DPP-02

LINEAR FIRST ORDER

1. The solution of the equation $x\frac{dy}{dx} + 3y = x$ is-

(A) $x^3y + \frac{x^4}{4} + c = 0$

(B) $x^3y = \frac{x^4}{4} + c$

(C) $x^3y + \frac{x^4}{4} = 0$

(D) None of these

Ans. (B)

Sol.

$$\frac{dy}{dx} + \frac{3y}{x} = 1$$

$$\text{I.f.} = e^{\int \frac{3}{x}dx} = x^3$$

$$\Rightarrow y.\, x^3 = \int x^3\, dx$$

$$yx^3 = \frac{x^4}{4} + c$$

2. The solution of $(1 + y^2)dx = (\tan^{-1} y - x)dy$ is -

(A) $xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

(B) $xe^{\tan^{-1} y} = (\tan^{-1} y + 1) - c$

(C) $xe^{\tan^{-1} y} = (\tan^{-1} y - 1) + c$

(D) None of these

Ans. (A)

Sol.

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F. } e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$$\Rightarrow x.\, e^{\tan^{-1} y} = \int e^{\tan^{-1} y}.\frac{\tan^{-1} y}{1+y^2}\, dy$$

$$\Rightarrow xe^{\tan^{-1} y} = e^{\tan^{-1} y}\,(\tan^{-1} y - 1) + c$$

3. The solution of the differential equation $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is

(A) $x + y = ce^{2x}$

(B) $y^2 = 2x^3 + c$

(C) $xy^2 = 2y^5 + c$

(D) $x(y^2 + xy) = 0$

Ans. (C)

Sol.

The given equation can be written as:

$\frac{dx}{dy} + \frac{2}{y}x = 10y^2$

...(1)

[Linear Equation in x]

Here 'P' = 2/y and 'Q' = $10y^2$.

I.F. $= e^{\int P.dy} = e^{\int \frac{2}{y}dy} = e^{2\log|y|}$

$= e^{\log y^2} = y^2$

Multiplying (1) by y^2, we get :

$y^2\frac{dx}{dy} + 2yx = 10y^4 \Rightarrow \frac{d}{dy}(x.y^2) = 10y^4$

Integrating, $xy^2 = 10\int y^4\,dy + c$

$\Rightarrow xy^2 = 2y^5 + c$ which is required solution.

4. The solution of the differential equation $\frac{dy}{dx} + y = \cos x$ is-

(A) $y = 1/2(\cos x + \sin x) + ce^{-x}$

(B) $y = 1/2(\cos x - \sin x) + ce^{-x}$

(C) $y = \cos x + \sin x + ce^{-x}$

(D) None of these

Ans. (A)

Sol.

$\frac{dy}{dx} + y = \cos x$

I.F. $= e^{\int dx} = e^x$

$y.\, e^x = \int e^x \cos x\, dx$

$y.\, e^x = e^x \cos x + \int e^x \sin x\, dx$

$\Rightarrow y.\, e^x = e^x \cos x + e^x \sin x - y.\, e^x + c$

$\Rightarrow 2y = (\cos x + \sin x) + 2ce^{-x}$

$\Rightarrow y = \frac{1}{2}(\cos x + \sin x) + ce^{-x}$

5. The solution of the differential equation, $\frac{dy}{dx} + \frac{y}{x} = x^2$ is-

(A) $4xy = x^4 + c$ (B) $xy = x^4 + c$

(C) $\frac{1}{4}xy = x^4 + c$ (D) $xy = 4x^4 + c$

Ans. (A)

Sol.

$\frac{dy}{dx} + \frac{y}{x} = x^2$

I.F. $= e^{\int \frac{1}{x} dx} = x$

$y.\, x = \int x^3\, dx$

$\Rightarrow xy = \frac{x^4}{4} + c \Rightarrow 4xy = x^4 + c$

6. The solution of the differential equation $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$, is-

(A) $xe^{2\tan^{-1} y} = e^{\tan^{-1} y} + k$ (B) $(x - 2) = ke^{-\tan^{-1} y}$

(C) $2xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + k$ (D) $xe^{\tan^{-1} y} = \tan^{-1} y + k$

Ans. (C)

Sol.

Given differential equation can be rewritten as

$$(1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

or $$\frac{dx}{xy}+\frac{1}{a+y^2}x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

$\therefore$ $$\text{IF}=e^{\int P dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Therefore, required solution is

$$xe^{\tan^{-1}y}=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+c_1$$

$\Rightarrow$ $$xe^{\tan^{-1}y}=\frac{1}{2}e^{2\tan^{-1}y}+c_1$$

$\Rightarrow$ $$2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c$$

7. The solution of the differential equation, $e^x(x+1)dx+(ye^y-xe^x)dy=0$ with initial condition $f(0)=0$, is

(A) $xe^x+2y^2e^y=0$ (B) $2xe^x+y^2e^y=0$

(C) $xe^x-2y^2e^y=0$ (D) $2xe^x-y^2e^y=0$

Ans. (B)

Sol.

$e^x(x+1)\,dx+(ye^y-xe^x)\,dy=0$

Put $xe^x=t$ $\Rightarrow$ $e^x(x+1)\,dx=dt$

$dt+(ye^y-t)\,dy=0$

$$\frac{dy}{dt}=\frac{1}{t-ye^y}\Rightarrow\frac{dt}{dy}=t-ey^y$$

$$\frac{dt}{dy}-t=-ye^y$$

I.F. $=e^{-y}$

$$(xe^x)(e^{-y})=\frac{-y^2}{2}+C;\quad(0,0)\Rightarrow C=0$$

$$(xe^x)(e^{-y})=\frac{-y^2}{2}$$

$2xe^x+y^2e^y=0$

8. The solution of $y^5x + y - x\frac{dy}{dx} = 0$ is

(A) $x^4/4 + 1/5(x/y)^5 = C$

(B) $x^5/5 + (1/4)(x/y)^4 = C$

(C) $(x/y)^5 + x^4/4 = C$

(D) $(xy)^4 + x^5/5 = C$

Ans. (B)

Sol.

$$y^5x + y - x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - y - y^5x = 0$$

$$\frac{dy}{dx} - \frac{y}{x} = y^5 \qquad \text{(LDE)}$$

$$\frac{1}{y^5}\frac{dy}{dx} - \frac{1}{y^4x} = 1$$

$$\frac{1}{y^4} = t \qquad \Rightarrow \frac{1}{y^5}\frac{dy}{dx} = -\frac{1}{4}\frac{dt}{dx}$$

$$-\frac{1}{4}\frac{dt}{dx} - \frac{t}{x} = 1$$

$$\frac{dt}{dx} + \frac{4t}{x} = -4 \qquad \text{(LDE)}$$

$$\text{I.F.} = e^{\int \frac{4}{x}dx} = x^4$$

$$x^4 . t = -\frac{4x^5}{5} + C$$

$$\frac{x^4}{y^4} = -\frac{4x^5}{5} + C$$

$$\frac{x^5}{5} + \frac{1}{4}\left(\frac{x}{y}\right)^4 = k$$

9. The solution of the differential equation, $x^2\frac{dy}{dx}\cdot\cos\frac{1}{x}-y\sin\frac{1}{x}=-1$, where $y\to-1$ as $x\to\infty$ is

(A) $y=\sin\frac{1}{x}-\cos\frac{1}{x}$ (B) $y=\frac{x+1}{x\sin\frac{1}{x}}$

(C) $y=\cos\frac{1}{x}+\sin\frac{1}{x}$ (D) $y=\frac{x+1}{x\cos\frac{1}{x}}$

Ans. (A)

Sol.

$$x^2\frac{dy}{dx}\,.\cos\frac{1}{x}-y\sin\frac{1}{x}=-1$$

$$\frac{dy}{dx}-\frac{y}{x^2}\tan\frac{1}{x}=\frac{-1}{x^2}\sec\frac{1}{x}$$

$$\text{I.F.}=e^{-\int\frac{1}{x^2}\tan\frac{1}{x}dx}=\sec\frac{1}{x}$$

$$y\left(\sec\frac{1}{x}\right)=\int\frac{-1}{x^2}\sec^2\frac{1}{x}\,dx$$

$$y\left(\sec\frac{1}{x}\right)=\tan\frac{1}{x}+C$$

$$C=-1$$

$$y\left(\sec\frac{1}{x}\right)=\tan\frac{1}{x}-1$$

$$y=\sin\frac{1}{x}-\cos\frac{1}{x}$$

10. The solution of $\frac{dy}{dx}+y\tan x=\sec x$ is-

(A) $y\sec x=\tan x+c$ (B) $y\tan x=\sec x+c$

(C) $\tan x=y\tan x+c$ (D) $x\sec x=y\tan y+c$

Ans. (C)

Sol.

$$\frac{dy}{dx}+y\tan x=\sec x$$

$$\text{I.F.}=e^{\int \tan x dx}=e^{\log\sec x}=\sec x$$

$$=\frac{1}{\cos x}$$

$$y \cdot \frac{1}{\cos x} = \int \frac{\sec x}{\cos x} dn + c$$

ysecx = tanx + 1

11. The solution of the equation $\frac{dy}{dx} + y\tan\ x = x^m \cos\ x$ is-

(A) $(m+1)y = x^{m+1}\cos\ x + c(m+1)\cos\ x$

(B) $my = (x^m + c)\cos\ x$

(C) $y = (x^{m+1} + c)\cos\ x$

(D) None of these

Ans. **(A)**

Sol.

$$\frac{dy}{dx} + y \tan x = x^m \cos x$$

$$\text{I.F.} = e^{\int \tan x} = \sec x$$

$$y.\sec x = \int x^m . \cos x . \sec x dx$$

$$y.\sec x = \frac{x^{m+1}}{m+1} + C$$

$$\Rightarrow (m+1)\ y = x^{m+1} \cos x + C(m+1) \cos x$$

12. The solution of the differential equation $\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$ is -

(A) $y(1+x^3) = x + 1/2\sin\ 2x + c$

(B) $y(1+x^3) = cx + 1/2\sin\ 2x$

(C) $y(1+x^3) = cx - 1/2\sin\ 2x$

(D) $y(1+x^3) = \frac{x}{2} - \frac{1}{4}\sin\ 2x + c$

Ans. **(D)**

Sol.

