HYPERBOLA

The **HYPERBOLA** is a conic whose eccentricity is greater than unity. $(e > 1)$.

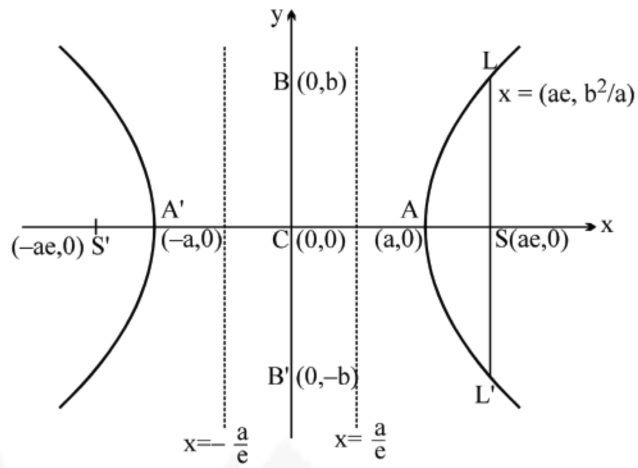
1. STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Where $b^2 = a^2(e^2 - 1)$

or $a^2 e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{C.A}{T.A}\right)^2$



FOCI :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

EQUATIONS OF DIRECTRICES :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES : $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

l **(Latus rectum)** $= \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$.

Note : l(L.R.) = 2e (distance from focus to the corresponding directrix)

TRANSVERSE AXIS : The line segment $A'A$ of length $2a$ in which the foci S' & S both lie is called the **T.A. OF THE HYPERBOLA.**

CONJUGATE AXIS : The line segment $B'B$ between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **C.A. OF THE HYPERBOLA.**

The T.A. & the C.A. of the hyperbola are together called the Principal axes of the hyperbola.

2. FOCAL PROPERTY :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left||PS| - |PS'|\right| = 2a$. The distance SS' = focal length.

3. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **CONJUGATE HYPERBOLAS** of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

Note That :

(a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

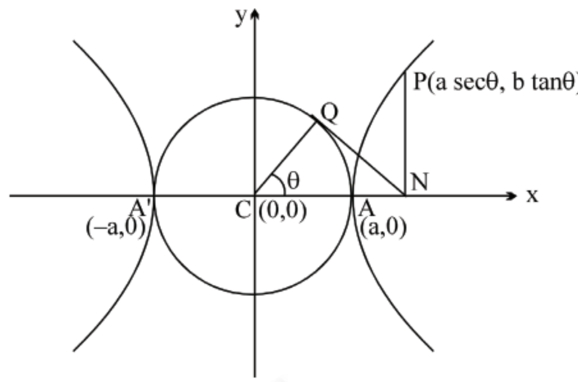
4. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an EQUILATERAL HYPERBOLA. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.

5. AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **AUXILIARY CIRCLE** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"CORRESPONDING POINTS "** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.



Note : The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations : $x = a \cosh\phi$, $y = b \sinh\phi$ also represents the same hyperbola.

General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

6. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or without the curve.

7. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as: $c^2 > = < a^2m^2 - b^2$.

8. TANGENTS AND NORMALS :

TANGENTS :

(a) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$.

<u>**Note**</u>: In general two tangents can be drawn from an external point $(x_1\ y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1\ y_1)$ to the hyperbola.

(b) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is x = a $\dfrac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$, y = b $\dfrac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$

(c) $y = mx \pm \sqrt{a^2m^2-b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Note that there are two parallel tangents having the same slope m.

(d) Equation of a chord joining α & β is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

NORMALS:

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point P(x_1, y_1) on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(b) The equation of the normal at the point P (a secθ, b tanθ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a\,x}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

(c) Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in ellipse.

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is :

$$x^2 + y^2 = a^2 - b^2.$$

If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

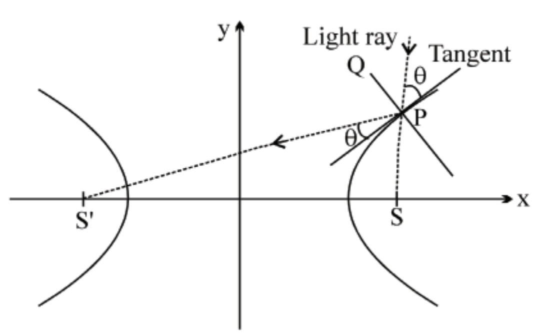
If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. HIGHLIGHTS ON TANGENT AND NORMAL :

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot$ (semi C ·A)^2

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H–3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ (a > k > b > 0) Xare confocal and therefore orthogonal.

