Workout 2

Scientific Computing III

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1.

1a)

The weak form is given by multiplying by v and integrating, and subsequenly using partial integration to get rid of the second derivative. Using the definitions given

$$\begin{aligned}(f,v) &= (au - u'', v) \\ &= a(u,v) - (u'', v) \\ &= a(u,v) + (u', v') \quad \text{(see lecture notes)}\end{aligned} \tag{1.1}$$

1b)

Find $u_h \in V_h$ such that

$$(f, v_h) = a(u_h, v_h) + (u_h', v_h') \quad \forall v_h \in V_h \tag{1.2}$$

Or, being more explicit:

$$\int_0^1 f(x)\phi_j(x)\ \mathrm{d}x = \sum_{i=1}^{n-1} c_i \int_0^1 \big(a\phi_i(x)\phi_j(x) + \phi_i'(x)\phi_j'(x)\big)\ \mathrm{d}x \quad \forall j = 1, \dots, n-1. \tag{1.3}$$

1c)

let $\vec{c}$ be the vector with elements c_i, let $\vec{b}$ be the vector whose i'th element is given by $\int_0^1 f(x)\phi_i(x)\ \mathrm{d}x$, finally let K be the matrix given by $K_{i,j} = \int_0^1 \big(a\phi_i(x)\phi_j(x) + \phi_i'(x)\phi_j'(x)\big)\ \mathrm{d}x$. Note that

$$\begin{aligned}\phi_j'(x) &= \frac{1}{h_j} \quad \forall x \in [x_{j-1}, x_j], \\ &= -\frac{1}{h_{j+1}} \quad \forall x \in [x_j, x_{j+1}], \\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{1.4}$$

From this it follows that

$$\begin{aligned}\int_0^1 \phi_i'(x)\phi_j'(x)\,\mathrm{d}x &= \frac{1}{h_i^2} + \frac{1}{h_{i+1}^2} \quad \text{if } i = j \\ &= -\frac{1}{h_i} \quad \text{if } j = i-1 \\ &= -\frac{1}{h_{i+1}} \quad \text{if } j = i+1\end{aligned} \tag{1.5}$$

which implies

$$K_{i,i} = \frac{2a}{3n} + \frac{1}{h_i^2} + \frac{1}{h_{i+1}^2} \tag{1.6}$$

$$K_{i,i-1} = \frac{a}{6n} - \frac{1}{h_i} \tag{1.7}$$

$$K_{i,i+1} = \frac{a}{6n} - \frac{1}{h_{i+1}} \tag{1.8}$$