

(MATHEMATICS)

MATRICES

1. INTRODUCTION :

A rectangular array of mn numbers (which may be real or complex) in the form of ' m ' horizontal lines (called rows) and ' n ' vertical lines (called columns), is called a matrix of order m by n , written as $m \times n$ matrix.

Such an array is enclosed by $[]$ or $()$. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The number a_{11}, a_{12}, \dots etc are known as the elements of the matrix A , a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{\text{th}}$ element of the matrix $A = [a_{ij}]$.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements :

$$a_{11} = 1, a_{12} = 2, a_{13} = 3, a_{21} = 0, a_{22} = -1, a_{23} = 9.$$

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \dots \dots \dots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : ($A = \mathbf{0}_{m \times n}$) An $m \times n$ matrix whose all entries are zero.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a } 3 \times 2 \text{ null matrix \& } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is } 3 \times 3 \text{ null matrix}$$

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$

$$\text{e.g. } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(e) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$ e.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : If number of rows = number of columns \Rightarrow matrix is a square matrix.
If number of rows = number of columns = n then, matrix is of the order ' n '.

Note : The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

3. TRACE OF MATRIX :

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A i.e. $(\text{tr } (A))$. Thus, if $A = [a_{ij}]_{n \times n}$, then $\text{tr } (A) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$

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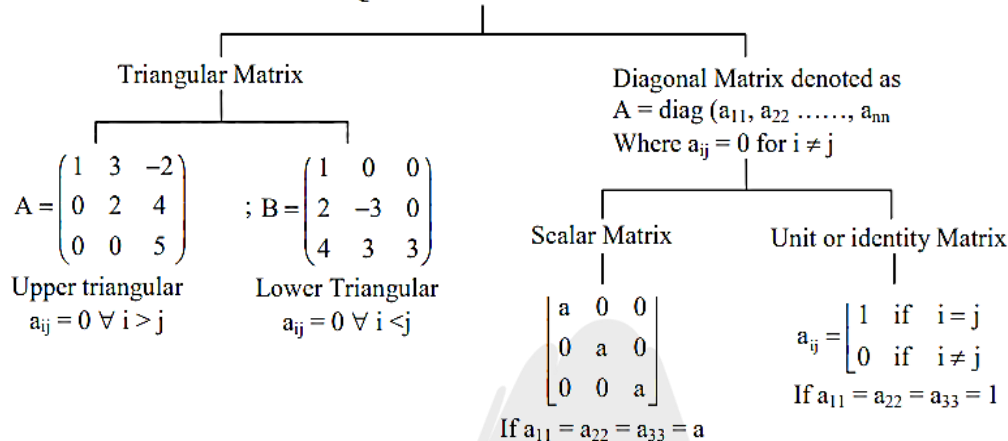
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Properties of trace of a matrix :

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar then

(i) $\text{tr}(\lambda A) = \lambda \text{tr}(A)$ (ii) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ (iii) $\text{tr}(AB) = \text{tr}(BA)$

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4. Note :

- (i) Minimum number of zeros in triangular matrix of order $n = n(n - 1)/2$.
 (ii) Minimum number of zero in a diagonal matrix of order $n = n(n - 1)$.

5. EQUALITY OF MATRICES :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if

- (a) both have the same order. (b) $a_{ij} = b_{ij}$ for each pair of i & j .

Illustration 1: Find the value of x, y, z and w which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$$

Solution : As the given matrices are equal so their corresponding elements are equal.

$$\begin{aligned} x+3 &= -x-1 & \Rightarrow 2x &= -4 \\ \therefore x &= -2 & \Rightarrow 2y+x &= 0 & \Rightarrow 2y-2 &= 0 \text{ [from (i)]} \\ \Rightarrow y &= 1 & \Rightarrow z-1 &= 3 & \Rightarrow z &= 4 \\ 4w-8 &= 2w & \Rightarrow 2w &= 8 & \therefore w &= 4 \end{aligned}$$

Ans.

Do yourself-1:

- (i) Find 2×3 matrix $[a_{ij}]_{2 \times 3}$, where $a_{ij} = i + 2j$
 (ii) Find the minimum number of zeroes in a triangular matrix of order 4.
 (iii) Find minimum number of zeros in a diagonal matrix of order 6.
 (iv) If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the value of x, y, a and b .

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6. ALGEBRA OF MATRICES :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) Addition of matrices is commutative : i.e. $A + B = B + A$

(b) Matrix addition is associative : $(A + B) + C = A + (B + C)$

(c) Additive inverse : If $A + B = \mathbf{O} = B + A$, then B is called additive inverse of A.

(d) Existence of additive identity : Let $A = [a_i]$ be an $m \times n$ matrix and \mathbf{O} be an $m \times n$ zero matrix, then $A + \mathbf{O} = \mathbf{O} + A = A$. In other words, \mathbf{O} is the additive identity for matrix addition.

(e) Cancellation laws : hold good in case of addition of matrices. If A, B, C are matrices of the same order, then $A + B = A + C \Rightarrow B = C$ (left cancellation law)

and $B + A = C + A \Rightarrow B = C$ (right-cancellation law)

Note: The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Illustration 2 : If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and $A + B - D = \mathbf{O}$ (zero matrix), then D matrix will be

Solution : Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow -a = 0 \Rightarrow a = 0, 1 - b = 0 \Rightarrow b = 1$$

$$3 - c = 0 \Rightarrow c = 3, 7 - d = 0 \Rightarrow d = 7,$$

$$5 - e = 0 \Rightarrow e = 5, 6 - f = 0 \Rightarrow f = 6 \quad \therefore D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix} \quad \text{Ans. (C)}$$

Do yourself -2 :

(i) If $A = \begin{bmatrix} 2 & 3 & 9 \\ 8 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 & 2 \\ 6 & 4 & 8 \end{bmatrix}$, then find a matrix C such that $A - B + C = \mathbf{O}$ and

also find the order of the matrix C.

(ii) If $A = \begin{bmatrix} 8 & 9 \\ 7/2 & 8 \\ 1 & -1 \end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive

inverse will be the matrix itself.

7. MULTIPLICATION OF A MATRIX BY A SCALAR :

$$\text{If } A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}; \quad kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$$

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Properties of scalar multiplication:

(a) If A and B are two matrices of the same order and 'k' be a scalar then $k(A + B) = kA + kB$.

(b) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1 + k_2)A = k_1 A + k_2 A$.

(c) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1 k_2)A = k_1(k_2 A) = k_2(k_1 A)$

8. MULTIPLICATION OF MATRICES (Row by Column) :

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$, then the matrix multiplication AB is possible if and only if $n = p$ and matrices are said to be conformable for multiplication. In the product AB, A is called pre-factor and B is called post factor.

\Rightarrow AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is $m \times p$ & $(AB)_{ij} = \sum_{r=1}^n a_{ir} b_{rj}$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3 \times 4}$ Then order of AB is 2×4 .

$$(AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{r=1}^3 a_{1r}b_{r1}$$

$$(AB)_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{r=1}^3 a_{2r}b_{r3}$$

$$\text{In general } a_{1j}b_{1j} + a_{12}b_{2j} + a_{13}b_{3j} = \sum_{r=1}^3 a_{ir}b_{rj}$$

Illustration 3 : If $\left[\begin{array}{ccc|c} 2 & 3 & 1 & x \\ 1 & 2 & 0 & -1 \\ 0 & 4 & 2 & -1 \\ 0 & 3 & 2 & -1 \end{array} \right] = 0$, then the value of x is :-

(A) -1

(B) 0

(C) 1

(D) 2

Solution : The LHS of the equation

$$= \begin{bmatrix} 2 & 4x+9 & 2x+5 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = [2x + 4x + 9 - 2x - 5] = 4x + 4$$

$$\text{Thus } 4x + 4 = 0 \Rightarrow x = -1$$

Ans. (A)

Illustration 4 : If A, B are two matrices such that $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$, then find AB.

