Lecture notes for
Hadron and Quark Physics

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Contents

Literature:

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[Srednicki] M. Srednicki, Quantum Field Theory, Cambridge University Press

[Peskin] M. E. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory, Perseus

[Bjorken] J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, Mc Graw-Hill

[Mulders] P. J. Mulders, Quantum Field Theory, http://www.nat.vu.nl/∼mulders/QFT-0.pdf

[PDG] Review of Particle Physics by the Particle Data Group http://pdg.lbl.gov/

The first section, 2, is — more or less — based on [Mosel]. The latter sections, 4-6, are based on [Scherer]. Section 3 will be presented without much justification as a list of rules how to calculate decay rates and cross sections based on Feynman rules. Derivations can be found in any book on quantum field theory, e.g., [Brown,Srednicki,Peskin,Bjorken,Mulders].

1 Motivation

1.1 Elementary particles

Since quite a while we know that atoms are not pointlike objects but consist of electrons (e^-), protons (p) and neutrons (n). The protons and neutrons as the building blocks of the atomic nucleus are also called "nucleons". A consequence of quantum field theory (the development combining the ideas of quantum mechanics and special relativity, see section 2) is the prediction that every particle should have a corresponding antiparticle. After Dirac's prediction, positrons (e^+), antiprotons ($\bar{p}$) and antineutrons ($\bar{n}$) have been discovered. For basically all elementary particles, which we know, the corresponding antiparticle has also been found. New particles appear in decays and reactions, e.g.,

$$n \to p + e^- + \bar{\nu}_e \tag{1.1.1}$$

and

$$p + \bar{p} \to m\,\pi \tag{1.1.2}$$

where ν_e denotes an electron neutrino and $\bar{\nu}_e$ the corresponding antineutrino.[1] π denotes the pion (see table 2 below) and m is an integer number.

Classification of forces: Four fundamental forces are known. The weakest one, gravity, does not play a role in laboratory reactions of elementary particles.[2] The other forces are the electromagnetic, the weak and the strong interaction.

- Electromagnetic interaction: Most of the phenomena that we experience is caused by this interaction. The force carrier is the photon. It couple to the electric charge of the particles. The corresponding theory is quantum electrodynamics (see section 2.3 below).

- Weak interaction: It is responsible, e.g., for the neutron decay (1.1.1). The force carriers are the W and Z bosons. They couple to the weak isospin. However, this statement is oversimplified, since the weak interaction is interwoven with electromagnetism. The corresponding unifying framework is the electroweak theory. Also the Higgs sector (see section 5.3.3 below) is part of this theory. The Higgs field provides masses for most of the elementary particles. The excitation of the Higgs field, the Higgs boson, might also be counted among the force carriers. But one cannot easily point out a reaction experienced "in daily life" where the Higgs boson acts as a force carrier.

- Strong interaction: It is responsible for the stability of the atomic nuclei. It also governs many particle reactions like (1.1.2). The fundamental force carriers are the gluons. They couple to "color". The corresponding theory is quantum chromodynamics (QCD).

Classification of particles: Apparently we make here a distinction between forces and particles. Indeed there are two different classical concepts: Particles are the objects that build matter; forces are experienced when objects/particles attract or repell each other over a distance. From the point of view of quantum field theory, however, the force carriers are also particles. And there are particle reactions where "matter particles" act as force carriers. Finally there are many particles that are only created in particle collisions. They are not directly building blocks of ordinary matter. Nonetheless it makes sense to distinguish the previously introduced fundamental force carriers from all the other particles. In

[1] Note that it is a matter of active research to find out whether neutrinos are their own antiparticles (Majorana vs. Dirac fermions).

[2] On a large (astrophysical) scale gravity is the only relevant force since the strong and the weak force are short-ranged. The electromagnetic force is only long-ranged for charged objects (electric monopoles), but short-ranged for neutral objects like atoms (which typically possess only higher electromagnetic multipoles).

quarks and gluons			
	name	mass (MeV/c^2)	electric charge
u	up	1 - 3	$+2/3$
d	down	4 - 6	$-1/3$
s	strange	90 - 100	$-1/3$
c	charm	$1.3 \cdot 10^3$	$+2/3$
b	bottom	$4.7 \cdot 10^3$	$-1/3$
t	top	$1.7 \cdot 10^5$	$+2/3$
g	gluon	0	0

Table 1: Some properties of quarks (u, d, s, c, b, t) and gluons (g) [PDG]. The electric charge is given in multiples of the positron or proton charge.

the following paragraph the phrase "particle" means exactly this: all the other particles.

Particles that do not interact strongly (i.e. which are not subject to the strong interaction) are called **leptons**. All others particles that reach detectors or are identified as resonances in scattering reactions are called **hadrons**. There are only a few types of leptons, namely the electron, e^-, the muon, μ^-, and the tau-lepton, τ^-, the corresponding uncharged electron-, mu- and tau-neutrinos, ν_e, ν_μ, ν_τ, and the corresponding antiparticles, e^+, μ^+, τ^+, $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$. On the other hand, there is a whole zoo of hadrons. We will list some hadrons below. All hadrons that have been identified so far are collected in the "Review of Particle Physics" [PDG].

Nowadays we know that hadrons are not elementary. One has identified quarks in the interior of hadrons; see table 1. Interestingly there are as many quark as lepton types. Indeed the electroweak theory would lead to inconsistencies if there was a mismatch between the number of quarks and the number of leptons.[3] However, one would like to understand on a deeper, less technical level, why the number of quarks and of leptons agrees. Closely connected to this issue is the question why the charge of the proton is exactly equal to the charge of the positron, in other words: why the proton and the electron charge exactly cancel each other.[4] At present there are no satisfying answers to these questions. Some proposed extensions of the standard model of elementary particles tackle these questions. Often a finite lifetime of the proton is predicted by such models, but so far proton decay has not been observed yet.

While the fundamental force carriers have integer spin, the quarks and leptons have spin 1/2. Quantum field theory predicts that particles with integer spin behave as bosons, while particles with half-integer spin are fermions. The corresponding Pauli principle is a cornerstone for the understanding of the properties

[3] The so-called Bell-Adler-Jackiw anomaly would render some calculations meaningless.

[4] If atoms were not neutral our world would probably look very different.

of atoms, molecules and chemistry.

1.2 Standard model of particle physics

The questions raised above can be regarded as a teaser to look deeper into the framework that describes the properties and dynamics of elementary particles. Nearly all observations (and there are many!) can be explained *quantitatively* by the so-called "Standard model of particle physics". It is a merger of the electroweak theory (also called Glashow-Salam-Weinberg model) and QCD. The standard model constitutes a quantum field theory; the fundamental objects are fields. The excitations of the quantized fields can be interpreted as particles. Both parts of the standard model have in common that symmetries, in particular local gauge symmetries (see sections 2.3 and 4), play a central role.

The standard model provides equations of motion for the fields that cannot be solved exactly for most of the cases of interest (scattering reactions, decay processes, formation of composite states). However, in many cases perturbation theory can be applied. Wherever such a perturbative treatment is possible, the standard model reaches an extremely high accuracy and predictive power. It is the best tested theory ever. This is, of course, not only a statement about the high quality of the theory, but also about the high quality of the corresponding experiments that strive to falsify the standard model or to find its limitations.

The standard model describes three of the four known fundamental forces; it does not contain gravity. The deeper reason is that we do not have a working theory of quantum gravity. The perturbative treatment that works so well for the standard model does not work for the quantized version of Einstein's theory of gravity. We do not know right now how to give a well-defined meaning to the expressions that perturbation theory provides for gravity theory. The solution might be a non-perturbative treatment of gravity as a quantum field theory. But it is also conceivable that one has to go beyond quantum field theory.

In the following some features and problems of the standard model will be briefly reviewed. It turns out to be useful to discuss these issues separately for the electroweak theory and for QCD and to contrast these two parts of the standard model. In this sense a given number of the following enumerated lists always refers to the same issue, once concerning the electroweak theory, once QCD.

Electroweak theory:

1. **Purpose:** It describes the electromagnetic and the weak interaction.

2. **Coupling constants:** They characterize the strengths of the interactions. In constrast to the name, these couplings are not really constant, but depend on the typical energy/momentum that is relevant for the considered process. One uses the notion of "running coupling constants". However,

the coupling constants of the electroweak theory are rather "crawling". For all energies that are of practical relevance, the coupling constants of the electroweak theory are so small, that perturbation theory can be used. As a result, predictions are obtained as power series in the coupling constants. Incredible accuracies can be reached.

3. **Asymptotic behavior:** At (very) high momenta some of the coupling constants become so large that perturbation theory breaks down. We do not have a proper definition of the theory in the high-energy limit. Thus the theory is incomplete from a conceptual point of view. In practice this might not matter, as the energy scale, where the quantum aspects of gravity become important, is actually smaller than the energy scale where the electroweak perturbation theory breaks down. But in principle the electroweak theory is not a complete theory.

4. **Parameters:** The electroweak theory (and simple extensions thereof) has on the order of 10 parameters[5] that must be adjusted to experimental data before one has predictive power. What is disturbing is that some of these parameters are much smaller than some others. For instance, the ratio between the coupling g_{eH} of the electron to the Higgs field and the coupling g_{tH} of the top quark to the Higgs field is of the order 10^{-5}. (This ratio translates to the ratio of masses of electron and top quark.) In general, it is one of the high ambitions of physics to explain why some quantity/effect is much smaller than another one. A pure statement of the fact that one needs some rather small parameters to describe the experiments is unsatisfying for a fundamental theory.

5. **Shortcomings:** Besides the already mentioned shortcomings it seems that the electroweak theory cannot explain quantitatively the baryon asymmetry in the universe, i.e. the fact that there is more matter than antimatter. It also does not provide an explanation of the origin of dark matter.

   While these issues concern cosmological problems (albeit connected to particle physics) there is one shortcoming that concerns directly the properties of elementary particles: In the electroweak theory the neutrinos are massless, but we know that they have (small, but) non-vanishing masses. Naively one might think that one can easily extend the electroweak theory by giving masses to the neutrinos just like the theory provides masses for all the charged fermions. Such a minimal extension could then form an improved standard model by patching it together with quantum chromodynamics. However, there are ambiguities in such an extension. All fundamental fermions besides the neutrinos have non-vanishing electric charge. This makes them distinct from their antiparticles, which carry the opposite

[5] This is the order of magnitude.

charge. The neutrinos, however, are chargeless and it is therefore unknown if they are their own antiparticles (Majorana fermions) or if particles and antiparticles are distinct from each other (Dirac fermions). To cover all possibilities induces a bunch of new parameters in a possible extension of the electroweak theory. In addition, given the smallness of the neutrino masses, the question, why some parameters are much smaller than some other, becomes even more pressing.

6. **Challenges:** The present challenge in particle physics is to look for traces of physics beyond the standard model. These searches go in two directions characterized by "higher energies" and "higher precision". At the highest reachable collision energies one tries to create and identify new particles not covered by the standard model. Since the standard model provides such an extremely high accuracy in its predictions, a complementary search direction is to perform high-precision tests, mostly at comparatively low energies. Here one is aiming at a potential difference between the result of a high-precision experiment and the corresponding standard-model prediction. If one could unambigiously identify a discrepancy then one would have found a trace of new physics.

   As already pointed out, the standard model is not exactly solvable. Predictions are typically provided by perturbation theory as power expansions in the coupling constants. In principle these predictions can be improved by calculating the contributions from higher powers. At some point, however, the influence of quarks becomes relevant and the running coupling constant of the strong interaction is not always small (see below). Typically the limiting factor in the accuracy of a standard-model prediction is our limited understanding of the strong interaction. This will be discussed next. The take-home message from the present discussion is: To reveal new physics one needs to understand the old physics (standard model) very well.

Quantum chromodynamics:

1. **Purpose:** It describes the strong interaction. The primary objects, i.e. the fundamental degrees of freedom are the quarks and the gluons. But the particles that reach detectors or are identified as resonances in scattering reactions are composite objects, the hadrons. The lightest fermionic hadrons, the protons and neutrons, form the atomic nuclei. This ties hadron physics and QCD very closely to nuclear physics.

2. **Coupling constant:** Since the strong interaction is so much constrained by symmetries (see section 4 below) there is only one coupling constant. In contrast to the coupling constants of the electroweak theory, the coupling "constant" of the strong interaction changes significantly with energy/momentum.

3. **Asymptotic behavior:** The coupling constant of the strong interaction decreases with increasing energy and vanishes asymptotically. This phenomenon is called "asymptotic freedom". From a practical point of view it implies that the strong interaction is actually "weak" at high energies. One can use perturbation theory to describe high-energy reactions. From a conceptual point of view asymptotic freedom implies that quantum chromodynamics is an intrinsically consistent, complete theory. The backside of the same coin is that the coupling constant becomes large at low energies. Perturbation theory cannot be used anymore and one can imagine that new phenomena might show up. Indeed, the fundamental degrees of freedom build composite objects, the hadrons. To understand the properties of hadrons requires non-perturbative approaches to QCD. The main part of these lecture notes is devoted to these developments. At present there are still many aspects of the strong interaction at low energies that we have not fully understood.

   We call the electroweak theory an incomplete theory, because it is not well-defined at (extremely) high energies. At low energies (at all energies of practical relevance) it is well-defined by perturbation theory. QCD is called a complete theory. It is defined by perturbation theory at high energies, but our knowledge is incomplete at low energies. Why is it then called a complete theory? The point is that there is a difference between "conceptually well-defined" and "solvable in practice". QCD can be defined in a space-time discretized world, in technical terms this is the path-integral formulation of quantum field theory. Low momenta correspond to large distances. If the discretization is fine enough, it does not matter for the description of low-energy phenomena whether the fields are defined on a grid or in continuous space-time. At high momenta, corresponding to very small distances, it does matter. But there asymptotic freedom and perturbation theory guarantee that everything is sufficiently well-defined.[6] If one defined the electroweak theory on a grid, then one could not guarantee that one has a proper continuum limit.

4. **Parameters:** In principle, QCD has one universal coupling constant and as many quark masses as there are quark types (we know of six). However, the explanation where the quark masses come from is part of the electroweak theory. In this sense the strong interaction has only one intrinsic parameter, the universal running coupling constant.

5. **Problems:** Perturbation theory is the best developed practical tool of quantum field theory. In high-energy reactions perturbative QCD has been

[6] It should be stressed that this reasoning is qualitative and pragmatic. From the mathematical point of view of axiomatic quantum mechanics the path integrals of quantum field theory are not mathematically well-defined.

tested to an extent (though not with the incredible precision of electroweak observables) that it is regarded as the correct theory of the strong interaction. For everything that has not been explained quantitatively yet, one does not think that the right theory is missing. Instead one blames our lack of knowledge how to solve the equations of motion of QCD in the energy regime where perturbation theory does not apply.

6. **Challenges:** The ongoing research in hadron physics aims at studying/solving QCD in the non-perturbative regime. In general this includes not only the challenge to understand how quarks and gluons form hadrons, but also how hadrons form atomic nuclei. It is rather clear by now that there is not a single method that can provide *the* solution. The present toolbox of QCD includes systematic approaches like lattice QCD for static quantities, effective field theories for low-energy quantities and perturbative QCD for high-energy quantities. For selected observables one also utilizes some fundamental properties of quantum field theory like analyticity (related to locally interacting fields), unitarity (related to the probabalistic interpretation of quantum theory) and dispersion relations (related to causality). Finally at present there are still many observables where phenomenological models are used. The phrases "systematic", "effective theory" and "phenomenological model" will be explained in section 1.3 below. It is a matter of active research to improve and extend the toolbox of QCD and the present lecture notes make an attempt to describe the current status of at least part of this toolbox.

There are several reasons why QCD is interesting: From the point of view of high-energy particle physics QCD is the prototype of a complete theory. Theories that improve the electroweak theory and/or unify it with QCD might be conceptually similar to QCD. In particular the prospect to have an extended symmetry structure that allows only for one universal coupling constant could provide an answer to the question why the proton and the positron have exactly the same charge. The keyword here is "grand unified theories". Essentially all such theories are based on a framework that resembles QCD.

From the point of view of low-energy particle physics QCD and hadron physics are important to provide the hadronic contribution to high-accuracy standard-model predictions. This enters the high-precision searches for physics beyond the standard model.

From the point of view of hadron and nuclear physics QCD is the fundamental theory of the strong interaction. The ultimative task is to explain all relevant and interesting features of hadrons and of atomic nuclei in a framework that is entirely based on QCD.

The general questions are: What are the fundamental building blocks of matter and how do these building blocks form matter? QCD plays an important role in addressing these questions because it is of direct relevance (strong interaction)

and because it serves as a prototype for a future theory that will encompass and supersede the standard model.

1.3 Phenomenological models, effective theories and microscopic theories

The logical next step seems to be that these lecture notes should become more specific and focus on the particles that are subject to the strong interaction. This expectation will be met soon. First, however, let us take one step back and have a look on physics in general and on the development of theories.

The first step to develop a theory is the observation of a pattern in experimental data. It gives rise to a **phenomenological model**. As an example let us study a spring that we try to elongate. Hooke's law is the very successful, i.e. predictive phenomenological model that relates the achieved elongation x to the size of the required force F:

$$F/x = \text{constant} \qquad \text{(Hooke's law)}. \tag{1.3.1}$$

The constant depends on the used spring, but it is the same for the same spring, it does not depend on the elongation x. This means that one needs one measurement for a given spring (one pair of values x and F) and then one has predictive power for any other value of x. The constant is the *parameter* of the developed phenomenological model.

It is an important step to develop such models. Often simple patterns/relations can be buried under a pile of side effects. For instance, Newton's "simple" statement, that objects that move with constant speed keep on moving, is entirely against the daily-life experience that moving objects get slower and slower and eventually stop. One has to strip off the omnipresent side effect of friction to recover Newton's beautifully simple laws.

If one has found a phenomenological model and tested sufficiently its predictive power, one could in principle be happy and study its applications. However, (fundamental) physics is more ambitious. After figuring out *how* something is, we want to know *why* it is like that. In physics one has a very specific way of asking why-questions. One is eventually interested in quantitative questions. To make the why-question concrete, we ask "how big" and "to which extent". The latter question concerns the quality of the developed model: To which extent does the model work that we have established? The first question concerns the size of the model parameters: How big are they? Can one calculate them?

The phenomenological model does not answer these deeper questions. It does not point out its own limitations. This makes the phenomenological model distinct from an **effective theory**. It also does not explain the size of its parameters. This is the purpose of an underlying, in general more **microscopic theory**.

Let us start with the less known concept, that of an effective theory. Before we spell out its generic properties, it is useful to give an example. Let us come back

to the spring and Hooke's law. A force that scales linearly with the elongation x corresponds to a potential energy V that is quadratic in x. Suppose we do not know anything about the spring, but assume that we work in the elastic regime where the spring does not break and is not permanently deformed. Then we can characterize the system by a potential energy $V(x)$. The elongation is measured relative to the state where no force acts on the spring. Therefore $x = 0$ corresponds to the situation where the potential energy has a minimum. Let us Taylor expand the otherwise unspecified potential energy around its minimum:

$$V(x) = V(0) + \frac{1}{2} V''(0)\, x^2 + \frac{1}{3!} V'''(0)\, x^3 + \dots \,. \tag{1.3.2}$$

The right-hand side of this equation defines the full effective theory. If one took it literally, it would not have any predictive power: One would need to know all derivatives $V^{(n)}(0)$, i.e. one would need infinitely many parameters. The proper way how to use an effective theory is to truncate the series in (1.3.2). The feature of the effective theory — that makes it distinct from a phenomenological model — is the fact that one can truncate it anywhere and that one can use the truncation to obtain an accuracy estimate. If one wants to achieve a larger accuracy, one truncates "later", i.e. includes more terms of the series expansion. An effective theory can be systematically improved. Of course, one needs more parameters to achieve this improvement. In this sense an effective theory is less predictive than a phenomenological model, but a truncated effective theory can tell to which extent, i.e. up to which accuracy it holds. A phenomenological model cannot tell about its own limitations.

The effective theory can tell us first of all *why* the phenomenological model works at all: If x is not very large, then the quadratic term in the expansion (1.3.2) should be sufficient. This is the range of applicability of Hooke's law. The effective theory can make these statements *quantitative*: The leading (non-trivial) order of the expansion in (1.3.2) corresponds to Hooke's law. We call this approximation V_{LO} where "LO" stands for "leading order":

$$V_{\mathrm{LO}}(x) := V(0) + \frac{1}{2} V''(0)\, x^2 \,. \tag{1.3.3}$$

So, how good is Hooke's law? To which extent does it hold? Suppose that we truncate one order higher, i.e. at "next-to-leading order" (NLO):

$$V_{\mathrm{NLO}}(x) := V(0) + \frac{1}{2} V''(0)\, x^2 + \frac{1}{3!} V'''(0)\, x^3 \,. \tag{1.3.4}$$

We perform a couple of measurements to pin down values for $V''(0)$ and $V'''(0)$. If the experiments had infinite precision, two measurements would be enough. In reality one has to account for the accuracy of the experiment, not only for the accuracy of the theory. Once we know our two parameters of the effective theory at NLO, we can make the following estimates: Hooke's law provides the force

$F = V''(0)\,x$. The NLO effective theory corrects this force by $\Delta F \approx |V'''(0)|\,x^2/2$. Thus the error that one makes when using Hooke's law is approximately given by

$$\frac{\Delta F}{F} \approx \frac{|V'''(0)|\,x}{2V''(0)}\,. \tag{1.3.5}$$

If one is willing to live with a relative uncertainty of y, then Hooke's law is accurate enough up to elongations

$$x \approx \frac{2V''(0)\,y}{|V'''(0)|}\,. \tag{1.3.6}$$

It should have become clear by now that one can go to next-to-next-to-leading order (NNLO) to estimate the accuracy of the NLO theory. For a given range of elongations the NLO effective theory is more accurate than the LO effective theory. For a given accuracy one can reach out to larger elongations with the NLO theory. In general one can *systematically* improve the framework until one is satisfied with the achieved accuracy. The price to pay is that one needs to determine more parameters before one obtains predictive power. But one gains the possibility to estimate the accuracy of the theory prediction and one knows how to improve it.

The final limitation of the effective theory, the need to determine more and more parameters, can be lifted if one can develop a better, often more **microscopic theory**. Concerning the spring this would be a material theory that calculates/predicts for a given material the parameters $V^{(n)}(0)$. Then one can perform all accuracy estimates without explicit measurements. But why should one do this, if one has now a better theory at hand? Aren't the phenomenological model and the effective theory obsolete once one has found a better theory? Very often this is not true and it would be extremely *in-effective* to abandon the effective theory in favor of the microscopic theory.

Another example might clarify this statement: Suppose one wants to describe the motion of a macroscopic object that is thrown. The object consists of atoms (microscopic theory). But one would not describe the motion of each of the atoms to figure out how an object flies. A phenomenological model does the main job: One treats the object as a point particle. The only parameter that one needs is the mass of the object. One can systematically refine the description: First one would introduce the tensor that contains the moments of inertia. In that way one can account for rotations of the object. And one can quantify how well the point-particle description works that does not account for the rotations. Next one could describe the vibrations of the object based on quantities like stiffness. In none of the "parameters" (mass, moments of inertia, stiffness) the notion of atoms has appeared yet. One rather constructs several layers of effective theories (or if not available phenomenological models). Each of these theories/models contains parameters that are calculated at the next more refined/microscopic level.

The final aim of physics is not to calculate the properties of a table from the standard model of particle physics or from an even more microscopic theory. Instead the purpose of a microscopic theory is to describe physics on the length scale for which the theory has been designed and, in addition, to provide the parameters for the effective theory that operates on a coarser scale. We are still far away to successfully bridge the gap quantitatively from macroscopic physics to particle physics, but the use of effective theories is a core concept in this endeavor.

For particle physics this way of thinking has two consequences, one for the standard model and one for QCD: Concerning the standard model of particle physics the modern point of view is that the standard model constitutes the lowest order of an effective field theory. Just like the LO expression (1.3.3) coincides with Hooke's phenomenological model. In principle, corrections can be systematically formulated, but so far one has not found observable effects from these corrections. In other words, all fits of NLO parameters are compatible with zero within the accuracy of the experimental determinations. The NLO parameters are truly very small. Naturally we would like to understand why these parameters are so small. Of course, one needs a better, more microscopic theory to answer this question.

In the realm of the strong interaction one has to face several scales. We will come back to them in the main part of these lecture notes. One framework is not enough to cover the physics of all scales that are relevant for the strong interaction. The concepts of scale separation and of effective field theories are used to deal with the multitude of scales. First of all this helps us to understand the features of the strong interaction and the formation of matter at its first levels of complexity. However, what one develops in the realm of the strong interaction might also be of relevance for improved theories of particle physics in general.

1.4 Hadrons and quarks

As already mentioned there is a plethora of hadrons. A by far non-exhaustive list is given in the tables 2 and 3 below. We know nowadays that the hadrons are not elementary, but consist out of quarks. Some of their properties are collected in table 1.

The tables 1-3 give rise to a lot of questions some of which are collected here:

- Why do hadrons not decay into quarks (and gluons)?
- Why are the hadrons (built from u, d, s quarks) so much heavier than the sum of their constituents?
- Why can protons, neutrons, kaons, ... not decay into pions?
- Why is the mass of proton and neutron (of $\pi^{\pm}$ and π^0, ...) nearly the same?
- Why are pions (kaons, etas) so light?

"stable" hadrons (selection)			
	mass (MeV/c^2)	life time (10^{-21}s)	minimal quark content
π^0	135	$8 \cdot 10^4$	$u\bar{u} - d\bar{d}$
$\pi^\pm$	140	$2.6 \cdot 10^{13}$	$u\bar{d}, d\bar{u}$
$K^\pm$	494	$1.2 \cdot 10^{13}$	$u\bar{s}, s\bar{u}$
$K^0, \bar{K}^0$	498	$0.9 \cdot 10^{11}(K^0_S)$, $5.2 \cdot 10^{13}(K^0_L)$	$d\bar{s}, s\bar{d}$
p	938	$\infty(> 10^{25}y)$	uud
n	940	$8.9 \cdot 10^{23}$	udd
Λ	1116	$2.6 \cdot 10^{11}$	uds
Σ^+	1189	$0.8 \cdot 10^{11}$	uus
Σ^0	1193	$0.7 \cdot 10^2$	uds
Σ^-	1197	$1.5 \cdot 10^{11}$	dds
$D^0, \bar{D}^0$	1865	$0.4 \cdot 10^9$	$c\bar{u}, u\bar{c}$
$D^\pm$	1869	$1.1 \cdot 10^9$	$c\bar{d}, d\bar{c}$
$D^\pm_s$	1969	$0.5 \cdot 10^9$	$c\bar{s}, s\bar{c}$
$B^\pm$	5279	$1.7 \cdot 10^9$	$u\bar{b}, b\bar{u}$
$B^0, \bar{B}^0$	5279	$1.5 \cdot 10^9$	$d\bar{b}, b\bar{d}$

Table 2: Some properties of hadrons which are stable with respect to the strong interaction [PDG].

hadronic resonances (selection)			
	mass (MeV/c^2)	life time (10^{-21}s)	minimal quark content
η	547	$0.7 \cdot 10^3$	$c_1(u\bar{u} + d\bar{d}) + c_2 s\bar{s}$
$\rho^{\pm,0}$	770	$4.7 \cdot 10^{-3}$	$u\bar{d}, d\bar{u}, u\bar{u} - d\bar{d}$
ω	782	$7.9 \cdot 10^{-2}$	$\approx u\bar{u} + d\bar{d}$
η'	958	3.3	$c'_1(u\bar{u} + d\bar{d}) + c'_2 s\bar{s}$
ϕ	1020	0.15	$\approx s\bar{s}$
Δ^{++}	1232	$5 \cdot 10^{-3}$	uuu
η_c	2980	$5 \cdot 10^{-2}$	$c\bar{c}$
J/ψ	3097	7.7	$c\bar{c}$
Υ	9460	13	$b\bar{b}$

Table 3: Some properties of hadronic resonances, i.e. hadrons which decay into other hadrons on account of the strong interaction [PDG].

- (Why is the η' so heavy?)
- (Why does the J/ψ live so long?)
- How can we explain in general the masses of the hadrons, their reaction cross sections, the life times of the resonances, their decay channels?

Some of these questions will be answered in the present lecture, but most of them are matter of active research of hadron physics. (The questions in parentheses arise if one already has answered some of the other questions.)

The insight that hadrons consist out of quarks essentially came from two sources: The classification of hadrons into multiplets (quark model) and (later) the observation of quarks (and gluons) in high-energy reactions. Forming groups of hadrons with similar masses and properties, Gell-Mann and collaborators found that the obtained structures can be interpreted as representations of the special unitary group $\mathbf{SU}(3)$ (see section 4 below). The, at that time, fictitious particles of the fundamental representation were called "quarks". In that picture it turned out that the known hadrons are either states which consist out of three quarks, called baryons, or states which consist out of a quark and an antiquark, called mesons (see the assignments in tables 2, 3). Other combinations, like states formed by two quarks or in particular the quarks themselves, have not been found as (isolated) entities. To explain the properties of the known hadrons (e.g., their electric charge and their spin) one has to assume that the quarks are spin-1/2 objects with fractional electric charge (where the unit of electric charge is given by the modulus of the electron charge). In spite of a lot of experimental activity searching for states lighter than the known hadrons and/or states which carry fractional electric charge, one never has found such single, isolated quarks. In the beginning one possible interpretation was that the quarks are not real but merely serve as part of an abstract classification scheme for hadrons. This picture changed with the observation of quarks (and later gluons) in high-energy reactions. Nowadays, the fact that quarks cannot be isolated over large distances is called "confinement", one of the not fully understood phenomena of the strong interaction.

1.5 Quarks in high-energy reactions

The discovery that the proton is not pointlike but has an intrinsic structure came from electron-proton reactions. As a matter of fact, the resolution which one can achieve is limited by the available collision energy/momentum (momentum translates to inverse wave length which limits the resolution). So far we have not found any intrinsic structure of the electron. In other words, e.g., elastic electron-electron scattering can be successfully explained by quantum electrodynamics where the electron is treated as an elementary object.

If also the proton was pointlike it should be possible to explain elastic electron-proton scattering by the same framework since the electrons do not interact strongly.[7] More generally, since we understand the electron very well we can use it as a diagnostic probe to explore, e.g., the proton. In Fig. 1 the result of elastic electron-proton scattering is displayed. The curve labeled "Mott Scattering" would emerge if the proton was pointlike. Obviously the proton has a finite size. The logically next question was whether the proton has an amorphous structure

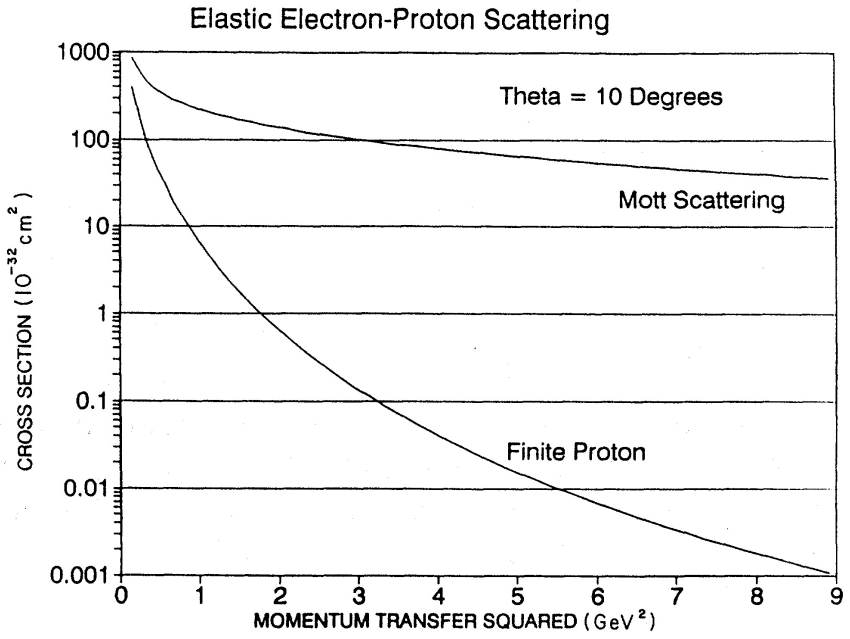


FIG. 4. Elastic scattering cross sections for electrons from a "point" proton and for the actual proton. The differences are attributable to the finite size of the proton.

Figure 1: Figure taken from H.W. Kendall, Rev. Mod. Phys. 63, 597 (1991) (Nobel prize lecture 1990).

("jelly") or pointlike constituents. It turned out that deep inelastic scattering reactions,

$$e^- + p \to e^- + \text{anything}, \tag{1.5.1}$$

at large energies could be understood if one assumed pointlike and basically non-interacting constituents with spin 1/2, called partons. Improved measurements

[7] In the energy region that is of relevance here the weak interaction is much smaller than the electromagnetic interaction and therefore not relevant.

revealed modifications to the picture that the partons do not interact. These modifications ("violations of Bjorken scaling") can be quantitatively explained by quantum chromodynamics, the theory in which the partons are identified with the quarks. An additional ingredient are the gluons which mediate the interactions between the quarks.

1.6 Confinement and asymptotic freedom

On the one hand, we have never found a single quark isolated from others over a large distance. Instead we can only isolate hadrons. On the other hand, protons are extended objects and high-energy reactions, e.g., of electrons scattering on protons, can be explained by the assumption of pointlike and (nearly) non-interacting constituents which form the proton. How do these observations fit together? One has to realize that the two observations concern different resolution scales: Short distances are resolved by large energy/momentum transfer (high resolution). Here quarks and gluons behave like (nearly) non-interacting single particles. This finding is called **asymptotic freedom**. An effect at large distances would concern the isolation of a single quark from the rest of a hadron. The phenomenological finding is that this is not possible. One observes the **confinement** of quarks and gluons. Consequently, at low momentum transfer the interaction between quarks and gluons seems to be strong (and attractive).

Thus the strong interaction is actually "weak" at large energies, but truly strong at small energies. Indeed, quantum field theories typically yield an effective interaction strength which depends on the momentum transfer. Most theories, however, predict a strength which grows with momentum. Only "non-abelian gauge theories" (see section 4) yield an effective interaction which decreases with momentum. Quantum chromodynamics (QCD) is such a non-abelian gauge theory. It explains the phenomenon of asymptotic freedom, which is exact at infinitely high energies, and explains also quantitatively the change of the interaction strength with lowering of the momentum. The interaction strength is depicted in Fig. 2, including a comparison to data. There is, however, an energetic limit below of which the corresponding calculations (perturbative QCD) loose their accuracy. The phenomenon of confinement cannot be rigorously explained yet, but at least QCD makes it plausible that new effects — as compared to the weakly[8] interacting quarks and gluons at high energies — appear at low energies.

One might say that in high-energy reactions we can "see" quarks and gluons (in particular in "jet events"), but we cannot "cut" them out (isolate them). If one tries to kick out a quark from a hadron, one produces a bunch of new hadrons instead. The lectures will provide an introduction to the theoretical

[8]Again: here "weakly" does not refer to the weak interaction, but to the energy regime where the strong interaction is not very strong anymore.

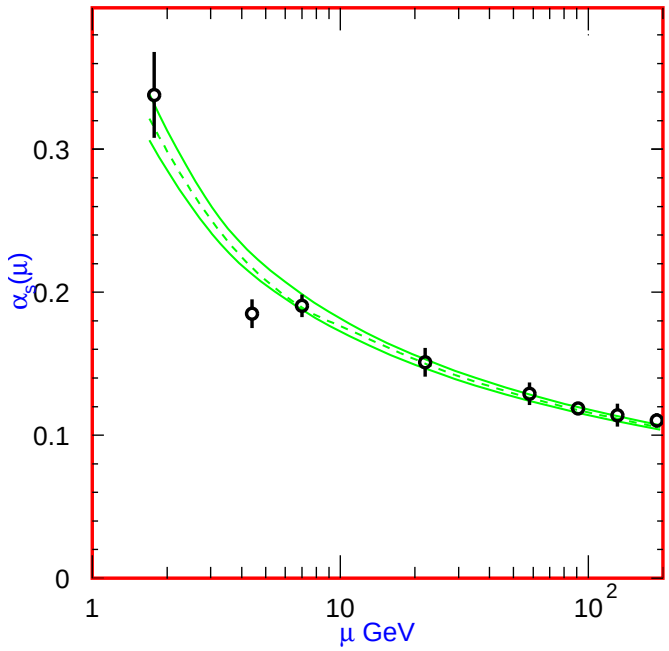


Figure 2: The effective interaction strength of QCD as a function of the typical energy μ. Figure taken from the Particle Data Group [PDG].

description of hadrons and to QCD as the underlying microscopic theory. The main emphasis will be on the low-energy regime where hadrons are the relevant degrees of freedom.

2 Lagrangians and Symmetries

2.0 Conventions

It is common practice in particle physics to use natural units: $c = 1 = \hbar$, where c denotes the velocity of light (in vacuum) and $\hbar$ Planck's quantum of action. A useful relation for calculations is $\hbar\, c \approx 197\,\mathrm{MeV\,fm}$.

Dealing with relativistic systems it is useful to introduce four-vectors which have well-defined transformation properties with respect to Lorentz transformations (rotations and boosts). The contravariant space-time four-vector for time t and position in space $\vec{r}$ is given by[9]

$$(x^\mu) = (x^0, x^1, x^2, x^3) := (t, \vec{r})\,. \tag{2.0.1}$$

To form Lorentz invariant products it is useful to introduce also the covariant four-vector

$$(x_\mu) = (x_0, x_1, x_2, x_3) := (t, -\vec{r})\,. \tag{2.0.2}$$

[9]Note that time differences and space distances have the same units since $c = 1$.

Also energy and momentum form a four-vector:

$$(p^\mu) = (p^0, p^1, p^2, p^3) := (E, \vec{p}\,), \qquad (p_\mu) := (E, -\vec{p}\,). \tag{2.0.3}$$

Care must be taken with the four-derivative operator: The object which transforms like the contravariant four-vectors is given by

$$(\partial^\mu) := (\partial^\mu_x) := \left(\frac{\partial}{\partial x_\mu}\right) = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \tag{2.0.4}$$

whereas

$$(\partial_\mu) := \left(\frac{\partial}{\partial x^\mu}\right) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right). \tag{2.0.5}$$

The relations between derivatives and energy-momentum already known from (non-relativistic) quantum mechanics can be conveniently formulated with four-vectors:

$$p^\mu \to i\partial^\mu, \qquad \text{i.e.} \quad E \to i\partial_t, \quad \vec{p} \to -i\vec{\nabla}. \tag{2.0.6}$$

The metric tensor, defined by

$$(g_{\mu\nu}) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =: (g^{\mu\nu}), \tag{2.0.7}$$

serves to rise or lower the indices, i.e. for an arbitrary four-vector a:

$$a^\mu = \sum_{\nu=0}^{3} g^{\mu\nu} a_\nu =: g^{\mu\nu} a_\nu \tag{2.0.8}$$

where in the last step we have introduced Einstein's sum convention: *If an index appears twice, an implicit summation over this index is assumed.*

Sometimes we will have to distinguish between the zeroth component ("time component") and the spatial components of a four-vector. In this case we use Latin indices to denote the spatial components. Together with Einstein's sum convention this implies that a Greek index which appears twice is to be summed from 0 to 3 whereas a Latin index is to be summed from 1 to 3 (if not stated otherwise).

The metric tensor transforms like a tensor of rank two with respect to Lorentz transformations, i.e. like a (direct) product of two four-vectors $a^\mu b^\nu$. We can use it to introduce

$$g_\mu{}^\nu := g_{\mu\alpha}\, g^{\alpha\nu} \qquad \Rightarrow \qquad (g_\mu{}^\nu) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.0.9}$$

Lorentz invariant scalar products between two four-vectors a and b can be written down in many ways, e.g.:

$$\begin{aligned} a \cdot b &:= a_\mu \, b^\mu = a^\mu \, b_\mu = a^\mu \, g_{\mu\nu} \, b^\nu = a_\mu \, g^{\mu\nu} \, b_\nu \\ &= a^0 \, b_0 + a^j \, b_j = a^0 \, b^0 - a^j \, b^j = a^0 \, b^0 - \vec{a} \cdot \vec{b} \end{aligned} \tag{2.0.10}$$

Sometimes one also uses

$$a^2 := a \cdot a \, . \tag{2.0.11}$$

This should only be used if it is clear from the context that one talks about the square of a and not about the second component of a^μ. A single free particle with mass m satisfies the energy-momentum relation

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^{\,2} = m^2 \, . \tag{2.0.12}$$

Finally we introduce the totally antisymmetric Levi-Civita tensor, $\epsilon^{\mu\nu\alpha\beta}$. In four dimensions it is defined by the following properties:

- sign change under exchange of any of the two indices,
- $\epsilon^{0123} = 1$.

This implies

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for} \quad (\mu,\nu,\alpha,\beta) = (0,1,2,3),(2,3,0,1),(3,2,1,0),\ldots \\ -1 & \text{for} \quad (\mu,\nu,\alpha,\beta) = (1,2,3,0),\ldots \\ 0 & \text{if} \quad \text{two indices agree.} \end{cases} \tag{2.0.13}$$

Sometimes we will also need the three-dimensional counterpart:

$$\epsilon^{jkl} = \begin{cases} +1 & \text{for} \quad (j,k,l) = (1,2,3),(2,3,1),(3,1,2)\,, \\ -1 & \text{for} \quad (j,k,l) = (1,3,2),(2,1,3),(3,2,1)\,, \\ 0 & \text{else.} \end{cases} \tag{2.0.14}$$

2.1 Lagrangians and quantization

2.1.1 Mechanics and field theory

In classical mechanics where one deals with a finite number, n, of degrees of freedom one can express the content of a dynamical system by specifying the "Lagrange function". It is a function of the (generalized) coordinates, q_j, and their time derivatives, $\dot{q}_j$, where $j = 1, \ldots, n$ enumerates the degrees of freedom:

$$L = L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) =: L(q_j, \dot{q}_j) \, . \tag{2.1.1}$$

The Lagrange function is so popular because it is perfectly suited to study the symmetry properties of a system — as we will see. From the Lagrange function one determines the action

$$S := \int dt\, L\left(q_j(t), \dot{q}_j(t)\right) \,. \tag{2.1.2}$$

The classical system evolves such that the action, S, acquires an extremal value (*Hamilton's principle of minimal action*). This is equivalent to solving the Euler-Lagrange equations of motion (e.o.m.'s)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0\,, \qquad j = 1, \ldots, n\,. \tag{2.1.3}$$

The conjugate momentum for the generalized coordinate q_j is defined by

$$p_j := \frac{\partial L}{\partial \dot{q}_j}\,. \tag{2.1.4}$$

Typically one can use (2.1.4) to express the velocities, $\dot{q}_j$, in term of the momenta and coordinates.[10] Then one can construct the Hamiltonian

$$H(q_j, p_j) = p_k\, \dot{q}_k(q_j, p_j) - L(q_l, \dot{q}_l(q_j, p_j))\,. \tag{2.1.5}$$

where summation over the number of degrees of freedom, enumerated by k, is implicit. From the Euler-Lagrange e.o.m.'s (2.1.3) one can derive the Hamilton e.o.m.'s,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}\,. \tag{2.1.6}$$

While the Lagrange and the Hamilton formulation are very popular there is an alternative description which is much less known. It starts from a Hamiltonian but derives the e.o.m.'s from a principle of minimal action. Since we will need the field theoretical counterpart of that formulation, we shall briefly introduce it here. In the Lagrange function (2.1.1) the velocities appear typically in quadratic order (kinetic energy). Indeed this enables to express the velocities in terms of the momenta via (2.1.4). The alternative formulation starts from a Lagrange function in first-order form given by

$$L(q_j, \dot{q}_j, p_j) = p_k\, \dot{q}_k - H(q_j, p_j) \tag{2.1.7}$$

where, again, summation over k is implicit. Thus, the Lagrange function is a function of the coordinates, their time derivatives *and* the momenta. Demanding again that the corresponding action (2.1.2) acquires an extreme value for the physically realized coordinates and momenta, one gets indeed the e.o.m.'s (2.1.6).

[10] This does not always work. If it does not, one has a system with *constraints*.

At the same time the relation (2.1.4) is still true. We will come back to that first-order form of the Lagrange function when discussing the Dirac fields.

The standard way to quantize a system is to demand

$$[q_j(t), p_k(t)] = i\delta_{jk}\,, \tag{2.1.8}$$

where the commutator of two operators or matrices, A and B, is defined by $[A, B] := AB - BA$. Note that we work here in the Heisenberg picture where the operators are time dependent. This is the picture where the generalization to field theory is most straightforward.

In a field theory each space point can be excited. The excitation amplitudes correspond to the previously discussed degrees of freedom. In a continuous space (e.g., our three-dimensional space) one has now infinitely many degrees of freedom and they form a continuum. The index j which enumerated the (finite) number of degrees of freedom for a mechanical system is now replaced by the continuous variable $\vec{r}$ which characterizes the position in space. The generalized coordinate is replaced by the field amplitude:

$$\begin{aligned} j &\rightarrow \vec{r} \\ q_j(t) &\rightarrow \phi(t, \vec{r}) = \phi(x)\,, \end{aligned} \tag{2.1.9}$$

where we have introduced the space-time four-vector x again (cf. section 2.0). Even more generally one might consider at the very same point several (m) field amplitudes, $\phi_a(x)$, $a = 1, \ldots, m$.

As usual when changing from a discrete to a continuous description the quantities one studied before become densities. Thus the central object is now the Lagrange density or just the "Lagrangian". Both from the point of view of Lorentz invariance and of having a continuum of degrees of freedom it is appropriate that the Lagrangian does not only depend on the amplitudes and their time derivatives but rather also on the spatial derivatives. Hence the generic dependence of the Lagrangian is

$$\mathcal{L} = \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))\,. \tag{2.1.10}$$

The Lagrange function is obtained by a volume integral,

$$L = \int d^3x\, \mathcal{L}\,, \tag{2.1.11}$$

and the action by a further time integral:

$$S = \int dt\, L = \int d^4x \mathcal{L}\,. \tag{2.1.12}$$

In relativistic systems the Lagrangian and therefore also the action are Lorentz invariant. One further demands that the Lagrangian is real-valued (after quantization: hermitian) to make sure that the Hamiltonian, which one obtains from the Lagrange function, is also hermitian.

The equations of motion (e.o.m.'s), which lead to field configurations which extremize the action, are now given by

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \qquad \text{for all } a. \tag{2.1.13}$$

The conjugate momentum (density) for the amplitude ϕ_a is defined by

$$\pi_a(x) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}\,. \tag{2.1.14}$$

The quantization corresponds to the discrete case (2.1.8). One has to distinguish, however, between bosons and fermions:

$$\text{for bosons:} \qquad [\phi_a(t,\vec{r}), \pi_b(t,\vec{r}\,')] = i\,\delta_{ab}\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,, \tag{2.1.15}$$

$$\text{for fermions:} \qquad \{\phi_a(t,\vec{r}), \pi_b(t,\vec{r}\,')\} = i\,\delta_{ab}\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,, \tag{2.1.16}$$

with the anticommutator for arbitrary operators, A and B, defined by $\{A,B\} := AB + BA$. Note that the (anti-)commutation rule applies to fields and momenta on equal time t. The physical content of the quantization procedure is to promote the field amplitudes to operators in Fock space, which is the space consisting of one-particle, two-particle, three-particle, ... wave functions (plus the no-particle wave function), see, e.g., [Brown] for an excellent introduction. In that context the commutation (anticommutation) relation ensures that a many-particle wave function remains unchanged (changes its sign) with respect to an exchange of two identical bosons (fermions). It turns out that a reasonable relativistic quantum field theory can only be achieved if fields with integer spin are quantized as bosons and fields with half-integer spin as fermions ("spin-statistics theorem").

2.1.2 Bosonic fields

In the following we will present some important examples for Lagrangians for relativistic systems.

The Lagrangian for the simplest field, an **uncharged scalar non-interacting boson** with mass m, is given by

$$\mathcal{L}_{\rm KG} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,m^2\,\phi^2 = \frac{1}{2}\partial_\alpha\phi\,g^{\alpha\beta}\,\partial_\beta\phi - \frac{1}{2}\,m^2\,\phi^2\,. \tag{2.1.17}$$

The final rewriting has been performed to display that the object $\partial_\mu\phi$, which shows up in the e.o.m. (2.1.13), actually appears twice in (2.1.17). The e.o.m. is obtained as

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}_{\rm KG}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathcal{L}_{\rm KG}}{\partial\phi} = \frac{1}{2}\,\partial_\mu\left(g_\alpha{}^\mu\, g^{\alpha\beta}\,\partial_\beta\phi + \partial_\alpha\phi\, g^{\alpha\beta}\, g_\beta{}^\mu\right) + m^2\,\phi \\ &= \partial_\mu\partial^\mu\phi + m^2\,\phi = \partial_t^2\phi - \vec{\nabla}^2\phi + m^2\,\phi \end{aligned} \tag{2.1.18}$$

which is indeed the Klein-Gordon equation. The conjugate momentum (2.1.14) is just

$$\pi = \frac{\partial \mathcal{L}_{\mathrm{KG}}}{\partial \dot{\phi}} = \dot{\phi}\,. \tag{2.1.19}$$

Therefore the quantization condition (2.1.15) leads to

$$[\phi(t,\vec{r}), \dot{\phi}(t,\vec{r}\,')] = i\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,. \tag{2.1.20}$$

As a linear partial differential equation with constant coefficients one can easily write down the most general solution of (2.1.18). To ensure that the Lagrangian is real-valued (hermitian after quantization) the field amplitude, ϕ, must also be hermitian. Under that restriction the general solution of (2.1.18) is given by

$$\phi(x) = \phi(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3\,2E_k} \left(a(\vec{k})\, e^{-i(E_k t - \vec{k}\cdot\vec{r})} + a^\dagger(\vec{k})\, e^{+i(E_k t - \vec{k}\cdot\vec{r})} \right)\,, \tag{2.1.21}$$

where the energy-momentum relation

$$E_k := \sqrt{\vec{k}^2 + m^2} \tag{2.1.22}$$

follows as a consequence of the equation of motion (2.1.18). The Fock space operator $a^\dagger(\vec{k})$ creates a single-particle mode with fixed three-momentum $\vec{k}$ while $a(\vec{k})$ annihilates this mode. The Klein-Gordon Lagrangian together with the quantization condition can describe systems with arbitrary many indistinguishable and non-interacting spin-0 (i.e. scalar) bosons.

Next we turn to an example which includes an interaction, the $\boldsymbol{\phi^4}$ **theory** ("phi-four theory"). The Lagrangian is given by

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,m^2\,\phi^2 - \frac{\lambda}{4!}\,\phi^4\,. \tag{2.1.23}$$

The last term is called a self-interaction term because it only involves one type of field. The parameter λ is called coupling constant and parametrizes the strength of the interaction. The appearance of the factor 1/4! is pure convention. One also could have introduced a coupling constant $\tilde{\lambda} = \lambda/4!$ instead. The reason for introducing the factor 1/4! will become clear in section 3. In an extended form (with four scalar fields and with spontaneous symmetry breaking, cf. section 5) the ϕ^4 theory constitutes the Higgs sector of the standard model of elementary particle physics (see, e.g., [Mosel,Peskin]). The e.o.m. following from (2.1.13) is

$$\partial_\mu\partial^\mu\phi + m^2\,\phi + \frac{\lambda}{3!}\,\phi^3 = 0\,. \tag{2.1.24}$$

Quantization is achieved in the same way as for the free Klein-Gordon theory, i.e. equations (2.1.19) and (2.1.20) hold again, since the interaction term does

not involve time derivatives. Obviously the differential equation (2.1.24) is not solved by plane waves. The interaction term causes a non-linearity in the equation of motion. In general, if a Lagrangian has a term with more than two fields, then the corresponding equation of motion has a non-linear term, an interaction. Typically such equations do not have exact analytical solutions and the corresponding quantum theory does not have exact solutions at all. One frequently used tool to obtain approximate solutions is perturbation theory. It can work if the involved coupling constants are small enough. What this statement means quantitatively must be worked out for each theory separately. We will come back to perturbation theory in section 3.

