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Definition 1 (Manifold and Chart). *The manifold is a set of points M. At any point $p_0 \in M$, there exists an open neighbourhood $U(p_0)$ such that for all $p \in U(p_0)$,*

$$\phi(p) = (x^1, ..., x^n), \quad \text{for some } (x^1, ..., x^n) \in \mathbb{R}^n. \tag{1}$$

A chart is defined as $(U(p_0), \phi)$.

Definition 2 (Tangent Space). *The tangent space T_pM is defined as*

$$T_pM := \{\text{All derivations defined on } M \text{ at } p\} \tag{2}$$

Theorem 1. *Let $v_p = v^j e_j$ and $D_v|_p = v^j \partial_j|_p$:*
(1). For $v_p \in \mathbb{R}^n_p$, $D_v|_p \in T_p\mathbb{R}^n$ is a derivation;
(2). $v_p \in \mathbb{R}^n_p \mapsto D_v|_p \in T_p\mathbb{R}^n$ is an isomorphism.

Proof. (1) is trivial. For (2): They are all vectors.
Injectivity: If $D_v|_p f = 0$ for any f, let $f = x^j$ then $D_v|_p x^j = v^j = 0$.
Surjectivity: Let $\omega \in T_p\mathbb{R}^n$, we would expect $v^j = \omega(x^j)$. Now,

$$f = f(p) + \partial_j f(p)(x^j - p^j) + (x^i - p^i)(x^j - p^j) \int_0^1 \partial_{i,j} f(p)(p + t(x - p))dt. \tag{3}$$

So

$$\omega(f) = \partial_j f(p)\omega(x^j) = v^j \partial_j f|_p. \tag{4}$$

□

Corollary 1.1.

$$\left\{ \frac{\partial}{\partial x^j}\bigg|_p \right\}_{j=1}^n \quad \text{is a basis for } T_p\mathbb{R}^n. \tag{5}$$

Proof. Since $\{e_j\}_{j=1}^n$ is a basis for $\mathbb{R}^n$, from the isomorphism we know that

$$\{D_{e_j}\}_{j=1}^n = \left\{ \frac{\partial}{\partial x^j}\bigg|_p \right\}_{j=1}^n \tag{6}$$

is a basis for $T_p\mathbb{R}^n$. □

Corollary 1.2. *Let $f \in C^\infty(\mathbb{R}^n)$. For $v = v^j \partial_j|_p \in T_p\mathbb{R}^n$, $D_{v^j e_j}|_p f = v(f)$.*

Definition 3. *Let $F : M \to N$, the differential (or pushforward) of F w.r.t. $v \in T_pM$ is $dF_p : T_pM \to T_pN$,*

$$dF_p(v)(f) := v(f \circ F), \text{ for any } f \in C^\infty(N). \tag{7}$$

Comment 1. *Let $f \in C^\infty(\mathbb{R}^n)$, $v \in \mathbb{R}^n$. We would have*

$$D_{dF(v)}|_{F(p)}(f) = D_v|_p(f \circ F) \text{ if } F(p+tv) := F(p) + dF(v)|_p t + o(t). \tag{8}$$

Because

$$D_v|_p(f \circ F) = \lim_{t \to 0} \frac{f(F(p+vt)) - f(F(p))}{t} = \lim_{t \to 0} \frac{f(F(p) + dF(v)t) - f(F(p))}{t} = D_{dF(v)}|_{F(p)}(f). \tag{9}$$

Moreover,

$$dF_p(v)(f) = v(f \circ F) \tag{10}$$

for $v \in T_p\mathbb{R}^n$.

Proposition 1. *Properties of differentials(pushforwards).*
(1). The identity:

$$d(Id_M) = Id_{T_pM}; \tag{11}$$

(2). The chain rule:

$$d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p; \tag{12}$$

(3). $(dF)_p^{-1} = dF_{F(p)}^{-1}$ and is an isomorphism if F is diffeomorphic.

Proof.
(1). $d(Id_M)(v)(f) = v(f) \Rightarrow d(Id_M)(v) = v$.
(2). $d(G \circ F)|_p(v)(f) = v(f \circ G \circ F) = v((f \circ G) \circ F) = dF|_p(v)(f \circ G)$. $dF|_p(v)$ is a tangent vector at $F(p)$, so this equals

$$dG|_{F(p)} \circ dF|_p(v)(f). \tag{13}$$

(3). From the result obtained in (1),

$$d(Id_M) = d(F^{-1} \circ F) = dF_{F(p)}^{-1} \circ dF_p = Id_{T_pM}. \tag{14}$$

□

Corollary 1.3. *A basis for T_pM is by isomorphism*

$$\left\{ (d\phi^{-1}) \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) \right\}_{j=1}^n, \tag{15}$$

and

$$(d\phi^{-1}) \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) f = \left. \frac{\partial f(\phi^{-1})}{\partial x^j} \right|_p. \tag{16}$$

We usually omit the $(d\phi^{-1})$ in the expression.

The proof is obvious.

Definition 4. *The cotangent space T_p^*M is the dual space for T_pM. Its elements are the cotangent vectors, or covectors in short.*

Proposition 2. *The function $\tilde{df}|_p : T_pM \to \mathbb{R}, v \mapsto df|_p(v) \circ Id_{\mathbb{R}}$ is a covector.*

Proof. It is obviously linear. And

$$\tilde{df}_p(v) = df|_p(v) Id_{\mathbb{R}} = v(Id_{\mathbb{R}} \circ f) = v(f) \in \mathbb{R}. \tag{17}$$

□

Proposition 3.

$$\left. \tilde{df} \right|_p = \left. \frac{\partial}{\partial x^j} \right|_p (f) \left. \lambda^j \right|_p. \tag{18}$$

Proof. Let the basis for T_p^*M be $\left\{ \lambda^j\big|_p \right\}_{j=1}^n$ such that $\lambda^i(\frac{\partial}{\partial x^j}) = \delta^i_j$. Suppose $\omega = \omega_j \lambda^j|_p \in T_p^*M$ then

$$\omega(\frac{\partial}{\partial x^j}|_p) = \omega_i \lambda^i|_p(\left.\frac{\partial}{\partial x^j}\right|_p) = \omega_i \delta^i_j = \omega_j. \tag{19}$$

Now

$$\tilde{df} = \tilde{df}(\frac{\partial}{\partial x^j})\lambda^j = \left.\frac{\partial}{\partial x^j}\right|_p (f)\ \lambda^j\big|_p. \tag{20}$$

□

Corollary 1.4. *The above basis for T_p^*M is in fact*

$$\left\{ \tilde{dx}^j\Big|_p \right\}_{j=1}^n, \tag{21}$$

and hence

$$\tilde{df}\Big|_p = \left.\frac{\partial}{\partial x^j}\right|_p (f)\ \tilde{dx}^j\Big|_p. \tag{22}$$

Proof. Let $f = x^j : M \to \mathbb{R}$, $\phi(p) = (x^1, ..., x^n)$ then $p \mapsto x^j$. Then

$$\tilde{dx}^i\Big|_p = \left.\frac{\partial}{\partial x^j}\right|_p (x^i)\ \lambda^j\big|_p = \delta^i_j\ \lambda^j\big|_p = \lambda^i\big|_p. \tag{23}$$

□

Comment 2. *For functions $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\tilde{df} = \frac{\partial f}{\partial x^j}\tilde{dx}^j. \tag{24}$$

Definition 5. *Let $f : M \to N$. The pullback by F at p is: $d^*F|_p : T_p^*N \to T_p^*M$,*

$$d^*F|_p(\omega)(v) := \omega(dF|_p(v)), \quad \omega \in T_{F(p)}N. \tag{25}$$

Comment 3. *Let $X \in T_pM$ and its dual be X^*. Let $f(X) = Y \in T_pN$ and its dual be Y^* (**Being the dual of** means $X^*(X) = 1$). It is natural to have (it is induced that) $d^*F|_p(Y^*) = X^*$. Applying $v \in T_pM$ then*

$$d^*F|_p(Y^*)(v) = X^*(v). \tag{26}$$

It is direct to check that $X^*(v) = Y^*(dF(v))$ by noticing that $X^*(X) = Y^*(dF(X)) = Y^*Y$. Hence

$$d^*F|_p(Y^*)(v) = Y^*(dF(v)) \tag{27}$$

which is just our definition.

