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HOMEWORK-03(Solution)

(H.P. & A.M. – G.M. – H. M.)

1. If $\mathbf{r}$ is one **AM** and $\mathbf{p, q}$ are two GM's between two given numbers, then $\mathbf{p^3 + q^3}$ is equal to

(A) $\mathbf{2pqr}$ (B) $\mathbf{2p^2q^2r^2}$ (C) $\mathbf{2pq/r}$ (D) none of these

Ans. **(A)**

Sol. Let given numbers be $\mathbf{x}$ and $\mathbf{y}$. Then

$$r = \frac{1}{2}(x+y) \quad .....(1)$$

$$\left.\begin{array}{l} p = (x^2y)^{1/3} \\ q = (xy^2)^{1/3} \end{array}\right] \Rightarrow pq = xy \quad ....(2)$$

$$\therefore\ p^3 + q^3 = x^2y + xy^2 = xy(x+y) = pq(2r) = 2pqr.$$

2. If $\mathbf{A_1, A_2; G_1, G_2}$ and $\mathbf{H_1, H_2}$ are respectively two AM's, two GM's and two HM's between two numbers, then $\frac{A_1+A_2}{H_1+H_2}$ equals

(A) $\frac{H_1H_2}{G_1G_2}$ (B) $\frac{G_1G_2}{H_1H_2}$ (C) $\frac{H_1H_2}{A_1A_2}$ (D) $\frac{G_1G_2}{A_1A_2}$

Ans. **(B)**

Sol. Let given two numbers be $\mathbf{a}$ and $\mathbf{b}$. Then

$$A_1 + A_2 = a + b,\ G_1G_2 = ab \quad .....(1)$$

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} \quad .....(2)$$

$$\Rightarrow \frac{H_1+H_2}{H_1H_2} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{H_1+H_2}{H_1H_2} = \frac{A_1+A_2}{G_1G_2} \quad \text{[from (1)]}$$

$$\therefore \frac{A_1+A_2}{H_1+H_2} = \frac{G_1G_2}{H_1H_2}. \quad [2]$$

3. If $\mathbf{a_1, a_2, a_3, \ldots, a_n}$ are positive real numbers such that their product is a fixed number $\mathbf{c}$, then minimum value of $\mathbf{a_1 + a_2 + a_3 + \cdots + 2a_n}$ is equal to

(A) $\mathbf{n(2c)^{1/n}}$ (B) $\mathbf{(n+1)c^{1/n}}$ (C) $\mathbf{2nc^{1/n}}$ (D) $\mathbf{(n+1)(2c)^{1/n}}$

Ans. **(A)**

Sol. For given positive numbers $\mathbf{a_1, a_2, \ldots, (2a_n)}$; using AM > GM.

$$\frac{a_1 + a_2 + \cdots + 2a_n}{n} > \left(a_1a_2 \ldots (2a_n)\right)^{1/n}$$

$$\Rightarrow a_1 + a_2 + \cdots + 2a_n > n\left(2a_1a_2 \ldots (a_n)\right)^{1/n} \Rightarrow a_1 + a_2 + \cdots + 2a_n > n(2c)^{1/n}.$$

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4. If $\alpha \in (0, \pi/2)$, then $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

(A) $2\tan\alpha$ (B) 1 (C) 2 (D) $\sec^2\alpha$

Ans. **(A)**

Sol. $\because \sqrt{x^2+x}$ and $\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ are positive numbers, so using $AM > GM$, we have

$$\frac{1}{2}\left(\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}\right) \geq \left(\sqrt{x^2+x}\cdot\frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)^{1/2}$$

$\Rightarrow \sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}} \geq 2\tan\,\alpha.$ [1]

5. If **AM** and **GM** of two roots of a quadratic equation are 9 and 4 respectively, then this equation is

(A) $x^2-18x+16=0$ (B) $x^2+18x-16=0$

(C) $x^2+18x+16=0$ (D) $x^2-18x-16=$

Ans. **(A)**

Sol. Let roots be α and β. Then as given

$\frac{\alpha+\beta}{2}=9$ and $\sqrt{\alpha\beta}=4$

$\Rightarrow \alpha+\beta=18$ and $\alpha\beta=16$

So required equation will be $x^2-18x+16=0$. [1]

6. Statement I: For every natural number $n \geq 2$ $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} > \sqrt{n}$.

Statement II : For every natural number $n \geq 2$ $\sqrt{n(n+1)} < n+1$

For above statements :

(A) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I.

