PARTIAL DERIVATIVES

Aim:

- To write Matlab codes to find Partial derivative of a given function f(x,y) at a given point (x_1,y_1) and also visualize it

We define,

If f' is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$F_x(x,y)=\lim_{h\to 0}[f(x+h,y)-f(x,y)]/h$$
$$F_y(x,y)=\lim_{h\to 0}[f(x,y+h)-f(x,y)]/h$$

Notation:

Let z=f(x,y).

The partial derivative fx(x,y) can also be written as

$$\frac{\partial f}{\partial x}(x,y) \quad \text{Or} \quad \frac{\partial f}{\partial x}$$

Similarly, fy(x,y) can also be written as

$$\frac{\partial f}{\partial y}(x,y) \quad \text{Or} \quad \frac{\partial f}{\partial y}$$

The partial derivative fy(x,y) evaluated at the point (x0,y0) can be expressed in several ways:

$$fx(x0,y0), \frac{\partial f}{\partial x}|(x0,y0), \text{ or } \frac{\partial f}{\partial x}(x0,y0).$$

There are analogous expressions for fy(x0,y0).

Geometrical Meaning

Suppose the graph of z=f(x,y) is the surface shown. Consider the partial derivative of f with respect to x at a point (x0,y0).

Holding y constant and varying x, we trace out a curve that is the intersection of the surface with the vertical plane y=y0.

The partial derivative fx(x0,y0) measures the change in z per unit increase in x along this curve. That is, fx(x0,y0) is just the slope of the curve at (x0,y0). The geometrical interpretation of fy(x0,y0) is analogous.

MATLAB Syntax Used:

diff(f,x)	Differentiate the function with respect to x symbolically
R = subs(S, old, new)	Replaces old value with new value in the symbolic expression S.
line(X,Y,Z)	Creates a line object in the current axes with default values x = [0 1] and y = [0 1]. You can specify the color, width, line style, and marker type, as well as other characteristics.
Y = ones(n)	Returns an n-by-n matrix of 1s. An error message appears if n is not a scalar.

set(H,'*PropertyName*',PropertyValue,...)	Sets the named properties to the specified values on the object(s) identified by H. H can be a vector of handles, in which case set sets the properties' values for all the objects

MATLAB Code:

Partial Derivatives for functions two variables

1. Initialization:

```
 clc
clear all
format compact
syms x y
z = input('Enter the two dimensional function f(x,y): ');
x1 = input('enter the x value at which the derivative has to be
evaluated: ');
y1 = input('enter the y value at which the derivative has to be
evaluated: ');
```

2. Slope Calculation:

```
z1 = subs(subs(z,x,x1),y,y1)
ezsurf(z,[x1-2 x1+2])
f1 = diff(z,x)
slopex = subs(subs(f1,x,x1),y,y1);
```

3. Visualization of the plane in which the partial derivative is sought:

```
[x2,z2]=meshgrid(x1-2:.25:x1+2,0:0.5:10);
y2=y1*ones(size(x2));
hold on
h1=surf(x2,y2,z2);
set(h1,'FaceColor',[0.7,0.7,0.7],'EdgeColor','none')
```

4. The Tangent line:

```
t=linspace(-1,1);
x3=x1+t;
y3=y1*ones(size(t));
z3=z1+slopex*t;
line(x3,y3,z3,'color','blue','linewidth',2)
```

Practice Problems:

1) Find the partial derivatives of F(x,y)=x^3+y^3+6xy-1 with respect to y at the point (1,1)

Output:

Enter the two dimensional function f(x,y):
x^3+y^3+6*x*y-1

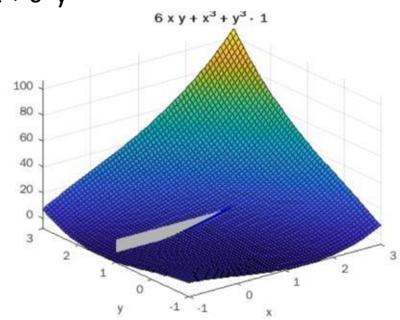
enter the x value at which the derivative has to be evaluated: 1

enter the y value at which the derivative has to be evaluated: 1

z1= 7

f1 =

3*x^2 + 6*y



2) Find the partial derivative of F(x,y)=4-x^2-2y^2 with respect to y at the point (1,1)

Output:

