Workout 1

Scientific Computing III

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1.

1a)

Note that

$$A = \begin{pmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ & 1 & 2 \end{pmatrix} \implies D^{-1} = \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix},\ R = \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \tag{1.1}$$

and that

$$b = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix},\ u^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.2}$$

Then Jacobi iteration gives

$$\begin{aligned} u^{(1)} &= D^{-1}\left(b - Ru^{(0)}\right) \\ &= \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} - \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{\alpha}{2} \\ 0 \\ \frac{\beta}{2} \end{pmatrix} \end{aligned} \tag{1.3}$$

and subsequently

$$\begin{aligned} u^{(2)} &= D^{-1}\left(b - Ru^{(1)}\right) \\ &= \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} - \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} \frac{\alpha}{2} \\ 0 \\ \frac{\beta}{2} \end{pmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} \alpha \\ \frac{-(\alpha+\beta)}{2} \\ \beta \end{pmatrix} \end{aligned} \tag{1.4}$$

1b)

$D_{\text{inv}} \Leftarrow diag(1./diag(\text{A}));$
$R \Leftarrow A - diag(diag(A));$
$u \Leftarrow u^{(0)};$
repeat
  $u \Leftarrow D_{\text{inv}} * (b - R * u);$
until *termination condition not met*;

Algorithm 1: Jacobi Iteration for non-descript termination condition

2.

Iterative methods converge iff $|\rho(G)| < 1$. To see that this is the case Subtract the result of step i from the result of step $i+1$ to get an expression of the error. This amounts to

$$\begin{aligned} Gu^{(k)} + c - u^{(k)} &= Gu^{(k)} + c - \left(Gu^{(k-1)} + c\right) \\ &= G\Delta^{(k)} \end{aligned} \tag{2.1}$$

where $\Delta^{(k)} = u^{(k)} - u^{(k-1)}$. This converges if and only for random u if all the eigenvalues of G are less than unity in absolute value (thus causing all components of the error in eigenvector space to reduce at each iteration).

2a)

Fits the above criteria and will thus converge.

2b)

contains a term equal to unity and will therefore not converge.

3.

The power method can be stated as: "repeatedly multiply-by-A-and-normalize". Doing so gives the values $u^{(1)} = [0.700001, -0.714142]^T$ and $u^{(2)} = [-1., 0.]^T$. This gives us an eigenvalue estimation of $Au^{(2)} \cdot u^{(2)} = 0.495$ with eigenvector $u^{(2)}$. This is obviously not a correct value, which can be attributed to the matrix having complex eigenvalues.