Solution : Given $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (i) & $A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$ (ii)

Adding (i) & (ii)

$$2A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

Subtracting (ii) from (i)

$$2B = \begin{bmatrix} -2 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\text{Now } AB = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{Ans.}$$

9. PROPERTIES OF MATRIX MULTIPLICATION:

(a) Matrix multiplication is not commutative: i.e. $AB \neq BA$

Here both AB & BA exist and also they are of the same type but $AB \neq BA$.

Example:

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ \& } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ then } AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow AB \neq BA \text{ (in general)}$$

(b) $AB = \mathbf{O} \Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$ (in general)

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ \& } B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note:

If A and B are two non-zero matrices such that $AB = \mathbf{O}$ then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = -BA$ then A and B are said to anticommute

(c) Matrix Multiplication Is Associative:

If A, B & C are conformable for the product AB & BC , then $(AB)C = A(BC)$

(d) Distributive:

$$\begin{aligned} A(B + C) &= AB + AC \\ (A + B)C &= AC + BC \end{aligned} \quad \text{Provided } A, B \text{ \& } C \text{ are conformable for respective products}$$

Illustration 5 : Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ & $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$ be the matrices

then, prove that in matrix multiplication cancellation law does not hold.

Solution : We have to show that $AB = AC$; though B is not equal to C .

$$\text{We have } AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3 \times 4}$$

$$\text{Now, } AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3 \times 4}$$

Here, $AB = AC$ though B is not equal to C . Thus cancellation law does not hold in general.

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Do yourself -3 :

- (i) If $A = \begin{bmatrix} 2 & 9 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 6 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 9 & -7 \\ -2 & 4 \end{bmatrix}$, then show that $A(B + C) = AB + AC$.
- (ii) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, then prove that $(A - B)^2 \neq A^2 - 2AB + B^2$.
- (iii) Find the value of x: $2 \begin{bmatrix} 3 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix} + x \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ -8 & -14 & -2 \end{bmatrix}$

10. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX:

For a square matrix A, $A^n = \underbrace{A \cdot A \cdot A \dots A}_{\text{upto n times}}$, A, where $n \in \mathbb{N}$

Note:

- (i) $A^m \cdot A^n = A^{m+n}$
- (ii) $(A^m)^n = A^{mn}$, where $m, n \in \mathbb{N}$
- (iii) If A and B are square matrices of same order and $AB = BA$ then
- $$(A + B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \dots + {}^nC_n B^n$$

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in \mathbb{N}$.

Do yourself -4 :

- (i) If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ where n is positive integer
- (ii) If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i = \sqrt{-1}$ and $x \in \mathbb{N}$, then A^{4x} equals-

(A) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

1. SPECIAL SQUARE MATRICES:

- (a) **Idempotent Matrix:** A square matrix is idempotent provided $A^2 = A$. For idempotent matrix note the following:
- (i) $A^n = A \forall n \geq 2, n \in \mathbb{N}$. (ii) Determinant value of idempotent matrix is either 0 or 1
- (b) **Periodic Matrix:** A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is periodic matrix. The period of the matrix is the least value of K for which this holds true.
- Note that period of an idempotent matrix is 1 .
- (c) **Nilpotent Matrix:** A square matrix of the order ' n ' is said to be nilpotent matrix of order m, $m \in \mathbb{N}$, if $A^m = \mathbf{O}$ & $A^{m-1} \neq \mathbf{O}$.
- (d) **Involutory Matrix:** If $A^2 = I$, the matrix is said to be an involutory matrix. i.e. square roots of identity matrix is involutory matrix.

Note: The determinant value of involutory matrix is 1 or -1 .

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Illustration 6: Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution : $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2 \cdot 2 + (-2) \cdot (-1) + (-4) \cdot 1 & 2(-2) + (-2) \cdot 3 + (-4) \cdot (-2) & 2 \cdot (-4) + (-2) \cdot 4 + (-4) \cdot (-3) \\ (-1) \cdot 2 + 3 \cdot (-1) + 4 \cdot 1 & (-1) \cdot (-2) + 3 \cdot 3 + 4 \cdot (-2) & (-1) \cdot (-4) + 3 \cdot 4 + 4 \cdot (-3) \\ 1 \cdot 2 + (-2) \cdot (-1) + (-3) \cdot 1 & 1 \cdot (-2) + (-2) \cdot 3 + (-3) \cdot (-2) & 1 \cdot (-4) + (-2) \cdot 4 + (-3) \cdot (-3) \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence the matrix A is idempotent.

Illustration 7: Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

$$\therefore A^3 = \mathbf{0} \text{ i.e., } A^k = \mathbf{0}$$

Here $k = 3$

Hence A is nilpotent of order 3.

Illustration 8: Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Solution : $A^2 = A \cdot A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad \text{Hence the given matrix A is involutory.}$$

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Illustration 9: Show that a square matrix A is involutory, iff $(I - A)(I + A) = \mathbf{0}$

Solution : Let A be involutory

Then $A^2 = I$

$$(I - A)(I + A) = I^2 + IA - AI - A^2 = I + A - A - A^2 = I - A^2 = \mathbf{0}$$

Conversely, let $(I - A)(I + A) = \mathbf{0}$

$$\Rightarrow I^2 + IA - AI - A^2 = \mathbf{0} \Rightarrow I + A - A - A^2 = \mathbf{0}$$

$$\Rightarrow I - A^2 = \mathbf{0} \Rightarrow A \text{ is involutory}$$

Do yourself -5 :

(i) The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(A) Idempotent matrix

(B) Involutory matrix

(C) Nilpotent matrix

(D) Periodic matrix

(ii) If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x

12. THE TRANSPOSE OF A MATRIX: (Changing rows & columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ji}]$ for $1 \leq i \leq m$ & $1 \leq j \leq n$ of order $n \times m$ Properties of transpose:

If A^T & B^T denote the transpose of A and B,

(a) $(A + B)^T = A^T + B^T$; note that A & B have the same order.

(b) $(AB)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

Note: In general: $(A_1 \cdot A_2, \dots, A_n)^T = A_n^T \dots A_2^T \cdot A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

Illustration 10: If A and B are matrices of order $m \times n$ and $n \times m$ respectively, then order of matrix

$B^T(A^T)^T$ is-

(A) $m \times n$

(B) $m \times m$

(C) $n \times n$

(D) Not defined

Solution : Order of B is $n \times m$ so order of B^T will be $m \times n$

Now $(A^T)^T = A$ & its order is $m \times n$. For the multiplication $B^T(A^T)^T$

Number of columns in prefactor \neq Number of rows in post factor.

Hence this multiplication is not defined.

Ans.(D)

13. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $AA^T = I$ Note:

(i) The determinant value of orthogonal matrix is either 1 or -1 .

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1 b_1 + a_2 b_2 + a_3 b_3 & a_1 c_1 + a_2 c_2 + a_3 c_3 \\ b_1 a_1 + b_2 a_2 + b_3 a_3 & b_1^2 + b_2^2 + b_3^2 & b_1 c_1 + b_2 c_2 + b_3 c_3 \\ c_1 a_1 + c_2 a_2 + c_3 a_3 & c_1 b_1 + c_2 b_2 + c_3 b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

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If $AA^T = I$, then

$$\sum_{i=1}^3 a_i^2 = \sum_{i=1}^3 b_i^2 = \sum_{i=1}^3 c_i^2 = 1 \text{ and } \sum_{i=1}^3 a_i b_i = \sum_{i=1}^3 b_i c_i = \sum_{i=1}^3 c_i a_i = 0$$

Illustration 11: Determine the values of α, β, γ , when is $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ orthogonal.

