KEY CONCEPTS

1. DEFINITION :

If f & g are functions of x such that $g'(x) = f(x)$ then the function g is called a **Primitive OR Antiderivative** Or INTEGRAL of f(x) w.r.t. x and is written symbolically as

$\int f(x)dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the constant of integration.

2. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \; n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a} ln(ax+b) + c$

(iii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

(iv) $\int a^{px+q}\, dx = \frac{1 a^{px+q}}{p} (a > 0) + c$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a} ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a} ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \text{cosec}^2(ax+b)dx = \frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b)\cdot\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \text{cosec}(ax+b)\cdot\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b) + c$

(xiii) $\int \sec dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \text{cosec}\, x\, dx = \ln(\text{cosec} x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\text{cosec} x + \cot x)$

(xv) $\int \sinh x dx = \cosh x + c$

(xvi) $\int \cosh x dx = \sinh x + c$

(xvii) $\int \text{sech}^2 x\, dx = \tanh x + c$

(xviii) $\text{cosech}^2 \int x\, dx = -\coth x + c$

(xix) $\int \text{sech} x \cdot \tanh x dx = -\text{sech} x + c$

(xx) $\int \text{cosech} x \cdot \coth x dx = -\text{cosech} x + c$

(xxi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xxiii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxiv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a} + c$

(xxv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x + \sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a} + c$

(xxvi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} + c$

(xxvii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$

(xxviii) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxix) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left[x + \sqrt{x^2+a^2}\right] + c$

(xxx) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left[x + \sqrt{x^2-a^2}\right] + c$

(xxxi) $\int e^{ax}\cdot\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxxii) $\int e^{ax}\cdot\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c$

3. TECHNIQUES OF INTEGRATION :

(i) Substitution or change of independent variable.

Integral $I=\int f(x)\,dx$ is changed to $\int f(\phi(t))f'(t)dt$, by a suitable substitution $x=\phi(t)$ provided the later integral is easier to integrate.

(ii) **Integration by part :** $\int u.v\,dx=u\int v\,dx-\int\left[\frac{du}{dx}\int v dx\right]dx$ where u & v are differentiable function.

Note: While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & (b) $\int\left[\frac{du}{dx}\int v dx\right]dx$ is simple to integrate.

This is generally obtained, by keeping the order of u&v as per the order of the letters in ILATE, where ; I-Inverse function, L-Logarithmic function,

A-Algebraic function, T-Trigonometric function & E - Exponential function

(iii) Partial fraction, spiliting a bigger fraction into smaller fraction by known methods.

4. INTEGRALS OF THE TYPE :

(i) $\int[f(x)]^n f'(x)dx$ OR $\int\frac{f'(x)}{[f(x)]^n}dx$ put $f(x)=t$ & proceed.

(ii) $\int\frac{dx}{ax^2+bx+c},\int\frac{dx}{\sqrt{ax^2+bx+c}},\int\sqrt{ax^2+bx+c}dx$

Express ax^2+bx+c in the form of perfect square & then apply the standard results .

(iii) $\int\frac{px+q}{ax^2+bx+c}dx,\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx.$

Express $px+q=A$ (differential co-efficient of denominator) $+B$.

(iv) $\int e^x[f(x)+f'(x)]dx=e^x\cdot f(x)+c$ **(v)** $\int[f(x)+xf'(x)]dx=xf(x)+c$

(vi) $\int\frac{dx}{x(x^n+1)}$ $n\in N$ Take x^n common & put $1+x^{-n}=t$.

(vii) $\int\frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n\in N$, take x^n common & put $1+x^{-n}=t^n$

(viii) $\int\frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1+x^{-n}=t$.

(ix) $\int\frac{dx}{a+b\sin^2x}$ **OR** $\int\frac{dx}{a+b\cos^2x}$ **OR** $\int\frac{dx}{a\sin^2x+b\sin x\cos x+\cos^2x}$

Multiply N^r & D^r by $\sec^2x$ & put $\tan x=t$.

(x) $\int\frac{dx}{a+b\sin x}$ OR $\int\frac{dx}{a+b\cos x}$ OR $\int\frac{dx}{a+b\sin x+c\cos x}$

Hint :Convert sines & cosines into their respective tangents of half the angles, put $\tan\frac{x}{2}=t$

(xi) $\int\frac{a\cdot\cos x+b\cdot\sin x+c}{\ell\cos x+m\cdot\sin x+n}dx$. Express $Nr\equiv A(Dr)+B\frac{d}{dx}(Dr)+c$ & proceed.

(xii) $\int \frac{x^2+1}{x^4+Kx^2+1}dx$ **OR** $\int \frac{x^2-1}{x^4+Kx^2+1}dx$ where K is any constant.

Hint: Divide Nr & Dr by x^2 & proceed.

(xiii) $\int \frac{dx}{(ax+b)\sqrt{px+q}} \& \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px + q = t^2$.

(xiv) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax + b = \frac{1}{t}$; $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x = \frac{1}{t}$

(xv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\, dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\, dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

DEFINITE INTEGRAL

1. $\int_a^b f(x)dx = F(b) - F(a)$ where $\int f(x)dx = F(x) + c$

VERY IMPORTANT NOTE : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

2. Properties Of Definite Integral :

P-1 $\int_a^b f(x)dx = \int_a^b f(t)dt$ provided f is same **P − 2** $\int_a^b f(x)dx = -\int_b^a f(x)dx$

P-3 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside or outside the interval $[a, b]$. This property to be used when f is piecewise continuous in (a, b).

P-4 $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function i.e. $f(x) = -f(-x)$. $= 2\int_0^a f(x)dx$ if $f(x)$ is an even function i.e. $f(x) = f(-x)$.

P-5 $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, In particular $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

P-6 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$
$= 0$ if $f(2a - x) = -f(x)$

P-7 $\int_0^f f(x)dx = n\int_0^a f(x)dx$; where'a'is the period of the function i.e. $f(a + x) = f(x)$

P-8 $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$ where $f(x)$ is periodic with period T & $n \in I$.

P-9 $\int_{ma}^{na} f(x)dx = (n - m)\int_0^a f(x)dx$ if $f(x)$ is periodic with period 'a'.

P-10 If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

P-11 $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

P-12 If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)dx \ge 0$.

