On the attractor of Piecewise Linear Iterated Function Systems

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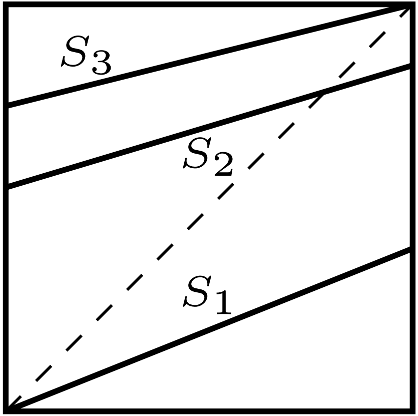
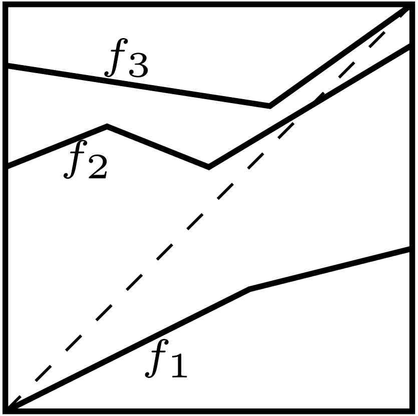


Self-similar IFS
$\mathcal{S} = \{S_1, S_2, S_3\}$



CPLIFS
$\mathcal{F} = \{f_1, f_2, f_3\}$

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a finite list of strict contractions on $\mathbb{R}$. We call it Iterated Function system (IFS). The attractor $\Lambda^{\mathcal{F}}$ of the IFS $\mathcal{F}$ is the unique non-empty compact set

$$\Lambda^{\mathcal{F}} = \bigcup_{k=1}^{m} f_k(\Lambda^{\mathcal{F}}). \tag{1}$$

Let $I^{\mathcal{F}}$ be the smallest non-empty compact interval such that $f_i(I^{\mathcal{F}}) \subset I^{\mathcal{F}}$ for all $i \in [m] := \{1, \dots, m\}$.

$$\Lambda^{\mathcal{F}} = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in[m]^n} I^{\mathcal{F}}_{i_1\dots i_n}, \tag{2}$$

where $I^{\mathcal{F}}_{i_1\dots i_n} := f_{i_1\dots i_n}(I^{\mathcal{F}})$ are the `cylinder intervals`, and we use the common shorthand notation $f_{i_1\dots i_n} := f_{i_1} \circ \dots \circ f_{i_n}$.

(3) $$\Phi^{\mathcal{F}}(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \dots i_n} |I^{\mathcal{F}}_{i_1\dots i_n}|^s.$$

It is easy to see that we can obtain $\Phi^{\mathcal{F}}(s)$ above as a special case of the non-additive upper capacity topological pressure introduced by Barreira in [1, p. 5]. $s \mapsto \Phi^{\mathcal{F}}(s)$ is strictly decreasing, continuous, $\Phi^{\mathcal{F}}(0) = \log m$ and tends to $-\infty$ as $s \to \infty$. So, the zero of $\Phi^{\mathcal{F}}(s)$ is well defined

(4) $$s_{\mathcal{F}} := (\Phi^{\mathcal{F}})^{-1}(0).$$

(5) $$\overline{\dim}_{\mathrm{B}} \Lambda^{\mathcal{F}} \leqslant \min\{1, s_{\mathcal{F}}\}.$$

If

$$\mathcal{F} = \{f_k(x) = r_k \cdot x + t_k\}_{k=1}^{m}$$

is self-similar then $s_{\mathcal{F}}$ is the similarity dimension. That is in the self-similar case

$$\sum_{k=1}^{m} {r_k}^{s_{\mathcal{F}}} = 1. \tag{6}$$

We can thus say that $s_{\mathcal{F}}$ is a generalization of the similarity dimension.

