerator and a string of 9's as the denominator. The number of 9's in the string is equal to the number of repeating digits in the numerator.

5. Take the fraction from Step 4 and divide it by 10 raised to the power of the number from Step 1. This number, worked out as a fraction, is the fraction equivalent to the original repeating decimal.

To illustrate the steps convert $0.00666\ldots$ to a fraction.
Step 1. The number of places from the decimal point to the repeating string of 6's is two.

Step 2. The result is the decimal $0.666\ldots$.
Step 3. The whole number is 0 .
Step 4. There is one repeating digit, a 6 , so the result is $0+\frac{6}{9}=\frac{2}{3}$.
Step 5. Dividing two thirds by $10^2 = 100$ gives

$$\frac{2}{3}\div 10^2=\frac{2}{3}\div 100=\frac{2}{3}\times\frac{1}{100}=\frac{\cancel{2}\times 1}{\cancel{2}\times 50\times 3}=\frac{1}{150}.\text{ So } 0.00\overline{6}=\frac{1}{150}.$$

As a more complicated example consider converting $3.1527272727\ldots$ to a fraction.
Step 1. The number of places from the decimal point to the repeating string of 27's is two.
Step 2. The result is the decimal $315.272727\ldots$.
Step 3. The whole number is 315 .
Step 4. There are two repeating digits, 27 , so the result is $315+\frac{27}{99}=315+\frac{\cancel{9}\times 3}{\cancel{9}\times 11}=315\frac{3}{11}$.
Step 5. Dividing the answer of Step 4 by $10^2 = 100$ gives

$$315\frac{3}{11} \div 100 = \frac{3468}{11} \times \frac{1}{100} = \frac{\cancel{4} \times 867}{11} \times \frac{1}{\cancel{4} \times 25} = \frac{867}{275} = 3\frac{42}{275}.$$

Using a calculator we can verify that $3\frac{42}{275} = 3 + 42 \div 275 = 3.15272727... = 3.15\overline{27}$.

Often we wish to approximate a decimal number by finding another decimal roughly equal to the first number, but expressed with less digits. This process is called **rounding**. To round use the following procedure:

1. From the problem determine the decimal place to which the number is to be rounded.

2. If the digit to the right of this decimal place is less than 5, then replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

3. If the digit to the right of the decimal place is 5 or greater, then increase the digit in this decimal place by 1 and replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

As an example, consider rounding 10,547.395 to the different decimal places shown in the following table.

10,547.395 rounded to	Decimal Place of Rounding	Result
2 places	hundredth's place	10,547.40
1 place	tenth's place	10,547.4
the nearest unit	one's place	10,547.
the nearest ten	ten's place	10,550
the nearest hundred	hundred's place	10,500
the nearest thousand	thousand's place	11,000

Raising numbers to powers or exponents occurs in many applications. Recall that b^n means a product of n factors of b . The number b is called the base, and n is the power or exponent.

So $1.574^5 = 1.574 \times 1.574 \times 1.574 \times 1.574 \times 1.574 = 9.661034658$.

This result is correct to as many places as your calculator will display. To perform this calculation on some calculators use the keystrokes 1.574 $\boxed{\wedge}$ 5 $\boxed{=}$, while other calculators enter 1.574 $\boxed{y^x}$ 5 $\boxed{=}$ or 1.574 $\boxed{x^y}$ 5 $\boxed{=}$.

Newer and/or graphing calculators generaly use the “carrot” symbol ^ for exponents.

Exponents of two and three are very common and have special names; b^2 is called "b **squared**" and b^3 is called "b **cubed**". Many calculators have $\boxed{x^2}$ keys to square a number. When evaluating an expression, the standard order of operations as discussed in **Section 1.1** requires that bases be raised to powers before any multiplications or divisions are performed. This hierarchy is built into scientific calculators. For example, consider evaluating $3.54\times 7.21^3-(10.7\times 6.28)^2\div 3.56$. On some calculators this is done with the following keystrokes:

$$3.54\ \boxed{\times}\ 7.21\boxed{\wedge}\ 3\ \boxed{-}\ \boxed{(}\ 10.7\boxed{\times}\ 6.28\ \boxed{)}\ \ \boxed{x^2}\ \boxed{\div}\ 3.56\ \boxed{=}.$$

The display shows the answer as 58.46760266 . The keystrokes on other calculators are identical except that the $\boxed{y^x}$ key (or $\boxed{x^y}$ key) is used instead of the $\boxed{\wedge}$ key. Some calculators have a dedicated $\boxed{x^3}$ key, and this key could have been used instead of $\boxed{\wedge}$ 3 above.

Consider evaluating 25^{12}. Entering 25 $\boxed{\wedge}$ 12 $\boxed{=}$ on the some calculators the answer is displayed $5.960464477\text{x}10^{16}$, while on others the result is $5.960464478\ ^{16}$.

Because of the large size of the number both calculators have expressed the result in scientific notation. In scientific notation we express the answer as a decimal number between 1 and 10 times ten to a power. In the example above the number between 1 and 10 is 5.960464478 and the power on 10 is 16. In ordinary decimal notation, which the calculator can't display for lack of space, this answer would be written as 59,604,644,780,000,000 . If you try to work with these large decimal numbers, the advantages of scientific notation soon become obvious!

Note: The result $5.960464478\ ^{16}$ seems to suggest that the exponent applies to 5.960464478 . This is not true. The exponent is on ten, but to save space in the display some calculators do not show the 10, it is "understood".

Now consider $(0.04)^{12}$. Most calculators display either 1.6777216×10^{-17} or $1.6777216\ ^{-17}$. The result is in scientific notation with a negative exponent on 10. In ordinary decimal notation this result would be 0.000000000000000016777216 .The left-most non-zero digit, a 1, is 16 or 17–1 decimal places to the right of the decimal point. Thus, in scientific notation a positive exponent on 10 gives the number of decimal places the decimal point must move to the right to get the ordinary decimal answer, while a negative exponent on 10 gives the number of decimal places the decimal point must move to the left to get the ordinary or "fixed point" decimal answer.

To enter a number in scientific notation some calculators use the $\boxed{\text{EXP}}$ key. For example, to enter 6×10^{23} use the following keystrokes: 6.02 $\boxed{\text{EXP}}$ 23 . A very small number like

7.15×10^{-12} is entered with 7.15 $\boxed{\text{EXP}}$ $\boxed{(-)}$ 12 . Here $\boxed{(-)}$ is the minus key. The procedure used on some other calculators is identical except that the $\boxed{\text{EE}}$ key is used instead of the $\boxed{\text{EXP}}$ key and the change sign key is $\boxed{+ \Leftrightarrow -}$ is used instead of $\boxed{(-)}$. Scientific notation will be covered more thoroughly in **Section 2.3**.

Consider a table of squares of the whole numbers.

N	N^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144

If we reverse this table, i.e., start with N^2 and get the value of N , the table would look like.

N^2	N
0	0
1	1
2	1.41421356
3	1.732050808
4	2
5	2.236067977
6	2.449489743
7	2.645751311
8	2.828427125
9	3
10	3.16227766
11	3.31662479
12	3.464101615

The second number is called the square root of the first. In symbols $N = \sqrt{N^2}$, for example, $3 = \sqrt{9}$. Recll that the square root symbol acts as a grouping symbol. Any operations inside the square root need to be completed before the root is taken. For example, $\sqrt{116-16} = \sqrt{100} = 10$. To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol. On the some calculators enter $\boxed{\sqrt{\ }}$ $\boxed{(}$ 116 $\boxed{-}$ 16 $\boxed{)}$ $\boxed{=}$, while $\boxed{(}$ 116 $\boxed{-}$ 16 $\boxed{)}$ $\boxed{\sqrt{\ }}$ $\boxed{=}$ are the corresponding keystrokes on older calculators such as the TI-30Xa calculators.

A table of the cubes of whole numbers can also be formed.

N	N^3
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512

If we reverse this table, i.e., start with $\boldsymbol{N}^3$ and get the value of $\boldsymbol{N}$, the table would look like.

N^3	N
0	0
1	1
2	1.25992105
3	1.44224957
4	1.587401052
5	1.709975947
6	1.817120593
7	1.912931183
8	2

The second number in the right table above is called the cube root of the first number in that table. In symbols $N = \sqrt[3]{N^3}$, for example, $3 = \sqrt[3]{27}$. The cube root, like the square root, acts as a grouping symbol. Any operations inside the cube root need to be completed before the root is taken. For example, $\sqrt[3]{85 \times 2 - 45} = \sqrt[3]{170-45} = \sqrt[3]{125} = 5$.

To perform this computation on the calculator parentheses need to be inserted around the expression inside the cube root symbol. On some calculators you will enter

$\boxed{\text{SHIFT}}$ $\boxed{\sqrt[3]{\ }}$ $\boxed{(}$ 85 $\boxed{\times}$ 2 $\boxed{-}$ 45 $\boxed{)}$ $\boxed{=}$, or use the key marked $\boxed{y^{1/x}}$ with x =3 . The keystrokes on the TI-30Xa are $\boxed{(}$ 85 $\boxed{\times}$ 2 $\boxed{-}$ 45 $\boxed{)}$ $\boxed{2nd}$ 0 $\boxed{=}$.

Your Turn!!

Perform the indicated operations giving answers to the stated number of decimal places:

$7.11643 + 3.3489$	(four places)	1) ________
$27.32 - 6.972$	(two places)	2) ________
0.25×0.4333	(two places)	3) ________
$7.123 \div 1.48$	(three places)	4) ________
$\$57.23 - \$7.89$	(two places, i.e., nearest penny)	5) ________
1.79^2	(two places)	6) ________
$\sqrt{0.144}$	(three places)	7) ________
$2.53^2 \times 1.96 - 5.36^2 \div 2.89$	(two places)	8) ________
$\sqrt{17^2 - 13^2}$	(one place)	9) ________
$\sqrt[3]{7 \times 8 - 11}$	(two places)	10) ________
$11.17^2 \times 5.10 - 5.97^3 \div 2.17$	(one place)	11) ________

Write the following fractions as decimals:

$2\frac{11}{16}$ 12) ________________

$1\frac{7}{12}$ 13) ________________

Write the following decimals as a fraction in lowest terms:

0.45 14) ________________

8.84 15) ________________

Solve the following problems:

A stack of eighteen pieces of lumber is 31.50 inches thick. How thick would a stack of thirty-three such sheets be?

16) ______________

A delivery truck gets 11.3 miles per gallon of gasoline. If gas costs $4.12 per gallon, what will be the cost of the gasoline needed to drive 189 miles? (Round to the nearest penny.)

17) ______________

A machinist earns $12.50 an hour plus time and a half for overtime (hours worked beyond 40). What is the machinist's gross pay for a 53.75 hour work week? (Round to the nearest penny.)

18) ______________

A welder earned $468.75 (gross pay before deductions) for 37.5 hours of work. Find her hourly rate of pay.

19) ______________

A cubic foot holds 7.481 gallons. A car has a gas tank which holds 14.5 gallons. To three decimal places, how many cubic feet is this?

20) ______________

Section 1.5 Significant Digits

In the previous section, each problem told you how to round your answers. In this section, you will learn how to determine for yourself how to round answers when you are working with calculations involving measurements. This will be possible because the way we write a number from a measurement carries information about the precision of the measurement. That, in turn, tells us how to round answers from calculations that involve measurements.

By the end of this section, you should be able to identify the number of significant digits in a measurement number, write the result of a measurement with the correct number of significant digits, and round calculations involving measurements to the correct number of significant digits.

Identifying and Writing Significant Digits

There is a shortcut that scientists and engineers use to indicate the precision of a measurement when stating the result.

- For example, if you measure the width of a box of chalk with a standard ruler, you might get an answer of 6.1 cm. Since the ruler is only marked in increments of 0.1 cm, you really can't measure any finer than half that increment, which is ±0.05 cm. Thus, the best way to state your measurement is as 6.1 cm ± 0.05 cm. That way, people know exactly how precise your measurement is, and they know not to expect or use a result stated to a higher precision, like 6.139 cm. Note: 6.1 cm ± 0.05 cm = 61 mm ± 0.5 mm.
- If you measure the width with a high-precision instrument like a digital caliper, then you might discover the answer to be 61.22 mm. Again, the uncertainty would be ± one-half of the smallest measurement possible, which would be ± 0.005 mm. The measurement is expressed as 61.22 mm ± 0.005 mm.
- However, there is a more efficient way to write a measurement like 6.1 cm ± 0.05 cm. Instead, just write 6.1 cm, and it is understood that the precision is ± one-half of the farthest-right digit. In this case, ***significant digits*** are being used to indicate the precision of the measurement.
- The ***significant digits*** (also called significant figures and abbreviated "sig figs") of a number are those digits that carry meaning contributing to its precision.

Example: Suppose that someone tells you the width of a piece of paper is 21.59 cm. However, you know that they measured it with a ruler only marked in increments of 0.1 cm. Write the measurement correctly using significant digits.

Solution: The measurement can only be stated to the nearest 0.1 cm. , i.e., 21.6 cm.

Example: Suppose you measure the length of a board to be $13\frac{5}{8}$ inches, and your tape measure is marked in eighths of an inch. Write the measurement as a decimal using significant digits.
Solution: Using a calculator, $(13+5\div 8)\text{in.} = 13.625 \text{ in}$. The uncertainty in the measurement is $\pm\frac{1}{16}\text{in} \approx .0625 \text{ in}$. Thus, the answer is $13.625 \text{ in.} \pm .0625 \text{ in}$, which stated using significant figures is 13.6 in. In other words, since (1/8)in. = 0.125 in., we round to the nearest tenth.

Example: Suppose you measured the length of a section of a running track for a sprint race to be 100 meters using a tape that measures to the nearest centimeter (i.e., hundredth of a meter). Write the measurement using the correct number of significant digits.
Solution: 100.00 m; You have to have all those zeros to show that after the hundreds place, you measured nothing but zeros down to the nearest 0.01 m.

Example: If you made the same measurement, but now the tape only measures to the nearest decimeter (i.e., tenth of a meter), write the measurement using significant digits.
Solution: 100.0 m ; There is one less zero in the answer to show you only measured to the nearest 0.1 m.

Example: What if the tape only measures to the nearest meter?
Solution: 100. m ; The position of the decimal point indicates accuracy to only the nearest 1 m.

These track length examples embody one of the most confusing parts of working with significant digits: measurements that have a lot of zeros in them. When you want to show that a zero is significant, you go to the extra effort of putting more zeros after the decimal point to show the finest increment of measurement, for example, 100.00 m . To show that a measurement has been made only to the nearest ones place put a decimal immediately after the ones place as in 100. m .

The rule for identifying the significant digits in a measurement can be stated in two ways:

1. All digits are significant except a string consisting of **only** zeros that hold place after a decimal point or a string consisting of **only** zeros that hold place to the right of the last non zero digit in a whole number.

2. If there is a decimal point present, find the left-most nonzero digit, and then count digits toward the right. If there is no decimal point in the number, find the right-most nonzero digit and count toward the left. In both cases, keep counting digits until you reach the other end of the number.

Examples: State the number of significant digits in each measurement below

1. 467.24 ft. has 5 significant digits
2. 0.006020 m has 4 significant digits; the leading zeros are place holders, but the trailing zero is not; it shows you made a measurement down to the nearest 0.000001 m.
3. 1250 lb. has 3 significant digits; the trailing zero is a place holder. The stated measurement is thus only precise to the nearest 10 pounds. If the measurement were accurate to the nearest pound, we would write it as 1250. lb.
4. 0.00003 g has 1 significant digit
5. 93,000,000 miles has 2 significant digits. This is more apparent if the result is stated in scientific notation as 9.3×10^{7} miles
6. 800 in. has only 1 significant digit
7. 800. in. has 3 significant digits

Significant Digits after Adding or Subtracting

- When you add or subtract measurements, then you have to round your answer to the precision of the least accurate measurement. That's because the uncertainty in the least precise measurement overwhelms the precision of the other measurements.
- A good practice as you are learning how to work with significant figures is to write down what your calculator gives for an answer, then write the correctly rounded answer.

Examples:

1. 22.85 g – 13.35 g = 9.50 g : The answer is rounded to a precision of 0.01 g
2. 114.37 cm + 3.080 cm +27.3 cm = 144.7 cm : The answer is rounded to a precision of 0.1 cm
3. 13 ft – 5.811 ft = 7 ft : The answer is rounded to a precision of 1 ft
4. 1250.27 mi + 3367.7 mi + 2257 mi + 4800 mi = 11,900 mi :
   The answer is rounded to a precision of 100 mi .

Significant Digits after Multiplying or Dividing

- Rule: Determine the smallest number of significant digits in any of the measurements involved in the calculation. Round the answer to that number of significant digits.

This rule is based on how the uncertainties propagate in computing the final answer. For example, if the floor of a room is measured at 14.12 feet long and 9.8 feet wide, our calculators would give us an area of $14.12 \text{ ft.} \times 4.8 \text{ ft.} = 67.776 \text{ ft}^2$ (sq. ft). However, length = 14.12 ft. really means $14.12 \text{ ft.} \pm 0.005 \text{ ft.}$ The actual length could range from a low of 14.115 ft. to a high of 14.125 ft. Similarly, the width = $4.8 \text{ ft.} \pm 0.05 \text{ ft}$, so the actual width is between 4.75 ft. and 4.85 ft. The largest possible area would therefore be $14.125 \text{ ft} \times 4.85 \text{ ft} = 68.50625 \text{ ft}^2$ and the smallest possible area would be $14.115 \text{ ft} \times 4.75 \text{ ft} = 67.04625 \text{ ft}^2$. Hence, based on the measurements the uncertainty in the area is in the one's place, i.e., we can only reliably estimate the area to the nearest square foot. So we round 67.776 sq. ft to 68 ft^2. There are only two significant digits in the calculated area which is due to there only being two significant digits in the width measurement. The four significant digits of the length measurement "are wasted" due to the limitations in our knowledge of the least significant measurement, namely the width.

In calculations "pure numbers", i.e., numbers without a unit of measurement attached, are considered exact with no limitation in accuracy and essentially have an infinite number of significant digits. They are ignored when determining the number of significant digits present in the answer. For example, $10 \times 22.85 \text{ cm} = 228.5 \text{ cm}$.

Examples:

1) $2.54 \text{ in.} \times 3 \text{ in.} = 7.62 \text{ in}^2$ ➔ round to 1 sig. fig. ➔ Answer = 8 in^2.

2) $310.2 \text{ cm} \times 51.05 \text{ cm} \times 0.100 \text{ cm} = 1{,}583.571 \text{ cm}^3$ ➔ round to 3 sig. fig.
   ➔ Answer = 1,580 cm^3.

3) $22.85 \text{ ft}^2 / 4 \text{ ft.} = 5.7125 \text{ ft}$ ➔ round to 1 sig. fig. ➔ Answer = 6 ft.

4) $358.9 \text{ m}^3 / 22 \text{ m}^2 = 16.313\,636\,36 \text{ m}$ ➔ round to 2 sig. fig. ➔ Answer = 16 m.

5) 45.40 g / 7.1000 mL = 6.394366197 g/mL ➔ round to 4 sig. fig.
   ➔ Answer = 6.394 g/mL.

6) $52.5 \text{ mi/hr} \times 3.8752 \text{ hr} = 203.448 \text{ mi}$ ➔ round to 3 sig. fig. ➔ Answer = 203 mi.

Your Turn!!

Identify the number of significant digits shown in each of the following measurements.

1. 400 m
2. 203 cm
3. 218 lb
4. 320. kg
5. 0.0001 in
6. 635.000 mm
7. 22,000 s (s is a second)
8. 5201 sq. ft

Perform the following computations using a calculator but rounding the answer to the correct number of significant digits. Your answer should include the correct unit of measurement.

Examples:

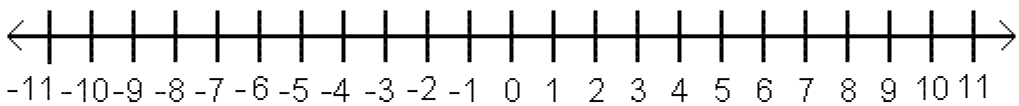
7.857 cm + 5.23 cm = 13.087 cm $\Rightarrow$ $\boxed{\text{Answer = 13.09 cm}}$ (round to nearest 0.01)

12.14 ft. $\times$ 2.2 ft. = 26.708 sq.ft. $\Rightarrow$ $\boxed{\text{Answer = 27 sq.ft.}}$ (round to 2 sig figs.)

9.	4.60 cm + 3 cm	= ________	22.	13.7 mL $\times$ 4	= ________
10.	0.008 in^2 + 0.05 in^2	= ________	23.	200 $\times$ 3.58 cu. in.	= ________
11.	22.4420 g + 56.981 g	= ________	24.	0.00003 m $\times$ 727 m	= ________
12.	200 lb − 87.3 lb	= ________	25.	5003 ft^3 / 3.781 ft	= ________
13.	67.5 s − 0.009 s (s is a second)	= ________	26.	5000 ft^3 / 3.781 ft	= ________
14.	71.86 hr − 13.1 hr	= ________	27.	5.00 $\times 10^3$ ft^3/3.781 ft	= ________
15.	357.89 in + 0.002 in	= ________	28.	300 mi./hr $\times$ 10.6 hr	= ________
16.	17.95 in + 32.42 in + 50 in	= ________	29.	0.059 min $\times$ 6.95 mi./min	= ________
17.	5.5 m^2 + 3.7m^2 + 2.97m^2	= ________	30.	3.14 / 400 $\times$ (5.6 cm)^2	= ________
18.	84.675 gal − 3 gal	= ________	31.	80 mi. / 0.675 hr	= ________
19.	75 cu. ft - 2.55 cu. ft	= ________	32.	8.5 g / 0.356 mL	= ________
20.	10 L - 9.9 L	= ________	33.	0.003 g / 106 mL	= ________
21.	13.7 ft $\times$ 2.5 ft	= ________	34.	3.00 g / 106000 mL	= ________

Section 1.6 Signed Numbers

When we are using numbers to express the change in a quantity, such as the amount of money in a checking account or a running back's total yards, we soon find that the quantities under study don't always increase. Bank accounts sometimes decline and running backs can lose yards! To represent a change which decreases we use negative numbers, while positive numbers represent an increase. A convenient way to visualize positive and negative numbers is the number line shown below. Here the positive (or "ordinary") numbers are to the right of zero and the negative numbers are to the left of zero.



The opposite of 5 is –5 since $-5 + 5 = 0$. For example, if you lose \$5 then make \$5, you're back to zero. By the same argument the opposite of –5 is 5 . If a running back gains 10 yards, then loses 7, his net yardage is 3 . In symbols, $10 + (-7) = 3$. So adding a negative 7 is the same as subtracting a positive 7. Also $10 + (-7) = (-7) + 10 = 3$. In general, $a + (-b) = a - b$, i.e., subtracting is the same as adding the opposite and visa versa .

Suppose a running back loses 3 yards every time he carried the ball. If he had four carries, his net yardage is $-3 + (-3) + (-3) + (-3) = -12$. You may have noticed that we don't write + –3, but rather + (–3) , this is just to avoid the potential confusion of two adjacent operation symbols. However, using the definition of whole number multiplication as repeated addition, we see that $4(-3) = -3 + (-3) + (-3) + (-3) = -12$. So a positive number times a negative number should result in a negative number. What about a negative times a negative? One of the fundamental rules of arithmetic is called the distributive property. It says that

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad .$$

For example,

$$5 \cdot (4 + 7) = 5 \cdot 4 + 5 \cdot 7 = 20 + 35$$

$$5 \cdot 11 \;=\; 55 \,.$$

Now,

$$(-1)(1 + (-1)) = (-1) \cdot 1 + (-1)(-1)$$

$$(-1)(0) = -1 + (-1)(-1)$$

$$0 = -1 + (-1)(-1)$$

So, (–1)(–1) added to –1 gives zero. But only 1 added to –1 makes zero. So we conclude that

$$\mathbf{(-1)(-1) = 1} \,.$$

In general, a negative number times a negative number gives a positive number. We have analogous statements in English. If I say "I am not dishonest", the double negative makes the sentence equivalent to saying "I am honest".

We now have another way of forming the opposite of any number, simply multiply by -1, i.e., $-b = (-1)b$. The standard order of operations requires that we square before multiplication. This means that $-5^2 = -1(5^2) = -25$, while $(-5)^2 = (-5)(-5) = 25$. Consider now subtracting a negative number as in $10 - (-8) = 10 + (-1)(-8) = 10 + 8 = 18$. So subtracting a negative is the same as adding the positive.

Finally, division is the opposite operation to multiplication. Since $(-5)(6) = -30$ and $(-5)(-6) = 30$, it then follows that

$$(-30)\div(-5)=6 \qquad (-30)\div 6=-5 \qquad 30\div(-5)=-6 \qquad 30\div(-6)=-5.$$

In general, a negative number divided by a negative number is a positive number, while a negative divided by a positive or a positive divided by a negative is negative.

To enter a negative number on some calculators use the minus sign key $\boxed{(-)}$ **before** the number just as it is written. For example, to evaluate

$$\frac{6\times(-10)}{-3-2}=\frac{-60}{-5}=12$$

enter the following keystrokes : $\boxed{(}$ 6 $\boxed{\times}$ $\boxed{(-)}$ 10 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{-}$ 2 $\boxed{)}$ $\boxed{=}$.

On the TI-30Xa you enter the negative numbers "backwards", first the value, then the change sign key $\boxed{+\Leftrightarrow-}$.

Your Turn!!

Calculate the following:

1. $-56\div 8$ = ____________
2. $(-8)^2\div(-16)$ = ____________
3. $-8^2\div(-16)$ = ____________
4. $-8^2\div 16$ = ____________
5. $\dfrac{(-5)(-4)}{-10}$ = ____________
6. $\dfrac{(-2)^3(-3)}{-6}$ = ____________

7. In the first six months of the year, Precise Auto Body had the following profit and loss record:

January	\$8,736.52	profit	April	\$478.68	profit
February	\$12,567.34	profit	May	\$179.66	loss
March	\$1,282.72	loss	June	\$1,257.23	profit

Find the total profit or loss for this six month period. ____________

Chapter 1 Sample Test

_______________/100

Each Problem is worth 4 points.

Perform the following operations.

1. $206 \times 115 - 19349$ = ____________

2. $9 \times 12 - 32 \times 5 \div 10 \times 8$ = ____________

3. $\dfrac{4 \times 5 - 6 \times 7}{7 \times 3 - 4 \times 8}$ = ____________

4. Write as an improper fraction. $4\frac{5}{32}$ **=** ____________

5. Reduce to lowest terms. $\frac{15}{24}$ = ____________

6. Supply the missing numerator. $\frac{11}{16} = \frac{?}{64}$**;** **?** **=** ____________

7. Indicate which number is larger. $\frac{15}{24}$ or $\frac{21}{28}$**;** ____________

Perform the indicated operations and express the answer as a fraction in lowest terms:

8. $\frac{5}{9} \times 4\frac{4}{5}$ = ____________

9. $4 \div \frac{5}{8}$ = ____________

10. $\frac{3}{4} + \frac{5}{16}$ = ____________

11. $3\frac{3}{8}-2\frac{3}{4}$ = ____________

12. $5\frac{1}{4}\div 2\frac{5}{8}$ = ____________

13. $11\frac{5}{32}-3\frac{3}{8}-6\frac{3}{16}$ = ____________

14. $4\frac{3}{8}+1\frac{2}{3}\times 2\frac{7}{16}$ = ____________

Perform the indicated operations giving answers to the stated number of decimal places:

15. 3.176^2 = ____________ (two places)

16. $\sqrt{13^2-12^2}$ = ____________ (one place)

17. $\sqrt[3]{3.14159\times 2.67^2-10.34}$ = ____________ (two places)

18. Write the following fraction as a decimal: $6\frac{7}{16}$ = ____________

19. Write the following decimal as a fraction in lowest terms: 0.375 = ____________

20. Calculate the following: $(-6)^2\div(-12)\times(-2)$ = ____________

21. A steam fitter earns $19.40 per hour plus time and a half for overtime (more than 40 hours in one week). If she works $56\frac{1}{2}$ hours one week, what is her gross pay for that week?

____

Perform the following calculations using measurements stating the answer to the correct number of significant digits.

22. 137.76 miles – 8 miles = ________________

23. $3.14159 \times 2 \times 1.28$ cm = ________________

24. $\dfrac{211.7 \text{ g}}{3.14159 \times (2.00 \text{ cm})^2 \times 6.5 \text{ cm}}$ = ________________

25. A car averages 31.7 miles per gallon and has an 11. gallon gas tank. What is the furthest distance the car can travel without stopping for gas? State your answer using the correct number of significant digits

____

Chapter 1 Sample Test Solutions

1.	**2.**	**3.**	**4.**	**5.**
4341	–20	2	$\frac{133}{32}$	$\frac{5}{8}$
6.	**7.**	**8.**	**9.**	**10.**
? = 44 (i.e., $\frac{44}{64}$)	$\frac{21}{28} = 0.75$	$\frac{8}{3} = 2\frac{2}{3}$	$\frac{32}{5} = 6\frac{2}{5}$	$\frac{17}{16} = 1\frac{1}{16}$
11.	**12.**	**13.**	**14.**	**15.**
$\frac{5}{8}$	2	$\frac{51}{32} = 1\frac{19}{32}$	$\frac{135}{16} = 8\frac{7}{16}$	10.09
16.	**17.**	**18.**	**19.**	**20.**
5.0	2.29	6.4375	$\frac{3}{8}$	6
21.	**22.**	**23.**	**24.**	**25.**
$1256.15	130 miles	8.04 cm	2.6 g /cm^3	350 miles

Chapter 2 Algebra

Algebra, despite its fearsome reputation, is really just "generalized" arithmetic. The name itself comes from the title of a ninth century Arabic mathematics text *Ilm al-jabr wa'l mugabalah* written by Al-Khowarozmi in the court of Al-Maman, the Caliph of Bagdad. Algebra concerns itself with the basic operations of arithmetic:

1. Adding/Subtracting
2. Multiplying/Dividing
3. Raising to powers/Taking roots

Notice that in each pair above, the second operation is the inverse (does the opposite) of the first. The use of letters or variables to represent numbers enables us to summarize infinitely many arithmetic facts with just a few algebraic formulas. For example, the different addition results $5+3=3+5$, $11+19=19+11$, $25+13=13+25$, $83+(-18)=-18+83$, can all be summarized by the one algebraic statement $a+b=b+a$ for any numbers a and b. This property of numbers is called the commutative law of addition, but the essential fact that it clarifies is that the answer of an addition problem does not depend on the order in which the numbers are added.

In **Section 1.6** the properties of "signed" numbers were discussed. The use of exponents is very important in algebraic expressions. Positive integer exponents were introduced in **Section 1.2**. In this same section the order of operations rules required to evaluate formulas are presented. For any number a the opposite or negative of a is given by $-a=-1\cdot a$. Combining this fact with the order of operations rule that raising a number to a power is done before any multiplications or divisions enables us to properly evaluate the following expressions.

$$-6^2=-1\cdot 6^2=-1\cdot 36=-36\text{, while } (-6)^2=(-6)\cdot(-6)=36.$$

In the first calculation, since squaring is done before multiplication, the result is a negative 36. In the second calculation the use of parenthesizes means that it is a negative 6 times a negative 6 which equals a positive 36. In other words, $-6^2=-36$ because a power only applies to “what it’s attached to” (i.e., the 6). It does not apply to the leading negative sign.

Section 2.1 Rules of Exponents

Recall that for a base x and a positive whole number n, the expression x raised to a an exponent or power of n means repeated multiplication of n factors of x. In symbols,

$$x^n=\overbrace{x\cdot x\cdot\ldots\cdot x}^{n}$$

Rule 1: When multiplying expressions containing the same base, add the exponents.

$$x^n\cdot x^m=\overbrace{x\cdot x\cdot\ldots\cdot x}^{n}\cdot\overbrace{x\cdot x\cdot\ldots\cdot x}^{m}=\overbrace{x\cdot x\cdot\ldots\cdot x}^{n+m}=x^{n+m}$$

For example, $x^2\cdot x^3=x^{2+3}=x^5$.

Rule 2: When dividing expressions containing the same base, subtract the exponent of the denominator from the exponent of the numerator.

$$\frac{x^n}{x^m} = \frac{\overbrace{x \cdot x \cdot \ldots \cdot x}^{n}}{x^m} = \frac{\overbrace{x \cdot x \cdot \ldots \cdot x}^{m} \cdot \overbrace{x \cdot x \cdot \ldots \cdot x}^{n-m}}{x^m} = \frac{x^m}{x^m} \cdot x^{n-m} = x^{n-m}$$

For example, $\frac{x^5}{x^2} = x^{5-2} = x^3$.

Rule 3: When raising a product to a power, each factor in the product is raised to that same power.

$$(x \cdot y)^n = \overbrace{(x \cdot y) \cdot (x \cdot y) \cdot \ldots \cdot (x \cdot y)}^{n} = \overbrace{x \cdot x \cdot \ldots \cdot x}^{n} \cdot \overbrace{y \cdot y \cdot \ldots \cdot y}^{n} = x^n y^n$$

For example, $(x \cdot y)^3 = x^3 y^3$.

Rule 4: When raising a quotient to a power, each factor in the quotient is raised to that same power.

$$\left(\frac{x}{y}\right)^n = \frac{\overbrace{x \cdot x \cdot \ \ldots \ \cdot x}^{n}}{\underbrace{y \cdot y \cdot \ldots \cdot y}_{n}} = \frac{x^n}{y^n}$$

For example, $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$.

Rule 5: When a base raised to a power is itself raised to another power, the answer is the base raised to the product of the two powers.

$$(x^m)^n = \overbrace{(x^m) \cdot (x^m) \cdot \ldots \cdot (x^m)}^{n} = \overbrace{\overbrace{x \cdot x \cdot \ldots \cdot x}^{m} \cdot \overbrace{x \cdot x \cdot \ldots \cdot x}^{m} \cdot \ldots \cdot \overbrace{x \cdot x \cdot \ldots \cdot x}^{m}}^{n} = x^{n \cdot m} = (x^n)^m$$

For example, $\left(x^2\right)^3 = x^{2 \cdot 3} = x^6$.

The rules of exponents are used to simplify the following expressions.

Rule 5: $-\left((-2)^2\right)^3 = -(-2)^{2 \cdot 3} = -(-2)^6 = -(64) = -64$

Rules 3 and 4, then Rule 5: $\left(\frac{-2y^2}{x^4}\right)^3 = \frac{(-2)^3\left(y^2\right)^3}{\left(x^4\right)^3} = \frac{-8\cdot y^{2\cdot 3}}{x^{4\cdot 3}} = \frac{-8\cdot y^6}{x^{12}}$

All five rules:

$$\left(3x^2y^3\right)^3\cdot\left(\frac{x^4}{2y^3}\right)^2 = 3^3\left(x^2\right)^3\left(y^3\right)^3\frac{\left(x^4\right)^2}{\left(2y^3\right)^2} = \frac{3^3}{2^2}x^{2\cdot 3}\cdot x^{4\cdot 2}\frac{y^{3\cdot 3}}{y^{3\cdot 2}} = \frac{27}{4}x^6\cdot x^8\frac{y^9}{y^6}$$

$$= \frac{27}{4}x^{6+8}\cdot y^{9-6} = \frac{27}{4}x^{14}y^3$$

The following table summarizes this section.

Rules of Exponents
1. $x^n\cdot x^m = x^{n+m}$ 2. $\frac{x^n}{x^m} = x^{n-m}$ 3. $(x\cdot y)^n = x^n y^n$ 4. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ 5. $\left(x^m\right)^n = x^{m\cdot n}$

Section 2.2 The Distributive Property

Monomials or "terms" are algebraic expressions consisting of a product of a number with variables raised to whole number exponents. The number part of the monomial is called the coefficient and is understood to be 1 if it is not stated. For example, $5x^2y, 3x^3yz^2, a^2b$ are all monomials. Two monomials are said to be "like terms" if they have the same variables raised to exactly the same exponents. For example, $3x^2y$ and $11x^2y$ are like terms, while $3x^2y$ and $11xy^2$ are "unlike" terms. The importance of like terms is that under the operations of addition or subtraction they can be combined into a single term. The distributive property of multiplication over addition says that for any numbers a, b, and c, $a\cdot(b+c) = a\cdot b + a\cdot c$. For example,

$$5\cdot(7+4) = 5\cdot 7 + 5\cdot 4$$

$$5\cdot 11 = 55 = 35 + 20$$

Using the distributive property "backwards" (called "factoring"), allows us to reduce the total number of operations in an expression. For example, $a \cdot b + a \cdot c = a \cdot (b + c)$, there are three operations on the left side (two multiplications and one addition), but only two operations on the right side (one addition and one multiplication). Thus, factoring usually results in a more efficient yet equivalent algebraic formula. The following examples show how to combine like terms using the distributive property.

$$3x^2y + 11x^2y = (3 + 11)x^2y = 14x^2y$$
$$3x^2y - 11x^2y = (3 - 11)x^2y = -8x^2y.$$

Note: "Unlike" terms can not be combined into a single monomial. The best one can do is factor out the largest common monomial, $3x^2y + 11xy^2 = xy(3x + 11y)$.

The definition of subtraction is the addition of the opposite and the definition of division is multiplication by the reciprocal. $a - b = a + (-b)$; $a \div b = a \cdot \frac{1}{b} = \frac{a}{b}$. The use of common denominators in adding fractions is really just an application of the distributive property. For example, $\frac{1}{4} + \frac{1}{3} = \frac{4}{12} + \frac{3}{12} = \frac{1}{12} \cdot 4 + \frac{1}{12} \cdot 3 = \frac{1}{12} \cdot (4 + 3) = \frac{1}{12} \cdot 7 = \frac{7}{12}$.

If we have a single term denominator we can proceed in the opposite direction.

$$\frac{a + c}{d} = \frac{1}{d} \cdot (a + c) = \frac{1}{d} \cdot a + \frac{1}{d} \cdot c = \frac{a}{d} + \frac{c}{d} .$$

That is, the *d* divides **both** the *a* and the *c* when they are separated by an addition or subtraction. In contrast, if the operation between *a* and *c* is multiplication, we can use *d* to divide just the *a*, or *d* to divide just the *c*, or *d* to divide the product of *a* and *c*, but we can not divide **both** the *a* and the *c*.

$$\frac{a \cdot c}{d} = \frac{1}{d} \cdot (a \cdot c) = \left(\frac{1}{d} \cdot a\right) \cdot c = \left(\frac{a}{d}\right) \cdot c = a \cdot \left(\frac{1}{d} \cdot c\right) = a \cdot \left(\frac{c}{d}\right) .$$

Consider the following examples.

$$\frac{a^2 + 2ab + b^2}{ab} = \frac{a^2}{ab} + 2 \cdot \frac{ab}{ab} + \frac{b^2}{ab} = \frac{a^2}{a} \cdot \frac{1}{b} + 2 \cdot 1 + \frac{b^2}{b} \cdot \frac{1}{a} = a \cdot \frac{1}{b} + 2 + b \cdot \frac{1}{a} = \frac{a}{b} + 2 + \frac{b}{a}$$
$$\frac{a^2 \cdot 2ab \cdot b^2}{ab} = a^2 \cdot 2 \cdot \frac{ab}{ab} \cdot b^2 = 2a^2b^2.$$

As was stated earlier the opposite of any quantity a, $-a$, is the same as $-1 \cdot a$. Therefore, in the following example the minus sign is distributed across the addition.

$$-(x + y) = -1 \cdot (x + y) = -1 \cdot x + (-1) \cdot y = -x + (-y) = -x - y .$$

In particular,

$$-(x - y) = -1 \cdot (x - y) = -1 \cdot [x + (-1) \cdot y] = (-1) \cdot x + (-1) \cdot (-1) \cdot y = -x + y = y - x$$

For example, $-(15 - 7) = 7 - 15 = -8$.

More complicated examples can be simplified by repeated use of the distributive property.

$$8\left(3x^2-7x+5\right)-4\left(2x^2-6x+7\right)=24x^2-56x+40-8x^2+24x-28=16x^2-32x+12$$

This result can be checked by plugging in numbers for x into both the starting expression and the final simplified form. If no algebra mistakes were made, the two answers must agree. Substitute $x=2$ (when checking avoid using 0 or 1, these values will often cause non-equivalent expressions to give the same answer) into $8\left(3x^2-7x+5\right)-4\left(2x^2-6x+7\right)$. This results in the following calculation:

$$8\left(3(2)^2-7\cdot 2+5\right)-4\left(2(2)^2-6\cdot 2+7\right)=8\cdot(3\cdot 4-14+5)-4\cdot(2\cdot 4-12+7)$$
$$=8(12-14+5)-4(8-12+7)=8\cdot 3-4\cdot 3=24-12=12$$

At $x=2$ the final simplified expression, $16x^2-32x+12$, computes as

$$16(2)^2-32\cdot 2+12=16\cdot 4-64+12=64-64+12=12$$

Thus, both expressions agree with a value of 12 when $x=2$. While this does not guarantee that the algebra is correct, it does show that an algebra mistake was extremely unlikely.

Your Turn!!

Evaluate and/or simplify the following expressions:

1. $17-33$ = ________
2. $-(17-33)$ = ________
3. $1.56-7.12+2.15^2$ = ________
4. $-7-\dfrac{12}{-3}(-4)^3$ = ________
5. $-6^2-(-4)^3$ = ________
6. $(-6)^2-(-4)^3$ = ________
7. $-\left(-(-3)^3\right)$ = ________
8. $-\left(-(-3)^3\right)^2$ = ________
9. $x^5\left(-x^3\right)^2$ = ________
10. $5x^3y^5\left(-\dfrac{x^3}{y^2}\right)^2$ = ________
11. $\left(-3x^2y\right)^2$ = ________
12. $\left(-3x^2y\right)^3$ = ________
13. $\dfrac{z^9y^7}{z^4y^5}$ = ________
14. $\left(3ab^2\right)\left(3a^2b\right)(3ab)^2$ = ________
15. $-\left(-5a^3b^5\right)^3$ = ________
16. $\dfrac{8x^4}{(2x)^2}$ = ________
17. $\dfrac{\left(-3x^2y\right)^2}{(-2x)^3}$ = ________
18. $\dfrac{\left(-2x^5y^4\right)^3}{\left(-5x^2y^5\right)^2}$ = ________

Perform the following operations and simplify the result.

19. $-(a+2b)$ = ______________

20. $-(a-2b)$ = ______________

21. $ab^4\left(\frac{5a^2}{b}+\frac{3a^3}{b^2}-\frac{7}{b^4}\right)$ = ______________

22. $4(3x^2-5x+2)-7(2x^2-6x-3)$ = ______________

23. $\frac{12x^4-8x^2+16x}{4x}$ = ______________

24. $\frac{49x^3y^4-28x^2y^2+14xy}{7xy}$ = ______________

Evaluate the following expressions at the values indicated.

25. At $x = 3$ and $y = -2$, $x^3+y^2-x^2-5y+3$ = _________

26. At $d = 2$ and $z = -1$, $z(d^2-z+d)-2d^2-5$ = _________

27. At $t = 3$ and $v = 12$ and $g = -32$, $\frac{1}{2}gt^2+vt+100$ = _________

28. At $x = -2$ and $y = -5$, $\frac{x^2+4y^2+2x}{4x^2-3y^2+9}$ = _________

29. At $a = 5.5$ cm , $b = 2.5$ cm , and $h = 6.5$ cm , $\frac{h(a+b)}{2}$ = _________

Section 2.3 Zero, Negative and Fractional Exponents

Recall that the second rule of exponents states that $\frac{x^n}{x^m}=x^{n-m}$. Suppose we were to apply this rule to $\frac{x^n}{x^n}=x^{n-n}=x^0$. But clearly anything divided by itself is 1. So we have the "reasonable" definition that for any number x different from zero (Remember division by zero is **never** meaningful. See the discussion **Section 1.3**), $x^0=1$.

Furthermore, $\frac{x}{x^2} = \frac{x}{x} \cdot \frac{1}{x} = 1 \cdot \frac{1}{x} = \frac{1}{x}$. But again by the second rule of exponents, $\frac{x}{x^2} = x^{1-2} = x^{-1}$. So x^{-1}must mean $\frac{1}{x}$. By a similar argument we conclude that for any *p*.

$$x^{-p} = \frac{1}{x^p} = \left(\frac{1}{x}\right)^p$$

A negative exponent means to take a reciprocal of the base! For example: $2^{-1} = \frac{1}{2^1} = 0.5$, $5^{-2} = \frac{1}{5^2} = \frac{1}{25} = 0.04$, $\left(-\frac{1}{3}\right)^{-3} = (-3)^3 = -27$. All of these results can be verified using a scientific calculator. The values of 10 raised to negative exponents are shown below. The use of positive and negative exponents on a base of 10 allows us to express numbers and measurements in scientific notation.

$$10^{-1} = \frac{1}{10^1} = 0.1 \qquad 10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001 \qquad 10^{-n} = \frac{1}{10^n} = \frac{1}{1\underbrace{00...0}_{n}} = 0.\overbrace{00...0}^{n-1}1$$

A typical number written in scientific notation looks like 3.4×10^5. The number has a decimal part (3.4, in this case) that is multiplied by a power of 10. A positive power on 10 always gives a 1 followed by a number of zeroes equal to the exponent. Now multiply this by the decimal part. For example, $3.4 \times 100{,}000 = 340{,}000$. Note that it is standard to use "×" for "times" in scientific notation which is perhaps the only place "×" is used in algebra. Since a negative power means we need to divide by the factors of 10, the decimal place is shifted to the left instead.

Example: $9 \times 10^{-4} = 0.0009$. The decimal place was shifted 4 places to the left.
Example: $2.3 \times 10^{-6} = 0.0000023$. The decimal point was shifted 6 places to the left.

To write a number in scientific notation, move the decimal place far enough so that there is exactly one nonzero digit to the left of the decimal point. Then multiply by a power of 10. The exponent on 10 is determined by the number of places you moved the decimal point. It is a positive exponent if the decimal point moved left, and a negative exponent if the decimal point moved right. This representation is equal to the number we started with since multiplying the decimal part by the power of 10 simply moves the decimal point back to where it originally was.

Example: $45{,}000{,}000 = 4.5 \times 10^7$ $\quad -0.000764 = -7.64 \times 10^{-4}$ $\quad 0.000000319 = 3.19 \times 10^{-7}$

When working with measurements, scientific notation removes the ambiguity associated with zeros in determining the number of significant digits, In scientific notation all reported digits in the decimal part are significant. Using scientific notation and the exponent rules also simplifies calculations involving large or small numbers.

Example: Express (320,000)(50,000,000) in Scientific Notation.

Solution:

$= (3.2 \times 10^5)(5 \times 10^7)$	Convert both numbers to scientific notation
$= (3.2)(5) \times 10^5 \times 10^7$	Multiply coefficients
$=16 \times 10^{5+7} = 16 \times 10^{12}$	Add exponents on the base of 10
$= 1.6 \times 10^{13}$	Move the decimal point one position to the right

The five rules of exponents apply to negative exponents as well as to positive exponents. This is illustrated in the following example.

$$\frac{10^5 \cdot \left(2 \cdot 10^4\right)^{-2}}{10^{-6}} = \frac{10^5 \cdot 2^{-2} \cdot \left(10^4\right)^{-2}}{10^{-6}} = \frac{1}{4} \cdot \frac{10^5 \cdot 10^{-8}}{10^{-6}} = \frac{1}{4} \times 10^{5-8-(-6)} = 0.250 \times 10^3 = 250$$

Example: Express $\left(6.02 \times 10^{23}\right)\left(1.6 \times 10^{-19}\right)$ in Scientific Notation.

Solution:

$= (6.02)(1.6) \times 10^{23} \times 10^{-19}$	Multiply coefficients
$= 9.6 \times 10^{23-19} = 9.6 \times 10^4$	Add exponents on the base of 10

Note: in accordance to the rules of significant figures only two digits appear in the decimal part.

Fractions can also be used as exponents. For example, $a^{\frac{1}{2}}$. The natural question is one of interpretation. By the fifth rule of exponents, $\left(a^{\frac{1}{2}}\right)^2 = a^{\frac{1}{2} \cdot 2} = a^1$. This means that when $a^{\frac{1}{2}}$ is squared the answer is *a*. From this we conclude that $a^{\frac{1}{2}} = \sqrt{a}$. By similar reasoning we arrive at the results $a^{\frac{1}{q}} = \sqrt[q]{a}$ and $a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$. These equations are useful in simplifying expressions involving roots as the following example illustrates.

$$\sqrt{49x^6 y^{10}} = \left(7^2 x^6 y^{10}\right)^{\frac{1}{2}} = 7^{\frac{2}{2}} x^{\frac{6}{2}} y^{\frac{10}{2}} = 7^1 x^3 y^5 = 7x^3 y^5$$

This result is easily verified since $\left(7x^3 y^5\right) \cdot \left(7x^3 y^5\right) = 49x^6 y^{10}$.

Thus, we have these three additional rules for exponents.

Additional Rules of Exponents
6. $x^{-p} = \dfrac{1}{x^p} = \left(\dfrac{1}{x}\right)^p$ 7. $a^{\frac{1}{2}} = \sqrt{a}$ 8. $a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$

Your Turn!!

In problems 1 to 6 express the answer in scientific notation to the correct number of significant figures. In problems 7 to 26 evaluate and simplify the given expression.

1.	$928{,}631$	= ______	11.	$\left(10^4 \cdot 10^2\right)^2$	= ______
2.	0.0000000103	= ______	12.	$10^{-5} \cdot 10^9$	= ______
3.	423×10^6	= ______	13.	$10^4 \cdot \left(10^{-3}\right)^3$	= ______
4.	0.00917×10^{-3}	= ______	14.	$\left(\dfrac{2 \times 10^{-4}}{10^{-6}}\right)^2$	= ______
5.	$\left(4.12 \times 10^4\right)^2 \cdot 3.01 \times 10^{-2}$	= ______	15.	$(8)^{\frac{1}{3}}$	= ______
6.	$12.0\text{ g} \div \left(6.02 \times 10^{23}\right)$	= ______	16.	$(-8)^{\frac{1}{3}}$	= ______
7.	$10^7 \cdot 10^{11}$	= ______	17.	$(-8)^{\frac{2}{3}}$	= ______
8.	$10^5 \cdot \left(2 \cdot 10^2\right)^3$	= ______	18.	$(-8)^{-\frac{1}{3}}$	= ______
9.	$x^5\left(-x^3\right)^2$	= ______	19.	$(-8)^{-\frac{2}{3}}$	= ______
10.	$\left(10^4\right)^3 \cdot 10^5$	= ______	20.	$\left(\dfrac{25}{49}\right)^{-\frac{3}{2}}$	= ______

Simplify the following expressing the answers as monomials with only positive exponents on occurring variables.

21. $\dfrac{24x^4y^{-3}}{4x^{-4}y^3}$ = ______

22. $\dfrac{12xy^2}{4xy^{-1}}$ = ______

23. $\left(\dfrac{xy^2}{z}\right)^{-1}$ = ________________

24. $\left(\dfrac{xy^{-2}}{z}\right)^{-1}$ = ________________

25. $\dfrac{\left(-2x^5y^4\right)^3}{\left(-5x^2y^5\right)^2}$ = ________________

26. $\dfrac{\left(-3x^2y\right)^2}{(-2x)^3}$ = ________________

27. $\left(\dfrac{x^{-2}y^{-3}}{z}\right)^{-2}\left(\dfrac{3x^2y^{-1}}{2z^2}\right)^{-2}$ = ________________

28. $\dfrac{4x^2y^{-2}}{\left(3x^2y^{-3}\right)^3}\left(\dfrac{2x}{9y^2}\right)^{-2}$ = ________________

29. $\sqrt{36x^8y^{12}}$ = ________________

Section 2.4 Solving Equations and Rearranging Formulas

An equation that contains a variable is an "open" sentence. This means that the truth or falsehood of the equation depends on the value substituted into the equation. For example, $8x + 12 = 28$ is false when $x = 3$, since $8 \cdot 3 + 12 = 24 + 12 = 36 \neq 28$. In fact, this equation is only true when $x = 2$. We say that 2 "solves" $8x + 12 = 28$. To solve an equation means to find the set of values that make the equation true. Some equations, like $x + 1 = x$, have no solutions, since no number can ever make the statement true. Other equations like $x + x = 2x$ work for any number, so the solution set contains infinitely many numbers.

In order to solve an equation like $8x + 12 = 28$, we need a set of rules that allow us to transform equations yet still guarantees that the set of solutions remains unchanged. There are two such rules.

1. Addition Rule: Given expressions A, B, and C, if $A = B$, then $A + C = B + C$ is an equivalent equation, i.e., an equation with exactly the same solutions. In summary, **equals added to equals are equal**.

2. Multiplication Rule: Given expressions A, B, and a number c, with $c \neq 0$, if $A = B$, then $c \cdot A = c \cdot B$ is an equivalent equation. In summary, **equals times equals are equal**.

Note: The addition rule includes subtraction, since subtraction is just adding the opposite. Similarly, the multiplication rule includes division, since division is multiplying by the reciprocal. In the multiplication rule it is important that $c \neq 0$. For example, if $x = 3$, the set of solutions is obviously just the number 3. However, the equation $0 \cdot x = 0 \cdot 3$ is true for any number. Thus, multiplying through an equation by zero changes the set of solutions.

Just knowing these rules does not enable us to solve equations anymore than just knowing the rules of chess wins chess matches. What is needed is a strategy. The goal is to get an equation of the form $x = \text{a number}$. We use the rules in such a way as "to isolate" the variable all alone on one side of the equal sign with a coefficient of 1. For example to solve $8x + 12 = 28$, first subtract 12 from both sides,

$$8x + 12 - 12 = 28 - 12$$

$$8x + 0 = 16$$

$$8x = 16$$

then, divide both sides by 8.

$$\frac{8x}{8} = \frac{16}{8}$$

$$1 \cdot x = 2$$

$$x = 2$$

To solve $3(2x-2)-3(x-5)=8-(x+11)$, first use the distributive property to combine like terms on both sides of the equation.

$$\begin{gathered}3(2x-2)-3(x-5)=8-(x+11)\\3\cdot 2x-3\cdot 2-3\cdot x+(-3)\cdot(-5)=8-x+(-1)\cdot 11\\6x-6-3x+15=8-x-11\\6x-3x+(-6)+15=8-11-x\\3x+9=-3-x\end{gathered}$$

Then add x to both sides.

$$\begin{aligned}3x+9&=-3-x\\3x+9+x&=-3-x+x\\3x+x+9&=-3+0\\4x+9&=-3\end{aligned}$$

Next subtract 9 from both sides to get the x term by itself.

$$\begin{aligned}4x+9-9&=-3-9\\4x+0&=-12\\4x&=-12\end{aligned}$$

Finally, divide both sides by 4.

$$\begin{aligned}\frac{4x}{4}&=\frac{-12}{4}\\1\cdot x&=-3\\x&=-3\end{aligned}$$

So the solution is -3, which can be verified by substitution into the original equation.

$$\begin{gathered}3[2(-3)-2]-3(-3-5)=8-(-3+11)\\3\cdot(-6-2)-3\cdot(-8)=8-8\\3\cdot(-8)+24=0\\-24+24=0\\0=0\end{gathered}$$

As was noted earlier not every equation we can write has a solution. Consider solving $3(2x-1)-5(x-2)=2x-(x-4)$. The following steps attempt to isolate x.

$$\begin{gathered}3(2x-1)-5(x-2)=2x-(x-4)\\3\cdot 2x-3\cdot 1-5\cdot x+(-5)\cdot(-2)=2x-x+(-1)\cdot(-4)\\6x-3-5x+10=x+4\\x+7=x+4\\-x+x+7=-x+x+4\\7=4\end{gathered}$$

The last statement 7 = 4 is never true. It's a contradiction. So no value of x will ever make $3(2x-1)-5(x-2)=2x-(x-4)$ a true statement. The equation has **no** solutions.

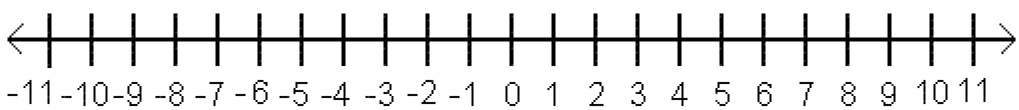
Conclusion: If in trying to isolate the variable, the variable drops out and an impossible equation results, then the original equation has no solutions. The key idea is that **both** the variable drops out and an impossible equation results. To conclude that there are no solutions it is necessary but not sufficient that the variable drops out as the next example illustrates.

Consider solving $3(2x-1)-5(x-2)=2x-(x-7)$. The following steps attempt to isolate x.

$$\begin{aligned}
3(2x-1)-5(x-2) &= 2x-(x-7)\\
3\cdot 2x-3\cdot 1-5\cdot x+(-5)\cdot(-2) &= 2x-x+(-1)\cdot(-7)\\
6x-3-5x+10 &= x+7\\
x+7 &= x+7\\
-x+x+7 &= -x+x+7\\
7 &= 7
\end{aligned}$$

The variable dropped out, but the resulting statement, 7 = 7, is always true! This means any number on the number line is a solution!

Conclusion: If in trying to isolate the variable, the variable drops out and an identity results, then the original equation has infinitely many solutions. Any number on the number line is a solution.



The same techniques used to isolate a variable in solving an equation can be used in rearranging a formula. For example, the area A of a triangle is given by the formula $A=\frac{1}{2}b\cdot h$, where b is the base and h is the height of the triangle. This formula is fine if we know b and h and want to calculate A. Suppose, on the other hand, that we want to know the height required for a given base to achieve a desired area. What we need now is a formula for h in terms of b and A. Starting with $A=\frac{1}{2}b\cdot h$, we multiply both sides by $\frac{2}{b}$ to isolate h.

$$\begin{gathered}
A=\frac{1}{2}b\cdot h=\frac{b}{2}\cdot h\\
\frac{2}{b}\cdot A=\frac{2}{b}\cdot\frac{b}{2}\cdot h=\frac{\cancel{2}}{\cancel{b}}\cdot\frac{\cancel{b}}{\cancel{2}}\cdot h=1\cdot h\\
\frac{2A}{b}=h \quad \text{or} \quad h=\frac{2A}{b}
\end{gathered}$$

So we obtain the equivalent rearranged formula, $h=\frac{2A}{b}$.

As a more complicated example, consider the formula $y=\frac{a(x+b)}{2}$. Suppose we need to rearrange this formula to solve for x in terms of y, a, and b. First multiply both sides by $h=\frac{2}{a}$.

$$y=\frac{a(x+b)}{2}$$

$$\frac{2}{a}\cdot y=\frac{2}{a}\cdot\frac{a(x+b)}{2}=\frac{\cancel{2}}{\cancel{a}}\cdot\frac{\cancel{a}(x+b)}{\cancel{2}}=1\cdot(x+b)$$

$$\frac{2y}{a}=x+b$$

Next subtract b from both sides.

$$\frac{2y}{a}-b=x+b-b$$

$$\frac{2y}{a}-b=x+0$$

$$\frac{2y}{a}-b=x \quad \text{or} \quad x=\frac{2y}{a}-b$$

As an example involving roots, consider the equation for the area of a circle, $A=\pi r^2$. To solve for r in terms of A, first divide both sides by π.

$$\frac{A}{\pi}=\frac{\pi r^2}{\pi}=1\cdot r^2$$

$$\text{so, } r^2=\frac{A}{\pi}$$

To solve for r, raise both sides of this equation to the power of $\frac{1}{2}$.

$$r=\left(r^2\right)^{\frac{1}{2}}=\left(\frac{A}{\pi}\right)^{\frac{1}{2}}=\sqrt{\frac{A}{\pi}}$$

Your Turn!!