Enter the two dimensional function f(x,y):
4-x^2-2*y^2

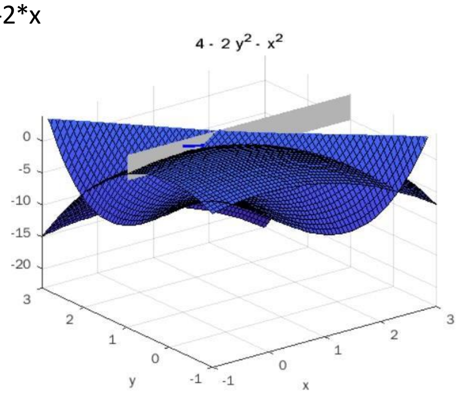
enter the x value at which the derivative has to be evaluated: 1

enter the y value at which the derivative has to be evaluated: 1

z1= 1

f1 =

-2*x



LOCAL MAXIMA AND MINIMA

Aim:

- To find the Maximum and Minimum values(Extreme values) for the given function f(x,y) using MATLAB

Mathematical form:

Let z=f(x,y) be the given function. Critical points are points in the xy-plane where the tangent plane is horizontal. The tangent plane is horizontal, if its normal vector points in the z direction. Hence, critical points are solutions of the equations: $f_x(x,y)=0$ and $f_y(x,y)=0$.

Procedure for finding the maximum or minimum values of f(x,y):

(1) For the given function f(x,y) find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and equate it to zero and solve them to find the roots (x1,y1),(x2,y2),- - - - - These points may be maximum or minimum points.

(2) Find the values $r=\frac{\partial^2 f}{\partial x^2},\ s=\frac{\partial^2 f}{\partial x \partial y}, t=\frac{\partial^2 f}{\partial y^2}$ at these points.

(3) (a) If $rt-s^2>0$ and $r<0$ at a certain point, then the function is maximum at that point
(b)If $rt-s^2>0$ and $r>0$ at a certain point , then the function is minimum at that point
(c) If $rt-s^2<0$ for a certain point, then the function is neither maximum nor minimum at that point. This point is known as saddle point.
(d) If $rt-s^2=0$ at a certain point, then nothing can be said whether the function is maximum or minimum at that point. In this case further investigation are required

Example:

Obtain the maximum and minimum values of $f(x,y)=2(x^2-y^2)-x^4+y^4$

Solution: $f(x,y)= 2(x^2-y^2)-x^4+y^4$

$$\frac{\partial f}{\partial x}=4x-4x^3$$

$$\frac{\partial f}{\partial y}=4y^3-4y$$

$$r=\frac{\partial^2 f}{\partial x^2}=4-12x^2$$

$$s=\frac{\partial^2 f}{\partial x \partial y}=0$$

$$t=\frac{\partial^2 f}{\partial y^2}=12y^2-4y$$

$$\frac{\partial f}{\partial x}=0 \Rightarrow 4x-4x^3=0, \frac{\partial f}{\partial y}=0 \Rightarrow 4y^3-4y=0$$

S.No	Critical Points	r	D=rs-t²	Remarks
1	(0,0)	4	-16 < 0	Saddle Point
2	(0,1)	4	32	Minimum
3	(0,-1)	4	32	Minimum
4	(1,0)	-8	32	Maximum
5	(1,1)	-8	-64 < 0	Saddle Point

6	(1,-1)	-8	-64 < 0	Saddle Point
7	(-1,0)	-8	32	Maximum
8	(-1,1)	-8	-64 < 0	Saddle Point
9	(-1,-1)	-8	-64 < 0	Saddle Point

The Minimum value of f(x,y) is -1 at (0,1) & (0,-1) and the Maximum value for f(x,y) is +1 at (1,0) & (-1,0)

MATLAB Syntax used:

Diff	diff(expr) differentiates a symbolic expression expr with respect to its free variable as determined by symvar.
Solve	Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings
Size	Dimensions of data and model objects and to access a specific size output.
Imag	Imaginary part of complex number, Y=imag(Z) returns the imaginary part of the elements of array Z.
Figure	Create figure graphics object, Figure objects are the individual windows on the screen in which the MATLAB software displays graphical output
Double	Convert to double precision, double(x) returns the double-precision value for X. If X is already a double-precision array, double has no effect.
Sprint	Format data into string. It applies the *format* to all elements of array *A* and any additional array arguments in column order, and returns the results to string *str*.
Solve	Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings.
Ezsurf	Easy-to-use 3-D colored surface plotter, ezsurf(fun) creates a graph of fun(x,y) using the surf function. fun is plotted over the default domain: -2π <x< 2π, -2π <y< 2π.
plot3	The plot3 function displays a three-dimensional plot of a set of data points.