(B) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(C) Statement I is true, Statement II is false.

(D) Statement I is false, Statement II is true

Ans. **(B)**

Sol. $\because$ AM > GM

$$\Rightarrow \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{n} > \left(\frac{1}{\sqrt{1}}\cdot\frac{1}{\sqrt{2}}\cdot\ldots\cdot\frac{1}{\sqrt{n}}\right)^{1/n}$$

$$\Rightarrow \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} > n\left(\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}\cdot\ldots\cdot\frac{1}{\sqrt{n}}\right)^{1/n}$$

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$> n\left(\frac{1}{n^{n/2}}\right)^{1/n} > n\frac{1}{\sqrt{n}} = \sqrt{n}$

So statement I is true. Obviously statement (B) is true.

[may be verified for $n = 2, 3, 4, \ldots$]

Further from statement II, we have

$\sqrt{n(n+1)} < n+1$

$\Rightarrow \sqrt{n} < \sqrt{n+1}$

$\Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ for $n \geq 2$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 1 + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$

$> 1 + \frac{n-1}{\sqrt{n}} > 1 + \sqrt{n} - \frac{1}{\sqrt{n}}$

$> \sqrt{n}.\ \left[\because 1 - \frac{1}{\sqrt{n}} > 0\right]$

This shows that statement II is a correct explanation for I.

[1]

7. Let a, b, c be positive ntegers such that b/a is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b+2$, then the vlaue of

$\frac{a^2+a-14}{a+1}$ is

Ans. 4

Sol. Let common ratio of the given GP be r and so $b = ar, c = ar^2$. Since a, b, c are positive integers and b/a is an integer, so $b/a = r$ is a positive integer greater than 1 .

As given AM of a, b, c is $b+2$, so we have

$\frac{a+b+c}{3} = b + 2 \Rightarrow a - 2b + c = 6$

$\Rightarrow a - 2ar + ar^2 = 6$

$\Rightarrow a(r-1)^2 = 6$

$\Rightarrow a = 6$ and $r = 2$

$\therefore\ [\because r, a \in N$ and $r \geq 2]$

$\therefore\ \frac{a^2+a-14}{a+1} = \frac{36+6-14}{6+1} = \frac{28}{7} = 4.$

8. If G_1, G_2 be two GM's and A be one AM between two numbers, then $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$ is equal to

(A) $4/2$ (B) A (C) $2A$ (D) A^2

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Ans. **(C)**

Sol. $= \frac{(a^2b)^{2/3}}{(ab^2)^{1/3}} + \frac{(ab^2)^{2/3}}{(a^2b)^{2/3}} = \frac{(a^2b)^{2/3}a^{1/3}+b^{1/3}(ab^2)^{2/3}}{(ab)^{2/3}}$

$= \frac{(ab)^{2/3}(a+b)}{(ab)^{2/3}} = 2A$

9. If the **HM** and **GM** of two positive numbers are in the ratio $\mathbf{4:5}$, then the numbers are in the ratio

(A) $\mathbf{4:1}$ (B) $\mathbf{4:3}$ (C) $\mathbf{2:3}$ (D) $\mathbf{3:5}$

Ans. **(A)**

Sol. Let numbers be $\mathbf{a}$ and $\mathbf{b}$. Then as given

$\frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{4}{5} \Rightarrow \frac{2\sqrt{ab}}{a+b} = \frac{4}{5} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3}{1} \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{4}{2}$

$\therefore \frac{a}{b} = \frac{4}{1}$

10. If the ratio between **AM** and **GM** of two numbers be $\mathbf{m:n}$, then the ratio between these numbers is

(A) $\mathbf{m+\sqrt{m^2-n^2} : m-\sqrt{m^2-n^2}}$ (B) $\mathbf{m+\sqrt{m^2+n^2} : m-\sqrt{m^2+n^2}}$

(C) $\mathbf{m-\sqrt{m^2-n^2} : m+\sqrt{m^2-n^2}}$ (D) none of these

Ans. **(A)**

Sol. Let $\mathbf{A = \lambda m}$ and $\mathbf{G = \lambda n}$. Now since numbers are

$\mathbf{A + \sqrt{A^2 - G^2}}$ and $\mathbf{A - \sqrt{A^2 - G^2}}$

$\therefore$ required ratio $= \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}$.