Definition 6 (A change of notation)**.** *The pullback by F of ω is*

$$(F^*\omega)_p := d^*F|_p(\omega). \tag{28}$$

And it acts on a vector $v \in T_pM$ in the following way:

$$(F^*\omega)_p(v) = d^*F|_p(\omega)(p) = \omega(dF|_p(v)). \tag{29}$$

Comment 4. *It is just the X^* in the role of $d^*F|_p(Y^*) = X^*$. We here just changed the notation.*

Proposition 4. *Properties of pullbacks.*
(1). Let ω be a covector field on N, and u a function on N, then

$$F^*(u\omega) = (u \circ F)F^*\omega. \tag{30}$$

(2).

$$F^*(\tilde{du}) = d(u \tilde{\circ} F). \tag{31}$$

(3).

$$(F \circ G)^*\omega = G^*F^*\omega. \tag{32}$$

Proof.
(1). $(F^*(u\omega))_p = d^*F|_p (u\omega)_{F(p)} = dF^*|_p (u(F(p))\omega_{F(p)}) = u(F(p))\, dF^*|_p \omega_{F(p)} = u(F(p))\, (F^*\omega)_p$.
(2). Let $v \in T_pM$. $F^*(\tilde{du})(v) = \tilde{du}(dF|_p(v)) = dF|_p(v)(u) = v(u \circ F) = d(u \tilde{\circ} F)$.
(3). $((F \circ G)^*\omega)(v) = \omega(d(F \circ G)(v)) = \omega(dF \circ dG(v)) = (F^*\omega)(dG(v)) = (G^*F^*\omega)(v)$. $\square$

Corollary 1.5. *Let $\omega = \omega_j \tilde{dx}^j$, then*

$$F^*\omega = (\omega_j \circ F)d(x^j \tilde{\circ} F). \tag{33}$$

Therefore we can see that what F^* does to ω is simply a change of parameter.

Definition 7. *If $\omega = f(x)\tilde{dx}$ in $U \supset [a, b]$, the integral of ω over $[a, b]$ is:*

$$\int_{[a,b]} \omega := \int_a^b f(x)dx. \tag{34}$$

Proposition 5 (Change of variable of integrals)**.** *If $\phi : [c, d] \to [a, b]$ and is increasing,*

$$\int_{[c,d]} \phi^*\omega = \int_{[a,b]} \omega. \tag{35}$$

Proof.

$$\int_{[c,d]} \phi^*\omega = \int_{[c,d]} (f\circ\phi)d(x \tilde{\circ} \phi) = \int_{[c,d]} (f\circ\phi)\frac{\partial(x \circ \phi)}{\partial x}\tilde{dx} = \int_c^d f(\phi(x))\phi'(x)dx = \int_a^b f(x)dx = \int_{[a,b]} \omega. \tag{36}$$

$\square$

Definition 8 (Line integrals)**.** *Given a curve $\gamma : [a, b] \to M$ on which the integral of ω is performed over, this integral is defined as*

$$\int_\gamma \omega := \int_{[a,b]} \gamma^*\omega. \tag{37}$$

Comment 5. *For $\gamma([a, b]) \in U$ where $\omega = f(t)dt$, this definition leads to*

$$\int_\gamma \omega = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{38}$$

Proposition 6 (Change of variable of paths)**.** *Let $\Gamma : [c, d] \to M$ and $\Gamma = \gamma \circ \phi$ wtih $\phi : [c, d] \to [a, b]$ increasing, then*

$$\int_\Gamma \omega = \int_\gamma \omega. \tag{39}$$

Proof.

$$\int_\Gamma \omega = \int_{[c,d]} (\gamma \circ \phi)^*\omega = \int_{[c,d]} \phi^*\gamma^*\omega = \int_{[a,b]} \gamma^*\omega = \int_\gamma \omega. \tag{40}$$

$\square$

Definition 9. *A tensor on V of type (k,l) is a multilinear function T, whose space consists of*

$$\bigotimes_{i=1}^{k} V \otimes \bigotimes_{j=1}^{l} V^{*}. \tag{41}$$

Definition 10. *A covariant k-tensor is in the space which consists of*

$$\bigotimes_{j=1}^{k} V^{*}. \tag{42}$$

Definition 11. *Let $I = (i_1, ..., i_k)$, $I_\sigma := (i_{\sigma(1)}, ..., i_{\sigma(k)})$. If T is a covariant k-tensor,*

$$T(v_I) := T(v_{i_1}, ..., v_{i_k}). \tag{43}$$

Definition 12. *A covariant k-tensor T that satisfies*

$$T(v_I) = \operatorname{sgn}(\sigma) T(v_{I_\sigma}) \tag{44}$$

is called an alternating k-tensor. The space of alternative k-tensors is denoted $\Lambda^k(V^*)$.

Definition 13. *Let $(\varepsilon^1, ..., \varepsilon^n)$ be a basis for V^*, the elementary tensor is defined as*

$$\varepsilon^I = \varepsilon^{i_1,...,i_n} := \det([\varepsilon(v)]^p_q) = \det(\varepsilon^p(v_q)). \tag{45}$$

Proposition 7. *Let $\{E_j\}_{j=1}^n$ be a basis for V and $\{\varepsilon^j\}_{j=1}^n$ be the dual basis, then*

$$\varepsilon^I(E_{J_1}, ..., E_{J_k}) = \delta^I_J = \det(\delta^{i_p}_{j_q}) \text{ and } = (sgn)(\sigma) \text{ if } J = I_\sigma. \tag{46}$$

Moreover,

$$\alpha(E_I)\delta^I_J = \alpha(E_J). \tag{47}$$

The proof is obvious.

Proposition 8. *$\{\varepsilon^I : I$ is a list of increasing indices of length $k\}$ is a basis for $\Lambda^k(V^*)$. And hence $\dim \Lambda^k(V^*) = C(n,k)$ for $\dim V = n$.*

Proof. We prove by showing that this basis is spanning and L.I.
Spanning: Let $\alpha \in \Lambda^k(V^*)$. Now let α act on any set of vectors $E_J : \alpha(E_J)$. Then

$$\sum_I \alpha(E_I)\varepsilon^I \text{ acting on } E_J \text{ is } \sum_I \alpha(E_I)\varepsilon^I(E_J) = \sum_I \alpha(E_I)\delta^I_J = \alpha(E_J). \tag{48}$$

L.I. : Let $\sum_I \alpha_I \varepsilon^I = 0$,and we act it on E_J for any J:

$$0 = 0(E_J) = \sum_I \alpha_I \varepsilon^I(E_J) = \alpha_J. \tag{49}$$

□

Proposition 9. *Let T be a linear operator on V and $\omega \in \Lambda^k(V^*)$ we have*

$$\omega(Tv_1, ..., Tv_k) = (\det T)\omega(v_1, ..., v_k). \tag{50}$$

The proof is the same as deducing the expression for det.