3. WALLI'S FORMULA :

$$\int_0^{\pi/2} \sin^n x \cdot \cos^m x \, dx = \frac{[(n-1)(n-3)(n-5)\ldots 1 or 2][(m-1)(m-3)\ldots 1 or 2]}{(m+n)(m+n-2)(m+n-4)\ldots \text{ or } 2} K$$

Where $K = \frac{\pi}{2}$ if both m and n are even $(m, n \in N)$
$= 1$ otherwise

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

If $h(x)$&$g(x)$ are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$$

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)dx = \text{Limit}_{n\to\infty} h[f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + \overline{n-1}h)]$$
$$= \text{Limit}_{h\to 0} h\sum_{r=0}^{n-1} f(a + rh) \text{ where } b - a = nh$$

If $a = 0$ & $b = 1$ then, $\underset{n\to\infty}{\text{Limit}}\, h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$ OR

$$\text{Limit}_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$$

6. ESTIMATION OF DEFINITE INTEGRAL:

(i) For a monotonic decreasing function in (a, b); $f(b).(b-a) < \int_a^b f(x)dx < f(a).(b\text{-}a)$ &

(ii) For a monotonic increasing function in (a, b); $f(a).(b\text{-}a) < \int_a^b f(x)dx < f(b).(b\text{-}a)$

(iii) **Cauchy Schwarz Inequality :** If f and g are continuous functions on $[a, b]$, then

$$\left(\int_a^b (f(t)g(t))dt\right)^2 \le \left(\int_a^b (f(t))^2 dt\right)\left(\int_a^b (g(t))^2 dt\right)$$

7. SOME IMPORTANT EXPANSIONS :

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots .\infty = \ln 2$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots ..\infty = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots ..\infty = \frac{\pi^2}{12}$

(iv) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots ..\infty = \frac{\pi^2}{8}$

(v) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots \infty = \frac{\pi^2}{24}$

EXERCISE–I

1. $\int \frac{\tan 2\theta}{\sqrt{\cos^6\theta + \sin^6\theta}} d\theta$

2. $\int \frac{5x^4+4x^5}{(x^5+x+1)^2} dx$

3. $\int \frac{\cos^2 x}{1+\tan x} dx$

4. $\int \left(\frac{\sin x + \sin 3x + \sin 5x + \sin 7x + \sin 9x + \sin 11x + \sin 13x + \sin 15x}{\cos x + \cos 3x + \cos 5x + \cos 7x + \cos 9x + \cos 11x + \cos 13x + \cos 15x}\right) dx$

5. $\int \frac{\ln\left(\ln\left(\frac{1+x}{1-x}\right)\right)}{1-x^2} dx$

6. $\int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right] \ln x \, dx$

7. $\int \cos 2\theta \cdot \ln \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} d\theta$

8. $\int \frac{a^2\sin^2 x + b^2\cos^2 x}{a^4\sin^2 x + b^4\cos^2 x} dx$

9. $\int \frac{dx}{\left(x+\sqrt{x(1+x)}\right)^2}$

10. $\int \frac{x^3+x+1}{x^4+x^2+1} dx$

11. $\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}} dx$

12. $\int (\sin x)^{-11/3}(\cos x)^{-1/3} dx$

13. $\int \frac{\cot x \, dx}{(1-\sin x)(\sec x+1)}$

14. $\int \sin^{-1}\sqrt{\frac{x}{a+x}} dx$

15. $\int \left[\frac{\sqrt{x^2+1}\left[\ln(x^2+1)-2\ln x\right]}{x^4}\right] dx$

16. $\int \frac{x+1}{x(1+xe^x)^2} dx$

17. Let f(x) is a quadratic function such that $f(0) = 1$ and $\int \frac{f(x)\,dx}{x^2(x+1)^3}$ is a rational function, find the value of $f'(0)$

18. Integrate $\frac{1}{2}f'(x)$ w.r.t. x^4, where $f(x) = \tan^{-1}x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$

19. $\int \frac{(\sqrt{x}+1)dx}{\sqrt{x}(\sqrt[3]{x}+1)}$

20. $\int \frac{dx}{\sin\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}}$

21. $\int \frac{x^2+x}{(e^x+x+1)^2} dx$

22. $\int \sqrt{\frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x + \cot x}} \cdot \frac{\sec x}{\sqrt{1+2\sec x}} dx$

23. $\int \frac{\cos x - \sin x}{7 - 9\sin 2x} dx$

24. $\int \frac{dx}{\sec x + \operatorname{cosec} x} dx$

25. $\int \frac{dx}{\sin x + \sec x}$

26. $\int \tan x \cdot \tan 2x \cdot \tan 3x \, dx$

27. $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

28. $\int \frac{x^2}{(x\cos x - \sin x)(x\sin x + \cos x)} dx$

29. $\int \frac{3 + 4\sin x + 2\cos x}{3 + 2\sin x + \cos x} dx$

30. $\int \frac{x^5+3x^4-x^3+8x^2-x+8}{x^2+1} dx$

31. $\int \frac{\sqrt{\sin^4 x + \cos^4 x}}{\sin^3 x \cos x} dx, x \in \left(0, \frac{\pi}{2}\right)$

32. $\int \frac{3x^2+1}{(x^2-1)^3} dx$

33. $\int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x} dx$

34. $\int \frac{(ax^2-b)dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

35. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$

36. $\int \frac{x}{(7x-10-x^2)^{3/2}}dx$

37. $\int \frac{x\ln x}{(x^2-1)^{3/2}}dx$

38. $\int \sqrt{\frac{(1-\sin x)(2-\sin x)}{(1+\sin x)(2+\sin x)}}dx$

39. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx$

40. $\int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$

41. $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

42. $\int \frac{dx}{\cos^3 x-\sin^3 x}$

43. $\int \frac{\sqrt{\cos 2x}}{\sin x}dx$

44. $\int \frac{(1+x^2)dx}{1-2x^2\cos\alpha+x^4}, \alpha \in (0,\pi)$

45. Evaluate the integral $\int \frac{\cos^2 x+\sin 2x}{(2\cos x-\sin x)^2}dx$, where $x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

46. Evaluate the integral $\int \frac{(x^2-x^3)\,dx}{(x+1)(x^3+x^2+x)^{3/2}}$

47. Evaluate $\int \frac{\sin^3 x\,dx}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}$

48. Evaluate $\int \frac{(x\cos x+1)\,dx}{\sqrt{2x^3e^{\sin x}+x^2}}$

49. $\int \frac{\sqrt[3]{(1+\sqrt[4]{x})}}{\sqrt{x}}dx$

Match the Column:

50.

Column-I	Column-II
(A) $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx$	(P) $\ln\left(\frac{(x^2+1)+\sqrt{x^4+1}}{x}\right)+C$
(B) $\int \frac{x^2-1}{x\sqrt{1+x^4}}dx$	(Q) $C-\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{x^4+1}-\sqrt{2}x}{(x^2-1)}\right)$
(C) $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}dx$	(R) $C-\tan^{-1}\left(\sqrt{\sqrt{1+\frac{1}{x^4}}-1}\right)$
(D) $\int \frac{1}{(1+x^4)\sqrt{\sqrt{1+x^4}-x^2}}dx$	(S) $\frac{\sqrt{x^4+x^2+1}}{x}+C$

EXERCISE–II

1. Evaluate: $\int_0^1 e^{ln \tan^{-1}x} \cdot \sin^{-1}(\cos x)dx$.

2. Prove that

(a) $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx = \frac{(\beta-\alpha)^2\pi}{8}$

(b) $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx = (\beta-\alpha)\frac{\pi}{2}$

(c) $\int_\alpha^\beta \frac{dx}{x\sqrt{(x-\alpha)(\beta-x)}} = \frac{\pi}{\sqrt{\alpha\beta}}$, where $\alpha, \beta > 0$

(d) $\int_\alpha^\beta \frac{x \cdot dx}{\sqrt{(x-\alpha)(\beta-x)}} = (\alpha+\beta)\frac{\pi}{2}$, where $\alpha < \beta$

3. (a) Evaluate $I_n = \int_1^e (ln^n x)\,dx$ hence find I_3.

(b) Determine a positive integer $n \le 5$, such that $\int^1 e^x(x-1)^n\,dx = 16 - 6e$.

