DPP-01(ELEMENTARY EXERCISE)

1. Let $\int_0^1 \frac{dx}{\sqrt{16+9x^2}} + \int_0^2 \frac{dx}{\sqrt{9+4x^2}} = \ln\ a$. Find a.

Ans. $\mathbf{2^{1/3} \cdot 3^{1/2}}$

Sol. $\int_0^1 \frac{dx}{\sqrt{16+9x^2}} + \int_0^2 \frac{dx}{\sqrt{9+4x^2}} = \frac{1}{3}\int_0^1 \frac{dx}{\sqrt{\frac{16}{9}+x^2}} + \frac{1}{2}\int_0^2 \frac{dx}{\sqrt{\left(\frac{9}{4}\right)+x^2}}$

$$= \frac{1}{3}\left[\ln\left(x+\sqrt{x^2\frac{16}{9}}\right)\right]_0^1 + \frac{1}{2}\left[\ln\left(x+\sqrt{x^2+\frac{9}{4}}\right)\right]_0^2 = \frac{1}{3}\ln\ 2 + \frac{1}{2}\ln\ 3$$

$= \ln\ 2^{\frac{1}{3}} \cdot 3^{\frac{1}{2}} = \ln\ a \quad \therefore a = 2^{\frac{1}{3}}3^{\frac{1}{2}}$

2. $\int_0^{\ell n2} xe^{-x}dx$

Ans. $\mathbf{\frac{1}{2}\ln\left(\frac{e}{2}\right)}$

Sol. $I = \int_0^{\ln\ 2} xe^{-x}dx$

$I = \left[x\int e^{-x}dx\right]_0^{\ln\ 2} - \int_0^{\ln\ 2} 1 \cdot \frac{e^{-x}}{-1}dx$

$I = -[xe^{-x}]_0^{\ln\ 2} + [-e^{-x}]_0^{\ln\ 2}$

$I = -\frac{\ln\ 2}{2} + \frac{1}{2} I = \frac{1}{2}(\ln\ e - \ln\ 2) I = \frac{1}{2}\left(\ln\ \frac{e}{2}\right)$

3. $\int_1^e \left(\frac{1}{\sqrt{x\ln\ x}} + \sqrt{\frac{\ln\ x}{x}}\right)dx$

Ans. $\mathbf{2\sqrt{e}}$

Sol. $I = \int_1^e \left(\frac{1}{\sqrt{x\ln\ x}} + \frac{\sqrt{\ln\ x}}{x}\right)dx \qquad I = \int_1^e \frac{1}{\sqrt{x\ln\ x}} + \int_1^e \sqrt{\frac{\ln\ x}{x}}dx$

$I_1\ I_2 \qquad I_2 = \int_1^e \sqrt{\frac{\ln\ x}{\sqrt{x}}}dx$

$= \sqrt{\ln\ x}\int \frac{x}{\sqrt{x}}\Big|_1^e - \int_1^e \frac{2\cdot\sqrt{x}}{2\sqrt{\ln\ x}\cdot x}dx \qquad = [\sqrt{\ln\ x}\cdot 2\sqrt{x}]_1^e - \int_1^e \frac{dx}{\sqrt{x\ln\ x}}$

$= 2\sqrt{e\ln\ e} - I_1 \qquad \therefore I = I_1 + 2\sqrt{e} - I_1 \qquad = 2\sqrt{e}$

4. Given $f'(x) = \frac{\cos\ x}{x}, f\left(\frac{\pi}{2}\right) = a, f\left(\frac{3\pi}{2}\right) = b$. Find the value of the definite integral $\int_{\pi/2}^{3\pi/2} f(x)dx$.

Ans. $\mathbf{2 - \frac{\pi}{2}(a - 3b)}$

Sol. Given $f'\cos = \frac{\cos x}{x}, f\left(\frac{x}{2}\right) = a$

$f\left(\frac{3x}{2}\right) = b$ $\quad I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 \cdot f(x)dx$

$I = [f(x)\int dx]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x \cdot f'(x)dx$ $\quad I = [xf(x)]$

$I = [xf(x)]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x \cdot \frac{\cos x}{x}dx$ $\quad I = \left(\frac{3\pi}{2}f\left(\frac{3\pi}{2}\right) - \frac{\pi}{2}f\left(\frac{\pi}{2}\right)\right) - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$I = \frac{\pi}{2}(3b - a) + 2 = 2 - \frac{\pi}{2}(a - 3b)$

