Exercise session 1 – Modeling Systems

1 Transitions Systems

We first recall that a *transition system* is a tuple $\mathcal{T} = (S, Act, \longrightarrow, I, AP, L)$ with

- S the set of states,
- Act the set of actions,
- $\longrightarrow \subseteq S \times Act \times S$ the transition relation,
- $I \subseteq S$ the set of initial states,
- AP the set of atomic propositions, and
- $L\colon S \to 2^{AP}$ the labeling function.

Let $\mathcal{T}_1 = (\{q_1, \ldots, q_5\}, \{a, b, c\}, \longrightarrow, \{p, q\}, L)$ be the transition system given below.

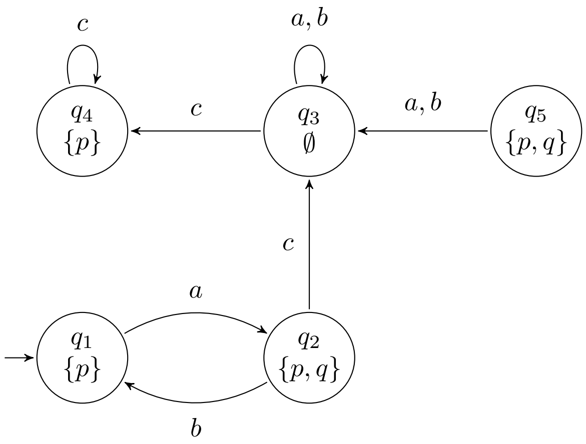

Figure 1: The transition system $\mathcal{T}_1$

Exercise 1. Compute the sets $Post^*(\{q_2\})$ and $Pre^*(\{q_3\})$.

Exercise 2. Which states are reachable?

Exercise 3. Describe the possible executions of the system.

Exercise 4. What are their traces?

Exercise 5. Give the TS of a traffic light that switches the lights in the sequence: Green, Yellow, Red, Green, ..., unless interrupted by an officer. In the case when an officer interferes, he can choose Immediate_Green or Immediate_Red, and in both cases the system must go to the corresponding state and continue from then.

2 Trace inclusion and equivalence

Exercise 6. Compare the traces of $\mathcal{T}_1$ from the previous section and $\mathcal{T}_2$ given below.

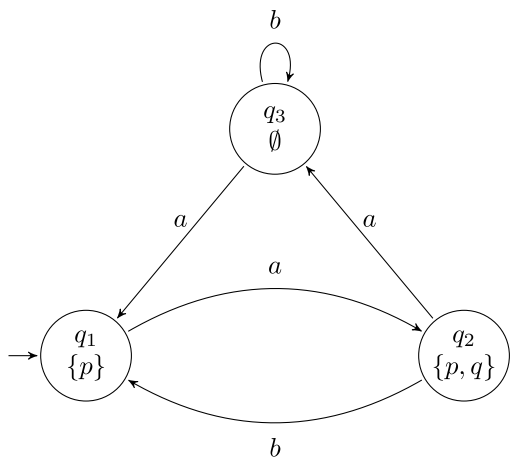

Figure 2: The transition system $\mathcal{T}_2 = (S_2, Act_2, \longrightarrow_2, I_2, AP_2, L_2)$.

Exercise 7. Same question for $\mathcal{T}_3$ and $\mathcal{T}_4$.