11. If **AM** and **GM** of two numbers are $\mathbf{75/4}$ and 15 respectively, then the greater number is

(A) 30 (B) 25 (C) 15 (D) $\mathbf{15/2}$

Ans. **(D)**

Sol. $\mathbf{a + b = 75/2, ab = 225 \Rightarrow a = 30,\ b = 15/2.}$

12. Let $\mathbf{a_1, a_2, a_3, \ldots}$ be in harmonic progression with $\mathbf{a_1 = 5}$ and $\mathbf{a_{20} = 25}$. The least positive integer $\mathbf{n}$ for which $\mathbf{a_n < 0}$ is

(A) 22 (B) 23 (C) 24 (D) 25

Ans. **(D)**

Sol. $a_{20} = 25, a_1 = 5 \Rightarrow \frac{1}{25} = \frac{1}{5} + 19d \Rightarrow d = -\frac{4}{19\times25}$

Now $a_n < 0 \Rightarrow \frac{1}{5} + (n-1)\left(-\frac{4}{19\times25}\right)201\% \Rightarrow \frac{(n-1)4}{19\times5} > 1 \Rightarrow n > \frac{19\times5}{4} + 1 \Rightarrow n \geq 25.$

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13. If $\mathbf{m}$ is the A.M. of two distinct real numbers $\mathbf{l}$ and $\mathbf{n}(\mathbf{l}, \mathbf{n} > 1)$ and $\mathbf{G_1}, \mathbf{G_2}$ and $\mathbf{G_3}$ are three geometric means between $\mathbf{l}$ and $\mathbf{n}$, then $\mathbf{G_1^4 + 2G_2^4 + G_3^4}$ equals.

(A) $\mathbf{4l^2mn}$ (B) $\mathbf{4lm^2n}$ (C) $\mathbf{4lmn^2}$ (D) $\mathbf{4l^2m^2n^2}$

Ans. **(B)**

Sol. $\mathbf{l + n = 2m}$

(1) $\mathbf{G_1 = l\left(\frac{n}{l}\right)^{\frac{1}{4}}, G_2 = l\left(\frac{n}{l}\right)^{\frac{2}{4}}, G_3 = l\left(\frac{n}{l}\right)^{\frac{3}{4}}}$

$\therefore$ Exp. $= \mathbf{l^4 \cdot \frac{n}{l} + 2l^2\left(\frac{n}{l}\right)^2 + l^4\left(\frac{n}{l}\right)^3 = nl^3 + 2n^2l^2 + n^3l}$

$= \mathbf{2n^2l^2 + nl(n^2 + l^2) = 2n^2l^2 + nl\{(n + l)^2 - 2nl\}}$

$= \mathbf{nl(n + l)^2 = nl. 4m^2 = 4lm^2n}$

14. If $\mathbf{H_1, H_2}$ are two HM's between $\mathbf{a}$ and $\mathbf{b}$, then $\frac{\mathbf{H_1H_2}}{\mathbf{H_1+H_2}}$ is equal to

(A) $\frac{\mathbf{a+b}}{\mathbf{ab}}$ (B) $\frac{\mathbf{a+b}}{\mathbf{2ab}}$ (C) $\frac{\mathbf{2ab}}{\mathbf{a+b}}$ (D) $\frac{\mathbf{ab}}{\mathbf{a+b}}$

Ans. **(D)**

Sol. $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in AP

$$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} \Rightarrow \frac{H_1H_2}{H_1 + H_2} = \frac{ab}{a + b}.$$

15. $\mathbf{n}$ AM's are inserted between 1 and 51 . if ratio between 4 th and 7th AM's is $\mathbf{3:5}$, then $\mathbf{n}$ equals