The natural projection

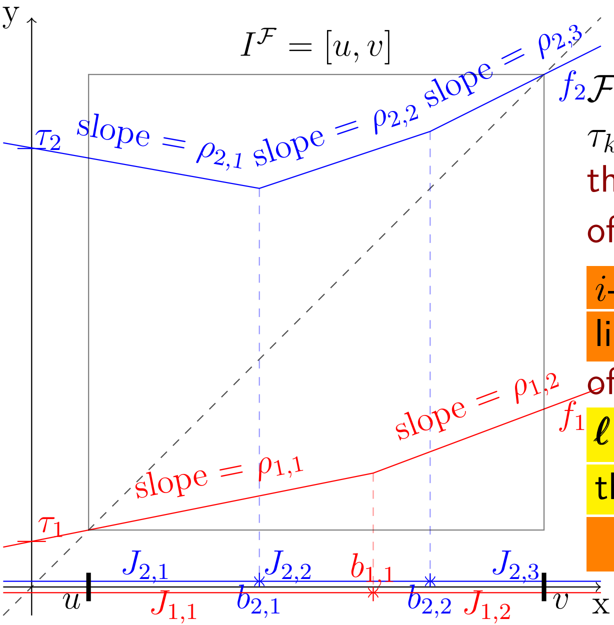
The points of the attractor Λ are coded by the elements of the symbolic space Σ

$$\boxed{\Sigma := \{\mathbf{i} = (i_1, i_2, \dots) : i_k \in [m]\}},$$

where we write $[m] := \{1, \dots, m\}$ by the natural coding (or natural projection) $\Pi : \Sigma \to \Lambda$

$$\boxed{\Pi^{\mathcal{F}}(\mathbf{i}) := \lim_{n\to\infty} f_{i_1\dots i_n}(x) = \bigcap_{n=1}^{\infty} I^{\mathcal{F}}_{i_1\dots i_n}}, \tag{7}$$

where $x \in \Lambda$ is arbitrary. Clearly, $\Pi^{\mathcal{F}}(\Sigma) = \Lambda$.



$\mathcal{F} = \{f_k\}_{k=1}^m$, $\Sigma := [m]^{\mathbb{N}}$. $\tau_k := f_k(0)$, $k \in [m]$. $l(k)$ is the number of breaking points of f_k: $\{b_{k,1}, \ldots, b_{k,l(k)}\}$. The i-th interval of linearity of f_k is $J_{k,i}$. Slope of f_k over $J_{k,i}$ is $\rho_{k,j}$. $\boldsymbol{\ell} = (l(1), \ldots, l(m))$ is the type of $\mathcal{F}$

$$L := l(1) + \cdots + l(m).$$

Let

$$\mathcal{A} := \{(k,i) : k \in [m], i \in [l(k)+1]\}, \quad \rho := \max_{(k,j)\in\mathcal{A}} \rho_{k,j}.$$

We say that $\mathcal{F}$ is small if

- $\rho < \frac{1}{2}$, when all functions of $\mathcal{F}$ are injective,
- $\rho < \frac{1}{3}$, otherwise.

Fix a type $\boldsymbol{\ell} = (l(1), \ldots, l(m)) \in \mathbb{N}^m$. $\mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$ is the set of vectors

$$\boldsymbol{\rho} = \left(\rho_{1,1}, \ldots, \rho_{1,l(1)+1}, \ldots \rho_{m,1}, \ldots, \rho_{m,l(1)+1}\right) \in \left((-1,1)\backslash\{0\}\right)^{L+m}$$

for which conditions (a) and (b) hold. That is $\mathcal{F}$ is small if the vector of the slopes of $\mathcal{F}$ satisfies $\boldsymbol{\rho}_{\mathcal{F}} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$. A CPLIFS $\mathcal{F}$ is uniquely determined by the triple of vectors $(\boldsymbol{\rho}, \mathfrak{b}, \boldsymbol{\tau})$, where

$$\mathfrak{b} = (\underbrace{b_{1,1}, \ldots, b_{1,l(1)}}_{L^1}, \underbrace{b_{2,1}, \ldots, b_{2,l(2)}}_{L^2}, \ldots, \underbrace{b_{m,1}, \ldots, b_{m,l(m)}}_{L^m}) \in \mathfrak{B}^{\boldsymbol{\ell}} \subset \mathbb{R}^L,$$

$$\mathfrak{B}^{\boldsymbol{\ell}} := \left\{\mathbf{x} \in \mathbb{R}^L : x_i < x_j \text{ if } i < j \text{ and } \exists k \in [m] \text{ with } i, j \in L^k\right\}.$$

$$\boldsymbol{\tau} := \left(\tau_1, \ldots, \tau_m\right) \in \mathbb{R}^m.$$

Terminology: Let $\mathfrak{P}$ be a property that makes sense for every CPLIFS. Fix a contraction vector $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\mathrm{small}}$. We consider the (exceptional) set of those $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \subset \mathbb{R}^{L+m}$ such that for the associated CPLIFS $\mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$ property $\mathfrak{P}$ does not hold:

(8)

$$E^{\boldsymbol{\rho}}_{\boldsymbol{\ell}} =: \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m : \mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \text{ does not have property } \mathfrak{P}\right\}.$$

We say that property $\mathfrak{P}$ holds $\dim_{\mathrm{P}}$-typically if for all type $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and for all $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\mathrm{small}}$ we have

(9)
$$\dim_{\mathrm{P}} E^{\boldsymbol{\rho}}_{\boldsymbol{\ell}} < L + m,$$

where $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and $L = \sum_{k=1}^{m} l(k)$ as above.

We always fix a $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$ and consider $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ as parameters. So, our parameter space is $L + m$ dimensional.

Definition 1.1

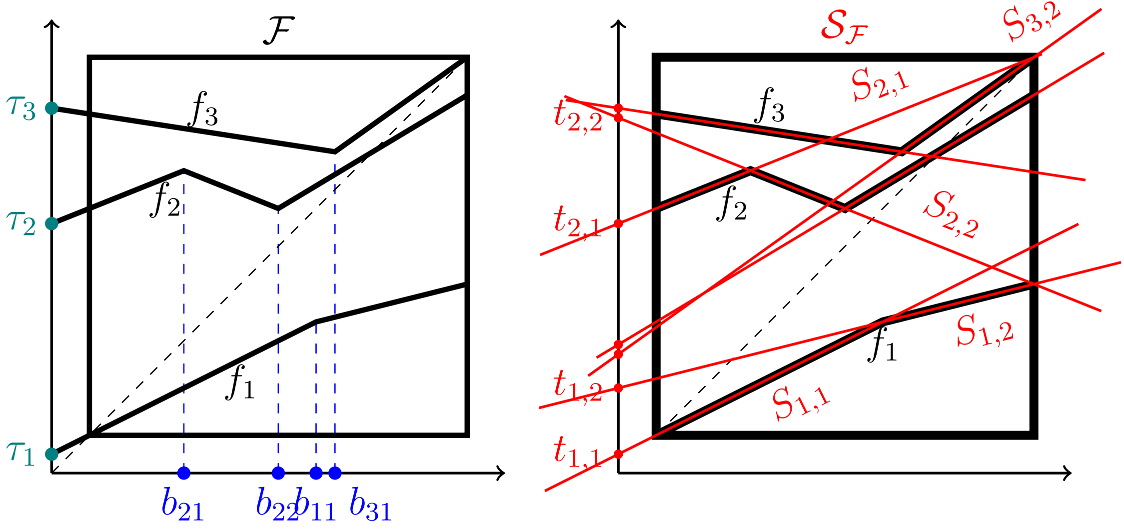
For a CPLIFS $\mathcal{F}$ the generated self-similar IFS

$$S_{\mathcal{F}} = \{S_a(x) = \rho_a x + t_a\}_{a \in \mathcal{A}}$$

consists of those similarity mappings on $\mathbb{R}$ whose graph coincide with the graph of f_k for some $k \in [m]$ on some interval of linearity $J_{k,i}$, $i \in l(k) + 1$ of f_k. That IFS

$$\boxed{S_a|_{J_{k,i}} = f_k|_{J_{k,i}}} \qquad \text{if } a = (k, i) \in \mathcal{A}.$$

The generated self-similar IFS



Definition 1.2

We say that a small CPLIFS $\mathcal{F}$ is regular if its attractor $\Lambda_{\mathcal{F}}$ does not contain any of the breaking points.

For a regular CPLIFS $\mathcal{F}$ there is a smallest N such that

$$\text{there are no breaking points in } \bigcup_{i_1 \dots i_N} I_{i_1 \dots i_N}. \tag{10}$$

Then we say that $\mathcal{F}$ is regular of order N.

Theorem 2.1 (Main Theorem I)

For a $\dim_{\mathrm{P}}$*-typical small CPLIFS* $\mathcal{F}$ *we have*

$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = \min\{1, s_{\mathcal{F}}\}. \tag{11}$$

To prove this result we verify the following theorem.