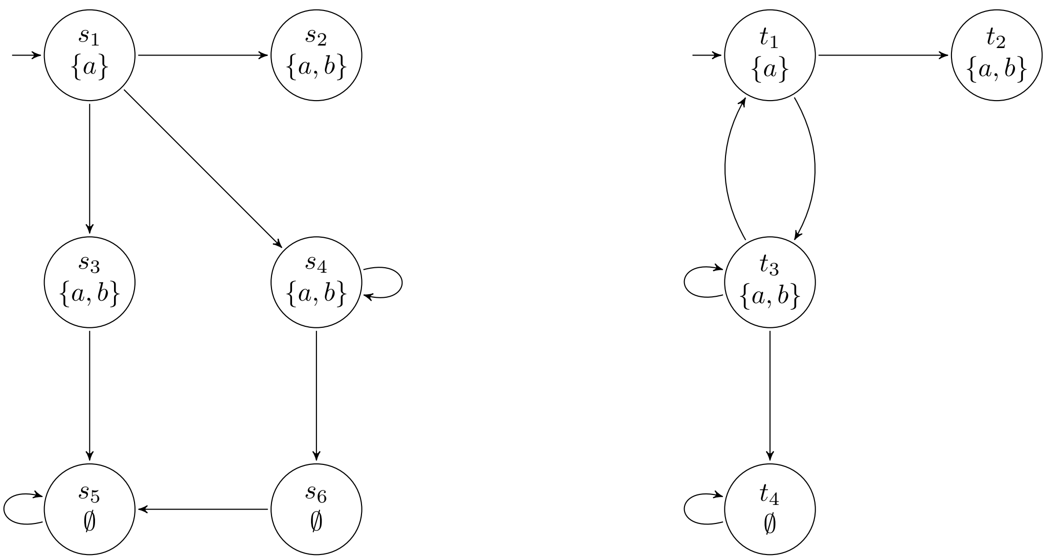

Figure 3: The transition systems $\mathcal{T}_3$ (left) and $\mathcal{T}_4$ (right).

3 Bisimulation

Definition 1. *Let* $\mathcal{T}_i = (S_i, Act_i, \longrightarrow_i, I_i, AP, L_i)$, $i = 1, 2$, *be TSs over AP. A* ***bisimulation*** *for* $(\mathcal{T}_1, \mathcal{T}_2)$ *is a binary relation* $\mathcal{R} \subseteq S_1 \times S_2$ *s.t.*

(A) $\forall\, s_1 \in I_1,\ \exists\, s_2 \in I_2,\ (s_1, s_2) \in \mathcal{R}$ *and* $\forall\, s_2 \in I_2,\ \exists\, s_1 \in I_1,\ (s_1, s_2) \in \mathcal{R}$

(B) for all $(s_1, s_2) \in \mathcal{R}$ *it holds:*

(1) $L_1(s_1) = L_2(s_2)$

(2) $s'_1 \in Post(s_1) \Longrightarrow \big(\exists\, s'_2 \in Post(s_2) \ \wedge\ (s'_1, s'_2) \in \mathcal{R}\big)$

(3) $s'_2 \in Post(s_2) \Longrightarrow \big(\exists\, s'_1 \in Post(s_1) \ \wedge\ (s'_1, s'_2) \in \mathcal{R}\big)$.

$\mathcal{T}_1$ and $\mathcal{T}_2$ are bisimulation-equivalent, or bisimilar, denoted $\mathcal{T}_1 \sim \mathcal{T}_2$, if there exists a bisimulation $\mathcal{R}$ for $(\mathcal{T}_1, \mathcal{T}_2)$.

Exercise 8. Are the two structures from Figure 4 bisimilar?

