Portfolio Optimization

Using the Markowitz Framework

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Markowitz Framework for Portofolio Composition

- Assume the rate of return r of an asset is modeled by known multivariate normal distributions.
- Maximize the profit $\bar{r} = \vec{r}^T \vec{w}$ subject to $\sum w_i = 1$.
  - If short selling is not allowed: $w_i > 0$.
- Variance of $\bar{r}$ is given by $\bar{\sigma} = \vec{w}^T \boldsymbol{\sigma} \vec{w}$.

Data Used

$$\vec{r} = 10^{-2} \begin{pmatrix} 13 \\ 5.3 \\ 10.5 \\ 5 \\ 12.6 \end{pmatrix}, \boldsymbol{\sigma} = 10^{-}2 \begin{pmatrix} 4.01 & -1.19 & 0.60 & 0.74 & -0.21 \\ & 1.12 & 0.21 & 0.54 & 0.55 \\ & & 3.04 & 0.77 & 0.29 \\ & & & 3.74 & -1.04 \\ & & & & 3.8 \end{pmatrix}$$

Maximizing Profit

Assume we want to maximize our profits with reckless abandon.

- If short selling is not allowed: Put eveything on the best performing asset, same variance as said asset.
- If short selling is allowed: infite returns, even more infinite risk.

Minimizing Variance

Play it safe, try to get the most 'stable' rate-of-return.

- Minimize $\vec{w}^T \boldsymbol{\sigma} \vec{w}$ with respect to $\vec{w}$, under the constraint that $\sum w_i = 1$.
  - Write constraint as $\mathbf{A}\vec{w} = b$, where $\mathbf{A} = \vec{1}$, $b = 1$.
- With short selling: solve the KKT system:

$$\begin{pmatrix} 2\boldsymbol{\sigma} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix} = \begin{pmatrix} \vec{x}^* \\ \lambda \end{pmatrix}$$

- Without short selling: additional constraint, solve with Matlab's *quadprog*.

Minimizing Variance

With short-selling:

$$\vec{x}^* = \vec{w} = \begin{pmatrix} 0.3198 \\ 0.7213 \\ 0.0350 \\ -0.0708 \\ -0.0052 \end{pmatrix}, \quad \bar{r} = 0.0792 \pm 0.0627$$

Minimizing Variance

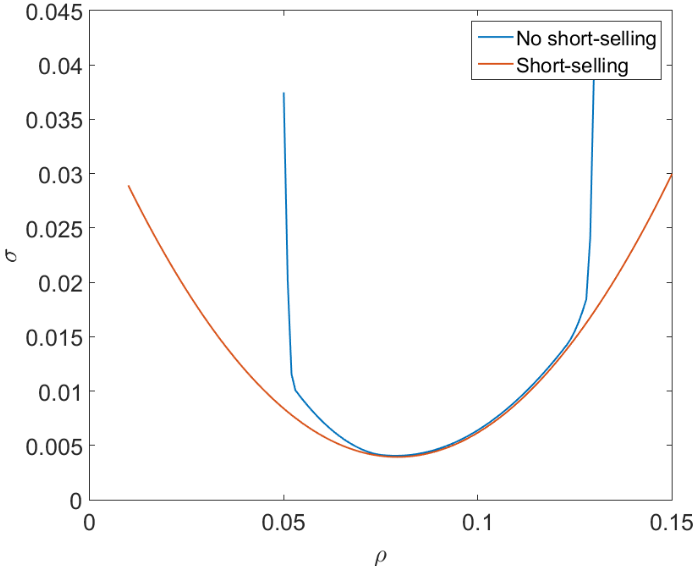
Without short-selling:

$$\vec{w} = \begin{pmatrix} 0.2922 \\ 0.6540 \\ 0.0278 \\ -0.0000 \\ 0.0259 \end{pmatrix}, \quad \bar{r} = 0.0788 \pm 0.0636$$