The next example concerns a **charged scalar non-interacting boson** field. The main purpose is to stress that differently looking Lagrangians can contain the very same physics. We consider a Lagrangian for two Klein-Gordon fields with the same mass:

$$\mathcal{L}_{2\times\mathrm{KG}} = \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\,m^2\,\phi_1^2 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}\,m^2\,\phi_2^2\,. \tag{2.1.25}$$

The e.o.m.'s follow from (2.1.13) for $a = 1, 2$:

$$\partial_\mu\partial^\mu\phi_1 + m^2\,\phi_1 = 0\,, \tag{2.1.26}$$
$$\partial_\mu\partial^\mu\phi_2 + m^2\,\phi_2 = 0\,, \tag{2.1.27}$$

and the canonical momentum (2.1.14) is just

$$\pi_a = \frac{\partial\mathcal{L}_{2\times\mathrm{KG}}}{\partial\dot\phi_a} = \dot\phi_a \qquad \text{for } a = 1, 2\,. \tag{2.1.28}$$

The non-trivial commutators (2.1.15) are

$$[\phi_1(t,\vec r), \dot\phi_1(t,\vec r\,')] = i\,\delta^{(3)}(\vec r - \vec r\,') = [\phi_2(t,\vec r), \dot\phi_2(t,\vec r\,')] \tag{2.1.29}$$

whereas all other equal-time commutators between the field amplitudes and/or momenta vanish. In particular we note:

$$[\phi_1(t,\vec r), \dot\phi_2(t,\vec r\,')] = 0 = [\phi_2(t,\vec r), \dot\phi_1(t,\vec r\,')]\,, \tag{2.1.30}$$

which means that the particles described by ϕ_1 and ϕ_2, respectively, are *distinguishable*. To avoid misunderstanding: *One* quantized Klein-Gordon field describes the physics of arbitrary many *indistinguishable* bosons. The bosons described by ϕ_1 are supposed to be *distinguishable* from the ones described by ϕ_2, in spite of the fact that they have the same mass by construction. To distinguish particles in practice one needs an interaction. This is not included in (2.1.25). We are heading for the description of a charged particle and its antiparticle (hence same mass). They are distinguishable by their charge, but to

experience the charge one needs an electromagnetic field, i.e. an interaction. This will be introduced below in section 2.3.2. We now introduce new fields[11]

$$\phi = \frac{1}{\sqrt{2}}\,(\phi_1 + i\,\phi_2) \qquad \Leftrightarrow \qquad \phi^\dagger = \frac{1}{\sqrt{2}}\,(\phi_1 - i\,\phi_2) \tag{2.1.31}$$

and we claim that the theory expressed in terms of ϕ and $\phi^\dagger$ is equivalent to the theory described by ϕ_1 and ϕ_2. Note that in spite of the same name the ϕ field introduced here is different from the one introduced in (2.1.17). The new ϕ field is *not* hermitian. Therefore, with its real and imaginary part it has a larger content of information. By inverting the relations (2.1.31) the Lagrangian (2.1.25) can now be rewritten in terms of ϕ and $\phi^\dagger$:

$$\mathcal{L}_{\text{charged KG}}(\phi, \partial_\mu\phi, \phi^\dagger, \partial_\mu\phi^\dagger) := \mathcal{L}_{2\times\text{KG}}(\phi_1, \partial_\mu\phi_1, \phi_2, \partial_\mu\phi_2) = \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\,\phi^\dagger\,\phi\,. \tag{2.1.32}$$

We treat ϕ and $\phi^\dagger$ as independent variables and derive the e.o.m.'s:

$$0 \;=\; \partial_\mu \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial\phi} = \partial_\mu\partial^\mu\phi^\dagger + m^2\,\phi^\dagger\,, \tag{2.1.33}$$

$$0 \;=\; \partial_\mu \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial(\partial_\mu\phi^\dagger)} - \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial\phi^\dagger} = \partial_\mu\partial^\mu\phi + m^2\,\phi\,. \tag{2.1.34}$$

Since the Lagrangian is still hermitian, an alternative derivation of the second equation (2.1.34) is to take the adjoint of the first equation (2.1.33).[12] The conjugate momenta are given by

$$\pi = \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial\dot\phi} = \dot\phi^\dagger\,, \qquad \pi^\dagger = \frac{\partial\mathcal{L}_{\text{charged KG}}}{\partial\dot\phi^\dagger} = \dot\phi\,, \tag{2.1.35}$$

which leads to the non-trivial commutators

$$[\phi(t,\vec r), \dot\phi^\dagger(t,\vec r\,')] = i\,\delta^{(3)}(\vec r - \vec r\,') = [\phi^\dagger(t,\vec r), \dot\phi(t,\vec r\,')]\,. \tag{2.1.36}$$

By explicit calculation (try it!) using the definition (2.1.31) one can easily show that the equations for ϕ_1 and ϕ_2, namely (2.1.26), (2.1.27), (2.1.28) and (2.1.29), are completely equivalent to the equations for ϕ (and $\phi^\dagger$), i.e. (2.1.33), (2.1.34), (2.1.35) and (2.1.36). The e.o.m. (2.1.34) for ϕ has the following general solution

$$\phi(x) = \phi(t,\vec r) = \int \frac{d^3k}{(2\pi)^3\,2E_k} \left(a(\vec k)\,e^{-i(E_k t - \vec k\cdot\vec r)} + b^\dagger(\vec k)\,e^{+i(E_k t - \vec k\cdot\vec r)} \right). \tag{2.1.37}$$

It is illuminating to compare this solution to the one for the simplest (hermitian) Klein-Gordon field, (2.1.21), but to recall that ϕ in (2.1.37) is *not* hermitian. For

[11] Note that the fields ϕ_1 and ϕ_2 are hermitian.

[12] Basically this demonstrates the intrinsic consistency of the approach which assumes that one can treat ϕ and $\phi^\dagger$ as independent variables.

completeness one can also write down the general solution for $\phi^\dagger$ which one gets by taking the adjoint of equation (2.1.37):

$$\phi^\dagger(x) = \phi^\dagger(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3\,2E_k} \left(b(\vec{k})\, e^{-i(E_k t - \vec{k}\cdot\vec{r})} + a^\dagger(\vec{k})\, e^{+i(E_k t - \vec{k}\cdot\vec{r})} \right) . \quad (2.1.38)$$

The interpretation is that $a^\dagger$ creates a particle while $b^\dagger$ creates an antiparticle. It is pure convention which state is called "particle" and which "antiparticle". We have presented the very same theory in two different ways, i.e. by the two Lagrangians (2.1.25), (2.1.32). One purpose is to demonstrate that the same physics can be expressed by different Lagrangians and different fields. We will frequently come back to that point. Concerning the present case of two bosons with the same mass, one reasonable question is which Lagrangian is more economic to work out the physical content of the theory. For non-interacting fields there is actually no particular preference. All equations tell that the particles move freely without interactions and with an energy-momentum relation given by (2.1.22). Once one includes an interaction, e.g., electromagnetism, this changes. As we will see in section 2.3.2 in the formulation using ϕ_1 and ϕ_2 every interaction with an electromagnetic field, a photon field, changes a ϕ_1 mode to a ϕ_2 mode (with different momentum) and vice versa. This makes the description very uneconomic (but not wrong!). In contrast, a mode described by ϕ remains a ϕ mode. The reason is that the electric charge is a conserved quantity and the modes generated or annihilated by ϕ have well-defined charge. In contrast, a mode generated, e.g., by ϕ_1 does not have a well-defined charge. It is a superposition of states with fixed charges. Of course, one can also say in turn that a mode with a fixed charge is a superposition of states created by ϕ_1 and ϕ_2. The whole discussion resembles the corresponding one in quantum mechanics where one looks for eigen states of a (complete) set of commuting observables (e.g., the Hamiltonian and the total angular momentum and the parity and ...). It would not be wrong but extremely uneconomic to ignore the conserved quantum numbers. And one would miss an essential point of the physics (and the fun/aesthetics of studying the corresponding symmetries, cf. section 2.2 below).

In present-day research such a discussion matters for neutrinos and their **oscillations/mixing**. Before we briefly touch this issue we should stress that there are at least two aspects which are different from the previously discussed case: First, neutrinos are fermions, not bosons, and, second, the discussion does not concern particles and antiparticles (which necessarily have the same mass) but neutrinos with different masses (all of them particles). Nonetheless, the previous considerations apply in the following sense: Neutrinos interact only very weakly. Therefore it makes sense to consider the (macroscopic) time between two interaction points. During that time neutrinos move freely without interactions (by construction) which is described by plane waves or wave packets with a well-defined energy-momentum relation, i.e. in particular by a well-defined mass. At

present we know of three types of neutrinos and we may call their corresponding fields ν_1, ν_2 and ν_3 (which corresponds to our fields ϕ_1 and ϕ_2). Their masses are different from each other. Neutrinos, being elementary (pointlike) and chargeless leptons, are only subject to the weak interaction (besides gravity). However, a state created, e.g., by ν_1 is not an eigen states of the weak-interaction Hamiltonian. Concerning the interaction, in particular the creation of neutrinos and their detection by scattering, it is more economic to classify the neutrinos as electron-, mu- and tau-neutrinos. The corresponding fields are called ν_e, ν_μ and ν_τ (which corresponds to our fields ϕ and $\phi^\dagger$). Thus at the points of production and detection the description in terms of ν_e, ν_μ and ν_τ is more economic, while for the propagation one would prefer ν_1, ν_2 and ν_3. The common choice is actually ν_e, ν_μ and ν_τ which leads to the notion of neutrino oscillation: Suppose electron-neutrinos are created in a weak-interaction process. The corresponding wave function is a superposition of states created by ν_1, ν_2 and ν_3. Having different masses they propagate with different velocities. In other words, their interference pattern changes with time and space. At the point of measurement (which is typically different from the point of creation) one therefore can have a finite probability to observe a mu- or tau-neutrino, since, e.g., ν_1 is a superposition of ν_e, ν_μ and ν_τ. This has been observed indeed. The effect would not take place for neutrinos having the same mass. Then there would be no reason to distinguish between ν_1, ν_2 and ν_3 on the one hand and ν_e, ν_μ and ν_τ on the other hand. The observation of neutrino oscillations has shown in particular that neutrinos cannot be massless (at least not all of them). Otherwise their masses would be the same (zero) and no oscillations would be observed. In an alternative description (same physics!) one could stick to ν_1, ν_2 and ν_3. With a certain probability, e.g., a ν_1 state is created in a weak-interaction process involving, e.g., the electron-neutrino. The considered ν_1 state propagates freely to the point of detection. There, with a certain probability it is measured, e.g., as a tau-neutrino. Obviously in that language the mixing takes place at the points of creation and annihilation while the propagation is unspectacular. We stress again that oscillation during propagation and mixing at the interaction points describes the *same* physics. It leads for a given Lagrangian to the same predictions for observable quantities, e.g., for the probability to observe a specific particle given a specific initial creation process. The question is merely whether the Lagrangian is given in terms of the fields ν_1, ν_2 and ν_3 or in terms of ν_e, ν_μ and ν_τ. Just like our two Lagrangians (2.1.25) and (2.1.32) look differently but can be rewritten into each other.

Next we turn to the **photon field**. The corresponding Maxwell equations in Lorentz covariant form can be obtained from the Lagrangian

$$\mathcal{L}_{\mathrm{M}}(A_\alpha, \partial_\beta A_\alpha) = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - j_\alpha A^\alpha \,. \tag{2.1.39}$$

The four fields A_α, with $\alpha = 0, 1, 2, 3$, denote the electromagnetic vector potential

which is related to the electromagnetic field strengths by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \text{for} \quad \mu,\nu = 0,1,2,3\,. \tag{2.1.40}$$

The zeroth component, A^0 is well-known from electrostatics as the potential from which the electric field strength, $\vec{E}$, can be obtained. The three-vector $\vec{A}$ is the vector potential from which the magnetic field strength, $\vec{B}$, is obtained (see (2.1.42) below). Finally we have introduced an external electromagnetic four-current $(j^\mu) = (\rho, \vec{j})$ with the electric charge density ρ and the electric current density $\vec{j}$. In section 2.3 we will replace/specify this current by an expression in terms of charged matter fields. The following relations hold

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{2.1.41}$$

which together with (2.1.40) implies

$$\vec{B} = \vec{\nabla} \times \vec{A}\,, \qquad \vec{E} = -\frac{\partial}{\partial t}\,\vec{A} - \vec{\nabla} A^0\,. \tag{2.1.42}$$

The relations (2.1.40) automatically ensure that the homogeneous Maxwell equations hold. The inhomogeneous Maxwell equations in Lorentz covariant form follow as e.o.m.'s from (2.1.39):

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}_{\rm M}}{\partial(\partial_\mu A_\nu)} - \frac{\mathcal{L}_{\rm M}}{\partial A_\nu} = -\frac{1}{2}\,\partial_\mu \left(F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \right) + j^\nu \\ &= -\frac{1}{2}\,\partial_\mu \left(F^{\alpha\beta} \left(g_\alpha{}^\mu\, g_\beta{}^\nu - g_\alpha{}^\nu\, g_\beta{}^\mu \right) \right) + j^\nu = -\frac{1}{2}\,\partial_\mu \left(F^{\mu\nu} - F^{\nu\mu} \right) + j^\nu \\ &= -\partial_\mu F^{\mu\nu} + j^\nu\,, \end{aligned} \tag{2.1.43}$$

where in the last step we have made use of the fact that the field strength is antisymmetric with respect to an exchange of its indices,

$$F^{\mu\nu} = -F^{\nu\mu}\,, \tag{2.1.44}$$

as can be directly obtained from (2.1.40). The quantization of the photon field is somewhat tricky because of the gauge symmetry. We will define the notion of "gauge symmetry" in section 2.3, but we will not elaborate on the way how to quantize the photon field (see, e.g., [Peskin,Bjorken]).

Finally we write down the free Lagrangian for a massive (uncharged) vector field, called Proca field:

$$\mathcal{L}_{\rm Proca}(V_\alpha, \partial_\beta V_\alpha) = -\frac{1}{4}\,V_{\alpha\beta} V^{\alpha\beta} + \frac{1}{2}\,m_V^2\,V_\alpha V^\alpha \tag{2.1.45}$$

with $V_{\alpha\beta} := \partial_\alpha V_\beta - \partial_\beta V_\alpha$.

2.1.3 Dirac fields

The purpose of the present section is to present the Lagrangian which leads to the Dirac equation and to discuss the notions of helicity and chirality. A detailed discussion of the most general solution of the Dirac equation is not intended here (see, e.g., [Peskin,Bjorken]).

There are different ways how to write down a Lagrangian which leads to the Dirac equation. In the following we will present first a Lagrangian which is hermitian and comment on alternative formulations afterwards.

We start with

$$\mathcal{L}_{\rm D}(\psi,\psi^\dagger,\partial_\mu\psi,\partial_\mu\psi^\dagger) = \bar\psi\left(\frac{1}{2}\,i\,(\overrightarrow{\not\partial}-\overleftarrow{\not\partial}) - m\right)\psi \tag{2.1.46}$$

where ψ denotes a four-component spinor and its adjoint enters the definition

$$\bar\psi := \psi^\dagger\gamma_0\,. \tag{2.1.47}$$

Here γ_0 is one of the four 4×4 gamma matrices, γ_μ, $\mu=0,1,2,3$. They satisfy the anticommutation relation

$$\{\gamma_\mu,\gamma_\nu\} = 2\,g_{\mu\nu}\,\mathbb{1}_{4\times4} \qquad \text{for} \quad \mu,\nu=0,1,2,3\,. \tag{2.1.48}$$

For later use we also define

$$\gamma_5 := -\frac{i}{4!}\,\epsilon^{\mu\nu\alpha\beta}\,\gamma_\mu\,\gamma_\nu\,\gamma_\alpha\,\gamma_\beta = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \tag{2.1.49}$$

with the Levi-Civita tensor defined in (2.0.13). Using (2.1.48) one can show that

$$\{\gamma_\mu,\gamma_5\} = 0 \qquad \text{and} \qquad \gamma_5\,\gamma_5 = \mathbb{1} \tag{2.1.50}$$

hold. Any "slashed" four-vector, a^μ, is defined as

$$\not a := \gamma_\mu a^\mu\,. \tag{2.1.51}$$

An arrow over a partial derivative (which does *not* indicate a three-vector) points in the direction of the objects on which the partial derivative is supposed to act on, i.e.

$$A\,\overrightarrow{\partial}_\mu B := A\,(\partial_\mu B)\,, \qquad A\overleftarrow{\partial}_\mu\,B := (\partial_\mu A)\,B\,. \tag{2.1.52}$$

The motivation to introduce $\overleftarrow{\partial}$ comes, on the one hand, from the fact that the gamma matrices cannot be interchanged with the spinors and, on the other hand, from the desire for a compact notation:

$$\bar\psi\,\overleftarrow{\not\partial}\,\psi = \bar\psi\,\overleftarrow{\partial_\mu}\gamma^\mu\,\psi = (\partial_\mu\bar\psi)\,\gamma^\mu\,\psi\,. \tag{2.1.53}$$

Finally m denotes the mass of the particle. We note in passing that the use of $\bar{\psi}$ instead of $\psi^\dagger$ ensures that the Lagrangian (2.1.46) is Lorentz invariant (see, e.g., [Peskin] for details).

The gamma matrices are not fully specified by the conditions (2.1.48). However, one can show that each choice is physically equivalent — provided that (2.1.48) is satisfied (see, e.g., [Peskin]). Common choices are the Dirac-Pauli representation which is particularly suited to discuss the non-relativistic limit (see, e.g., [Bjorken]) and the Weyl representation which is useful to discuss the limit of massless particles (ultrarelativistic systems, see, e.g., [Peskin]). Here we choose the Weyl representation. The gamma matrices are then given by

$$\gamma^0 = \begin{pmatrix} 0_{2\times 2} & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0_{2\times 2} & \sigma^j \\ -\sigma^j & 0_{2\times 2} \end{pmatrix}, \qquad j = 1,2,3\,, \tag{2.1.54}$$

where σ^j denote the Pauli matrices defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.55}$$

The latter satisfy the relations

$$[\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l \tag{2.1.56}$$

and

$$\{\sigma^j, \sigma^k\} = 2\delta^{jk}\,\mathbb{1} \tag{2.1.57}$$

which can be combined to

$$\sigma^j\,\sigma^k = \delta^{jk}\,\mathbb{1} + i\epsilon^{jkl}\sigma^l\,. \tag{2.1.58}$$

Here the Levi-Civita symbol in three dimensions has been defined in (2.0.14). It is straightforward to show that the gamma matrices given in (2.1.54) indeed satisfy (2.1.48). In the Weyl representation the γ_5 matrix introduced in (2.1.49) is diagonal:

$$\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \mathbb{1}_{2\times 2} \end{pmatrix}. \tag{2.1.59}$$

From (2.1.46) we derive the e.o.m.'s

$$0 = \partial_\mu \frac{\partial \mathcal{L}_{\rm D}}{\partial(\partial_\mu\psi)} - \frac{\partial \mathcal{L}_{\rm D}}{\partial\psi} = \partial_\mu\bar{\psi}\,\frac{1}{2}\,i\,\gamma^\mu - \bar{\psi}\left(-\frac{1}{2}\,i\,\overleftarrow{\partial\!\!\!/} - m\right) = \bar{\psi}\,(i\,\overleftarrow{\partial\!\!\!/} + m) \tag{2.1.60}$$

and

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}_{\rm D}}{\partial(\partial_\mu\psi^\dagger)} - \frac{\partial \mathcal{L}_{\rm D}}{\partial\psi^\dagger} = \partial_\mu\left(-\frac{1}{2}\,i\,\gamma_0\,\gamma^\mu\,\psi\right) - \gamma_0\left(\frac{1}{2}\,i\,\overrightarrow{\partial\!\!\!/} - m\right)\psi \\ &= -\gamma_0\,(i\partial\!\!\!/ - m)\,\psi\,. \end{aligned} \tag{2.1.61}$$

Both (2.1.60) and (2.1.61) are equivalent to the Dirac equation,

$$(i\partial\!\!\!/ - m)\,\psi = 0\,. \tag{2.1.62}$$

This can be seen as follows: Using (2.1.48) for $\mu = \nu = 0$ one can convince oneself that $\gamma_0\,\gamma_0 = \mathbb{1}$. Multiplying (2.1.61) by $-\gamma_0$ leads immediately to the Dirac equation (2.1.62). On the other hand, taking the adjoint of (2.1.60) yields

$$0 = (-i\gamma_\mu^\dagger\,\partial^\mu + m)\,\gamma_0^\dagger\,\psi = -\gamma_0^\dagger\,(i\gamma_0^\dagger\gamma_\mu^\dagger\gamma_0^\dagger\,\partial^\mu - m)\,\psi = -\gamma_0\,(i\partial\!\!\!/ - m)\,\psi \tag{2.1.63}$$

where in the last step we have used

$$\gamma_0^\dagger = \gamma_0 \qquad \text{and} \qquad \gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu \tag{2.1.64}$$

which can be derived from the explicit representation (2.1.54). Obviously (2.1.63) is the same as (2.1.61) which should not be surprising: Since the Lagrangian (2.1.46) is hermitian one must get (2.1.61) from the adjoint of (2.1.60), but it is always reassuring to see the intrinsic consistency of the theory.

In the literature one often finds a different Lagrangian for the Dirac theory:

$$\tilde{\mathcal{L}}_{\rm D}(\psi,\psi^\dagger,\partial_\mu\psi,\partial_\mu\psi^\dagger) = \bar\psi\,\left(i\partial\!\!\!/ - m\right)\psi\,. \tag{2.1.65}$$

Obviously this Lagrangian is not hermitian. On the other hand, the e.o.m.'s are deduced from the action, not directly from the Lagrangian. In that sense what matters is the action and this quantity is actually identical for both Lagrangians (2.1.46) and (2.1.65). The point is that the difference is a total derivative which drops out in the calculation of the action and which does not matter for the e.o.m.'s:

$$\begin{aligned}
\tilde S_{\rm D} - S_{\rm D} &= \int d^4x(\tilde{\mathcal{L}}_{\rm D} - \mathcal{L}_{\rm D}) \\
&= \int d^4x\left[\bar\psi\,\left(i\partial\!\!\!/ - m\right)\psi - \bar\psi\,\left(\frac{1}{2}\,i\,(\overrightarrow{\partial\!\!\!/} - \overleftarrow{\partial\!\!\!/}) - m\right)\psi\right] \\
&= \int d^4x\,\frac{1}{2}\,i\,\bar\psi\,(\overrightarrow{\partial\!\!\!/} + \overleftarrow{\partial\!\!\!/})\psi = \frac{1}{2}\,i\int d^4x\,\partial_\mu(\bar\psi\,\gamma^\mu\,\psi) = 0\,.
\end{aligned} \tag{2.1.66}$$

It is straightforward to show that the same Dirac equation (2.1.62) is obtained by applying the usual machinery (2.1.13) to (2.1.65).

As a last step before we will construct solutions of the Dirac equation we shall define the quantization. It requires the introduction of a conjugate momentum. Inspecting the Lagrangian (2.1.65) we find that it is of linear order in the time derivative. For mechanical systems this resembles the first-order formulation of the Lagrange function introduced in (2.1.7). Note that the Lagrangian (2.1.46) neither resembles the standard second-order formulation (2.1.1), where the time derivatives appear quadratically, nor the first-order formulation (2.1.7), where the

time derivative acts on one quantity, but not on both. However, for all Lagrange formulations only the action matters for the derivation of the e.o.m.'s from the principle of minimal action. In that sense (2.1.46) and (2.1.65) are equivalent, but from the Lagrangian (2.1.65) one can directly read off the momentum (2.1.14):

$$\pi_j = \frac{\partial \tilde{\mathcal{L}}_{\mathrm{D}}}{\partial \dot{\psi}_j} = \bar{\psi}_j \, i\gamma_0 = i\psi_j^\dagger \tag{2.1.67}$$

where ψ_j denotes the jth spinor component of ψ. On account of (2.1.16) one finds

$$\{\psi_j(t,\vec{r}), \psi_k^\dagger(t,\vec{r}\,')\} = \delta_{jk}\, \delta^{(3)}(\vec{r}-\vec{r}\,') \,. \tag{2.1.68}$$

Next we will construct solutions of the Dirac equation (2.1.62). Dealing with coupled linear partial differential equations with constant coefficients we know that the solutions are superpositions of plane waves. Anticipating already some of the results which we will find below, the general solution is

$$\psi(x) = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3\, 2E_p} \left(a(\vec{p},s)\, u(\vec{p},s)\, e^{-ip\cdot x} + b^\dagger(\vec{p},s)\, v(\vec{p},s)\, e^{+ip\cdot x} \right) \tag{2.1.69}$$

with $p\cdot x = p_\mu x^\mu = Et - \vec{p}\cdot\vec{r}$ and the energy $p^0 = E = E_p \geq 0$. The summation in (2.1.69) concerns the spin variable where $s = +$ ($s = -$) is related to spin up (down). The Fock space operator $a(\vec{p},s)$ annihilates modes with momentum $\vec{p}$ and spin variable s while $b^\dagger(\vec{p},s)$ creates antiparticles with momentum $\vec{p}$ and spin variable s. Finally u and v are four-component spinors (where each component is an in general complex-valued number). In other words with the ansatz (2.1.69) we have managed to decompose the Fock-space structure (contained in a, $b^\dagger$), the spinor structure (contained in u and v) and the x dependence (contained in the plane-wave exponentials). Of course, this is only possible because there are no interactions (the Lagrangian (2.1.46) contains only terms of quadratic order in the fields).

The quantity (2.1.69) solves the Dirac equation (2.1.62) if the spinors u and v satisfy

$$(\not{p} - m)\, u(\vec{p},s) = 0 \qquad \text{and} \qquad (\not{p} + m)\, v(\vec{p},s) = 0 \,. \tag{2.1.70}$$

We will first construct solutions for u: Using the Weyl representation (2.1.54) for the gamma matrices one finds

$$\not{p} = p^0\, \gamma_0 + p^j\, \gamma_j = p^0\, \gamma^0 - p^j\, \gamma^j = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} - \begin{pmatrix} 0 & \vec{p}\cdot\vec{\sigma} \\ -\vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} . \tag{2.1.71}$$

It is convenient to decompose u in terms of two two-component spinors, χ and φ:

$$u = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} . \tag{2.1.72}$$

This yields

$$0 = (\not{p} - m)\, u(\vec{p}, s) = \begin{pmatrix} -m & E - \vec{p}\cdot\vec{\sigma} \\ E + \vec{p}\cdot\vec{\sigma} & -m \end{pmatrix} \begin{pmatrix} \chi(\vec{p}, s) \\ \varphi(\vec{p}, s) \end{pmatrix} \tag{2.1.73}$$

or in components

$$-m\,\chi + (E - \vec{p}\cdot\vec{\sigma})\,\varphi = 0 \qquad \text{and} \qquad -m\,\varphi + (E + \vec{p}\cdot\vec{\sigma})\,\chi = 0\,. \tag{2.1.74}$$

The first equation can be used to express χ in terms of φ:

$$\chi = \frac{1}{m}\,(E - \vec{p}\cdot\vec{\sigma})\,\varphi \tag{2.1.75}$$

which can be plugged into the second equation in (2.1.74). One gets

$$-m\,\varphi + (E + \vec{p}\cdot\vec{\sigma})\,\frac{1}{m}\,(E - \vec{p}\cdot\vec{\sigma})\,\varphi = 0\,. \tag{2.1.76}$$

Multiplying by m and rearranging things one ends up with

$$0 = \left(E^2 - (\vec{p}\cdot\vec{\sigma})^2 - m^2\right)\,\varphi = (E^2 - \vec{p}^{\,2} - m^2)\,\varphi\,, \tag{2.1.77}$$

where in the last step we have used (2.1.58) to get

$$(\vec{p}\cdot\vec{\sigma})^2 = p^j\,\sigma^j\,p^k\,\sigma^k = p^j\,p^k\,(\delta^{jk} + i\epsilon^{jkl}\sigma^l) = p^j\,p^k\,\delta^{jk} + i\,\underbrace{p^j\,p^k}_{\text{sym.}}\,\underbrace{\epsilon^{jkl}}_{antisym.}\,\sigma^l = \vec{p}^{\,2}\,. \tag{2.1.78}$$

The abbreviations "sym." and "antisym." are supposed to indicate that the respective expression is symmetric or antisymmetric, respectively, with respect to an exchange of the indices j and k. Note that the product of a symmetric and an antisymmetric expression, once summed over the indices, vanishes.

Non-trivial solutions for (2.1.77) can only be obtained for $E^2 - \vec{p}^{\,2} - m^2 = 0$, i.e. if the relativistic energy-momentum relation of a free particle,

$$E = E_p = \sqrt{\vec{p}^{\,2} + m^2} \tag{2.1.79}$$

is satisfied. Then the two-component spinor φ in (2.1.77) is arbitrary. One can choose a basis to construct the most general solution for φ and therefore also for u. Recall that once φ is fixed one can get u from (2.1.72), (2.1.75) and (2.1.79).

Instead of constructing the most general u we will study a specific example: the propagation of the plane wave in the z-direction, i.e.

$$\vec{p} = (0, 0, p_z) \qquad \text{with} \quad p_z \geq 0\,. \tag{2.1.80}$$

In principle, all solutions can be constructed from this example by applying a rotation to the found solutions. (However, we have not specified how rotations

act on the spinors:-) In the following we are aiming at definitions for helicity and chirality. Therefore, it will be sufficient to study the proposed example. Choosing the z-direction is convenient, because also the Pauli matrices (2.1.55) have been chosen such that the third one is a diagonal matrix.

We choose the basis states for φ as follows:

$$\varphi(\vec{p}=(0,0,p_z), s=+) := \begin{pmatrix} \sqrt{E_p+p_z} \\ 0 \end{pmatrix} \tag{2.1.81}$$

and

$$\varphi(\vec{p}=(0,0,p_z), s=-) := \begin{pmatrix} 0 \\ \sqrt{E_p-p_z} \end{pmatrix} . \tag{2.1.82}$$

From (2.1.75) we can get the other two components of u. For our example (2.1.80) the following relations hold:

$$E - \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix} = \begin{pmatrix} E-p_z & 0 \\ 0 & E+p_z \end{pmatrix} \tag{2.1.83}$$

and

$$\sqrt{E_p+p_z}\,\sqrt{E_p-p_z} = \sqrt{E_p^2-p_z^2} = \sqrt{E_p^2-\vec{p}^{\,2}} = m\,. \tag{2.1.84}$$

This leads to

$$\begin{aligned} u(\vec{p}=(0,0,p_z), s=+) &= \begin{pmatrix} \sqrt{E_p-p_z} \\ 0 \\ \sqrt{E_p+p_z} \\ 0 \end{pmatrix} , \\ u(\vec{p}=(0,0,p_z), s=-) &= \begin{pmatrix} 0 \\ \sqrt{E_p+p_z} \\ 0 \\ \sqrt{E_p-p_z} \end{pmatrix} . \end{aligned} \tag{2.1.85}$$

Let us discuss two special cases: For a particle at rest, $p_z = 0$, the four-spinors become

$$\begin{aligned} u(\vec{p}=0, s=+) &= \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \\ u(\vec{p}=0, s=-) &= \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} . \end{aligned} \tag{2.1.86}$$

On the other hand, for a massless particle (or for a particle which moves with ultrarelativistic speed such that the mass can be neglected relative to the momentum) the spinors obtain a very simple form:

$$\begin{aligned} u(\vec{p}=(0,0,p_z),s=+) &\overset{m\to 0}{\to} \sqrt{2p_z}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \\ u(\vec{p}=(0,0,p_z),s=-) &\overset{m\to 0}{\to} \sqrt{2p_z}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \end{aligned} \tag{2.1.87}$$

Therefore the Weyl representation is very convenient for massless states.

The very same considerations lead to the following solutions for the spinors v which satisfy the second equation in (2.1.70):

$$\begin{aligned} v(\vec{p}=(0,0,p_z),s=+) &= \begin{pmatrix}0\\\sqrt{E_p+p_z}\\0\\-\sqrt{E_p-p_z}\end{pmatrix}, \\ v(\vec{p}=(0,0,p_z),s=-) &= \begin{pmatrix}-\sqrt{E_p-p_z}\\0\\\sqrt{E_p+p_z}\\0\end{pmatrix}. \end{aligned} \tag{2.1.88}$$

Note the subtle sign differences between u and v by comparing (2.1.85) and (2.1.88). For the case of massless particles also the expressions for the v's simplify:

$$\begin{aligned} v(\vec{p}=(0,0,p_z),s=+) &\overset{m\to 0}{\to} \sqrt{2p_z}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \\ v(\vec{p}=(0,0,p_z),s=-) &\overset{m\to 0}{\to} \sqrt{2p_z}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \end{aligned} \tag{2.1.89}$$

Finally for an antiparticle at rest one finds

$$\begin{aligned} v(\vec{p}=0, s=+) &= \sqrt{m} \begin{pmatrix} 0 \\ +1 \\ 0 \\ -1 \end{pmatrix} , \\ v(\vec{p}=0, s=-) &= \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix} . \end{aligned} \tag{2.1.90}$$

Next we will introduce the concepts of helicity and chirality. We will see in section 4 that QCD is a theory which possesses an approximate chiral symmetry. To understand what is meant by such a statement we shall discuss this issue first for the non-interaction case. The helicity operator is defined by

$$\Sigma(\vec{p}) := \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix} , \tag{2.1.91}$$

where $\vec{p}$ denotes the momentum. Acting on a Dirac spinor with fixed momentum it asks about the spin projection in flight direction. We note in passing that $\Sigma(\vec{p})$ can be expressed in terms of Dirac's gamma matrices: Introducing

$$\sigma_{\mu\nu} := \frac{i}{2} [\gamma_\mu, \gamma_\nu] \tag{2.1.92}$$

one finds

$$\Sigma(\vec{p}) = \frac{1}{2\,|\vec{p}|}\, \epsilon^{jkl}\, \sigma^{jk}\, p^l \,. \tag{2.1.93}$$

We restrict ourselves again to the case where the momentum of the considered state points in z-direction. In this case we simply get

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{for} \quad \vec{p} = (0,0,p_z),\ p_z \geq 0 \,. \tag{2.1.94}$$

The spinors u and v specified in (2.1.85) and (2.1.88), respectively, are eigen states of Σ. We restrict the discussion to the particle states u. Obviously the state $u(\vec{p}, s=+)$ has positive helicity while the state $u(\vec{p}, s=-)$ has negative helicity:

$$\Sigma\, u(\vec{p}=(0,0,p_z), s) = s\, u(\vec{p}=(0,0,p_z), s) \,. \tag{2.1.95}$$

One can show that the helicity is conserved for free Dirac fields. There is one subtlety, however: The helicity does not characterize a state in an absolute way

but only relative to a frame of reference. If one boosts to a system which moves faster than the considered particle, then $\vec{p}$ changes its sign while the spin does not. Consequently the helicity flips. This indicates that the helicity property cannot directly enter an interaction term (which must be Lorentz invariant). There is one exception, however, namely if the considered particle is massless. In this case the particle moves with the speed of light and one cannot boost to a frame of reference which moves faster than the particle.

Indeed for a massless particle the helicity agrees with the chirality which we shall introduce next. Also for massive particles the chirality can be defined in a Lorentz invariant way. Therefore a state with well-defined chirality can enter an interaction term. However, in contrast to the helicity, the chirality is not conserved in time (for massive states). Chirality (handedness) is defined via the projection operators

$$P_R := \frac{1}{2}\left(\mathbb{1} + \gamma_5\right), \qquad P_L := \frac{1}{2}\left(\mathbb{1} - \gamma_5\right) \tag{2.1.96}$$

where γ_5 has been defined in (2.1.49). From the properties of γ_5, given in (2.1.50), it is easy to show that P_R and P_L are indeed projectors, satisfying:

$$\begin{aligned} P_R\, P_R &= P_R\,, \qquad & P_R\, P_L &= 0\,, \\ P_L\, P_R &= 0\,, \qquad & P_L\, P_L &= P_L\,. \end{aligned} \tag{2.1.97}$$

In the Weyl representation the projectors become very simple:

$$P_R = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \mathbb{1}_{2\times 2} \end{pmatrix}, \qquad P_L = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \tag{2.1.98}$$

i.e. they project out the upper or lower two components, respectively.

For massless states one finds for the case of particle states propagating in the z-direction, cf. (2.1.85):

$$\begin{aligned} P_R\, u(\vec{p}=(0,0,p_z), s=+) &= u(\vec{p}=(0,0,p_z), s=+) && \text{right-handed state} \\ P_R\, u(\vec{p}=(0,0,p_z), s=-) &= 0 \\ P_L\, u(\vec{p}=(0,0,p_z), s=+) &= 0 \\ P_L\, u(\vec{p}=(0,0,p_z), s=-) &= u(\vec{p}=(0,0,p_z), s=-) && \text{left-handed state} \end{aligned} \tag{2.1.99}$$

For massive particles characterized in general by the Dirac spinor ψ, e.g., a right-handed state can still be defined by $\psi_R := P_R\,\psi$. However, in this case, ψ_R

does *not* satisfy the Dirac equation:

$$\begin{aligned}(i\partial\!\!\!/ - m)\,\psi_R &= (i\gamma_\mu\partial^\mu - m)\,\frac{1}{2}\,(\mathbb{1}+\gamma_5)\,\psi = \frac{1}{2}\,(i\gamma_\mu\partial^\mu - m + i\gamma_\mu\partial^\mu\,\gamma_5 - \gamma_5\,m)\,\psi \\ &= \frac{1}{2}\,(i\gamma_\mu\partial^\mu - m - \gamma_5\,i\gamma_\mu\partial^\mu - \gamma_5\,m)\,\psi \\ &= \frac{1}{2}\,(i\gamma_\mu\partial^\mu\psi - m\,\psi - \gamma_5\,i\gamma_\mu\partial^\mu\psi - \gamma_5\,m\,\psi) \\ &= \frac{1}{2}\,(m\,\psi - m\,\psi - \gamma_5\,m\,\psi - \gamma_5\,m\,\psi) \\ &= -\gamma_5\,m\,\psi \neq 0\,, \end{aligned} \tag{2.1.100}$$

where we have used the anticommutation relation (2.1.50) for γ_5 and the Dirac equation for ψ in the form: $i\gamma_\mu\partial^\mu\psi = m\,\psi$. The calculation (2.1.100) demonstrates that chirality is not a conserved quantity except for massless particles.

Note that, in contrast to helicity, chirality is rather unintuitive for massive states (in spite of the seemingly intuitive name "handedness"). Chirality plays a central role in the weak interaction where the W bosons couple only to left-handed states. It will also be important when we study the symmetry properties of the QCD Lagrangian.

2.1.4 Further examples of field theories with interactions

Of course, one can also couple different types of particles. As an example we discuss different versions of the Yukawa theory. Consider the Lagrangian

$$\mathcal{L}_{\rm Y} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar\psi\,\left(i\partial\!\!\!/ - m - g\,\phi\right)\psi \tag{2.1.101}$$

for a hermitian scalar field ϕ with mass M. The Lagrangian contains the propagation of ϕ and of a fermion field with mass m, described by ψ, and it contains an interaction term with coupling constant g. The e.o.m.'s are readily obtained:

$$(\partial_\mu\partial^\mu + M^2)\,\phi = -g\,\bar\psi\psi \tag{2.1.102}$$

and

$$\left(i\partial\!\!\!/ - m - g\,\phi\right)\psi = 0\,. \tag{2.1.103}$$

Thus the fermions constitute a source for the boson field. In turn the boson field influences the fermion propagation. Due to the non-linearities the e.o.m.'s cannot be solved analytically. In the standard model of particle physics the coupling of the Higgs fields to the fermions is of Yukawa type. We note in passing that for the Yukawa interaction the conjugate momenta, (2.1.19) and (2.1.67), and hence the quantization rules for the bosons, (2.1.20), and fermions, (2.1.68), remain unchanged from the non-interaction case.

As a second example, we slightly modify the interaction term:

$$\tilde{\mathcal{L}}_{\mathrm{Y}} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar{\psi}\,\left(i\partial\!\!\!/ - m - i\,\tilde{g}\,\phi\gamma_5\right)\psi \tag{2.1.104}$$

with the spinor matrix γ_5 defined in (2.1.49). We have introduced this example here, because we will come back to it when discussing the parity transformation.

A Lagrangian relevant for pion-nucleon interaction (here in a simplified version) is

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar{\psi}\,\left(i\partial\!\!\!/ - m - h\,(\partial^\mu\phi)\,\gamma_\mu\gamma_5\right)\psi \tag{2.1.105}$$

with a coupling constant h.

Of course, one can also involve more than one boson field in the interaction term, e.g.,

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar{\psi}\,\left(i\partial\!\!\!/ - m - \tilde{h}\,\phi^2\right)\psi\,, \tag{2.1.106}$$

or couple fermions to each other, e.g.,

$$\mathcal{L} = \bar{\psi}\,\left(i\partial\!\!\!/ - m\right)\psi - h'\,(\bar{\psi}\psi)^2\,. \tag{2.1.107}$$

Obviously one can invent infinitely many interaction terms. There is one restriction, however: In any relativistic field theory fermion fields must appear in pairs. Otherwise the Lagrangian would not be a scalar quantity, since nothing would compensate the half-integer spin.[13] There is no such restriction for boson fields. To avoid misunderstanding: The two fermion fields need not be the same. The weak interaction, e.g., couples electrons to neutrinos (and the W boson) and in the realm of the strong interaction the Δ with spin 3/2 can couple to a nucleon with spin 1/2 (and a pion).

2.2 Symmetries and Noether currents

2.2.1 Symmetries in mechanics and quantum mechanics

A symmetry of a mechanical system implies on a formal level that the Lagrange function remains unchanged if the coordinates are modified in a specific way. This modification is called symmetry transformation. At least theoretical physicists regard the symmetry of a system as something beautiful. Even if one does not share the feeling of beauty what one should at least appreciate is the fact that symmetries typically simplify the description of the system. Consider as an example a two-body problem with a potential which depends only on the distance

[13] Actually the field theory need not be relativistic, i.e. Lorentz invariant. Rotational invariance is sufficient here.

between the two bodies and not on the direction. This system possesses rotational invariance. According to Noether's theorem (see also section 2.2.3 below) each continuous symmetry leads to a conserved quantity which in our example is the angular momentum. As a consequence the relative motion of the two-body system is restricted to a plane. In other words, the in general three-dimensional motion is reduced to a two-dimensional one. Even if the problem cannot be solved analytically, the numerical effort is sizably reduced.

In quantum mechanics, symmetries lead to selection rules and often to a degeneracy of states: Consider a system characterized by a Hamiltonian H and an operator, $\mathcal{O}$, which commutes with H:

$$[H, \mathcal{O}] = 0\,. \tag{2.2.1}$$

Then one can find basis states, $|\psi_j\rangle$, which are at the same time eigen states of H and of $\mathcal{O}$. Transitions between $|\psi_1\rangle$ and $|\psi_2\rangle$ are forbidden, i.e.

$$\langle\psi_1|\psi_2\rangle = 0\,, \tag{2.2.2}$$

if $o_1 \neq o_2$, where o_j denotes the respective eigen value, i.e.

$$\mathcal{O}\,|\psi_j\rangle = o_j\,|\psi_j\rangle\,. \tag{2.2.3}$$

Degenerate states appear if one has more than one conserved operator and if these operators have a non-vanishing commutator: Consider for example two operators, L_1 and L_2, which satisfy

$$[L_j, H] = 0 \qquad \text{for} \quad j = 1, 2 \tag{2.2.4}$$

and

$$[L_1, L_2] \neq 0\,. \tag{2.2.5}$$

Let $|\psi\rangle$ denote an eigen state of H and of L_1:

$$H\,|\psi\rangle = E\,|\psi\rangle\,, \qquad L_1\,|\psi\rangle = l_1\,|\psi\rangle\,. \tag{2.2.6}$$

Then also the state $L_2\,|\psi\rangle$ is an eigen state of H with the very same energy E:

$$H\,(L_2\,|\psi\rangle) = L_2\,H\,|\psi\rangle = L_2\,E\,|\psi\rangle = E\,(L_2\,|\psi\rangle)\,. \tag{2.2.7}$$

On the other hand, (2.2.5) ensures that there are states $L_2\,|\psi\rangle$ which are different from multiples of $|\psi\rangle$, i.e. one really finds *different* states which have the same energy. In quantum field theory such a feature translates to the situation where one finds distinguishable particles with the same mass.

2.2.2 Parity

The parity transformation, P, with respect to a given coordinate system changes the position vector to its negative. The time component is left untouched:

$$\begin{aligned} t &\xrightarrow{P} t' = t \\ \vec{r} &\xrightarrow{P} \vec{r}\,' = -\vec{r}\,. \end{aligned} \tag{2.2.8}$$

For convenience we introduce the transformed four-vector x' with components t' and $\vec{r}\,'$.

A "scalar" field is a field which does not change under a parity transformation, i.e. the new field values are obtained from the old ones at the transformed position:

$$\phi(x) \xrightarrow{P} \phi'(x') = \phi(x) \qquad \text{for scalar field} \tag{2.2.9}$$

which implies $\phi'(t, \vec{r}) = \phi(t, -\vec{r})$.

A "pseudoscalar" field changes its sign under a parity transformation:

$$\phi(x) \xrightarrow{P} \phi'(x') = -\phi(x) \qquad \text{for pseudoscalar field.} \tag{2.2.10}$$

For particles with spin moving with momentum $\vec{p}$ one demands that the parity transformation should change the momentum to $-\vec{p}$, but should not change the spin. The reason is that the spin is an angular momentum which is not a vector, but an axial-vector. Recall that the classical orbital angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. Both vectors $\vec{r}$ and $\vec{p}$ change their sign under a parity transformation, but then $\vec{L}$ does not. A Dirac field changes according to

$$\psi(x) \xrightarrow{P} \psi'(x') = \pm\gamma_0\, \psi(x) \qquad \text{for Dirac field} \tag{2.2.11}$$

with an in principle arbitrary sign. This sign does not matter for one type of particle but a relative sign between different particle types becomes a physical issue if the considered interaction conserves parity. As we will see below this concerns quantum electrodynamics and quantum chromodynamics. In the hadronic sector one defines that protons have positive parity, i.e. one chooses the plus version in (2.2.11). As we will see this fixes the parity to be positive for neutrons and negative for antiprotons and antineutrons. There are spin 1/2 resonances where the *particles* have opposite parity to the proton, e.g., the $N^*(1535)$ which plays an important role in the production of eta mesons.

We shall motivate formula (2.2.11) for the case discussed above, a particle or antiparticle moving in the z-direction. In addition, we will show that the parity of a spin 1/2 particle and of its antiparticle are opposite to each other. For

$$\psi(x) \sim u(\vec{p} = (0, 0, p_z), s)\; e^{-ip\cdot x} \tag{2.2.12}$$

name of current	definition	parity
scalar	$\bar{\psi}\psi$	$+1$
pseudoscalar	$i\,\bar{\psi}\gamma_5\psi$	-1
vector (zeroth comp.)	$\bar{\psi}\gamma_0\psi = \psi^\dagger\psi$	$+1$
vector (spatial comp.)	$\bar{\psi}\gamma_j\psi$	-1
axial-vector (zeroth comp.)	$\bar{\psi}\gamma_0\gamma_5\psi$	-1
axial-vector (spatial comp.)	$\bar{\psi}\gamma_j\gamma_5\psi$	$+1$

Table 4: Behavior of fermion bilinears with respect to the parity transformation.

one finds (from now on for the + sign in (2.2.11))

$$\begin{aligned} \psi'(x) &= \gamma_0\,\psi(x') \sim \gamma_0\,u(\vec{p}=(0,0,p_z),s)\;e^{-ip\cdot x'} \\ &= u(\vec{p}=(0,0,-p_z),s)\;e^{-i(Et-(-\vec{p})\cdot\vec{r})} \end{aligned} \tag{2.2.13}$$

where we have used (2.1.54) and (2.1.85). Thus the plane wave moves in the opposite direction, but the spin remains unchanged. For $\vec{p}=0$ the spinor u, given in (2.1.86), is an eigen state of the parity operator with eigen value $+1$. The considered fermion particle has positive parity. The corresponding antiparticle then has negative parity which can be seen as follows: Starting with

$$\psi(x) \sim v(\vec{p}=(0,0,p_z),s)\;e^{+ip\cdot x} \tag{2.2.14}$$

we get

$$\begin{aligned} \psi'(x) &= \gamma_0\,\psi(x') \sim \gamma_0\,v(\vec{p}=(0,0,p_z),s)\;e^{+ip\cdot x'} \\ &= -v(\vec{p}=(0,0,-p_z),s)\;e^{+i(Et-(-\vec{p})\cdot\vec{r})} \end{aligned} \tag{2.2.15}$$

where we have used (2.1.88). Thus for $\vec{p}=0$ the spinor v, given in (2.1.90), is an eigen state of the parity operator with eigen value -1.

As we have already seen in section 2.1.4, fermion bilinears, $\bar{\psi}\ldots\psi$, play an important role in the construction of Lagrangians. We will also see below that they show up in the definition of conserved currents. In table 4 we have collected various bilinears and specified their behavior with respect to the parity transformation. All these currents either remain unchanged ("+1", positive parity) or flip their sign ("−1", negative parity). This can easily be shown using (2.2.11) and the anticommutation relations (2.1.48) and (2.1.50). Note that the bilinears in table 4 have been constructed such that they are hermitian. In particular this is ensured by the i in the definition of the pseudoscalar.

Let us turn to some examples for Lagrangians which do or do not conserve parity, respectively. The free Dirac Lagrangian (2.1.46) or (2.1.65) is invariant

under the parity transformation defined in (2.2.8), (2.2.11), i.e.

$$\mathcal{L}_{\mathrm{D}}(\psi'(x')) = \mathcal{L}_{\mathrm{D}}(\psi(x))\,. \tag{2.2.16}$$

This can easily be seen from table 4 and from

$$\begin{aligned} \frac{\partial}{\partial t} &\overset{P}{\to} \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \\ \vec{\nabla}_r &\overset{P}{\to} \vec{\nabla}_{r'} = -\vec{\nabla}_r\,. \end{aligned} \tag{2.2.17}$$

In particular one has:

$$\begin{aligned} \bar{\psi}'(x')\,\partial\!\!\!/_{x'}\,\psi'(x') &= \bar{\psi}'(x')\,\gamma_0\partial^0_{x'}\,\psi'(x') + \bar{\psi}'(x')\,\gamma_j\partial^j_{x'}\,\psi'(x') \\ &= \bar{\psi}(x)\,\gamma_0\partial^0_x\,\psi(x) + (-1)\,\bar{\psi}(x)\,\gamma_j(-\partial^j_x)\,\psi(x) \\ &= \bar{\psi}(x)\,\partial\!\!\!/_x\,\psi(x)\,. \end{aligned} \tag{2.2.18}$$

The Yukawa Lagrangian given in (2.1.101) is invariant under the parity transformation

$$\begin{aligned} t &\overset{P}{\to} t' = t \\ \vec{r} &\overset{P}{\to} \vec{r}\,' = -\vec{r} \\ \phi(x) &\overset{P}{\to} \phi'(x') = \phi(x) \\ \psi(x) &\overset{P}{\to} \psi'(x') = \pm\gamma_0\,\psi(x)\,, \end{aligned} \tag{2.2.19}$$

i.e.

$$\mathcal{L}_{\mathrm{Y}}(\phi'(x'),\psi'(x')) = \mathcal{L}_{\mathrm{Y}}(\phi(x),\psi(x))\,. \tag{2.2.20}$$

Obviously the boson field is a scalar field.

The Lagrangian (2.1.104) is invariant under the following parity transformation:

$$\begin{aligned} t &\overset{P}{\to} t' = t \\ \vec{r} &\overset{P}{\to} \vec{r}\,' = -\vec{r} \\ \phi(x) &\overset{P}{\to} \phi'(x') = -\phi(x) \\ \psi(x) &\overset{P}{\to} \psi'(x') = \pm\gamma_0\,\psi(x)\,. \end{aligned} \tag{2.2.21}$$

For such an interaction the involved boson field is a pseudoscalar. Note that from the free Klein-Gordon Lagrangian one cannot tell whether ϕ is a scalar or pseudoscalar field. If there was no other interaction where the field ϕ appears, one can conclude that ϕ has a well-defined (here negative) parity. The same is of course true for the previous example. We will see below that QED and QCD conserve parity. Since the photon takes only part in the electromagnetic interaction and the gluons only in the strong interaction, one can conclude that both types of fields/particles have a well-defined (negative) parity.

Consider, however, the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar{\psi}\,\left(i\partial\!\!\!/ - m - g\,\phi(1+i\gamma_5)\right)\psi\,. \tag{2.2.22}$$

There is no transformation of type (2.2.19) or (2.2.21) which leaves this Lagrangian unchanged. The interaction violates parity conservation. A similar situation happens actually for the weak interaction. The W bosons couple in a way to the fermions that violates parity conservation. Therefore one does not assign parity to the W bosons.

Besides parity there are two other discrete symmetries connected to the space-time structure and to the particle-antiparticle symmetry, namely time reversal and charge conjugation. The free Dirac, Klein-Gordon and Maxwell fields are invariant with respect to parity, time reversal and charge conjugation. The strong and the electromagnetic interactions conserve all three discrete symmetries. The weak interaction conserves none of them separately. A combined operation of parity, time reversal and charge conjugation is a symmetry respected by all relativistic local field theories. We do not discuss the symmetries of time reversal and charge conjugation here but instead refer to textbooks on quantum field theory.

2.2.3 Continuous symmetries and Noether currents

The Noether theorem applied to field theory states:
Each continuous symmetry of the action leads to a conserved current j^ν, i.e. the current satisfies a continuity equation

$$\partial_\nu j^\nu = 0\,. \tag{2.2.23}$$

The spatial integral over the zeroth component,

$$Q := \int d^3r\, j^0(t,\vec{r})\,. \tag{2.2.24}$$

yields a conserved "charge", i.e.

$$\frac{d}{dt}\,Q = 0\,. \tag{2.2.25}$$

The last equation, (2.2.25), is a direct consequence of the continuity equation (2.2.23) using Gauss' law:

$$\frac{d}{dt}\,Q = \int d^3r\,\frac{\partial}{\partial t}\,j^0(t,\vec{r}) = -\int d^3r\,\vec{\nabla}\cdot\vec{j}(t,\vec{r}) = -\int\limits_{\text{surface}} d^2r\,\vec{n}\cdot\vec{j}(t,\vec{r}) = 0\,, \tag{2.2.26}$$

where it is assumed that the surface term (at infinity) vanishes ($\vec{n}$ denotes a unit vector perpendicular to the surface). In the following we will explicitly construct

the conserved current and derive the conservation law from the Euler-Lagrange e.o.m.'s (2.1.13).

As a first, very important example we consider translational invariance in space and time. We will derive the conservation laws for energy and momentum. Consider a change of space-time coordinates

$$x \to x' = x + \Delta x \tag{2.2.27}$$

for any Δx. In the most general case the Lagrangian can explicitly depend on space and time and implicitly via its dependence on the fields:

$$\mathcal{L} = \mathcal{L}(x, \phi_a(x), \partial_x^\nu \phi_a(x)) \,. \tag{2.2.28}$$

An example are external sources of the fields (e.g. detectors which produce macroscopic electromagnetic fields). The change of the fields, corresponding to (2.2.27), is given by

$$\phi_a(x) \to \phi'_a(x') = \phi_a(x) \,. \tag{2.2.29}$$

In this case the modified action is

$$S' = \int d^4x' \, \mathcal{L}(x', \phi'_a(x'), \partial^\nu_{x'} \phi'_a(x')) = \int d^4x \, \mathcal{L}(x + \Delta x, \phi_a(x), \partial_x^\nu \phi_a(x)) \,, \tag{2.2.30}$$

which in general is different from the original action S. However, if the Lagrangian does not explicitly depend on x, but only via the fields, i.e.

$$\mathcal{L} = \mathcal{L}(\phi_a(x), \partial_x^\nu \phi_a(x)) \,, \tag{2.2.31}$$

then the action remains unchanged. This is the situation which is actually of interest: reactions of elementary particles with each other in the absence of external (macroscopic) disturbances.

