1) consider the pdf $f(x_1, x_2) = \begin{cases} \frac{1}{2} x_1 + \frac{3}{2} x_2 & ; \; 0 \le x_1 , x_2 \le 1 \\ 0 & ; \text{ otherwise} \end{cases}$

i) The marginal density functions of x_1 & x_2

Marginal pdf of x_1 $\Big\}$ $= f_{x_1}(x_1) = \int f(x_1, x_2)\, dx_2$

$$= \int_0^1 \left(\frac{1}{2} x_1 + \frac{3}{2} x_2\right) dx_2$$

$$= \left[\frac{1}{2} x_1 x_2 + \frac{3}{2} \frac{x_2^2}{2}\right]_0^1$$

$$= \left[\frac{x_1}{2} + \frac{3}{4}\right] + [0+0]$$

$$= \frac{4x_1 + 6}{8}$$

$$\boxed{f_{x_1}(x_1) = \frac{2x_1 + 3}{4} \quad ; \quad 0 \le x_1 \le 1}$$

Marginal pdf of x_2 $\Big\}$ $= f_{x_2}(x_2) = \int f(x_1, x_2)\, dx_1$

$$= \int_0^1 \left(\frac{1}{2} x_1 + \frac{3}{2} x_2\right) dx_1$$

$$= \left[\frac{x_1^2}{4} + \frac{3}{2} x_1 x_2\right]_0^1$$

$$= \frac{1}{4} + \frac{3x_2}{2}$$

$$= \frac{2 + 12x_2}{8}$$

$$\boxed{f_{x_2}(x_2) = \frac{6x_2 + 1}{4} \quad ; \quad 0 \le x_2 \le 1}$$

ii) Conditional density function of X_2 given X_1

$$f_{x_2/x_1}\left(x_2/x_1\right) = \frac{f(x_1, x_2)}{f_{x_1}(x_1)}$$

$$= \frac{\frac{x_1+3x_2}{2}}{\frac{2x_1+3}{4}}$$

$$\boxed{f_{x_2/x_1}\left(\frac{x_2}{x_1}\right) = \frac{2x_1+6x_2}{2x_1+3}}$$

iii) Conditional density function of x_1 given x_2

$$f_{x_1/x_2}\left(x_1/x_2\right) = \frac{f(x_1, x_2)}{f_{x_2}(x_2)}$$

$$= \frac{\frac{x_1+3x_2}{2}}{\frac{6x_2+1}{4}}$$

$$\boxed{f_{x_1/x_2}\left(x_1/x_2\right) = \frac{2x_1+6x_2}{6x_2+1}}$$

iv) $E(x_1)$

$$E(x_1) = \int x_1 f_{x_1}(x_1)\,dx_1$$

$$= \int_0^1 x_1\left(\frac{2x_1+3}{4}\right)dx_1$$

$$= \frac{1}{4}\left[\frac{2x_1^3}{3} + \frac{3x_1^2}{2}\right]_0^1 = \frac{1}{4}\left[\frac{2}{3}+\frac{3}{2}\right] = \boxed{\frac{13}{24}}$$

$E(x_2)$

$$E(x_2) = \int x_2 f_{x_2}(x_2)\, dx_2$$

$$= \int_0^1 x_2 \left(\frac{6x_2+1}{4}\right) dx_2$$

$$= \frac{1}{4}\left[\frac{\cancel{6}^{2} x_2^3}{\cancel{3}} + \frac{x_2^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[2 + \frac{1}{2}\right]$$

$$\boxed{E(x_2) = \frac{5}{8}}$$

$E(x_1^2)$

$$E(x_1^2) = \int x_1^2 f_{x_1}(x_1)\, dx_1$$

$$= \int_0^1 x_1^2 \left(\frac{2x_1+3}{4}\right) dx_1$$

$$= \frac{1}{4}\left[\frac{\cancel{2} x_1^4}{\cancel{4}\,2} + \frac{\cancel{3} x_1^3}{\cancel{3}}\right]_0^1$$

$$= \frac{1}{4}\left[\frac{1}{2} + 1\right]$$

$$\boxed{E(x_1^2) = \frac{3}{8}}$$

$E(x_2^2)$

$$E(x_2^2) = \int x_2^2 f_{x_2}(x_2)\, dx_2$$

$$= \int_0^1 x_2^2 \left(\frac{6x_2+1}{4}\right) dx_2$$

$$= \frac{1}{4}\left[\frac{6x_2^4}{4} + \frac{x_2^3}{3}\right]_0^1 = \frac{1}{4}\left[\frac{3}{2} + \frac{1}{3}\right] = \boxed{\frac{11}{24}}$$