Solve the following equations.

1. $5x+11=-9$

2. $x-3(2x-1)=-12$

3. $7(x-1)-5(3x-2)=10-3(x-1)$

4. $5x-3(2x+4)=-17$

5. $5x-3(2x+4)=10-x$

6. $5x-3(2x-4)=12-x$

7. $4(2x-3)-3(x-2)=x-3(6-x)+11$

8. $4(2x-3)-3(x-2)=2x-3(6-x)+11$

9. $4(2x-3)-3(x-2)=2x-3(6-x)+12$

10. Given that $P=VI$ solve for I in terms of P and V.

11. Given that $A=\dfrac{h(a+b)}{2}$ solve for b in terms of A, a, and h.

12. Given that $V = \frac{4\pi}{3} r^3$, solve for r in terms of V.

13. Given that $2a = b + 3c$, solve for b in terms of a and c.

14. Given that $2a = \frac{b^2}{c} - a$, solve for a in terms of b and c.

15. Given that $2a = \frac{b^2 + ab}{c}$, solve for a in terms of b and c.

16. Given that $\frac{a}{a+h} = \frac{r_2}{r_1}$, solve for a in terms of h, r_1, and r_2.

17. Given that $V = \frac{\pi}{3} r^2 h$, solve for h in terms of V and r.

18. Given that $V = \frac{\pi}{3} r^2 h$, solve for r in terms of V and h.

19. Given that $V = \pi d \left({r_2}^2 - {r_1}^2 \right)$, solve for d in terms of V, r_1, and r_2.

20. Given that $V = \pi d \left({r_2}^2 - {r_1}^2 \right)$, solve for r_2 in terms of V, r_1, and d.

21. Given that $V = \pi d \left({r_2}^2 - {r_1}^2 \right)$, solve for r_1 in terms of V, r_2, and d.

22. Given that $V = \frac{4\pi}{3} \left({r_2}^3 - {r_1}^3 \right)$, solve for r_2 in terms of V and r_1.

Section 2.5 Solving Linear Inequalities in One Variable

Inequalities are algebraic expressions related by >, <, ≥, and ≤. The symbol ≥ means greater than or equal and ≤ means less than or equal. They are used when you specify that the result of a calculation to be greater than or less than a certain answer. Linear inequalities are solved exactly the same as linear equalities, except that if you multiply by or divide by a negative number, you have to reverse the direction of the inequality. This fact is demonstrated below.

$$\begin{aligned} 2 &< 5 \\ (-1)(2) &> (-1)(5) \\ -2 &> -5 \end{aligned} \qquad\qquad \begin{aligned} -x &< 7 \\ (-1)(-x) &> (-1)(7) \\ x &> -7 \end{aligned}$$

To **solve an inequality** means to find *all* values of the variable that make the inequality true. Any of these values is a **solution** to the inequality, and the set of all possible solutions is called the **solution set**.

Properties of Inequalities: An inequality can be transformed into an equivalent inequality by:

- adding or subtracting any quantity to both sides (the addition property of inequalities), or
- multiplying or dividing by any positive quantity (the multiplication property of inequalities).
- If both sides are multiplied or divided by a negative quantity, then **the direction of the inequality symbol gets reversed**.

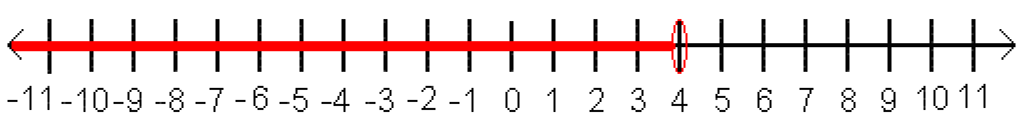
Steps for Solving a Linear Inequality:

1. Simplify each side separately.
2. Isolate the variable terms on one side using the addition property of inequalities.
3. Isolate the variable using the multiplication property of inequalities.

Example: Solve $8x-5<27$.
Solution: Isolate x in the inequality.

$$8x-5<27 \Rightarrow 8x<27+5 \Rightarrow 8x<32 \Rightarrow x<\frac{32}{8}=4$$

The solution set consists of all the numbers on the number line less than (to the left of) 4. This is shown graphically below by overlaying all the numbers to the left of 4. The open oval about 4 indicates that 4 is **not** in the solution set.



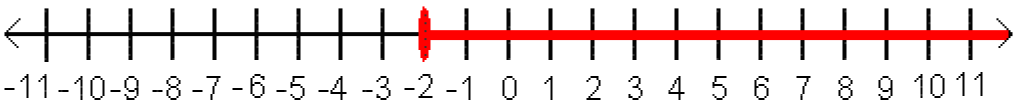
Example: Solve $4x-2(x-1)\le 5x+8$.

Solution: Isolate x in the inequality.

$$4x-2(x-1)\le 5x+8 \Rightarrow 4x-2x+2\le 5x+8 \Rightarrow 2x-5x\le 8-2 \Rightarrow -3x\le 6 \Rightarrow \frac{-3x}{-3}\ge\frac{6}{-3}$$

$$\Rightarrow x\ge -2$$

Note that when both sides of the inequality were divided by **–**3, the $\le$ reversed direction to become $\ge$. The solution set consists of all the numbers on the number line at or to the right of **–**2. This is shown graphically below by overlaying all the numbers to the right of **–**2. The shaded or "closed" oval about **–**2 indicates that **–**2 **is** in the solution set.



Example: Solve $7x-2(x-4)>2(x+5)+3(x+2)$.

Solution: Isolate x in the inequality.

$$7x-2(x-4)>2(x+5)+3(x+2) \Rightarrow 7x-2x+8>2x+10+3x+6 \Rightarrow 5x+8>5x+16$$

$$5x-5x+8>16 \Rightarrow 8>16$$

The variable dropped out and we are left with the contradiction that 8 is a larger number than 16. This means there are no solutions and so there is nothing to graph!

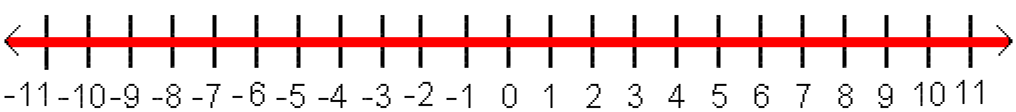
Example: Solve $7x-2(x-4)<2(x+5)+3(x+2)$.

Solution: Isolate x in the inequality.

$$7x-2(x-4)<2(x+5)+3(x+2) \Rightarrow 7x-2x+8<2x+10+3x+6 \Rightarrow 5x+8<5x+16$$

$$5x-5x+8<16 \Rightarrow 8<16$$

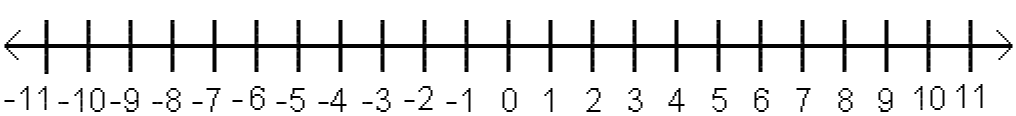
The variable dropped out and we are left with the always true statement that 8 is a smaller number than 16. This means that any number on the number line is a solution. Thus we overlay the entire line!



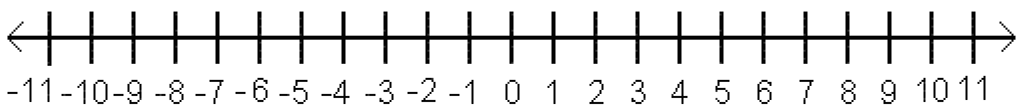
Your Turn!!

Solve and graph the solution of the following inequalities.

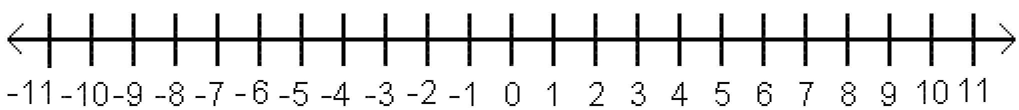
1. $4x+11<-9$



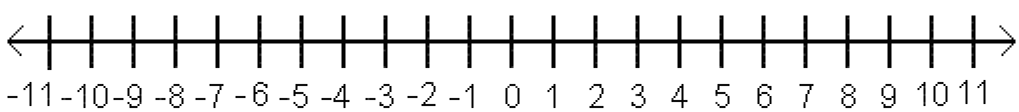
2. $x-2(3x-1)\ge -13$



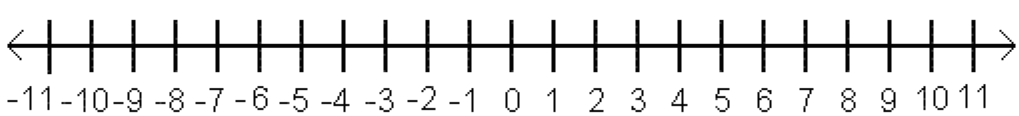
3. $5(x-1)-5(2x-2)>8-3(x-1)$



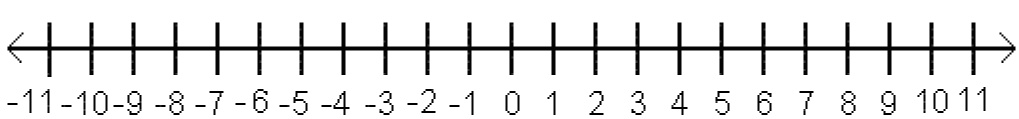
4. $5x-3(2x-7)\le 19$



5. $5x-3(2x+4)<6-x$



6. $5x-3(2x-4)\ge 25-x$



Chapter 2 Sample Test

_______________/100

Each Problem is worth 4 points.

Simplify each of the following.

1. $-3^2 - 6(-5 + 2)$ = _______________

2. $(-3)^2 - 6(-5) + 2$ = _______________

3. $3(2x - 7)$ = _______________

4. $9u - 5v + 12u - 10 - 2v + 4$ = _______________

5. $-5(3w - 4v - 7)$ = _______________

6. $(5x^2 - 7x + 9) - 2(x^2 + 2x - 5)$ = _______________

Evaluate each of the following expressions with the given values.

7. Evaluate the polynomial $x^2 - 3x - 9$ when $x = -4$.

Answer = _______________

8. Evaluate the formula $f = \dfrac{F}{d - F}$ to find the value of f when $F = 45.$ cm and $d = 60.$ cm

Answer = _______________

Solve the following equations.

9. $8x = 40$

Answer = _______________

10. $6t - 12 = 6$

Answer = _______________

11. $8y + 5 = 3y - 20$

Answer = _______________

12. $3(4z - 2) = 2(z + 27)$

Answer = _______________

13. $25 - 8(3c - 5) = 3(6c - 5) - 4$

Answer = _______________

14. $2(4x - 7) - 5(3x - 3) = x - 4(2x + 3)$

Answer = _______________

15. Write 2.1365×10^4 in standard (decimal) notation.

Answer = _______________

16. Write 0.00000001463 in scientific notation.

Answer = _______________

17. Simplify and write the answer in scientific notation to the correct number of significant figures. $\dfrac{\left(4.8 \times 10^{8}\right)\left(2.24 \times 10^{-6}\right)}{9.6 \times 10^{-6}}$.

Answer = _______________

Use the Exponent Rules to simplify each of the following expressions.

18. $x^{12} \cdot x^{6}$ = ______________

19. $\left(-x^{4}\right)^{3}$ = ______________

20. $\left(-2x^{4}y^{3}\right)\cdot\left(-3x^{6}y^{5}\right)$ = ______________

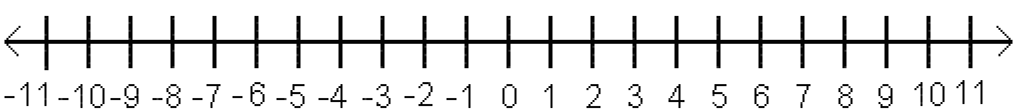
21. $\left(\dfrac{-2}{3}\right)^{-2}$ = ______________

22. $\left(\dfrac{8}{125}\right)^{-\frac{2}{3}}$ = ______________

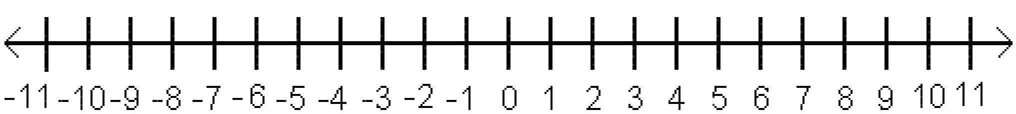
23. $\dfrac{28\left(x^{4}y^{-2}\right)^{2}}{7x^{8}y^{-8}}$ = ______________

Solve and graph the solution of the following inequalities:

24. $5(x-7)-3>4(x-8)-2(x+10)+5$



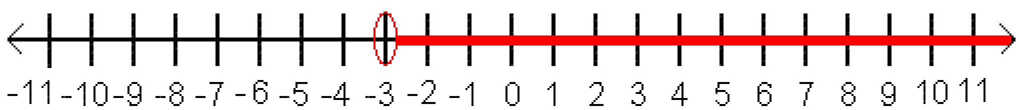
25. $2(x-7)-3(x-5)\leq 2x-3-2(x+1)$



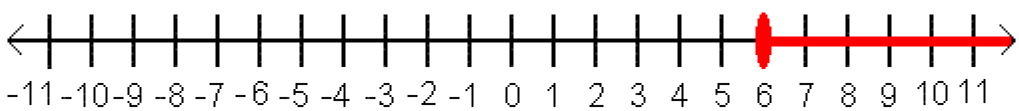
Chapter 2 Sample Test Solutions

1.	**2.**	**3.**	**4.**	**5.**
9	41	$6x-21$	$21u-7v-6$	$-15w+20v+35$
6.	**7.**	**8.**	**9.**	**10.**
$3x^2-11x+19$	19	3	$x=5$	$t=3$
11.	**12.**	**13.**	**14.**	**15.**
$y=-5$	$z=6$	$c=2$	No Solution	21365 or 21,365
16.	**17.**	**18.**	**19.**	**20.**
1.463×10^{-8}	1.1×10^{-8}	x^{18}	$-x^{12}$	$6x^{10}y^8$
21.	**22.**	**23.**	**24.**	**25.**
$\frac{9}{4}=2.25$	$\frac{25}{4}=6.25$	$4y^4$	$x>-3$	$x\geq 6$

24.

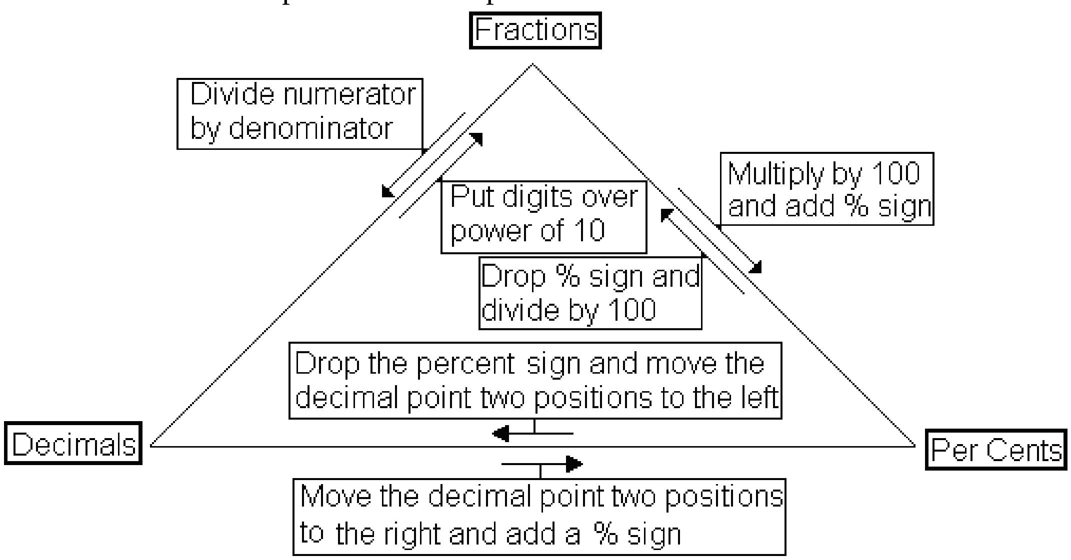


25.



Section 3.1 Percent Problems

The word percent comes from Latin and literally means divide (per) by 100 (cent). Even the % symbol, if you look closely at it, resembles /100 . Thus, percents are just another way of writing fractions or decimal fractions. In the chart below are shown the conversions between fractions, decimals, and percents. A more detailed discussion for converting fractions into decimals and decimals into fractions was provided in Chapter 1.



Examples:

Converting a decimal to a percent $$0.145 = 14.5\% = 14\frac{1}{2}\%$$

Converting a decimal to a fraction $$0.145 = \frac{145}{1000} = \frac{29 \times \cancel{5}}{200 \times \cancel{5}} = \frac{29}{200}$$

Converting a percent to a decimal $$28\% = 28 \div 100 = 0.28$$

Converting a percent to a fraction $$28\% = \frac{28}{100} = \frac{7}{25}$$

Converting a fraction to a decimal $$\frac{1}{6} = 1 \div 6 = 0.166\overline{6}$$

Converting a fraction to a percent $$\frac{1}{6} = \frac{1}{6} \times 100\% = \frac{50}{3}\% = 16\frac{2}{3}\%$$ $$= 16.667\% \text{ (a percent to 3 places)}$$

Converting a percent to a decimal $$\frac{1}{4}\% = 0.25\% = 0.25 \div 100 = 0.0025$$

Converting a percent to a fraction $$\frac{1}{4}\% = \frac{1}{4} \div 100 = \frac{1}{4} \times \frac{1}{100} = \frac{1}{400}$$

Example: The reduction percentage for automotive paint is the amount (volume) of thinner added divided by the amount (volume) of paint originally present. For a 150% reduction how many gallons of thinner must be added to 7.5 gallons of paint?
Solution:

$$150\% = \frac{\text{volume thinner}}{\text{volume paint}} = \frac{\text{volume thinner}}{7.5 \text{ gallons}} = \text{volume thinner} \times \frac{1}{7.5 \text{ gallons}}$$

$$\Rightarrow \text{volume thinner } = 150\% \times 7.5 \text{ gallons} = 1.50 \times 7.5 \text{ gallons} = 11.25 \text{ gallons} = 11\frac{1}{4} \text{ gallons}$$

There are three basic types of percent problems.

1. Missing part.

2. Missing total amount.

3. Missing percent.

All three types of problems can be solved by the same strategy of translating word problems into math and a "little bit" of algebra. The translation is as follows:

what	means	the unknown, often called x
is	means	=
of	means	$\times$ (multiplication)

The following six problems illustrate the method.

Example: What is 16% of \$140 ? (A missing part problem.)
Solution: $x = 16\% \times \$140 = 0.16 \times \$140 = \$22.40$.

Example: 27 is 54% of what value? (A missing total amount.)
Solution: $27 = 54\%(x) = 0.54x \quad \Rightarrow \quad x = 27 \div 0.54 = 50$.

Example: What percent of \$160 is \$200? (A missing percent.)
Solution: $x(\$160) = \$200 \quad \Rightarrow \quad x = \$200 \div \$160 = 1.25 = 125\ \%$.

Example: What percent of $\frac{1}{4}$ is $\frac{3}{32}$? (A missing percent.)

Solution: $x\left(\frac{1}{4}\right) = \frac{3}{32} \quad \Rightarrow \quad x = \frac{3}{32} \div \frac{1}{4} = \frac{3}{32} \times \frac{4}{1} = \frac{3}{8} = \frac{3}{8} \times 100\ \% = 37\frac{1}{2}\ \%$.

Example: 125% of what value is \$423 ? (A missing total amount.)
Solution: $125\%\left(x\right)=\$423 \quad\Rightarrow\quad x=\$423\div 1.25=\$338.40$.

There are many common percent applications. In most communities consumers pay sales tax on certain purchases. If this tax is 5.5%, then the amount of sales tax is (0.055)(sales price). The checkout price is then 100% of sales price + 5.5% of sales price or 105.5% of sales price. For example, suppose we purchase an item listed at \$89.99 .

The sales price					=	\$89.99
The sales tax	=	(0.055)(\$89.99)	=	\$4.94945	=	\$ 4.95
				Checkout	=	\$94.94

A faster way is to compute this is 105.5% of \$89.99 = (1.055)(\$89.99) = \$94.93945 which, rounded up to the next penny, is \$94.94 .

Example: The checkout price of a new TV was \$258.48 . If 5.5% sales tax was charged, what was the sales price?
Solution:
$$\text{Purchase Price}=105.5\%\times\text{Sales Price}=1.055\times\text{Sales Price}$$
$$\Rightarrow\text{Sales Price}=\text{Purcahse Price}\div 1.055\Rightarrow\text{Sales Price}=\$258.48\div 1.055=\$245.00$$
Occasionally retailers offer discounts on items. A 10% discount means that 10% of the list price has been removed in setting the sales price. If there had been **no** discount, the sales price would be 100% of the list price. The discount of 10% means that the new sales price is 100% – 10% = 90% of list price. This illustrates an important principle, when considering a percent change, the "base line" is 100%. Thus, a 12% increase means that the new value is 112% of the old value. Working backwards we get the following formula:

$$\text{Percent change} = \frac{\text{new value}-\text{old value}}{\text{old value}}\times 100\% \quad .$$

Note: multiplying by 100% converts the fractional change into a percent change. If the percent change is positive, this is called a percent increase, if negative, a percent decrease.

Example: A TV regularly priced at \$199.99 is discounted 20%. What is the new selling price?
Solution: $\text{New Sales Price } =\left(100\%-20\%\right)\times\$199.99=0.80\times\$199.99=\$160.00$.

Example: If 5.5% sales tax is included, what is the checkout price of a TV discounted 20% from its regular price of \$199.99?
Solution: $\text{Checkout Price } =\left(105.5\%\right)\times\$160.00=1.055\times\$160.00=\$168.80$. Here, the final answer was rounded up to the next penny.

Example: New car prices of a particular make went from \$13,499 to \$14,200. What was the percent increase?
Solution: $\text{Percent Increase } =\left(14,200-13,499\right)\div 13,499\times 100\%=5.19\%$.

When a measurement is specified on a precision part, such as a diameter of 0.150 in plus or minus 0.001 in , the 0.001 in is called the tolerance. This means that any measured diameter between 0.149 in and 0.151 in is acceptable. Another way of specifying the tolerance is the percent tolerance defined by the following formula:

$$\text{Percent Tolerance} = \frac{\text{tolerance}}{\text{specified value}} \times 100\% \quad .$$

Again multiplying by 100% changes the fractional tolerance into a percent tolerance.

Example: A resistor is rated at $85\ \Omega \pm 2\ \Omega$. What is the percent tolerance?
Solution: $\text{Percent Tolerance} = 2\ \Omega \div 85\ \Omega \times 100\% \ = 2.35\%$.

Example: An acceptable voltage reading is 15.0 V with a percent tolerance of 3%. Express this tolerance in volts.
Solution: $\text{Tolerance} = 3\% \times 15.0\text{ V} = 0.45\text{ V}$. So, the voltage is $15.0\text{ V} \pm 0.45\text{ V}$.

Percent error is similar to percent tolerance. When an actual measurement is made and compared to the “true” or “specified” value, the percent error is defined by the following formula:

$$\text{Percent Error} = \frac{\text{measured value} - \text{specified value}}{\text{specified value}} \times 100\% \quad .$$

Usually if the calculated percent error is negative the minus sign is ignored. This is because we are concerned with “how close” we are to the specified value and not whether we are above or below it.

Example: A part is specified as having a diameter of 0.250 in . The manufactured part measures 0.254 in . What is the percent error?
Solution: $\text{Percent Error} = \left(0.254\text{ in} - 0.250\text{ in}\right) \div 0.250\text{ in} \times 100\% \ = 1.60\%$

Example: Some sales personnel earn part or all of their salary on a commission basis. This means that they are paid a percentage of their total sales. A sales person earns \$250 per week plus a 2.5% commission. If the person desires a week’s gross pay of \$900, how much merchandise must be sold?
Solution: The amount of commission = \$900 – \$250 = \$650 .
$2.5\% \text{ of sales} = 0.025 \times \text{sales} = \$650 \ \Rightarrow \ \text{sales} = \$650 \div 0.025 = \$26{,}000$

Financial Applications: Loans

When getting a loan from a lender, we must pay back both the amount borrowed (the **principal**) plus **interest** for the temporary use of the lender's money. In a simple interest loan everything is paid in one lump sum. For example, if you borrow \$1500 at an interest of 5.0%, then you must pay back 100% of the principal plus 5.0% of the principal = 105% of the principal or (1.05)(\$1500) = \$1575 .

Most conventional loans are paid back in a sequence of monthly payments. In each payment a portion is used to pay the interest owed on the remaining debt and what is left over is used to reduce the debt, i.e., is paid against principal. A conventional loan is characterized by the following four parameters.

1. The initial principal symbolized by P.

2. The annual percentage rate (APR) of interest symbolized by R.

3. The number of years over which the loan is paid off (the period of amortization) symbolized by N.

4. The monthly payment symbolized by M.

To calculate M knowing P, R, and N use the formula: $M = \dfrac{P \cdot R}{12 \cdot \left[1 - 1 \div (1 + R/12)^{12 \cdot N}\right]}$

To calculate N knowing P, R and M use the formula: $N = \dfrac{\log\left[1 - \dfrac{R \cdot P}{12 \cdot M}\right]}{-12 \cdot \log\left[1 + \dfrac{R}{12}\right]}$

To calculate P knowing M, R, and N use the formula: $P = \dfrac{12 \cdot M}{R} \cdot \left[1 - 1 \div (1 + R/12)^{12 \cdot N}\right]$

In the formula for N, **log** is the logarithm function, which is found on any scientific calculator. These formulas are rather complicated although they can be easily computed using a spreadsheet program such as Excel. With patience they can also be done on a scientific calculator. To illustrate these calculations consider the following three examples.

Example: If you borrow \$12,000 at an annual percentage rate of 2.3% to be paid off over 4 years, what is your monthly payment?

Solution: P = \$12,000, R = 0.023 and N = 4, so we need to use the formula for M. Because the number of instructions that can be stored on older calculators such as the TI-30Xa is less than that on newer models, the keystrokes for the TI-30Xa must be entered differently.

On newer calculators determine M with the following keystrokes:

$$\mathrm{P}\boxed{\times}\mathrm{R}\boxed{\div}\boxed{(}12\boxed{\times}\boxed{(}1\boxed{-}1\boxed{\div}\boxed{(}1+\mathrm{R}\boxed{\div}12\boxed{)}\boxed{\wedge}\boxed{(}12\boxed{\times}\mathrm{N}\boxed{)}\boxed{)}\boxed{)}\boxed{=}$$

For the values given for P, R, and N this works out to be \$261.92, if rounded up to the next penny. Remember that some older calculators use the symbol $\boxed{y^x}$ or $\boxed{x^y}$ for raising a base to an exponent instead of $\boxed{\wedge}$.

The corresponding keystrokes for the TI-30Xa are

$$\boxed{(}1\boxed{-}1\boxed{\div}\boxed{(}1\boxed{+}\mathrm{R}\boxed{\div}12\boxed{)}\boxed{y^x}\boxed{(}12\boxed{\times}\mathrm{N}\boxed{)}\boxed{)}\boxed{1/x}\boxed{\times}\mathrm{P}\boxed{\times}\mathrm{R}\boxed{\div}12$$

The $\boxed{1/x}$ key is in the third row, second column.

Example: If you borrow \$18,000 at an annual percentage rate of 3.1% and make monthly payments of \$324.24, how long does it take to pay off the loan?

Solution: P = \$18,000, R = 0.031 and M = \$324.24, so we need to use the formula for N.

On newer calculators determine N with the following keystrokes:

$$\boxed{\log}\boxed{(}1\boxed{-}\mathrm{R}\boxed{\times}\mathrm{P}\boxed{\div}\boxed{(}12\boxed{\times}\mathrm{M}\boxed{)}\boxed{)}\boxed{\div}\boxed{(}\boxed{-}12\boxed{\times}\boxed{\log}\boxed{(}1\boxed{+}\mathrm{R}\boxed{\div}12\boxed{)}\boxed{)}\boxed{=}$$

For the values given for P, R, and M the computed value for N is 4.99949 or 5 years.

The corresponding keystrokes for the TI-30Xa are

$$\boxed{(}1\boxed{-}\mathrm{R}\boxed{\times}\mathrm{P}\boxed{\div}\boxed{(}12\boxed{\times}\mathrm{M}\boxed{)}\boxed{)}\boxed{\log}\boxed{\div}\boxed{(}12\boxed{+\Leftrightarrow-}\boxed{\times}\boxed{(}1\boxed{+}\mathrm{R}\boxed{\div}12\boxed{)}\boxed{\log}\boxed{)}\boxed{=}$$

Here $\boxed{+\Leftrightarrow-}$ stands for the “change sign” key which is in the bottom row, fourth column.

Example: If you can afford monthly payments of $350 over a three year period, what is the most money you can borrow at an annual percentage rate of 4.2%?
Solution: M = $350, R = 0.042 and N = 3, so we need to use the formula for P.

On newer calculators determine P with the following keystrokes:

$$12\boxed{\times}M\boxed{\div}R\boxed{\times}\boxed{(}1\boxed{-}1\boxed{\div}\boxed{(}1\boxed{+}R\boxed{\div}12\boxed{)}\boxed{\wedge}\boxed{(}12\boxed{\times}N\boxed{)}\boxed{)}\boxed{=}$$

Plugging in the values given for M, R, and N we calculate that P, if rounded up to the next penny, is $11,819.12 . On a practical level you can afford to borrow about $11,800.

The corresponding keystrokes for the TI-30Xa are

$$\boxed{(}1\boxed{-}1\boxed{\div}\boxed{(}1\boxed{+}R\boxed{\div}12\boxed{)}\boxed{y^x}\boxed{(}12\boxed{\times}N\boxed{)}\boxed{)}\boxed{\times}12\boxed{\times}M\boxed{\div}R\boxed{=}$$

Your Turn!!

Write each of the following numbers as a percent:

0.37 1) ________________

0.012 2) ________________

2.59 3) ________________

$\frac{3}{8}$ 4) ________________

$1\frac{2}{3}$ 5) ________________

Write each percent as a decimal number:

22.5% 6) ________________

211% 7) ________________

Write each percent as a fraction reduced to lowest terms:

125% 8) ________________

$1\frac{5}{6}\%$ 9) ________________

Solve for the following unknowns:

What is 16.5% of 128?

10) ________________

125% of $950 is what amount?

11) ________________

53 is what percent of 120?

12) ________________

$125 is 36% of what amount? (Round to the nearest penny.)

13) ________________

What percent of $\frac{3}{8}$ is $\frac{5}{16}$?

14) ________________

A license cost $120. If the cost increases 7.5%, what is the new cost of this license?

15) ________________

If sales tax is 5.5%, what would be the check out price of a band saw with a list price of $289?

16) ________________

From a 15.0 lb cylinder 1.6 lb of material is removed during machining. What percent of material was removed?

17) ________________

An electrical resistor is rated at 85 ohms plus or minus 5%. Express this tolerance in ohms.

18) ________________

A piston is to have a diameter of 0.787 in ∀ 0.003 in . What is the percent tolerance?

19) ________________

Specifications call for a pin to be 1.500 in long. If the finished pin measures 1.504 in., what is the percent error?

20) ________________

If an electric drill usually selling for $89.95 is on sale at a discount of 25%, what is the new list

price?

21) ________________

A new car is advertised as selling for $14,220. This price reflects a 9% discount. What was the original (list) price? (Round to the nearest penny.)

22) ________________

An assembly line is shut down for inspection if the fraction of defective products exceeds 0.5%. If the normal day's production is 10,500 units, at most how many defective units can there be if the line is not to be shut down?

23) ________________

A salesperson is paid $380 per week plus a 2.2% commission. What is the person's sales total if the gross pay for a given week is $1,395? (Round to the nearest penny.)

24) ________________

What is the monthly payment required to pay off a $12,000 loan in two years at an annual percentage rate of 2.7%? (Round to the nearest penny.)

25) ________________

What is the largest amount which can be borrowed over three years at 4.5% APR if the largest affordable monthly payment is $279? (Round to the nearest ten dollars.)

26) ________________

How long would it take to pay off $15,000 at 5.2% APR if the monthly payment is $450? (Round to the nearest tenth of a year.)

27) ________________

You can afford 15% of your monthly income of $2300 on car payments. If the quoted annual percentage rate of the loan is 2.5% over three years, what is the most you can borrow? (Round to the nearest ten dollars.)

28) ________________

Section 3.2 Applied Problems

Word problems are often frustrating to solve particularly for beginning students. The problem is essentially one of translating statements made in an imprecise language like English into the very precise language of mathematics.

The following steps are given as a general strategy for solving word problems.

1. Read through the problem quickly to determine just what it is that's to be calculated.

2. Assign variable names to the relevant quantities, which are needed to calculate the answer.

3. Make a schematic of the problem, a table or picture that helps organize or visualize the relationships between the variables.

4. Carefully reread the problem and translate sentences into equations. This translation is often the hardest step in the solution. To aid in this translation the "dictionary" of English/Math equivalents presented below may prove helpful.

English	Math
is	$=$
of (with a fraction or %)	$\times$ (multiply)
sum of	$+$
difference	$-$
x more than y	$x+y$
x less than y	$y-x$
increase	$+$
decrease	$-$
twice x or 2 times x	$2x$
n times x	nx

5. Solve the equation(s) of step 4 to obtain the solution.

6. Check that the answer does **indeed** solve the stated problem. Does the answer make sense in the context of the stated problem? If not, even if it is a solution of the equation(s), it can not be a solution to the original word problem.

As an example of this last point, consider the following problem.

Example: The sum of two consecutive integers is 50. What are the numbers?

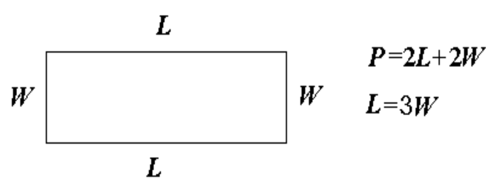
Solution: Let x be the first integer, then consecutive means that the next integer must be $x + 1$. Hence, the sum of the two numbers is 50, translates into the equation, $x+(x+1)=50$.
We can solve this equation by the following steps.

$$\begin{aligned}
&x+(x+1)=50\\
&x+x+1=50\\
&2x+1=50\\
&2x+1-1=50-1\\
&2x=49\\
&\frac{2x}{2}=\frac{49}{2}=24\frac{1}{2}=24.5
\end{aligned}$$

So the "solution" is $x=24.5$. But the original problem specified that x was to be an integer! Thus, we conclude that there is **no** solution to this problem.

Example: A rectangle is three times as long as it is wide. The perimeter is 112 cm. What are the rectangle's dimensions?

Solution: First, we see that we need to find the dimensions, i.e., the length and the width of the rectangle. Let W stand for the width and L stand for the length. From basic geometry the perimeter of a rectangle is given by $P = 2L + 2W$. Furthermore, the statement, " three times as long as it is wide", translates as $L = 3W$. Thus, in any equation in which L occurs it can be replaced by $3W$. These facts are summarized below.



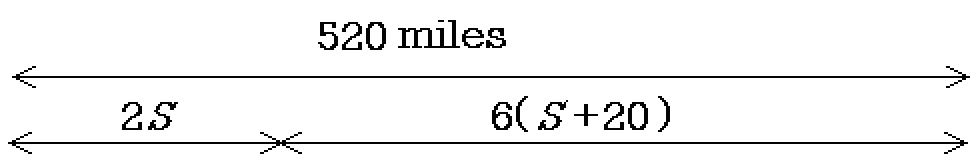
The perimeter equation can be solved for W as follows:

$$2L + 2W = 112$$
$$2\left(3W\right) + 2W = 112$$
$$6W + 2W = 112$$
$$8W = 112$$
$$\frac{8W}{8} = \frac{112}{8} = 14$$
$$W = 14$$

Since $L = 3W$, we conclude that $L = 3 \cdot 14 = 42$. Thus, the rectangle's dimensions are $14\text{ cm} \times 42\text{ cm}$, a completely "reasonable" solution which solves the stated problem.

Example: John drove for eight hours straight and traveled 520 miles. For two hours he was in heavy traffic and had to reduce his usual highway speed by 20 mph. The remaining time he drove his usual highway speed. What is John's usual highway speed?

Solution: We are supposed to determine John's usual driving speed. Let S stand for John's speed in heavy traffic. Then his usual highway speed is $S + 20$ .The fundamental relationship we must use is that $\text{speed} \times \text{time traveled} = \text{distance traveled}$. The speed S has a time traveled of 2 hours, while the usual speed (S + 20) has a time traveled of 6 hours (8−2). The problem is diagramed below.



The total distance traveled is the sum of $2S$ and $6(S + 20)$. This becomes the equation, $2S + 6(S + 20) = 520$. The following steps solve this equation.

$$2S + 6S + 6 \cdot 20 = 520 \qquad 8S = 400$$
$$8S + 120 = 520 \qquad \Rightarrow \qquad \frac{8S}{8} = \frac{400}{8} = 50$$
$$8S + 120 - 120 = 520 - 120 \qquad S = 50$$

The **actual** solution to the stated problem is $S + 20 = 50 + 20 = 70$, since we were supposed to find John's usual highway speed, not his reduced speed in traffic.
Example: How many grams of 10% tin solder must be added to 800 g of 15% tin solder in order to make a final solder which is 12% tin?

Solution: This is a typical "mixture" problem. Let x stand for the amount of 10% tin solder. The fundamental principle behind this problem is conservation of matter. When we combine the x grams of the 10% tin solder with the 800 g of 15% solder, the final 12% tin solder must contain 800 + x grams. The following table serves as a useful tool in putting this information together.

	10% Tin		15% Tin		12% Tin
Amount of Solder	x	+	800 g	=	x+800 g
% Tin	10%		15%		12%
Amount of Tin	0.10x		0.15(800) = 120		0.12(x+800)
	0.10x	+	120	=	0.12(x+800)

The tin in the final 12% solder must all have come either from the 10% tin solder or the 15% tin solder. This translates into the equation: $0.10x + 120 = 0.12(800 + x)$, which we solve for x.

$$
\begin{aligned}
&0.10x + 120 = 0.12(800 + x) \\
&0.10x + 120 = 0.12(800) + 0.12x \\
&0.10x + 120 = 96 + 0.12x \\
&0.10x - 0.10x + 120 = 96 + 0.12x - 0.10x \\
&120 = 96 + (0.12 - 0.10)x \\
&\text{or} \\
&96 + 0.02x = 120 \\
&-96 + 96 + 0.02x = -96 + 120 \\
&0.02x = 24 \\
&\frac{0.02x}{0.02} = \frac{24}{0.02} \\
&x = 1200
\end{aligned}
$$

This answer can be verified by plugging it into the previous table, where indeed it is true that 120 + 120 =0.12(2000).

	10% Tin		15% Tin		12% Tin
Amount of Solder	1200 g	+	800 g	=	1200 g + 800 g
% Tin	10%		15%		12%
Amount of Tin	.10(1200)		0.15(800) = 120		0.12(1200+800)
	120	+	120	=	0.12(2000) = 240

Your Turn!!

1. The sum of two consecutive odd integers is 444. What are the two numbers?

2. A rectangle is three times as long as it is wide. Its perimeter is 48 in. What are the rectangle's dimensions?

3. It takes you four hours to drive to a city. If you had driven 15 mph faster, you would have gotten there in only three hours. How far did you drive?

4. John's age is one year less than twice Mary's age. Their combined age is 74 years. How old are they?

5. In Sarah's pocket there is $2.51 in change. There is one more dime than quarters, but one less dime than twice the number of nickels. The number of pennies is three less than twice the number of dimes. How many of each type of coin are there?

6. How many grams of 16% tin solder must be added to 200 g of 25% tin solder to make a final solder which is 20% tin?

Section 3.3 Ratios. Proportions and Variation

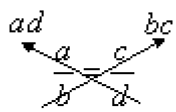
A ratio is just another name for a fraction, i.e., a numerator over a denominator. When we equate two ratios the resulting equation is called a proportion. The proportion "a is to b like c is to d" is shown below.

$$\frac{a}{b} = \frac{c}{d}$$

An equation equivalent to $\frac{a}{b} = \frac{c}{d}$ is obtained by multiplying both sides by bd.

$$bd\frac{a}{b} = bd\frac{c}{d}$$
$$ad\frac{b}{b} = bc\frac{d}{d}$$
$$ad = bc$$

The products ad and bc are called "cross products" since we have crossed diagonally across the equal sign of the proportion.



Thus, whenever we have $\frac{a}{b} = \frac{c}{d}$, we also have $ad = bc$ and visa versa. For example, $\frac{3}{4} = \frac{51}{68}$ is a valid proportion since $3 \cdot 68 = 4 \cdot 51 = 204$. In a proportion if a single numerator or a denominator is unknown, and the other three are known, we can solve for the unknown by using the cross product. For example,

$$\frac{9}{x} = \frac{5}{6}$$

$$\text{So, } 6 \cdot 9 = 5x \Rightarrow 5x = 54 \Rightarrow x = \frac{54}{5} = 10.8$$

Note: The steps were first to multiply to form the cross product and second to divide to get the answer. Hence this method is sometimes called "cross multiply and divide".

To verify this answer, substitute 10.8 into the original proportion.

$$\frac{9}{10.8} \stackrel{?}{=} \frac{5}{6}$$

$$\text{i.e., does } 0.8333... = \frac{5}{6}\text{ ?}$$

Since 5 divided by 6 is 0.8333…, it checks.

Variation is a specialized vocabulary, which allows us to describe the relations between quantities using sentences rather than equations. This is an important tool for communicating in situations where the type setting of equations is either difficult or impossible.

If $y = kx$, for some fixed k (we call k the "constant of proportionality"), we say y "varies directly as x" or "y is directly proportional to x". In a direct variation, if k is positive, as x increases so does y and as x decreases so does y. Stated differently, in a direct variation the two variables "move" in the same direction.

If $y = kx^n$, for some fixed k and fixed n, we say y "varies directly as x to the n" or "y is directly proportional to x to the n". For example, at fixed load or resistance, the electric power varies directly as the square of the current. Letting P stand for the electric power and i stand for the current this statement is equivalent to the equation $P = ki^2$.

If $y = \frac{k}{x}$, for some fixed k, we say y "varies inversely as x" or "y is inversely proportional to x". In an inverse variation, if k is positive, as x increases y decreases and as x decreases y increases. Stated differently, in an inverse variation the two variable "move" in opposite directions.

If $y = \frac{k}{x^n}$, for some fixed k, we say y "varies inversely as x to the n" or "y is inversely proportional to x to the n". For example, the intensity of a light source varies inversely as the square of the distance from the source. In symbols, $I = \frac{k}{d^2}$, where I is the intensity of the light source and d is the distance from the light source.

Often a quantity depends on more than one variable. For example, the pressure of a gas depends on the amount of gas present (the number of "moles", n), the absolute temperature (T), and the volume occupied by the gas (V). The relationship between these variables is that the pressure of a gas varies as the product of the amount of gas and the absolute temperature and inversely as the volume occupied by the gas. The phrase, "varies as the product", means the same thing as varies directly as the product. Thus, the relationship is described by the following formula.

$$P = \frac{RnT}{V}$$

Here, by standard usage, the proportionality constant in this example is called R.

Problems involving variation are often of the form of providing two different sets of data with an unknown quantity in one of the data sets. The simplest way to solve this kind of problem is to convert it into a proportion as the following examples illustrate.

Example: A drive gear having 80 teeth and rotating at 40 rpm meshes with a second gear having 16 teeth, How fast does the second gear rotate?

Solution: This is a proportion problem since for two meshing gears the number teeth per minute that pass the contact point must be the same for both gears. Let the number of teeth be *N* and the rate of rotation (measured in rpm) be *r*, then $N \cdot r = k$ (*k* a constant). Stated differently, $\frac{Nr}{N} = r = \frac{k}{N}$. This means that the rate of rotation for meshing gears is inversely proportional to the number of teeth. The fewer teeth, the faster the gear rotates. Let $N_1 = 80$, $r_1 = 40\text{ rpm}$, and $N_2 = 16$. The use of subscripts enables us to label the values that belong together in the same data set. Since $r_1 = \frac{k}{N_1}$ and $r_2 = \frac{k}{N_2}$, by taking the ratio of r_2 to r_1 the constant *k* drops out.

$$\frac{r_2}{r_1} = r_2 \div r_1 = \frac{k}{N_2} \div \frac{k}{N_1} = \frac{\cancel{k}}{N_2} \cdot \frac{N_1}{\cancel{k}} = \frac{N_1}{N_2}$$

$$\frac{r_2}{r_1} = \frac{N_1}{N_2}$$

Since *r* is inversely proportional to *N*, the subscripts on *N* are opposite (the numerator and the denominator are switched) to those on *r*. Plugging in the values for the variables gives the proportion, $\frac{r_2}{40\text{ rpm}} = \frac{80}{16}$, which we solve for r_2.

$$\frac{r_2}{40\text{ rpm}} = \frac{80}{16}$$

$$16r_2 = 40\text{ rpm} \cdot 80$$

$$r_2 = 40\text{ rpm} \cdot 80 \div 16 = 200\text{ rpm}$$

Example: If *y* varies as the product of *x* and the square of *z*, what is *x* when *y* = 4 and *z* = 2, if *y* = 12 when *x* = 6 and *z* = 3 ?

Solution: Let $x_1 = 6$, $z_1 = 3$, $y_1 = 12$, and $y_2 = 4$, $z_2 = 2$. Then it is x_2 we need to calculate. The variation statement translates into the equation, $y = kxz^2$. Dividing y_2 by y_1 results in the following proportion.

$$\frac{y_2}{y_1} = \frac{kx_2{z_2}^2}{kx_1{z_1}^2} = \frac{x_2{z_2}^2}{x_1{z_1}^2} \quad \text{or} \quad \frac{y_2}{y_1} = \frac{x_2{z_2}^2}{x_1{z_1}^2}$$

Plugging in the known values gives the proportion, $\frac{4}{12} = \frac{x_2 \cdot 2^2}{6 \cdot 3^2}$, which we solve for x_2.

$$\frac{4}{12} = \frac{x_2 \cdot 4}{6 \cdot 9} \quad \Rightarrow \quad \frac{1 \cdot 4}{3 \cdot 4} = \frac{4x_2}{54} \quad \Rightarrow \quad \frac{1}{3} = \frac{2x_2}{27} \quad \Rightarrow \quad 6x_2 = 27 \quad \Rightarrow \quad x_2 = \frac{27}{6} = \frac{9}{2} = 4.5$$

Example: For fixed amount of gas the pressure varies as directly as the absolute temperature and inversely as the volume occupied by the gas. If the pressure is $\frac{26.9 \text{ lb}}{\text{in}^2}$ when the absolute temperature is 320°K and the volume is 38.6 liters, what is the absolute temperature when the pressure is $\frac{38.0 \text{ lb}}{\text{in}^2}$ and the volume is 33.7 liters?

Solution: Let $P_1 = 26.9$ psi , $T_1 = 320$K , $V_1 = 38.6$ L , and $P_2 = 38.0$ psi , $V_2 = 33.7$ L . Then it is T_2 we need to calculate. The variation statement translates into the equation, $P = \frac{kT}{V}$. Dividing P_2 by P_1 results in the following proportion.

$$P_2 \div P_1 = \frac{kT_2}{V_2} \div \frac{kT_1}{V_1} = \frac{kT_2}{V_2} \cdot \frac{V_1}{kT_1}$$
$$\frac{P_2}{P_1} = \frac{\cancel{k}T_2}{V_2} \cdot \frac{V_1}{\cancel{k}T_1} = \frac{T_2}{T_1} \cdot \frac{V_1}{V_2}$$
$$\frac{P_2}{P_1} = \frac{T_2}{T_1} \cdot \frac{V_1}{V_2}$$

Since P is directly proportional to T and inversely proportional to V, in the above equation the subscripts for P and T match across numerator and denominator, while the subscripts for P and V are "switched". Plugging in the values for the variables gives the proportion, $\frac{38.0 \text{ psi}}{26.9 \text{ psi}} = \frac{T_2}{320\text{K}} \cdot \frac{38.6 \text{ L}}{33.7 \text{ L}}$, which we solve for T_2.

$$\frac{38.0 \text{ psi}}{26.9 \text{ psi}} = \frac{T_2}{320\text{K}} \cdot \frac{38.6 \text{ L}}{33.7 \text{ L}} \Rightarrow \frac{38.0}{26.9} = \frac{38.6T_2}{33.7(320\text{K})}$$
$$\Rightarrow 38.6T_2 = 33.7(320\text{K})\frac{38.0}{26.9} \Rightarrow T_2 = 320\text{K}\frac{33.7(38.0)}{38.6(26.9)} = 394.7\text{K}$$

So the final temperature of the gas is 395°K.

Note: By solving this problem as a proportion the unwanted units of measure "cancel". You can even mix metric and English units as long as the same quantity is measured in the same units. The value of the proportionality constant is never really needed.

Your Turn!!

1. The index of refraction of a material is the ratio of the speed of light in vacuum, $3.00\times10^{8}\ \frac{\text{m}}{\text{s}}$, to the speed of light in the material. The index of refraction of water is 1.33 . What is the speed of light in water?

2. A drive gear having 100 teeth and rotating at 40 rpm meshes with a second gear having 20 teeth. How fast does the second gear rotate?

3. If y is directly proportional to x and $x = 10$ when $y = 4$, what is x when $y = 12$?

4. If x is inversely proportional to y and $x = 24$ when $y = 3$, what is x when $y = 6$?

5. If y varies directly as the product of x^2 and t, what is t when $y = 10$ and $x = 5$, if y is 15 when $x = 3$ and $t = 5$?

The pressure of a gas, P, varies directly as the absolute temperature T and inversely as the volume, V.

6. If P is 10.5 psi when T is 400 K and V is 3.5 L , what is P when T is 380 K and V is 3.0 L ?

7. If P is 25.0 psi when T is 400 K and V is 6.00 cubic feet, what is T when P is 20.0 psi and V is 9.00 cubic feet?

8. If P is 1.0 atmosphere when T is 300 K and V is 22.0 L , what is V when T is 450 K and P is 3.2 atmospheres?

9. The resistance, R, of a wire varies directly as the length, l, and inversely as the diameter, d, squared. If R is 3.64 ohms when $l = 5.0$ m and $d = 1.25$ mm , what is the resistance of a wire made of the same material that is 20.0 m long and has a diameter of 0.250 cm ?

Chapter 3 Sample Test

______________/100

Each Problem is worth 5 points.

1. Express $\frac{5}{16}$ in percent notation.

2. Express 48% as a fraction in lowest terms.

3. Express 58.6% as a decimal number.

4. 19 is what percent of 50?

5. 140 is what percent of 80?

6. What is 34% of 436?

7. The sales tax rate in a city is 5.5%. Find the check out amount for an item with a list price of $250.

8. 58% of employees at a company opt for a managed health care plan. If the company employees 450 people, how many **did not** enroll in the managed health care plan?

9. An item which normally sells for $189 is discounted 27%. What is the new sales price?

10. On a fixed rate 30 year house mortgage you borrow $160,000 at an annual percentage rate of 3.5%. What is your monthly payment? Round your answer to the nearest dollar.

11. You can afford 20% of your monthly income of $2870 on payments for an SUV. If the quoted annual percentage rate of the loan is 3.0% over five years, what is the most you can borrow? Round your answer to the nearest ten dollars.

12. A rectangle is three times as long as it is wide. Its perimeter is 60 in. What are the rectangle's dimensions? **Note:** The perimeter is the sum of all **four sides** of a rectangle.

13. It takes you three hours to drive to a city. If you had driven 15 mph slower, it would have taken you an hour longer. How far did you drive?

14. Susan and Mary are sisters born exactly four years apart, i.e., they share the same birthday. On their birthday in 2014 there combined age was 46 years. In what year was the younger sister born?

15. How many grams of 15% tin solder must be added to 200 g of 25% tin solder to make a final solder which is 20% tin?

16. If y varies directly as x and if $y = 150$ when $x = 8$, then what is y when $x = 20$?

17. If y varies jointly as x and the square of z, and if $y = 400$ when $x = 12$ and $z = 4$, then what is y when $x = 8$ and $z = 6$?

18. The time it takes to complete a task varies indirectly with the number of people performing the task. Jim's lawn service using two mowers can cut a large lawn in 3.0 hours. How long would it take to cut the grass on the same lawn if 5 mowers were used?

19. The pressure of a gas, P, varies directly as the absolute temperature T and inversely as the volume, V. If P is 18.5 psi when T is 300 K and V is 1.5 L , what is P when T is 390 K and V is 2.0 L ?

20. The resistance, R, of a wire varies directly as the length, l, and inversely as the diameter, d, squared. If R is 1.50 ohms when $l = 2.0$ m and $d = 1.5$ mm, what is the resistance of a wire made of the same material that is 8.0 m long and has a diameter of 3.0 mm?

Chapter 3 Sample Test Solutions

1.	**2.**	**3.**	**4.**
$31\frac{1}{4}\% = 31.25\%$	$\frac{12}{25}$	0.586	38%
5.	**6.**	**7.**	**8.**
175%	148.24	\$263.75	189 people
9.	**10.**	**11.**	**12.**
\$137.97	$\$718.47 \approx \$718$	$\$31,944.45 \approx \$31,944$	$7.5 \text{ in } \times 22.5 \text{ in}$
13.	**14.**	**15.**	**16.**
180 miles	1993	200 g of 15% tin	375
17.	**18.**	**19.**	**20.**
600	$\frac{6}{5} \text{ hour} = 1.2 \text{ hour}$	$18.0375 \text{ psi} \approx 18.0 \text{ psi}$	$1.5\ \Omega$ = 1.5 ohms

Chapter 4 Algebra and Graphs of Lines

Section 4.1 Graphing a Linear Equation Using a Table of Values

Big Skill: You should be able to graph linear algebraic equations by creating a table of values and then plotting the points and connecting the dots to form a straight line. You should also be able to read a given graph for approximate values.

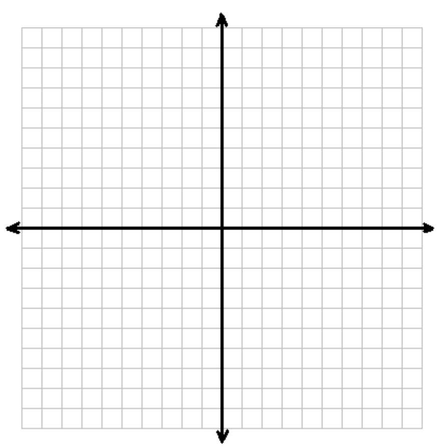
When there is a relationship between two quantities, it is helpful to understand that relationship with a picture. The main way we picture mathematical relationships is with a graph. A graph is a picture formed by dividing the plane into four regions, called quadrants, with a pair of number lines that intersect at right angles. We put a point on the graph to represent the relationship for a single pair of values by moving horizontally along the first number line, and then vertically parallel to the second number line. The collection of all such points for the relationship is called the graph of the relationship. When we graph “A vs. B”, the vertical axis represents the amount of A, and the horizontal axis represents the amount of B. B is called the ***independent variable***, and A is called the ***dependent variable***.

Technique #1 for graphing a line: USING A TABLE OF VALUES

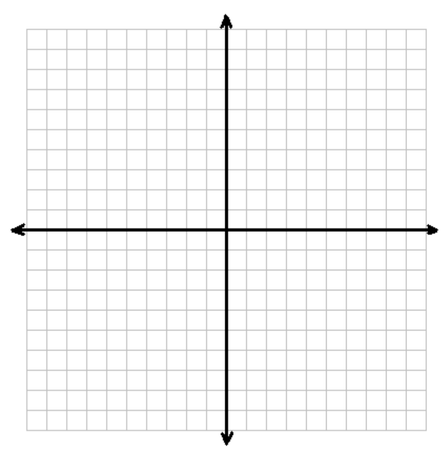
1. Solve the linear equation for the dependent variable (y if y and x are the only two variables in the equation.)
2. Pick some random values of the independent variable (x if y and x are the only two variables in the equation).
3. Calculate the values for the dependent variable using the equation; make a table of the values.
4. Graph the points and connect them with a straight line.

Your Turn!! Graph the following six linear equations.

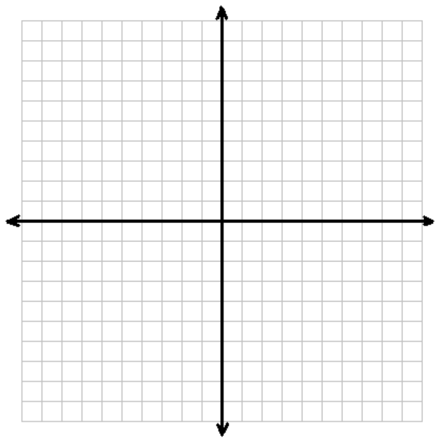
Problem 1 $y = 2x$



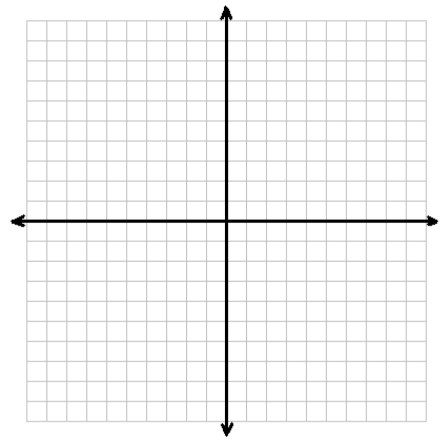
Problem 2 $y = -\frac{1}{2}x$



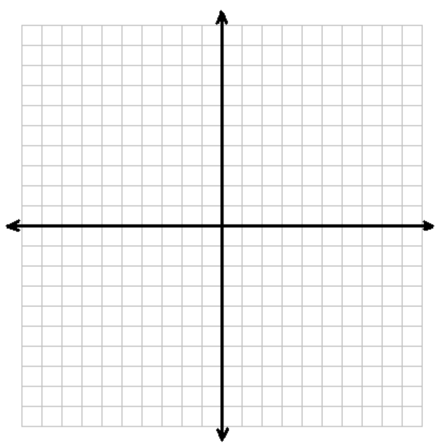
Problem 3 $y = x + 2$



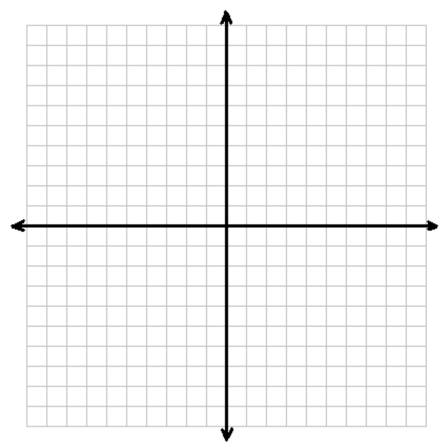
Problem 4 $y = -2x + 4$



Problem 5 $y = -\frac{1}{2}x + 4$



Problem 6 $y = \frac{x-4}{2}$



Section 4.2 Graphing a Linear Equation Using the Slope Intercept Method

Big Idea: A shortcut for graphing lines is to understand that when you solve for the dependent variable, then the constant term is where the line crosses the vertical axis, and the number multiplying the dependent variable is the "slope," which tells you how far up and over to move to get to the next point.