5. $\int_{-1}^{1} \frac{xdx}{\sqrt{5-4x}}$

Ans. $\frac{1}{6}$

Sol. $I = \int_{-1}^{1} \frac{xdx}{\sqrt{5-4x}}$

Put $5 - 4x = t \Rightarrow x = \frac{5-t}{4} - 4dx = dt$ $\quad dx = \frac{-dt}{4}$

$I = \int_9^1 \frac{5-t}{4.4\sqrt{t}}(-dt)$ $\quad I = \frac{1}{16}\left[\int_1^9 \frac{5}{\sqrt{t}}dt\right]$

$$I = \frac{1}{16}\left[\int_1^9 \frac{5}{\sqrt{t}}dt - \int_1^9 \frac{t}{\sqrt{t}}dt\right]$$

$$I = \frac{1}{16}\left(5.2\left[t^{\frac{1}{2}}\right]_1^9 - \frac{2}{3}\left[t^{\frac{3}{2}}\right]_1^9\right)$$

$I = \frac{1}{16}\left[5.2.2 - \frac{2}{3}(27 - 1)\right]$

$$I = \frac{1}{16}\left(20 - \frac{52}{3}\right)$$

$$I = \frac{1}{16} \times \frac{8}{3} = \frac{1}{6}$$

6. $\int_2^e \left(\frac{1}{\ln x} - \frac{1}{\ell n^2 x}\right) dx$

Ans. $e - \frac{2}{\ln 2}$

Sol. $I = \int_2^e \left(\frac{1}{\ln x} - \frac{1}{\ln^2 x}\right) dx$ $\quad I = \int_2^e \frac{dx}{\ln x} - \int_2^e \frac{dx}{\ln^2 x}$

$I_1 = \int_2^e \frac{dx}{\ln x} - \left[\frac{1}{\ln x}\int dx\right]_2^e - \int_2^e \frac{x \cdot \left(0 - \frac{1}{x}\right)}{\ln^2 x}dx$

$I_1 = \left[\frac{x}{\ln x}\right]_2^e + \int_2^e \frac{dx}{\ln^2 x}$ $\quad \therefore I = \left[e - \frac{2}{\ln 2}\right] + I_2 - I_2 \Rightarrow I = e - \frac{2}{\ln 2}$

7. $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$

Ans. $\frac{\pi}{4}$

Sol. $I=\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{\sin^4 x+\cos^4 x}dx$ $\quad I=\int_0^{\frac{\pi}{4}}\frac{\sin x}{1-\frac{\sin^2 x}{2}dx}$

$I=2\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{2-\sin^2 2x}$ $\quad I=2\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{1+\cos^2 2x}dx$

Put$\cos 2x=t-2\sin 2xdx=dt$

$I=\int_1^0\frac{dt}{2(1+t^2)}$ $\quad I=\int_0^1\frac{dt}{2(1+t^2)}$

$I=[\tan^{-1} t]_0^1$ $\quad I=\frac{\pi}{4}-0=\frac{\pi}{4}$

8. $\int_0^{\pi/2}\frac{\cos xdx}{(1+\sin x)(2+\sin x)}$

Ans. $\ln\frac{4}{3}$

Sol. $\int_0^{\frac{\pi}{2}}\frac{\cos xdx}{(1+\sin x)(2+\sin x)}$ $\quad$ Put $1+\mathrm{isn}\ x=t$ $\quad \cos xdx=dt$

$I=\int_1^2\frac{dt}{t(1+t)}$ $\quad I=\int_1^2\left(\frac{1}{t}-\frac{1}{1+t}\right)dt$

$I=[\ln t-\ln(1+t)]_1^2$ $\quad I=\left[\ln\frac{t}{1+t}\right]_1^2$ $\quad I=\left[\ln\frac{2}{3}-\ln\frac{1}{2}\right]$

$I=[\ln 2-\ln 3+\ln 2]$ $\quad I=\left[\ln\frac{4}{3}\right]$

9. $\int_0^{\pi/4}\frac{\sin^2 x\cdot\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$

Ans. $\frac{1}{6}$

Sol. $I=\int_0^{\frac{\pi}{4}}\frac{\sin^2 x\cdot\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$ $\quad I=\int_0^{\frac{\pi}{4}}\frac{\sin^2 x\cdot\cos^2 x}{\sin^6 x+\cos^6 x+2\sin^3 x\cos^3 x}dx$

$I=\int_0^{\frac{\pi}{4}}\frac{\tan^2 x\sec^2 x}{\tan^6 x+1+2\tan^3 x}dx$ $\quad$ Put $\tan_2 xdx=t$

