S. Leupold **October – December 2015**

Hadron and Quark Physics

Hand-in problem 1'

Obviously most of you (though not all) got stuck with the evaluation of the double differential decay width (last part of task 2 of hand-ins 1 - this part is worth **5 points**). Since the main aspect of this course is that you learn something, I decided to provide more hints and to offer you the possibility to hand in this part again. This means:
1. If you got stuck and would like to try it again, I will not grade your previous attempt but your new one. Just hand in a new version - deadline 1. December 2015.
2. If you got stuck but are sick about this task, then just hand in again what you have already written. I then will grade this.
3. Naturally, those few, who managed to solve the task without additional hints, should be honored. They will receive the 5 points for the correct solution plus 5 extra points.

Here is the problem again: Calculate the double-differential decay rate

$$\frac{d\Gamma}{dm_{12}^2\, dm_{23}^2} = \frac{1}{2M}\, |\mathcal{M}|^2 \int \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3\, 2E_i}\, (2\pi)^4\, \delta(p - p_1 - p_2 - p_3) \\ \times \delta(m_{12}^2 - (p_1+p_2)^2)\, \delta(m_{23}^2 - (p_2+p_3)^2)\,. \tag{1}$$

You should find

$$\frac{d\Gamma}{dm_{12}^2\, dm_{23}^2} = \frac{1}{(2\pi)^3}\, \frac{1}{32\, M^3}\, |\mathcal{M}|^2\,. \tag{2}$$

A related problem has been presented in the question session of the lectures. The corresponding (hand-written) notes are attached here.

To be handed in at 1.12.2015 at the latest.

ways to hand it in: during the lectures; via email to stefan.leupold@physics.uu.se; in-box of "hadrons and quarks" on top of the old mail boxes in house 1, second floor, close to house 8

Pure 3-body phase space

$$I = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta(P - p_1 - p_2 - p_3)$$

with $p_i^0 = E_i = \sqrt{m_i^2 + \vec{p}_i^{\,2}}$, $i = 1, 2, 3$

want to know I for $P = (M, \vec{0})$

insert $1 = \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(q - p_1 - p_2)$

note: q^0 integration can be limited to $q^0 \in [m_1+m_2, +\infty[$

since δ function enforces $q^0 = p_1^0 + p_2^0 = E_1 + E_2 \geq m_1 + m_2$

$$I = \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(P - q - p_3) \cdot J$$

with $J = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(q - p_1 - p_2)$ 2-body phase space

J is Lorentz invariant $\rightsquigarrow$ can be evaluated in any frame

$\rightsquigarrow$ choose frame where $\vec{q} = 0$ $\rightsquigarrow$ $q^2 = (q^0)^2$

(nice extra exercise to show that one can choose this frame, i.e. that $q^2 > 0$)

$$J = \frac{1}{4(2\pi)^2} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \delta(q^0 - E_1 - E_2)\, \delta(-\vec{p}_1 - \vec{p}_2)$$

$$= \frac{1}{16\pi^2} \int \frac{d^3p_1}{E_1 E_2(E_1)} \delta(q^0 - E_1 - E_2(E_1))$$

with $E_2(E_1) = \sqrt{m_2^2 + \vec{p}_2^{\,2}} = \sqrt{m_2^2 + \vec{p}_1^{\,2}} = \sqrt{m_2^2 + E_1^2 - m_1^2}$

choose spherical coordinates and change $|\vec{p}_1|$ integration to E_1 integration $\rightsquigarrow$ $\vec{p}_1^{\,2} + m_1^2 = E_1^2 \Rightarrow |\vec{p}_1|\, d|\vec{p}_1| = E_1\, dE_1$

$d^3p_1 = d\Omega\, |\vec{p}_1|^2 d|\vec{p}_1| = d\Omega \sqrt{E_1^2 - m_1^2}\, E_1\, dE_1$

$\int d\Omega = 4\pi$

$$J = \frac{1}{4\pi} \int_{m_1}^{\infty} dE_1 \frac{\sqrt{E_1^2 - m_1^2}}{E_2(E_1)} \delta(q^0 - E_1 - E_2(E_1))$$

rewrite δ function to $\delta(E_1 - \tilde{E}_1)$ where $\tilde{E}_1$ is solution of
$f(\tilde{E}_1) = q^0 - \tilde{E}_1 - E_2(\tilde{E}_1) = 0$

$$\delta(f(E_1)) = \frac{\delta(E_1 - \tilde{E}_1)}{|f'(\tilde{E}_1)|}$$

$$q^0 - \tilde{E}_1 - E_2(\tilde{E}_1) = 0$$

$$\Rightarrow (q^0 - \tilde{E}_1)^2 = m_2^2 + \tilde{E}_1^2 - m_1^2$$

$$\Leftrightarrow q^{0\,2} - 2q^0 \tilde{E}_1 = m_2^2 - m_1^2$$

$$\Leftrightarrow \tilde{E}_1 = \frac{q^{0\,2} + m_1^2 - m_2^2}{2q^0}$$

$$\left(\Rightarrow E_2(\tilde{E}_1) = q^0 - \tilde{E}_1 = \frac{q^{0\,2} + m_2^2 - m_1^2}{2q^0}\right)$$