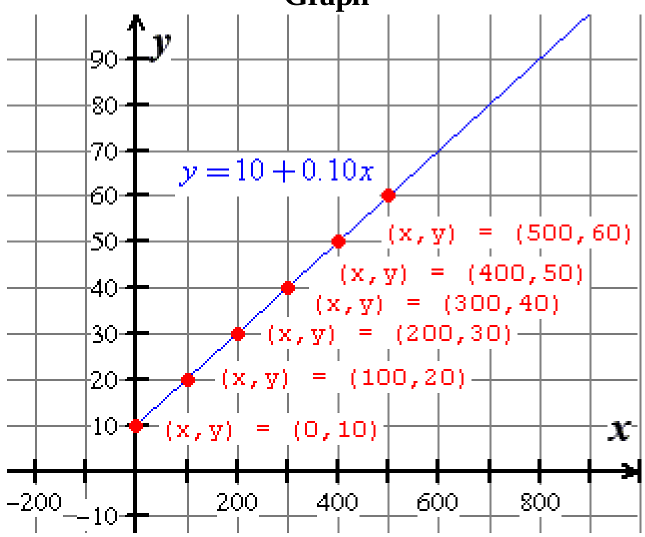
One thing to notice when making a table of values to graph a line is that if you pick your "random" values of the independent variable to change by the same amount every time, then the dependent variable also changes by the same amount every time. Here is an example that illustrates this:

A cell phone package has a flat fee of $10 per month plus a cost 0f 10 cents a minute for every minute you talk. Write an equation for the cost of this package and then graph the relationship. Let *y* be the dependent or output variable which in this problem is the cost in dollars. Let *x* be the independent or input variable which in this problem is the number of minutes. Then $y = 10 + 0.10x$. This can be used to generate the following table of values and graph shown below.

Table of Values

Time, *x*, in minutes	**Cost, *y*, in dollars:** $y = 10 + 0.10x$
0	10
100	20
200	30
300	40
400	50
500	60

Graph



This constant rate of change for both variables is captured in a ratio called the "slope" of the line. The slope of a line is how much the dependent or vertical variable changes divided by how much the independent or horizontal variable changes. For the example above:

$$\text{slope} = \frac{\text{change in cost}}{\text{change in minutes}} = \frac{\$10}{100 \text{ minutes}} = \frac{\$0.10}{1 \text{ minute}}$$

Notice that the slope is just the rate from the original statement of the problem, and also that it is the number multiplying x, the independent variable. Simply put, the slope converts a change in minutes to a change in dollars in this example.

A second thing to notice is that when $x = 0$ minutes, the cost was $y = \$10$, and that \$10 was the constant term in the equation we got when we solved for the dependent variable:

$$y = \frac{\$.10}{1 \text{ minute}} x + \boxed{\$10}$$

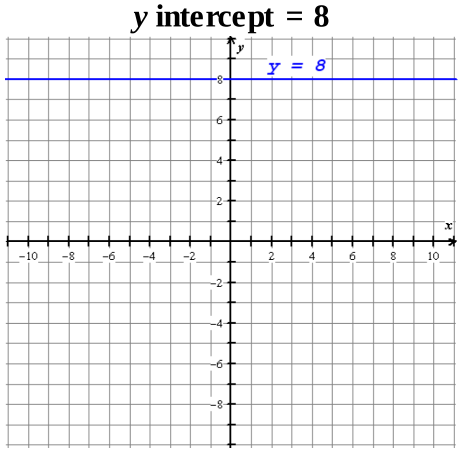
The point on the graph (0 minutes, \$10) is called the "vertical intercept," or more commonly the "*y* intercept," because most math books are so boring that they only ask you to graph *x* and *y* as the independent and dependent variables, respectively.

<u>Technique #2 for graphing a line: SLOPE INTERCEPT METHOD</u>

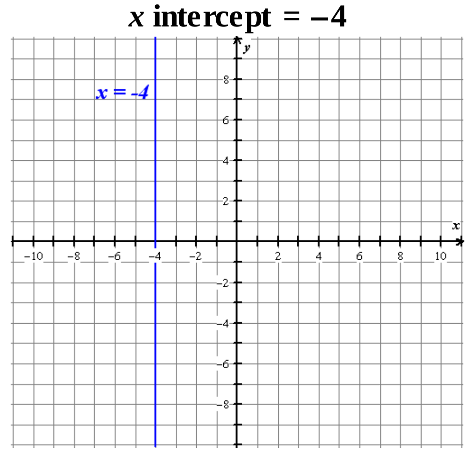
- Solve the linear equation for the dependent variable (i.e., solve for *y* if *x* and *y* are the only two variables in the equation)
- The constant term is the vertical intercept, or the *y*-intercept. Plot it.
- The factor of the *x* term is the slope. Use it to count up and over (or down and over if the slope is negative) to plot more points on the line.

What kind of a line is intended if only **one** variable is present? For example $9y = 72$ or $4x = -16$. The first equation is equivalent to $y = 8$, with the value of *x* not specified. In the plane this describes a horizontal line that contains points $(x, 8)$ for every possible value of *x*. The slope of this line is zero since the rate of change of *y* due to changes in *x* is zero; *y* doesn't change. **Note** the equation can also be written as $y = 0x + 8$. The second equation, $4x = -16$, is equivalent to $x = -4$, with the value of *y* not specified. In the plane this describes a vertical line that contains points $(-4, y)$ for every possible value of *y*. The slope of this line is not defined since there is never a change in *x* to relate to a change in *y*.

The line $y = 8 = 0x + 8$; slope = 0; y intercept = 8



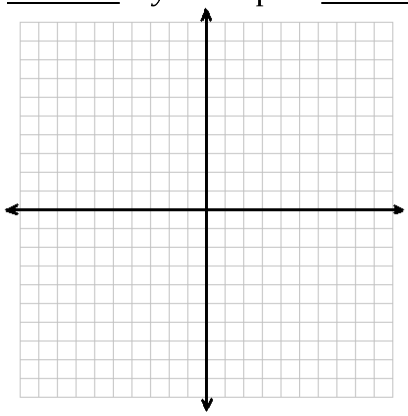
The line $x = -4$; slope not defined; x intercept = -4



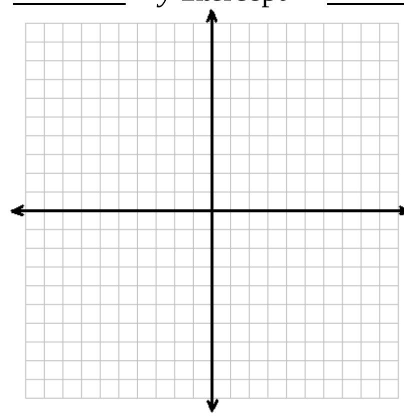
Your Turn!!

Use the slope and intercept to graph the following:

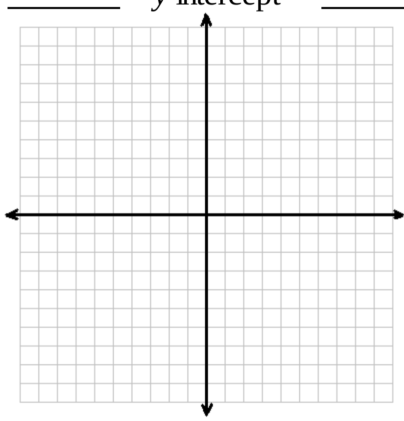
Problem 1 Graph $-6x + 5y = -10$
Slope = ________ y intercept = ________



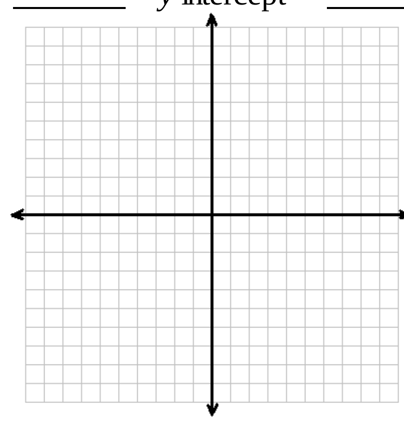
Problem 2 Graph $5x - 8y = 64$
Slope = ________ y intercept = ________



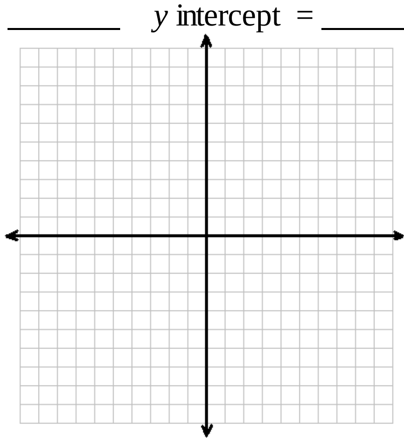
Problem 3 Graph $10x - 5y = 30$
Slope = ________ y intercept = ________



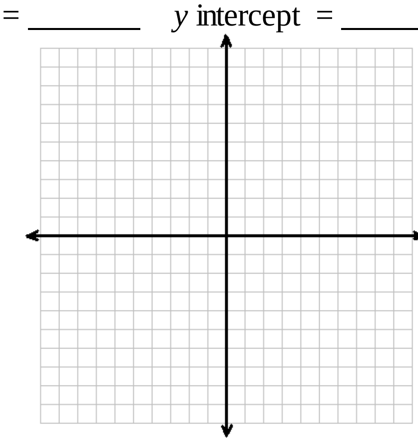
Problem 4 Graph $8x - 6y = -54$
Slope = ________ y intercept = ________



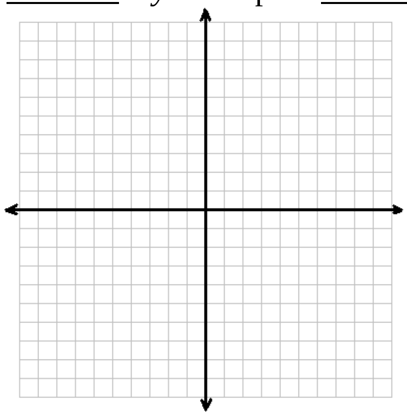
Problem 5 Graph $-3x + 7y = -56$
Slope = ________ y intercept = ________



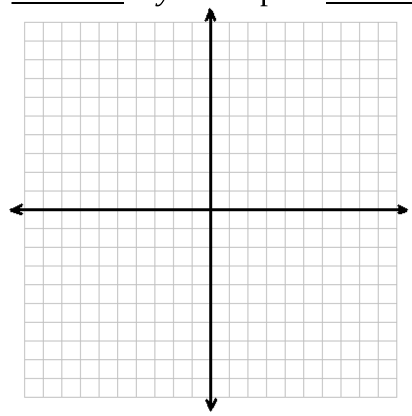
Problem 6 Graph $5x - 7y = -56$
Slope = ________ y intercept = ________



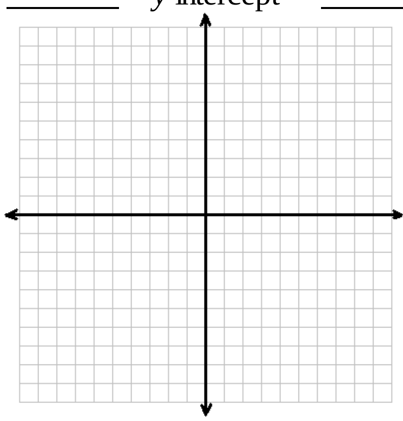
Problem 7 Graph $-x + 9y = -81$
Slope = ________ y intercept = ________



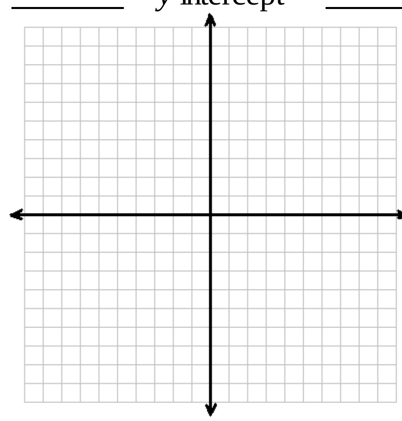
Problem 8 Graph $6x + 4y = 28$
Slope = ________ y intercept = ________



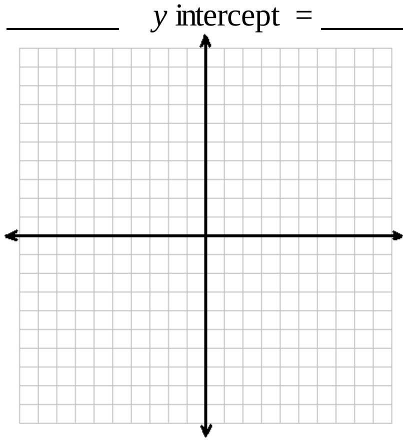
Problem 9 Graph $-3y = 15$
Slope = ________ y intercept = ________



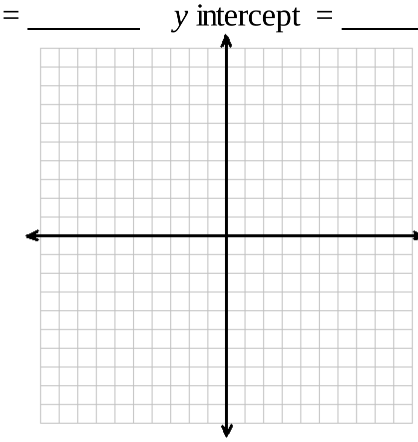
Problem 10 Graph $-2x + 2y = 16$
Slope = ________ y intercept = ________



Problem 11 Graph $-8x + 5y = -45$
Slope = ________ y intercept = ________



Problem 12 Graph $-10x = -30$
Slope = ________ y intercept = ________



Section 4.3 Graphing a Linear Equation Using Intercepts

Technique #3 for graphing a line: INTERCEPT METHOD

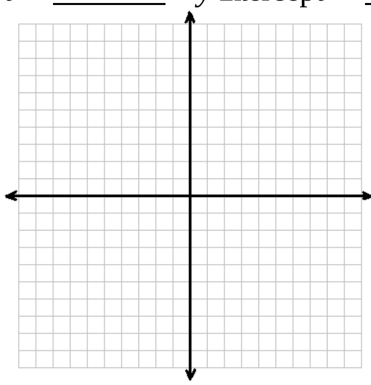
- Set the independent variable to zero, then solve the linear equation for the dependent variable (i.e., set $x = 0$ and solve for y if x and y are the only two variables in the equation).
- Plot the point $(0, y)$. This is called the y-intercept.
- Set the dependent variable to zero, then solve the linear equation for the independent variable (i.e., set $y = 0$ and solve for x if x and y are the only two variables in the equation).
- Plot the point $(x, 0)$. This is called the x-intercept.
- Draw the line between the two intercepts.

Your Turn!!

Use the intercept method to graph the following:

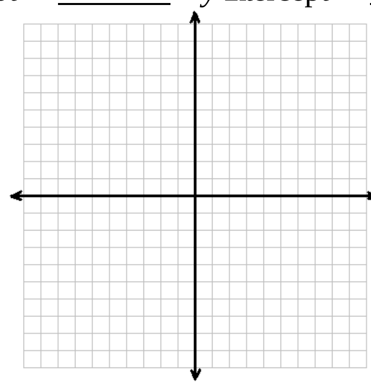
Problem 1 Graph $-x - 8y = -72$

x intercept = _______ y intercept = _______



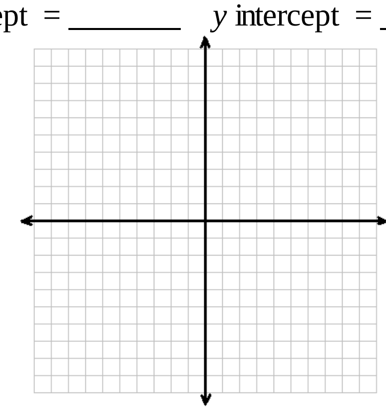
Problem 2 Graph $2x + 3y = -15$

x intercept = _______ y intercept = _______



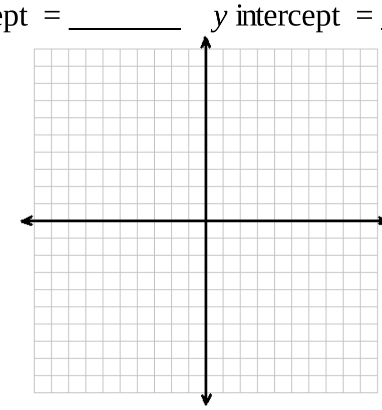
Problem 3 Graph $7x - y = 4$

x intercept = _______ y intercept = _______



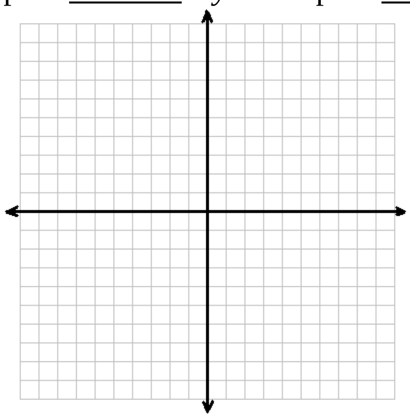
Problem 4 Graph $6x - 10y = -50$

x intercept = _______ y intercept = _______



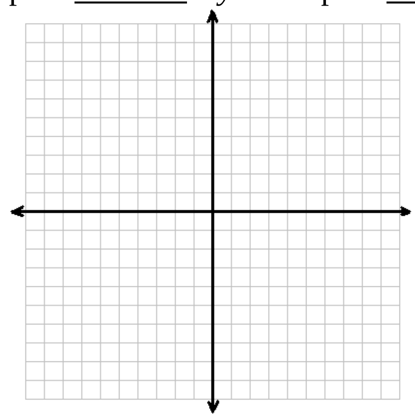
Problem 5 Graph $6x - 2y = 12$

x intercept = ________ y intercept = ________



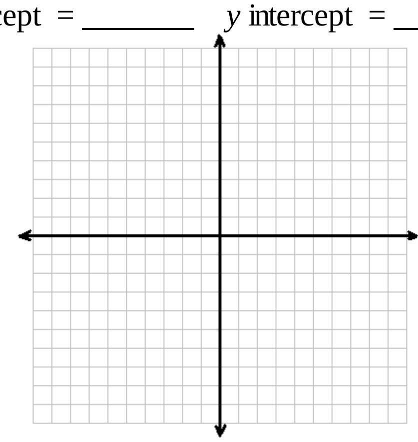
Problem 6 Graph $2y = -8$

x intercept = ________ y intercept = ________



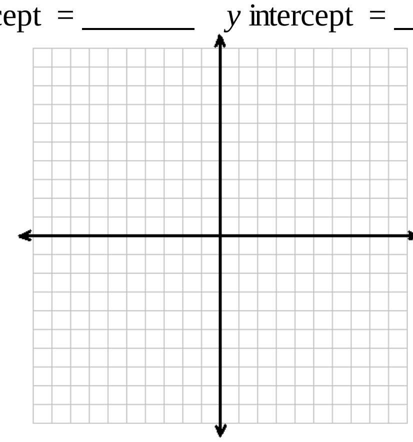
Problem 7 Graph $-5x + 5y = -40$

x intercept = ________ y intercept = ________



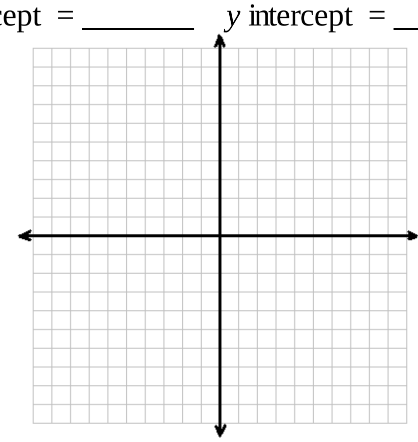
Problem 8 Graph $9x + 3y = 15$

x intercept = ________ y intercept = ________



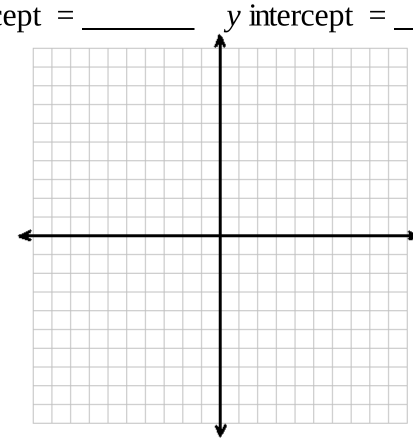
Problem 9 Graph $-5x + 6y = -30$

x intercept = ________ y intercept = ________



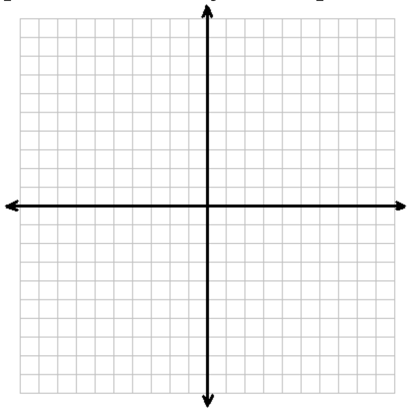
Problem 10 Graph $-10x + 5y = 40$

x intercept = ________ y intercept = ________



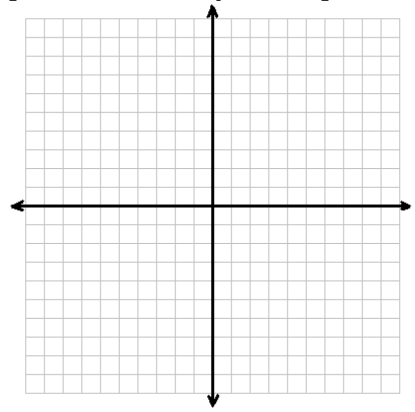
Problem 11 Graph $-x + 4y = 0$

x intercept = ________ y intercept = ________



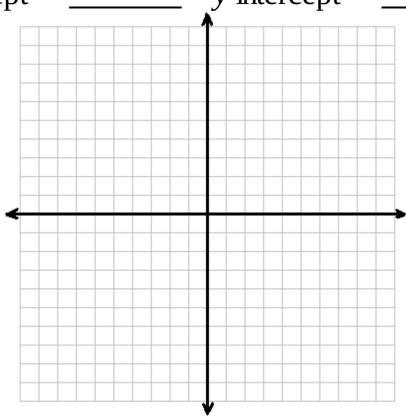
Problem 12 Graph $x + 3y = 24$

x intercept = ________ y intercept = ________



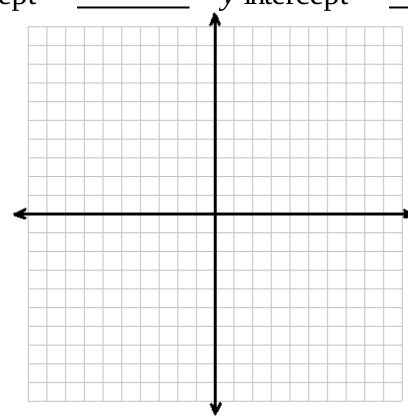
Problem 13 Graph $6x = 30$

x intercept = ________ y intercept = ________



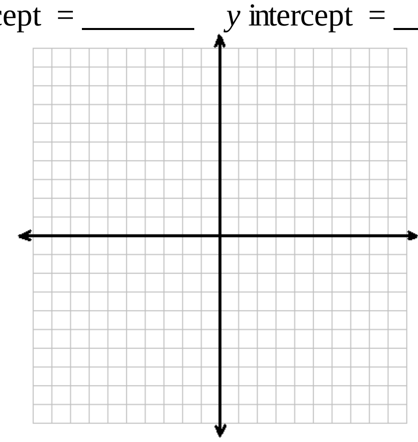
Problem 14 Graph $-3x + 4y = -36$

x intercept = ________ y intercept = ________



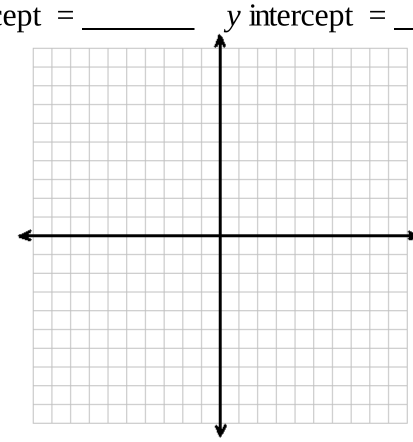
Problem 15 Graph $-2x - 4y = 20$

x intercept = ________ y intercept = ________



Problem 16 Graph $6x - 7y = -42$

x intercept = ________ y intercept = ________



Section 4.4 Graphing a Linear Inequality

Big Skill: You should be able to graph inequalities by drawing either a solid or dotted line, and then shading in one side of the line or the other.

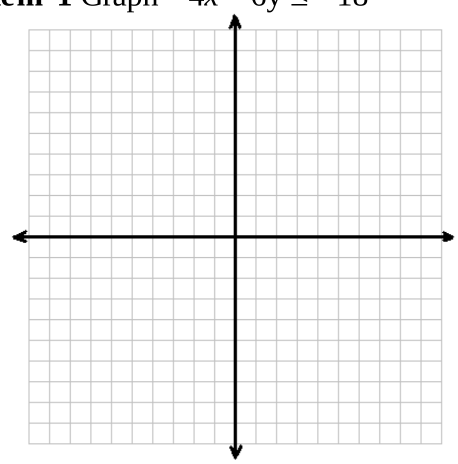
To graph a linear inequality:

1. Graph the boundary line by solving for the dependent variable.
   a. If the inequality uses ≥ or ≤, then draw a solid line to show that the line itself satisfies the inequality.
   b. If the inequality uses just < or >, then draw a dashed line to show that the line does not satisfies the inequality.
2. Shade the appropriate side.
   a. If the inequality is < or ≤, shade the graph below the line.
   b. If the inequality is > or ≥, shade the graph above the line.

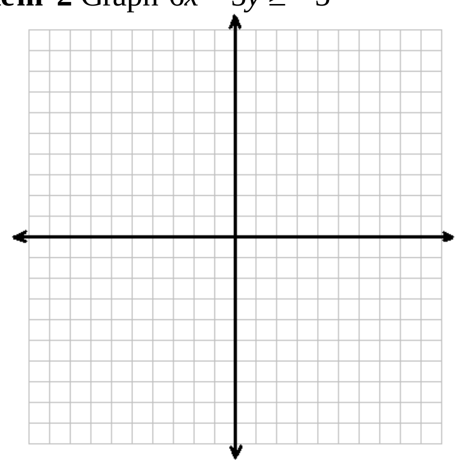
Your Turn!!

Graph the following inequalities.

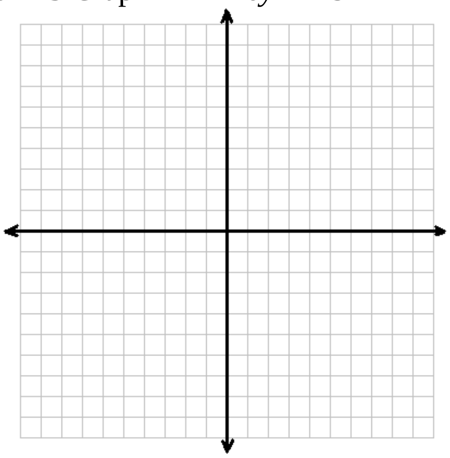
Problem 1 Graph $-4x - 6y \le -18$



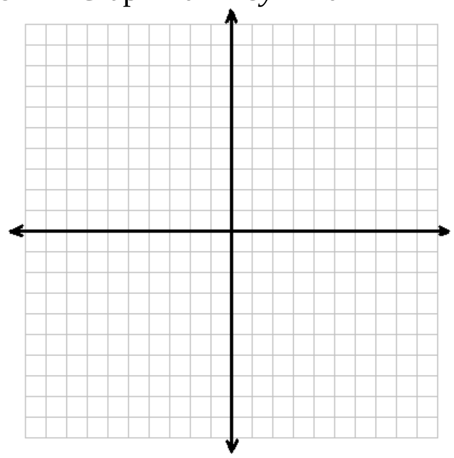
Problem 2 Graph $6x - 5y \ge -5$



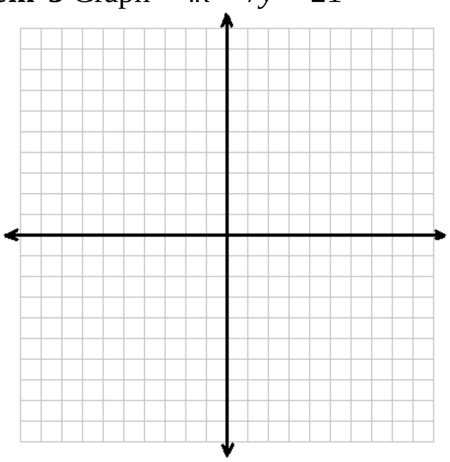
Problem 3 Graph $-4x + 6y < -54$



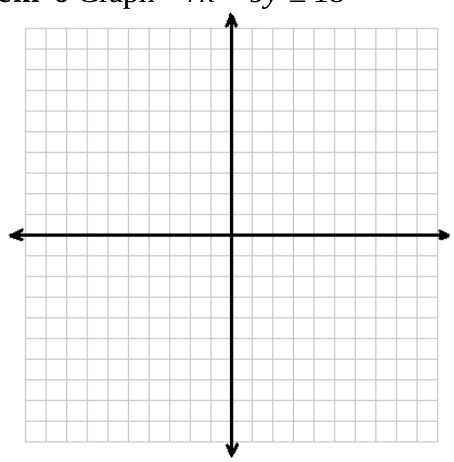
Problem 4 Graph $10x + 8y < 40$



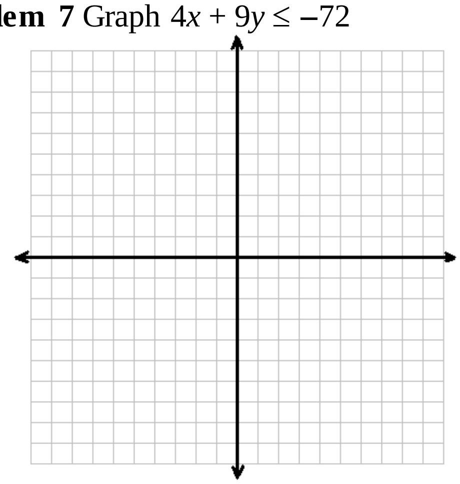
Problem 5 Graph $-4x + 7y < 21$



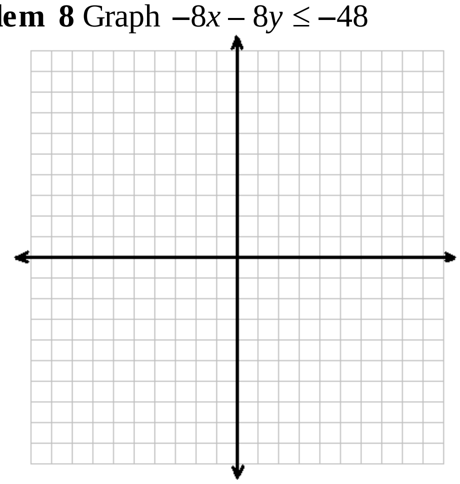
Problem 6 Graph $-7x + 9y \leq 18$



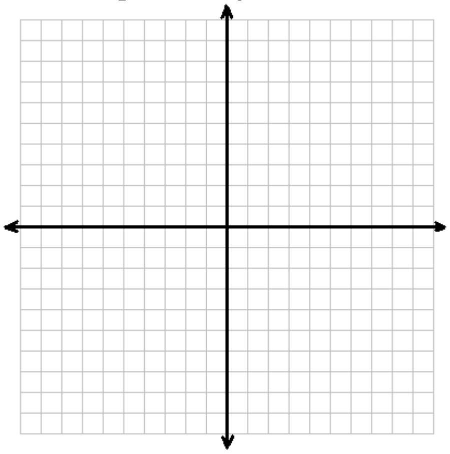
Problem 7 Graph $4x + 9y \leq -72$



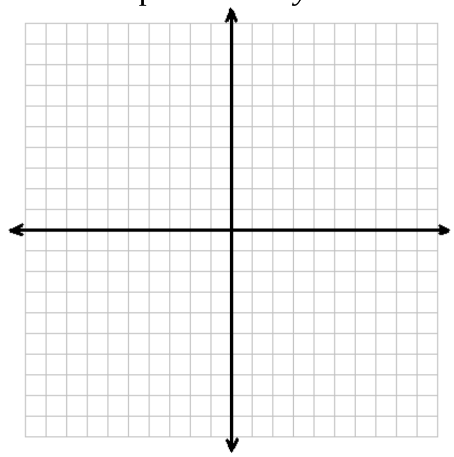
Problem 8 Graph $-8x - 8y \leq -48$



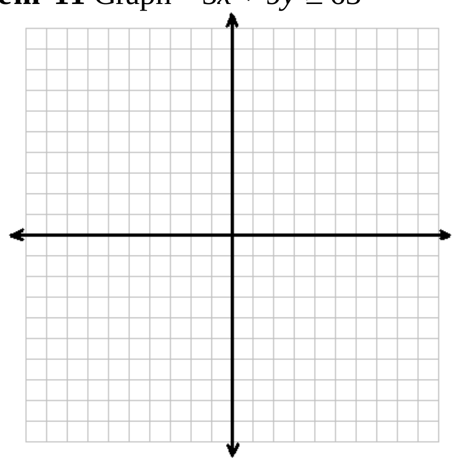
Problem 9 Graph $5x + 3y \le 27$



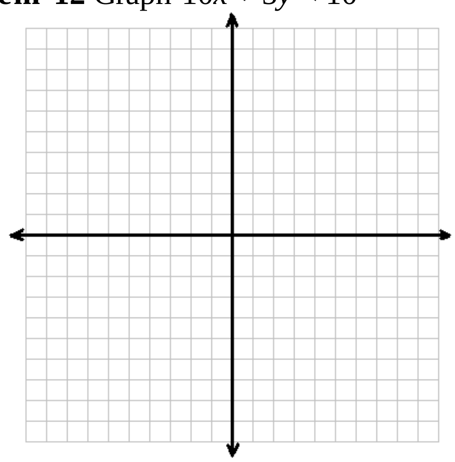
Problem 10 Graph $-10x - 4y > 32$



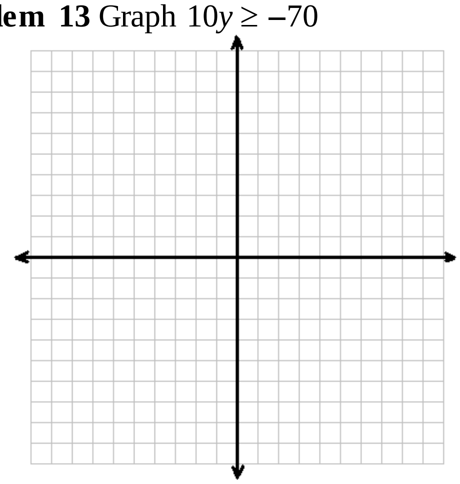
Problem 11 Graph $-5x + 9y \ge 63$



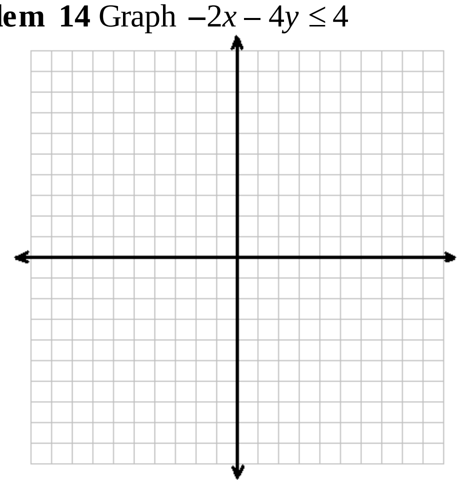
Problem 12 Graph $10x + 5y < 10$



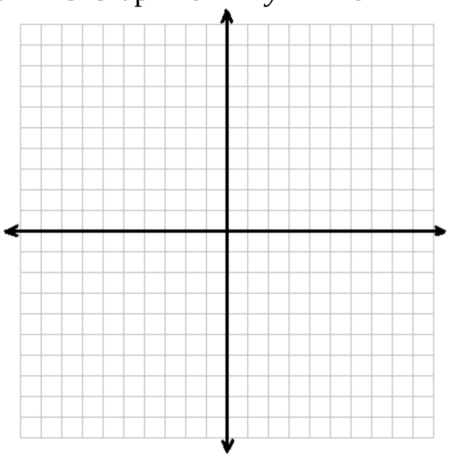
Problem 13 Graph $10y \ge -70$



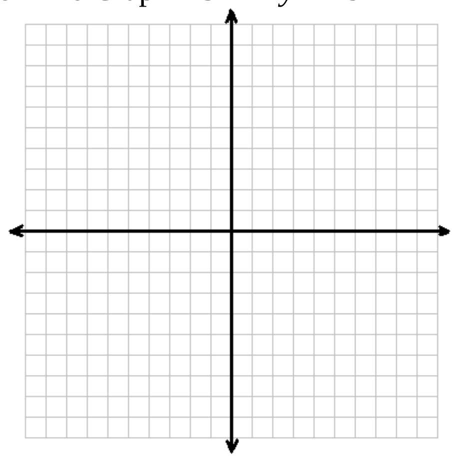
Problem 14 Graph $-2x - 4y \le 4$



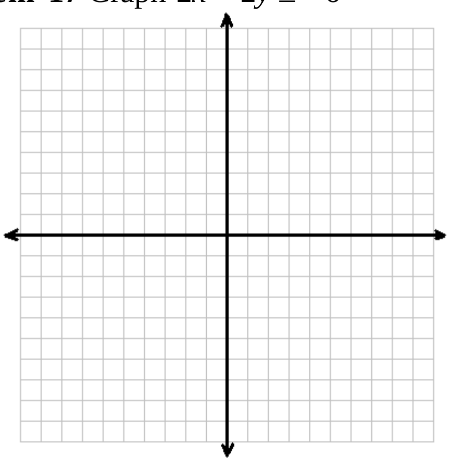
Problem 15 Graph $-8x + 7y < -28$



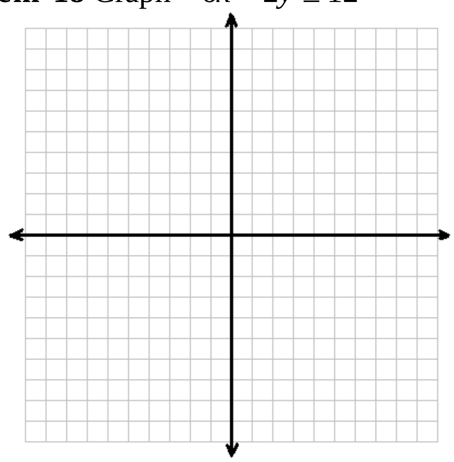
Problem 16 Graph $-5x + 7y < 49$



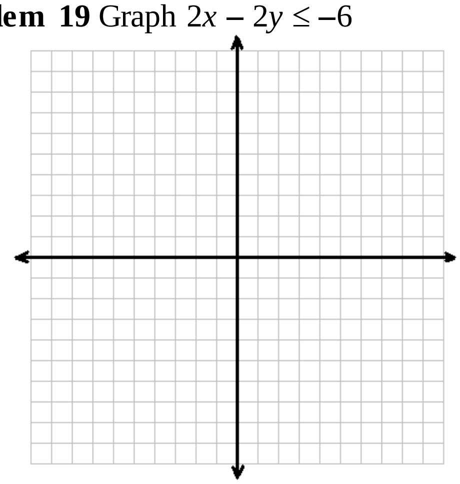
Problem 17 Graph $2x + 2y \le -6$



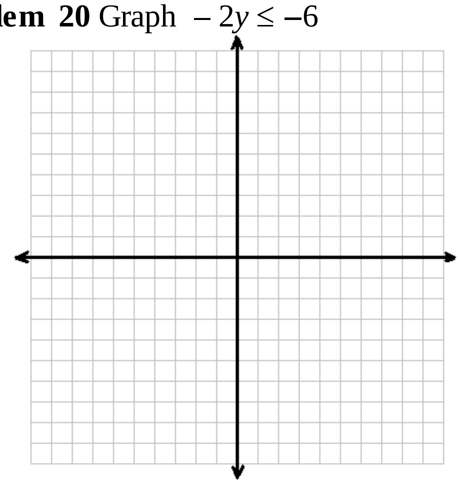
Problem 18 Graph $-6x - 2y \le 12$



Problem 19 Graph $2x - 2y \le -6$



Problem 20 Graph $-2y \le -6$

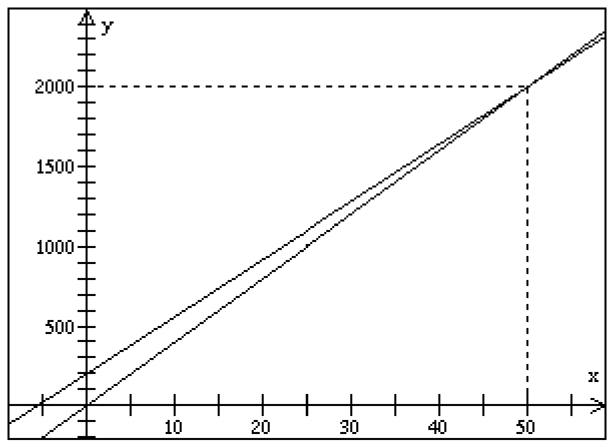


Section 4.5 Solving a System of Two Linear Equations by Graphing

Big Skill: You should be able to solve a system of two linear equations in two unknowns by graphing the lines and reading the intersection point off the graph.

Example of the type of problem the skills in this chapter help us solve:
Suppose that one country club has a $200 membership fee, and the golf costs $36 per round. A second country club has no membership fee and the gold costs $40 per round. How many games would you have to play at each club so that the cost was the same?
Let n = the number of games you play
Let c = the cost of playing that number of games
For club #1: $c = 36n + 200$
For club #2: $c = 40n$

➔ Graph the lines $\begin{cases} y = 36x + 200 \\ y = 40x \end{cases}$ and see where they cross…



From the graph, it looks like if we play 50 games at either club, the cost will be $2,000.00.

A **system of linear equations** is a grouping of two or more linear equations, each of which contains the same variables.

Examples: $\begin{cases} y = 36x + 200 \\ y = 40x \end{cases}$ $\begin{cases} 2x + 3y = 4 \\ 3x - y = -5 \end{cases}$

In Chapter 3 when we had just a single linear equation (in one variable) our goal was to find the **single** number that made the equation be true. That number was the solution of the equation.
Example: The equation $2x – 5 = 15$ has a solution of $x = 5$, because $2(5) – 5 = 10 – 5 = 15$

Now for this chapter, our goal is to find the **solution to a system of linear equations**, which consists of values for both of the variables x and y that make both equations in the system be true.

Example:

The system of equations $\begin{cases} y = 36x + 200 \\ y = 40x \end{cases}$ has the solution $(x, y) = (50,\ 2000)$ because:

2000 = 36(50) + 200	**AND**	2000 = 40(50)
2000 = 1800 + 200		2000 = 2000
2000 = 2000		

Solving a System of Two Linear Equations by Graphing

- Graph both the lines.
- Read the coordinates of the intersection point off the graph.
- Check to see if those coordinates are the solution.

We see from these examples that there are three different cases for the solution to a system of two equations in two variables. We describe these cases using the words:

- **Consistent**, which means that there is at least one solution (no solutions ➔ **inconsistent**)
- **Dependent**, which means that the graphs of the lines are the same (different lines ➔ **independent**)

Three Possible Cases for Solutions of a System of Two Linear Equations in Two Variables:

Intersecting Lines: The lines intersect at one point, and thus the system has exactly one solution. This type of system is called **consistent** and the equations are called **independent**.

Parallel lines: The lines never intersect (i.e., they are parallel to one another), and thus the system has no solutions. This type of system is called **inconsistent** and the equations are called **independent**.

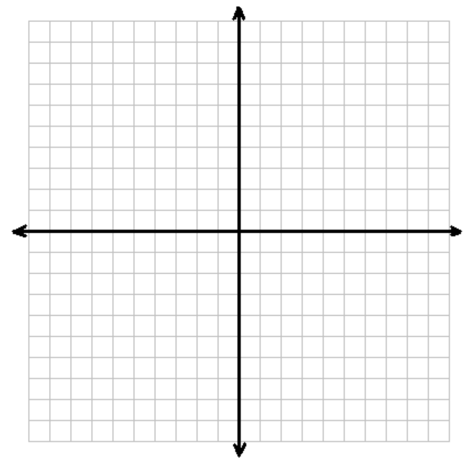
Coincidental Lines: The lines lie on top of each other, and thus the system has infinitely many solutions. This type of system is called **consistent** and the equations are called **dependent**.

We can identify which of these three cases a system of equations will fall into (without graphing) by putting both equations into slope-intercept form, and comparing the slopes and *y*-intercepts.

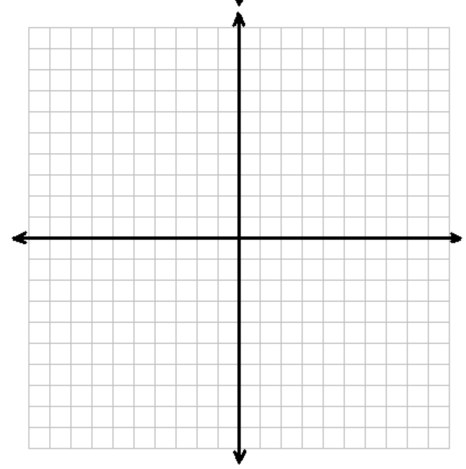
Your Turn!!

Solve the following systems of equations by graphing. Verify your solutions by plugging them back into the original system.

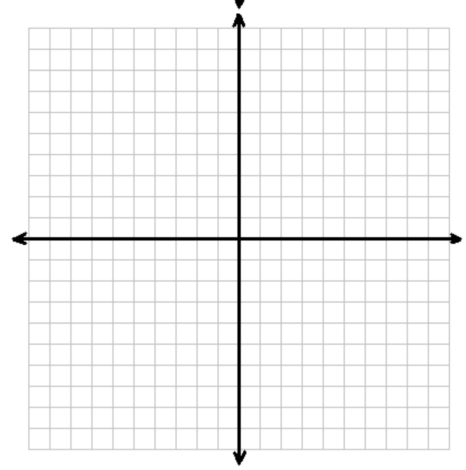
1) $$\begin{aligned} -8x + 4y &= 24 \\ 4x - 2y &= -12 \end{aligned}$$



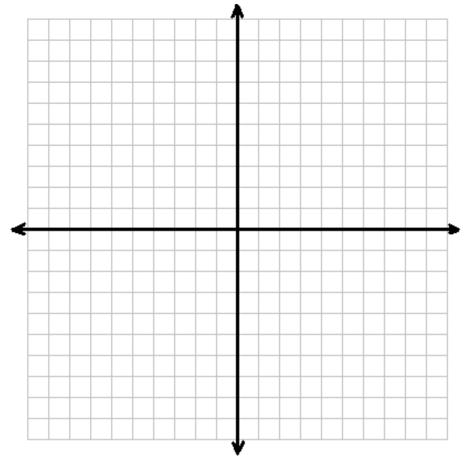
2) $$\begin{aligned} 2x - 7y &= 82 \\ -x - y &= 4 \end{aligned}$$



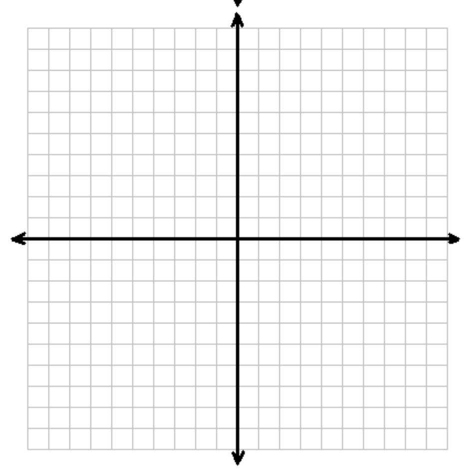
3) $$\begin{aligned} -2x - 2y &= -14 \\ -4x + 4y &= -12 \end{aligned}$$



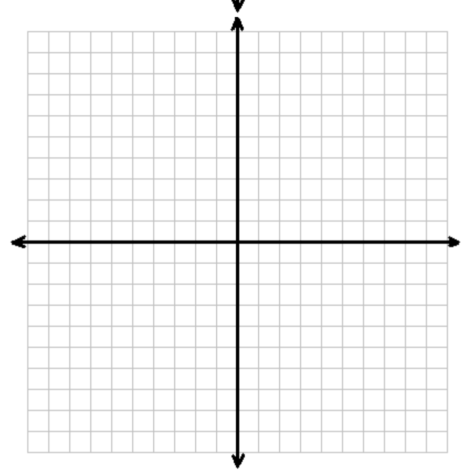
4)
$$\begin{aligned} 3x - 4y &= -12 \\ -3x - 2y &= 30 \end{aligned}$$



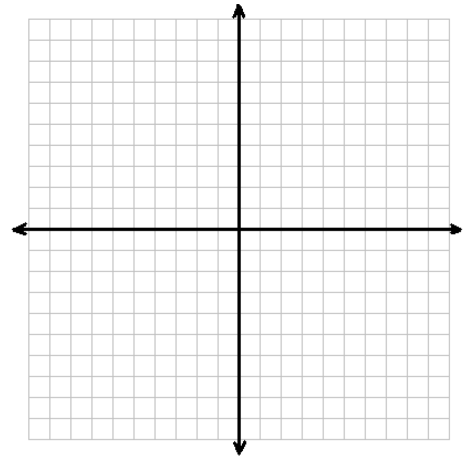
5)
$$\begin{aligned} 6x + 5y &= 58 \\ -7x + 8y &= -40 \end{aligned}$$



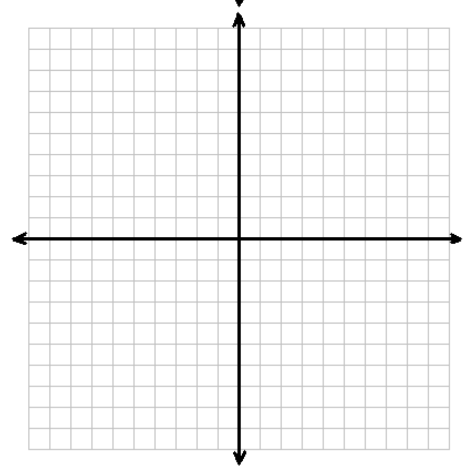
6)
$$\begin{aligned} 6x + 8y &= -40 \\ -2x + 4y &= -40 \end{aligned}$$



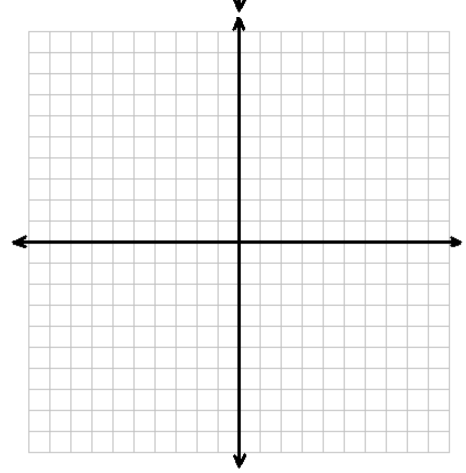
7) $$\begin{aligned} -7x - 4y &= -91 \\ -7x + 3y &= -42 \end{aligned}$$



8) $$\begin{aligned} 4x - y &= 27 \\ -7x + 6y &= -26 \end{aligned}$$



9) $$\begin{aligned} -2x - 3y &= -11 \\ 4x - 4y &= -48 \end{aligned}$$



Section 4.6 Solving a System of Two Linear Equations by Algebraic Methods

Big Skill: You should be able to solve a system of two linear equations in two unknowns by solving one equation for one of the variables, then replacing that variable in the other equation with its expression.

In the previous section, we saw that the lines $\begin{cases} y = 36x + 200 \\ y = 40x \end{cases}$ cross at the point (50, 2000), which means that if you play 50 games at either club, the cost will be \$2,000. Here is how to solve this system using a purely algebraic technique called "substitution".

Since the first equation already tells us that $y = 36x + 200$, we can replace the y in the second equation with
$36x + 200$:

$$\begin{aligned} y &= 40x \\ 36x + 200 &= 40x \\ 36x + 200 - 36x &= 40x - 36x \\ 200 &= 4x \\ \frac{200}{4} &= \frac{4x}{4} \quad \Rightarrow \quad \boxed{50 = x} \end{aligned}$$

Now that we know $x = 50$, we can replace the x in $y = 36x + 200$ to get:

$$\begin{aligned} y &= 36x + 200 \\ &= 36(50) + 200 \\ &= 1800 + 200 \quad \Rightarrow \quad \boxed{y = 2000} \end{aligned}$$

So, we just derived that the solution to this system is (50, 2000).

Solving a System of Two Linear Equations Using the Substitution Method

- Solve one of the equations for one of the variables; pick the easiest variable to solve for.
- Replace that variable in the other equation with the expression you just derived.
- Solve your new equation; it should have only one variable in it.
- Substitute this answer into the first equation to find the value of the other variable.
- Check to see if those coordinates are the solution.

An alternate, often easier, algebraic method is called elimination. The idea is to multiply through each equation by a pair of numbers so that if the two multiplied equations are added, one of the variables "cancels" or is eliminated.

For example, consider the following system:

$$\begin{aligned} 5x - 3y &= 19 \\ 2x + 7y &= -17 \end{aligned}$$

If we multiply the top equation through by -2 and the bottom equation through by 5, then when the two multiplied equations are added, the x variable is eliminated.

$$\begin{aligned} -2(5x-3y) &= -2(19) \\ 5(2x+7y) &= 5(-17) \end{aligned} \Rightarrow \begin{aligned} -10x+6y &= -38 \\ 10x+35y &= -85 \end{aligned}$$

$$\Rightarrow \quad 0x+41y = -124 \Rightarrow y = \frac{-124}{41} = -3$$

Then the value of $y = -3$ is substituted into either of the original equations to determine x.

$$5x - 3(-3) = 19 \Rightarrow 5x+9=19 \Rightarrow 5x = 19-9 = 10 \Rightarrow x = \frac{10}{5} = 2$$

$$\text{or, } 2x+7(-3) = -17 \Rightarrow 2x-21 = -17 \Rightarrow 2x = 21-17 = 4 \Rightarrow x = \frac{4}{2} = 2.$$

This same problem could be solved by first eliminating y by multiplying the top equation through by 7 and the bottom equation through by 3, and then adding the two multiplied equations.

$$\begin{aligned} 7(5x-3y) &= 7(19) \\ 3(2x+7y) &= 3(-17) \end{aligned} \Rightarrow \begin{aligned} 35x-21y &= 133 \\ 6x+21y &= -51 \end{aligned}$$

$$\Rightarrow \quad 41x+0y = 82 \Rightarrow x = \frac{82}{41} = 2$$

Then the value of $x = 2$ is substituted into either of the original equations to determine y.

$$5(2) - 3y = 19 \Rightarrow 10-3y = 19 \Rightarrow -3y = 19-10 = 9 \Rightarrow y = \frac{9}{-3} = -3$$

$$\text{or, } 2(2)+7y = -17 \Rightarrow 4+7y = -17 \Rightarrow 7y = -4-17 = -21 \Rightarrow y = \frac{-21}{7} = -3.$$

Finally, consider the following system:

$$\begin{aligned} 2x-3y &= 11 \\ -8x+12y &= -14 \end{aligned}$$

If we multiply the top equation through by 4 and the bottom equation through by 1 (which leaves it unchanged), then when the two multiplied equations are added, both x and y are eliminated and an impossible or inconsistent equation results. This means that the two lines are parallel and there are no solutions.

$$\begin{aligned} 4(2x-3y) &= 4(11) \\ 1(-8x+12y) &= 1(-14) \end{aligned} \Rightarrow \begin{aligned} 8x-12y &= 44 \\ -8x+12y &= -14 \end{aligned} \Rightarrow \quad 0x+0y = 30 \Rightarrow 0 = 30$$

In contrast, in the system

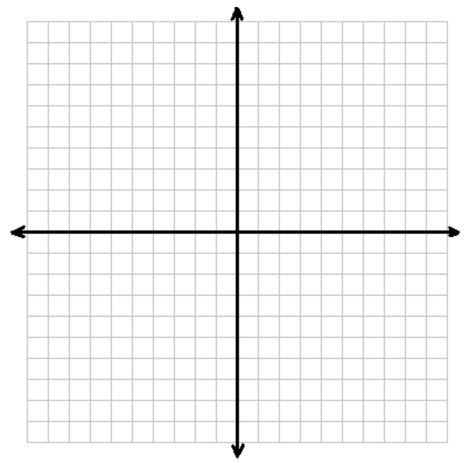
$$\begin{aligned} 2x-3y &= 11 \\ -8x+12y &= -44 \end{aligned}$$

leads to the consistent statement that 0 = 0. This system is consistent, but dependent and has infinitely many solutions or pairs of (x, y) values on the coincidental line.

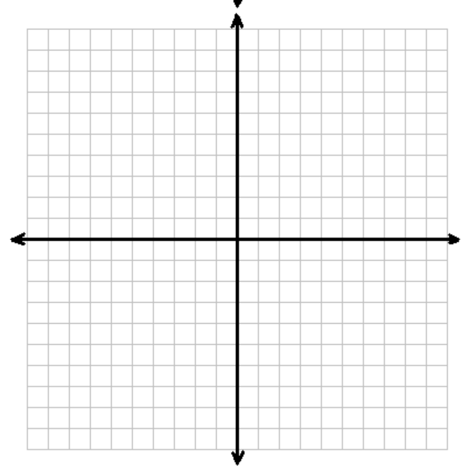
Your Turn!!

Solve the following systems of equations using algebra. Verify your solutions by plugging them back into the original system and by graphing the lines.