4. $\int_0^{\pi/2} \sin 2x \cdot \arctan(\sin x)\,dx$

5. If $P = \int_0^\infty \frac{x^2}{1+x^4}dx$; $Q = \int_0^\infty \frac{xdx}{1+x^4}$ and $R = \int_0^\infty \frac{dx}{1+x^4}$ then prove that

(a) $Q = \frac{\pi}{4}$, (b) $P = R$, (c) $P - \sqrt{2}\,Q + R = \frac{\pi}{2\sqrt{2}}$

6. $\int_1^2 \frac{(x^2-1)dx}{x^3 \cdot \sqrt{2x^4-2x^2+1}} = \frac{u}{v}$ where u and v are in their lowest form. Find the value of $\frac{(1000)u}{v}$.

7. Evaluate $\int_0^{\pi/2} \frac{\sin^6 x\,dx}{\sin x + \cos x}$

8. For $a \ge 2$, if the value of the definite integral $\int_0^\infty \frac{dx}{a^2+(x-(1/x))^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

9. If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$$\int_0^{\pi/2} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\,dx = 2$$

then find the value of $1000(a_1^2 + a_2^2 + a_3^2)$.

10. Let $u = \int_0^{\pi/4}\left(\frac{\cos x}{\sin x + \cos x}\right)^2 dx$ and $v = \int_0^{\pi/4}\left(\frac{\sin x + \cos x}{\cos x}\right)^2 dx$. Find the value of $\frac{v}{u}$.

11. $\int_0^{\pi/2} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}\,dx$

12. $\int_{-2}^{2} \frac{x^2-x}{\sqrt{x^2+4}}dx$

13. $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}dx$

14. $\int_0^{\pi/4} \frac{xdx}{\cos x\,(\cos x + \sin x)}$

15. $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$

16. $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\frac{1}{x}\right)dx$

17. $\text{Lim}_{n\to\infty} n^2\int_{-1/n}^{1/n}(2007\sin x + 2008\cos x)|x|dx$.

18. Find the value of the definite integral $\int_0^\pi |\sqrt{2}\sin x + 2\cos x|dx$.

19. If $\int_0^\pi \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}dx$ has the value equal to $\left(\frac{\pi}{k} + \sqrt{w}\right)$ where k and w are positive integers find the value of $(k^2 + w^2)$.

20. $\int_0^1 \frac{1-x}{1+x} \cdot \frac{dx}{\sqrt{x+x^2+x^3}}$

21. $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin\left(\frac{\pi}{4}+x\right)}dx$

22. $\int_0^1 \frac{x^2 \cdot \ln x}{\sqrt{1-x^2}}dx$

23. If $\int_{\pi/4}^{\pi/3} \frac{(\sin^3\theta - \cos^3\theta - \cos^2\theta)(\sin\theta + \cos\theta + \cos^2\theta)^{2007}}{(\sin\theta)^{2009}(\cos\theta)^{2009}} d\theta = \frac{(a+\sqrt{b})^n - (1+\sqrt{c})^n}{d}$

where a, b, c and d are all positive integers. Find the value $(a + b + c + d)$.

24. $\int_0^{\sqrt{3}} \sin^{-1}\frac{2x}{1+x^2} dx$ **25.** $\int_0^{\pi} \frac{(ax+b)\sec x \tan x}{4+\tan^2 x} dx (a, b > 0)$ 26. $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)} dx$

27. $\int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$ **28.** $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\, dx$ **29.** $\int_0^{2\pi} \frac{dx}{2+\sin 2x}$

30. $\int_0^{a} \frac{\ln(1+ax)}{1+x^2} dx, a \in N$ **31.** $\int_0^{\frac{\ln 3}{2}} \frac{e^x+1}{e^{2x}+1} dx$ **32.** $\int_0^{2\pi} \frac{x^2 \sin x}{8+\sin^2 x} dx$

33. Let α, β be the distinct positive roots of the equation $\tan x = 2x$ then evaluate $\int_0^1 (\sin\alpha x \cdot \sin\beta x) dx$, independent of α and β.

34. Show that $\int_0^{p+q\pi} |\cos x| dx = 2q + \sin p$ where $q \in N$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$

35. Show that the sum of the two integrals $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$ is zero.

36. If $\int_0^{\pi} \frac{x\sin^3 x}{4-\cos^2 x} dx = \pi\left(1 - \frac{a\ln b}{c}\right)$ where a and b are prime and $c \in N$, find the value of $(a + b + c)$.

37. $\int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] dx$

38. $\int_{\sqrt{\frac{3a^2+b^2}{2}}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x.dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$

39. Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t + k| dt$ depending on the value of $k \in R$.

40. Evaluate the definite integral, $\int_{-1}^{1} \frac{(2x^{332} + x^{998} + 4x^{1668} \cdot \sin x^{691})}{1 + x^{666}} dx$

41. $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cdot\cos x\right)}{2x-\pi} dx$

42. **(a)** Show that $\int_0^{\infty} \frac{dx}{x^2 + 2x\cos\theta + 1} = 2\int_0^1 \frac{dx}{x^2 + 2x\cos\theta + 1}$

(b) Evaluate: $f(\theta) = \int_0^{\infty} \frac{\tan^{-1}x}{x^2 + 2x\cos\theta + 1} dx, \theta \in (0, \pi)$

43. Evaluate: $\text{Lim}_{n\to\infty} \frac{1}{n^2}\sum_{k=0}^{n-1}\left[k\int_k^{k+1}\sqrt{(x-k)(k+1-x)}\, dx\right]$

44. Show that $\int_0^{\infty} f\left(\frac{a}{x} + \frac{x}{a}\right) \cdot \frac{\ln x}{x} dx = \ln a \cdot \int_0^{\infty} f\left(\frac{a}{x} + \frac{x}{a}\right) \cdot \frac{dx}{x}$

45. Let $y = f(x)$ be a quadratic function with $f'(2) = 1$. Find the value of the integral $\int_{2-\pi}^{2+\pi} f(x) \cdot \sin\left(\frac{x-2}{2}\right) dx$.

46. Prove that $\int_0^{\pi} \frac{\ln(1-\lambda^2 \sin^2 x)}{\sin x} dx = -2(\sin^{-1}\lambda)^2$ if $0 \le \lambda \le 1$.

47. Evaluate $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)}{e^x + 1} dx$

48. Prove that $\int_0^x \left(\int_0^u f(t)dt\right) du = \int_0^x f(u)(x-u) du$

49. Prove that $\int_0^{\pi/4} \frac{x^2}{(x\sin x + \cos x)^2} dx = \frac{4-\pi}{4+\pi}$

EXERCISE–III

1. If the derivative of f(x) wrt x is $\frac{\cos x}{f(x)}$ then show that f(x) is a periodic function.

2. Find the range of the function, $f(x)=\int_{-1}^{1}\frac{\sin x\,dt}{1-2t\cos x+t^2}$.

3. A function f is defined in $[-1,1]$ as $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x};\ x\neq 0;\ f(0)=0;\ f(1/\pi)=0$. Discuss the continuity and derivability of f at $x=0$.

4. Let $f(x)=\begin{cases}-1 & \text{if } -2\le x\le 0\\ |x-1| & \text{if } 0<x\le 2\end{cases}$ and $g(x)=\int_{-2}^{x}f(t)dt$. Define g(x) as a function of x and test the continuity and differentiability of g(x) in $(-2,2)$.