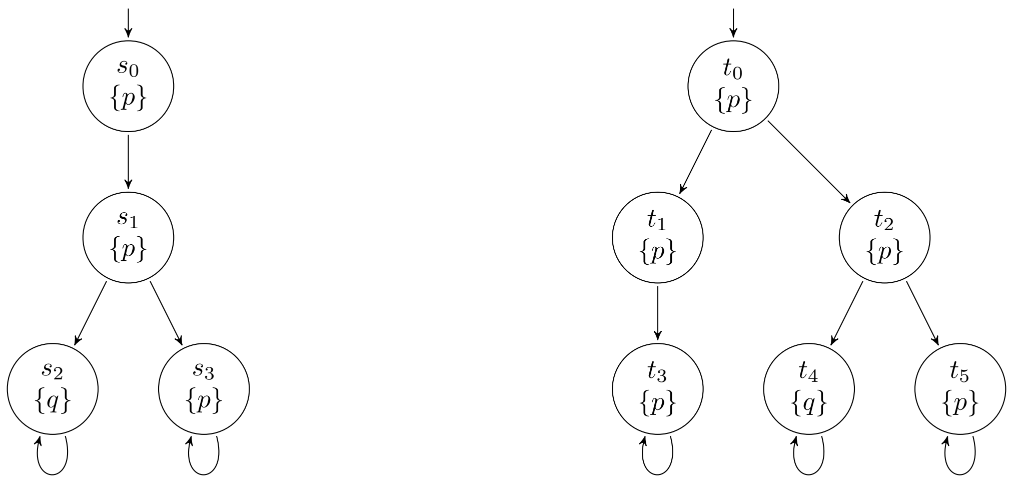

Figure 4: The transition systems $\mathcal{T}_5$ (left) and $\mathcal{T}_6$ (right).

Exercise 9. Give the greatest relation that satisfies conditions 1 to 3 of bisimulation relation (again, for the TSs in Figure 4).

Exercise 10. Same questions with the structures $\mathcal{T}_5$ (above) and $\mathcal{T}_7$ (Figure 5).

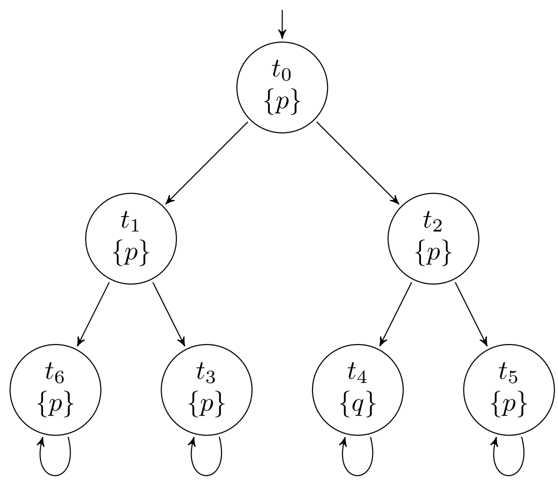

Figure 5: The transition system $\mathcal{T}_7$.

Exercise 11. Determine if $\mathcal{T}_6 \sim \mathcal{T}_7$ by quotienting the composite TS $\mathcal{T} = \mathcal{T}_6 \oplus \mathcal{T}_7$.

Exercise 12. Compute the quotient of $\mathcal{T}_6$ by the coarsest equivalence relation ρ compatible with its sets of transitions (use the algorithm presented in the lecture notes) to compute ρ.

Exercise 13. Prove that $\sim$ is an equivalence relation: reflexive, transitive, symmetric.

Exercise 14. Prove that $\mathcal{T} \sim \mathcal{T}/\sim$.

4 Simulation

Definition 2. *Let $\mathcal{T}_i = (S_i, Act_i, \longrightarrow_i, I_i, AP, L_i)$, $i = 1, 2$, be TSs over AP. A **simulation** for $(\mathcal{T}_1, \mathcal{T}_2)$ is a binary relation $\mathcal{R} \subseteq S_1 \times S_2$ s.t.*

(A) $\forall\, s_1 \in I_1,\ \exists\, s_2 \in I_2,\ (s_1, s_2) \in \mathcal{R}$

(B) for all $(s_1, s_2) \in \mathcal{R}$ it holds:

(1) $L_1(s_1) = L_2(s_2)$

(2) $s_1' \in Post(s_1) \implies \big(\exists\, s_2' \in Post(s_2)\ \wedge\ (s_1', s_2') \in \mathcal{R}\big)$

$\mathcal{T}_1$ is simulated by $\mathcal{T}_2$, or equivalently $\mathcal{T}_2$ simulates $\mathcal{T}_1$, denoted $\mathcal{T}_1 \preceq \mathcal{T}_2$, if there exists a simulation $\mathcal{R}$ for $(\mathcal{T}_1, \mathcal{T}_2)$.