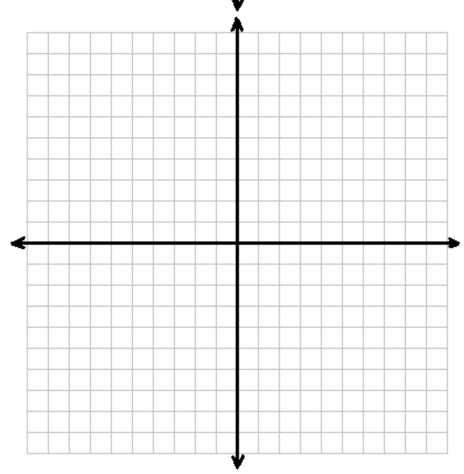
1) $\begin{aligned} 4x - y &= 27 \\ -7x + 6y &= -26 \end{aligned}$



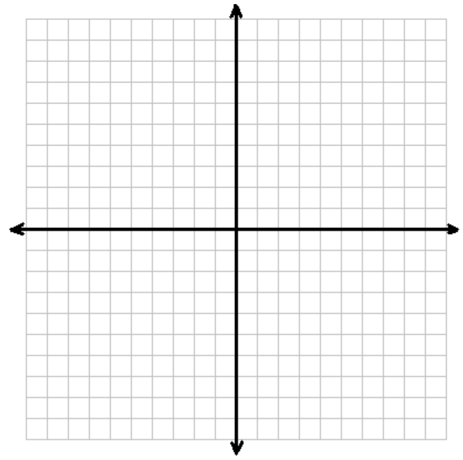
2) $\begin{aligned} -2x - y &= -16 \\ 4x - y &= 14 \end{aligned}$



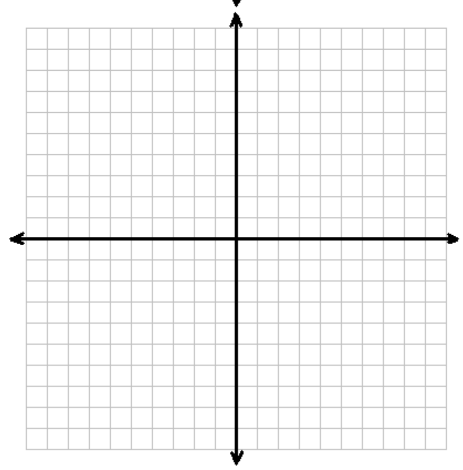
3) $\begin{aligned} -5x - 2y &= 38 \\ -7x - 2y &= 46 \end{aligned}$



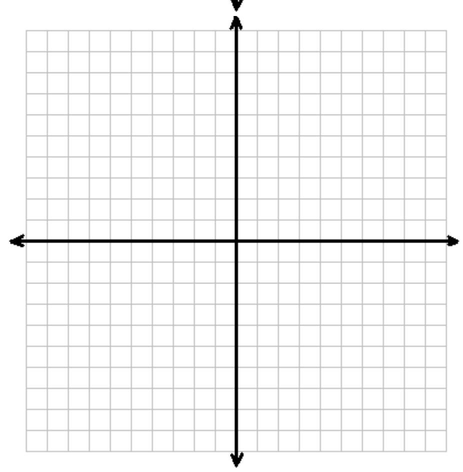
4) $$\begin{aligned} 4x + 6y &= 36 \\ -2x - 3y &= 12 \end{aligned}$$



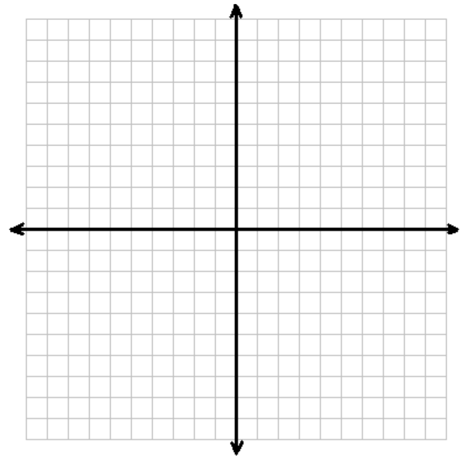
5) $$\begin{aligned} 4x - 8y &= 16 \\ 3x - 6y &= -12 \end{aligned}$$



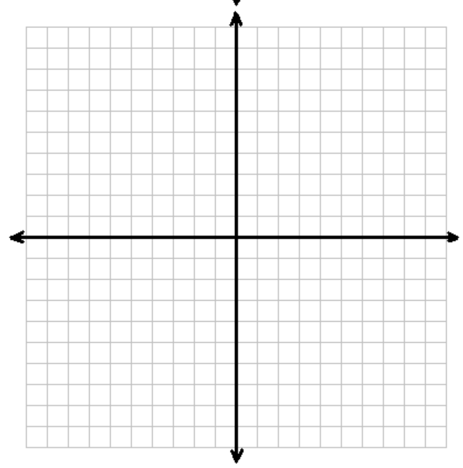
6) $$\begin{aligned} 3x + y &= 1 \\ -2x + 4y &= 18 \end{aligned}$$



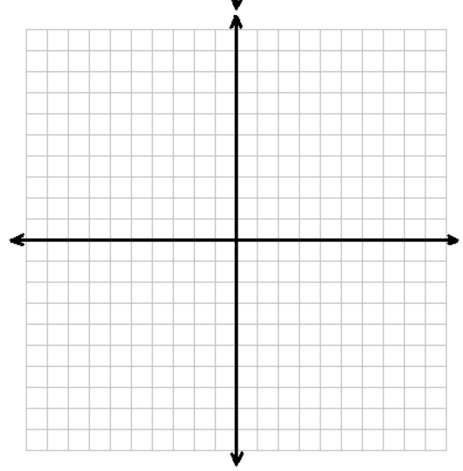
7) $$\begin{aligned} x - y &= 5 \\ 3x - 7y &= 7 \end{aligned}$$



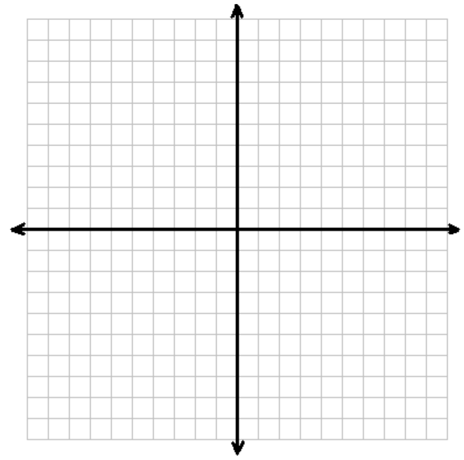
8) $$\begin{aligned} -4x + 8y &= 8 \\ 3x - 6y &= -6 \end{aligned}$$



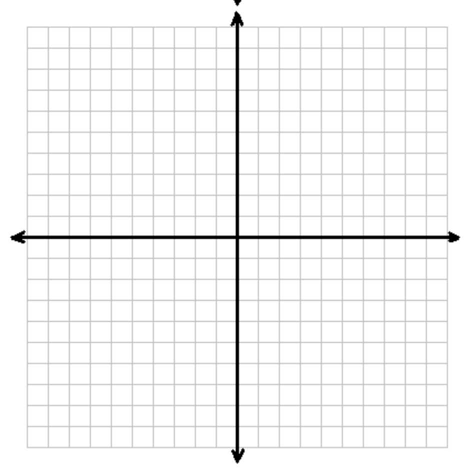
9) $$\begin{aligned} -3x + 5y &= 57 \\ 6x + 3y &= 3 \end{aligned}$$



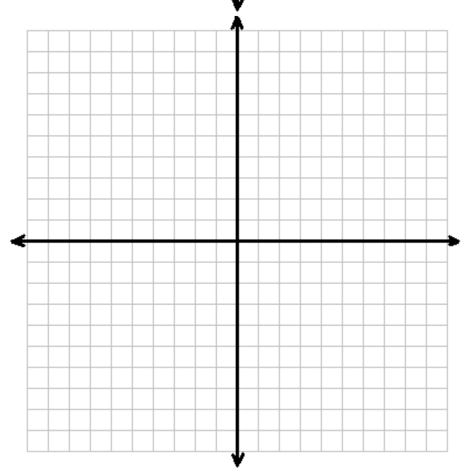
10) $$\begin{aligned} 3x - 3y &= 9 \\ -5x + 5y &= -15 \end{aligned}$$



11) $$\begin{aligned} 5x - 5y &= -35 \\ -4x - 6y &= -52 \end{aligned}$$



12) $$\begin{aligned} -3x + 3y &= 15 \\ -x + 3y &= 17 \end{aligned}$$



Chapter 4 Sample Test

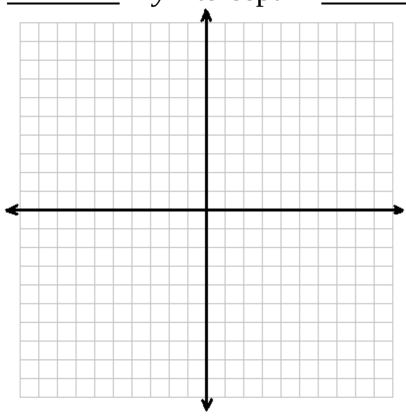
______________/100

Problems 1 through 12 are each 5 points. Problems 13 through 16 are each 10 points.

For each equation below state the requested information and graph the line.

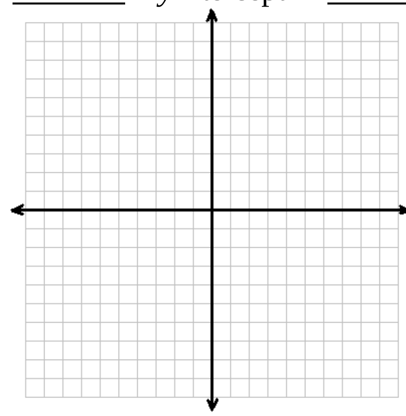
Problem 1 $2x - 3y = 24$

Slope = _______ y intercept = _______



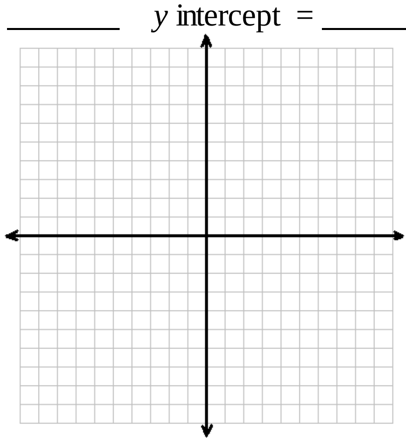
Problem 2 $10x - 4y = 20$

Slope = _______ y intercept = _______



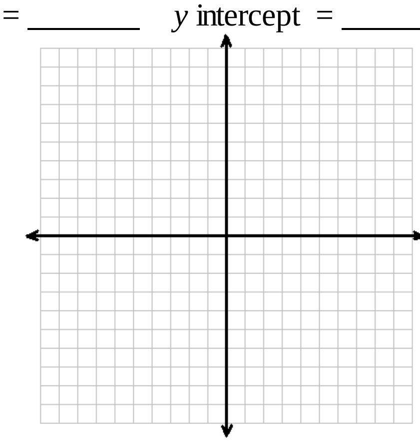
Problem 3 $2x = -12$

Slope = _______ y intercept = _______



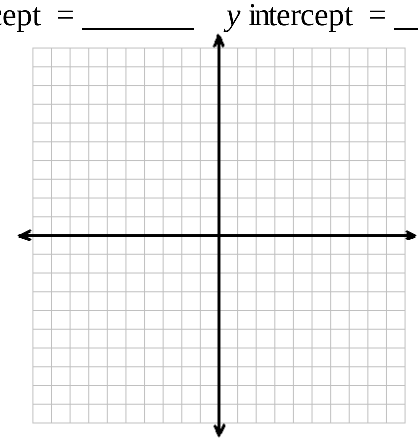
Problem 4 $4y = 16$

Slope = _______ y intercept = _______



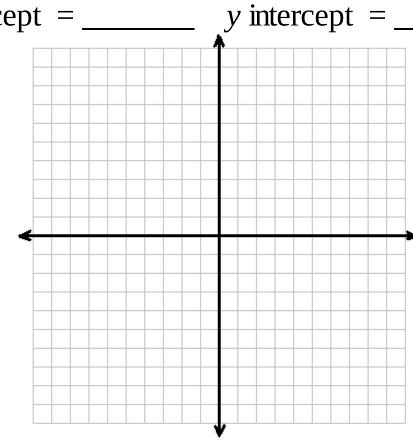
Problem 5 $-6x + 2y = -18$

x intercept = _______ y intercept = _______



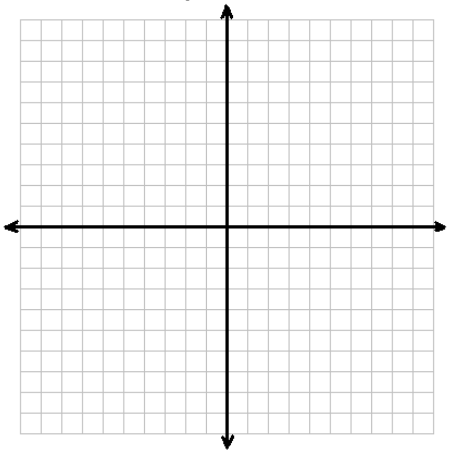
Problem 6 $3x + 6y = 18$

x intercept = _______ y intercept = _______

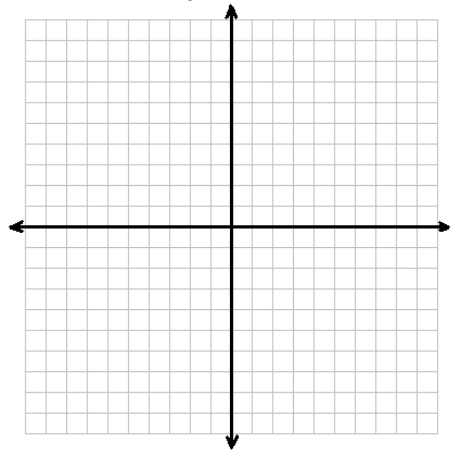


Graph the solution of each of the following inequalities by shading the appropriate region.

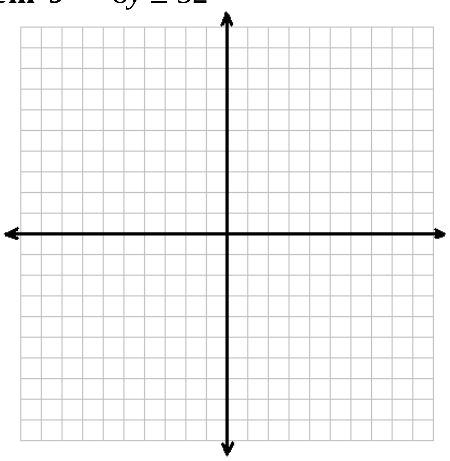
Problem 7 $-2x + 4y \geq 8$



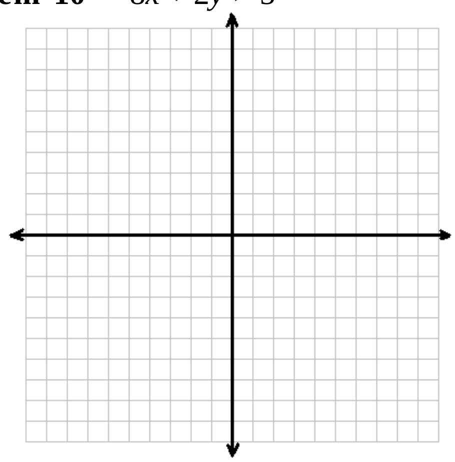
Problem 8 $-6x - 3y \leq -12$



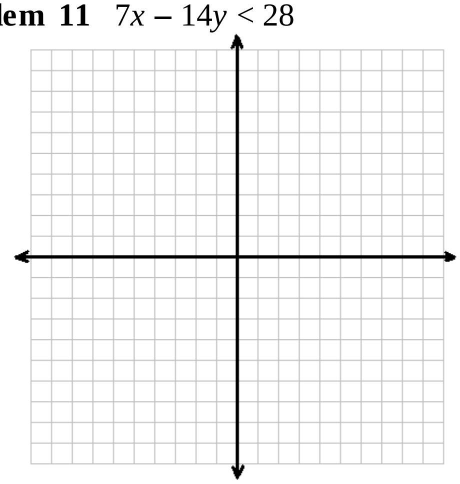
Problem 9 $8y \geq 32$



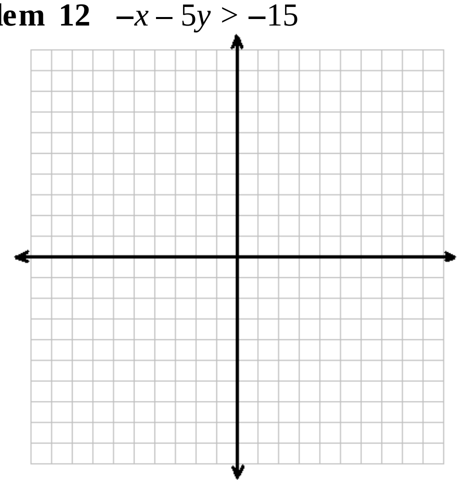
Problem 10 $-8x + 2y > 5$



Problem 11 $7x - 14y < 28$



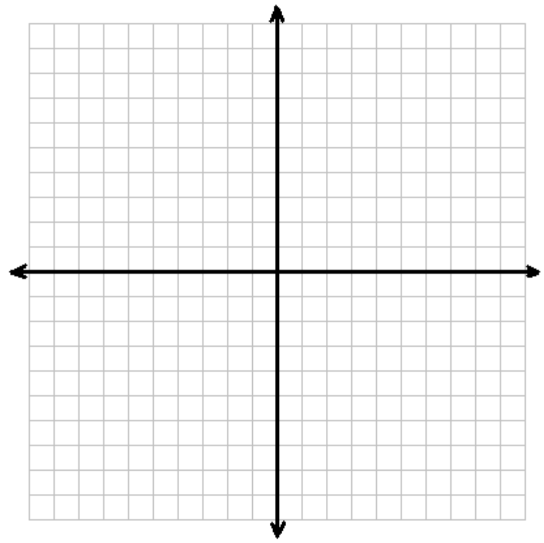
Problem 12 $-x - 5y > -15$



Solve the following systems of equations using algebra. Verify your solutions by plugging them back into the original system and graphing the two lines.

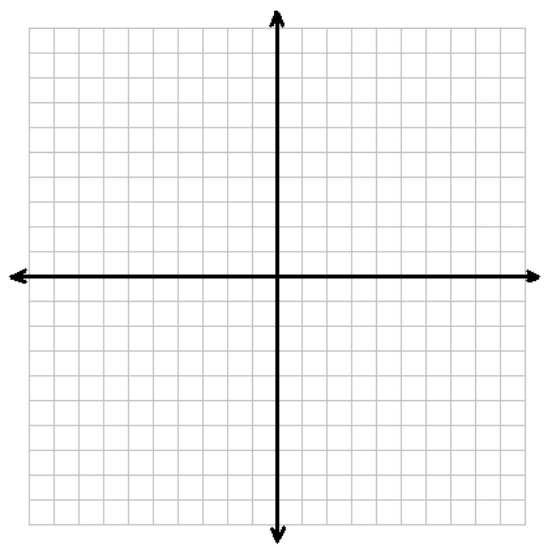
Problem 13

$$\begin{aligned} -2x + 3y &= -1 \\ 6x - 5y &= 7 \end{aligned}$$



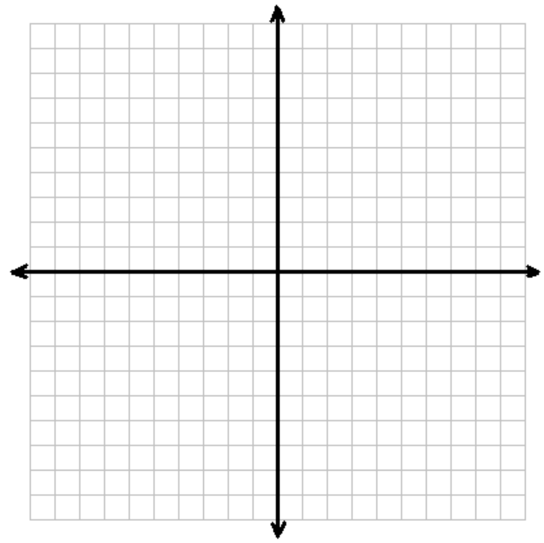
Problem 14

$$\begin{aligned} 5x + 2y &= 1 \\ 2x + 5y &= 13 \end{aligned}$$



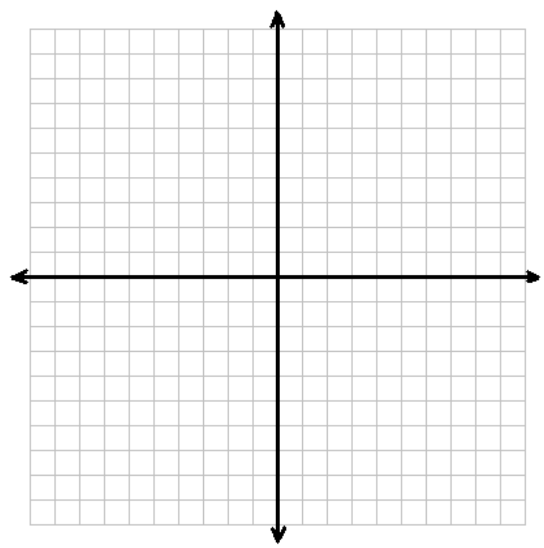
Problem 15

$$\begin{aligned} -4x + 6y &= -8 \\ 6x - 9y &= 7 \end{aligned}$$



Problem 16

$$\begin{aligned} 5x - 4y &= -1 \\ 2x + 3y &= 18 \end{aligned}$$

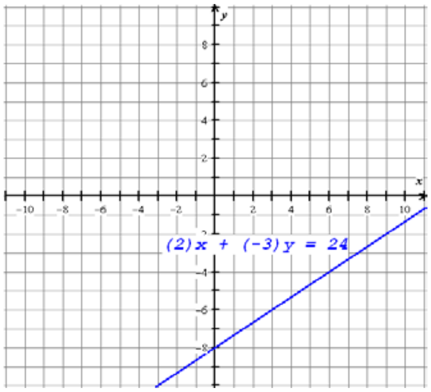


Chapter 4 Sample Test Solutions

Problem 1

$$2x-3y=24 \Rightarrow 3y+24=2x$$
$$\Rightarrow 3y=2x-24 \Rightarrow y=\frac{2}{3}x-8$$

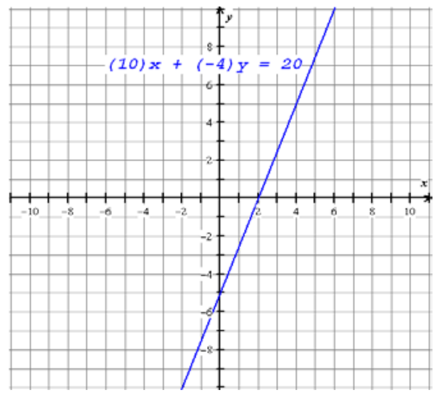
Slope = $\frac{2}{3}=0.666\ldots$ y intercept = -8



Problem 2

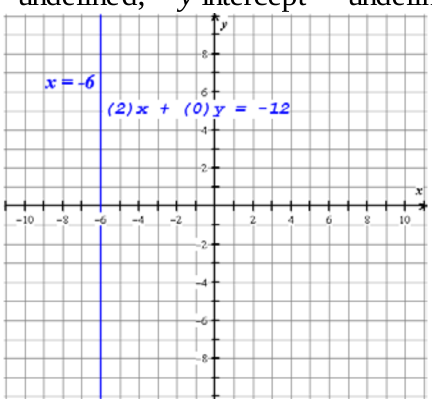
$$10x-4y=20 \Rightarrow 4y+20=10x$$
$$\Rightarrow 4y=10x-20 \Rightarrow y=\frac{5}{2}x-5$$

Slope = $\frac{5}{2}=2,5$ y intercept = -5



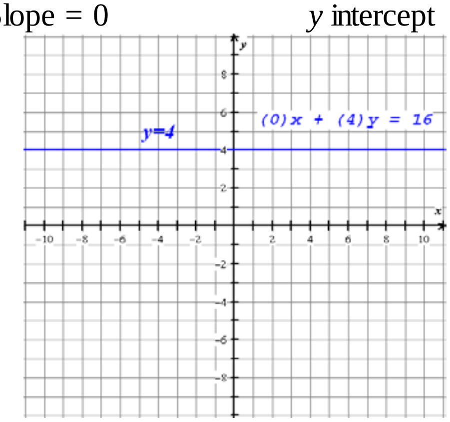
Problem 3 $2x=-12 \Rightarrow x=-6$; vertical line

Slope = undefined, y intercept = undefined



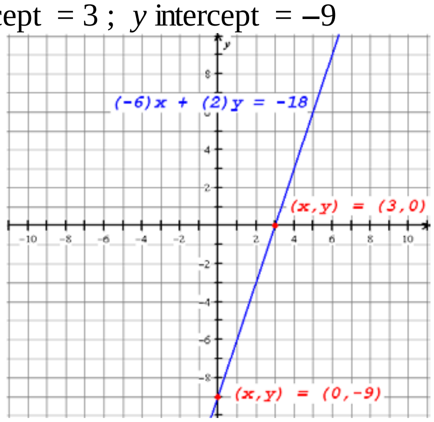
Problem 4 $4y=16 \Rightarrow y=4$; a horizontal line

Slope = 0 y intercept = 4



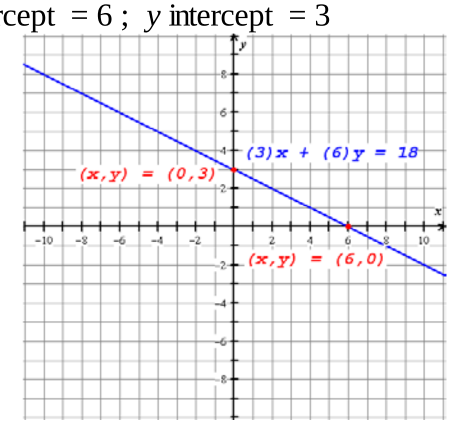
Problem 5 $-6x+2y=-18 \Rightarrow y=3x-9$

x intercept = 3 ; y intercept = -9



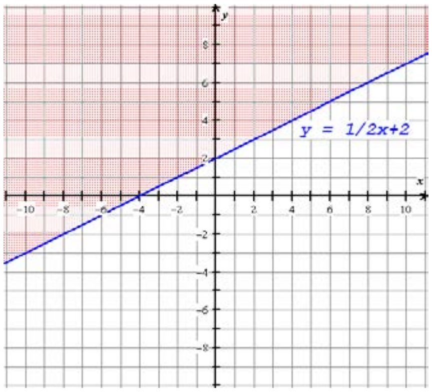
Problem 6 $3x+6y=18 \Rightarrow y=3-x/2$

x intercept = 6 ; y intercept = 3

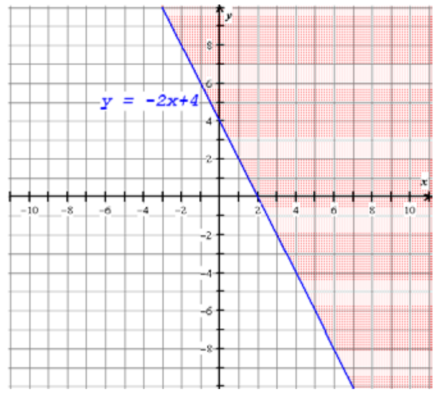


Graph the solution of each of the following inequalities by shading the appropriate region.

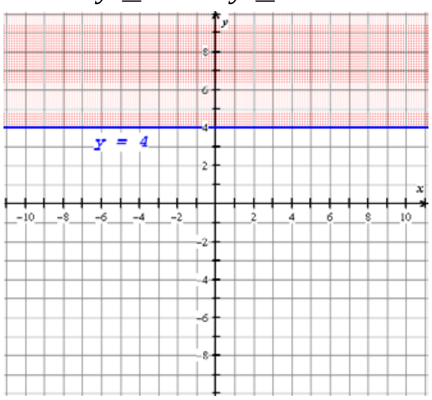
Problem 7 $-2x+4y \ge 8 \Rightarrow y \ge \dfrac{1}{2}x+2$



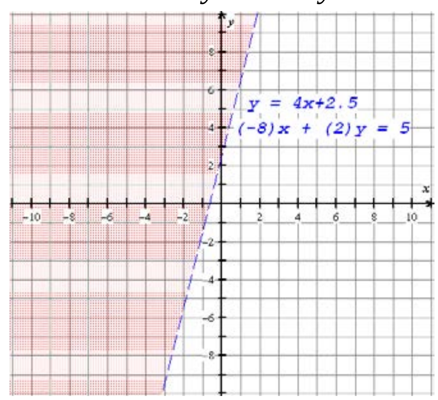
Problem 8 $-6x-3y \le -12 \Rightarrow 2x+y \ge 4 \Rightarrow y \ge -2x+4$



Problem 9 $8y \ge 32 \Rightarrow y \ge 4$

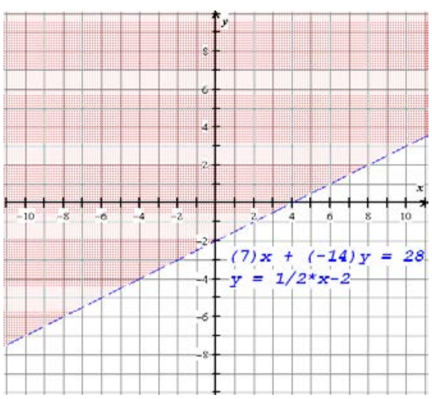


Problem 10 $-8x+2y > 5 \Rightarrow y > 4x+2.5$



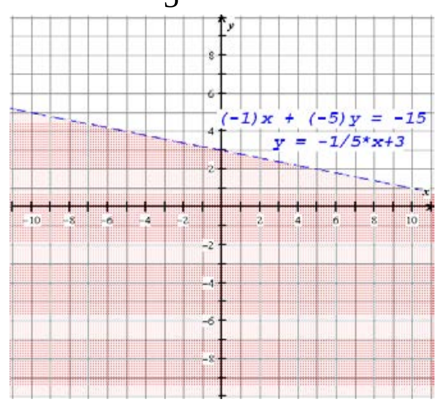
Problem 11

$7x-14y < 28 \Rightarrow -x+2y > -4 \Rightarrow y > \dfrac{1}{2}x-2$



Problem 12

$-x-5y > -15 \Rightarrow \dfrac{1}{5}x+y < 3 \Rightarrow y < -\dfrac{1}{5}x+3$



Solve the following systems of equations using algebra. Verify your solutions by plugging them back into the original system and graphing the two lines.

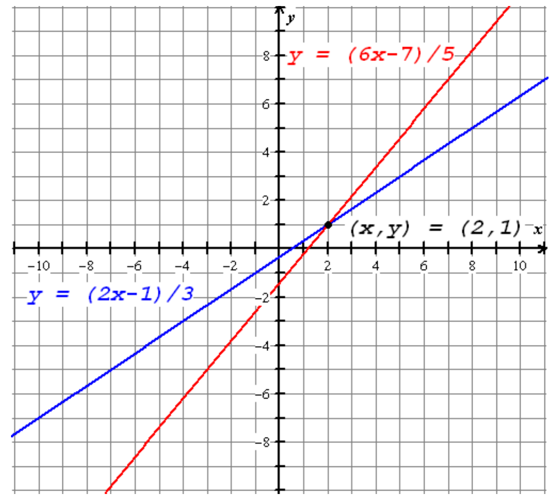
Problem 13

$$\begin{aligned} -2x + 3y &= -1 \\ 6x - 5y &= 7 \end{aligned} \Rightarrow \begin{aligned} 3(-2x + 3y) &= 3(-1) \\ 6x - 5y &= 7 \end{aligned}$$

$$\Rightarrow \begin{aligned} -6x + 9y &= -3 \\ 6x - 5y &= 7 \end{aligned}$$

$$\Rightarrow 4y = 4 \Rightarrow y = 1 \Rightarrow -2x + 3(1) = -1 \Rightarrow -2x = -1 - 3 = -4$$

$$\Rightarrow x = 2$$



So solution is $(x, y) = (2, 1)$

Check: $$\begin{aligned} -2(2) + 3(1) &= -4 + 3 = -1 \\ 6(2) - 5(1) &= 12 - 5 = 7 \end{aligned}$$

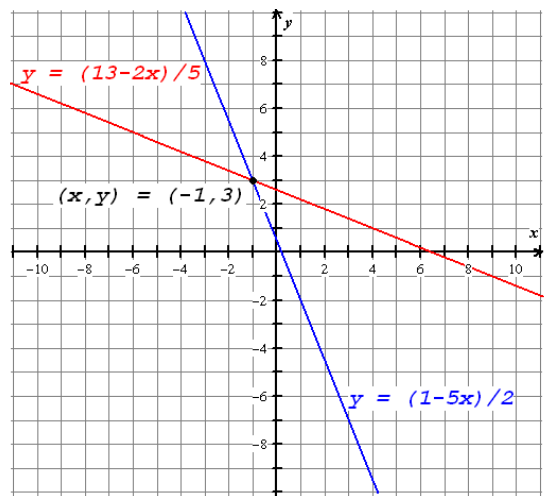
Problem 14

$$\begin{aligned} 5x + 2y &= 1 \\ 2x + 5y &= 13 \end{aligned} \Rightarrow \begin{aligned} -5(5x + 2y) &= -5(1) \\ 2(2x + 5y) &= 2(13) \end{aligned}$$

$$\Rightarrow \begin{aligned} -25x - 10y &= -5 \\ 4x + 10y &= 26 \end{aligned}$$

$$\Rightarrow -21x = 21 \Rightarrow x = -1 \Rightarrow 5(-1) + 2y = 1 \Rightarrow 2y = 6$$

$$\Rightarrow y = 3$$



So solution is $(x, y) = (-1, 3)$

Check: $$\begin{aligned} 5(-1) + 2(3) &= -5 + 6 = 1 \\ 2(-1) + 5(3) &= -2 + 15 = 13 \end{aligned}$$

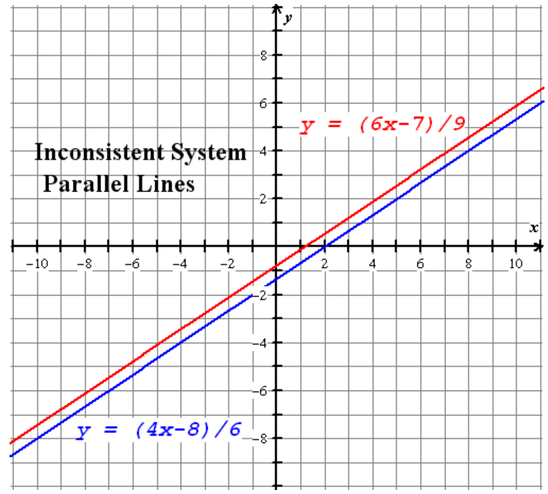
Problem 15

$$\begin{array}{rcl} -4x + 6y &=& -8 \\ 6x - 9y &=& 7 \end{array} \Rightarrow \begin{array}{rcl} 3(-4x + 6y) &=& 3(-8) \\ 2(6x - 9y) &=& 2(7) \end{array}$$

$$\Rightarrow \begin{array}{rcl} -12x + 18y &=& -24 \\ 12x - 18y &=& 14 \end{array}$$

$$\Rightarrow 0x + 0y = -10 \Rightarrow 0 = -10$$

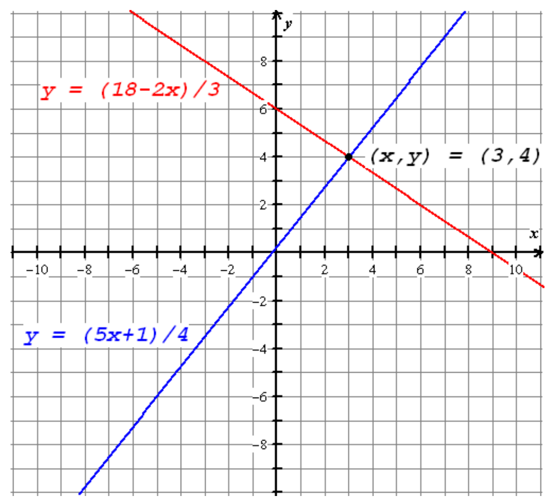
This is an inconsistency. So there are **no solutions**, the lines are parallel.



Problem 16 $\begin{array}{rcl} 5x - 4y &=& -1 \\ 2x + 3y &=& 18 \end{array}$

So solution is $(x, y) = (3, 4)$

Check: $\begin{array}{rcl} 5(3) - 4(4) &=& 15 - 16 = -1 \\ 2(3) + 3(4) &=& 6 + 12 = 18 \end{array}$



Chapter 5 Measurement

Section 5.1 Units of Measurements and Conversions

All measurements consist of two parts, a number part and a unit. For example, if we measure a table with a tape measure and report its width as 37.5 in, 37.5 is the number part, in is the unit part. The connection between the number part and unit part is the operation of multiplication. So when we write 37.5 in, we really mean (37.5)(1 in), i.e., 37.5 multiples of the base unit of one inch. The use of the word "base" in base unit does not imply that it is being raised to a power.

There are many different kinds of things we can measure, but all common physical measurements can be reduced to just a few kinds of things. The four basic quantities are often listed as follows:

1. Length measured in units such as feet, miles, meters, etc.

2. Time measured in units such as seconds, minutes, hours, etc.

3. Mass (or weight) measured in units such as grams, kilograms, pounds, etc.

4. Electric charge measured in coulombs or amp-seconds.

All other measurements can be expressed as a combination of these. It should really be noted that mass and weight are not identical. Weight is the gravitational force acting on a mass and varies with position in space. For example, the weight of an object on the surface of the earth is about six times the weight of that same object on the surface of the moon. However, the mass of the object is the same in both locations. Despite this difference, in these notes we will not distinguish between mass and weight.

It is not sensible to add or subtract measurements of different kinds of things. For example, 15 lb + 7 ft is a meaningless operation. This is just the old adage that it's impossible to add apples and oranges! To add or subtract measurements requires the same kind of quantities, as in 9 ft + 8 ft = 17 ft . Note, we just add the numbers and carry the factor of the unit. This is just the distributive property discussed in Chapter 3. Suppose we have 8ft + 36 in. Here, the quantities to be added are both lengths so the operation makes sense, but we can't actually perform the addition until we get the units to agree as in 8 ft + 36 in = 8 ft + 3ft = 11 ft. Here we "converted" 36 in to 3 ft. The operation could also have been done by converting 8 ft to 72 in , and adding 72 in + 36 in = 108 in.

While adding or subtracting different kinds of measurements is impossible, multiplying or dividing measurements is always possible. For example, $5\text{ lb}\times 4\text{ ft} = 20\text{ ft}\cdot\text{lb}$, where a $\text{ft}\cdot\text{lb}$ is a "foot pound" which is a unit of either energy or torque. As a second example, $100\text{ miles}\div 4\text{ gallons} = 25\text{ mpg}$, where mpg or miles per gallon measures fuel economy.

Measurement conversion is a necessary skill since the same set of units is often not used throughout a calculation. The basis of measurement conversion is the unit fraction. Any quantity Q remains unchanged when multiplied by 1.

$$Q = Q \cdot 1$$

The catch is that 1 has infinitely many "aliases". For example,

$$1 = \frac{12\text{ in}}{1\text{ ft}} = \frac{1\text{ ft}}{12\text{ in}} = \frac{4\text{ qt}}{1\text{ gal}} = \frac{1\text{ gal}}{4\text{ qt}} = \frac{60\text{ sec}}{1\text{ min}} = \frac{1\text{ min}}{60\text{ sec}} \quad .$$

All of these represent unit fractions, since the numerator is the same amount as the denominator. The trick is to use the "proper" aliases to cancel the units you don't want and get the units you do want. For example, to convert 18 in to ft we could use the following procedure.

$$18\text{ in} = \frac{18\text{ }\cancel{\text{in}}}{1} \times \frac{1\text{ ft}}{12\text{ }\cancel{\text{in}}} = \frac{18}{12}\text{ ft} = \frac{3}{2}\text{ ft} = 1.5\text{ ft} \quad .$$

Here the in unit was cancelled by appearing in both numerator and denominator.

Consider rounding 0.434 in to the nearest 64'th of an inch. The trick is to use the unit fraction

$$\frac{64}{64} \text{ as follows}: 0.434\text{ in} \times \frac{64}{64} = 0.434 \times 64 \times \frac{1\text{ in}}{64} = 27.776 \times \frac{1\text{ in}}{64} \approx \frac{28}{64}\text{ in} = \frac{7}{16}\text{ in} \quad .$$

So 0.434 in is seven sixteenth's of an inch to the nearest 64'th of an inch.

More complicated conversions can involve more than one unit fraction. The speed 100. feet per second can be converted to miles per hour correct to 1 decimal place by the following:

$$100.\frac{\text{ft}}{\text{s}} = 100.\frac{\text{ft}}{\text{s}} \times \frac{60\text{ s}}{1\text{ min}} \times \frac{60\text{ min}}{1\text{ hr}} \times \frac{1\text{ mile}}{5280\text{ ft}}$$

$$100.\frac{\text{ft}}{\text{s}} = 100\frac{\cancel{\text{ft}}}{\cancel{\text{s}}} \times \frac{60\text{ }\cancel{\text{s}}}{1\text{ }\cancel{\text{min}}} \times \frac{60\text{ }\cancel{\text{min}}}{1\text{ hr}} \times \frac{1\text{ mile}}{5280\text{ }\cancel{\text{ft}}} = \frac{100 \times 60 \times 60}{5280} \; \frac{\text{mile}}{\text{hr}} = 68.2\text{ mph} \quad .$$

As an aide in setting up conversion calculations a set of equivalent measurements is presented on the following page. This list also includes the formulas for converting temperature from Fahrenheit ($°\text{F}$) to Celsius ($°\text{C}$). For example, to find the Fahrenheit equivalent of $40°\text{C}$, we calculate as follows:

$$\text{Temp } °\text{F} = 40 \times \frac{9°\text{F}}{5} + 32°\text{F} = 104°\text{F}.$$

The Celsius equivalent of minus $10°\text{F}$ is computed using the formula

$$\text{Temp } °\text{C} = (-10 - 32) \times \frac{5°\text{C}}{9} = -23°\text{C}.$$

Some Conversion Relations for English and Metric Units

Linear Measure:

1 ft = 12 in
1 yd = 3 ft
1 mile = 5280 ft
1 rod = 16.5 ft
1 furlong = 220 yd
1 in = 2.54 cm
1 ft = 0.3048 m
1 yd = 0.9144 m
1 mile = 1.609344 km

Area Measure:

1 acre = 160 sq rods
1 sq mile = 640 acres
1 cm^2 = 0.15500031 in^2
1 m^2 = 1.195990046 yd^2
1 km^2 = 0.3861021585 sq mile

Volume Measure:

16 oz = 1 pt
2 pt = 1 qt
4 qt = 1 gal
1 gal = 0.13368056 ft^3
1 gal = 231 in^3
1 gal = 3.78541178 L
1 ft^3 = 7.48051948 gal
1 ft^3 = 28.31684659 L
1 L = 0.26417205 gal
1 L = 1.056688209 qt
1 L = 61.02374409 in^3
1 L = 0.001 m^3
1mL = 1 cm^3

Weight Measure:
16 oz = 1 lb
1 oz = 28.348 g
1 ton = 2000 lb
1 lb = 453.568 g
1 kg = 2.20474 lb

Time Measure:
1 min = 60 s
1 hr = 60 min
1 hr = 3600 s
1 day = 24 hours

Temperature Conversions:

$$\text{Temp}^\circ\text{C} = \left(\text{Temp}^\circ\text{F} - 32\right) \times \frac{5}{9}$$

$$\text{Temp}^\circ\text{F} = \text{Temp}^\circ\text{C} \times \frac{9}{5} + 32^\circ\text{F}$$

Section 5.2 Length, Area and Volume

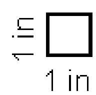
There is a definite relationship between length, area and volume measurements. Area is the amount of "two-dimensional space" inside of a planar figure. For example, a 2 ft by 6 ft rectangle has an area, A = to the product (2 ft)(6 ft) = 12 ft^2. Here, the unit ft^2 is one square foot (sq ft) which literally means a one foot by one foot square. When we say that the area of the rectangle is 12 ft^2, we mean that we could fit exactly 12 one foot by one foot squares inside this rectangle.

Care must be taken when converting units of area. Suppose we want to calculate how many square inches are in an area of 1.60 ft^2. We need the unit fraction between square inches and square feet.

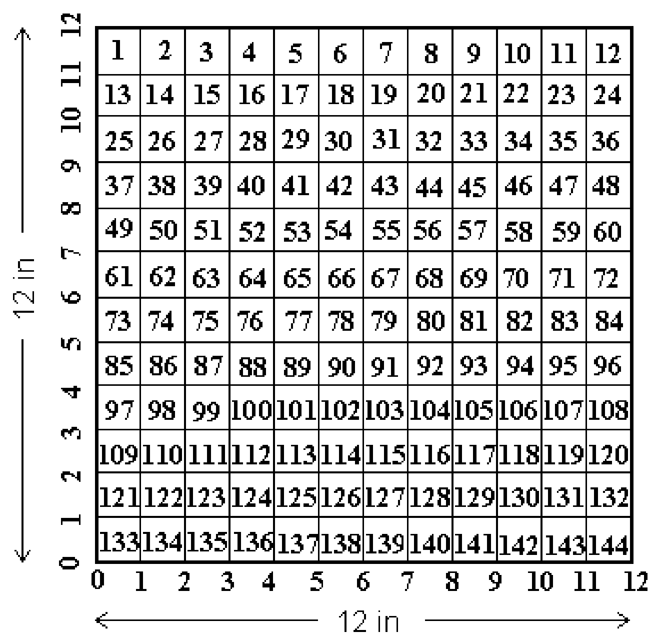
$$1\,\text{ft}^2 = (12\,\text{in})^2 = 12\,\text{in} \times 12\,\text{in} = 144\,\text{in}^2.$$

Note: when we evaluate (12 in)^2 we square **both** the 12 **and** the in . This is illustrated below.

1 square inch =



1 square ft=144 square in =

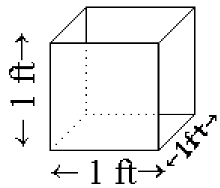


So the following calculation performs the conversion of 1.60 ft^2 to in^2.

$$1.60\ \text{ft}^2 = 1.60\ \cancel{\text{ft}}^2 \times \frac{144\ \text{in}^2}{1\ \cancel{\text{ft}}^2} = 1.60 \times 144\ \text{in}^2 = 230.\ \text{in}^2.$$

Volume is the amount of "three-dimensional space" inside of a solid. For example, a 2 ft by 3 ft by 2 ft box has a volume, V = (2 ft)(3 ft)(2 ft) = 12 ft^3**.** Here the unit ft^3 is one cubic foot (cu ft), which literally means a one foot by one foot by one foot cube as shown below.

$$1\,\text{ft}^3 = 1\,\text{ft} \times 1\,\text{ft} \times 1\,\text{ft} =$$

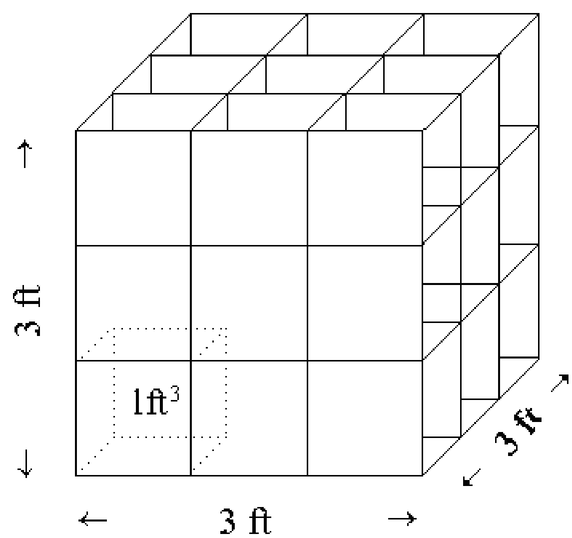


When we say that the volume of the box is 12. ft^3, we mean that we could fit exactly 12 one foot by one foot by one foot cubes inside this box. Like area conversions, volume conversions require careful setup. Suppose we wish to convert 12. ft^3 to cubic yards.

$$1\text{yard} = 3\text{ ft} \quad \Rightarrow 1\text{ cu yd} = 1\text{ yd}^3 = (3\text{ft})^3 = 3\text{ft} \times 3\text{ft} \times 3\text{ft} = 27\text{ ft}^3.$$

Note: when we evaluate $(3\text{ ft})^3$ we cube **both** the 3 **and** the ft . This is illustrated below.

$$1\text{yd}^3 = 3\text{ft} \times 3\text{ft} \times 3\text{ft} = 27\text{ft}^3 =$$



The following calculation performs the conversion of 12 ft^3 to cubic yards.

$$12.\text{ ft}^3 = 12\,\cancel{\text{ft}}^3 \times \frac{1\text{ yd}^3}{27\,\cancel{\text{ft}}^3} = \frac{12}{27}\text{yd}^3 = 0.44\text{yd}^3.$$

Section 5.3 The Metric System

One of the consequences of the French Revolution of 1789 was the development of the metric system of measurement. This system was designed to replace the earlier French system, which like its English counterpart had its origins in medieval society and royal institutions. Three features make the metric system very attractive.

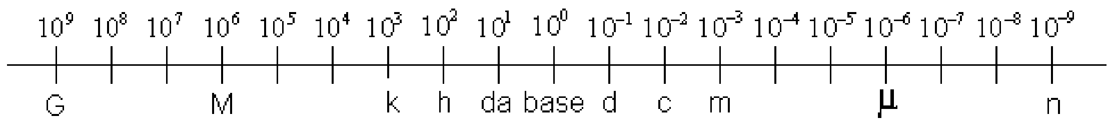
1. It is built on powers of 10, just like our decimal number system. Every unit is a multiple of 10 of some other unit. Thus, "strange" English unit multipliers like 3, 12, and 16 are banished!

2. A deliberate effort was made to coordinate different measures. For example, the fundamental unit of volume, the liter symbolized by L, is simply related to the fundamental unit of length, the meter symbolized by m, through the equation $1\ m^3 = 1000\ L$. Contrast this with the English system where $1\ gal = 231\ in^3 = 0.134\ ft^3$.

3. It is "universal". It can be used with **any** kind of measurement in exactly the same way.

It is interesting to note, that the metric system was so well accepted and in place that when electrical measurements began some 180 years ago only metric units were developed and have survived. The customary electric units we are all know, the volt, V, amp, A, and ohm, Ω are all metric.

The metric system uses a two-part representation of all measurements. The first character or prefix indicates the power of 10 used, while the remainder of the measurement is the base unit. This is illustrated below.



$10^{-9} = 0.000000001$
$10^{-8} = 0.00000001$
$10^{-7} = 0.0000001$
$10^{-6} = 0.000001$
$10^{-5} = 0.00001$
$10^{-4} = 0.0001$
$10^{-3} = 0.001$
$10^{-2} = 0.01$
$10^{-1} = 0.1$
$10^{0} = 1$
$10^{1} = 10$
$10^{2} = 100$
$10^{3} = 1000$
$10^{4} = 10{,}000$
$10^{5} = 100{,}000$
$10^{6} = 1{,}000{,}000$
$10^{7} = 10{,}000{,}000$
$10^{8} = 100{,}000{,}000$
$10^{9} = 1{,}000{,}000{,}000$

Prefix Names
G=giga
M=mega
k=kilo
h=hecto
da=deca
d=deci
c=centi
m=milli
μ=micro
n=nano

Important Metric Base Units
length - meter abbreviated m
time - second abbreviated s
mass - gram abbreviated g
volume - liter abbreviated L
(Note: the metric system has been set up so 1 mL= 1 cm^3)
power - watt - abbreviated w
energy - joule - j
force - newton -N
frequency - hertz abbreviated Hz
electric current - ampere abbreviated A
electric potential - volt abbreviated V
electric resistance - ohm abbreviated Ω
electric capacitance - farad abbreviated F
electric inductance - henry abbreviated H

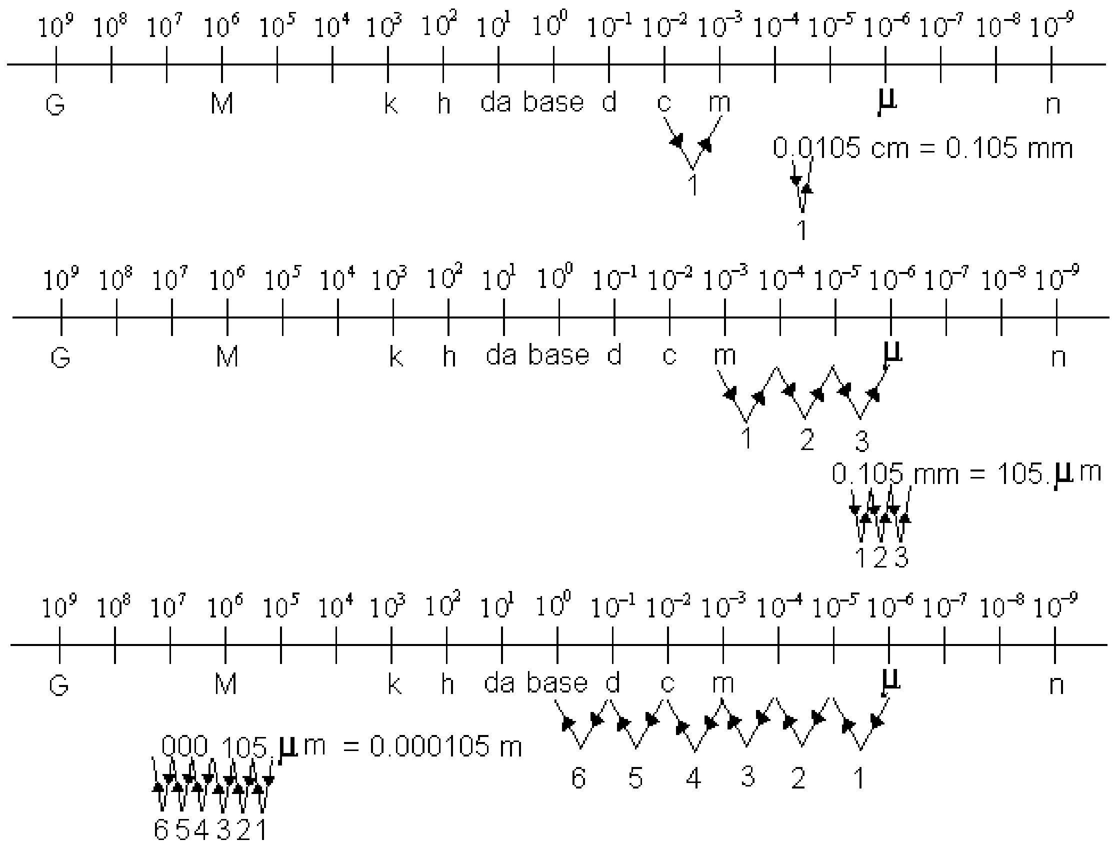
Conversions within the metric system are particularly easy. The steps are as follows:

1. Lay out a chart as shown below.

2. Locate the starting unit position and the final unit position on this chart and note the direction from the starting unit to the final unit.

3. Count the number of positions on the chart from the starting unit space to the final unit space.

4. Move the decimal point of the number part of the measurement the same number of decimal places as the count in Step 3 and in the same direction as noted in Step 2.



As before area and volume conversions within the metric system require careful setup. For example, suppose we want to convert 0.042 m^2 to square cm. The calculation can be setup as follows:

$$0.042\,\text{m}^2 = 0.042\,\text{m}^2 \times \left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^2 = 0.042\,\cancel{\text{m}}^2 \times \frac{10000\,\text{cm}^2}{1\,\cancel{\text{m}}^2} = 0.042 \times 10000\,\text{cm}^2 = 420\,\text{cm}^2.$$

As second example, to convert 187 mm^3 to mL, we proceed as shown below:

$$187\text{ mm}^3 = 187\text{ mm}^3 \times \left(\frac{1\text{ cm}}{10\text{ mm}}\right)^3 \times \frac{1\text{ mL}}{1\text{ cm}^3} = 187\ \cancel{\text{mm}}^3 \times \frac{1\ \cancel{\text{cm}}^3}{1000\ \cancel{\text{mm}}^3} \times \frac{1\text{ mL}}{1\ \cancel{\text{cm}}^3}$$
$$= \frac{187}{1000}\text{ mL} = 0.187\text{ mL}.$$

Conversions between metric and English units require the use of conversion factors.

Example: Convert 1.80 gallons per minute to m^3 per hour.

$$1.80\,\frac{\text{gal}}{\text{min}} = 1.80\,\frac{\cancel{\text{gal}}}{\cancel{\text{min}}} \times \frac{60\ \cancel{\text{min}}}{1\text{ hr}} \times \frac{3.7854\ \cancel{\text{L}}}{1\ \cancel{\text{gal}}} \times \frac{0.001\text{ m}^3}{1\ \cancel{\text{L}}} = 1.80 \times 60 \times 3.7854 \times 0.001\,\frac{\text{m}^3}{\text{hr}}$$
$$= 0.409\,\frac{\text{m}^3}{\text{hr}}.$$

Example: Convert 83.0 km per hour to ft per second.

$$83.0\ \frac{\text{k}\cancel{\text{m}}}{\text{hr}} = 83.0\ \frac{\text{k}\cancel{\text{m}}}{\text{h}\cancel{\text{r}}} \times \frac{1\text{ h}\cancel{\text{r}}}{3600\text{ s}} \times \frac{1000\ \cancel{\text{m}}}{1\text{ k}\cancel{\text{m}}} \times \frac{1\text{ ft}}{0.3048\ \cancel{\text{m}}} = 83.0 \div 3600 \times 1000 \div 0.3048\,\frac{\text{ft}}{\text{s}}$$
$$= 75.6\,\frac{\text{ft}}{\text{s}}\ .$$

Note: In both of these examples, the final answer has the same number of significant digits as the original data.

Your Turn!!

Perform the following calculations with measurement numbers.

12 ft 3 in – 8 ft 8 in

1) ________________

$4.2\text{ cm} \times 3.5\text{ cm}^2$

2) ________________

$530\text{ miles } \div 35\text{ mpg (mpg} = \text{miles per gallon)}$

3) ________________

Convert the following measurements as indicated giving answers to the correct number of significant digits (see **Section 1.4**). Write answers in the blank space provided.

4. 185°F = ________________ °C

5. 17.6 liters = ________________ gal

6. 132 kg = ________________ lb

7. 127 ft^3 = ________________ yd^3

8. 129 mA = ________________ A

9. 0.589 m = ________________ mm

10. 58 cm = ________________ m

11. 127μV = ________________ V

12. 98 km/hour = ________________ mph

13. 10°C = ________________ °F

14. 2.56 sq ft = ________________ cm^2

15. 2.49 gal = ________________ mL

16. 179 cm^3 = ________________ in^3

17. 83 mph = ________________ ft/sec

18. 19.5 gal/min = ________________ ft^3/hour

19. 25.8 mpg = ________________ km/L

Round to the nearest 32'nd of an inch: 0.165 in =

20. ________________

Round to the nearest 64'th of an inch: 0.645 in =

21. ________________

What size bolt, to the nearest 64'th of an inch, will fit a hole 12 mm in diameter?

