

# Product of determinants

$$\begin{vmatrix}
 \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\
 b_1 & b_2 & b_3 \\
 c_1 & c_2 & c_3
 \end{vmatrix} \begin{vmatrix}
 \bar{d}_1 & \bar{d}_2 & \bar{d}_3 \\
 e_1 & e_2 & e_3 \\
 f_1 & f_2 & f_3
 \end{vmatrix} = \begin{vmatrix}
 a_1 d_1 + a_2 e_1 + a_3 f_1 & a_1 d_2 + a_2 e_2 + a_3 f_2 & a_1 d_3 + a_2 e_3 + a_3 f_3 \\
 - & - & - \\
 - & - & -
 \end{vmatrix}$$

Row      Column  
 Row      Row  
 Row      Row  
 Column      Column  
 Column      Column

$a_{ij}$  -

1. Simplify

$$\begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 & a_1 l_3 + b_1 m_3 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 & a_2 l_3 + b_2 m_3 \\ a_3 l_1 + b_3 m_1 & a_3 l_2 + b_3 m_2 & a_3 l_3 + b_3 m_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\underline{Q} \cdot \begin{vmatrix} (a_1 - b_1)^2 & (a_1 - b_2)^2 & (a_1 - b_3)^2 \\ (a_2 - b_1)^2 & (a_2 - b_2)^2 & (a_2 - b_3)^2 \\ (a_3 - b_1)^2 & (a_3 - b_2)^2 & (a_3 - b_3)^2 \end{vmatrix} = \begin{vmatrix} a_1^2 & -2a_1 & 1 \\ a_2^2 & -2a_2 & 1 \\ a_3^2 & -2a_3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}$$

$$a_1^2 - 2a_1 b_1 + b_1^2 = 2 \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \begin{vmatrix} 1 & b_1 & b_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix}$$

$$= 2 (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) (b_1 - b_2)(b_2 - b_3)(b_3 - b_1)$$

$$\frac{3}{(-a_1b_1+a_1^2b_1^2)} \begin{vmatrix} \frac{1-a_1^3b_1^3}{1-a_1b_1} & \frac{1-a_1^3b_2^3}{1-a_1b_2} & \frac{1-a_1^3b_3^3}{1-a_1b_3} \\ \frac{1-a_2^3b_1^3}{1-a_2b_1} & \frac{1-a_2^3b_2^3}{1-a_2b_2} & \frac{1-a_2^3b_3^3}{1-a_2b_3} \\ \frac{1-a_3^3b_1^3}{1-a_3b_1} & \frac{1-a_3^3b_2^3}{1-a_3b_2} & \frac{1-a_3^3b_3^3}{1-a_3b_3} \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \begin{vmatrix} b_1 & b_1^2 & b_1^3 \\ b_2 & b_2^2 & b_2^3 \\ b_3 & b_3^2 & b_3^3 \end{vmatrix}$$

$$= (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1).$$

# System of Equations (Cramer's rule)

$$\Delta_3 = z \Delta$$

$$\Delta_2 = y \Delta$$

$$\Delta_1 = x \Delta$$

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\uparrow c_1 \rightarrow c_1 - yc_2 - zc_3$   
 $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$   
 $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$   
 $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

# System of equations

$$\Delta_1 = x\Delta$$

$$\Delta_2 = y\Delta$$

$$\Delta_3 = z\Delta$$

Unique solution  
 $\Delta \neq 0$

Infinite solution  
 $\Delta = 0 = \Delta_1 = \Delta_2 = \Delta_3$

No solution  
 $\Delta = 0$   
& at least one  
of  $\Delta_1, \Delta_2, \Delta_3$   
 $\neq 0$ .

$$(x, y, z) = \left( \frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta} \right)$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 4 & -2 & -2 \\ 6 & -3 & -3 \end{pmatrix} = 0$$

$$\begin{aligned} 2x - y - z &= 1 \\ 4x - 2y - 2z &= 3 \\ 6x - 3y - 3z &= 3 \end{aligned}$$

exception  
no solution

$$a_1x + a$$

$$(x, y, z) = (3k, 2k, 5) \\ k \in \mathbb{R}.$$

$$\text{Let } z = k$$

$$a_1x + b_1y = d_1 - c_1k$$

$$a_2x + b_2y = d_2 - c_2k$$

$x = f(k)$
$y = g(k)$

Consistent  $\rightarrow$  A system of eqns is said  
to be consistent if it has at least  
one solution.