Definition 14. *The wedge product (or exterior product) of two covectors $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ is defined as:*

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta), \tag{51}$$

where

$$\operatorname{Alt}(\alpha)(v_1, ..., v_k) = \frac{1}{k!} \sum_\sigma \operatorname{sgn}(\sigma) \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}). \tag{52}$$

Proposition 10.

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \ \textit{where } IJ = (i_1, ..., i_k, j_1, ..., j_l). \tag{53}$$

Proof. We show the equality of them acting on the basis of $\Lambda^{k+l}(V^*)$. We consider one non-trivial case: E_P, where $P = IJ$.

$$\varepsilon^I \wedge \varepsilon^J(E_{IJ}) = \frac{1}{k!l!} \sum_{\sigma \in S_k; \sigma' \in S_l} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')\varepsilon^I(E_{\sigma(i_1)}, ..., E_{\sigma(i_k)})\varepsilon^J(E_{\sigma'(j_1)}, ..., E_{\sigma'(j_l)}) \tag{54}$$

$$= \frac{1}{k!l!} \sum_{\sigma \in S_k; \sigma' \in S_l} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')\varepsilon^I(E_{I_\sigma})\varepsilon^J(E_{J_{\sigma'}}) = \frac{1}{k!l!} \sum_{\sigma \in S_k; \sigma' \in S_l} \mathrm{sgn}(\sigma)^2\mathrm{sgn}(\sigma')^2\varepsilon^I(E_I)\varepsilon^J(E_J) \tag{55}$$

$$= \frac{1}{k!l!} \sum_{\sigma \in S_k; \sigma' \in S_l} 1 = \frac{1}{k!l!}(k!l!) = 1 = \varepsilon^{IJ}(E_{IJ}) = 1. \tag{56}$$

Now the rest of the non-trivial cases is when $P = I_\sigma J_{\sigma'}$. This case is trivial, for

$$\mathrm{sgn}(\sigma \otimes \sigma') = \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma'). \tag{57}$$

□

Proposition 11. *Properties of wedge product:*
(1).

$$\omega \wedge \eta = (-1)^{k+l}\eta \wedge \omega. \tag{58}$$

(2).

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^{i_1, ..., i_k}. \tag{59}$$

(3).

$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, ..., v_k) = \det(\omega^i(v_j)). \tag{60}$$

The proof is trivial.

Corollary 1.6. *Any k-covector ω can be written as*

$$\omega = \sum_I{}' \omega_I \varepsilon^I := \sum_{I: Increasing} \omega_I \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}, \quad \omega_I \in \mathbb{R}. \tag{61}$$

Definition 15. *The interior multiplication by v map $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ is defined as:*

$$(i_v\omega)(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1}). \tag{62}$$

An alternative notation is $i_v\omega = v \lrcorner \omega$.

Lemma 1.

$$i_v(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^k (-1)^{i-1}\omega^i(v)\omega^1 \wedge \cdots \wedge \cancel{\omega^i} \wedge \cdots \wedge \omega^k. \tag{63}$$

Proof.
$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, ..., v_k) = \det(\omega^i(v_j))$. Apply the cofactor expansion:

$$\det(\omega^i(v_j)) = \sum_{i=1}^k (-1)^{i-1}\omega^i(v_1) \det(\omega^i(v_j))\big|_{i,j\geq 2} = \sum_{i=1}^k (-1)^{i-1}\omega^i(v_1)\omega^1 \wedge \cdots \cancel{\omega^i} \wedge \cdots \wedge \omega^k(v_2, ..., v_k). \tag{64}$$

Fix $v_1 = v$ then

$$i_v(\omega^1 \wedge \cdots \wedge \omega^k)(v_2, ..., v_k) = \sum_{i=1}^k (-1)^{i-1}\omega^i(v_1)\omega^1 \wedge \cdots \cancel{\omega^i} \wedge \cdots \wedge \omega^k(v_2, ..., v_k). \tag{65}$$

□

Proposition 12. *Properties of interior multiplication:*
(1).

$$i_v \circ i_v = 0. \tag{66}$$

(2).

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta). \tag{67}$$

Proof.
(1). $(i_v \circ i_v\omega)(v_1, ..., v_{k-2}) = \omega(v, v, v_1, ..., v_{k-2}) = 0.$
(2). ω and η can be decomposed into a linear combination of products of covectors. We only need to show that this holds for $\omega = a^1 \wedge \cdots \wedge a^k$ and $\eta = a^{k+1} \wedge \cdots \wedge a^{k+l}$. So now

$$i_v(\omega \wedge \eta) = \sum_{i=1}^{k+l} (-1)^{i-1} a^i(v) a^1 \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^{k+l}; \tag{68}$$

$$(i_v\omega) \wedge \eta = \sum_{i=1}^{k} (-1)^{i-1} a^i(v) a^1 \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^k \wedge a^{k+1} \wedge \cdots \wedge a^{k+l}; \tag{69}$$

$$(-1)^k \omega \wedge (i_v\eta) = \sum_{i=1}^{l} (-1)^{i-1}(-1)^k a^{k+i}(v) a^1 \wedge \cdots \wedge a^{k+l} \wedge a^{k+1} \wedge \cdots \wedge \cancel{a^{k+i}} \wedge \cdots \wedge a^{k+l} \tag{70}$$

$$= \sum_{i=k+1}^{k+l} (-1)^{i-1} a^i(v) a^1 \wedge \cdots \wedge a^{k+l} \wedge a^{k+1} \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^{k+l}. \tag{71}$$

And since

$$\sum_{i=1}^{k+l} (-1)^{i-1} a^i(v) a^1 \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^{k+l}$$

$$= \sum_{i=1}^{k} (-1)^{i-1} a^i(v) a^1 \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^{k+l} + \sum_{i=k+1}^{k+l} (-1)^{i-1} a^i(v) a^i(v) a^1 \wedge \cdots \wedge \cancel{a^i} \wedge \cdots \wedge a^{k+l}, \tag{72}$$

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta). \tag{73}$$

□

Definition 16. *A section on $\Lambda^k(T^*M)$ is called a k-form. In other words, a k-form at a point $p \in M$ is an element in $\Lambda^k(T_p^*M)$. The vector space of all k-forms given M is denoted $\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$.*

In the following paragraphs, $\tilde{df}$ will be notationally replaced by df for convenience for any function f. We automatically distinguish them.