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$$

$$\text{I.F.} = e^{\int \frac{3x^2}{1+x^3} dx} = 1 + x^3$$

$$\Rightarrow \quad y.(1+x^3) = \int \sin^2 x dx$$

$$\Rightarrow \quad y(1+x^3) = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$\Rightarrow \quad y(1+x^2) = \frac{1}{2}x - \frac{1}{4} \sin 2x + c$$

13. The solution of the equation $(1-x^2)dy + xydx = xy^2dx$ is-

(A) $(y-1)^2(1-x^2) = 0$
(B) $(y-1)^2(1-x^2) = c^2y^2$
(C) $(y-1)^2(1+x^2) = c^2y^2$
(D) None of these

Ans. (B)

Sol.

$(1-x^2)dy + xy\,dx = xy^2dy$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{(1-x^2)}\cdot\frac{1}{y} = \frac{x}{1-x^2}$

Let $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad \frac{dt}{dx} - \frac{x}{1-x^2}t = \frac{-x}{1-x^2}$

I.F. $= e^{-\int\frac{x}{1-x^2}dx} = (1-x^2)^{1/2}$

$\Rightarrow \quad t.(1-x^2)^{1/2} = -\int\frac{x}{(1-x^2)^{1/2}}dx$

$\Rightarrow \quad \frac{(1-x^2)^{1/2}}{y} = (1-x^2)^{1/2} - c$

$\Rightarrow \quad (1-x^2)(y-1)^2 = c^2y^2$

14. The graph of the function $y = f(x)$ passing through the point $(0,1)$ and satisfying the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ is such that

(A) it is a constant function
(B) it is periodic
(C) it is neither an even nor an odd function
(D) it is continuous & differentiable for all x.

Ans. (ABD)

Sol.

$$\frac{dy}{dx} + y\cos x = \cos x$$

$$\text{I.F.} = e^{\int \cos x\,dx} = e^{\sin x}$$

$$y\,e^{\sin x} = \int e^{\sin x}\cos x\,dx = e^{\sin x} + C$$

$$y = 1 + ce^{-\sin x} \quad ; \quad C = 0 \quad \text{as } (0, 1)$$

$$y = 1$$

15. The solution of $\left(\frac{dy}{dx}\right)(x^2y^3 + xy) = 1$ is

(A) $1/x = 2 - y^2 + Ce^{-y^2/2}$

(B) the solution of an equation which is reducible to linear equation.

(C) $2/x = 1 - y^2 + e^{-y/2}$

(D) $\frac{1-2x}{x} = -y^2 + \text{C}e^{-y^2/2}$

Ans. (ABD)

Sol.

$$\frac{dx}{dy} = xy + x^2y^2 \Rightarrow \frac{1}{x^2}\frac{dy}{dx} - \frac{y}{x} = y^3$$

$$\text{Let } -\frac{1}{x} = t$$

$$\text{or } \frac{1}{x^2}\,dx = dt \text{ or } \frac{dt}{dy} + yt = y^3$$

$$\text{I.F.} = e^{\int y\,dy} = e^{y^2/2} \text{ or}$$

$$te^{y^2/2} = \int y^3e^{y^2/2}\,dy + c$$

$$\text{or } te^{y^2/2} = y^2e^{y^2/2} - 2e^{y^2/2} + c$$

$$\text{or } -\frac{1}{x} = y^2 - 2 + ce^{-y^2/2}$$

$$\text{or } \frac{1}{x} = 2 - y^2 + ce^{-y^2/2}$$

16. Let $y = y(t)$ be a solution to the differential equation $y' + 2ty = t^2$, then find $\text{Lim}_{t\to\infty} \frac{y}{t}$.

Ans. $\frac{1}{2}$

Sol.

$$y = y(t)$$

$$\frac{dy}{dt} + 2yt = t^2$$

$$\text{I.F.} = e^{\int 2t\,dt} = e^{t^2}$$

$$= \frac{\int t^2 e^{t^2} dt}{e^{t^2}}$$

$$\lim_{t\to\infty} \frac{y}{t} = \lim_{t\to\infty} \frac{\int t^2 e^{t^2} dt}{t\,e^{t^2}} = \lim_{t\to\infty} \frac{t^2 e^{t^2}}{e^{t^2} + 2t^2 e^{t^2}}$$

$$= \lim_{t\to\infty} \frac{t^2}{2t^2+1} = \frac{1}{2}$$

17. $(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$

Ans. $y = c(1-x^2) + \sqrt{1-x^2}$

Sol.

$$(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$$

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2} dx} = e^{-\ell n(1-x^2)} = \frac{1}{1-x^2}$$

$$\frac{y}{1-x^2} = \int \frac{x}{(1-x^2)^{3/2}} dx$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C$$

$$y = C(1-x^2) + \sqrt{1-x^2}$$

18. $(1+y^2)dx = (\tan^{-1} y - x)dy$

Ans. $x = ce^{-\arctan y} + \arctan y - 1$

Sol.

$(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

Put $\quad \tan^{-1}y = t \quad \Rightarrow \left(\frac{1}{1+y^2}\right) dy = dt$

$dx = (t - x)\, dt \quad \Rightarrow \frac{dx}{dt} + x = t$

I.F. $= e^{\int 1.dt} = e^t$

$x\, e^t = \int t\, e^t\, dt$

$xe^t = te^t - e^t + c$

$x = t - 1 + ce^{-t}$

$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

19. $\frac{dy}{dx} - y\ln 2 = 2^{\sin x} \cdot (\cos x - 1)\ln\ 2$, y being bounded when $x \to +\infty$.

Ans. $y = 2^{\sin x}$

Sol.

I.f. $= e^{\int \ell n\, 2dx} = (1/2)^x$

so $y(1/2)^x = \int 2^{(\sin x - x)}(\cos x - 1)\, \ell n\, 2\, dx$

$\sin x - x = t \qquad$ (let)

$\Rightarrow y \cdot \left(\frac{1}{2}\right)^x = 2^{\sin x - x} + c$

$c = 0$ as $x \to \infty$ so $y = 2^{\sin x}$

20. Consider the differential equation, $\frac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of $u(x)$ and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If $w(x)$ is the third particular solution different from u(x) and v(x) then find the ratio $\frac{v(x)-u(x)}{w(x)-u(x)}$.

Ans. (i) $y = u(x) + K(u(x) - v(x))$ where K is any constant ;
(ii) $\alpha + \beta = 1$;
(iii) constant

Sol.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

(i) $u' + Pu = Q$
$v' + Pv = Q$

subtract $Q = \frac{uv'-vu'}{u-v}$

$$\int \frac{u'-v'}{u-v} = \int -P$$

$$\ell n\,(u-v) = -\int P\,dx$$

$$u - v = e^{-\int P\,dx}$$

$$IF = e^{\int P\,dx} = \frac{1}{u-v}$$

$$y\left(\frac{1}{u-v}\right) = \int \frac{uv'-vu'}{(u-v)^2}dx$$

$$= \int \frac{d(v/u)}{\left(1-\frac{v}{u}\right)^2}$$

$$y\left(\frac{1}{u-v}\right) = \frac{u}{u-v} + k$$

$$y = u + k\,(u-v)$$

(ii) $\alpha = 1 + k, \quad \beta = -k$
$\alpha + \beta = 1$

(iii) $u - v = e^{-\int p dx}$

$$u - w = e^{-\int p dx} \Rightarrow \frac{1}{u-w} \Rightarrow e^{\int p dx}$$

$$\frac{u-v}{u-w} = e^{-\int p dx} \times e^{\int p dx} = \text{constant}$$

21. $(1-x^2)^2 dy + \left(y\sqrt{1-x^2} - x - \sqrt{1-x^2}\right)dx = 0.$

Ans. $y = \frac{x}{\sqrt{1-x^2}} = ce^{-\frac{x}{\sqrt{1-x^2}}}$

Sol.

$$\frac{dy}{dx} + y\frac{1}{(1-x^2)^{3/2}} = \frac{x+\sqrt{1-x^2}}{(1-x^2)^{3/2}}$$

$$\text{I.F.} = e^{\int \frac{1}{(1-x^2)^{3/2}}dx} = \frac{x}{e^{\sqrt{1-x^2}}}$$

$$y\frac{x}{e^{\sqrt{1-x^2}}} = \int \frac{x}{e^{\sqrt{1-x^2}}}\frac{x+\sqrt{1-x^2}}{(1-x^2)^{5/2}}dx$$

$$y\frac{x}{e^{\sqrt{1-x^2}}} =$$

Let $\frac{x}{\sqrt{1-x^2}} = t$

22. Find the integral curve of the differential equation, $x\,(1 - x\ell ny)\cdot\frac{dy}{dx} + y = 0$ which passes through $\left(1,\frac{1}{e}\right)$