Inconsistent  $\rightarrow$  No Solution

# Homogeneous System of Equations

$$a_1x + b_1y + c_1z = 0$$

$$(0, 0, 0)$$

Condition for given

System to have

$$a_2x + b_2y + c_2z = 0$$

non trivial solutions

$$a_3x + b_3y + c_3z = 0$$

also

$$\Delta_1 = \Delta_2 = \Delta_3 = 0$$

Trivial Solution

$$(x, y, z) = (0, 0, 0)$$

$$\boxed{\Delta = 0}$$

$$\Delta = 0$$

$$\Delta \neq 0$$

Non Trivial Solution

At least one of  $x, y, z \neq 0$

L. Find  $p, q$  so that equations

$$q=3, p \in \mathbb{R} - \{2\}.$$

$$2x + py + 6z = 8$$

$$x + 2y + 9z = 5$$

$$x + y + 3z = 4$$

has (i) no solution

(ii) unique solution

$$\text{Ans} \rightarrow p \in \mathbb{R} - \{2\}, q \in \mathbb{R} - \{3\}$$

(iii) infinite soln.

$$\Delta_3 = -p+2$$

$$\Delta = (q-3)(p-2)$$

$$\Delta_1 = (p-2)(4q-15)$$

$$\Delta_2 = 0$$

$$\text{Ans} \rightarrow p=2, q \in \mathbb{R}$$

$$\begin{aligned} \Delta &= 0 \\ \Rightarrow p &= 2 \text{ or } q = 3 \end{aligned}$$

2. Find  $\theta$  for which system of eqns

$$(\sin 3\theta)x - y + z = 0$$

$$(\cos 2\theta)x + 4y + 3z = 0$$

$$2x + 7y + 7z = 0$$

has non trivial solutions

$$\theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in \mathbb{Z}$$

$$(2\sin\theta + 3)(2\sin\theta - 1) = 0$$

$$\begin{vmatrix} -1 & 1 \\ 4 & 3 \\ 7 & 7 \end{vmatrix} = 0 = 7\sin 3\theta + 14\cos 2\theta - 14$$

$$\begin{cases} \sin\theta = 0 \\ \sin\theta = \frac{1}{2} \end{cases}$$

$$\begin{aligned} 3\sin\theta - 4\sin^3\theta - 4\sin^2\theta &= 0 \\ 4\sin^2\theta + 4\sin\theta - 3 &= 0 \\ -2\sin\theta + 6\cos\theta &= 0 \end{aligned}$$

3. If  $A, B, C$  are angles of triangle, then P.T.

$$\begin{matrix} \pi - (A+B) \\ \pi - (A+C) \end{matrix}$$

$$(\sin 2A)x + (\sin C)y + (\sin B)z = 0$$

$$(\sin C)x + (\sin 2B)y + (\sin A)z = 0$$

$$(\sin B)x + (\sin A)y + (\sin 2C)z = 0$$

possesses non  
trivial solutions

$$\begin{vmatrix} \sin(A+B) & \sin(A+C) & \sin(A+B) \\ \sin(A+B) & \sin(B+C) & \sin(B+C) \\ \sin(C+A) & \sin(C+B) & \sin(C+A) \end{vmatrix} = \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix} \begin{vmatrix} \cos A & \cos B & \cos C \\ \sin A & \sin B & \sin C \\ 0 & 0 & 0 \end{vmatrix} = 0.$$