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Workout — FEM

Mandatory exercises

1. Consider the following boundary value problem

$$\begin{aligned} -u'' + au &= f, \quad 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned}$$

where a is a real positive constant. Define $(f,g) = \int_0^1 f(x)g(x)\,dx$ and $V = \{v | v \text{ cont. on } [0,1],\ v' \text{ bounded and piecewise cont. on } [0,1],\ v(0) = v(1) = 0\}$.

   (a) Derive the weak form of the differential equation.

   (b) Define the piecewise linear hat functions $\{\phi_i(x)\}_{i=1}^{n-1}$ by $\phi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, and the linear space $V_h = \{v \in V | v(x) \text{ linear on } [x_i, x_{i+1}],\ i = 0, \dots, n-1\}$, where $x_i = ih$, $h = 1/n$. Then every function $v_h \in V_h$ can be written as a linear combination of the hat functions. In particular, $u_h = \sum_{i=1}^{n-1} c_i \phi_i(x)$ for some coefficients c_i. Define the finite element method using these basis functions.

   (c) Derive the matrix K in the linear system $K\vec{c} = \vec{b}$ that gives the coefficients c_i in the finite element solution. Note that,

$$\int_0^1 \phi_i(x)\phi_j(x)\,dx = \begin{cases} \frac{2}{3n}, & i = j, \\ \frac{1}{6n}, & |i-j| = 1, \\ 0, & |i-j| > 1. \end{cases}$$

Non-mandatory exercises

2. You are going to solve the PDE

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in \Omega, \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

for two different domains Ω. In each case, you can choose between using the finite difference method and the finite element method. Make your choice and motivate it by pointing out the advantages or disadvantages of the respective methods.

(a) Ω is the unit square.

(b) Ω has the shape of a gear wheel (kugghjul).

3. Consider the boundary value problem in exercise 1. If $a = a(x)$ is a function of x, what effects does that have on the computations?

4. The stationary heat equation for a metal rod with one end at a fixed temperature, a constant heat flux at the other end, and a heat source function $f(x)$ is given by

$$\begin{cases} -u''(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u'(1) &= 1. \end{cases}$$

(a) Derive the weak formulation of the problem. The space

$$V^0 = \{v(x) \mid \; v(0) = 0,\; v \text{ is piecewise continuously differentiable on } 0 \le x \le 1\},$$

can be used both for the weak solution $u(x)$ and for the test functions $v(x)$.

(b) Introduce a uniform grid $x_j = jh$, $j = 0, \ldots, n$, where $h = 1/n$. Discretize the weak form of the PDE using the space

$$V_h^0 = \{v(x) \in V^0 \mid v(x) \text{ is linear on } [x_j,\, x_{j+1}],\ j = 0, \ldots, n-1\},$$

and derive the finite element method using linear hat functions as your basis functions. Give your final result as a linear system of equations, where the matrix elements are given explicitly, but the right hand side may contain integrals with the function $f(x)$.
Hint: Make a figure of your hat functions in order to get *all* the integrals right.