Self-similar sets, dimension drop and Okamoto's function

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Self-similar IFSs

Okamoto's function

Our results

An Iterated Function System (IFS) $\mathcal{F} = \{f_k\}_{k=1}^m$ is a finite list of strict contractions on $\mathbb{R}^d$.

The **attractor** of the IFS $\mathcal{F}$ is the unique non-empty compact set that satisfies

$$\Lambda = \bigcup_{k=1}^{m} f_k(\Lambda). \tag{1}$$

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(1) $$\Lambda = \bigcup_{k=1}^m f_k(\Lambda).$$

By iterating formula (1), one obtains

(2) $$\Lambda = \bigcup_{(i_1,\dots,i_n)\in[m]^n} f_{i_1\dots i_n}(\Lambda).$$

Here we used the common notation $f_{i_1\dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$.

Assume we are on the line and let I be the smallest non-empty compact interval such that $f_i(I) \subset I$ for all $i \in [m] := \{1, \ldots, m\}$.

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in[m]^n} I_{i_1\ldots i_n}, \tag{3}$$

where $I_{i_1\ldots i_n} := f_{i_1\ldots i_n}(I)$ are the cylinder intervals.
Thus these intervals form a natural cover of the attractor. In higher dimensions cylinder sets can be defined analogously.

The Hausdorff dimension

The t-dimensional Hausdorff measure of a set E is

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : E \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| \leqslant \delta \right\} \right\}, \tag{4}$$

where the infimum is taken over all $\{A_i\}$ covers.

The Hausdorff dimension

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where the infimum is taken over all $\{A_i\}$ covers.
The **Hausdorff dimension** of Λ is defined as

(5) $$\dim_{\mathrm{H}} E = \inf\{t : \mathcal{H}^t(E) = 0\} = \sup\{t : \mathcal{H}^t(E) = \infty\}.$$

The packing dimension

The t-dimensional packing measure of a set E is

$$\text{(6)} \quad \tilde{\mathcal{P}}^t(E) = \lim_{\delta \to 0} \left\{ \sup \left\{ \sum_{i=1}^{\infty} |B_i|^t : \{\bar{B}_i\} \text{ is a } \delta\text{-packing of } E \right\} \right\},$$

$$\text{(7)} \quad \mathcal{P}^t(E) = \inf \left\{ \sum_{i=1}^{\infty} \tilde{\mathcal{P}}^t(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

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The **packing dimension** of Λ is defined as

$$(8)\quad \dim_{\mathrm{P}} \Lambda = \inf\{t : \mathcal{P}^t(\Lambda) = 0\} = \sup\{t : \mathcal{P}^t(\Lambda) = \infty\}.$$

Box dimension

Let $E \subset \mathbb{R}^d$ be a bounded set. For $\delta > 0$, let $N_\delta(E)$ be the minimal number of sets of diameter δ needed to cover E. The **lower and upper box dimensions** of E are defined by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta},$$

$$\overline{\dim}_{\mathrm{B}} E := \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

If the limit exists, we call it the **box dimension** of E and denote it with $\dim_{\mathrm{B}} E$.

Similarity dimension

If our iterated function system is of the form

$$\mathcal{F} = \{f_k(x) = r_k \cdot x + t_k\}_{k=1}^m$$

then $\mathcal{F}$ is called self-similar. By using the cylinder intervals as a natural covering system of the attractor, one obtains

$$\sum_{\bar{\imath}\in[m]^n} |I_{\bar{\imath}}|^s = \sum_{\bar{\imath}\in[m]^n} |r_{i_1} \cdots r_{i_n}|^s = \left(\sum_{i=1}^m |r_i|^s\right)^n. \tag{9}$$

The **similarity dimension** of $\mathcal{F}$ is the unique number s_0 defined as

$$\sum_{i=1}^m |r_i|^{s_0} = 1.$$

Affinity dimension

Let $\mathcal{F}$ be a planar self-affine IFS of the form

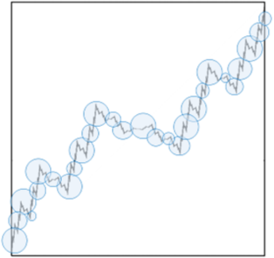
$$\mathcal{F} = \{f_i(x,y) = (\alpha_i x, \beta_i y) + (t_{i,1}, t_{i,2})\}_{i=1}^m.$$