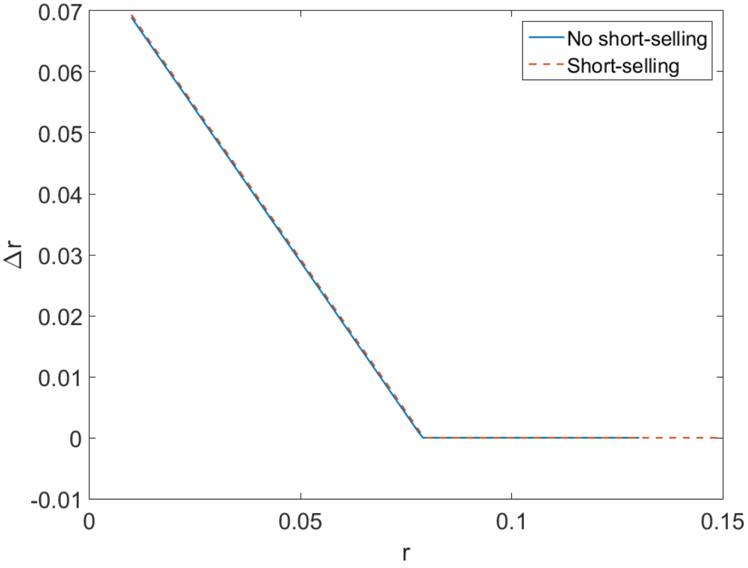
Fixed Target Returns

- Want an effective interest rate of ρ
- Expand $\mathbf{A}$ and $\vec{b}$ so that $\mathbf{A} = \begin{pmatrix} \vec{1} \\ \vec{r} \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}$.
- Solve like before
- For $\rho = 0.1$:
  - With shorting: $\bar{r} = 0.1 + -0.0785$.
  - Without shorting: $\bar{r} = 0.1 + -0.08$.
- Without shorting no solution for $\rho \notin [\min(\vec{r}), \max(\vec{r})]$.

Fixed Target Returns



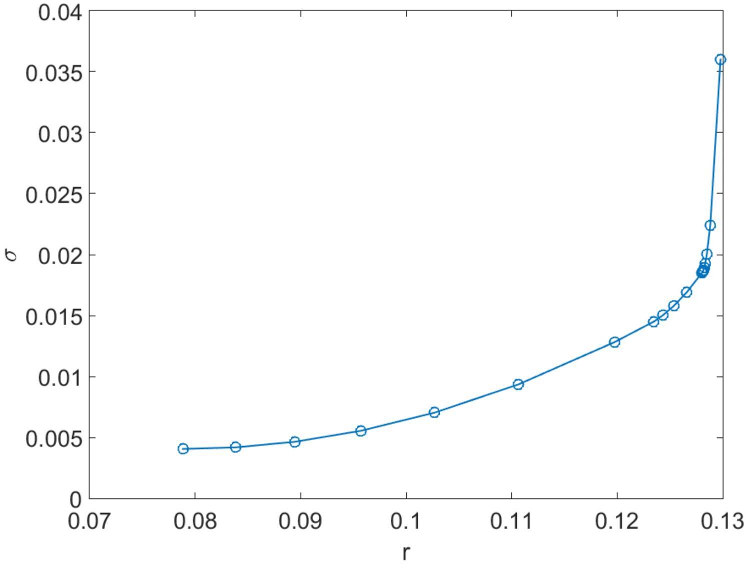
Minimum Target Returns



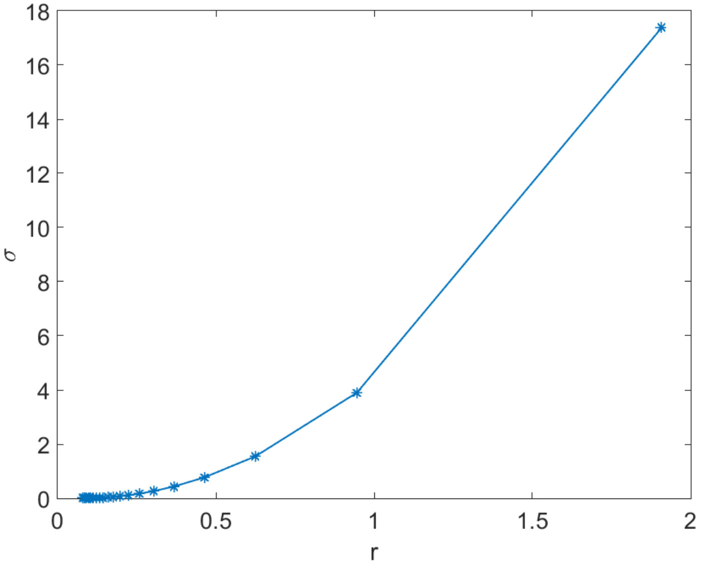
Balancing Volatility Returns

- Want to balance volatility with expected returns
- Minimize $\alpha \vec{w}^T \boldsymbol{\sigma} \vec{w} - (1-\alpha)\vec{r}$ for $\alpha \in [0,1]$ instead.
  - $\alpha = 1 \to$ minimize volatility.
  - $\alpha = 0 \to$ maximize profit.
- Solving for different values of α gives a graph of $\bar{r}$ vs. $\bar{\sigma}$.

$\bar{r}$ vs $\bar{\sigma}$ (no short-selling)



$\bar{r}$ vs $\bar{\sigma}$ (with short-selling)



Any Questions?