$E(x_1 x_2)$

$$E(x_1 x_2) = \int\int x_1 x_2 \, f(x_1, x_2) \, dx_1 \, dx_2$$

$$= \int_0^1\int_0^1 x_1 x_2 \left(\frac{x_1}{2} + \frac{3x_2}{2} \right) dx_1 \, dx_2$$

$$= \int_0^1\int_0^1 \frac{x_1^2 x_2}{2} + \frac{3x_1 x_2^2}{2} \; dx_1 \, dx_2$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x_1^3 x_2}{3} + \frac{3x_1^2 x_2^2}{2} \right]_0^1 dx_2$$

$$= \frac{1}{2}\int_0^1 \frac{x_2}{3} + \frac{3x_2^2}{2} \; dx_2$$

$$= \frac{1}{2}\left[\frac{x_2^2}{6} + \frac{3x_2^3}{6} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{6} + \frac{3}{6} \right]$$

$$\boxed{E(x_1 x_2) = \frac{1}{3}}$$

v) Covariance Matrix :

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$\sigma_1^2 = E(x_1^2) - [E(x_1)]^2$$

$$= \frac{3}{8} - \left(\frac{13}{24}\right)^2$$

$$= \frac{3}{8} - \frac{169}{576} = \frac{1728 - 1352}{4608} = \frac{376}{4608}.$$

$$= \frac{47}{576}$$

$$\sigma_2^2 = E(x_2^2) - [E(x_2)]^2$$

$$= \frac{11}{24} - \left(\frac{5}{8}\right)^2$$

$$= \frac{11}{24} - \frac{25}{64}$$

$$= \frac{704 - 600}{1536} = \frac{104}{1536}$$

$$= \frac{13}{192}$$

$$\sigma_{12} = E(x_1 x_2) - E(x_1)E(x_2)$$

$$= \frac{1}{3} - \left(\frac{13}{24}\right)\left(\frac{5}{8}\right)$$

$$= \frac{1}{3} - \frac{65}{200}$$

$$= \frac{200 - 195}{600} = \frac{5}{600}$$

$$= \frac{1}{120}$$

$$\boxed{\Sigma = \begin{bmatrix} \frac{47}{576} & \frac{1}{120} \\ \frac{1}{120} & \frac{13}{192} \end{bmatrix}}$$

2) If X is distributed as $N_6(\mu, \Sigma)$ find the distribution of $\begin{bmatrix} x_3 \\ x_6 \end{bmatrix}$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}$$

$$\Sigma = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & \sigma_{56} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_{66} \end{pmatrix} \end{array}$$

$$X_{new} = \left(\begin{array}{c} x_3 \\ x_6 \\ \hline x_1 \\ x_2 \\ x_4 \\ x_5 \end{array}\right) \begin{array}{l} \to x^{(1)} \\ \\ \\ \\ \to x^{(2)} \\ \\ \end{array} \qquad \mu_{new} = \left(\begin{array}{c} \mu_3 \\ \mu_6 \\ \hline \mu_1 \\ \mu_2 \\ \mu_4 \\ \mu_5 \end{array}\right) \begin{array}{l} \to \mu^{(1)} \\ \\ \\ \\ \to \mu^{(2)} \\ \\ \end{array}$$

Σ_{11}

$$\Sigma_{new} = \begin{array}{c} \\ x_3 \\ x_6 \\ x_1 \\ x_2 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cc|cccc} x_3 & x_6 & x_1 & x_2 & x_4 & x_5 \end{array} \\ \left(\begin{array}{cc|cccc} \sigma_{33} & \sigma_{36} & \sigma_{31} & \sigma_{32} & \sigma_{34} & \sigma_{35} \\ \sigma_{63} & \sigma_{66} & \sigma_{61} & \sigma_{62} & \sigma_{64} & \sigma_{65} \\ \hline \sigma_{13} & \sigma_{16} & \sigma_{11} & \sigma_{12} & \sigma_{14} & \sigma_{15} \\ \sigma_{23} & \sigma_{26} & \sigma_{21} & \sigma_{22} & \sigma_{24} & \sigma_{25} \\ \sigma_{43} & \sigma_{46} & \sigma_{41} & \sigma_{42} & \sigma_{44} & \sigma_{45} \\ \sigma_{56} & \sigma_{56} & \sigma_{51} & \sigma_{52} & \sigma_{54} & \sigma_{55} \end{array}\right) \end{array} \to \Sigma_{12}$$

