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INTERMEDIATE FAQ

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1 Output

Pre-climate change output $Y(t)$ depends on a (constant-returns-to-scale) Cobb-Douglas production function in technology, capital and labor, denoted by $A(t), K(t)$ and $L(t)$, respectively.

$$Y(t) = A(t)K(t)^{\gamma}L(t)^{1-\gamma}$$

The capital share, γ, is set to 30%. Final usuable output, $Q(t)$, is pre-damages output reduced by a fraction $\Omega(t) < 1$ due to harms from climate change, and by resources spent on abatement, $\Lambda(t) < 1$.

$$Q(t) = \Omega(t)[1 - \Lambda(t)]A(t)K(t)^{\gamma}L(t)^{1-\gamma}.$$

Each of the components of output is discussed below.

1.1 Population growth

$$L(t) = L(0) + [L(Tmax) - L(0)]\left(1 - \frac{1}{e^{L_g \times (t-1)}}\right).$$

The population in period t, $L(t)$ evolves according to the weighted average formula between the initial population in year 2005, $L(0)$, and the asymptotic population in the last year we consider, $L(Tmax)$. Users choose the asymptotic population $L(Tmax)$ given the population growth rate, L_g and the initial population in 2005, $L(0)$.

1.2 Capital

The amount of capital available for use depends on the depreciated capital from the prior period and the amount saved.

$$K(t) = I(t) + (1 - \delta_K)K(t-1),$$

where δ_K is the rate of depreciation of the capital stock. The default level of this δ_K is set to 10%. Users can change this to any value between 0.08 and 0.2.

Individuals save a fixed portion of their output, $I(t) = sQ(t)$, and output is split between consumption and savings.

$$Q(t) = C(t) + I(t).$$

The model assumes that individuals save 20% of output but users can change this to any value between 15% and 25%. Some versions of DICE solve for the optimal savings rate, but Nordhaus reports that the amount is relatively insensitive to assumptions so we leave it as a user-chosen parameter.

1.3 Total Factor Productivity

Total factor productivity (TFP, or $A(t)$) describes how efficiently capital and labor produce output. The initial value, $A(0)$, is set to produce 2005 output given the observed 2005 capital and labor inputs. TPF is assumed to evolve according the equation:

$$A(t) = \frac{A(t-1)}{1 - A_g(t-1)},$$

for $t \geq 1$ given the inital technology level, $A(0)$. The model assumes that productivity growth will slow down in the coming decades. In particular, the TFP growth rate is:

$$A_g(t) = A_g(0) \times exp(-\Delta_a \times (t-1)),$$

where users choose Δ_a which measures the decline rate of this productivity given the initial TFP level of $A_g(0)$.

More formally, Δ_a is percentage growth rate of $A_g(t)$, the growth rate of TFP. To see this note that

$$\frac{d \ln A_g(t)}{dt} = \underbrace{\frac{A_g(t+1) - A_g(t)}{A_g(t)}}_{\text{growth rate in \%}} = -\Delta_a$$

Figure 1 shows how the evoluation of TFP changes for a choice of Δ_a.

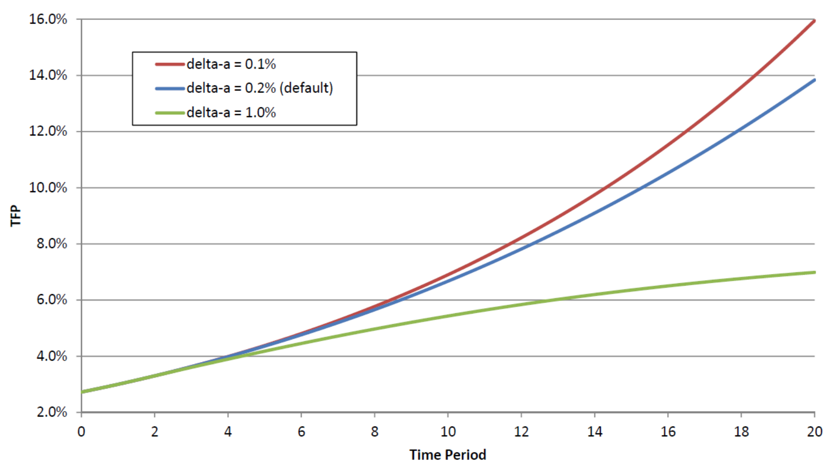

Figure 1: Evolution of TFP for choice of Δ_a

1.4 Damages

The damage function measures the share of usable output after subtracting the harms from temperature increases due to climate change. The default model assumes that the damage function has the following functional form:

$$\Omega(t) = \frac{1}{1 + \pi_2 T_{AT}(t)^{\varepsilon}}.$$

Users choose π_2 and ε. ε is the exponent on the global mean surface temperature and set to 2.0 as a default. Users can set ε in the range $[1, 3]$. π_2 can be choosen to be any value between 0.002 and 0.0035 with the default value of 0.0028. Under these default values, the harm from a 2.5 degree temperature change is 1.7% of output, which is based on the estimates in DICE.

In future versions of webDICE, users will be able to choose alternative forms of the damage function such as exponential damages and also choose how damages affect the economy. The existing implementation, following DICE, treats damages as creating unusable output. Harms from climate change affect capital only through a reduction in savings and they do not change the productivity of the economy. Future versions will allow users to choose damages that reduce the capital stock directly or that change the productivity growth.

1.5 Abatement costs

$$\Lambda(t) = \theta_1(t)\pi(t)\mu(t)^{\theta_2} = \theta_1(t)\varphi(t)^{1-\theta_2}\mu(t)^{\theta_2}.$$

The abatement cost $\Lambda(t)$ is the fraction of output spent on keeping emissions below a specified level chosen under a climate policy. It is the cost of reducing emissions of GHGs, measured as a fraction of total output.

There are three components to abatement costs: the emissions control rate $\mu(t)$, the participation fraction $\varphi(t)$, and the cost of replacing all fossil fuels with carbon-free energy (the backstop technology) $\theta_1(t)$. Abatement costs go up with the control rate, go up as participation goes down, and go down as low-carbon energy technology improves.

1.5.1 The emisisons control rate

The emissions control rate $\mu(t)$ is the emissions reductions expressed as a fraction of total emissions, $\mu(t) \in [0, 1]$. To represent increasing marginal costs, $\mu(t)$ is raised to the power θ_2. The default value of θ_2 is 2.8. Users can change this number to anything between 2 and 4.

Under the business as usual scenarios, $\mu(t) = 0$, because those scenarios assume no controls on emissions. Abatement costs will be positive in two circumstances: if the model is run in optimization mode or if the user sets emissions controls. In optimization mode, the model solves for the path of $\mu(t)$ that maximizes the discounted sum of utility over time.

Users have the option of simulating a climate treaty by setting carbon caps in three years, 2050, 2100, and 2150, denoted by $e2050cap$, $e2100cap$, and $e2150cap$, respectively. The caps are specified as a percentage of emisssions in 2005. For example, if $e2050cap = 150\%$ emissions in the years 2050 to 2100 can be no more than 150% of 2005 emissions, and if $e2100cap = 70\%$, emissions in the years 2100 to 2150 can be no more than 70% of 2005 emissions. Users can set the caps anywhere from 100% (which caps emissions at the 2005 levels) to 0% (which reduces emissions to zero).

The model implements the caps by solving for $\mu(t)$ to produce the specified levels of emissions. Specifically, the model induces the controlled emissions rate $\mu(t) > 0$ from the carbon caps set by users whenever emissions of a year exceeds the cap in the threshold years of 2050-2100, 2100-2150, and 2150-$Tmax$. That is, given the equation for industrial emissions (see section 1.6) $E_{ind}(t) = \sigma(t)[1-\mu(t)]A(t)K(t)^{\gamma}L(t)^{1-\gamma}$, we set $\mu(t)$ as:

$$\mu(t) = \begin{cases} 0 & if \quad E_{ind}(t) < E_{ind}(2005) \times ecap \\ 1 - \frac{E_{ind}(2005) \times ecap}{\sigma(t)A(t)K(t)^{\gamma}L(t)^{1-\gamma}} & if \quad E_{ind}(t) > E_{ind}(2005) \times ecap \end{cases}$$

For instance, consider the year 2070. This year should obey the emissions cap of the year 2050. Thus, $\mu(2070) = 1 - \frac{E_{ind}(2005) \times e2050cap}{\sigma(2070)A(2070)K(2070)^{\gamma}L(2070)^{1-\gamma}}$ if $E_{ind)}(2070) > E_{ind}(2005) \times e2050cap$. If we have more emissions than the chosen amount for a given year, we would like to decrease it by controlling the reduction rate of $\mu(t)$.

