Method of Differentiation

① Ab-Initio Method = First Principle

If $y = f(x) \rightarrow$ Ⓐ

Ⓑ－Ⓐ $y + \delta y = f(x + \delta x) \rightarrow$ Ⓑ

$$\delta y = f(x + \delta x) - f(x)$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\boxed{\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = f'(x) = \frac{dy}{dx}} = Dy$$

$\rightarrow$ Notation Used for 1st diff

② 2nd Derivative $= \dfrac{d^2 y}{d x^2} = f''(x)$

③ $$\boxed{\frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)} = \frac{1}{\lim_{\delta y \to 0} \dfrac{\delta x}{\delta y}}} \rightarrow \text{Ratio}$$

$\dfrac{3}{4} = \dfrac{1}{\dfrac{4}{3}}$

④ $\dfrac{d^2 y}{d x^2} = \dfrac{1}{\left(\dfrac{d^2 x}{d y^2}\right)}$ ✗

$$\frac{d^2 y}{d x^2} = \frac{d\left(\dfrac{dy}{dx}\right)}{dx}$$

Q

$$y = \ln x \quad \text{find} \frac{dy}{dx} = ?$$

$$y = \ln x \longrightarrow A$$

$$y + \delta y = \ln(x + \delta x) \to B$$

$B - A$

$$\delta y = \ln(x + \delta x) - \ln x$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\ln(x + \delta x) - \ln x}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\ln\left(1 + \frac{\delta x}{x}\right)}{x \times \left(\frac{\delta x}{x}\right)} = \frac{1}{x}$$

Q 2 $\quad y = \sin x \quad \text{find} \frac{dy}{dx} = ? \longrightarrow \frac{d(\sin x)}{dx}$

$$y = \sin x \longrightarrow A$$

$$y + \delta y = \sin(x + \delta x) \to B$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\frac{d(\sin x)}{dx} \underset{B - A}{\overbrace{\left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x}\right)}} = \lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos\left(x + \frac{\delta x}{2}\right) \cdot \sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \cos(x + 0) \times 1$$

$$\frac{dy}{dx} = \cos x$$

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

3) $y = \tan^{-1}x$ then $\frac{dy}{dx} = ?$

$\downarrow$

$x = \tan y \longrightarrow Ⓐ$

$x + \delta x = \tan(y + \delta y) \longrightarrow Ⓑ$

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$Ⓑ - Ⓐ \quad \delta x = \tan(y + \delta y) - \tan y.$

$\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y} = \lim\limits_{\delta y \to 0} \frac{\tan(y + \delta y) - \tan y}{\delta y}$

$= \lim\limits_{\delta y \to 0} \left| \frac{\sin(\delta y)}{(\delta y)} \middle| \frac{}{(\cos(y + \delta y) \cdot \cos y)} \right.$

$\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y} = 1 \times \frac{1}{\cos y \cdot \cos y} = \frac{1}{\cos^2 y} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$

$\frac{dy}{dx} = \frac{1}{\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y}} = \frac{1}{1 + x^2}$

$y = \tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$

$\underline{Doubt} \rightarrow \lim\limits_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{f'(x)?}{\boxed{\frac{dy}{dx}}}$

Proof of Product Rule.

1) $\dfrac{d(\sin x)}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\sin(x+\delta x)-\sin x}{\delta x}$

2) $\dfrac{d(\ln x)}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\ln(x+\delta x)-\ln x}{\delta x}$