22. ________________

A box has dimensions of 5 ft 6 in H 3 ft 9 in H 3 ft 4 in . How many cubic meters is this?

23. ________________

A car has a gas tank with a capacity of 50 L. If the car gets 33.5 miles per gallon, how many km can the car travel on a full tank?

24. ________________

Chapter 5 Sample Test ______________/100

Each problem is worth 5 points. Give answers to the correct number of significant digits.

1. Perform the following calculation: 10 ft 5 in – 7 ft 9 in
2. Perform the following calculation: $289 \text{ miles} \div 10.7 \text{ gallons}$
3. Perform the following calculation: $6.25 \text{ ft}^2 \times 3.15 \text{ ft}$
4. Perform the following calculation: $1060 \text{ km} \div 110. \text{ km per hour}$
5. Convert $41.0°\text{C}$ to $°\text{F}$
6. Convert 36.5 liters to gallons
7. Convert 189 lb. to kg
8. Convert 976 in^3 to ft^3
9. Convert 245 mV to V
10. Convert 0.088 m to mm

11. Convert 75. mi/hr to m/s

12. Convert 54.5 ft^2 to yd^2

13. Convert 800.0 ft^3 to gallons

14. Convert 2.25 L to in^3

15. Convert 508.0 in^3 to cm^3

16. What size bolt, to the nearest 64'th of an inch, fits a hole that is 0.225 inches in diameter?

17. A garden rectangular pool has dimensions of 2.0 yd by 3.0 yd by 1.5 yd. How many gallons can it hold?

18. To pour a concrete driveway you must order the concrete in cubic yards. If the driveway is to be 12 feet wide, 45 feet long and 4 inches thick, how many cubic yards of concrete are needed? Round the answer to one decimal place.

19. Moving at 70.0 mph how long to the nearest tenth of an hour does it take to travel 500. km?

20. A car has a gas tank with a capacity of 50.0 L. If the car gets 60.0 kilometers per gallon, how many miles can the car travel on a full tank?

Chapter 5 Sample Test Solutions

1.	**2.**	**3.**	**4.**	**5.**
2 ft 8 in = 32 in	27.0 mpg	19.7 ft^3	9.64 hr	106°F
6.	**7.**	**8.**	**9.**	**10.**
9.64 gal	85.7 kg	0.565 ft^3	0.245 V	88 mm
11.	**12.**	**13.**	**14.**	**15.**
33.5 m/s	6.06 yd^2	5984 gal	127 in^3	8325 cm^3
16.	**17.**	**18.**	**19.**	**20.**
$\frac{7}{32}$ in	1800 gal	6.7 yd^3	4.4 hr	493 miles

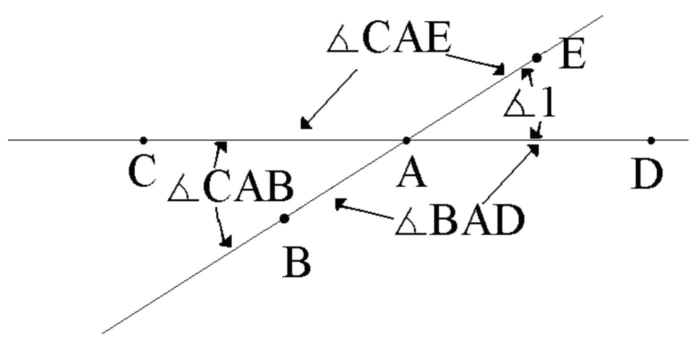
Geometry is a word from Greek which combines the word *geo* meaning *Earth* in Greek and the word *metry* which means to *measure*, so geometry literally means measuring the Earth. More generally geometry is the study of shapes, their properties, and how to measure them.

Section 6.1 Plane Geometry

In this section we are considering plane or flat two-dimensional geometry. (Solid geometry studies objects in three dimensions.) The fundamental notions used to describe planar objects are points and lines.

Basic Facts About Points and Lines:

- Two different points define one and only one line that passes through both of them
- We shall use capital letters such as **A** to label points and a pair of letters such as **AB** to label the segment of the line between **A** and **B.**
- Line segments are measured in units of length such as feet or meters.
- The shortest distance between the two points is along this line.
- Two lines in a plane either meet or in other words intersect at a single point or are parallel, meaning that the two lines never touch.
- Where the lines meet four "openings" or angles are formed as shown below. The symbols $\angle$ and $\measuredangle$ are both used for the word angle. (See figure below)



Angles

In order to discuss angles, several technical terms are used. As shown in the above picture, the point where the lines intersect **A** is called the **vertex** of the four angles. The separate angles are then named by indicating a point on each of the two lines which act as sides of the angle. Thus $\angle$**BAD** is the angle with vertex at **A** and with sides that include segments **AB** and **AD**. This same angle could also be labeled as $\angle$**DAB**. A less precise but more convenient notation is to label angles by a single letter or number, provided that the symbol is placed appropriately in the diagram. For example, in the above diagram $\angle$**1** = $\angle$**EAD**.

Measuring Angles

The most commonly used measure for angles is the degree system developed by the Babylonians over 3000 years ago. They decided for reasons of the number of days in a year and easy divisibility to divide a circle into 360 equal units of angle called a degree and signified by the symbol **E**. For precise surveying or problems in astronomy a smaller unit is required. This lead to the minute which is 1/60'th of a degree and the second which is 1/60'th of a minute. The symbol for minute is **N**, while the symbol for second is **O**.

Thus, we have $1' = \frac{1}{60}°$ and $1'' = \frac{1}{60}' = \frac{1}{60} \times \frac{1}{60}° = \frac{1}{3600}°$. An angle to the nearest second could be specified as $\angle 4 = 37°29'54''$ this notation is called DMS for Degree Minute Second. This same angle could be given in DD (Decimal Degrees) as $\angle 4 = (37 + \frac{29}{60} + \frac{54}{3600})° = 37.4983333...°$. To go back to DMS,

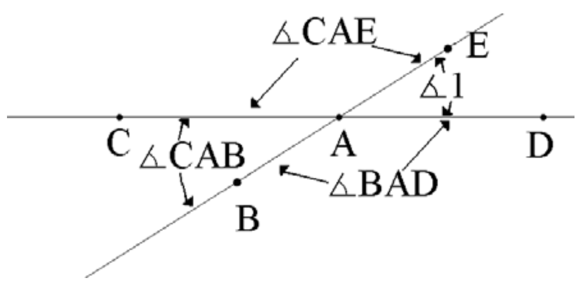
$$\angle 4 = 37.4983333...° = 37° + \frac{60'}{1°} \times .498333...° = 37°29.9' = 37°29' + .9' \times \frac{60''}{1'} = 37°29'54''.$$

These conversions have been built into many calculators. Since each of you may have different calculators, *please ask your instructor for help if you're not sure how to use your calculator.*

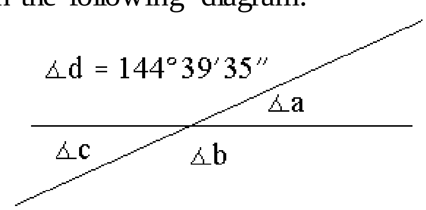
Facts about measuring angles

- Since there are $360°$ in a circle, the angular measure of a straight line is $180°$
- In a square or right angle there are $90°$.
- Two angles whose sum is $180°$ make a straight line and are called **supplementary**.

If two lines intersect as shown below: $\angle\mathbf{EAD} + \angle\mathbf{CAE} = \angle\mathbf{EAD} + \angle\mathbf{BAD} = 180°$. Since $\angle\mathbf{EAD}$ is on both sides of this equation, we conclude that $\angle\mathbf{CAE} = \angle\mathbf{BAD}$ and similarly that $\angle\mathbf{CAB} = \angle\mathbf{EAD}$. These equal angles formed by two intersecting lines are called **vertex (or vertical) angles**.

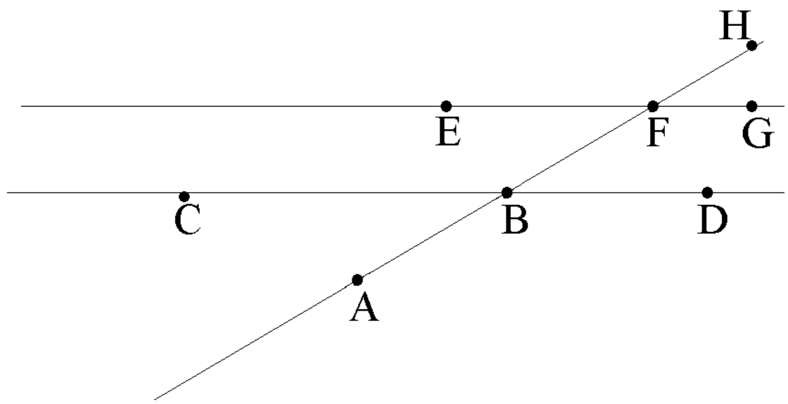


Example: Determine the angles in the following diagram:

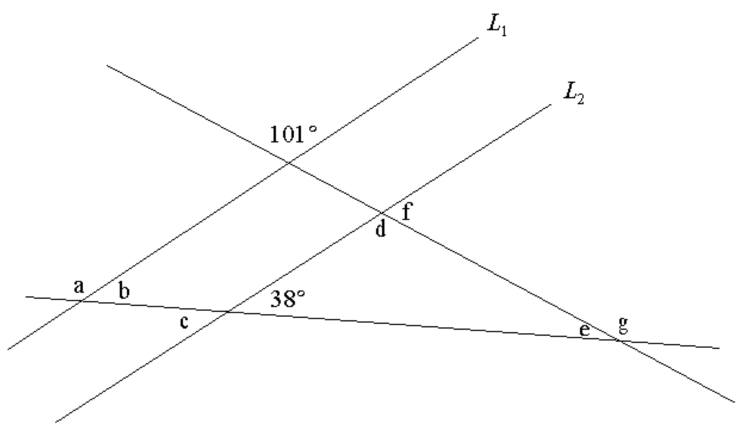


Solution: Once the measure of $\angle\mathbf{d}$ is known, the remaining three can be determined. Since they are vertex angles, $\angle\mathbf{b} = \angle\mathbf{d} = 144°39'35''$. Since $\angle\mathbf{a}$ is the supplement to $\angle\mathbf{d}$ and a vertex angle to $\angle\mathbf{c}$ we have $\angle\mathbf{a} = \angle\mathbf{c} = 180° - 144°39'35'' = 179°59'60'' - 144°39'35'' = 35°20'25''$.
This same calculation can also be done on most scientific calculators using a few keystrokes.

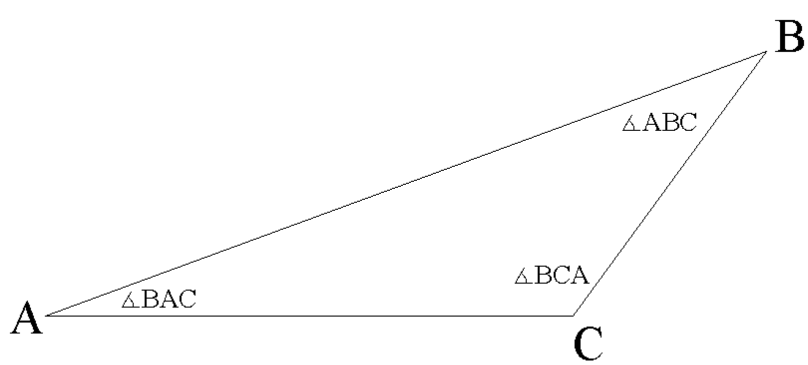
Consider a pair of parallel lines crossed by a third line (called the **transversal**) as shown below. If point **B** is superimposed, or in other words, moved so that it is directly over point **F** by moving segment **CD** onto the line through **EG**, the **corresponding** angles $\angle\mathbf{EFB}$ and $\angle\mathbf{CBA}$ are equal. Similarly, $\angle\mathbf{EFH} = \angle\mathbf{CBF}$, $\angle\mathbf{ABD} = \angle\mathbf{BFG}$, and $\angle\mathbf{FBD} = \angle\mathbf{HFG}$.From the equality of vertex angles, the **alternating interior** (alternate sides of the transversal, inside the two parallel lines) are equal, i.e., $\angle\mathbf{EFB} = \angle\mathbf{FBD}$ and $\angle\mathbf{CBF} = \angle\mathbf{BFG}$. Similarly, the **alternating exterior** (alternate sides of the transversal, outside the two parallel lines) are equal, i.e., $\angle\mathbf{EFH} = \angle\mathbf{ABD}$ and $\angle\mathbf{CBA} = \angle\mathbf{HFG}$.



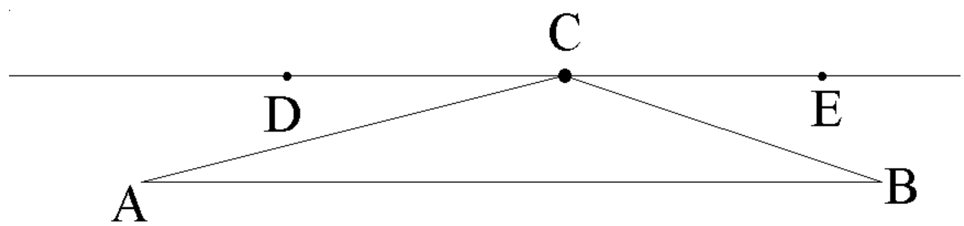
Example: In the figure below L_1 and L_2 are parallel, so ⊇**d =**101E, ⊇**f =**180E– ⊇**d =** 79E, ⊇**b =** ⊇**c =** 38E,⊇**a =**180E– ⊇**b =** 142E, ⊇**e =**180E– ⊇**d** –38E= 41E, and ⊇**g =**180E– ⊇**e =** 139E**.**



Polygons are closed figures in the plane whose sides are line segments. The simplest polygon is the three-sided **triangle.** The points at the corners **A**, **B**, and **C** are the **vertices** of the triangle, and the angles $\angle\mathbf{BAC}$, $\angle\mathbf{BCA}$, and $\angle\mathbf{ABC}$ are called the interior angles of the triangle. The triangle is often then labeled as triangle **ABC**.

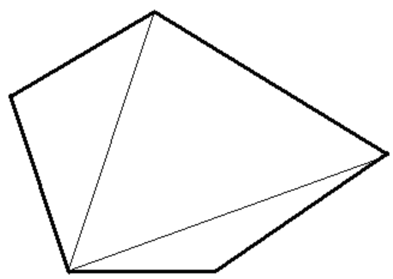


Consider the triangle **ACB** shown below. Through the vertex **C** we construct a line parallel to the segment **AB.** Since they are alternating interior angles, $\angle\mathbf{DCA} = \angle\mathbf{CAB}$ and $\angle\mathbf{ECB} = \angle\mathbf{CBA}$. However, $\angle\mathbf{DCA} + \angle\mathbf{ACB} + \angle\mathbf{ECB} = 180^\circ$, so we have $\angle\mathbf{CAB} + \angle\mathbf{ACB} + \angle\mathbf{CBA} = 180^\circ$.

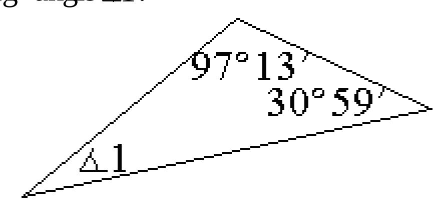


Important Fact: The interior angles of any triangle always add up to $180°$.

Consider the five-sided pentagon shown below. By drawing three triangles from a vertex, we see that the sum of the internal angles of the pentagon is the sum of the internal angles in three triangles or $3(180°) = 540°$. A similar argument for an *n*-sided polygon shows that the sum of the internal angles is $(n-2)(180°)$.



Example: In the triangle below calculate the missing angle $\measuredangle 1$.

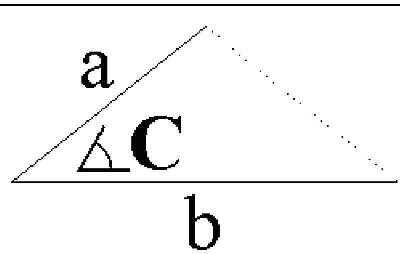
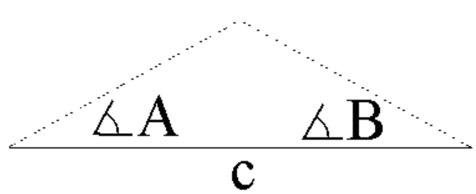
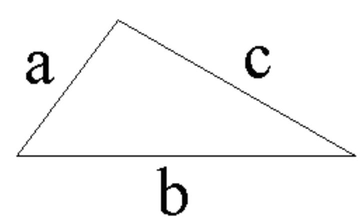


Solution: $\measuredangle 1 = 180° - 97°13' - 30°59' = 51°48'$

Example: What is the sum of the interior angles in a 7 sided polygon?

Solution: In this case *n* = 7, so $(n-2)(180°) = 5(180°) = 900°$.

What determines a triangle? Every triangle has three sides and three angles, so six numbers (three angle measures and three side lengths) are associated with every triangle. Two triangles are called **congruent** if one can be superimposed on top of the other so that it fits exactly. In other words, the triangles have exactly the same shape and size. Essentially, triangles that are congruent are the "same". There are three rules for congruency when using triangles:

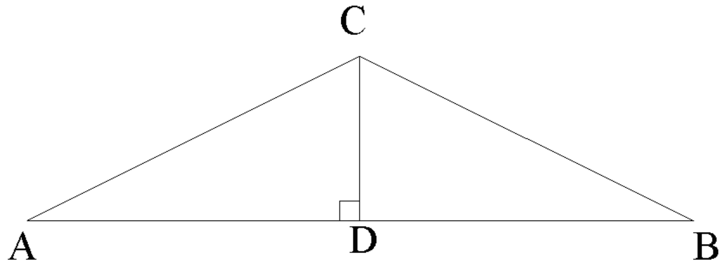
Side Angle Side abbreviated **SAS**	**Angle Side Angle** abbreviated **ASA**	**Side Side Side** abbreviated **SSS**
		

The notation **SAS** means the length of two sides and the angle between these two sides are known; the remaining side and two angles are to be found out. The notation **ASA** (or **AAS**, since if any two angles in a triangle are known, the third can be determined from $180°$ minus the sum of the other two) means two angles and the side between them have been specified. Of course, the sum of the two specified angles must be less than $180°$. Finally, **SSS** means all three sides have been specified. Since the shortest distance between any two vertices is the side joining them, the longest side of a triangle **must be shorter** than the sum of the other two sides.

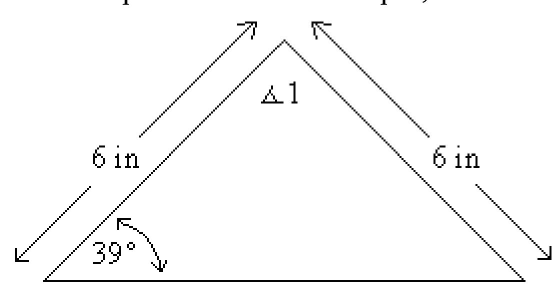
Note: AAA, (meaning that all three angles of the triangle are known, but none of the lengths of the sides are known) determines a triangle's shape, but **NOT** its size. Since two triangles can have the same shape but differ in size, **AAA** is **not** a rule of congruence.

Types of Triangles

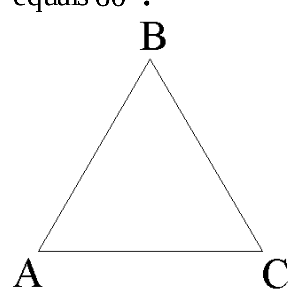
A triangle is called isosceles if two sides are equal. Consider the isosceles triangle **ACB** with **AC** = **CB**, from **C** construct **CD** to **D** the mid point (i.e., **AD** = **DB**) of **AB** . Now by **SSS** triangle **ADC** is congruent to triangle **BDC** . Thus, $\angle \mathbf{ADC} = \angle \mathbf{BDC}$, and since these two angles sum to $180°$, **CD** is perpendicular (makes a right angle) to **AB** . This is indicated by the "little" box at **D** . In addition, the angles opposite to the equal sides are also equal, $\angle \mathbf{CAB} = \angle \mathbf{CBA}$.



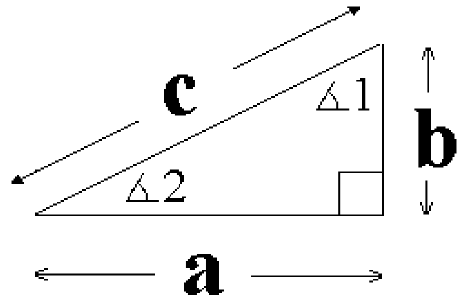
Example: Determine $\angle 1$ in the following isosceles triangle.
Solution: The angles opposite the equal sides must be equal, $\angle 1 = 180° - 2(39°) = 102°$.



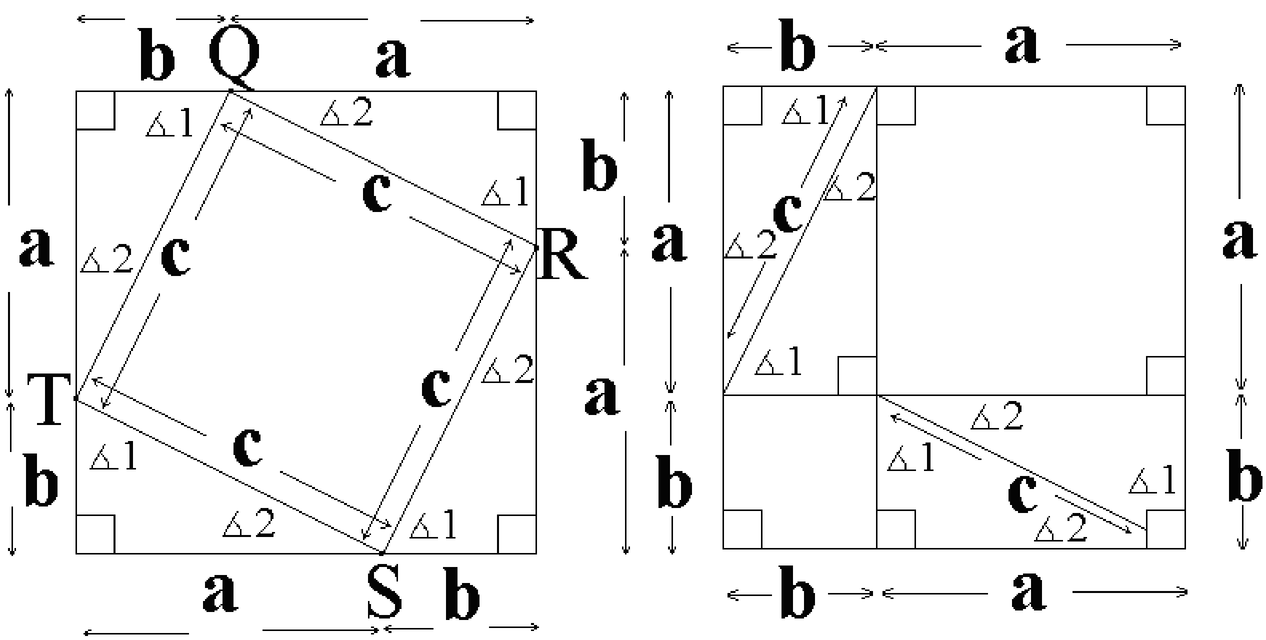
A triangle is called **equilateral** if all three sides are equal. An equilateral triangle is also isosceles, so $\angle \mathbf{BAC} = \angle \mathbf{BCA}$ and $\angle \mathbf{BAC} = \angle \mathbf{ABC}$. Since all three angles are equal and since they sum to $180°$, each internal angle equals $60°$**.**



A **right triangle** is a triangle with a $90°$ interior angle. The large side opposite the $90°$ angle is called the **hypotenuse** while the remaining two sides are called **legs**. In the diagram shown, the hypotenuse is **c** and the legs are **a** and **b**. The two **acute** (less than $90°$) angles are $\measuredangle 1$ and $\measuredangle 2$ with $\measuredangle 1$ opposite to the side of length **a** and $\measuredangle 2$ opposite to the side of length **b**. The sum of the three interior angles is $\measuredangle 1 + \measuredangle 2 + 90° = 180°$, so we have the result that the two acute angles in a right triangle are **complementary**, i.e., $\measuredangle 1 + \measuredangle 2 = 90°$.



Construct a right triangle with legs **a** and **b** and hypotenuse **c** and $\measuredangle 1$ opposite to the side of length **a** and $\measuredangle 2$ opposite to the side of length **b**. Consider the two ways of constructing a square of side **a + b** shown below.



In the figure on the left we have $\measuredangle 1 + \measuredangle \mathbf{TQR} + \measuredangle 2 = 180°$. Since $\measuredangle 1 + \measuredangle 2 = 90°$, $\measuredangle \mathbf{TQR} = 90°$. A similar argument shows that $\measuredangle \mathbf{QTS} = \measuredangle \mathbf{QRS} = \measuredangle \mathbf{RST} = 90°$. So the figure in the center on the left is a square of side **c**, i.e., a square of the hypotenuse. Thus, the area of the square of side **a + b** is the square of **c** plus the area of the four right triangles. From the figure on the right we see that the area of the square of side **a + b** is the square of **a** plus the square of **b** plus the area of the same four right triangles. Since the area of the two figures must be the same we conclude that the square of the hypotenuse is the sum of the squares of the legs, or in symbols:

$$c^2 = a^2 + b^2$$

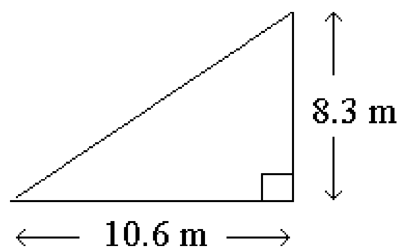
This result is called the **Pythagorean Theorem** and is probably the most famous and useful result in geometry! The result can be stated a number of different ways. To calculate the hypotenuse knowing the lengths of the legs, we take the square root and get

$$c = \sqrt{a^2 + b^2}$$

To calculate a leg, say **a**, knowing the hypotenuse and the other leg, **b**, the formula is rearranged as follows. **Remember** to use parantheses since the square root symbol is a grouping symbol.

$$a = \sqrt{c^2 - b^2} \qquad \text{leg} = \sqrt{\text{hypotenuse}^2 - \text{other leg}^2}$$

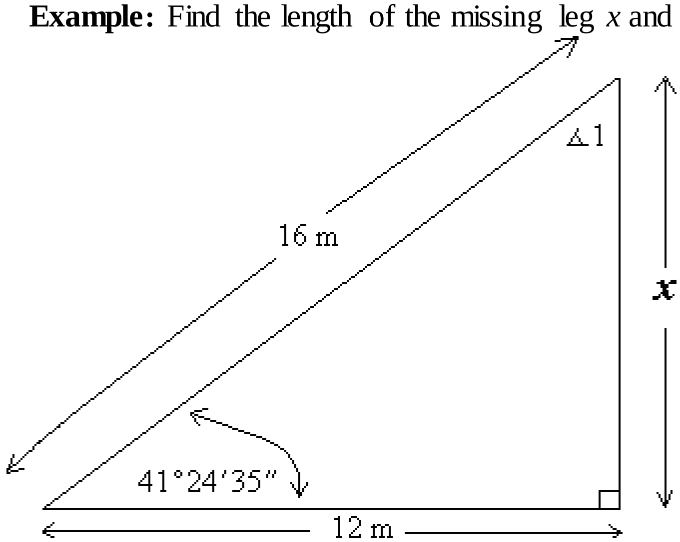
Example: Find the hypotenuse of the right triangle pictured below.



Solution: $c = \sqrt{\left((8.3\text{m})^2 + (10.6\text{m})^2\right)} = \sqrt{68.89\text{m}^2 + 112.36\text{m}^2} = \sqrt{181.25\text{m}^2} = 13.5 \text{ m}$

Note: The implied parenthesis inside the square root has been made explicit as required to get the correct answer on the calculator. The units work out to be linear units as required for a length since $\sqrt{\text{m}^2} = \text{m}$.

Example: Find the length of the missing leg *x* and the missing angle below.



Solution:

$$x = \sqrt{\left((16\text{m})^2 - (12\text{m})^2\right)} = \sqrt{112\text{m}^2} = 10.6\text{m}.$$

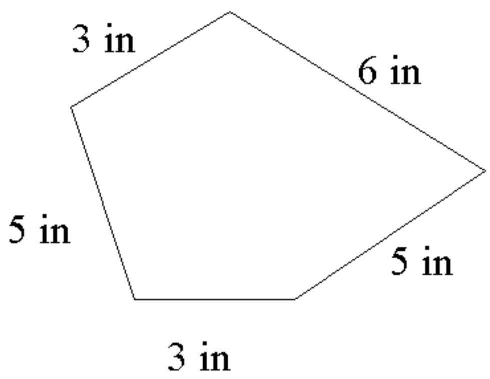
The missing angle is complementary to 41°24'35", so

$$\angle 1 = 90° - 41°24'35'' = 48°35'25''.$$

Perimeter and Area

The last topic in plane geometry we consider is the **perimeter** and area of a figure in a plane. The **perimeter** is the total linear distance around the boundary of a polygon.

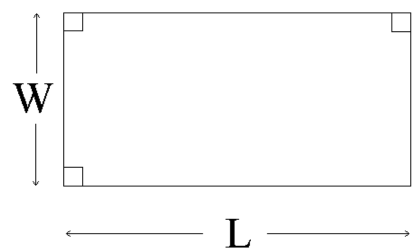
Example: Determine the perimeter of the following pentagon.



Solution:
P = 3 in + 6 in + 5 in + 3 in + 5 in = 22 in.

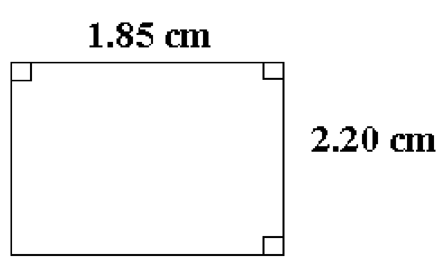
Area and Perimeter of a Rectangle

A rectangle has width W and length L. The perimeter is computed as $P=W+W+L+L=2W+2L$.



The area of a rectangle, which measures the amount of "two dimensional space" inside the rectangle, is given by $A=W\cdot L$. **Note:** perimeter always has units of length, while area always has units of length^2.

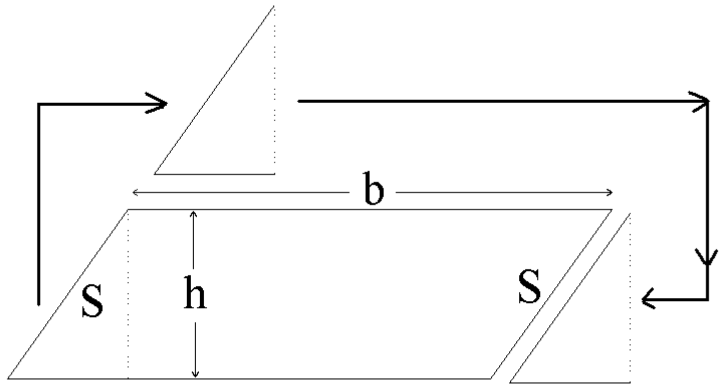
Example: Determine the perimeter and area of the rectangle shown below.



Solution: P = 2(1.85 cm) + 2(2.20 cm) = 8.10 cm and $A = 1.85\text{ cm} \times 2.20\text{ cm} = 4.07\text{ cm}^2$.

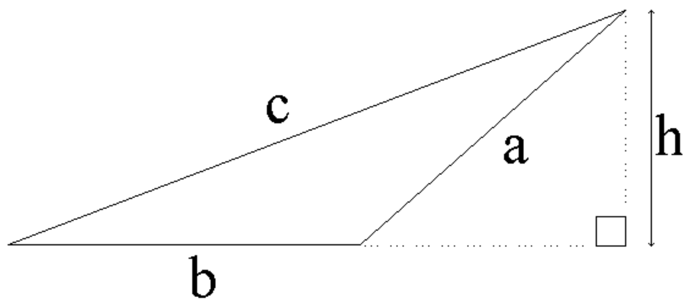
Area of a Parallelogram

A **parallelogram** is a four sided polygon (or quadralateral) with opposite sides parallel. All rectangles are parallelograms, but in a generic parallelogram the angle between adjacent sides is not necessarily $90°$. Consider the parallelogram shown below with base **b** and perpendicular distance **h** between the top and bottom sides. Imagine cutting off a right triangle from the left end and moving it to the right end. Since the left and right sides are parallel, this right triangle fits perfectly to make a rectangle of dimensions **b** and **h**. So we have for a parallelogram: $A = b \cdot h$.

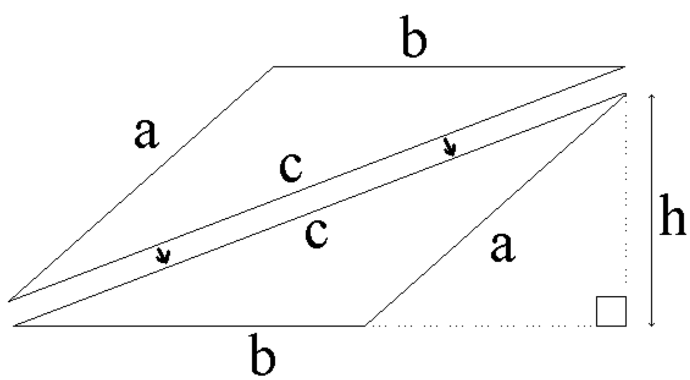


Area of a Triangle

Next consider a generic triangle triangle with bottom side (base) of length **b** and perpendicular distance (height) **h** from the base to the top vertex.



Imagine making an exact copy of this triangle and joining it to the original triangle as shown.



The result is a parallelogram of base **b** and height **h** . Since the area of the original triangle is half of the area of this parallelogram, we arrive at the result that the area of a triangle is given by

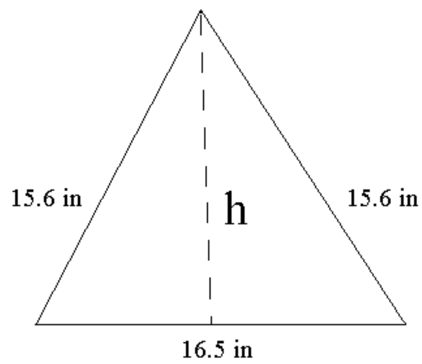
$$A = \frac{1}{2} b \cdot h \ .$$

A more detailed argument based on the pythagorean theorem shows that for a triangle with sides **a** , **b** , and **c** the area can be calculated from Heron's formula:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

with the semi-perimeter S given by the formula: $S = \dfrac{a+b+c}{2}$.

Example: Calculate the perimeter and area of the following triangle.



Solution: $P = 15.6 \text{ in} + 15.6 \text{ in} + 16.5 \text{ in} = 47.7 \text{ in}$. To calculate the area, we could calculate the height by dropping a perpendicular from the top vertex. Since the triangle is isosceles, this bisects the 16.5 in base. Using the Pythagorean Theorem we compute h as follows:

$$h = \sqrt{(15.6 \text{ in})^2 - (16.5 \text{ in} \div 2)^2} = \sqrt{175.30 \text{ in}^2} = 13.2 \text{ in}$$

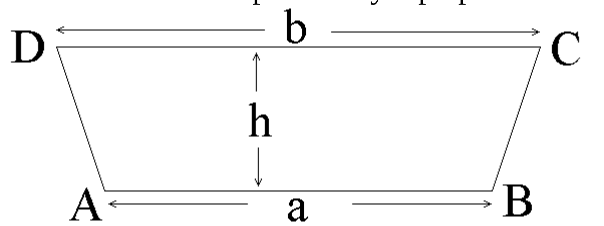
Then the area is calculated as half the base times the height. $A = 0.5 \times 16.5 \text{ in} \times 13.2 \text{ in} = 109 \text{ in}^2$

Another way to calculate the area is to use Heron's formula. $S = \dfrac{47.7 \text{ in}}{2} = 23.85 \text{ in}$

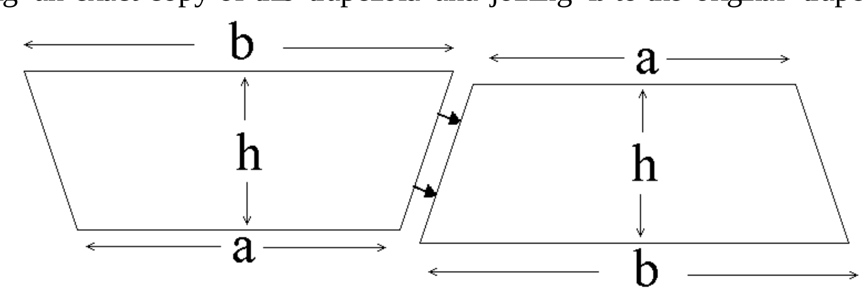
$$A = \sqrt{23.85 \text{ in}(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 16.5 \text{ in})} = \sqrt{11931 \text{ in}^4} = 109 \text{ in}^2$$

Area of a Trapezoid

A quadrilateral with two opposite sides parallel is called a trapezoid. Suppose that the two parallel faces have lengths **a** and **b** and are separated by a perpendicular distance **h**.



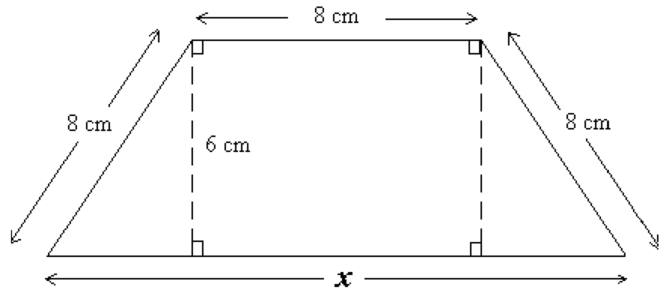
Imagine making an exact copy of this trapezoid and joining it to the original trapezoid as shown.



The result is a parallelogram of base **a + b** and height **h** . Since the area of the original trapezoid is half of the area of this parallelogram, we arrive at the result that the area of a trapezoid is given by the folowing formula:

$$A = \frac{1}{2}(a + b) \cdot h$$

Example: Calculate the perimeter and area of the following trapezoid.



Solution: The first step is to calculate the length x. Using the Pythagorean Theorem the length of the base of the right triangles that form the sides of the trapezoid is computed as follows:

$$\sqrt{(8\text{ cm})^2-(6\text{ cm})^2}=\sqrt{28\text{ cm}^2}=5.29\text{ cm}$$

Then $x=8\text{ cm}+2\times 5.29\text{ cm} = 18.58\text{ cm}$. The perimeter is just the sum of the lenghts of the four sides. $\text{P}=18.6\text{ cm}+8\text{ cm}+8\text{ cm}+8\text{ cm}=42.6\text{ cm}$

To calculate the area, we could add the area of the two right triangles that form the sides of the trapezoid to the area of the 8 cm by 6 cm central rectangle.

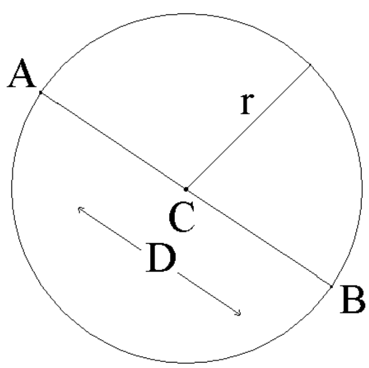
$$\text{A} = 2\times\frac{1}{2}\times 5.29\text{ cm}\times 6\text{cm}+8\text{ cm}\times 6\text{ cm} = 79.7\text{ cm}^2$$

We get the same result by using the formula for the area of a trapezoid .

$$\text{A}=\frac{1}{2}\times(18.58\text{ cm}+8\text{ cm})\times 6\text{cm}=79.7\text{ cm}^2$$

Circles

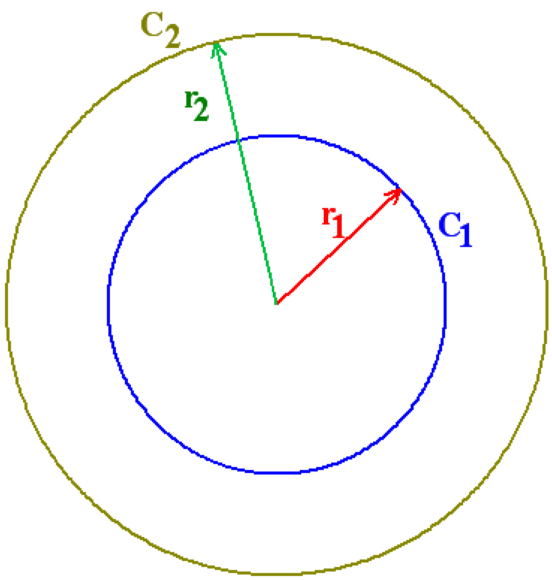
A circle is formed by generating all points in a plane which are a fixed distance called the **radius** from a center point here labeled as **C**. A line segment with end points on the circle that passes through the center is called a **diameter**. **D** and **r** symbolize the lengths of the diameter and radius, respectively. Since **AC = CB = r** and **D = AB = AC + CB =** 2**r** , we have the following:



$$\text{D}=2\cdot\text{r}$$
$$\text{r}=\frac{\text{D}}{2}$$

All circles are **similar**. By this we mean that all circles have the same shape. The distance around the boundary of a circle is called its **circumference** which is like the perimeter of a polygon.

Imagine that we have two circles. The first labeled as **1** has radius $\mathbf{r_1}$, diameter $\mathbf{D_1}$ and circumference $\mathbf{C_1}$. The second labeled as **2** has radius $\mathbf{r_2}$, diameter $\mathbf{D_2}$ and circumference $\mathbf{C_2}$. Since all circles are scale models of each, the ratio of circumference to diameter is the same for both circles. The value of this ratio is symbolized by the Greek letter lower case pi, π.

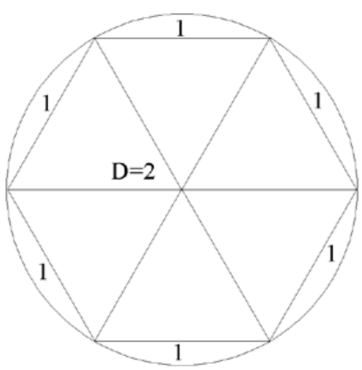


$$\frac{C_2}{D_2} = \frac{C_1}{D_1} = \pi \quad \text{or} \quad \begin{array}{l} C_2 = \pi \cdot D_2 \\ C_1 = \pi \cdot D_1 \end{array}$$

According to the above argument then, the distance around the outside of a circle or in other words the "perimeter "of a circle, is given by the formula.

$$C = \pi \cdot D = 2\pi \cdot r$$

The "perimeter" or a circle is usually called the **circumference** of the circle. The existence and value of pi strike many people as mysterious. However, a simple argument illustrated below shows that pi is slightly larger than 3. Form six equlateral triangles each of side 1 (the units of length don't matter). Since all of the internal angles equal to 60E, the six equilateral triangles can be joined next to each other with a common vertex as shown. The union of the six equilateral triangles is a regular **hexagon** with a perimeter of 6. Making the common vertex the center of a circle of radius 1, we see that the circumference of this circle is slightly bigger than 6.

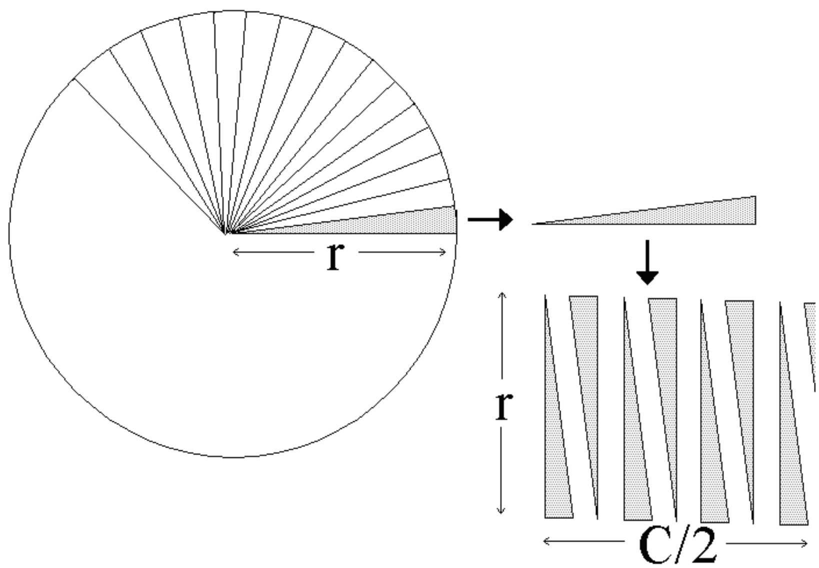


$$\begin{array}{l} C > 6 \\ \pi = C \div D > 3 \\ C \approx 6 \\ \pi \approx 6 \div D \approx 3 \end{array}$$

In fact the decimal approximation to pi is known to over a billion digits! There is a pi button on most calculators. Pressing it gives $\pi \approx 3.141592654$.

Area of a Circle

The determination of the formula for the area of a circle was done by Archimedes over 2200 years ago. To get some understanding of it, observe the following diagram. A circle is sliced up into an even number of thin slices. The slices "almost" look like triangles. The slices are then rearranged, half pointing up, the rest pointing down. Then they are pushed against each other until they just touch. The resulting figure resembles a rectangle, except the top and bottom have little bumps, but by making the number of slices bigger and bigger (this technique is called the method of "exhaustion"), the bumps get smaller and smaller until we do "get" a rectangle. The height of the rectangle is **r** and the base is half of the circumference (since half of the slices faced up and the rest down). Thus we get the following "famous" formula for the area of a circle.



$$A = r \cdot \frac{C}{2} = r \cdot \frac{2\pi r}{2}$$
$$= r\,\pi r = \pi r^2$$

Example: Determine the circumference and area of a circle of diameter 2.500 in

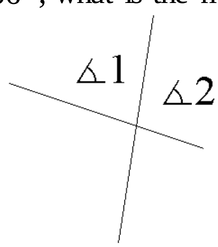
Solution: $C = \pi \times 2.500 \text{ in} = 7.854 \text{ in}$

$r = \frac{D}{2} = 1.250 \text{ in} \Rightarrow A = \pi \times (1.250 \text{ in})^2 = 4.909 \text{ in}^2$

.

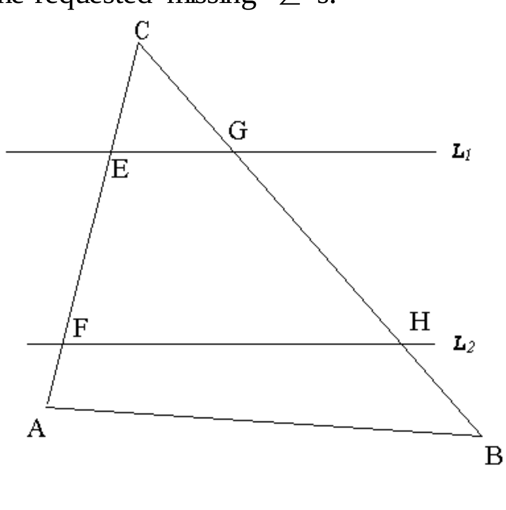
Your Turn!!

If $\angle 1 = 85°19'56"$, what is the measure of $\angle 2$?



1. $\angle 2$ =____________

If line $\boldsymbol{L_1}$ is parallel to line $\boldsymbol{L_2}$ and $\angle \mathbf{CAB} = 76°59'$ and $\angle \mathbf{CBA} = 46°29'$ and $\angle \mathbf{CEG} = 61°29'$, find the requested missing $\angle$'s.

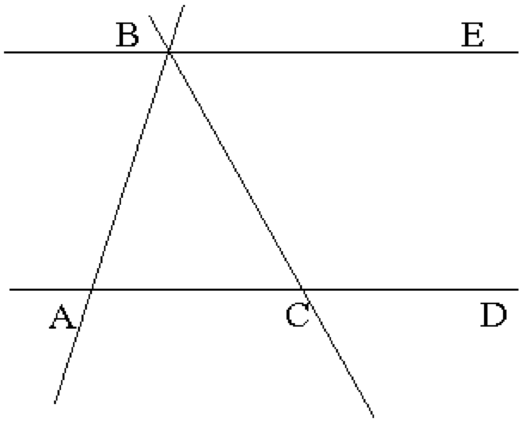


2. $\angle$ACB =____________

3. $\angle$CFH =____________

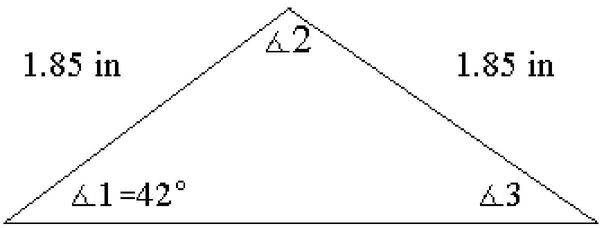
4. $\angle$CGE =____________

If **BE** is parallel to **AD** and $\angle ACB = 64°17'$ and $\angle BAC = 66°35'$, find the missing $\angle$'s.

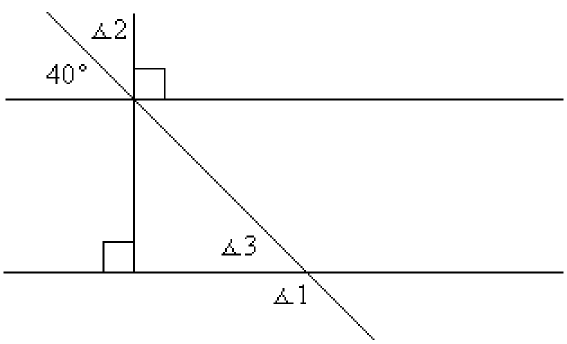


5. $\angle ABC =$____________
6. $\angle BCD =$____________
7. $\angle CBE =$____________

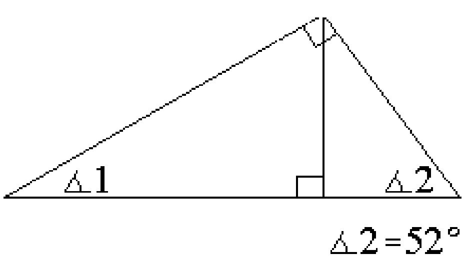
Find the measure of the missing angles.



8. $\angle 2 =$____________
9. $\angle 3 =$____________

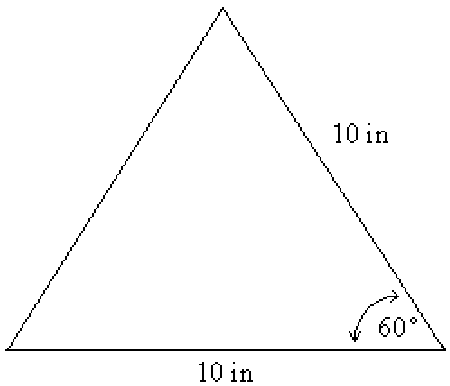


10. $\angle 1 =$____________
11. $\angle 2 =$____________
12. $\angle 3 =$____________

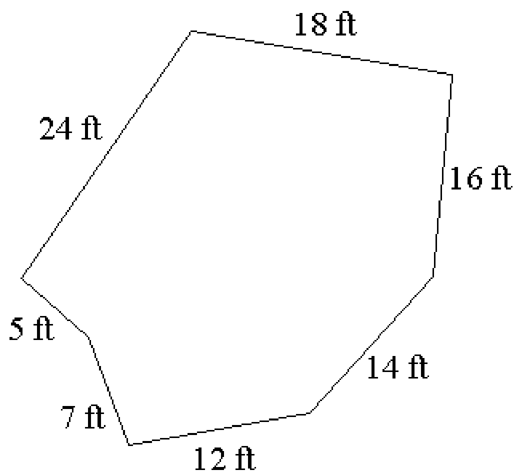


13. $\angle 1 =$____________

For each figure below calculate the distance around (perimeter) P.

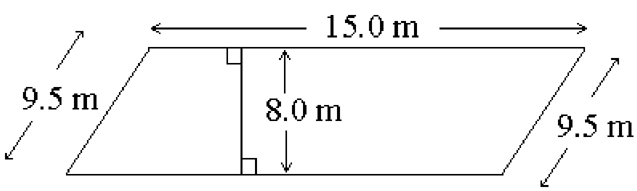


14. P =____________



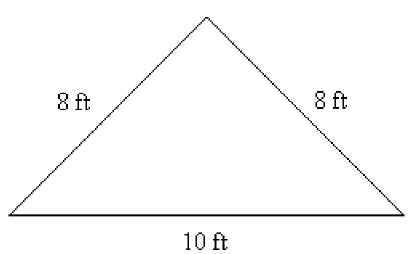
15. P =____________

For each figure below calculate both the area, A, and the distance around (perimeter or circumference), P or C.



16. P =____________

17. A =____________

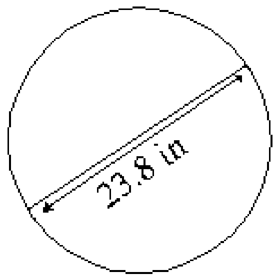


18. P =____________

19. A =____________

20. P =____________

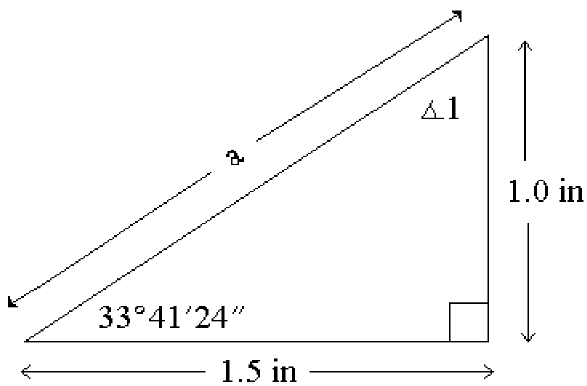
21. A =____________



22. C = ____________

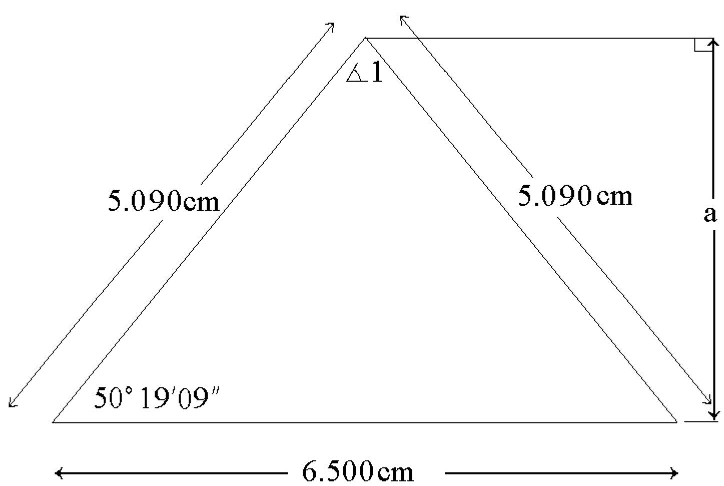
23. A = ____________

For each figure below calculate the requested missing information.



24. $\angle 1 =$ ____________

25. a = ____________



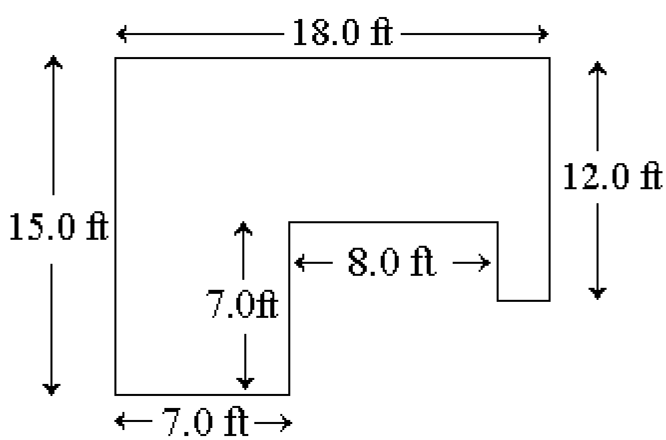
26. $\angle 1 =$ ____________

27. a = ____________

28. P = ____________

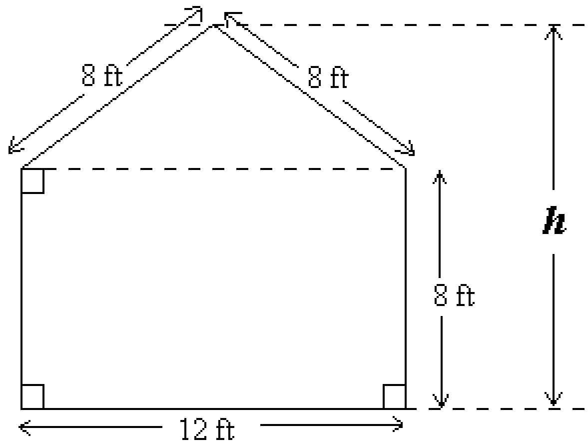
29. A = ____________

How many square feet of flooring does the following room have?



30. ______________

The cross section of a shed is shown below. Determine the height *h* above the ground and A, the total area of the building's cross section.



31. ***h*** = ____________

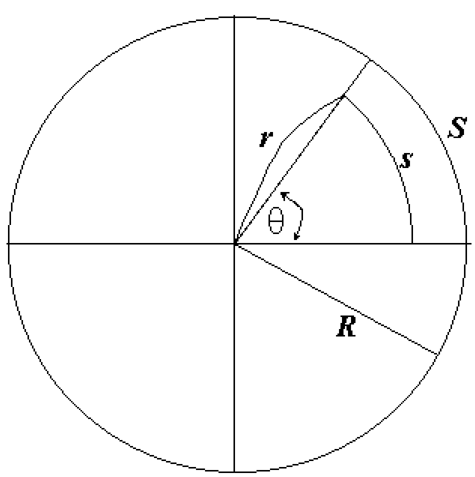
32. **A** = ____________

A 2.25 cm diameter hole is drilled in a 4.5 cm diameter circle. What area is left after the hole has been drilled?

33. ____________

Section 6.2 Radian Measure and its Applications

In addition to DMS (Degrees, Minutes, and Seconds) and DD (Decimal Degrees), there is another system for measuring angles called **radian measure**. The basis of this system is that all sectors of a circle having the same central angle θ (theta) are similar. Thus, for a given central angle, the ratio of the arc length of the sector to the radius of the circle is a constant. We define this constant as the radian measure of the angle. We say that the arc length is "cut-off" or subtended by the central angle theta at a given radius *r*.



$$\theta(\text{measured in radians}) = \frac{s}{r} = \frac{S}{R}$$

$$\text{Arc length} = \text{radius} \times \text{central angle in radians}$$
$$s = r \cdot \theta \quad (\theta \text{ measured in radians})$$

$$C = 2\pi r = r \cdot (\text{full revolution angle in radians})$$
so a full revolution of $360° = 2\pi$ radians
or more simply π rads $= 180°$

To convert decimal degree measurements to radians we use the conversion factor $1 = \dfrac{\pi \text{ radians}}{180°}$.

To convert radian measure to decimal degrees we use the conversion factor $1 = \dfrac{180°}{\pi \text{ radians}}$.

