S. Leupold **October – December 2015**

Hadron and Quark Physics
Hand-in problems 1

1. **Relativity and electrodynamics** *10 points*

   (a) **Four-derivative:**
   The contravariant four-derivative is given by

$$(\partial^\mu) := \left(\frac{\partial}{\partial t}, -\vec{\nabla}_x\right) = \left(\frac{\partial}{\partial x_\mu}\right) . \tag{1}$$

   The minus appearing for the spatial components might be disturbing in view of the fact that for the contravariant four-vector of position there is no minus sign:

$$(x^\mu) = (t, \vec{x}) . \tag{2}$$

   Given an arbitrary other (x-independent) four-vector b show that the expression

$$\partial^\mu (x \cdot b) = \partial^\mu (x^\nu b_\nu) \tag{3}$$

   is indeed equal to b^μ, i.e., a contravariant four-vector as it should be.

   (b) **Relativistic formulation of the Maxwell equations:**
   Given the Lorentz tensor of the electromagnetic field strength

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} , \tag{4}$$

   the four-vector of the current $j = (j^\mu) = (\rho, \vec{j})$ and the Levi-Civita symbol in four dimensions,

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even permutations of } (\mu,\nu,\alpha,\beta) = (0,1,2,3) , \\ -1 & \text{for even permutations of } (\mu,\nu,\alpha,\beta) = (1,2,3,0) , \\ 0 & \text{else,} \end{cases} \tag{5}$$

   show that the equations

$$\partial_\mu F^{\mu\nu} = j^\nu , \quad \epsilon^{\mu\nu\alpha\beta}\, \partial_\nu F_{\alpha\beta} = 0 \tag{6}$$

   and $\partial_\mu j^\mu = 0$ are equivalent to the Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \text{Gauss' law} \tag{7}$$

$$\frac{\partial}{\partial t} \vec{E} = \vec{\nabla} \times \vec{B} - \vec{j} \qquad \text{Ampere's law} \tag{8}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{no magnetic monopoles} \tag{9}$$

$$\frac{\partial}{\partial t} \vec{B} = -\vec{\nabla} \times \vec{E} \qquad \text{Faraday's law} \tag{10}$$

and the current conservation

$$\frac{\partial}{\partial t}\,\rho + \vec{\nabla}\cdot\vec{j} = 0\,. \tag{11}$$

(c) **Electromagnetic potential:**
Given again the Lorentz tensor of the electromagnetic field strength, equation (4), and the (four-)vector potential $A = (A^\mu) = (\phi, \vec{A})$ show that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{12}$$

is equivalent to

$$\vec{B} = \vec{\nabla}\times\vec{A}\,, \qquad \vec{E} = -\frac{\partial}{\partial t}\,\vec{A} - \vec{\nabla}\phi\,. \tag{13}$$

Note that the four-derivative is given by equation (1).

(d) **Principle of minimal substitution:**
Using the principle of minimal substitution

$$p^\mu \to p^\mu - qA^\mu \tag{14}$$

to introduce interactions with the electromagnetic field one obtains from the energy-momentum relation of a free (here non-relativistic) particle, i.e., from the free Hamiltonian $H_{\text{free}} = \vec{p}^{\,2}/(2m)$ the following expression:

$$H = \frac{[\vec{p} - q\vec{A}(t,\vec{x})]^2}{2m} + q\phi(t,\vec{x})\,. \tag{15}$$

Derive Newton's equation of motion with the Lorentz force,

$$m\,\frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{d\vec{x}}{dt}\times\vec{B}\right)\,, \tag{16}$$

from the Hamilton equations

$$\dot{x}^i = \frac{\partial H}{\partial p^i}\,, \qquad \dot{p}^i = -\frac{\partial H}{\partial x^i}\,. \tag{17}$$

2. **Three-body phase space** *10 points*

Solve task 1 from section 3.4 of the Lecture Notes

To be handed in at 16.11.2015 (10am) at the latest.

ways to hand them in: during the lectures; via email to stefan.leupold@physics.uu.se; in-box of "hadrons and quarks" on top of the old mail boxes in house 1, second floor, close to house 8