Comment 6. *In any chart, a k-form ω can be written as*

$$\omega = \sum_I{}' \omega_I dx^I = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{74}$$

Definition 17 (Extension of pullbacks)**.** *Let $F : M \to N$ and $\omega \in \Omega^k(N)$, we define*

$$(F^*\omega)\big|_p (v_1, ..., v_k) := \omega(\,dF|_p\,(v_1), ..., dF|_p\,(v_k)). \tag{75}$$

Proposition 13. *Properties of pullback on forms:*
(1)

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta). \tag{76}$$

(2)In a chart

$$F^*(\sum_I{}' \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}) = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F). \tag{77}$$

Proof.
(1).

$$F^*(\omega \wedge \eta)(v_1, ..., v_{k+l}) = (\omega \wedge \eta)(dF(v_1), ..., dF(v_{k+l})) = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta)(dF(v_1), ..., dF(v_{k+l})) \tag{78}$$

while

$$(F^*\omega) \wedge (F^*\eta) = \frac{(k+l)!}{k!l!}\mathrm{Alt}(F^*\omega \otimes F^*\eta)(v_1, ..., v_k). \tag{79}$$

Now

$$(k+l)!\mathrm{Alt}(F^*\omega \otimes F^*\eta)(v_1, ..., v_k) = \sum_\sigma \mathrm{sgn}(\sigma) F^*\omega \otimes F^*\eta(v_{\sigma(1)}, ..., v_{\sigma(2)}) \tag{80}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma) F^*\omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) F^*\eta(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}) \tag{81}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma)\omega(dF(v_{\sigma(1)}), ..., dF(v_{\sigma(k)}))\eta(dF(v_{\sigma(k+1)}), ..., dF(v_{\sigma(k+l)})) \tag{82}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma)(\omega \otimes \eta)(dF(v_{\sigma(1)}), ..., dF(v_{\sigma(k+l)})) = (k+l)!\mathrm{Alt}(\omega \otimes \eta)(dF(v_1)), ..., dF(v_{k+l})). \tag{83}$$

(2). This follows directly from (1). □

Proposition 14. *Given $F : M \to N$, x^j be the local coordinate for M and y^j N. We have*

$$F^*(u dy^1 \wedge \cdots dy^n) = (u \circ F)(\det(dF)) dx^1 \wedge \cdots \wedge dx^n \tag{84}$$

where $[dF]$ is represented in the corresponding basis ∂_{x^i} and ∂_{y^j}.

Proof. At each point p,

$$F^*(u dy^1 \wedge \cdots \wedge dy^n)(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) = (u \circ F)(d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F))(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) \tag{85}$$

$$= (u \circ F)(d(F^1) \wedge \cdots \wedge d(F^n))(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) = (u \circ F)\det(dF^i(\frac{\partial}{\partial x^j})) = (u \circ F)\det(\frac{\partial F^i}{\partial x^j}). \tag{86}$$

□

Definition 18. *Given $\omega = \sum_I{}' \omega_I dx^I$, the exterior differential of ω is defined as*

$$d\omega := \sum_I{}' d\omega_I \wedge dx^I. \tag{87}$$

Here $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

Comment 7.

$$\sum_I{}' d\omega_I \wedge dx^I = \sum_I{}' \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I. \tag{88}$$

For a 1-form $\omega = f_i dx^i$,

$$d\omega = d(f_i dx^i) = \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i<j}(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j}) dx^i \wedge dx^j. \tag{89}$$

Proposition 15. *Properties of d:*
(1).

$$d \circ d = 0. \tag{90}$$

(2).

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \tag{91}$$

(3).

$$dF^*(\omega) = F^*(d\omega). \tag{92}$$

Proof.
(1). Locally,

$$d\circ d\omega = d\left(\sum_I{}'\frac{\partial\omega_I}{\partial x^j}dx^j\wedge dx^I\right) = \sum_I{}'\frac{\partial^2\omega_I}{\partial x^i\partial x^j}dx^i\wedge dx^j\wedge dx^I \tag{93}$$

$$= \sum_{i<j}\sum_I{}'\left(\frac{\partial^2\omega_I}{\partial x^i\partial x^j}-\frac{\partial^2\omega_I}{\partial x^j\partial x^i}\right)dx^i\wedge dx^j\wedge dx^I = 0. \tag{94}$$

(2). Due to the linearity of d, let us consider the case with $\omega_I dx^I$ and $\eta_J dx^J$:

$$d(\omega_I dx^I\wedge\eta_J dx^J) = d(\omega_I\eta_J)\wedge dx^I\wedge dx^J = (\eta_J d\omega_I+\omega_I d\eta_J)\wedge dx^I\wedge dx^J \tag{95}$$

$$= d\omega_I\wedge dx^I\wedge\eta_J dx^J + \omega_I d\eta_J\wedge dx^I\wedge dx^J = d(\omega_I dx^I)\wedge\eta + (-1)^k\omega_I\wedge d(\eta_J dx^J). \tag{96}$$

(3). Again, let us consider $\omega = \omega_I dx^I$:

$$dF^*(\omega_I\wedge dx^I) = d\left((\omega_I\circ F)\wedge d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F)\right) = d(\omega_I\circ F)\wedge d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F). \tag{97}$$

$$F^*(d\omega_I\wedge dx^I) = d(\omega_I\circ F)\wedge d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F). \tag{98}$$

□

Definition 19. *The integral of an n-form ω on $D\subset\mathbb{R}^n$ is defined as*

$$\int_D\omega = \int_D f dx^1\wedge\cdots\wedge dx^n := \int_D f dx^1\cdots dx^n. \tag{99}$$

Definition 20. *The integral of an n-form ω on a chart (U,ϕ) of an n-dimensional manifold M is defined as*

$$\int_U\omega = \pm\int_{\phi(U)}(\phi^{-1})^*\omega, \tag{100}$$

where $\pm = +$ if the chart is positively oriented and $-$ otherwise.

Comment 8. *Suppose M is embedded in a bigger space, say $\mathbb{R}^{2n}$. Under whichever parametrization(atlas) ϕ with $\phi(M)=D$, if $p\in\partial M$, $\phi(p)\in\partial D$. Since if not, an open neighbourhood $\subset D$ containing $\phi(p)$ would be mapped back to some open neighbourhood $\subset M$ containing p.*

Definition 21. *An n-dimensional manifold with regular boundary is a manifold with $\phi_i(U_i)\subset\mathbb{H}^n$ $\forall i$, the half space of $\mathbb{R}^n$. A point $p\in M$ is in the boundary ∂M if for $p\in U_i$, we have*

$$\phi_i(p) = (0, x_2, ..., x_n). \tag{101}$$

Definition 22. *Suppose N is an outward pointing normal on ∂M, the orientation of ∂M is*

$$[N, E_1, ..., E_{n-1}],\ \textit{where } (E_1,...,E_{n-1}) \textit{ is a basis for } T_pM. \tag{102}$$

Lemma 2. *Let ω be a compactly supported $(n-1)$ form on $\mathbb{H}^n$, we have*

$$\int_{\mathbb{H}^n}d\omega = \int_{\partial\mathbb{H}^n}\omega. \tag{103}$$

Proof. Let ω be compactly supported on $A=[0,R]\times[-R,R]^{n-1}$, then ω vanishes at $x^i=\pm R$ for $i=2,...,n$(while ω might not vanish at $x^1=0$). We have

$$\omega = \sum_{i=1}^n\omega_i dx^1\wedge\cdots\wedge\cancel{dx^i}\wedge\cdots\wedge dx^n, \tag{104}$$

so

$$d\omega = \sum_{i=1}^n d\omega_i\wedge dx^1\wedge\cdots\wedge\cancel{dx^i}\wedge\cdots\wedge dx^n = \sum_{i=1}^n\frac{\partial\omega_i}{\partial x^j}dx^j\wedge dx^1\wedge\cdots\wedge\cancel{dx^i}\wedge\cdots\wedge dx^n \tag{105}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n. \tag{106}$$

Therefore

$$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^{n} (-1)^{n-1} \int_0^R dx^1 \cdots \int_{-R}^{R} dx^n \frac{\partial \omega_i}{\partial x^i}$$

$$= \sum_{i=2}^{n} (-1)^{n-1} \int_0^R dx^1 \cdots \cancel{\int_{-R}^{R} dx^i} \cdots \int_0^R dx^n [\omega_i]_{x^i=-R}^{x^i=R} + \int_{-R}^{R} dx^2 \cdots \int_{-R}^{R} dx^n [\omega_1]_{x^1=0}^{x^1=R} \tag{107}$$

$$= 0 - \int_{-R}^{R} dx^2 \cdots \int_{-R}^{R} dx^n \omega_1(0, x^2, ..., x^n). \tag{108}$$