3) $\dfrac{d(f(x)\cdot g(x))}{dx} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x)\cdot g(x)}{\delta x}$

$$= \lim\limits_{\delta x \to 0} \dfrac{\left(f(x+\delta x)\cdot g(x+\delta x) - f(x+\delta x)\cdot g(x) + f(x+\delta x)\cdot g(x) - f(x)\cdot g(x)\right)}{\delta x}$$

$$= \lim\limits_{\delta x \to 0} \quad f(x+\delta x)\left\{\dfrac{g(x+\delta x)-g(x)}{\delta x}\right\} + g(x)\left\{\dfrac{f(x+\delta x)-f(x)}{\delta x}\right\}$$

$$= \boxed{\dfrac{d(f(x)\cdot g(x))}{dx} = f(x)\cdot g'(x) + g(x)\, f'(x)}$$

(5) Product Rule

$$(U \cdot V)' = U \cdot V' + V \cdot U'$$

(6) $(f_1 f_2 f_3 \cdots f_n)' = ?$

$$f_1'(f_2 f_3 \cdots f_n) + (f_2 \cdot f_3 \cdot f_4 \cdots f_n)' \cdot f_1$$

$$f_1(f_2 f_3 \cdots f_n) + f_1(f_2'(f_3 \cdot f_4 \cdots f_n) + (f_3 \cdot f_4 \cdots f_n)' f_2)$$

$$f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdot f_4 \cdots f_n + f_1 f_2(f_3'(f_4 f_5 \cdots f_n) + (f_4 f_5 \cdots f_n)' \cdot f_3)$$

$$\frac{F'(x)}{F(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \frac{f_3'(x)}{f_3(x)} + \cdots + \frac{f_n'(x)}{f_n(x)} \implies$$

$$(U \cdot V \cdot W)' = U' V W + U V' W + U V \cdot W'$$

Q $\quad \dfrac{F'(x)}{F(x)} = \dfrac{1}{(x-1)} + \dfrac{1}{(x-2)} + \dfrac{1}{(x-3)} + \cdots + \dfrac{1}{(x-n)}$ then $F(x) = ?$

Upar Wala ka product

$$= \frac{(x-1)'}{(x-1)} + \frac{(x-2)'}{(x-2)} + \frac{(x-3)'}{(x-3)} \cdots + \frac{(x-n)'}{(x-n)}$$

$$\implies F(x) = (x-1)(x-2) \cdots (x-n)$$

$$F(x) = \prod_{r=1}^{n} f_r(x)$$

$$\frac{F'(x)}{F(x)} = \sum_{r=1}^{n} \frac{f_r'(x)}{f_r(x)}$$

☆

$$\boxed{\dfrac{F'(x)}{F(x)} = \sum_{r=1}^{n.} \dfrac{f_0'(x)}{f_r(x)} \quad \text{thun} \quad F(x) = \prod_{r=1}^{n.} f_r(x)}$$

$$\dfrac{\left(x-n\right)^{n-2}}{(x-n)^n} = \dfrac{(x-n)^n}{(x-n)^2 \cdot (x-n)^n}$$

$$A^{n-2} = \dfrac{A^n}{A^2}$$

Q $\text{If } f(x) = \prod_{n=1}^{100} (x-n)^{n(101-n)} \quad \text{then find } \dfrac{f(101)}{f'(101)} = ?$

$$\dfrac{f'(x)}{f(x)} = \sum_{n=1}^{100} \dfrac{|n|(101-n)(x-n)^{n(101-n)-1}}{(x-n)^{n(101-n)}}$$

$$x^n \to n(x)^{n-1}$$

$$(x-1)^n \to n(x-1)^{n-1}$$

$$(x-1)^{101n} \to 101n(x-1)^{101n-1}$$

$$(x-1)^{n(101-n)} \to n(101-r)(x-1)^{n(101-n)-1}$$

$$\dfrac{f'(x)}{f(x)} = \sum_{n=1}^{100} \dfrac{(n)(101-n)}{(x-n)}$$

$$\dfrac{f'(101)}{f(101)} = \sum_{n=1}^{100} \dfrac{(n)(101-n)}{(101-n)} = 5050 \Rightarrow \dfrac{f(101)}{f'(101)} = \dfrac{1}{5050}$$