We define the pressure function

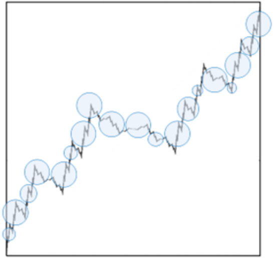
$$P_{\mathcal{F}}(s) = \begin{cases} \max\left\{\sum_{i=1}^m |\alpha_i|^s, \sum_{i=1}^m |\beta_i|^s\right\}, \text{ if } 0 \leqslant s < 1 \\ \max\left\{\sum_{i=1}^m |\alpha_i||\beta_i|^{s-1}, \sum_{i=1}^m |\beta_i||\alpha_i|^{s-1}\right\}, \text{ if } 0 \leqslant s < 1 \\ \sum_{i=1}^m \left(|\alpha_i||\beta_i|\right)^{s/2}, \text{ if } 2 \leqslant s. \end{cases}$$

The **affinity dimension** of $\mathcal{F}$ is the unique s_0 for which

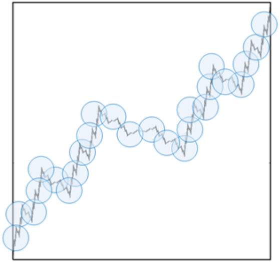
$$P_{\mathcal{F}}(s_0) = 1.$$



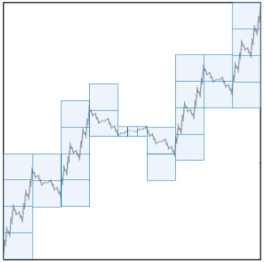
Hausdorff dimension ($\dim_H$)



Packing dimension ($\dim_P$)



Box dimension ($\dim_B$)



Affinity dimension ($\dim_{Aff}$)

It follows from the definitions, that

$$\dim_{\mathrm{H}} \Lambda \leqslant \dim_{\mathrm{P}} \Lambda \leqslant \overline{\dim}_{\mathrm{B}} \Lambda \leqslant \min\{d, s_0\}. \tag{10}$$

Under what condition do we have equality?

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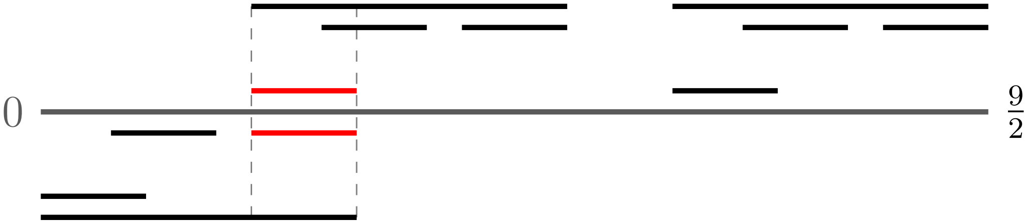
Under what condition do we have equality?

It is easy to see that $\dim_{\mathrm{H}} \Lambda < s_0$ if some cylinder intervals are identical. Thus we need to require some separation for the system.

Dimension drop phenomena



Let Λ be the attractor of $\mathcal{F} = \{f_i(x) = \frac{1}{3}x + i\}_{i\in\{0,1,3\}}$. Due to the exact overlappings, $\dim_{\mathrm{H}} \Lambda$ is strictly smaller than the similarity dimension

$$\dim_{\mathrm{H}} \Lambda \approx 0.876 < 1 = s_0.$$

Separation Conditions

Consider the self-similar IFS $\mathcal{F} = \{f_k\}_{k=1}^m$.
We say that $\mathcal{F}$ satisfies the **Strong Separation Property** if

$$\forall i, j \in [m], i \neq j : f_i(\Lambda) \cap f_j(\Lambda) = \varnothing.$$

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$$\forall i, j \in [m], i \neq j : f_i(\Lambda) \cap f_j(\Lambda) = \varnothing.$$

We say that $\mathcal{F}$ satisfies the **Open Set Condition** (OSC) if $\exists U$ open set such that $\forall i \in [m] : f_i(U) \subset U$ and

$$\forall i, j \in [m], i \neq j : f_i(U) \cap f_j(U) = \varnothing.$$

Both of these conditions guarantee that $\dim_{\mathrm{H}} \Lambda = s_0$.