The considered translational invariance is continuous, because the symmetry (2.2.27) exists for any value of Δx. In particular, it will be useful to study small changes with infinitesimal δx. For the derivation of the conserved quantities we will study changes of the coordinates without changing the fields. In this case the change of the Lagrangian with respect to an infinitesimal translation can be calculated in two ways: First, simply by

$$\delta\mathcal{L} = \mathcal{L}(x + \Delta x) - \mathcal{L}(x) = \partial^\mu \mathcal{L} \, \delta x_\mu \,. \tag{2.2.32}$$

This is generally true. The second way of expressing the change of the Lagrangian makes use of the fact that the x dependence is only implicit:

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}\left(\phi_a(x + \Delta x), \partial_x^\nu \phi_a(x + \Delta x)\right) - \mathcal{L}\left(\phi_a(x), \partial_x^\nu \phi_a(x)\right) \\ &= \frac{\partial \mathcal{L}}{\partial \phi_a} \, \partial^\mu \phi_a \, \delta x_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \, \partial^\mu(\partial_\nu \phi_a) \, \delta x_\mu \,. \end{aligned} \tag{2.2.33}$$

Note that the summation over a, i.e. over all types of fields, is implicit here. Now we can use the Euler-Lagrange e.o.m.'s (2.1.13) to rewrite (2.2.33) in the following way:

$$\begin{aligned}\delta\mathcal{L} &= \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\partial^\mu\phi_a\,\delta x_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\partial_\nu(\partial^\mu\phi_a)\,\delta x_\mu \\ &= \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\partial^\mu\phi_a\right)\,\delta x_\mu\,. \end{aligned} \tag{2.2.34}$$

Comparing (2.2.32) and (2.2.34) one gets

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\partial^\mu\phi_a - g^{\mu\nu}\mathcal{L}\right)\,\delta x_\mu = 0\,. \tag{2.2.35}$$

This relation holds for any δx (as long as it is infinitesimally small to justify the truncation of the Taylor expansion in linear order). Therefore, we must have

$$\partial_\nu T^{\mu\nu} = 0 \tag{2.2.36}$$

with the *energy-momentum tensor*

$$T^{\mu\nu} := \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\partial^\mu\phi_a - g^{\mu\nu}\mathcal{L}\,. \tag{2.2.37}$$

Obviously, for a given value of $\mu = 0, 1, 2, 3$ the four-component object $T^{\mu\nu}$ is a current in the sense of (2.2.23). Therefore the integrated quantities

$$H := \int d^3r\, T^{00} \qquad \text{and} \qquad P^j := \int d^3r\, T^{j0} \tag{2.2.38}$$

are conserved. It is easy to see that H is indeed the total energy, Hamiltonian, of the system:

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\,\partial^0\phi_a - \mathcal{L} = \pi^a\,\dot{\phi}_a - \mathcal{L} \tag{2.2.39}$$

which is just the Hamiltonian density, the counterpart of (2.1.5). In (2.2.39) we have introduced again the conjugate momentum (2.1.14). In (2.2.38) the three-momentum, which corresponds to the total energy, is introduced such that $T^{\mu 0}$ forms a four-vector. In addition, it is a straightforward exercise to show that for a free-field case the expressions in (2.2.38) just add up the energy or momenta, respectively, of the single-particle modes (apart from a pure number which one can subtract to ensure that the vacuum state has vanishing energy).

As a second example of Noether's theorem we study modifications of the fields themselves which are supposed to leave the Lagrangian invariant:

$$\phi_a \to \Lambda_{ab}\,\phi_b \tag{2.2.40}$$

with a given set of matrices Λ with constant components Λ_{ab}. We demand that these matrices which mix the fields are continuously connected to unity, i.e. that there are in particular infinitesimally small deviations from the case of no modification:

$$\Lambda_{ab} = \delta_{ab} + \delta\alpha\, T_{ab} \tag{2.2.41}$$

with a given T_{ab} and an arbitrary (but infinitesimally small) parameter $\delta\alpha$. We will present several examples below how such field transformations look like. The corresponding conserved quantities will be, e.g., electric charge, baryon number, isospin, strangeness, ... Here we will discuss the general case: If the Lagrangian remains unchanged, we find

$$\begin{aligned} 0 &= \delta\mathcal{L} = \mathcal{L}(\Lambda_{ab}\,\phi_b) - \mathcal{L}(\phi_a) = \mathcal{L}(\phi_a + \delta\alpha\, T_{ab}\,\phi_b) - \mathcal{L}(\phi_a) \\ &= \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\alpha\, T_{ab}\,\phi_b + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\delta\alpha\, T_{ab}\,\partial_\nu\phi_b \\ &= \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\delta\alpha\, T_{ab}\,\phi_b + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,\delta\alpha\, T_{ab}\,\partial_\nu\phi_b \\ &= \delta\alpha\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,T_{ab}\,\phi_b\right) \end{aligned} \tag{2.2.42}$$

Note that we have not displayed explicitly the dependence of the Lagrangian on the derivatives of the fields to keep the notation compact. We have made use, again, of the e.o.m.'s (2.1.13) and of the fact that $\delta\alpha$, and T_{ab} do not depend on x. Since the parameter $\delta\alpha$ was supposed to be arbitrary one finds the conserved current

$$j^\nu := \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\,T_{ab}\,\phi_b \tag{2.2.43}$$

for a given matrix T_{ab}.

2.3 Quantum electrodynamics and extensions

2.3.1 QED for electrons

The Lagrangian of quantum electrodynamics (QED) for electrons is obtained from the Dirac and Maxwell Lagrangians, (2.1.65) and (2.1.39), respectively, together with the principle of minimal substitution. The latter is well known from classical electrodynamics where the coupling of an electromagnetic field to a particle with energy $p^0 = E$ and momentum $\vec{p}$ is achieved by the replacement rule

$$p^\mu \to p^\mu + eA^\mu \qquad \text{(principle of minimal substitution)} \tag{2.3.1}$$

where e denotes the modulus of the electron charge, i.e. the charge of the electron is $-e$. The vector potential A^μ has already been introduced in (2.1.39). In quantum theory the rule (2.3.1) translates to a modification of derivatives:

$$\partial^\mu \to D^\mu := \partial^\mu - ieA^\mu \qquad \text{(principle of minimal substitution).} \tag{2.3.2}$$

The operator D^μ is called "gauge covariant derivative".

The QED Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm QED} &= -\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + \bar\psi\,\left(i\not{D} - m\right)\psi \\ &= -\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + \bar\psi\,\left(i\not{\partial} - m\right)\psi + A^\mu\, e\,\bar\psi\gamma_\mu\psi \qquad (2.3.3)\end{aligned}$$

with the field strengths (cf. (2.1.40))

$$F^{\mu\nu} := \frac{i}{e}\,[D^\mu, D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \qquad (2.3.4)$$

The first form given in (2.3.3) is the most compact version which demonstrates how one comes from the free Dirac Lagrangian to the interaction case via the principle of minimal substitution. On the other hand, the last form given in (2.3.3) displays the interaction term explicitly.

The definition (2.3.4) of the field strengths in terms of the gauge covariant derivatives will be useful below for the generalization to QCD and gluons. Note that in this definition the derivatives act on *everything* to the right of it, in particular on not displayed arbitrary quantities. To illustrate this issue we consider an arbitrary function $f(x)$ and calculate

$$\begin{aligned}[D^\mu, D^\nu]\,f &= ((\partial^\mu - ieA^\mu)(\partial^\nu - ieA^\nu) - (\partial^\nu - ieA^\nu)(\partial^\mu - ieA^\mu))\;f \\ &= -ie\;(\partial^\mu(A^\nu\,f) + A^\mu\,\partial^\nu f - \partial^\nu(A^\mu\,f) - A^\nu\,\partial^\mu f) \\ &= -ie\;((\partial^\mu A^\nu) - (\partial^\nu A^\mu))\;f \qquad (2.3.5)\end{aligned}$$

where in the last expression the derivatives do not act on f but only on the photon fields. This shows that the expression $[D^\mu, D^\nu]$ formally looks like a derivative operator acting on an arbitrary object, e.g., the function f. But in reality it contains only derivatives of fields which just *multiply* the arbitrary function f. When we deal with derivatives in the following we always have to state very clearly on which quantities they are supposed to act.

The QED Lagrangian (2.3.3) possesses several symmetries. We will discuss parity, global phase transformations which lead to the conserved electric current and local gauge symmetry which leads to the demand that photons must be massless.

Parity: The Lagrangian (2.3.3) is invariant with respect to the parity transformation

$$\begin{aligned} t &\stackrel{P}{\to} t' = t \\ \vec r &\stackrel{P}{\to} \vec r\,' = -\vec r \\ A_0(x) &\stackrel{P}{\to} A_0'(x') = A_0(x) \\ A_j(x) &\stackrel{P}{\to} A_j'(x') = -A_j(x) \\ \psi(x) &\stackrel{P}{\to} \psi'(x') = \pm\gamma_0\,\psi(x)\,. \qquad (2.3.6)\end{aligned}$$

Note that the photon field transforms like a four-vector. It is a *vector* field (in distinction to an axial-vector field). It is an easy exercise to prove this invariance. In particular, for the interaction term we recall the transformation properties of the fermion bilinears given in table 4.

Charge conservation: Obviously the Lagrangian (2.3.3) is invariant with respect to the global continuous transformations

$$\begin{aligned} \psi(x) &\to e^{ie\Lambda}\,\psi(x) \\ A_\mu(x) &\to A_\mu(x) \end{aligned} \tag{2.3.7}$$

with $\Lambda \in \mathbb{R}$. For completeness we also display the transformation property of $\psi^\dagger$ which is a consequence of the transformation property of ψ:

$$\psi^\dagger(x) \to \psi^\dagger(x)\, e^{-ie\Lambda}\,. \tag{2.3.8}$$

Note that here Λ does not depend on x. One therefore calls the transformations "global".

The transformations $U := e^{ie\Lambda}$ form the *unitary group* $\mathbf{U}(1)$. More generally, all $n \times n$ matrices U which satisfy $U\,U^\dagger = \mathbb{1}$, i.e. $U^\dagger = U^{-1}$, form the unitary group $\mathbf{U}(n)$. If, in addition, one demands $\det U = 1$ one has the *special unitary group* $\mathbf{SU}(n)$. Such groups play an important role for the strong and for the weak interaction for $n > 1$. For our present case all these matrices essentially boil down to numbers. We note in passing that $\mathbf{SU}(1)$ is a trivial group, it only consists of one element, namely unity. What is more interesting, is the fact that every unitary transformation can be formed by a product of a $\mathbf{U}(1)$ transformation and a special unitary transformation, i.e. $\mathbf{U}(n) = \mathbf{U}(1) \times \mathbf{SU}(n)$.

The appearance of the charge e in the exponent in (2.3.7) is a pure convention; the parameter $\tilde{\Lambda} := e\Lambda$ would also be an arbitrary real constant. However, the convention ensures to end up with the electric charge measured in multiples of e. Indeed, we can apply Noether's law and determine the conserved current and the associated charge: Infinitesimally small transformations of type (2.3.7), (2.3.8) are given by

$$\begin{aligned} \psi(x) &\to \psi(x) + ie\delta\Lambda\,\psi(x) \\ \psi^\dagger(x) &\to \psi^\dagger(x) - ie\delta\Lambda\,\psi^\dagger(x)\,. \end{aligned} \tag{2.3.9}$$

Therefore the conserved current (2.2.43) is given by

$$j^\nu = \frac{\partial\mathcal{L}_{\rm QED}}{\partial(\partial_\nu\psi)}\,(+ie)\,\psi + \psi^\dagger\,(-ie)\,\frac{\partial\mathcal{L}_{\rm QED}}{\partial(\partial_\nu\psi^\dagger)} = -e\,\bar{\psi}\gamma^\nu\psi\,. \tag{2.3.10}$$

It would be a nice exercise to prove the continuity equation, $\partial_\nu j^\nu = 0$, directly by using the QED e.o.m.'s which follow from (2.3.3).

The conserved charge is given by

$$Q = \int d^3r'\, j^0(t', \vec{r}') = -e \int d^3r'\, \bar{\psi}\gamma^0\psi = -e \int d^3r\, \psi^\dagger\psi\,. \tag{2.3.11}$$

Using the conjugate momentum (2.1.67), which actually is the same for free Dirac fields and for QED, one can express the charge as

$$Q = ie \int d^3r'\, \pi(t', \vec{r}')\, \psi(t', \vec{r}')\,. \tag{2.3.12}$$

We shall show now that Q measures indeed the electric charge. We will do that by proving that the application of ψ on an arbitrary n-particle system adds the charge $+e$ to this system. Recall the meaning of ψ for the free-field case, i.e. the discussion after equation (2.1.69). In that sense the part of ψ, which creates (antiparticle) states, adds the charge $+e$, while the part of ψ, which annihilates (particle) states, subtracts the charge $-e$. Correspondingly, $\psi^\dagger$ adds the charge $-e$, i.e. particles (created by $\psi^\dagger$) and antiparticles (created by ψ) have opposite charge.

Suppose that we have an n-particle system, characterized by the state vector $|n\rangle$, which is an eigen state of Q:

$$Q\,|n\rangle = q_n\,|n\rangle\,. \tag{2.3.13}$$

Then we want to check whether $\psi\,|n\rangle$ is also an eigen state of the charge operator and, if it is, we want to know the difference of its eigen value and q_n. In other words, we want to check whether $\psi\,|n\rangle$ is an eigen state of $(Q - q_n)$ and we want to know the corresponding eigen value. Therefore we calculate

$$(Q - q_n)\psi|n\rangle = Q\,\psi|n\rangle - \psi\, q_n\,|n\rangle = Q\,\psi|n\rangle - \psi\, Q\,|n\rangle = [Q, \psi]\,|n\rangle\,. \tag{2.3.14}$$

If the commutator yields a constant times ψ we have reached our goal. The commutator is given by

$$\begin{aligned}
[Q, \psi_j(t, \vec{r})] &= ie \int d^3r'\, [\pi_k(t, \vec{r}')\, \psi_k(t, \vec{r}'), \psi_j(t, \vec{r})] \\
&= ie \int d^3r'\, \left(\pi_k(t, \vec{r}')\, \{\psi_k(t, \vec{r}'), \psi_j(t, \vec{r})\} - \{\pi_k(t, \vec{r}'), \psi_j(t, \vec{r})\}\, \psi_k(t, \vec{r}')\right) \\
&= e \int d^3r'\, \delta_{kj}\, \delta^{(3)}(\vec{r} - \vec{r}')\, \psi_k(t, \vec{r}') = e\, \psi_j(t, \vec{r})\,.
\end{aligned} \tag{2.3.15}$$

In the first step we have chosen the time variable for Q to be t. This is legitimate since Q is a constant of motion. Therefore, one can use any time. Choosing it to be t enabled the use of the *equal-time* anticommutation relations (2.1.16). They come into play via the relation

$$[AB, C] = A\,\{B, C\} - \{A, C\}\,B \tag{2.3.16}$$

which hold for arbitrary operators/matrices A, B, C as can be easily proven from the definitions for commutators and anticommutators.

The adjoint of (2.3.15) yields

$$-e\,\psi^\dagger = -[Q,\psi]^\dagger = -[\psi^\dagger, Q] = [Q,\psi^\dagger]\,, \tag{2.3.17}$$

where we have used the fact that Q is hermitian as can be easily seen from the last expression in (2.3.11). This proves that indeed ψ adds a charge $+e$ to the system while $\psi^\dagger$ adds a charge $-e$.

Before we continue we note that there is one aspect which is slightly unpleasant: The action of Q on the vacuum state does not yield zero. This is related to the need for a proper ordering of the fields which after quantization (2.1.16) become Fock space operators. Indeed, in the derivation of the conserved current (2.2.43) we have not payed much attention to the ordering of the fields. One could fix that problem by modifying the definition of the current j^ν. In the end this would lead to a modified charge which deviates from (2.3.11) by a constant. However, as we have seen, what matters are commutators of the charge operator with other quantities. Here such a constant always drops out.

Local gauge invariance: The QED Lagrangian shows a larger symmetry which includes the previously discussed case (2.3.7): QED is invariant with respect to *local* "gauge" transformations

$$\begin{aligned} \psi(x) &\rightarrow e^{ie\Lambda(x)}\,\psi(x) \\ A^\mu(x) &\rightarrow A^\mu(x) + \partial_x^\mu \Lambda(x)\,. \end{aligned} \tag{2.3.18}$$

The gauge transformation for the vector potential A_μ might be known from classical electrodynamics where it is discussed as a symmetry of the Maxwell equations which does not change the electric and magnetic field strength. Thus the symmetry transformations have no observable consequences. Only gauge invariant objects, i.e. quantities which are independent of $\Lambda(x)$, are observable.

A very elegant notation is given by

$$\begin{aligned} \psi(x) &\rightarrow U(x)\,\psi(x) \\ \psi^\dagger(x) &\rightarrow \psi^\dagger(x)\,U^\dagger(x) \\ A^\mu(x) &\rightarrow U(x)\left(A^\mu(x) + \frac{i}{e}\,\partial_x^\mu\right) U^\dagger(x) \end{aligned} \tag{2.3.19}$$

with $U(x) \in \mathbf{U}(1)$, i.e.

$$U^\dagger(x)\,U(x) = 1\,. \tag{2.3.20}$$

For the present case (2.3.19) is equivalent to (2.3.18), but it can be generalized to other symmetry groups beyond $\mathbf{U}(1)$.

We shall show now that these local continuous transformations indeed do not change the QED Lagrangian (2.3.3): Actually it is sufficient to show that the gauge covariant derivative (2.3.2) transforms according to

$$D^\mu(x) \to U(x)\, D^\mu(x)\, U^\dagger(x) \tag{2.3.21}$$

where the derivative in D^μ does not only act on $U^\dagger(x)$ but also further to the right. Using (2.3.21) and (2.3.19) one finds

$$F^{\mu\nu} = \frac{i}{e}\,[D^\mu, D^\nu] \to \frac{i}{e}\,[U\, D^\mu\, U^\dagger, U\, D^\nu\, U^\dagger] = \frac{i}{e}\, U\,[D^\mu, D^\nu]\, U^\dagger = U\, F^{\mu\nu}\, U^\dagger = F^{\mu\nu}\,, \tag{2.3.22}$$

$$m\,\bar\psi\psi \to m\,\bar\psi\, U^\dagger\, U\,\psi = m\,\bar\psi\psi\,, \tag{2.3.23}$$

and

$$\bar\psi\, \not{D}\,\psi \to \bar\psi\, U^\dagger\, U\, \not{D}\, U^\dagger\, U\,\psi = \bar\psi\, \not{D}\,\psi\,. \tag{2.3.24}$$

Thus the QED Lagrangian remains unchanged provided we can show that (2.3.21) is true. Indeed, one finds:

$$\begin{aligned} U(x)\, D^\mu(x)\, U^\dagger(x) &= U(x)\,(\partial_x^\mu - ieA^\mu(x))\, U^\dagger(x) \\ &= U(x)\,(\partial_x^\mu\, U^\dagger(x)) + U(x)\, U^\dagger(x)\, \partial_x^\mu - ie\, U(x)\, A^\mu(x)\, U^\dagger(x) \\ &= \partial_x^\mu - ie\, U(x)\, \left(A^\mu(x) + \frac{i}{e}\,\partial_x^\mu\right) U^\dagger(x) \end{aligned} \tag{2.3.25}$$

where the very last derivative acts only on $U^\dagger(x)$. We have recovered the transformed photon field. Thus (2.3.21) is indeed a consequence of (2.3.19).

We have stressed before that a gauge transformation should not have an observable consequence. All observables are unchanged by a gauge transformation. This does not mean, however, that the requirement that a theory is gauge invariant should not have an observable consequence. Indeed, for QED this requirement forbids photons to have a mass. Suppose we would add a "mass" term $+\frac{1}{2}\, m_\gamma^2\, A_\mu A^\mu$ to the QED Lagrangian. It is a simple exercise to show that such a term would lead to an energy-momentum relation $E = \sqrt{\vec{p}^2 + m_\gamma^2}$ for free photons. However, such a term is not gauge invariant. Applying the transformation rule (2.3.18) the mass term does not remain invariant:

$$\frac{1}{2}\, m_\gamma^2\, A_\mu A^\mu \to \frac{1}{2}\, m_\gamma^2\, (A_\mu + \partial_\mu\Lambda)(A^\mu + \partial^\mu\Lambda) \neq \frac{1}{2}\, m_\gamma^2\, A_\mu A^\mu\,. \tag{2.3.26}$$

Thus, in the presence of a mass term for the vector field A_μ the theory would not be gauge invariant. Demanding gauge invariance leads to the prediction that the vector field is massless, i.e. describes massless quanta.

Actually this is a feature not restricted to QED. In gauge theories, i.e. theories which possess a local gauge invariance, the fields which appear in the gauge covariant derivatives must be massless. These fields are also called "gauge bosons".

All known fundamental theories (the electroweak theory, QCD and even general relativity) are gauge theories. The corresponding gauge bosons (photons, gluons, gravitons) are indeed massless — except for the W and Z bosons of the weak interaction. Initially they are also massless, but obtain a dynamically generated mass by the effect of spontaneous symmetry breaking (see section 5). Concerning the electroweak theory this effect is called the Higgs mechanism. But also in the realm of QED one can observe the spontaneous breaking of the gauge symmetry. It appears in superconducting materials and leads to the fact that magnetic fields cannot enter a superconducting region. This is known as the Meissner effect. The photons get a Meissner mass (the inverse of the London penetration depth) which leads to an exponential decrease of the magnetic field inside of a superconductor.

2.3.2 Scalar QED

Of course, also bosons can be subject to the electromagnetic interaction, e.g., charged pions or nuclei with integer spin. We will discuss here the case of scalar fields. In section 2.1.2 we have claimed that a hermitian boson field describes uncharged particles. Indeed, replacing the partial derivatives in (2.1.17) by gauge covariant ones according to (2.3.2) would lead to a Lagrangian which is no longer hermitian. Thus we need at least two scalar fields which also makes sense from the point of view that a particle and an antiparticle should have opposite charge, i.e. are distinguishable. In section 2.1.2 we have introduced two equivalent Lagrangians, (2.1.25) and (2.1.32), to describe two non-interacting boson fields with the same mass. Applying the principle of minimal substitution (2.3.2) literally to (2.1.25) would also yield a non-hermitian Lagrangian. However, applying the minimal substitution to (2.1.32) in the form

$$\partial_\mu \phi \to D_\mu \phi = (\partial_\mu - iqA_\mu)\,\phi\,, \qquad \partial_\mu \phi^\dagger \to (D_\mu \phi)^\dagger = (\partial_\mu \overset{!}{+} iqA_\mu)\,\phi^\dagger \quad (2.3.27)$$

leads to the hermitian Lagrangian of scalar QED,

$$\mathcal{L}_{\text{scalar QED}} = -\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + (D_\mu\phi)^\dagger\,D^\mu\phi - m^2\,\phi^\dagger\,\phi\,. \quad (2.3.28)$$

Note that we have replaced the positron charge e by a generic charge q, i.e.

$$D^\mu := \partial^\mu - iqA^\mu\,, \quad (2.3.29)$$

to keep things somewhat more general.

One can now split the Lagrangian into the non-interaction and the interaction part,

$$\begin{aligned}\mathcal{L}_{\text{scalar QED}} \;=\; & -\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\,\phi^\dagger\,\phi \\ & - iq(\partial_\mu\phi^\dagger)\,A^\mu\,\phi + iq\,\phi^\dagger\,A_\mu\,\partial^\mu\phi + q^2\,\phi^\dagger\,A_\mu A^\mu\,\phi\,, \quad (2.3.30)\end{aligned}$$

and one might rewrite the Lagrangian in terms of the real and imaginary part of ϕ by introducing ϕ_1 and ϕ_2 according to (2.1.31):

$$\begin{aligned}\mathcal{L}_{\text{scalar QED}} = &-\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\,m^2\,\phi_1^2 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}\,m^2\,\phi_2^2 \\ &+ q\,(\partial_\mu\phi_1)\,A^\mu\,\phi_2 - q\,(\partial_\mu\phi_2)\,A^\mu\,\phi_1 + \frac{1}{2}\,q^2\,\phi_1^2\,A_\mu A^\mu + \frac{1}{2}\,q^2\,\phi_2^2\,A_\mu A^\mu\,. \quad (2.3.31)\end{aligned}$$

The latter Lagrangian contains the same information as the one given in (2.3.30), but it is a rather uneconomic way of expressing the same physics: At each interaction of the boson fields with the electromagnetic field a ϕ_1 particle changes to a ϕ_2 particle and vice versa. In contrast, the Lagrangian (2.3.30) describes, e.g., how a particle is disturbed by the electromagnetic field. In the interaction with the electromagnetic field the field ϕ destroys an incoming particle mode and the field $\phi^\dagger$ creates a new particle (not antiparticle!) mode. Effectively the particle just changes its momentum in the interaction with the electromagnetic field. Thus the particle does not switch to an antiparticle in the course of the interaction with the photon, while a ϕ_1 particle changes to a ϕ_2 particle. Below we will show that ϕ and $\phi^\dagger$ create or annihilate eigen states of the electric charge while the states created by ϕ_1 or ϕ_2 are not eigen states of the charge operator. The conclusion is that in the presence of electromagnetic interactions the language using the non-hermitian field ϕ and its conjugate is much more economic than the language using its real and imaginary part ϕ_1 and ϕ_2.

The Lagrangian (2.3.28) of scalar QED is invariant with respect to the local $\mathbf{U}(1)$ gauge transformations

$$\begin{aligned}\phi(x) &\rightarrow U(x)\,\phi(x) \\ A^\mu(x) &\rightarrow U(x)\left(A^\mu(x) + \frac{i}{q}\,\partial_x^\mu\right)U^\dagger(x) \quad (2.3.32)\end{aligned}$$

with $U(x) \in \mathbf{U}(1)$. These transformations imply

$$\begin{aligned}\phi^\dagger(x) &\rightarrow \phi^\dagger(x)\,U^\dagger(x) \\ D^\mu(x) &\rightarrow U(x)\,D^\mu(x)\,U^\dagger(x)\,. \quad (2.3.33)\end{aligned}$$

The proof of gauge invariance of the Lagrangian (2.3.28) is essentially the same as in fermionic QED.

As part of the local transformations $U(x)$ one can also study global, i.e. x-independent transformations and here infinitesimally ones. Using Noether's theorem one can construct the corresponding conserved current. Infinitesimal global transformations, $U(x) \rightarrow e^{iq\,\delta\Lambda} \approx 1 + iq\,\delta\Lambda$, change the fields according to

$$\begin{aligned}\phi &\rightarrow \phi + iq\,\delta\Lambda\,\phi \\ \phi^\dagger &\rightarrow \phi^\dagger - iq\,\phi^\dagger\,\delta\Lambda \\ A^\mu &\rightarrow A^\mu\,. \quad (2.3.34)\end{aligned}$$

The corresponding conserved current is given by

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}_{\text{scalar QED}}}{\partial(\partial_\mu \phi)}\, iq\,\phi + \phi^\dagger\,(-iq)\,\frac{\partial \mathcal{L}_{\text{scalar QED}}}{\partial(\partial_\mu \phi^\dagger)} = (D^\mu\phi)^\dagger\, iq\,\phi - iq\,\phi^\dagger\, D^\mu\phi \\ &=: -iq\,\phi^\dagger\,(\overrightarrow{D^\mu} - \overleftarrow{D^\mu}^\dagger)\,\phi = -iq\,\phi^\dagger\,(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})\,\phi - 2\,q^2\,\phi^\dagger\, A^\nu\,\phi\,. \end{aligned} \tag{2.3.35}$$

Note that this current and therefore also the charge, $Q = \int d^3r\, j^0$, are hermitian which is a necessary requirement for an observable quantity.

The conjugate momenta

$$\pi = \frac{\partial \mathcal{L}_{\text{scalar QED}}}{\partial \dot\phi} = (D_0\phi)^\dagger\,, \qquad \pi^\dagger = \frac{\partial \mathcal{L}_{\text{scalar QED}}}{\partial \dot\phi^\dagger} = D_0\phi \tag{2.3.36}$$

differ from the free-field expressions (2.1.35). Note that this was not the case for fermionic QED. There are more remarkable differences (which all are interrelated): Like the canonical momentum also the conserved current (2.3.35) of scalar QED involves the photon field. In addition, in the interaction terms in the Lagrangian (2.3.30) the photon field appears not only in linear order, but also in quadratic order. While fermionic QED has only "three-point" interactions, scalar QED has three- and four-point interactions.

Using (2.3.36) and (2.3.35) one can write the conserved charge as

$$Q = \int d^3r\, j^0 = iq \int d^3r\,(\pi\,\phi - \phi^\dagger\,\pi^\dagger)\,. \tag{2.3.37}$$

See also the corresponding expression (2.3.12) for the fermionic case. It is a simple exercise using the commutation relation (2.1.15) to show that

$$[Q,\phi] = q\,\phi \qquad \Leftrightarrow \qquad [Q,\phi^\dagger] = -q\,\phi^\dagger\,. \tag{2.3.38}$$

which shows that the modes created or annihilated by ϕ or $\phi^\dagger$ are eigen states of the electric charge operator Q. In particular, the particle state has charge $(-q)$ and the antiparticle state $(+q)$. It is a generic property that all charges (electric charge, baryon number, strangeness, ...) have opposite signs for particles and antiparticles.

Using the real and imaginary part of ϕ, one finds, e.g.,

$$[Q,\phi_1] = iq\,\phi_2 \tag{2.3.39}$$

which can be easily seen with the help of (2.3.38) by expressing ϕ_1 and ϕ_2 in terms of ϕ or $\phi^\dagger$. This can be achieved by inverting the relations (2.1.31). From (2.3.39) we deduce that the states created or destroyed by ϕ_1 are *not* eigen states of the charge operator. This brings us back to the statement that for a charged boson the use of ϕ_1 and ϕ_2 is uneconomic as soon as the electromagnetic interaction is switched on.

2.3.3 Toy model for the weak theory

We have seen that the demand of local gauge invariance forbids a mass for the photon. It only could be generated dynamically by spontaneous symmetry breaking. In the following we will show that a *chiral* local gauge invariance even forbids a mass for the fermions. We will use a simple extension of QED to demonstrate that. As it stands, it is a toy model which is not realized in nature. However, the electroweak theory is a somewhat more complicated version of this toy model: For the electroweak theory the symmetry group of the gauge transformations is more complicated and the Higgs mechanism provides a mass for the gauge bosons (W and Z) and for the fermions (e.g., the quarks).

The toy model contains a fermion field ψ and a gauge field V_μ. It is defined by the following Lagrangian:

$$\mathcal{L}_{\text{tw}} = \bar{\psi}\,\left(i\partial\!\!\!/ + g V\!\!\!/\, P_L\right)\psi - \frac{1}{4}\,V_{\alpha\beta}V^{\alpha\beta} \tag{2.3.40}$$

with the field strength

$$V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu\,, \tag{2.3.41}$$

the coupling constant g, and the projector $P_L = \frac{1}{2}\,(\mathbb{1} - \gamma_5)$ on left-handed states introduced in (2.1.96).

Before we discuss global and local continuous symmetries we note that parity is *not* a symmetry of (2.3.40). This is caused by the appearance of the projector P_L in the interaction term. If it was $\mathbb{1}$ instead of P_L, then parity would be a symmetry with the gauge field V_μ transforming like a vector under parity (cf. (2.3.6)). If it was γ_5 instead of P_L, then parity would be a symmetry with the gauge field V_μ transforming like an axial-vector under parity. All this can be easily deduced using the results of table 4. Since P_L appears in the interaction, there is no transformation property for V_μ which would lead to parity as a symmetry. Parity is broken (this is also the case for the weak theory).

We can use the projection operators (2.1.96) to define left- and right-handed states:

$$\psi_R := P_R\,\psi\,, \quad \psi_L := P_L\,\psi\,, \quad \bar{\psi_R} := (\psi_R)^\dagger\,\gamma_0\,, \quad \bar{\psi_L} := (\psi_L)^\dagger\,\gamma_0\,. \tag{2.3.42}$$

Using the fact that γ_5 is hermitian and, according to (2.1.50), anticommutes with the four gamma matrices γ^μ one can show:

$$P_R\,\gamma^\mu = \gamma^\mu\,P_L \tag{2.3.43}$$

and the corresponding relation exchanging L and R. Therefore, e.g.,

$$\bar{\psi_R} = \psi^\dagger\,P_R\,\gamma_0 = \psi^\dagger\,\gamma_0\,P_L = \bar{\psi}\,P_L\,. \tag{2.3.44}$$

This leads to

$$\begin{aligned}\bar{\psi}\,\gamma^\mu\,\psi &= \bar{\psi}\,\gamma^\mu\,(P_R+P_L)\,\psi = \bar{\psi}\,\gamma^\mu\,P_R\,\psi + \bar{\psi}\,\gamma^\mu\,P_L\,\psi \\ &= \bar{\psi}\,\gamma^\mu\,P_R\,P_R\,\psi + \bar{\psi}\,\gamma^\mu\,P_L\,P_L\,\psi = \bar{\psi}\,P_L\,\gamma^\mu\,P_R\,\psi + \bar{\psi}\,P_R\,\gamma^\mu\,P_L\,\psi \\ &= \bar{\psi_R}\,\gamma^\mu\,\psi_R + \bar{\psi_L}\,\gamma^\mu\,\psi_L\,. \end{aligned} \tag{2.3.45}$$

Thus the Lagrangian (2.3.40) can be decomposed into its left- and right-handed part:

$$\mathcal{L}_{\rm tw} = \bar{\psi_R}\,i\partial\!\!\!/\,\psi_R + \bar{\psi_L}\,iD\!\!\!/\,\psi_L - \frac{1}{4}\,V_{\alpha\beta}V^{\alpha\beta} \tag{2.3.46}$$

with the gauge covariant derivative

$$D_\mu := \partial_\mu - igV_\mu\,. \tag{2.3.47}$$

Obviously, the right-handed fermion fields do not take part in the interaction. Of course, this has been achieved by the appearance of the projector P_L in the interaction term in (2.3.40). The right- and left-handed fields are not mixed in any way. This is achieved by the fact that there is no mass term. Recall that for free *massless* Dirac fields chirality is conserved. This is exactly expressed by the decomposition in (2.3.46). We will come back to the discussion of a mass term below.

The Lagrangian (2.3.46) has a global *plus* a local symmetry: It remains invariant with respect to *global* $\mathbf{U}(1)$ transformations of the right-handed fields, i.e.

$$\begin{aligned}\psi_R(x) &\to U_R\,\psi_R(x)\,, \qquad U_R \in \mathbf{U}(1) \\ \psi_L(x) &\to \psi_L(x) \\ V^\mu(x) &\to V^\mu(x)\,, \end{aligned} \tag{2.3.48}$$

and it remains invariant with respect to *local* $\mathbf{U}(1)$ transformations of the left-handed fields and the gauge fields, i.e.

$$\begin{aligned}\psi_R(x) &\to \psi_R(x) \\ \psi_L(x) &\to U_L(x)\,\psi_L(x)\,, \qquad\qquad U_L(x) \in \mathbf{U}(1) \\ V^\mu(x) &\to U_L(x)\left(V^\mu(x) + \frac{i}{g}\,\partial_x^\mu\right) U_L^\dagger(x)\,. \end{aligned} \tag{2.3.49}$$

Symmetries which separately concern the left- and right-handed components of fermionic fields are called "chiral symmetries". For the toy model discussed here, the symmetry group is $\mathbf{U}_{L,\rm local}(1) \times \mathbf{U}_{R,\rm global}(1)$. We will see below, that QCD has an approximate chiral symmetry.

If one demands a chiral symmetry then the fermions cannot have a mass. The reason is that a mass term does not remain unchanged with respect to a

transformation which concerns only one part of the fermionic fields, e.g., only the left-handed part:

$$\begin{aligned} m\,\bar{\psi}\psi \quad &= \quad m\,\bar{\psi}\,(P_R + P_L)\,\psi = m\,\bar{\psi}_L\psi_R + m\,\bar{\psi}_R\psi_L \\ &\overset{\mathbf{U}_{L,\mathrm{local}}(1)}{\rightarrow} \quad m\,\bar{\psi}_L\,U_L^\dagger\,\psi_R + m\,\bar{\psi}_R\,U_L\,\psi_L \neq m\,\bar{\psi}\psi\,. \end{aligned} \tag{2.3.50}$$

It is straightforward to show that the symmetries of the Lagrangian (2.3.46) lead to separately conserved charges for the left- and for the right-handed states. In addition, the symmetries forbid a mass for the gauge fields (local gauge symmetry) and also forbid a mass for the fermions (chiral symmetry).

2.4 Exercises

1. **Different Lagrangians for the Dirac equation:**

   Show that the following two Lagrangians lead to the same equation of motion (the Dirac equation):

   $$\mathcal{L}_{\mathrm{D}}(\psi,\psi^\dagger,\partial_\mu\psi,\partial_\mu\psi^\dagger) = \bar{\psi}\,\left(\frac{1}{2}\,i\,(\overrightarrow{\partial\!\!\!/} - \overleftarrow{\partial\!\!\!/}) - m\right)\psi \tag{2.4.1}$$

   and

   $$\tilde{\mathcal{L}}_{\mathrm{D}}(\psi,\psi^\dagger,\partial_\mu\psi,\partial_\mu\psi^\dagger) = \bar{\psi}\,\left(i\partial\!\!\!/ - m\right)\psi\,. \tag{2.4.2}$$

   Recall: In general, the equations of motion are obtained from

   $$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0 \qquad \text{for all } a. \tag{2.4.3}$$

2. **Mixing:**

   The following exercise is the field theoretical counterpart for the problem how to find the eigen modes of coupled harmonic oscillators:

   Given two hermitian bosonic fields, ϕ_1 and ϕ_2, and the Lagrangian

   $$\mathcal{L} = \frac{1}{2}\,\partial_\alpha\phi^T\cdot\partial^\alpha\phi - \frac{1}{2}\,\phi^T\cdot\mathcal{M}\cdot\phi \tag{2.4.4}$$

   where the fields are collected in a two-component structure

   $$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad \Rightarrow \qquad \phi^T = (\phi_1\ ,\ \phi_2) \tag{2.4.5}$$

   and $\mathcal{M}$ denotes a symmetric mass matrix

   $$\mathcal{M} = \begin{pmatrix} m_{11}^2 & m_{12}^2 \\ m_{12}^2 & m_{22}^2 \end{pmatrix} \tag{2.4.6}$$

   with $\det\mathcal{M} > 0$.

(a) Derive the equations of motion for ϕ_1 and ϕ_2.

(b) Obviously the equations of motion mix the two fields ϕ_1 and ϕ_2. On the other hand, the Lagrangian is quadratic in the fields (and the equations of motion are linear), so there can be no real interaction. Show that one can find modified fields, ϕ_1' and ϕ_2', for which the Lagrangian and the equations of motion are decoupled.

Comment: A similar situation emerges in the neutrino sector (with the difference that neutrinos are fermions, not bosons). The production and observation of neutrinos (via the weak interaction) distinguishes, e.g., between electron and muon neutrinos — the counterparts of our fields ϕ_1 and ϕ_2. On the other hand, the mass eigen states — corresponding to our fields ϕ_1' and ϕ_2' — are superpositions of the electron and the muon neutrino. Since the mass eigen states propagate independently of each other, this implies that even if only electron neutrinos are produced at first place, after some time more and more muon neutrinos emerge via the mixing phenomenon. After a characteristic time (depending on the mixing angle) one has a maximum of muon neutrinos. One more after the same time they have vanished again and so forth (oscillations). This can be studied in our example by solving the equations of motion for ϕ_1 and ϕ_2 with the initial condition that at $t = 0$ there is a wave packet for ϕ_1 and that ϕ_2 vanishes. One can then see that there are characteristic distances and times (connected by the velocity of the center of the wave packet) where the ϕ_2 amplitude has a maximum.

3 Scattering theory and Feynman rules

In the following we will discuss decay and scattering processes and we will present a set of rules how to calculate decay rates and scattering cross sections.

3.1 Decay processes

The characteristic quantity is the *decay rate* Γ. Given an ensemble of uncorrelated identical unstable particles, the survival probability after time t is given by

$$p_{\text{survival}} = e^{-\Gamma t}\,, \tag{3.1.1}$$

i.e. Γ is the inverse "life time". Due to Lorentz dilation it matters whether the decaying particle is moving. The decay rate considered in the following concerns the eigen time of the unstable particle, i.e. it is measured in the frame where the unstable particle is at rest.

Suppose that there is more than one decay channel for the considered unstable particle. The "partial" decay rate for channel number k may be denoted by

$\Gamma_{\text{partial},k}$. Assuming that all channels are uncorrelated, the total survival probability is given by

$$e^{-\Gamma_{\text{partial},1}\,t}\; e^{-\Gamma_{\text{partial},2}\,t}\; \ldots \stackrel{!}{=} e^{-\Gamma_{\text{tot}}\,t}\,, \tag{3.1.2}$$

i.e. the total decay rate is given by the sum of all partial decay rates,

$$\Gamma_{\text{tot}} = \sum_k \Gamma_{\text{partial},k}\,. \tag{3.1.3}$$

In quantum field theory one considers the differential decay rate for a particle with mass M which decays into n particles with masses m_j and momenta $\vec{p}_j$ in the volume elements d^3p_j, $j = 1, \ldots, n$. The corresponding energies of the emerging particles are $E_j = \sqrt{m_j^2 + \vec{p}_j^{\,2}}$. The differential decay rate is given by

$$d\Gamma = \frac{1}{2M}\;|\mathcal{M}_{1\to n}|^2 \underbrace{(2\pi)^4\,\delta^{(4)}\left(P - \sum_{j=1}^{n} p_j\right) \prod_{j=1}^{n} \frac{d^3p_j}{(2\pi)^3\,2E_j}}_{\text{“phase space”}} \tag{3.1.4}$$

with the four-momenta of the initial particle (which is at rest) $P = (M, \vec{o})$ and of the final particles $p_j = (E_j, \vec{p}_j)$.

The only quantity where details of the interactions between the initial and the final particles enter is the Lorentz invariant “Feynman matrix element” $\mathcal{M}_{1\to n}$. (The word “matrix” refers here to the reaction matrix, the so-called “S matrix”; see, e.g., [Brown,Peskin,Bjorken,Mulders]) We will see below how this quantity can be obtained from a given Lagrangian.

In principle, the Feynman matrix element $\mathcal{M}_{1\to n}$ can depend on all four-momenta p_j. However, the respective energy is fixed by the three-momentum. The masses are not additional variables but constants. Therefore it seems that the Feynman matrix element can depend on $3\,(n+1)$ variables. But since it is Lorentz invariant, it can only depend on Lorentz invariant combinations of the four-momenta. How many independent Lorentz invariant variables does one have? A Lorentz invariant expression cannot change if we change the frame of reference. We have rotations and boosts available to change this frame. This reduces the number of independent relevant variables by 6 (three directions for rotations, three for boosts). In addition we have the conservation of energy and of three-momentum. This constrains the free variables further. In total we have (at most) $3\,(n+1) - 10 = 3n - 7$ independent variables. How this dependence looks like is dictated by the Lagrangian.

3.1.1 Example: Two-body decay

For $n = 2$ we expect that there is *no* free variable on which the Feynman matrix element $\mathcal{M}_{1\to 2}$ might depend on. Indeed, the only Lorentz invariant combination

we can build is $p_1 \cdot p_2$, but this quantity can be rewritten as

$$p_1 \cdot p_2 = \frac{1}{2}\left((p_1+p_2)^2 - p_1^2 - p_2^2\right) = \frac{1}{2}\left(P^2 - p_1^2 - p_2^2\right) = \frac{1}{2}\left(M^2 - m_1^2 - m_2^2\right) . \tag{3.1.5}$$

Thus, for two-body decays the Feynman matrix element $\mathcal{M}_{1\to 2}$ is a constant. Consequently, the differential decay rate (3.1.4) can be integrated for any $\mathcal{M}_{1\to 2}$, i.e. for any Lagrangian. Only the absolute size of $\mathcal{M}_{1\to 2}$ depends on the Lagrangian, but there is no dependence on the integration variables $\vec{p}_j$. We get

$$\begin{aligned}\Gamma &= \frac{1}{2M}\left|\mathcal{M}_{1\to 2}\right|^2 \int \frac{d^3p_1}{(2\pi)^3\,2E_1}\,\frac{d^3p_2}{(2\pi)^3\,2E_2}\,(2\pi)^4\,\delta^{(4)}\left(P - p_1 - p_2\right) \\ &= \frac{1}{8M}\left|\mathcal{M}_{1\to 2}\right|^2 \frac{1}{(2\pi)^2}\int \frac{d^3p_1}{E_1}\,\frac{d^3p_2}{E_2}\,\delta(M - E_1 - E_2)\,\delta^{(3)}\left(-\vec{p}_1 - \vec{p}_2\right)\end{aligned} \tag{3.1.6}$$

where in the last step we have used the fact that the decaying particle is at rest.

The delta functions for the three-momenta can be used to eliminate $\vec{p}_2$. The energy of the second final particle, E_2, becomes a function of $|\vec{p}_1|$ or, equivalently, a function of the energy of the first final particle:

$$E_2 = \sqrt{m_2^2 + \vec{p}_2^{\,2}} = \sqrt{m_2^2 + \vec{p}_1^{\,2}} = \sqrt{m_2^2 + E_1^2 - m_1^2} =: E_2(E_1)\,. \tag{3.1.7}$$

Thus

$$\begin{aligned}\Gamma &= \frac{1}{8M}\left|\mathcal{M}_{1\to 2}\right|^2 \frac{1}{(2\pi)^2}\int \frac{d^3p_1}{E_1\,E_2(E_1)}\,\delta(M - E_1 - E_2(E_1)) \\ &= \frac{1}{8M}\left|\mathcal{M}_{1\to 2}\right|^2 \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta\,\sin\vartheta \\ &\quad \times \int_0^{\infty} dp_1\,\frac{p_1^2}{E_1\,E_2(E_1)}\,\delta(M - E_1 - E_2(E_1)) \\ &= \frac{1}{8M}\left|\mathcal{M}_{1\to 2}\right|^2 \frac{1}{\pi}\int_{m_1}^{\infty} dE_1\,\frac{p_1(E_1)}{E_2(E_1)}\,\delta(M - E_1 - E_2(E_1))\end{aligned} \tag{3.1.8}$$

where we have introduced spherical coordinates for $\vec{p}_1$ and changed the $p_1 := |\vec{p}_1|$ integration to an energy integration (from $E_1^2 = m_1^2 + p_1^2$ one gets $E_1\,dE_1 = p_1\,dp_1$).

The delta function fixes the energy variable in terms of the masses M, m_1, and m_2. We find:

$$\begin{aligned}&M - E_1 - E_2(E_1) = 0 \quad \Leftrightarrow \quad M - E_1 = E_2(E_1) \\ &\Rightarrow \quad (M - E_1)^2 = m_2^2 + E_1^2 - m_1^2 \quad \Leftrightarrow \quad E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}\,.\end{aligned} \tag{3.1.9}$$

The energy of the second final particle is then given by

$$E_2 = M - E_1 = \frac{M^2 + m_2^2 - m_1^2}{2M}\,. \tag{3.1.10}$$

The same result is obtained from (3.1.7) as one can easily check.

To evaluate the delta function in (3.1.8) we use

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x - x_i) \tag{3.1.11}$$

where the x_i denote the zeros of the function $f(x)$, i.e. $f(x_i) = 0$. Thus we have to calculate

$$\begin{aligned}\frac{d}{dE_1}\left(M - E_1 - E_2(E_1)\right) &= -1 - \frac{2E_1}{2\sqrt{m_2^2 + E_1^2 - m_1^2}} = -1 - \frac{E_1}{E_2(E_1)} \\ &= -\frac{E_1 + E_2(E_1)}{E_2(E_1)}\,.\end{aligned} \tag{3.1.12}$$

This yields

$$\begin{aligned}\delta(M - E_1 - E_2(E_1)) &= \frac{E_2(E_1)}{E_1 + E_2(E_1)}\,\delta\left(E_1 - \frac{M^2 + m_1^2 - m_2^2}{2M}\right) \\ &= \frac{E_2}{M}\,\delta\left(E_1 - \frac{M^2 + m_1^2 - m_2^2}{2M}\right)\end{aligned} \tag{3.1.13}$$

with E_2 given in (3.1.10).

For the decay rate (3.1.8) this implies

$$\Gamma = \frac{p_{\rm cm}}{8\pi M^2}\,|\mathcal{M}_{1\to 2}|^2 \tag{3.1.14}$$

where $p_{\rm cm} = p_1(E_1) = |\vec{p}_1| = |\vec{p}_2|$ is the modulus of the three-momentum of the first final particle which is also the modulus of the three-momentum of the second final particle, since we are in the frame where the decaying particle is at rest. At the same time this is the frame where the center of mass of the two final particles is at rest. Therefore $p_{\rm cm}$ is called center-of-mass momentum. In terms of the

masses it is given by

$$\begin{aligned} p_{\rm cm} &= p_1(E_1) = \sqrt{E_1^2 - m_1^2} = \left(\left(\frac{M^2 + m_1^2 - m_2^2}{2M} \right)^2 - m_1^2 \right)^{1/2} \\ &= \frac{1}{2M} \left(\left(M^2 + m_1^2 - m_2^2 \right)^2 - 4M^2 m_1^2 \right)^{1/2} \\ &= \frac{1}{2M} \left((M^2 + m_1^2 - m_2^2 + 2Mm_1)(M^2 + m_1^2 - m_2^2 - 2Mm_1) \right)^{1/2} \\ &= \frac{1}{2M} \left(\left((M + m_1)^2 - m_2^2 \right) \left((M - m_1)^2 - m_2^2 \right) \right)^{1/2} \\ &= \frac{1}{2M} \left((M + m_1 + m_2)(M + m_1 - m_2)(M - m_1 + m_2)(M - m_1 - m_2) \right)^{1/2} \\ &= \frac{1}{2M} \left(\left(M^2 - (m_1 + m_2)^2 \right) \left(M^2 - (m_1 - m_2)^2 \right) \right)^{1/2} . \end{aligned} \tag{3.1.15}$$

The last form is the most common one. Here it is manifest that the masses of the two final particles enter in a symmetric way.

There is one issue where so far we have not payed much attention to: The energy integral in (3.1.8) does not extend from $-\infty$ to $+\infty$ but has a lower limit at m_1. Consequently the delta function (3.1.13) in the energy can only yield a non-vanishing result if the solution (3.1.9) for E_1 is inside the integration range, i.e.

$$E_1 \geq m_1 \quad \Leftrightarrow \quad \frac{M^2 + m_1^2 - m_2^2}{2M} \geq m_1 \quad \Leftrightarrow \quad M \geq m_1 + m_2 \,. \tag{3.1.16}$$

Therefore the equation (3.1.14) is a little bit sloppy. The correct version is

$$\Gamma = \begin{cases} 0 & \text{for} \quad M < m_1 + m_2 \\ \frac{p_{\rm cm}}{8\pi M^2} \, |\mathcal{M}_{1\to 2}|^2 & \text{for} \quad M \geq m_1 + m_2 \,. \end{cases} \tag{3.1.17}$$

Note that $p_{\rm cm}$, as given in (3.1.15), vanishes for $M = m_1 + m_2$. Therefore, the decay rate Γ is a continuous function of the mass M of the decaying particle.

3.2 Scattering processes

The characteristic quantity is the *cross section* σ. Consider two particles with four-momenta p_1 and p_2 which scatter into n other particles with four-momenta $p_3, \ \ldots, \ p_{n+2}$ in the respective volume elements d^3p_j, $j = 3, \ldots, n+2$. The corresponding masses and energies are denoted by m_j and E_j, respectively, with $j = 1, \ldots, n+2$. The fully differential cross section is given by

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \, |\mathcal{M}_{2\to n}|^2 \\ &\quad \times (2\pi)^4 \, \delta^{(4)} \left(p_1 + p_2 - \sum_{j=3}^{n+2} p_j \right) \prod_{j=3}^{n+2} \frac{d^3 p_j}{(2\pi)^3 \, 2E_j} \,. \end{aligned} \tag{3.2.1}$$

In principle, a cross section is defined in the so-called laboratory frame (lab frame) which is defined by $\vec{p}_2 = 0$. However, a boost to the center-of-mass frame of the two initial particles does not change the cross section because the latter is defined as an area perpendicular to the beam direction $\vec{p}_1$ in the lab frame. The boost to the center-of-mass frame is along $\vec{p}_1$ and leaves all distances orthogonal to the boost direction unchanged. Only distances along the longitudinal direction would be subject to a Lorentz contraction. Therefore, the formula (3.2.1) is also valid in the center-of-mass frame which is defined by $\vec{p}_1 + \vec{p}_2 = 0$.

It is useful to introduce the Lorentz invariant quantity

$$s := (p_1 + p_2)^2 = 2\, p_1 \cdot p_2 + m_1^2 + m_2^2 \,. \tag{3.2.2}$$

In the center-of-mass frame one has $s = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 = (E_1 + E_2)^2$ since by definition $\vec{p}_1 + \vec{p}_2 = 0$. Thus, $\sqrt{s}$ is the total energy of the reaction in the center-of-mass frame. The sum of the initial four-momenta and therefore also the sum of the final four-momenta is given by

$$p_1 + p_2 = \sum_{j=3}^{n+2} p_j = \left(\sqrt{s}, \vec{o}\right) \qquad \text{in center-of-mass frame.} \tag{3.2.3}$$

In the center-of-mass frame the modulus of the three-momentum of $\vec{p}_1$ (and therefore also of $\vec{p}_2$) is given by

$$|\vec{p}_1| = |\vec{p}_2| = p_{\text{cm,in}} := \frac{1}{2\sqrt{s}} \left(\left(s - (m_1 + m_2)^2\right) \left(s - (m_1 - m_2)^2\right) \right)^{1/2} . \tag{3.2.4}$$

The corresponding energies of the initial particles are

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \,, \qquad E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \,. \tag{3.2.5}$$

These are basically the same relations as (3.1.15), (3.1.9), and (3.1.10) just replacing M by $\sqrt{s}$. The reason is that we have the same kinematical situation: Two particles are considered in their center-of-mass frame. The total energy is given by M or $\sqrt{s}$, respectively.

Using (3.2.2) we can rewrite the so-called flux factor, i.e. the square root in (3.2.1):

$$\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2\, m_2^2 &= \left(\frac{1}{2} \left(s - m_1^2 - m_2^2\right) \right)^2 - m_1^2\, m_2^2 \\
&= \frac{1}{4} \left(s - m_1^2 - m_2^2 - 2\, m_1 m_2\right) \left(s - m_1^2 - m_2^2 + 2\, m_1 m_2\right) \\
&= \frac{1}{4} \left(s - (m_1 + m_2)^2\right) \left(s - (m_1 - m_2)^2\right) \\
&= (p_{\text{cm,in}}\, \sqrt{s})^2 \,.
\end{aligned} \tag{3.2.6}$$

Thus,

$$d\sigma = \frac{1}{4\,p_{\mathrm{cm,in}}\,\sqrt{s}}\,|\mathcal{M}_{2\to n}|^2\,(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-\sum_{j=3}^{n+2}p_j\right)\prod_{j=3}^{n+2}\frac{d^3p_j}{(2\pi)^3\,2E_j}\,. \quad (3.2.7)$$

In principle, the Feynman matrix element $\mathcal{M}_{2\to n}$ can depend on all four-momenta p_j. But since it is Lorentz invariant, it can only depend on Lorentz invariant combinations of the four-momenta. The same considerations as for the decay processes show that the Feynman matrix element $\mathcal{M}_{2\to n}$ depends on $3n-4$ independent variables. How this dependence looks like is dictated by the Lagrangian.