H–4 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

11. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

Let y = mx + c is the asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solving these two we get the quadratic as

$(b^2-a^2m^2)\,x^2-2a^2mcx-a^2(b^2+c^2)=0$ ....(1)

In order that y = mx + c be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are :

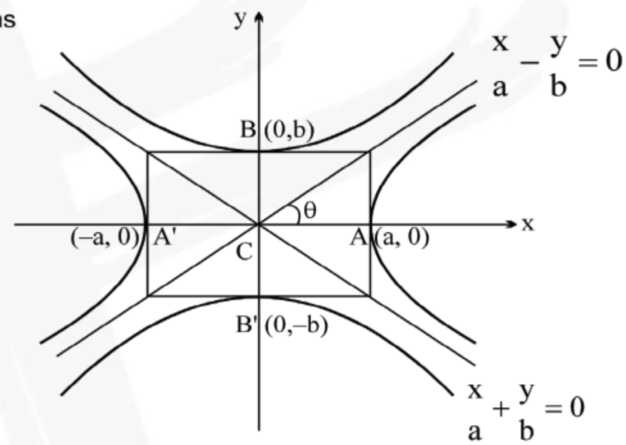
coeff of x^2 = 0 & coeff of x = 0.

$\Rightarrow$ $b^2-a^2m^2=0$ or $m=\pm\frac{b}{a}$ &

$a^2mc=0 \Rightarrow c=0$.

$\therefore$ equations of asymptote are $\frac{x}{a}+\frac{y}{b}=0$

and $\frac{x}{a}-\frac{y}{b}=0$.



combined equation to the asymptotes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

PARTICULAR CASE :

When b = a the asymptotes of the rectangular hyperbola.

$x^2-y^2=a^2$ are, $y=\pm\ x$ which are at right angles.

Note :

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let f (x, y) = 0 represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

gives the centre of the hyperbola.

12. HIGHLIGHTS ON ASYMPTOTES:

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

H–3 The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

13. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the points P (t_1) & Q(t_2) is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$.

(c) Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal : $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

<u>Suggested problems from Loney</u>: **Exercise-36** (Q.1 to 6, 16, 22), **Exercise-37** (Q.1, 3, 5, 7, 12)

EXERCISE-I

1. Find the equation to the hyperbola whose directrix is $2x + y = 1$, focus $(1, 1)$ & eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

2. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

3. For the hyperbola $\frac{x^2}{100} - \frac{y^2}{25} = 1$, prove that

(i) eccentricity $= \sqrt{5}/2$ (ii) SA. S′A = 25, where S & S′ are the foci & A is the vertex.

4. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

5. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

6. Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point $(0, 5/2)$. Find their equations.

7. If C is the centre of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, S, S′ its foci and P a point on it.

Prove that SP. S′P = $CP^2 - a^2 + b^2$.

8. If θ_1 & θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $\tan\frac{\theta_1}{2} \cdot \tan\frac{\theta_2}{2} + \frac{e-1}{e+1} = 0$.

9. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x –axis. If $\tan\theta. \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

10. If two points P & Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C be such that CP is perpendicular to CQ & $a < b$, then prove that $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

11. An ellipse has eccentricity 1/2 and one focus at the point P (1/2, 1). Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. Find the equation of the ellipse in the standard form.

12. The tangents & normal at a point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the y – axis at A & B. Prove that the circle on AB as diameter passes through the foci of the hyperbola.

13. The perpendicular from the centre upon the normal on any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets at R. Find the locus of R.

14. If the normal at a point P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the x – axis at G, show that SG = e. SP, S being the focus of the hyperbola.

15. Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4\,a^2x^2y^2$.

16. If a chord joining the points P $(a\sec\theta, a\tan\theta)$ & Q $(a\sec\phi, a\tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$.

17. Chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are tangents to the circle drawn on the line joining the foci as diameter. Find the locus of the point of intersection of tangents at the extremities of the chords.