Exponential Separation Condition

The distance of two similarity mappings $g_1(x) = r_1 x + \tau_1$ and $g_2(x) = r_2 x + \tau_2$, $r_1, r_2 \in (-1,1) \setminus \{0\}$, on $\mathbb{R}$.

$$\operatorname{dist}(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } r_1 = r_2; \\ \infty, & \text{otherwise.} \end{cases}$$

We say that the self-similar IFS $\mathcal{F}$ satisfies the **Exponential Separation Condition** (ESC) if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^{\infty}$ such that

$$\operatorname{dist}(f_{\overline{\imath}}, f_{\overline{\jmath}}) \geqslant c^{n_\ell} \quad \text{for all } \ell \text{ and for all } \overline{\imath}, \overline{\jmath} \in \{1, \dots, m\}^{n_\ell}, \ \overline{\imath} \neq \overline{\jmath}.$$

Theorem (Hochman [6])

Let $\mathcal{S}^{\mathbf{t}} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^{m}$ *be a family of self-similar IFSs with* $|\lambda_i| < 1, \lambda_i \neq 0, \mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$*. Write* $\Lambda^{\mathbf{t}}$ *for the attractor of* $\mathcal{S}^{\mathbf{t}}$*. If* $\mathcal{S}^{\mathbf{t}}$ *satisfies the ESC, then*

$$\dim_{\mathrm{H}} \Lambda = \min\{1, s_0\}.$$

Further, for most parameters the ESC holds

$$\dim_{\mathrm{P}} \left\{ \boldsymbol{t} \in \mathbb{R}^m : \mathcal{S}^{\boldsymbol{t}} \textit{ does not satisfy the ESC } \right\} = m - 1.$$

Self-similar IFSs

Okamoto's function

Our results

Okamoto's function

Okamoto [8] introduced and studied a one-parameter family of self-affine functions

$T_a : [0,1] \to [0,1]$ for $a \in (0,1)$.

Let $a \in \left(\frac{1}{2}, 1\right)$, and consider the planar self-affine IFS $\mathcal{F}$. We often refer to $\mathcal{F}$ as the Okamoto IFS.

$$F_1(x,y) = \left(\frac{x}{3}, ay\right)$$

$$F_2(x,y) = \left(\frac{x+1}{3}, (1-2a)y + a\right)$$

$$F_3(x,y) = \left(\frac{x+2}{3}, ay + 1 - a\right)$$

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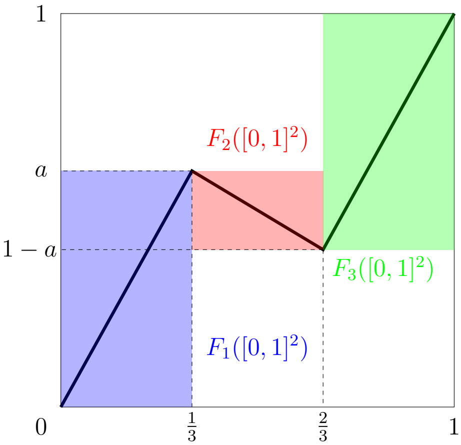

Properties of T_a

$T_{2/3}$ and $T_{5/6}$ were studied by Perkins and Bourbaki repsectively. Okamoto was the first who investigated the whole family.

Theorem (Okamoto 2005 [8])

Let $a_0 \approx 0.5592$ be the unique real root of $54a^3 - 27a^2 = 1$.

(i) $\frac{2}{3} \leqslant a < 1 : T_a$ *is nowhere differentiable,*

(ii) $a_0 < a < \frac{2}{3} : T_a$ *is nondifferentiable at almost every $x \in [0, 1]$, but differentiable at uncountably many points,*

(iii) $0 < a < a_0 : T_a$ *is differentiable almost everywhere, but nondifferentiable at uncountably many points.*

Properties of T_a

Let $\mathcal{D}_\infty(a) = \{x \in [0,1] : T_a'(x) = \pm\infty\}$.

Theorem (Allaart '16 [1])

Let $\hat{a} \approx 0.5598$ *and* $\phi = \frac{\sqrt{5}-1}{2}$.

(i) $\phi \leqslant a < 1 : \mathcal{D}_\infty(a)$ *is empty,*

(ii) $\hat{a} < a < \phi : \mathcal{D}_\infty(a)$ *is countably infinite, containing only rational points,*

(iii) $\frac{1}{2} < a < \hat{a} : \mathcal{D}_\infty(a)$ *is uncountable with* $\dim_{\mathrm{H}} \mathcal{D}_\infty(a) > 0$.

Level sets of T_a

We may gain more insight on the structure of Okamoto's function if look at its level sets.

Theorem (Baker, Bender '23 [3])

Let $a_9 \approx 0.50049$.