MATLAB Code:

INPUT	Enter the function f(x,y)
OUTPUT	Maxima , Minima values for the given function and Graph for visualizing its solution

```
% Local maxima and minima for two variables
clc
clear all
close all
format compact
%%
syms x y real
f =input('Enter the function f(x,y): ');
fx = diff(f,x)
fy = diff(f,y)
[ax ay] = solve(fx,fy)
fxx = diff(fx,x)
D = fxx*diff(fy,y) - diff(fx,y)^2%ln-m^2

%% Collecting critical points
```

```
r=1;
for k=1:1:size(ax)
if ((imag(ax(k))==0)&&(imag(ay(k))==0))
ptx(r)=ax(k);
pty(r)=ay(k);
r=r+1;
end
end

%% Visulalizing the function

a1=max(double(ax))
a2=min(double(ax))
b1=max(double(ay))
b2=min(double(ay))
ezsurf(f,[a2-.5,a1+.5,b2-.5,b1+.5])
colormap('summer');
shading interp
hold on

%% Finding the maximum and minimum values of the function and their
visulaization

for r1=1:1:(r-1)
T1=subs(subs(D,x,ptx(r1)),y,pty(r1))
T2=subs(subs(fxx,x,ptx(r1)),y,pty(r1))
if (double(T1) == 0)
sprintf('The point (x,y) is (%d,%d) and need further investigation',
double(ptx(r1)),double(pty(r1)))
elseif (double(T1) < 0)
T3=subs(subs(f,x,ptx(r1)),y,pty(r1))
sprintf('The point (x,y) is (%d,%d) a saddle point', double(ptx(r1)),double(pty(r1)))
plot3(double(ptx(r1)),double(pty(r1)),double(T3),'b.','markersize',30);
else
if (double(T2) < 0)
sprintf('The maximum point(x,y) is (%d, %d)', double(ptx(r1)),double(pty(r1)))
T3=subs(subs(f,x,ptx(r1)),y,pty(r1))
sprintf('The value of the function is %d', double(T3))
plot3(double(ptx(r1)),double(pty(r1)),double(T3),'r+','markersize',30);
else
sprintf('The minimum point(x,y) is (%d, %d)', double(ptx(r1)),double(pty(r1)))
T3=subs(subs(f,x,ptx(r1)),y,pty(r1))
sprintf('The value of the function is %d', double(T3))
plot3(double(ptx(r1)),double(pty(r1)),double(T3),'m*','markersize',30);
end
end
end
```

Practice Problems:

1) Find the maximum and minimum value of F(x,y)= $2x^3$+xy^2+$5x^2$+y^2

Output:

```
Enter the function f(x,y): 2*x^3+x*y^2+5*x^2+y^2
a1 =
     0
a2 =
```

```
    -1.6667
b1 =
      2
b2 =
    -2
ans =
    'The minimum point(x,y) is (0, 0)' T3 =
0
ans =
    'The value of the function is 0' T3 =
3
```

ans =

'The point (x,y) is (-1,-2) a saddle point' T3 =

3

ans =

'The point (x,y) is (-1,2) a saddle point' ans =

'The maximum point(x,y) is (-1.666667e+00, 0)' T3 =

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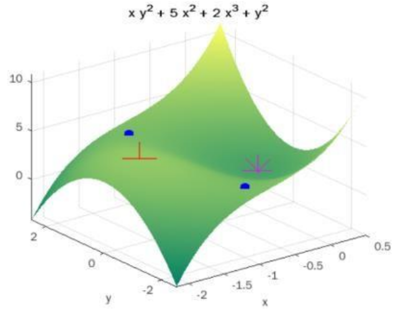
ans =

'The value of the function is 4.629630e+00'



2) Find the local maximum and minimum values of $f(x,y)=2(x^2-y^2)-x^4+y^4$

Output:

Enter the function f(x,y): 2*(x^2-
y^2)-x^4+y^4

a1 =

1

a2 =

-1

b1 =

1

b2 =

-1

T3 = 0

ans =

'The point (x,y) is (0,0) a saddle point' ans =

'The maximum point(x,y) is (-1, 0)' T3 =

1

ans =

'The value of the function is 1'

