KEY CONCEPTS

1. DEFINITION :

Complex numbers are definited as expressions of the form a + ib where a, b $\in$ R & i = $\sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

EVERY COMPLEX NUMBER CAN BE REGARDED AS

Purely real	Purely imaginary	Imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complete Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is both purely real as well as purely imaginary but not imaginary.

(c) i = $\sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{a\,b}$ only if atleast one of either a or b is non-negative.

2. CONJUGATE COMPLEX :

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z}$ = a – ib.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

3. ALGEBRAIC OPERATIONS :

The algebraic operations on complex numbers are similiar to those on real numbers treating *i* as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless .

However in real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ but in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

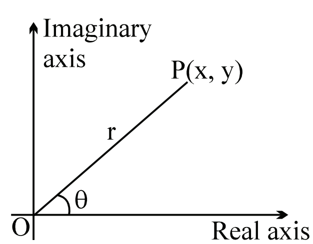
4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometric Representation) :

Every complex number z = x + i y can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude .

eg. $|z| = \sqrt{x^2 + y^2}$ &

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

NOTE :

(i) $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is **not correct**

(ii) Argument of a complex number is a many valued function . If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(iii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is given by its modulus.

(vi) There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

(b) **Trignometric / Polar Representation :**

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note: $\cos\theta + i\sin\theta$ is also written as $CiS\,\theta$.

Also $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix}-e^{-ix}}{2}$ are known as Euler's identities.

(c) **Exponential Representation :**

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

6. **IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :**

If $z, z_1, z_2 \in C$ then ;

(a) $z + \bar{z} = 2\,\text{Re}(z)$; $z - \bar{z} = 2i\,\text{Im}(z)$; $\overline{(\bar{z})} = z$; $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$;

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ $\quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \ne 0$

(b) $|z| \ge 0$; $|z| \ge \text{Re}(z)$; $|z| \ge \text{Im}(z)$; $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$;

$|z_1 z_2| = |z_1| \cdot |z_2|$; $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \ne 0$, $|z^n| = |z|^n$;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$ **[TRIANGLE INEQUALITY]**

(c) **(i)** $\text{amp}(z_1 \cdot z_2) = \text{amp}\, z_1 + \text{amp}\, z_2 + 2k\pi$. $\quad k \in I$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2 + 2k\pi$; $k \in I$

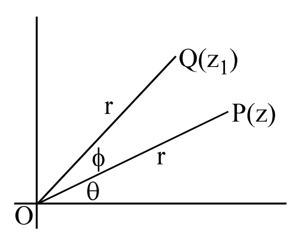
(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$.

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(7) **VECTORIAL REPRESENTATION OF A COMPLEX :**

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

NOTE :



(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z \cdot e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$

(ii) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then

AB || CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary]

(iii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(a) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3 z_0^2$

8. DEMOIVRE'S THEOREM :

Statement : $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$ ¥ $n \in Q$. The theorem is very useful in determining the roots of any complex quantity

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

9. CUBE ROOT OF UNITY :

(i) The cube roots of unity are 1 , $\dfrac{-1+i\sqrt{3}}{2}$, $\dfrac{-1-i\sqrt{3}}{2}$.

(ii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$. In general $1 + w^r + w^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :

$\cos 0 + i \sin 0$; $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$, $\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

(v) The following factorisation should be remembered :

($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

10. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ &

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2) \ldots\ldots (1 - \alpha_{n-1}) = n$ &

$(1 + \alpha_1)(1 + \alpha_2) \ldots\ldots (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \ldots\ldots \alpha_{n-1} = 1$ or -1 according as n is odd or even.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\dfrac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(A) If z_1 & z_2 are two complex numbers then the complex number $z = \dfrac{nz_1 + mz_2}{m+n}$ divides the joins of z_1 & z_2 in the ratio m : n.

Note:

(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1 , z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then :

(a) Centroid of the Δ ABC $= \dfrac{z_1 + z_2 + z_3}{3}$:

(b) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \quad \textbf{OR} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(c) Incentre of the Δ ABC $= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(d) Circumcentre of the Δ ABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(B) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(C) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(D) The equation of a line joining z_1 & z_2 is given by ;
$z = z_1 + t\,(z_1 - z_2)$ where t is a perameter.

(E) $z = z_1\,(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to oz_1.

(F) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as
$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}$ = 0. This is also the condition for three complex numbers to be collinear.

(G) Complex equation of a straight line through two given points z_1 & z_2 can be written as
$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha \bar{z} + r = 0$
where r is real and α is a non zero complex constant.

(H) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form $z\bar{z} + \bar{\alpha} z + \alpha\bar{z} + r = 0$, r is real centre $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha} - r}$.

Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.

