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1TD184 Optimisation
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Assignment 2 – challenging option
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1 Structural shape optimisation

1.1 Background

Solid mechanics is one of the first application areas where shape optimisation came into use. In essence, the objective is simple: to make a structure as light, strong or stiff as possible under a set of constraints. The constraints can be that the maximum stress in the structure must not exceed a certain value, or that the maximum deformation of the structure should be kept under a certain limit. In this project, we look at a simple one-dimensional beam supported in its both ends. The objective is to make the beam as light as possible under various constraints. Although the problem does not require deep knowledge in solid mechanics – every day experience and some sound judgement is sufficient – it illustrates how familiarity with the application is necessary in order to successfully carry out an optimisation task and to interpret the results.

1.2 The problem

Consider a beam of length L supported in its both ends. The beam is charged with a distributed load $q(x)$ along its length, but it is also subject to its own weight, summing up to a total load of $Q(x)$ ([N/m]). Apparently, the beam will flex under the load and attain an equilibrium state as in Fig. 1. The shape $w(x)$ of the beam will depend on the type of supports (eg. simple supports as in the figures, or clamped supports), the load distribution $q(x)$, the elasticity module E of the material and a geometric parameter I called the moment of inertia, which is determined by the shape of the cross section of the beam. For a rectangular beam, I is given by $I = ba^3/12$. The attained shape $w(x)$ is given by the solution to the classical *Euler-Bernoulli equation*

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d^2w}{dx^2}\right) = Q(x) \tag{1}$$

$$w(0) = w(L) = 0 \tag{2}$$

Since this is a fourth order boundary value problem, four boundary conditions are necessary. The two first ones are given in the equation and represent the position of the ends. The two other conditions are given by the type of support. For simple supports like in the figure, $w''(0) = w''(L) = 0$, whereas for clamped supports, $w'(0) = w'(L) = 0$.

The objective now is to determine a cross sectional shape that minimises the amount of material in the beam under different constraints. In order to keep the problem as simple as possible (In a real situation it would be wasteful to use a thin but wide beam, or in fact a rectangular beam at all) we restrict ourselves to a situation where a is constant and the width $b(x)$ is the shape to be determined. Also, let the load $q(x)$ have a constant value of 10 N/m.

For the given data, this one-dimensional problem can be tackled by analytical methods, but for more general cases, the solution gets increasingly complicated and quickly goes out of hand. The shape optimisation suggested in this project does not have this limitation, and once you have a working program, investigating arbitrary load cases $q(x)$ is straightforward.

Solve as much of the problem as you can, starting with 1.2.1. The last part 1.2.2 is more exploratory than 1.2.1 and not mandatory for the assignment.

There are several ways to attack the problem, each of which has its proper merits and disadvantages. A few hints will be given in the following; Eq. 1 is a model example for more complicated problems, and it is the intention that it is solved numerically. The method *par excellence* in solid mechanics is the finite element method, but for this one dimensional problem, a finite difference approximation is quite satisfactory. A centred, second order difference approximation of Eq. 1 along with the applicable boundary conditions yields the required relation between $w(x)$ and $Q(x)$. The unknown function $b(x)$ appears in the expression for $I(x)$, and has to be given a discrete approximation. The most straightforward option is to discretise $[0, L]$ in N equidistant points x_j on which w and q, as well as b, are approximated.

Since this is a non-linearly constrained optimisation problem, `lsqnonlin` is the algorithm of choice. Try to implement the gradients of the constraint function, but before you do so, it is advisable to use the built-in numerical differencing option to verify that your proposed formulation is over-all sound.

1.2.1 Maximum deflection constraint

In the first part of the project, minimise the amount of material while keeping the maximum deflection below a certain threshold $|w|_{max} \leq 12.0$ mm. Let a have the value 3.0 mm. Investigate the two cases where the beam is simply supported beam and has clamped supports, respectively.

Give a qualitative interpretation of your results. Are there any points of extreme stress? Discuss the validity of your solutions.

1.2.2 Maximum stress constraint

The previous task did not take stress into account, but it is of course essential to consider this factor in order to avoid structural collapse. According the the Euler-Bernoulli beam theory, it can be shown that the tensile stress experienced by the beam under consideration can be expressed as

$$\sigma = -zE\frac{d^2w}{dx^2}, \tag{3}$$

where z is the distance from the axis of symmetry of the cross section of the beam. σ is positive or negative depending on if the stress is tensile (stretching) or compressive. At each position x, the maximum stress is found at the top and the bottom surfaces of the beam and being equal in magnitude. A typical value of the maximum allowed tensile stress σ_{ma} for construction steel is 110 MPa. The maximum allowed compressive stress is usually higher, so it is sufficient to monitor the value of

$$\sigma = \frac{a}{2} E \frac{d^2 w}{dx^2}. \tag{4}$$

In this part of the project, start by calculating the stress at the surface of the beam to see if there is any position where the maximum allowed value is exceeded. Then, reformulate the minimisation as in 1.2.1 taking stress into account.

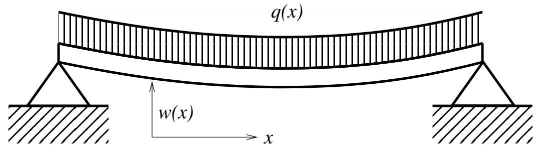

Figure 1: Simply supported beam under a static and homogeneous load $q(x)$. $w''(0) = w''(L) = 0$

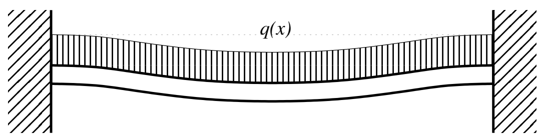

Figure 2: Clamped beam; $w'(0) = w'(L) = 0$.

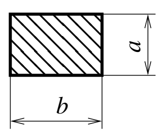

Figure 3: Cross section of beam.

Notation	Entity	Value	Unit
E	Elasticity module	210	GPa
σ_{ma}	Maximum allowed tensile stress	110	MPa
ρ	Density of steel	7800	kg·m^{-3}
L	Length of beam	1000	mm
a	Thickness of beam	3.0	mm

Table 1: List of material properties.