ans =

'The maximum point(x,y) is (1, 0)' T3 =

1

ans =

'The value of the function is 1' ans =

'The minimum point(x,y) is (0, -1)' T3 =

-1

ans =

'The value of the function is -1' ans =

'The minimum point(x,y) is (0, 1)' T3 =

-1

ans =

'The value of the function is -1' T3 =

0

ans =

'The point (x,y) is (-1,-1) a saddle point' T3 =

0

ans =

'The point (x,y) is (1,-1) a saddle point' T3 =

0

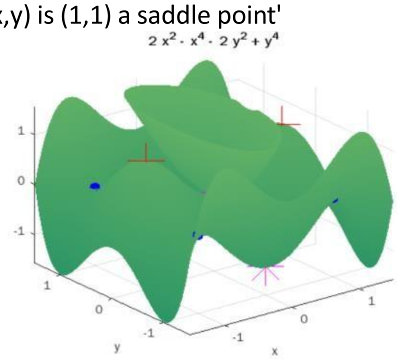
ans =

'The point (x,y) is (-1,1) a saddle point' T3 =

0

ans =

'The point (x,y) is (1,1) a saddle point'



Lagrange Multipliers

1. Initialization:

```
syms x y lam real

f= x^2+2*y^2 %input('Enter f(x,y) to be extremized : ');

g= x^2+y^2-1 %%input('Enter the constraint function g(x,y) : ');
```

2. Computing Partial derivatives and finding the critical points:

```
F=f-lam*g
```

```
Fx=diff(F,x)
Fy=diff(F,y)
[ax,ay,alam]=solve([Fx,Fy,g],x,y,lam)
ax=double(ax)
ay=double(ay)
%% Collecting critical points
r=1;
for k=1:1:size(ax)
      if ((imag(ax(k))==0)&&(imag(ay(k))==0))
      ptx(r)=ax(k);
      pty(r)=ay(k);
      r=r+1;
      end
end
```

3. Computing the values at the critical points

```
ax=ptx
ay=pty
T = subs(f,{x,y},{ax,ay})
T=double(T)
epx=3
epy=3
figure (1)
for i = 1:length(T)
D=[ax(i)-epx ax(i)+epx ay(i)-epy ay(i)+epy]
fprintf('The critical point (x,y) is (%1.3f,%1.3f).',ax(i),ay(i))
fprintf('The value of the function is %1.3f\n',T(i))
ezcontour(f,D)
hold on
h = ezplot(g,D);
set(h,'Color',[1,0.7,0.9])
plot(ax(i),ay(i),'k.','markersize',15+2*i)
end
```

4. Finding the Maximum and minimum at those points:

```
f_min=min(T)
f_max=max(T)
```

Practice Problems:

1. **Find the extreme values of the function $(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.**

Output:

Enter f(x,y) to be extremized : 3*x+4*y

Enter the constraint function g(x,y) : x^2+y^2-1

Fd =

[3 - 2*lam*x, 4 - 2*lam*y, - x^2 - y^2 + 1] ax =

-3/5

3/5

ay =

-4/5

4/5 alam

=

-5/2

5/2

T =

-5

5

T =

-5

5

epyu =

0.8000

D =

-1.1000 1.1000 -1.3000 1.3000

The critical point (x,y) is (-0.600,-0.800).The value of the function is -5.000 The critical point (x,y) is (0.600,0.800).The value of the function is 5.000

TT =

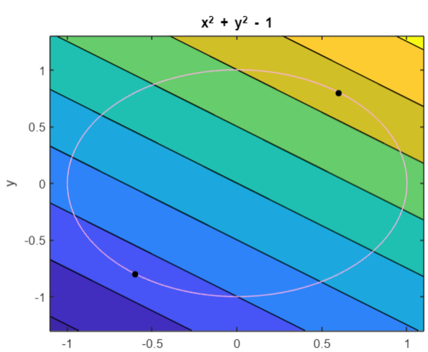
-5

5

f_min =

-5

f_max = 5



2. Find the extreme values of the function $(x,)$=xy on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Output:

Enter f(x,y) to be extremized :
x*y
Enter the constraint function g(x,y) :
(x^2)/8+(y^2)/2-1

Fd =

[y - (lam*x)/4, x - lam*y, - x^2/8 - y^2/2 + 1] ax =

2
-2
-2
2

ay =

-1
1
-1
1

alam =

-2
-2
2
2

T =

-2
-2
2
2

T =

-2
-2
2
2

epyu =

1

D =

-2.5000 2.5000 -1.5000 1.5000

The critical point (x,y) is (2.000,-1.000).The value of the function is -2.000 The critical point (x,y) is (-2.000,1.000).The value of the function is -2.000 The critical point (x,y) is (-2.000,-1.000).The value of the function is 2.000 The critical point (x,y) is (2.000,1.000).The value of the function is 2.000 TT =

-2

-
2
2
2

f_min =

-2

f_max =

2