(I) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is :

(i) $\arg \dfrac{z - z_2}{z - z_1} = \pm \dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

(J) Condition for four given points z_1 , z_2 , z_3 & z_4 to be concyclic is, the number
$\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow \dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$

13.(a) Reflection points for a straight line :
Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if ; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) Inverse points w.r.t. a circle :
Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :
(i) the point O, P, Q are collinear and on the same side of O. **(ii)** OP . OQ = ρ^2.
Note that the two points z_1 & z_2 will be the inverse points w.r.t. the circle
$z\bar{z} + \bar{\alpha} z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha} z_1 + \alpha\bar{z}_2 + r = 0$.

14. PTOLEMY'S THEOREM :
It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.
i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

15. LOGARITHM OF A COMPLEX QUANTITY :

(i) $\text{Log}_e(\alpha + i\beta) = \dfrac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$.

VERY ELEMENTARY EXERCISE

1. Simplify and express the result in the form of $a + bi$

(a) $\left(\frac{1+2i}{2+i}\right)^2$ (b) $-i\,(9 + 6\,i)\,(2 - i)^{-1}$ (c) $\left(\frac{4i^3 - i}{2i+1}\right)^2$ (d) $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$ (e) $\frac{(2+i)^2}{2-i} - \frac{(2-i)^2}{2+i}$

(f) A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$.

2. Given that $x, y \in R$, solve : (a) $(x + 2y) + i\,(2x - 3y) = 5 - 4i$ (b) $(x + iy) + (7 - 5i) = 9 + 4i$
(c) $x^2 - y^2 - i\,(2x + y) = 2i$ (d) $(2 + 3i)\,x^2 - (3 - 2i)\,y = 2x - 3y + 5i$

3. Find the square root of : (a) $9 + 40\,i$ (b) $-11 - 60\,i$ (c) $50\,i$

4. (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$
(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$

5. Among the complex numbers z satisfying the condition $\left|z + 3 - \sqrt{3}\,i\right| = \sqrt{3}$, find the number having the least positive argument.

6. Solve the following equations over C and express the result in the form $a + ib$, $a, b \in R$.
(a) $ix^2 - 3x - 2i = 0$ (b) $2\,(1 + i)\,x^2 - 4\,(2 - i)\,x - 5 - 3\,i = 0$

7. Locate the points representing the complex number z on the Argand plane:

(a) $|z + 1 - 2i| = \sqrt{7}$; **(b)** $|z - 1|^2 + |z + 1|^2 = 4$; **(c)** $\left|\frac{z-3}{z+3}\right| = 3$; **(d)** $|z - 3| = |z - 6|$

8. If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'.

9. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle z_1, z_2, z_3 is equilateral.

10. For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex?

11. Find the modulus, argument and the principal argument of the complex numbers.

(i) $6\,(\cos 310° - i \sin 310°)$ (ii) $-2\,(\cos 30° + i \sin 30°)$ (iii) $\frac{2+i}{4i + (1+i)^2}$

12. If $(x + iy)^{1/3} = a + bi$; prove that $4\,(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

13. Let z be a complex number such that $z \in c\backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

14. Prove the identity, $|1 - z_1\bar{z}_2|^2 - |z_1 - z_2|^2 = \left(1 - |z_1|^2\right)\left(1 - |z_2|^2\right)$

15. Prove the identity, $|1 + z_1\bar{z}_2|^2 + |z_1 - z_2|^2 = \left(1 + |z_1|^2\right)\left(1 + |z_2|^2\right)$

16. For any two complex numbers, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. Also give the geometrical interpretation of this identity.

17. (a) Find all non–zero complex numbers Z satisfying $\bar{Z} = i\,Z^2$.
(b) If the complex numbers $z_1, z_2, \ldots\ldots\ldots\ldots z_n$ lie on the unit circle $|z| = 1$ then show that
$|z_1 + z_2 + \ldots\ldots\ldots + z_n| = |z_1^{-1} + z_2^{-1} + \ldots\ldots\ldots + z_n^{-1}|$.

18. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z-4|+|z+4| = 16$.

19. Let 'A' denotes the real part of the complex number $z = \frac{19+7i}{9-i} + \frac{20+5i}{7+6i}$
and 'B' denotes the sum of the imaginary parts of the roots of the equation
$z^2 - 8(1-i)z + 63 - 16i = 0$
and 'C' denotes the sum of the series, $1 + i + i^2 + i^3 + \ldots + i^{2008}$ where $i = \sqrt{-1}$.
Find the value of $(A - B + C)$.

20. Let $z = (0, 1) \in C$. Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

EXERCISE–I

1. Simplify and express the result in the form of $a + bi$:

(a) $-i(9+6i)(2-i)^{-1}$ (b) $\left(\frac{4i^3 - i}{2i+1}\right)^2$ (c) $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$

(d) $\frac{(2+i)^2}{2-i} - \frac{(2-i)^2}{2+i}$ (e) $\sqrt{i} + \sqrt{-i}$

2. Find the modulus, argument and the principal argument of the complex numbers.

(i) $z = 1 + \cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)$ (ii) $(\tan 1 - i)^2$

(iii) $z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ (iv) $\frac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

3. Given that $x, y \in R$, solve :

(a) $(x+2y) + i(2x-3y) = 5 - 4i$ (b) $\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$

(c) $x^2 - y^2 - i(2x+y) = 2i$ (d) $(2+3i)x^2 - (3-2i)y = 2x - 3y + 5i$

(e) $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

4. (a) Let Z is complex satisfying the equation, $z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root, then find the value of *m*.