(A) 48 (B) 42 (C) 36 (D) 24

Ans. **(D)**

Sol. Corresponding $\mathbf{d} = \frac{\mathbf{51-1}}{\mathbf{n+1}} = \frac{\mathbf{50}}{\mathbf{n+1}}$. So as given

$$\frac{A_4}{A_7} = \frac{1 + 4\left(\frac{50}{n + 1}\right)}{1 + 7\left(\frac{50}{n + 1}\right)} = \frac{3}{5} \Rightarrow \frac{n + 201}{n + 351} = \frac{3}{5} \Rightarrow n = 24.$$

16. If AM of two numbers $\mathbf{a}$ and $\mathbf{b}$ is twice of their GM, then $\mathbf{a:b}$ is equal to

(A) $\mathbf{\sqrt{3} + 1 : \sqrt{3} - 1}$ (B) $\mathbf{2 + \sqrt{3} : 2 - \sqrt{3}}$

(C) $\mathbf{3:2}$ (D) none of these

Ans. **(B)**

Sol. As given $\mathbf{A = 2G \Rightarrow a + b = 4\sqrt{ab}}$

$$\Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1} \Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{2+1}{2-1} = \frac{3}{1}$$

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$$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{3}}{1} \Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$\therefore\ \mathbf{a}:\mathbf{b}=(\sqrt{3}+1)^2:(\sqrt{3}-1)^2=(2+\sqrt{3}):(2-\sqrt{3})$

17. If $\mathbf{A}$ be one $\mathbf{AM}$ and $\mathbf{p},\mathbf{q}$ be two GM's between two numbers, then $\mathbf{2A}$ is equal to

(A) $\frac{p^3+q^3}{pq}$ (B) $\frac{p^3-q^3}{pq}$ (C) $\frac{p^2+q^2}{2}$ (D) $\frac{pq}{2}$

Ans. **(A)**

Sol. Let numbers be $\mathbf{x},\mathbf{y}$. Then $\mathbf{2A}=\mathbf{x}+\mathbf{y}$.

Also $\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{y}$ are in GP $\Rightarrow \mathbf{p}^2=\mathbf{xq},\ \mathbf{q}^2=\mathbf{yp}$

$$\Rightarrow x=\frac{p^2}{q}, y=\frac{q^2}{p} \Rightarrow 2A=\frac{p^3+q^3}{pq}.$$

18. If $\mathbf{a},\mathbf{b},\mathbf{c}$ are in GP and $\mathbf{x},\mathbf{y}$ are AM's between $\mathbf{a},\mathbf{b}$ and $\mathbf{b},\mathbf{c}$ respectively, then

(A) $\frac{1}{x}+\frac{1}{y}=2$ (B) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$ (C) $\frac{1}{x}+\frac{1}{y}=\frac{2}{a}$ (D) $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$

Ans. **(D)**

Sol. $\mathbf{b}^2=\mathbf{ac}, \mathbf{2x}=\mathbf{a}+\mathbf{b}, \mathbf{2y}=\mathbf{b}+\mathbf{c}$

$\Rightarrow b^2=(2x-b)(2y-b)$

$\Rightarrow 2xy=b(x+y)$

$\Rightarrow \frac{x+y}{xy}=\frac{2}{b}$

19. If $\mathbf{AM}$ and $\mathbf{HM}$ of the roots of a quadratic equation are $\mathbf{3/2}$ and $\mathbf{4/3}$ respectively, then that equation is

(A) $x^2+3x+2=0$ (B) $x^2-3x+2=0$

(C) $x^2-3x-4=0$ (D) none of these

Ans. **(A)**

Sol. If roots are α,β; then $\alpha+\beta=3$ and $\frac{2\alpha\beta}{\alpha+\beta}=\frac{4}{3} \Rightarrow \alpha\beta=2$.

20. If $\mathbf{A}$ is one AM between two numbers $\mathbf{a}$ and $\mathbf{b}$, and the sum of $\mathbf{n}$ AM's between them is $\mathbf{S}$, then $\mathbf{S/A}$ depends on 3)

(A) $\mathbf{n},\mathbf{a},\mathbf{b}$ (B) $\mathbf{n},\mathbf{b}$ (C) $\mathbf{n},\mathbf{a}$ (D) $\mathbf{n}$

Ans. **(D)**

Sol. $\frac{S}{A}=\frac{n(A)}{A}=\mathbf{n} \Rightarrow$ it depends on $\mathbf{n}$.