Now consider

$$\int_{\partial \mathbb{H}^n} \omega = \int_{\partial \mathbb{H}^n \cap A} \omega = \sum_{i=1}^{n} \int_{\partial \mathbb{H}^n \cap A} \omega_i(0, x^2, ..., x^n) dx^1 \wedge \cdots \wedge \cancel{dx^i} \wedge \cdots \wedge dx^n. \tag{109}$$

Since $x^1 \equiv 0$ on $\partial \mathbb{H}^n \cap A$, $dx^1(v) = v(x^1) = v^i \partial_{x^i}(x^1) = 0$. Then the above integral is

$$\int_{\partial \mathbb{H}^n \cap A} \omega_1(x^2, ..., x^n) dx^2 \wedge \cdots \wedge dx^n. \tag{110}$$

Since $\mathbb{H}^n \cap A$ being an $(n-1)$-dimensional manifold, is not yet a subset of $\mathbb{R}^{n-1}$, we have to use apply pullback $(\phi^{-1})^*$ on our form. Since $\phi : (0, x^2, ..., x^n) \mapsto (x^1, ..., x^{n-1})$, we have

$$d(x^i \circ \phi^{-1}) = dx^i; \quad (\omega_1 \circ \phi^{-1}) = \omega_1(x^2, ..., x^n). \tag{111}$$

The above integral becomes just

$$\int_{\partial \mathbb{R}^n \cap A} \mathcal{O} \omega_1(x^2, ..., x^n) dx^2 \wedge \cdots \wedge dx^n. \tag{112}$$

Now we examine $\mathcal{O}$, which is the orientation-preserving index for ϕ^{-1}. The outward pointing normal is $-E_1$, and $\{-E_1, E_2, ..., E_n\}$ is negatively oriented so the index is -1. Then the integral is

$$-\int_{\partial \mathbb{R}^n \cap A} \omega_1(x^2, ..., x^n) dx^2 \wedge \cdots \wedge dx^n = -\int_{-R}^{R} dx^2 \cdots \int_{-R}^{R} dx^n \omega_1(x^2, ..., x^n) = \int_{\mathbb{H}^n} d\omega. \tag{113}$$

□

Theorem 2 (Stokes Theorem)**.** *Let ω be an $(n-1)$-form on an n-dimensional manifold M, we have*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{114}$$

Proof. Let $\{(U_i \in \mathbb{H}^n, \phi_i)\}$ be positively oriented charts for M, and ω be compactly supported on U_i then

$$\int_M d\omega = \int_{U_i} d\omega = \int_{\phi_i(U_i)} (\phi_i^{-1})^* d\omega = \int_{\mathbb{H}^n} (\phi_i^{-1})^* d\omega = \int_{\mathbb{H}^n} d\left((\phi_i^{-1})^* \omega\right) = \int_{\partial \mathbb{H}^n} (\phi_i^{-1})^* \omega = \int_{\partial M} \omega. \tag{115}$$

The last equality follows from that $d\phi$ takes outward pointing vectors on ∂M to outward ones on $\partial \mathbb{H}^n$, and hence preserves orientation. Now, consider ψ_j as the partition of unity of $\{U_i\}$. Then

$$\int_{\partial M} \omega = \sum_j \int_{\partial M} \psi_j \omega = \sum_j \int_M d(\psi_j \omega) = \sum_j \int_M d\psi_j \wedge \omega + \sum_j \int_M \psi_j d\omega \tag{116}$$

$$= \int_M d \sum_j \psi_j \wedge \omega + \int_M \sum_j \psi_j d\omega = \int_M d(1) \wedge \omega + \int_M 1 d\omega = \int_M d\omega. \tag{117}$$

□

Comment 9. *The convention for orientations on ∂M was just to make Stokes theorem hold.*

Definition 23. *A form ω is closed if*

$$d\omega = 0. \tag{118}$$

It is exact if

$$\omega = d\eta \text{ for some } \eta. \tag{119}$$

The set of closed k-forms is

$$\mathcal{Z}^k := \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) \tag{120}$$

and the set of exact k-forms is

$$\mathcal{B}^k := \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)). \tag{121}$$

Definition 24. *The de Rham cohomology group in degree k is defined as*

$$H^k_{\mathrm{dR}}(M) := \mathcal{Z}^k / \mathcal{B}^k. \tag{122}$$

Comment 10. *Since $\mathcal{Z}^k$ and $\mathcal{B}^k$ are vector spaces, $H^k_{\mathrm{dR}}(M)$ is also a vector space, whose elements are equivalent classes $[\omega]$. The equivalence relation is:*

$$\omega \equiv \eta \text{ if } \omega - \eta = d\beta. \tag{123}$$

In other words, $H^k_{\mathrm{dR}}(M) = \{[\omega] : \omega \in \Omega^k(M)\}$ is a vector space over $\mathbb{R}$, subjected to $[d\beta] = 0$.

Proposition 16. *Let $F : M \to N$, the (linear) map $F^* : \Omega^k(N) \to \Omega^k(M)$ also sends $\mathcal{Z}^k(N)$ to $\mathcal{Z}^k(M)$ and similarly $\mathcal{B}^k(N)$ to $\mathcal{B}^k(M)$. Hence, it can be extended on $H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$. This is called the cohomology map, whose action is obviously*

$$F^*[\omega] = [F^*\omega]. \tag{124}$$

Then naturally we still have $(F \circ G)^* = G^* \circ F^*$.

Proof.

$$dF^*(\omega) = F^*(d\omega) = 0; \quad F^*(d\beta) = dF^*(\beta). \tag{125}$$

This function is well defined: If $[\omega] = [\omega']$, we have $\omega' = \omega + d\beta$, so

$$F^*[\omega'] = [F^*\omega'] = [F^*\omega + F^*d\beta] = [F^*\omega + d(F^*\beta)] = [F^*\omega]. \tag{126}$$

□

Corollary 2.1. *If M and N are diffeomorphic, $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$.*

Proof. Let $\phi : M \to N$. Obviously $\dim \mathcal{B}^k(M) = \dim \mathcal{B}^k(N)$, so $\phi^*\omega = d\beta$ must imply $\omega \in \mathcal{B}^k(N)$ and so

$$\ker(\phi^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)) = \{[0]\}. \tag{127}$$

□

Lemma 3. *We define $i_t : M \to [0,1] \times M$ as:*

$$i_t(x) := (t, x). \tag{128}$$

Then we have $i_0^* = i_1^*$ where

$$i^*_{t_0}\omega(t, \cdot) := \omega(t_0, \cdot). \tag{129}$$

Proof. We only need to show $i_0^*\omega - i_1^*\omega = d\eta$. Any form on $[0,1]\times M$ can be written as

$$\omega(t,x) = \alpha(t,x) + \beta(t,x)\wedge dt, \quad \alpha \text{ does not involve } dt. \tag{130}$$

We only consider the case where ω is closed:

$$d\omega = i_t^* d\alpha + \partial_t\alpha \wedge dt + d\beta \wedge dt = 0 \Rightarrow i_t^* d\alpha = 0; \text{ and } \partial_t \alpha = d\beta = i_t^* d\beta. \tag{131}$$