Theorem 2.2

Let $\mathcal{F}$ *be a regular CPLIFS for which the generated self-similar IFS satisfies the Exponential Separation Condition (ESC). Then*

$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = \min\{1, s_{\mathcal{F}}\}. \tag{12}$$

Theorem 2.3 (Main Theorem II)

Fix a type $\boldsymbol{\ell}$ and a small slope vector $\boldsymbol{\rho}$ with positive entries. For $\mathcal{L}_{m+L}$-almost every $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ we have

(13) $$s_{\mathcal{F}} > 1 \implies \mathcal{L}_1(\Lambda^{(\mathfrak{b},\boldsymbol{\tau})}) > 0,$$

where $\Lambda^{(\mathfrak{b},\boldsymbol{\tau})}$ denotes the attractor of $\mathcal{F}^{(\boldsymbol{\varrho},\mathfrak{b},\boldsymbol{\tau})}$.

The distance of two similarity mappings $g_1(x) = r_1 x + \tau_1$ and $g_2(x) = r_2 x + \tau_2$, $r_1, r_2 \in (-1,1) \setminus \{0\}$, on $\mathbb{R}$.

$$\text{(14)} \qquad \boxed{\operatorname{dist}(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } r_1 = r_2; \\ \infty, & \text{otherwise.} \end{cases}}$$

Definition 3.1

Given a self-similar IFS $\mathcal{S} = \{S_k(x)\}_{k=1}^{M}$ on $\mathbb{R}$. We say that $\mathcal{F}$ satisfies the Exponential Separation Condition (ESC) if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^{\infty}$ such that

$$\boxed{\operatorname{dist}(S_{\bar{\imath}}, S_{\bar{\jmath}}) \geqslant c^{n_\ell}} \text{ for all } \ell \text{ and for all } \bar{\imath}, \bar{\jmath} \in \{1, \dots, M\}^{n_\ell}, \ \bar{\imath} \neq \bar{\jmath}.$$

Theorem 3.2 (Hochman [2] and Jordan, Rapaport [4])

Let $\mathcal{S}$ be a self-similar IFS on the line as above. We assume that $\mathcal{S}$ satisfies the so called Exponential Separation Condition (ESC).

(a) *If μ is a self simar measure then* $\dim_{\mathrm{H}} \Pi_* \mu = \min \left\{ 1, \frac{h_\mu}{\chi(\mu)} \right\}$.

(b) $\dim_{\mathrm{H}} \Lambda = \min \{1, s\}$, *where s is the similarity dimension.*

(c) *If μ is an ergodic invariant probability measure then*

$$\dim_{\mathrm{H}} \Pi_* \mu = \min \left\{ 1, \frac{h_\mu}{\chi(\mu)} \right\}.$$

Fix a vector of contractions $\boldsymbol{r} := (r_1, \ldots, r_M) \in ((-1,1) \setminus \{0\})^M$. For a $\boldsymbol{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$ we consider the self-similar IFS associated to the vector of translations $\boldsymbol{t}$:

$$\mathcal{S}^{\boldsymbol{t}} := \{S_k(x) = r_k x + t_k\}_{k=1}^{M}.$$

The following theorem was proved in [3, Theorem 1.10]

Theorem 3.3 (Multiparameter Hochman Theorem)

$$\dim_{\mathrm{P}} \left\{ \boldsymbol{t} \in \mathbb{R}^M : \mathcal{S}^{\boldsymbol{t}} \textit{ does not satisfy the ESC} \right\} = M - 1. \tag{15}$$

Lemma 3.4

Let $\mathcal{F}$ be a CPLIFS, and let $\mathcal{S}_{\mathcal{F}}$ be the generated self-similar IFS. As above let $\boldsymbol{t} := (t_a)_{a\in\mathcal{A}} \in \mathbb{R}^{L+m}$ be the vector formed from the translation part of the generated self-similar IFS $\mathcal{S}_{\mathcal{F}}$. There is a non-singular linear transformation $F : \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \to \mathbb{R}^{L+m}$ which depends only on $\boldsymbol{\rho}$ such that

$$F_{\boldsymbol{\rho}}(\mathfrak{b}, \boldsymbol{\tau}) = \boldsymbol{t}.$$

This implies that an assertion holds for packing dimension typical $\mathcal{F}$ if and only if it holds for packing dimension typical generated self-similar system $\mathcal{S}_{\mathcal{F}}$.

Proof of Main Theorem I

The Multiparameter Hochman Theorem implies that ESC is a $\dim_{\mathrm{P}} -typical$ property of self-similar systems. Combined with Lemma 3.4 we obtain that the generated self-similar IFS of a typical CPLIFS satisfies the ESC.