One drawback is that the model uses step function setting caps in each threshold year and assuming that those caps apply for future years, until another threshhold year comes. Abrupt reduction of emissions might be more costly to implement than smooth reduction in practice. The step function, however, reflects common practice in treaties which specify target years.

1.5.2 The participation fraction

The participation fraction, $\varphi(t) \in [01]$ is the fraction of emissions that are subject to control. For example, if less than all countries participate in an emissions control regime or some industries are exempt, $\varphi(t) < 1$. To represent the increasing marginal costs of controlling emissions when participation is less than full, this rate is raised to $1 - \theta_2$, which (after adjusting for the sign) is the same marginal costs as the emissions control rate. We can think of $\varphi(t)^{1-\theta_2}$ as the participation markup in abatement costs.

Users specify the participation fractions for the years 2015 and 2205. The model interpolates an increasing fraction of participation for the period beween 2015 and 2205 according to:

$$\varphi(t) = \begin{cases} \varphi(1) & if \quad t = 1 \\ \varphi(21) + [\varphi(2) - \varphi(21)] \times exp(-DFE \times (t-2)) & if \quad t = 2, .., 24 \\ \varphi(21) & if \quad t = 25, ... \end{cases}$$

The participation fraction in DICE-2007 was motivated by the Kyoto Protocol that was initially adopted in year 1997 and entered into force in 2005. Under the Kyoto, 37 Annex I contries committed themselves to a reduction of greenhouse gases in phase I. The participation fraction in 2005 allows DICE to model this. The fraction can change in 2015 to represent a possible extension and modification of the Kyoto protocol. The final target allows the model to consider long term effects with gradually increasing participation.

The participation fraction is set to 1 in the current version of webDICE. In future versions, users will be able to modify the fraction.

1.5.3 Backstop technology

The final component, $\theta_1(t)$, is the cost of replacing all fossil fuels with clean energy at time t, called the backstop technology. Its initial cost of the backstop technology is set to $\$1,170/ton\ CO_2$. That is, if $\mu(0)$ were set to be 1, abatment costs would be emissions multiplied by $\$1,170$. The cost of the backstop technology is assumed to decline by $\$585/ton\ CO_2$ to a final value that is half the initial value under the default. The price of the backstop technology evolves according to:

$$\theta_1(t) = \left[\frac{BC(0) \times \sigma(t)}{\theta_2}\right] \times \left[\frac{ratio - 1 + \exp(-BC_g(0) \times (t-1))}{ratio}\right].$$

If $BC(reduct)$ is the dollar decline in the price of the backstop technology between the current period and the final period, $ratio = BC(0)/BC(reduct)$. (This formulation, while awkward, folows the notation in DICE 2007.) Users can set $ratio$ and the rate of decline, $BC_g(0)$, based on their views about how fast we will develop clean energy. The defaults are $rataio = 2$ and $BC_g(0) = 5\%$.

The marginal cost of abatement at a control rate of 100% is equal to the backstop price for each year. To see, this note that:

$$\begin{aligned} BC(t) = MC(\Lambda)|_{\mu(t)=1} &= \frac{\partial \Lambda}{\partial \mu(t)}|_{\mu(t)=1} = \theta_1(t)\theta_2 \\ &= \underbrace{BC(0)}_{(1)} \times \underbrace{\sigma(t)}_{(2)} \times \underbrace{\left[\frac{ratio - 1 + \exp(-BC_g(0) \times (t-1))}{ratio}\right]}_{(3)}. \end{aligned}$$

The backstop cost in period t is proportional to the three factors. Factor (1) is a initial condition that can be based on data. Factor (2) is the carbon intensity, and is decreasing over time; as economy gets less carbon intense, the marginal cost of shifting all the fossil fuels into alternative energy decreases. Factor (3) represents improvement in the backstop technology, such as less expensive solar power. Notice $\frac{\partial(3)}{\partial t} < 0$ so that (3) declines in time, reducing the cost of the backstop.