21. The AM of two numbers exceeds their GM by 15 & HM by 27 . Find the numbers.

Ans. **120,30**

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Sol. Let the numbers be a, b and their A.M., G.M., and H.M. be denoted by A,G, and H respectively. Also we know that A.G.H are in G.P.

or $G^2 = AH...(1)$

Since $A - G = 15$ and $A - H = 27$

$(A-15)^2 = G^2 = AH$ by $(1) = A(A-27)$

or $-30A + 225 = -27A$ or $3A = 225$

$\therefore A = 75 = \frac{a+b}{2} \therefore a + b = 150...(2)$

Since $A - G = 15 \therefore 75 - G = 15$

or $G = 60 = \sqrt{ab}$

$ab = 3600 \ldots (3)$

Hence from (2) and (3) we conclude that a and b are the roots of $t^2 - 150t + 3600 = 0$

or $(t-120)(t-30) = 0 \therefore t = 120, 30$

Hence the two numbers are 120 and 30 .

22. The A.M. between two positive numbers exceeds the G.M. by 5 , and the G.M. exceeds the H.M. by 4. Find the numbers

Ans. 40,10

Sol. Given

$A.M = 5 + GM$ - (1)

$G.M = 4 + H.M$ - (2)

We know

$G.M^2 = AM \cdot HM$

$GM^2 = (5 + GM)(GM - 4)$

$GM^2 = 5GM - 20 + GM^2 - 4GM$

$\therefore GM = 20$

$AM = 25(\text{ from } (1))$

$\sqrt{ab} = 20, \frac{a+b}{2} = 25 \Rightarrow a + b = 50$

$\Rightarrow ab = 400 \ldots (3)$

$\Rightarrow (a-b)^2 = (a+b)^2 - 4ab$

$= 50^2 - 4 \times 400$

$= 2500 - 1600$

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$= 900$

$\therefore a - b = \sqrt{900} = 30 \ldots (4)$

from $(3)\&(4)$

$a + b = 50$

$a - b = 30$

$2a = 80$

$a = 40 \& b = 10$

$\therefore$ The numbers are $(40, 10)$.

23. If G be the geometric mean of x and y, then prove that $\frac{1}{(G^2-x^2)} + \frac{1}{(G^2-y^2)} = \frac{1}{G^2}$.

Sol. As given $G = \sqrt{xy}$

$\therefore \frac{1}{G^2-x^2} + \frac{1}{G^2-y^2} = \frac{1}{xy-x^2} + \frac{1}{xy-y^2}$

$= \frac{1}{x-y}\left[-\frac{1}{x} + \frac{1}{y}\right] = \frac{1}{xy} = \frac{1}{G^2}$

24. Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

Ans. 64 and 4

Sol. Let the two numbers be a and b such that $a > b$

Then, $\frac{a+b}{2} = 34$ and $\sqrt{ab} = 16$

$a + b = 68$

(1) and $ab = 256$

$\therefore (a-b)^2 = (a+b)^2 - 4ab$

$(a-b)^2 = (68)^2 - 4 \times 256 = 3600$

$a - b = 60$

solving eqn (1) & (2), we get,

$a = 64$ and $b = 4$

Hence, required numbers are 64 and 4 .

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25. If a is the A.M. of b & c, and the two geometric means between b & c are G_1 and G_2, then prove that ${G_1}^3 + {G_2}^3 = 2abc$.

Sol. We have G_1 and G_2 as two geometric means between b and c i.e. b, G_1, G_2, c are in G.P