(b) a, b, c are real numbers in the polynomial, $P(Z) = 2Z^4 + aZ^3 + bZ^2 + cZ + 3$
If two roots of the equation $P(Z) = 0$ are 2 and i, then find the value of 'a'.

5. (a) Find the real values of x & y for which $z_1 = 9y^2 - 4 - 10ix$ and $z_2 = 8y^2 - 20i$ are conjugate complex of each other.

(b) Find the value of $x^4 - x^3 + x^2 + 3x - 5$ if $x = 2 + 3i$

6. Solve the following for z :
$z^2 - (3-2i)z = (5i-5)$

7. (a) If $iZ^3 + Z^2 - Z + i = 0$, then show that $|Z| = 1$.

(b) Let z_1 and z_2 be two complex numbers such that $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$.

(c) Let $z_1 = 10 + 6i$ & $z_2 = 4 + 6i$. If z is any complex number such that the argument of, $\frac{z - z_1}{z - z_2}$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

8. Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]......\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$$ is equal to $\left(1 - \frac{1}{2^{2^n}}\right)(1+i)$ where $n \geq 2$.

9. Let z_1, z_2 be complex numbers with $|z_1| = |z_2| = 1$, prove that $|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| \geq 2$.

10. Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ (b) $\text{Re}\left(\frac{z+2i}{iz+2}\right) \leq 4$ $(z \neq 2i)$

(c) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$ (d) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

11. Let $A = \{a \in R \mid$ the equation $(1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0\}$ has at least one real root. Find the value of $\sum_{a \in A} a^2$.

12. P is a point on the Aragand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 . \cos 2\theta = Z_1 . Z_3 \cos^2\theta$.

13. Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundry.

14. For $x \in (0, \pi/2)$ and $\sin x = \frac{1}{3}$, if $\sum_{n=0}^{\infty} \frac{\sin(nx)}{3^n} = \frac{a + b\sqrt{b}}{c}$ then find the value of $(a + b + c)$, where a, b, c are positive integers. (You may Use the fact that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$)

15. Find all real values of the parameter a for which the equation $(a - 1)z^4 - 4z^2 + a + 2 = 0$ has only pure imaginary roots.

16. Let $A \equiv z_1$; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that

$$z_1\bar{z}_2 + \bar{z}_1z_2 = z_2\bar{z}_3 + \bar{z}_2z_3 = z_3\bar{z}_1 + \bar{z}_3z_1$$

hence show that the Δ ABC is a right angled triangle $\Leftrightarrow z_1\bar{z}_2 + \bar{z}_1z_2 = z_2\bar{z}_3 + \bar{z}_2z_3 = z_3\bar{z}_1 + \bar{z}_3z_1 = 0$

17. If the complex number P(w) lies on the standard unit circle in an Argand's plane and $z = (aw + b)(w - c)^{-1}$ then, find the locus of z and interpret it. Given a, b, c are real.

18. (a) Without expanding the determinant at any stage , find $K \in R$ such that

$$\begin{vmatrix} 4i & 8+i & 4+3i \\ -8+i & 16i & i \\ -4+Ki & i & 8i \end{vmatrix}$$ has purely imaginary value.

(b) If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$$ where $i = \sqrt{-1}$ then find the value of D.

19. If w is an imaginary cube root of unity then prove that :

(a) $(1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8)$ ..... to 2n factors = 2^{2n}.

(b) If w is a complex cube root of unity, find the value of $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ ..... to n factors.

20. Prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\dfrac{n\pi}{2}-n\theta\right) + i\sin\left(\dfrac{n\pi}{2}-n\theta\right)$. Hence deduce that

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5 = 0$$

21. If $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -3/2$ then prove that:

(a) $\Sigma \cos 2\alpha = 0 = \Sigma \sin 2\alpha$

(b) $\Sigma \sin(\alpha+\beta) = 0 = \Sigma \cos(\alpha+\beta)$

(c) $\Sigma \sin^2\alpha = \Sigma \cos^2\alpha = 3/2$

(d) $\Sigma \sin 3\alpha = 3\sin(\alpha+\beta+\gamma)$

(e) $\Sigma \cos 3\alpha = 3\cos(\alpha+\beta+\gamma)$

(f) $\cos^3(\theta+\alpha) + \cos^3(\theta+\beta) + \cos^3(\theta+\gamma) = 3\cos(\theta+\alpha).\cos(\theta+\beta).\cos(\theta+\gamma)$ where $\theta \in R$.

22. Resolve $Z^5 + 1$ into linear & quadratic factors with real coefficients. Deduce that : $4\cdot\sin\dfrac{\pi}{10}\cdot\cos\dfrac{\pi}{5} = 1$.

23. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ & $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

24. If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z-2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

25. **(a)** Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x) = \dfrac{1}{x-i}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which $f(a)$, $f(b)$, $f(c)$ and $f(d)$ form a square on the complex plane. Find the area of the square.

EXERCISE–II

1. If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p, q, r are the moduli of non–zero complex numbers u, v, w respectively, prove that, $\arg\dfrac{w}{v} = \arg\left(\dfrac{w-u}{v-u}\right)^2$.

2. Let $Z = 18 + 26i$ where $Z_0 = x_0 + iy_0$ $(x_0, y_0 \in R)$ is the cube root of Z having least positive argument. Find the value of $x_0y_0(x_0 + y_0)$.

3. Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the co-ordinate axes in the Argand's plane. Further show that $|z| = \sqrt{\dfrac{-\text{Im}(z)}{2\,\text{Re}(z)\,\text{Im}(z)+1}}$

4. If ω is the fifth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

5. Prove that, with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p, p', q, q' are all real.
(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.
(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.
State whether these equal roots are real or complex.

6. If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots, z_7$, find the value of

(a) $\sum_{r=1}^{7} \mathrm{Re}(Z_r)$ and (b) $\sum_{r=1}^{7} \mathrm{Im}(Z_r)$

7. Find the roots of the equation $Z^n = (Z + 1)^n$ and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation $\left(2\sin\frac{m\pi}{n}\right)^2 \overline{Z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2 \overline{Z} + 1 = 0$.

8. Dividing f(z) by $z - i$, we get the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

9. Let z_1 & z_2 be any two arbitrary complex numbers then prove that :

$$|z_1 + z_2| \geq \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|.$$

10. If Z_r, r = 1, 2, 3, ... 2m, $m \in N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

11.(i) Let Cr's denotes the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$. If the integers
$a_n = C_0 + C_3 + C_6 + C_9 + \ldots\ldots$
$b_n = C_1 + C_4 + C_7 + C_{10} + \ldots\ldots$
and $c_n = C_2 + C_5 + C_8 + C_{11} + \ldots\ldots$, then
prove that **(a)** $a_n^3 + b_n^3 + c_n^3 - 3a_nb_nc_n = 2^n$, **(b)** $(a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2$.

(ii) Prove the identity: $(C_0 - C_2 + C_4 - C_6 + \ldots)^2 + (C_1 - C_3 + C_5 - C_7 + \ldots)^2 = 2^n$

12. Let z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that :
(i) $2z_2 = (1 + i)z_1 + (1 - i)z_3$ & (ii) $2z_4 = (1 - i)z_1 + (1 + i)z_3$

13. Show that all the roots of the equation $\left(\frac{1 + ix}{1 - ix}\right)^n = \frac{1 + ia}{1 - ia}$ $a \in R$ are real and distinct.

14. Prove that:

(a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots + {}^nC_n \cos(n + 1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos\left(\frac{n + 2}{2}\right)x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots + {}^nC_n \sin(n + 1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin\left(\frac{n + 2}{2}\right)x$

(c) $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + ... + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When $n \in N$.

15. Show that all roots of the equation $a_0z^n + a_1z^{n-1} + ... + a_{n-1}z + a_n = n$, where $|a_i| \le 1$, $i = 0, 1, 2, ..., n$ lie outside the circle with centre at the origin and radius $\frac{n-1}{n}$.

16. The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi-\alpha)$. Show that

$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

17. Evaluate: $\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$.

18. Let a, b, c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$. Find the value of k.

19. Let α, β be fixed complex numbers and z is a variable complex number such that,

$|z-\alpha|^2 + |z-\beta|^2 = k$.

Find out the limits for 'k' such that the locus of z is a circle. Find also the centre and radius of the circle.

20. C is the complex number. $f : C \to R$ is defined by $f(z) = |z^3 - z + 2|$. Find the maximum value of $f(z)$ if $|z| = 1$.

21. Let $f(x) = \log_{\cos 3x}(\cos 2ix)$ if $x \ne 0$ and $f(0) = K$ (where $i = \sqrt{-1}$) is continuous at $x = 0$ then find the value of K.

22. If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with $a, b, c > 0$; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that

$$\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$$

23. Find the set of points on the argand plane for which the real part of the complex number $(1 + i)z^2$ is positive where $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$.

24. If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer. Find N.

25. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ $(a, b, c, d \in R)$ has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of 'b'.

EXERCISE–III

1. (a) If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to :

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

(b) For complex numbers z & ω, prove that, $|z|^2\,\omega - |\omega|^2\,z = z - \omega$ if and only if, $z = \omega$ or $z\overline{\omega} = 1$ **[JEE '99, 2 + 10 (out of 200)]**

2. (i) If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then find the value of, $f(x) + f(\alpha x) + \ldots\ldots + f(\alpha^6 x)$ independent of α. **[REE '99, 6]**

(ii) Let $\alpha + i\beta$; $\alpha, \beta \in R$, be a root of the equation $x^3 + qx + r = 0$; $q, r \in R$. Find a real cubic equation, independent of α & β, whose one root is 2α. **[REE '99, 3]**

3.(a) If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is :

(A) equal to 1 (B) less than 1 (C) greater than 3 (D) equal to 3

(b) If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

[JEE 2000 (Screening) 1 + 1 out of 35]

4. Given, $z = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are, $\alpha = z + z^3 + \ldots\ldots + z^{2n-1}$ & $\beta = z^2 + z^4 + \ldots\ldots + z^{2n}$.