(i) $\frac{1}{2} \leqslant a < a_9$: *there exists* $y \in [0,1]$ *for which* $|T_a^{-1}(y)| = 3$,

(ii) *Assume that for every* $y \in [0,1]$, $|T_a^{-1}(y)|$ *is either uncountable or* 1. *Then if* $|T_a^{-1}(y)|$ *is uncountable,* $\exists s > 0$ *such that* $\dim_{\mathrm{H}} T_a^{-1}(y) \geqslant s$.

Dimension theory of the graph

In Example 11.4 of Falconer's book [4], the box dimension of the graph of general self-affine functions, in particular the box dimension of $\mathrm{graph}(T_a)$, was calculated.

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The closed formula for the box-counting dimension of $\mathrm{graph}(T_a)$ was published by McCollum [7], who also claimed that the Hausdorff and box-counting dimension of the graph are equal.

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The closed formula for the box-counting dimension of $\mathrm{graph}(T_a)$ was published by McCollum [7], who also claimed that the Hausdorff and box-counting dimension of the graph are equal.

However, his argument was incorrect.

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Our results

Our main result

For $a \in \left(\frac{1}{2}, 1\right)$ set $s_0(a) = 1 + \frac{\ln(4a-1)}{\ln(3)}$, and write $\mathcal{O}_a = \mathrm{graph}(T_a)$.

Theorem (B. Bárány, P. '24)

There exists an $\mathcal{E} \subset \left(\frac{1}{2}, 1\right)$ *with* $\dim_{\mathrm{H}} \mathcal{E} = 0$ *such that for every* $a \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$ *the following statements hold.*

- $\dim_{\mathrm{H}} \mathcal{O}_a = \dim_{\mathrm{B}} \mathcal{O}_a = \dim_{\mathrm{A}} \mathcal{O}_a = s_0(a)$,
- $\forall y \in [0,1] : \dim_{\mathrm{H}} T_a^{-1}(y) \leqslant s_0(a) - 1$,
- *For* $\mathcal{L}$*-almost every* $y \in [0,1]$:

$$\dim_{\mathrm{H}} T_a^{-1}(y) = s_0(a) - 1.$$

Main tools

Let $\mathcal{S} = \{\mathbf{A}_i\mathbf{x} + \mathbf{t}_i\}_{i=1}^m$ be a self-affine iterated function system on the plane, where $\mathbf{A}_i$ is a diagonal matrix for all $i \in \{1, \ldots, m\}$.

Let Λ be the attractor of $\mathcal{S}$, and let μ be the projection of a Bernoulli measure on the symbolic space defined by the probability vector $\mathbf{p}$.

Write proj_y for the projection to the y-axis and L_y for the level set of Λ corresponding to y

$$L_y = \{x \in [0,1] : (x,y) \in \Lambda\}.$$

Theorem (Feng-Hu '09 [5])

Let $\mu_y^{\mathrm{proj}_y^{-1}}$ *denote the conditional measure of* μ *with respect to* L_y *for* $y \in [0,1]$. *Let* $h_{\mathbf{p}}$ *be the entropy of* $\mathbf{p}$ *and* $0 < \chi_1 < \chi_2$ *be the Lyapunov exponents of* μ. *If* x *is the dominating direction, then*

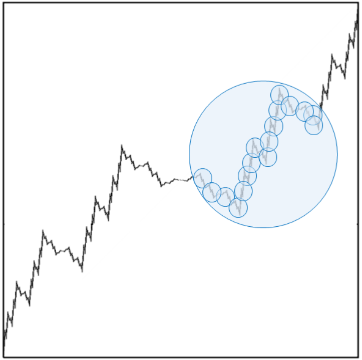
(i) $\dim_{\mathrm{H}} \mu_y^{\mathrm{proj}_y^{-1}} + \dim_{\mathrm{H}}(\mathrm{proj}_y)_*\mu = \dim_{\mathrm{H}} \mu$ *for* $(\mathrm{proj}_y)_*\mu$*-almost all* $y \in [0,1]$,

(ii) $\dim_{\mathrm{H}} \mu = \frac{h_{\mathbf{p}}}{\chi_2} + \left(1 - \frac{\chi_1}{\chi_2}\right) \cdot \dim_{\mathrm{H}}(\mathrm{proj}_y)_*\mu$.

Assoud dimension

Let $E \subset \mathbb{R}^d$ be a non-empty set. The **Assouad dimension** of E is

$$\dim_{\mathrm{A}} E = \inf \left\{ \alpha \;\middle|\; \exists C > 0, \forall x \in E, \right.$$
$$\forall 0 < r < R, \forall x \in E :$$
$$\left. N_r(B(x,R) \cap E) \leqslant C \left(\frac{R}{r}\right)^{\alpha} \right\}$$



Assouad dimension ($\dim_{\mathrm{A}}$)

To show that the Assouad dimension of the graph of the function is also equal to the affinity dimension, we used the following result.