The next three exmaples use these principles.

Example: A car on a circular track of radius 0.4 miles is travelling 125 mph. In 2 seconds through what central angle measured from the center of the track does the car travel?

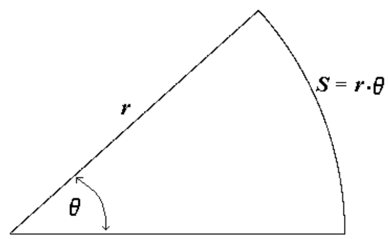
Solution: The arc length subtended by the central angle is just the distance travelled by the car. Since distance is speed multiplied by time, we have

$$S=\left(\frac{125\text{ miles}}{1\text{ hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ sec}}\right)\cdot 2\text{ sec}=.0694\text{ miles}$$

$$\theta(\text{in radians})=\frac{S}{r},\text{ so}$$

$$\theta\ (\text{in radians})=\frac{0.0694\text{ miles}}{0.4\text{ miles}}=0.174\text{ rads}=0.174\text{ rads}\cdot\frac{180°}{\pi\text{ rads}}=9.97°\approx 10°$$

Example: What is the area of the sector of a circle of diameter 12.0 m subtended by 42 degrees?
Solution: A sector of a circle is "wedge" with a vertex at the circle's center as shown below.



The fraction of a full circle that this sector represents is $\frac{42°}{360°}=0.1167$. Thus, the area will be 11.67% of the area of a full circle of radius 6.0 m: $A=0.1167\cdot\pi\cdot(6.0\text{ m})^2=13.2\text{ m}^2$.

Example: How fast is a car moving if its 34.0 in diameter tire makes 400 rpm and never slips?

Solution: If the tire never slips the distance the tire travels along the ground in one revolution is equal to the circumference of the tire. If the wheel makes 400. revolutions in one minute, then it is moving at a speed of 400 cirumferences per minute.

$$\text{Speed}=\frac{400\text{ rev}}{1\text{ min}}\cdot\frac{\pi\cdot 34.0\text{ in}}{1\text{ rev}}=42,700\frac{\text{in}}{\text{min}}=42,700\frac{\text{in}}{\text{min}}\cdot\frac{1\text{ ft}}{12\text{ in}}=3560\frac{\text{ft}}{\text{min}}$$

$$=3560\frac{\text{ft}}{\text{min}}\cdot\frac{60\text{ min}}{1\text{ hr}}\cdot\frac{1\text{ mile}}{5280\text{ ft}}=40.6\text{ mph}$$

Your Turn!!

1. Convert the following angles from degree measure to radians and revolutions. Give answers to four decimal places.

	Radians	Revolutions
$30°$ =	______________	______________
$120°$ =	______________	______________
$300°$ =	______________	______________

2. Convert the following angles from radian measure to degree measure and revolutions. Give answers to four decimal places.

		Dec Degrees	DMS	Revolutions
$\frac{\pi}{2}$	=	______________	______________	_____________
$\frac{\pi}{12}$	=	______________	______________	_____________
0.50	=	______________	______________	_____________

3. To three digits find the arclength subtended by an angle of 43.5° and a radius of 31.7 in.

4. Find the angle in both radians and degrees subtended by an arc of 40.0 cm in a circle of diameter 50.0 cm .

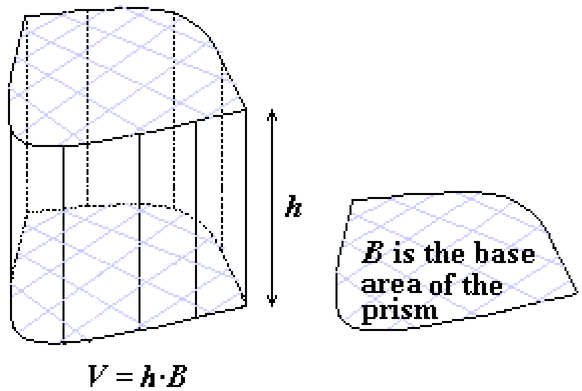
5. A car on a circular race track with a radius of 0.25 miles is travelling 110 mph. In six seconds through what central angle from the center of the track does the car turn?

6. Find the area of a sector of a circle of diameter 40.0 ft subtended by an angle of 16.0 degrees.

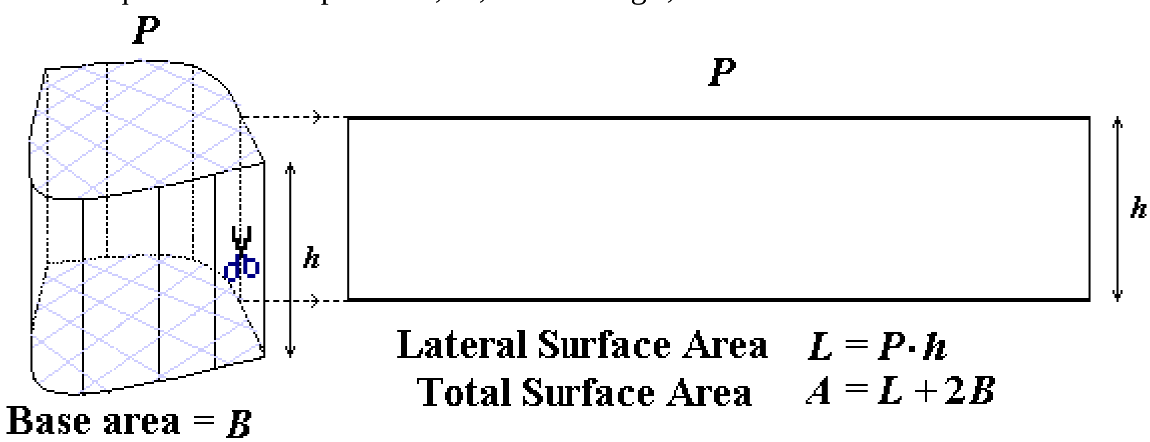
7. A bicyclist pedals at such a rate that both wheels rotate at 205 rpm. The outside wheel diameter is 26.0 in . Assuming that the tires never slip against the ground, what is the bicyclist's speed in mph?

Section 6.3 The Volume and Surface Area of a Solid

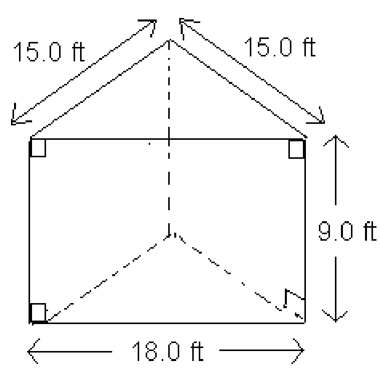
The core idea of a **volume** is an area moved through a third dimension. Consider the following prism. This is a solid figure with two parallel faces and lateral (sideways) sides perpendicular to these faces. The parallel face is called the prism's base. If the area of the base is *B* and the perpendicular height between the two parallel faces is *h*, then the volume of the prism is given by the formula $V = h \cdot B$. Volume as the product of a length times an area **must have units of cubic length,** such as cubic inches, cubic feet etc.



The area occupied by the lateral sides of the prism is called the **lateral surface area**, L . The sum of the lateral surface area with the top and bottom base areas is called the **total surface area,** A . If you imagine "cutting" along the side of the prism perpendicular to the base and then "unfolding" and laying flat the lateral surface area, the resulting figure is a rectangle with dimensions equal to the base perimeter, P, and the height, h.



Example: consider calculating the volume, lateral and total surface areas of the following prism.



Solution: The base is a triangle of sides 15.0 ft, 15.0 ft and 18.0 ft. The base area can be calculated using Heron's formula.

$$S = \frac{15.0+15.0+18.0}{2}\text{ ft} = 24.0\text{ ft}$$

$$B = \sqrt{24.0(24.0-15.0)(24.0-15.0)(24.0-18.0)}\text{ ft}^2$$

$$B = 108\text{ ft}^2$$

$$V = 108\text{ ft}^2(9.0\text{ ft}) = 972\text{ ft}^3$$

$$L = 48.0\text{ ft}(9.0\text{ f})\text{t} = 432\text{ ft}^2$$

$$A = 432\text{ ft}^2 + 2\left(108\text{ ft}^2\right) = 648\text{ ft}^2$$

An alternate approach to calculating this base area uses the fact that the base is an isosceles triangle. Dropping a perpendicular from the top vertex will then bisect the 18.0 ft side. The height of the resulting right triangle is then calculated by use of the Pythagorean Theorem.

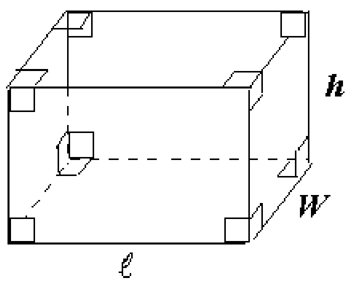
$$\sqrt{15^2 - 9^2}\ \text{ft} = \sqrt{144}\ \text{ft} = 12.0\ \text{ft}$$

$$B = \frac{1}{2} 18.0\ \text{ft} \times 12.0\ \text{ft} = 108\ \text{ft}^2$$

The formulas for the volumes and surface areas of solids, which are not prisms, require more elaborate geometric reasoning. These are included in the results presented below.

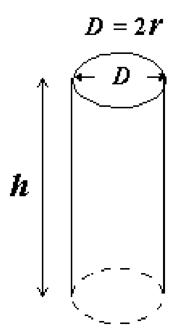
Volume and Surface Area of a Rectangular Prism

Probably the most recognizable prism is the rectangular prism or "box". The volume is simply the product of the lengths of the three sides.



$$B = \ell \cdot W$$
$$P = 2\ell + 2W$$
$$V = B \cdot h = \ell \cdot W \cdot h$$
$$L = P \cdot h = 2\ell \cdot h + 2W \cdot h$$
$$A = L + 2B = 2\ell \cdot h + 2W \cdot h + 2\ell \cdot W$$

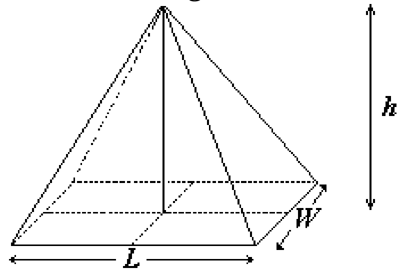
A right circular cylinder is also a prism. Here the circumference of the base circle is used in calculating the lateral surface area.



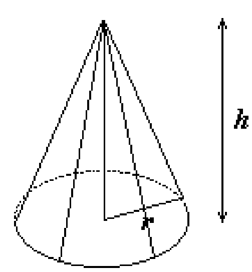
$$B = \pi r^2 = \frac{\pi D^2}{4}$$
$$C = \pi D = 2\pi r$$
$$V = B \cdot h = \pi r^2 h = \frac{\pi D^2 h}{4}$$
$$L = C \cdot h = \pi D h = 2\pi r h$$
$$A = L + 2B = 2\pi r h + 2\pi r^2$$

Volume of a Pyramid and Cone

Right **pyramids** and **cones** both have a volume equal to *one third* that of the corresponding prism having the same base and height.

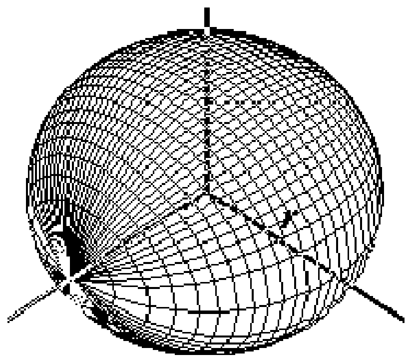


$$V = \frac{h \cdot B}{3} = \frac{h \cdot L \cdot W}{3}$$



$$V = \frac{h \cdot B}{3} = \frac{\pi r^2 h}{3}$$

Volume and Total Surface Area, A, of a Sphere



$$V = \frac{4\pi r^3}{3} = \frac{\pi D^3}{6}$$

$$A = 4\pi r^2 = \pi D^2$$

Example: A storage tank is in the shape of a right circular cylinder capped with a hemispherical dome. The inner diameter of the tank is 15.0 ft and the height from the floor to the inner top of the hemisphere is 25.0 ft. The entire inside area of the tank needs to be coated with a sealant. One gallon of this sealant is required for every 150 square ft. What is the capacity of the tank in gallons? How many gallons of sealant are required to cover the inside surface of the tank?

Solution: The radius of the hemisphere must match the radius of the cylinder, r = 7.5 ft. The height of the cylinder is the total height, 25 ft, minus this radius, which amounts to 17.5 ft. The capacity or volume of the tank is the sum of the cylinder's volume plus half of the volume of a sphere with a 7.5 ft radius. (To convert cubic ft to gallons see Chapter 5 on Measurement.)

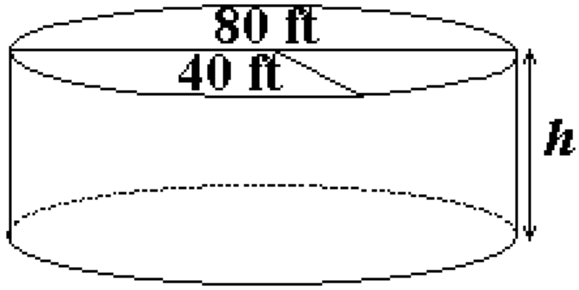
$$V = \pi r^2 h + \frac{1}{2}\cdot\frac{4\pi r^3}{3} = \pi\left(7.5\text{ ft}\right)^2 \cdot 17.5\text{ ft } + \frac{2\pi\left(7.5\text{ ft}\right)^3}{3}$$
$$= 3976\text{ ft}^3 = 3976\text{ ft}^3 \cdot \frac{1\text{ gal}}{0.13368\text{ ft}^3} = 29{,}743\text{ gal}$$

Given that the tank's dimensions are only good to three digits, we should probably report the tank's capacity as 29,700 gallons. To determine how many gallons of sealant are required we need to calculate the total internal surface area of the tank. This consists of the circular floor, the lateral surface of the cylinder and half the surface area of a sphere with a 7.5 ft radius.

$$A = \pi r^2 + 2\pi r \cdot h + \frac{1}{2}\cdot 4\pi r^2 = \pi\left(7.5\text{ ft}\right)^2 + 2\pi \cdot 7.5\text{ ft} \cdot 17.5\text{ ft } + 2\pi\left(7.5\text{ ft}\right)^2 = 1355\text{ ft}^2$$

One gallon of sealant covers 150 square feet of surface, so we will require $1355\text{ ft}^2 \times \frac{1\text{ gal}}{150\text{ ft}^2} = 9.03\text{ gal}$. However, to be safe and to cover waste 10 gallons should probably be ordered.

Example: As a variation on the above problem, suppose we were designing a cylindrical storage tank (no hemispherical cap this time!) to have a capacity of 1 million gallons and a base diameter of 80 ft, how tall must the tank be?



Solution: First, we convert the million gallons into cubic feet.

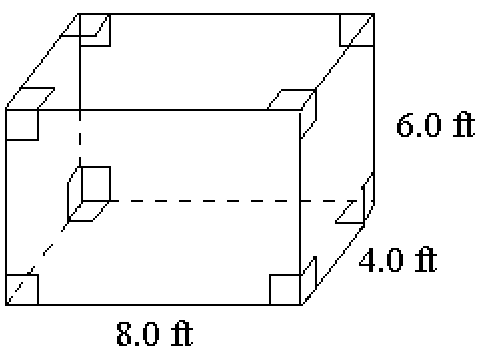
$$1{,}000{,}000 \text{ gal} = 10^6 \text{ gal} \cdot \frac{0.13368 \text{ ft}^3}{1 \text{ gal}} = 133{,}680\text{ft}^3 = 134{,}000 \text{ ft}^3$$

Then from the formula for the volume we solve for the height h.

$$V = B \cdot h = \pi r^2 \cdot h \quad \Rightarrow \quad h = \frac{V}{\pi r^2} = \frac{134000 \text{ ft}^3}{\pi (40 \text{ ft})^2} = 26.7\text{ft}$$

Your Turn!!

Find the lateral surface area, L, the total surface area, A, and the the volume, V, of the following solids.1.

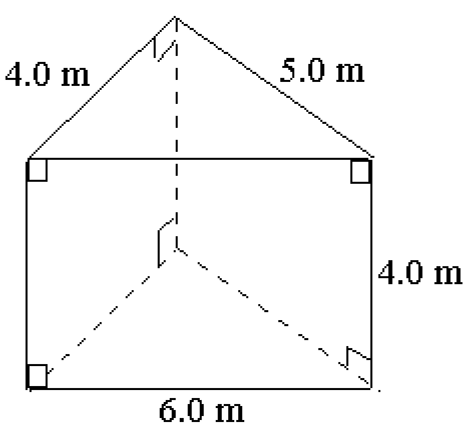


$L =$ ______________

$A =$ ______________

$V =$ ______________

2.

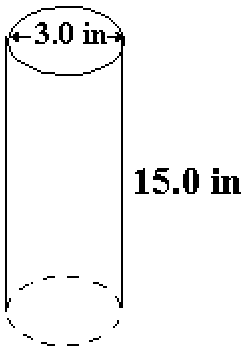


$L =$ ______________

$A =$ ______________

$V =$ ______________

3.

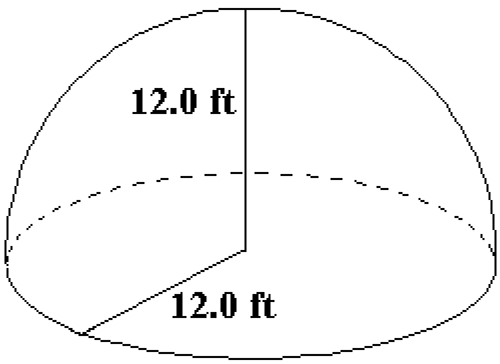


$L =$ ______________

$A =$ ______________

$V =$ ______________

4.



$L =$ ______________

$A =$ ______________

$V =$ ______________

A cylindrical holding tank has an inner diameter of 40.0 ft and walls that are 18 in thick. The tank is designed to hold 350,000 gallons.

5. What is the tank's inside perimeter?

6. What is the tank's outside perimeter?

7. What is the area of the tank floor?

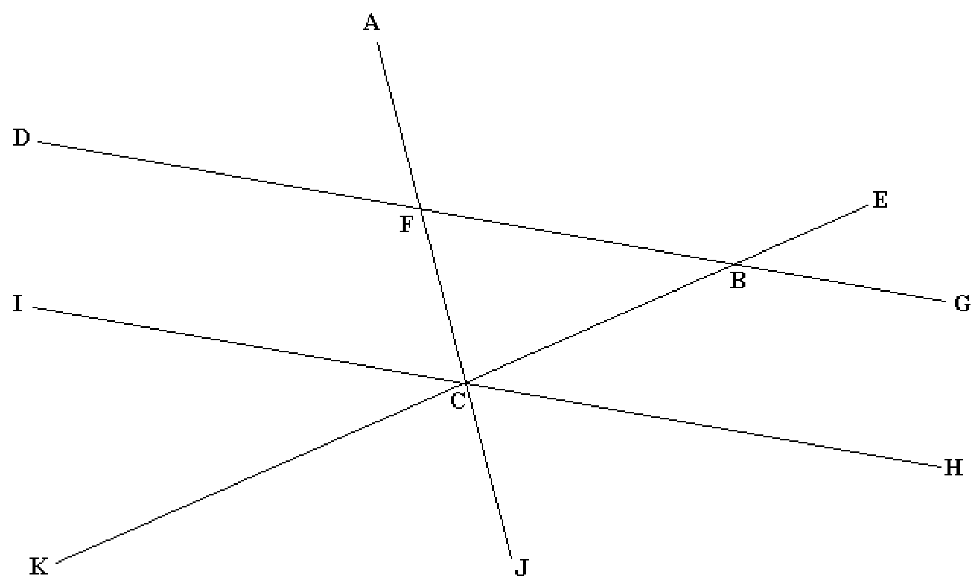
8. What is the height of the tank in feet?

9. A swimming pool is in the shape of a trapezoidal prism. The shallow end is 3.0 ft deep, the deep end is 8.0 ft, the width of the pool is 25 ft and the length from the shallow end to the deep end is 50 ft .How many gallons of water does it take to fill the pool?

Chapter 6 Sample Test ______________/100

All problems are each worth 5 points. Give answers to the correct number of significant digits.

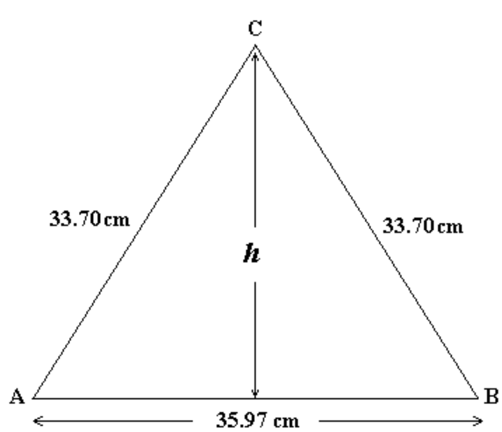
1. If the line passing through the segment **DG** is parallel to the line passing through the segment **IH**, $\angle DFA = 58°43'$, and $\angle BCH = 43°29'$, determine the requested missing angles.



1. $\angle AFB =$ ____________ 2. $\angle FBC =$ ____________

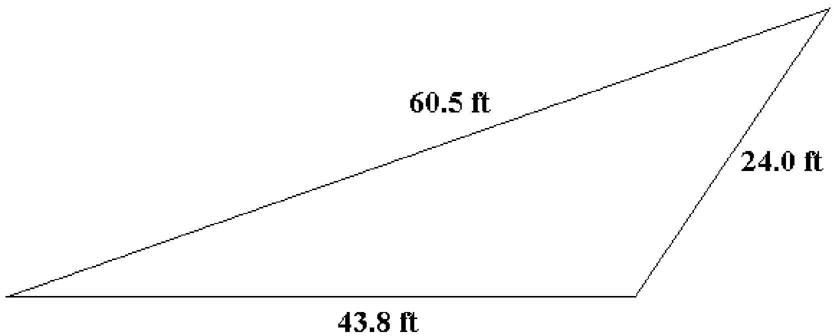
3. $\angle FCB =$ ____________ 4. $\angle KCJ =$ ____________

In triangle **ABC**, $\angle$**CBA** = $57°44'44''$. Determine the measure of $\angle$**ACB** to the nearest minute



5. The measure of $\angle$**ACB** to the nearest minute = ____________
6. The perimeter of triangle **ABC** to the nearest .01 cm = ____________
7. The altitude ***h*** to the nearest .01 cm = ____________
8. The area of triangle **ABC** to the nearest sq cm = ____________

Calculate both the perimeter and area of the following triangle. Give the perimeter to the nearest tenth of a ft and the area to the nearest square foot.



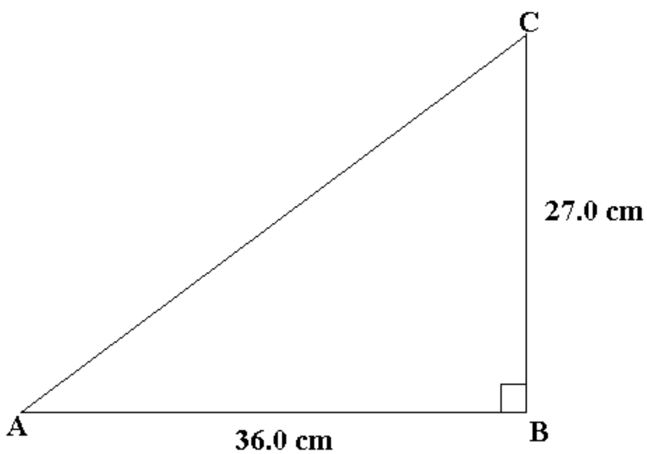
9. P = ____________

10. A = ____________

A circle has a diameter of 11.375 inches. Determine the circumference to the nearest thousandth of an inch and the area to the nearest 0.1 sq in .

11. C = ____________ 12. A = ____________

In triangle **ABC**, $\angle$**ACB** = $53°07'48''$ determine the requested missing information



13. $\angle$CAB to the nearest second = ____________

14. Length of **AC** to the nearest 0.1 cm = ____________

15. Area of triangle **ABC** to the nearest sq cm = ____________

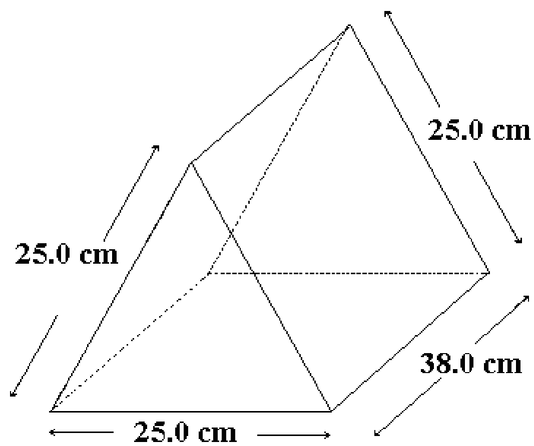
16. A circle has a diameter of 12.0 in . Calculate to the nearest hundredth of an inch the arclength of a sector of this circle which is subtended by an angle of $17°30'$.

Arclength = ____________________

17. A bicyclist pedals at such a rate that both wheels rotate at 235. rpm. The outside wheel diameter is 26.0 in . Assuming that the tires never slip against the ground, what is the bicyclist's speed in mph ?

Speed = ________________ (to the nearest tenth)

Find the total surface area, A, and the the volume, V, of the following prism.



18. A to the nearest square cm = ____________

19. V to the nearest tenth of a liter = ____________

A cylindrical holding tank has an inner diameter of 50.0 ft. The tank is designed to hold 500,000 gallons.

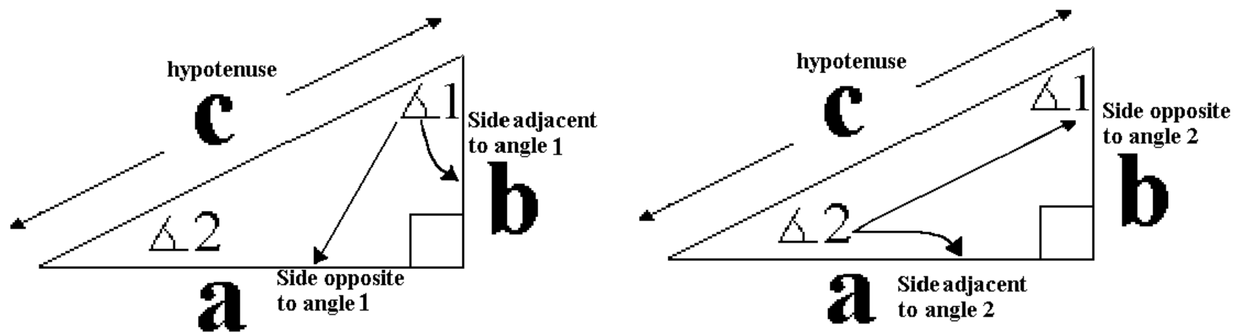
20. Determine the height of the tank to the nearest foot. Height = ________________

Chapter 6 Sample Test Solutions

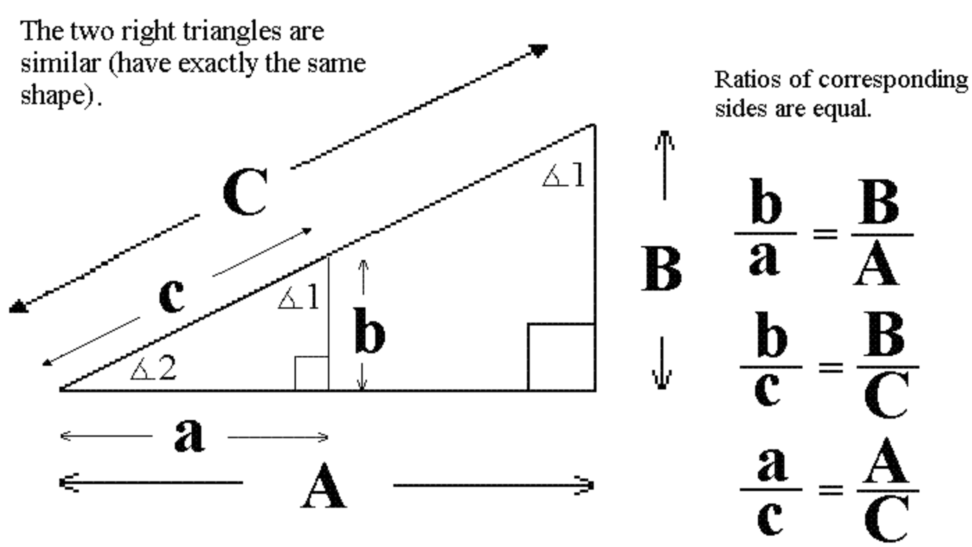
1.	**2.**	**3.**	**4.**
$\angle\mathbf{AFB} = 121°17'$	$\angle\mathbf{FBC} = 43°29'$	$\angle\mathbf{FCB} = 77°48'$	$\angle\mathbf{KCJ} = 77°48'$
5.	**6.**	**7.**	**8.**
$\angle\mathbf{ACB} = 64°31'$	P = 103.37 cm	h = 28.50 cm	A = 513 cm^2
9.	**10.**	**11.**	**12.**
P = 128.3 ft	A = 437 ft^2	C = 35.736 in	A = 101.6 in^2
13.	**14.**	**15.**	**16.**
$\angle\mathbf{CAB} = 36°52'12''$	**AC** = 45.0 cm	A = 486 cm^2	Arclength = 1.83 in
17.	**18.**	**19.**	**20.**
Speed = 18.2 mph	A = 3391 cm^2	V = 10.3 L	Height = 34.0 ft

Section 7.1 Sine, Cosine and Tangent

In this section we will restrict our attention to right triangles. With respect to the acute angles (the two angles less than 90 degrees) in a right triangle, we can classify the two legs (the sides which are not the largest side, the hypotenuse) as being either **opposite** to or **adjacent** to the given angle. This is shown below in the following two diagrams.



Since angles 1 and 2 are complementary, the value of one angle determines the second. Two triangles with the same set of angles are similar. They must have exactly the same shape. This shape then depends entirely on the value of one of the two acute angles. We say the shape is a "function" of the acute angle. The angle determines the shape and hence the various proportions between the triangle's three sides. Thus these proportions are functions of the acute angle. We call these functions trigonometric functions (trig functions for short) and for a given acute angle there are three of special importance. These are the "sine", "cosine" and "tangent" functions.



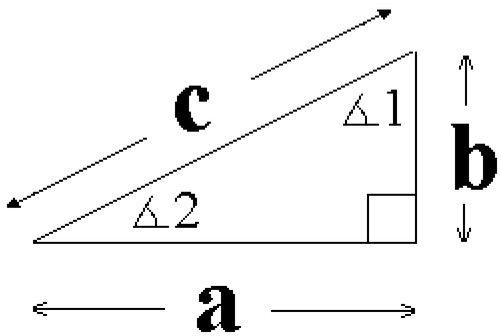
The common abbreviation for sine is "sin" (still pronounced sine). The cosine means the sine of the complement (see the picture above) and is abbreviated "cos", while tangent is written as "tan". Standard notation is to separate the name of the trig function from the name of the acute angle (called the "argument" of the function) with parenthesis. The idea is that, if you "input" the value of the acute angle to the trig function, it "outputs" the requested ratio of sides.

Definition of the Trigonometric Functions

$$\sin(\text{acute } \angle) = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos(\text{acute } \angle) = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan(\text{acute } \angle) = \frac{\text{opposite side}}{\text{adjacent side}}$$

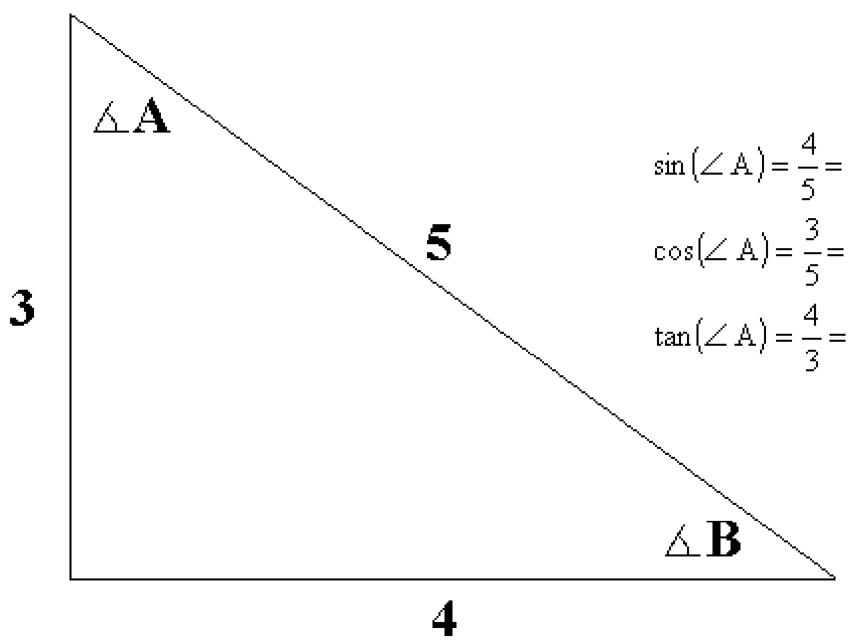


$$\sin(\angle 2) = \frac{b}{c} \quad ; \quad \sin(\angle 1) = \frac{a}{c} = \cos(\angle 2)$$

$$\cos(\angle 2) = \frac{a}{c} \quad ; \quad \cos(\angle 1) = \frac{b}{c} = \sin(\angle 2)$$

$$\tan(\angle 2) = \frac{b}{a} \quad ; \quad \tan(\angle 1) = \frac{a}{b} = \frac{1}{\tan(\angle 2)}$$

Consider the "famous" 3, 4, 5 right triangle. The values of the three trig functions are computed below for both of the acute angles A and B.



$$\sin(\angle A) = \frac{4}{5} = 0.8 \quad ; \quad \sin(\angle B) = \frac{3}{5} = 0.6$$

$$\cos(\angle A) = \frac{3}{5} = 0.6 \quad ; \quad \cos(\angle B) = \frac{4}{5} = 0.8$$

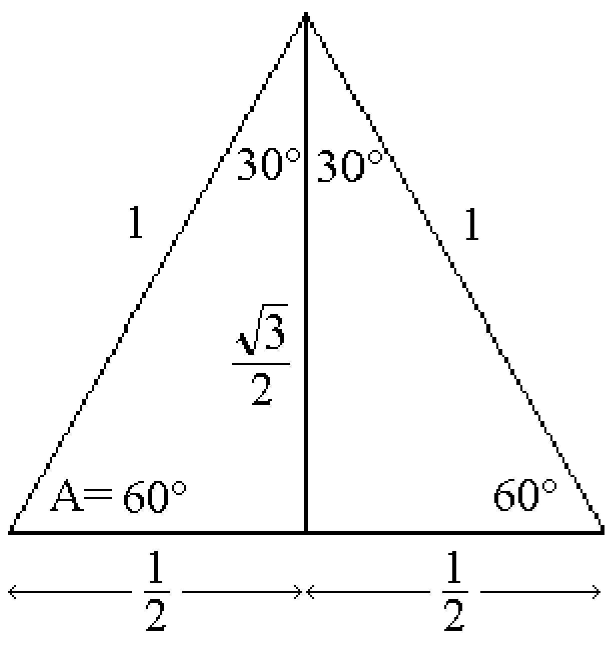
$$\tan(\angle A) = \frac{4}{3} = 1.333... \quad ; \quad \tan(\angle B) = \frac{3}{4} = 0.75$$

To determine the values of these acute angles we need the "inverse" trig functions. A given acute angle uniquely determines the shape and thus the values of the ratios of the sides). So we can reverse this process and determine the acute angle from a given ratio of the sides. The angle is then a function of the ratio. These functions are most commonly designated as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. On most calculators they are accessed by using a **2nd**, **SHIFT**, or **INV** key in combination with the primary trig function key. Using the inverse trig functions, angle A in the 3, 4, 5 triangle can be calculated by the various methods illustrated below. The results are shown to the nearest ten thousandth decimal degree and to the nearest second.

$$A = \sin^{-1}(0.8) = 53.1301^\circ = 53^\circ 07' 48''$$
$$A = \cos^{-1}(0.6) = 53.1301^\circ = 53^\circ 07' 48''$$
$$A = \tan^{-1}\left(\frac{4}{3}\right) = 53.1301^\circ = 53^\circ 07' 48''$$
$$A = 90^\circ - B = 90^\circ - \sin^{-1}(0.6) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$
$$A = 90^\circ - B = 90^\circ - \cos^{-1}(0.8) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$
$$A = 90^\circ - B = 90^\circ - \tan^{-1}(0.75) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$

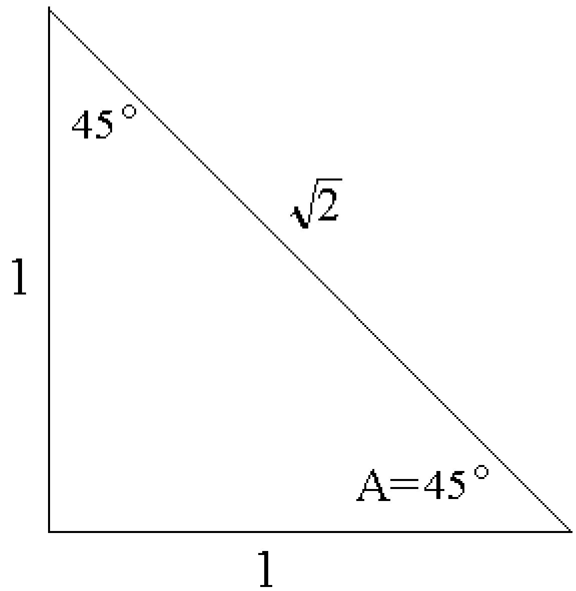
The following two examples using “special” triangles confirm that the inverse trig functions give the correct answer for the appropriate angle.

Consider an equilateral triangle of side 1. Dropping a perpendicular from the top vertex to the opposite side creates two right triangles with legs $\frac{1}{2}$ and $\sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$, and a hypotenuse of 1.



$$\sin(A) = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} = 0.8660254$$
$$\cos(A) = \frac{\frac{1}{2}}{1} = \frac{1}{2} = 0.5$$
$$\tan(A) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = 1.7320508$$
$$A = \sin^{-1}(0.8660254) = 60.0000^\circ$$
$$A = \cos^{-1}(0.5) = 60.0000^\circ$$
$$A = \tan^{-1}(1.7320508) = 60.0000^\circ$$

Now consider an isosceles right triangle with legs of length 1 and a hypotenuse of $\sqrt{1^2+1^2} = \sqrt{2}$.



$$\sin(A) = \frac{1}{\sqrt{2}} = 0.707106781187$$
$$\cos(A) = \frac{1}{\sqrt{2}} = 0.707106781187$$
$$\tan(A) = \frac{1}{1} = 1$$
$$A = \sin^{-1}(0.707106781187) = 45.0000°$$
$$A = \cos^{-1}(0.707106781187) = 45.0000°$$
$$A = \tan^{-1}(1) = 45.0000°$$

Your Turn!!

1. Convert the following angles from decimal degrees to DMS notation.

DMS

$36.35° = $ ____________

$152.19° = $ ____________

$262.25° = $ ____________

$0.157° = $ ____________

2. Convert the following angles from DMS notation to decimal degrees with four decimal places.

Decimal Degree

$15°42'19'' = $ ____________

$121°\ 37'53'' = $ ____________

$1°16'29'' = $ ____________

$3'14'' = $ ____________

For each angle give the values of the three trig functions to four decimal places.

3. $\theta = 49.76°$

$\cos(\theta) = $ ____________

$\sin(\theta) = $ ____________

$\tan(\theta) = $ ____________

4. $\theta = 16°11'49''$

$\cos(\theta) = $ ____________

$\sin(\theta) = $ ____________

$\tan(\theta) = $ ____________

5. Given that $\sin(\theta) = 0.55$ and $0° < \theta < 90°$, determine the following:

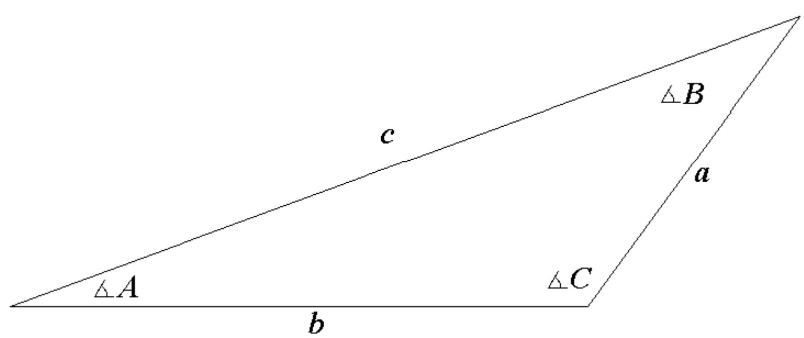
	DD (4 places)	DMS
$\theta =$	____________	____________
$\cos(\theta) =$	____________	
$\sin(\theta) =$	____________	
$\tan(\theta) =$	____________	

6. Given that $\tan(\theta) = 1.07$ and $0° < \theta < 90°$, determine the following:

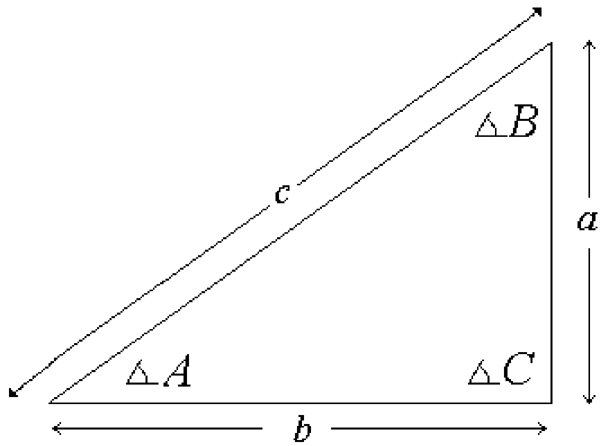
	DD (4 places)	DMS
$\theta =$	____________	____________
$\cos(\theta) =$	____________	
$\sin(\theta) =$	____________	
$\tan(\theta) =$	____________	

Section 7.2 Solving Right Triangles

With the three trig functions and their inverses we can solve for the missing information (lengths of sides and measure of angles) in any right triangle provided that we know either two sides or one side and one of the acute angles. To facilitate such problems the following notation will be used to describe any triangle. The **CAPITAL** letters ***A***, ***B***, and ***C*** will designate the three angles. The corresponding lower case letter will represent the side opposite to each angle. This is illustrated in the figure below.



For a right triangle $\angle C = 90°$ and c is the hypotenuse.



In **any** triangle the largest side is opposite the largest angle and the smallest side is opposite the smallest angle.

The following three examples demonstrate the solution of right triangle problems.

Example:

a	= ________	b	= ________	c	= 7.56 m
A	= 36.5°	B	= ________	C	= 90°

Solution: Since angles A and B are complementary, $B = 90° - 36.5° = 53.5° = 53°30'$.
From the definition of the sine,

$$\sin(36.5°) = \frac{\text{opp}}{\text{hyp}} = \frac{a}{7.56\text{ m}} \Rightarrow \text{solving for } a, \quad \frac{a \cdot 7.56\text{ m}}{7.56\text{ m}} = 7.56\text{ m} \cdot \sin(36.5°) = 4.497\text{ m}$$

so $a = 4.497$ m.
The length of side b can now be determined using the cosine function.

$$\cos(36.5°) = \frac{\text{adj}}{\text{hyp}} = \frac{b}{7.56\text{ m}} \Rightarrow \text{solving for } b, \quad \frac{b \cdot 7.56\text{ m}}{7.56\text{ m}} = 7.56\text{ m} \cdot \cos(36.5°) = 6.077\text{ m}$$

so $b = 6.077$ m

There are two other methods of solution for b which use the calculated value of a and either the Pythagorean Theorem or the tangent function. However, any method which uses calculated values rather than the initial "given" values (the values of c, C, and A) risks unintended inaccuracy due to rounding errors. Since the solution shown above uses **only** the given information, it is the preferred method of solution.

Example:

a	= 12.0 ft	b	= ________	c	= ________
A	= ________	B	= 19°23′39″	C	= 90°

Solution: Angle A is the complement of angle B, so
$A = 90° - B = 90° - 19°23'39'' = 89°59'60'' - 19°23'39'' = 70°36'21''$.
Next we use the tangent function to calculate the length of side b, then the cosine to calculate the hypotenuse, c.

$$19°23'39'' = 19.39417°$$

$$\tan(19.39417°) = \frac{b}{12.0\text{ ft}} \Rightarrow b = 12.0\text{ ft} \cdot \tan(19.39417°) = 4.224\text{ ft}.$$

$$\cos(19.39417°) = \frac{a}{c} = \frac{12.0\text{ ft}}{c} \Rightarrow c \cdot \cos(19.39417°) = \frac{12.0\text{ ft} \cdot \cancel{c}}{\cancel{c}} \Rightarrow c = \frac{12.0\text{ ft}}{\cos(19.39417°)} = 12.722\text{ ft}.$$

Again other solution methods can be used, but the approach given above solves for all unknowns solely in terms of the initial given information.

Example:

a	= 23.3 cm	b	= ___________	c	= 37.8 cm
A	= ___________	B	= ___________	C	= 90°

Solution: First use the Pythagorean Theorem to calculate the length of side b, then use the inverse sine to calculate one of the missing acute angles.

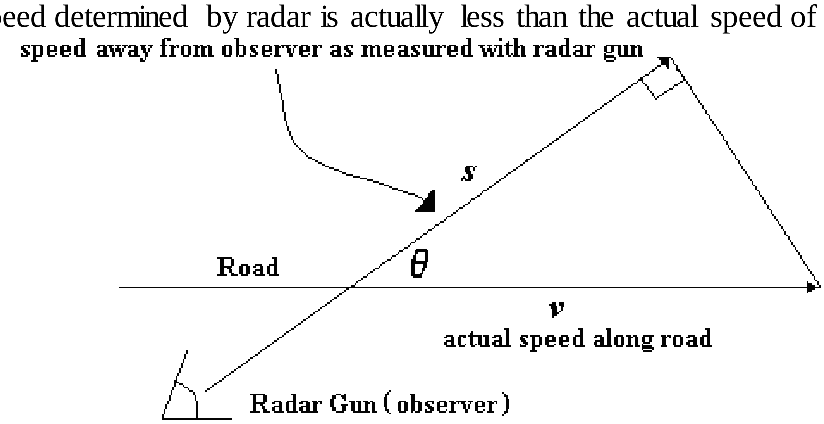
$$b = \sqrt{c^2 - a^2} = \sqrt{37.8^2 - 23.3^2}\text{ cm} = 29.7649\text{ cm}.$$

$$\sin(A) = \frac{a}{c} = \frac{23.3}{37.8} = 0.61640 \Rightarrow \quad A = \sin^{-1}(0.61640) = 38.0539° = 38°03'14''.$$

$$B = 90° - A = 90° - 38.0539° = 51.9461° = 51°56'46''$$

In the above examples **more** digits were reported in the computed answers than the precision of the given data warranted. This was done to illustrate the use of the calculator in solving these problems and to avoid round off discrepancies when comparing different approaches. In practice, the computed answers would be reported to a fewer number of digits.

In law enforcement the solution of right triangles is used to calculate the actual speed of a vehicle from the radar reading of its speed. This is often referred to as the “cosine angle error”. Consider the diagram shown below. The angle between the direction of the radar gun and the road along which the vehicle is moving is given as theta. The radar gun measures the vehicle’s speed along its direction of sight. So unless theta is zero (i.e., unless the gun is aimed parallel to the road), the speed determined by radar is actually less than the actual speed of the vehicle.



Let s stand for the speed measured with the radar gun and let v be the actual speed of the vehicle along the road. Then from trigonometry we have the relations stated below.

$$s = v \cdot \cos(\theta) \quad \text{or} \quad v = \frac{s}{\cos(\theta)}$$

For example, if a the radar gun makes a 20 degree angle with the road and reads 85 mph, the actual speed of the car road is calculated as follows:

$$v = \frac{85 \text{ mph}}{\cos(20°)} = 90.5 \text{ mph}$$

Similarly, if the car is travelling at 65 mph and the radar gun makes a 45 degree angle with the road, then the speed as read from the radar unit is given by

$$s = 65 \text{ mph} \cdot \cos(45°) = 46.0 \text{ mph}$$

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. $a = 16.0$ cm $\quad b = 12.0$ cm $\quad c =$ __________
   $A =$ __________ $\quad B =$ __________ $\quad C = 90.00°$

2. $a =$ __________ $\quad b =$ __________ $\quad c = 5.0\,0$ cm
   $A = 40.00°$ $\quad B =$ __________ $\quad C = 90.00°$

3. $a = 25.4$ ft $\quad b =$ __________ $\quad c =$ __________
   $A = 17.21°$ $\quad B =$ __________ $\quad C = 90.00°$

4. A radar gun makes a 32 degree angle with a road and measures a speed of 59.4 mph. How fast is the car actually traveling?

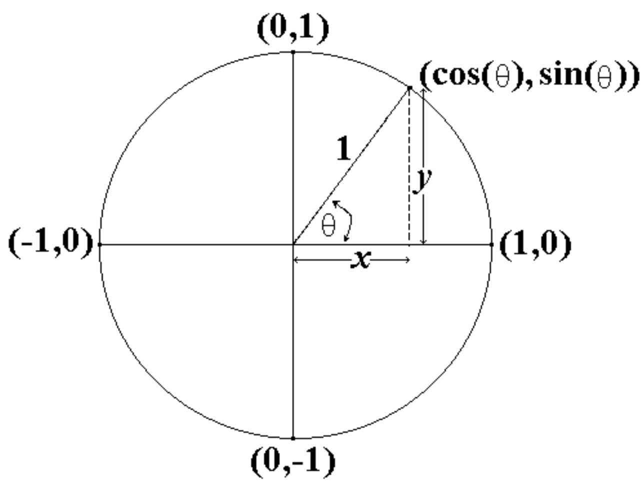
5. If a radar gun is positioned at a 35 degree angle with respect to the road and the speed limit is 35 mph, what minimum radar reading means an actual speed more than 10 mph over the speed limit?

6. If a car's actual speed is 60 mph, fill in the following table. Give answers to the nearest tenth of a mile per hour.

Angle between direction of radar gun and road	**Radar reading of car's speed**
$0°$	
$10°$	
$20°$	
$30°$	
$40°$	
$50°$	
$60°$	
$70°$	
$80°$	
$90°$	

Section 7.3 The Law of Sines and the Law of Cosines

In order to apply the trigonometric functions to a wider variety of problems we need ways of solving oblique (i.e., non-right) triangles. The two tools that allow us to proceed are the Law of Sines and the Law of Cosines. However, we first need to extend the definition of the trig functions from acute angles to any angle. To do this we construct a circle of radius 1 centered at the origin of an $x-y$ set of coordinates. As shown below this circle crosses the axes at the points $(1,\ 0)$, $(0,\ 1)$, $(-1,\ 0)$, $(0,\ -1)$.



From the origin construct a line segment that intersects the circle at a point with coordinates (x, y). The angle theta is measured from the positive x axis to this segment. The trig functions are then defined as follows:

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0$$

If both x and y are positive theta is an acute angle and the definitions given above are just the results for a right triangle with adjacent leg x, opposite leg y, and a hypotenuse of 1. If either x or y is negative or zero, then theta is not an acute angle and we obtain an extension of our earlier definitions. For example, we now have the following results.

$\cos(270°) = 0$	$\cos(180°) = -1$	$\cos(0°) = 1$	$\cos(90°) = 0$
$\sin(270°) = -1$	$\sin(180°) = 0$	$\sin(0°) = 0$	$\sin(90°) = 1$
$\tan(270°)$ undefined	$\tan(180°) = 0$	$\tan(0°) = 0$	$\tan(90°)$ undefined

If an angle is negative, it is measured "down" from the x axis, i.e., the angle moves "clockwise" around the circle. For negative inputs, the inverse sine and inverse tangent functions will return an angle between negative ninety degrees and zero degrees, meaning that the angle is in that part of the coordinate system where x is positive and y is negative. On the other hand, for a negative input, the inverse cosine function will return an angle between positive ninety degrees and positive one hundred eighty degrees (that is, an obtuse angle). This means that such an angle is in that part of the coordinate system where x is negative and y is positive. **Note:** Even though the inverse cosine function can return an obtuse angle, the inverse sine **cannot** do so. Thus, we get the following results.

$$\cos^{-1}(-0.707106781) = 135°$$
$$\sin^{-1}(-0.707106781) = -45°$$
$$\tan^{-1}(-1) = -45°$$

As the diagram illustrates if

$\cos(\theta) = -x$

$\sin(\theta) = y$

then

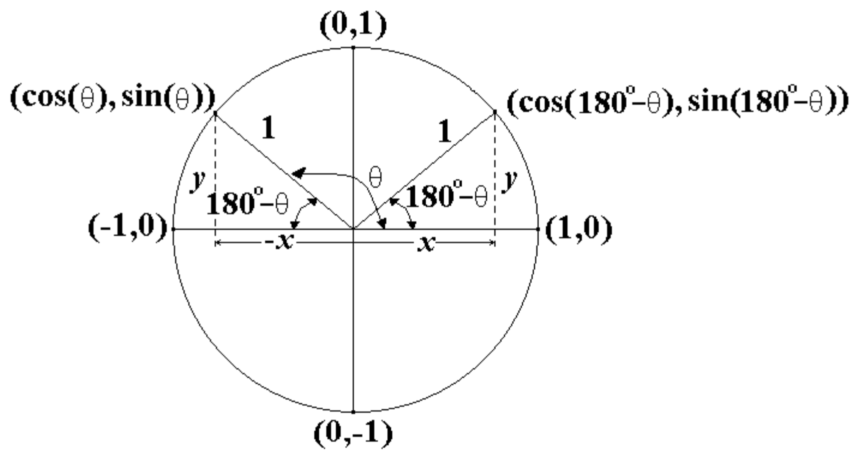
$\cos(180° - \theta) = x$

$\sin(180° - \theta) = y$

or

$\cos(180° - \theta) = -\cos(\theta)$

$\sin(180° - \theta) = \sin(\theta)$.



Finally, by the Pythagorean Theorem, for any point with coordinates (x, y) on this circle, we have the equation.

$$x^2 + y^2 = 1$$

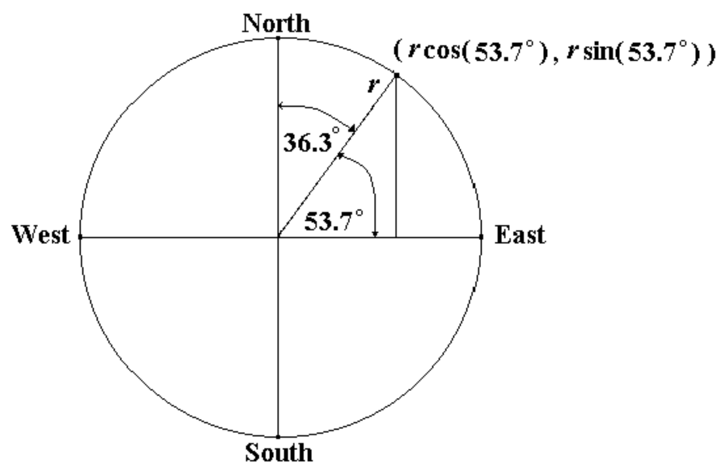
or for any angle θ

$$[\cos(\theta)]^2 + [\sin(\theta)]^2 = 1 \ .$$

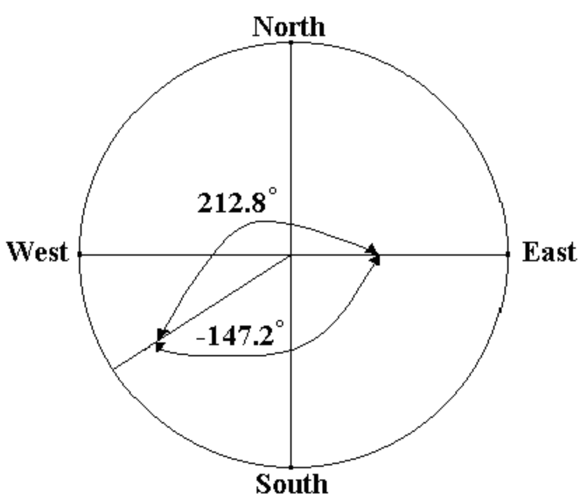
The use of trig functions defined for all angles is useful in navigation. The standard notation is to take North as the direction of the positive y axis and East as the direction of the positive x axis. If the angle theta is measured from East, then the positions east and north of the origin (the starting position) are given respectively by

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta)$$

Where *r* is the distance from the origin. If *x* is negative, the position is to the west and if *y* is negative the position is to the south. A special notation is also used to specify angles. For example, in the diagram below the orientation could be specified as a direction-angle-direction such as $\mathrm{E}53.7°\mathrm{N}$ or $\mathrm{N}36.3°\mathrm{E}$. This is also sometimes stated as 53.7 degrees north of east or 36.3 degrees east of north.



In the diagram below the orientation could be specified in any of the following equivalent ways.



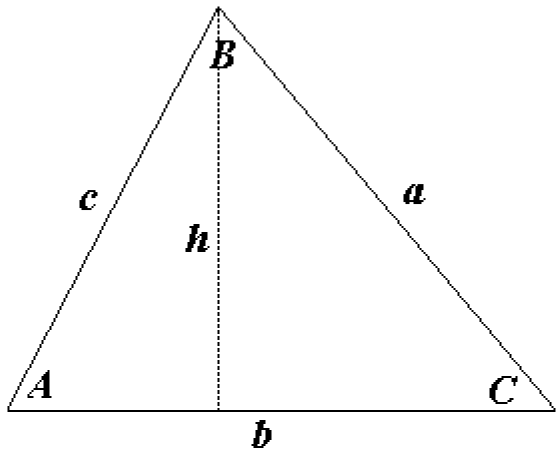
$$\theta = 212.8°$$
$$\theta = -147.2°$$
$$\mathrm{W}(212.8° - 180°)\mathrm{S} = \mathrm{W}32.8°\mathrm{S}$$
$$\mathrm{S}(147.2° - 90°)\mathrm{W} = \mathrm{S}(57.2°)\mathrm{W}$$

Example: A plane flies $\mathrm{W}18.6°\mathrm{N}$ at a speed of 347 mph for 33 minutes, how far west and north has the plane traveled?
Solution:

$$\theta = 180° - 18.6° = 161.4°$$
$$r = 347 \frac{\text{miles}}{\text{hr}} \cdot 33\,\text{min} \cdot \frac{1\text{ hr}}{60\text{ min}} = 191\text{ miles}$$
$$x = r \cdot \cos(\theta) = -181\text{ miles}$$
$$x = r \cdot \sin(\theta) = 60.9\text{ miles}$$

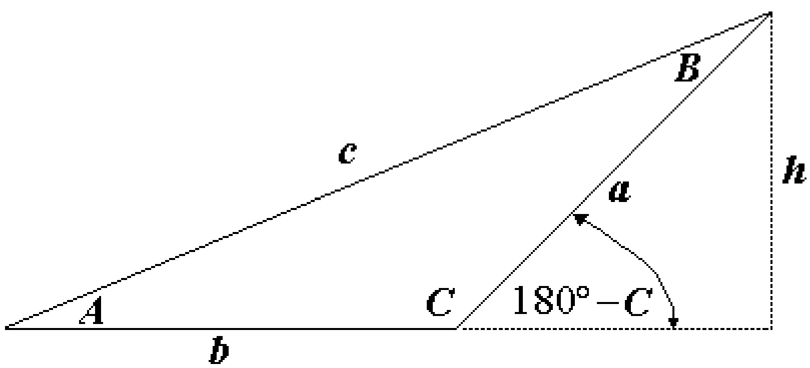
So the plane has traveled a total distance of 191 miles. It is 181 miles west and 60.9 miles north of its original position.

