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1. Let $x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}$. Then $\frac{d^2y}{dx^2} = ky$, find k.

$$\frac{dx}{dy} = 1 \times \frac{1}{\sqrt{1+4y^2}} - 0$$

$$\frac{dy}{dx} = \sqrt{1+4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\sqrt{1+4y^2}\right)\frac{dy}{dx} = \frac{4y}{\sqrt{1+4y^2}}\sqrt{1+4y^2} = 4y$$

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2. If $x = \int_1^{t^2} z \ln z \, dz$, $y = \int_{t^2}^{1} z^2 \ln z \, dz$

find $\frac{dy}{dx}$

$$= \frac{0 - 2t\left(t^4 \ln t^2\right)}{2t\left(t^2 \ln t^2\right) - 0} = -t^2.$$

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3.

$$\frac{d}{dx}\left(\int_x^{x^2}\frac{dt}{x^2+t^2}\right), x>0.$$

$$\frac{d}{dx}\left(\frac{1}{x}\tan^{-1}\frac{t}{x}\Big|_x^{x^2}\right)=\frac{d}{dx}\left(\frac{1}{x}\left(\tan^{-1}x-\frac{\pi}{4}\right)\right)$$

$$=-\frac{1}{x^2}\left(\tan^{-1}x-\frac{\pi}{4}\right)+\frac{1}{x}\frac{1}{(1+x^2)}$$

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4.

$$\lim_{x\to 0}\left(\frac{\int_0^x x\, e^{t^2}\,dt}{1-e^{x^2}}\right) = \lim_{x\to 0}\frac{\frac{\int_0^x e^{t^2}\,dt}{x}}{\left(\frac{1-e^{x^2}}{x^2}\right)}$$

$$= \lim_{x\to 0}\left(\frac{x^2}{1-e^{x^2}}\right)\lim_{x\to 0}\left(\frac{e^{x^2}}{1}\right) = -1$$

$$\frac{\int_0^x e^{t^2}\,dt + x e^{x^2}}{-2x\, e^{x^2}} = \frac{e^{x^2}+e^{x^2}+2x^2 e^{x^2}}{-2\left(e^{x^2}+2x^2 e^{x^2}\right)}$$

$$= -1.$$

5. If $x \in \left[0, \frac{\pi}{2}\right]$, P.T.

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt = \frac{\pi}{4}.$$

$$f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

$$f'(x) = 2\sin x\cos x \; \overbrace{\cancel{\sin^{-1}\sin x}}^{x} - 0 + (-2\cos x\sin x)\,\overbrace{\cancel{\cos^{-1}(\cos x)}}^{x} - 0$$

$$= 0 \qquad \forall\, x \in \left[0, \frac{\pi}{2}\right]$$

$$f(x) = \text{const.} = f\left(\frac{\pi}{4}\right) = \int_0^{\frac{1}{2}} \left(\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t}\right)dt$$

$$= \frac{\pi}{4}.$$

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6. $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\,dx$

$\boxed{I(0) = 0}$

$I'(a) = \int_0^1 \left(\frac{\partial}{\partial a}\left(\frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\right)\right)dx = \int_0^1 \frac{\cancel{x}\,dx}{\cancel{x}\sqrt{1-x^2}\,(1+a^2x^2)}$

$x = \sin\theta$

$\int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{1+(1+a^2)\tan^2\theta} = \int_0^{\frac{\pi}{2}} \frac{\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}\,(1+a^2\sin^2\theta)}$

$= \frac{1}{(1+a^2)}\int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{\frac{1}{1+a^2}+\tan^2\theta} = \frac{\sqrt{1+a^2}}{1+a^2}\tan^{-1}\left(\sqrt{1+a^2}\tan\theta\right)\Big|_0^{\pi/2}$

$= \frac{\pi}{2\sqrt{1+a^2}}$

$I'(a) = \frac{\pi}{2\sqrt{1+a^2}} \Rightarrow I(a) = \frac{\pi}{2}\ln\left|a+\sqrt{1+a^2}\right| + C$

$\Rightarrow a=0, \quad 0 = \frac{\pi}{2}\ln 1 + C \Rightarrow \boxed{C=0}$

$= \frac{\pi}{2}\ln\left(a+\sqrt{1+a^2}\right)$

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$$\int_0^1 \frac{\tan^{-1}(2x)\,dx}{x\sqrt{1-x^2}} = \frac{\pi}{2}\ln\left(2+\sqrt{5}\right)$$

$$\int_0^1 \frac{\tan^{-1}(ax)\,dx}{x\sqrt{1-x^2}}$$

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7. $\displaystyle\int_0^1 \frac{(x^2-1)}{\ln x}\,dx = \ln 3$

$I(a) = \ln(a+1)$

$$I(a) = \int_0^1 \frac{x^a-1}{\ln x}\,dx \Rightarrow \boxed{I(0)=0}$$

$$I'(a) = \int_0^1 \frac{x^a \ln x - 0}{\ln x}\,dx = \int_0^1 x^a\,dx$$

$$= \left.\frac{x^{a+1}}{a+1}\right|_0^1$$

$$= \frac{1}{a+1}$$

$$\frac{d}{da}\left(a^x\right)$$

$$I'(a) = \frac{1}{a+1} \Rightarrow I(a) = \ln(a+1) + c$$

$$I(0) = 0 \Rightarrow \boxed{c = 0}$$

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D·I as Limit of Sum

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + r\left(\frac{b-a}{n}\right)\right) \qquad b = a + nh}$$

$\int_a^b \sin x\,dx = ?$

$$\int_a^b e^x\,dx = \lim_{h\to 0} h\left(e^a + e^{a+h} + e^{a+2h} + \cdots + e^{a+(n-1)h}\right)$$

$$= \lim_{h\to 0} h\,\frac{e^a\left(e^{nh}-1\right)}{e^h - 1} = \lim_{h\to 0} \frac{h e^a\left(e^{b-a}-1\right)}{\left(e^h-1\right)}$$

$$= e^a\left(e^{b-a}-1\right)$$

$$e^b - e^a$$