3.2.1 Example: Two-body final state

For $n=2$ we expect two independent variables. One of them is $p_1\cdot p_2$, but equivalently one can take $s=(p_1+p_2)^2$ which we have already introduced in (3.2.2). It is useful to introduce the other so-called Mandelstam variables

$$t := (p_1-p_3)^2 = m_1^2+m_3^2-2\,p_1\cdot p_3\,, \quad (3.2.8)$$
$$u := (p_2-p_3)^2 = m_2^2+m_3^2-2\,p_2\cdot p_3\,. \quad (3.2.9)$$

The Lorentz invariant products which one can build are $p_1\cdot p_2$, $p_1\cdot p_3$, and $p_2\cdot p_3$. p_4 is fixed by $p_4=p_1+p_2-p_3$. Obviously, all Lorentz invariant quantities can be expressed in terms of the Mandelstam variables s, t, and u. They are, however, not independent:

$$\begin{aligned} s+t+u &= (p_1+p_2)^2+(p_1-p_3)^2+(\underbrace{p_2-p_3}_{=p_4-p_1})^2 \\ &= m_1^2+m_2^2+2\,p_1\cdot p_2+m_1^2+m_3^2-2\,p_1\cdot p_3+m_4^2+m_1^2-2\,p_1\cdot p_4 \\ &= 3\,m_1^2+m_2^2+m_3^2+m_4^2+2\,p_1\cdot(\underbrace{p_2-p_3-p_4}_{=-p_1}) \\ &= 3\,m_1^2+m_2^2+m_3^2+m_4^2-2\,m_1^2 \\ &= m_1^2+m_2^2+m_3^2+m_4^2\,. \end{aligned} \quad (3.2.10)$$

Thus, s and t, or equivalently s and u, are the independent variables of a $2\to 2$ scattering reaction.

If we want to calculate the total cross section from (3.2.7) the integrals over the final momenta do not concern the variable s since the latter is solely given by the initial momenta. In other words, s is fixed from outside by the collision energy. The only dependence of the Feynman matrix element $\mathcal{M}_{2\to 2}$ which influences the integrations over the momenta of the final states is the t dependence. Actually this is equivalent to the dependence on the scattering angle ϑ defined in the center-of-mass frame. The latter is the angle between the direction of the incoming

three-momentum $\vec{p}_1$ and the direction of the outgoing three-momentum $\vec{p}_3$. The relation between the Mandelstam variable t and the scattering angle ϑ is given by

$$\begin{aligned} t &= (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2\,E_1\,E_3 + 2\,\vec{p}_1 \cdot \vec{p}_3 \\ &= m_1^2 + m_3^2 - 2\,E_1\,E_3 + 2\,|\vec{p}_1|\,|\vec{p}_3|\,\cos\vartheta \\ &= m_1^2 + m_3^2 - 2\,E_1\,E_3 + 2\,p_{\rm cm,in}\,p_{\rm cm,out}\,\cos\vartheta\,. \end{aligned} \tag{3.2.11}$$

We have introduced the modulus of the momentum of each of the two outgoing particles in the center-of-mass frame, cf. (3.2.4),

$$|\vec{p}_3| = |\vec{p}_4| = p_{\rm cm,out} := \frac{1}{2\sqrt{s}} \Big(\big(s - (m_3 + m_4)^2\big)\,\big(s - (m_3 - m_4)^2\big) \Big)^{1/2}\,. \tag{3.2.12}$$

Note that the center-of-mass frame for the two incoming particles is at the same time the center-of-mass frame for the two outgoing particles. In that frame the energies of the latter are given by, cf. (3.2.5),

$$E_3 = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}\,, \qquad E_4 = \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}\,. \tag{3.2.13}$$

In the following we will calculate the differential cross section $d\sigma/dt$ and the total cross section σ. Obviously, the former can be related to the differential cross section $d\sigma/d\vartheta$ which is frequently used, e.g., in non-relativistic scattering theory.

We first determine

$$\begin{aligned} \frac{d\sigma}{dt} &= \int d\sigma\,\delta\left(t - (p_1 - p_3)^2\right) \\ &= \frac{1}{4\,p_{\rm cm,in}\,\sqrt{s}} \int \frac{d^3p_3}{(2\pi)^3\,2E_3}\,\frac{d^3p_4}{(2\pi)^3\,2E_4}\,|\mathcal{M}_{2\to2}(s,t)|^2 \\ &\qquad \times (2\pi)^4\,\delta^{(4)}\left(p_1 + p_2 - p_3 - p_4\right)\,\delta\left(t - (p_1 - p_3)^2\right)\,. \end{aligned} \tag{3.2.14}$$

Since the whole expression is Lorentz invariant we can evaluate it in any frame. We choose the center-of-mass frame and in that frame we choose the coordinate system such that $\vec{p}_1$ points in the z-direction. Then the delta functions for the three-momenta can be used to eliminate $\vec{p}_4 = -\vec{p}_3$. For the $\vec{p}_3$ integrations we can introduce spherical coordinates $(p_3, \varphi, \vartheta)$. The integration variable ϑ is the angle between $\vec{p}_3$ and the z-direction, but since $\vec{p}_1$ points in the z-direction, the angle ϑ is at the same time the scattering angle introduced in (3.2.11). Finally we change again the momentum integration to an energy integration in E_3 (cf. (3.1.8)). We get

$$\begin{aligned} \frac{d\sigma}{dt} = \frac{1}{16\,p_{\rm cm,in}\,\sqrt{s}}\,|\mathcal{M}_{2\to2}(s,t)|^2\,\frac{1}{(2\pi)^2}\int_0^{2\pi} d\varphi \int_{m_3}^{\infty} dE_3 \int_0^{\pi} d\vartheta\,\sin\vartheta\,\frac{p_3(E_3)}{E_4(E_3)} \\ \times\delta\left(\sqrt{s} - E_3 - E_4(E_3)\right)\,\delta\left(t - m_1^2 - m_3^2 + 2\,E_1\,E_3 - 2\,p_{\rm cm,in}\,p_3(E_3)\,\cos\vartheta\right) \end{aligned} \tag{3.2.15}$$

with E_1 given in (3.2.5), $E_4(E_3) = \sqrt{m_4^2 + \vec{p}_4^{\,2}} = \sqrt{m_4^2 + \vec{p}_3^{\,2}} = \sqrt{m_4^2 + E_3^2 - m_3^2}$, and $p_3(E_3) = \sqrt{E_3^2 - m_3^2}$.

The integration over ϑ can be rewritten:

$$\int_0^\pi d\vartheta \, \sin\vartheta \ldots = -\int_1^{-1} d(\cos\vartheta) \ldots = \int_{-1}^{1} d(\cos\vartheta) \ldots \tag{3.2.16}$$

The last delta function in (3.2.15) fixes the variable ϑ to the relation given in (3.2.11). This leads to

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{16\, p_{\mathrm{cm,in}}\, \sqrt{s}} \, |\mathcal{M}_{2\to2}(s,t)|^2 \, \frac{1}{2\pi} \\ &\quad \times \int_{m_3}^{\infty} dE_3 \, \frac{p_3(E_3)}{E_4(E_3)} \, \delta\left(\sqrt{s} - E_3 - E_4(E_3)\right) \, \frac{1}{2\, p_{\mathrm{cm,in}}\, p_3(E_3)} \,. \end{aligned} \tag{3.2.17}$$

The remaining delta function can be used to perform the E_3 integration. This is completely equivalent to the considerations which entered (3.1.13). The result is

$$\frac{d\sigma}{dt} = \frac{1}{64\,\pi\, p_{\mathrm{cm,in}}^2\, s} \, |\mathcal{M}_{2\to2}(s,t)|^2 \,. \tag{3.2.18}$$

Similar to the considerations which led to (3.1.17) we have to make sure that the finite integration boundaries of ϑ and E_3 in (3.2.15) are properly accounted for. This places constraints on s and t: From $E_3 \geq m_3$ one gets using (3.2.13):

$$\sqrt{s} \geq m_3 + m_4 \,. \tag{3.2.19}$$

In other words, the reaction can only happen if there is enough energy to produce the particles 3 and 4.

From $0 \leq \vartheta \leq \pi$ one obtains using (3.2.11):

$$\begin{aligned} & -2\, p_{\mathrm{cm,in}}\, p_{\mathrm{cm,out}} \leq t - m_1^2 - m_3^2 + 2\, E_1\, E_3 \leq 2\, p_{\mathrm{cm,in}}\, p_{\mathrm{cm,out}} \\ \Leftrightarrow \quad & -(p_{\mathrm{cm,in}} + p_{\mathrm{cm,out}})^2 \leq t \underbrace{-m_1^2 - p_{\mathrm{cm,in}}^2}_{=-E_1^2} \underbrace{-m_3^2 - p_{\mathrm{cm,out}}^2}_{=-E_3^2} + 2\, E_1\, E_3 \leq -(p_{\mathrm{cm,in}} - p_{\mathrm{cm,out}})^2 \end{aligned}$$

$$\Leftrightarrow \quad -(p_{\mathrm{cm,in}} + p_{\mathrm{cm,out}})^2 \leq t - (E_1 - E_3)^2 \leq -(p_{\mathrm{cm,in}} - p_{\mathrm{cm,out}})^2 \,. \tag{3.2.20}$$

The energies can be expressed in terms of s using (3.2.13). We finally get

$$t_1 \leq t \leq t_0 \tag{3.2.21}$$

with

$$t_1 := \left(\frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{2\sqrt{s}}\right)^2 - (p_{\mathrm{cm,in}} + p_{\mathrm{cm,out}})^2 \,, \tag{3.2.22}$$

$$t_0 := \left(\frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{2\sqrt{s}}\right)^2 - (p_{\mathrm{cm,in}} - p_{\mathrm{cm,out}})^2 \,. \tag{3.2.23}$$

Therefore the exact statement which should replace the sloppy version (3.2.18) is

$$\frac{d\sigma}{dt} = \begin{cases} \dfrac{1}{64\,\pi\,p_{\mathrm{cm,in}}^2\;s}\;|\mathcal{M}_{2\to2}(s,t)|^2 & \text{for} \quad \sqrt{s} \geq m_3 + m_4 \quad \text{and} \quad t_1 \leq t \leq t_0 \\ 0 & \text{else.} \end{cases} \tag{3.2.24}$$

Finally the total cross section is given by

$$\sigma = \int_{t_1}^{t_0} dt\,\frac{d\sigma}{dt} = \frac{1}{64\,\pi\,p_{\mathrm{cm,in}}^2\;s}\;\int_{t_1}^{t_0} dt\;|\mathcal{M}_{2\to2}(s,t)|^2\;. \tag{3.2.25}$$

Here the result depends on the explicit form of the Feynman matrix element $\mathcal{M}_{2\to2}(s,t)$ due to its t dependence.

3.3 Feynman rules

Typically the Feynman matrix element for a reaction of interest cannot be calculated exactly from a given Lagrangian. If one has an argument why interactions are small, then one can use perturbation theory to calculate the Feynman matrix element. The input are the solutions for the free-field equations and the interaction part of the Lagrangian is treated as a perturbation. This works if there is a small parameter assigned to the interaction Lagrangian. This small parameter can be a coupling constant — or several coupling constants and an idea how their sizes are relative to each other. The result of perturbation theory is a power series in the small parameter(s). Calculating the whole power series would be equivalent to solving the quantum field theory exactly. On the other hand, both for comparisons to experiments and for predictions a finite accuracy of the calculation is sufficient in practice. Then a finite number of terms in the power series should be sufficient to reach the desired accuracy.[14]

In QED, in the electroweak theory and also in high-energy QCD one uses perturbation theory in powers of the respective coupling constants since the coupling constants are small (for QCD see figure 2). We shall call such approaches "standard" perturbation theories. For the mentioned examples one can expect to get a reasonable result from a few terms in the power series. In hadron physics this is different: For low-energy QCD the relevant entities are the hadrons, i.e. composite objects made out of quarks and gluons. If the involved energies and momenta are so low that the intrinsic quark-gluon structure is not resolved, it makes sense to formulate a quantum field theory with fields representing the

[14] Actually the convergence of the power series is not granted. Quite contrary one can show at least for some theories that the power series is only an asymptotic series and not a convergent one. Still one might get a result reasonably close to the exact one by a finite number of terms provided the expansion parameter is small enough.

relevant hadron degrees of freedom. However, it turns out that the coupling constants are typically *not* small. Then it would be interesting to find another small parameter with which one can develop an alternative power series. This is the purpose of *effective field theories*. Examples are chiral perturbation theory and heavy-quark effective theory. For energies where only the few lightest hadrons are relevant, it turns out that one can expand in powers of the involved energies and momenta. This scheme is called chiral perturbation theory. The reason why one can formulate a systematic power expansion in powers of momenta is intimately connected to the symmetries of QCD as we will see in section 6. For hadrons which contain a heavy quark (charm or even bottom quark), it turns out that one can expand in inverse powers of the heavy quark mass. This effective field theory will not be discussed here.

All the previously discussed approaches are systematic in the sense that one knows at least in principle how to improve an obtained result: One just has to calculate the next order(s) of the power series.[15] In addition, the last calculated term in the power series, i.e. the one with the highest power of the small parameter, can be used to estimate the accuracy of the achieved result. Such systematic approaches (standard perturbation theories, effective field theories) are different from phenomenological *models*. In the realm of hadron physics hadronic models are frequently used and are often phenomenologically very successful. They are constructed such that they contain the symmetries inherent to QCD, but use hadrons as elementary degrees of freedom. In particular in the energy region where a lot of different hadron species become relevant as active degrees of freedom (the region of hadronic resonances) at present there is no systematic effective field theory. Then using hadronic models is the best one can do — and/or aiming at an extension of the applicability region of effective field theories. Models are not solely characterized by a Lagrangian but by additional ad-hoc rules which specify which contributions should be taken into account. Typically one considers the lowest-order contributions in powers of the coupling constants, i.e. one drops higher-order contributions by hand in spite of the fact that the involved couplings are not small. It must be stressed that nothing is wrong in principle in having a model instead of a full quantum field theory (the latter is defined solely by a Lagrangian and the quantization rules). Adjusting the free parameters, e.g., the coupling constants, to experiments and predicting the outcome of other experiments can be successful. However, one cannot anticipate how successful the model will be, because one cannot estimate the intrinsic accuracy of the calculation. The point is that in contrast to a systematic approach one does not know how to improve a model in a systematic way. There is just no expansion parameter which allows for systematic improvements and accuracy estimates.

In the following we will demonstrate how one uses Feynman rules to approximately calculate a Feynman matrix element. We shall use standard perturbation

[15] In practice this can be tedious.

theory, i.e. we assume that all the involved coupling constants are small. As already announced in the beginning of these lecture notes we will not provide a derivation of the Feynman rules. This can be found in any textbook on quantum field theory. The keywords are time-dependent perturbation theory and Wick expansion.

The basic idea is that every contribution to the power series in the coupling constant(s) can be visualized by a couple of diagrams. Each piece in a diagram translates to a part of a formula. The final formula constitutes one contribution to the Feynman matrix element. Every interaction term in a given Lagrangian translates into a vertex, i.e. a crossing point of lines. The number of lines attached to a vertex is the number of fields in the interaction term. The Feynman diagrams consist out of such vertices and lines connecting them. Each vertex is proportional to the corresponding coupling constant. Therefore only a finite number of vertices must be considered since one wants to determine "only" a finite number of terms in the coupling-constant power series.

Suppose we want to determine the Feynman matrix element $\mathcal{M}_{m\to n}$ for the reaction of the initial particles with momenta $p_1, \ldots, p_m$ into the final particles with momenta $p_{m+1}, \ldots, p_{m+n}$. The general Feynman rules are

1. Given an interaction part, $\mathcal{L}_{\text{int}}$, of a Lagrangian for generic fields ϕ_a, rewrite the corresponding action $i\,S_{\text{int}} := i\int d^4x\,\mathcal{L}_{\text{int}}$ into momentum space by introducing the Fourier decomposition

$$\phi_a(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\,\phi_a(q)\,. \tag{3.3.1}$$

   Read off the corresponding vertex.

2. Connect vertices to each other and to the external particles by lines. There are internal lines ("propagators") connecting the vertices and external lines where only one end is attached to a vertex. There is one external line for each initial or final particle. Each line carries a momentum. The momentum of an external line is just the momentum of the corresponding initial or final particle. Total energy-momentum conservation applies to each vertex which fixes some of the other momenta assigned to lines.

3. Draw all possible connected diagrams up to the order of interest in powers of the coupling constant(s). The (approximation to the) Feynman matrix element, $i\mathcal{M}_{m\to n}$, is obtained as the sum of all diagrams.

4. To get one diagram
   - multiply
     (a) all propagators,
     (b) all vertices,

(c) all external lines,
(d) a factor (-1) for each closed fermion loop,
(e) a relative minus sign between diagrams where only two fermion lines are interchanged,
(f) a symmetry factor which accounts for identical lines at a vertex and for identical vertices,

- sum over all indices not fixed,
- integrate over all momenta k not fixed by energy-momentum conservation, $\int \frac{d^4k}{(2\pi)^4} \ldots$.

The symmetry factor mentioned above is obtained by counting all possibilities how to connect the vertices such that the considered diagram is obtained. If there are m identical vertices the result must be divided by $1/(m!)$.

We will now specify these general Feynman rules by providing some generic examples. The first example concerns a hermitian scalar field ϕ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2 \underbrace{+\,\frac{\kappa}{3!}\,\phi^3 - \frac{\lambda}{4!}\,\phi^4}_{=:\,\mathcal{L}_{\rm int}} . \tag{3.3.2}$$

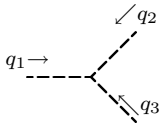
Using (3.3.1) one finds, e.g.,

$$\begin{aligned} & i\int d^4x\,\frac{\kappa}{3!}\,[\phi(x)]^3 \\ &= i\int d^4x\,\frac{\kappa}{3!}\int \frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1\cdot x}\,\phi(q_1)\int \frac{d^4q_2}{(2\pi)^4}\,e^{-iq_2\cdot x}\,\phi(q_2)\int \frac{d^4q_3}{(2\pi)^4}\,e^{-iq_3\cdot x}\,\phi(q_3) \\ &= \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3)\,\phi(q_1)\,\phi(q_2)\,\phi(q_3)\,\frac{i\kappa}{3!} . \end{aligned} \tag{3.3.3}$$

Together with the corresponding calculation for the ϕ^4 term one obtains for the interaction part of the action:

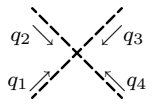
$$\begin{aligned} i\,S_{\rm int} \;=\; & \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3)\,\phi(q_1)\,\phi(q_2)\,\phi(q_3)\,\frac{i\kappa}{3!} \\ & + \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3+q_4) \\ & \qquad \times \phi(q_1)\,\phi(q_2)\,\phi(q_3)\,\phi(q_4)\,\frac{-i\lambda}{4!} . \end{aligned} \tag{3.3.4}$$

One reads off the vertices



$$= i\,\frac{\kappa}{3!} \qquad (\text{with} \quad q_1+q_2+q_3=0) \tag{3.3.5}$$

and



$$= -i\,\frac{\lambda}{4!} \qquad (\text{with} \quad q_1+q_2+q_3+q_4=0)\,. \tag{3.3.6}$$

Note that the delta functions in (3.3.4) ensure energy-momentum conservation at the vertices. This is a consequence of the locality of the action, i.e. of the fact that all fields have the same space-time argument. In that way the space-time integration, e.g., in (3.3.3) produces the delta function for energy and momentum.

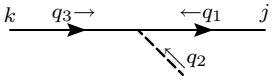
As a second example we consider a Lagrangian for fermions and bosons which contains a derivative in the interaction term:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2 + \bar\psi\,\left(i\partial\!\!\!/ - m + g\,\gamma_\mu\gamma_5\,(\partial^\mu\phi)\right)\psi\,. \tag{3.3.7}$$

We note in passing that this interaction resembles the one for a pion coupled to nucleons. The interaction part of the action becomes

$$\begin{aligned} i\,S_{\rm int} &= i\int d^4x\,\int\frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1\cdot x}\,\bar\psi_j(q_1) \\ &\quad\times\int\frac{d^4q_2}{(2\pi)^4}\,e^{-iq_2\cdot x}\,g\,(\gamma_\mu\gamma_5)_{jk}\,\left(\partial^\mu e^{-iq_2\cdot x}\right)\phi(q_2)\int\frac{d^4q_3}{(2\pi)^4}\,e^{-iq_3\cdot x}\,\psi_k(q_3) \\ &= \int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3) \\ &\quad\times\bar\psi_j(q_1)\,\phi(q_2)\,\psi_k(q_3)\;ig\,(\gamma_\mu\gamma_5)_{jk}\,(-i)\,q_2^\mu \end{aligned} \tag{3.3.8}$$

where we have displayed the spinor indices explicitly. We read off the vertex



$$= g\,(\not{q}_2\gamma_5)_{jk}\,. \tag{3.3.9}$$

The arrow on the line indicates that the field ψ is different from the field $\bar\psi$.

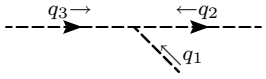
A third example concerns the case of charged bosons where one also has to distinguish between the field and its adjoint:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,M^2\,\varphi^2 + \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\,\phi^\dagger\phi + \kappa\,\varphi\,\partial_\mu\phi^\dagger\,\partial^\mu\phi \tag{3.3.10}$$

with a hermitian boson field φ and a non-hermitian boson field ϕ. One finds

$$\begin{aligned} i\,S_{\rm int} &= \int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3) \\ &\quad\times\varphi(q_1)\,\phi^\dagger(q_2)\,\phi(q_3)\;i\kappa\,(-q_2\cdot q_3) \end{aligned} \tag{3.3.11}$$

and the vertex reads

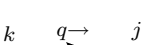


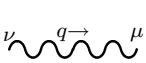
$$= -i\kappa\,q_2\cdot q_3\,. \tag{3.3.12}$$

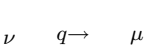
Next we turn to the propagators, i.e. the internal lines which connect the vertices with each other. We collect here the rules for scalar bosons, for Dirac spin-1/2 fermions, and for massless and massive spin-1 (vector) bosons:

$$= \frac{i}{q^2 - m^2 + i\epsilon}$$ for an uncharged scalar boson with mass m,


$$= \frac{i}{q^2 - m^2 + i\epsilon}$$ for a charged scalar boson with mass m,


$$= \frac{i\,(\not{q} + m)_{jk}}{q^2 - m^2 + i\epsilon}$$ for a Dirac spin-1/2 fermion with mass m,


$$= \frac{-i\,g_{\mu\nu}}{q^2 + i\epsilon}$$ for a massless vector boson,


$$= \frac{-i}{q^2 - m^2 + i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{i}{m^2}\frac{q_\mu q_\nu}{q^2}$$ for a vector boson with mass m.

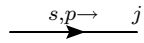
(3.3.13)

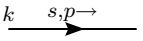
Here the $\epsilon \to 0^+$ is introduced to make the case well-defined when the rest of the denominator becomes zero, i.e. when the momentum q of the internal line satisfies the energy-momentum relation of free particles, $q^2 = m^2$.

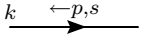
Finally the external lines correspond to the following expressions:

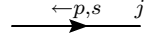
$= 1$ for an uncharged scalar boson with mass m,

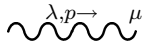
$= 1$ for a charged scalar boson with mass m,

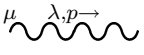

$= u_j(\vec{p}, s)$ for an initial Dirac spin-1/2 fermion with mass m,


$= \bar{u}_k(\vec{p}, s)$ for a final Dirac spin-1/2 fermion with mass m,


$= \bar{v}_k(\vec{p}, s)$ for an initial Dirac spin-1/2 antifermion with mass m,


$= v_j(\vec{p}, s)$ for a final Dirac spin-1/2 antifermion with mass m,


$= \epsilon_\mu(\vec{p}, \lambda)$ for an initial vector boson,


$= \epsilon^*_\mu(\vec{p}, \lambda)$ for a final vector boson.

(3.3.14)

The quantities u and v are the spinors used to construct solutions (2.1.69) of the

free Dirac equation. In practice all one actually needs to know is

$$(\not{p} - m)\, u(\vec{p}, s) = 0\,, \qquad (\not{p} + m)\, v(\vec{p}, s) = 0\,, \tag{3.3.15}$$

$$\sum_s u_j(\vec{p}, s)\, \bar{u}_k(\vec{p}, s) = (\not{p} + m)_{jk} = p_\mu\, (\gamma^\mu)_{jk} + m\, \delta_{jk} \tag{3.3.16}$$

and

$$\sum_s v_j(\vec{p}, s)\, \bar{v}_k(\vec{p}, s) = (\not{p} - m)_{jk} = p_\mu\, (\gamma^\mu)_{jk} - m\, \delta_{jk}\,. \tag{3.3.17}$$

We have displayed explicitly the spinor indices in the last two equations. The mass of the Dirac fermion is m and $p_0 = \sqrt{\vec{p}^{\,2} + m^2}$ is its energy.

For external vector bosons we have introduced the polarization vector $\epsilon^\mu(\vec{p}, \lambda)$ where λ denotes the different polarizations a vector particle can have. There are three polarizations ($\lambda = -1, 0, +1$) for massive and two ($\lambda = \pm 1$) for massless vectors.[16] In practice all one needs to know is

$$p^\mu\, \epsilon_\mu(\vec{p}, \lambda) = 0 \tag{3.3.18}$$

and

$$\sum_\lambda \epsilon^*_\mu(\vec{p}, \lambda)\, \epsilon_\nu(\vec{p}, \lambda) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) \tag{3.3.19}$$

for a massive vector boson. Here m denotes its mass and $p_0 = \sqrt{\vec{p}^{\,2} + m^2}$ its energy. The corresponding relation for massless vector bosons is

$$\sum_\lambda \epsilon^*_\mu(\vec{p}, \lambda)\, \epsilon_\nu(\vec{p}, \lambda) = -g_{\mu\nu}\,. \tag{3.3.20}$$

3.4 Exercises

1. **Three-body phase space:**

   Consider the decay of a particle with four-momentum p and mass M, i.e. $p^2 := p_\mu p^\mu = M^2$, into three particles with four-momenta p_1, p_2 and p_3. The corresponding masses are denoted by m_1, m_2 and m_3, respectively. If one neglects or averages over spins the square of the decay matrix element $\mathcal{M}$ is Lorentz invariant. Therefore it can only depend on Lorentz invariant combinations of the four-momenta. It is useful to introduce the following variables

   $$m_{12}^2 = (p_1 + p_2)^2\,, \qquad m_{23}^2 = (p_2 + p_3)^2\,. \tag{3.4.1}$$

[16] Recall from classical electrodynamics that light cannot be polarized longitudinally to its flight direction. The only polarizations are circularly left and circularly right.

Using energy-momentum conservation, show that all Lorentz invariant combinations of momenta can be rewritten in terms of these two-body variables:

$$\begin{aligned} p_1 \cdot p_2 &= \frac{1}{2}\left(m_{12}^2 - m_1^2 - m_2^2\right), && (3.4.2)\\ p_2 \cdot p_3 &= \frac{1}{2}\left(m_{23}^2 - m_2^2 - m_3^2\right), && (3.4.3)\\ p_1 \cdot p_3 &= \frac{1}{2}\left(M^2 - m_{12}^2 - m_{23}^2 + m_2^2\right), && (3.4.4)\\ (p_1+p_3)^2 &= M^2 - m_{12}^2 - m_{23}^2 + m_1^2 + m_2^2 + m_3^2, && (3.4.5) \end{aligned}$$

with $p^2 = M^2$, $p_i^2 = m_i^2$.
Show that the kinematical boundaries for m_{12} are given by

$$(m_1 + m_2)^2 \le m_{12}^2 \le (M - m_3)^2 . \tag{3.4.6}$$

Show that for given m_{12} the kinematical boundaries for m_{23} are

$$(m_{23}^2)_{\min} \le m_{23}^2 \le (m_{23}^2)_{\max} \tag{3.4.7}$$

with

$$\begin{aligned} (m_{23}^2)_{\min} &:= (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2 , && (3.4.8)\\ (m_{23}^2)_{\max} &:= (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2 , && (3.4.9) \end{aligned}$$

where

$$E_2^* := \frac{m_{12}^2 - m_1^2 + m_2^2}{2\, m_{12}} \qquad \text{and} \qquad E_3^* := \frac{M^2 - m_{12}^2 - m_3^2}{2\, m_{12}} \tag{3.4.10}$$

are the energies of particles 2 and 3, respectively, in the frame defined by $\vec{p}_1 + \vec{p}_2 = 0$.

Calculate in the rest frame of the decaying particle the double-differential decay rate

$$\begin{aligned} \frac{d\Gamma}{dm_{12}^2\, dm_{23}^2} = \frac{1}{2M}\,|\mathcal{M}|^2 \int \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3\, 2E_i}\,(2\pi)^4\,\delta(p - p_1 - p_2 - p_3) \\ \times \delta(m_{12}^2 - (p_1+p_2)^2)\,\delta(m_{23}^2 - (p_2+p_3)^2)\,. \end{aligned} \tag{3.4.11}$$

You should find

$$\frac{d\Gamma}{dm_{12}^2\, dm_{23}^2} = \frac{1}{(2\pi)^3}\,\frac{1}{32\, M^3}\,|\mathcal{M}|^2 . \tag{3.4.12}$$

Further hints: The calculation of a three-body phase space

$$\int \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3\,2E_i}\,(2\pi)^4\,\delta(p-p_1-p_2-p_3) \\ \times\delta(m_{12}^2-(p_1+p_2)^2)\,\delta(m_{23}^2-(p_2+p_3)^2) \tag{3.4.13}$$

can be carried out by rewriting it to two times a two-body phase space. This is achieved by inserting

$$1 = \int d^4q\,\delta(q-p_1-p_2) \tag{3.4.14}$$

into (3.4.13) and then performing the integrations over p_1 and p_2:

$$\begin{aligned} & \int \frac{d^3p_1}{(2\pi)^3\,2E_1}\,\frac{d^3p_2}{(2\pi)^3\,2E_2}\,(2\pi)^4\,\delta(p-p_1-p_2-p_3) \\ & \qquad \times\delta(m_{12}^2-(p_1+p_2)^2)\,\delta(m_{23}^2-(p_2+p_3)^2)\,\delta(q-p_1-p_2) \\ = & \int \frac{d^3p_1}{(2\pi)^3\,2E_1}\,\frac{d^3p_2}{(2\pi)^3\,2E_2}\,(2\pi)^4\,\delta(p-q-p_3) \\ & \qquad \times\delta(m_{12}^2-q^2)\,\delta(m_{23}^2-(p_2+p_3)^2)\,\delta(q-p_1-p_2) \\ = & \,(2\pi)^4\,\delta(p-q-p_3)\,\delta(m_{12}^2-q^2) \\ & \times \int \frac{d^3p_1}{(2\pi)^3\,2E_1}\,\frac{d^3p_2}{(2\pi)^3\,2E_2}\,\delta(q-p_1-p_2)\,\delta(m_{23}^2-(p_2+p_3)^2)\,. \end{aligned} \tag{3.4.15}$$

If you formally replace $p_1 \to p_4$, $p_2 \to p_3$, $p_3 \to -p_1$, $m_{23}^2 \to t$ and $q \to p_1+p_2$ (i.e. $q^2 \to s$) this is exactly the type of mutiple integral which has been calculated in the lecture notes for the $2 \to 2$ scattering problem. Redo these calculations for the variables at hand. Finally you have to perform the integrations over p_3 and q. This is again similar to a two-body phase space. In an intermediate step it might be necessary to change the frame of reference. An energy E_i in the frame $\vec{q}=0$ can be written in a manifestly Lorentz invariant form as $E_i = (p_i\cdot q)/\sqrt{q^2}$. The corresponding modulus of the three-momentum is then given by $|\vec{p}_i| = \sqrt{E_i^2-m_i^2} = \sqrt{(p_i\cdot q)^2-q^2\,m_i^2}/\sqrt{q^2}$.

2. **Decay of the neutral pion:**

   Given the following Lagrangian for a photon field A_μ and a neutral-pion field π^0:

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial^\mu\pi^0\,\partial_\mu\pi^0 - \frac{1}{2}\,m_\pi^2\,(\pi^0)^2 - \frac{\alpha}{8\pi f}\,\varepsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}\,F_{\alpha\beta}\,\pi^0 \tag{3.4.16}$$

with the usual electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here $f \approx 92\,\mathrm{MeV}$ is the so-called pion-decay constant and $\alpha \approx 1/137$ the fine-structure constant of QED. Note: The structure and size of the interaction term in (3.4.16) is a prediction from chiral perturbation theory.

Calculate at tree level the decay width of the process $\pi^0 \to 2\gamma$. (It is common practice to denote photons by γ.) Compare your result to the experimental value given by the particle data group (web address: pdglive.lbl.gov/).
Note: For two indistinguishable particles in the final state (here two photons) the decay width is **one half** of the integrated differential decay width. Otherwise one would count each photon twice.

Show that the Lagrangian (3.4.16) is gauge invariant with respect to electromagnetic gauge transformations. (Hint: The pion is neutral. Its field is hermitian. Do not think too complicated.)

Show that the Lagrangian (3.4.16) possesses a parity invariance. How must the pion transform with respect to parity to obtain this invariance?

Hints: You can also find the mass of the neutral pion at the web page of the particle data group. The neutral pion decays mainly into two photons. Therefore, for practical purposes, the pion life time is the inverse of the decay width of $\pi^0 \to 2\gamma$.
The Levi-Civita tensor satisfies the equation

$$\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon_{\kappa\lambda\alpha\beta} = -2\,\left(g^\mu{}_\kappa\, g^\nu{}_\lambda - g^\mu{}_\lambda\, g^\nu{}_\kappa\right)\,. \tag{3.4.17}$$

3. **Decay of a neutral scalar particle:**

Given the following Lagrangian for a photon and the scalar meson f_0:

$$\mathcal{L}_s = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial^\mu f_0\,\partial_\mu f_0 - \frac{1}{2}\,m_{f_0}^2\,(f_0)^2 - k\,F_{\mu\nu}\,F^{\mu\nu}\,f_0\,. \tag{3.4.18}$$

Calculate at tree level the decay width of the process $f_0 \to 2\gamma$ for the scalar meson $f_0(980)$ (see the web site of the particle data group). Determine the coupling constant k such that the experimental decay width is reproduced.

Repeat the analysis concerning parity of the previous task.

Hint: The branching fraction for a specific decay branch is the corresponding width divided by the total width of the decaying particle.

4. **Decay of a vector meson:**

Given the following Lagrangian for a neutral pion π^0, a charged kaon $K^\pm$

and a charged strange vector meson $K^{*\pm}$:

$$\begin{aligned}\mathcal{L}_v &= -\frac{1}{2}\,K^{*+}_{\mu\nu}\,(K^{*-})^{\mu\nu} + m^2_{K^*}\,K^{*+}_{\mu}\,(K^{*-})^{\mu} + \partial^\mu K^+\,\partial_\mu K^- - m^2_K\,K^+\,K^- \\ &\quad + \frac{1}{2}\,\partial^\mu\pi^0\,\partial_\mu\pi^0 - \frac{1}{2}\,m^2_\pi\,(\pi^0)^2 + ig\,K^{*+}_\mu\,(\pi^0\,\partial^\mu K^- - K^-\,\partial^\mu\pi^0) \\ &\quad - ig\,K^{*-}_\mu\,(\pi^0\,\partial^\mu K^+ - K^+\,\partial^\mu\pi^0)\,. \qquad (3.4.19)\end{aligned}$$

with $K^{*+}_\mu := (K^{*-}_\mu)^\dagger$, $K^+ := (K^-)^\dagger$. The field strength of the vector meson is again given by $K^{*+}_{\mu\nu} = \partial_\mu K^{*+}_\nu - \partial_\nu K^{*+}_\mu$.

Calculate at tree level the decay width for the process $K^{*+} \to \pi^0 K^+$ in terms of the coupling constant g.

Show that the Lagrangian (3.4.19) is hermitian (if the coupling constant g is real-valued).

Show that the Lagrangian (3.4.19) is invariant with respect to charge conjugation

$$\begin{aligned}K^{*+}_\mu &\to -K^{*-}_\mu\,, \\ K^+ &\to K^-\,, \qquad (3.4.20)\\ \pi^0 &\to \pi^0\,.\end{aligned}$$

Suppose one would replace the first ig in (3.4.19) by g' and the second ig by $-g'$. Show that the new Lagrangian would still be hermitian if g' is real-valued, but that the invariance with respect to charge conjugation would be spoiled.

Hint: Note that the names of the particles in the interaction Lagrangian concern the states moving towards a vertex. The corresponding antiparticles formally move away from the vertex.

4 Quantum chromodynamics and its symmetries

4.0 Primer on special unitary groups

In the following we will find symmetries which amount to the mixing of n fields such that the overall strength of the amplitudes is not changed. Such transformations can be represented by unitary $n \times n$ matrices. The transformations form the group $\mathbf{U}(\mathrm{n})$. As already mentioned, a unitary group can be decomposed as

$$\mathbf{U}(n) = \mathbf{U}(1) \times \mathbf{SU}(n)\,. \qquad (4.0.1)$$

The special unitary group $\mathbf{SU}(n)$ consists of transformations which are unitary and have determinant 1:

$$U \in \mathbf{SU}(n) \qquad \Leftrightarrow \qquad U\,U^\dagger = \mathbb{1} \quad \text{and} \quad \det U = 1\,. \qquad (4.0.2)$$

The decomposition (4.0.1) is easily proven: Take any $V \in \mathbf{U}(n)$ and build

$$U := (\det V)^{-1/n}\, V \qquad \Leftrightarrow \qquad V = (\det V)^{1/n}\, U\,. \tag{4.0.3}$$

We will show that $(\det V)^{1/n} \in \mathbf{U}(1)$ and $U \in \mathbf{SU}(n)$. Since V is unitary one finds:

$$1 = \det(\mathbb{1}) = \det(V\,V^\dagger) = (\det V)\;(\det V)^* = |\det V|^2 \tag{4.0.4}$$

and therefore $\det V = e^{i\alpha}$ with some $\alpha \in \mathbb{R}$. Thus $\det V \in \mathbf{U}(1)$ and therefore also $(\det V)^{1/n} = e^{i\alpha/n} \in \mathbf{U}(1)$. The quantity U has the following properties:

$$U\,U^\dagger = (\det V)^{-1/n}\, V\,V^\dagger\, \left((\det V)^{-1/n}\right)^* = |\det V|^{-2/n}\, \mathbb{1} = \mathbb{1} \tag{4.0.5}$$

and

$$\det U = \det\left((\det V)^{-1/n}\, V\right) = \left((\det V)^{-1/n}\right)^n\, \det V = 1\,. \tag{4.0.6}$$

Therefore $U \in \mathbf{SU}(n)$ which completes the proof that any unitary matrix can be written as a product of a phase and a special unitary matrix. This decomposition (4.0.1) implies that one can discuss separately the symmetry properties associated with $\mathbf{U}(1)$ and the symmetry properties associated with $\mathbf{SU}(n)$. Therefore we restrict our considerations to special unitary groups.

One can show that any transformation $U \in \mathbf{SU}(n)$ can be represented as

$$U = e^{iA} \tag{4.0.7}$$

where A denotes a hermitian and traceless $n \times n$ matrix. The exponential of a matrix M is defined by

$$e^M := \sum_{n=0}^{\infty} \frac{1}{n!}\, M^n \qquad \text{with} \qquad M^0 := \mathbb{1}\,. \tag{4.0.8}$$

The hermiticity of A implies the unitarity of U and that A is traceless leads to a unit determinant for U:

$$U\,U^\dagger = e^{iA}\, e^{-iA^\dagger} = e^{iA}\, e^{-iA} = \mathbb{1} \tag{4.0.9}$$

The first and the last equality can be shown from the definition (4.0.8).

$$\det U = \det e^{iA} = e^{i\,\mathrm{tr} A} = e^0 = 1\,. \tag{4.0.10}$$

The relation between the determinant of an exponential and the exponential of the trace can be shown by diagonalizing A using unitary transformations.

If one wants to represent *all* special unitary matrices in the way of (4.0.7), one has to find a basis for the hermitian and traceless $n \times n$ matrices. It is easy

to see that there are $n^2 - 1$ such matrices which are linearly independent: An arbitrary complex $n \times n$ matrix has $2\,n^2$ independent quantities — the real and the imaginary part of every entry. Demanding hermiticity means that the diagonal entries must be real-valued (n independent quantities) and that the entries below the diagonal are fixed by the entries above. The number of entries above the diagonal is $\frac{n^2-n}{2}$. Counting real and imaginary parts we find $n^2 - n$ independent quantities for the off-diagonal entries. Thus in total a hermitian matrix has n^2 independent quantities. Therefore one would need n^2 linearly independent matrices to form a basis. The demand that the matrices in addition should be traceless reduces the number of basis matrices to $n^2 - 1$.

The representation (4.0.7) can now be rewritten as

$$U = e^{i\,c_a t_a} \tag{4.0.11}$$

with $c_a \in \mathbb{R}$, $a = 1, \ldots, n^2 - 1$ (and as always summation over a is implicitly assumed). The t_a are $n^2 - 1$ hermitian, traceless and linearly independent $n \times n$ matrices. One can choose them orthogonal and normalized by demanding, e.g.,

$$\mathrm{tr}(t^a\,t^b) = \frac{1}{2}\,\delta^{ab}\,. \tag{4.0.12}$$

Note that the indices are arbitrarily up or down.

Because of the property (4.0.11) one calls the t^a "generators" of the group $\mathbf{SU}(n)$. The commutator of two generators is again related to the generators (the generators form a "Lie algebra"):

$$[t_b, t_c] = i\,f_{abc}\,t_a \tag{4.0.13}$$

with the number-valued objects f_{abc} which are called "structure constants". The equation (4.0.13) expresses the fact that $i\,[t_b, t_c]$ is also hermitian and traceless and therefore a superposition of the basis elements t_a. It is easy to see that this is true. Using (4.0.12) and (4.0.13) it is easy to show that the structure constants are completely antisymmetric under exchanging any pair of indices,

$$f^{abc} = -f^{bac} = -f^{cba} = \ldots\,. \tag{4.0.14}$$

For QCD we will need two special unitary groups, namely $\mathbf{SU}(2)$ and $\mathbf{SU}(3)$. Explicit representations of the basis matrices and of the structure constants are

- for $\mathbf{SU}(2)$:

$$t^a = \frac{1}{2}\,\sigma^a\,, \qquad a = 1, 2, 3 \tag{4.0.15}$$

  with the Pauli matrices σ^a given in (2.1.55). On account of (2.1.56) the structure constants are given by the Levi-Civita symbol: $f^{abc} = \epsilon^{abc}$.

- for $\mathbf{SU}(3)$:

$$t^a = \frac{1}{2}\,\lambda^a\,, \qquad a = 1, \ldots, 8 \tag{4.0.16}$$

with the Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\,, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{4.0.17}$$

and the structure constants

$$\begin{aligned}
&f^{123} = 1\,, \qquad f^{458} = f^{678} = \frac{\sqrt{3}}{2}\,, \\
&f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}\,.
\end{aligned} \tag{4.0.18}$$

Any structure constant not fixed by the relations (4.0.14) and (4.0.18) vanishes.

As an application of all these considerations let us prove that

$$\mathrm{tr}\left(U^\dagger(x)\,\partial_\mu U(x)\right) = 0 \tag{4.0.19}$$

holds for an arbitrary special unitary matrix $U(x) \in \mathbf{SU}(n)$: As pointed out in (4.0.7), we can write $U(x) = e^{iA(x)}$ with A being hermitian and traceless. This

can be used to write

$$\begin{aligned}
\operatorname{tr}\left(U^\dagger\,\partial_\mu U\right) &= \operatorname{tr}\left(U^\dagger\,\partial_\mu \sum_{n=0}^{\infty}\frac{i^n}{n!}\,A^n\right) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\operatorname{tr}\left(U^\dagger\,\partial_\mu A^n\right)\\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}\operatorname{tr}\left(U^\dagger\sum_{m=0}^{n-1}A^m\,(\partial_\mu A)\,A^{n-m-1}\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\frac{i^n}{n!}\operatorname{tr}\left(A^{n-m-1}\,U^\dagger\,A^m\,\partial_\mu A\right)\\
&\overset{(*)}{=} \sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\frac{i^n}{n!}\operatorname{tr}\left(U^\dagger\,A^{n-m-1}\,A^m\,\partial_\mu A\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\frac{i^n}{n!}\operatorname{tr}\left(U^\dagger\,A^{n-1}\,\partial_\mu A\right) = \sum_{n=0}^{\infty}n\,\frac{i^n}{n!}\operatorname{tr}\left(U^\dagger\,A^{n-1}\,\partial_\mu A\right)\\
&= i\operatorname{tr}\left(U^\dagger\,U\,\partial_\mu A\right) = i\operatorname{tr}\left(\partial_\mu A\right) = 0
\end{aligned}\tag{4.0.20}$$

where in step $(*)$ we have used $A\,U^\dagger = A\,e^{-iA} = e^{-iA}\,A = U^\dagger\,A$. In the last step we have used that A is traceless.

4.1 Quantum chromodynamics

In its most compact version the Lagrangian of quantum chromodynamics (QCD) is given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \bar{\psi}\left(i\not{D} - \mathcal{M}_q\right)\psi\,. \tag{4.1.1}$$

Here all quark fields are encoded in ψ. It is an object with three indices, ψ_{cfs}, with the color index $c = 1,2,3$ (or "red", "green", "blue"), the flavor index $f = 1,\ldots,6$ (or "up", "down", "strange", "charm", "bottom", "top") and — as usual for fermions — the spinor index $s = 1,2,3,4$. The mass matrix $\mathcal{M}$ is a diagonal matrix in flavor space containing the masses of the different quarks, cf. table 1:

$$\mathcal{M}_q = \begin{pmatrix} m_u & 0 & 0 & 0 & 0 & 0\\ 0 & m_d & 0 & 0 & 0 & 0\\ 0 & 0 & m_s & 0 & 0 & 0\\ 0 & 0 & 0 & m_c & 0 & 0\\ 0 & 0 & 0 & 0 & m_b & 0\\ 0 & 0 & 0 & 0 & 0 & m_t \end{pmatrix}. \tag{4.1.2}$$

The gauge covariant derivative

$$D_\mu := \partial_\mu - igA_\mu \tag{4.1.3}$$

contains the gluon-field color matrix A_μ and the QCD coupling constant g. The field-strength color matrix is given by

$$F_{\mu\nu} := \frac{i}{g}\,[D_\mu, D_\nu]\,. \tag{4.1.4}$$

In (4.1.1) the trace over the color matrices is denoted by "tr".

The color matrix structure is disentangled from the field content by introducing $A_\mu = A^a_\mu\, t^a$, with $a = 1, \ldots, 8$ and the 3×3 matrices defined in (4.0.16), (4.0.17). The A^a_μ are the eight gluon fields. One can work out the corresponding field strength from the definition (4.1.4):

$$\begin{aligned} F_{\mu\nu} &= \frac{i}{g}\,[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] \\ &= \partial_\mu A^a_\nu\, t^a - \partial_\nu A^a_\mu\, t^a - ig\, A^b_\mu\, A^c_\nu\, [t^b, t^c] \\ &= \partial_\mu A^a_\nu\, t^a - \partial_\nu A^a_\mu\, t^a + g\, f^{abc}\, A^b_\mu\, A^c_\nu\, t^a \\ &= \underbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc}\, A^b_\mu\, A^c_\nu\right)}_{=:\, F^a_{\mu\nu}}\, t^a \end{aligned} \tag{4.1.5}$$

where we have used (4.0.13). This expression should be compared to the corresponding QED formula (2.3.4). The QCD field strength contains a term with two gluon fields (which involves the coupling constant g). Such a term is not present in QED. In the Lagrangian (4.1.1) such an additional term gives rise to interaction terms for three and for four gluons. In QED there are no photon-photon interactions on the elementary level of the Lagrangian. One can show in QCD perturbation theory that it is the gluon-gluon interaction which causes the effect of asymptotic freedom, cf. section 1.6.

Displaying all indices explicitly (and assuming as always a summation over all indices which appear twice) the QCD Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\rm QCD} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \\ &\quad + \bar\psi_{cfs}\left(i\,(\gamma_\mu)_{s\,s'}(\delta_{c\,c'}\,\partial^\mu - igA^\mu_a\,(t^a)_{c\,c'}) - \delta_{c\,c'}\,\delta_{s\,s'}\,m_f\right)\psi_{c'fs'}\,. \end{aligned} \tag{4.1.6}$$

Obviously the gluons do not discriminate between different flavors while the mass term does not discriminate between different colors (and spins). In other words, the interaction between two quarks as mediated by gluons has always the same strength no matter which flavors the quarks have. For a given flavor differently colored quarks have the same mass. Both effects are a consequence of the non-abelian local gauge symmetry to which we turn next.

4.2 Local gauge symmetry

QCD is invariant with respect to the following local continuous color transformations (cf. the corresponding QED case (2.3.19)):

$$\psi(x) \;\to\; U(x)\,\psi(x) \qquad \Rightarrow \qquad \bar\psi(x) \to \bar\psi(x)\,U^\dagger(x) \tag{4.2.1}$$

$$A^\mu(x) \;\to\; U(x)\left(A^\mu(x) + \frac{i}{g}\,\partial_x^\mu\right) U^\dagger(x) \tag{4.2.2}$$

with the special unitary 3×3 color matrices $U(x) \in \mathbf{SU}_c(3)$. In components the transformation property of the quark fields (4.2.1) reads

$$\psi_{cfs}(x) \;\to\; [U(x)]_{c\,c'}\,\psi_{c'fs}(x) \tag{4.2.3}$$

i.e. only the color structure is touched by the transformations, not the flavor or spin.

The proof that the QCD Lagrangian is indeed invariant with respect to the gauge transformations proceeds exactly along the lines of the corresponding proof for the gauge invariance of QED, cf. (2.3.19)-(2.3.24). From the transformation property (4.2.2) for the gluon fields one can show:

$$D^\mu(x) \to U(x)\,D^\mu(x)\,U^\dagger(x)\,. \tag{4.2.4}$$

This works in the same way as in (2.3.25). From that we can derive

$$F^{\mu\nu} = \frac{i}{g}\,[D^\mu, D^\nu] \to \frac{i}{g}\,[U\,D^\mu\,U^\dagger, U\,D^\nu\,U^\dagger] = \frac{i}{g}\,U\,[D^\mu, D^\nu]\,U^\dagger = U\,F^{\mu\nu}\,U^\dagger \tag{4.2.5}$$

and

$$\bar\psi\,\not{D}\,\psi \to \bar\psi\,U^\dagger\,U\,\not{D}\,U^\dagger\,U\,\psi = \bar\psi\,\not{D}\,\psi\,. \tag{4.2.6}$$

The gluon field-strength term in the QCD Lagrangian (4.1.1) becomes

$$\begin{aligned}\mathrm{tr}\,(F_{\mu\nu}\,F^{\mu\nu}) \to \mathrm{tr}\left(U\,F_{\mu\nu}\,U^\dagger\,U\,F^{\mu\nu}\,U^\dagger\right) &= \mathrm{tr}\left(F_{\mu\nu}\,U^\dagger\,U\,F^{\mu\nu}\,U^\dagger\,U\right) \\ &= \mathrm{tr}\,(F_{\mu\nu}\,F^{\mu\nu})\end{aligned} \tag{4.2.7}$$

where we have used the cyclicity of the trace.

Finally the quark mass term transforms as

$$\bar\psi_{cfs}\,\mathcal{M}_{ff'}\,\psi_{cf's} \to \bar\psi_{cfs}\,U^\dagger_{cc'}\,\mathcal{M}_{ff'}\,U_{c'c''}\psi_{c''f's} = \bar\psi_{cfs}\,\mathcal{M}_{ff'}\,\psi_{cf's}\,. \tag{4.2.8}$$

This completes the proof that the QCD Lagrangian is invariant under the local non-abelian[17] gauge transformations $U(x) \in \mathbf{SU}_c(3)$. QCD is called a non-abelian gauge theory.

[17] "Non-abelian" refers to the fact that two transformations do not commute: In general $U_1\,U_2 \neq U_2\,U_1$ for $U_{1,2} \in \mathbf{SU}(\mathrm{n})$.

This gauge symmetry has the following consequences: Of course, like in QED, the gauge bosons, here the gluons, cannot have a mass. On the other hand, the most notable effect of a massless carrier of a force is a long-range potential (like the Coulomb potential between two static sources). However, due to confinement, gluons cannot be exchanged over long distances. The strong interaction is short ranged in spite of the fact that gluons are massless.

More important is the fact that only gauge invariant objects can be observable. Therefore the quanta created or annihilated by the elementary fields ψ and A^a_μ are not observable. In principle, this also applies to QED, but there at least the field strength $F_{\mu\nu}$ is gauge invariant. For the electrons one can construct the object

$$\tilde{\psi}(t,\vec{r}) := \exp\left(ie\int d^3r' G(\vec{r}-\vec{r}')\,\vec{\nabla}\cdot\vec{A}(t,\vec{r}')\right)\psi(t,\vec{r}) \tag{4.2.9}$$

with the quantity $G(\vec{r}-\vec{r}')$ satisfying

$$\vec{\nabla}^2_r G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')\,. \tag{4.2.10}$$

From the gauge transformations (2.3.18) it is easy to see that this quantity is gauge invariant. The physical meaning is that it creates a positron (and annihilates an electron) together with the appropriate Coulomb field arround it. From elementary considerations in scattering theory one knows that one needs the Coulomb phase to appropriately describe a charged state even if it is asymptotically far away from all other particles. Exactly this Coulomb phase is created in addition to the fermion via the construction (4.2.9). In a non-abelian gauge theory this construction does not produce a gauge invariant quantity.

The demand that all observable quantities must be gauge invariant (one calls them "color white") explains naturally the appearance of quark-antiquark and three-quark states as physical objects (hadrons!). Indeed, for arbitrary flavor and spin structures the following combinations are gauge invariant:

$$\bar{\psi}_{cfs}(x)\,\psi_{cf's'}(x) \to \bar{\psi}_{cfs}(x)\,U^\dagger_{cc'}(x)\,U_{c'c''}(x)\,\psi_{c''f's'}(x) = \bar{\psi}_{cfs}(x)\,\psi_{cf's'}(x) \tag{4.2.11}$$

and

$$\begin{aligned}
&\epsilon_{cde}\,\psi_{cfs}(x)\,\psi_{df's'}(x)\,\psi_{ef''s''}(x)\\
&\quad\to \epsilon_{cde}\,U_{cc'}(x)\,U_{dd'}(x)\,U_{ee'}(x)\,\psi_{c'fs}(x)\,\psi_{d'f's'}(x)\,\psi_{e'f''s''}(x)\\
&\quad\quad= \epsilon_{c'd'e'}\,\psi_{c'fs}(x)\,\psi_{d'f's'}(x)\,\psi_{e'f''s''}(x)
\end{aligned} \tag{4.2.12}$$

where we have used the following property of 3×3 matrices:[18]

$$\epsilon_{cde}\,U_{cc'}\,U_{dd'}\,U_{ee'} = \det U\,\epsilon_{c'd'e'}\,. \tag{4.2.13}$$

[18]The proof is straightforward: Just write down the definition of the determinant and compare left and right hand side.