Ans. $x(ey + \ell ny + 1) = 1$

Sol.

$$x(1 - x\,\ell n\,y).\frac{dy}{dx} + y = 0$$

$$\Rightarrow y.\frac{dy}{dx} + x = x^2 \log y$$

divding by $x^2 y$

$$\frac{1}{x^2}\frac{dx}{dy} + \frac{1}{xy} = \frac{1}{y}\log y$$

$$\text{Let } \frac{1}{x} = t \Rightarrow -\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$$

$$\Rightarrow -\frac{dt}{dy} + \frac{t}{y} = \frac{1}{y}\log y \Rightarrow \frac{dt}{dy} - \frac{t}{y} = -\frac{1}{y}\log y$$

$$\text{If} \Rightarrow e^{-\int \frac{1}{y}dy} = e^{-\log y} = 1/y$$

$$\text{sot } .1/y = \int -\frac{1}{y^2}\log y\,dy = \frac{1}{y}\log y - \int \frac{1}{y}.\frac{1}{y}dy$$

$$\Rightarrow t = \log y + 1 + cy \Rightarrow \log ex + cy$$

$$\Rightarrow \frac{1}{x} = \log e\,y + cy \Rightarrow x\,(\log ey + cy) = 1$$

23. A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If ' m ' grams of salt are present in the tank after t minute, express ' m ' in terms of t and find the amount of salt present after 10 minutes.

Ans. $y=5t\left(1+\frac{50}{50+t}\right) gms; 91\frac{2}{3}$ gms

Sol.

$$\frac{dm}{dt}=10-\left(\frac{m}{50+t}\right)$$

$$\frac{dm}{dt}+\frac{m}{50+t}=10$$

$$IF=e^{\int\frac{dt}{50+t}}=50+t$$

$$m(50+t)=\int 10(50+t)dt$$

$$m(50+t)=10\left(50t+\frac{t^2}{2}\right)$$

$$m=5\left(\frac{100t+t^2}{50+t}\right)$$

$$=5t\left(\frac{100+t}{50+t}\right)$$

$$=5t\left(1+\frac{50}{50+t}\right)$$

at $t=10$

$$m=91\frac{2}{3}$$

24. Find all functions f(x) defined on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with real values and has primitive $F(x)$ such that $f(x)+\cos x\, F(x)=\frac{\sin 2x}{(1+\sin x)^2}$. Find $f(x)$

Ans. $f(x)=-\frac{2\cos x}{(1+\sin x)^2}-Ce^{-\sin x}\cdot\cos x$

Sol.

Given that $F(x) = \int f(x)dx \Rightarrow f(x) = F'(x)$

$$F'(x) + \cos x.\ F(x) = \frac{\sin 2x}{(1+\sin x)^2}$$

$$\frac{dy}{dx} + y\cos x = \frac{\sin 2x}{(1+\sin x)^2}$$

$$\text{I.F.} = e^{\int \cos x\, dx} = e^{\sin x}$$

$$y(e^{\sin x}) = \int \frac{2\sin \cos x}{(1+\sin)^2}\, e^{\sin x}\, dx$$

put $\sin x = t$

$$= 2\int \frac{t}{(1+t)^2}\, e^t\, dt$$

$$y\,(e^{\sin x}) = \frac{2e^{\sin x}}{1+\sin x} + c$$

$$F(x) = \frac{2}{1+\sin x} + c\, e^{-\sin x}$$

$$F'(x) = \frac{-2\cos x}{(1+\sin x)^2} - c\, e^{-\sin x}\cos x$$

25. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition y (1) = 1 is- +

[AIEEE 2008]

(A) $y = x\ln x + x^2$ (B) $y = xe^{(x-1)}$

(C) $y = x\ln x + x$ (D) $y = \ln x + x$

Ans. (C)

Sol.

Given equation can be rewritten as

$$\frac{dy}{dx} - \frac{1}{x}y = 1$$

Now, $\text{IF} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\therefore$ Required solution is

$$y\left(\frac{1}{x}\right) = \int \frac{1}{x}dx = \log x + c$$

Since, $y(1) = 1 \Rightarrow 1 = \log 1 + c$

$\Rightarrow c = 1$

$\therefore y = x\log x + x$

26. Solution of the differential equation $\cos xdy = y(\sin x - y)dx, 0 < x < \frac{\pi}{2}$ is -

[AIEEE 2010]

(A) $\sec x = (\tan x + c)y$ (B) $y\sec x = \tan x + c$

(C) $y\tan x = \sec x + c$ (D) $\tan x = (\sec x + c)y$

Ans. (A)

Sol.

Since, $\cos x\, dy = y\sin x\, dx - y^2\, dx$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Put, $-\frac{1}{y} = a \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{dz}{dx} + (\tan x)z = -\sec x$$

This is a linear differential equation.
Therefore

$$\text{IF} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is

$$z\cdot(\sec x) = \int -\sec x \cdot \sec x\, dx + c_1$$

$$\Rightarrow -\frac{1}{y}\sec x = -\tan x + c_1$$

$$\Rightarrow \sec x = y(\tan x + c)$$

27. Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx} + y = 2x\log x, (x \geq 1)$. Then $y(e)$ is equal to: **[JEE Main 2015]**

(A) 2 (B) 2e (C) e (D) 0

Ans. **(A)**

Sol.

$$\frac{dy}{dx} + \frac{1}{x\log x} \cdot y = 2$$

$$\text{I.F.} = e^{\int \frac{1}{x\log x}} = e^{\log(\log x)} = \log x$$

$$y \cdot \log x = \int 2 \cdot \log x \, dx$$

$$y \log x = 2(x \log x - x) + c$$

$$x = 1 \Rightarrow \quad c = 2$$

$$x = e \Rightarrow \quad y = 2(e - e) + 2 = 2$$

28. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to:[JEE Main 2016]

(A) $-\frac{4}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $-\frac{2}{5}$

Ans. **(C)**

Sol.

$$y(1 + xy)dx = x\,dy$$

Passes through $(1, -1)$

$$\frac{dy}{dx} = \frac{y}{x}(1 + xy)$$

$$\frac{dy}{dx} = \frac{y}{x} + y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{-1}{y} = t \qquad \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \frac{t}{x} = 1$$

$$\text{If} = e^{\int \frac{1}{x}dx} = x$$

$$tx = \int x\,dx$$

$$tx = \frac{x^2}{2} + C$$

$$\frac{-x}{y} = \frac{x^2}{2} + C$$

$x = 1, \quad y = -1$

$$1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

$$f\left(\frac{-1}{2}\right) \Rightarrow \frac{1}{2y} = \frac{1}{8} + \frac{1}{2}$$

$$\frac{1}{y} = \frac{1}{4} + 1 \quad y = \frac{4}{5}$$

ORTHOGONAL AND ISOGONAL CURVES

29. The differential equation representing the orthogonal trajectories of the family of curves

$xy = k^2$ is

(A) $xdy - ydx = 0$ (B) $xdy + ydx = 0$

(C) $xdx - ydy = 0$ (D) $xdx + ydy = 0$

Ans. (C)

Sol.

$xy = k^2$

Differentiating the above relation we get $xdy + ydx = 0$ The orthogonal trajectory to the above differential equation family is $xdx - ydy = 0$

30. Orthogonal trajectories of family of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, where a is any arbitrary constant, is

(A) $x^{2/3} - y^{2/3} = c$ (B) $x^{4/3} - y^{4/3} = c$

(C) $x^{4/3} + y^{4/3} = c$ (D) $x^{1/3} - y^{1/3} = c$

Ans. (B)

Sol.

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \frac{dx}{dy} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$x^{4/3} - y^{4/3} = c$

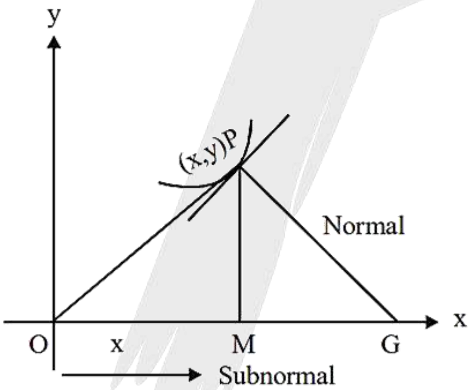
31. The curve for which the normal at any point (x, y) and the line joining origin to that point form an isosceles triangle with the x-axis as base is

(A) an ellipse
(B) a rectangular hyperbola
(C) a circle
(D) none of these

Ans. (B)

Sol.

It is given that the triangle OPG is an isoseceles triangle



Therefore, OM = MG = sub-normal

$\Rightarrow x = y\frac{dy}{dx} \Rightarrow x\,dx = y\,dy$

On integration, we get $x^2 - y^2 = C$, which is a rectangular hyperbola

32. A perpendicular drawn from any point P of the curve on the x-axis meets the x-axis at A. Length of the perpendicular from A on the tangent line at P is equal to ' a '. If this curve cuts the y-axis orthogonally, find the equation to all possible curves, expressing the answer explicitly.

Ans. $y = \pm a\frac{e^{x/a}+e^{-x/a}}{2}$ & $y = \pm a$

Sol.