To understand how setting $ratio$ determines the final price, note that a $t = 1$, $BC(1) = BC(0) \times \sigma(1)$, which is the current backstop price multiplied by the emissions intensity. As $t \to \infty$, $BC(t) \to [BC(0) - BC(reduct)] \times \sigma(t)$. The greater the reduction in costs, the lower the $ratio$. When $ratio = 1$, the final cost of the backstop technology is \$0. When $ratio = 2$, (as in the default), the final cost of the backstop technology is half the initial cost, and similarly for higher values of $ratio$. Figure 2 illustrates.

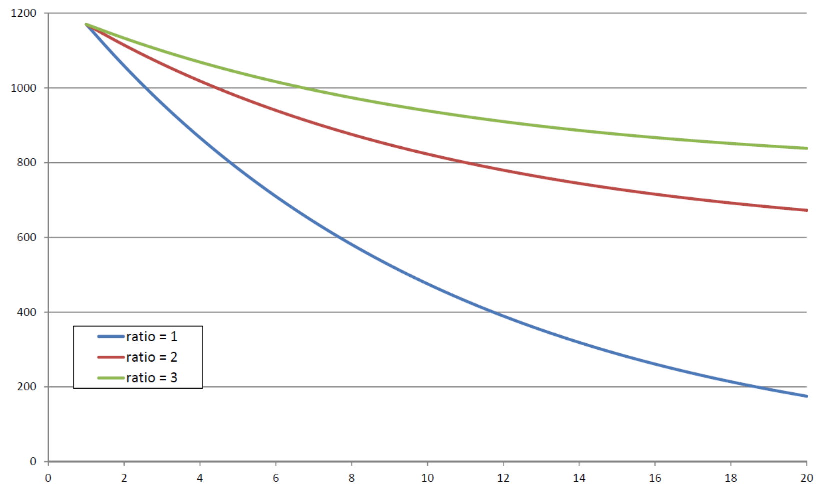

Figure 2: Evolution of backstop price (in \$1,000) for choice of $ratio$, $BC_g(0) = 10\%$

1.6 Carbon intensity and emissions

Economic activity produces emissions. Emissions are equal to the carbon intensity of the economy, $\sigma(t)$ multiplied by output, and reduced by emissions controls $\mu(t)$.

$$E_{Ind}(t) = \sigma(t)[1 - \mu(t)]A(t)K(t)^{\gamma}L(t)^{1-\gamma}.$$

The carbon intensity of the economy is assumed to decline (even without emissions policies) due to improvements in energy efficiency, according to:

$$\sigma(t) = \frac{\sigma(t-1)}{1 - \sigma_g(t)},$$
$$\sigma_g(t) = -\sigma_g(0) \times exp(-\sigma_{d1} \times (t-1)),$$

given $\sigma(0)$. $\sigma_g(t) < 0$ is the rate of decarbonization. This rate of carbon-saving technological change is assumed to slow down at the user-chosen value of σ_{d1}. The default is 0.003. Figure 3 shows the evolution of carbon intensity for the choice of σ_{d1}.