Assuming that involving more fields leads to a higher energy of the formed state it appears natural that the lightest states which we find in nature are structures of the types (4.2.11) and (4.2.12). It must be stressed that this is a plausibility argument and not at all a proof. We will see below that these structures (4.2.11) and (4.2.12) lead to particular combinations of quantum numbers. Interestingly, all states which have been found so far are in accordance with these combinations.

One might reformulate the confinement problem as the question: Why is it impossible to construct a white state from a single quark and infinitely many gluons? (A finite number of gluons would not lead to a gauge invariant structure.) So far we have no satisfactory answer to this question.

A further consequence of the non-abelian gauge invariance is the fact that there is only one universal coupling constant g. Indeed, the coupling between gluons which appears in the last line of (4.1.5) must be the same as the coupling between quarks and gluons which appears in (4.1.6). To say it more explicitly, the coupling of one flavor-type of quarks to gluons must be the same as the gluon-gluon coupling. Therefore all quark flavors couple with the same strength to gluons. Otherwise the theory would not be gauge invariant.

The same effect appears in the weak theory. There is one universal coupling constant between the W bosons and the fermion fields. The underlying non-abelian gauge symmetry is based on an $\mathbf{SU}(2)$ group. This is different, however, in QED: In principle, two electrically charged fields could have an arbitrary charge ratio. The point is that QED is an abelian gauge theory which implies that the photons do not couple to themselves on an elementary level.[19] The appearance of the coupling constant in the definition of the field strength is artificial in (2.3.4) as one can see by inspecting the last expression. In contrast, the appearing of the coupling constant in the QCD field strength (4.1.5) is real. On the other hand, in reality the ratios between the electric charges of the elementary particles are not arbitrary. All charges are integer multiples of the down-quark charge.[20] This suggests that there is a non-abelian symmetry in the physics of elementary particles which contains as a subgroup the $\mathbf{U}(1)$ gauge symmetry of QED. So far this non-abelian symmetry has not been isolated. The keyword here is "grand unified theory", one of the theoretical aims of particle physics. An interesting attempt which collected all known gauge symmetry groups from QCD ($\mathbf{SU}(3)$) and the electroweak theory ($\mathbf{SU}(2) \times \mathbf{U}(1)$) as subgroups of an $\mathbf{SU}(5)$ gauge group could explain the quantization of the electric charge, but leads to a prediction for a finite proton life time which has been disproven by experiment.

Let us come back to QCD: As a consequence of the universality of the coupling constant, QCD has only a small number of parameters: the quark masses and one coupling constant. This provides high predictive power and therefore the possibil-

[19] Electron loops can mediate photon-photon interactions, but there is no photon-photon coupling term in the QED Lagrangian.

[20] As a consequence the proton and the electron charge *exactly* cancel each other. Only this effects leads to the atomic physics and the chemistry of our world.

ity to test QCD. So far all these tests — carried out especially in the high-energy regime where the effective coupling strength of QCD is small — have confirmed that QCD is the correct theory of the strong interaction. Indeed, also this smallness of the effective coupling strength at high energies (asymptotic freedom) is a consequence of the non-abelian gauge symmetry. One can show that in four space-time dimensions only non-abelian gauge theories can lead to asymptotic freedom. Historically the experimental finding of quasi-free partons inside the proton (cf. section 1.5) and the theoretical finding that non-abelian local gauge theories provide asymptotic freedom have initiated the development of QCD as a quantum field theory. In addition, the symmetry considerations (concerning global symmetries) developed by analyzing the mass and reaction patterns in the hadron sector have isolated quarks as the building blocks of hadrons. Identifying partons with quarks has led to QCD as the theory of the strong interaction which contains the non-abelian local gauge symmetry and in addition several global symmetries. We now turn to these global symmetries which will provide us with selection rules and degenerate states.

4.3 Exact global symmetries

4.3.1 Baryon number conservation

The QCD Lagrangian (4.1.1) is invariant with respect to the global continuous transformations

$$\begin{aligned} \psi &\rightarrow U\,\psi = e^{i\alpha}\,\psi \\ A_\mu &\rightarrow A_\mu \end{aligned} \tag{4.3.1}$$

with $U \in \mathbf{U}(1)$, i.e. $\alpha \in \mathbb{R}$. Thus *all* quark fields (independent of flavor, color, and spin) obtain the same additional phase. From (4.3.1) one can also extract that the antiquarks obtain the opposite phase,

$$\bar\psi \rightarrow \bar\psi\, U^\dagger = \bar\psi\, e^{-i\alpha}\,. \tag{4.3.2}$$

According to Noether's theorem this continuous symmetry is connected to a conserved current,

$$j_B^\mu \sim \bar\psi\gamma^\mu\psi = \bar\psi_{cfs}(\gamma^\mu)_{ss'}\psi_{cfs'}\,. \tag{4.3.3}$$

It is fixed up to an overall constant coefficient which for historical reasons is chosen to be 1/3:

$$j_B^\mu := \frac{1}{3}\,\bar\psi\gamma^\mu\psi\,. \tag{4.3.4}$$

The associated conserved charge is called "baryon number". It is given by

$$B := \int d^3r\, j_B^0(t,\vec r) = \frac{1}{3}\int d^3r\,\psi^\dagger(t,\vec r)\,\psi(t,\vec r) = \frac{1}{3}\int d^3r\,\psi^\dagger_{cfs}(t,\vec r)\,\psi_{cfs}(t,\vec r)\,. \tag{4.3.5}$$

From the last expression we see that all quarks contribute in the same way. In section 2.3.1 we have derived a similar quantity, the electric charge (2.3.11). We have shown there that the operator of electric charge counts electrons minus positrons (times the charge $-e$). In the same way, the baryon number counts quarks minus antiquarks times the factor 1/3.

If one calculates the commutator between the baryon number operator (4.3.5) and the gauge invariant structures (4.2.11) and (4.2.12) — cf. (2.3.15) for the corresponding QED case — one finds that the quark-antiquark structure has baryon number 0 while a three-quark structure has baryon number 1 (annihilating three quarks produces, of course, -1). A three-quark structure is called "baryon" and a quark-antiquark structure is called "meson":

$$\bar{\psi}_{cfs}(x)\,\psi_{cf's'}(x) \qquad \text{meson,} \tag{4.3.6}$$

$$\epsilon_{cde}\,\psi_{cfs}(x)\,\psi_{df's'}(x)\,\psi_{ef''s''}(x) \qquad \text{baryon.} \tag{4.3.7}$$

Some mesons and baryons, in particular the lightest ones, are collected in the tables 2 and 3.

The conservation of baryon number forbids the decay of baryons into mesons, e.g.,

$$p \not\to \pi^+\,\pi^0\,, \tag{4.3.8}$$

but this is also forbidden from angular momentum conservation. The decay of nuclei with integer spin into mesons is not forbidden by angular momentum conservation. Here baryon number conservation is at work to guarantee the stability of atomic nuclei. For example, an alpha particle, i.e. a helium nucleus, cannot decay into pions,

$$\alpha \not\to \pi^+\,\pi^+\,\pi^0\,. \tag{4.3.9}$$

4.3.2 Flavor number conservation

Since each flavor of quark fields appears pairwise in the QCD Lagrangian (4.1.6) one can pick one individual flavor, e.g. strangeness, and change the phase of only these quarks:

$$\begin{aligned} u := \psi_1 &\to u \\ d := \psi_2 &\to d \\ s := \psi_3 &\to e^{i\alpha_s}\,s \\ c := \psi_4 &\to c \\ b := \psi_5 &\to b \\ t := \psi_6 &\to t \\ A_\mu &\to A_\mu \end{aligned} \tag{4.3.10}$$

with $\alpha_s \in \mathbb{R}$. Note that only the flavor index has been displayed explicitly for ψ_f. The strange antiquarks transform as

$$\bar{s} \to \bar{s}\, e^{-i\alpha_s} \tag{4.3.11}$$

while all other antiquark fields remain unchanged. The transformation (4.3.10), (4.3.11) leaves the QCD Lagrangian (4.1.6) invariant. The conserved current is defined by

$$j_S^\mu := -\bar{s}\,\gamma^\mu\, s = -\bar{\psi}_{c3t}\,\gamma^\mu\,\psi_{c3t} \tag{4.3.12}$$

and the corresponding charge, the "strangeness", is given by

$$S = \int d^3r\, j_S^0 = -\int d^3r\, s^\dagger s\,. \tag{4.3.13}$$

Here the historically motivated convention is such that strange antiquarks have strangeness $+1$, while it is -1 for strange quarks.

The conservation of strangeness allows the pairwise production of strange and antistrange quarks, because then the net strangeness is still zero. Therefore strange particles can be copiously produced in reactions based on the strong interaction, e.g.,

$$p + p \to p + \Lambda + K^+\,. \tag{4.3.14}$$

However, strangeness conservation forbids the decay of a hadron carrying strangeness into hadrons without strangeness, e.g.,

$$K^+ \not\to \pi^+\,\pi^0\,, \qquad \text{decay forbidden in strong interaction.} \tag{4.3.15}$$

While strangeness is conserved by the strong interaction, it is not by the weak interaction. Therefore hadrons with strangeness do decay, but they live much longer than hadrons which decay due to the strong interaction, cf. tables 2 and 3.

Since the strange quark hass been randomly chosen to be the one where the transformation (4.3.10) is non-trivial, one finds a conserved current and a corresponding charge for each flavor separately. The convention is such that quarks with positive/negative electric charge, cf. table 1, have positive/negative flavor number. Thus the conserved charges are "upness", $U := +\int d^3r\, u^\dagger u$, "downness", $D := -\int d^3r\, d^\dagger d$, "charmness", $Ch := +\int d^3r\, c^\dagger c$, "bottomness", $Bo := -\int d^3r\, b^\dagger b$, and "topness", $To := +\int d^3r\, t^\dagger t$.[21]

The meson and baryon structures in (4.3.6) and (4.3.7), respectively, give rise to particular combinations, e.g., of strangeness and electric charge: For example,

[21] We have used double letters for the latter three flavors since C is reserved for charge conjugation, B is already introduced as the baryon number and T is sometimes used in textbooks for isospin.

a meson of type (4.3.6) with positive strangeness can only have an electric charge of +1 or 0 (times the positron charge). From our present understanding nothing indicates that QCD would dictate that all hadronic structures must be of types (4.3.6) or (4.3.7). Nonetheless, all quantum number combinations of hadrons which have been found so far are in accordance with the assignments (4.3.6) and (4.3.7).

4.4 Approximate global symmetries

By inspecting table 1 one finds that two of the quark flavors, up and down, have very small masses — as compared to the masses of hadrons, cf. tables 2 and 3. In addition, the strange quark has a moderately small mass. We will explore in the following the consequences of the smallness of the masses of the first two/three flavors. For QCD the quark masses are just input parameters. Therefore the fact that some are small is accidental from the QCD point of view. In the electroweak theory the quarks get their mass from the interaction with the Higgs fields and the effect of spontaneous symetry breaking. But also in the electroweak theory one has one parameter per quark type to tune the respective quark mass. In principle, we would like to understand the origin of the quark masses and why they are so different. This is a field of active research of elementary particle physics. QCD can contribute to this question by providing the quark masses as accurate as possible. Indeed, the numbers given in table 1 for the light quarks are based in part on chiral perturbation theory, cf. section 6.

4.4.1 Isospin symmetry

Suppose we ignore the mass *difference* between up- and down-quark masses, i.e. we approximate the QCD Lagrangian (4.1.1) by

$$\mathcal{L}_{=,2} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=s,c,b,t} \bar{f}\left(i\not{D} - m_f\right) f + \bar{q}\,(i\not{D} - m_q)\,q \qquad (4.4.1)$$

with the two-component "isospinor"

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad (4.4.2)$$

and the averaged quark mass $m_q = (m_u + m_d)/2$. The index of the Lagrangian $\mathcal{L}_{=,2}$ refers to setting the two lightest quark masses equal to each other.

The Lagrangian (4.4.1) is invariant with respect to $\mathbf{SU}(2)$ "isospin" transfor-

mations:

$$\begin{aligned} q &\rightarrow U\,q \qquad \text{(in components:} \quad q_{cfs} \rightarrow U_{ff'}\,q_{cf's}) \\ s &\rightarrow s \\ c &\rightarrow c \\ b &\rightarrow b \\ t &\rightarrow t \\ A_\mu &\rightarrow A_\mu \end{aligned} \tag{4.4.3}$$

with $U \in \mathbf{SU}(2)$. As a consequence of (4.4.3) the adjoint isospinor $\bar{q}$ transforms as

$$\bar{q} \;\rightarrow\; \bar{q}\,U^\dagger\,. \tag{4.4.4}$$

Using the representation (4.0.11) for small numbers c_a one can derive the conserved Noether currents from the formalism developed in equations (2.2.40)-(2.2.43). One finds that each of the three currents

$$j_a^\mu := \bar{q}\,\gamma^\mu\,t_a\,q = \bar{q}_{cfs}\,(\gamma^\mu)_{ss'}\,(t_a)_{ff'}\,q_{cf's'}\,, \quad a = 1,2,3\,, \tag{4.4.5}$$

satisfies a continuity equation (2.2.23). The three 2×2 flavor matrices t_a are given by (4.0.15). The three conserved charges are the isospin operators

$$\begin{aligned} I_1 &:= \int d^3r\,q^\dagger\,t_1\,q = \frac{1}{2}\int d^3r\,(d^\dagger u + u^\dagger d)\,, \\ I_2 &:= \int d^3r\,q^\dagger\,t_2\,q = \frac{i}{2}\int d^3r\,(d^\dagger u - u^\dagger d)\,, \\ I_3 &:= \int d^3r\,q^\dagger\,t_3\,q = \frac{1}{2}\int d^3r\,(u^\dagger u - d^\dagger d)\,. \end{aligned} \tag{4.4.6}$$

They form the "isospin vector" $\vec{I} := (I_1, I_2, I_3)$. As a consequence of the conservation of $\vec{I}$, also $\vec{I}^{\,2}$ is conserved. From the definitions (4.4.6) and the quantization condition (2.1.16) it is easy to show that the following commutation relations hold

$$[I_a, I_b] = i\,\epsilon^{abc}\,I_c \tag{4.4.7}$$

with $a, b = 1, 2, 3$ (and summation over $c = 1, 2, 3$ is implicitly assumed). Therefore isospin has the same mathematical group structure as spin (angular momentum). This has caused the name "isospin". The knowledge which has been developed in quantum mechanics for spin can be immediately translated to isospin:

- The square, $\vec{I}^{\,2}$, of the isospin vector commutes with all its components. Therefore physical states can be constructed as eigen states of the three commuting observables energy, $\vec{I}^{\,2}$ and I_3.

- The eigen values of $\vec{I}^2$ are $I\,(I+1)$ where I must be integer or half-integer.
- The eigen values of I_3 range from $-I$ to I in unit steps, i.e. for a given I there are $2I+1$ states which form a multiplet.
- The "ladder" operators $I_\pm := I_1 \pm iI_2$ can be used to systematically construct all eigen states of $\vec{I}^2$ and I_3. Indeed, applying I_1 or I_2 or $I_\pm$ to an eigen state of $\vec{I}^2$ and I_3 leads in general to a state which is different from the original one (otherwise $I_{1,2}$ would commute with I_3). However, the new state has the same quantum number I *and the same energy*. (These are exactly the considerations presented in section 2.2.1.) Thus we expect degenerate states in the spectrum of the strongly interacting particles.

All the previous considerations about isospin concerned the Lagrangian (4.4.1) which is "only" an approximation to the QCD Lagrangian (4.1.1). In reality the up- and down-quark masses are different from each other. However, this difference is very small on a hadronic scale. Therefore we expect to find nearly degenerate states and this is exactly what is happening (cf. tables 2, 3): Proton and neutron are nearly degenerate in mass (isospin doublet). So are the neutral and the charged pions (isospin triplet).[22] Also the rho mesons form a triplet.

It is easy to see that the operator of strangeness, (4.3.13), commutes with all isospin operators (4.4.6). Therefore, states can be classified according to their energy, strangeness and isospin. The kaons with positive strangeness, K^+ and K^0, form an isospin doublet. They are also nearly degenerate in mass. The same is, of course, true for the corresponding antiparticles, K^- and $\bar{K}^0$, which have negative strangeness.

The isospin operators (4.4.6) also commute with charmness, bottomness, topness and with the baryon number operator (4.3.5). (The latter requires some calculations.) Actually only particular combinations of quantum numbers are allowed. To see this we express the electric charge in terms of the conserved "charges" of the strong interaction:

$$\begin{aligned} Q &= e\int d^3r\,\left(\frac{2}{3}\,u^\dagger u - \frac{1}{3}\,d^\dagger d - \frac{1}{3}\,s^\dagger s + \frac{2}{3}\,c^\dagger c - \frac{1}{3}\,b^\dagger b + \frac{2}{3}\,t^\dagger t\right) \\ &= e\int d^3r\,\left(\frac{1}{6}\,(u^\dagger u + d^\dagger d + s^\dagger s + c^\dagger c + b^\dagger b + t^\dagger t)\right. \\ &\qquad \left. + \frac{1}{2}\,(u^\dagger u - d^\dagger d) - \frac{1}{2}\,s^\dagger s + \frac{1}{2}\,c^\dagger c - \frac{1}{2}\,b^\dagger b + \frac{1}{2}\,t^\dagger t\right) \\ &= e\left(\frac{1}{2}\,B + I_3 + \frac{1}{2}\,S + \frac{1}{2}\,Ch + \frac{1}{2}\,Bo + \frac{1}{2}\,To\right)\,. \end{aligned} \tag{4.4.8}$$

[22] Note that the masses of the two charged pions are exactly equal to each other because of particle-antiparticle symmetry.

Suppose, e.g., we find a meson ($B = 0$) without strangeness, charmness, bottomness and topness. If it has a non-vanishing electric charge we conclude from (4.4.8) that the isospin is non-vanishing, too. But then, closeby in mass, there must be other states with different electric charges. Together they form an isospin multiplet. Obviously this provides a strong prediction of QCD: If one finds one state under the mentioned conditions there also must be others in the studied mass range. So far the isospin partners have always been found. Their absence would be a strong argument against QCD being the correct theory of the strong interaction. In turn, an isosinglet meson without strangeness, charmness, bottomness and topness must have vanishing electric charge. Such states have been found, e.g., the pseudoscalar eta meson and the vector meson ω.

The quark-antiquark and three-quark structures (4.2.11) and (4.2.12), respectively, restrict the combinations of quantum numbers even more: In the meson sector the isospin, I, cannot get bigger than one. Thus we expect only triplets, doublets and singlets in the meson sector. The triplets appear only if the quantum numbers of strangeness, charmness, bottomness and topness are zero. The doublets appear only if one of these quantum numbers is non-zero. In the baryon sector there are, e.g., only quartets and doublets in the absence of strangeness, charmness, bottomness and topness. If there is, e.g., one strange, charm, bottom or top quark together with two up or down quarks in the baryon structure (4.2.12), then there are only isospin singlets and triplets. Finding hadrons with quantum numbers which disagree with these asignments would point towards structures beyond the ones constructed in (4.2.11) and (4.2.12). One calls such structures "exotic". To find exotic hadrons is not necessarily in disagreement to QCD. We rather would learn about a new aspect of QCD. The search for hadrons with exotic quantum numbers is one of the challenges of hadron physics.

As one more example let us look at a quartet ($I = 3/2$) in the baryon sector, the Delta baryons. They have baryon number $+1$ and no strangeness or other quantum numbers from heavier quarks. Therefore, on acount of (4.4.8) there must be Delta baryons with electric charges -1, 0, $+1$ and $+2$ (times e), in complete accordance with experiment. Indeed, among the baryons constructed by (4.2.12) an electric charge $+2$ (and no heavy-quark content) is only possible within a quartet, i.e. for each such baryon with electric charge $+2$ there must be three other nearly degenerate states with electric charges -1, 0, and $+1$. This is exactly what has been experimentally observed.

There is one more interesting aspect related to the state Δ^{++}: It turns out that the Δ quartet has not only isospin 3/2 but also a spin of 3/2. Therefore there exists a Δ^{++} with magnetic quantum number (z-component of spin) $+3/2$. From the structure (4.2.12) this is most easily constructed by using three quarks, each with spin z-component of $+1/2$, i.e. spin up, $\uparrow$. On the other hand, the Δ^{++} contains three up quarks (now "up" denotes flavor, not spin). Therefore, the spin-isospin content of such a Δ^{++} state is $u_\uparrow(x)\, u_\uparrow(x)\, u_\uparrow(x)$ where the letter denotes the flavor up and the index denotes the spin up. Obviously this struc-

ture is completely symmetric in exchanging two (fermion!) fields. If this was the whole structure such a state would not exist. Historically, color was introduced exactly for the purpose to ensure the spin-statistics theorem. Indeed, the structure (4.2.12) becomes $\epsilon^{abc}\, u_{a\uparrow}(x)\, u_{b\uparrow}(x)\, u_{c\uparrow}(x)$. This quantity is completely antisymmetric in exchanging two fields and therefore the corresponding composite state does not vanish. After the finding that non-abelian gauge theories lead to asymptotic freedom, one was looking for the appropriate global symmetry which should be promoted to a local gauge symmetry. It was suggestive to use the color quantum number because its only purpose was to ensure the spin-statistics theorem, otherwise it was unobservable.

Back to isospin symmetry: As a consequence of the difference between the up- and down-quark mass, the quantities I_1 and I_2, and therefore also $\vec{I}^2$, are not exactly conserved in QCD. The third component, $I_3 = \frac{1}{2}(U + D)$, is fully conserved because the upness U and the downness D are exactly conserved, as shown in section 4.3.2. One consequence of the non-conservation of $\vec{I}^2$ is the fact that some physical hadrons are mixtures of states with different values for the isospin I. Since this mixing is caused by the quark-mass difference it is not very large. As an example one can look at the pions: They can only mix with isospin singlet states. But the latter have vanishing electric charge. Thus the charged pions do not mix; they are pure isospin-1 states. The physical state π^0 is dominantly the isospin partner of the charged states, but has a small admixture of an isosinglet state. The physical state η is dominantly an isosinglet state and has a small admixture of the electrically neutral isospin-1 state. The same statement applies to the vector-meson states ω and ρ^0.

In reality the mass difference between up and down quark is not the only noticable source of isospin breaking. Also the electromagnetic interaction breaks isospin since the electric charges of up and down quark are different. On a formal level the QED (not QCD!) interaction term for these quarks is given by

$$\begin{aligned} \mathcal{L}_{\text{int,QED,quarks}} &= -\frac{2}{3}\, e\, \bar{u}\, \gamma_\mu A^\mu\, u + \frac{1}{3}\, e\, \bar{d}\, \gamma_\mu A^\mu\, d \\ &= -\frac{1}{6}\, e\, \bar{q}\, \gamma_\mu A^\mu\, q - e\, \bar{q}\, \gamma_\mu A^\mu\, t_3\, q\,. \end{aligned} \tag{4.4.9}$$

The very last term, $\sim \bar{q}\, t_3\, q$, is not invariant with respect to the isospin transformations (4.4.3). Of course, the electromagnetic interaction between the quarks within the hadrons leads to attraction and repulsion effects which modify the masses of the composite objects. These effects have the same order of magnitude as the quark-mass difference. It is a hot topic of present-day research to disentangle the isospin breaking effects caused by the quark-mass difference from those caused by the electromagnetic interaction. It provides one of the challenges towards an accurate determination of the light quark masses.

4.4.2 Flavor SU(3) symmetry

In view of the success of isospin symmetry in explaining the qualitative pattern of the hadron spectrum (and countless reaction rates of hadrons and properties of nuclei which we have not discussed) we are now even more brave and neglect the mass differences between the lightest *three* quarks. Thus we approximate the QCD Lagrangian (4.1.1) by

$$\mathcal{L}_{=,3} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=c,b,t} \bar{f}\left(i\not{D} - m_f\right) f + \bar{q}\,(i\not{D} - m_q)\,q \qquad (4.4.10)$$

with the three-component object

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad (4.4.11)$$

and the averaged quark mass $m_q = (m_u + m_d + m_s)/3$. The index of the Lagrangian $\mathcal{L}_{=,3}$ refers to setting the three lightest quark masses equal to each other. Note that we use again the same letter q to collect the lightest quark flavors. One should be careful not to mix up (4.4.11) with (4.4.2). In the following we only use (4.4.11).

The Lagrangian (4.4.10) is invariant with respect to $\mathbf{SU}(3)$ flavor transformations:

$$\begin{aligned} q &\to U\,q \qquad \text{(in components:} \quad q_{cfs} \to U_{ff'}\,q_{cf's}) \\ c &\to c \\ b &\to b \\ t &\to t \\ A_\mu &\to A_\mu \end{aligned} \qquad (4.4.12)$$

with $U \in \mathbf{SU}(3)$. As a consequence of (4.4.12) the adjoint object $\bar{q}$ transforms as

$$\bar{q} \;\to\; \bar{q}\,U^\dagger\,. \qquad (4.4.13)$$

Note that these transformations concern flavor, not color. The mathematical group structure is formally the same, but the physics is different. On a formal level one can see the difference most easily by comparing the transformations displayed in components, i.e. compare (4.2.3) and (4.4.12).

Using the representation (4.0.11) for small numbers c_a one can derive the conserved Noether currents from the formalism developed in equations (2.2.40)-(2.2.43). One finds that each of the eight currents

$$j_a^\mu := \bar{q}\,\gamma^\mu\,t_a\,q = \bar{q}_{cfs}\,(\gamma^\mu)_{ss'}\,(t_a)_{ff'}\,q_{cf's'}\,, \quad a = 1,\ldots,8\,, \qquad (4.4.14)$$

satisfies a continuity equation (2.2.23). The eight 3×3 flavor matrices t_a are given by (4.0.16), (4.0.17). Note that the first three of them agree with the isospin currents (4.4.5). This is one aspect of the fact that the flavor **SU**(3) transformations contain as a subgroup the previously discussed isospin **SU**(2) transformations.

The eight conserved charges are

$$I_a := \int d^3r\, q^\dagger\, t_a\, q = \frac{1}{2} \int d^3r\, q^\dagger\, \lambda_a\, q\,. \tag{4.4.15}$$

In the following we will be mainly interested in the quantities

$$I_3 = \frac{1}{2} \int d^3r\, q^\dagger\, \lambda_3\, q = \frac{1}{2} \int d^3r\, (u^\dagger u - d^\dagger d) \tag{4.4.16}$$

and

$$I_8 = \frac{1}{2\sqrt{3}} \int d^3r\, (u^\dagger u + d^\dagger d - 2s^\dagger s)\,. \tag{4.4.17}$$

The first of them is again the isospin operator already introduced in (4.4.6). From the other quantity one constructs the hypercharge

$$\begin{aligned} Y &:= \frac{2}{\sqrt{3}}\, I_8 = \frac{1}{3} \int d^3r\, (u^\dagger u + d^\dagger d - 2s^\dagger s) \\ &= \frac{1}{3} \int d^3r\, (u^\dagger u + d^\dagger d + s^\dagger s) - \int d^3r\, s^\dagger s\,. \end{aligned} \tag{4.4.18}$$

Thus in the absence of heavier quarks (charm, bottom, top) one finds $Y = B + S$ with the baryon number B and the strangeness S. Obviously, I_3 and Y are conserved.[23]

The conserved flavor charges (4.4.15) satisfy the commutation relations

$$[I_a, I_b] = i f_{abc}\, I_c \tag{4.4.19}$$

which defines an **SU**(3) algebra. Like for the case of spin and isospin (cf. the items at page 93f) one can use the commutation relations (4.4.19) to construct multiplets and determine their properties. We do not go into details here (see, e.g., [Mosel]) but just collect the most important facts:

- Corresponding to $\vec{I}^{\,2}$ for the two-flavor case one can construct two operators which commute with all eight I_a and which characterize a given multiplet. These "Casimir" operators are not very much used in practical applications. Therefore we do not present them here explicitly.

[23] Even if the **SU**(3) flavor symmetry is only approximate.

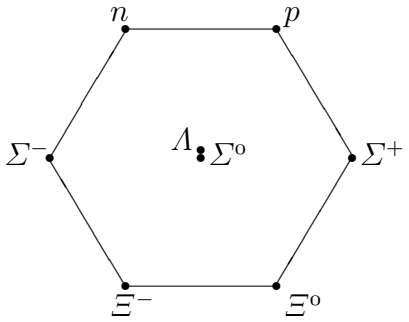
- The two flavor-charge operators I_3 and I_8 (or Y) commute with each other. One can construct physical states which are eigen states of the energy and the operators I_3 and Y (and of the Casimir operators).

- The other six flavor-charge operators can be used to construct the multiplets. It turns out that the lowest dimensional multiplets are singlets, triplets, sextets, octets and decuplets.[24]

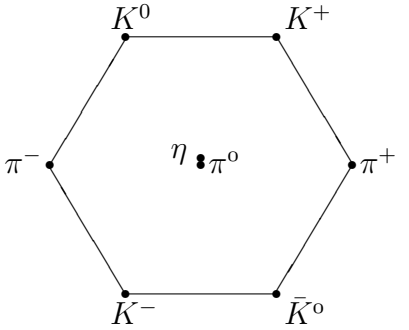
- Different eigen values for I_3 and Y are assigned to different members of one multiplet. If the **SU**(3) flavor symmetry was exact, all members of one multiplet would have the same mass.

The low-lying hadrons can be assigned to the following multiplets:

- baryon octet:

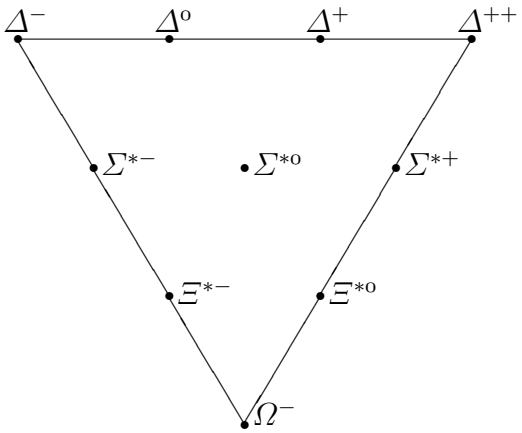


- meson octet:



- baryon decuplet:

[24] In **SU**(2) the dimensionalities of the multiplets are just all positive integer numbers, since for any $m \in \mathbb{N}$ one can find an integer or half-integer isospin I such that $2I + 1 = m$. **SU**(3) is more restrictive in that respect. There are, e.g., no doublets, quartets or quintets.



Historically, this classification has been introduced by Gell-Mann and coworkers. That time the Ω^- (the state at the bottom of the decuplet) has not been observed yet. Its existence has been predicted and later confirmed by experiments. Finding octets and decuplets has been interpreted as the consequence of an **SU**(3) structure, but the fundamental representations (triplets and antitriplets) have not been observed at that time.[25] The hypothetical members of the fundamental triplet were called "quarks". It has been suggested that the known hadrons are composed of these unobserved quarks. In the beginning it was unclear whether this was just a formal mathematical framework to explain the hadrons or whether the quarks were real.

We note in passing that also the sextets have never been observed as low-energy entities. While the octets and the decuplets (and singlets) emerge from the structures (4.2.11) and (4.2.12), a sextet contains two quarks. Also for a two-quark structure it is not so easy (impossible?) to construct a gauge invariant combination.

The members of the multiplets displayed above are not very close in mass. This could have been expected from the fact that the strange-quark mass is sizably different from the up- and down-quark masses. Nonetheless the approximate Lagrangian (4.4.10) can be used as a starting point for a perturbative expansion in powers of the quark-mass differences. In spite of the fact that one cannot determine the hadron masses easily from QCD, the approximate **SU**(3) symmetry provides predictions for the mass-splitting pattern within one multiplet by relating special combinations of hadron-mass differences to each other (see, e.g., [Mosel]). In that way, e.g., the mass of the Ω^- has been predicted.

[25] For spin (or isospin) this would be like finding spin-1, spin-3/2, ... structures but no spin-1/2 states.

4.5 Chiral symmetry

We have seen that it is very useful to study the equal-mass case for the two (or three) lightest flavors. On the other hand, the quark masses are not really the same: The up-quark mass is $m_u \approx 3\,\text{MeV}$ while the down quark has a mass of $m_d \approx 6\,\text{MeV}$. Thus, the difference of up- and down-quark mass, $\Delta m = m_d - m_u \approx 3\,\text{MeV}$, is *not* small compared to the absolute values of the quark masses. However, the mass difference, Δm, is very small compared to the masses of the hadrons which build up the isospin multiplets: The lightest hadrons are the pions which already have a mass of $\approx 140\,\text{MeV} \gg \Delta m$. As we will see below, the pions are actually unnaturally light for being hadrons. Therefore, the proper scale to compare to are the masses of ρ mesons or nucleons, i.e. a scale of about 1 GeV which is more than two orders of magnitude larger than the quark masses. Even for the strange quark there is still about one order of magnitude between the strange-quark mass and the hadrons which contain strange quarks.[26]

But let us stick to up and down quarks for the moment. We have seen that isospin multiplets do *not* emerge because up- and down-quark masses are similar on an absolute scale, but because both quark masses — and therefore also their difference — are *very small* on a hadronic scale. As a consequence of these considerations it it worth to study the limit of massless quarks. This approximation should not be worse than neglecting the quark-mass difference. As we will see next, QCD in the limit of massless quarks has even more symmetries.

We approximate the QCD Lagrangian (4.1.1) by

$$\mathcal{L}_{0,N_f} = -\frac{1}{2}\,\text{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=(s,)c,b,t} \bar{f}\left(i\not{D} - m_f\right) f + \bar{q}\,i\not{D}\,q \tag{4.5.1}$$

where we have neglected the masses of the lightest N_f flavors. In practice we are interested in $N_f = 2$ and $N_f = 3$. The field q is then given by (4.4.2) or (4.4.11), respectively. For $N_f = 2$ we expect to find symmetry properties comparable in quality to isospin. For $N_f = 3$ we expect to see significant deviations from our ideal massless limit but the pattern of these deviations might still remember the symmetries of the massless limit.

To study the symmetries of (4.5.1) it is useful to rewrite it first. To this end we make use of the projection operators on right- and left-handed states introduced in (2.1.96). Using their projection properties (2.1.97) we can rewrite the light-quark term of the Lagrangian (4.5.1):

$$\begin{aligned}\bar{q}\,i\not{D}\,q &= \bar{q}\,i\not{D}\,\mathbb{1}\,q = \bar{q}\,i\not{D}\,(P_R + P_L)\,q = \bar{q}\,i\not{D}\,P_R\,q + \bar{q}\,i\not{D}\,P_L\,q \\ &= \bar{q}\,i\not{D}\,P_R\,P_R\,q + \bar{q}\,i\not{D}\,P_L\,P_L\,q = \bar{q}\,i\not{D}\,P_R\,q_R + \bar{q}\,i\not{D}\,P_L\,q_L\,,\end{aligned} \tag{4.5.2}$$

where we have introduced the purely right- and left-handed fields, $q_R := P_R\,q$ and $q_L := P_L\,q$, respectively. Next we shall move the remaining projection operators

[26] Again, we exclude the lightest state, the kaon, from our considerations for reasons which will become clear below.

to the left. Recalling the anticommutation relation (2.1.50) between γ_5 and any of the four other gamma matrices one finds:

$$\begin{aligned} \bar{q}\, i\not{D}\, P_R\, q_R &= q^\dagger\, \gamma_0\, i\gamma_\mu D^\mu\, P_R\, q_R = q^\dagger\, \gamma_0\, P_L\, i\gamma_\mu D^\mu\, q_R = q^\dagger\, P_R\, \gamma_0\, i\gamma_\mu D^\mu\, q_R \\ &= \bar{q_R}\, i\not{D}\, q_R \end{aligned} \tag{4.5.3}$$

with $\bar{q_R} := q_R^\dagger\, \gamma_0$. In the same way one gets

$$\bar{q}\, i\not{D}\, P_L\, q_L = \bar{q_L}\, i\not{D}\, q_L \tag{4.5.4}$$

and therefore the Lagrangian of massless fermions splits up in a purely right-handed and a purely left-handed part:

$$\bar{q}\, i\not{D}\, q = \bar{q_R}\, i\not{D}\, q_R + \bar{q_L}\, i\not{D}\, q_L\,. \tag{4.5.5}$$

The approximation (4.5.1) to the QCD Lagrangian, which now obtains the form

$$\mathcal{L}_{0,N_f} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=(s,)c,b,t} \bar{f}\left(i\not{D} - m_f\right) f + \bar{q_R}\, i\not{D}\, q_R + \bar{q_L}\, i\not{D}\, q_L\,, \tag{4.5.6}$$

is invariant with respect to flavor transformations which can act *independently* on left- and right-handed quarks: The transformations[27]

$$\begin{array}{rcl} q_R & \to & U_R\, q_R \\ q_L & \to & U_L\, q_L \end{array} \tag{4.5.7}$$

with $U_R, U_L \in \mathbf{SU}(N_f)$ do not change (4.5.6). For obvious reasons one calls U_R a right-handed transformation and U_L a left-handed transformation. Note that special versions of (4.5.7) are

$$\begin{array}{rcl} q_R & \to & U_R\, q_R \\ q_L & \to & q_L \end{array} \tag{4.5.8}$$

where we have chosen $U_L = \mathbb{1}$ and

$$\begin{array}{rcl} q_R & \to & q_R \\ q_L & \to & U_L\, q_L \end{array} \tag{4.5.9}$$

where we have chosen $U_R = \mathbb{1}$. On the other hand, the invariance with respect to (4.5.8) *and* with respect to (4.5.9) is equivalent to the invariance with respect to (4.5.7). Mathematically we have a direct product of the two flavor-transformation groups. All this can be summarized by the statement that the

[27] For simplicity we do not display that all the heavy-quark fields and also the gluons remain unchanged.

Lagrangian (4.5.6) is invariant with respect to flavor transformations of the "chiral" group $\mathbf{SU}_L(N_f)\times \mathbf{SU}_R(N_f)$. The indices just indicate on which quarks (left- or right-handed) the respective flavor transformation is applied.

The chiral group can also be rewritten in a different way which makes contact with the isospin or flavor transformations which we have introduced before. Indeed, one special version of (4.5.7) is the choice $U_R = U_L =: U_V$ which just yields

$$\left.\begin{array}{rcl} q_R & \to & U_V\, q_R \\ q_L & \to & U_V\, q_L \end{array}\right\} \qquad \Leftrightarrow \qquad q \to U_V\, q\,. \tag{4.5.10}$$

We will motivate the index V in a moment. Obviously, (4.5.10) is nothing but the flavor transformation (4.4.3) or (4.4.12) which we have found before for the case of equal masses. Of course, our case of massless quarks is one special case of having quarks with the same mass. So it is not surprising that we recover the symmetries which we have identified before. However, we now have a richer symmetry structure. The additional symmetry which one does not have for equal, but non-vanishing quark masses is recovered by choosing

$$U_R = U_L^\dagger =: U\,. \tag{4.5.11}$$

Now we use (4.0.7), i.e. the fact that every special unitary matrix U can be expressed via a hermitian and traceless matrix B as $U = e^{iB}$. In addition, we introduce the matrix $U_A := e^{iB\gamma_5}$. Also the index A will be motivated later. Note that U_A is a matrix which acts on flavor *and* spinor indices. Choosing (4.5.11) for (4.5.7) we find

$$\left.\begin{array}{rcl} q_R & \to & U_R\, q_R = U\, q_R = e^{iB}\, q_R \stackrel{*}{=} P_R\, e^{iB\gamma_5}\, q = P_R\, U_A\, q \\ q_L & \to & U_L\, q_L = U^\dagger\, q_L = e^{-iB}\, q_L \stackrel{*}{=} P_L\, e^{iB\gamma_5}\, q = P_L\, U_A\, q \end{array}\right\} \quad \Leftrightarrow \quad q \to U_A\, q\,. \tag{4.5.12}$$

The last transformation reads in components

$$q_{cfs} \to (U_A)_{fsf's'}\, q_{cf's'} \tag{4.5.13}$$

with

$$(U_A)_{fsf's'} = \left(e^{iB\gamma_5}\right)_{fsf's'} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\, (B^n)_{ff'}\, (\gamma_5^n)_{ss'}\,, \tag{4.5.14}$$

i.e. the transformation U_A mixes flavors and spinor components.

So far we have not shown how one can bring the γ_5 into the game in (4.5.12), i.e. we have not proven the equalities denoted by *. We can use again the Taylor expansion of the exponential together with

$$P_{R,L}\, \gamma_5^{2m} = P_{R,L}\, \mathbb{1} = P_{R,L}\,, \tag{4.5.15}$$

$$P_{R,L}\, \gamma_5^{2m+1} = P_{R,L}\, \gamma_5 = \pm P_{R,L}\,, \tag{4.5.16}$$

for any $m \in \mathbb{N}_0$. The upper sign of $\pm$ refers to P_R and the lower sign to P_L. These relations can be written in a compact form as

$$P_{R,L}\,\gamma_5^n = P_{R,L}\,(\pm 1)^n\,, \tag{4.5.17}$$

for any $n \in \mathbb{N}_0$. In this way we obtain

$$P_{R,L}\,e^{iB\gamma_5} = P_{R,L}\sum_{n=0}^{\infty}\frac{1}{n!}\,(iB\gamma_5)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\,(\pm iB)^n\,P_{R,L} = e^{\pm iB}\,P_{R,L}\,. \tag{4.5.18}$$

These two relations (the one for P_R with the upper sign and the one for P_L with the lower sign) are used in (4.5.12) to produce the compact transformation rule on the right hand side with the matrix U_A.

The invariance of (4.5.6) with respect to the independent flavor(-spinor) transformations (4.5.10) and (4.5.12) can be expressed as an invariance with respect to the direct group $\mathbf{SU}_V(N_f)\times\mathbf{SU}_A(N_f)$. The transformations U_V and U_A can be obtained from the purely right-handed and purely left-handed transformations U_R and U_L, respectively, and vice versa. Mathematically this means $\mathbf{SU}_L(N_f)\times\mathbf{SU}_R(N_f) = \mathbf{SU}_V(N_f)\times\mathbf{SU}_A(N_f)$.

Finally we have to motivate the indices V and A: Applying Noether's theorem, expressed by the equations (2.2.40)-(2.2.43), to the transformation (4.5.12) one finds the conserved currents

$$(j_A)^\mu_a := \bar{q}\,\gamma^\mu\gamma_5\,t_a\,q = \bar{q}_{cfs}\,(\gamma^\mu\gamma_5)_{ss'}\,(t_a)_{ff'}\,q_{cf's'}\,,\quad a = 1,\ldots,N_f^2-1\,, \tag{4.5.19}$$

where t_a denote the generators of the special unitary group $\mathbf{SU}(N_f)$. The conserved currents for the transformations (4.5.10) have already been determined in the previous subsections. They are given by

$$(j_V)^\mu_a := \bar{q}\,\gamma^\mu\,t_a\,q = \bar{q}_{cfs}\,(\gamma^\mu)_{ss'}\,(t_a)_{ff'}\,q_{cf's'}\,,\quad a = 1,\ldots,N_f^2-1\,. \tag{4.5.20}$$

The currents $(j_V)^\mu_a$ transform like (four-)vectors with respect to Lorentz transformations. In particular, this implies that the spatial components change their sign with respect to a parity transformation while the zeroth component remains unchanged.[28] The currents $(j_A)^\mu_a$ transform like axial-vectors. With respect to parity the spatial components remain unchanged.[29] The zeroth component changes sign.[30] This will become important in a moment when we discuss the expected

[28] Just imagine how the four-momentum would change with respect to a parity transformation.

[29] The prototype of a three-dimensional classical axial-vector is the angular momentum, $\vec{L} = \vec{r}\times\vec{p}$. Obviously, $\vec{L}$ remains unchanged with respect to a parity transformation, because *both* $\vec{r}$ and $\vec{p}$ flip sign.

[30] This fact might be unintuitive. However, it can be motivated easily: The product of a (four-)vector and an axial-vector should have a well-defined behavior with respect to Lorentz transformations. It transforms as a scalar with respect to rotations and boosts. With respect to parity it should flip its sign. Anything else would not fit together with the behavior of the spatial components. To achieve this the zeroth component of an axial-vector has to flip sign, because the zeroth component of the vector does not. The product of a vector and an axial-vector transforms like a pseudoscalar.

multiplet structure. Since the corresponding conserved current behaves like a vector, one calls the transformations (4.5.10) of the flavor group $\mathbf{SU}_V(N_f)$ "vector flavor transformations". Correspondingly, the transformations (4.5.12) are called "axial-vector flavor transformations".

Of course, one can also construct conserved currents for the purely right- or purely left-handed transformations, (4.5.8) or (4.5.9), respectively. Apart from an arbitrary normalization they are just given by the sum or difference of the vector and axial-vector flavor currents:

$$(j_R)^\mu_a := \bar{q_R}\,\gamma^\mu\,t_a\,q_R = \bar{q}\,\gamma^\mu\,t_a\,P_R\,q = \frac{1}{2}\,(j_V)^\mu_a + \frac{1}{2}\,(j_A)^\mu_a \tag{4.5.21}$$

and

$$(j_L)^\mu_a := \bar{q_L}\,\gamma^\mu\,t_a\,q_L = \bar{q}\,\gamma^\mu\,t_a\,P_L\,q = \frac{1}{2}\,(j_V)^\mu_a - \frac{1}{2}\,(j_A)^\mu_a \tag{4.5.22}$$

with $a = 1, \ldots, N_f^2 - 1$. Obviously, conservation of $(j_V)^\mu_a$ *and* of $(j_A)^\mu_a$ implies conservation of $(j_L)^\mu_a$ *and* of $(j_R)^\mu_a$, and vice versa. This is another manifestation of $\mathbf{SU}_L(N_f) \times \mathbf{SU}_R(N_f) = \mathbf{SU}_V(N_f) \times \mathbf{SU}_A(N_f)$.

Now, let's see which degeneracies are implied by this chiral flavor group. We have already analyzed the consequences of the (approximate) invariance with respect to $\mathbf{SU}_V(N_f)$ in the previous subsections. Therefore, we can concentrate on $\mathbf{SU}_A(N_f)$. For $N_f = 2$ we have found hadron multiplets of $\mathbf{SU}_V(N_f)$ which deviate in their masses by only a few MeV. Thus, we expect the same quality concerning the multiplets which emerge from the full group $\mathbf{SU}_V(2) \times \mathbf{SU}_A(2)$. We will not work out the full multiplet structure for a reason which will become clear in a moment. Instead, let us just construct a state along the line of reasoning presented in (2.2.4)-(2.2.7) and analyze its properties: The charges which emerge, e.g. for $N_f = 2$, from the currents (4.5.20) and (4.5.19),

$$(I_V)_a := \int d^3r\,(j_V)^0_a(t,\vec{r}) = \int d^3r\,q^\dagger\,t_a\,q\,, \quad a = 1, \ldots, 3\,, \tag{4.5.23}$$

and

$$(I_A)_a := \int d^3r\,(j_A)^0_a(t,\vec{r}) = \int d^3r\,q^\dagger\,\gamma_5\,t_a\,q\,, \quad a = 1, \ldots, 3\,, \tag{4.5.24}$$

are (approximately) conserved in QCD, i.e. they commute with the Hamiltonian $H_{0,2}$ which emerges from (4.5.6) for $N_f = 2$. On the other hand, one can easily check that $(I_V)_a$ and $(I_A)_b$ in general do not commute, $a, b = 1, 2, 3$. We have already seen that we can classify hadrons with respect to the isospin operators $\vec{I}^2_V := \sum_a (I_V)_a\,(I_V)_a$ and $(I_V)_3$. Since these isospin operators have a well-defined behavior with respect to parity (they are invariant), one can classify the hadrons simultaneously concerning their isospin and parity (and, of course, energy/mass).

Suppose we have such a hadronic state, $|h\rangle$, with energy E_h, i.e. $H_{0,2}\,|h\rangle = E_h\,|h\rangle$. Then, the state $(I_A)_a\,|h\rangle$ is in general (non-vanishing and) different from the original state $|h\rangle$, because $(I_A)_a$ does not commute with the isospin operators. However, the new state has the same energy as the original one:

$$H_{0,2}\,(I_A)_a\,|h\rangle = (I_A)_a\,H_{0,2}\,|h\rangle = (I_A)_a\,E_h\,|h\rangle = E_h\,(I_A)_a\,|h\rangle\,. \qquad (4.5.25)$$

For QCD this equality is only approximate, but we would expect it to be accurate on the level of a few MeV. Now comes the crucial point: The operator $(I_A)_a$ changes to its negative with respect to a parity transformation, because it is constructed from the zeroth component of an axial-vector current. Therefore, the state $(I_A)_a\,|h\rangle$ has *opposite* parity from the state $|h\rangle$. Consequently, we expect to find *parity doublets*, i.e. states with approximately the same energy but different parities. However, it seems that such states do not exist: There are no mesons in the mass region of the pions (about 140 MeV) which have positive parity.[31] There are also no other baryons in the mass region of the nucleons (about 1 GeV) which have negative parity. The lightest meson with positive parity is the σ meson with a mass of about 600 MeV. The lightest baryon with negative parity is the $N^*(1520)$ with a mass of about 1520 MeV (hence the name) [PDG]. The mass deviations between these supposed-to-be parity doublets is on the order of their respective masses and not a small correction.

So something is wrong in our line of reasoning. Possible explanations, are, of course, that we just have not found the corresponding hadrons or that we just have no clue at all what QCD is about. Since we have not found *any* parity doublet we exclude the first possibility. Indeed, there are good experimental reasons to believe that we have not just missed to find the degenerate parity partners. Our present-day understanding is *not* that we simply do not understand QCD. Instead, there is a new physical effect at work here. Recovering this effect will turn the present disappointment into a success and actually open a new fascinating theoretical playground (which has been honored by the Nobel prize to Y. Nambu in 2008). What we observe, isospin partners but no parity partners, can be explained by the phenomenon of spontaneous symmetry breaking. We will understand why the parity partners are not needed (at the single-hadron level) and, even more, why the pions are so much lighter than the other hadrons.

4.6 Exercises

1. **Lie algebras:**

   (a) Show for arbitrary $n \times n$ matrices A, B, C:

   $$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0\,. \qquad (4.6.1)$$

   Hint: Use the definition of the commutator.

[31] The pions have negative parity.

(b) Given an SU(n) group with generators t_a and the completely antisymmetric structure constants f_{abc} introduced via

$$[t_a, t_b] = i\, f_{abc}\, t_c\,, \tag{4.6.2}$$

with $a, b, c = 1, \ldots, n^2 - 1$. (Einstein's sum convention is applied throughout.)
Show that the structure constants satisfy

$$f_{abe}\, f_{cde} + f_{ace}\, f_{dbe} + f_{ade}\, f_{bce} = 0\,. \tag{4.6.3}$$

Hint: Use $A = t_a$, $B = t_b$, $C = t_c$ in (4.6.1) together with (4.6.2).

(c) Show that the $(n^2 - 1) \times (n^2 - 1)$ matrices T_a with components

$$(T_a)_{bc} := -i\, f_{abc} \tag{4.6.4}$$

satisfy the Lie algebra (4.6.2), i.e.

$$[T_a, T_b] = i\, f_{abc}\, T_c\,. \tag{4.6.5}$$

Hint: Write down (4.6.5) in components and use (4.6.3).

(d) Consider an SU(n) gauge theory where the $n^2 - 1$ gauge fields A^μ_a, $a = 1, \ldots, n^2 - 1$ transform according to

$$A^\mu(x) \to U(x) \left(A^\mu(x) + \frac{i}{g}\, \partial^\mu_x \right) U^\dagger(x)\,. \tag{4.6.6}$$

Here the $n \times n$ matrix $U(x)$ is an element of SU(n), i.e.

$$U(x) = e^{i\beta_a(x)\, t_a} \tag{4.6.7}$$

with $\beta_a(x) \in \mathbb{R}$ and the $n \times n$ matrices t_a denoting the generators of SU(n), which satisfy (4.6.2). The gauge fields are collected in the (traceless) $n \times n$ matrix $A^\mu(x) := A^\mu_a(x)\, t_a$.
The field strengths $F^{\mu\nu}_a$ related to the gauge fields via

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g\, f_{abc}\, A^\mu_b\, A^\nu_c \tag{4.6.8}$$

can be collected in a (traceless) $n \times n$ matrix $F^{\mu\nu} := F^{\mu\nu}_a\, t_a$ or in an $n^2 - 1$ component vector $\mathcal{F}^{\mu\nu}$ (with components $(\mathcal{F}^{\mu\nu})_a := F^{\mu\nu}_a$).
Show that the following transformation properties hold:

$$F^{\mu\nu}(x) \to U(x)\, F^{\mu\nu}(x)\, U^\dagger(x) \tag{4.6.9}$$

and

$$\mathcal{F}^{\mu\nu}(x) \to \mathcal{U}(x)\, \mathcal{F}^{\mu\nu}(x) \tag{4.6.10}$$

with $U(x)$ given in (4.6.7),

$$\mathcal{U}(x) := e^{i\beta_a(x)\,T_a} \tag{4.6.11}$$

and the $(n^2-1)\times(n^2-1)$ matrices T_a defined in (4.6.4).

Hint: To prove (4.6.10) it is sufficient to consider infinitesimal transformations (you do not need to prove that this is sufficient). Write down (4.6.9) and (4.6.10) for infinitesimal transformations in components and compare.

2. **Mass splitting in meson multiplets:**
(following the footsteps of Gell-Mann and Okubo)

Given a free Lagrangian for vector and pseudoscalar mesons

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\,\mathrm{tr}\,(\partial_\mu\Phi\,\partial^\mu\Phi) - \frac{1}{4}\,\mathrm{tr}\,(\mathcal{M}_p\,\Phi\,\Phi) \\ &\quad - \frac{1}{8}\,\mathrm{tr}\,(V_{\mu\nu}\,V^{\mu\nu}) + \frac{1}{4}\,\mathrm{tr}\,(\mathcal{M}_V\,V_\mu\,V^\mu)\ .\end{aligned} \tag{4.6.12}$$

The meson fields are collected in flavor matrices. The eight lightest pseudoscalar mesons are contained in

$$\Phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta \end{pmatrix} \tag{4.6.13}$$

and the nine lightest vector mesons in

$$V_\mu = \begin{pmatrix} \rho^0_\mu + \omega_\mu & \sqrt{2}\rho^+_\mu & \sqrt{2}K^{*+}_\mu \\ \sqrt{2}\rho^-_\mu & -\rho^0_\mu + \omega_\mu & \sqrt{2}K^{*0}_\mu \\ \sqrt{2}K^{*-}_\mu & \sqrt{2}\bar{K}^{*0}_\mu & \sqrt{2}\varphi_\mu \end{pmatrix}\ . \tag{4.6.14}$$

These states are made out of a quark — u, d, s — and an antiquark — $\bar{u}$, $\bar{d}$, $\bar{s}$. One can think about the matrices (4.6.13), (4.6.14) as obtained from a direct product

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes (\bar{u}, \bar{d}, \bar{s}) = \begin{pmatrix} u\,\bar{u} & u\,\bar{d} & u\,\bar{s} \\ d\,\bar{u} & d\,\bar{d} & d\,\bar{s} \\ s\,\bar{u} & s\,\bar{d} & s\,\bar{s} \end{pmatrix}\ . \tag{4.6.15}$$

For instance, the K^+ consists of a u quark and a strange antiquark $\bar{s}$.