[REE 2000 (Mains) 3 out of 100]

5. Find all those roots of the equation $z^{12} - 56z^6 - 512 = 0$ whose imaginary part is positive.

[REE 2000, 3 out of 100]

6. (a) The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(A) of area zero (B) right-angled isosceles
(C) equilateral (D) obtuse – angled isosceles

(b) Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form

(A) 4k + 1 (B) 4k + 2 (C) 4k + 3 (D) 4k

[JEE 2001 (Scr) 1 + 1 out of 35]

7.(a) Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

(A) 3ω (B) $3\omega(\omega - 1)$ (C) $3\omega^2$ (D) $3\omega(1 - \omega)$

(b) For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(A) 0 (B) 2 (C) 7 (D) 17
[JEE 2002 (Scr) 3+3]

(c) Let a complex number α , $\alpha \neq 1$, be a root of the equation
$z^{p+q} - z^p - z^q + 1 = 0$ where p, q are distinct primes.
Show that either $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$, but not both together.
[JEE 2002, (5)]

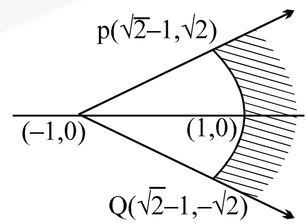
8.(a) If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1 \bar{z}_2}{z_1 - z_2}\right| < 1$.

(b) Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$.
[JEE-03, 2 + 2 out of 60]

9.(a) ω is an imaginary cube root of unity. If $(1 + \omega^2)^m = (1 + \omega^4)^m$, then least positive integral value of m is
(A) 6 (B) 5 (C) 4 (D) 3
[JEE 2004 (Scr)]

(b) Find centre and radius of the circle determined by all complex numbers z = x + i y satisfying $\left|\frac{(z-\alpha)}{(z-\beta)}\right| = k$,
where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex and $k \neq 1$. **[JEE 2004, 2 out of 60]**

10.(a) The locus of z which lies in shaded region is best represented by



(A) $z : |z + 1| > 2, |\arg(z + 1)| < \pi/4$
(B) $z : |z - 1| > 2, |\arg(z - 1)| < \pi/4$
(C) $z : |z + 1| < 2, |\arg(z + 1)| < \pi/2$
(D) $z : |z - 1| < 2, |\arg(z - 1)| < \pi/2$

(b) If a, b, c are integers not all equal and w is a cube root of unity ($w \neq 1$), then the minimum value of $|a + bw + cw^2|$ is
(A) 0 (B) 1 (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$
[JEE 2005 (Scr), 3 + 3]

(c) If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of square. **[JEE 2005 (Mains), 4]**

11. If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\frac{w - \bar{w}z}{1 - z}$ is purely real, then the set of values of z is **[JEE 2006, 3]**
(A) $\{z : |z| = 1\}$ (B) $\{z : z = \bar{z}$) (C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

12.(a) A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is
(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$ (C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

(b) If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on
(A) a line not passing through the origin (B) $|z| = \sqrt{2}$
(C) the x-axis (D) the y-axis **[JEE 2007, 3+3]**

13.(a) A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by

(A) $6 + 7i$ (B) $-7 + 6i$ (C) $7 + 6i$ (D) $-6 + 7i$

(b) **Comprehension (3 questions together)**

Let A, B, C be three sets of complex numbers as defined below **[JEE 2008, 3 + 4 + 4 + 4]**

$A = \{z : \operatorname{Im} z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \operatorname{Re}((1-i)z) = \sqrt{2}\}$

(i) The number of elements in the set $A \cap B \cap C$ is

(A) 0 (B) 1 (C) 2 (D) ∞

(ii) Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

(iii) Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6 (C) –6 and 6 (D) –3 and 9

14. Let $z = \cos\theta + i \sin\theta$. Then the value of $\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$ at $\theta = 2°$ is **[JEE 2009]**

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$ (C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

15. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is **[JEE 2009]**

(A) 48 (B) 32 (C) 40 (D) 80

16. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non-zero complex numbers w, then **[JEE 2010]**

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (B) $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (D) $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z_2 - z_1)$

17. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to **[JEE 2010]**

18. [Note : Here z takes the values in the complex plane and Im z and Rez denote, respectively, the imaginary part and the real part of z] **[JEE 2010]**