$$f'(E_1) = -1 - \frac{dE_2(E_1)}{dE_1} = -1 - \frac{E_1}{\sqrt{m_2^2 + E_1^2 - m_1^2}} = -1 - \frac{E_1}{E_2(E_1)}$$

$$= \frac{-E_2(E_1) - E_1}{E_2(E_1)}$$

$$\Rightarrow E_2(\tilde{E}_1) \cdot |f'(\tilde{E}_1)| = |E_2(\tilde{E}_1) + \tilde{E}_1| = |q^0| = q^0$$

$$\sqrt{\tilde{E}_1^2 - m_1^2} = \frac{1}{2q^0} \sqrt{\lambda((q^0)^2, m_1^2, m_2^2)}$$

with Källén function $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$

$$J = \frac{1}{4\pi} \frac{1}{2q^0} \sqrt{\lambda((q^0)^2, m_1^2, m_2^2)} \frac{1}{q^0} \Theta(\tilde{E}_1 - m_1)$$

$\tilde{E}_1 \geq m_1 \Leftrightarrow q^{0\,2} + m_1^2 - m_2^2 \geq 2q^0 m_1 \Leftrightarrow (q^0 - m_1)^2 \geq m_2^2$
this is anyway satisfied since $q^0 \geq m_1 + m_2$
$\rightsquigarrow$ keep (for later use) $\Theta(q^0 - m_1 - m_2)$

$$J = \frac{1}{8\pi (q^0)^2} \sqrt{\lambda((q^0)^2, m_1^2, m_2^2)}\ \Theta(q^0 - m_1 - m_2)$$

but J is Lorentz invariant

⇝ rewrite it in manifestly Lorentz invariant form
(recall $(q^0)^2 = q^2$ in frame $\vec{q} = 0$)

$$J = \frac{1}{8\pi q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)}\ \Theta(\sqrt{q^2} - m_1 - m_2)$$

$$I = \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int d^4 q\ \delta(P - q - p_3) \frac{1}{8\pi q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)}\ \Theta(\sqrt{q^2} - m_1 - m_2)$$

(it would not make sense to have an expression for J that is only valid for $\vec{q} = 0$, if one wants to integrate over $d^4 q$)

In principle also I is Lorentz invariant
⇝ evaluate it in frame $\vec{P} = 0$
(this is what one wants anyway)

$$I = \frac{1}{2^7 \pi^4} \int d^4 q\ \frac{1}{q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)}\ \overset{\Theta(\sqrt{q^2} - m_1 - m_2)}{\vee} \int \frac{d^3 p_3}{E_3}\ \delta(M - q^0 - E_3)\ \delta(-\vec{q} - \vec{p}_3)$$

$$= \frac{1}{2^7 \pi^4} \int d^4 q\ \frac{1}{q^2} \sqrt{\lambda}\ \delta(M - q^0 - E_3(|\vec{q}|)) \frac{1}{E_3(|\vec{q}|)}\ \Theta(\sqrt{q^2} - m_1 - m_2)$$

$$E_3(|\vec{q}|) = \sqrt{m_3^2 + \vec{p}_3^{\,2}} = \sqrt{m_3^2 + \vec{q}^{\,2}}$$

$$d^4 q = dq^0\ |\vec{q}|^2\ d|\vec{q}|\ d\Omega$$

⇝ perform angular and q^0 integrations
⇝ q^2 is replaced by $\tilde{q}^2 := (M - E_3(|\vec{q}|))^2 - |\vec{q}|^2$
⇝ find limits of final $|\vec{q}|$ integral, ...

alternative:

start again with

$$I = \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int d^4 q \; \delta(P - q - p_3) \underbrace{\frac{1}{8\pi q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)}\, \Theta\left(\sqrt{q^2} - m_1 - m_2\right)}_{= J(q^2)}$$

insert $1 = \int_0^\infty dm^2 \; \delta(q^2 - m^2)$ (recall: $\sqrt{q^2} \geq m_1 + m_2 \geq 0$)

$$I = \int_0^\infty dm^2 \, J(m^2) \int d^4 q \; \delta(q^2 - m^2) \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \; \delta(P - q - p_3)$$

use $\delta(q^2 - m^2) = \delta((q^0)^2 - \vec{q}^{\,2} - m^2) = \delta((q^0)^2 - E_q^2)$, $E_q := \sqrt{\vec{q}^{\,2} + m^2}$

to perform q^0 integration

(and recall $q^0 \geq m_1 + m_2 \geq 0$)

$$\delta((q^0)^2 - E_q^2) = \frac{1}{2E_q} \delta(q^0 - E_q)$$

$$I = \int_0^\infty dm^2 \, J(m^2) \underbrace{\int \frac{d^3 q}{2E_q} \frac{d^3 p_3}{(2\pi)^3 2E_3} \; \delta(P - q - p_3)}_{=: K}$$

where $q^0 = E_q$ in δ function

$$K = \frac{1}{2\pi} \underbrace{\int \frac{d^3 q}{(2\pi)^3 2E_q} \frac{d^3 p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta(P - q - p_3)}_{\text{2-body phase space with masses } m \text{ and } m_3}$$

$\rightsquigarrow$ take formula for J and replace $\begin{cases} q \to P \\ m_1 \to m \\ m_2 \to m_3 \end{cases}$

$\rightsquigarrow$ perform m^2 integration (numerically)