The matrix of pseudoscalar fields in (4.6.13) is hermitian. This means that the fields π^0 and η are hermitian while the fields with electric charge and/or strangeness are adjoint to each other: $(\pi^-)^\dagger =: \pi^+$, $(K^-)^\dagger =: K^+$, $(K^0)^\dagger =: \bar{K}^0$. The same remark applies to the matrix of vector fields

in (4.6.14): The fields ρ^0, ω and φ are hermitian while the fields with electric charge and/or strangeness are adjoint to each other: $(\rho^-)^\dagger =: \rho^+$, $(K^{*-})^\dagger =: K^{*+}$, $(K^{*0})^\dagger =: \bar{K}^{*0}$. The field strength of the massive vector fields is given by $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$.

The mass matrices are given by

$$\mathcal{M}_p := 2\,B\,\mathcal{M}_q \tag{4.6.16}$$

and

$$\mathcal{M}_V := m_V^2\,\mathbb{1} + c_m\,\mathcal{M}_q \tag{4.6.17}$$

with the quark-mass matrix

$$\mathcal{M}_q = \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{4.6.18}$$

and three constants m_V, B and c_m (which all have the dimension of mass).

For the present purpose isospin breaking is ignored. Therefore m_u and m_d have been replaced by a common m_q in (4.6.18). If all quark masses were zero, the vector mesons still would have a mass. All vector mesons would have the same mass m_V (you will see this in the following by explicit calculation). The pseudoscalar mesons would be massless (this will be motivated later in the lectures). The non-vanishing quark masses split up the masses of different states in a multiplet. This mass splitting provided by the quark-mass matrix has been included in (4.6.16) and (4.6.17) in linear order — if the quark masses are small, the quadratic order should be small and has been neglected.

Finally "tr" denotes the trace in (4.6.12). In principle, one could imagine to add a term with a double trace like $\mathrm{tr}(V_\mu)\,\mathrm{tr}(V^\mu)$. Phenomenologically it turns out that this is not necessary.

After this long introduction here is the task: Calculate from (4.6.12) the masses for the pseudoscalar states (kaons, pions, eta). You have two independent parameters: $B\,m_q$ and $B\,m_s$. Determine these two parameters from two of your mass relations. Use the rest of your mass relations to derive equations which only contain the meson masses.

Calculate from (4.6.12) the masses for the vector states (rho, omega, K-star, phi). You have two more parameters: m_V and $c_m\,m_q$. (The product $c_m\,m_s$ can be obtained from the other four parameters.) Determine these two parameters from two of your mass relations. Use the rest of your mass relations to derive equations which only contain the meson masses.

Check how well the obtained relations are fulfilled phenomenologically.

Hint: You should find

$$
\begin{aligned}
&m_\pi := m_{\pi^0} = m_{\pi^\pm} \qquad \text{(isospin symmetry)},\\
&m_K := m_{K^\pm} = m_{K^0} \qquad \text{(isospin symmetry)},\\
&m_\eta^2 = \frac{4}{3}\, m_K^2 - \frac{1}{3}\, m_\pi^2 \,,\\
&\\
&m_\rho := m_{\rho^0} = m_{\rho^\pm} \qquad \text{(isospin symmetry)},\\
&m_{K^*} := m_{K^{*\pm}} = m_{K^{*0}} \qquad \text{(isospin symmetry)},\\
&m_\omega = m_\rho \,,\\
&m_{K^*}^2 = \frac{1}{2}\, \left(m_\rho^2 + m_\varphi^2\right) \,.
\end{aligned}
\tag{4.6.19}
$$

3. **Symmetries of hadrons:**

Consider the following phenomenological Lagrangian for vector mesons and pseudoscalar mesons:

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{4}\,\mathrm{tr}\,(\partial_\mu \Phi\, \partial^\mu \Phi) - \frac{1}{4}\,\mathrm{tr}\,(\mathcal{M}_p\, \Phi\, \Phi) - \frac{1}{8}\,\mathrm{tr}\,(V_{\mu\nu}\, V^{\mu\nu}) + \frac{1}{4}\,\mathrm{tr}\,(\mathcal{M}_V\, V_\mu\, V^\mu)\\
& + ig\,\mathrm{tr}\,(V_\mu\, [\Phi, \partial^\mu \Phi])
\end{aligned}
\tag{4.6.20}
$$

with the mass matrices

$$
\mathcal{M}_p = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2\, m_K^2 - m_\pi^2 \end{pmatrix}
\tag{4.6.21}
$$

and

$$
\mathcal{M}_V = \begin{pmatrix} m_\rho^2 & 0 & 0 \\ 0 & m_\rho^2 & 0 \\ 0 & 0 & m_\varphi^2 \end{pmatrix} \,.
\tag{4.6.22}
$$

For the present purpose we will ignore isospin breaking. The mass of all pions is then $m_\pi \approx 140\,\mathrm{MeV}$ and all kaons have a mass of $m_K \approx 500\,\mathrm{MeV}$. The mass of all ρ mesons is $m_\rho \approx 770\,\mathrm{MeV}$ and the mass of the φ meson is $m_\varphi \approx 1020\,\mathrm{MeV}$.

The hermitian matrix

$$
\Phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta \end{pmatrix}
\tag{4.6.23}
$$

contains the pseudoscalar fields. Here the fields π^0 and η are hermitian while the fields with electric charge and/or strangeness are adjoint to each other: $(\pi^-)^\dagger =: \pi^+$, $(K^-)^\dagger =: K^+$, $(K^0)^\dagger =: \bar{K}^0$.

The hermitian matrix

$$V_\mu = \begin{pmatrix} \rho_\mu^0 + \omega_\mu & \sqrt{2}\rho_\mu^+ & \sqrt{2}K_\mu^{*+} \\ \sqrt{2}\rho_\mu^- & -\rho_\mu^0 + \omega_\mu & \sqrt{2}K_\mu^{*0} \\ \sqrt{2}K_\mu^{*-} & \sqrt{2}\bar{K}_\mu^{*0} & \sqrt{2}\varphi_\mu \end{pmatrix} \tag{4.6.24}$$

contains the nine vector mesons. Also here the fields ρ^0, ω and φ are hermitian while the fields with electric charge and/or strangeness are adjoint to each other: $(\rho^-)^\dagger =: \rho^+$, $(K^{*-})^\dagger =: K^{*+}$, $(K^{*0})^\dagger =: \bar{K}^{*0}$. The field strength of the massive vector fields is given by $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$.

Finally "tr" denotes the trace and g is a coupling constant.

(a) Flavor symmetry:
Consider continuous transformations of "flavor" $\mathbf{SU}(3)$:

$$\begin{aligned} \Phi &\to U\,\Phi\,U^\dagger \\ V_\mu &\to U\,V_\mu\,U^\dagger \end{aligned} \tag{4.6.25}$$

with $U \in \mathbf{SU}(3)$, i.e. U is a 3×3 matrix which satisfies $U\,U^\dagger = \mathbb{1}$ and $\det U = 1$. Such matrices can be generated from $U = e^{i\beta_a t_a}$ with eight linearly independent hermitian 3×3 matrices t_a and eight arbitrary real numbers $\beta_a \in \mathbb{R}$, $a = 1, 2, \ldots, 8$.
Show that all terms in the Langrangian (4.6.20) are invariant with respect to (4.6.25) except for the mass terms. Show that also the mass terms would be invariant, if there was no difference between the pion and the kaon mass and no difference between the ρ and the φ mass.
Remark: In particular, this means that the interaction term has been constructed such that flavor symmetry is not broken.

(b) Isospin symmetry:
Even when the kaon mass is different from the pion mass and the ρ mass is different from the φ mass, there is an $\mathbf{SU}(2)$ subgroup of the flavor $\mathbf{SU}(3)$ group which constitutes a symmetry of the Lagrangian (4.6.20):
Given the "generators" of the isospin group

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.6.26}$$

show that the Langrangian (4.6.20) is invariant with respect to the transformations

$$\begin{aligned} \Phi &\to \tilde{U}\,\Phi\,\tilde{U}^\dagger \\ V_\mu &\to \tilde{U}\,V_\mu\,\tilde{U}^\dagger \end{aligned} \tag{4.6.27}$$

with $\tilde{U} = e^{i\beta_c t_c}$ with three arbitrary real numbers $\beta_c \in \mathbb{R}$, $c = 1, 2, 3$. Hint: Show that $\tilde{U}$ is a flavor $\mathbf{SU}(3)$ matrix. You can use $\det(e^A) = e^{\mathrm{tr}A}$ to prove that. Since you have already shown that the Lagrangian (4.6.20) is invariant with respect to all $\mathbf{SU}(3)$ transformations (4.6.25) — except for the mass terms —, all what you have to do now, is to show that the transformations (4.6.27) do not change the mass terms.

(c) Conservation of electric charge:
Consider the continuous transformations

$$\begin{aligned} &\pi^+ \to e^{-i\alpha}\,\pi^+\,, \quad \pi^- \to e^{i\alpha}\,\pi^-\,, && K^+ \to e^{-i\alpha}\,K^+\,, \quad K^- \to e^{i\alpha}\,K^-\,, \\ &\pi^0 \to \pi^0\,, \quad \eta \to \eta\,, && K^0 \to K^0\,, \quad \bar{K}^0 \to \bar{K}^0\,, \\ &\rho^+ \to e^{-i\alpha}\,\rho^+\,, \quad \rho^- \to e^{i\alpha}\,\rho^-\,, && K^{*+} \to e^{-i\alpha}\,K^{*+}\,, \quad K^{*-} \to e^{i\alpha}\,K^{*-}\,, \\ &\rho^0 \to \rho^0\,, \quad \omega \to \omega\,, \quad \varphi \to \varphi\,, && K^{*0} \to K^{*0}\,, \quad \bar{K}^{*0} \to \bar{K}^{*0} \end{aligned}$$

with $\alpha \in \mathbb{R}$. Show that the Lagrangian (4.6.20) is invariant with respect to these transformations.
Hint: One can formulate this transformation as a flavor transformation.

(d) Strangeness conservation:
Consider the continuous transformations

$$\begin{aligned} &\pi^+ \to \pi^+\,, \quad \pi^- \to \pi^-\,, && K^+ \to e^{-i\beta}\,K^+\,, \quad K^- \to e^{i\beta}\,K^-\,, \\ &\pi^0 \to \pi^0\,, \quad \eta \to \eta\,, && K^0 \to e^{-i\beta}\,K^0\,, \quad \bar{K}^0 \to e^{i\beta}\,\bar{K}^0\,, \\ &\rho^+ \to \rho^+\,, \quad \rho^- \to \rho^-\,, && K^{*+} \to e^{-i\beta}\,K^{*+}\,, \quad K^{*-} \to e^{i\beta}\,K^{*-}\,, \\ &\rho^0 \to \rho^0\,, \quad \omega \to \omega\,, \quad \varphi \to \varphi\,, && K^{*0} \to e^{-i\beta}\,K^{*0}\,, \quad \bar{K}^{*0} \to e^{i\beta}\,\bar{K}^{*0} \end{aligned}$$

with $\beta \in \mathbb{R}$. Show that the Lagrangian (4.6.20) is invariant with respect to these transformations.

(e) Charge conjugation:
Consider the discrete transformation which exchanges particles and antiparticles:

$$\Phi \to \Phi^T\,, \qquad V_\mu \to -V_\mu^T\,. \tag{4.6.28}$$

Show that the Lagrangian (4.6.20) is invariant with respect to this transformation. Show that an interaction term $\sim \mathrm{tr}(V_\mu\,\{\Phi, \partial^\mu\Phi\})$

would not be invariant with respect to this transformation. Here $\{\ldots,\ldots\}$ denotes the anticommutator. Which meson fields are eigen states of this transformation and which are the corresponding eigen values?
Hint: $\mathrm{tr}A^T = \mathrm{tr}A$.

(f) G-parity:
One can combine charge conjugation with a discrete isospin transformation (isospin flip) to define the G-parity operation:

$$\Phi \to e^{i\pi t_2}\,\Phi^T\,e^{-i\pi t_2}\,, \qquad V_\mu \to -e^{i\pi t_2}\,V_\mu^T\,e^{-i\pi t_2}\,. \tag{4.6.29}$$

Which meson fields are eigen states of this transformation and which are the corresponding eigen values?
Hint: You can assume without proof $e^{i\pi t_2} = 2i\,t_2 + P_3$ with the projector on the third component $P_3 = \mathrm{diag}(0,0,1)$.

4. **Chiral symmetry:**

(a) **Conserved currents:**
Given the two-flavor chirally symmetric QCD-type Lagrangian

$$\mathcal{L}_{0,2} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=s,c,b,t} \bar{f}\left(i\not{D} - m_f\right) f + \bar{q}\,i\not{D}\,q \tag{4.6.30}$$

with the two-component "isospinor"

$$q = \begin{pmatrix} u \\ d \end{pmatrix}\,. \tag{4.6.31}$$

This Lagrangian is invariant with respect to the chiral $\mathrm{SU}_V(2)\times\mathrm{SU}_A(2)$ transformations (you do not need to show this)

$$q \to \exp\left[i\left(\alpha_a^V t_a + \alpha_a^A\,t_a\gamma_5\right)\right]\,q \tag{4.6.32}$$

where the summation index a runs from 1 to 3, $\alpha_a^V, \alpha_a^A \in \mathbb{R}$, and $t_a = \sigma_a/2$ with the three 2×2 Pauli matrices σ_a. Based on Noether's theorem determine the six conserved currents.

(b) **Nearly conserved currents:**
Given the *full* QCD Lagrangian (i.e. with masses for up and down quarks reinstalled)

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \sum_{f=u,d,s,c,b,t} \bar{f}\left(i\not{D} - m_f\right) f\,, \tag{4.6.33}$$

show that the six currents $j^\mu_{a,V} = \bar{q}\gamma^\mu t_a q$ and $j^\mu_{a,A} = \bar{q}\gamma^\mu\gamma_5 t_a q$ satisfy

$$\partial_\mu j^\mu_{a,V} = i\,\bar{q}\,[\mathcal{M}, t_a]\,q\,, \qquad \partial_\mu j^\mu_{a,A} = i\,\bar{q}\,\{\mathcal{M}, t_a\}\gamma_5\,q \tag{4.6.34}$$

with the commutator $[A, B] := AB - BA$, the anti-commutator $\{A, B\} := AB + BA$ and the two-flavor mass matrix

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{4.6.35}$$

Hint: Derive first the equations of motion for the quark fields, i.e. show

$$i\gamma_\mu\partial^\mu f + g\gamma_\mu A^\mu f - m_f f = 0 \tag{4.6.36}$$

and

$$i\,\partial^\mu\bar{f}\gamma_\mu - \bar{f}g\gamma_\mu A^\mu + \bar{f}\,m_f = 0\,. \tag{4.6.37}$$

(c) **Commutation relations:**
The charges of chiral $\mathrm{SU}_V(2)\times\mathrm{SU}_A(2)$ symmetry

$$I^V_a = \int d^3r\, q^\dagger(t,\vec{r})\,t_a\,q(t,\vec{r})\,, \qquad I^A_a = \int d^3r\, q^\dagger(t,\vec{r})\,\gamma_5 t_a\,q(t,\vec{r}) \tag{4.6.38}$$

satisfy

$$\begin{aligned} [I^V_a, I^V_b] &= i\,\epsilon^{abd}\,I^V_d\,, \\ [I^A_a, I^A_b] &= i\,\epsilon^{abd}\,I^V_d\,, \\ [I^V_a, I^A_b] &= i\,\epsilon^{abd}\,I^A_d\,. \end{aligned} \tag{4.6.39}$$

Prove at least one of these three relations (your choice).

Hints: Introduce explicitly the color (c), flavor (f) and spinor (s) indices by writing, e.g., $q^\dagger\gamma_5 t_a q = q^\dagger_{cfs}\,(\gamma_5)_{ss'}\,(t_a)_{ff'}\,q_{cf's'}$ and use that the fermion fields q_{cfs} satisfy the following anticommutation relations (quantization conditions)

$$\begin{aligned} \{q_{cfs}(t,\vec{r}), q^\dagger_{c'f's'}(t,\vec{r}\,')\} &= \delta_{cc'}\,\delta_{ff'}\,\delta_{ss'}\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,, \\ \{q_{cfs}(t,\vec{r}), q_{c'f's'}(t,\vec{r}\,')\} &= 0\,, \\ \{q^\dagger_{cfs}(t,\vec{r}), q^\dagger_{c'f's'}(t,\vec{r}\,')\} &= 0\,. \end{aligned} \tag{4.6.40}$$

Prove first the following relations

$$\left[\int d^3r'\, q^\dagger_{c'fs}(t,\vec{r}\,')\,q_{c'f's'}(t,\vec{r}\,')\,,\, q^\dagger_{cf''s''}(t,\vec{r})\right] = q^\dagger_{cfs}(t,\vec{r})\,\delta_{f'f''}\,\delta_{s's''}\,, \tag{4.6.41}$$

$$\left[\int d^3r'\, q^\dagger_{c'fs}(t,\vec{r}\,')\, q_{c'f's'}(t,\vec{r}\,')\,,\, q_{cf'''s'''}(t,\vec{r})\right] = -q_{cf's'}(t,\vec{r})\,\delta_{ff'''}\,\delta_{ss'''}\,, \tag{4.6.42}$$

and

$$\left[\int d^3r'\, q^\dagger_{c'fs}(t,\vec{r}\,')\, q_{c'f's'}(t,\vec{r}\,')\,,\, q^\dagger_{cf''s''}(t,\vec{r})\, q_{cf'''s'''}(t,\vec{r})\right] = q^\dagger_{cfs}(t,\vec{r})\,\delta_{f'f''}\,\delta_{s's''}\, q_{cf'''s'''}(t,\vec{r}) - q^\dagger_{cf''s''}(t,\vec{r})\, q_{cf's'}(t,\vec{r})\,\delta_{ff'''}\,\delta_{ss'''}\,, \tag{4.6.43}$$

by using (4.6.40) together with $[A,BC] = [A,B]C + B[A,C]$ and $[AB,C] = A\{B,C\} - \{A,C\}B$.

Final hint: The ϵ^{abd} structure in (4.6.39) is caused by $[t_a, t_b]$.

(d) **Chiral quartet:**

The pseudoscalar isovector quark current $\bar{d}(x)i\gamma_5 u(x)$ has the quantum numbers of the positively charged pion, π^+. Determine all the commutators between this quark current and any of the six charges (4.6.38).

The hints of the previous task apply also here. Express the quark current as $\bar{d}i\gamma_5 u = \bar{q}i\gamma_5 t q$ with a 2×2 matrix t and show $t = t_1 - it_2$. Note that the charges (4.6.38) are conserved (for the Lagrangian (4.6.30)). Therefore one can use any time argument for the fields, in particular the same time argument. Only this justifies the use of the equal-time quantization conditions (4.6.40).

Show that the results involve the pseudoscalar isovector quark current $\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d$ (quantum numbers of π^0) and the scalar isoscalar quark current $\bar{u}u + \bar{d}d$ (quantum numbers of f_0). If one performs the corresponding calculation for these quark currents (you do not need to do this), the result involves in addition the pseudoscalar isovector quark current $\bar{u}i\gamma_5 d$ (quantum numbers of π^-). In total the four quark currents form a chiral quartet. Using the particle listings at http://pdg.lbl.gov/ find the π- and f_0-mesons with the respective lowest masses. Check that only the pions are approximately degenerate but not the pions with any f_0.

(e) **Chiral sextet:**

The vector isovector quark current $\bar{d}\gamma_\mu u$ has the quantum numbers of the positively charged rho-meson, ρ^+. Determine all the commutators between this quark current and any of the six charges (4.6.38). Show that the results involve the vector isovector quark current $\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d$ (quantum numbers of ρ^0) and the axial-vector isovector quark current $\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$ (quantum numbers of the neutral a_1-meson). If one

continues the corresponding calculations with the respective obtained currents until one does not obtain any new quark currents (you do not need to do this), then one finds in total the three vector isovector quark currents (quantum numbers of the ρ triplet) and the three axial-vector isovector quark currents (quantum numbers of the a_1 triplet), i.e. a chiral sextet. Using the particle listings at http://pdg.lbl.gov/ find the ρ- and a_1-mesons with the respective lowest masses.

5 Spontaneous symmetry breaking

As we will see, the missing of parity doublets and other features of low-energy QCD can be explained by the spontaneous breakdown of the chiral symmetry. Concerning symmetry considerations we have so far only studied the Lagrangian. This, however, is not enough to obtain a complete picture of the symmetry pattern. Spontaneous symmetry breaking is defined by the situation that a symmetry is present for the Lagrangian but not for the ground state of the system. In our case the ground state is the vacuum which possesses a rather non-trivial structure. In the following we will explain the issue of spontaneous symmetry breaking in several levels of sophistication: We will start with a system which has nothing to do with physics — a round dinner table with salad plates. Next we will turn to an example from solid-state physics — the Heisenberg magnet. Then we will come back to the spontaneous breakdown of chiral symmetry and translate on a qualitative level what we have learned before to QCD. As a next step we will introduce a field theoretical example which shows spontaneous symmetry breaking. This example can be expanded to discuss the Higgs mechanism which concerns the spontaneous breaking of a local gauge symmetry. Finally, in the next section, we will introduce the concepts of chiral perturbation theory which deals with chiral symmetry breaking of QCD on a quantitative level.

Imagine several people sitting on a round dinner table.[32] Each person finds one salad plate to the right and one to the left at equal distance. So far the situation is symmetric with respect to a parity transformation. What is the "ground state" of the system? The ground state is the most favored one (minimal energy). Of course at dinner people are in favor of starting to eat. So one person chooses randomly a salad plate, e.g., the one to his/her left. The neighbor to the left has no choice any more and also picks her/his left one and so on. In the end we find a situation which is no longer parity symmetric: A parity transformation yields a situation where everyone has picked his/her right salad plate. If we draw arrows pointing from the people to the respective chosen salad plates (in our case the arrows point inwards to the left) we graphically find a kind of flow

[32]This example (or a similar one) probably goes back to A. Salam; see, e.g., Brout, Englert, Spontaneous symmetry breaking in gauge theories: A historical survey, 1998, http://arxiv.org/abs/hep-th/9802142.

pattern with points clockwise. The parity transformation yields a flow pattern pointing counter clockwise. While the whole setup of a dinner table with salad plates ("Lagrangian") does not prefer any direction for the flow pattern, there is a favored direction in the "ground state", the actual dinner situation. The Lagrangian has a symmetry which the ground state does not share. This is spontaneous symmetry breaking.

5.1 An example from solid-state physics: the Heisenberg magnet

Let us come back to physics and study a Heisenberg magnet which serves as a model for a ferromagnet.[33] It describes a solid state of interacting microscopic magnetic dipoles (spins). The interaction Hamiltonian is given by

$$H_{\text{int}} = g \sum_{i \neq j} \vec{s}_i \cdot \vec{s}_j \,. \tag{5.1.1}$$

The interaction between the spins does not prefer any direction. We find rotational invariance at the level of the Hamiltonian/Lagrangian. In contrast the ground state of the system (lowest-energy state), which is the unexcited solid state, has a preferred direction as shown in Fig. 3 (left). The ground state breaks

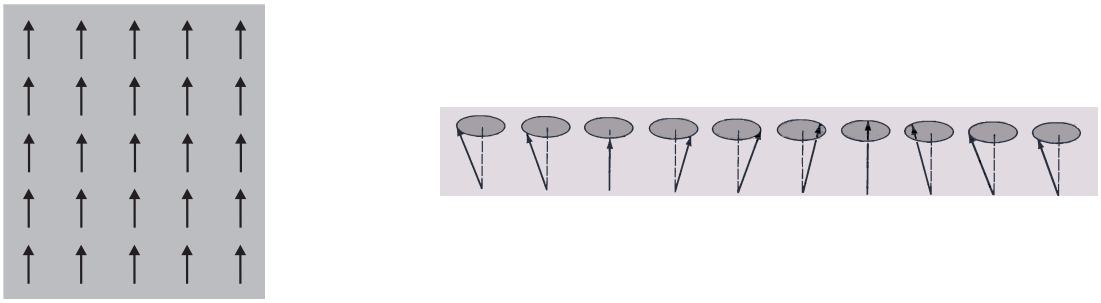


Figure 3: Schematic pictures of a Heisenberg magnet in its ground state (left) and of a spin wave (right). (Right figure taken from Ashcroft/Mermin, Solid state physics, Saunders, 1976.)

the rotational invariance, a further example of the spontaneous breakdown of a symmetry. The following properties are connected with this breakdown:

1. **Ground-state expectation value:** We find a macroscopic magnetization

$$\vec{M} = \langle \vec{s}_i \rangle \,, \tag{5.1.2}$$

i.e. a non-vanishing expectation value. This corresponds to the non-vanishing flow pattern of the round-dinner-table system.

[33] see, e.g., Ashcroft/Mermin, Solid state physics, Saunders, 1976

2. **Goldstone bosons:** Suppose we excite the solid state a little bit, e.g. by kicking some of the spins. Due to the interaction we observe a spin wave propagating through the solid state. This is depicted in Fig. 3 (right). If we study these spin waves we find a gapless excitation spectrum. "Gapless" just means that the (dispersion) relation $\omega(\vec{k})$ between frequency ω and wave number $\vec{k}$ satisfies

$$\lim_{\vec{k}\to 0} \omega(\vec{k}) = 0\,. \tag{5.1.3}$$

In a (relativistic) quantum theory, where frequency translates to energy and wave number to momentum (de Broglie), a gapless excitation spectrum just translates to a massless particle. Such a gapless excitation/massless particle is called a (Nambu-)Goldstone boson. One can intuitively understand why the excitation spectrum is gapless, i.e. why one can excite the solid state with arbitrary less energy: Suppose we study a plane-wave excitation of the spin system shown in Fig. 3 (left) and go to larger and larger wavelengths and low frequencies. In the infinite wavelength limit and for zero frequency all spins are aligned again, but point in a somewhat different direction. If we imagine the spins as microscopically small and densely distributed, then one can obtain the same situation by simply rotating the whole solid state to the new direction as depicted in Fig. 4. In the absence of external forces/fields

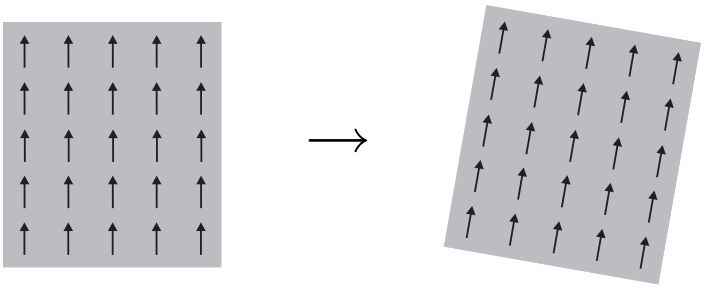


Figure 4: An extremely long-wavelength excitation is equivalent to rotating the whole solid state.

this does not cost any energy. Hence we find a continuous connection of the spin-wave excitations to the zero-energy case, i.e. a gapless excitation spectrum.

How many Goldstone bosons do we have? The number is just the number of broken continuous[34] symmetries: If we consider the solid state of Fig. 3 in three-dimensional space, there are two spin-wave excitations: one horizontal and one perpendicular to the paper[35] plane. Their long-wavelength limits just correspond to the rotations which are *not* symmetry transformations of the ground state. On the other hand, there is still one subgroup of all

[34]The breaking of *discrete* symmetries does not lead to Goldstone bosons.

[35]I assume that you have printed the lecture notes. Otherwise it is the screen plane.

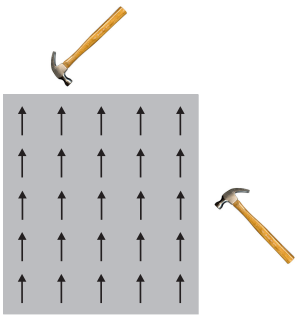


Figure 5: Phonon excitations in vertical and horizontal direction.

rotations of three-dimensional space left which is not spontaneously broken: The rotations around the vertical axis leave the ground state invariant. There are no spin waves in this direction.

3. **Non-degeneracy of states:** Next let us discuss other excitations, different from the spin waves. For example, consider phonon excitations (vibrations) as depicted in Fig. 5. If there was no spontaneous symmetry breaking, i.e. if rotational invariance was a symmetry of the interactions *and* of the ground state, then a phonon excitation in vertical direction would cost the same energy as the corresponding one in horizontal direction. On the other hand, in the presence of spontaneous symmetry breaking it is easy to imagine that the two phonon excitations in general will have rather different dispersion relations. They are not degenerate any more.

4. **Additional explicit symmetry breaking:** We have already learned that we find a non-vanishing expectation value, the magnetization $\vec{M}$. Without external forces, however, we cannot observe it. Only if we introduce an external magnetic field $\vec{B}$ into our Hamiltonian,

$$H_{\text{int}} = g \sum_{i \neq j} \vec{s}_i \cdot \vec{s}_j + \vec{B} \cdot \sum_i \vec{s}_i \,, \tag{5.1.4}$$

we can measure the size and direction of $\vec{M}$. It is important to realize that $\vec{B}$ breaks the rotational symmetry *explicitly*. The new Hamiltonian (5.1.4) is no longer invariant under rotations but for small enough $\vec{B}$ there is an approximate symmetry. We also note that from the term which explicitly breaks the symmetry we can read off the operator which gets a non-trivial expectation value: It is just $\vec{s}_i$ which on the one hand multiplies $\vec{B}$ in the Hamiltonian (5.1.4) and on the other hand enters the definition of the magnetization (5.1.2).

5. **Pseudo Goldstone bosons:** In the presence of $\vec{B}$ the excitation spectrum of spin waves is no longer gapless: In the ground state all spins point in

Figure 6: A phonon excitation in horizontal direction and a phonon excitation in vertical direction with a rotated solid state.

the direction of $\vec{B}$. It now costs energy to rotate the whole solid state in a different direction (infinite-wavelength excitation). Typically the gap scales with some power of $|\vec{B}|$ as the gap has to vanish for $\vec{B} \to 0$.

6. **Hidden symmetry:** Instead of talking about a system with spontaneous "breaking" of a symmetry, some authors prefer to call it a system with a "hidden" symmetry. To understand what this means let us come back to our phonon excitations depicted in Fig. 5. We have seen already that the two phonon excitations are in general not degenerate. However, there is still a degeneracy left: Consider the rotation of the solid state and (afterwards) a vertical phonon excitation. Clearly the dispersion relation is identical to the dispersion relation of a horizontal phonon excitation (without rotating the solid state). This is visualized in Fig. 6. On the other hand, we recall that a rotation of the solid state is nothing but a long-wavelength spin wave. Therefore, the simultaneous excitation of a phonon and a spin wave is degenerate to a single phonon excitation. In a system where the ground state shares the symmetry of the interactions, the degeneracy of states is realized at the single-excitation (single-particle) level. In a system with spontaneously broken (hidden!) symmetry, the degeneracy is realized by an excitation and an excitation plus a spin wave. Obviously this makes only sense if the spin wave is gapless, i.e. a Goldstone boson.

5.2 Translation to QCD

Let us come back to QCD. All experimental facts indicate that in QCD the $\mathbf{SU}_V(N_f)$ symmetry is (approximately) realized in an ordinary way while the $\mathbf{SU}_A(N_f)$ symmetry is spontaneously broken/hidden. As we will see, such an assumption naturally explains the non-existence of parity doublets (at the single-particle level) and also the appearance of rather light states (pions, kaons, η mesons). We shall now translate everything (number by number) which we have learned from the previous examples to QCD:

1. **Vacuum expectation value:** Also here we find a non-trivial expectation value with respect to the ground state, which in the case of quantum field theory is the vacuum. As we will further motivate below, the non-vanishing vacuum expectation value is the quark condensate (chiral condensate) with an estimated size of

$$\langle \bar{q}q \rangle \approx -(240\,\mathrm{MeV})^3 \times N_f\,. \tag{5.2.1}$$

   Here q is given by (4.4.2) or (4.4.11), depending on the number of considered light flavors.

2. **Goldstone bosons:** If the chiral symmetry was an exact symmetry we would find a gapless excitation spectrum, i.e. we would find states which were exactly massless. As the broken symmetry group $\mathbf{SU}_A(N_f)$ has N_f^2-1 degrees of freedom the number of Goldstone bosons is also N_f^2-1.

3. **Non-degeneracy of states:** As already pointed out in the previous section there are no parity doublets on the level of single hadrons. Indeed, we do not expect them, if $\mathbf{SU}_A(N_f)$ is spontaneously broken.

4. **Additional explicit symmetry breaking:** It is suggestive (but not a proof and not always true) that the non-vanishing vacuum expectation value can be deduced from the terms in the Lagrangian which explicitly break the symmetry. In QCD these are the mass terms of the N_f light quarks. In analogy to the case of a Heisenberg magnet placed in an external magnetic field we read off from the QCD Lagrangian (4.1.1) the operators which get non-trivial vacuum expectation values: For $N_f = 2$ one has $\bar{u}u$ and $\bar{d}d$. One can show (in chiral perturbation theory) that the difference between the vacuum expectation values is driven by the (small) quark masses. Therefore, one finds $\langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle$. This has been implicitly assumed in (5.2.1). Note that we did not give a proof but only a plausibility argument that it is $\bar{q}q$ which gets a non-trivial (and sizable) vacuum expectation value. We present here the standard picture of chiral symmetry breaking which forms the basis of chiral perturbation theory. But in principle the situation can be more complicated. The consequences have been worked out in "generalized chiral perturbation theory". For more details we refer to Knecht/Moussallam/Stern/Fuchs, Nucl. Phys. B471, 445, 1996, http://arxiv.org/abs/hep-ph/9512404. Concerning our previous solid-state example one can imagine to have a more complicated ground-state situation than the one depicted in Fig. 3: One might have layers of spins where all spins in one layer point in the same direction, but the spins in the neighboring layer point in the opposite direction. In this case the "vacuum" expectation value of $\vec{s}_i$ vanishes. Such a system is an idealization of an anti-ferromagnet. The corresponding situation is in principle conceivable in

QCD, but leads to predictions which differ from the predictions of standard chiral perturbation theory. Modern experiments (e.g., DIRAC@CERN) determining the low-energy properties of the pion-pion interaction seem to confirm the standard scenario with a sizable quark condensate.

5. **Pseudo Goldstone bosons:** For non-vanishing quark masses (explicit symmetry breaking) the Goldstone bosons are no longer exactly massless, but light. This explains the appearance of light states: for $N_f = 2$ the three pions and for $N_f = 3$ the eight states pions, kaons, η. Their masses scale with a power of the quark mass as expressed by the Gell-Mann–Oakes–Renner relation (here for $N_f = 2$)

$$m_\pi^2 \, f_\pi^2 = -m_q \, \langle \bar{q} q \rangle \tag{5.2.2}$$

where m_π denotes the pion mass, f_π the pion decay constant, and m_q the average light quark mass. We will come back to this relation below. At present we only want to give a qualitative overview.

6. **Hidden symmetry:** The $\mathbf{SU}_A(N_f)$ symmetry is hidden in the sense that there are no parity doublets at the level of single hadrons, but there are approximately degenerate states: For example a proton on the one hand and a neutron plus a π^+ on the other hand have comparable masses, because the pions are so much lighter than the other hadrons. Also the parity is different, because the pions have negative parity. Otherwise they would not qualify as Goldstone bosons of spontaneously broken $\mathbf{SU}_A(2)$. We should note, however, that the "degeneracy" is not as good as for the isospin symmetry where the mass differences are only a few MeV. Nonetheless, the concept of spontaneous symmetry breaking yields a qualitatively satisfying explanation of the low-energy hadron spectrum.

5.3 An example from quantum field theory

5.3.1 Exact symmetry

We consider the Lagrangian

$$\mathcal{L}(\phi^\dagger, \phi) = \partial_\mu \phi^\dagger \, \partial^\mu \phi + m^2 \, \phi^\dagger \, \phi - \frac{\lambda}{4} \, (\phi^\dagger \phi)^2 \tag{5.3.1}$$

with a complex (non-hermitian) scalar field ϕ and a coupling constant $\lambda > 0$. Note that the sign in front of m^2 is different from the usual mass term, cf. (2.1.32). The Lagrangian (5.3.1) has a $\mathbf{U}(1)$ symmetry: It remains unchanged with respect to the transformations

$$\phi \to e^{i\alpha} \, \phi \qquad \Rightarrow \qquad \phi^\dagger \to e^{-i\alpha} \, \phi^\dagger \tag{5.3.2}$$

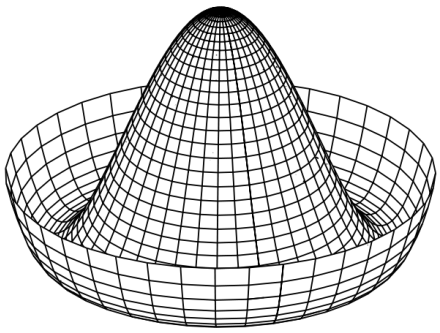


Figure 7: The "mexican hat" potential as a function of (the real and imaginary part of) the classical field φ. (Figure taken from Wikipedia, http://en.wikipedia.org/wiki/Spontaneous_symmetry_breaking.)

with arbitrary $\alpha \in \mathbb{R}$.

The vacuum state is by definition the state with the smallest energy. Therefore, we determine the Hamiltonian and search for the field configuration which minimizes it. Introducing the canonically conjugate momenta $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^\dagger$ and $\pi^\dagger = \partial_0 \phi$ the Hamiltonian (density) is given by

$$\mathcal{H} = \pi\pi^\dagger + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + \mathcal{V}(\phi^\dagger\phi) = \partial_0\phi^\dagger\partial_0\phi + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + \mathcal{V}(\phi^\dagger, \phi) \tag{5.3.3}$$

with the "effective potential"

$$\mathcal{V}(\phi^\dagger, \phi) := -m^2\,\phi^\dagger\,\phi + \frac{\lambda}{4}\,(\phi^\dagger\phi)^2\,. \tag{5.3.4}$$

Obviously all terms in (5.3.3) with derivatives appear in squares and contribute positively to the Hamiltonian. Therefore, one can minimize them by a field configuration which is constant. What is left to do is to minimize the effective potential (5.3.4). A constant field configuration does not correspond to a particle excitation. Therefore, for the following considerations we can replace ϕ in (5.3.4) by its classical counterpart, i.e. by its expectation value $\varphi := \langle\phi\rangle = \langle \mathrm{vac}|\phi|\mathrm{vac}\rangle$. The effective potential $\mathcal{V}(\varphi^*, \varphi)$ is displayed in Fig. 7. It is easy to see that $\mathcal{V}$ has a maximum at $\varphi = 0$ and degenerate minima for all φ which satisfy[36]

$$|\varphi| = \phi_{\mathrm{vac}} := \sqrt{\frac{2m^2}{\lambda}}\,. \tag{5.3.5}$$

Note that these non-trivial minima would not appear, if the m^2 term in (5.3.1) had the opposite sign. Note also that the sign of λ is important: We have chosen it to be positive. A negative λ would lead to an effective potential and therefore

[36] Obviously, $\mathcal{V}(\varphi^*, \varphi)$ depends only on $|\varphi|$. Just solve for the minimum conditions $\frac{d\mathcal{V}}{d|\varphi|} = 0$ and $\frac{d^2\mathcal{V}}{d|\varphi|^2} > 0$.

to a Hamiltonian which would not be bounded from below. This would be an unphysical system if there was no well-defined lowest-energy (vacuum) state.

Now we choose one explicit minimum solution, i.e. we demand that the vacuum state is, e.g., given by

$$\langle\phi\rangle = \phi_{\text{vac}} = \sqrt{\frac{2m^2}{\lambda}}\,. \tag{5.3.6}$$

This corresponds to the "choice" of the ferromagnet for the (in principle arbitrary) direction where the magnetization points to. After such a choice the symmetry is broken because we have picked one of the minima to be our vacuum state. In practice this choice is often not made "by hand" but rather by an explicit symmetry breaking. For our field theoretical model we will come back to this issue below. For the ferromagnet it is typically a disturbance of the rotationally invariant system like, e.g., impurities in the crystal structure which make one direction of the spin alignment energetically preferable or the magnetic field of the earth.

Physical particles are (quantized) field excitations on top of the vacuum. The simplest excitations one can imagine are the ones which are spatially homogeneous, i.e. which have no $\vec{r}$ dependence. These are just the ones which have no momentum. In this case the effective potential $\mathcal{V}$ shown in Fig. 7 can be regarded as the potential energy for two degrees of freedom, $\text{Re}\phi(t)$ and $\text{Im}\phi(t)$. Like in classical mechanics one can look for oscillations in the vicinity of the minimum. Obviously, one can imagine two types of excitations: One in the radial direction of the effective potential and one in the azimuthal direction. The latter is related to the symmetry (5.3.2) which is spontaneously broken. Now we recall from our example of the Heisenberg magnet that the excitations in the directions which have to do with the broken symmetry are the Goldstone bosons. Therefore, we expect that the azimuthal excitations are the massless Goldstone bosons. Let us see whether our formalism leads to the same result.

To this end, we parametrize $\phi(x)$ in the following way:

$$\phi(x) = \left(\phi_{\text{vac}} + \frac{1}{\sqrt{2}}\,\rho(x)\right)\, e^{i\alpha(x)} \tag{5.3.7}$$

with hermitian fields $\rho(x)$ and $\alpha(x)$. Note that on purpose we choose for the azimuthal field, $\alpha(x)$, the same letter as we have used for the symmetry transformations in (5.3.2). They are intimately connected. The factor $\frac{1}{\sqrt{2}}$ in front of the radial excitation, ρ, is pure convention. As we will see, it ensures that we end up with the appropriate kinetic term in the final Lagrangian. Finally, we note that the particle content is still the same as before: We had two independent fields ϕ and $\phi^\dagger$ which are replaced now by the two new fields $\rho(x)$ and $\alpha(x)$. We want to insert (5.3.7) into the Lagrangian (5.3.1). As intermediate steps we determine

$$\partial_\mu\phi(x) = \left[\frac{1}{\sqrt{2}}\,\partial_\mu\rho(x) + i\left(\phi_{\text{vac}} + \frac{1}{\sqrt{2}}\,\rho(x)\right)\,\partial_\mu\alpha(x)\right]\, e^{i\alpha(x)} \tag{5.3.8}$$

and

$$\partial_\mu \phi^\dagger \, \partial^\mu \phi = \frac{1}{2} \, \partial_\mu \rho \, \partial^\mu \rho + \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}} \, \rho \right)^2 \, \partial_\mu \alpha \, \partial^\mu \alpha \, . \tag{5.3.9}$$

Obviously, the effective potential does not depend on α. We finally obtain for the Lagrangian (5.3.1):

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \, \partial_\mu \rho \, \partial^\mu \rho + \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}} \, \rho \right)^2 \, \partial_\mu \alpha \, \partial^\mu \alpha \\ &\quad + m^2 \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}} \, \rho \right)^2 - \frac{\lambda}{4} \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}} \, \rho \right)^4 \\ &= m^2 \, \phi_{\rm vac}^2 - \frac{\lambda}{4} \, \phi_{\rm vac}^4 \\ &\quad + \sqrt{2} \, m^2 \, \phi_{\rm vac} \, \rho - \frac{\lambda}{\sqrt{2}} \, \phi_{\rm vac}^3 \, \rho \\ &\quad + \frac{1}{2} \, \partial_\mu \rho \, \partial^\mu \rho + \frac{1}{2} \, m^2 \, \rho^2 - \frac{3}{4} \, \lambda \, \phi_{\rm vac}^2 \, \rho^2 + \phi_{\rm vac}^2 \, \partial_\mu \alpha \, \partial^\mu \alpha \\ &\quad - \frac{\lambda}{2\sqrt{2}} \, \phi_{\rm vac} \, \rho^3 + \sqrt{2} \, \phi_{\rm vac} \, \rho \, \partial_\mu \alpha \, \partial^\mu \alpha \\ &\quad - \frac{\lambda}{16} \, \rho^4 + \frac{1}{2} \, \rho^2 \, \partial_\mu \alpha \, \partial^\mu \alpha \, . \end{aligned} \tag{5.3.10}$$

In the last step we have sorted the terms in powers of fields to distinguish the interaction terms from the kinetic and mass terms. We have also obtained terms which do not depend on the fields at all. They are irrelevant because they do not contribute to the equations of motion. Apparently we find also terms of linear order in the field ρ. However, these terms cancel each other:

$$\sqrt{2} \, m^2 \, \phi_{\rm vac} - \frac{\lambda}{\sqrt{2}} \, \phi_{\rm vac}^3 = \sqrt{2} \, m^2 \sqrt{\frac{2m^2}{\lambda}} - \frac{\lambda}{\sqrt{2}} \left(\sqrt{\frac{2m^2}{\lambda}} \right)^3 = 0 \, , \tag{5.3.11}$$

where we have used (5.3.6).

The kinetic term of the (hermitian) ρ field $\sim \partial_\mu \rho \, \partial^\mu \rho$ shows already its standard form, cf. (2.1.17). The mass term of the ρ field should have the form $-\frac{1}{2} \, m_\rho^2 \, \rho^2$ so that one can read off the mass m_ρ. By comparison with (5.3.10) we find

$$-\frac{1}{2} \, m_\rho^2 = \frac{1}{2} \, m^2 - \frac{3}{4} \, \lambda \, \phi_{\rm vac}^2 = \frac{1}{2} \, m^2 - \frac{3}{4} \, \lambda \left(\sqrt{\frac{2m^2}{\lambda}} \right)^2 = -m^2 \, . \tag{5.3.12}$$

Thus, the mass is given by

$$m_\rho = \sqrt{2} \, m \, . \tag{5.3.13}$$

Obviously, the excitation α does not have a mass term. We have recovered the massless Goldstone boson. The kinetic term does not have the standard form, but a rescaling

$$\alpha := \frac{\tilde{\alpha}}{\sqrt{2}\,\phi_{\rm vac}} \tag{5.3.14}$$

leads to a field with a standard kinetic term. Finally we obtain the new Lagrangian

$$\begin{aligned} \mathcal{L}(\rho,\tilde{\alpha}) &= \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{2}\,m_\rho^2\,\rho^2 + \frac{1}{2}\,\partial_\mu\tilde{\alpha}\,\partial^\mu\tilde{\alpha} \\ &\quad - \frac{\lambda}{2\sqrt{2}}\,\phi_{\rm vac}\,\rho^3 + \frac{1}{\sqrt{2}\,\phi_{\rm vac}}\,\rho\,\partial_\mu\tilde{\alpha}\,\partial^\mu\tilde{\alpha} - \frac{\lambda}{16}\,\rho^4 + \frac{1}{4\,\phi_{\rm vac}}\,\rho^2\,\partial_\mu\tilde{\alpha}\,\partial^\mu\tilde{\alpha}\,, \end{aligned} \tag{5.3.15}$$

where, strictly speaking, the equality hold only up to the irrelevant constant terms. The new Lagrangian (5.3.15) does not show the symmetry (5.3.2) any more. However, since it is equivalent to the original Lagrangian (5.3.1) the symmetry is only hidden.

In the absence of spontaneous symmetry breaking, i.e. if the m^2 term in (5.3.15) had the opposite sign, the two excitations ϕ and $\phi^\dagger$ would be degenerate (with mass m). In contrast, in the presence of spontaneous symmetry breaking there are no degenerate fields any more. Instead one has one massless mode, the Goldstone boson, and one massive mode, the radial excitation. The appearance of the Goldstone boson is generic for spontaneous symmetry breaking. The other mode is specific for the considered model Lagrangian.

As already stressed, the Goldstone boson, $\tilde{\alpha}$, does not have a mass term. Even more, all its interactions displayed in (5.3.15) come with derivatives of the field. This means that very soft, i.e. low-momentum excitations interact only very weakly. Indeed, this is a generic feature of Goldstone bosons that their interactions vanish with vanishing momentum.

5.3.2 Approximate symmetry

The Lagrangian (5.3.1) has an *exact* symmetry (5.3.2) which is spontaneously broken. Let us now add a term which breaks the symmetry *explicitly*. This can be achieved, e.g., by adding a term $\sim {\rm Re}\phi$. The extended Lagrangian is then given by

$$\mathcal{L}_\kappa(\phi^\dagger,\phi) = \partial_\mu\phi^\dagger\,\partial^\mu\phi + m^2\,\phi^\dagger\,\phi - \frac{\lambda}{4}\,(\phi^\dagger\phi)^2 + \kappa\,(\phi+\phi^\dagger)\,. \tag{5.3.16}$$

Obviously in the presence of the additional term the Lagrangian is no longer symmetric with respect to the transformations (5.3.2). We treat κ as a small parameter to ensure that we still have an approximate symmetry. We will also demand

$$\kappa > 0 \tag{5.3.17}$$

without any loss of generality.[37]

The interesting thing to observe is that the vacuum, i.e. the field configuration with the lowest energy is now unique. The effective potential

$$\mathcal{V}_\kappa(\varphi^*,\varphi) := -m^2\,\varphi^*\,\varphi + \frac{\lambda}{4}\,(\varphi^*\varphi)^2 - \kappa\,(\varphi+\varphi^*) \tag{5.3.18}$$

has exactly one absolute minimum. Actually the (small) κ term slightly tilts the mexican hat potential displayed in Fig. 7. This can be seen as follows: Every extremum satisfies

$$\frac{\partial\mathcal{V}_\kappa(\varphi^*,\varphi)}{\partial\varphi} = -m^2\,\varphi^* + \frac{\lambda}{2}\,(\varphi^*)^2\,\varphi - \kappa \stackrel{!}{=} 0 \tag{5.3.19}$$

and

$$\frac{\partial\mathcal{V}_\kappa(\varphi^*,\varphi)}{\partial\varphi^*} = -m^2\,\varphi + \frac{\lambda}{2}\,\varphi^*\,\varphi^2 - \kappa \stackrel{!}{=} 0\,. \tag{5.3.20}$$

Obviously $\varphi = 0$ is not a solution of these equations. Therefore we can multiply the first equation by φ^* and the second by φ and subtract the two equations. We get

$$\kappa\,(\varphi-\varphi^*) = 0\,. \tag{5.3.21}$$

Thus the value for φ where $\mathcal{V}_\kappa$ has a minimum must be real. If we insert this information into (5.3.19) we obtain an algebraic equation of third order:

$$-m^2\,\varphi + \frac{\lambda}{2}\,\varphi^3 - \kappa = 0\,. \tag{5.3.22}$$

In principle such equations are analytically solvable, but the results are complicated and maybe not very illuminating. However, we can do better: Since κ is small, we know that the minimum is close to one of the minima of our symmetric problem ($\kappa = 0$). There we have found that the field configuration φ satisfies (5.3.5). From the condition that φ should be real and that it should be close to the solutions found in (5.3.5) we can conclude that the absolute minimum is either at

$$\varphi \stackrel{?}{=} \sqrt{\frac{2m^2}{\lambda}} + \epsilon =: \phi_{\text{vac}} \qquad \text{for } \epsilon \text{ small}\,. \tag{5.3.23}$$

or at

$$\varphi \stackrel{?}{=} -\sqrt{\frac{2m^2}{\lambda}} + \tilde{\epsilon} =: \tilde{\phi}_{\text{vac}} \qquad \text{for } \tilde{\epsilon} \text{ small}\,. \tag{5.3.24}$$

or that both are absolute minima. Now let us look at the effective potential (5.3.18) for real values of φ:

$$\mathcal{V}_\kappa(\varphi) = -m^2\,\varphi^2 + \frac{\lambda}{4}\,\varphi^4 - 2\kappa\,\varphi\,. \tag{5.3.25}$$

[37] For negative κ just rename the fields $\phi \to -\phi$.

Assume that the absolute minimum is at (5.3.24), i.e. that $\varphi = \tilde{\phi}_{\rm vac}$ is *negative*. Let us calculate — just for fun;-) — the value of the effective potential for $\tilde{\varphi} = -\tilde{\phi}_{\rm vac} > 0$:

$$\mathcal{V}_\kappa(\tilde{\varphi}) := -m^2\,\tilde{\varphi}^2 + \frac{\lambda}{4}\,\tilde{\varphi}^4 - 2\kappa\,\tilde{\varphi} < -m^2\,\tilde{\varphi}^2 + \frac{\lambda}{4}\,\tilde{\varphi}^4 + 2\kappa\,\tilde{\varphi} = \mathcal{V}_\kappa(-\tilde{\varphi}) = \mathcal{V}_\kappa(\tilde{\phi}_{\rm vac})\,. \tag{5.3.26}$$

Thus we have found a field $\varphi = \tilde{\varphi}$ where the effective potential obtains a smaller value than for the assumed absolute minimum. This is a contradiction. Hence, there is only one absolute minimum and it is at $\varphi = \phi_{\rm vac}$ given in (5.3.23).

If we ignore terms of order ϵ^2 we can approximately determine the minimum position from (5.3.22):

$$\begin{aligned} 0 &= -m^2\,\phi_{\rm vac} + \frac{\lambda}{2}\,\phi_{\rm vac}^3 - \kappa = -m^2\left(\sqrt{\frac{2m^2}{\lambda}} + \epsilon\right) + \frac{\lambda}{2}\left(\sqrt{\frac{2m^2}{\lambda}} + \epsilon\right)^3 - \kappa \\ &\approx -m^2\left(\sqrt{\frac{2m^2}{\lambda}} + \epsilon\right) + \frac{\lambda}{2}\left(\left(\sqrt{\frac{2m^2}{\lambda}}\right)^3 + 3\,\frac{2m^2}{\lambda}\,\epsilon\right) - \kappa \end{aligned} \tag{5.3.27}$$

which yields

$$\epsilon \approx \frac{\kappa}{2m^2} \qquad \Rightarrow \qquad \phi_{\rm vac} \approx \sqrt{\frac{2m^2}{\lambda}} + \frac{\kappa}{2m^2}\,. \tag{5.3.28}$$

Since ϵ scales with κ which was assumed to be a small quantity, it was reasonable to neglect terms of order ϵ^2. On the other hand, we also get an idea now what "small κ" means quantitatively: The corrections caused by κ should be small. For the vacuum expectation value $\phi_{\rm vac}$ this implies

$$\frac{\kappa}{2m^2} \ll \sqrt{\frac{2m^2}{\lambda}} \qquad \Rightarrow \qquad \kappa \ll \frac{2\sqrt{2}\,m^3}{\sqrt{\lambda}}\,. \tag{5.3.29}$$

For the case of the Heisenberg magnet we have found that the operator which obtains a non-vanishing vacuum (ground state) expectation value is the one which appears together with the explicit symmetry breaking term (the external magnetic field). In the present example we have two fields, ϕ and $\phi^\dagger$, or expressed differently, Reϕ and Imϕ. It is Re$\phi = \frac{1}{2}\,(\phi + \phi^\dagger)$ which appears together with κ in (5.3.16). And, indeed, the vacuum expectation value (5.3.23) is real. In other words: $\langle {\rm Re}\phi\rangle = \phi_{\rm vac} \neq 0$ while $\langle {\rm Im}\phi\rangle = 0$.