Column-I	**Column-II**
(A) The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	(P) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\lvert z + 4\rvert + \lvert z - 4\rvert = 10$ is contained in or equal to	(Q) the set of points z satisfying Im z = 0
(C) If $\lvert w\rvert = 2$, then the set of points $z = w - 1/w$ is contained in or equal to	(R) the set of points z satisfying $\lvert \operatorname{Im} z\rvert \leq 1$
(D) If $\lvert w\rvert = 1$, then the set of points $z = w + 1/w$ is contained in or equal to	(S) the set of point z satisfying $\lvert \operatorname{Re} z\rvert \leq 2$
	(T) the set of points z satisfying $\lvert z\rvert \leq 3$

19. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b, and c is either ω or ω^2. Then the number of distinct matrices in the set S is

(A) 2 (B) 6 (C) 4 (D) 8

[JEE 2011]

20. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is

[JEE 2011]

21. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that **[JEE 2011]**

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z.$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

22. Match the statements given in Column I with the values given in Column II **[JEE 2011]**

Column-I	**Column-II**
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(P) $\dfrac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$, then the value of $f\left(\dfrac{\pi}{6}\right)$ is	(Q) $\dfrac{2\pi}{3}$
(C) The value of $\dfrac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is	(R) $\dfrac{\pi}{3}$
(D) The maximum value of $\left|\mathrm{Arg}\left(\dfrac{1}{1-z}\right)\right|$ for $|z| = 1, z \neq 1$ is given by	(S) π
	(T) $\pi/2$

23. Match the statements given in Column I with the intervals/union of intervals given in Column II

[JEE 2011]

Column-I	**Column-II**
(A) The set $\left\{\mathrm{Re}\left(\dfrac{2iz}{1-z^2}\right) : z \text{ is a complex number}, |z| = 1, z \neq \pm 1\right\}$	(P) $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1 - 3^{2(x-1)}}\right)$ is	(Q) $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \dfrac{\pi}{2}\right\}$ is	(R) $[2, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then f(x) is increasing in	(S) $(-\infty, -1] \cup [1, \infty)$
	(T) $(-\infty, 0] \cup [2, \infty)$

24. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value **[JEE 2012]**

(A) -1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

25. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals :

(A) $\pi - \theta$ (B) $-\theta$ (C) $\frac{\pi}{2} - \theta$ (D) θ

[IIT JEE Main 2013]

26. Let complex numbers α and $\frac{1}{\alpha}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ **[IIT JEE Advance 2013]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

27. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[IIT JEE Advance 2013]**

(A) 57 (B) 55 (C) 58 (D) 56

28. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[IIT JEE Advance 2013]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Comprehension (Q.29 to Q.30) **[IIT JEE Advance 2013]**

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left[z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right]$ and $S_3 = \{z \in C : \text{Re}\, z > 0\}$.

29. $\min\limits_{z \in S} |1 - 3i - z| =$

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

30. Area of S =

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

31. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$ **[IIT JEE Main 2014]**

(A) Is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$ (B) Is equal to $\frac{5}{2}$

(C) Lies in the interval (1, 2) (D) Is strictly greater than $\frac{5}{2}$

32. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ...9. **[IIT JEE Advanced 2014]**

	List-I		List-II
(P)	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	(1)	True
(Q)	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	(2)	False
(R)	$\frac{|1-z_1||1-z_2|.............|1-z_9|}{10}$ equals	(3)	1
(S)	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	(4)	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

33. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a: **[JEE Mains 2015]**

(A) circle of radius $\sqrt{2}$ (B) straight line parallel to x-axis

(C) straight line parallel to y-axis (D) circle of radius 2

34. For any integer k, let $a_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}|a_{k+1}-a_k|}{\sum_{k=1}^{3}|a_{4k-1}-a_{4k-2}|}$$ is **[JEE Advanced 2015]**

35. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE Mains 2016]**

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (D) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

36. Let a, b $\in$ R and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in C : z = \frac{1}{a+ibt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If z = x + iy and z $\in$ S, then (x, y) lies on **[JEE Advanced 2016]**

(A) The circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for a > 0, b $\neq$ 0

(B) The circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for a < 0, b $\neq$ 0

(C) The x-axis for a $\neq$ 0, b = 0

(D) The y-axis for a = 0, b $\neq$ 0

37. Let $z = \frac{-1+\sqrt{3}i}{2}$, where i = $\sqrt{-1}$, and r, s $\in$ {1, 2, 3}. Let P = $\begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs(r, s) for which $P^2 = -I$ is : **[JEE Advanced 2016]**

38. Let ω be a complex number such that $2\omega + 1 = z$ where z = $\sqrt{-3}$.

If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix}$ = 3k, then k is equal to : **[JEE Mains 2017]**

(A) –1 (B) 1 (C) – z (D) z

39. Let a, b, x and y be real numbers such that a – b = 1 and y $\neq$ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x? **[JEE Advanced 2017]**

(A) $-1+\sqrt{1-y^2}$ (B) $1-\sqrt{1+y^2}$ (C) $1+\sqrt{1+y^2}$ (D) $-1-\sqrt{1-y^2}$

40. If $\alpha, \beta \in$ C are the distinct roots, of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to : **[JEE Main 2018]**

(A) 2 (B) –1 (C) 0 (D) 1

41. For a non-zero complex number z, let arg(z) denote the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is(are) **FALSE**? **[JEE Advanced 2018]**

(A) arg(–1 –i) = $\frac{\pi}{4}$, where i = $\sqrt{-1}$

(B) The function f : R $\to$ ($-\pi$, π], defined by f(t) = arg(–1 + it) for all t $\in$ R, is continuous at all points of R, where i = $\sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

42. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ ($x, y \in \mathbb{R}$, $i = \sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE?