Let the curve is y = – f(x) & point P is (x, y)

so point A is (x, 0)

equation of tangent at P is

$$Y - y = f'(x)(X - x) \quad .....(1)$$

Length of $\perp^r$ from (x, 0) to tangent is 'a'

so $\left|\frac{y}{\sqrt{(f'(x))^2+1}}\right| = a$

on squaring : $y^2 = a^2 (f'(x))^2 + a^2$

$$f'(x) = \pm\sqrt{\frac{y^2}{a^2}-1} \quad .....(2)$$

solve by taking +ve & –ve sign seprately also on y-axis, x = 0

& angle b/w curve & y-axis is $\frac{\pi}{2}$

so $\Rightarrow f'(x) = 0 \quad .....(3)$

33. Find the orthogonal trajectories for the given family of curves when ' a ' is the parameter.

(i) $y = ax^2$

(ii) $\cos y = ae^{-x}$

(iii) $x^k + y^k = a^k$

(iv) Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which makes with it an angle of 45°.

Ans. (i) $x^2 + 2y^2 = c$,
(ii) $\sin y = ce^{-x}$,
(iii) $y = cx$ if $k = 2$ and $\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$ if $k \neq 2$

(iv) $x^2 - y^2 + 2xy = c; x^2 - y^2 - 2xy = c$

Sol.

(i) $y = ax^2 \Rightarrow \frac{dy}{dx} = 2ax$

$-\frac{dx}{dy} = 2ax$

$-\frac{dx}{dy} = \frac{2y}{x} \quad \Rightarrow -x\,dx = 2y\,dy$

integrate $\quad \Rightarrow x^2 + 2y^2 = k$

(ii) $\cos y = ae^{-x}$

$-\sin y \frac{dy}{dx} = -ae^{-x}$

$\sin y \frac{dy}{dx} = \cos y$

$\sin y \left(-\frac{dy}{dx}\right) = \cos y$

$-dx = \cot y\,dy$

$\ln(\sin y) = -x + c$

$\sin y = ke^{-x}$

MIXED PROBLEMS

34. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through $(1,0)$. The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(A) $x = 0$ (B) $y = 0$

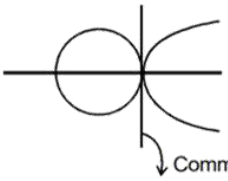
(C) $y = x + 1$ (D) $x + y + 1 = 0$

Ans. (A)

Sol.

$$Y - y = \frac{-dx}{dy}(X - x)$$

$$0 - y = \frac{-dx}{dy}(1 - x)$$



$ydy = (1 - x)\,dx$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C$$

passing through (0, 0)

$\Rightarrow C = 0$

$x^2 + y^2 - 2x = 0$

$y^2 = 4x$

$x = 0 \rightarrow$ is common target

35. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = 1/15$ then the time to drain the tank if the water is 4 meter deep to start with is

(A) 30 min (B) 45 min (C) 60 min (D) 80 min

Ans. (C)

Sol.

$$\frac{dy}{dt} = k\sqrt{y} \Rightarrow \int_0^y \frac{dy}{\sqrt{y}} = \int_0^t k\,dt$$

$$2\sqrt{y} = kt \qquad \Rightarrow t = \frac{2\sqrt{y}}{k}$$

$t = 2 \times 2 \times 15 = 60$ min

36. The solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}(e^x + e^{-x}) + 1 = 0$ is

(A) $y + e^{-x} = c$ (B) $y - e^{-x} = c$

(C) $y + e^x = c$ (D) $y - e^x = c$

Ans. (AD)

Sol.

$$\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)(e^x + e^{-x}) + 1 = 0$$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x}) \pm \sqrt{(e^x + e^{-x})^2 - 4}}{2}$$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x}) \pm (e^x - e^{-x})}{2}$$

+ve $\quad \frac{dy}{dx} = e^x \quad \Rightarrow y = e^x + C_1$

–ve $\quad \frac{dy}{dx} = e^{-x} \quad \Rightarrow y = -e^{-x} + C_2$

37. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$ is

(A) $f(x) = c \cdot e^{(2-\sqrt{3})x}$
(B) $f(x) = c \cdot e^{(2+\sqrt{3})x}$
(C) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$
(D) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$

Ans. (CD)

Sol.

$$y^2 + 4y'y + (y')^2 = 0$$

$$y' = \frac{-4y \pm \sqrt{16y^2 - 4y^2}}{2}$$

$$\frac{dy}{dx} = \frac{-4y \pm 2\sqrt{3}y}{2}$$

+ve $\quad \frac{dy}{y} = (-2 + \sqrt{3})\,dx$

$\ell ny = (-2 + \sqrt{3})\,x + \ell nC$

$y = k\,e^{(-2+\sqrt{3})x}$

–ve $\quad y = k\,e^{(-2-\sqrt{3})x}$

38. Let C be a curve such that the normal at any point P on it meets x-axis and y-axis at A and B respectively. If BP: PA $= 1:2$ (internally) and the curve passes through the point (0,4), then which of the following alternative(s) is/are correct?

(A) The curve passes through $(\sqrt{10}, -6)$

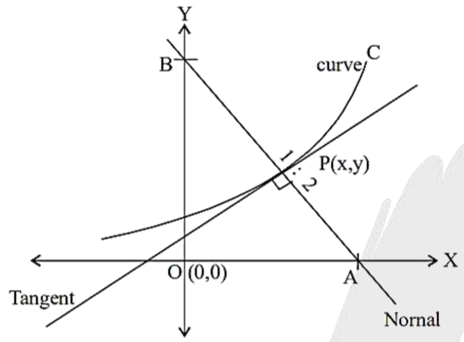
(B) The equation of tangent at $(4, 4\sqrt{3})$ is $2x + \sqrt{3}y = 20$

(C) The differential equation for the curve is $yy' + 2x = 0$

(D) The curve represent a hyperbola.

Ans. (AD)

Sol.



The equation of normal at P(x, y) is (Y − y)

$$= \frac{-1}{\frac{dy}{dx}}(X - x)$$

$$\therefore\ A\left(x + y\frac{dy}{dx}, 0\right) \text{ and } B\left(0, y + \frac{x}{\frac{dy}{dx}}\right)$$

EXACT DIFFERENTIAL EQUATION

39. The equation of the curve passing through (3,4)& satisfying the differential equation,

$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ can be

(A) $x - y + 1 = 0$

(B) $x^2 + y^2 = 25$

(C) $x^2 + y^2 - 5x - 10 = 0$

(D) $x + y - 7 = 0$

Ans. (AB)

Sol.

$$\frac{dy}{dx} = \frac{-(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$$

+ve $\quad \dfrac{dy}{dx} = \dfrac{-(x-y)+(x+y)}{2y}$

$$\frac{dy}{dx} = 1 \Rightarrow y = x + C$$

passes through (3, 4)

$y - x = 1$

–ve $\quad \dfrac{dy}{dx} = \dfrac{-(x-y)-(x+y)}{2y}$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$\Rightarrow \quad ydy + xdx = 0$

$y^2 + x^2 = 25$

40. The orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2 = \frac{4}{x}$ are

(A) $9(y+c)^2 = x^3$

(B) $y + c = \frac{-x^{3/2}}{3}$

(C) $y + c = \frac{x^{3/2}}{3}$

(D) all of these

Ans. (ABCD)

Sol.

$$\left(\frac{dy}{dx}\right)^2 = \frac{4}{x} \Rightarrow \frac{dy}{dx} = \pm\frac{2}{\sqrt{x}}$$

Orthogonal Trajectory

$$\frac{dy}{dx} = \mp\frac{\sqrt{x}}{2}$$

By integrating

$$y + C = \pm\frac{x^{3/2}}{3} \quad \text{or} \quad 9(y+C)^2 = x^3$$

SUBJECTIVE | JEE ADVANCED

41. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal.

Sol.

Assume two curves

$$\frac{dy}{dx} = \frac{a_1x + a_2y}{a_3x + a_4y} \quad \& \quad \frac{dy}{dx} = \frac{b_1x + b_2y}{b_3x + b_3y}$$

Solve this differential equation by putting y = tx
and then solve with y = mx to get P_1 and P_2
& get the slope at P_1 & P_2

42. A normal is drawn at a point P(x, y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through (0, k).

Ans. $x^2 + y^2 = k^2$

Sol.

L_N (Length of the normal) = $y\sqrt{1+m^2}$

$y\sqrt{1+m^2} = k \Rightarrow 1 + m^2 = \frac{k^2}{y^2}$

$m^2 = \frac{k^2 - y^2}{y^2} \Rightarrow m = \pm\frac{\sqrt{k^2-y^2}}{y}$

$y\frac{dy}{dx} = \pm\sqrt{k^2-y^2} \Rightarrow \frac{y\,dx}{\sqrt{k^2-y^2}} = \pm\,dx$

$-\sqrt{k^2-y^2} = \pm x + C$

(0, k) $\Rightarrow$ C = 0

$-\sqrt{k^2-y^2} = \pm x$

$k^2 - y^2 = x^2$

$x^2 + y^2 = k^2$ Circle

43. Find the curve for which the sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co-ordinates of the point of tangency, the proportionality factor is equal tok.