The last equality follows from that $d(\beta\wedge dt) = i_t^* d\beta \wedge dt + \partial_t\beta\wedge dt \wedge dt = i_t^* d\beta \wedge dt$. Now consider

$$\mathcal{K}\omega(t,x) := \int_{[0,t]} \beta(s,x)\wedge ds. \tag{132}$$

Then by chain rule the differential is simply

$$d\mathcal{K}\omega(t,x) = \int_{[0,t]} i_s^* d\beta(s,x)\wedge ds + \beta(t,x)\wedge dt \tag{133}$$

$$= \int_{[0,t]} \partial_s\alpha(s,x)\wedge ds + \beta(t,x)\wedge dt = \alpha(t,x) - \alpha(0,x) + \beta(t,x)\wedge dt. \tag{134}$$

Therefore let $\eta = \mathcal{K}\omega(1,x)$ then we have

$$d\eta = d\mathcal{K}\omega(1,x) = \alpha(1,x) - \alpha(0,x) = i_1^*\omega(t,x) - i_0^*\omega(t,x). \tag{135}$$

□

Proposition 17. *If f_0, f_1 are homotopic, $f_0^* = f_1^*$.*

Proof. We have $H(t,x)$ a continuous(smooth) map such that $f_0(x) = H(t,x)\circ i_0$ and $f_1(x) = H(t,x)\circ i_1$. Now

$$f_0^* = (H\circ t_0)^* = i_0^*\circ H^* = i_1^* \circ H^* = (H\circ t_1)^* = f_1^*. \tag{136}$$

□

Corollary 2.2 (Cauchy integral theorem). *If γ_1 and γ_2 are homotopic, we have for any $\omega \in \Omega^1(\mathbb{C})$,*

$$\oint_{\gamma_1}\omega = \oint_{\gamma_2}\omega. \tag{137}$$

Proof. $d\omega = 0$. So

$$\oint_{\gamma_1}\omega = \int_{[0,1]}\gamma_1^*\omega = \int_{[0,1]}\gamma_2^*\omega + d\mathcal{K}(H^*\omega)(1,t) = \int_{[0,1]}\gamma_2^*\omega + d\mathcal{K}(\gamma_2^*\omega)(t) = \oint_{\gamma_2}\omega. \tag{138}$$

The last equality follows from that $\gamma_2(0) = \gamma_2(1)$. □

Theorem 3 (Poincaré Lemma). *On any contractible manifold M, $H^k_{\mathrm{dR}}(M) = \{[0]\}$. This is to say, any closed form is exact.*

Proof. Let $f_0 = p\in M$ and $f_1 = \mathrm{id}$. We have

$$f_0^*\omega(v) = \omega(df_0(v)) = 0 \Rightarrow f_0^*\omega = 0, \forall\omega \Rightarrow f_0^* = 0 \Rightarrow f_0^*(H^k_{\mathrm{dR}}(M)) = \{[0]\}. \tag{139}$$

And now

$$f_1^*\omega(v) = \omega(df_1(v)) = \omega(v) \Rightarrow f_1^* = \mathrm{id} \Rightarrow f_1^*(H^k_{\mathrm{dR}}(M)) = H^k_{\mathrm{dR}}(M). \tag{140}$$

Since M is contractible, f_0 and f_1 are homotopic so $f_0^* = f_1^*$. Therefore

$$H^k_{\mathrm{dR}}(M) = f_1^*(H^k_{\mathrm{dR}}(M)) = f_0^*(H^k_{\mathrm{dR}}(M)) = \{[0]\}. \tag{141}$$

□

Theorem 4 (Pushforward Correspondence)**.** *Γ is a 1-manifold. Now consider a smooth curve $\gamma(t) \in \Gamma$. The velocity of the curve γ is*

$$\gamma' := d\gamma \frac{\partial}{\partial t} \in T_{\gamma(t)}\Gamma. \tag{142}$$

Moreover,

$$dF(v) = (F \circ \gamma)', \quad \textit{where } \gamma(0) = p, \quad \gamma' = v. \tag{143}$$

Proof.

$$(F \circ \gamma)' g = d(F \circ \gamma) \frac{\partial}{\partial t} g = \frac{\partial}{\partial t}(g \circ F \circ \gamma) = d\gamma \frac{\partial}{\partial t}(g \circ F) = v(g \circ F) = dF(v)(g). \tag{144}$$

□

Definition 25. *On a manifold M, we define a symmetric, covariant 2-tensor field g as the metric on M. (M, g) is called a Riemannian manifold. Then in a local coordinate neighbourhood,*

$$g = g_{ij}(x_1, ..., x_n) dx^i \otimes dx^j, \quad g_{ij} = g_{ji}. \tag{145}$$

Hence

$$g_{ij}(x_1, ..., x_n) = g(\partial_i, \partial_j). \tag{146}$$

Directly,

$$g_{ij} dx^i \otimes dx^j = \frac{1}{2} g_{ij}(dx^i \otimes dx^j + dx^j \otimes dx^i) = g_{ij}\mathrm{Sym}(dx^i \otimes dx^j) =: g_{ij} dx^i dx^j. \tag{147}$$

For vectors X, Y,

$$g(X, Y) := \langle X, Y \rangle = \langle Y, X \rangle. \tag{148}$$

Definition 26. *Let $E \to M$ be a smooth real vector bundle. A connection(or covariant derivative) operator $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ is defined by the following properties:*

$$\begin{aligned} \nabla(fX) &= (df) \otimes X + f\nabla X; \\ \nabla(X \oplus Y) &= \nabla X \oplus \nabla Y; \\ \nabla(X \otimes Y) &= \nabla X \otimes Y + X \otimes \nabla Y. \end{aligned} \tag{149}$$

Then we define $\nabla_v : \Gamma(E) \to \Gamma(E)$ for $v \in TM$ by

$$\nabla_v(fX) := \nabla(fX)(v) = df(v) \otimes X + f\nabla X(v). \tag{150}$$

Comment 11. *This definition of connection arises from the properties of directional derivative in Euclidean spaces:*

$$D_v(fX) = (D_v f)X + fD_v X = df(v)X + fD_v X \Rightarrow D(fX) = df \otimes X + fDX. \tag{151}$$

Definition 27. *The Christoffel symbols Γ is defined as*

$$\nabla_{\partial_i} \partial_j := \Gamma^k_{ij} \partial_k. \tag{152}$$

Hence $\nabla \partial_j = \Gamma^k_{ij} \partial_k \otimes dx^i$.

