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Lab - Eigenvalues and eigenvectors

Introduction

The objectives of this lab are to show different contexts where eigenvalues are of interest and to introduce you to the simplest method for computing eigenvalues and eigenvectors, the power method.

The power method

For a given matrix A, the power method can be used for computing the eigenvalue with the largest magnitude and the corresponding eigenvector.

The theory behind the power method

Assume that the $n \times n$ matrix A has the eigenvalues λ_j, $j = 1, \ldots, n$ and that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|. \tag{1}$$

Also assume that the eigenvectors x_j, $j = 1, \ldots, n$ are linearly independent. This means that any vector z can be written as

$$z = \sum_{j=1}^{n} c_j x_j,$$

and

$$Az = \sum_{j=1}^{n} c_j A x_j = \sum_{j=1}^{n} c_j \lambda_j x_j.$$

Repeated multiplication with A leads to

$$A^k z = \sum_{j=1}^{n} c_j \lambda_j^k x_j = \lambda_1^k \left(c_1 x_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k x_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k x_n \right).$$

Because of (1) this behaves asymptotically as $\lambda_1^k c_1 x_1$ (as $k \to \infty$), i.e., a multiple of the eigenvector x_1.

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The power method algorithm

Choose an arbitrary initial guess z_0 for the eigenvector.
For $k = 0, 1, \ldots$ repeat

$$\begin{cases} y_k &= \dfrac{z_k}{||z_k||_2}, \\ z_{k+1} &= Ay_k, \\ \mu_k &= y_k^H z_{k+1}, \end{cases}$$

until $|\mu_k - \mu_{k-1}| < \text{tol}$.

The vectors y_k converge to the normalized eigenvector x_1. This means that $\mu_k = y_k^H z_{k+1} = y_k^H A y_k \approx y_k^H \lambda_1 y_k = \lambda_1 ||y_k||_2 = \lambda_1$.

Note that all vectors in the algorithm (z_k and y_k) must be column vectors.

Laboration

- Write a MATLAB function `[lambda,x]=mypower(A)` that applies the power method to a matrix A and returns the eigenvalue λ_1 and the eigenvector x_1. Apply your algorithm to random matrices of different sizes. You can use the MATLAB function `eig` to check your results.

- Try some different ways of choosing the inital guess z_0. How much does it affect the number of iterations?

- Create a matrix A with `A=inv(C)*diag(lambda)*C` where `lambda` is a vector with the eigenvalues you want to give matrix A and C is a random matrix. Experiment with different eigenvalues and study how the convergence of the power method depends on the eigenvalues.

Pay attention to:

- Convergence of the power method depends on the eigenvalues.