Ans. $y = \frac{1}{k}$ en $|c(k^2x^2 - 1)|$

Sol.

$$\frac{y\sqrt{1+m^2}}{m} + \frac{y}{m} = kxy$$

$$\frac{\sqrt{1+m^2}+1}{m} = kx$$

$$\sqrt{1+m^2} = (kxm - 1)$$

$$1 + m^2 = k^2x^2m^2 + 1 - 2kxm$$

$$m = 0\ ;\ m = \frac{-2kx}{1-k^2x^2} \Rightarrow \frac{dy}{dx} = \frac{-2kx}{1-k^2x^2}$$

$$dy = \frac{1}{k}\left[\frac{-2k^2x}{1-k^2x^2}\right]dx$$

$$y = \frac{1}{k}\ \ell n\,|(1 - k^2x^2)| + \ell nC$$

44. Find the curve $y = f(x)$ where $f(x) \geq 0, f(0) = 0$, bounding a curvilinear trapezoid with the base $[0, x]$ whose area is proportional to $(n+1)^{\text{th}}$ power of $f(x)$. It is known that $f(1) = 1$.

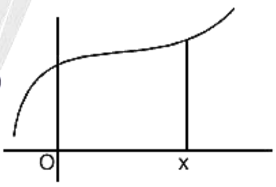
Ans. $y = x^{1/n}$

Sol.

$$A = \int_0^x f(x)\,dx = k\,[f(x)]^{n+1}$$

Use Leibnitz's theorem

$f(x)=k\,(n+1)\,[f(x)]^n.f'(x)$



$$\frac{1}{k(n+1)} = [f(x)]^{n-1}\,.\,f'(x)$$

$$\frac{x}{k(n+1)} = \frac{[f(x)]^n}{n} + C$$

$x = 0, f(0) = 0 \Rightarrow C = 0$

$x = 1, f(1) = 1 \Rightarrow k = \frac{1}{n+1}$

$[f(x)]^n = x \Rightarrow f(x) = x^{1/n}$

45. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

Ans. $\frac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \ell n\left|\left(y \pm \sqrt{y^2-x^2}\right)\cdot\frac{c^2}{x^3}\right|$, where same sign has to be taken

Sol.

$x = |y\sqrt{1+m^2}|$

$y\sqrt{1+m^2} = \pm x$

$\sqrt{1+m^2} = \pm \frac{x}{y} \Rightarrow 1 + m^2 = \frac{x^2}{y^2}$

$m^2 = \frac{x^2 - y^2}{y^2} \Rightarrow \frac{dy}{dx} = \pm\sqrt{\frac{x^2-y^2}{y^2}}$

Put $y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$\frac{t\,dt}{\sqrt{1-t^2} - t^2} = \frac{dx}{x}$

After integration

$\frac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \ell n\left[(y \pm \sqrt{y^2-x^2})\frac{c^2}{x^3}\right]$

46. Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point and the tangent at this point equals half the square of its abscissa.

Ans. $y = cx^2 \pm x$

Sol.

Tangent

$Y - y = \frac{dy}{dx}(x - x)$

put $X = 0 \Rightarrow Y = \left(y - x\frac{dy}{dx}\right)$

$A = \frac{1}{2}x^2$

$\left|\frac{1}{2}\left[2y - x\frac{dy}{dx}\right].x\right| = \frac{1}{2}x^2$

$2y - x\frac{dy}{dx} = \pm x$

$\frac{dy}{dx}\ \frac{2}{x}y = \pm 1$

$IF = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

$\left(\frac{y}{x^2}\right) = \int \pm \frac{1}{x^2}dx$

$\frac{y}{x^2} = \pm \frac{1}{x} + c$

$y = cx^2 \pm x$

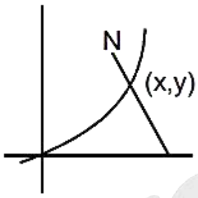
47. Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Ans. $y^2 = 2x + 1 - e^{2x}$

Sol.

Equation of normal

$Y\text{-}y = \dfrac{-1}{dy/dx}(X - x)$



Mid point (h, k)

$= \left(\dfrac{y\,dy/dx + 2x}{2}, \dfrac{y}{2}\right)$

put in curve : $y.\dfrac{dy}{dx} - y^2 - 2x$

Let $y^2 = t$ & solve the L.D.E.

$y^2 = t$

$2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\dfrac{1}{2}\dfrac{dt}{dx} - t = -2x$

$\dfrac{dt}{dx} - 2b = -4x$

If $e^{\int -2dx} = e^{-2x}$

$4te^{-2x} = \int e^{2x}4x$

$-4\left[x\dfrac{e^{-2x}}{-2} + \dfrac{1}{2}\int e^{-2x}dx\right]$

$= -4\left[\dfrac{xe^{-2x}}{-2} - \dfrac{1}{4}e^{-2x}\right] + C$

$y^2 = 4\left[\dfrac{x}{-2} - \dfrac{1}{4}\right] + Ce^{2x}$

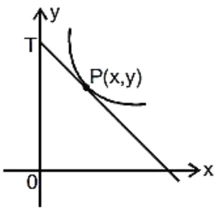
$y^2 = 2x + 1 - Ce^{2x}$

48. Find the curve for which the portion of y-axis cutoff between the origin and the tangent varies as cube of the abscissa of the point of contact.

Ans. $2y + Kx^3 = cx$

Sol.

$$Y - y = \frac{dy}{dx}(X - x)$$



$$OT = y - x\frac{dy}{dx}$$

$$\& \;\; y - x\frac{dy}{dx} = Kx^3 \Rightarrow \frac{x\,dy - y\,dx}{x^2} = -\,kx\,dx$$

on integrating

$$\frac{y}{x} = -\frac{-kx^2}{2} + C$$

49. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at $t = 0$, the mass of the radius was m_0 and during time $t_0 \alpha\%$ of the original mass of radium decay.

Ans. $m = m_0 e^{-kt}$ where $k = -\frac{1}{t_o}\ln\left(1 - \frac{\alpha}{100}\right)$

Sol.

$$\frac{-dm}{dt} = Km$$

$$\ell n\ m = -\ kt + \ell nc$$

$$\frac{m}{c} = e^{-kt} \qquad \Rightarrow m = ce^{-t}$$

---

at $t = 0$, $m = m_0$ $\qquad \Rightarrow C = m_0$

$m = m_0 e^{-kt}$

at $t = t_0$, $m = m_0\left(1 - \frac{\alpha}{100}\right)$

$$m_0\left(1 - \frac{\alpha}{100}\right) = m_0\, e^{-kt_0} \Rightarrow k = \frac{-1}{t_0}\,\ell n\left(1 - \frac{\alpha}{100}\right)$$

$m = m_0\, e^{-kt}$

Where

$$K = \frac{-1}{t_0}\,\ell n\left(1 - \frac{\alpha}{100}\right)$$

50. Let the function ln $f(x)$ is defined where $f(x)$ exists for x ≥ 2 and k is fixed positive real number, prove that if $\frac{d}{dx}(x.f(x)) \le -kf(x)$ then $f(x) \le Ax^{-}1-\text{k}$ where A is independent of x.

Sol.

$$\frac{d}{dx}(x\, f(x)) \le -k\, f(x)$$

$x\, f'(x) + f(x) \le -k\, f(x)$

$xf'(x) + (1+k)\, f(x) \le 0$

multiply by x^k

$x^{k+1} f'(x) + (k+1)\, x^k f(x) \le 0$

$$\frac{d}{dx}(x^{k+1}.\, f(x)) \le 0$$

Let $F(x) = x^{k+1} f(x)$

F(x) is decreasing for $x \ge 2$

$F(x) \le F(2)$

$F(x) \le A$

$x^{k+1} f(x) \le A$

$f(x) \le A.\, x^{-1-k}$

51. Find the differentiable function which satisfies the equation

$f(x) = -\int_0^x f(t)\tan\, tdt + \int_0^x \tan\,(t-x)dt$ where $x \in (-\pi/2, \pi/2)$

Ans. $\cos\, x - 1$

Sol.

$$f(x) = -\int_0^x f(t)\ \tan t\, dt + \int_0^x \tan(t-x)\, dt$$

put $t - x = z \Rightarrow dt = dz$

$$f(x) = -\int_0^x f(t)\ \tan t\, dt + \int_{-x}^0 \tan z\ dz$$

Now use leibnitz

$f'(x) = -f(x)\tan x - \tan x$

$$\frac{dy}{dx} + y\tan x = -\tan x$$

$$\text{I.F.} = e^{\int \tan x\, dx} = \sec x$$

$$y(\sec x) = -\int \tan x \sec x\, dx = -\sec x + c$$

$y = c\cos x - 1$

$y(0) = 0 \Rightarrow c = 1$

$y = \cos x - 1$

52. A tank contains 100 litres of fresh water. A solution containing 1gm/ litre of soluble lawn fertilizer runs into the tank at the rate of 1lit/min and the mixture is pumped out of the tank of 3 litres /min. Find the time when the amount of fertilizer in the tank is maximum.

Ans. $27\frac{7}{9}$ minutes

Sol.

Rate at which fertilizer added = 1 × 1=1 gm/min.
volume of solution at time t = 100 + (1 – 3) t
= 100 – 2t

$$\Rightarrow \frac{dy}{dt} = 1 - \left(\frac{3y}{100-2t}\right)$$

$$\frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 1$$

$$\Rightarrow y = (100 - 2t) + c\,(100 - 2t)^{3/2}$$

$$\text{at } t = 0,\ y = 0 \Rightarrow c = -\frac{1}{10}$$

$$y = (100 - 2t) - \frac{1}{10}(100 - 2t)^{3/2}$$

$$\frac{dy}{dt} = 0 \Rightarrow t = 27\frac{7}{9}$$

53. A tank with a capacity of 1000 litres originally contains 100 gms of salt dissolved in 400 litres of water. Beginning at time $t = 0$ and ending at time $t = 100$ minutes, water containing 1gm of salt per litre enters the tank at the rate of 4 litre/minute and the wheel mixed solution is drained from the tank at a rate of 2 litre/minute. Find the differential equation for the amount of salt y in the tank at time t.