Exercise 15. Show that $\preceq$ is transitive and reflexive.

Exercise 16. The relation $\preceq$ between transition systems is a preorder. Explain why it is, in general, not an order. Give a counter-example *(we assume that two transition systems $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal if they are isomorphic).*

Exercise 17. We recall that two TSs $\mathcal{T}_1$ and TS_2 are similar, denoted $\mathcal{T}_1 \simeq \mathcal{T}_2$, if $\mathcal{T}_1 \preceq \mathcal{T}_2$ and $\mathcal{T}_2 \preceq \mathcal{T}_1$. Prove that $\mathcal{T} \simeq (\mathcal{T}/\simeq)$.

Exercise 18. Build a TS $\mathcal{T}$ such that $\mathcal{T} \not\sim \mathcal{T}/\simeq$.

Exercise 19. Prove that $\mathcal{T}_1 \sim \mathcal{T}_2 \implies \mathcal{T}_1 \simeq \mathcal{T}_2$ and show that the converse does not hold, in general, using an example different from the one given during the lecture.

Exercise 20. Determine whether $\mathcal{M}_1 \simeq \mathcal{M}_2$ (see Figure 6) using the simulation preorder algorithm.

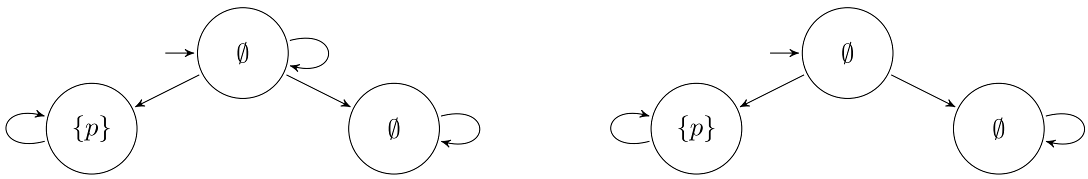

Figure 6: The transition systems $\mathcal{M}_1$ (left) and $\mathcal{M}_2$ (right). The set of propositions is $AP = \{p, q\}$.

Exercise 21. Compare the TSs from Figure 6 with $\preceq$.

Exercise 22. Compute $\mathcal{T}/\simeq$ for the TS from Figure 7 using the simulation preorder algorithm.

Exercise 23. Simulation implies language inclusion, i.e. $\mathcal{T}_1 \preceq \mathcal{T}_2$ implies $Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2)$, for all transition systems $\mathcal{T}_1, \mathcal{T}_2$ without terminal states. Prove this statement and show that the converse does not hold.

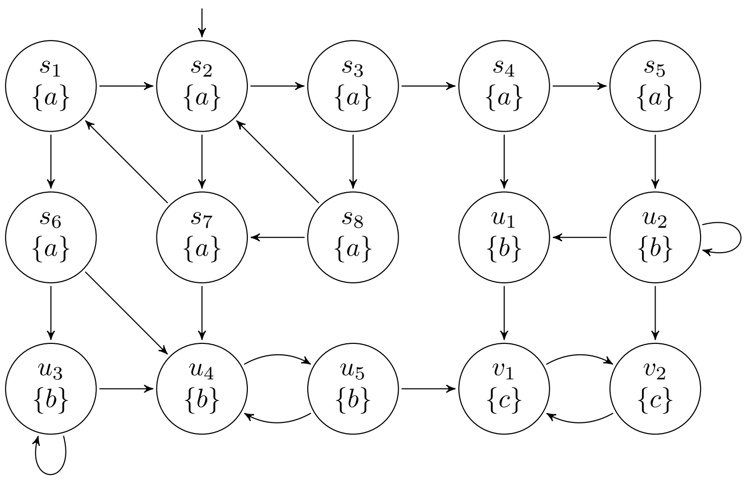

Figure 7: Complex TS with propositions $AP = \{a, b, c\}$.