Theorem (Antilla-Bárány-Käenmäki '23 [2])

If $\mathcal{S}$ satisfies the OSC, and its dominating direction is x, then

$$\dim_{\mathrm{A}} \Lambda \leqslant \max\{\dim_{\mathrm{H}} \Lambda, 1 + \sup_{y \in [0,1]} \dim_{\mathrm{H}} L_y\}.$$

Why not every $a > 1/2$?

The projection of $\mathcal{O}_a$ to the y-axis contains heavy overlaps, which might result in dimension drop.

We can describe the projection of $\mathcal{O}_a$ to the y-axis with the help of the self-similar IFS

$$\Phi_a = \{f_1(x) = ax,\ f_2(x) = (1-2a)x + a,\ f_3(x) = ax + (1-a)\}.$$

Theorem (B. Bárány, P. '24)

There exists an $\mathcal{E} \subset \left(\frac{1}{2}, 1\right)$ *with* $\dim_{\mathrm{H}} \mathcal{E} = 0$ *such that for every* $a \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$ *the IFS* Φ_a *is exponentially separated.*

Proof sketch

To make the formulas simpler, we work with a conjugate of Φ_a

$$\widetilde{\Phi}_b = \left\{ \phi_1(x) = \frac{1+b}{2}x - 1, \phi_2(x) = (-b)x, \phi_3(x) = \frac{1+b}{2}x + 1 \right\}.$$

Write $\Sigma = \{1,2,3\}^{\mathbb{N}}$ and let $\Pi_b : \Sigma \to \mathbb{R}$ denote the natural projection of $\widetilde{\Phi}_b$.

$$\forall \bar{\imath} = i_1 i_2 \cdots \in \Sigma : \Pi_b(\bar{\imath}) := \lim_{n\to\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_n}(0).$$

Proof sketch

We need to show that outside of a set of exceptional parameters $\mathcal{E}$ with $\dim_{\mathrm{H}} \mathcal{E} = 0$ we have

$$\exists \varepsilon > 0, \exists N \geqslant 1, \forall n \geqslant N : \min_{\bar{\imath} \neq \bar{\jmath} \in \Sigma_n} \left| \Pi_b(\bar{\imath}) - \Pi_b(\bar{\jmath}) \right| > \varepsilon^n .$$

To deal with the overlaps, we partition $\Sigma \times \Sigma$ into three parts

$$A_1 := \left\{ (\bar{\imath}, \bar{\jmath}) \in \Sigma \times \Sigma : (i_1, j_1) \neq (1,3) \right\}$$
$$A_2 := \left\{ (\bar{\imath}, \bar{\jmath}) \in \Sigma \times \Sigma : (i_1, j_1) \neq (3,1) \right\}$$
$$A_3 := \left\{ (\bar{\imath}, \bar{\jmath}) \in \Sigma \times \Sigma : (i_1, j_1) \in \{(1,3),(3,1)\} \right\}.$$

Proof sketch

Now define the mappings

$$F^1_{\bar\imath,\bar\jmath}(b) := b\,\Pi_b(\bar\imath) + \frac{1+b}{2}\,\Pi_b(\bar\jmath) - 1,$$
$$F^2_{\bar\imath,\bar\jmath}(b) := b\,\Pi_b(\bar\imath) + \frac{1+b}{2}\,\Pi_b(\bar\jmath) + 1,$$
$$F^3_{\bar\imath,\bar\jmath}(b) := \Pi_b(\bar\imath) - \Pi_b(\bar\jmath).$$

With the help of these functions, one can show that $\widetilde{\Phi}_b$ satisfies the exponential separation condition for typical parameters.

Proof sketch

Proposition

There exists an exceptional set of parameters $\mathcal{E} \subset (0, 1/2)$ *with* $\dim_{\mathrm{H}} \mathcal{E} = 0$ *such that for all* $b \in (0, 1/2) \backslash \mathcal{E}$

$$\exists \delta > 0, \exists N \in \mathbb{N}, \forall l \geqslant N, \forall (\bar{\imath}, \bar{\jmath}) \in (\Sigma_l \times \Sigma_l) \cap A_k : \left| F^k_{\bar{\imath},\bar{\jmath}}(b) \right| > \delta^l,$$

for every $k \in \{0, 1, 3\}$.

Thank you for your attention!

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