The Law of Sines can be explained by the following argument. Consider an acute (i.e., the largest angle is less than a right angle) triangle. Drop a perpendicular of length h from the top vertex to the bottom base.



$$h = c \cdot \sin(A)$$
$$h = a \cdot \sin(C)$$
$$\text{So } c \cdot \sin(A) = a \cdot \sin(C)$$
$$\frac{\cancel{c} \cdot \sin(A)}{a \cdot \cancel{c}} = \frac{\cancel{a} \cdot \sin(C)}{\cancel{a} \cdot c}$$
$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

Since the labeling of the sides and opposite angles is completely arbitrary this last ratio must also equal the ratio of the sine of angle B to the length of side b. If the triangle were obtuse (i.e., the largest angle is larger than a right angle) the same result is obtained as shown below.

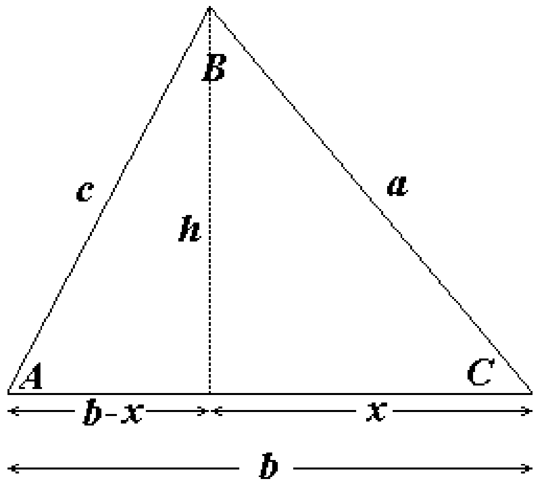


$$h = c \cdot \sin(A)$$
$$h = a \cdot \sin(180^\circ - C)$$
$$\text{but, } \sin(180^\circ - C) = \sin(C)$$
$$\text{So } c \cdot \sin(A) = a \cdot \sin(C)$$
$$\frac{\cancel{c} \cdot \sin(A)}{a \cdot \cancel{c}} = \frac{\cancel{a} \cdot \sin(C)}{\cancel{a} \cdot c}$$
$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

Thus, for **any** triangle we have that the ratio of the sine of any internal angle to the length of the opposite side is a constant for **that** specific triangle. The Law of Sines is summarized below.

$$\boxed{\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad \text{or} \quad \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}}$$

To justify the Law of Cosines again consider the acute triangle shown. Drop a perpendicular of length h to the bottom base. Using the Pythagorean Theorem on both of the right triangles formed yields the following equations.



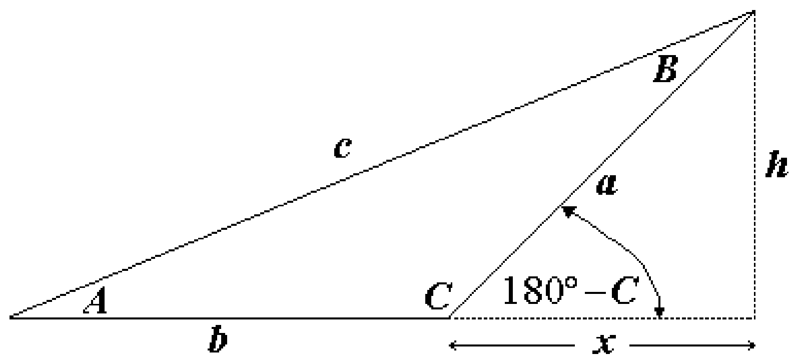
$$
\begin{aligned}
c^2 &= (b-x)^2 + h^2 \\
(b-x)^2 &= (b-x)\cdot(b-x) \\
&= (b-x)\cdot b + (b-x)\cdot(-x) \\
&= b\cdot(b-x) + (-x)\cdot(b-x) \\
&= b\cdot b + b\cdot(-x) + (-x)\cdot b + (-x)\cdot(-x) \\
&= b^2 - b\cdot x - x\cdot b + x^2 = b^2 - 2b\cdot x + x^2
\end{aligned}
$$

But, $x = a\cdot\cos(C)$ and $x^2 + h^2 = a^2$

So $c^2 = b^2 - 2b\cdot x + x^2 + h^2 = b^2 - 2b\cdot a\cdot\cos(C) + a^2$

$c^2 = a^2 + b^2 - 2a\cdot b\cdot\cos(C)$

Again since the labeling of sides and angles was arbitrary, similar results for the square of sides *a* and *b* must also be true. If the triangle is obtuse, we still obtain the same results as shown below.



$$
\begin{aligned}
c^2 &= (b+x)^2 + h^2 \\
(b+x)^2 &= (b+x)(b+x) = b(b+x) + x(b+x) \\
&= b\cdot b + b\cdot x + b\cdot x + x\cdot x = b^2 + 2b\cdot x + x^2
\end{aligned}
$$

but, $x = a\cdot\cos(180° - C) = -a\cos(C)$ and $x^2 + h^2 = a^2$.

So, $c^2 = b^2 + 2bx + x^2 + h^2 = b^2 + 2b\cdot(-a\cos(C)) + a^2$

$= a^2 + b^2 - 2a\cdot b\cos(C)$

In summary, the Law of Cosines states that for **any** triangle the length of any side squared equals the sum of the squares of the other two sides minus twice the product of the other two sides with the cosine of the opposite angle. The Law of Cosines generalizes the Pythagorean Theorem to any kind of triangle, not just those with a right angle.

$$
\boxed{
\begin{aligned}
c^2 &= a^2 + b^2 - 2a\cdot b\cdot\cos(C) \Rightarrow c = \sqrt{\left(a^2 + b^2 - 2a\cdot b\cdot\cos(C)\right)} \\
b^2 &= a^2 + c^2 - 2a\cdot c\cdot\cos(B) \Rightarrow b = \sqrt{\left(a^2 + c^2 - 2a\cdot c\cdot\cos(B)\right)} \\
a^2 &= b^2 + c^2 - 2b\cdot c\cdot\cos(A) \Rightarrow a = \sqrt{\left(b^2 + c^2 - 2b\cdot c\cdot\cos(A)\right)}
\end{aligned}
}
$$

Rearranging these formulas to solve for the relevant angles gives the following results.

$$c^2 = a^2 + b^2 - 2a \cdot b \cdot \cos(C) \Rightarrow c^2 - a^2 - b^2 = -2a \cdot b \cdot \cos(C)$$

$$\frac{c^2 - a^2 - b^2}{-2a \cdot b} = \frac{-2a \cdot b \cdot \cos(C)}{-2a \cdot b}$$

$$\cos(C) = -\frac{c^2 - a^2 - b^2}{2a \cdot b} = \frac{(-1) \cdot \left(c^2 - a^2 - b^2\right)}{2a \cdot b} = \frac{a^2 + b^2 - c^2}{2a \cdot b}$$

$$\boxed{\begin{aligned} \cos(C) &= \frac{a^2 + b^2 - c^2}{2a \cdot b} \Rightarrow \quad C = \cos^{-1}\left(\frac{(a^2 + b^2 - c^2)}{(2a \cdot b)}\right) \\ \cos(B) &= \frac{a^2 + c^2 - b^2}{2a \cdot c} \Rightarrow \quad B = \cos^{-1}\left(\frac{(a^2 + c^2 - b^2)}{(2a \cdot c)}\right) \\ \cos(A) &= \frac{b^2 + c^2 - a^2}{2b \cdot c} \Rightarrow \quad A = \cos^{-1}\left(\frac{(b^2 + c^2 - a^2)}{(2b \cdot c)}\right) \end{aligned}}$$

The unneeded parentheses were added to the formulas for the sides a, b, c and the angles A, B and C to emphasize that both the square root symbol and the fraction bar act as implied grouping symbols. Failure to provide these parentheses on a scientific calculator will result in a wrong answer.

Since the inverse cosine function returns angles in the range $0°$ to $180°$, the Law of Cosines can be used to find **any** angle of a triangle when the lengths of all three sides are known. In contrast, since the inverse sine function can't return angles between $90°$ to $180°$, the Law of Sines will be unable to directly compute the measure of an obtuse angle in a triangle.

Your Turn!!

A plane travels W35°N (i.e., 35 degrees north of west) for 15 minutes at 350 mph.

1. In these 15 minutes how far did the plane travel?

2. In these 15 minutes how far west did the plane travel?

3. In these 15 minutes how far north did the plane travel?

4. Given that $\sin(\theta) = 0.35$ and $90° < \theta < 180°$, determine the following:

	DD (4 places)	DMS
$\theta =$	____________	______________
$\cos(\theta) =$	_____________	
$\sin(\theta) =$	_____________	
$\tan(\theta) =$	_____________	

5. Given that $\cos(\theta) = -0.55$ and $90° < \theta < 180°$, determine the following:

	DD (4 places)	DMS
$\theta =$	___________	_____________
$\cos(\theta) =$	____________	
$\sin(\theta) =$	____________	
$\tan(\theta) =$	____________	

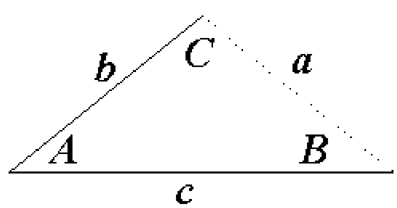
6. Given that $\tan(\theta) = 1.25$ and $180° < \theta < 270°$, determine the following:

	DD (4 places)	DMS
$\theta =$	___________	_____________
$\cos(\theta) =$	____________	
$\sin(\theta) =$	____________	
$\tan(\theta) =$	____________	

Section 7.4 Solving Oblique Triangles

With the sine and cosine functions and their inverses we can solve for the three missing items of information (lengths of sides or measure of angles) in **any** triangle provided that we have the three relevant inputs. The three relevant inputs are based on the congruency criteria discussed in Chapter 6. Specifically, these are SAS, ASA, and SSS. The procedure for solving a triangle in each case is presented below.

1. SAS



Given two sides b and c and the angle A between these sides:

A. Use the Law of Cosines to calculate the length of the side opposite the given angle.

$$a^2 = b^2 + c^2 - 2b\cdot c\cdot\cos(A) \Rightarrow a = \sqrt{b^2 + c^2 - 2b\cdot c\cdot\cos(A)}$$

B. Use the Law of Sines to calculate the measure of the angle opposite to the smaller of sides a and b. This is done to avoid trying to compute an obtuse angle using the inverse sine function. Calculate the third missing angle by subtracting the sum of the two known angles from $180°$.

If $b > c$ $\quad \dfrac{\sin(C)}{c} = \dfrac{\sin(A)}{a} \Rightarrow \sin(C) = \dfrac{c \cdot \sin(A)}{a} \Rightarrow C = \sin^{-1}\left(\dfrac{c \cdot \sin(A)}{a}\right)$

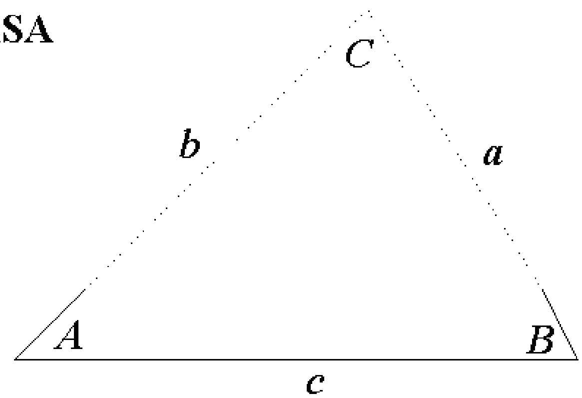
$$B = 180° - A - C$$

If $c > b$ $\quad \dfrac{\sin(B)}{b} = \dfrac{\sin(A)}{a} \Rightarrow \sin(B) = \dfrac{b \cdot \sin(A)}{a} \Rightarrow B = \sin^{-1}\left(\dfrac{b \cdot \sin(A)}{a}\right)$

$$C = 180° - A - B$$

Note: In the formulas above we used the version of the Law of Sines with the sines of the unknown angles in the numerator. This is not at all necessary, but it does make the algebra of solving for the unknown angles a little easier to follow.

2. ASA



Given two angles A and B and c the side between them:

A. Calculate the missing angle by subtracting the sum of the two given angles from 180 degrees.
B. Use the Law of Sines to calculate the two missing sides.

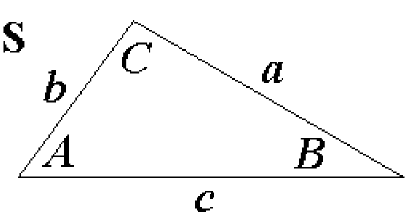
$$C = 180° - A - B$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \cdot \sin(A)}{\sin(C)}$$

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \cdot \sin(B)}{\sin(C)}$$

Note: In the formulas above we used the version of the Law of Sines with the unknown sides in the numerator.

3. SSS



Given the three sides, a, b, c, determine the three angles A, B, C:

A. Use the Law of Cosines to calculate the measure of the angle opposite to the longest side. This is the only angle in the triangle that could be obtuse. Hence this angle can always be calculated using the inverse cosine function, but may not be directly computable using the inverse sine function: $C=\cos^{-1}\left(\frac{\left(a^2+b^2-c^2\right)}{(2a\cdot b)}\right)$

B. Use the Law of Sines to calculate the measure of one of the remaining two unknown angles. Calculate the third missing angle by subtracting the sum of the two known angles from $180°$.

$$\frac{\sin(A)}{a}=\frac{\sin(C)}{c}\Rightarrow \sin(A)=\frac{a\cdot\sin(C)}{c}\Rightarrow A=\sin^{-1}\left(\frac{a\cdot\sin(C)}{c}\right)$$

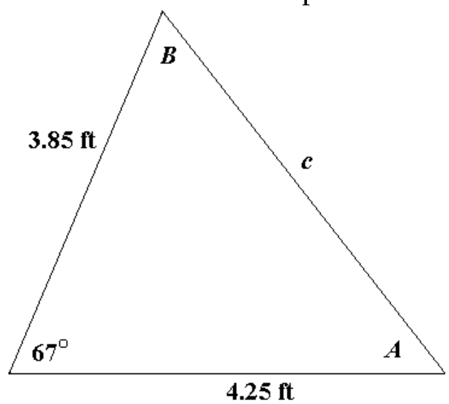
$$B=180°-A-C$$

Example: Given a triangle with the following information, solve for the missing side and angles.

a = 3.85 ft b = 4.25 ft c = ____________

A = ____________ B = ____________ C = 67°

Solution: This is a SAS problem. First we calculate the missing side c, then the angles A and B.



$$c=\sqrt{a^2+b^2-2a\cdot b\cdot\cos(C)}$$
$$=\sqrt{3.85^2+4.25^2-2(3.85)(4.25)\cos(67°)}\text{ ft}$$
$$c=4.483\text{ ft}$$
$$\frac{\sin(A)}{3.85\text{ ft}}=\frac{\sin(67°)}{4.483\text{ ft}}\Rightarrow \sin(A)=\frac{3.85\text{ ft}\sin(67°)}{4.483\text{ ft}}=0.7905$$
$$\Rightarrow A=\sin^{-1}(0.7905)=52.23°$$
$$B=180°-A-C=180°-52.23°-67°=60.77°$$

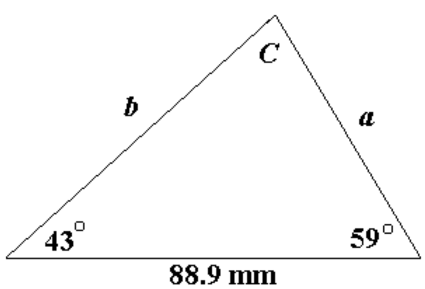
Example: Given a triangle with the following information, solve for the missing sides and angle.

a = ____________ b = ____________ c = 88.9 mm

A = 43° B = 59° C = ____________

Solution: This is an ASA problem. First we calculate the missing angle C, then the sides a and b.

$$C=180°-43°-59°=78°$$



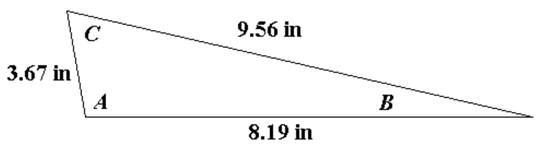
$$\frac{a}{\sin(A)}=\frac{c}{\sin(C)}$$
$$\Rightarrow a=\frac{c\sin(A)}{\sin(C)}=\frac{88.9\text{ mm}\sin(43°)}{\sin(78°)}=61.98\text{ mm}$$
$$\frac{b}{\sin(B)}=\frac{c}{\sin(C)}\Rightarrow$$
$$b=\frac{c\sin(B)}{\sin(C)}=\frac{88.9\text{ mm}\sin(59°)}{\sin(78°)}=77.90\text{ mm}$$

Example: Given a triangle with the following information, solve for the three missing angles.

a = 9.56 in b = 3.67 in c = 8.19 in

A = ___________ B = ___________ C = ___________

Solution: This is an SSS problem. First we calculate the missing angle A (the largest angle), then the angles C and B.



$$A = \cos^{-1}\left(\frac{\left(b^2 + c^2 - a^2\right)}{\left(2 \cdot b \cdot c\right)}\right)$$

$$A = \cos^{-1}\left(\frac{\left(3.67^2 + 8.19^2 - 9.56^2\right)}{\left(2 \times 3.67 \times 8.19\right)}\right)$$

$$A = \cos^{-1}\left(-0.18046\right) = 100.40° \qquad \frac{\sin(C)}{c} = \frac{\sin(A)}{a}$$

$$\Rightarrow \sin(C) = \frac{c\sin(A)}{a} = \frac{8.19 \text{ in} \times \sin(100.40°)}{9.56 \text{ in}} = 0.84263$$

$$C = \sin^{-1}(0.84263) = 57.42°$$

$$B = 180° - A - C = 180° - 100.40° - 57.42° = 22.18°$$

.

Note: If first we had calculated either angle B or C from the Law of Cosines and then had attempted to calculate angle A using the Law of Sines, we would have instead obtained A's supplement. The reason for this is again the fact that for positive input, the inverse sine function always returns an angle between zero and ninety degrees. This illustrates the need to set up the calculation in such a way so that the Law of Sines is not used to calculate the angle opposite to the **longest side**. Such an **incorrect** solution is shown below.

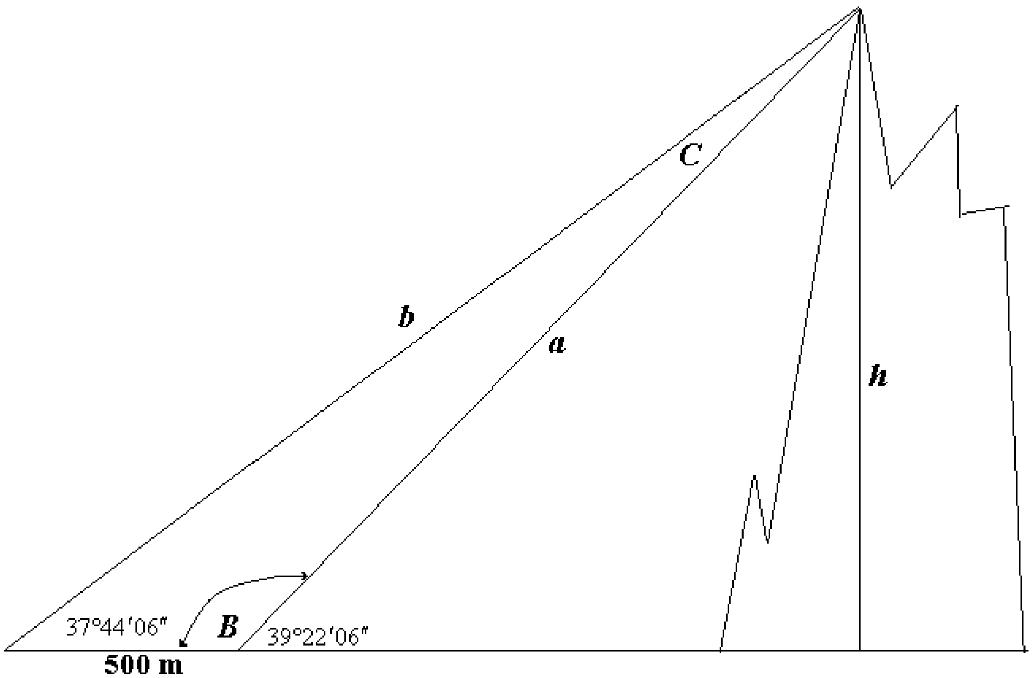
$$B = \cos^{-1}\left(\frac{\left(a^2 + c^2 - b^2\right)}{\left(2 \cdot a \cdot c\right)}\right) = \cos^{-1}\left(\frac{\left(9.56^2 + 8.19^2 - 3.67^2\right)}{\left(2 \times 9.56 \times 8.19\right)}\right) = \cos^{-1}(0.92597) = 22.18°$$

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \Rightarrow \sin(A) = \frac{a\sin(B)}{b} = \frac{9.56 \text{ in} \times \sin(22.18°)}{3.67 \text{ in}} = 0.98358$$

$$\text{So, } "A" = \sin^{-1}(0.98358) = 79.60° = 180° - 100.40°$$

Using this flawed procedure we obtained the supplement of the correct answer. Since $\sin(79.60°) = \sin(100.40°) = 0.98358$, the inverse sine function gave the "wrong" answer.

Example:
An explorer is taking measurements on a distant mountain peak. She initially measures an angle of elevation of $37°44'06''$. She then advances 500 meters closer and measures a new angle of elevation of $39°22'06''$. How high is the mountain peak above the horizontal of the explorer?



Solution: Angles of elevation are measured from the horizontal "up" to a distant object. In a similar way angles of depression are measured from the horizontal "down" to a distant object. To determine the height *h*, we need to determine either one of the lengths *a* or *b* and use right triangle trigonometry. The determination of *a* or *b* means solving an ASA triangle. First we calculate the angle C, then use the Law of Sines to calculate either *a* or *b*.

$$A = 37°44°06'' = 37.735°$$
$$B = 180° - 39°22'06'' = 140°37'54'' = 140.63167°$$
$$C = 180° - (A + B) = 180° - 178°22'' = 1°38'' = 1.6333°$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c\sin(A)}{\sin(C)} = \frac{500\text{m}\sin(37.735°)}{\sin(1.6333°)} = 10,736.\text{m}$$

So, $h = a \cdot \sin(39.3683°) = 10,736\text{m}\sin(39.3683°) = 6810.\text{m}$

or

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c\sin(B)}{\sin(C)} = \frac{500\text{m}\times\sin(140.6317°)}{\sin(1.6333°)} = 11{,}127.\text{m}$$

So, $h = b \cdot \sin(37.735°) = 11{,}127\text{m}\sin(37.735°) = 6810.\text{m}$

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. $a = 17.2$ cm $\quad b = 21.3$ cm $\quad c =$ ____________
   $A =$ ____________ $\quad B =$ ____________ $\quad C = 45.00°$

2. $a = 11.1$ in $\quad b = 21.5$ in $\quad c = 16.2$ in
   $A =$ ____________ $\quad B =$ ____________ $\quad C =$ ____________

3. $a =$ ____________ $\quad b =$ ____________ $\quad c = 14.3$ ft
   $A = 50.00°$ $\quad B = 30.00°$ $\quad C =$ ____________

4. A geologist sights a distant hilltop with an angle of elevation of 23 degrees 37 minutes. She then advances 1000 m closer to the mountain and measures a new angle of elevation of 29 degrees 10 minutes. How high is the hilltop above the level of the geologist?

5. Two airplanes leave the same airport at the same time. The planes are travelling 400 mph and 360 mph respectively. After one and one half-hours the planes are 200 miles apart. To two decimal places what is the angle between the airplanes' courses of flight?

Chapter 7 Sample Test

______________/100

Problems 1 through 12 are each worth 7 points. Problems 13 and 14 are each worth 8 points.

1. Convert the following angles from decimal degrees to DMS notation and from DMS to decimal degrees to the nearest ten-thousandth.

Decimal Degree		DMS	
$11°30'$	= ____________	$41.18°$	= ____________
$83°\ 19'10''$	= ____________	$129.25°$	= ____________
$50'50''$	= ____________	$41.33°$	= ____________
$42°15'30''$	= ____________	$0.022°$	= ____________

2. For each angle give the values of the three trig functions to four decimal places.

$\theta = 52°16'49''$		$\theta = 20.25°$	
$\cos(\theta)$	= ____________	$\cos(\theta)$	= ____________
$\sin(\theta)$	= ____________	$\sin(\theta)$	= ____________
$\tan(\theta)$	= ____________	$\tan(\theta)$	= ____________

3. Given that $\sin(\theta) = 0.25$ and $0° < \theta < 90°$, determine the following:

	DD (4 places)	DMS
θ	= ____________	____________
$\cos(\theta)$	= ____________	
$\sin(\theta)$	= ____________	
$\tan(\theta)$	= ____________	

4. Given that $\tan(\theta) = 0.85$ and $0° < \theta < 90°$, determine the following:

	DD (4 places)	DMS
θ	= ____________	____________
$\cos(\theta)$	= ____________	
$\sin(\theta)$	= ____________	
$\tan(\theta)$	= ____________	

5. Given that $\cos(\theta) = -.85$ and $90° < \theta < 180°$, determine the following:

	DD (4 places)	DMS
θ	= ____________	____________
$\cos(\theta)$	= ____________	
$\sin(\theta)$	= ____________	
$\tan(\theta)$	= ____________	

6. Given that $\tan(\theta) = 0.75$ and $180° < \theta < 270°$, determine the following:

	DD (4 places)	DMS
$\theta =$	____________	______________
$\cos(\theta) =$	_____________	
$\sin(\theta) =$	_____________	
$\tan(\theta) =$	_____________	

Solve for the missing sides (3 significant digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

7. $a = 23.0$ cm $\quad b = 18.5$ cm $\quad c =$ _____________
$A =$ _____________ $\quad B =$ _____________ $\quad C = 90.00°$

8. $a =$ _____________ $\quad b =$ _____________ $\quad c = 86.7$ ft
$A = 23.75°$ $\quad B =$ _____________ $\quad C = 90.00°$

9. $a = 3.68$ m $\quad b =$ _____________ $\quad c =$ _____________
$A = 55.86°$ $\quad B =$ _____________ $\quad C = 90.00°$

10. $a = 25.0$ cm $\quad b = 33.6$ cm $\quad c =$ _____________
$A =$ _____________ $\quad B =$ _____________ $\quad C = 120.00°$

11. $a = 42.0$ in $\quad b =$ _____________ $\quad c =$ _____________
$A = 38.41°$ $\quad B =$ _____________ $\quad C = 91.50°$

12. $a = 19.6$ cm $\quad b = 28.6$ cm $\quad c = 43.2$ cm
$A =$ _____________ $\quad B =$ _____________ $\quad C =$ _____________

13. A radar gun makes a 45.0 degree angle with a road and measures a speed of 64.8 mph. To the nearest tenth of a mph, how fast is the car actually traveling?

14. Two airplanes leave the same airport at the same time. The planes are traveling 500. mph and 480. mph respectively. After one half-hour the planes are 160. miles apart. To two decimal places what is the angle between the planes' lines of flight.

Chapter 7 Sample Test Solution

1. Decimal Degree $11°30' = 11.5°$ $83°\,19'10'' = 83.3194°$ $50'50'' = 0.8472°$ $42°15'30'' = 42.2583°$	DMS $41.18° = 41°10'$ $129.25° = 129°15'$ $41.33° = 41°19'48''$ $0.022° = 0°1'19''$
2. $\theta = 52°16'49''$ $\cos(\theta) = 0.6118$ $\sin(\theta) = 0.7910$ $\tan(\theta) = 1.2929$	$\theta = 20.25°$ $\cos(\theta) = 0.9382$ $\sin(\theta) = 0.3461$ $\tan(\theta) = 0.3689$
3. DD DMS $\theta = 14.4775°$ $14°28'39''$ $\cos(\theta) = 0.9682$ $\sin(\theta) = 0.25$ $\tan(\theta) = 0.2582$	4. DD DMS $\theta = 40.3645°$ $40°21'52''$ $\cos(\theta) = 0.7619$ $\sin(\theta) = 0.6476$ $\tan(\theta) = 0.85$
5. DD DMS $\theta = 148.2117°$ $148°12'42''$ $\cos(\theta) = -0.85$ $\sin(\theta) = 0.5267$ $\tan(\theta) = -0.6197$	6. DD DMS $\theta = 216.8699°$ $216°52'12''$ $\cos\theta = -0.8$ $\sin\theta = -0.6$ $\tan\theta = 0.75$
7. $a = 23.0$ cm $b = 18.5$ cm $c = 29.5$ cm $A = 51.19°$ $B = 38.81°$ $C = 90.00°$	8. $a = 34.9$ ft $b = 79.4$ ft $c = 86.7$ ft $A = 23.75°$ $B = 66.25°$ $C = 90.00°$
9. $a = 3.68$ m $b = 2.50$ m $c = 4.45$ m $A = 51.19°$ $B = 34.14°$ $C = 90.00°$	10. $a = 25.0$ cm $b = 33.6$ cm $c = 50.9$ cm $A = 25.16°$ $B = 34.84°$ $C = 120.00°$
11. $a = 42.0$ in $b = 51.9$ in $c = 67.6$ in $A = 38.41°$ $B = 50.09°$ $C = 91.50°$	12. $a = 19.6$ cm $b = 28.6$ cm $c = 43.2$ cm $A = 21.44°$ $B = 32.23°$ $C = 126.33°$
13. $64.8 \text{ mph}/\cos(45°) = 91.6 \text{ mph}$	14. $\cos^{-1}\left(\frac{250^2 + 240^2 - 160^2}{2\times 250\times 240}\right) = \cos^{-1}(.875) = 38.05°$

Section 8.1 Types of Data and Frequency Distributions

Statistics is the science of collecting, organizing, analyzing, presenting, and interpreting data. The data that is used in statistics can come from many sources. It can be measured directly by the experimenter or it can be obtained from some other secondary source like a government agency, a business, a commercial or scientific publication.

Two major classifications exist as to the type of data collected. These are qualitative versus quantitative data. Qualitative data is data that is a name or category. Examples of qualitative data are a person's gender, hair or eye color, type of car, the letter grade received in a class, etc. Qualitative data can use numbers as a label, such as ranking movies from 1 to 5. However, computations on the numbers and even comparisons between different values generally have no meaning. For example, a zip code of 9200 is not really twice a zip code of 4600. Zip codes are just references to specific geographical locations in the United States. Zip codes and phone numbers are numeric in nature but there would be little meaning to the result of arithmetical computation on their numerical values.

Quantitative data is data that is numerical and for which it makes sense to add, subtract, multiply or divide data values. For example, weight, height and time to complete a task are examples of quantitative data. We can further distinguish between absolute scale and relative scale quantitative data. The standard example for this distinction is temperature measurements. The Celsius and Fahrenheit scales are both relative scale (relative in fact to the freezing and boiling points of water). A temperature of $0°\text{F}$ does not indicate the absence of heat. If that were the case temperatures below $0°\text{F}$ would be impossible. It does make sense to do some arithmetic on relative scale data. If the temperature at 7 am is $29°\text{F}$ and the temperature at noon is $45°\text{F}$, the subtraction $45°\text{F} - 29°\text{F} = 16°\text{F}$ has a sensible interpretation as how much the temperature has changed in those five hours. However, it is not true that $45°\text{F} \div 29°\text{F} = 1.552$ gives any valid indication of how hotter it has become! The Kelvin and Rankine temperature scales are examples of absolute quantitative data. The relationships are given by the formulas:

$$[°\text{R}] = [°\text{F}] + 459.67 \qquad [°\text{K}] = [°\text{C}] + 273.15$$

A temperature of $45°\text{F}$ corresponds to an absolute temperature of 504.67°R and a temperature of $29°\text{F}$ corresponds to an absolute temperature of 488.67°R. The ratio $504.7°\text{R} \div 488.7°\text{R} = 1.033$ does give a valid indication of how much hotter $45°\text{F}$ is compared to $29°\text{F}$.

Qualitative data generally consists of a frequency count. This means how many occurrences of a given category were observed.

Example: You are taking Introduction to College Mathematics and wonder how other people in your program have done who have taken the course from the same instructor. You survey 50 people and generate the following qualitative data:

B	B	A	D	D	C	BC	AB	D	B	C	AB	AB
BC	C	B	AB	B	A	C	AB	BC	C	B	BC	AB
BC	BC	BC	BC	D	C	D	B	BC	B	C	B	BC
B	BC	B	B	B	BC	B	BC	C	C	C		

This generates the following frequency distribution.

Grade	Frequency	Relative Frequency	Relative Frequency $\times 360°$
A	2	4%	14.4°
AB	6	12%	43.2°
B	14	28%	108°
BC	13	26%	93.6°
C	10	20%	72°
D	5	10%	36°
F	0	0%	0°

The relative frequency is the proportion of the total that each category represents. The sum of the frequency column is the total number of measurements, called the **sample size** and usually designated as ***n***. The sum of the relative frequency column must always be 1 or 100%. Frequency is commonly symbolized by *f*, so relative frequency $= f \div n$ and is often expressed as a percent. The last column will be used in making a pie chart of this data in the next section.

Example: Traffic Control has been monitoring the number of cars that cross a particular intersection of two streets during fourteen "non-rush" one hour periods during the day. The number of cars in this example is quantitative data. Twenty cars passing in one hour is twice as many as ten cars per hour. Over the ten days they have recorded the following 140 observations.

20	19	20	23	23	17	16	19	20	22	22	16
17	20	19	21	17	23	24	25	21	16	25	24
22	19	21	23	23	25	19	23	24	23	22	21
23	21	20	25	17	23	15	19	16	25	25	21
23	17	16	17	23	22	17	24	19	23	17	25
19	22	17	24	25	22	16	17	23	16	17	17
24	20	16	20	18	20	19	18	18	16	23	25
24	16	21	19	16	23	23	20	16	21	21	17
15	21	17	22	22	23	18	25	20	24	15	24
24	25	17	20	21	23	21	24	19	22	24	20
18	23	15	19	17	22	21	22	22	24	19	24
15	20	15	20	20	25	20	24				

In order to make sense of this data we organize it into a frequency distribution. Now each category is the value of the absolute quantitative score, designated by x, which equals the number of cars per hour that cross the given intersection.

x	f	Cumulative f	Relative f
15	6	6	4.29%
16	12	18	8.57%
17	16	34	11.43%
18	5	39	3.57%
19	13	52	9.29%
20	16	68	11.43%
21	13	81	9.29%
22	13	94	9.29%
23	19	113	13.57%
24	15	128	10.71%
25	12	140	8.57%

The column labeled cumulative frequency is a "running total" of the frequencies. For example, there where 113 different hours of observation when the number of cars passing through the intersection was less than or equal to 23. In order to gain greater clarity with large amounts quantitative data it is sometimes advantageous to group the data into classes or categories that represent a range of scores. For example, referring to the traffic data, the data has been grouped into pairs of *x* scores. The class **18.5 – 20.5** consists of the scores 19 and 20. The artificial use of the decimal in the class boundary removes any ambiguity as to which class a score of 20 belongs. The class frequency *f* is the number of scores in the original ("ungrouped") data that fall into a given class.

Cars / hour	***x* = Mid Point**	***f***	**Cumulative *f***	**Relative *f***
14.5 – 16.5	15.5	18	18	12.86%
16.5 – 18.5	17.5	21	39	15.00%
18.5 – 20.5	19.5	29	68	20.71%
20.5 – 22.5	20.5	26	94	18.57%
22.5 – 24.5	22.5	34	128	24.29%
24.5 – 26.5	24.5	12	140	8.57%

In this example the difference between consecutive lower (or upper class boundaries) is the constant value of 2. This is called the class width. The column labeled mid point (also called the class mark) is the average of the upper and lower class boundaries for a given class. The class mid point is here called *x* since it is representative of the scores in a given class. It is a typical score of the class. Grouping the data and replacing the individual scores by the class mid point loses some information, but gains clarity. The goal is to choose classes in such a way that this trade off enhances our understanding of the data set.

Your Turn!!

1. Covance was performing one of their medical studies and put out a call for healthy people between the ages of 18 and 40. The study would consist of a 5 day stay at their facility for which the participant would get $800. The ages of the 107 people who applied to be in the study are given below.

22	23	22	19	18	20	27	19	19	24	35	25
25	35	21	33	19	24	18	22	30	20	29	18
31	28	27	21	30	27	32	33	23	29	25	31
25	28	30	33	28	22	21	30	22	22	20	19
33	31	26	18	23	28	19	21	27	29	38	29
22	22	26	32	25	21	25	24	25	35	23	31
28	29	26	32	32	21	20	24	21	28	27	28
29	19	24	29	25	27	25	21	19	27	32	24
32	19	18	27	34	25	18	32	31	26	23	

a. Form an ungrouped frequency distribution for the ages given above

***x* = age**	***f***	**Cumulative *f***	**Relative *f***
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			
31			
32			
33			
34			
35			
36			
37			
38			

b. Form a grouped frequency distribution for the ages above using 7 classes starting with 17.5 – 20.5 as the first class.

Age Class	***x* = Mid Point**	***f***	**Cumulative *f***	**Relative *f***
17.5 – 20.5				
20.5 – 23.5				
23.5 – 26.5				
26.5 – 29.5				
29.5 – 32.5				
32.5 – 35.5				
35.5 – 38.5				

2. A 100 gram bag of M & M plain candy contained 115 pieces of candy in the following colors: red (**R**), orange (**O**), yellow (**Y**), brown (**N**), blue (**B**), and green (**G**). The candies came out of the bag in the following order.

Y	O	G	G	R	Y	O	R	N	B	B	R
R	Y	N	R	B	N	Y	B	Y	Y	Y	Y
R	N	B	N	R	N	Y	O	R	R	G	B
Y	B	N	Y	O	B	R	R	Y	R	N	N
N	G	R	B	B	B	R	N	N	N	Y	N
Y	N	Y	B	B	O	N	G	N	Y	B	R
Y	G	N	N	B	O	N	N	R	R	R	R
O	N	R	N	G	Y	Y	N	O	G	N	Y
R	Y	N	N	R	N	N	Y	N	N	Y	N
N	O	N	Y	Y	Y	R					

Construct a frequency distribution based upon the color of the candy that gives both frequency and relative frequency for the color of the candy.

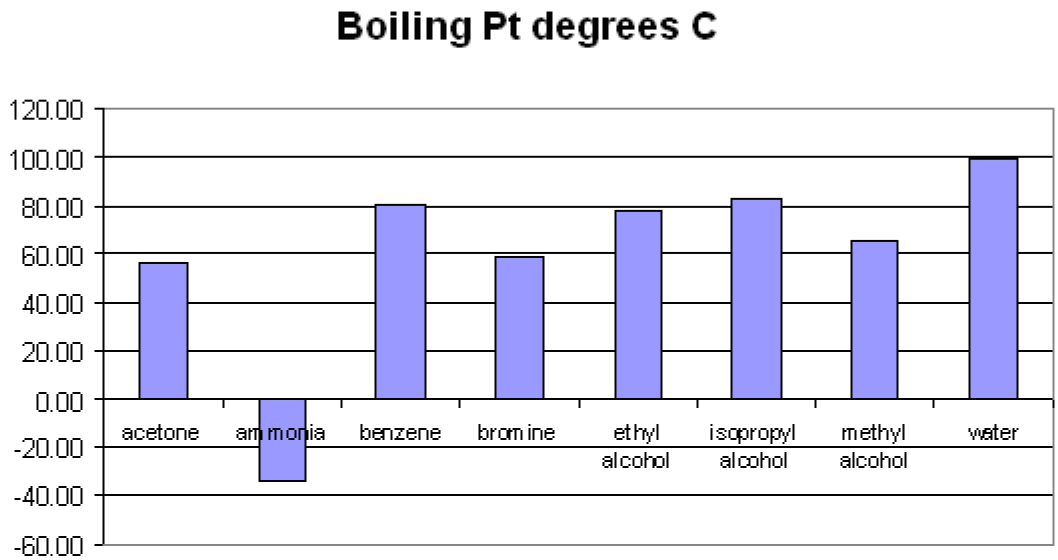
Grade	Frequency	Relative Frequency	Relative Frequency $\times$ 360°
R			
O			
Y			
N			
B			
G			

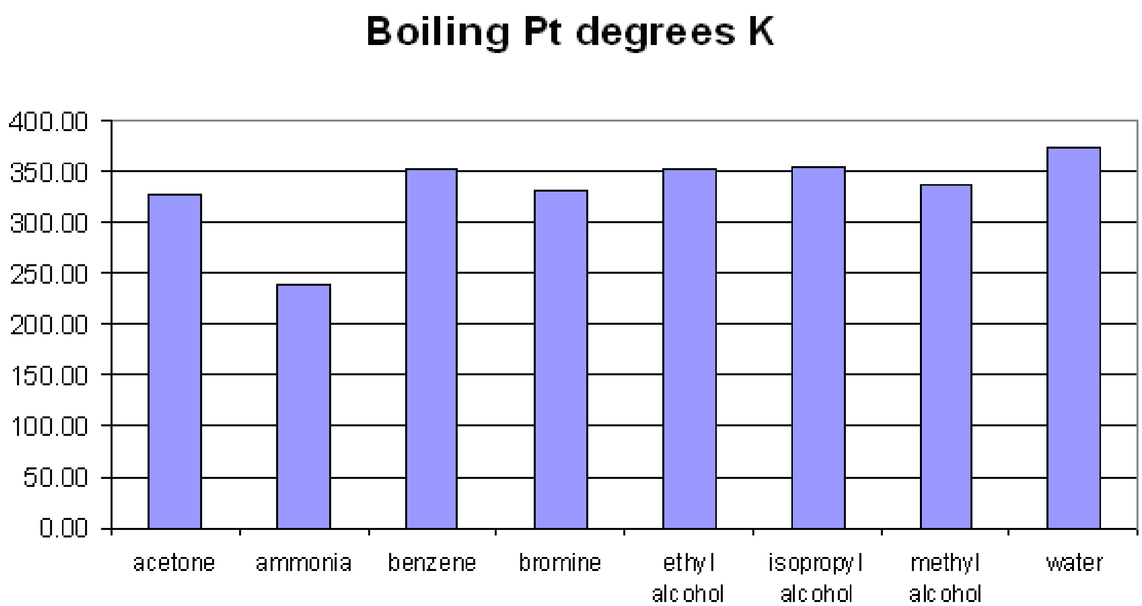
Section 8.2 Graphing Data and Interpreting Graphs

The graphical representation of data allows for the presentation and effective summary of facts and relationships. Graphs in two dimensions can be divided into categorical charts and relational line or curve plots. One type of a categorical chart is a bar chart where a quantitative variable is plotted against a category or quality. In general, the categories can be presented in any order. Standard practice is to plot the categories along the horizontal axis and the quantity along the vertical axis. For example, the table below gives the boiling point at atmospheric pressure for eight different liquids. The boiling points are expressed both in degrees Celsius and in degrees Kelvin (i.e., absolute temperature).

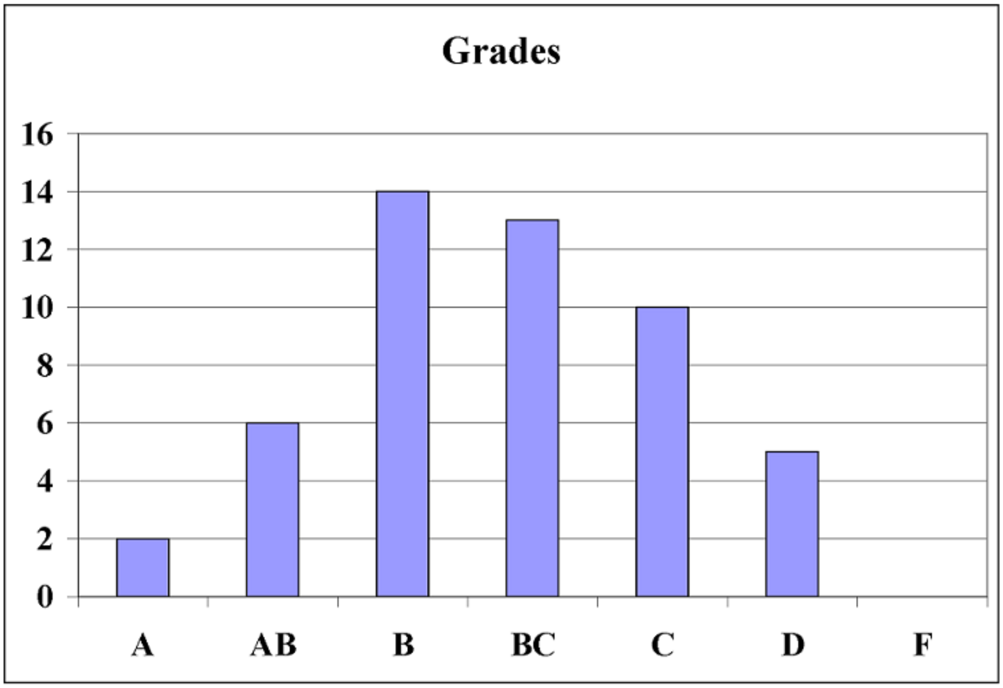
Liquid	Boiling Pt degrees C	Boiling Pt degrees K
acetone	56.20	329.36
ammonia	-33.35	239.81
benzene	80.10	353.26
bromine	58.78	331.94
ethyl alcohol	78.50	351.66
isopropyl alcohol	82.40	355.56
methyl alcohol	64.96	338.12
water	100.00	373.16

This Celsius data is used to construct the following bar chart. Using absolute temperatures gives a less exaggerated view of the same data. Which chart is most appropriate depends on what information we wish to emphasize. The differences between boiling points are easier to see on the first chart, while the second gives a more accurate comparison of the different liquids.

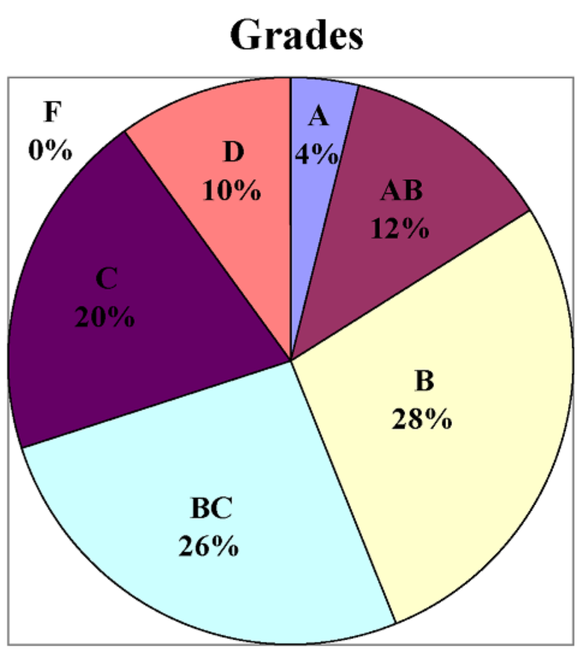




For qualitative data such as the example in the last section of grades received in Introduction to College Mathematics the bar chart presents either the frequency or relative frequency of each category.



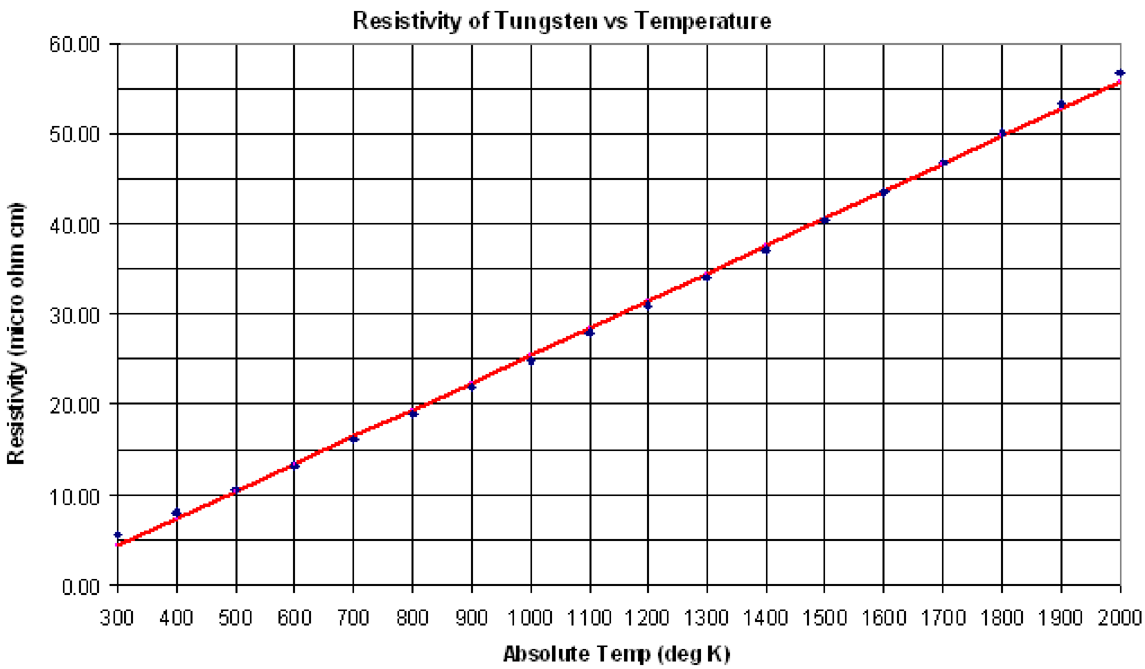
Another very visual way to present the same data is with a pie or circle chart. Here the fraction of the total each category makes up is represented as a sector of a circle. The central angle in degrees of the sector corresponding to a given category is computed by the formula $\textbf{Relative Frequency} \times \mathbf{360^\circ}$. In practice, bar and pie charts such as the one shown here are easily generated in a spread sheet program such as MS Excel.



In a relational line or curve plot, both the horizontal x and vertical variables y represent quantities. For example, data on the electrical resistivity of tungsten versus absolute temperature is given in the following table.

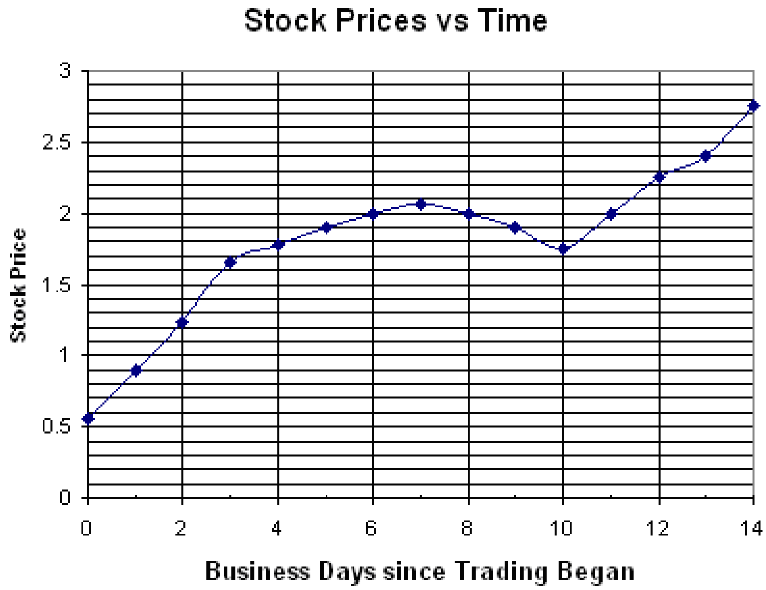
Temp Degree K	Resistivity micro ohm cm
300	5.65
400	8.06
500	10.56
600	13.23
700	16.09
800	19.00
900	21.94
1000	24.93
1100	27.94
1200	30.98
1300	34.08
1400	37.19
1500	40.36
1600	43.55
1700	46.78
1800	50.05
1900	53.35
2000	56.67

One first determines which variable is "*x*", and which is "*y*". We usually imagine that *x* is the independent or input variable (whose value we control or set), and that *y* is the dependent or output variable (whose value we measure or record).Each data point is then plotted from this table of values as was discussed in Chapter 4. From this table we construct the graph.

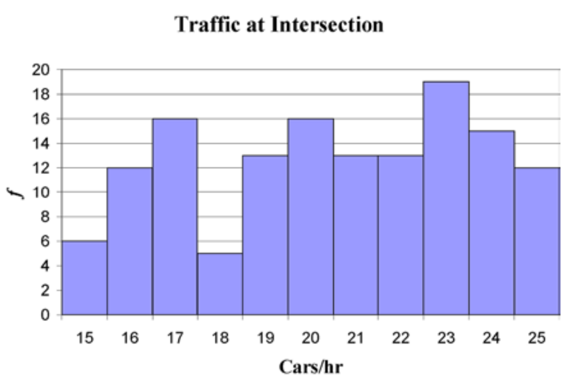


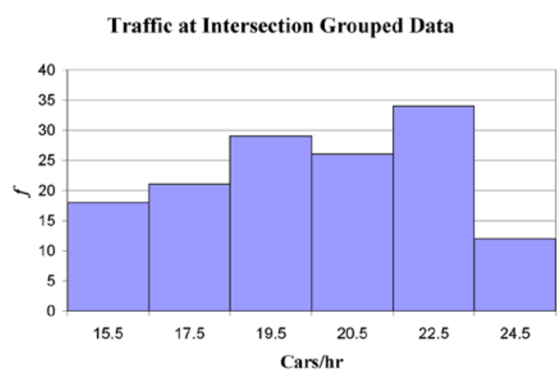
We can see from the graph that these data points are well approximated by the straight "trend" line drawn in the graph. We say that the relationship between the two variables is "approximately linear". This line can be used to estimate values of the resistivity that were never measured. For example, when the temperature is 1050 degrees K, we can estimate a value of about 27 micro ohm cm for the resistivity of tungsten.

When a curve has y "going up" as x moves from left to right, we say that y is "increasing", while if the value of y goes down as x moves from left to right, y is "decreasing". As an example, if we plot stock market prices versus time, stocks are increasing in value when the curve moves up going from left to right. In the graph shown on the next page stock prices are increasing on the days in the interval 0 to 7 and the days in the interval 10 to 14. Stock prices are decreasing on the days in the interval 7 to 10. The stock prices "peaked" or had a "local maximum" at day 7, when the price was $2.05. The stock prices "bottomed out" or had a "local minimum" at day 10, when the price was $1.75. A local maximum always happens at a transition from increasing to decreasing. Similarly, a local minimum occurs at a transition from decreasing to increasing.



From a frequency distribution of a quantitative variable we can construct a bar chart of frequency (or relative frequency) versus the values of the scores. Such a bar chart is called a frequency histogram and gives a concise visual representation of the distribution of scores. This can also be done for grouped data where the class midpoints are used on the horizontal axis. These histograms are illustrated below for the traffic data presented in the previous section. Both histograms agree in showing that the frequency does not vary drastically with the number of cars per hour that pass through the intersection. In other words in any given hour studied it was almost as likely to see 20 cars per hour as 17 or 24.



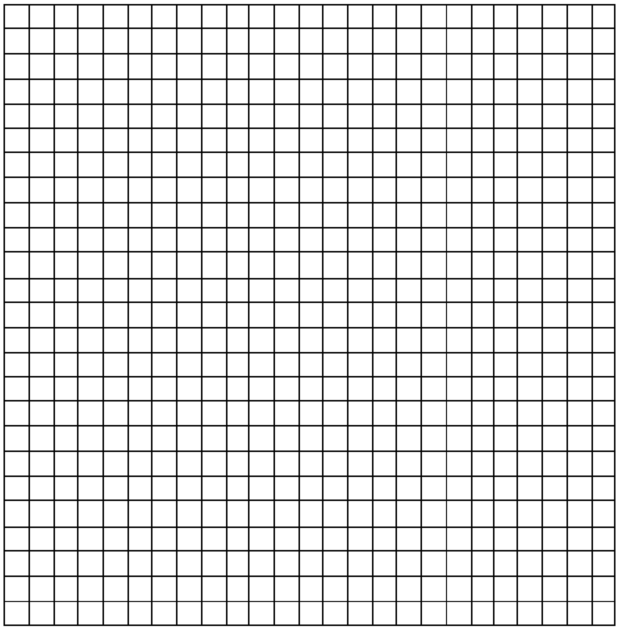


Your Turn!!

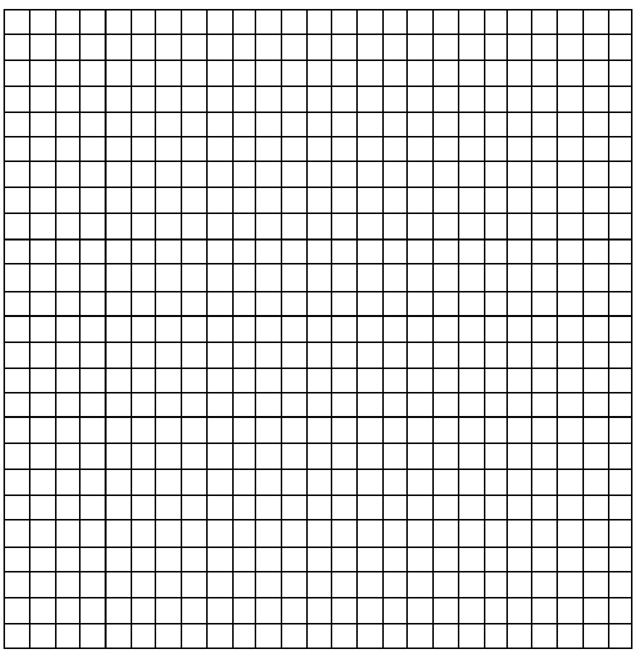
1. Below is a table which gives the electrical resistivity (good conductors have less resistivity) in micro-ohm-cm of various metallic elements designated by their chemical symbol.

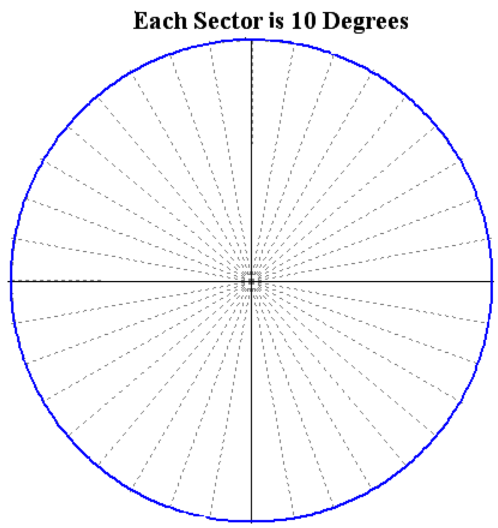
Element	**Al**	**Cu**	**Au**	**Pb**	**Ni**	**Pt**	**Ag**	**Zn**
Resistivity	2.65	1.67	2.35	20.64	6.84	10.6	1.59	5.92

Construct a bar graph that displays this information.