Solution : Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ $\therefore A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

But given A is orthogonal. $\therefore AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$4\beta^2 + \gamma^2 = 1 \quad \dots\dots\dots(i)$$

$$2\beta^2 - \gamma^2 = 0 \quad \dots\dots\dots(ii)$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \dots\dots\dots(iii)$$

From (i) and (ii), $6\beta^2 = 1$ $\therefore \beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm \frac{1}{\sqrt{2}}, \beta = \pm \frac{1}{\sqrt{6}}$ and $\gamma = \pm \frac{1}{\sqrt{3}}$ **Ans.**

Do yourself - 6 :

(i) If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$, then show that $(AB)^T = B^T \cdot A^T$.

(ii) If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

(iii) If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & 1/2 & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

(iv) Show that the matrix $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ is an orthogonal matrix.

14. SYMMETRIC & SKEW SYMMETRIC MATRIX:

(a) **Symmetric matrix:**

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji} \forall i \& j$ (conjugate elements are equal).

Hence for symmetric matrix $A = A^T$.

Note: Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) **Skew symmetric matrix:**

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji} \forall i \& j$ (the pair of conjugate elements are additive inverse of each other). For a skew symmetric matrix $A = -A^T$.

Note:

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \forall i$. Thus the diagonal elements

of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

(c) **Properties of symmetric & skew symmetric matrix :**

(i) A is symmetric if $A^T = A$ & A is skew symmetric if $A^T = -A$

(ii) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(iii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iv) If A & B are symmetric matrices then,

(1) $AB + BA$ is a symmetric matrix (2) $AB - BA$ is a skew symmetric matrix.

(v) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}} \quad \text{and} \quad A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Illustration 12 : If A is symmetric as well as skew symmetric matrix, then A is -

(A) diagonal matrix

(B) null matrix

(C) triangular matrix

(D) None of these

Solution :

Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

for $i = j$, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

for $i \neq j$, $a_{ij} = -a_{ji}$ [$\because A$ is skew symmetric] & $a_{ij} = a_{ji}$ [$\because A$ is symmetric]

$\therefore a_{ij} = 0$ for all $i \neq j$

so, $a_{ij} = 0$ for all 'i' and 'j' i.e. A is null matrix.

Ans. (B)

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Do yourself -7 :

- (i) If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x .
- (ii) Express matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.

15. ADJOINT OF A SQUARE MATRIX:

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A , denoted by $\text{adj } A$, is defined as the transpose of the cofactor matrix.

$$\text{Then, } \text{adj } A = [C_{ij}]^T \Rightarrow \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Theorem: $A(\text{adj } A) = (\text{adj } A) \cdot A = |A|I_n$.

Proof :

$$A \cdot (\text{adj } A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A \cdot (\text{Adj } A) = |A|I$$

(whatever may be the value only $|A|$ will come out as a common element) If $|A| \neq 0$, then

$$\frac{A \cdot (\text{adj } A)}{|A|} = I = \text{unit matrix of the same order as that of } A$$

Properties of adjoint matrix:

If A be a square matrix of order n , then

- (i) $\text{adj } |A| = |A|^{n-1}$ (ii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$, where $|A| \neq 0$
- (iii) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$, where $|A| \neq 0$ (iv) $\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$
- (v) $\text{adj } (KA) = K^{n-1}(\text{adj } A)$, K is a scalar (vi) $\text{adj } A^T = (\text{adj } A)^T$

Method of find adjoint of a 2×2 square matrix, directly:

Let A be a 2×2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

$$\text{e.g. If } A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Illustration 13 : If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then $\text{adj } A$ is equal to-

$$(A) \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$$

$$(B) \begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & 14 & 4 \end{bmatrix}$$

$$(C) \begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$$

(D) None of these

Solution : $\text{adj } A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ Ans. (A)

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Illustration 14 : If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then $\text{adj}(\text{adj } A)$ is equal to-

- (A) 8 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (B) 16 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (C) 64 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (D) None of these

Solution : $|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now $\text{adj}(\text{adj } A) = |A|^{3-2} A = 8 \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = 16 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans. (B)**

Do yourself -8 :

- (i) For any 2×2 matrix, if $A(\text{Adj } A) = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$, then $|A|$ is equal-
 (A) 20 (B) 625 (C) 15 (D) 0
- (ii) Which of the following is/are incorrect?
 (A) Adjoint of a symmetric matrix is skew symmetric matrix.
 (B) Adjoint of a diagonal matrix is a diagonal matrix.
 (C) $A(\text{Adj } A) = (\text{Adj } A)A = |A|I$
 (D) Adjoint of a unit matrix is a diagonal matrix
- (iii) If A be a square matrix of the order 5 and $B = \text{Adj } (A)$ then find $\text{Adj } (5A)$.
- (iv) If A be a square matrix of order 4 and $|A| = 3$ then find $\text{adj}(\text{adj } A)$.

16. INVERSE OF A MATRIX (Reciprocal Matrix):

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, $AB = I = BA$.

B is called the inverse (reciprocal) of A and is denoted by A^{-1} . Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

We have, $A \cdot (\text{adj } A) = |A|I_n$

$$|A^{-1} \cdot A(\text{adj } A) = A^{-1}I_n|A|$$

$$I_n(\text{adj } A) = A^{-1}|A|I_n$$

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that

$$|A| \neq 0$$

Properties of inverse:

(i) If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Note: If A_1, A_2, \dots, A_n are all invertible square matrices of order n then

$$(A_1A_2 \dots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \dots A_2^{-1}A_1^{-1}$$

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- (ii) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.
- (iii) If A is invertible, (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}; k \in \mathbb{N}$
- (iv) If A is non-singular matrix, then $|A^{-1}| = |A|^{-1}$
- (v) If idempotent matrix is invertible then its inverse will be identity matrix.
- (vi) A nilpotent matrix will not be invertible because its determinant value is zero.
- (vii) Orthogonal matrix A is always invertible and $A^{-1} = A^T$.
- (viii) $A = A^{-1}$ for an involutory matrix.

Cancellation law : Let A, B, C be square matrices of the same order 'n'.

If A is a non-singular matrix, then

- (a) $AB = AC \Rightarrow B = C$ (Left cancellation law)
- (b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

Note that these cancellation laws hold only if the matrix 'A' is non-singular (i.e. $|A| \neq 0$).

Illustration 15 : Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution : $A^T = A$

[\because A is a symmetric matrix]

$$(A^T)^{-1} = A^{-1}$$

[Since A is non-singular matrix] $\Rightarrow (A^{-1})^T = A^{-1}$ Hence proved

Illustration 16 : $\begin{bmatrix} 1 & -\tan \theta/2 \\ \tan \theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan \theta/2 \\ -\tan \theta/2 & 1 \end{bmatrix}^{-1}$ is equal to-

(A) $\begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$

(B) $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

(C) $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

(D) None of these

Solution. $\begin{bmatrix} 1 & \tan \theta/2 \\ \tan \theta/2 & 1 \end{bmatrix}^{-1} = \frac{1}{\sec^2 \theta/2} \begin{bmatrix} 1 & -\tan \theta/2 \\ \tan \theta/2 & 1 \end{bmatrix}$

$$\therefore \text{Product} = \frac{1}{\sec^2 \theta/2} \begin{bmatrix} 1 & -\tan \theta/2 \\ \tan \theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan \theta/2 \\ -\tan \theta/2 & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 \theta/2} \begin{bmatrix} 1 - \tan^2 \theta/2 & -2\tan \theta/2 \\ 2\tan \theta/2 & 1 - \tan^2 \theta/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta/2 \sin^2 \theta/2 & -2\sin \theta/2 \cos \theta/2 \\ 2\sin \theta/2 \cos \theta/2 & \cos^2 \theta/2 - \sin^2 \theta/2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ Ans.(C)}$$

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Illustration 17: If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $M = AB$, then M^{-1} is equal to-

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$ (C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Solution : $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, $\text{adj } M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} \therefore M^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ **Ans. (C)**

Do yourself -9 :

(i) If 'A' is a square matrix such that $A^2 = I$ then A^{-1} is equal to-

(A) $A + I$ (B) A (C) 0 (D) $2A$

(ii) If 'A' is an orthogonal matrix, then A^{-1} equals-

(A) A (B) A^T (C) A^2 (D) None of these

(iii) If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to-

(A) $\frac{1}{27} \begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (B) $\frac{1}{27} \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ (C) $\frac{1}{27} \begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (D) $\frac{1}{27} \begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

17. MATRIX POLYNOMIAL:

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_nx^0$, then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI^n$.