$Q.$ $Y = \sin x \cdot e^{\sqrt{\sin x}} \cdot \ln x$ find $\frac{dy}{dx} = ?$

Q $Y = \sec^3 \sqrt{ax+b}$ then $\frac{dy}{dx} = ?$

$t = \sec \sqrt{ax+b}$

① ③ ② ④

$\frac{dy}{dx} = 3 \left(\sec \sqrt{ax+b}\right)^2$

$\times \sec \sqrt{ax+b} \cdot \tan \sqrt{ax+b}$

$\times \frac{1}{2\sqrt{ax+b}} \times (a)$

$\frac{y'}{y} = \left(\cot x + \frac{\cos x}{2\sqrt{\sin x}} + \frac{1}{x \cdot \ln x}\right)$

$y' = y \left\{ \quad\quad '' \quad\quad \right\}$

$$F(x) = f_1 \cdot f_2 \cdot f_3$$

$$\frac{F'(x)}{F(x)} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}$$

① Power
② Outer fxn
③ Power of Inside fxn
④ Inside fxn.
⑤ Power of more Inside fxn
⑥ Inside fxn

$(6)^*$ Quotient Rule.

$$Y = \frac{f(x)}{g(x)} \text{ then } \frac{dy}{dx} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$$

(7) Chain Rule $\rightarrow$ $Y = f(u)$, $U = g(v)$, $v = h(x)$ then $\frac{dy}{dx} = ?$

$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{du}{dv} \times \frac{dv}{dx} = f'(u) \cdot g'(v) \cdot h'(x) \cdot 1$

$$Y = f(g(h(x)))$$

$$\frac{dy}{dx} = f'(g(h(x)) \times g'(h(x)) \times h'(x) \times 1$$

till last fxn is diff

(8) $\dfrac{d}{dx}(K \cdot f(x)) = K \cdot f'(x)$

(9) $\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

(10) **Trigo formula.**

A) $1 - \cos 2\theta = 2\sin^2\theta$

B) $1 + \cos 2\theta = 2\cos^2\theta$

C) $1 - \sin 2\theta = (\cos\theta - \sin\theta)^2$

D) $1 + \sin 2\theta = (\cos\theta + \sin\theta)^2$

E) $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$, $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

(11) **Formula list**

① $(x^n)' = n(x)^{n-1}$

(2) $\left(\dfrac{1}{x}\right)' \to \dfrac{-1}{x^2}$

(3) $\left(\dfrac{1}{x^2}\right)' = \dfrac{-2}{x^3}$

(4) $\left(\dfrac{1}{x^{17}}\right)' = \dfrac{-17}{x^{18}}$

(5) $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$

(6) $e^x \to e^x$

(7) $(a^x)' \to a^x \ln a$

$(2^{-x})' = 2^{-x} \ln 2 \times -1$

$(2^{-x^2})' \to 2^{-x^2} \ln 2 \times -2x$

$\left(2^{x + \frac{1}{x}}\right)' = 2^{x + \frac{1}{x}} \cdot \ln 2 \times \left(1 - \dfrac{1}{x^2}\right)$

$\left(2^{\sin x}\right)' \to 2^{\sin x} \ln 2 \times (\cos x)$

$\left(2^{\sqrt{\sin x}}\right)' \to 2^{\sqrt{\sin x}} \ln 2 \times \dfrac{1}{2\sqrt{\sin x}} \times \cos x$

(8) $(\log_e x)' = \dfrac{1}{x}$

(9) $(\log_a x)' = \left(\dfrac{\log_e x}{\log_e a}\right)' = \dfrac{1}{\log_e a} \times \dfrac{1}{x}$

(10)

10) $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$

$(\csc x)' = -\csc x \cot x$

11) $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$

$(\cos^{-1} x)' = \dfrac{-1}{\sqrt{1-x^2}}$

$(\tan^{-1} x)' = \dfrac{1}{1+x^2}$

$\cot^{-1} x = \dfrac{-1}{1+x^2}$

$(\sec^{-1} x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

$(\csc^{-1} x)' = \dfrac{-1}{|x|\sqrt{x^2-1}}$

12) $\boxed{\dfrac{d}{dx} |x| = \dfrac{|x|}{x}}$