(A) If L has exactly one element, then $|s| \neq |t|$ **[JEE Advanced 2018]**

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) Let L has more than one element, then L has infinitely many elements

43. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

[JEE Advanced 2019]

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

44. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$\{|a + b\omega + c\omega^2|^2 : a, b, c$ distinct non-zero integers$\}$

equals________ **[JEE Advanced 2019]**

45. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ? **[JEE Advanced 2020]**

(A) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$

(B) $|z| \leq 2$ for all $z \in S$

(C) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

46. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____. **[JEE Advanced 2020]**

47. Let $\theta_1, \theta_2, \ldots\ldots\theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \ldots.. + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for k = 2,3, ...,10, where $i = \sqrt{-1}$.

Consider the statements P and Q given below :

$P : |z_2 - z_1| + |z_3 - z_2| + \ldots\ldots. + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$

$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \ldots.. + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$

Then, **[JEE Advanced 2021]**

(A) P is **TRUE** and Q is **FALSE**

(B) Q is **TRUE** and P is **FALSE**

(C) both P and Q are **TRUE**

(D) both P and Q are **FALSE**

48. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z + \alpha}{z + \beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0$$

Then which of the following statements is (are) **TRUE** ? **[JEE Advanced 2021]**

(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

ANSWER KEY
VERY ELEMENTARY EXERCISE

Q.1 **(a)** $\frac{7}{25}+\frac{24}{25}$ i; **(b)** $\frac{21}{5}-\frac{12}{5}$ i; **(c)** 3 + 4i; **(d)** $-\frac{8}{29}$ + 0i; **(e)** $\frac{22}{5}$ i; **(f)** 15

Q.2 **(a)** x =1, y = 2; **(b)** (2, 9); **(c)** (–2 , 2) or $\left(-\frac{2}{3},-\frac{2}{3}\right)$; **(d)** (1 ,1) $\left(0,\frac{5}{2}\right)$

Q.3 **(a)** ± (5 + 4i) ; **(b)** ± (5 – 6i) **(c)** ± 5(1 + i)

Q.4 **(a)** –160 ; **(b)** – (77 +108 i)

Q.5 $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$

Q.6 **(a)** – i , – 2i **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

Q.7 **(a)** on a circle of radius $\sqrt{7}$ with centre (–1, 2) **;** **(b)** on a unit circle with centre at origin
(c) on a circle with centre (–15/4, 0) & radius 9/4 **;** **(d)** a straight line

Q.8 a = b = 2 – $\sqrt{3}$;

Q.9 $z_3 = \sqrt{3}(1-i)$ and $z'_3=\sqrt{3}(-1-i)$

Q.10 x = 1, y = – 4 or x = – 1, y = – 4

Q.11 **(i)** Modulus = 6 , Arg = $2k\pi+\frac{5\pi}{18}$ (K ∈ I) , Principal Arg = $\frac{5\pi}{18}$ (K ∈ I)

(ii) Modulus = 2 , Arg = $2k\pi+\frac{7\pi}{6}$, Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\frac{\sqrt{5}}{6}$, Arg = $2k\pi-\tan^{-1}2$ (K ∈ I) , Principal Arg = $-\tan^{-1}2$

Q.17 **(a)** $\frac{\sqrt{3}}{2}-\frac{i}{2},-\frac{\sqrt{3}}{2}-\frac{i}{2}$, i

Q.18 $\frac{x^2}{64}+\frac{y^2}{48}=1$

Q.19 13

Q.20 $\begin{cases}(1,0) & \text{for } n=4k\\ (1,1) & \text{for } n=4k+1\\ (0,1) & \text{for } n=4k+2\\ (0,0) & \text{for } n=4k+3\end{cases}$

EXERCISE–I

Q.1 (a) $\frac{21}{5}-\frac{12}{5}$ i **(b)** 3 + 4 i **(c)** $-\frac{8}{29}$ + 0 i **(d)** $\frac{22}{5}$i **(e)** $\pm\sqrt{2}+0i$ or $0\pm\sqrt{2}\,i$

Q.2 **(i)** Principal Arg z = $-\frac{4\pi}{9}$; |z| = 2 cos $\frac{4\pi}{9}$; Arg z = $2k\pi-\frac{4\pi}{9}$ k ∈ I