Where A is the given square matrix. If $f(A)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

18. CHARACTERISTIC EQUATION:

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of

A & the equation $|A - xI| = 0$ is called as characteristic equation of A. After solving the characteristic polynomial the values of 'x' are said to be characteristic roots of the polynomial.

Note: (i) Sum of the roots of the characteristic equation is equal to trace of the matrix.

(ii) Product of the roots of the characteristic equation is equal to the determinant value.

(iii) The degree of characteristic equation is same as the order of the matrix.

Illustration 18: If $f(x) = x^2 - 3x + 3$ and $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ be a square matrix then prove that $f(A) = 0$.

Hence find A^4 .

Solution : $A^2 = A \cdot A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix}$

Hence $A^2 - 3A + 3I = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Alter: $\therefore |A - xI| = 0 \Rightarrow \begin{vmatrix} 2-x & 1 \\ -1 & 1-x \end{vmatrix} = 0$

$\Rightarrow (2-x)(1-x) + 1 = 0 \Rightarrow x^2 - 3x + 3 = 0$

(characteristic polynomial)

By Cayley-Hamilton Theorem $A^2 - 3A + 3I = \mathbf{0}$.

Hence proved.

Now $A^2 = 3A - 3I$

Squaring on both the sides

$$A^4 = 9(A^2 - 2A + I)$$

$$= 9 \left(\begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 2 \begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 9 \begin{bmatrix} 3-4+1 & 3-2 \\ -3+2 & -2+1 \end{bmatrix}$$

$$= 9 \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix}$$

19. CAYLEY-HAMILTON THEOREM:

Every square matrix A satisfy its characteristic equation i.e. $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$

is the characteristic equation of A, then $a_0 A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = \mathbf{0}$

Note : This theorem is helpful to find the inverse of any non-singular square matrix. i.e.

$$a_0 A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = \mathbf{0}$$

On multiplying by A^{-1} on both the sides of above equation, we get

$$A^{-1} = -\frac{1}{a_n} (a_0 A^{n-1} + a_1 A^{n-2} + \dots + a_{n-1} I)$$

Illustration 19: If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, show that $5 A^{-1} = A^2 + A - 5I$

Solution : We have the characteristic equation of A.

$$|A - xI| = 0$$

$$\text{i.e. } \begin{vmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = 0$$

$$\text{i.e. } x^3 + x^2 - 5x - 5 = 0$$

Using Cayley - Hamilton theorem

$$A^3 + A^2 - 5A - 5I = \mathbf{0} \Rightarrow 5I = A^3 + A^2 - 5A$$

Multiplying by A^{-1} , we get $5 A^{-1} = A^2 + A - 5I$

Do yourself -10 :

(i) Determine the characteristic roots of the matrix A. Hence find the trace the determinant value

of A. Where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and also prove that $A^3 - 18A^2 + 45A = \mathbf{0}$.

20. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss-Jordan method:

$$a_1x + b_1y + c_1z = d_1 \Rightarrow a_2x + b_2y + c_2z = d_2 \Rightarrow a_3x + b_3y + c_3z = d_3$$

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$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B$$

Multiplying $\text{adj } A$ on both the sides of (i) $\Rightarrow (\text{adj } A)AX = (\text{adj } A)B \Rightarrow |A|X = (\text{adj } A)B$

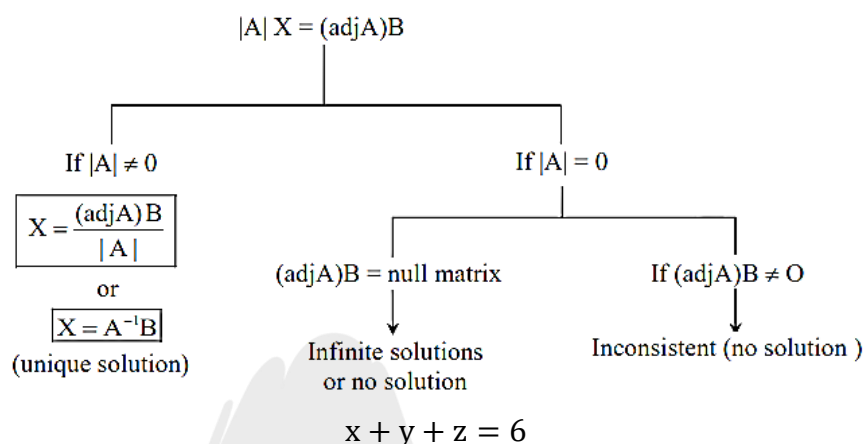


Illustration 20 : Solve the system $x - y + z = 2$ using matrix method.

$$2x + y - z = 1$$

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$ Then the system is $AX = B$.

$$|A| = 6, \text{ hence } A \text{ is non singular, } \Rightarrow \text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x = 1, y = 2, z = 3$$

$$\text{i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{Ans.}$$

Do yourself -11 :

- (i) The system of equations $x + 2y - 3z = 1, x - y + 4z = 0, 2x + y + z = 1$ has-
- (A) only two solutions (B) only one solution
- (C) no solution (D) infinitely many solutions
- (ii) The system of equations $x + y + z = 8, x - y + 2z = 6, 3x + 5y - 7z = 14$ has-
- (A) Unique solution (B) infinite number of solutions
- (C) no solution (D) none of these

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ANSWERS FOR DO YOURSELF

1: (i) $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ (ii) 6 (iii) 30 (iv) $x = 2, y = -1, a = 1, b = -3$

2: (i) $\begin{bmatrix} -7 & -10 & -7 \\ -2 & 6 & 3 \end{bmatrix} \times 2$ (ii) $\begin{bmatrix} -8 & -9 \\ -7/2 & -8 \\ -1 & 1 \end{bmatrix}$

3: (iii) $x = -2$

4: (ii) C

5: (i) C (ii) $x = 0$

6: (ii) $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

7: (i) -4 (ii) $\begin{bmatrix} 2 & 7 & 4 \\ 7 & -7 & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & \frac{3}{2} \\ -3 & -\frac{3}{2} & 0 \end{bmatrix}$

8: (i) C

9: (i) B (ii) A (iii) 625 B (iv) 9 A

10: (i) $\lambda = 0, 3$ and $15 \text{tr}(A) = 18, |A| = 0$

11: (i) D (ii) A

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EXERCISE - 1

1. Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$, then $\text{Tr}(A) - \text{Tr}(B)$ has the value equal to
 (A) 0 (B) 1 (C) 2 (D) none
2. If $\begin{bmatrix} x & 3x - y \\ zx + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, then
 (A) $x = 3, y = 7, z = 1, w = 14$ (B) $x = 3, y = -5, x = -1, w = -4$
 (C) $x = 3, y = 6, z = 2, w = 7$ (D) None of these
3. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ equals:
 (A) $32 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $4 \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$
 (C) $4 \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$ (D) $32 \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ [JEE-MAIN 2013]
4. If $M = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$, then M^{2011} is -
 (A) $10^{1005}M$ (B) $10^{1005}N$
 (C) $10^{2010}M$ (D) $10^{2011}M$
5. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - kA - I_2 = 0$, then value of k is-
 (A) 4 (B) 2 (C) 1 (D) -4
6. Let three matrices are $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then
 $\text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \dots + \infty$ is equal to -
 (A) 6 (B) 9 (C) 12 (D) none
7. For a matrix $A = \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$, the value of $\prod_{r=1}^{50} \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$ is equal to -
 (A) $\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 4950 \\ 0 & 1 \end{bmatrix}$
 (C) $\begin{bmatrix} 1 & 5050 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2500 \\ 0 & 1 \end{bmatrix}$
8. A and B are two given matrices such that the order of A is 3×4 , if $A'B$ and BA' are both defined then
 (A) order of B' is 3×4 (B) order of $B'A$ is 4×4
 (C) order of $B'A$ is 3×3 (D) $B'A$ is undefined