The same can be proved for regularity:

Proposition 3.5

A $\dim_{\mathrm{P}}$-typical small CPLIFS is regular.

This proposition can be proved by a transversality like argument, we use the restrictions on the slopes here.

Proof of Theorem 2.2 in steps

- First we associate a self-similar IFS $\mathcal{S}_{\mathcal{F}}$ to every $\mathcal{F} \in \mathrm{CPLIFS}_{\boldsymbol{\ell},N}$ which is the relevant subsystem of $\mathcal{S}_{\mathcal{F}}^{N}$, that consists of all N-fold iterations of functions in $\mathcal{S}_{\mathcal{F}}$.
- Then, we construct a graph-directed self-similar IFS $\mathcal{F}^{\mathcal{G}}$ such that
  - The functions of $\mathcal{F}^{\mathcal{G}}$ are elements of the associated IFS $\mathcal{S}_{\mathcal{F}}$.
  - The attractor $\Lambda^{\mathcal{F}^{\mathcal{G}}}$ of this self-similar GDIFS coincides with $\Lambda^{\mathcal{F}}$.

Proof of Theorem 2.2 in steps (Cont.)

- Let ν be the Markov measure defined by a stochastic matrix made from the slopes of the functions in $\mathcal{F}^{\mathcal{G}}$. We show that

$$\frac{h_\mu}{\chi(\mu)} = s_{\mathcal{F}}.$$

- Finally we apply Jordan-Rapaport Theorem for this measure, to show that $s_{\mathcal{F}}$ is also a lower bound.

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, and define the family of self-similar graph directed iterated function systems $\mathcal{F}^{\mathbf{t}} := \{f_e(x) = \lambda_e x + t_e\}_{e\in\mathcal{E}}$ parametrized by the vector of translations $\mathbf{t}$. We will use the following result on GDIFSs.

Theorem 4.1 (Keane, Simon, Solomyak, 2003)

Suppose that $\mathcal{G}$ is strongly connected and $\lambda_e > 0$ for all $e \in \mathcal{E}$. Set $N = |\mathcal{E}|$. Then, for $\mathcal{L}_N$-almost every $\mathbf{t} \in \mathbb{R}$ we have

(a) $\dim_{\mathrm{H}} \Lambda = \min\{1, s_{\mathcal{F}}\}$,

(b) *if $s_{\mathcal{F}} > 1$, then $\mathcal{L}_1(\Lambda^{\mathbf{t}}) > 0$.*

Let $\mathcal{T}^{\boldsymbol{\rho}} \subset \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ be the set of those $(\mathfrak{b}, \boldsymbol{\tau})$ parameters for which the associated CPLIFS $\mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau})}$ is regular.

By Proposition 3.5, $\mathcal{T}^{\boldsymbol{\rho}}$ has total Lebesgue measure. Thus, it is enough to prove the theorem for $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathcal{T}^{\boldsymbol{\rho}}$.

Observe that for each $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathcal{T}^{\boldsymbol{\tau}}$ there exists a closed neighbourhood $\mathcal{T}^{\boldsymbol{\tau}}_{(\mathfrak{b},\boldsymbol{\tau})}$ such that

$$\forall(\widehat{\mathfrak{b}}, \widehat{\boldsymbol{\tau}}) \in \mathcal{T}^{\boldsymbol{\tau}}_{(\mathfrak{b},\boldsymbol{\tau})} : \mathcal{G}^{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}})} \equiv \mathcal{G}^{(\mathfrak{b},\boldsymbol{\tau})}, \tag{16}$$

where $\mathcal{G}^{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}})}$ is the directed graph of the associated GDIFS of $\mathcal{F}^{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}})}$.

The result follows from Theorem 4.1 and Lemma 3.4.

Thank you for your attention!

References

[1] Luis M Barreira.
A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems.
Ergodic Theory and Dynamical Systems, 16(5):871–928, 1996.

[2] Michael Hochman.
On self-similar sets with overlaps and inverse theorems for entropy.
Annals of Mathematics, pages 773–822, 2014.

[3] Michael Hochman.
On self-similar sets with overlaps and inverse theorems for entropy in $\mathbb{R}^d$, 2015.

References (cont.)

[4] Thomas Jordan and Ariel Rapaport.
Dimension of ergodic measures projected onto self-similar sets with overlaps.
Proceedings of the London Mathematical Society, 2020.