18. Let 'p' be the perpendicular distance from the centre C of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the tangent drawn at a point R on the hyperbola. If S & S′ are the two foci of the hyperbola, then show that $(RS + RS')^2 = 4\,a^2\left(1+\frac{b^2}{p^2}\right)$.

19. Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.

20. An ellipse and a hyperbola have their principal axes along the coordinate axes and have a common foci separated by a distance $2\sqrt{13}$, the difference of their focal semi axes is equal to 4. If the ratio of their eccentricities is 3/7. Find the equation of these curves.

EXERCISE-II

1. Prove that the locus of the middle point of the chord of contact of tangents from any point of the circle $x^2 + y^2 = r^2$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by the equation $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\frac{(x^2+y^2)}{r^2}$.

2. The graphs of $x^2 + y^2 + 6x - 24y + 72 = 0$ & $x^2 - y^2 + 6x + 16y - 46 = 0$ intersect at four points. Compute the sum of the distances of these four points from the point (– 3, 2).

3. Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

4. A line through the origin meets the circle $x^2 + y^2 = a^2$ at P & the hyperbola $x^2 - y^2 = a^2$ at Q. Prove that the locus of the point of intersection of the tangent at P to the circle and the tangent at Q to the hyperbola is curve $a^4(x^2 - a^2) + 4x^2y^4 = 0$.

5. A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P & Q. Prove that the middle point of PQ lies on another parabola.

6. The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ drawn at an extremity of its latus rectum is parallel to an asymptote.

   Show that the eccentricity is equal to the square root of $(1+\sqrt{5})/2$.

7. Ascertain the co-ordinates of the two points Q & R, where the tangent to the hyperbola $\frac{x^2}{45} - \frac{y^2}{20} = 1$ at the point P(9, 4) intersects the two asymptotes. Finally prove that P is the middle point of QR. Also compute the area of the triangle CQR where C is the centre of the hyperbola.

8. A point P divides the focal length of the hyperbola $9x^2 - 16y^2 = 144$ in the ratio S′P : PS = 2 : 3 where S & S′ are the foci of the hyperbola. Through P a straight line is drawn at an angle of 135° to the axis OX. Find the points of intersection of this line with the asymptotes of the hyperbola.

9. Find the length of the diameter of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ perpendicular to the asymptote of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ passing through the first & third quadrants.

10. The tangent at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the asymptote in Q. Show that the locus of the mid point of PQ is a similiar hyperbola.

11. A transversal cuts the same branch of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in P, P' and the asymptotes in Q, Q'. Prove that (i) PQ = P'Q' & (ii) PQ' = P'Q

12. A series of hyperbolas is drawn having a common transverse axis of length 2a. Prove that the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymtote, is the curve $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$.

13. From any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to another hyperbola which has the same asymptotes. Show that the chord of contact cuts off a constant area from the asymptotes.

14. Through any point P of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a line QPR is drawn with a fixed gradient m, meeting the asymptotes in Q & R. Show that the product, $(QP) \cdot (PR) = \frac{a^2b^2(1+m^2)}{b^2 - a^2m^2}$.

15. If a rectangular hyperbola have the equation, $xy = c^2$, prove that the locus of the middle points of the chords of constant length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$.

16. A triangle is inscribed in the rectangular hyperbola $xy = c^2$. Prove that the perpendiculars to the sides at the points where they meet the asymptotes are concurrent. If the point of concurrence is (x_1, y_1) for one asymptote and (x_2, y_2) for the other, then prove that $x_2y_1 = c^2$.

17. Prove that infinite number of triangles can be inscribed in the rectangular hyperbola, $xy = c^2$ whose sides touch the parabola, $y^2 = 4ax$.

18. The normals at three points P, Q, R on a rectangular hyperbola $xy = c^2$ intersect at a point on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.

19. Tangents are drawn from any point on the rectangular hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Prove that these tangents are equally inclined to the asymptotes of the hyperbola.

20. P & Q are two variable points on a rectangular hyperbola $xy = c^2$ such that the tangent at Q passes through the foot of the ordinate of P. Show that the locus of the point of intersection of tangent at P & Q is a hyperbola with the same asymptotes as the given hyperbola.