Ans. $\frac{dy}{dt} = 4 - \frac{y}{200+t}$

Sol.

Rate at which salt added = 4 × 1 = 4 gm/min.
volume of solution at time t = 40 + (4–2)t = 40+2t

$$\frac{dy}{dt} = 4 - \left(\frac{y}{40+2t}\right)2$$

$$\frac{dy}{dt} = 4 - \frac{y}{200+t}$$

54. $\frac{dy}{dx} = y + \int_0^1 y dx$ given $y = 1$, where $x = 0$

Ans. $y = \frac{1}{3-e}(2e^x - e + 1)$

Sol.

$$\frac{dy}{dx} = y + \int_0^1 y\,dx$$

Let $A = \int_0^1 y\,dx$

$$\frac{dy}{dx} = y + A \Rightarrow \frac{dy}{dx} - y = A$$

$\text{If} = e^{\int -1.dx} = e^{-x}$

$y(e^{-x}) = A e^{-x} + c \Rightarrow y = A + c e^x$

$y = 1 \Rightarrow x = 0 \Rightarrow c = 1 + A$

$y = A + (1 + A) e^x$

$$A = \int_0^1 y\,dx \Rightarrow A = \int_0^1 A + (1 + A)e^x dx$$

$$\Rightarrow A = \frac{e-1}{3-e}$$

$$y = \frac{e-1}{3-e} + \left(1 + \frac{e-1}{3-e}\right) e^x$$

$$\Rightarrow y = \frac{1}{3-e}(2e^x - e + 1)$$

55. Find the continuous function which satisfies the relation,

$\int_0^x tf(x-t)dt = \int_0^x f(t)dt + \sin\ x + \cos\ x - x - 1$, for all real number x.

Ans. $f(x) = e^x - \cos\ x$

Sol.

.47 $\int_0^x t(f(x-t)dt = \int_0^x f(t)dt + \sin x + \cos x - x - 1$

Let $x - t = z \Rightarrow dt = -dz$

$$\int_0^x (x-z)f(z)\,dz = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

$$x\int_0^x f(z)\,dz - \int_0^x zf(z)\,dz = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

Use liebnitz

$$\int_0^x f(z)\,dz + x\,f(x) - x\,f(x) = f(x) + \cos x - \sin x - 1$$

again leibnitz

$f(x) = f'(x) - \sin x - \cos x$

$\frac{dy}{dx} - y = \sin x + \cos x$

$IF = e^{-x}$

$y(e^{-x}) = \int e^{-x}(\sin x + \cos x)\,dx$

$y\,(e^{-x}) = -e^{-x}\cos x + c$

$y = c\,e^{x} - \cos x$

$x = 0 \Rightarrow y = 0$

$c = 1$

$y = e^{x} - \cos x$

56. A curve passing through (1,0) such that the ratio of the square of the intercept cut by any tangent off the y-axis to the subnormal is equal to the ratio of the product of the co-ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

COMPREHENSION

Let $y = f(x)$ and $y = g(x)$ be the pair of curves such that

(i) the tangents at point with equal abcissae intersect on y-axis

(ii) the normals drawn at points with equal abscissae intersect on x-axis and

(iii) curve $f(x)$ passes through (1,1) and $g(x)$ passes through (2,3) then

Ans. $x = e^{2\sqrt{y/x}}; x = e^{-2\sqrt{y/x}}$

Sol.

Let the curve is y = f(x) & point of tangent is (x, y)

Equation of tangent : $Y - y = f'(x)(X - x)$ .....(1)

at y-axis, intercept = $y - x f'(x)$ .....(2)

subnormal = $y.f'(x)$ .....(3)

Slope of tangent = $f'(x)$ .....(4)

Subtangent = $\frac{y}{f'(x)}$ .....(5)

according to question :

$$\frac{(y - x f'(x))^2}{y.f'(x)} = \frac{x.y}{(f'(x))^2\left(\frac{y}{f'(x)}\right)} \quad .....(6)$$

Solve equation (6) & for C, use point (1, 0)

57. The curve f(x) is given by -

(A) $\frac{2}{x} - x$

(B) $2x^2 - \frac{1}{x}$

(C) $\frac{2}{x^2} - x$

(D) none of these

Ans. **(A)**

Sol.

(x, 0)

$y - f(x) = f'(x)(x - x)$ ...(1)

$y - g(x) = g'(x)(x - x)$ ...(2)

$y - f - x f'(x) = g - - xg'$

$f - g = x(f' - g')$

$\int \frac{f' - g'}{f - g} = \int \frac{1}{x}$

$\ln(f - g) = \ln x + c$

$f - g = xk$ ...(1)

equation (1) & (2)

$y - f = \frac{-1}{f'}(x - n)$

$y - g = \frac{-1}{g'}(x - n)$

$x = ff' + x = gg' + x$

$f^2 = g^2 + c$

$f^2 - g^2 = c$

$f + g = \frac{c}{xn}$ ...(2)

$f = \frac{1}{2}\left(kx + \frac{c}{kx}\right)$

$g = \frac{1}{2}\left(\frac{c}{kx} - kx\right)$

$f(1) = 1$

$g(2) = 3$

$1 = \frac{1}{2}\left(k + \frac{c}{k}\right)$

$3 = \frac{1}{2}\left(\frac{c}{2k} - 2k\right)$

58. The curve g(x) is given by -

(A) $x - \frac{1}{x}$ (B) $x + \frac{2}{x}$ (C) $x^2 - \frac{1}{x^2}$ (D) none of these

Ans. (B)

59. The value of $\int_1^2 (g(x) - f(x))dx$ is -

(A) 2 (B) 3 (C) 4 (D) $4\ell n2$

Ans. (B)

MATCH THE COLUMN

60. **Column-I** | **Column-II**

(A) A curve passing through (2,3) having the property that length of the radius vector of any of its point P is equal to the length of the tangent drawn at this point, can be — (P) Straight line

(B) A curve passing through (1,1) having the property that any tangent intersects the y-axis at the point which is equidistant from the point of tangency and the origin, can be — (Q) Circle

(C) A curve passing through (1,0) for which the length of normal is equal to the radius vector, can be — (R) Parabola

(D) A curve passes through the point (2,1) and having the property that the segment of any of its tangent between the point of tangency and the x-axis is bisected by the y-axis, can be — (S) Hyperbola

Ans. A-P,S; B-Q; C-Q,S; D-R

Sol.

(A) RV = $\sqrt{x^2+y^2} = \frac{y}{m}\sqrt{1+m^2}$

$\Rightarrow m^2(x^2+y^2) = y^2(1+m^2)$

$\Rightarrow m^2 = \frac{y^2}{x^2} \Rightarrow m = \pm\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ or $\frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$ or $\int\frac{dy}{y} = \int\frac{-dx}{x}$

$\Rightarrow y = x\,c_1$ or $y = \frac{C_2}{x} \Rightarrow xy = C_2$

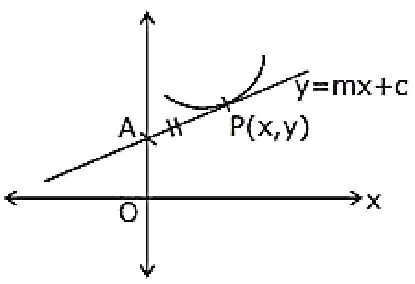
Straight line Hyperbola

(B) A(0, y – mx)

OA = OP

$\Rightarrow (y-mx)^2 = x^2 + m^2x^2$

$\Rightarrow m = \frac{dy}{dx} = \frac{y^2-x^2}{2xy} = \frac{(y/x)^2-1}{2(y/x)}$



Put y = vx

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

PREVIOUS YEAR JEE MAIN

61. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is - **[AIEEE 2008]**

(A) $(y-2)y'^2 = 25-(y-2)^2$

(B) $(y-2)^2y'^2 = 25-(y-2)^2$

(C) $(x-2)^2y'^2 = 25-(y-2)^2$

(D) $(x-2)y'^2 = 25-(y-2)^2$

Ans. (B)

Sol.

The equation of family of circles with centre on $y = 2$ and of radius 5 is

$$(x-\alpha)^2 + (y-2)^2 = 5 \quad ...(i)$$

$$\Rightarrow \quad x^2 + \alpha^2 - 2\alpha + y^2 + 4 - 4y = 25$$

On differentiating w.r.t. x, we get

$$2x - 2\alpha + 2y\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \alpha = x + \frac{dy}{dx}(y-2)$$

On putting the value of α in Eq. (i), we get

$$\left(x - x - \frac{dy}{dx}(y-2)\right)^2 + (y-2)^2 = 5^5$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2(y-2)^2 = 25-(y-2)^2$$

or $\quad (y')^2(y-2)^2 = 25-(y-2)^2$

62. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is: **[AIEEE 2011]**

(A) $T^2 - \frac{1}{k}$

(B) $I - \frac{KT}{2}$

(C) $I - \frac{k(T-t)^2}{2}$

(D) $e - Kt$

Ans. (B)

Sol.