$I=\int_0^1\frac{t^2dt}{t^6+1+2t^3}$ $\quad I=\int_0^1\frac{t^2}{(t^3+1)^2}dt$

Put $t^3=z$ $\quad$ st $^2dt=dz$

$I=\frac{1}{3}\int_0^1\frac{dz}{(z+1)^2}$ $\quad I=\frac{1}{3}\left[\frac{-1}{z+1}\right]_0^1$

$I=-\frac{1}{3}\left[\frac{1}{2}-1\right]=\frac{1}{6}$

10. $\int_{1/3}^3\frac{\sin^{-1}\frac{x}{\sqrt{1+x^2}}}{x}dx$

Ans. $\frac{\pi\ln 3}{2}$

Sol. $I=\int_{\frac{1}{3}}^3\frac{\sin^{-1}\frac{x}{\sqrt{1+x^2}}}{x}dx$ $\quad$ Let $x\to\frac{1}{x}$

$dx=\frac{-1}{t^2}dt$ $\quad I=\int_{\frac{1}{3}}^3\frac{\sin^{-1}\frac{1}{\sqrt{1+x^2}}}{x}dt$

$\sin^{-1}\frac{1}{\sqrt{1+x^2}}=\cos^{-1}\frac{x}{\sqrt{1+x^2}}I=\int_{\frac{1}{3}}^{3}\frac{\cos^{-1}\frac{x}{\sqrt{1+x^2}}}{x}dx$

(i) + (ii)

$2I=\int_{\frac{1}{3}}^{3}\frac{1}{x}\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}+\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right)dx$

$2I=\frac{x}{2}\int_{\frac{1}{3}}^{3}\frac{dx}{x}=\frac{x}{2}\left(\ln\ 3-\ln\ \frac{1}{3}\right)=\ln\ 3\qquad\Rightarrow I=\frac{\ln\ 3}{2}$

11. $\int_{2}^{3}\frac{dx}{\sqrt{(x-1)(5-x)}}$

Ans. $\frac{\pi}{6}$

Sol. $I=\int_{2}^{3}\frac{dx}{(x-1)(5-x)}$

Put $x-1=t$ $\qquad dx=dt$

$I=\int_{1}^{2}\frac{dt}{\sqrt{t(4-t)}}\qquad I=\int_{1}^{2}\frac{dt}{\sqrt{2^2-(t-2)^2}}$

Put $t-2=z$ $\qquad dt=dz$

$I=\int_{-1}^{0}\frac{dz}{\sqrt{2^2-z^2}}\qquad I=\left[\sin^{-1}\frac{z}{2}\right]_{-1}^{0}$

$I=\left[0-\left(-\frac{\pi}{6}\right)\right]\qquad I=\frac{\pi}{6}$

12. $\int_{3/2}^{2}\left(\frac{x-1}{3-x}\right)^{1/2}dx$

Ans. $\frac{\sqrt{3}}{2}-1+\frac{\pi}{6}$

Sol. $I=\int_{\frac{3}{2}}^{2}\left(\frac{x-1}{3-x}\right)^{\frac{1}{2}}dx$

put $3-x=t^2\Rightarrow 2-t^2=x-1-dx=2tdt\qquad I=-\int_{\frac{\sqrt{3}}{2}}^{1}\frac{\sqrt{2-t^2}}{t}2tdt$

$I=2\int_{1}^{\frac{\sqrt{3}}{2}}\sqrt{2-t^2}dt\qquad I=2\left[\frac{t}{2}\sqrt{2-t^2}+\frac{2}{2}\sin^{-1}\frac{t}{\sqrt{2}}\right]_{1}^{\frac{\sqrt{3}}{2}}$

$I=2\left[\frac{1}{2}\cdot\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{1}{23}}+\sin^{-1}\sqrt{\frac{3}{4}}\right]_{-}\left(\frac{1}{2}+\frac{\pi}{4}\right)\qquad I=\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-1-\frac{\pi}{2}$

$I=\frac{\sqrt{3}-2}{2}+\frac{\pi}{6}=\frac{1}{2}\left(\sqrt{3}-2+\frac{\pi}{12}\right)\qquad I=\frac{\sqrt{3}}{2}-1+\frac{\pi}{12}$

13. $\int_{0}^{\pi/4}x\cos\ x\cos\ 3xdx$

Ans. $\frac{\pi-3}{16}$

Sol. $I=\int_{0}^{\frac{\pi}{4}}x\cos\ x\cos\ 3xdx\qquad I=\frac{1}{2}\int_{0}^{\frac{\pi}{4}}x2\cos\ x\cos\ 3xdx$