(ii) Modulus = $\sec^2 1$, Arg = $2n\pi+(2-\pi)$, Principal Arg = $(2-\pi)$

(iii) Principal value of Agr z = $-\frac{\pi}{2}$ & |z| = $\frac{3}{2}$; Principal value of Arg z = $\frac{\pi}{2}$ & |z| = $\frac{2}{3}$

(iv) Modulus = $\frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$, Arg z = $2n\pi+\frac{11\pi}{20}$, Principal Arg = $\frac{11\pi}{20}$

Q.3(a) x = 1, y = 2; **(b)** x = 1 & y = 2 ; **(c)** (–2 , 2) or $\left(-\frac{2}{3},-\frac{2}{3}\right)$; **(d)** (1 ,1) $\left(0,\frac{5}{2}\right)$; **(e)** x =K, y = $\frac{3K}{2}$ K∈R

Q.4 **(a)** 2, **(b)** – 11/2

Q.5 **(a)** [(– 2, 2) ; (– 2, – 2)] **(b)** – (77 +108 i)

Q.6 z = (2 + i) or (1 – 3i)

Q.7 **(b)** 2

Q.10 **(a)** The region between the co encentric circles with centre at (0 , 2) & radii 1 & 3 units

(b) region outside or on the circle with centre $\frac{1}{2}$ + 2i and radius $\frac{1}{2}$.

(c) semi circle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$ **(d)** a ray emanating from the point (3 + 4i) directed away from the origin & having equation $\sqrt{3}\,x - y + 4 - 3\sqrt{3} = 0$

Q.11 18 **Q.14** 41 **Q.15** [–3 , –2] **Q.17** $(1 - c^2)\,|z|^2 - 2(a + bc)(\text{Re } z) + a^2 - b^2 = 0$

Q.18 (a) K = 3 , **(b)** – 4 **Q.19** **(b)** one if n is even ; $-w^2$ if n is odd

Q.22 $(Z + 1)(Z^2 - 2Z\cos 36° + 1)(Z^2 - 2Z\cos 108° + 1)$ **Q.24** 4

Q.25 **(a)** $\pi - 2$; **(b)** 1/2

EXERCISE–II

Q.2 12 **Q.6** (a) $-\frac{7}{2}$, (b) zero **Q.8** $\frac{iz}{2} + \frac{1}{2} + i$ **Q.17** 48(1 - i)

Q.18 $-\omega$ or $-\omega^2$ **Q.19** $k > \frac{1}{2}|\alpha - \beta|^2$

Q.20 | f (z) | is maximum when z = ω, where ω is the cube root unity and | f (z) | = $\sqrt{13}$

Q.21 K = – 4/9

Q.23 required set is constituted by the angles without their boundaries, whose sides are the straight lines $y = (\sqrt{2} - 1)\,x$ and $y + (\sqrt{2} + 1)\,x = 0$ containing the x – axis

Q.24 198 **Q.25** 51

EXERCISE–III

Q.1 (a) C

Q.2 (i) $7A_0 + 7A_7x^7 + 7A_{14}x^{14}$; (ii) $x^3 + qx - r = 0$

Q.3 (a) A **(b)** A **Q.4** $z^2 + z + \frac{\sin^2 n\theta}{\sin^2\theta} = 0$, where $\theta = \frac{2\pi}{2n + 1}$

Q.5 $\pm 1 + i\sqrt{3}$, $\frac{(\pm\sqrt{3} + i)}{\sqrt{2}}$, $\sqrt{2}\,i$ **Q.6 (a)** C, **(b)** D **Q.7** **(a)** B ; **(b)** B

Q.9 **(a)** D ; **(b)** Centre $\equiv \frac{k^2\beta - \alpha}{k^2 - 1}$, Radius = $\frac{1}{(k^2 - 1)}\sqrt{|\alpha - k^2\beta|^2 - (k^2.|\beta|^2 - |\alpha|^2)(k^2 - 1)}$

Q.10 **(a)** A, **(b)** B, **(c)** $z_2 = -\sqrt{3}\,i$; $z_3 = (1 - \sqrt{3}) + i$; $z_4 = (1 + \sqrt{3}) - i$ **Q.11** D

Q.12 **(a)** D ; **(b)** D

Q.13 **(a)** D ; **(b)** (i) B; (ii) C; (iii) D

Q14. D **Q15.** A

16. A, C, D **17.** 1

18. A – Q, R ; B – P ; C – P, S, T ; D – Q, R, S, T **19.** A **20.** 5

21. **Bonus**

22. A → Q ; B → P ; C → S ; D → T ; **23.** A → S ; B → T ; C → R ; D → R ;

24.	D	**25.**	D	**26.**	C	**27.**	B, C, D	**28.**	C, D	**29.**	C
30.	B	**31.**	C	**32.**	C	**33.**	D	**34.**	4	**35.**	D
36.	A,C,D	**37.**	1	**38.**	C	**39.**	AD	**40.**	D	**41.**	A,B,D
42.	A,C,D	**43.**	C	**44.**	3.00	**45.**	BC	**46.**	8	**47.**	C
48.	B, D										