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9. If the product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \dots \dots \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is equal to the matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$ then the value of n is equal to -
 (A) 26 (B) 27 (C) 377 (D) 378
10. Consider a matrix $A(\theta) = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix}$ then,
 (A) $A(\theta)$ is symmetric (B) $A(\theta)$ is skew symmetric
 (C) $A^{-1}(\theta) = A(\pi - \theta)$ (D) $A^2(\theta) = A\left(\frac{\pi}{2} - 2\theta\right)$
11. If p, q, r are 3 real number satisfying the matrix equation, $[pqr] \begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3 \ 0 \ 1]$, then
 $2p + q - r$ equals :- [JEE-MAIN 2013]
 (A) -1 (B) 4 (C) -3 (D) 2
12. If A, B and C are $n \times n$ matrices and $\det(A) = 2, \det(B) = 3$ and $\det(C) = 5$, then the value of the $\det(A^2BC^{-1})$ is equal to
 (A) $\frac{6}{5}$ (B) $\frac{12}{5}$ (C) $\frac{18}{5}$ (D) $\frac{24}{5}$
13. Which of the following is an orthogonal matrix -
 (A) $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$ (B) $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$
 (C) $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$ (D) $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$
14. Matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, if $xyz = 60$ and $8x + 4y + 3z = 20$, then $A(\text{adj } A)$ is equal to:-
 (A) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ (B) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$
 (C) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ (D) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$
15. The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is a
 (A) non-singular (B) Idempotent
 (C) Nilpotent (D) Orthogonal

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16. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then A is
 (A) Involutory matrix (B) Idempotent matrix
 (C) Nilpotent matrix (D) None of these
17. If A and B are symmetric matrices, then ABA is -
 (A) symmetric matrix (B) skew symmetric matrix
 (C) diagonal matrix (D) scalar matrix
18. Let $A = \begin{pmatrix} 0 & \sin \alpha & \sin \alpha \sin \beta \\ -\sin \alpha & 0 & \cos \alpha \cos \beta \\ -\sin \alpha \sin \beta & -\cos \alpha \cos \beta & 0 \end{pmatrix}$, then-
 (A) |A| is independent of α and β (B) A^{-1} depends only on α
 (C) A^{-1} depends only on β (D) none of these
19. Number of real values of λ for which the matrix $A = \begin{bmatrix} \lambda - 1 & \lambda & \lambda + 1 \\ 2 & -1 & 3 \\ \lambda + 3 & \lambda - 2 & \lambda + 7 \end{bmatrix}$ has no inverse
 (A) 0 (B) 1 (C) 2 (D) infinite
20. If $A = [a_{ij}]_{2 \times 2}$ where $a_{ij} = \begin{cases} i + j & i \neq j \\ i^2 - 2j & i = j \end{cases}$, then A^{-1} is equal to -
 (A) $\frac{1}{9} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ (B) $\frac{1}{9} \begin{bmatrix} 0 & -3 \\ 3 & -1 \end{bmatrix}$ (C) $\frac{1}{9} \begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$ (D) $\frac{1}{3} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

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EXERCISE - 2

- Let $M = \begin{bmatrix} a & -360 \\ b & c \end{bmatrix}$, where a, b and c are integers. Find the smallest positive value of b such that $M^2 = \mathbf{O}$, where \mathbf{O} denotes 2×2 null matrix.
- Find the number of 2×2 matrix satisfying following conditions:
(i) a_{ij} is 1 or -1 ; (ii) $a_{11}a_{21} + a_{12}a_{22} = 0$
- Find the value of x and y that satisfy the equations $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such that $AB = B$ and $a + d = 5050$. Find the value of $(ad - bc)$.
- Define $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Find a vertical vector V such that $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ (where I is the 2×2 identity matrix).
- If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the matrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.
- If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\frac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha - B & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$
- If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of $f(a)$, where $f(x) = x - x^2$, when $bc = 1/4$. Hence otherwise evaluate a .
- If the matrix A is involutory, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A) \cdot \frac{1}{2}(I - A) = \mathbf{O}$.
- Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent matrix. Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$
- $A = \begin{pmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{pmatrix}$ is Symmetric and $B = \begin{pmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{pmatrix}$ is skew symmetric, then find AB . Is AB a symmetric, skew symmetric or neither of them. Justify your answer.
- Express the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also express the matrix as a sum of a symmetric and a skew symmetric matrix.

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13. (a) A is a square matrix of order n.
 ℓ = maximum number of distinct entries if A is a triangular matrix
 m = maximum number of distinct entries if A is a diagonal matrix
 p = minimum number of zeroes if A is a triangular matrix
 If $\ell + 5 = p + 2m$, find the order of the matrix.
 (b) Let A be the set of all 3×3 skew symmetric matrices whose entries are either $-1, 0$ or 1 .
 If there are exactly three 0's, three 1's and three (-1) 's, then find the number of such matrices.
14. If A is an idempotent non-zero matrix and I is an Identity matrix of the same order, find the value of n, $n \in \mathbb{N}$, such that $(A + I)^n = I + 127A$.
15. Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and $AB \neq I$ (where I is an identity matrix of order 3×3). Find the value of $\text{Tr.}(AB + (AB)^2 + (AB)^3 + \dots + (AB)^{100})$, where $\text{Tr.}(A)$ denotes the trace of matrix A.
16. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that value of f and g satisfying the matrix equation $A^2 + fA + gI = \mathbf{0}$ are equal to $-\text{tr}(A)$ and determinant of A respectively. Given a, b, c, d are non zero reals and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
17. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [abc]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+2)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$ be three given matrices, where a, b, c and $x \in \mathbb{R}$. Given that $\text{tr}(AB) = \text{tr}(C) \forall x \in \mathbb{R}$, where $\text{tr}(A)$ denotes trace of A. Find the value of $(a + b + c)$

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EXERCISE - 3 (JM)

1. If A and B are square matrices of size $n \times n$ such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true- [AIEEE-2006]
 - (A) $AB = BA$
 - (B) Either of A or B is a zero matrix
 - (C) Either of A or B is an identity matrix
 - (D) $A = B$
2. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{N}$. Then [AIEEE-2006]
 - (A) there exist more than one but finite number of B's such that $AB = BA$
 - (B) there exist exactly one B such that $AB = BA$
 - (C) there exist infinitely many B's such that $AB = BA$
 - (D) there cannot exist any B such that $AB = BA$
3. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$, Then $|\alpha|$ equals-
 - (A) 5^2
 - (B) 1
 - (C) $1/5$
 - (D) 5
4. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denoted by $\text{tr}(A)$, the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement-1: If $A \neq I$ and $A \neq -I$, then $\det A = -1$ [AIEEE-2008]

Statement-2: If $A \neq I$ and $A \neq -I$, then $\text{tr}(A) \neq 0$.

