Hogwarts Quests - Assignment 2

1. You are tracking the distance to the Hogwarts Express. A magical instrument reports it's 100 leagues away. Before the reading, your belief about the distance D was a Gaussian $D \sim N(\mu = 98, \sigma^2 = 16)$. The instrument's reading is the true distance plus Gaussian noise ($N(0, 4)$).

   a. What is the PDF of your prior belief of the train's true distance?

   b. What is the probability density of seeing a reading of 100 leagues, given the true distance is t?

   c. What is the PDF of your posterior belief (after the reading) of the train's true distance? (You can leave a constant and don't need to simplify).

2. On average, 5.5 owls arrive at the Owlery per minute. What is the probability that:

   a. More than 7 owls will arrive in the next minute?

   b. More than 13 owls will arrive in the next 2 minutes?

   c. More than 15 owls will arrive in the next 3 minutes?

3. The median of a continuous random variable (like the height of a gnome) having cumulative distribution function F is the value m such that $F(m) = 0.5$. Find the median of X (in terms of distribution parameters) if:

   a. $X \sim \text{Uni}(a, b)$ (Uniform distribution, like the spread of Floo powder).

   b. $X \sim N(\mu, \sigma^2)$ (Normal distribution, like scores on the O.W.L.s).

4. Let X_i be the number of students visiting the Hogwarts library in week i, where $X_i \sim N(2200, 52900)$. Assume weekly visits X_i are independent.

   a. What is the probability that the total number of visitors in the next two weeks exceeds 5000?

   b. What is the probability that the weekly number of visitors exceeds 2000 in at least 2 of the next 3 weeks?

5. Let X, Y, and Z be independent random variables representing the magical power levels of three Hogwarts students, where $X \sim N(\mu_1, {\sigma_1}^2)$ (Gryffindor), $Y \sim N(\mu_2, {\sigma_2}^2)$ (Hufflepuff), and $Z \sim N(\mu_3, {\sigma_3}^2)$ (Ravenclaw).

   a. Let $A = X + Y$. What is the distribution of the combined power A?

   b. Let $B = 5X + 2$. What is the distribution of B (perhaps after a power-enhancing charm)?

   c. Let $C = aX - bY + c^2Z$, where a, b, and c are real-valued constants representing spell modifiers. What is the distribution (and parameters) for C? Show how you derived your answer.

6. The joint probability density function of continuous random variables X (skill in Potions) and Y (skill in Charms) is given by $f_{X,Y}(x,y) = c\frac{y}{x}$ where $0 < y < x < 1$.

   a. What is the value of c for this to be a valid probability density function?

   b. Are Potion skill (X) and Charm skill (Y) independent? Explain.

   c. What is the marginal density function of X?

   d. What is the marginal density function of Y?

7. Choose a number X at random from the set of house points $\{1, 2, 3, 4, 5, 6\}$ awarded by Professor McGonagall. Now choose a number Y at random from the subset of points no larger than X, $\{1, \ldots, X\}$.

   a. Determine the joint probability mass function of X (initial points) and Y (second random selection).

   b. Determine the conditional mass function $P(X = j|Y = i)$ as a function of i and j.

   c. Are X and Y independent? Explain.