

1. Draw the graph of $f(x) = \left(1 + \frac{1}{x}\right)^x$.

$$(-\infty, -1) \cup (0, \infty)$$

$$f'(x) = \underbrace{\left(1 + \frac{1}{x}\right)^x}_{>0} \left(\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1} \right) > 0$$



$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$$

$$g'(x) = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2}$$

$$= \frac{-1}{x(x+1)^2}$$

$$g \uparrow \boxed{x \in (-\infty, -1)} \Rightarrow g(x) > \lim_{x \rightarrow -\infty} g(x) = 0 \quad \forall x \in (-\infty, -1)$$

$$\downarrow x \in (0, \infty) \Rightarrow g(x) > \lim_{x \rightarrow \infty} g(x) = 0, \quad \forall x \in (0, \infty)$$

$$\lim_{x \rightarrow -\infty} f(x) = e$$

$$\lim_{x \rightarrow -1} f(x) = \infty$$

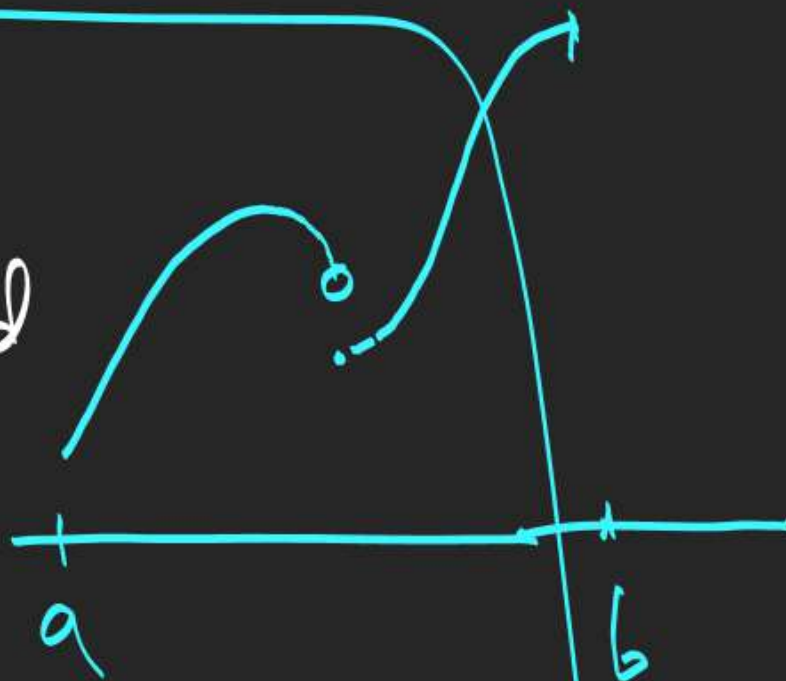
$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow 0^+} e^{x \ln\left(1 + \frac{1}{x}\right)} = \lim_{x \rightarrow 0^+} \frac{x \ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right) = \infty$$

$$= \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right) = \infty$$

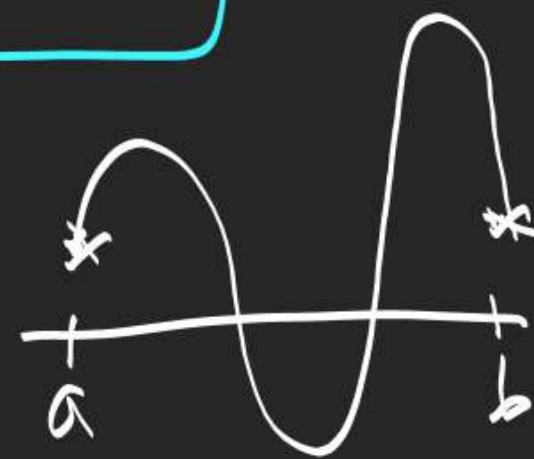
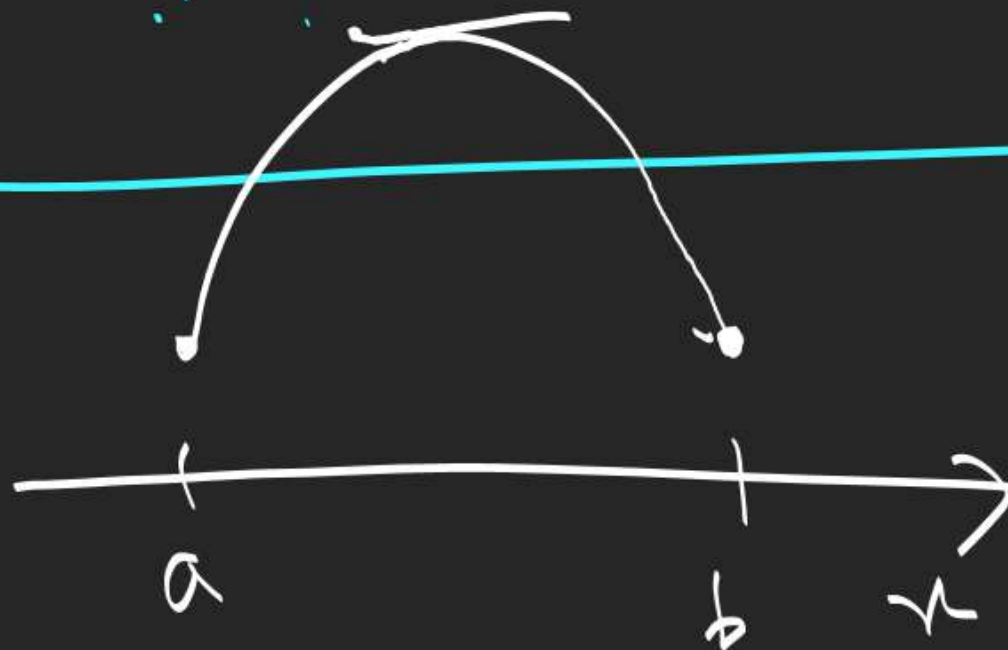
$$\mathcal{R}_f = (1, e) \cup (e, \infty)$$