 - (A) Statement -1 is false, Statement -2 is true.
 - (B) Statement-1 is true, Statement- 2 is true; Statement- 2 is a correct explanation for Statement-1.
 - (C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
 - (D) Statement-1 is true, Statement-2 is false
5. Let A be a square matrix all of whose entries are integers. Then which one of the following is true?
 - (A) If $\det A = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers
 - (B) If $\det A \neq \pm 1$, then A^{-1} exists and all its entries are non-integers
 - (C) If $\det A = \pm 1$, then A^{-1} exists and all its entries are integers
 - (D) If $\det A = \pm 1$, then A^{-1} need not exist [AIEEE-2008]
6. Let A be a 2×2 matrix [AIEEE-2009]

Statement-1 : $\text{adj}(\text{adj } A) = A$

Statement-2 : $|\text{adj } A| = |A|$

 - (A) Statement-1 is true, Statement-2 is false.
 - (B) Statement-1 is false, Statement- 2 is true.
 - (C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
 - (D) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for statement -1 .

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7. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is :
 (A) Less than 4 (B) 5 (C) 6 (D) At least 7

[AIEEE-2010]

8. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define $\text{Tr}(A)$ = sum of diagonal elements of A and $|A|$ = determinant of matrix A .

Statement-1 : $\text{Tr}(A) = 0$.

Statement-2: $|A| = 1$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(C) Statement-1 is true, Statement-2 is false.

(D) Statement-1 is false, Statement-2 is true.

[AIEEE-2010]

9. Let A and B be two symmetric matrices of order 3.

[AIEEE-2011]

Statement-1 : $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative.

(A) Statement-1 is true, Statement-2 is false.

(B) Statement-1 is false, Statement-2 is true

(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(D) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

10. Statement-1 : Determinant of a skew-symmetric matrix of order 3 is zero.

Statement-2 : For any matrix A , $\det(A^T) = \det(A)$ and $\det(-A) = -\det(A)$.

Where $\det(B)$ denotes the determinant of matrix B . Then :

[AIEEE-2011]

(A) Statement-1 is true and statement-2 is false (B) Both statements are true

(C) Both statements are false

(D) Statement-1 is false and statement-2 is true.

11. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

then $u_1 + u_2$ is equal to :

[AIEEE-2012]

(A) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(B) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(C) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

(D) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

12. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

(A) 4

(B) 11

(C) 5

(D) 0

[JEE(Mains)-2013]

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13. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1} A'$, the BB' equals :

- (A) $I + B$ (B) I (C) B^{-1} (D) $(B^{-1})'$

[JEE(Mains)-2014]

14. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix,

then the ordered pair (a, b) is equal to :

[JEE(Mains)-2015]

- (A) (2,1) (B) $(-2, -1)$ (C) (2, -1) (D) $(-2, 1)$

15. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{adj } A = AA^T$, then $5a + b$ is equal to :

[JEE(Mains)-2016]

- (A) 13 (B) -1 (C) 5 (D) 4

16. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then $\text{adj } (3A^2 + 12A)$ is equal to :

[JEE(Mains)-2017]

- (A) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (B) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (C) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (D) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

17. Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I_3$, then $|p|$ is:

(JEE Mains 2019)

- (A) $\frac{1}{\sqrt{5}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{\sqrt{6}}$

18. Let $A = \begin{bmatrix} 2 & 2 \\ 9 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then value of $10A^{-1}$ is:

(JEE Mains 2020)

- (A) $4I - A$ (B) $6I - A$ (C) $A - 4I$ (D) $A - 6I$

19. If for the matrix, $A = \begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}$, $AA^T = I_2$, then the value of $\alpha^4 + \beta^4$ is: (JEE Mains 2021)

- (A) 1 (B) 2 (C) 3 (D) 4

20. Let $A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$. Then the number of elements in the set

$\{(n, m): n, m \in \{1, 2, \dots, 10\} \text{ and } nA^n + mB^m = I\}$ is

(JEE Mains 2022)

21. If A and B are two non-zero $n \times n$ matrices such that $A^2 + B = A^2 B$, then (JEE Mains 2023)

- (A) $AB = I$ (B) $A^2 B = I$ (C) $A^2 = I$ or $B = I$ (4) $A^2 B = BA^2$

EXERCISE - 4 (JA)

Comprehension (3 questions)

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if U_1, U_2 , and U_3 are columns matrices satisfying. $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and U is 3×3 matrix whose columns are U_1, U_2, U_3 then answer the following questions-

- (a) The value of $|U|$ is -
 (A) 3 (B) -3 (C) $3/2$ (D) 2
- (b) The sum of the elements of U^{-1} is -
 (A) -1 (B) 0 (C) 1 (D) 3 [JEE 2006]
- (c) The value of $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is - [JEE 2006]
 (A) [5] (B) $[5/2]$ (C) [4] (D) $[3/2]$

2. Match the Statement / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I

Column II

- | | |
|---|-------|
| (A) The minimum value of $\frac{x^2+2x+4}{x+2}$ is | (P) 0 |
| (B) Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A+B)(A-B) = (A-B)(A+B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB , then the possible values of k are | (Q) 1 |
| (C) Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than | (R) 2 |
| (D) If $\sin \theta = \cos \phi$, then the possible values of $\frac{1}{\pi} \left(\theta - \phi - \frac{\pi}{2} \right)$ are | (S) 3 |
- [JEE 2008, 6]

Comprehension (3 questions)

3. Let \mathcal{A} be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.
- (a) The number of matrices in \mathcal{A} is -
 (A) 12 (B) 6 (C) 9 (D) 3

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- (b) The number of matrices A in \mathcal{A} for which the system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, is -
- (A) less than 4 (B) at least 4 but less than 7
(C) at least 7 but less than 10 (D) at least 10

- (c) The number of matrices A in \mathcal{A} for which the system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent, is -
- (A) 0 (B) more than 2 (C) 2 (D) 1

4. (a) The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ has exactly two distinct solutions, is}$$

- (A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

- (b) Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If $\det(\text{adj } A) + \det(\text{adj } B) = 10^6$, then $[k]$ is equal to

[Note : $\text{adj } M$ denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k].

- (c) Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \dots, p-1\} \right\}$$

- (i) The number of A in T_p such that A is either symmetric or skew-symmetric or both, and $\det(A)$ divisible by p is -

- (A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p-1$

- (ii) The number of A in T_p such that the trace of A is not divisible by p but $\det(A)$ is divisible by p is

[Note : The trace of a matrix is the sum of its diagonal entries.]

- (A) $(p-1)(p^2 - p + 1)$ (B) $p^3 - (p-1)^2$
(C) $(p-1)^2$ (D) $(p-1)(p^2 - 2)$

- (iii) The number of A in T_p such that $\det(A)$ is not divisible by p is -

[JEE 2010]

- (A) $2p^2$ (B) $p^3 - 5p$ (C) $p^3 - 3p$ (D) $p^3 - p^2$

5. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P , then $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ is equal to -

[JEE 2011, 4]

- (A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

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6. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each of a, b and c is either ω or ω^2 . Then the number of distinct matrices in the set S is -
 (A) 2 (B) 6 (C) 4 (D) 8
7. Let M be 3×3 matrix satisfying $M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$. Then the sum of the diagonal entries of M is [JEE 2011, 4]
8. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is - [JEE 2012, 3M, -1M]
9. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that [JEE 2012, 3M, -1M]
 (A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$ (C) $PX = 2X$ (D) $PX = -X$
10. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) - (A) -2 (B) -1 (C) 1 (D) 2 [JEE 2012, 4M]
11. For 3×3 matrices M and N , which of the following statement(s) is (are) NOT correct?
 (A) $^T MN$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric
 (B) $MN - NM$ is skew symmetric for all symmetric matrices M and N
 (C) MN is symmetric for all symmetric matrices M and N
 (D) $(\text{adj } M)(\text{adj } N) = \text{adj } (MN)$ for all invertible matrices M and N [JEE-Advanced 2013, 4, (-1)]
12. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if
 (A) the first column of M is the transpose of the second row of M
 (B) the second row of M is the transpose of the first column of M
 (C) M is a diagonal matrix with nonzero entries in the main diagonal [JEE(Advanced)-2014, 3]
 (D) the product of entries in the main diagonal of M is not the square of an integer
13. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then
 (A) determinant of $(M^2 + MN^2)$ is 0 [JEE(Advanced)-2014, 3]
 (B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix
 (C) determinant of $(M^2 + MN^2) \geq 1$
 (D) for a 3×3 matrix U , if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

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14. Let X and Y be two arbitrary, 3×3 , non-zero, skew-symmetric matrices and Z be an arbitrary 3×3 , non-zero, symmetric matrix. Then which of the following matrices is (are) skewsymmetric ?