Finally we want to determine the mass of the pseudo Goldstone boson. In general, we can insert the parametrization (cf. (5.3.7) and (5.3.14))

$$\phi(x) = \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}}\,\rho(x)\right)\,e^{i\tilde{\alpha}(x)/(\sqrt{2}\phi_{\rm vac})} \tag{5.3.30}$$

into the Lagrangian (5.3.16) and determine all kinetic, mass and interaction terms. However, we restrict ourselves here to the pseudo Goldstone boson field

$\tilde{\alpha}$ and determine its kinetic and mass term. In other words, we do not need $\rho(x)$. Thus we just replace

$$\phi(x) \to \phi_{\text{vac}}\, e^{i\tilde{\alpha}(x)/(\sqrt{2}\phi_{\text{vac}})} \tag{5.3.31}$$

in the Lagrangian (5.3.16):

$$\mathcal{L}_\kappa \to \frac{1}{2}\,\partial_\mu\tilde{\alpha}\,\partial^\mu\tilde{\alpha} + 2\kappa\,\phi_{\text{vac}}\,\cos\frac{\tilde{\alpha}}{\sqrt{2}\phi_{\text{vac}}} \to \frac{1}{2}\,\partial_\mu\tilde{\alpha}\,\partial^\mu\tilde{\alpha} - \frac{1}{2}\,\frac{\kappa}{\phi_{\text{vac}}}\,\tilde{\alpha}^2\,, \tag{5.3.32}$$

where we have dropped irrelevant constant terms and expanded in powers of the field $\tilde{\alpha}$. Obviously the kinetic term has its standard form[38] and we can read off the mass of the pseudo Goldstone boson as

$$m_G = \sqrt{\frac{\kappa}{\phi_{\text{vac}}}}\,. \tag{5.3.33}$$

Note that it is m_G^2 and not m_G which scales linearly with the symmetry breaking parameter κ. This is qualitatively the same behavior as in the Gell-Mann–Oakes–Renner relation (5.2.2).

5.3.3 Local symmetry — the Higgs mechanism

We have now everything at hand to discuss the spontaneous breaking of a *local* symmetry. Physically this concerns the electroweak theory and (as far as we know[39]) not QCD. But since we are so close it would be a shame not to make this further detour.

Consider the toy model for the electroweak theory introduced in subsection 2.3.3. We combine it with the Lagrangian (5.3.1) in the following way:

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}\,V_{\alpha\beta}V^{\alpha\beta} + \bar{\psi}\,\left(i\partial\!\!\!/ + gV\!\!\!/\,P_L\right)\psi - \frac{1}{2}\,h\,\bar{\psi}\,(\phi+\phi^\dagger)\,\psi - \frac{1}{2}\,h\,\bar{\psi}\,(\phi-\phi^\dagger)\,\gamma_5\,\psi \\ & + (D_\mu\phi)^\dagger\,D^\mu\phi + m^2\,\phi^\dagger\,\phi - \frac{\lambda}{4}\,(\phi^\dagger\phi)^2\end{aligned} \tag{5.3.34}$$

with the field strength

$$V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \tag{5.3.35}$$

and the covariant derivative

$$D_\mu := \partial_\mu - ig\,V_\mu\,. \tag{5.3.36}$$

Obviously, this Lagrangian contains in total 8 degrees of freedom: 4 fermionic (spin up and down for particle and antiparticle), 2 "scalar" and 2 "vector" (as already discussed, a massless vector field has 2 polarizations). The quotation

[38] Otherwise we had to rescale the field $\tilde{\alpha}$ which would change the expression for the mass!

[39] There are ideas where this phenomenon also enters non-perturbative QCD.

marks are meant to indicate that the Lagrangian (5.3.34) is not parity invariant. Therefore, there is no point in distinguishing vector from axial-vector fields or scalars from pseudoscalars. In the following we will see that the particle spectrum of this theory is *not* a massless fermion, a massless vector field and two degenerate scalars. We know already that in the presence of spontaneous symmetry breaking (of a global symmetry) the two degenerate scalars change to one massive and one massless mode. However, as we will see now, if a *local* symmetry is spontaneously broken, one obtains a *massive* fermion, a *massive* vector field and one massive scalar, i.e. *no* Goldstone boson. Still, the total number of degrees of freedom remains the same: 4 fermionic, 1 scalar and 3 vector degrees of freedom (a massive vector field has 3 polarizations); in total again 8 degrees of freedom. As we will see, the Goldstone boson changes to one of the vector degrees of freedom. One says that the Goldstone mode is eaten up by the vector boson. The formalism which provides mass for chiral fermions and for gauge bosons is called Higgs mechanism.

Introducing left- and right-handed fermion fields we can rewrite the Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4} V_{\alpha\beta} V^{\alpha\beta} + \bar{\psi}_R \, i\partial\!\!\!/ \, \psi_R + \bar{\psi}_L \, iD\!\!\!/ \, \psi_L - h \, \bar{\psi}_L \, \phi \, \psi_R - h \, \bar{\psi}_R \, \phi^\dagger \, \psi_L \\ &\quad + (D_\mu \phi)^\dagger \, D^\mu \phi + m^2 \, \phi^\dagger \, \phi - \frac{\lambda}{4} \, (\phi^\dagger \phi)^2 \, . \end{aligned} \tag{5.3.37}$$

In this way it is easy to see that the theory is invariant with respect to global (right-handed) phase transformations,

$$\begin{aligned} \psi_R(x) &\to U_R \, \psi_R(x) \\ \phi(x) &\to \phi(x) \, U_R^\dagger \\ \psi_L(x) &\to \psi_L(x) \\ V^\mu(x) &\to V^\mu(x) \, , \end{aligned} \tag{5.3.38}$$

$U_R \in \mathbf{U}(1)$, and it remains invariant with respect to *local* $\mathbf{U}(1)$ transformations of the left-handed fields, the scalars and the gauge fields, i.e.

$$\begin{aligned} \psi_R(x) &\to \psi_R(x) \\ \phi(x) &\to U_L(x) \, \phi(x) \\ \psi_L(x) &\to U_L(x) \, \psi_L(x) \\ V^\mu(x) &\to U_L(x) \left(V^\mu(x) + \frac{i}{g} \, \partial_x^\mu \right) U_L^\dagger(x) \, , \end{aligned} \tag{5.3.39}$$

$U_L(x) \in \mathbf{U}(1)$. Both symmetries include phase transformations for the scalar field. On the other hand, we know from subsection 5.3.1 that the effective potential of the scalar fields is constructed such that this symmetry is spontaneously broken.

As a next step, we will determine again the vacuum, i.e. the state of lowest energy. The vacuum expectation values of the fields ψ and V_μ must vanish.

Otherwise this would specify a preferred direction (where the spin points to) and Lorentz invariance would not be manifest any more. If we drop ψ and V_μ from the Lagrangian and also all derivative terms, we end up with the very same effective potential given in (5.3.4) and depicted in Fig. 7. Again we choose the vacuum expectation value to be real and positive and parametrize the scalar field as

$$\phi(x) = \left(\phi_{\rm vac} + \frac{1}{\sqrt{2}}\,\rho(x)\right)\, e^{i\alpha(x)} \tag{5.3.40}$$

with hermitian fields $\rho(x)$ and $\alpha(x)$ and the vacuum expectation value

$$\langle\phi\rangle = \phi_{\rm vac} = \sqrt{\frac{2m^2}{\lambda}}\,. \tag{5.3.41}$$

As a first step we are interested in the particle content of this theory. To this end we insert (5.3.40) into the Lagrangian (5.3.34) and expand in powers of all fields. Keeping only terms up to (including) second order we find

$$\begin{aligned}\mathcal{L} \to \mathcal{L}_{\rm quadratic} \;&=\; -\frac{1}{4}\,V_{\alpha\beta}V^{\alpha\beta} + \bar\psi\,(i\partial\!\!\!/ - h\,\phi_{\rm vac})\,\psi + \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho \\ &\quad + g^2\,\phi_{\rm vac}^2\left(V_\mu - \frac{1}{g}\,\partial_\mu\alpha\right)\left(V^\mu - \frac{1}{g}\,\partial^\mu\alpha\right) - m^2\,\rho^2\,. \end{aligned}\tag{5.3.42}$$

Obviously, the fields V_μ and α mix with each other. Therefore, we introduce a new field

$$\tilde V_\mu := V_\mu - \frac{1}{g}\,\partial_\mu\alpha\,. \tag{5.3.43}$$

Remarkably it has the same field strength as the original field:

$$\tilde V_{\mu\nu} := \partial_\mu\tilde V_\nu - \partial_\nu\tilde V_\mu = \underbrace{\partial_\mu V_\nu - \partial_\nu V_\mu}_{=V_{\mu\nu}} - \frac{1}{g}\,\underbrace{(\partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha)}_{=0} = V_{\mu\nu}\,. \tag{5.3.44}$$

Therefore, the quadratic part of the Lagrangian obtains the final form

$$\begin{aligned}\mathcal{L}_{\rm quadratic} \;&=\; \bar\psi\,(i\partial\!\!\!/ - h\,\phi_{\rm vac})\,\psi - \frac{1}{4}\,\tilde V_{\alpha\beta}\tilde V^{\alpha\beta} + \frac{1}{2}\,2g^2\,\phi_{\rm vac}^2\,\tilde V_\mu\,\tilde V^\mu \\ &\quad + \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{2}\,2m^2\,\rho^2\,. \end{aligned}\tag{5.3.45}$$

Thus, we have a massive fermion with mass $h\,\phi_{\rm vac}$, a massive vector boson with mass $\sqrt{2}\,g\,\phi_{\rm vac}$, a massive scalar boson — the Higgs boson — with mass $\sqrt{2}\,m$ and *no (massless) Goldstone boson.*

One might think that the Goldstone boson field, α, could still appear in the interaction terms which we have not worked out yet. However, we shall show now that this is not the case. The field α can be completely eliminated from

the theory by field redefinitions. Actually it is just the local symmetry (5.3.39) (which is spontaneously broken) which allows for the field redefinition — another incarnation of the fact that the symmetry is just hidden. To see this we rewrite the parametrization (5.3.40) in a more compact form

$$\phi(x) = U_G(x)\, H(x) \tag{5.3.46}$$

with the Goldstone-boson field encoded in $U_G(x) \in \mathbf{U}(1)$ and the hermitian Higgs field $H(x) := \phi_{\rm vac} + \frac{1}{\sqrt{2}}\,\rho(x)$. By construction, one has $\langle H\rangle = \phi_{\rm vac}$. This time we insert the decomposition (5.3.46) into (5.3.37). Let us discuss separately the part which contains fermions,

$$\begin{aligned} \mathcal{L}_\psi &:= \bar\psi_R\, i\partial\!\!\!/\, \psi_R + \bar\psi_L\, iD\!\!\!/\, \psi_L - h\,\bar\psi_L\,\phi\,\psi_R - h\,\bar\psi_R\,\phi^\dagger\,\psi_L \\ &= \bar\psi_R\, i\partial\!\!\!/\, \psi_R + \bar\psi_L\, iD\!\!\!/\, \psi_L - h\,\bar\psi_L\, U_G\, H\,\psi_R - h\,\bar\psi_R\, H\, U_G^\dagger\,\psi_L\,, \end{aligned} \tag{5.3.47}$$

the kinetic part for the scalars,

$$\mathcal{L}_{\phi,\rm kinet} := (D_\mu\phi)^\dagger\, D^\mu\phi = (D_\mu\, U_G\, H)^\dagger\, (D^\mu\, U_G\, H)\,, \tag{5.3.48}$$

the effective potential of the scalars,

$$\mathcal{V} = -m^2\,\phi^\dagger\,\phi + \frac{\lambda}{4}\,(\phi^\dagger\phi)^2 = -m^2\, H^2 + \frac{\lambda}{4}\, H^4\,, \tag{5.3.49}$$

and the kinetic part of the vector field,

$$\mathcal{L}_{V,\rm kinet} := -\frac{1}{4}\, V_{\alpha\beta} V^{\alpha\beta}\,. \tag{5.3.50}$$

Obviously, our complete Lagrangian of (5.3.37) is just the sum of these terms:

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_{\phi,\rm kinet} - \mathcal{V} + \mathcal{L}_{V,\rm kinet}\,. \tag{5.3.51}$$

Now we introduce the new fields

$$\psi'_L := U_G^\dagger\,\psi_L \tag{5.3.52}$$

and

$$\tilde V_\mu := U_G^\dagger \left(V_\mu + \frac{i}{g}\,\partial_\mu\right) U_G\,. \tag{5.3.53}$$

Note that the latter definition is identical to (5.3.43) on account of $U_G = e^{i\alpha}$. Here we have just written it in a way which can be generalized to more complicated groups than $\mathbf{U}(1)$.

The fermionic part obtains the form

$$\mathcal{L}_\psi = \bar\psi_R\, i\partial\!\!\!/\, \psi_R + \bar\psi'_L\, U_G^\dagger\, iD\!\!\!/\, U_G\,\psi'_L - h\,\bar\psi'_L\, H\,\psi_R - h\,\bar\psi_R\, H\,\psi'_L\,, \tag{5.3.54}$$

where the derivative contained in D_μ acts on both fields U_G and ψ'_L. The only term where U_G seems to appear explicitly can be rewritten as

$$\begin{aligned} U_G^\dagger\, D_\mu\, U_G\, \psi'_L &= \partial_\mu \psi'_L + U_G^\dagger\, (\partial_\mu U_G)\, \psi'_L - ig\, U_G^\dagger\, V_\mu\, U_G\, \psi'_L \\ &= \partial_\mu \psi'_L - ig\, \tilde{V}_\mu\, \psi'_L = \tilde{D}_\mu \psi'_L\,, \end{aligned} \tag{5.3.55}$$

where we have introduced the gauge covariant derivative

$$\tilde{D}_\mu := \partial_\mu - ig\, \tilde{V}_\mu \tag{5.3.56}$$

for the new gauge field $\tilde{V}_\mu$. The final form of the fermionic part of the Lagrangian is

$$\begin{aligned} \mathcal{L}_\psi(\bar{\psi_R}, \psi_R, \bar{\psi'_L}, \psi'_L, \tilde{V}_\mu, H) &= \bar{\psi_R}\, i\partial\!\!\!/\, \psi_R + \bar{\psi'_L}\, i\gamma^\mu \tilde{D}_\mu\, \psi'_L \\ &\quad - h\, \bar{\psi'_L}\, H\, \psi_R - h\, \bar{\psi_R}\, H\, \psi'_L\,, \end{aligned} \tag{5.3.57}$$

and does *not* explicitly contain the Goldstone boson encoded in U_G.

Essentially in the same way one can rewrite the kinetic part (5.3.48) of the scalar fields:

$$\mathcal{L}_{\phi,\mathrm{kinet}}(\tilde{V}_\mu, H) = (\tilde{D}_\mu H)^\dagger\, \tilde{D}^\mu H\,. \tag{5.3.58}$$

Also this term does not contain U_G any more. The same is, of course, true for the effective potential (5.3.49) which depends only on H.

Finally, it is easy to show that the field strength of the new gauge field $\tilde{V}_\mu$ is identical to the field strength of the original field V_μ. We have shown this already in (5.3.44). Therefore, also the last term

$$\mathcal{L}_{V,\mathrm{kinet}}(\tilde{V}_\mu) := -\frac{1}{4}\, \tilde{V}_{\alpha\beta} \tilde{V}^{\alpha\beta}\,. \tag{5.3.59}$$

of the Lagrangian (5.3.51) has been rewritten such that no trace of the Goldstone-boson field is left.

To conclude we have obtained the identity

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\, V_{\alpha\beta} V^{\alpha\beta} + \bar{\psi_R}\, i\partial\!\!\!/\, \psi_R + \bar{\psi_L}\, iD\!\!\!\!/\, \psi_L - h\, \bar{\psi_L}\, \phi\, \psi_R - h\, \bar{\psi_R}\, \phi^\dagger\, \psi_L \\ &\quad + (D_\mu \phi)^\dagger\, D^\mu \phi + m^2\, \phi^\dagger\, \phi - \frac{\lambda}{4}\, (\phi^\dagger \phi)^2 \\ &= -\frac{1}{4}\, \tilde{V}_{\alpha\beta} \tilde{V}^{\alpha\beta} + \bar{\psi_R}\, i\partial\!\!\!/\, \psi_R + \bar{\psi'_L}\, i\gamma^\mu \tilde{D}_\mu\, \psi'_L - h\, \bar{\psi'_L}\, H\, \psi_R - h\, \bar{\psi_R}\, H\, \psi'_L \\ &\quad + (\tilde{D}_\mu H)^\dagger\, \tilde{D}^\mu H + m^2\, H^2 - \frac{\lambda}{4}\, H^4 \end{aligned} \tag{5.3.60}$$

The upper lines contain the original fields and show all symmetries (5.3.38) and (5.3.39) explicitly. The lower lines contain the new fields and display the particle

content in the presence of spontaneous symmetry breaking. This would be even more explicit if we inserted $H(x) = \phi_{\rm vac} + \frac{1}{\sqrt{2}}\,\rho(x)$. Essentially we have made the changes

$$\begin{aligned} \phi(x) = U_G(x)\, H(x) &\;\to\; H(x) \\ \psi_L(x) &\;\to\; \psi'_L(x) = U_G^\dagger(x)\, \psi_L(x) \\ V_\mu(x) &\;\to\; \tilde{V}_\mu(x) = U_G^\dagger(x) \left(V_\mu(x) + \frac{i}{g}\,\partial_\mu \right) U_G(x)\,. \end{aligned} \tag{5.3.61}$$

to obtain the final expression in (5.3.60) from the original one. But this is just like a gauge transformation (5.3.39). The (hidden) local symmetry ensures that the Goldstone boson disappears when one introduces the new fields H, ψ'_L and $\tilde{V}_\mu$. The Goldstone boson does appear for a spontaneously broken *global* symmetry, but for a *local* symmetry it is eaten up by the gauge boson (and a phase change of the other fields). The gauge boson obtains a mass by this Higgs mechanism.

5.4 Exercises

1. **A downgraded version of the weak theory:**

   Given the Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm weak} &= -\frac{1}{4}\, W^a_{\alpha\beta} W_a^{\alpha\beta} + \overline{\Psi_L}\, i\not{D}\, \Psi_L + \overline{\psi_R}\, i\not{\partial}\, \psi_R + (D_\mu \Phi)^\dagger\, (D^\mu \Phi) \\ &\quad + m^2 \Phi^\dagger \Phi - \frac{\lambda}{4}\, (\Phi^\dagger \Phi)^2 - h\, \left(\overline{\psi_R}\, \Phi^\dagger\, \Psi_L + \overline{\Psi_L}\, \Phi\, \psi_R \right)\,. \end{aligned} \tag{5.4.1}$$

   Here Ψ_L is a two-component left-handed fermion field with components $(\Psi_L)_f$, $f = 1, 2$. One says that Ψ_L is a weak-isospin doublet. ψ_R is a one-component right-handed fermion field. (Concerning weak isospin it is a singlet.) Φ is a two-component (non-Hermitian) scalar field, it is also a weak-isospin doublet. Finally, there is a gauge field for the weak isospin. This three-component gauge field is contained in the 2×2 matrix $W_\mu = W^a_\mu\, t_a$ where t_a are the generators of SU(2) for the two-dimensional representation. The field strength is given by

$$W^{\mu\nu}_a\, t_a = W^{\mu\nu} = \frac{i}{g}\, [D^\mu, D^\nu]\,, \tag{5.4.2}$$

   with the gauge covariant derivative

$$D^\mu := \partial^\mu - igW^\mu\,. \tag{5.4.3}$$

   Note that the W^μ_a are Hermitian fields and that the pure left- or right-handedness of the fermion fields implies

$$\overline{\psi_R}\, \psi_R = 0\,, \quad \left(\overline{\Psi_L} \right)_1 (\Psi_L)_1 = 0\,, \quad \left(\overline{\Psi_L} \right)_2 (\Psi_L)_2 = 0\,. \tag{5.4.4}$$

For clarity some parts of the Lagrangian shall be given with explicit indices for the two-component weak-isospin structure — these indices are called f and g in the following — and the spinor structure — these indices are called j, k:

$$\overline{\Psi_L}\, i\not{D}\, \Psi_L = \left(\overline{\Psi_L}\right)_{fj} i(\gamma_\mu)_{jk}\, \left(\partial^\mu\, \delta_{fg} - ig\, W_a^\mu\, (t_a)_{fg}\right)\, (\Psi_L)_{gk}\,, \tag{5.4.5}$$

$$\overline{\psi_R}\, \Phi^\dagger\, \Psi_L = \left(\overline{\psi_R}\right)_j\, \left(\Phi^\dagger\right)_f\, (\Psi_L)_{fj}\,. \tag{5.4.6}$$

This theory is a down-graded version of the weak theory. It has only one doublet of fermions instead of 6. But it has a Higgs field which — as we will see — gives mass to the vector mesons and to one of the fermions.

Show that the Lagrangian (5.4.1) is invariant under local SU(2) transformations (gauge symmetry of weak isospin)

$$\begin{aligned} \Psi_L(x) &\rightarrow U(x)\, \Psi_L(x) \\ \Phi(x) &\rightarrow U(x)\, \Phi(x) \\ W^\mu(x) &\rightarrow U(x)\, \left(W^\mu(x) + \frac{i}{g}\, \partial_x^\mu\right) U^\dagger(x) \\ \psi_R(x) &\rightarrow \psi_R(x) \end{aligned} \tag{5.4.7}$$

with arbitrary $U(x) \in \mathrm{SU}(2)$. Derive the corresponding Noether current for these transformations (for the Noether current it is sufficient to study global continuous transformations).

Show that mass terms for the fermions and for the gauge bosons, like e.g.

$$\frac{1}{2}\, m_W^2\, W_a^\mu\, W_\mu^a - m_f\, \left(\overline{\psi_R}\, (\Psi_L)_1 + \left(\overline{\Psi_L}\right)_1\, \psi_R\right) \tag{5.4.8}$$

are incompatible with the gauge symmetry.

The ground state of the system (vacuum) is the field configuration with the smallest energy. Assume for simplicity that for this configuration all fields vanish except for $\Phi = \Phi_{\min} := \binom{v}{0}$ with v a real positive number. Determine v. Replace in (5.4.1) the field Φ by $\Phi_{\min}$ and show that mass terms have been dynamically created for the gauge fields W_μ^a and for the fermion field $\psi := \psi_R + (\Psi_L)_1$. Determine these masses.

One can show that any $\Phi(x)$ can be written as (it is a nice exercise, but you do not need to prove it)

$$\Phi(x) = \tilde{U}(x) \begin{pmatrix} v + \frac{1}{\sqrt{2}}\, \rho(x) \\ 0 \end{pmatrix} \tag{5.4.9}$$

with $\tilde{U} \in \mathrm{SU}(2)$ and ρ a Hermitian field. Find new fields

$$\begin{aligned} \psi'(x) &:= \psi_R(x) + (U'(x)\,\Psi_L(x))_1 \\ \psi_L(x) &:= (U'(x)\,\Psi_L(x))_2 \\ W'_\mu(x) &:= U'(x)\left(W_\mu(x) + \frac{i}{g}\,\partial_\mu\right) U'^\dagger(x) \end{aligned} \tag{5.4.10}$$

such that the Lagrangian (5.4.1) can be expressed entirely in terms of the four fields ψ', ψ_L, W'_μ and ρ, i.e. such that $\tilde{U}$ does not appear any more. Determine the masses of the four new fields. (Hint: You should find $U' = \tilde{U}^\dagger$.)

6 Effective field theories

As emphasized in the introductory section the theory of the strong interaction, QCD, has a coupling constant which significantly varies with the energy of the considered reaction, see Fig. 2. For high energies the coupling is small enough and a perturbative expansion in powers of the QCD coupling constant makes sense. For low energies, on the other hand, the perturbation series breaks down and there is the confinement phenomenon which expresses the fact that one observes hadrons and not quarks as freely moving entities. We call the hadrons "relevant degrees of freedom" of low-energy QCD.

We have seen that some features of the hadron spectrum can be understood by analyzing the symmetry pattern of QCD. Can we get more than a qualitative understanding of the low-energy region of hadrons? As we shall see in this section, also a quantitative description with predictive power is possible — at least for energies which are so low that no other hadrons except for the Goldstone bosons are excited.[40] This framework is called "chiral perturbation theory".

The general framework how to describe the dynamics of the relevant degrees of freedom is an *effective (field) theory*. The relevant degrees of freedom can be very different from the elementary/microscopic ones. In particular one might consider bound states of particles. Suppose that one studies reactions of such bound states where the involved energies and momenta are much lower than the binding energy. In this case it is rather intuitive that the relevant degrees of freedom are the bound states and not the particles which are bound together. If the bound states scatter one would like to determine the scattering cross section as a function of the kinematic variables. If one fully understands the particles which form a bound state and their interactions, then one can calculate the scattering cross section microscopically. One the other hand, even without this microscopic knowledge, a

[40] Actually one can also include the nucleon multiplet (see page 99) in a systematic way, see [Scherer] for details. Extensions to other hadrons is much more complicated and a topic of active research.

few parameters related to the bound state are typically enough to determine the cross section, e.g. the size of the bound state, its compressibility, its polarizability, ... In this case, one can fix these parameters by some measurements and make predictions for other kinematical situations. Clearly this effective theory has a natural limit at energies where the bound states break apart. The effective theory is restricted to low energies, but there it can have predictive power.

Another example might illustrate the idea of an effective theory even more: Consider a macroscopic piece of matter moving under the influence of an external force (e.g. gravity). Is it necessary to describe the motion of every single atom inside of this piece of matter? Of course not. An effective description is provided by regarding the piece of matter as a point particle. One parameter enters here, the total mass of the body. If one knows its decomposition in terms of atoms one can calculate the total mass. But also without this knowledge one can make predictions for the behavior of the considered piece of matter under the influence of forces. One just needs one measurement to fix the total mass. If one wants to improve the picture one can consider in addition the intrinsic rotation of this piece of matter. Still one does not need to look at all the single atoms but one can introduce the moment of inertia as an additional parameter. One only has a few relevant degrees of freedom (translational and rotational motion of the piece of matter as a whole) as compared to the huge number of microscopic degrees of freedom (translational motion of all the atoms).

The basic ideas of an effective field theory are:

- use (only) the relevant degrees of freedom;
- write down all possible interactions;
- do not expand in powers of the coupling constants, but instead in powers of the involved energies and momenta ("power counting").

From the previous examples it should become clear that it is not so complicated to isolate the relevant degrees of freedom. What is, however, complicated is to formulate a theory with predictive power for these relevant degrees of freedom. Especially the idea to write down *all* possible interactions can imply that one has to write down infinitely many types of interactions. This seems to be impossible in practice. Even if one has just many different interactions and not infinitely many, the predictive power seems to be marginal. Here, the symmetries of the system under consideration are extremely important. Indeed, chiral perturbation theory as the effective field theory of low-energy QCD has the following properties:

- The Goldstone bosons are regarded as the relevant degrees of freedom. Therefore, the theory is — at best — applicable for low energies where non-Goldstone bosons are not excited.
- Chiral symmetry reduces the number of free parameters and sorts the remaining types of interactions in terms of importance.

- In general the importance of a process/Feynman diagram is classified by the Taylor expansion in powers of energies, momenta and masses.
- As already noted, the nucleon multiplet can be included, but taking other hadrons into account is complicated.

In the following we will make these ideas more quantitative. We will study several simpler systems to get familiar with the concepts of effective (field) theories. Finally we will present the basics of chiral perturbation theory.

6.1 Mechanical example

Consider a particle P in a complicated potential V_1 shown in Fig. 8. Suppose that the energy of P is low, i.e. that P is close to the absolute minimum of V_1. In this case one can approximate V_1 by the quadratic potential V_2. This is

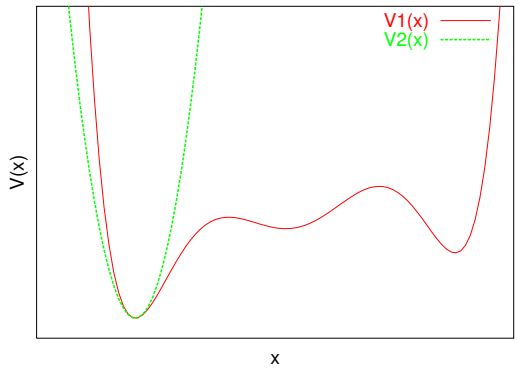


Figure 8: Complete (V_1, red, full line) and effective (V_2, green, dashed line) description for a mechanical potential.

the harmonic approximation known from classical mechanics. The complicated potential V_1 and the quadratic potential V_2 are in general completely different but *effectively* agree for particle P. The potential V_2 is only a polynomial of second order. Therefore it has three parameters (the location in x, the height of the minimum and the curvature). Even if we do not know anything about V_1 we can determine V_2 with a few measurements. For example, one can observe the oscillation of P for a given energy and determine in this way the parameters of V_2. Then we can make predictions for the oscillations of P for different energies — provided the energy is not too large.

6.2 Resonant scattering

The second example concerns a field-theoretical description of the scattering of two particles by forming a resonance. The modulus of the scattering amplitude

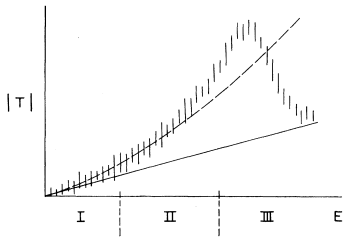


Figure 9: Schematic example for the low-energy approximations for a resonant scattering. The modulus of the scattering amplitude is displayed as a function of the reaction energy $E = \sqrt{s}$. The vertical lines are supposed to be data. Figure taken from Donoghue/Ramirez/Valencia, Phys. Rev. D39, 1947 (1989).

(essentially the Feynman matrix element) is displayed in Fig. 9. The toy-model Lagrangian is given by

$$\mathcal{L}[\phi,\chi] = -\frac{1}{2}\,\chi\,(\Box + M^2)\chi - \frac{1}{2}\,\phi\,(\Box + m^2)\phi - \frac{1}{2}\,g\,\chi\,\phi^2 \tag{6.2.1}$$

with $\Box := \partial_\mu \partial^\mu$. The two particles which scatter are described by the field ϕ. Their interaction happens by coupling to the "resonance" χ with coupling strength g. We assume that the mass m of the ϕ particles is much smaller than the mass M of the resonance, $m \ll M$.

Let us briefly explain the peak in Fig. 9 — albeit it will not be our main interest. If one calculates the process $\phi + \phi \to \phi + \phi$ from (6.2.1) at tree level, one has a contribution to the Feynman matrix element which is proportional to $\frac{1}{s-M^2}$. This contribution becomes very large for $s \approx M^2$. The resonance χ with mass M is excited. Loop corrections take care that the cross section does not blow up to infinity at the resonance position. Physically this is the rescattering process which provides a finite life time for the resonance. For the classical analogue this is the damping of the resonance.

Suppose we are not interested in the energy regime around M where the resonance is excited, but rather in the low-energy regime significantly below M (regimes I and II in Fig. 9). Let us rewrite the Lagrangian (6.2.1) for our purposes:

$$\begin{aligned}
\mathcal{L}[\phi,\chi] &= -\frac{1}{2}\,\chi\,(\Box + M^2)\chi - \frac{1}{2}\,\phi\,(\Box + m^2)\phi - \frac{1}{2}\,g\,\chi\,\phi^2 \\
&= -\frac{1}{2}\left(\chi + \frac{1}{2}\,g\,\phi^2\,\frac{1}{\Box + M^2}\right)(\Box + M^2)\left(\chi + \frac{1}{2}\,g\,\frac{1}{\Box + M^2}\,\phi^2\right) \\
&\quad - \frac{1}{2}\,\phi\,(\Box + m^2)\phi + \frac{1}{8}\,g^2\phi^2\,\frac{1}{\Box + M^2}\,\phi^2 \\
&= -\frac{1}{2}\,\tilde{\chi}\,(\Box + M^2)\,\tilde{\chi} - \frac{1}{2}\,\phi\,(\Box + m^2)\phi + \frac{1}{8}\,g^2\phi^2\,\frac{1}{\Box + M^2}\,\phi^2
\end{aligned} \tag{6.2.2}$$

where we have introduced the new field $\tilde{\chi} := \chi + \frac{1}{2}\,g\,\frac{1}{\Box + M^2}\,\phi^2$. The operator

$G := \frac{1}{\Box+M^2}$ is defined by

$$(\Box_x + M^2)\, G(x,y) = \delta(x-y) \tag{6.2.3}$$

and $\frac{1}{\Box+M^2}\,\phi^2$ is a short-hand notation for $\int d^4y\, G(x,y)\,\phi^2(y)$.

Since we are interested in the scattering of ϕ particles we can ignore the $\tilde{\chi}$ part of the Lagrangian (6.2.2). If one rewrites the Lagrangian into momentum space (as appropriate for the Feynman rules, cf. section 3) the operator $\Box$ turns into a square of some four-momenta. If one restricts the considerations to low momenta (much) smaller than the resonance mass M, then one can expand the non-local operator $\frac{1}{\Box+M^2}$ in a local power series:

$$\frac{1}{\Box + M^2} = \sum_{n=0}^{\infty} \left(\frac{-\Box}{M^2}\right)^n . \tag{6.2.4}$$

Thus at low energies we have to deal with the Lagrangian

$$\mathcal{L} \to -\frac{1}{2}\,\phi\,(\Box + m^2)\phi + \underbrace{\frac{g^2}{8M^2}\,\phi^2 \sum_{n=0}^{\infty} \left(\frac{-\Box}{M^2}\right)^n \phi^2}_{=:\mathcal{L}_{\rm int}} . \tag{6.2.5}$$

The procedure to get rid of the "irrelevant" degrees of freedom which are not excitable — here χ — is called "integrating out degrees of freedom".[41]

Suppose now that we are interested in *very* low energies (region I in Fig. 9). In this case it is legitimate to neglect all terms in the interaction part $\mathcal{L}_{\rm int}$ of (6.2.5) which contain the operator $\Box$. The reason is that this operator translates to momenta. If they are negligibly small as compared to M one can neglect such terms. The interaction part $\mathcal{L}_{\rm int}$ of (6.2.5) is then approximately given by

$$\mathcal{L}_{\rm int} \approx \frac{g^2}{8M^2}\,\phi^4 =: \mathcal{L}^{(1)}_{\rm eff} \qquad \text{for very low energies.} \tag{6.2.6}$$

If one calculates the scattering amplitude from this *effective* Lagrangian $\mathcal{L}^{(1)}_{\rm eff}$ one obtains the full line plotted in Fig. 9.[42] Obviously it gives a reasonable description of the data in energy region I.

At somewhat higher energies one might consider in addition the term linear in the operator $\Box$, but neglect higher orders of this operator:

$$\mathcal{L}_{\rm int} \approx \frac{g^2}{8M^2}\,\phi^4 - \frac{g^2}{8M^4}\,\phi^2\Box\phi^2 =: \mathcal{L}^{(2)}_{\rm eff} \qquad \text{for low energies.} \tag{6.2.7}$$

[41] This phrase stems from the path integral formulation of quantum field theory which we do not discuss here.

[42] Note that the figure is schematic. It is meant to visualize the discussion, not to quantitatively reproduce the results of a calculation.

The result for the scattering amplitude is visualized in Fig. 9 as the dashed line. Obviously it provides a reasonable description of the data in the regions I and II.

Concerning region II the previous approximation, using just $\mathcal{L}_{\text{eff}}^{(1)}$ from (6.2.6), did not provide a good description. One needs $\mathcal{L}_{\text{eff}}^{(2)}$ given in (6.2.7). Concerning region I one can use $\mathcal{L}_{\text{eff}}^{(1)}$, but if one is unsatisfied with the accuracy one can improve the description by using $\mathcal{L}_{\text{eff}}^{(2)}$. In that way one can use more and more extended/complicated Lagrangians which allow to describe data in a larger energy regime or with a larger accuracy. Of course, there is a limit for the whole approach. If the involved energies and momenta become comparable to the resonance mass M, the expansion of (6.2.4) breaks down. An effective theory breaks down if not considered degrees of freedom, here χ, become relevant.

So far we have assumed that we know a theory (6.2.1) which fully describes our scattering problem with "data" shown in Fig. 9. But suppose that we do not know the model parameters g and M very well and that our accelerator for the ϕ-ϕ scattering does not provide enough energy to reach the "high-energy" resonance regime III. Also in this case we can use the idea of an effective field theory (if we have data which are accurate enough): We write more generally

$$\mathcal{L}_{\text{eff}} = c_1\,\phi^4 + c_2\,\phi^2\Box\phi^2 \tag{6.2.8}$$

and fit the parameters c_1 and c_2 to the low-energy data. If we are sure that our full theory is given by (6.2.1) we can determine the "microscopic" parameters g and M from c_1 and c_2 by comparing (6.2.7) with (6.2.8).[43]

So far we have assumed that we know the general form of the microscopic theory (6.2.1) even if we do not know the microscopic parameters g and M. Suppose now that we do not know the microscopic theory at all. The only information which we keep is that ϕ should always appear in pairs, i.e. that the Lagrangian is invariant with respect to the change $\phi \to -\phi$. In this case we can still write down an effective low-energy theory. This is a theory with not too many derivatives, because derivatives translate to energies and momenta which are assumed to be small. The low-energy theory for ϕ-ϕ scattering now takes the form

$$\mathcal{L}_{\text{eff}} = c_1\,f_1(\phi^2) + c_2\,f_2(\phi^2)\,(\partial_\mu\phi)\,(\partial^\mu\phi) \tag{6.2.9}$$

with *arbitrary* functions f_1 and f_2. Of course one can also add terms like $\sim \phi^2\Box^2\phi^2$ to improve the accuracy of the description. But this provides only a correction for small enough energies. Note that our previous term of second order in derivatives, $\phi^2\Box\phi^2$ is equivalent to $-4\,\phi^2\,(\partial_\mu\phi)\,(\partial^\mu\phi)$. One can see this easily by partial

[43] Historically this resembles the determination of parameters of the standard model from Fermi's low-energy theory of the weak interaction. Also here the point was that the W and Z bosons were much heavier than the available reaction energy. Nonetheless they leave their imprint on the low-energy effective theory.

integration (which does not change the action). Therefore, the Lagrangian (6.2.8) is fully covered by our more general expression (6.2.9).

In contrast to (6.2.8) the more general Lagrangian (6.2.9) has lost most of its predictive power because the functions f_1 and f_2 are completely arbitrary. There was one constraint, however. The functions f_1 and f_2 depend on ϕ^2 and not just on ϕ. This gives rise to a selection rule: Processes like $2\phi \to 3\phi$ are forbidden. Obviously symmetries constrain the construction of an effective field theory. We shall study this in more detail with the next example.

6.3 Photon-photon scattering

Suppose we want to describe elastic photon-photon scattering, $\gamma + \gamma \to \gamma + \gamma$ at energies well below the mass of electrons. In principle, one can use quantum electrodynamics (QED). In lowest non-trivial order in the QED coupling constant the relevant diagrams are depicted in Fig. 10. One should get the idea from this

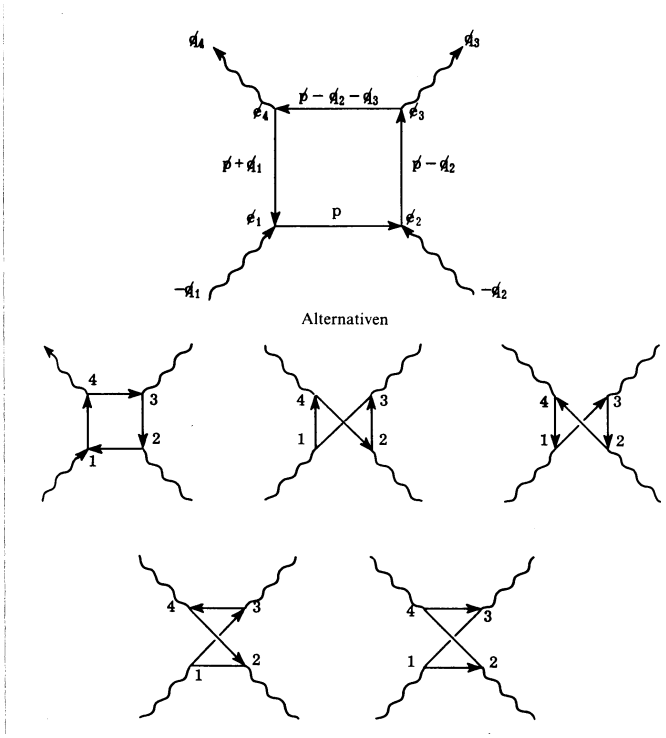


Figure 10: QED diagrams contributing to photon-photon scattering.

figure that the corresponding calculation takes some time. On the other hand, the only relevant degrees of freedom are the photons. Therefore, one can try to construct an effective field theory using just photon fields.[44] This might yield an

[44]This has been worked out first by W. Heisenberg and H. Euler in 1936. The full QED calculation is only from 1950 presented by R. Karplus and M. Neuman.

easier way how to calculate the scattering cross section. Also the effective theory has to obey the symmetries of QED, in particular gauge invariance (2.3.18). Essentially this implies that one can only use the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (and derivatives) for the construction. At low enough energies the first guess for an effective Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + a\, F_{\mu\nu}\Box F^{\mu\nu} + b\,(\partial^\mu F_{\mu\nu})(\partial_\alpha F^{\alpha\nu}) \\ &\quad + c\,(F_{\mu\nu}F^{\mu\nu})^2 + d\, F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} + O(\partial^6) \end{aligned} \tag{6.3.1}$$

which is the most general form up to corrections of order ∂^6. Notably there are only four parameters, a, b, c and d, and not unknown functions as it was the case for our previous example (6.2.9). The gauge invariance of QED together with the derivative expansion has reduced the effective Lagrangian to a finite number of terms. Thus symmetries can significantly reduce the number of allowed interaction terms.

We have reduced the possible interaction terms from infinitely many to four by

- restricting ourselves to the relevant degrees of freedom — here just photons,
- restricting ourselves to low energies — then terms with too many derivatives are negligible (power counting),
- using the symmetries of the system — here gauge invariance (and parity[45]).

Four parameters is a comfortably small number, but we can do even better: From (6.3.1) one can deduce the equation of motion

$$\partial_\mu F^{\mu\nu} - (4a - 2b)\,\Box\partial_\mu F^{\mu\nu} + O(\partial F \cdot F^2) + O(\partial^6) = 0\,. \tag{6.3.2}$$

Therefore we find: $\partial_\mu F^{\mu\nu} = O(\partial^4)$ and $\Box\partial_\mu F^{\mu\nu} = O(\partial^6)$. Consequently an equivalent equation of motion is given by

$$\partial_\mu F^{\mu\nu} + O(\partial F \cdot F^2) + O(\partial^6) = 0\,. \tag{6.3.3}$$

In other words we can drop the terms $\sim a, b$ in the Lagrangian (6.3.1). They look like terms of order ∂^4, but are in fact of order ∂^6. This order is not considered explicitly. Except for corrections of order ∂^6 the Lagrangian (6.3.1) is equivalent to

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + c\,(F_{\mu\nu}F^{\mu\nu})^2 + d\, F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} + O(\partial^6)\,. \tag{6.3.4}$$

Thus, in lowest non-trivial order (in derivatives) the theory of low-energy photon-photon scattering has only two free parameters! One calls such parameters of an effective theory *low-energy constants*.

[45] Otherwise terms with $\epsilon_{\mu\nu\alpha\beta}$ would show up in addition.

These low-energy constants can be obtained from the microscopic theory. One can calculate low-energy photon-photon scattering, on the one hand, from (6.3.4) and, on the other hand, from the diagrams shown in Fig. 10. By matching the calculations one determines c and d. But if one does not know QED or is too lazy/busy/... to calculate the corresponding diagrams one can still determine the two free parameters c and d by a few measurements. Then one has predictive power for other experiments — as long as one stays at low enough energies. This shows that one can use effective theories in two ways:

- If the microscopic theory is unknown or too complicated to provide solutions for the low-energy problems one is interested in, then one can determine the low-energy coefficients by a few measurements. Predictions for other low-energy processes are then possible.

- If the microscopic theory is known it is often still more economic to determine only the low-energy coefficients in the microscopic theory and calculate more complicated processes in the effective theory.

For all these considerations it is mandatory that one writes down the most general effective theory, i.e. *all* terms which are consistent with the symmetries of the system and which contribute to the desired order in the expansion in powers of the energies/momenta.

So far we have not specified what "low enough energies" means. For which energies, E, does the effective theory (6.3.4) apply? (E denotes here the typical energy of the photons, i.e. the energy of a scattering photon in the center-of-mass frame or the total collision energy $\sqrt{s}$.) For our example where we know the microscopic theory, QED, the answer is rather obvious: The electron mass, m_e, is the only scale and the electrons are the degrees of freedom which were neglected. Therefore, for dimensional reasons the expansion parameter is E/m_e. The effective theory breaks down, if E becomes as large as m_e. As long as one is well below this limit the effective theory (6.3.4) should work. Of course one can improve the quality of its description/predictions by including the terms of order ∂^6. If one does not know the microscopic theory it is not so obvious how to determine the breakdown scale of the effective theory. The only thing one can do is to determine the parameters of a few orders in the power expansion and study the convergence properties.

6.4 Basics of chiral perturbation theory

6.4.1 Building blocks and power counting

The task is to develop an effective field theory for QCD at low energy — to be more specific: for the description of the (pseudo) Goldstone bosons at low energies. An important point is again that the symmetries of the microscopic

theory, here QCD, should be respected by the effective theory. This reduces the number of possible interaction terms. The most important symmetry will be the chiral symmetry of QCD. Consequently, the emerging effective field theory is called "chiral perturbation theory" (χPT). We will deal in the following with three light flavors (up, down, strange). It should be clear that the developed theory would work even better for two flavors, just like isospin symmetry works better than three-flavor symmetry. Nonetheless, flavor symmetry turned out to be very useful already when developing the quark model. Therefore, we can expect that it makes sense to develop a chiral effective field theory for the three-flavor case in spite of the fact that the strange-quark mass is not extremely small.

This brings us to the issue how to deal with the fact that chiral symmetry is only an approximate symmetry of QCD and not an exact one. Indeed, there is an interesting way how to incorporate an approximate symmetry in a systematic way: Let us study the Lagrangian (cf. (4.5.1))

$$\begin{aligned}\mathcal{L}_{0,3,\text{ext}} \;:=\; & -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}(x)F^{\mu\nu}(x)) + \sum_{f=c,b,t} \bar{f}(x)\,(i\gamma_\mu D^\mu - m_f)\,f(x) \\ & + \bar{q}(x)\,i\gamma_\mu D^\mu\,q(x) + \bar{q}(x)\,\gamma_\mu v^\mu(x)\,q(x) + \bar{q}(x)\,\gamma_\mu\gamma_5 a^\mu(x)\,q(x) \\ & - \bar{q}(x)\,s(x)\,q(x) + i\,\bar{q}(x)\,\gamma_5 p(x)\,q(x) \end{aligned} \tag{6.4.1}$$

with

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{6.4.2}$$

and with external (classical) hermitian 3×3 flavor-matrix fields v_μ, a_μ, s and p. Concerning Lorentz invariance the external fields transform like vectors (v_μ), axial-vectors (a_μ), scalars (s) and pseudoscalars (p), respectively. For the external vector and axial-vector fields we assume a vanishing flavor trace,

$$\mathrm{tr}\, v_\mu = 0 = \mathrm{tr}\, a_\mu\,. \tag{6.4.3}$$

On the one hand, one obtains from (6.4.1) full QCD, (4.1.1), by setting $v_\mu, a_\mu, p \to 0$ and

$$s \to \mathcal{M}_{q,3} := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}\,. \tag{6.4.4}$$

On the other hand, the Lagrangian of (6.4.1) has a *local* (and exact) chiral symmetry. To see this we introduce the usual left- and right-handed quark fields, $q_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\,q$, and, in addition, the combinations

$$\begin{aligned} r_\mu &:= v_\mu + a_\mu\,, & l_\mu &:= v_\mu - a_\mu\,, \\ \mathcal{M} &:= s + ip \quad \Leftrightarrow & \mathcal{M}^\dagger &= s - ip\,. \end{aligned} \tag{6.4.5}$$

The new external fields are left- and right-handed vector fields, l_μ and r_μ, respectively, and a generalized mass term, $\mathcal{M}$.

We can rewrite the Lagrangian (6.4.1) in the following way:

$$\begin{aligned}\mathcal{L}_{0,3,\text{ext}} &= -\frac{1}{2}\,\text{tr}\,(F_{\mu\nu}(x)F^{\mu\nu}(x)) + \sum_{f=c,b,t}\bar{f}(x)\,(i\gamma_\mu D^\mu - m_f)\,f(x) \\ &\quad + \bar{q_R}(x)\,i\gamma_\mu D^\mu\,q_R(x) + \bar{q_R}(x)\,\gamma_\mu r^\mu(x)\,q_R(x) \\ &\quad + \bar{q_L}(x)\,i\gamma_\mu D^\mu\,q_L(x) + \bar{q_L}(x)\,\gamma_\mu l^\mu(x)\,q_L(x) \\ &\quad - \bar{q_R}(x)\,\mathcal{M}(x)\,q_L(x) - \bar{q_L}(x)\,\mathcal{M}^\dagger(x)\,q_R(x)\,. \end{aligned} \tag{6.4.6}$$

Now it is straightforward to show that (6.4.6) is invariant with respect to the local flavor transformations

$$\begin{aligned} q_R(x) &\to V_R(x)\,q_R(x)\,, \\ \bar{q_R}(x) &\to \bar{q_R}(x)\,V_R^\dagger(x)\,, \\ v_\mu(x)+a_\mu(x)=r_\mu(x) &\to V_R(x)\,(r_\mu(x)+i\partial_\mu)\,V_R^\dagger(x)\,, \\ q_L(x) &\to V_L(x)\,q_L(x)\,, \\ \bar{q_L}(x) &\to \bar{q_L}(x)\,V_L^\dagger(x)\,, \\ v_\mu(x)-a_\mu(x)=l_\mu(x) &\to V_L(x)\,(l_\mu(x)+i\partial_\mu)\,V_L^\dagger(x)\,, \\ s(x)+ip(x)=\mathcal{M}(x) &\to V_R(x)\,\mathcal{M}(x)\,V_L^\dagger(x)\,, \\ s(x)-ip(x)=\mathcal{M}^\dagger(x) &\to V_L(x)\,\mathcal{M}^\dagger(x)\,V_R^\dagger(x)\,, \end{aligned} \tag{6.4.7}$$

where $V_{R,L}(x) \in \mathbf{SU}(3)$. Thus, we have isolated an exact and local invariance with respect to the chiral flavor group $\mathbf{SU}_R(3)\times\mathbf{SU}_L(3)$. Note that we have not displayed explicitly that the gluons and the heavy quark fields are supposed to remain unchanged with respect to these light-flavor transformations.

The next step is to develop an effective field theory for (6.4.6) which possesses the same symmetries, in particular the local chiral symmetry (6.4.7). Assuming that this chiral symmetry is spontaneously broken, we want to develop an effective theory for the relevant low-energy degrees of freedom, the Goldstone bosons, i.e. the eight lightest pseudoscalar mesons π^0, $\pi^\pm$, $K^\pm$, K^0, $\bar{K}^0$, η. We collect these fields in an eight-component object, $\hat{\phi}_a$, $a=1,\ldots,8$. It is used to form a 3×3 flavor matrix

$$\hat{\phi} = \sum_{a=1}^{8}\hat{\phi}_a\,t^a \tag{6.4.8}$$

with the generators t^a of $\mathbf{SU}(3)$ given in (4.0.16), (4.0.17). Up to a normalization factor, which we will specify below, this flavor matrix is given by

$$\hat{\phi} \sim \begin{pmatrix} \pi^0+\frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0+\frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta \end{pmatrix}. \tag{6.4.9}$$

Note that $\hat{\phi}$ is hermitian

$$\hat{\phi}^\dagger = \hat{\phi} \tag{6.4.10}$$

since, e.g., π^- is supposed to be the antiparticle of π^+, i.e. $\pi^- = (\pi^+)^\dagger$.

Strictly speaking the fields π^0 and η are not the corresponding physical states. Instead these fields are eigen states of the isospin operators. In the following we will assume for simplicity exact isospin symmetry by neglecting the difference between the up- and the down-quark mass: $m_u, m_d \to m_q$. We will keep, however, the difference between m_q and the strange-quark mass: $m_s \neq m_q$.