Definition 28. *Given the connection ∇ acting on T_pM, the induced connection acts on T^*_pM by:*

$$\nabla(\omega(X)) := (\nabla\omega)(X) + \omega(\nabla X), \quad \omega \in T^*_pM; X \in T_pM. \tag{153}$$

Corollary 4.1.

$$\nabla dx^k = -\Gamma^k_{ij} dx^i \otimes dx^j \Leftrightarrow \nabla_{\partial_i} dx^k = -\Gamma^k_{ij} dx^j. \tag{154}$$

Proof.

$$0 = \nabla(\delta^i_j) = \nabla(dx^i(\partial_j)) = \nabla(dx^i)(\partial_j) + dx^i(\nabla\partial_j) = \nabla(dx^i)(\partial_j) + dx^i(\Gamma^l_{jk}\partial_l \otimes dx^k) = \nabla_{\partial_j} dx^i + \Gamma^i_{jk} dx^k. \tag{155}$$

Therefore $\nabla_{\partial_j} dx^i = -\Gamma^i_{jk} dx^k$. □

Theorem 5. *Let $X = X^i\partial_i; Y = Y^j\partial_j; Z = Z^k\partial_k$,*

$$X\langle Y, Z\rangle = (\nabla_X g)(Y, Z) + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle . \tag{156}$$

Proof.

$$(\nabla_X g)(Y,Z) = \nabla_X(g_{ij}dx^i \otimes dx^j)(Y,Z) = dg_{ij}(X)(dx^i \otimes dx^j)(Y,Z) + g_{ij}\nabla_X(dx^i \otimes dx^j)(Y,Z) \tag{157}$$

$$= X(g_{ij})Y^iZ^j + g_{ij}(\nabla_X dx^i \otimes dx^j)(Y,Z) + g_{ij}(dx^i \otimes \nabla_X dx^j)(Y,Z) \tag{158}$$

$$= X(g_{ij}Y^iZ^j) - g_{ij}X(Y^i)Z^j - g_{ij}Y^iX(Z^j) - g_{ij}X^l\Gamma^i_{lk}dx^k \otimes dx^j(Y,Z) - g_{ij}X^l\Gamma^j_{lk}dx^i \otimes dx^k(Y,Z) \tag{159}$$

$$= X\langle Y,Z\rangle - g_{ij}X(Y^i)Z^j - g_{ij}Y^iX(Z^j) - g_{ij}X^l\Gamma^i_{lk}Y^kZ^j - g_{ij}X^l\Gamma^j_{lk}Y^iZ^k. \tag{160}$$

Now

$$g_{ij}X(Y^i)Z^j + g_{ij}X^l\Gamma^i_{lk}Y^kZ^j = g_{ij}[X(Y^i) + X^l\Gamma^i_{lk}Y^k]Z^j = g_{ij}(\nabla_X Y)^iZ^j = \langle \nabla_X Y, Z\rangle \tag{161}$$

since

$$\nabla_X Y = \nabla_X(Y^i\partial_i) = dY^i(X)\partial_i + \nabla_X(Y^k\partial_k) = X(Y^i)\partial_i + X^lY^k\Gamma^i_{lk}\partial_i. \tag{162}$$

The case is similar for the other two terms. Therefore

$$(\nabla_X g)(Y,Z) = X\langle Y,Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle . \tag{163}$$

□

Theorem 6 (Fundamental theorem of Riemannian geometry)**.** *If* $\mathrm{Tor}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$ *and $\nabla g = 0$, there $\exists!\ \nabla$. This connection is called the Riemann connection.*

Proof.
Uniqueness:

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle . \end{aligned} \tag{164}$$

Adding the first two equations and subtracting the third one from it:

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle &= \left\langle \cancel{\mathrm{Tor}(X,Y)} - [X,Y], Z\right\rangle + 2\langle \nabla_Y X, Z\rangle \\ &+ \left\langle \cancel{\mathrm{Tor}(X,Z)} + [X,Z], Y\right\rangle + \left\langle \cancel{\mathrm{Tor}(Y,Z)} + [Y,Z], X\right\rangle \end{aligned} \tag{165}$$

Rearranging,

$$2\langle \nabla_Y X, Z\rangle = -\langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle + X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle . \tag{166}$$

Existence: We simply define the connection as given above. □

Below we will always use the Riemann connection.

Comment 12. *The form of Γ^k_{ij} in some coordinate is calculated from:*

$$g_{kl}\Gamma^k_{ij} = \langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = \partial_i\langle \partial_j, \partial_l\rangle - \langle \nabla_{\partial_i}\partial_l, \partial_j\rangle = \partial_i g_{jl} - g_{kj}\Gamma^k_{il}; \tag{167}$$

$$g_{ki}\Gamma^k_{jl} = \partial_j g_{li} - g_{kl}\Gamma^k_{ji}; \tag{168}$$

$$g_{kj}\Gamma^k_{li} = \partial_l g_{ij} - g_{ki}\Gamma^k_{lj}. \tag{169}$$

Going back to the expression for the connection,

$$\langle [\partial_i, \partial_j], \partial_l\rangle = \langle \partial_i\partial_j - \partial_j\partial_i, \partial_l\rangle = \langle 0, \partial_l\rangle = 0. \tag{170}$$

Therefore $\langle \nabla_{\partial_i}\partial_j, Z\rangle = \left\langle \nabla_{\partial_j}\partial_i, Z\right\rangle$ and so $\Gamma^k_{ij} = \Gamma^k_{ji}$. Now,

$$g_{kl}\Gamma^k_{ij} = \partial_i g_{jl} - g_{kj}\Gamma^k_{il} = \partial_i g_{jl} - \partial_l g_{ij} + g_{ki}\Gamma^k_{lj} = \partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{li} - g_{kl}\Gamma^k_{ji}. \tag{171}$$

Hence

$$g_{kl}\Gamma^k_{ij} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right). \tag{172}$$

Definition 29. *A curve γ is geodesic if*

$$\nabla_{\gamma'}\gamma' = 0. \tag{173}$$

In the following context, γ will always be referred to as some geodesic curve.

Definition 30. *Let V be a vector field and $V(t) \in T_{c(t)}M$. This means that V is defined on a curve c. In the neighbourhood of $c(t)$ the tangent vectors are $X_i(c(t))$ then we have*

$$V(t) = V^i(c(t))X_i(c(t)). \tag{174}$$

The covariant derivative along the curve is defined as

$$\nabla_{c'(t)}V(t) = V^{i\prime}(t)X_i(c(t)) + V^j(t)\nabla_{c'(t)}X_j(c(t)) \tag{175}$$

which is in accordance with the definition of ∇.

Proposition 18. *The geodesic equation in the coordinate form with $\phi(\gamma(t)) = (c^1(t), ..., c^n(t))$ is*

$$c^{k\prime\prime}(t) + \Gamma^k_{ij}c^{i\prime}(t)c^{j\prime}(t) = 0. \tag{176}$$

Moreover, this has a unique solution with any initial velocity $v \in T_{\gamma(0)}M$:

$$\gamma'_v(0) = \left.\frac{\partial}{\partial t}\right|_{\gamma(0)} = \left.\frac{\partial}{\partial x^i}\right|_{c(0)} \left.\frac{\partial x^i}{\partial t}\right|_0 = c^{i\prime}(0) \left.\frac{\partial}{\partial x^i}\right|_{c(0)} = v. \tag{177}$$

In fact locally, $\phi = c^{-1}$ for a non-constant curve γ.

Proof.

$$\nabla_{\gamma'}\gamma' = 0 \Rightarrow \nabla_{c^{i\prime}(t)\partial_i}c^{j\prime}(t)\partial_j = (c^{k\prime})'\partial_k + c^{j\prime}\nabla_{c^{i\prime}\partial_i}\partial_j = c^{k\prime\prime}\partial_k + c^{i\prime}c^{j\prime}\nabla_{\partial_i}\partial_j. \tag{178}$$

Using the coordinate expression for ∇, we have

$$0 = c^{k\prime\prime}\partial_k + c^{i\prime}c^{j\prime}\Gamma^k_{ij}\partial_k \Rightarrow c^{k\prime\prime} + c^{i\prime}c^{j\prime}\Gamma^k_{ij} = 0. \tag{179}$$

In fact if we let $c' = u$, the velocity we have

$$u^{k\prime} + \Gamma^k_{ij}u^iu^j = 0. \tag{180}$$

By Picard's theorem of ODE, we know that there exists a unique solution given $u(0)$. □