$$c = b(r)^{4-1}$$

$$c = br^3$$

$$r = \left(\frac{c}{b}\right)^{\frac{1}{2}}$$

$$G_1 = br = b\left(\frac{c}{b}\right)^{\frac{1}{3}}$$

$$G_1^3 = b^3\left(\frac{c}{b}\right) = b^2c$$

$$G_2 = br^2 = b\left(\frac{c}{b}\right)^{\frac{2}{3}}$$

$$G_2^3 = br^2 = b^3\left(\frac{c^2}{b^2}\right) = bc^2$$

As a is AM of b and c, hence

$$2a = b + c$$

$$G_1^3 + G_2^3 = b^2c + bc^2$$

$$= bc(b + c)$$

$$= 2bca \text{ [Using (i)]}$$

$$G_1^3 + G_2^3 = 2abc$$

26. Insert three arithmetic means between 3 and 19 .

Ans. 14, 9, 4 or 4, 9, 14

Sol. Since, 3 A.M.'s are inserted between between 3 and 19, so

$$d = \frac{19 - 3}{3 + 1} = 4$$

Therefore, the A.M.'s are

$$3 + 4, 3 + 2 \times 4, 3 + 3 \times 4$$

$$\Rightarrow 7, 11, 15$$

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27. If eleven A.M.'s are inserted between 28 and 10 , then find the number of integral A.M.'s.

Ans. 5, 10,15, 20

Sol. After inserting eleven A.M.s the new series would have a common diffference

$$\mathbf{d = \frac{28-10}{11+1} = \frac{3}{2}}$$

Hence every even A.M. would be integral as it would be of the form $10 + 2r \times \frac{3}{2}$ where r is an integer $= 10 + 3r$

$= \mathbf{13, 16, 19, 22, 25}$

Hence, 5 integral A.M.'s.

H.P.

1. If pth term of a HP be $\mathbf{q}$ and qth term be $\mathbf{p}$, then its $'(\mathbf{p+q})$ th term is

(A) $\frac{1}{p+q}$ (B) $\frac{1}{p} + \frac{1}{q}$ (C) $\frac{pq}{p+q}$ (D) $\mathbf{p + q}$

Ans. (C)

Sol. Let **a** and **d** be the first term and common difference of the corresponding AP, then its

$\mathbf{T_p = 1/q}$ and $\mathbf{T_q = 1/p}$

$\Rightarrow \mathbf{a + (p-1)d = \frac{1}{q}}$ and $\mathbf{a + (q-1)d = \frac{1}{p}}$

$\Rightarrow \mathbf{(p-q)d = \frac{1}{q} - \frac{1}{p} \Rightarrow d = \frac{1}{pq}}$.

Now $(\mathbf{p+q})$ th term of this AP

$$\mathbf{a + (p+q-1)d = [a + (p-1)d] + qd}$$

$$= \mathbf{\frac{1}{q} + q\left(\frac{1}{pq}\right) = \frac{1}{q} + \frac{1}{p} = \frac{p+q}{pq}}$$

$\therefore \mathbf{T_{p+q}}$ of HP $= \mathbf{\frac{pq}{p+q}}$.

Aliter. For the corresponding AP, use the result

$$\mathbf{\frac{T_{p+q} - T_p}{(p+q) - p} = \frac{T_p - T_q}{p-q}}.$$

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2. Example 9. Five numbers $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, e are such that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in $\mathbf{AP}$; $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are in GP and $\mathbf{c}, \mathbf{d}, \mathbf{e}$ are in HP. If $\mathbf{a} = \mathbf{2}, \mathbf{e} = \mathbf{18}$; then values of $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are

(A) $\mathbf{2, 6, 18}$ (B) $\mathbf{4, 6, 9}$ (C) $\mathbf{4, 6, 8}$ (D) $\mathbf{-2, -6, 18}$

Ans. **(B)**

Sol. $b = \frac{2+c}{2}$

$\mathbf{c^2 = bd}$

$\mathbf{d = \frac{36c}{c+18}}$.

Eliminate $\mathbf{d}$ from (B) and (C), we get $\mathbf{c = \pm 6}$.

Now from(A)

from (C)

$\mathbf{b = 4, -2}$

$\mathbf{d = 9, -18}$

$\mathbf{b = 4, c = 6, d = 9}$. [2]

3. Example 23. If $\mathbf{a}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}$ are in $\mathbf{AP}$, then $\mathbf{x + y + z = 15}$ and if $\mathbf{a}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b}$ are in HP, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$. Numbers $\mathbf{a}$ and $\mathbf{b}$ are

(A) 8,2 (B) 11,3 (C) 9,1 (D) none of these

Ans. **(C)**

Sol. By property of AP, $\mathbf{x + z = a + b}$ and $\mathbf{y = \frac{1}{2}(a + b)}$

$$\Rightarrow \mathbf{x + y + z = \frac{3}{2}(a + b) \Rightarrow a + b = 10}$$

Also $\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in AP, so as above

$$\mathbf{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{10}{9} \Rightarrow ab = 9}.$$