[JEE(Advanced)-2015, 4M, -2M]

- (A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3Y^4$ (D) $X^{23} + Y^{23}$

15. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where

$k \in \mathbb{R}, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then-

[JEE(Advanced)-2016, 4(-2)]

- (A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$
(C) $\det(\text{Padj}(Q)) = 2^9$ (D) $\det(Q\text{adj}(P)) = 2^{13}$

16. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that

$p^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals

[JEE(Advanced)-2016, 3(-1)]

- (A) 52 (B) 103 (C) 201 (D) 205

17. Which of the following is (are) NOT the square of a 3×3 matrix with real entries ?

[JEE(Advanced)-2017]

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

18. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of $M^T M$ is 5 ?

[JEE(Advanced)-2017]

- (A) 198 (B) 135 (C) 126 (D) 162

19. For a real number α , if the system $\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

[JEE(Advanced)-2017]

20. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1 \quad \Rightarrow \quad 2x - 4y + 3z = b_2 \quad \Rightarrow \quad x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has(have) at

least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

[JEE(Advanced)-2018]

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- (A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
 (B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
 (C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
 (D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$
21. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is [JEE(Advanced)-2018]
22. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? [JEE(Advanced)-2019]
- (A) $\det(\text{adj } M^2) = 81$ (B) $a + b = 3$
 (C) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$ (D) if $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$
23. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statements is/are ALWAYS TRUE? [JEE(Advanced)-2020]
- (A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$
24. For any 3×3 matrix M , let $|M|$ denote the determinant of M . Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) TRUE? [JEE(Advanced)-2021]
- (A) $|FE| = |I - FE||FGE|$ (B) $(I - FE)(I + FGE) = I$
 (C) $EFG = GEF$ (D) $(I - FE)(I - FGE) = I$
25. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal M^{2022} ? [JEE(Advanced)-2022]
- (A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ (B) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$
 (C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$ (D) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$
26. Let $M = (a_{ij})$, $i, j \in \{1, 2, 3\}$, be the 3×3 matrix such that $a_{ij} = 1$ if $j + 1$ is divisible by i , otherwise $a_{ij} = 0$. Then which of the following statements is(are) true? [JEE(Advanced)-2023]
- (A) M is invertible
 (B) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$
 (C) The set $\{X \in \mathbb{R}^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
 (D) The matrix $(M - 2I)$ is invertible, where I is the 3×3 identity matrix

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EXERCISE - 5

- Let A , other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let $\text{Tr}(A)$ be the sum of diagonal elements of A . [JEE-MAIN 2013]
Statement-1: $\text{Tr}(A) = 0$
Statement-2: $\det(A) = -1$
(A) Statement-1 and Statement-2 are true and statement-2 is not the correct explanation for statement-1
(B) Statement -1 and Statement-2 are true and statement-2 is a correct explanation for statement-1
(C) statement-1 is true and statement-2 is false
(D) statement-1 is false and statement-2 is true.
- Let P and Q are orthogonal matrices of order 3×3 such that $A = PQ^T$ & $B = QP^T$, then the value of $\text{tr}(AB)$ is equal to
(A) 1 (B) 3 (C) 5 (D) 7
- Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0,1,2\}, a_{11} = a_{22} \right\}$. Then the number of non-singular matrices in the set S is
(A) 24 (B) 10 (C) 20 (D) 27
- If $A = \begin{bmatrix} \cos^2 \alpha & \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha & \sin^2 \alpha \end{bmatrix}$; $B = \begin{bmatrix} \cos^2 \beta & \sin \beta \cos \beta \\ \sin \beta \cos \beta & \sin^2 \beta \end{bmatrix}$ are such that, AB is a null matrix, then which of the following should necessarily be an odd integral multiple of $\frac{\pi}{2}$.
(A) α (B) β (C) $\alpha - \beta$ (D) $\alpha + \beta$
- A is an involutory matrix given by $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ then the inverse of $\frac{A}{2}$ will be
(A) $2A$ (B) $\frac{A^{-1}}{2}$ (C) $\frac{A}{2}$ (D) A^2
- If A and B are invertible matrices, which one of the following statement is/are incorrect
(A) $\text{Adj}(A) = |A|A^{-1}$ (B) $\det(A^{-1}) = |\det(A)|^{-1}$
(C) $(A+B)^{-1} = B^{-1} + A^{-1}$ (D) $(AB)^{-1} = B^{-1}A^{-1}$
- Identify the incorrect statement in respect of two square matrices A and B conformable for sum and product-
(A) $t_r(A+B) = t_r(A) + t_r(B)$ (B) $t_r(\alpha A) = \alpha t_r(A), \alpha \in \mathbb{R}$
(C) $t_r(A^T) = t_r(A)$ (D) $t_r(AB) \neq t_r(BA)$
- If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then -
(A) $a = 1, c = -1$ (B) $a = 2, c = -\frac{1}{2}$
(C) $a = -1, c = 1$ (D) $a = \frac{1}{2}, c = \frac{1}{2}$

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9. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix}$ where $x, y, z \in \mathbb{R}$. If $B^T A B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ then the number of ordered triplet (x, y, z) is
 (A) 2 (B) 6 (C) 8 (D) 9
10. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10 B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of matrix A , then α is-
 (A) -2 (B) -1 (C) 2 (D) 5
11. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then let us define a function $f(x) = \det. (A^T A^{-1})$ then which of the following can not be the value of $\underbrace{f(f(f \dots \dots f(x)))}_{n \text{ times}}$ is ($n \geq 2$)
 (A) $f^n(x)$ (B) 1 (C) $f^{n-1}(x)$ (D) $nf(x)$
12. Let the matrix A and B be define as $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$ then the value of $\text{Det. } (2 A^9 B^{-1})$, is
 (A) 2 (B) 1 (C) -1 (D) -2
 MORE THAN ONE ARE CORRECT
13. Let $\det(\text{adj}(\text{adj } A)) = 14^4$ where $A = \begin{bmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $x \neq -\frac{11}{3}$, then
 (A) $x = 1$ (B) $\det(2 A) = 112$
 (C) $x = 2$ (D) $\det(2 A) = 256$
14. Let $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, then-
 (A) $7|A| = \frac{1}{2}$ (B) $|\text{adj } A| = \frac{1}{196}$
 (C) $\text{trace}(\text{adj } A) = -\frac{1}{7}$ (D) Matrix A is symmetric matrix
15. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, then which of the following is(are) true? (trace of A denotes sum of principal diagonal elements of A)
 (A) A is invertible (B) $\text{trace}(\text{adj}(A)) = 144$
 (C) $\text{trace}(\text{adj}(\text{adj}(A))) = 8$ (D) $|\text{adj } A|$ is less than 400
16. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$ then-
 (A) $k = 1, \alpha = 14$ (B) $\alpha = 14, \beta = 22$
 (C) $k = -1, \beta = 22$ (D) $\alpha = -14, \beta = -22$