Comment 13. *The solution to the geodesic equation with initial velocity αv is in fact, $\gamma_v(t/\alpha)$. To see this, let $\tau = t/\alpha$ then the geodesic equation becomes*

$$\cancel{\alpha^2}c^k(\tau)'' + \cancel{\alpha^2}c^i(\tau)'c^j(\tau)'\Gamma^k_{ij} = 0 \tag{181}$$

with the initial solution

$$c(\tau = 0)' = \alpha c(t = 0)'. \tag{182}$$

Definition 31. *The exponential map* $\exp : B_\varepsilon(0) \subset T_pM \to M$ *is defined as*

$$\exp_p(v) := \gamma_v(1). \tag{183}$$

Here $v \in T_pM$. Additionally, this leads to

$$\gamma_v(t) = \gamma_{tv}(t/t) = \exp_p(tv). \tag{184}$$

Moreover

$$v = \left.\frac{\partial}{\partial t}\right|_p = \exp_p{}'(tv)\ \Big|_{t=0}\ . \tag{185}$$

Lemma 4. $\exp_p$ *is a local diffeomorhism at $v = \mathbf{0}$.*

Proof. For any vector space $V \ni \vec{a}$, we have the following isomorphism :

$$T_{\vec{a}}V \simeq V, \quad D_v|_{\vec{a}} \mapsto v. \tag{186}$$

Then

$$d(\exp_p)|_{\mathbf{0}} : T_{\mathbf{0}}T_pM \to T_pM, \text{ by} \tag{187}$$

$$d(\exp_p)|_{\mathbf{0}}(D_v|_{\mathbf{0}})f = D_v|_{\mathbf{0}}(f \circ \exp_p) = \left.\frac{\partial}{\partial t}\right|_{t=0} f \circ \exp_p(\mathbf{0} + tv) = \left.\frac{\partial}{\partial t}\right|_{t=0} f \circ \gamma(t) = d\gamma\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\right) f = v(f). \tag{188}$$

Therefore

$$d(\exp_p)|_{\mathbf{0}} : D_v|_{\mathbf{0}} \mapsto v \tag{189}$$

induces an isomorphism. By inverse function theroem, this is what we want.

□

Definition 32. $\exp_p$ *maps* $B_\varepsilon(0)$ *to some neighbourhood* V *of* p*. Let* E *be the isomorphism* $T_pM \to \mathbb{R}^n$ *defined by*

$$E : \partial_i \mapsto (dx^1(e_i), ..., dx^n(e_i)), \tag{190}$$

where e_i *is a set of orthonormal basis of the tangent space. This can be done by diagonalizing the metric. Then let the chart map* $\phi_p := E \circ \exp_p^{-1}$ *with* $\phi(p) := (X_1, ..., X_n)$*.*

$$\{U; (X^1, ..., X^n)\} \tag{191}$$

is defined as the normal coordinate system.

Proposition 19. *Properties of the normal coordinate system.*
(1). $\gamma_v(t) = \exp_p(tv) = \exp_p(tv^i\partial_i)$ *has the following expression in this neighbourhood*

$$\phi_p(\gamma_v(t)) = (tv^1, ..., tv^n). \tag{192}$$

(2).

$$g_{ij}(p) = \delta_{ij}. \tag{193}$$

(3).

$$\Gamma^k_{ij}(p) = 0. \tag{194}$$

(4).

$$\partial_l g_{ij}(p) = 0. \tag{195}$$

Proof.

(1). $\gamma_v(t)$ in the normal coordinate is

$$\phi_p(\gamma_v(t)) = E \circ \exp_p^{-1} \exp_p(tv) = E(tv^i\partial_i) = (tv^1, ..., tv^n). \tag{196}$$

(2). Consider

$$\left.\frac{\partial}{\partial X^i}\right|_p = d(E \circ \exp_p^{-1})^{-1}\left(\left.\frac{\partial}{\partial X^i}\right|_0\right) = d\exp_p\big|_{\mathbf{0}} \circ dE^{-1}\left(\left.\frac{\partial}{\partial X^i}\right|_0\right) = d\exp_p(D_{e_i}) = e_i \tag{197}$$

where we used the fact that

$$dE^{-1}\left(\left.\frac{\partial}{\partial X^i}\right|_0\right) f = \left.\frac{\partial}{\partial X^i}\right|_0 (f \circ E^{-1}) = \lim_{t\to 0} f \circ E^{-1}(0, ..., t, ..., 0) \tag{198}$$

$$= \lim_{t\to 0} f \circ tE^{-1}(0, ..., 1, ..., 0) = \lim_{t\to 0} f(te_i) = D_{e_i} f. \tag{199}$$

Therefore

$$\left\langle \left.\frac{\partial}{\partial X^i}\right|_p, \left.\frac{\partial}{\partial X^j}\right|_p \right\rangle = \langle e_i, e_j \rangle = \delta_{ij}. \tag{200}$$

(3). Consider the special geodesic curve in (1): $c^i = tv^i$.

$$\Gamma^k_{ij}(\gamma_v(t))v^i v^j = 0, \forall v^i, v^j, k \Rightarrow \Gamma^k_{ij}(\gamma_v(t)) = 0. \tag{201}$$

(4). From the coordinate expression for Γ^k_{ij}, we have

$$\begin{matrix} \partial_l g_{ij} - \partial_i g_{jl} - \partial_j g_{li} = 0; \\ \partial_i g_{jl} - \partial_j g_{li} - \partial_l g_{ij} = 0. \end{matrix}, \text{ Adding them gives} \Rightarrow \partial_j g_{li} = 0. \tag{202}$$

□

Comment 14. *This is the frame of reference for the object moving at an acceleration identical to the one given by the geodesic equation.*

Definition 33. *A coordinate frame field of E on U is defined as*

$$\left\{dx^i \otimes s_i\right\}_{i=1,\dots,n}, \quad s_i \in \Gamma(E) \textit{ linearly independent.} \tag{203}$$

Obviously this is a basis for $\Gamma(T^*M \otimes E)$. So we have

$$\nabla s_i = T^k_{ij} dx^j \otimes s_k, \tag{204}$$

where T^k_{ij} is a smooth function on U. If $s_i = \partial_i$, we have $T^k_{ij} = \Gamma^k_{ij}$.
The connection matrix ω is defined by

$$\omega^k_i := T^k_{ij} dx^j, \tag{205}$$

so

$$\nabla s_i = \omega^k_i \otimes s_k. \tag{206}$$

Corollary 6.1. *Write $(\nabla s_1, ..., \nabla s_n)$ as ∇S, $(s_1, ..., s_n)$ as S and treat them as vectors. Then*

$$\nabla S = \omega \otimes S. \tag{207}$$

Proposition 20. *For another frame $S' = (s'_1, ..., s'_n) = A \cdot S$ where A is an invertible matrix, we have the corresponding connection matrix to be*

$$\omega' = dA \cdot A^{-1} + A \cdot \omega \cdot A^{-1}. \tag{208}$$

Proof. Consider

$$(DS')_i = \nabla s'_i = \nabla(A_{ij} s_j) = dA_{ij} \otimes s_j + A_{ij} \otimes \nabla s_j = dA_{ij} s_j + A_{ij} \omega^k_j \otimes s_k = dA \otimes S + A \cdot \omega \otimes S \tag{209}$$

$$= dA \cdot A^{-1} \cdot A \otimes S + A \cdot \omega \cdot A^{-1} \cdot A \otimes S = dA \otimes S' + A \cdot \omega \cdot A^{-1} \otimes S' = (dA + A \cdot \omega \cdot A^{-1}) \otimes S'. \tag{210}$$

□