5. If $\phi(x)=\cos x-\int_0^x(x-t)\phi(t)dt$. Then find the value of $\phi''(x)+\phi(x)$.

6. If $y=\frac{1}{a}\int_0^x f(t)\cdot\sin a(x-t)dt$ then prove that $\frac{d^2y}{dx^2}+a^2y=f(x)$.

7. If $y=x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at $x=e$

8. A curve C_1 is defined by: $\frac{dy}{dx}=e^x\cos x$ for $x\in[0,2\pi]$ and passes through the origin. Prove that the roots of the function $y=0$ (other than zero) occurs in the ranges $\frac{\pi}{2}<x<\pi$ and $\frac{3\pi}{2}<x<2\pi$.

9. **(a)** Let $g(x)=x^c\cdot e^{2x}$ & let $f(x)=\int_0^x e^{2t}\cdot(3t^2+1)^{1/2}dt$. For a certain value of 'c', the limit of $\frac{f'(x)}{g'(x)}$ as $x\to\infty$ is finite and non zero. Determine the value of ' c ' and the limit.

(b) Find the constants 'a' $(a>0)$ and 'b' such that, $\mathrm{Lim}_{x\to 0}\frac{\int_0^x\frac{t^2dt}{\sqrt{a+t}}}{bx-\sin x}=1$.

10. Evaluate: $\mathrm{Lim}_{x\to+\infty}\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}}\frac{3t^4+1}{(t-3)(t^2+3)}dt$

11. Determine a pair of number a and b for which $\int_0^1\frac{ax+b}{(x^2+3x+2)^2}dx=\frac{5}{2}$.

12. If $\int_0^\infty\frac{\ell nt}{x^2+t^2}dt=\frac{\pi\ell n2}{4}$ $(x>0)$ then show that there can be two integral values of 'x' satisfying this equation.

13. Evaluate: $\mathrm{Lim}_{a\to\infty}\frac{\int_0^a\sin^4x\,dx}{a}$

14. Prove that: (a) $I_{m,n}=\int_0^1x^m\cdot(1-x)^n dx=\frac{m!n!}{(m+n+1)!}$ $m,n\in N$.

(b) $I_{m,n}=\int_0^1x^m\cdot(\ln x)^n dx=(-1)^n\frac{n!}{(m+1)^{n+1}}m,n\in N$.

15. Find a positive real valued continuously differentiable functions f on the real line such that for all x

$$f^2(x)=\int_0^x((f(t))^2+(f'(t))^2)dt+e^2$$

16. Let f(x) be a continuously differentiable function then prove that,

$\int_1^x [t]f'(t)dt = [x].f(x) - \sum_{k=1}^{[x]} f(k)$ where ['] denotes the greatest integer function and $x > 1$.

17. Let $F(x) = \int_{-1}^{x} \sqrt{4+t^2}\,dt$ and $G(x) = \int_x^1 \sqrt{4+t^2}\,dt$ then compute the value of $(FG)'(0)$ where dash denotes the derivative.

18. Show that for a continuously thrice differentiable function f(x)

$$f(x) - f(0) = xf'(0) + \frac{f''(0)\cdot x^2}{2} + \frac{1}{2}\int_0^x f'''(t)(x-t)^2 dt$$

19. Evaluate: **(a)** $\text{Lim}_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$;

(b) $\text{Lim}_{n\to\infty} \frac{1}{n}\left[\frac{1}{n+1} + \frac{2}{n+2} + \cdots . + \frac{3n}{4n}\right]$

20. **(a)** $\text{Lim}_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n}$

(b) For positive integers n, let $A_n = \frac{1}{n}\{(n+1) + (n+2) + \cdots \dots (n+n)\}$,

$B_n = \{(n+1)(n+2)\dots\dots(n+n)\}^{1/n}$. If $\text{Lim}_{n\to\infty} \frac{A_n}{B_n} = \frac{ae}{b}$ where $a, b \in N$ and relatively prime find the value of $(a+b)$.

21. Let f be an injective function such that $f(x)f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x & y with $f'(0) = 0$ & $f'(1) = 2 \neq f(0)$. Find f(x) & show that, $3\int f(x)dx - x(f(x)+2)$ is a constant.

22. Let $I = \int_{1/2}^{2} \frac{\ln t}{1+t^n} dt$, find the sign of the integral for different values of $n \in N \cup \{0\}$.

23. Let f be a function such that $|f(u) - f(v)| \leq |u - v|$ for all real u & v in an interval [a, b] . Then:

(i) Prove that f is continuous at each point of [a, b].

(ii) Assume that f is integrable on [a, b]. Prove that, $\left|\int_a^b f(x)dx - (b-a)f(c)\right| \leq \frac{(b-a)^2}{2}$, where $a \leq c \leq b$

24. Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+m+1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{k+n+1}$

25. Let f and g be function that are differentiable for all real numbers x and that have the following properties:

(i) $f'(x) = f(x) - g(x)$; **(ii)** $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$; **(iv)** $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find f(x) and g(x).

26. If $f(x) = x + \int_0^1 (xy^2 + x^2y)f(y)dy$ where x and y are independent variable. Find f(x).

27. Prove that $\sin x + \sin 3x + \sin 5x + \cdots . + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x}, k \in N$ and hence prove that , $\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \ldots + \frac{1}{2k-1}$.

28. If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$, then show that $U_1, U_2, U_3, \ldots, U_n$ constitute an AP. Hence or otherwise find the value of U_n.

29. Suppose $f: R \to R^+$ be a differentiable function and satisfies $3f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$ with $f(1) = 6$. If $U = \text{Lim}_{n\to\infty} n\left(f\left(1+\frac{1}{n}\right) - f(1)\right)$ and $V = \int_0^3 f(x)dx$ then find

(a) the range of $f(x)$; (b) the value of U; (c) the value of the product UV

30. Prove the inequalities:

(a) $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$

(b) $2e^{-1/4} < \int_0^2 e^{x^2-x} dx < 2e^2$.

(c) $\frac{1}{3} < \int_0^1 x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$

(d) $\frac{1}{2} \le \int_0^2 \frac{dx}{2+x^2} \le \frac{5}{6}$

EXERCISE–IV

1. $\int \frac{dx}{\cos x+\sqrt{3}\sin x}$ is equal to **(AIEEE 2007)**

(A) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$ (B) $\frac{1}{2}\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$

(C) $\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$ (D) $\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$

2. Let $F(x)=f(x)+f\left(\frac{1}{x}\right)$, where $f(x)=\int_1^x \frac{\log t}{1+t}dt$. Then $F(e)$ is equal to : **(AIEEE 2007)**

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) 2

3. The value of $\sqrt{2}\int \frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is : **(AIEEE 2008)**

(A) $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+C$ (B) $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(C) $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$ (D) $x-\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

4. $\int_0^{\pi}[\cot x]dx$, where [.] denotes the greatest integer function, is equal to : **(AIEEE 2009)**

(A) $\frac{\pi}{2}$ (B) 1 (C) -1 (D) $-\frac{\pi}{2}$

5. Let $p(x)$ be a function defined on R such that $p'(x)=p'(1-x)$, for all $x\in[0,1]$, $p(0)=1$, and $p(1)=41$. Then $\int_0^1 p(x)dx$ is equal to : **(AIEEE 2010)**