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17. If A and B are 3×3 matrices and $|A| \neq 0$, then which of the following are true?
 (A) $|AB| = 0 \Rightarrow |B| = 0$ (B) $|AB| = 0 \Rightarrow B = 0$
 (C) $|A^{-1}| = |A|^{-1}$ (D) $|A + A| = 2|A|$
18. If D_1 and D_2 are two 3×3 diagonal matrices where none of the diagonal element is zero, then -
 (A) $D_1 D_2$ is a diagonal matrix (B) $D_1 D_2 = D_2 D_1$
 (C) $D_1^2 + D_2^2$ is a diagonal matrix (D) none of these
19. If A and B are two 3×3 matrices such that their product AB is a null matrix then
 (A) $\det. A \neq 0 \Rightarrow B$ must be a null matrix.
 (B) $\det. B \neq 0 \Rightarrow A$ must be a null matrix
 (C) If none of A and B are null matrices then atleast one of the two matrices must be singular.
 (D) If neither $\det. A$ nor $\det. B$ is zero then given statement is not possible.
20. Let $A = a_{ij}$ be a matrix of order 3 where $a_{ij} = \begin{cases} x & \text{if } i = j, x \in \mathbb{R} \\ 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$, then which of the following hold(s) good ?
 (A) for $x = 2$, A is a diagonal matrix
 (B) A is a symmetric matrix
 (C) for $x = 2$, $\det A$ has the value equal to 6
 (D) Let $f(x) = \det A$, then the function $f(x)$ has both the maxima and minima.

Paragraph for question nos. 21 to 22

Consider the system $AX = B$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

21. Sum of elements of $(\text{adj } A) B$ is -
 (A) -1 (B) 2 (C) -2 (D) -4
22. Value of $\text{tr}(XB^T)$ is (where $\text{tr}(A)$ denotes trace of matrix A)-
 (A) 0 (B) 1 (C) 2 (D) 3

Paragraph for question nos. 23 to 25

If A is a symmetric and B skew symmetric matrix and $A + B$ is nonsingular and $C = (A + B)^{-1}(A - B)$ then

23. $C^T(A + B)C =$
 (A) $A + B$ (B) $A - B$ (C) A (D) B
24. $C^T(A - B)C =$
 (A) $A + B$ (B) $A - B$ (C) A (D) B
25. $C^T A C$
 (A) $A + B$ (B) $A - B$ (C) A (D) B

EXERCISE - 6

- Let A be the 2×2 matrices given by $A = [a_{ij}]$, where $a_{ij} \in \{0,1,2,3,4\}$ such that $a_{11} + a_{12} + a_{21} + a_{22} = 4$
 - Find the number of matrices A such that the trace of A is equal to 4.
 - Find the number of matrices A such that A is invertible.
 - Find the absolute value of the difference between maximum value and minimum value of $\det(A)$.
 - Find the number of matrices A such that A is either symmetric or skew-symmetric or both and $\det(A)$ is divisible by 2
- For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2} .
- Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
 - Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$
- If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that $F(x) \cdot F(y) = F(x+y)$. Hence prove that $[F(x)]^{-1} = F(-x)$.
- Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$, then find the value of $(x+y+z)$ where I is a unit matrix of order 3.
- Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that $Cb = D$. Solve the matrix equation $Ax = b$.
- Let $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$ and $C_r = \begin{bmatrix} r \cdot 3^r & 2^r \\ 0 & (r-1)3^r \end{bmatrix}$ be 3 given matrices Compute the value of $\sum_{r=1}^{50} \text{tr} \cdot ((AB)^r C_r)$. (where $\text{tr}(A)$ denotes trace of matrix A)
- Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq \mathbb{N}$ then find the minimum value of $\sum (\cos^x \theta + \sin^x \theta)$, $\theta \in \mathbb{R}$.
- Consider the two matrices A and B where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Let $n(A)$ denotes the number of elements in A and $n(XY) = 0$, when the two matrices X and Y are not conformable for multiplication. If $C = (AB)(B'A)$; $D = (B'A)(AB)$ then, find the value of $\left(\frac{n(C)((|D|^2 + n(D)))}{n(A) - n(B)} \right)$.

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10. Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(a) $AX = A$

(b) $XA = I$

(c) $XB = \mathbf{0}$ but $BX \neq \mathbf{0}$.

11. Find the product of two matrices A & B , where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

and use it to solve the following system of linear equations, $x + y + 2z = 1$; $3x + 2y + z = 7$; $2x + y + 3z = 2$.

12. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(a) has a unique solution ;

(b) has no solution and

(c) has infinitely many solutions

13. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then solve the following matrix equation.

(a) $AX = B - I$

(b) $(B - I)X = IC$

(c) $CX = A$

14. $A_{3 \times 3}$ is a matrix such that $|A| = a$, $B = (\text{adj } A)$ such that $|B| = b$. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \dots$ up to ∞ and $a = 3$.

15. If A and B are square matrices of order 3, where $|A| = -2$ and $|B| = 1$, then find $|(A^{-1})\text{adj}(B^{-1})\text{adj}(2A^{-1})|$.

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ANSWER KEY

EXERCISE - 1

1. C 2. A 3. B 4. A 5. A 6. A 7. D
8. B 9. B 10. C 11. C 12. B 13. A 14. C
15. B 16. C 17. A 18. A 19. D 20. A

EXERCISE - 2

1. 10 2. 8 3. $x = \frac{3}{2}, y = 2$ 4. 5049 5. $V = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$
7. 1 8. $f(a) = 1/4, a = 1/2$ 10. $\begin{bmatrix} 1 & 0 \\ 40 & 14 \\ & 1 \end{bmatrix}$
11. AB is neither symmetric nor skew symmetric
12. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$
13. (a) 4, (b) 8 14. $n = 7$ 15. 100 16. $f = -(a + d); g = ad - bc$
17. 7

EXERCISE - 3 (JM)

1. A 2. C 3. C 4. D 5. C 6. D 7. D
8. D 9. D 10. A 11. A 12. B 13. B 14. B
15. C 16. C 17. C 18. D 19. A 20. 1 21. D

EXERCISE - 4 (JA)

1. (a) A, (b) B, (c) A 2. (A) R(B) Q, S (C) R, S(D) P, R 3. (a) A, (b) B, (c) B
4. (a) A, (b) 4; (c) (i) D, (ii) C, (iii) D 5. Bonus 6. A
7. 9 8. D 9. D 10. A, D 11. C, D 12. C, D 13. A, B
14. C, D 15. B, C 16. B 17. A & 18. (A) 19. 1 20. (A, D)
21. (4.00) 22. B, C, D 23. B, C, D 24. A, B, C 25. (A) 26. B, C

EXERCISE - 5

1. A 2. B 3. C 4. C 5. A 6. C 7. D
8. A 9. C 10. D 11. D 12. D 13. A, B 14. B, C, D
15. A, B, D 16. B, C 17. A, C 18. A, B, C 19. A, B, C, D 20. B, D
21. C 22. A 23. A 24. B 25. C

(MATHEMATICS)

MATRICES

EXERCISE - 6

1. (i) 5, (ii) 18, (iii) 8, (iv) 5
2. $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$
3. (a) $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$; (b) $\frac{1}{19} \begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$
5. 1
6. $x_1 = 1, x_2 = -1, x_3 = 1$
7. $3(49.3^{50} + 1)$
8. 2
9. 650
10. (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in \mathbb{R}$;
(ii) X does not exist;
(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ where $a, c \in \mathbb{R}$ and $3a + c \neq 0$; $3b + d \neq 0$
11. $x = 2, y = 1, z = -1$
12. (i) $a \neq -3, b \in \mathbb{R}$;
(ii) $a = -3$ and $b \neq 1/3$;
(iii) $a = -3, b = 1/3$
13. (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$
(b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$,
(c) no solution
14. 225
15. -8