Σ_{21} $\qquad$ Σ_{22}

By property of MND, the subsets of the components of x will follow a normal distribution as $N(\mu^{(1)}, \Sigma_{11})$

$$N\left(\begin{bmatrix} \mu_3 \\ \mu_6 \end{bmatrix}, \begin{bmatrix} \sigma_{33} & \sigma_{36} \\ \sigma_{63} & \sigma_{66} \end{bmatrix}\right)$$

3) Let $X \sim N_3(\mu, \Sigma)$ with $\mu' = [2 \quad 1 \quad 3]$ and $\Sigma = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Which of the following RV's are independent explain.

a) x_1 and x_2

From the given Σ,

$cov(x_1, x_2) = -1 \neq 0$.

$\therefore x_1$ and x_2 are not independent

b) x_2 and x_3

From the given Σ,

$cov(x_2, x_3) = 0$

$\therefore x_2$ and x_3 are independent.

c) x_3 and x_1

From the given Σ,

$cov(x_3, x_1) = 0$

$\therefore x_1$ and x_3 are independent.

d) $\frac{x_1 + x_3}{2}$ and x_2

Given 2 linear combinations of x_1, x_2 & x_3

$\frac{1}{2}x_1 + 0x_2 + \frac{1}{2}x_3$

$0x_1 + x_2 + 0x_3$

$$A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

By property of MND, several linear combinations of the components of X, will be normally distributed as

$$AX \sim N(A\mu, A\Sigma A')$$

$$A\Sigma A' = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/2 & 3/2 \\ -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 \\ -1/2 & 4 \end{pmatrix}$$

$$\Sigma_{11} = 1;\ \Sigma_{12} = -1/2;\ \Sigma_{21} = -1/2;\ \Sigma_{22} = 4$$

$$\Sigma_{12} = -1/2 \neq 0$$

$\therefore$ $\frac{x_1+x_3}{2}$ and x_2 are independent.

e) x_1 and $x_1 - \frac{1}{2}x_2 + x_3$

Given 2 linear combinations of x_1, x_2 & x_3

$x_1 + 0x_2 + 0x_3$

$x_1 - \frac{1}{2}x_2 + x_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 1 \end{bmatrix}$$

By property of MND, several linear combinations of the components of x, will be normally distributed as

$$AX \sim N(A\mu, A\Sigma A')$$

$$A\Sigma A' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1/2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 3/2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1/2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 3/2 \\ 3/2 & 6 \end{bmatrix}$$

$$\Sigma_{12} = \frac{3}{2} \neq 0$$

$\therefore$ x_1 and $x_1 - \frac{x_2}{2} + x_3$ are not independent.

f) Find the conditional distribution of x_3 given $X_1 = x_1$, $X_2 = x_2$

$$\Sigma_{new} = \begin{array}{c c} & \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_1 \\ x_2 \end{array} & \left(\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 1 & -1 \\ 0 & -1 & 4 \end{array}\right) \end{array}$$

$$\Sigma_{11} = 3 \; ; \; \Sigma_{12} = (0 \quad 0) \; ; \; \Sigma_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; ; \; \Sigma_{22} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$$

$$\mu_{new} = \begin{array}{c} x_3 \\ x_1 \\ x_2 \end{array} \left(\begin{array}{c} 3 \\ \hline 2 \\ 1 \end{array}\right) \qquad \mu^{(1)} = 3 \qquad \mu^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$X_{new} = \left(\begin{array}{c} x_3 \\ \hline x_1 \\ x_2 \end{array}\right) \begin{array}{l} \to x^{(1)} \\ \\ \to x^{(2)} \end{array}$$

The conditional distribution will be normally distributed with mean $= \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$ & covariance $= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$\text{mean} = 3 + (0 \;\; 0)\frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$$

$$= 3 + (0 \;\; 0)\frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} x_1 - 2 \\ x_2 - 1 \end{bmatrix}$$

$$= 3 + \frac{1}{4}\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix}$$

$$\text{mean} = 3$$

$$\text{covariance} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= 3 - (0 \;\; 0)\frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{covariance} = 3$$

Distribution of x_3 given $X_1 = x_1, X_2 = x_2$ is $N(3, 3)$