(A) 42 (B) $\sqrt{41}$ (C) 21 (D) 41

6. The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is : **(AIEEE 2011)**

(A) log2 (B) πlog2 (C) $\frac{\pi}{8}$log2 (D) $\frac{\pi}{2}$log2

7. For $x\in\left(0,\frac{5\pi}{2}\right)$, define $f(x)=\int_0^x \sqrt{t}\sin t\,dt$. Then f has : **(AIEEE 2011)**

(A) Local maximum at π and local minimum at 2π

(B) Local maximum at π and 2π

(C) Local minimum at π and 2π

(D) Local minimum at π and local maximum at 2π

8. If $g(x)=\int_0^{\pi}\cos^4 t\,dt$, then $g(x+\pi)$ equals : **(AIEEE 2012)**

(A) $\frac{g(x)}{g(\pi)}$ (B) $g(x)+g(\pi)$ (C) $g(x)-g(\pi)$ (D) $g(x)\cdot g(\pi)$

9. If $\int \frac{5\tan x}{\tan x-2}dx = x+a\ln(\sin x-2\cos x)+k$, where k is some constant, then a =

(A) -1 (B) -2 (C) 1 (D) 2 **(AIEEE 2012)**

10. If $\int f(x)dx=\Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to : **[JEE Main 2013]**

(A) $\frac{1}{3}\left[x^3\Psi(x^3)-\int x^3\Psi(x^3)dx\right]+C$ (B) $\frac{1}{3}\left[x^3\Psi(x^3)-\int x^2\Psi(x^3)dx\right]+C$

(C) $\frac{1}{3}x^3\Psi(x^3)-3\int x^3\Psi(x^3)dx+C$ (D) $\frac{1}{3}x^3\Psi(x^3)-\int x^2\Psi(x^3)dx+C$

11. **Statement-1**: The value of the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$ **[JEE Main 2013]**

Statement-2 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$.

(A) Statement-1 is false ; Statement-2 is true.

(B) Statement-1 is true ; Statement-2 is true ; Statement-2 is a **correct** explanation for Statement-1

(C) Statement-1 is true ; Statement-2 is true ; Statement- 2 is **not** a **correct** explanation for Statement-1

(D) Statement-1 is true ; Statement-2 is false

12. The integral $\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to **[JEE Main 2014]**

(A) $-xe^{x+\frac{1}{x}}+c$ (B) $(x-1)e^{x+\frac{1}{x}}+c$ (C) $xe^{x+\frac{1}{x}}+c$ (D) $(x+1)e^{x+\frac{1}{x}}+c$

13. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals: **[JEE Main 2014]**

(A) $4\sqrt{3}-4-\frac{\pi}{3}$ (B) $\pi-4$ (C) $\frac{2\pi}{3}-4-4\sqrt{3}$ (D) $4\sqrt{3}-4$

14. The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals: **[JEE Main 2015]**

(A) $-\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$ (B) $\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$

(C) $(x^4+1)^{1/4}+c$ (D) $-(x^4+1)^{1/4}+c$

15. The integral $\int_2^4 \frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to: **[JEE Main 2015]**

(A) 6 (B) 2 (C) 4 (D) 1

16. The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to: **[JEE Main 2016]**

(A) $\frac{-x^5}{(x^5+x^3+1)^2}+C$ (B) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$

(C) $\frac{x^5}{2(x^5+x^3+1)^2}+C$ (D) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$

where C is an arbitrary constant

17. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to : **[JEE Main 2016]**

(A) $\frac{18}{e^4}$ (B) $\frac{27}{e^2}$ (C) $\frac{9}{e^2}$ (D) $3\log 3-2$

18. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to: **[JEE Main 2017]**

(A) 4 (B) -1 (C) -2 (D) 2

19. Let $I_n = \int \tan^n x\,dx, (n > 1)$. $I_4 + I_6 = a\tan^5 x + b\,x^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to : **[JEE Main 2017]**

(A) $\left(\frac{1}{5}, -1\right)$ (B) $\left(-\frac{1}{5}, 0\right)$ (C) $\left(-\frac{1}{5}, 1\right)$ (D) $\left(\frac{1}{5}, 0\right)$

20. The area (in sq. units) of the region $\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is :

(A) $\frac{7}{3}$ (B) $\frac{5}{2}$ (C) $\frac{59}{12}$ (D) $\frac{3}{2}$ **[JEE Main 2017]**

21. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x} dx$ is : **[JEE Main 2018]**

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{8}$ (C) $\frac{\pi}{2}$ (D) 4π

22. The integral $\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$ is equal to: **[JEE Main 2018]**

(A) $\frac{-1}{1+\cot^3 x} + C$ (B) $\frac{1}{3(1+\tan^3 x)} + C$ (C) $\frac{-1}{3(1+\tan^3 x)} + C$ (D) $\frac{1}{1+\cot^3 x} + C$

(where C is a constant of integration)

EXERCISE–V

1. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+C$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+C$

(C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+C$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$ **[JEE 2006, 3]**

COMPREHENSION:

2. Suppose we define the definite integral using the following formula

$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$, for more accurate result for

$c \in (a,b) F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$. When $c = \frac{a+b}{2}$,

$\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$

(a) $\int_0^{\pi/2} \sin x\, dx$ is equal to

(A) $\frac{\pi}{8}(1+\sqrt{2})$ (B) $\frac{\pi}{4}(1+\sqrt{2})$ (C) $\frac{\pi}{8\sqrt{2}}$ (D) $\frac{\pi}{4\sqrt{2}}$

(b) If f(x) is a polynomial and if $\text{Lim}_{t\to a}\frac{\int_a^t f(x)dx - \frac{t-a}{2}(f(t)+f(a))}{(t-a)^3} = 0$ for all a then the degree of f(x) can atmost be

(A) 1 (B) 2 (C) 3 (D) 4

(c) If $f''(x) < 0, \forall\, x \in (a,b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which F(c) is maximum, then $f'(c)$ is equal to **[JEE 2006, 5 marks each]**

(A) $\frac{f(b)-f(a)}{b-a}$ (B) $\frac{2(f(b)-f(a))}{b-a}$ (C) $\frac{2f(b)-f(a)}{2b-a}$ (D) 0

3. Find the value of $\frac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ **[JEE 2006, 6]**

4.(a) $\text{Lim}_{x\to\frac{\pi}{4}}\frac{\int_2^{\sec^2 x} f(t)dt}{x^2-\frac{\pi^2}{16}}$ equals **[JEE 2007, 3+3+3+6]**

(A) $\frac{8}{\pi}f(2)$ (B) $\frac{2}{\pi}f(2)$ (C) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (D) 4f(2)

(b) Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(fof\, o...of)}_{f \text{ occurs n times}}(x)$. Then $\int x^{n-2} g(x)dx$ equals

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$ (B) $\frac{1}{(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$

(C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$ (D) $\frac{1}{(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$

(c) Let F(x) be an indefinite integral of $\sin^2 x$.

Statement-1: The function F(x) satisfies $F(x+\pi) = F(x)$ for all real x.

because

Statement-2: $\sin^2(x+\pi) = \sin^2 x$ for all real x.

(A) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

(C) Statement- 1 is true, statement-2 is false.