Given, $\frac{d\{V(t)\}}{dt} = -k(T-t)$

$\therefore$ $d\{V(t)\} = -k(T-t)dt,$

when $t = 0,\ V(t) = I$ ...(i)

$\Rightarrow$ $\int_0^T d\{v(t)\} = \int_0^T -k(T-t)dt$

$\Rightarrow$ $V(T) - V(0) = k\left\{\frac{(t-T)^2}{2}\right\}_0^T$

$\Rightarrow$ $V(T) - I = \frac{k}{2}\{(T-T)^2 - (0-T)^2\}$

$\therefore$ $V(T) = I - \frac{k}{2}T^2$

63. At present, a firm is manufacturing 2000 items. It is estimated that rate of change of production P w.r.t additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is: **[JEE Main 2013]**

(A) 3500 (B) 4500 (C) 2500 (D) 3000

Ans. (A)

Sol.

$$\frac{dp}{dx} = 100 - 12\sqrt{x}$$

$$\Rightarrow P = 100x - \frac{12x^{3/2}}{3/2} + c$$

If $x = 0$ then $P = 2000$

$\Rightarrow P = 100x - 8x^{3/2} + 2000$

If $x = 25$

$P = 2500 - 1000 + 2000 = 3500$

64. Let the population of rabbits surviving at a time t be governed by the differential equation

$\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200.$ [JEE Main 2014]

If $p(0) = 100$, then $p(t)$ equals:

(A) $400 - 300e^{t/2}$ (B) $300 - 200e^{-t/2}$

(C) $600 - 500e^{t/2}$ (D) $400 - 300e^{-t/2}$

Ans. (A)

Sol.

$$\int \frac{2dp(t)}{p(t)-400} = \int dt$$

$2\log|p(t) - 400| = t + c$

...(1)

$t = 0, p = 100$

$2\log(300) = C$

From (1)

$2\log|p(t) - 400| = t + 2\log(300)$

$|p(t) - 400| = e^{t/2} \cdot e^{\log(300)}$

$\boxed{p(t) = 400 - e^{t/2}(300)}$

65. If $(2+\sin x)\frac{dy}{dx} + (y+1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to:

[JEE Main 2017]

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{4}{3}$

Ans. (A)

Sol.

$$\frac{dy}{dx} + \frac{(y+1)(\cos x)}{2+\sin x} = 0$$

$$\int \frac{dy}{y+1} = -\int \frac{\cos x}{2+\sin x}dx$$

$\ln|y+1| = -\ln|2 + \sin x| + c$ ...(1)

$x = 0, y = 1$

$\boxed{c = \ln 4}$

$$\ln|y+1| = \ln\left(\frac{y}{2+\sin x}\right)$$

$x = \frac{\pi}{2}$

$y + 1 = \frac{4}{3}$	$y + 1 = -\frac{4}{3}$
$\boxed{y = \frac{1}{3}}$	$\boxed{y = -\frac{7}{3}}$

PREVIOUS YEAR JEE ADVANCED

66.

(a) Let f(x) be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\operatorname{Lim}_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is **[JEE 2007]**

(A) $\frac{1}{3x} + \frac{2x^2}{3}$

(B) $\frac{-1}{3x} + \frac{4x^2}{3}$

(C) $\frac{-1}{x} + \frac{2}{x^2}$

(D) $\frac{1}{x}$

(b) The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(A) variable radii and a fixed centre at (0,1)

(B) variable radii and a fixed centre at $(0, -1)$

(C) fixed radius 1 and variable centres along the x-axis.

(D) fixed radius 1 and variable centres along the y axis.

Ans. (A)

Sol.

(a)-A $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

$x^2 f'(x) - 2x f(x) + 1 = 0$

$\Rightarrow \quad f(x) = cx^2 + \frac{1}{3x} \quad f(1) = 1$

$c = \frac{2}{3}$

$f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

(b)-C $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{y}}$

$\int \frac{y}{\sqrt{1-y^2}} dy = dx$

$-\sqrt{1-y^2} = x + c$

$(x + c)^2 + y^2 = 1$

centre $(-c, 0)$ radius $= \sqrt{c^2 - c^2 + 1} = 1$

67. Let a solution $y = y(x)$ of the differential equation, $x\sqrt{x^2-1}\,dy = y\sqrt{y^2-1}dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$. STATEMENT-1 : $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ **[JEE 2008]**

and

STATEMENT-2 : $y(x)$ is given by

$$\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$$

(A) Statement- 1 is true, Statement- 2 is true ; Statement-2 is correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is true ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is true, Statement-2 is false.

(D) Statement-1 is false, Statement-2 is true.

Ans. (C)

Sol.

$$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$$

$$\frac{dx}{x\sqrt{x^2-1}} = \frac{dy}{y\sqrt{y^2-1}}$$

$$\sec^{-1} x = \sec^{-1} y + c$$

$$\sec^{-1} 2 = \sec^{-1}\frac{2}{\sqrt{3}} + c$$

$$c = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$$

$$y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$$

$$\cos^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y} + \frac{\pi}{6}$$

$$\cos^{-1}\frac{1}{y} = \cos^{-1}\frac{1}{x} - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$$

$$\frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$$

68. If. $y(x)$ satisfies the differential equation $y' - y\tan\ x = 2x\sec\ x$ and $y(0) = 0$, then [JEE 2012]

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$

(B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$

(D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Ans. **(AD)**

Sol.

$$\text{I.F.} = \cos x$$

$$y \cdot \cos x = \int 2x \sec x \cdot \cos x \cdot dx$$

$$y \cdot \cos x = x^2 + c,\ c = 0$$

$$y = x^2 \sec x$$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4} \cdot \sqrt{2}$$

$$y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$$

$$y'\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9} \cdot \sqrt{5} + \frac{2\pi}{3} \cdot 2$$

$$= \frac{2\pi^2}{5\sqrt{3}} + \frac{4\pi}{3}$$

69. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1,1)$ satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is **[JEE 2014]**

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

(B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

(D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Ans. **(B)**

Sol.

$$\frac{dy}{dx}+\frac{x}{x^2-1}y=\frac{x^4+2x}{\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int \frac{x}{x^2-1}dx} = \sqrt{1-x^2}$$

$$y.\sqrt{1-x^2} = \int \frac{x(x^3+2)}{\sqrt{1-x^2}}\sqrt{1-x^2}dx$$

$$y.\sqrt{1-x^2} = \int (x^4+2x)dx = \frac{x^5}{5}+x^2+c$$

$$f(0) = 0 \Rightarrow c = 0$$

$$f(x)\sqrt{1-x^2} = \frac{x^5}{5}+x^2$$

$$\text{Now, } \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$$

$$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx = 2\int_0^{\pi/3} \frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta$$

$$= 2\int_0^{\pi/3} \sin^2\theta\, d\theta = 2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi/3}$$

$$= 2\left(\frac{\pi}{6}\right)-2\left(\frac{\sqrt{3}}{8}\right) = \frac{\pi}{3}-\frac{\sqrt{3}}{4}$$

70. Let $y(x)$ be a solution of the differential equation $(1+e^x)y'+ye^x=1$. If $y(0)=2$, then which of the following statements is (are) true? **[JEE 2015]**

(A) $y(-4)=0$

(B) $y(-2)=0$

(C) $y(x)$ has a critical point in the interval $(-1,0)$

(D) $y(x)$ has no critical point in the interval $(-1,0)$

Ans. (AC)

Sol.

y(x)

$(1 + e^x)\, y^1 + e^x\, y = 1$

$$\frac{dy}{dx} + \frac{e^x}{1+e^x} y = \frac{1}{1+e^x}$$

Intergrating factor

$$e^{\int \frac{ex}{1+e^x}dx} = e^{\ln(1+e^x)} = 1 + e^x$$

$$\Rightarrow (1 + e^x)\, y = \int \frac{1+e^x}{1+e^x} dx + c$$

$(1 + e^x)y = x + c$

$y(0) = 2 \quad \Rightarrow (1 + 1)2 = 0 + c\ (c = 4)$

$$2 + 2(y')^2 + 2yy'' - \frac{2x + 2yy'}{1 + y'} y'' = 0$$

$2 + 2y' + 2(y')^2 + 2(y')^3 + 2yy'' + 2yy'y'' -$
$2 \times y'' - 2yy'y'' = 0$

$y''(2y - 2x) + y'(2 + 2y' + 2(y')^2) + 2 = 0$

$y''(y - x) + y'(1 + y' + (y')^2) + 1 = 0$

$P = y - x$

$Q = 1 + y' + (y')^2$

71. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 =, x > 0$ passes through the point (1,3). Then the solution curve **[JEE 2016]**

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersects $y = (x + 3)^2$

Ans. **(A,D)**

Sol.

