Example 3.7 - Line Fitting

Consider the problem of line fitting or given the observations

$$x[n] = A + Bn + w[n] \qquad n = 0, 1, \ldots, N-1$$

where $w[n]$ is WGN, determine the CRLB for the slope B and the intercept A. The parameter vector in this case is $\boldsymbol{\theta} = [A\, B]^T$. We need to first compute the 2×2 Fisher information matrix,

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} -E\left[\dfrac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial A^2}\right] & -E\left[\dfrac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial A \partial B}\right] \\ -E\left[\dfrac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial B \partial A}\right] & -E\left[\dfrac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial B^2}\right] \end{bmatrix}.$$

The likelihood function is

$$p(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A - Bn)^2\right\}$$

from which the derivatives follow as

$$\begin{aligned} \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial A} &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n] - A - Bn) \\ \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial B} &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n] - A - Bn)n \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial A^2} &= -\frac{N}{\sigma^2} \\
\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial A \partial B} &= -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} n \\
\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial B^2} &= -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} n^2.
\end{aligned}$$

Since the second-order derivatives do not depend on $\mathbf{x}$, we have immediately that

$$\begin{aligned}
\mathbf{I}(\boldsymbol{\theta}) &= \frac{1}{\sigma^2}\begin{bmatrix} N & \sum_{n=0}^{N-1} n \\ \sum_{n=0}^{N-1} n & \sum_{n=0}^{N-1} n^2 \end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix} N & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{N(N-1)(2N-1)}{6} \end{bmatrix}
\end{aligned}$$

where we have used the identities

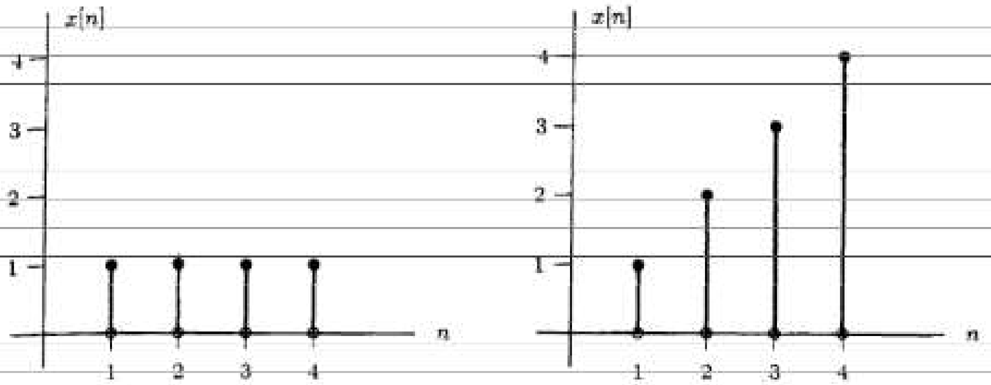
$$\begin{aligned}
\sum_{n=0}^{N-1} n &= \frac{N(N-1)}{2} \\
\sum_{n=0}^{N-1} n^2 &= \frac{N(N-1)(2N-1)}{6}.
\end{aligned} \tag{3.22}$$

Inverting the matrix yields

$$\mathbf{I}^{-1}(\boldsymbol{\theta}) = \sigma^2 \begin{bmatrix} \frac{2(2N-1)}{N(N+1)} & -\frac{6}{N(N+1)} \\ -\frac{6}{N(N+1)} & \frac{12}{N(N^2-1)} \end{bmatrix}.$$

It follows from (3.20) that the CRLB is

$$\begin{aligned}
\operatorname{var}(\hat{A}) &\geq \frac{2(2N-1)\sigma^2}{N(N+1)} \\
\operatorname{var}(\hat{B}) &\geq \frac{12\sigma^2}{N(N^2-1)}.
\end{aligned}$$



(a) $A = 0, B = 0$ to $A = 1, B = 0$ (b) $A = 0, B = 0$ to $A = 0, B = 1$

Figure 3.5 Sensitivity of observations to parameter changes—no noise

Some interesting observations follow from examination of the CRLB. Note first that the CRLB for $\hat{A}$ has increased over that obtained when B is known, for in the latter case we have

$$\operatorname{var}(\hat{A}) \geq -\frac{1}{E\left[\frac{\partial^2 \ln p(\mathbf{x}; A)}{\partial A^2}\right]} = \frac{\sigma^2}{N}$$

and for $N \geq 2$, $2(2N-1)/(N+1) > 1$. This is a quite general result that asserts that ***the CRLB always increases as we estimate more parameters*** (see Problems 3.11 and 3.12). A second point is that

$$\frac{\mathrm{CRLB}(\hat{A})}{\mathrm{CRLB}(\hat{B})} = \frac{(2N-1)(N-1)}{6} > 1$$

for $N > 3$. Hence, B is easier to estimate, its CRLB decreasing as $1/N^3$ as opposed to the $1/N$ dependence for the CRLB of A. These differing dependences indicate that $x[n]$ is *more sensitive* to changes in B than to changes in A. A simple calculation reveals

$$\begin{aligned}
\Delta x[n] &\approx \frac{\partial x[n]}{\partial A}\Delta A = \Delta A \\
\Delta x[n] &\approx \frac{\partial x[n]}{\partial B}\Delta B = n\Delta B.
\end{aligned}$$