(D) Statement- 1 is false, statement-2 is true.

MATCH THE COLUMN:

(d) Match the integrals in **Column I** with the values in **Column II.**

	Column I		**Column II**
(A)	$\int_{-1}^{1} \frac{dx}{1+x^2}$	**(P)**	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B)	$\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	**(Q)**	$2\log\left(\frac{2}{3}\right)$
(C)	$\int_{2}^{3} \frac{dx}{1-x^2}$	**(R)**	$\frac{\pi}{3}$
(D)	$\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	**(S)**	$\frac{\pi}{2}$

5.(a) Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx, J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$ **[JEE 2008, 3(-1)]**

Then, for an arbitrary constant C, the value of $J - I$ equals

(A) $\frac{1}{2}\ln\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right) + C$ (B) $\frac{1}{2}\ln\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right) + C$

(C) $\frac{1}{2}\ln\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right) + C$ (D) $\frac{1}{2}\ln\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right) + C$

(b) Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$, for $n = 1,2,3, \ldots ....$ Then,

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

[JEE 2008, 4]

6. Let f be a non-negative function defined on the interval [0,1]. If **[JEE 2009, 3(-1)]**

$\int_0^x \sqrt{1-(f'(t))^2}dt = \int_0^x f(t)dt, 0 \le x \le 1$, and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

7. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx, n = 0,1,2, \ldots \ldots$, then **[JEE 2009, 4(-1)]**

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$ (C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $\|_n = I_{n+1}$

8. Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of $f(\ln 5)$ is

[JEE 2009, 4(-1)]

9. The value of $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ is **[JEE 2010]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

10. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) **[JEE 2010]**

(A) $\frac{22}{7} - \pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

11. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$.

Then which of the following statement(s) is (are) true? **[JEE 2010]**

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

12. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [−10,10] by **[JEE 2010]**

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is

13. Let f be a real-valued function defined on the interval (−1,1) such that

$e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\, dt$, for all $x \in (-1,1)$ and let f^{-1} be the inverse function of f. Then

$(f^{-1})'(2)$ is equal to

(A) 1 (B) 1/3 (C) 1/2 (D) 1/e **[JEE 2010]**

14. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}dx$ is : **[JEE 2011]**

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (C) $\frac{1}{6}\ln\frac{3}{2}$

15. Let $f: [-1,2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let

$R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x), x = -1, x = 2$, and the x-axis. Then

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$ **[JEE 2011]**

16. Let $f: [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If

$6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \ge 1$, then the value of $f(2)$ is **[JEE 2011]**

17. The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}} dx$ equals (for some arbitrary constant K) **[JEE 2012]**

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

18. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **[JEE 2012]**

(A) 0 (B) $\frac{\pi^2}{2}-4$

(C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

19. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x)<2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE Advanced 2013]

(A) $(2e-1,2e)$ (B) $(e-1,2e-1)$

(C) $\left(\frac{e-1}{2},e-1\right)$ (D) $\left(0,\frac{e-1}{2}\right)$

20. For $a\in R$ (the set of all real numbers), $a\neq-1$, **[JEE Advanced 2013]**

$$\lim_{n\to\infty}\frac{(1^a+2^a+\cdots\ldots+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]}=\frac{1}{60}\quad \text{Then } a=$$

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

21. Let $f:[a,b]\to[1,\infty)$ be a continous function and let $g: R\to R$ be defined as

$$g(x)=\begin{cases}0 & \text{if } x<a\\ \int_a^x f(t)dt & \text{if } a\le x\le b.\\ \int_a^b f(t)dt & \text{if } x>b.\end{cases}$$

[JEE Advanced 2014]

Then

(A) g(x) is continous but not differentiable at a

(B) g(x) is differentiable on R

(C) g(x) is continous but not differentiable at b

(D) g(x) is continous and differentiable at either a or b but not both

22. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is **[JEE Advanced 2014]**

23. The following integral **[JEE Advanced 2014]**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec}x)^{17}dx$$

is equal to

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}du$ (B) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17}du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17}du$ (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}du$

24. Let f: $[0,2] \to R$ be a function which is continuous on $[0,2]$ and is differentiable on $(0,2)$ with $f(0) = 1$. Let

$$F(x) = \int_0^{x^2} f(\sqrt{t})dt$$

for $x \in [0,2]$. If $F'(x) = f'(x)$ for all $x \in (0,2)$, then $F(2)$ equals **[JEE Advanced 2014]**

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

Comprehension (Q.25 to Q.26)

Given that for each $a \in (0,1)$. **[JEE Advanced 2014]**

$$\lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$$

exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0,1)$.

25. The value of $g\left(\frac{1}{2}\right)$ is:

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

26. The value of $g'\left(\frac{1}{2}\right)$ is :

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

27.

	List -I	List – II	
(P)	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	**(1)**	8
(Q)	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value is	**(2)**	2
(R)	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)}dx$ equals	**(3)**	4
(S)	$\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	**(4)**	0

[JEE Advanced 2014]

Code:

	P	**Q**	**R**	**S**
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

28. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\ ?$$ **[JEE Advanced 2015]**

(A) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

29. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) **[JEE Advanced 2015]**

(A) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

30. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are **[JEE Advanced 2015]**

(A) $m = 13, M = 24$ (B) $m = \frac{1}{4}, M = \frac{1}{2}$ (C) $m = -11, M = 0$ (D) $m = 1, M = 12$

Paragraph for question no. 31 to 32 **[JEE Advanced 2015]**

Let $F : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0, F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$.

31. The correct statement(s) is(are)

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \ne 0$ for any $x \in (1,3)$

(D) $f'(x) = 0$ for some $x \in (1,3)$

32. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is(are)

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\int_1^3 f(x)dx = -12$

33. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12 + 9x^2}{1 + x^2}\right) dx$, **[JEE Advanced 2015]**

where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1 + \alpha| - \frac{3\pi}{4}\right)$ is

34. Let $f: R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1,2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|dt$ for all $x \in [-1,2]$. If $\lim_{x\to1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE Advanced 2015]**

35. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to: **[JEE Advanced 2016]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

36. Let $f(x) = \lim_{n\to\infty} \left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\ldots\ldots\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\ldots\ldots\left(x^2+\frac{n^2}{n^2}\right)} \right)^{\frac{x}{n}}$, for all $x > 0$. Then **[JEE Advanced 2016]**

(A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$ (C) $f'(2) \leq 0$ (D) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

37. The total number of distinct $x \in [0,1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is : **[JEE Advanced 2016]**

38. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then **[JEE Advanced 2017]**

(A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

39. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)dt$, then **[JEE Advanced 2017]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

40. Let f: ú → ú be a differentiable function such that $f(0) = 0, f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$.

[JEE Advanced 2017]

If $g(x) = \int_x^{\pi/2} [(\operatorname{cosec} t)f'(t) - (\cot t)(\operatorname{cosec} t)f(t)]dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to0} g(x) =$

41. Let f: ú → (0,1) be a continuous funciton. Then, which of the following function(s) has(have) the value zero at some point in the interval (0,1) ? **[JEE Advanced 2017]**

(A) $x - \int_0^{\frac{\pi}{2}-x} f(t) \cos t\, dt$ (B) $x^9 - f(x)$

(C) $e^x - \int_0^x f(t) \sin t\, dt$ (D) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t\, dt$

42. For each positive integer n, let $y_n = \frac{1}{n}((n+1)(n+2)\ldots\ldots(n+n))^{\frac{1}{n}}$. For $x \in \mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{x\to\infty} y_n = L$, then the value of $[L]$ is____________.