From (1) and (2), $\mathbf{a}, \mathbf{b}$ are 9,1 . [3]

4. If $\mathbf{a_1, a_2, a_3, \ldots, a_n}$ are in $\mathbf{HP}$, then $\mathbf{a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n}$ is equal to

(A) $\mathbf{na_1a_n}$ (B) $\mathbf{(n-1)a_1a_n}$ (C) $\mathbf{(n+1)a_1a_n}$ (D) none of these

Ans. **(B)**

Sol. Let $\mathbf{d}$ be common difference of the corresponding

AP. So $\mathbf{\frac{1}{a_2} - \frac{1}{a_1} = d, \frac{1}{a_3} - \frac{1}{a_2} = d, \ldots, \frac{1}{a_n} - \frac{1}{a_{n-1}} = d}$

$\mathbf{\Rightarrow a_1 - a_2 = d(a_1a_2), a_2 - a_3 = d(a_2a_3), \ldots, (a_{n-1} - a_n) = d(a_{n-1}a_n)}$

Adding these relations, we get

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$\mathbf{a_1 - a_n = d(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n)}$

Also $\frac{1}{a_n} = \mathbf{T_n} = \frac{1}{a_1} + (\mathbf{n-1})\mathbf{d}$

$\Rightarrow \dfrac{\mathbf{1}}{\mathbf{a_n}} - \dfrac{\mathbf{1}}{\mathbf{a_1}} = (\mathbf{n-1})\mathbf{d}$

$\Rightarrow \mathbf{a_1 - a_n = (n-1)d(a_1a_n)}$

From (1) and (2), we have

$\mathbf{(n-1)(a_1a_n) = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n}$ [2]

5. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in HP, then the value of $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$ is

(A) $\mathbf{2/bc + 1/b^2}$ (B) $\mathbf{3/c^2 + 2/ca}$

(C) $\mathbf{3/b^2 - 2/ab}$ (D) none of these

Ans. **(C)**

Sol. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in HP, so $\frac{2}{b} = \frac{1}{c} + \frac{1}{a}$. ......(1)

$= \frac{1}{c^2} - \left(\frac{1}{a}-\frac{1}{b}\right)^2 = \left(\frac{2}{b}-\frac{1}{a}\right)^2 - \left(\frac{1}{a}-\frac{1}{b}\right)^2$ [using (1)]

$= \frac{3}{b^2} - \frac{2}{ab}$

6. If $\mathbf{x}, \mathbf{1}, \mathbf{z}$ are in **AP** and $\mathbf{x}, \mathbf{2}, \mathbf{z}$ are in **GP**, then $\mathbf{x}, \mathbf{4}, \mathbf{z}$ are in

(A) **AP** (B) GP (C) **HP** (D) none of these

Ans. **(C)**

Sol. $\mathbf{x}, \mathbf{1}, \mathbf{z}$ are in AP $\Rightarrow \mathbf{x + z = 2}$ $\mathbf{x}, \mathbf{2}, \mathbf{z}$ are in GP $\Rightarrow \mathbf{xz = 4}$.

Now $\frac{2xz}{x+z} = \frac{8}{7} = \mathbf{4} \Rightarrow \mathbf{x}, \mathbf{4}, \mathbf{z}$ are in HP.

7. **If the roots of $\mathbf{10x^3 - cx^2 - 54x - 27 = 0}$ are in harmonic progression, then find c and all the roots.**

Ans. $\mathbf{C = 9}$; $\mathbf{(3, -3/2, -3/5)}$

Sol. **Given roots of equation $\mathbf{10x^3 - cx^2 - 54x - 27 = 0}$ are in H.P.**

Replacing x by $\frac{1}{x}$, then we get

$\mathbf{27x^3 + 54x^2 + cx - 10 = 0}$

Now, roots of equation (i) are in A.P.
So, let the roots be $\alpha - \beta, \alpha, \alpha + \beta$, then

$$\alpha - \beta + \alpha + \alpha + \beta = -\frac{54}{27} = -2$$

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$\Rightarrow \alpha = -\frac{2}{3}$

$\because \alpha = -\frac{2}{3}$ is a root of equation (i) then

$$27\left(-\frac{2}{3}\right)^3 + 54\left(-\frac{2}{3}\right)^2 + c\left(-\frac{2}{3}\right) - 10 = 0$$

$\Rightarrow c = 9$

8. An AP & an HP have the same first term, the same last term & the same number of terms; prove that the product of the r^{th} term from the beginning in one series & the r^{th} term from the end in the other is independent of r.