Now we recall from our toy model of subsection 5.3.1 that it is appropriate to parametrize the Goldstone bosons in the same way as the spontaneously broken symmetry, cf. (5.3.7) and (5.3.2). The symmetry which is spontaneously broken in QCD is $\mathbf{SU}_A(3)$. Therefore, we encode the Goldstone bosons in the special unitary 3×3 flavor matrix

$$U(x) = e^{i\hat{\phi}(x)}\,. \tag{6.4.11}$$

The effective theory does not contain the quarks and gluons any more. It does, however, contain the external fields v_μ, a_μ, s and p with their transformation properties given in (6.4.7). And, of course, it should contain $U(x)$ introduced in (6.4.11). We have to figure out how $U(x)$ transforms under the local chiral transformations. We will be rather heuristic here and refer to [Scherer] for the mathematical details. Expanding $U(x)$ up to linear order in the fields, $U \approx \mathbb{1} + i\hat{\phi}$ we find a structure "scalar plus i times pseudoscalar". The corresponding structure in (6.4.7) transforms like $V_R(x) \ldots V_L^\dagger(x)$. Therefore, we assume the same for U: With respect to local chiral flavor transformations we demand

$$U(x) \to V_R(x)\, U(x)\, V_L^\dagger(x)\,. \tag{6.4.12}$$

In accord with the local chiral symmetry (6.4.7), (6.4.12) our building blocks for the effective theory are

- the unitary matrix $U(x)$ which contains the Goldstone boson fields,
- the matrix $\mathcal{M}(x) = s(x) + ip(x)$ which contains the external scalar and pseudoscalar fields and which brings back ordinary full QCD by the replacement (cf. (6.4.4))

$$\mathcal{M}(x) \to \mathcal{M}_{q,3} \approx \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix}\,, \tag{6.4.13}$$

  where we have neglected again the mass difference between up- and down-quark mass in accordance with our particle assignments (6.4.9),

- a chiral gauge covariant derivative D_μ defined by its action on arbitrary objects A which transform as $A(x) \to V_R(x)\, A(x)\, V_L^\dagger(x)$:

$$D_\mu A := \partial_\mu A - i r_\mu\, A + i A\, l_\mu\,, \tag{6.4.14}$$

- field-strength tensors (cf. (4.1.5)) constructed from the left- and right-handed external vector fields:

$$\begin{aligned} f^R_{\mu\nu} &:= \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu]\,, \\ f^L_{\mu\nu} &:= \partial_\mu l_\nu - \partial_\nu l_\mu - i\,[l_\mu, l_\nu]\,. \end{aligned} \tag{6.4.15}$$

Note that the definition (6.4.14) of the chiral gauge covariant derivative D_μ ensures that $D_\mu A$ transforms like A (check it!):

$$A(x) \to V_R(x)\, A(x)\, V_L^\dagger(x) \qquad \Rightarrow \qquad D_\mu A(x) \to V_R(x)\, D_\mu A(x)\, V_L^\dagger(x)\,. \tag{6.4.16}$$

The field strength tensors transform homogeneously and according to their respective label (L or R):

$$\begin{aligned} f^R_{\mu\nu}(x) &\to V_R(x)\, f^R_{\mu\nu}(x)\, V_R^\dagger(x)\,, \\ f^L_{\mu\nu}(x) &\to V_L(x)\, f^L_{\mu\nu}(x)\, V_L^\dagger(x)\,. \end{aligned} \tag{6.4.17}$$

Next we have to define our power counting, i.e. we have to assign a degree of importance to our building blocks. In accord with our previous discussions we assume

$$D_\mu \sim O(Q) \tag{6.4.18}$$

where Q denotes a typical (small) energy or momentum, e.g. something in the order of the masses of the pseudo Goldstone bosons. The power counting will be organized as a Taylor expansion in powers of Q. Since field strengths are built from two gauge covariant derivatives we demand

$$f^{R/L}_{\mu\nu} \sim O(Q^2)\,. \tag{6.4.19}$$

In ordinary perturbation theory one assumes that the coupling constants for all interactions are small. In such a framework interaction terms involving a larger number of fields are typically smaller. For the case at hand we will not use ordinary perturbation theory. On the contrary, terms with a large number of fields can still be significant. Therefore, we do not assume any suppression for the Goldstone boson fields $\hat\phi$ or U:

$$U(x) \sim O(Q^0)\,. \tag{6.4.20}$$

Finally we have to assign powers of the typical momentum Q to the generalized mass term, the external field $\mathcal{M}$. We will see below that it is *not* the masses of

the Goldstone bosons which are proportional to $\mathcal{M}$. Instead it is the square of the Goldstone-boson masses. We have already presented such a relation, so far without justification — the Gell-Mann–Oakes–Renner relation (5.2.2). Below we will derive this relation. Anticipating these later results we demand

$$\mathcal{M} \sim m_G^2 \sim O(Q^2)\,, \tag{6.4.21}$$

where we have denoted the generic mass of a pseudo Goldstone boson by m_G.

Now we are ready to write down for given (small) n the most general Lagrangian of order Q^n which is built from U, D_μ, $f_{\mu\nu}^{R/L}$ and $\mathcal{M}$ and which is invariant with respect to local chiral transformations (and with respect to parity and charge-conjugation symmetry). At order Q^0 we only have U to build a Lagrangian and the result must be hermitian. The only conceivable structures are $U^\dagger U = \mathbb{1}$ and $\det U = 1$. Recall that U is by construction a special unitary matrix. We see that at order Q^0 there is no non-trivial contribution to the Lagrangian. Odd powers in Q^n do not appear for two reasons: First, because of Lorentz invariance. Lorentz indices and, therefore, derivatives must appear in squares. Second, all building blocks except for the derivatives are of even order in Q^n. The lowest non-trivial order is Q^2. Here we can build the three hermitian structures

$$\mathrm{tr}\left((D_\mu U)^\dagger D^\mu U\right)\,, \quad \mathrm{tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) \quad \text{and} \quad i\,\mathrm{tr}(U^\dagger \mathcal{M} - \mathcal{M}^\dagger U)\,. \tag{6.4.22}$$

The symbol "tr" denotes a flavor trace. Note that

$$\mathrm{tr}(U^\dagger D_\mu U) = 0 \tag{6.4.23}$$

for special unitary matrices U. For the partial derivative this has been proven in (4.0.19). For the external vector fields which come into play via (6.4.14) one has to use (6.4.3).

A Lagrangian which contains only three terms is already very simple. However, there is even a further constraint: We also want the effective Lagrangian to be parity invariant. With respect to a parity transformation the generalized mass matrix $\mathcal{M} = s + ip$ changes into $s - ip = \mathcal{M}^\dagger$. The same applies to U since the Goldstone bosons are pseudoscalars:

$$U(x) = e^{i\hat{\phi}(x)} \quad \overset{\text{parity}}{\longrightarrow} \quad e^{-i\hat{\phi}(x)} = U^\dagger(x)\,. \tag{6.4.24}$$

The requirement of parity invariance forbids the third structure in (6.4.22). The lowest-order Lagrangian of chiral perturbation theory is given by

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} := \frac{1}{4}\,F_0^2\,\left(\mathrm{tr}\left((D_\mu U)^\dagger D^\mu U\right) + 2B_0\,\mathrm{tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U)\right)\,. \tag{6.4.25}$$

Obviously it has only two free parameters, F_0 and B_0. Below we will work out some consequences of the lowest-order Lagrangian and we will construct the next-to-leading-order Lagrangian.

A final remark on our "trick", (6.4.1), (6.4.4), how the quark masses sneaked into a theory which is chirally symmetric by introducing external fields. We have motivated this technique as an interesting *possibility* how to incorporate an approximate symmetry. However, from the quantum field theoretical point of view it is actually a *necessity* to promote global to local symmetries. We will not go into detail here but just note that a fundamental quantity of quantum field theory is the "generating functional" from which all n-point (Green's) functions can be obtained. In turn this generating functional is based on the action obtained from the Lagrangian (6.4.1). Therefore, the promotion of a global to a local symmetry is automatic in a quantum field theory. What makes, e.g., chiral symmetry different from a dynamical local symmetry like the gauge symmetry of QED or the color-gauge symmetry of QCD is the fact that in the latter cases the gauge fields are not external classical fields but active quantum degrees of freedom with their own dynamics. In other words, our world contains photons and gluons, but no gauge particles assigned to the chiral symmetry.

6.4.2 Consequences of the leading-order Lagrangian

The Lagrangian in (6.4.25) looks very simple. In particular it is appealing that it contains only two free parameters, F_0 and B_0. Strictly speaking one should also count the quark masses as free parameters. Then one finds three parameter combinations: F_0, $B_0 m_q$ and $B_0 m_s$, cf. (6.4.13). On the other hand, the Lagrangian (6.4.25) is far from being trivial: The special unitary matrix U contains the Goldstone boson fields in arbitrary powers. Therefore, even this leading-order Lagrangian contains a lot of non-trivial interactions, all governed by the same few parameters. This should provide predictive power for low-energy processes, i.e. non-trivial checks of our understanding of chiral perturbation theory and in that way of QCD.

Let us start with the field content of (6.4.25), i.e. the free-field terms. To this end we expand $U = e^{i\hat{\phi}}$ in powers of $\hat{\phi}$ and keep the terms up to (including) second order in the fields. We drop the external vector fields r_μ and l_μ contained in the chiral gauge covariant derivative (6.4.14) and replace the generalized mass matrix $\mathcal{M}$ by the quark-mass matrix, (6.4.13). Except for constant terms which do not contribute to the equations of motion we obtain a kinetic and a mass term. The kinetic term is given by

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} \overset{\text{kinetic}}{\longrightarrow} \frac{1}{4} F_0^2 \,\mathrm{tr}(\partial_\mu \hat{\phi}\, \partial^\mu \hat{\phi})\,. \tag{6.4.26}$$

If we now choose (cf. (6.4.9))

$$\hat{\phi} = \frac{1}{F_0} \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix} \tag{6.4.27}$$

we obtain the proper kinetic terms for charged and neutral particles, respectively:

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} \stackrel{\text{kinetic}}{\to} \frac{1}{2}\,\partial_\mu\pi^0\,\partial^\mu\pi^0 + \partial_\mu\pi^-\,\partial^\mu\pi^+ + \frac{1}{2}\,\partial_\mu\eta\,\partial^\mu\eta + \partial_\mu K^-\,\partial^\mu K^+ + \partial_\mu\bar{K}^0\,\partial^\mu K^0\,. \tag{6.4.28}$$

The mass term is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\chi\mathrm{PT}} &\stackrel{\text{mass}}{\to} -\frac{1}{2}\,F_0^2\,B_0\,\mathrm{tr}(\mathcal{M}\,\hat{\phi}^2)\\ &= -\frac{1}{2}\,B_0\,\Big(2\,m_q\,(\pi^0)^2 + 4\,m_q\,\pi^+\,\pi^- + \frac{2}{3}\,(m_q+2m_s)\,\eta^2\\ &\qquad + 2\,(m_q+m_s)\,K^+\,K^- + 2\,(m_q+m_s)\,K^0\,\bar{K}^0\Big)\,.\end{aligned} \tag{6.4.29}$$

This yields a common pion mass (squared),

$$m_\pi^2 = 2\,m_q\,B_0\,, \tag{6.4.30}$$

a common kaon mass,

$$m_K^2 = (m_q+m_s)\,B_0\,, \tag{6.4.31}$$

and a mass for the eta meson

$$m_\eta^2 = \frac{2}{3}\,(m_q+2m_s)\,B_0\,. \tag{6.4.32}$$

Obviously we have three masses, m_π, m_K and m_η, expressed in terms of two parameter combinations, $B_0\,m_q$ and $B_0\,m_s$. In other words, we have a prediction for one of the masses in terms of the other two, e.g.

$$m_\eta^2 = \frac{4}{3}\,m_K^2 - \frac{1}{3}\,m_\pi^2\,. \tag{6.4.33}$$

Using the experimental numbers for the (isospin averaged) pion and kaon masses one obtains $m_\eta \approx 567\,\mathrm{MeV}$. This is very close to the experimental value of 547 MeV. Note that this is a leading-order calculation with the effective Lagrangian. Of course, there are corrections from higher orders in the Taylor expansion in powers of the typical momentum $Q \sim m_{\pi,K}$, i.e.

$$m_\eta^2 = \frac{4}{3}\,m_K^2 - \frac{1}{3}\,m_\pi^2 + O(m_{\pi,K}^4)\,. \tag{6.4.34}$$

There are further relations which one can deduce from (6.4.30)-(6.4.32). First, we see that indeed the squares of the masses, m_π, m_K and m_η, of the pseudo Goldstone bosons scale linearly with the quark masses. Thus, it was reasonable

to assign in (6.4.21) the order Q^2 to the quark-mass term $\mathcal{M}$. Second, we obtain a prediction for a quark-mass ratio:

$$\frac{m_s}{m_q} = 2\,\frac{m_K^2}{m_\pi^2} - 1 \approx 25\,. \tag{6.4.35}$$

Indeed, this prediction, refined by next-to-leading-order terms, enters the estimates for the quark masses given in [PDG]. In other words, chiral perturbation theory is a crucial ingredient for the determination of the fundamental properties of the elementary particles.

As already pointed out, our lowest-order effective Lagrangian (6.4.25) contains three parameters, F_0, $B_0\,m_q$ and $B_0\,m_s$. In (6.4.30), (6.4.31) we have related two of them to experimental data (masses of pseudoscalar mesons). Finally we shall comment on the third parameter, the "pion decay constant" F_0. Once all three parameters have been fixed one has full predictive power for all conceivable low-energy processes.

So far we have used the external fields $s(x)$, $p(x)$, $v_\mu(x)$ and $a_\mu(x)$ in (6.4.1) as a book-keeping device for chiral symmetry: The fields $v_\mu(x)$ and $a_\mu(x)$ promote the chiral symmetry to a local symmetry, the fields $s(x)$, $p(x)$ keep track of the quark masses in a way consistent with the chiral symmetry. The (axial-)vector fields v_μ and a_μ can be used for an additional purpose concerning real physics: If one replaces

$$v_\mu \to \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} e\,A_\mu \tag{6.4.36}$$

with the photon field A_μ, then one can describe the coupling of QED to the quarks. Note that the matrix in (6.4.36) contains the quark charges (relative to the positron charge e). Since v_μ also appears in the effective theory (6.4.25), on account of (6.4.14), (6.4.5), we have a description how QED couples to composite objects, the pseudoscalar hadrons.

The corresponding statement holds also for the weak interaction. Without going into details the crucial quantities are the vector fields v_μ and a_μ. The weak interaction is responsible for the finite life time of the charged pion (and for the life times of the kaons). Let us work out the coupling of a pion to the external vector fields based on the effective Lagrangian (6.4.25). To this end we expand the Lagrangian to linear order in $\hat{\phi}$ and keep only terms of linear order in the external vector fields. This yields

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} \overset{\text{pion decay}}{\to} -F_0^2\,\mathrm{tr}(a^\mu\,\partial_\mu\hat{\phi})\,. \tag{6.4.37}$$

Using (6.4.27) we find that the axial-vector current couples with a strength proportional to F_0 to the Goldstone bosons. Therefore, up to factors specific for

the weak interaction the pion-decay width is proportional to F_0^2, since the corresponding amplitude is proportional to F_0. Therefore, one calls F_0 the "pion decay constant". The remaining parameter of our leading-order effective field theory can be extracted from a weak-interaction process, the pion decay. Numerically one obtains $F_0 \approx 92\,\mathrm{MeV}$. If one improves the effective theory by including the terms of order Q^4 (see below), one obtains additional contributions to the pion decay. Still, the "pion decay constant" which one extracts from the pion decay data remains the same, but it is no longer just F_0. Thus it makes sense to distinguish conceptually between the physical pion decay constant, f_π, extracted from the data and the leading-order pion decay constant F_0. The difference is only relevant if one works out the effective Lagrangian of order Q^4 which is by order $Q^2 \sim m_{\pi,K}^2 \sim m_{q,s}$ suppressed relative to the leading-order effective Lagrangian (6.4.25), i.e.

$$f_\pi = F_0 + O(m_{q,s})\,. \tag{6.4.38}$$

Having determined all free parameters one can study whether there are further interaction processes which are governed by the lowest-order effective Lagrangian (6.4.25). Indeed, it does not only contain free-field terms and couplings to external fields. Expanding U in powers of $\hat{\phi}$ yields n-point interactions for arbitrary high n. In particular, the $2 \to 2$ scattering processes of Goldstone bosons into (the same or other) Goldstone bosons are obtained by expanding U up to (including) fourth order in $\hat{\phi}$. We do not go into detail here but just discuss a few qualitative facts: In the "chiral limit", i.e. if one neglects all quark masses, the interactions emerge from $\frac{1}{4} F_0^2 \,\mathrm{tr}\left((\partial_\mu U)^\dagger \partial^\mu U\right)$. Obviously, the interactions among Goldstone bosons contain derivatives, a fact generic for Goldstone bosons as already discussed in section 5.[46] Using (6.4.27) one finds that the 4-point interactions which lead to the $2 \to 2$ scattering processes scale with $1/F_0^2$. Actually, if one includes other hadrons in the effective theory[47] one finds that also their interaction with the Goldstone bosons is given by terms which scale with $1/F_0^2$ (in the chiral limit). No other parameters enter here. This is a prediction from chiral perturbation theory that the strength of the low-energy interaction of any hadron with the Goldstone bosons is dictated by only one parameter F_0. Even more this parameter which dictates the strong-interaction processes is related to a weak-interaction process, the pion decay. The discussed low-energy strong-interaction terms are called "Weinberg-Tomozawa" interactions. To the extent that it could be tested the experimental results are in agreement with the predicted behavior.

So far, we have treated B_0 as a free parameter of our effective theory, unrelated to the microscopic theory, QCD. However, we can relate it to the quark

[46] In the presence of explicit symmetry breaking, i.e. here for finite quark masses, there are additional interaction terms which scale with the symmetry breaking quark masses.

[47] In general, especially for hadronic resonances, this can be conceptually difficult, but for nucleons it is possible.

condensate in the following way. Suppose we evaluate the vacuum expectation value

$$\langle \bar{u}u + \bar{d}d \rangle = -\frac{\partial}{\partial m_q} \langle \mathcal{L}_{\rm QCD} \rangle \,. \tag{6.4.39}$$

Clearly, a non-vanishing vacuum expectation value of a local quantity is a low-energy phenomenon: Because of Lorentz invariance it is the same everywhere. Hence one does not need high resolution, i.e. large momenta, to resolve a vacuum expectation value. Instead of using the QCD Lagrangian we can use as well the effective Lagrangian of chiral perturbation theory which in lowest order is given by (6.4.25). Replacing $\mathcal{M}$ by the quark-mass matrix, as in (6.4.13), one obtains

$$\langle \bar{u}u + \bar{d}d \rangle = -\frac{1}{2} F_0^2 B_0 \,{\rm tr}(\langle U + U^\dagger \rangle P_2) \tag{6.4.40}$$

with the flavor projection operator on the upper two components

$$P_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\partial}{\partial m_q} \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix} , \tag{6.4.41}$$

where we have indicated on the right-hand side how this projection operator emerges.

Without proof we assume that the ground state of the effective theory is given by $\hat{\phi} = 0$, i.e. $\langle U \rangle = \mathbb{1}$. Then we obtain

$$\langle \bar{u}u + \bar{d}d \rangle = -2 F_0^2 B_0 \,. \tag{6.4.42}$$

Together with (6.4.30) this leads to the famous Gell-Mann–Oakes–Renner relation (in leading order)

$$m_\pi^2 F_0^2 = -m_q \langle \bar{u}u + \bar{d}d \rangle \tag{6.4.43}$$

which has already been presented in (5.2.2). This equation contains "macroscopic" (hadronic) quantities on the left-hand side, mass and decay constant of the pion, and microscopic (quark) quantities on the right-hand side, isospin averaged light-quark mass and quark condensate. At present, we still get information about the right-hand side from the left-hand side. If our knowledge about QCD at low energies grows, then one day in the future we can predict the macroscopic quantities from the microscopic ones.

So far we have restricted ourselves to the quark condensate for the two lightest flavors. However, lowest-order chiral perturbation theory (used for three flavors) provides also a prediction for the strange-quark condensate:

$$\begin{aligned} \langle \bar{s}s \rangle &= -\frac{\partial}{\partial m_s} \langle \mathcal{L}_{\rm QCD} \rangle \approx -\frac{\partial}{\partial m_s} \langle \mathcal{L}^{(2)}_{\chi{\rm PT}} \rangle \\ &\overset{(*)}{=} -\frac{1}{2} F_0^2 B_0 \,{\rm tr}\left(\langle U + U^\dagger \rangle (\mathbb{1} - P_2) \right) = -F_0^2 B_0 = \frac{1}{2} \langle \bar{u}u + \bar{d}d \rangle \,. \end{aligned} \tag{6.4.44}$$

At the step denoted by $(*)$ we have performed the usual replacement (6.4.13). Since we have assumed exact isospin symmetry, we have in addition

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \,. \tag{6.4.45}$$

In other words, there is a nonvanishing quark condensate for each light flavor. In lowest-order chiral perturbation theory these condensates are of same size:

$$\langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle \approx \langle \bar{s}s \rangle \approx \frac{1}{3} \langle \bar{q}q \rangle \,. \tag{6.4.46}$$

Corrections come at order $Q^2 \sim m_{q,s}$ relative to these leading-order estimates, i.e. for example

$$\langle \bar{u}u \rangle - \langle \bar{s}s \rangle = O(m_{s,q}) \,. \tag{6.4.47}$$

Note that such condensates break the chiral symmetry (4.5.7) of QCD. They are order parameters of spontaneous chiral symmetry breaking.

6.4.3 Beyond leading order — the importance of loops

So far we have restricted our considerations to tree-level processes which emerge from effective Lagrangians.[48] However, the Lagrangian (6.4.25) of lowest-order chiral perturbation theory also gives rise to loop diagrams. If all loop diagrams were of the same importance we would have the same problem as with QCD for low energies. Any calculation including a finite number of diagrams would be inappropriate. Fortunately this is not the case. We will show in the following that the more loops a diagram of chiral perturbation theory has, the less important it is, i.e. that it is of higher order in the Taylor expansion of the typical momentum scale Q. These considerations have been put forward by S. Weinberg.

This does not mean that loops are always negligible. Quite contrary loops are important if one wants to improve the accuracy of the calculation by going to higher powers in Q. However, one always has to include only a finite, well-defined set of loop diagrams, i.e. the task is manageable — just like it is the case for ordinary perturbation theory in powers of coupling constants. We will construct the next-to-leading-order Lagrangian of chiral perturbation theory and will point out which tree and which loop diagrams are necessary for next-to-leading-order calculations.

Let us start with an example illustrating at which order loop diagrams come in: Suppose we want to calculate the elastic scattering of two Goldstone bosons for low enough energies. The involved energies are of order Q and the idea is to determine the scattering amplitude as a power series in Q. A tree-level and a one-loop contribution are schematically shown in Fig. 11. We use the vertices

[48] Feynman diagrams without loops are called "tree level".

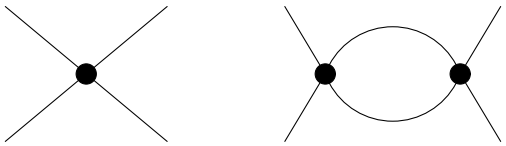
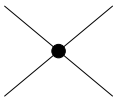


Figure 11: Some Feynman diagrams for elastic scattering of Goldstone bosons. The lines represent Goldstone bosons. The vertices are supposed to come from the lowest-order Lagrangian (6.4.25).

from lowest-order chiral perturbation theory, i.e. from (6.4.25). The derivatives translate to momenta in the Feynman rules. External boson lines contribute just with a factor 1. Therefore, the contribution of the tree-level diagram of Fig. 11 with a vertex from the derivative term in (6.4.25) contributes at order Q^2 to the two-body scattering amplitude. The vertex which emerges from the B_0 term of (6.4.25) is proportional to the quark masses which in turn are proportional to the squares of the Goldstone boson masses. Thus, also tree-level diagrams from this part of the leading-order Lagrangian contribute at order Q^2, i.e.

 $\sim Q^2\,.$ (6.4.48)

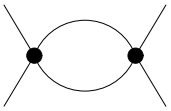
The loop diagram of Fig. 11 contains two vertices and two propagators. Formally loop diagrams diverge, but they can be renormalized with the help of so-called counter terms. Actually the next-to-leading-order Lagrangian of chiral perturbation theory, which we will present below, provides the appropriate counter terms. We will not go into details here, but suppose that the renormalization procedure has been carried out. Then the scale at which the loop is renormalized sets the typical scale for all momenta which appear in the loop. Since we are interested in external momenta of the order of Q, the renormalization is also carried out at around Q. A propagator of a (Goldstone boson) particle with mass m_G and for momentum k is then of order

$$\frac{1}{k^2 - m_G^2} \sim \frac{1}{Q^2}\,. \tag{6.4.49}$$

The loop causes a four-dimensional integration which for dimensional reasons scales like

$$\int d^4k \sim Q^4\,. \tag{6.4.50}$$

In total, the one-loop diagram of Fig. 11 contributes at order

 $\sim (Q^2)^2 \left(\frac{1}{Q^2}\right)^2 Q^4 \sim Q^4\,,$ (6.4.51)

where the first factor on the right-hand side stems from the two vertices, the second from the two propagators and the third from the loop integral. Obviously the contribution of the loop diagram is suppressed by order Q^2 relative to the tree-level diagram (6.4.48). As a next step we shall show that this is a generic result.

In general, it should have become clear from the previous example that an *arbitrary* diagram contributes at order Q^D where the "chiral dimension" D is given by

$$D = 4\,l - 2\,p + \sum_{n=1}^{\infty} 2\,n\,v_{2n}\,. \tag{6.4.52}$$

Here l denotes the number of independent loops of the considered diagram, p the number of propagators and v_{2n} the number of vertices from the Lagrangian of chiral perturbation theory of order Q^{2n}. For $n = 1$ this Lagrangian is given by (6.4.25). The $n = 2$ Lagrangian will be developed in subsection 6.4.4.

So far it seems that D can be small even for a large number, l, of loops as long as the number of propagators, p, is sufficiently large. However, there is a relation between l, p and the total number of vertices, $v := \sum v_{2n}$ for any connected diagram:

$$l - p + v = 1\,. \tag{6.4.53}$$

To avoid disrupting the main line of arguments we will postpone a sketch of the proof for this relation to the end of this subsection. If we solve (6.4.53) for p and plug the result into (6.4.52), we find

$$D = 2\,l + 2 + \sum_{n=1}^{\infty} 2\,(n-1)\,v_{2n}\,. \tag{6.4.54}$$

This relation is very satisfying: None of the contributions is negative. Therefore, we can ask which diagrams can contribute for a given D, i.e. which diagrams have the importance Q^D. The lowest possible value for D is $D = 2$. Then (6.4.54) implies $l = 0$, v_2 arbitrary and $v_{2n} = 0$ for $n > 1$. Thus only tree-level diagrams contribute and only the ones for $n = 1$. These are the diagrams emerging from the lowest-order Lagrangian (6.4.25). For a given process one only has to calculate a finite number of terms.

As an example let's consider a production reaction of Goldstone bosons, i.e. a $2 \to 4$ process (two Goldstone bosons scatter and produce two more Goldstone bosons). The contributing diagrams for the leading-order calculation are schematically displayed in Fig. 12.

We can also ask which diagrams contribute at order Q^4, i.e. the largest correction to the leading-order result. For $D = 4$ there are two possibilities: Either $l = 1$, v_2 arbitrary and $v_{2n} = 0$ for $n > 1$. These are one-loop diagrams emerging

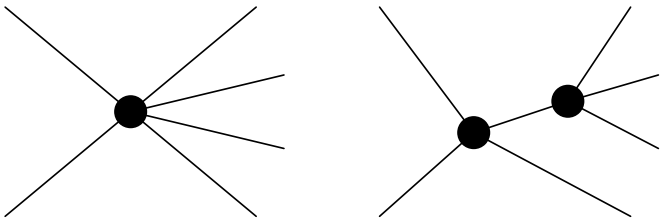


Figure 12: Sketch of Feynman diagrams contributing to a $2 \to 4$ reaction at leading order of the chiral power counting. The vertices are supposed to come from the lowest-order Lagrangian (6.4.25). The lines represent Goldstone bosons.

from the lowest-order Lagrangian (6.4.25). Or $l = 0$, v_2 arbitrary, $v_4 = 1$ and $v_{2n} = 0$ for $n > 2$. These are tree-level diagrams with *one* vertex emerging from the next-to-leading-order Lagrangian and arbitrary many other vertices emerging from (6.4.25). We will present the next-to-leading-order Lagrangian below. Anticipating that this Lagrangian has a finite number of terms, we see again that there is only a finite number of diagrams which need to be considered for a given process at a given order Q^D.

As an example we consider the scattering of two Goldstone bosons with the additional emission of a photon. Such processes can be deduced from the Lagrangians of chiral perturbation theory if one replaces the external vector sources by (6.4.36). Some diagrams contributing at next-to-leading order are schematically displayed in Fig. 13. Note that we do not display the leading-order contributions. These would be just tree-level diagrams with Q^2 vertices.

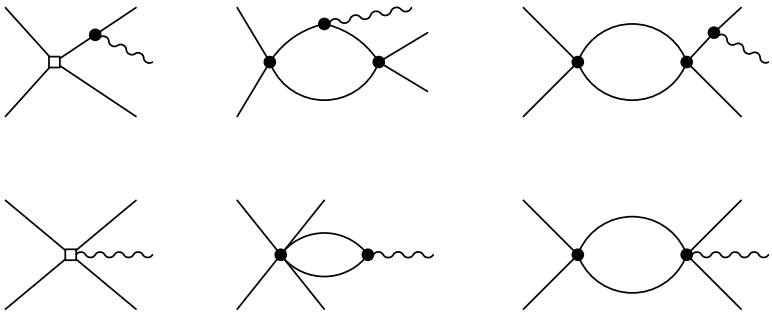


Figure 13: Sketch of some Feynman diagrams contributing to a $P+P \to P+P+\gamma$ reaction at next-to-leading order of the chiral power counting. Here P generically denotes a pseudoscalar Goldstone boson. These particles are represented in the diagrams by straight lines. The photon is denoted by γ and is represented by a wavy line. The filled-circle vertices are supposed to come from the lowest-order (Q^2) Lagrangian (6.4.25). The rectangular vertices are supposed to come from the next-to-leading-order (Q^4) Lagrangian given in (6.4.55) below.

Weinberg's power-counting formula (6.4.54) shows that diagrams with more loops are less important in the Taylor expansion in powers of Q. One aspect is

crucial here to yield this desirable result: The sum in (6.4.54) starts at $n = 1$, *not* at $n = 0$. If there were vertices of order Q^0, i.e. $v_0 \neq 0$, then there would be diagrams with (arbitrary many) loops which were as important as tree diagrams. Just consider the example of Fig. 11. If the vertices appearing there were of order Q^0, both diagrams would contribute at order Q^0 to the scattering amplitude; and so would infinitely many other diagrams with arbitrary many loops. Then we could not calculate the corresponding scattering amplitude, at least not by using Feynman diagrams.[49] To give an example for an interaction term which yields a Q^0 interaction: $\mathrm{tr}(\hat{\phi}^4)$ with $\hat{\phi}$ given in (6.4.27). Fortunately there are no such vertices of order Q^0 and the reason is that we are dealing with Goldstone bosons. All interactions between them either involve derivatives or the explicit symmetry breaking quark-mass terms. Both types of interactions are associated with positive powers in Q. This would be different for a theory which includes other particles besides the Goldstone bosons. We will come back to this point below.

Finally we shall sketch the proof of relation (6.4.53). It uses the idea of complete induction. Let us start with one vertex with arbitrary many legs. If no pair of legs is connected, we have $v = 1$, $l = 0 = p$ which satisfies (6.4.53). If we connect two of the legs we produce one loop and we have one propagator. Thus, $v = p = l = 1$ and (6.4.53) still holds. If we continue with connecting two more legs we always add one more loop and one more propagator. Hence, for one vertex the relation (6.4.53) is always fulfilled.

Next we suppose that (6.4.53) holds for v vertices. We now add one more. All Feynman diagrams of physical relevance are connected diagrams. And the relation (6.4.53) holds only for connected diagrams anyway. So we connect the additional vertex to the ensemble of v vertices. The simplest way is by having a propagator between the additional vertex and one of the others. In total we have increased v and p by 1. This does not change (6.4.53). By assumption it was valid before, so it still is. Of course, connecting the new vertex to the other ones by one line is not the only possibility. We can have a connection via m lines. Then we have m more propagators and $m - 1$ loops. In total we have increased v by 1, p by m and l by $m - 1$. Therefore, there is no change in the combination $v + l - p$. Relation (6.4.53) still holds. This completes the proof by complete induction.

6.4.4 Beyond leading order — higher-order Lagrangians

In (6.4.25) we have presented the complete leading-order Lagrangian (order Q^2) of chiral perturbation theory. In the same way one can construct the next-to-leading-order (Q^4) Lagrangian. We recall the building blocks, U, D_μ, $f_{\mu\nu}^{R/L}$, $\mathcal{M}$,

[49] At present, Feynman diagrams — perturbation theory in powers of coupling constants or in powers of the expansion parameter Q — are our best developed tool to approach quantum-field systems.

from page 147, their power counting, (6.4.20), (6.4.18), (6.4.19), (6.4.21), and the fact that the resulting Lagrangian should be invariant with respect to local chiral transformations, (6.4.12), (6.4.16), (6.4.17), (6.4.7), and with respect to parity and charge conjugation. The next-to-leading-order Lagrangian of chiral perturbation theory has been worked out by J. Gasser and H. Leutwyler. It can be written in the form (see [Scherer] for more details and references)

$$\begin{aligned}
\mathcal{L}^{(4)}_{\chi\mathrm{PT}} \;:=\; & L_1\,[\mathrm{tr}((D_\mu U)^\dagger D^\mu U)]^2 + L_2\,\mathrm{tr}((D_\mu U)^\dagger D_\nu U)\,\mathrm{tr}((D^\mu U)^\dagger D^\nu U) \\
& + L_3\,\mathrm{tr}((D_\mu U)^\dagger (D^\mu U)\,(D_\nu U)^\dagger D^\nu U) \\
& + L_4\,\mathrm{tr}((D_\mu U)^\dagger D^\mu U)\,\mathrm{tr}(\chi^\dagger U + \chi U^\dagger) \\
& + L_5\,\mathrm{tr}((D_\mu U)^\dagger D^\mu U\,(\chi^\dagger U + \chi U^\dagger)) + L_6\,[\mathrm{tr}(\chi^\dagger U + \chi U^\dagger)]^2 \\
& + L_7\,[\mathrm{tr}(\chi^\dagger U - \chi U^\dagger)]^2 + L_8\,\mathrm{tr}(\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger) \\
& - iL_9\,\mathrm{tr}(f^R_{\mu\nu}\,(D^\mu U)\,(D^\nu U)^\dagger + f^L_{\mu\nu}\,(D^\mu U)^\dagger D^\nu U) + L_{10}\,\mathrm{tr}(U^\dagger f^R_{\mu\nu} U f^{\mu\nu}_L) \\
& + H_1 \mathrm{tr}(f^R_{\mu\nu} f_R^{\mu\nu} + f^L_{\mu\nu} f_L^{\mu\nu}) + H_2 \mathrm{tr}(\chi^\dagger \chi)\,. \qquad (6.4.55)
\end{aligned}$$

We have introduced $\chi := B_0 \mathcal{M}$. The chiral Lagrangian of order Q^4 contains 10 physical free parameters, $L_1, \ldots, L_{10}$. The additional parameters, H_1, H_2, are of no physical significance because they only contain external fields. They are needed, along with the parameters L_i, for the renormalization of divergent one-loop diagrams.

The Lagrangian (6.4.55) does *not* contain all the terms which one naively would write down. Especially there are no terms like $\mathrm{tr}((D_\mu D^\mu U)^\dagger\, D_\nu D^\nu U)$. The reason is the same as the one which reduced (6.3.1) to (6.3.4), namely the use of equations of motion. For details we refer to [Scherer].

In addition, there is a second argument which reduces the number of terms. The alert reader might have noticed that there are some types of double-trace terms in (6.4.55) where there are no corresponding single-trace terms. For example, there is no term $\sim \mathrm{tr}((D_\mu U)^\dagger D_\nu U\,(D^\mu U)^\dagger D^\nu U)$ which would correspond to the L_2 term. The necessary ingredient here is a mathematical relation which connects traces of products of matrices with products of traces. Since it is always fun to use some linear algebra we shall go through this line of arguments.

The Cayley-Hamilton theorem states that any matrix A is a solution of its associated characteristic polynomial $p_A(t) := \det(t\,\mathbb{1} - A)$, i.e.

$$p_A(A) = 0\,. \qquad (6.4.56)$$

For 3×3 matrices the determinant of any matrix, B, is related to traces by

$$\det(B) = \frac{1}{3}\,\mathrm{tr}(B^3) - \frac{1}{2}\,\mathrm{tr}(B^2)\,\mathrm{tr}(B) + \frac{1}{6}\,[\mathrm{tr}(B)]^3\,. \qquad (6.4.57)$$

Thus, (6.4.56) is equivalent to

$$\begin{aligned} 0 &= A^3 - A^2\,\mathrm{tr}(A) - \frac{1}{2}\,A\,(\mathrm{tr}(A^2) - (\mathrm{tr}A)^2) \\ &\quad + \mathbb{1}\,\left(-\frac{1}{3}\,\mathrm{tr}(A^3) + \frac{1}{2}\,\mathrm{tr}(A^2)\,\mathrm{tr}(A) - \frac{1}{6}\,[\mathrm{tr}(A)]^3\right) \end{aligned} \tag{6.4.58}$$

for any matrix 3×3 matrix A. Let us choose $A = A_1 + A_2 + A_3$ with three arbitrary 3×3 matrices A_i, $i = 1, 2, 3$. From (6.4.58) we can deduce

$$\begin{aligned} 0 &= F_3(A_1, A_2, A_3) \\ &:= A_1\,\{A_2, A_3\} + A_2\,\{A_3, A_1\} + A_3\,\{A_1, A_2\} \\ &\quad - \mathrm{tr}(A_1)\,\{A_2, A_3\} - \mathrm{tr}(A_2)\,\{A_3, A_1\} - \mathrm{tr}(A_3)\,\{A_1, A_2\} \\ &\quad - \mathrm{tr}(A_1\,A_2)\,A_3 - \mathrm{tr}(A_2\,A_3)\,A_1 - \mathrm{tr}(A_3\,A_1)\,A_2 \\ &\quad + \mathrm{tr}(A_1)\,\mathrm{tr}(A_2)\,A_3 + \mathrm{tr}(A_2)\,\mathrm{tr}(A_3)\,A_1 + \mathrm{tr}(A_3)\,\mathrm{tr}(A_1)\,A_2 \\ &\quad - \mathrm{tr}(A_1\,A_2\,A_3)\,\mathbb{1} - \mathrm{tr}(A_3\,A_2\,A_1)\,\mathbb{1} \\ &\quad + \mathrm{tr}(A_1\,A_2)\,\mathrm{tr}(A_3)\,\mathbb{1} + \mathrm{tr}(A_2\,A_3)\,\mathrm{tr}(A_1)\,\mathbb{1} + \mathrm{tr}(A_3\,A_1)\,\mathrm{tr}(A_2)\,\mathbb{1} \\ &\quad - \mathrm{tr}(A_1)\,\mathrm{tr}(A_2)\,\mathrm{tr}(A_3)\,\mathbb{1}\,. \end{aligned} \tag{6.4.59}$$

Hence, for any four 3×3 matrices A_i, $i = 1, 2, 3, 4$ we find

$$0 = \mathrm{tr}(F_3(A_1, A_2, A_3)\,A_4)\,. \tag{6.4.60}$$

This can be used to relate terms with different number of (flavor) traces.

As an example let us study the structure

$$\mathrm{tr}((D_\mu U)^\dagger D_\nu U\,(D^\mu U)^\dagger D^\nu U) = \mathrm{tr}(\underbrace{(D_\mu U)^\dagger U}_{=:A_1}\,\underbrace{U^\dagger D_\nu U}_{=:A_2}\,\underbrace{(D^\mu U)^\dagger U}_{=:A_3}\,\underbrace{U^\dagger D^\nu U}_{=:A_4})\,. \tag{6.4.61}$$

By inserting $\mathbb{1} = U\,U^\dagger$ we have introduced four matrices, A_i, $i = 1, 2, 3, 4$, which all transform in the same way with respect to local chiral transformations and which are traceless, $\mathrm{tr}(A_i) = 0$; cf. (6.4.23). For this case equation (6.4.60) reduces to

$$\begin{aligned} &\mathrm{tr}(A_1\,A_2\,A_3\,A_4) + \mathrm{tr}(A_1\,A_3\,A_2\,A_4) + \mathrm{tr}(A_2\,A_3\,A_1\,A_4) + \mathrm{tr}(A_2\,A_1\,A_3\,A_4) \\ &\qquad + \mathrm{tr}(A_3\,A_1\,A_2\,A_4) + \mathrm{tr}(A_3\,A_2\,A_1\,A_4) \\ &= \mathrm{tr}(A_1\,A_2)\,\mathrm{tr}(A_3\,A_4) + \mathrm{tr}(A_2\,A_3)\,\mathrm{tr}(A_1\,A_4) + \mathrm{tr}(A_3\,A_1)\,\mathrm{tr}(A_2\,A_4)\,. \end{aligned} \tag{6.4.62}$$

This relates the structure (6.4.61) to structures which appear in (6.4.55):

$$\begin{aligned} 2\,\mathrm{tr}((D_\mu U)^\dagger D_\nu U\,(D^\mu U)^\dagger D^\nu U) &= -4\,\mathrm{tr}((D_\mu U)^\dagger D^\mu U\,(D_\nu U)^\dagger D^\nu U) \\ &\quad + 2\,\mathrm{tr}((D_\mu U)^\dagger D_\nu U)\,\mathrm{tr}((D^\mu U)^\dagger D^\nu U) \\ &\quad + \mathrm{tr}((D_\mu U)^\dagger D^\mu U)\,\mathrm{tr}((D_\nu U)^\dagger D^\nu U)\,. \end{aligned} \tag{6.4.63}$$

Such relations deduced from (6.4.60) become even more important for the construction of the Q^6 Lagrangian of chiral perturbation theory [Scherer].

The leading-order Lagrangian, (6.4.25), of chiral perturbation theory contains only two low-energy constants, F_0 and B_0. The next-to-leading-order Lagrangian, (6.4.55), contains 10 additional parameters (low-energy constants), L_i, $i = 1, \ldots, 10$. In principle, all these low-energy constants are related to the underlying microscopic theory, QCD. So far, however, these parameters reflect our inability to solve QCD in the low-energy regime. They must be fitted to experimental data. Clearly, the predictive power of the whole approach depends on the amount of free parameters. It should not be surprising that higher orders of chiral perturbation theory involve more and more parameters. One just can build more different structures with the building blocks U, D_μ, $f_{\mu\nu}^{R/L}$ and $\mathcal{M}$. The Q^6 Lagrangian of chiral perturbation theory has also been worked out by several groups (see [Scherer]). The order of magnitude of free parameters is about 100. For low enough energies and/or not too high accuracy this does not matter since one does not need such Q^6 terms which are only corrections to the Q^4 corrections. In addition, not every process depends on 100 parameters. Therefore, Q^6 calculations have been performed for selected quantities. Nonetheless, the dramatic increase of free parameters when going to higher orders of chiral perturbation theory sets a practical limit to such kind of calculations.

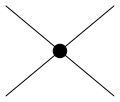
In addition, there is an energetic limit for the applicability of chiral perturbation theory as a whole. As for every effective theory the expansion in powers of energy/momentum breaks down for energies where not-considered physical states can be excited (recall the discussions in subsections 6.2 and 6.3). Therefore, the whole power series in Q breaks down if Q gets as big as the mass of other not-considered hadrons. In practice, this happens around the mass of the ρ meson which is $m_\rho \approx 770\,\mathrm{MeV}$.

The first impression might be that one can improve chiral perturbation theory, i.e. increase the energetic range of applicability, by just including more hadron types. This is a topic of active research. One severe complication emerges from the fact that interactions between other hadrons can be of order Q^0 which would mess up the power counting, as discussed after equation (6.4.54). The inclusion of nucleons is actually possible [Scherer], but other hadrons provide a challenge; in particular because there is an additional intrinsic scale of chiral perturbation theory which we will discuss next.

Indeed, there is one more scale which limits the application of chiral perturbation theory to higher energies. It is induced by the loops which provide a necessary ingredient to go beyond leading order. One has to ask the question for which energies the loop diagrams become numerically as important as the tree-level diagrams. At such an energy loop diagrams with arbitrary many loops contribute with the same strength. The Taylor expansion in Q breaks down. To find an estimate for this value of Q, the "breakdown scale" Λ_χ, we can concentrate here on vertices from the leading-order Lagrangian (6.4.25) and we only study the

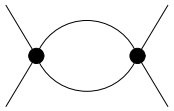
vertices with derivatives.[50] For definiteness, we study again the case depicted in Fig. 11. This time, however, we shall be more careful with numerical factors and dimensionful coefficients. A four-point interaction from the derivative term of (6.4.25) has the structure Q^2/F_0^2. Propagators are still of size $\sim 1/Q^2$. The loop integral is $\int \frac{d^4k}{(2\pi)^4}$ and generically contributes with $Q^4/(4\pi)^2$. A remark is in order concerning the difference between $1/(2\pi)^4$, which appears in the loop integral, and $1/(4\pi)^2$, which appears in the final size estimate: The difference originates from the "angular integral" over four dimensions. We use the quotation marks here because the concept of angles in a non-Euclidean space, here Minkowski space, is less intuitive than the word "angle" implies. However, technically the four-dimensional loop integral can be rewritten into a Euclidean integral (see any book on quantum field theory, the key word is "Wick rotation").

Now we compare the tree-level contribution,



$$\sim \frac{Q^2}{F_0^2}\,, \tag{6.4.64}$$

and the loop contribution (cf. equation (6.4.51))



$$\sim \left(\frac{Q^2}{F_0^2}\right)^2 \left(\frac{1}{Q^2}\right)^2 \frac{Q^4}{(4\pi)^2} = \frac{Q^2}{F_0^2}\,\frac{Q^2}{(4\pi F_0)^2}\,. \tag{6.4.65}$$

Obviously the breakdown scale, where tree-level and loop diagram become of equal size is at

$$\Lambda_\chi \approx 4\pi F_0 \approx 1\,\mathrm{GeV}. \tag{6.4.66}$$

Roughly this is the same scale as where additional degrees of freedom (ρ meson, ...) enter. It restricts the application of chiral perturbation theory to energies and momenta (and masses) significantly below 1 GeV.

Final remark: Of course, there are many topics, related to quarks and hadrons, which have not been covered. To name a few: the chiral anomaly, the Wess-Zumino-Witten action, perturbative QCD, electroweak interactions of quarks and hadrons, lattice QCD, quark models, hadronic models. These topics have to await an extended version of the lecture notes.

[50] The vertices proportional to the quark masses are less harmful because they do not grow with Q.

6.5 Exercises

1. **Pion-pion scattering in chiral perturbation theory:**

   The lowest-order Lagrangian of chiral perturbation theory is given by (neglecting external fields)

   $$\mathcal{L}_{\chi\mathrm{PT}} = \frac{F_0^2}{4}\,\mathrm{tr}\left(\partial_\mu U^\dagger\,\partial^\mu U\right) + \frac{F_0^2\,B_0\,m_q}{2}\,\mathrm{tr}\left(U^\dagger + U\right)\,, \tag{6.5.1}$$

   where we have restricted ourselves to two flavors and ignored isospin breaking effects. Consequently there is only one quark mass m_q. $F_0 \approx 92\,\mathrm{MeV}$ is the pion-decay constant and the combination $B_0\,m_q$ enters the pion mass. The matrix U contains the pseudoscalar pion fields:

   $$U = e^{i\Phi/F_0} \qquad \text{with} \qquad \Phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\,. \tag{6.5.2}$$

   Determine the pion mass m_π by expanding all terms in (6.5.1) up to quadratic order in the fields Φ.

   Determine the interactions relevant for pion-pion scattering by expanding all terms in (6.5.1) up to quartic order in the fields Φ.

   Calculate at leading order (LO) of chiral perturbation theory the scattering amplitude (Feynman matrix element) $\mathcal{M}_{\mathrm{LO}}$ for the elastic reaction $\pi^+\pi^+ \to \pi^+\pi^+$ and the corresponding differential and total cross sections, $\mathrm{d}\sigma_{\mathrm{LO}}/\mathrm{d}t$ and σ_{LO}, respectively. Express all quantities in terms of the Mandelstam variables s and t. Determine the scattering length a_{LO} (in femtometer) defined by

   $$4\pi a_{\mathrm{LO}}^2 = \lim_{s\to 4m_\pi^2} \sigma_{\mathrm{LO}}\,. \tag{6.5.3}$$

   Note: At leading order one just has to consider the tree-level diagrams emerging from (6.5.1).

   Calculate the same quantities for the elastic reaction $\pi^+\pi^- \to \pi^+\pi^-$.

   Hints: You should find the interaction terms

   $$\frac{1}{24F_0^2}\,\mathrm{tr}(\partial_\mu\Phi\,\Phi\,\partial^\mu\Phi\,\Phi - \partial_\mu\Phi\,\partial^\mu\Phi\,\Phi^2) + \frac{m_\pi^2}{48F_0^2}\,\mathrm{tr}(\Phi^4) \tag{6.5.4}$$

   and the matrix elements

   $$\mathcal{M}_{\pi^+\pi^+\to\pi^+\pi^+} = -\frac{1}{F_0^2}\,(s-2m_\pi^2)\,, \qquad \mathcal{M}_{\pi^+\pi^-\to\pi^+\pi^-} = -\frac{1}{F_0^2}\,(2m_\pi^2 - s - t)\,. \tag{6.5.5}$$

2. **Corrections from vector-meson exchange:**

   This is a continuation of the previous task. Given the phenomenological Lagrangian

$$\mathcal{L}_{\rm vec} \;=\; -\frac{1}{4}\,\rho_{\mu\nu}\,\rho^{\mu\nu} + \frac{1}{2}\,m_\rho^2\,\rho_\mu\,\rho^\mu - i\,\frac{g}{F_0^2}\,\rho_{\mu\nu}\,\partial^\mu\pi^+\,\partial^\nu\pi^- \qquad (6.5.6)$$

   for pions and the neutral rho meson ρ_μ in addition to (6.5.1). The field strength of the vector meson is given by $\rho_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu$. The interaction term in (6.5.6) has been constructed with derivatives acting on the pion fields. Such a structure is demanded by the Goldstone theorem.

   Calculate the decay width of the rho meson into two pions based on a tree-level calculation using (6.5.6). Determine the coupling constant g by comparing your calculation with the experimental decay width of the rho meson. (Hint: You should find $g \approx 0.18$.)

   Calculate at tree level the scattering amplitude $\mathcal{M}_{\rm LO+vec}$ for the elastic reaction $\pi^+\pi^+ \to \pi^+\pi^+$ based on $\mathcal{L} = \mathcal{L}_{\chi\rm PT} + \mathcal{L}_{\rm vec}$. Expand the obtained amplitude in a power series of pion momenta, generically called p, i.e. use $s, t, m_\pi^2 = O(p^2)$. You should find that the contribution from $\mathcal{L}_{\chi\rm PT}$ is of order $O(p^2)$ while the contribution from $\mathcal{L}_{\rm vec}$ starts only at order $O(p^6)$. Thus for small pion momenta the contribution from vector-meson exchange is just a correction. Calculate the total cross section $\sigma_{\rm LO+vec}$ for the reaction $\pi^+\pi^+ \to \pi^+\pi^+$. Determine the value of the total reaction energy $\sqrt{s}$ where the correction becomes as large as the leading-order result, i.e. determine $\sqrt{s}$ such that $\sigma_{\rm LO+vec} \approx 2\,\sigma_{\rm LO}$.

   Repeat the whole analysis for the reaction $\pi^+\pi^- \to \pi^+\pi^-$.

   (There is no practical use in an analytical calculation of the cross sections. It is recommended to perform the integral over t numerically as a function of s.)

3. **Life time of the η'-meson:**

   Collect from the particle listings at http://pdg.lbl.gov/ the mass of the η'-meson and all hadrons that are lighter than the η'. Consider all energetically possible hadronic two- and three-body decays of the η'. Write down at least 8 of them. Actually all hadronic two-body decays are forbidden by parity, charge conjugation, strangeness and/or isospin conservation. Pick an example of a hadronic two-body decay that is forbidden by strangeness conservation. Pick an example of a hadronic two-body decay that is forbidden by isospin conservation.

   This paragraph is for background information. It does not contain tasks: The hadronic three-body decay with the lightest sum of masses of the final

states is $\eta' \to 3\pi$. This decay is forbidden by a combination of isospin conservation and charge conjugation symmetry. The decay $\eta' \to f_0\pi\pi$ is forbidden by parity conservation. The only other hadronic three-body decays are $\eta' \to \eta\pi^0\pi^0$ and $\eta' \to \eta\pi^+\pi^-$. There is no conservation law that forbids these decays. However, spontaneous chiral symmetry breaking suppresses these decays. The rest of the task is devoted to understand this mechanism.

In the absence of chiral symmetry breaking the simplest interaction which one can write down is

$$\mathcal{L}_{\text{naive}} = g\,\eta'\,\eta\,\left(\pi^+\,\pi^- + \frac{1}{2}\,\pi^0\,\pi^0\right) \tag{6.5.7}$$

with a dimensionless coupling constant g. Note that the factor $1/2$ is the generic factor that appears between charged and neutral bosons. Show that the tree-level matrix element for both reactions $\eta' \to \eta\pi^0\pi^0$ and $\eta' \to \eta\pi^+\pi^-$ is the same. Show that the corresponding decay widths satisfy

$$\frac{\Gamma_{\eta'\to\eta\pi^0\pi^0}}{\Gamma_{\eta'\to\eta\pi^+\pi^-}} = \frac{1}{2} \tag{6.5.8}$$

if one neglects the mass difference between charged and neutral pions (isospin symmetry).

Spontaneous chiral symmetry breaking requires Goldstone bosons. The pions are regarded as these Goldstone bosons. Therefore the appropriate interaction terms are derivative interactions. Instead of (6.5.7) an interaction term that is in agreement with the concept of spontaneously broken chiral symmetry should read (for two-flavor chiral symmetry)

$$\mathcal{L}_{\text{chiral}} = \frac{\tilde{g}}{m_{\eta'}^2}\,\eta'\,\eta\,\left(\partial_\mu\pi^+\,\partial^\mu\pi^- + \frac{1}{2}\,\partial_\mu\pi^0\,\partial^\mu\pi^0\right) \tag{6.5.9}$$

with another dimensionless coupling constant $\tilde{g}$. Note that a typical hadronic mass scale $m_{\eta'}$ has been inserted to make $\tilde{g}$ dimensionless. Determine both decay widths for $\eta' \to \eta\pi^+\pi^-$ from tree-level matrix elements of (6.5.7) and (6.5.9), respectively. Assume that $\tilde{g} = g$ and determine numerically the ratio between these two widths. You should find that the width obtained from the realistic interaction (6.5.9) is much smaller than the naive width estimate from (6.5.7).

Note: In principle there are additional interaction terms that scale with the light quark masses (and are therefore small in size). Strictly speaking the η is also a Goldstone boson (of three-flavor chiral symmetry breaking).

Therefore an even more realistic, SU(3) symmetric interaction is given by

$$\begin{aligned} \sim \quad & \partial_\nu \eta' \, \partial^\nu \eta \left(\partial_\mu \pi^+ \, \partial^\mu \pi^- + \frac{1}{2} \, \partial_\mu \pi^0 \, \partial^\mu \pi^0 \right) \\ & + \partial_\nu \eta' \, \partial_\mu \eta \left(\partial^\mu \pi^0 \, \partial^\nu \pi^0 + \partial^\mu \pi^+ \, \partial^\nu \pi^- + \partial^\mu \pi^- \, \partial^\nu \pi^+ \right) . \end{aligned} \tag{6.5.10}$$

This would reduce the realistic decay width even (a little bit) more.