$$\left(x^2+2xy+4x+2y+4-xy\right)\frac{dy}{dx}-y^2=0$$

$x>0$

$$\left((x+2)^2+y(x+2)\right)\frac{dy}{dx}=y^2$$

$$(x+2)(x+2+y).dy=y^2dx$$

$$\frac{dy}{dx}=\frac{y^2}{(x+2)(x+2+y)}$$

$y=v(x+2)$

$$\frac{dy}{dx}=v+(x+2)\frac{dv}{dx}$$

$$(x+2)\frac{dv}{dx}=\frac{v^2}{1+v}-v$$

$$(x+2)\frac{dv}{dx}=\frac{v^2-v-v^2}{1+v}$$

$$\frac{1+v}{v}dv=-\frac{dx}{x+2}$$

$$\left(\frac{1}{v}+1\right)dv=-\frac{dx}{x+2}$$

$$\ln v+v=-\ln(x+2)+c$$

$$\ln((x+2).v)+v=c$$

$$\ln\{y\}+\frac{y}{x+2}=c \qquad (1,3)$$

$$\ln 3+\frac{3}{3}=c$$

$$c=1+\ln 3$$

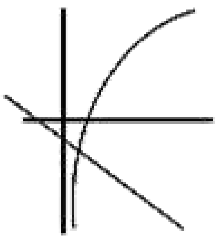
$$\ln y+\frac{y}{x+2}=1+\ln 3$$

$$\ln\left(\frac{y}{3}\right)=1-\frac{y}{x+2}$$

(A) $y=x+2$

$$\therefore \ln\left(\frac{y}{3}\right)=0$$

$y=3, \quad x=1$

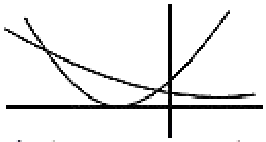


(C) $\ln\frac{y}{3} = 1-(x+2)$

$\ln\left(\frac{y}{3}\right) = -x-1$

(D) $\ln\frac{(x+3)^2}{3} = 1-\frac{(x+3)^2}{x+2}$

$\frac{(x+3)^2}{2} = e^{1-\frac{(x+3)^2}{x+2}}$



Both solution are negative though it is given $x > 0$

72. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = (\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1}dx, x > 0 \text{ and } y(0) = \sqrt{7}, \text{ then } y = (256) =$$

[JEE 2017]

(A) 3 (B) 16 (C) 9 (D) 80

Ans. (A)

Sol.

$$\sqrt{4+\sqrt{9+\sqrt{x}}} = t$$

$$\frac{1}{2\sqrt{4+\sqrt{9+\sqrt{x}}}} \times \frac{1}{2\sqrt{9+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx = dt$$

$\Rightarrow$ $dy = dt$

$y = t + \lambda$

$y = \sqrt{4+\sqrt{9+\sqrt{x}}} + \lambda$

$y(0) = \sqrt{7} + \lambda \Rightarrow \boxed{\lambda = 0}$

$\Rightarrow$ $y(256) = \sqrt{4+\sqrt{9+16}} = \sqrt{4+5}$

$= 3$

73. Let $f:[0,\infty)\to R$ be a continuous function such that $f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$ for all $x\in[0,\infty)$. Then which of the following statement (s) is (are) TRUE? [JEE Adv. 2018]

(A) The curve $y=f(x)$ passes through the point $(1,2)$

(B) The curve $y=f(x)$ passes throgh the point $(2,-1)$

(C) The area of the region $\{(x,y)\in[0,1]\times R: f(x)\le y\le\sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x,y)\in[0,1]\times R: f(x)\le y\le\sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

Ans. (B,C)

Sol.

$$f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$$

$$f(x)=1-2x+e^x\int_0^x e^{-t}f(t)dt \quad \text{.........(1)}$$

Differentiate both sides

$$f'(x)=0-2+e^x(e^{-x}f(x))+e^x\int_0^x e^{-t}f(t)dt$$

$$f'(x)=f(x)-2+f(x)-1+2x$$

$$f'(x)=2f(x)+2x-3$$

$$f'(x)-2f(x)=2x-3$$

$$\frac{dy}{dx}-2y=2x-3$$

I.f. $=e^{-2x}$

$$y.e^{-2x}=\int(2x-3)e^{-2x}dx$$

$$y=(1-x)+Ce^{2x} \quad \text{....(2)}$$

put x = 0 in equation (1) we get

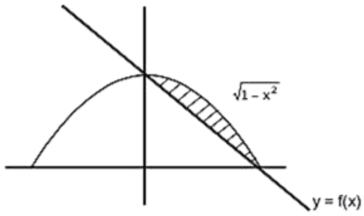
y = 1

Put x = 0 and y = 1 in equation (2)

1 = 1 + C

C = 0

y=1−x which posses through point (2,−1)



Now required area

$$=\frac{1}{4}\pi.(1)^2-\frac{1}{2}.1.1=\frac{\pi}{4}-\frac{1}{2}$$

74. Let Γ denotes a curve $y = y(x)$ which is in the first quadrant and let the point $(1,0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_p. If PY_p has length 1 for each point P on Γ, then Which of the following options is/are correct? [JEE Adv. 2019]

(A) $xy' - \sqrt{1-x^2} = 0$

(B) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(C) $xy' + \sqrt{1-x^2} = 0$

(D) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

Ans. **(CD)**

Sol.

Equation of Tangent at P

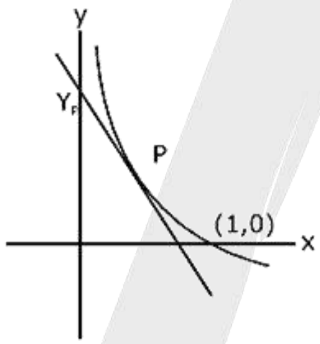
$$Y - y = \frac{dy}{dx}(X - x)$$

For $Y_p \Rightarrow (X = 0)$

$$Y_p = y - x\frac{dy}{dx}$$

distance $Y_pP = 1$

$$x^2 + \left(y - y + x\frac{dy}{dx}\right)^2 = 1$$



$$x^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^2} - 1$$

75. If $y(x)$ is the solution of the differential equation $xdy - (y^2 - 4y)dx = 0$ for $x > 0, y(1) = 2$, and the slope of the curve $y = y(x)$ is never zero, then the value of $10y(\sqrt{2})$. [JEE Adv. 2022]

Ans. **8**

76. For $x \in R$, let the function $y(x)$ be the solution of the differential equation

[JEE Adv. 2022]

$$\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right), y(0) = 0.$$

Then, which of the following statements is/are TRUE ?

(A) $y(x)$ is an increasing function

(B) y(x) is a decreasing function

(C) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points

(D) $y(x)$ is a periodic function

Ans. (C)

ANSWER KEY

1. (B) **2.** (A) **3.** (C) **4.** (A) **5.** (A) **6.** (C) **7.** (B)

8. (B) **9.** (A) **10.** (C)

11. The solution of the equation $\frac{dy}{dx}+y\tan x=x^{m}\cos x$ is-

(A) $(m+1)y=x^{m+1}\cos x+c(m+1)\cos x$

(B) $my=(x^{m}+c)\cos x$

(C) $y=(x^{m+1}+c)\cos x$

(D) None of these

12. (D) **13.** (B) **14.** (ABD) **15.** (ABD)

16. $\frac{1}{2}$ **17.** $y=c(1-x^2)+\sqrt{1-x^2}$ **18.** $x=ce^{-\arctan y}+\arctan y-1$

19. $y=2^{\sin x}$

20. **(i)** $y=u(x)+K(u(x)-v(x))$ where K is any constant ;

(ii) $\alpha+\beta=1$;

(iii) constant

21. $y=\frac{x}{\sqrt{1-x^2}}=ce^{-\frac{x}{\sqrt{1-x^2}}}$ **22.** $x(ey+\ell ny+1)=1$

23. $y=5t\left(1+\frac{50}{50+t}\right)$ gms; $91\frac{2}{3}$ gms **24.** $f(x)=-\frac{2\cos x}{(1+\sin x)^2}-Ce^{-\sin x}\cdot\cos x$

25. (C) **26.** (A) **27.** (A) **28.** (C) **29.** (C) **30.** (B)

31. (B) **32.** $y=\pm a\frac{e^{x/a}+e^{-x/a}}{2}$ & $y=\pm a$

33. **(i)** $x^2+2y^2=c$,

(ii) $\sin y=ce^{-x}$,

(iii) $y=cx$ if $k=2$ and $\frac{1}{x^{k-2}}-\frac{1}{y^{k-2}}=\frac{1}{c^{k-2}}$ if $k\neq 2$

(iv) $x^2-y^2+2xy=c$; $x^2-y^2-2xy=c$

34. (A) **35.** (C) **36.** (AD) **37.** (CD) **38.** (AD) **39.** (AB) **40.** (ABCD)

42. $x^2+y^2=k^2$ **43.** $y=\frac{1}{k}\text{en}\,|c(k^2x^2-1)|$

44. $y=x^{1/n}$

45. $\frac{y^2\pm y\sqrt{y^2-x^2}}{x^2}=\ell n\left|\left(y\pm\sqrt{y^2-x^2}\right)\cdot\frac{c^2}{x^3}\right|$, where same sign has to be taken

46. $y=cx^2\pm x$ **47.** $y^2=2x+1-e^{2x}$

48. $2y+Kx^3=cx$ **51.** $\cos x-1$

52.	$27\frac{7}{9}$ minutes						**53.**	$\frac{dy}{dt} = 4 - \frac{y}{200+t}$					
54.	$y = \frac{1}{3-e}(2e^x - e + 1)$						**55.**	$f(x) = e^x - \cos x$					
56.	$x = e^{2\sqrt{y/x}}; x = e^{-2\sqrt{y/x}}$						**57.**	(A)					
58.	(B)	**59.**	(B)	**60.**	A-P, S; B-Q; C-Q, S; D-R								
61.	(B)	**62.**	(B)	**63.**	(A)	**64.**	(A)	**65.**	(A)	**66.**	(A)	**67.**	(C)
68.	(AD)	**69.**	(B)	**70.**	(AC)	**71.**	(AD)	**72.**	(A)	**73.**	(BC)	**74.**	(CD)
75.	8	**76.**	(C)										