Sol. For the A.P., 1 st term $= p$, nth term $= q$

$q = p + (n-1)d$, where d is the common difference

$\Rightarrow \mathrm{d} = \dfrac{\mathrm{q}-\mathrm{p}}{(\mathrm{n}-1)}$

$(r-1)$ th term of the A.P. $= p + \frac{r(q-p)}{(n-1)} = \frac{p(n-1)+r(q-p)}{(n-1)}$

For H.P. 1st term $= p$, nth term $= q$

$\Rightarrow$ the corresponding A.P. has the first term $\frac{1}{p}$ and the nth term $\frac{1}{q}$

$\frac{1}{\mathrm{q}} = \frac{1}{\mathrm{p}} + (\mathrm{n}+1)\mathrm{D}$, whre D is the common difference

$$\Rightarrow D = \frac{\frac{1}{q} - \frac{1}{p}}{(n-1)} = \frac{(p-q)}{pq(n-1)}$$

$(n-r)$ th term of the corresponding A.P.

$= \frac{1}{p} + \frac{(n-r-1)(p-q)}{pq(n-1)} = \frac{q(n-1)+(n-r-1)(p-q)}{pq(n-1)}$

$= \frac{(n-r-1)p+qr}{pq(n-1)} = \frac{p(n-1)+r(q-p)}{pq(n-1)}$

Therefore. The $(n-r$ th term of the H.P. $= \frac{pq(n-1)}{p(n-1)+r(q-p)}$

From (1) and (2) we see that the product of the terms $= pq$ independent of r

9. If the 10^{th} term of an HP is 21 and 21^{st} term of the same HP is 10 , then find the 210^{th} term.

Ans. 1

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Sol. 10^{th} terms of H.P $=\frac{1}{a+9d}=21, t_{21}=\frac{1}{a+20d}=10$

$\rightarrow a+9d=\frac{1}{21}--(1)$

and $a+20d=\frac{1}{10}--(2)$

$(2)-(1)=11d=\frac{1}{10}-\frac{1}{21}=\frac{11}{210}\Rightarrow d=\frac{1}{210}$ and $a=\frac{1}{210}$

10. Given that $a^x=b^y=c^z=d^u$ & a, b, c, d are in GP, show that x, y, z, u are in HP.

Sol. Since, a, b, c, d are in G.P
let, $b=ar, c=ar^2, d=ar^3$
Given that, $b=ar\ c=ar^2\ d=ar^3$
Taking logarithm, we get

$b=\ \text{ar}\ c=ar^2\ d=ar^3$

$\Rightarrow x(\log a)=y(\log a+\log r)=z(\log a+2\log r)=u(\log a+3\log r)=k$

Therefore, $x=\frac{k}{\log a}; y=\frac{k}{\log a+\log r}; z=\frac{k}{\log a+2\log r}; u=\frac{k}{\log a+3\log r}$
Since, $\log a, \log a+\log r, \log a+2\log r, \log a+3\log r$ are in A.P
Therefore, x, y, z, u are in H.P

11. If a, b, c and d are in H.P. , then find the value of $\frac{a^{-2}-d^{-2}}{b^{-2}-c^{-2}}$

Ans. 3

Sol. If a, b, c and d are in H.P.

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in $A\cdot P$

as $\frac{1}{a}=A, \frac{1}{b}=B, \frac{1}{c}=C$

$\frac{1}{d}=D$

A, B, C, D are in $A\cdot P$

$\frac{a^{-2}-d^{-2}}{b^{-2}-c^{-2}}=\frac{A^2-D^2}{B^2-c^2}=\frac{A^2-(A+3m)^2}{(A+m)^2-(A+2m)^2}$ $B=A+m$

$C=A+2m$

$D=A+3m=3$