[JEE Advanced 2018]

43. The value of the integral $\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}} dx$ is **[JEE Advanced 2018]**

44. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27\, I^2$ equals **[JEE Advanced 2019]**

45. For $a \in R, |a| > 1$, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+\cdots+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+\cdots+\frac{1}{(an+n)^2}\right)}\right) = 54$

Then the possible value(s) of a is/are : **[JEE Advanced 2019]**

(A) -9 (B) 8 (C) 7 (D) -6

46. The value of the integral $\int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta}+\sqrt{\sin\theta})^5}d\theta$ equals________. **[JEE Advanced 2019]**

47. Which of the following inequalities is/are TRUE? **[JEE Advanced 2020]**

(A) $\int_0^1 x\cos x\, dx \geq \frac{3}{8}$ (B) $\int_0^1 x\sin x\, dx \geq \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\, dx \geq \frac{1}{2}$ (D) $\int_0^1 x^2\sin x\, dx \geq \frac{2}{9}$

48. Let $f: R \to R$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F: [0, \pi] \to R$ is defined by $F(x) = \int_0^x f(t)dt$, and if

$\int_0^{\pi}(f'(x) + F(x))\cos x\, dx = 2$, then the value of $f(0)$ is **[JEE Advanced 2020]**

49. Let $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R$ be a continuous function such

that $f(0) = 1$ and $\int_0^{\frac{\pi}{3}} f(t)dt = 0$

Then which of the following statements is (are) True? **[JEE Advanced 2021]**

(A) The equation $f(x) - 3\cos 3x = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(B) The equation $f(x) - 3\sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(C) $\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}} = -1$

(D) $\lim_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2} = -1$

Question Stem for Question Nos. 50 to 51

Question Stem **[JEE Advanced 2021]**

Let $g_i: \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbf{R}$, $i = 1,2$, and $f: \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbf{R}$ be functions such that

$$g_1(x) = 1, g_2(x) = |4x - \pi| \text{ and } f(x) = \sin^2 x, \text{ for all } x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$$

Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x)\cdot g_i(x)dx,\ i = 1,2$

50. The value of $\frac{16\, S_1}{\pi}$ is

51. The value of $\frac{48\, S_2}{\pi^2}$ is

52. For any real number x, let [x] denote the largest integer less than or equal to x. If $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of 9I is_____. **[JEE Advanced 2021]**

ANSWER KEY

EXERCISE–I

Q.1 $\ln\left(\frac{1+\sqrt{1+3\cos^2 2\theta}}{\cos 2\theta}\right)+C$

Q.2 $C-\frac{x+1}{x^5+x+1}$ or $C+\frac{x^5}{x^5+x+1}$

Q.3 $\frac{1}{4}\ln(\cos x+\sin x)+\frac{x}{2}+\frac{1}{8}(\sin 2x+\cos 2x)+C$

Q.4 $\frac{1}{8}\ln(\sec 8x)+C$

Q.5 $\frac{1}{2}\left[\ln\left(\frac{1+x}{1-x}\right)\cdot\ln\left(\ln\frac{1+x}{1-x}\right)-\ln\left(\frac{1+x}{1-x}\right)\right]+C$

Q.6 $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$

Q.7 $\frac{1}{2}(\sin 2\theta)\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln(\sec 2\theta)+C$

Q.8 $\frac{1}{a^2+b^2}\left(x+\tan^{-1}\left(\frac{a^2\tan x}{b^2}\right)\right)+C$

Q.9 $2\ln\frac{t}{2t+1}+\frac{1}{2t+1}+C$ when $t=x+\sqrt{x^2+x}$

Q.10 $\frac{1}{2}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$

Q.11 $\cos a\cdot\arccos\left(\frac{\cos x}{\cos a}\right)-\sin a\cdot\ln\left(\sin x+\sqrt{\sin^2 x-\sin^2 a}\right)+C$

Q.12 $-\frac{3(1+4\tan^2 x)}{8(\tan x)^{8/3}}+C$

Q.13 $\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+\frac{1}{4}\sec^2\frac{x}{2}+\tan\frac{x}{2}+$

Q.14 $(a+x)\arctan\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

Q.15 $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3\ln\left(1+\frac{1}{x^2}\right)\right]$

Q.16 $\ln\left(\frac{xe^x}{1+xe^x}\right)+\frac{1}{1+xe^x}+C$

Q.17 3

Q.18 - $\ln(1-x^4)+C$

Q.19 $6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\ln(1+t^2)-\tan^{-1}t\right]+C$ where $t=x^{1/6}$

Q.20 $\frac{4}{\sqrt{\cos\frac{x}{2}}}+2\tan^{-1}\sqrt{\cos\frac{x}{2}}-\ln\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}}+C$

Q.21 $C-\ln(1+(x+1)e^{-x})-\frac{1}{1+(x+1)e^{-x}}$

Q.22 $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)+C$

Q.23 $\frac{1}{24}\ln\frac{(4+3\sin x+3\cos x)}{(4-3\sin x-3\cos x)}+C$

Q.24 $\frac{1}{2}\left[\sin x-\cos x-\frac{1}{\sqrt{2}}\ln\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right]+C$

Q.25 $\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}+\arctan(\sin x+\cos x)+C$

Q.26 $\left[-\ell n(\sec x) - \frac{1}{2}\ell n(\sec 2x) + \frac{1}{3}\ell n(\sec 3x)\right] + C$

Q.27 $C - \frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}$

Q.28 $ln\left|\frac{x\sin x+\cos x}{x\cos x-\sin x}\right|$

Q.29 $2x - 3\arctan\left(\tan\frac{x}{2}+1\right) + C$

Q.30 $\frac{x^4}{4} + x^3 - x^2 + 5x + \frac{1}{2}ln(x^2+1) + 3\tan^{-1}x + C$

Q.31 $C - \frac{\sqrt{1+t^2}}{2} - \frac{1}{4}ln\frac{\sqrt{t^2+1}-1}{\sqrt{t^2+1}+1}$, where $t = \cot^2 x$

Q.32 $C - \frac{x}{(x^2-1)^2}$

Q.33 $C\text{-} -e^{\cos x}(x + \operatorname{cosec} x)$

Q.34 $\sin^{-1}\left(\frac{ax^2+b}{cx}\right) + k$

Q.35 $e^x\sqrt{\frac{1+x}{1-x}} + c$

Q.36 $\frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + C$ **Q. 37** $\operatorname{arcsec}x - \frac{\ln x}{\sqrt{x^2-1}} + C$

Q.38 $\sqrt{3}\ln\frac{t-\sqrt{3}}{t+\sqrt{3}} + 2\tan^{-1}(t) + C$ where $t = \sqrt{\frac{2-\sin x}{2+\sin x}}$