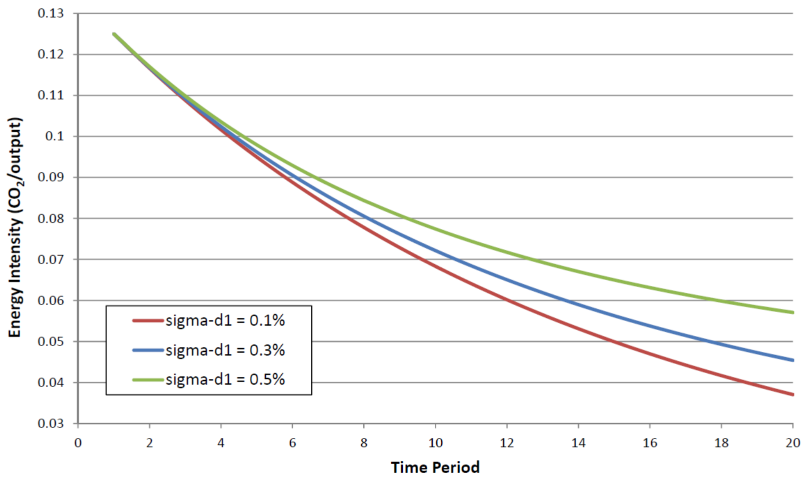

Figure 3: Evolution of energy intensity for choice of σ_{d1}

Total emissions are limited to the known fossil fuel reserves, set at a default value of 6,000 gigatons of carbon dioxide: $CCum \geq \sum_{t=0}^{Tmax} E_{Ind}(t)$. Users can increase this value based on their views about likely discoveries of additional fossil fuel reserves.

Emissions also arise from deforestation. Deforestation is currently exogenous to the model and is assumed to decline over time according to

$$E_{Land}(t) = E_{Land}(0)(1 - 0.1)^{t-1}.$$

Emissions can be reduced through a wide range of policies parameterized by the emission caps of years 2050, 2100, and 2150.

2 Carbon cycle

DICE simulates the carbon cycle using a linear three-reservoir model where the three reservoirs are the deep ocean, the upper ocean and the atmosphere. Each of these reservoirs is well-mixed in the short run and a transition matrix governs the transfer of carbon among the reservoirs. If $M_{i,t}$ is the mass of carbon (in gigatons) in reservoir i, then:

$$\begin{bmatrix} M_{A,t} \\ M_{U,t} \\ M_{L,t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} & 0 \\ 1-\phi_{11} & 1-\phi_{21}-\phi_{23} & \phi_{32} \\ 0 & \phi_{23} & 1-\phi_{32} \end{bmatrix} \begin{bmatrix} M_{A,t-1} \\ M_{U,t-1} \\ M_{L,t-1} \end{bmatrix} + \begin{bmatrix} E_{t-1} \\ 0 \\ 0 \end{bmatrix},$$

where the parameters $\phi_{i,j}$ represent the transfer rate from resevoir i to resevoir j (per time period), and E_t is emissions at time t. The model only includes for CO_2 in its emissions factor and atmospheric carbon concentration. Other greenhouse gases are assumed to be exogenous and enter the forcing equation separately (see below).

The relationship between greenhouse gases and radiative forcing $F(t)$ is given by

$$F_t = \eta \frac{\ln\left(M_{A,avg}\right) - \ln\left(M_A^{1750}\right)}{\ln 2} + F_{Ex,t},$$

where $F_{EX,t}$ represents forcing from gases other than CO_2. These other forcings are estimated in a given period by a linear function of the difference between forcings from non-CO_2 today and an estimate of those forcings in the year 2100:

$$F_{EX,t} = F_{EX,0} + 0.1\left(F_{EX,2100} - F_{EX,0}\right) \cdot t.$$

The forcing equation uses an average of the mass of carbon in the atmosphere determined based on emissions in the prior period, $M_{A,t}$ and the mass of carbon in the atmosphere based on emissions in the current period, $M_{A,t+1}$. That is, $M_{A,avg} = \left(M_{A,t} + M_{A,t+1}\right)/2$. Using current emissions to (in part) to determine forcing is an adjustment to correct for the relatively long 10-year time steps used in the model.

Radiative forcing leads the warming in the atmosphere, which then warms the upper ocean, gradually warming the deep ocean. The model is:

$$\begin{aligned} T_t &= T_{t-1} + \xi_1\left[F_t - \lambda T_{t-1} - \xi_2\left(T_{t-1} - T_{L,t-1}\right)\right] \\ T_{L,t} &= T_{L,t-1} + \xi_3\left(T_{t-1} - T_{L,t-1}\right) \end{aligned}$$