Rolle's Theorem

- ① $f(x)$ is continuous in $[a, b]$
- ② $f(x)$ is differentiable in (a, b) , and
- ③ $f(a) = f(b)$



$\Rightarrow \exists c \in (a, b)$, such that $f'(c) = 0$.



Lagrange's Mean Value Theorem (LMVT)

- If $f(x)$ is continuous in $[a, b]$, and
- $f(x)$ is differentiable in (a, b) ,

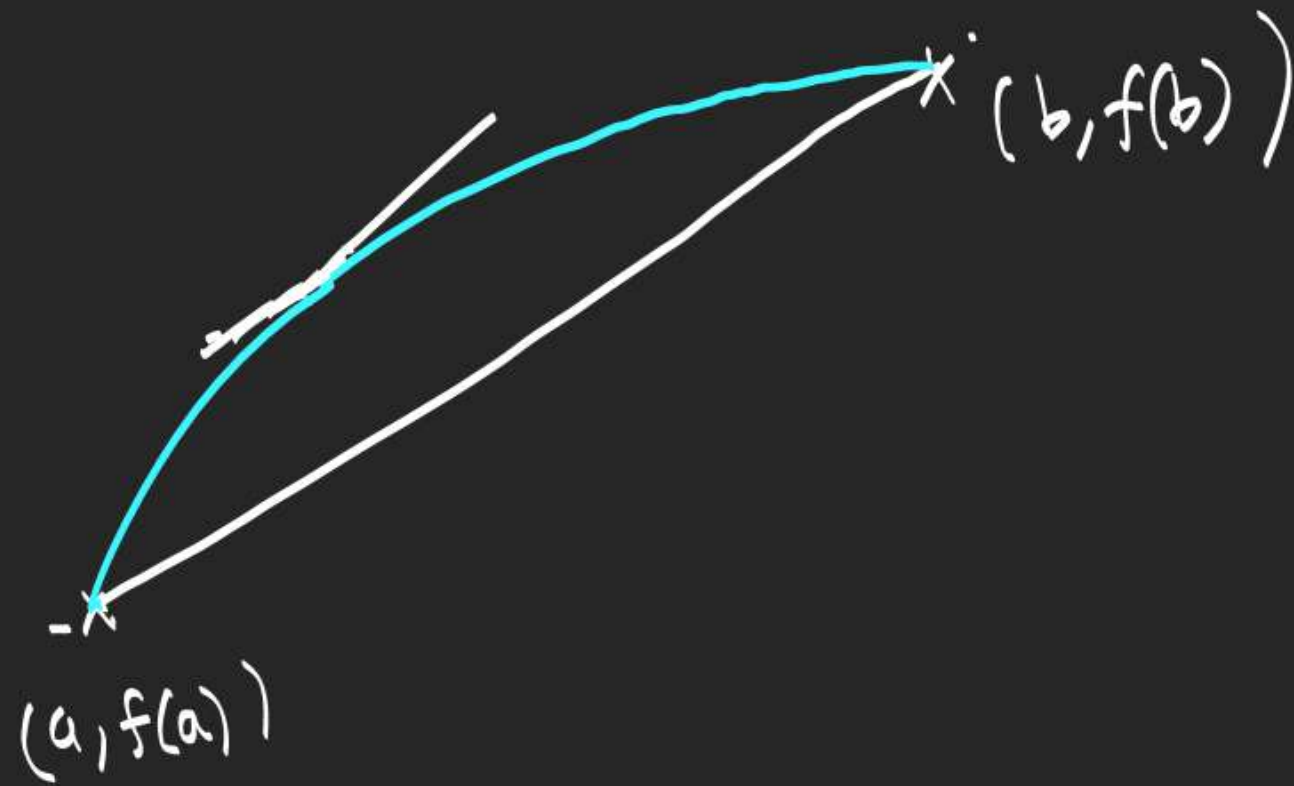
$$\Rightarrow \exists c \in (a, b), \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

\Rightarrow Using Rolle's, $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

$$g(b) - g(a) = (f(b) - f(a)) - \left(\frac{f(b) - f(a)}{b - a} \right) (b - a) = 0$$

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) x$$

g is cont. in $[a, b]$
 g is diff. in (a, b)
 $g(b) = g(a)$



$$\frac{f(b) - f(a)}{b - a}$$

Generalised Mean Value Theorem

(Cauchy's Mean Value Theorem)

11-15 (Ind.)

Diff. $\rightarrow 2x-4$ (1-10)

- $f(x), g(x)$ are cont. in $[a, b]$
- $f(x), g(x)$ are diff. in (a, b)

$$\Rightarrow \exists c \in (a, b) \text{ s.t. } (f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$