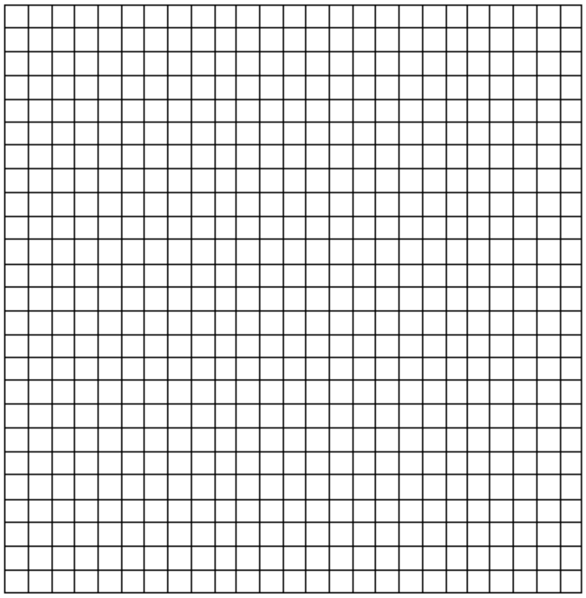
2. Using the data of Problem 2 in Section 8.1, construct both a bar chart and a pie chart showing the distribution of different color M & M's.



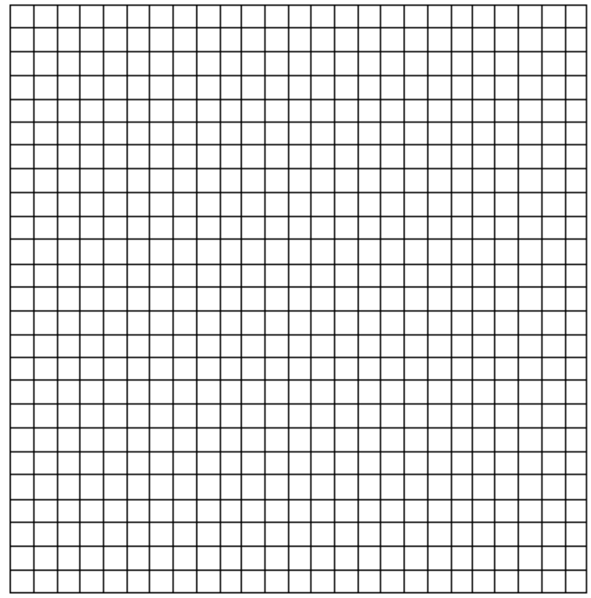


3. Using the two frequency distributions of Problem 1 in Section 8.1, construct a frequency histogram for both the ungrouped (original) and grouped data.

Ungrouped Data

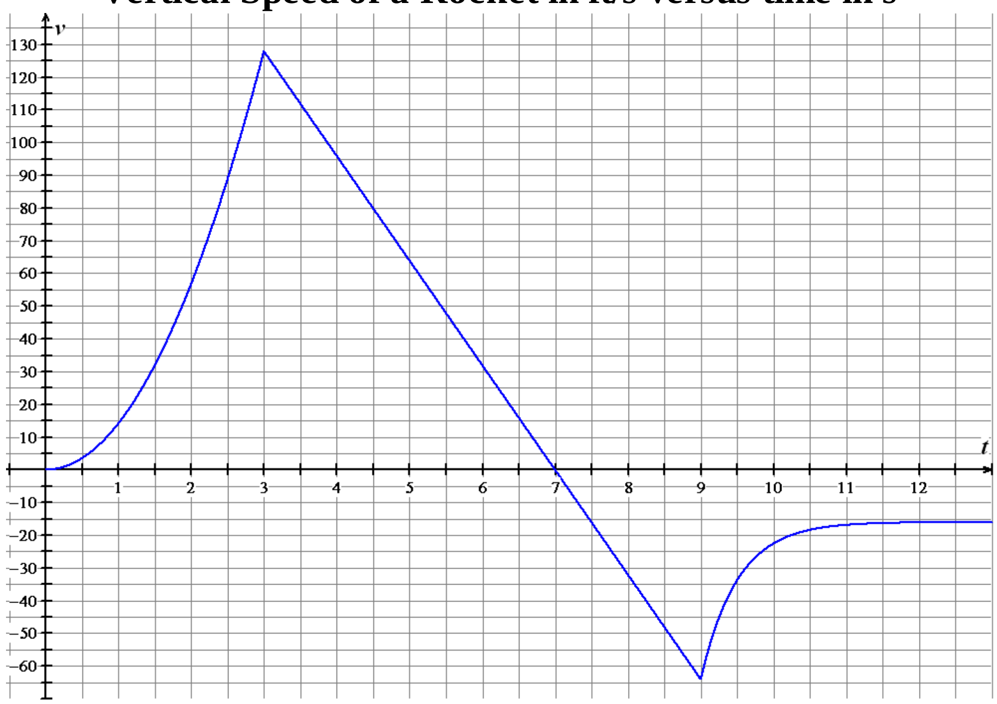


Grouped Data



4. The following graph gives the vertical speed of a model rocket in ft/s as a function of time after launch measured in seconds. A positive vertical speed means the rocket is moving up. A negative vertical speed means the rocket is moving down.

Vertical Speed of a Rocket in ft/s versus time in s



a. What is the model rocket's speed when the time is 0.0 seconds.

b. Estimate the model rocket's speed when the time is 5.0 seconds.

c. Estimate the times when the model rocket's speed is 90 ft/s.

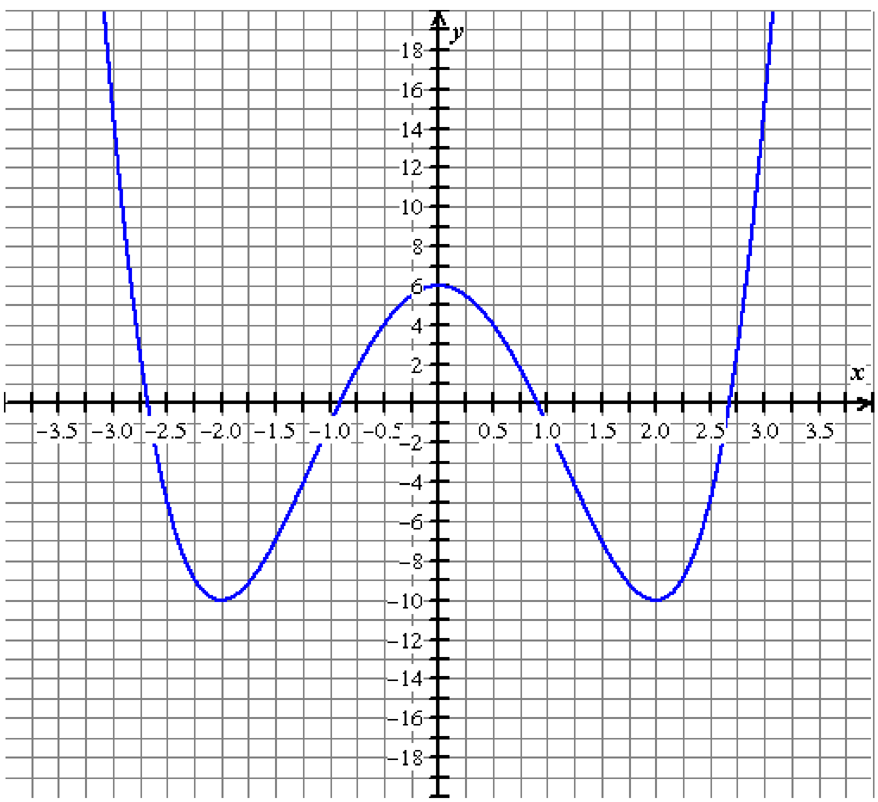
d. At what time does the model rocket reach its maximum height?

e. At what time does the model rocket's engine burn out?

f. At what time does the model rocket's parachute deploy?

5. Below is shown the graph of y versus x. Answer the following questions.

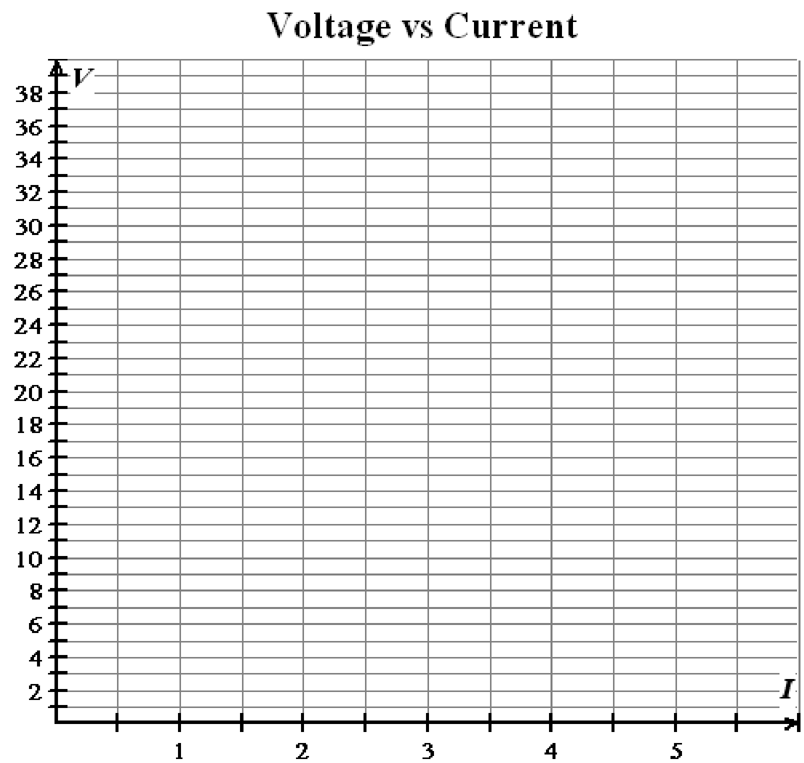
a. When $x = -1.5$, estimate the value of y.
b. When $x = 0$, estimate the value of y.
c. When $x = 3$, estimate the value of y.
d. When $x = 2$, estimate the value of y.
e. When $x = -2.5$, estimate the value of y.
f. For what values of x values is y increasing?
g. For what values of x is y decreasing?
h. At what value of x does y have a local maximum?



6. For a given circuit the amount of applied voltage, V, and the current in amps, I, are related by means of the following table.

V **Voltage**	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
I **Current**	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5

a. Construct a graph of Voltage versus Current from the following table:
b. From your graph estimate V when I is 5.0 amps.



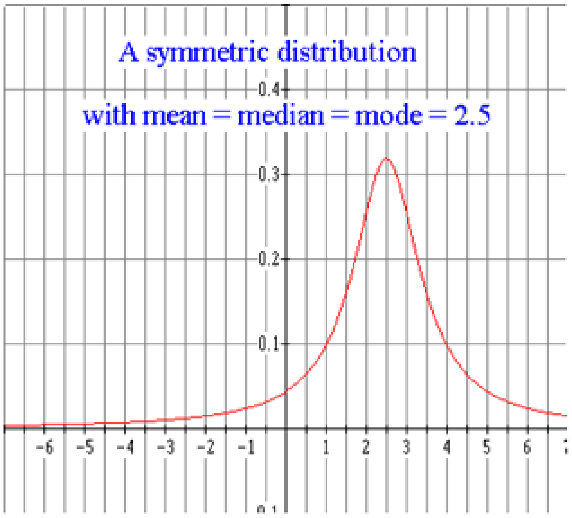
Section 8.3 Descriptive Statistics

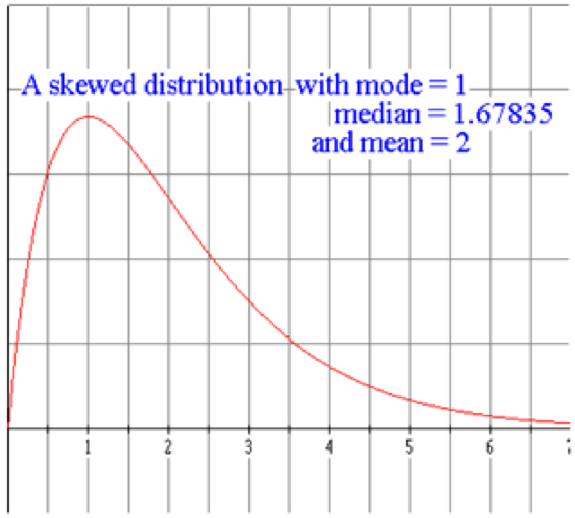
Descriptive statistics is the body of methods used to represent and summarize sets of numerical data. Descriptive statistics provides a means of describing how a set of measurements (for example, people's weights measured in pounds) is distributed. Descriptive statistics involves two different kinds of summary statistics. These summary statistics are called measures of center and measures of spread.

Measures of Center

Measures of center involve a "typical" data value about which all the other data values are distributed. The most commonly used measures of center are the mean, the median and the mode. The mean is sometimes called the arithmetic average and it is the sum of all the data values divided by the number of data values. If the data is arranged in order, high to low or low to high, then the median is the "middle" data value with as many data values above it as below it. The mode is the data value that occurs the most often (has the highest frequency). The mode is only meaningful for large sets of data where it is likely for the same score to be measured more than once.

If we were to graph a large data set and the mean, median, and the mode were all the same, our graph would look like the graph below on the left. A graph of this type is called symmetric: one side of the graph looks just like the other side of the graph. Symmetric graphs result from distributions of data in which the mean, median and mode are all equal. If any of these three are different from the other two or if all three are different the graph will no longer be symmetric and will become skewed. Therefore, any differences between these three measures are an indication of skewness or departure from a symmetric distribution.

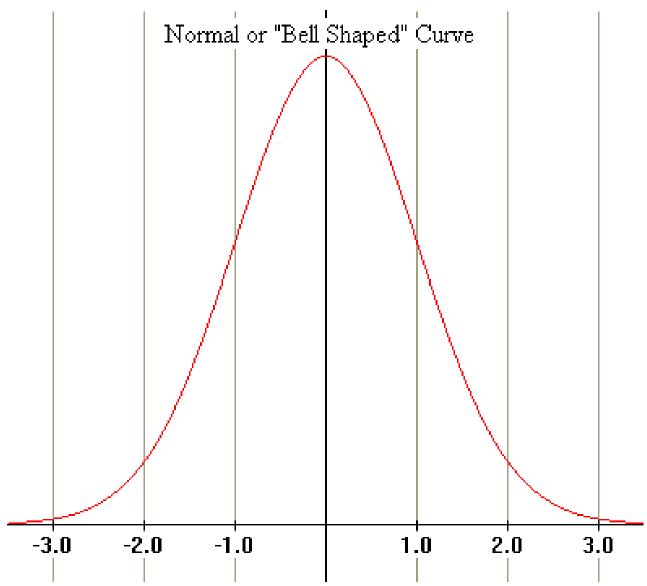


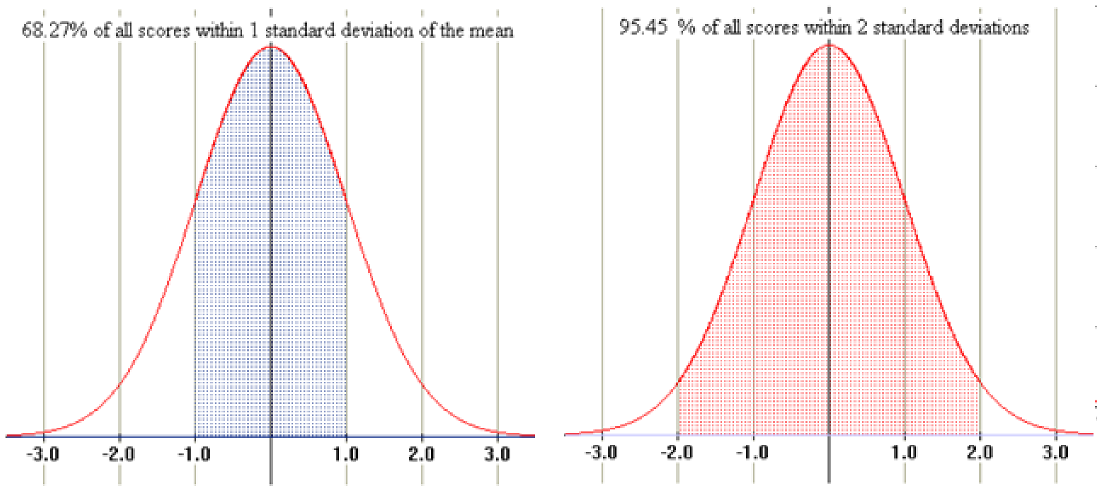


Measures of Spread

The most commonly used measures of spread are range, standard deviation, and inter-quartile range. The range is the difference between the highest data value (the maximum) and the lowest data value (the minimum). The standard deviation measures a "typical" distance of data values in the distribution from the mean. The inter-quartile range is the distance covered by the "middle half" of the data from the score that marks the first quarter point to the score that marks the third quarter point.

For the famous "bell-shaped" curve (what is generally called a normal distribution curve or a Gaussian distribution), 68% of all scores are found within one standard deviation of the mean and 95.5% of all scores are within two standard deviations of the mean. More generally, for any distribution of scores, **at least** 75% of all scores must be within two standard deviations of the mean.





Notation for Statistical Calculations

To make the calculation of these statistics easier the following notation will be used.
n represents the number of data values which make up the data set or the sample
x_i represents a subscripted score. The index i can take values from 1 to n and x_i is the i'th score in the data set. Furthermore, it will be assumed that the scores have been **sorted** from smallest to largest (an increasing sort). Thus, $\boldsymbol{x_1}$ **is the minimum** and $\boldsymbol{x_n}$ **is the maximum.**
f_i represents the frequency for a corresponding x_i. If the scores don't repeat $f_i = 1$. The **mode** of a frequency distribution is then just the score with the largest frequency.

$\sum$ is the upper case Greek letter sigma. This symbol stands for the mathematical operation of summation and means we take the sum of the data values: $x_1 + x_2 + x_3 + ... + x_n$. This sum can be more concisely written as $\sum_{i=1}^{n} x_i$. The variable i is called a "dummy" summation index and is just used to signify that various values of x are being summed. The designation $i = 1$ beneath the sigma indicates where the sum begins and the n above the sigma indicates where the sum ends. In statistics it is nearly always the case that sums begin at 1 and end at n, so the following abbreviated symbols are often employed.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + ... + x_n = \sum_i x_i = \sum x_i = \sum x \ .$$

Calculation of the Mean

The mean is the arithmetic average and it is usually indicated by $\bar{x}$ (called x bar). Sometimes a second symbol for the mean is used, called the population mean, when all possible data values have been measured. The population mean is represented by the Greek letter lower case mu, μ. The mean can be computed by adding together all of the data values and dividing by the number of data values that were added together. In terms of a formula, we write the mean calculation as

$$\bar{x} = \frac{\sum x_i}{n} .$$

Example: In the years 1992 – 1994 there were 18 space shuttle missions. The duration of these missions in days is given below. Compute the mean duration of these space shuttle missions.

8 9 9 14 8 8 10 7 6 9 7 8 10 14 11 8 14 11

Solution $\bar{x} = \dfrac{\sum x_i}{n} = \dfrac{8+9+9+14+8+8+10+7+6+9+7+8+10+14+11+8+14+11}{18} = 9.5$

If there is a lot of data, forming a frequency distribution reduces the number of computations since multiplication is repeated addition.

$$\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i} = \frac{\sum x_i \cdot f_i}{n}$$

If the data is ungrouped we use each data value for x_i in the formula. If the data is grouped then we use the midpoint of each class for x_i. The midpoint is the average of the upper and lower limits of the class

Example: A 15 question practice driver's education test was administered to a 120 people driver's ed students. The number of correct responses on this test for this group is shown below. Note: For this set of scores **Mode** = 9, since this score has the highest frequency.

***x* = Correct**	***f***	**Cumulative** *f*	$x \cdot f$	$x^2 \cdot f$
3	3	3	9	27
4	4	7	16	64
5	7	14	35	175
6	10	24	60	360
7	13	37	91	637
8	14	51	112	896
9	16	67	144	1296
10	14	81	140	1400
11	9	90	99	1089
12	10	100	120	1440
13	8	108	104	1352
14	7	115	98	1372
15	5	120	75	1125
Total	$\sum f_i = 120$		$\sum x_i f_i = 1103$	$\sum x_i^2 f_i = 11233$

Solution: Since each score occurred multiple times, we need to multiply each score by its frequency to obtain the correct total sum of occurring scores. The sample size n is the sum of the frequencies. $\bar{x} = \dfrac{\sum x_i \cdot f_i}{\sum f_i} = \dfrac{1103}{120} = 9.192$, so the mean or average number correct on the test was 9.19 .

Calculation of the Median, Q_1 and Q_3: Box Plots

The median of a set of scores is the "middle" score with as many scores above as below. It can be calculated by the following procedure. From the first score (i = 1) to the last score (i = n), there are n – 1 "changes of position". The median is the score halfway through these changes. Thus, the pointer or index to the median is given by $1 + \frac{1}{2}(n-1)$. Now, if n is odd (say 15), this index points to an actual score (x_8) which has as many scores above it as below. On the other hand, if n is even (say 16), this index points to a "non-existent" score ($x_{8.5}$) which is interpreted as a value that is half-way between x_8 and x_9. This is expressed in the following formula:

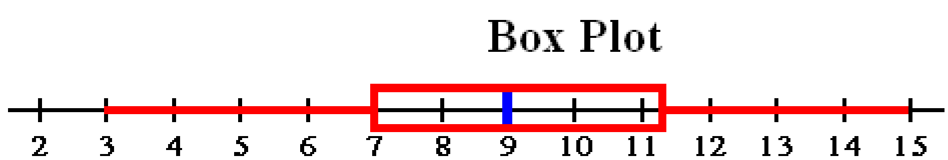
$$\text{for } n=16, \quad M_\text{d} = x_{8.5} = x_8 + \frac{1}{2}(x_9 - x_8) = \frac{2x_8 + x_9 - x_8}{2} = \frac{x_8 + x_9}{2}$$

This procedure can be generalized to calculate the quartiles Q_1 and Q_3 ($Q_2 = M_\text{d}$), which are the scores one quarter and three quarters of the way through the data respectively. The index of Q_1 (called the first quartile) is given by $1+\frac{1}{4}(n-1)$, and the index of Q_3 (called the third quartile) is given by $1+\frac{3}{4}(n-1)$. For example, if $n = 16$, the index to Q_1 is 4.75 and the index to Q_3 is 12.25, so that $Q_1 = x_{4.75} = x_4 + 0.75(x_5 - x_4)$ and $Q_3 = x_{12.25} = x_{12} + 0.25(x_{13} - x_{12})$.
The inter-quartile range or IQR is the distance from Q_1 to Q_3, i.e., IQR = $Q_3 – Q_1$. The IQR is a measure of spread. It covers the middle half of all scores.

The values of Q_1, M_d, and Q_3 are the basis of representing the data in what is called a "box plot" or a "box and whiskers plot". From Q_1 to Q_3 a box is drawn with the position of M_d marked as a vertical line within the box. Inner fences are defined on the left at $Q_1 - 1.5(\text{IQR})$ and on the right at $Q_3 + 1.5(\text{IQR})$. Outer fences are defined at $Q_1 - 3(\text{IQR})$ and $Q_3 + 3(\text{IQR})$. Any score between the inner fences and the outer fences is considered a moderate outlier. Any score beyond the outer fences is an extreme outlier. A straight line or whisker is drawn from Q_1 to the smallest score larger than the left inner fence. Any scores less than the inner fence are marked with a symbol highlighting them as outliers. Similarly, a whisker is drawn from Q_3 to the largest score larger than the right inner fence. Any scores larger than the right inner fence are also marked with a symbol highlighting them as outliers. If all scores are between the inner fences, then a whisker is drawn from the minimum score x_1 to Q_1 and a second whisker is drawn from Q_3 to the maximum score, x_n. This rather simple construction generates a graphical representation of the distribution of the scores.

Example: Construct a box plot of the scores on the driver's education test.
Solution:

	Index	**Formula**	**Result**
$\boldsymbol{Q_1}$	$1+0.25(120-1)=30.75$	$Q_1 = x_{30.75}$ $= x_{30} + 0.75(x_{31} - x_{30})$	$Q_1 = 7+0.75(7-7)$ $=7$
$\boldsymbol{M_\text{d}}$	$1+0.5(120-1)=60.5$	$M_\text{d} = x_{60.5}$ $= x_{60} + 0.50(x_{61} - x_{60})$	$M_\text{d} = 9+0.50(9-9)$ $=9$
$\boldsymbol{Q_3}$	$1+0.75(120-1)=90.25$	$Q_3 = x_{90.25}$ $= x_{90} + 0.25(x_{91} - x_{90})$	$Q_3 = 11+0.25(12-11)$ $=11.25$



The cumulative frequency makes it easy to locate a given score from its index. For example, in the above distribution to find score x_{31}, the values of the cumulative frequency indicate that scores 25 through 37 are all equal to 7, so $x_{31} = 7$.

Example: Using the data set on hourly traffic at an intersection from Section 8.1 determine the median and quartiles of both the ungrouped and the grouped data and generate box plots for both.

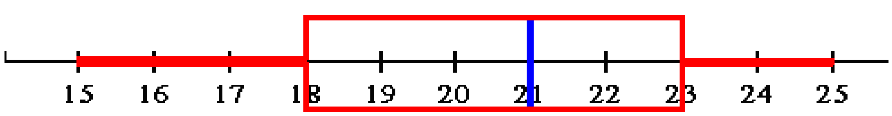
Solution:

Ungrouped Data

x	f	Cumulative f	xf	x^2f
15	6	6	90	1350
16	12	18	192	3072
17	16	34	272	4624
18	5	39	90	1620
19	13	52	247	4693
20	16	68	320	6400
21	13	81	273	5733
22	13	94	286	6292
23	19	113	437	10051
24	15	128	360	8640
25	12	140	300	7500
		Total	**2867**	**59975**

	Index	**Formula**	**Result**
Q_1	$1+0.25(140-1)=35.75$	$Q_1 = x_{35.75} = x_{35} + 0.75(x_{36} - x_{35})$	$Q_1 = 18 + 0.75(18-18) = 18$
M_d	$1+0.5(140-1)=70.5$	$M_d = x_{70.5} = x_{70} + 0.50(x_{71} - x_{70})$	$M_d = 21 + 0.50(21-21) = 21$
Q_3	$1+0.75(140-1)=105.25$	$Q_3 = x_{105.25} = x_{105} + 0.25(x_{106} - x_{105})$	$Q_3 = 23 + 0.25(23-23) = 23$

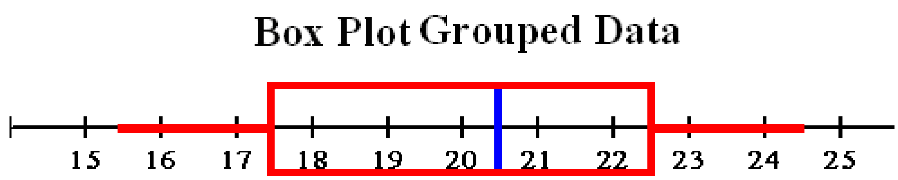
Box Plot Ungrouped Data



Grouped Data

Cars / hour	x = Mid Point	f	Cumulative f	xf	x^2f
14.5 – 16.5	15.5	18	18	279	4324.5
16.5 – 18.5	17.5	21	39	367.5	6431.25
18.5 – 20.5	19.5	29	68	565.5	11027.25
20.5 – 22.5	20.5	26	94	533	10926.5
22.5 – 24.5	22.5	34	128	765	17212.5
24.5 – 26.5	24.5	12	140	294	7203
			Total	**2804**	**57125**

	Index	Formula	Result
Q_1	$1+0.25(140-1)=35.75$	$Q_1 = x_{35.75} = x_{35}+0.75(x_{36}-x_{35})$	$Q_1 = 17.5+0.75(17.5-17.5) = 17.5$
M_d	$1+0.5(140-1)=70.5$	$M_d = x_{70.5} = x_{70}+0.50(x_{71}-x_{70})$	$M_d = 20.5+0.50(20.5-20.5 = 20.5$
Q_3	$1+0.75(140-1)=105.25$	$Q_3 = x_{105.25} = x_{105}+0.25(x_{106}-x_{105})$	$Q_3 = 22.5+0.25(22.5-22.5) = 22.5$



Calculation of the Standard Deviation

The standard deviation is a measure of the variability in quantitative data. If the standard deviation is large then there is a lot of variability. Most manufacturing processes strife to keep product variability to a minimum.

The standard deviation is usually calculated by the following formula: $s=\sqrt{\dfrac{\sum (x_i-\bar{x})^2}{n-1}}$. The symbol s indicates that this is a sample standard deviation. If all possible data values have been measured, then we have the slightly different formula (which gives a slightly smaller result) $\sigma=\sqrt{\dfrac{\sum (x_i-\mu)^2}{n}}$. The Greek letters σ (lower case sigma) and μ (lower case mu) designate that this standard deviation characterizes a population, i.e., all possible scores, as opposed to just a sample. Both of these formulas can be manipulated using algebra to give the computationally more efficient formulas shown below.

$$s=\sqrt{\frac{\sum (x_i-\bar{x})^2}{n-1}}=\sqrt{\frac{\sum x_i^{\,2}-\frac{\left(\sum x_i\right)^2}{n}}{n-1}} \quad ; \quad \sigma=\sqrt{\frac{\sum (x_i-\mu)^2}{n}}=\sqrt{\frac{\sum x_i^{\,2}-\frac{\left(\sum x_i\right)^2}{n}}{n}}.$$

Here order of operations is crucial. $\sum x_i^{\,2}$ means that each x is first squared and then summed. $\sum x_i^{\,2}$ is the sum of squares. In contrast, $\left(\sum x_i\right)^2$ means that the x's are first summed and then this answer is then squared. $\left(\sum x_i\right)^2$ is the square of the sum

Example: Compute both the sample and population standard deviations for $n=5$, $x_1=1$, $x_2=3$, $x_3=4$, $x_4=5$, and $x_5=7$.

Solution: $\sum x_i^{\,2}=1+9+16+25+49=100$; $\left(\sum x_i\right)^2=(1+3+4+5+7)^2=20^2=400$.

Regarding these scores as a sample gives the results:

$$\bar{x} = \frac{20}{5} = 4 \ ; \ s = \sqrt{\frac{100 - \frac{400}{5}}{5-1}} = \sqrt{5} = 2.2361$$

If these scores are a population (the set of all possible scores for the data set), then

$$\mu = \frac{20}{5} = 4 \ ; \ \sigma = \sqrt{\frac{100 - \frac{400}{5}}{5}} = \sqrt{4} = 2 \ .$$

Note that the means are the same but the standard deviations are a little different. The values of s and σ indicate that for this data set a typical distance of scores from the mean is about 2.

These formulas may seem **extremely complicated**. **Fortunately**, they are "built into" nearly every scientific calculator. It is **strongly** recommended that you learn how to properly use the statistical functions on your calculator so as to avoid having to perform the explicit summations. Information is available online for most makes and you may ask your instructor for assistance.

For large data sets organized into a frequency distribution, the formulas take advantage of the multiple occurance of scores as indicated by the frequencies. $n = \sum f_i$.

$$s = \sqrt{\frac{\sum f_i \left(x_i - \bar{x}\right)^2}{n-1}} = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n-1}} \qquad \sigma = \sqrt{\frac{\sum f_i \left(x_i - \mu\right)^2}{n}} = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n}}$$

Example: Using the data set on traffic at an intersection from Section 8.1 determine mean, sample, and population standard deviations for both the ungrouped and the grouped data.
Solution: The needed sums are computed in the previously listed frequency distribution tables.

Ungrouped Data	**Grouped Data**
$\bar{x} = \frac{2867}{140} = 20.479$	$\bar{x} = \frac{2804}{140} = 20.029$
$s = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n-1}}$ $= \sqrt{\frac{59975 - 2867^2/140}{139}} = 3.014$	$s = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n-1}}$ $= \sqrt{\frac{57125 - 2804^2/140}{139}} = 2.635$
$\sigma = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n}}$ $= \sqrt{\frac{59975 - 2867^2/140}{140}} = 3.003$	$\sigma = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n}}$ $= \sqrt{\frac{57125 - 2804^2/140}{140}} = 2.625$

.

Example: Forty runners were surveyed as to the number of miles per week they run. The results are given below.

Class	Frequency
5.5 – 10.5	2
10.5 – 15.5	4
15.5 – 20.5	6
20.5 – 25.5	10
25.5 – 30.5	8
30.5 – 35.5	6
35.5 – 40.5	4

Determine the following descriptive statistics and generate a frequency histogram and box plot for this data.

Descriptive Statistic

Minimum
Maximum
Range
Mode
Median, M_d
Mean, $\bar{x}$
Q_1
Q_3
IQR
Sample Standard Deviation, s
Population Standard Deviation, σ

We will let the class midpoint represent all the scores in a given class.

Class	Midpoint = x_i	f_i	Cumulative f	$x_i f_i$	$x_i^2 f_i$
5.5 – 10.5	8	2	2	16	128
10.5 – 15.5	13	4	6	52	676
15.5 – 20.5	18	6	12	108	1944
20.5 – 25.5	23	10	22	230	5250
25.5 – 30.5	28	8	30	224	6272
30.5 – 35.5	33	6	36	198	6534
35.5 – 40.5	38	4	40	152	5776
			Sums =	980	26620

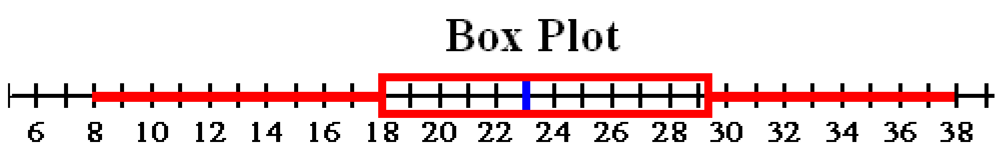
	Index	**Formula**	**Result**
Q_1	$1+0.25(40-1)=10.75$	$Q_1=x_{10.75} = x_{10}+0.75(x_{11}-x_{10})$	$Q_1=18+0.75(18-18) = 18$
M_d	$1+0.5(40-1)=20.5$	$M_d=x_{20.5} = x_{20}+0.50(x_{21}-x_{20})$	$M_d=23+0.50(23-23) = 23$
Q_3	$1+0.75(40-1)=30.25$	$Q_3=x_{30.25} = x_{30}+0.25(x_{31}-x_{30})$	$Q_3=28+0.25(33-28) = 29.25$

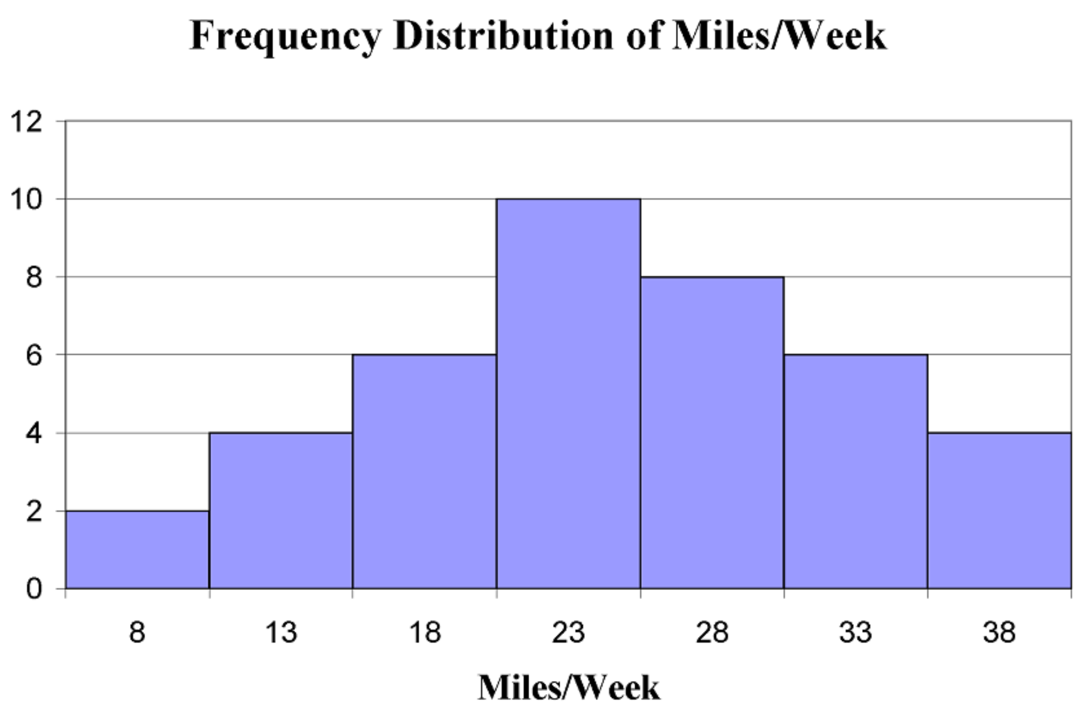
$$\bar{x} = \frac{980}{40} = 24.5$$

$$s = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n-1}} = \sqrt{\frac{26620 - 980^2/40}{39}} = 8.181$$

$$\sigma = \sqrt{\frac{\sum f_i x_i^2 - \frac{\left(\sum f_i x_i\right)^2}{n}}{n}} = \sqrt{\frac{26620 - 980^2/40}{40}} = 8.078$$

Descriptive Statistic	
Minimum	8
Maximum	38
Range	30
Mode	23
Median, M_d	23
Mean, $\bar{x}$	24.5
Q_1	18
Q_3	29.25
IQR	11.25
Sample Standard Deviation, s	8.181
Population Standard Deviation, σ	8.078





Your Turn!!

1. For the following data set of twelve scores compute the stated descriptive statistics.

18.1 , 19.7 , 17.8 , 19.1 , 16.5 , 20.0 , 18.5 , 17.9 , 20.9 , 20.3 , 19.4 , 18.3

a. The mean, $\bar{x}$ = ______________ e. The IQR = ______________

b. The median, M_d = ______________ f. The Range = ______________

c. The first quartile, Q_1 = ______________ g. The Stand Dev, s = ______________

d. The third quartile, Q_3 = ______________ h. The Stand Dev, σ = ______________

2. The following frequency distribution is based on the test scores on the Statistics chapter of Intro to College Math for three sections taught by the same instructor.

x	*f*	*x*	*f*	*x*	*f*	*x*	*f*	*x*	*f*
69	1	75	1	81	4	87	4	93	0
70	0	76	0	82	5	88	4	94	1
71	0	77	3	83	6	89	3	95	0
72	2	78	2	84	7	90	0	96	1
73	1	79	4	85	7	91	2	97	1
74	3	80	3	86	9	92	1		

Compute the following and generate both a box plot and frequency histogram of the scores.

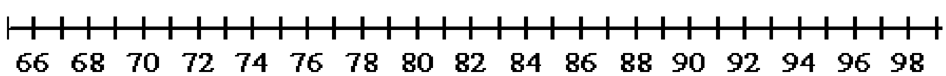
a. The mean, $\bar{x}$ = ______________ e. The IQR = ______________

b. The median, M_d = ______________ f. The Mode = ______________

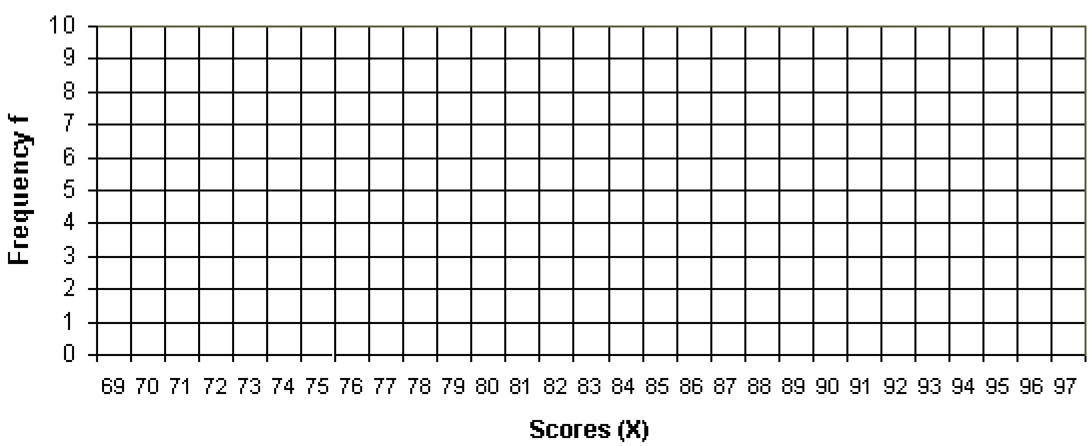
c. The first quartile, Q_1 = ______________ g. The Stand Dev, s = ______________

d. The third quartile, Q_3 = ______________ h. The Stand Dev, σ = ______________

Box Plot



Histogram of Data



Chapter 8 Sample Test ______________/100

Each problem is worth 5 points.

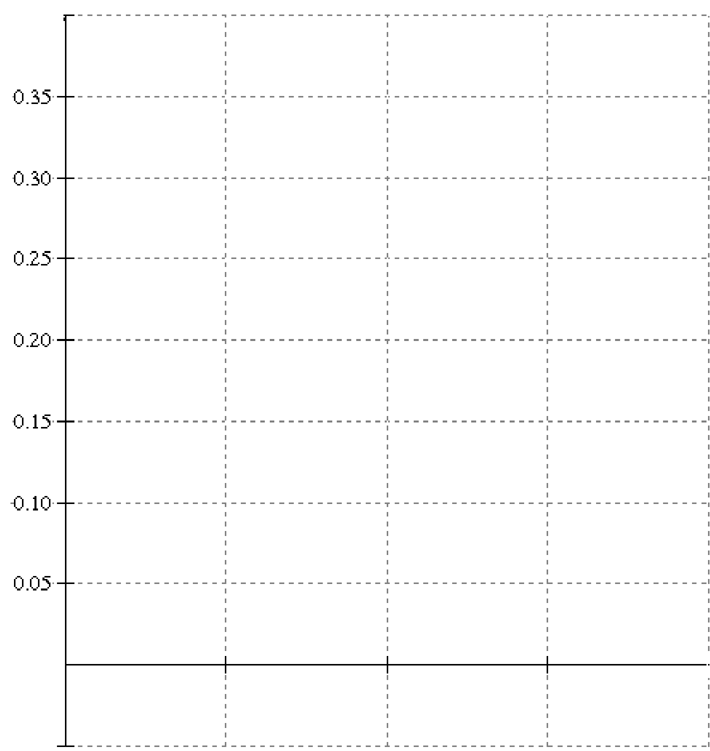
A sample of 50 residents in a community is surveyed as to the type of housing in which they dwell. The four mutually exclusive possible choices where apartment (A), condominium (C), Single Family Dwelling (i.e., a House) (H), or Other (N). The responses were as follows:

H	C	H	N	H	A	C	A	N	C
N	C	A	N	A	C	A	N	C	A
H	A	H	H	H	H	C	A	C	A
C	H	C	H	A	A	A	H	N	H
A	A	H	C	C	A	A	C	H	H

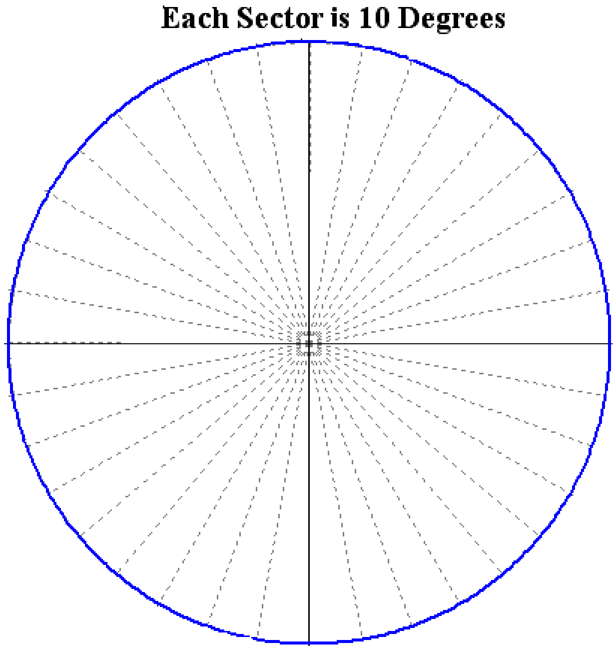
1. Form a frequency distribution for this data

Category	*f*	Relative *f*
A		
C		
H		
N		

2. Create a bar chart for this data with relative frequency along the vertical axis.

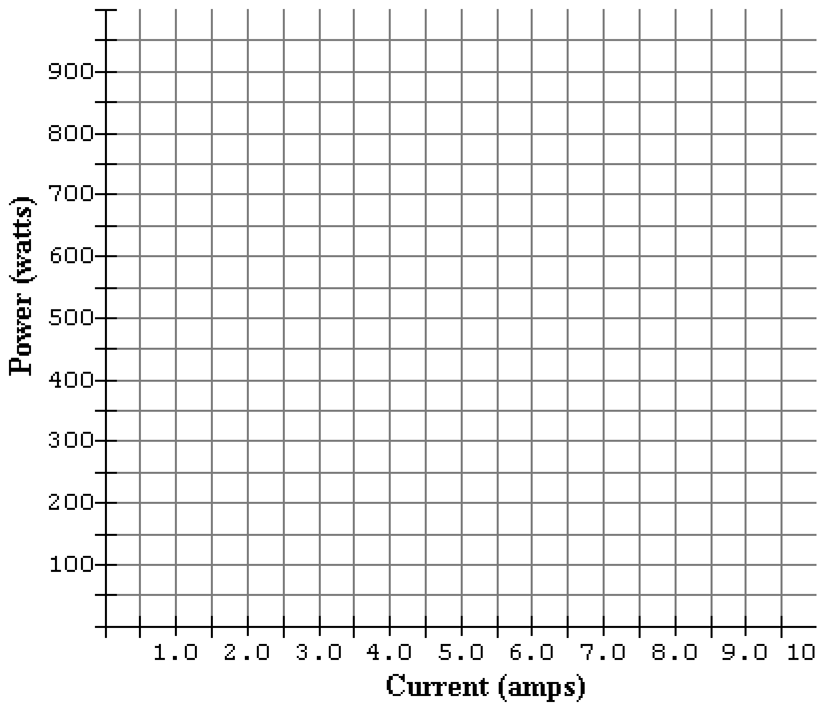


3. Create a circle graph for the data.



4. Construct a graph of electrical Power Consumed versus Current from the following table:

P **Power (watts)**	8.0	32.0	72.0	128.0	200.0	288.0	392.0	512.0	648.0
I **Current (amps)**	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0



5. From your graph estimate P when I is 2.5 amps.

For the following data set of ten scores compute the statistics requested.

18, 20, 24, 26, 28, 29, 30, 32, 34, 36

6. The mean, $\bar{x}$ $\bar{x}$ =_______________

7. The median, M_d M_d =_______________

8. The sample standard deviation, s to three decimal places s =_______________

9. The population standard deviation, σ to three decimal places σ =_______________

Measurements (microns) were made at various locations on a thermoforming polyamide layer. The data is summarized in the following frequency distribution. Calculate the sample statistics requested using the **class midpoint** to represent all **the scores** in a given class.

10. Fill in the missing columns.

Class	***x* = Class Mid Point**	***f***	**Cumulative *f***	***xf***	**x^2f**
8.5 – 9.0		**5**			
9.0 – 9.5		**3**			
9.5 – 10.0		**6**			
10.0 – 10.5		**13**			
10.5 – 11.0		**16**			
11.0 – 11.5		**8**			
11.5 – 12.0		**5**			
12.0 – 12.5		**2**			
			Sum =		

Using the frequency distribution in Problem 10 calculate the following. Give all numerical answers below to three decimal places

11. The mean, $\bar{x}$ $\bar{x}$ =_______________

12. The median, M_d M_d =_______________

13. The mode Mode =_______________

14. The first quartile, Q_1 Q_1 =_______________

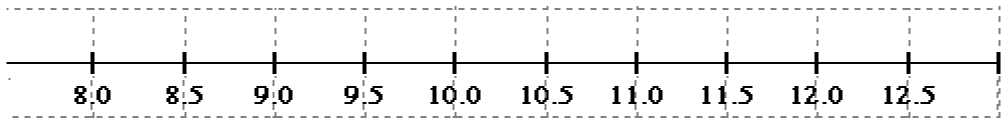
15. The third quartile, Q_3 Q_3 =_______________

16. The inter-quartile range IQR =_______________

17. Range Range =_______________

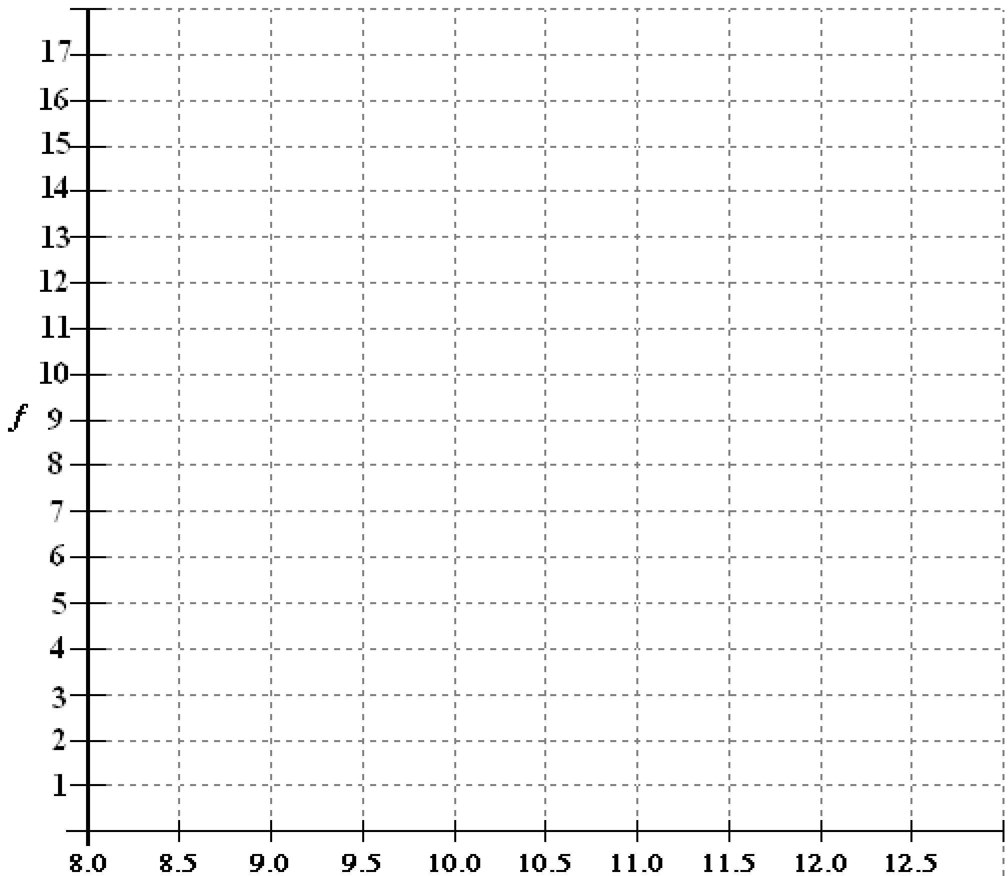
18. The sample standard deviation, s s =_______________

19. Generate a box plot of this data set.



20. Generate a frequency histogram of this data set.

Histogram for Problem 20

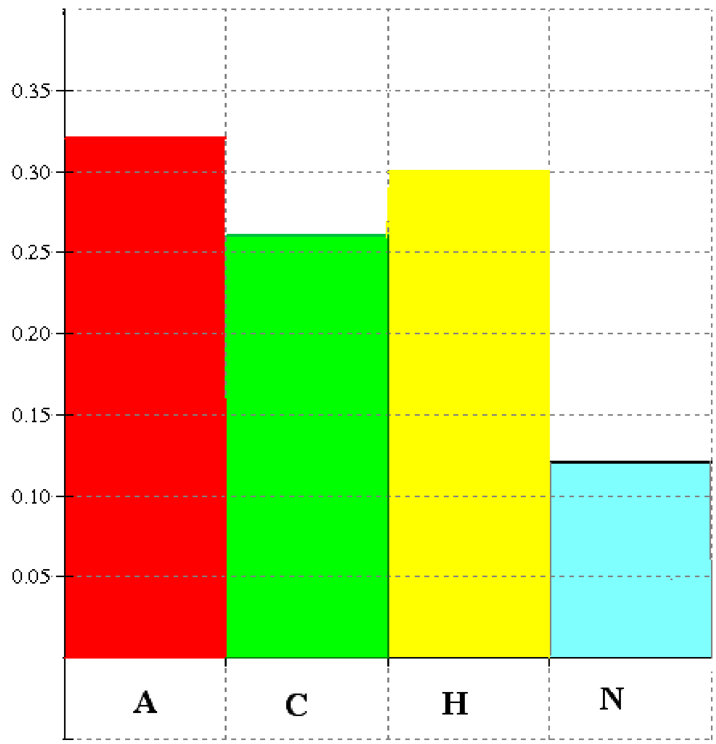


Chapter 8 Sample Test Solution

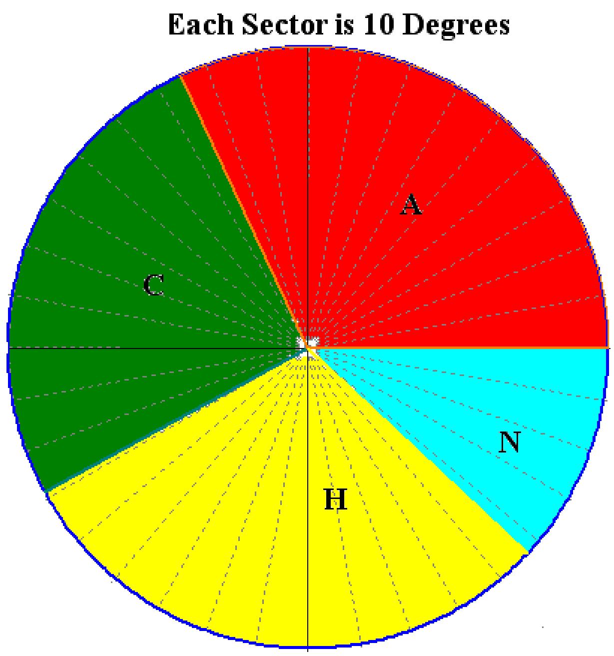
1.

Category	f	Relative f	Relative $f \times 360°$
A	16	32.00%	115.2°
C	13	26.00%	93.6°
H	15	30.00%	108°
N	6	12.00%	43.2°

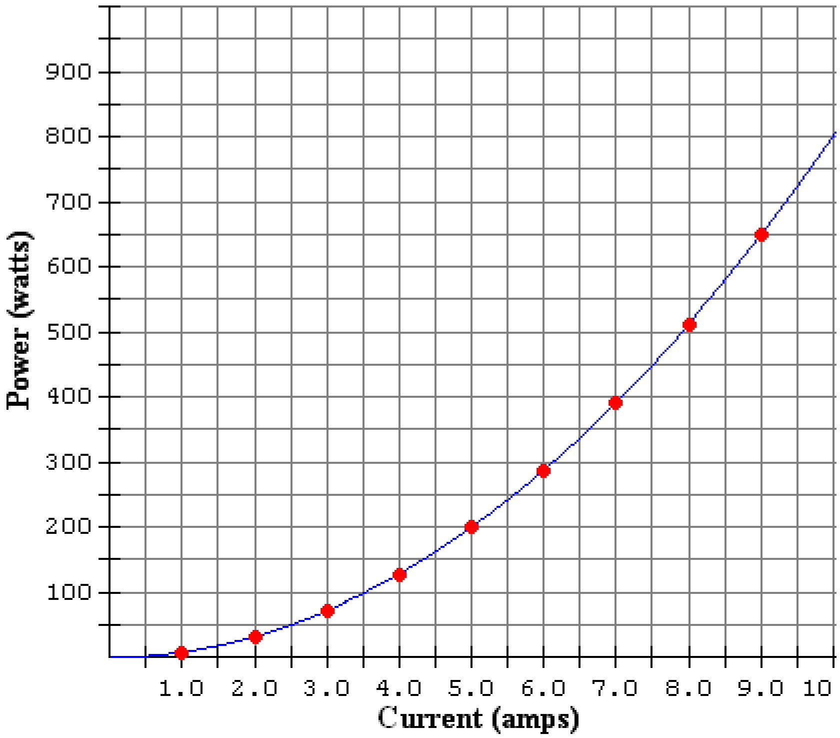
2.



3.



4.



5.
When I is 2.5 amps P is about 50 watts.

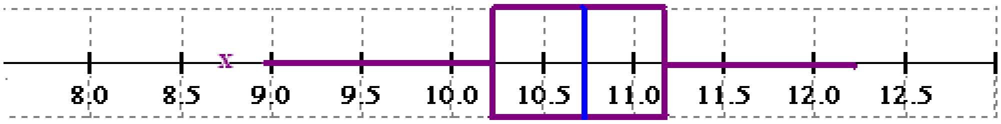
6. $\bar{x} = \mathbf{27.7}$ 7. $M_d = \mathbf{28.5}$ 8. $s = \mathbf{5.813}$ 9. $\sigma = \mathbf{5.515}$

10.

Class	x = Class Mid Point	f	Cumulative f	xf	x^2f
8.5 – 9.0	**8.75**	**5**	**5**	**43.75**	**382.8125**
9.0 – 9.5	**9.25**	**3**	**8**	**27.75**	**256.6875**
9.5 – 10.0	**9.75**	**6**	**14**	**58.50**	**570.3750**
10.0 – 10.5	**10.25**	**13**	**27**	**133.25**	**1365.8125**
10.5 – 11.0	**10.75**	**16**	**43**	**172.00**	**1849.0000**
11.0 – 11.5	**11.25**	**8**	**51**	**90.00**	**1012.5000**
11.5 – 12.0	**11.75**	**5**	**56**	**58.75**	**690.3125**
12.0 – 12.5	**12.25**	**2**	**58**	**24.50**	**300.1250**
			Sum =	**608.50**	**6427.6250**

11. $\bar{x} = 608.5 / 58 = \mathbf{10.491}$
12. M_d = **10.750**
13. Mode = **10.750**
14. Q_1 = **10.250**
15. Q_3 = **11.125**
16. IQR = **0.875**
17. Range = **3.500**
18. $s = \sqrt{\dfrac{6427.625 - 608.5^2 / 58}{57}} = \mathbf{0.875}$

Box Plot for Problem 19



Histogram for Problem 20