EXERCISE-III

1. (a) The curve described parametrically by, $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents:

(A) a parabola (B) an ellipse (C) a hyperbola (D) a pair of straight lines

(b) Let P ($a \sec \theta$, $b \tan \theta$) and Q ($a \sec \phi$, $b \tan \phi$), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P & Q, then k is equal to:

(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$ (C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

(c) If x = 9 is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents, is :

(A) $9x^2 - 8y^2 + 18x - 9 = 0$ (B) $9x^2 - 8y^2 - 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x - 9 = 0$ (D) $9x^2 - 8y^2 + 18x + 9 = 0$

[JEE '99, 2 + 2 + 2 (out of 200)]

2. The equation of the common tangent to the curve $y^2 = 8x$ and $xy = -1$ is

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$ **[JEE 2002 Screening]**

3. Given the family of hyperbols $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$ for $\alpha \in (0, \pi/2)$ which of the following does not change with varying α?

(A) abscissa of foci (B) eccentricity

(C) equations of directrices (D) abscissa of vertices **[JEE 2003 (Scr.)]**

4. The line $2x + \sqrt{6}\, y = 2$ is a tangent to the curve $x^2 - 2y^2 = 4$. The point of contact is

(A) $(4, -\sqrt{6})$ (B) $(7, -2\sqrt{6})$ (C) (2, 3) (D) $(\sqrt{6}, 1)$ **[JEE 2004 (Scr.)]**

5. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of midpoint of the chord of contact. **[JEE 2005 (Mains), 4]**

6. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axis coincides with the major and minor axis of the ellipse, and product of their eccentricities is 1, then

(A) equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$

(B) equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$

(C) focus of hyperbola (5, 0)

(D) focus of hyperbola is $\left(5\sqrt{3},\,0\right)$ **[JEE 2006, 5]**

Comprehension: (3 questions)

7. Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A

(a) If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

(b) A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(A) ellipse (B) hyperbola (C) parabola (D) parts of straight line

(c) A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

(A) 1/2 sq. units (B) 2/3 sq. units (C) 1 sq. unit (D) 2 sq. units

[JEE 2006, 5 marks each]

8. (a) A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

(A) $x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$

(B) $x^2\sec^2\theta - y^2\operatorname{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$

(D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$ **[JEE 2007, 3]**

(b) Match the statements in **Column I** with the properties in **Column II**.

	Column I		Column II
(A)	Two intersecting circles	(P)	have a common tangent
(B)	Two mutually external circles	(Q)	have a common normal
(C)	Two circles, one strictly inside the other	(R)	do not have a common tangent
(D)	Two branches of a hyperbola	(S)	do not have a common normal

[JEE 2007, 3 + 6]

9. (a) Let *a* and *b* be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(A) four straight lines, when $c = 0$ and a, b are of the same sign.

(B) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a.

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a.

(b) Consider a branch of the hyperbola, $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[JEE 2008, 3+3]**

(A) $1-\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}}-1$ (C) $1+\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}}+1$

10. Match the conics in **Column I** with the statements/expressions in **Column II.** **[JEE 2009]**

Column I	**Column II**
(A) Circle	(P) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2 + y^2 = 4$
(B) Parabola	(Q) Points z in the complex plane satisfying $\|z + 2\| - \|z - 2\| = \pm 3$
(C) Ellipse	(R) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(S) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(T) Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \|z\|^2 + 1$

11. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[JEE 2009]**

(A) Equation of ellipse is $x^2 + 2y^2 = 2$ (B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$ (D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Paragraph for Questions 12 to 13 **[JEE 2010]**

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

12. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

13. Equation of the circle with AB as its diameter is

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

14. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[JEE 2010]**

15. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is **[JEE 2011]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

16. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then **[JEE 2011]**

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$ (B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$ (D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

17. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contact of the tangents on the hyperbola are **[JEE 2012]**

(A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ (C) $\left(3\sqrt{3},-2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3},2\sqrt{2}\right)$

18. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (ℓ, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are) **[IIT JEE Advance - 2015]**

(A) $\frac{d\ell}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$

(C) $\frac{d\ell}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1 > 0$

19. The centres of those circles which touch the circle, $x^2 - y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on :

(A) a circle (B) an ellipse which is not a circle

(C) a hyperbola (D) a parabola **[JEE Main - 2016]**

20. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE Main - 2016]**

(A) $\frac{4}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) $\sqrt{3}$

21. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at (±2, 0). Then the tangent to this hyperbola at P also passes through the point : **[JEE Main - 2017]**

