Summary
Scientific Computing III
R.G.A. Deckers

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Partial Differential Equations

Iterative Methods

Jacobi

Jacobi's method is an iterative matrix-splitting method for solving linear systems of equations $Ax = b$.

The Method

Take $A = D - L - U$, and define $A = M - N$ where $M = D, N = L + U$ then solve

$$Du^+ = (L + U)u + b. \tag{2.1}$$

Or, define $G = M^{-1}$ and $c = M^{-1}b$ then

$$u^+ = Gu + c. \tag{2.2}$$

A fixed point of this expression (equilibrium) corresponds to the answer of the original equation.

Stability

Let u^* be the real solution to the equation. Then the error at step k is given by

$$
\begin{aligned}
e^{[k]} &= u^{[k]} - u^* \\
&= \left(Gu^{[k-1]} + c\right) - (Gu^* + c) \\
&= Ge^{[k-1]} \\
&= G^k e^{[0]}
\end{aligned}
\tag{2.3}
$$

From this we see that the method will converger if the spectral radius of $G < 1$.

Rate-of-Convergence

If the method converges, we can get the rate of convergence from the stability condition by looking at the 2-norm. Assume G can be diagonallized as

$$G^k = R\Lambda^k R^{-1} \tag{2.4}$$

then the the two-norm of the error is bound by

$$||e^{[k]}||_2 \leq ||\Lambda^k||_2||R||_2||R^{-1}||_2||e^{[0]}||_2 = \rho(G)^k||R||_2||R^{-1}||_2||e^{[0]}||_2. \tag{2.5}$$

If the matrix is a normal matrix (for example Hermitian or Skew-Hermitian) then the product of R-norms is unity and the error is bounded by the spectral radius. That is the method converges linearly proportional to the spectral radius of G.

Gauss-Seidel

Gauss-Seidel's method is an iterative matrix-splitting method for solving linear systems of equations $Ax = b$.

The Method

Take $A = D - L - U$, and define $A = M - N$ where $M = D - L, N = U$ then solve

$$(D - L)u^+ = Uu + b. \tag{2.6}$$

Which can be solved effetcively via forward subsitution. Or, define $G = M^{-1}$ and $c = M^{-1}b$ then

$$u^+ = Gu + c. \tag{2.7}$$

A fixed point of this expression (equilibrium) corresponds to the answer of the original equation.

Stability

See section a.2.

Rate-of-Convergence

See section a.3. In practice, the GS-method often performs better by about a factor of two, but has the same asymptotic behaviour.

Succesive Over-Relaxation (SOR)

SOR is an iterative matrix-splitting method for solving linear systems of equations $Ax = b$.

The Method

Take $A = D - L - U$, and define $A = M - N$ where $M = \frac{1}{\omega}(D - \omega L)$, $N = \frac{1}{\omega}((1-\omega)D + \omega U)$ then solve

$$Mu^{+} = Nu + b. \tag{2.8}$$

A more efficient way to look at this method is to compute the delta of a GS step and multiply it by ω.

Stability

This method is much harder to analyze than the other methods. One theorem states that if A is SPD and $D - \omega L$ is non-singular the method converges for all $0 < \omega < 2$.

Rate-of-Convergence

Rate of convergence is better than the other methods, determining the optimal or even stable ω can be quite difficult so in practice this method is only used for the special cases where ideal values are known or at the least the bounds.

Power Method

The power rmethod can be used to compute the dominant eigenvalue λ of a matrix A along with it's dominant eigenvector b.

The method, in it's simplest form is:

$$\begin{aligned} & b = b_0 \\ & \text{repeat:} \\ & \qquad b = \frac{Ab}{||Ab||} \\ & \qquad \lambda = \frac{b^{*}Ab}{b^{*}b} \\ & \text{until termination condition met} \end{aligned} \tag{2.9}$$

The method converges linearlly proportional to $|\lambda_1|/|\lambda_2|$. That is, the more dominant the first dominant eigenvalue is the quicker it converges.

Finite Element Method

Finite Difference Method