Q.39 $\tan^{-1}\left(\frac{\sqrt{2\sin 2x}}{\sin x+\cos x}\right) + C$

Q.40 $4\ln x + \frac{7}{x} + 6\tan^{-1}(x) + \frac{6x}{1+x^2} + C$

Q.41 $\frac{-2}{\alpha-\beta}\cdot\sqrt{\frac{x-\beta}{x-\alpha}} + C$

Q.42 $\frac{2}{3}\tan^{-1}(\sin x+\cos x) + \frac{1}{3\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin x+\cos x}{\sqrt{2}-\sin x-\cos x}\right| + C$

Q.43 $\frac{1}{\sqrt{2}}ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right) - \frac{1}{2}ln\left(\frac{1-t}{1+t}\right)$ where $t = \cos\theta$ and $\theta = \operatorname{cosec}^{-1}(\cot x)$

Q.44 $\frac{1}{2}\left(\operatorname{cosec}\frac{\alpha}{2}\right)\cdot\tan^{-1}\left(\left(\frac{x^2-1}{2x}\right)\operatorname{cosec}\frac{\alpha}{2}\right)$

Q.45 $\frac{1}{2-\tan x} - \frac{1}{5}x + \frac{2}{5}\ln\sec x - \frac{2}{5}\ln|2-\tan x| + C$

Q.46 $2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + \frac{2\sqrt{x}}{\sqrt{x^2+x+1}} + C$

Q.47 $\ln|\tan^{-1}(\sec x + \cos x)| + C$

Q.48 $\ln\left|\frac{\sqrt{2xe^{\sin x}+1}-1}{\sqrt{2xe^{\sin x}+1}+1}\right| + C$

Q.49 $\frac{12}{7}\left(1+x^{1/4}\right)^{7/3} - 3\left(1+x^{1/4}\right)^{4/3} + C$

Q.50 (A) S; (B) P ; (C) Q ; (D) R

EXERCISE–II

Q.1 $\frac{\pi^2}{8}-\frac{\pi}{4}(1+\ln 2)+\frac{1}{2}$ **Q.3** **(a)** $I_n = e - nI_{n-1}, I_3 = 6-2e$; **(b)** $n=3$

Q.4 $\frac{\pi}{2}-1$ **Q.6** 125 **Q.7** $\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{4}$ **Q.8** 2525

Q.9 5250 **Q.10** 4 **Q.11** $ln\ 2$ **Q.12** $4\sqrt{2}-4ln(\sqrt{2}+1)$

Q.13 $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$ **Q.14** $\frac{\pi}{8}\ln 2$ **Q.15** $\frac{\pi^2}{6\sqrt{3}}$ **Q.16** $\frac{\pi}{8}\ln 2$

Q.17 2008 Q.18 $2\sqrt{6}$ **Q.19** 153 **Q.20** $\frac{\pi}{3}$

Q.21 $\frac{\pi(a+b)}{2\sqrt{2}}$ **Q.22** $\frac{\pi}{8}(1-\ln 4)$ **Q.23** 2021 **Q.24** $\frac{\pi\sqrt{3}}{3}$

Q.25 $\frac{(a\pi+2b)\pi}{3\sqrt{3}}$ **Q.26** $\frac{\pi(\pi+3)}{2}$ **Q.27** $\frac{5\pi}{27}$ **Q.28** $\frac{16\pi}{3}-2\sqrt{3}$

Q.29 $\frac{2\pi}{\sqrt{3}}$ **Q.30** $\tan^{-1}(a)\cdot \ln\sqrt{1+a^2}$ **Q.31** $\frac{1}{2}\left[\frac{\pi}{6}+\ln 3-\ln 2\right]$

Q.32 $-\frac{2\pi^2}{3}\ln 2$ **Q.33** 0 **Q.36** 10 **Q.37** $\frac{3\pi^2}{16}$

Q.38 $\frac{\pi}{12}$ **Q.39** real & distinct $\forall k \in R$ **Q.40** $\frac{\pi+4}{666}$

Q.41 $\frac{8}{\pi}$ **Q.42** (b) $\frac{\pi\theta}{4\sin\theta}$ **Q.43** $\frac{\pi}{16}$ **Q.45** $I = 8$ as $\int_0^{\pi/2} y\sin y\,dy = 1$

Q.47 $\frac{\pi}{2\sqrt{3}}$

EXERCISE–III

Q.2 $\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$ **Q.3** cont. & der. at $x=0$

Q.4 $g(x)$ is cont. in $(-2,2)$; $g(x)$ is der. at $x=1$ & not der. at $x=0$. Note that ;

$$g(x) = \begin{bmatrix} -(x+2) & \text{for} & -2\le x\le 0 \\ -2+x-\dfrac{x^2}{2} & \text{for} & 0<x<1 \\ \dfrac{x^2}{2}-x-1 & \text{for} & 1\le x\le 2 \end{bmatrix}$$

Q.5 - cosx **Q.7** $1+e$

Q.9 (a) $c=1$ and $\text{Limit}_{x\to\infty}$ will be $\frac{\sqrt{3}}{2}$ (b) $a=4$ and $b=1$

Q.10 13.5 **Q.11** $a=15, b=\frac{45}{2}$ **Q.12** $x=2$ or 4

Q.13 $\frac{3}{8}$ **Q.15** $f(x)=e^{x+1}$ **Q.17** 0

Q.19 (a) $2e^{(1/2)(\pi-4)}$; (b) $3-\ln 4$ **Q.20** (a) $\frac{1}{e}$; (b) 11

Q.21 $f(x)=1+x^2$ **Q.22** for $n=1, l>0, n=2, l=0, n\ge 3, l<0$

Q.25 $f(x)=3+2e^{2x}; g(x)=3-2e^{2x}$ **Q.26** $f(x)=x+\frac{61}{119}x+\frac{80}{119}x^2$

Q.28 $U_n=\frac{n\pi}{2}$ **Q.29** (a) $(0,\infty)$; (b) $6\ ln\ 2$; (c) 126

EXERCISE–IV

1.	A	**2.**	A	**3.**	C	**4.**	D	**5.**	C	**6.**	B	**7.**	A
8.	B	**9.**	D	**10.**	D	**11.**	A	**12.**	C	**13.**	A	**14.**	A
15.	D	**16.**	D	**17.**	B	**18.**	D	**19.**	B	**20.**	B	**21.**	A
22.	C												

EXERCISE–V

1. D **2.** **(a)** A, **(b)** A, **(c)** A **3.** 5051

4. **(a)** A; **(b)** A; **(c)** D; **(d)** **(A)** S; **(B)** S; **(C)** P; **(D)** R **5.** **(a)** C; **(b)** A, D **6.** C

7.	A, B, C	**8.**	0	**9.**	B	**10.**	A	**11.**	B, C	**12.**	4	**13.**	B
14.	A	**15.**	C	**16.**	Bonus	**17.**	C	**18.**	B	**19.**	D	**20.**	B
21.	A, C	**22.**	2	**23.**	A	**24.**	B	**25.**	A	**26.**	D	**27.**	D
28.	A, C	**29.**	A, B	**30.**	D	**31.**	A, B, C	**32.**	C, D	**33.**	9	**34.**	7
35.	A	**36.**	B,C	**37.**	1	**38.**	B, C	**39.**	Bonus	40.	2	**41.**	AB
42.	1	**43.**	2	**44.**	4.00	**45.**	(A, B) 46. 0.50 47. ABD					**48.**	4.00
49.	A,B,C	**50.**	2.00	**51.**	1.50	**52.**	182.00						