(A) $(\sqrt{3},\sqrt{2})$ (B) $(-\sqrt{2},-\sqrt{3})$

(C) $(3\sqrt{2},2\sqrt{3})$ (D) $(2\sqrt{2},3\sqrt{3})$

22. If 2x – y + 1 = 0 is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle? **[JEE Advanced 2017]**

(A) a, 4, 2 (B) 2a, 8, 1 (C) 2a, 4, 1 (D) a, 4, 1

Answer Q. 23, Q. 24 and Q. 25 by appropriately matching the information given in the three columns of the following table. **[JEE Advanced 2017]**

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

	Column-1	Column-2	Column-3
(I)	$x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II)	$x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III)	$y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV)	$x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

23. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1, 1), then which of the following options is the only CORRECT combination for obtaining its equation?

(A) (II) (ii) (Q) (B) (III) (i) (P) (C) (I) (ii) (Q) (D) (I) (i)(P)

24. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(A) (III) (i) (P) (B) (II) (iv) (R) (C) (III) (ii) (Q) (D) (I) (ii) (Q)

25. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

(A) (IV) (iii) (S) (B) (II) (iv) (R) (C) (IV) (iv) (S) (D) (II) (iii) (R)

26. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is : **[JEE Main 2018]**

(A) $36\sqrt{5}$ (B) $45\sqrt{5}$ (C) $54\sqrt{3}$ (D) $60\sqrt{3}$

27. Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$. **[JEE Advanced 2018]**

	List - I		List - II
(P)	The length of the conjugate axis of H is	(1)	8
(Q)	The eccentricity of H is	(2)	$\frac{4}{\sqrt{3}}$
(R)	The distance between the foci of H is	(3)	$\frac{2}{\sqrt{3}}$
(S)	The length of the latus rectum of H is	(4)	4

The correct option is :

(A) P → 4; Q→ 2; R→ 1; S→3 (B) P → 4; Q→ 3; R→ 1; S→ 2

(C) P → 4; Q→ 1; R→ 3; S→ 2 (D) P → 3; Q→ 4; R→ 2; S→1

28. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point (1, 0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

[JEE Advanced 2020]

(A) $1 < e < \sqrt{2}$ (B) $\sqrt{2} < e < 2$ (C) $\Delta = a^4$ (D) $\Delta = b^4$

ANSWER KEY

EXERCISE–I

1. $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$; $\sqrt{\frac{48}{5}}$ **2.** $a^2 = 25/2$; $b^2 = 16$

4. $(-1, 2)$; $(4, 2)$ & $(-6, 2)$; $5x - 4 = 0$ & $5x + 14 = 0$; $\frac{32}{3}$; 6 ; 8 ; $y - 2 = 0$; $x + 1 = 0$; $4x - 3y + 10 = 0$; $4x + 3y - 2 = 0$.

5. $x + y \pm 3\sqrt{3} = 0$ **6.** $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$ **11.** $\frac{\left(x-\frac{1}{3}\right)^2}{1/9} + \frac{(y-1)^2}{1/12} = 1$

13. $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = x^2y^2 (a^2 + b^2)^2$ **17.** $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}$

20. $\frac{x^2}{49} + \frac{y^2}{36} = 1$; $\frac{x^2}{9} - \frac{y^2}{4} = 1$

EXERCISE–II

2. 40 **3.** $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

7. (15, 10) and (3, – 2) and 30 sq. units **8.** $(-4, 3)$ & $\left(-\frac{4}{7}, -\frac{3}{7}\right)$ **9.** $\frac{150}{\sqrt{481}}$

10. $4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 3$ **13.** ab **20.** $xy = \frac{8}{9}c^2$

EXERCISE–III

1. **(a)** A ; **(b)** D ; **(c)** B **2.** D **3.** A **4.** A **5.** $\frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2+y^2}{9}\right)^2$

6. A, C **7.** (a) A, (b) C, (c) C **8. (a)** A; **(b)** (A) P, Q; (B) P ,Q; (C) Q, R; (D) Q, R
9. **(a)** B; **(b)** B **10.** (A) → P ; (B) → S, T ; (C) → R ; (D) → Q, S **11.** A, B
12. B **13.** A **14.** 2 **15.** B **16.** B, D **17.** A, B
18. A, B, D **19.** D **20.** C **21.** D **22.** ABD
23. C **24.** A **25.** B **26.** B **27.** B **28.** A,D