where the ξ_i are the transfer coefficients reflecting the rates of flow and thermal capacities of the sinks. In particular, $1/\xi_1$ is the thermal capacity of the atmosphere and the upper oceans, $1/\xi_3$ is the transfer rate from the upper ocean to the deep ocean, and ξ_2 is the ratio of the thermal capacity of the deep oceans to the transfer rate from the shallow to deep ocean.
The key parameter is λ, or climate sensitivity, is a way of representing the equilibrium temperature from doubling the concentration of CO_2. If we solve the temperature equation for a constant temperature (i.e. equilibrium), we get $\Delta T/\Delta F = 1/\lambda$. If T_{2xCO_2} is the equilibrium impact of a doubling of CO_2 concentrations, we get $T_{2xCO_2} = \Delta F_{2xCO_2}/\lambda$ where ΔF_{2xCO_2} is the change in radiative forcing from a doubling of CO_2. Therefore, setting λ allows us to set the climate sensitivity.

The carbon cycle model in DICE is highly simplified compared to other commonly used geophysical models that concentrate their efforts on a more accurate representation of the physics and chemistry of the carbon-climate system. These simplifications were designed to allow the model to run quickly, a feature that is particularly important during optimization mode. The simplifications raise concerns about the accuracy of the model, particularly over long time periods. Future versions of webDICE will include more accurate models of the carbon cycle.

3 Optimization

Users can run the model in either simulation mode or optimization. In simulation mode, the model projects outcomes based on the chosen parameter values. In optimization mode, the model finds the emissions control rate $\mu(t)$ that maximizes the discounted sum of utility given the assumptions in the model about output, emissions, and harm. It then translates these controls into the equivalent carbon tax rate, which is:

$$\tau(t) = \theta_1(t)\mu(t)^{\theta_2 - 1},$$

(assuming participation is complete, as it would be under an optimal control regime).

3.1 Objective Function

The objective function in the model is the discounted sum of utility:

$$W = \sum_{t=1}^{Tmax} U[c(t), L(t)]R(t).$$

Utility is

$$U\left[c(t), L(t)\right] = L(t)\left[c(t)^{1-\alpha}/(1-\alpha)\right].$$

where α is the elasticity of marginal utility. α is set by default to be 2 but users can set it between 1 and 3. Higher values of α mean that marginal utility declines faster with increases in income. $c(t) = C(t)/L(t)$ is per capita consumption.

$R(t) = 1/[(1+\rho)^{(10\times(t-1))}]$ is the discount factor applied to utility. It is based on ρ, the pure rate of time preference or the utility discount rate. ρ is set by default to be 1.5% based on the value used in DICE, but users can set it to be any value between 0 and 4.

The discount rate is one of the most controversial components of climate policy as it in part determines the trade-offs between present and future generations. In addition, because it acts exponentially, small changes in its value can have large changes in policy. For this reason, we suggest users choose their preferred value but also to test how optimal policies change when they choose different values.

Our default values, $\rho = 1.5\%$ and $\alpha = 2$, combined with the implicit growth rate 2% using the default parameters, imply real return to capital of 5.5%. This can be changed by changing ρ and α as well as changing default parameters that affect growth.

Users should be aware that utilities are not comparable for different values of α, the elasticity of the marginal utility of consumption. Increasing or decreasing α may change utility for the same economic output even in a single period and comparisons of the raw utility values are not necessarily meaningful.

It is not clear that there is a method of normalizing utility for different values of α that makes the comparisons meaningful.

3.2 Computational considerations

The model is implemented in Python. We chose Python because it is widely used and free. The source code is available on the website. Optimization uses the Python implementation of open-source solver Ipopt using the linear solver MA27 from HSL. We use Amazon's EC2 to host the website, and are greatful to Amazon for a generous grant to support our use of EC2.

Nordhaus's original model was written in both GAMS and Excel. Our Python code has been verified against Nordhaus's GAMS code.

The model uses 10-year time steps and runs for 60 periods. We only display output for the first 20 time steps or 200 years. Simulating the economy for 200 years is highly speculative so displays of results this far in the future need to be understood as a way of understanding the possible effects of assumptions rather than a prediction about what will actually happen. The model runs for 600 years primarily so that in optimization mode, what happens in the final periods has little effect on earlier optimization choices.