$I=\frac{1}{2}\int_0^{\frac{\pi}{4}} x(\cos\ 4x+\cos\ 2x)dx$ $\quad$ $I=\frac{1}{2}\left[\int_0^{\frac{\pi}{4}} x\cos\ 4xdx+\int_0^{\frac{\pi}{4}} x\cos\ 2xdx\right]$

$I=\frac{1}{2}\left[x\frac{\sin\ 4x}{4}\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{\sin\ 4x}{4}dx+X\frac{\sin\ 2x}{2}\Big|_0^{\frac{\pi}{4}}-\frac{\pi}{4}\frac{\sin\ 2x}{2}dx\right]$

$I=\frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{16}[\cos\ 4x]_0^{\frac{\pi}{4}}\ I=\frac{\pi}{2}\left(0+\frac{\pi}{8}-\frac{3}{8}\right)I=\frac{1}{2}1+\frac{1}{4}[\cos\ 2x]_0^{\frac{\pi}{4}}\right]I+\frac{-2}{16}+\frac{\pi}{8}+\frac{1}{4}(0-1)\Big]$

14. $\int_0^{\pi/2}\frac{dx}{5+4\sin\ x}$

Ans. $\frac{2}{3}\tan^{-1}\frac{1}{3}$

Sol. $I=\int_0^{\frac{\pi}{2}}\frac{dx}{5+4\sin\ x}$ $\quad$ Put $\sin\ x=\frac{2\tan\frac{x}{2}}{1+\tan^3\frac{x}{2}}$

$I=\int_0^{\frac{\pi}{2}}\frac{1+\tan^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}dx$ $\quad$ Put $\tan\frac{x}{2}=t$

$\sec^2\frac{x}{2}\cdot\frac{1}{2}dx=dt\ I=\int_0^1\frac{2dt}{5+5t^2+8t}$

$I=\frac{2}{5}\int_0^1\frac{dt}{t^2+\frac{8}{5}t+1}$ $\quad$ $I=\frac{2}{5}\int_0^1\frac{dt}{\left(t+\frac{4^2}{5}\right)+1-\frac{4^2}{5^2}}$

$I=\frac{2}{5}\int_0^1\frac{dt}{\left(t+\frac{4}{5}\right)^2+\frac{3^2}{5^2}}$ $\quad$ $I=\frac{2}{5}\frac{1}{\frac{3}{5}}\left[\tan^{-1}\frac{t+\frac{4}{5}}{\frac{3}{5}}\right]_0^1$

$I=\frac{2}{3}\left[\tan^{-1}\ 3-\tan^{-1}\ \frac{4}{3}\right]$ $\quad$ $I=\frac{2}{3}\tan^{-1}\left(\frac{\frac{2}{3}}{5}\tan^{-1}\left(\frac{3-\frac{4}{3}}{1+3\cdot\frac{4}{3}}\right)\right.$ $\quad$ $I=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

15. $\int_2^3\frac{dx}{(x-1)\sqrt{x^2-2x}}$

Ans. $\frac{\pi}{3}$

Sol. $I=\int_2^3\frac{dt}{(x-1)\sqrt{x^2-2x}}$ $\quad$ $I=\int_2^3\frac{dt}{(x-1)\sqrt{(x-1)^2-1}}$

$I=[\sin^{-1}(x-1)]_2^3$ $\quad$ $I=[\sin^{-1}\ 2-\sec^{-1}\ 1]$ $\quad$ $I=\left(\frac{\pi}{3}-0\right)=\frac{\pi}{3}$

16. $\int_0^{\pi/2}\frac{dx}{1+\cos\ \theta\cos\ x}\theta\in(0,\pi)$

Ans. $\frac{\theta}{\sin\theta}$

Sol. $I=\int_0^{\frac{x}{2}}\frac{dx}{1+\cos\ \theta\cos\ x}\theta\in(0,x)$ $I=\int_0^{\frac{x}{2}}\frac{dx}{1+\cos\ \theta\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$ $\quad$ $I=\int_0^{\frac{x}{2}}\frac{\sec^2\frac{x}{2}dx}{1+\tan^2\frac{x}{2}+\cos'\ \theta-\cos\ \theta\tan^2\frac{x}{2}}$

$I=\int_0^{\frac{x}{2}}\frac{\sec^2\frac{x}{2}dx}{(1+\cos^2\ \theta)+\tan^2(1-\cos\ \theta)}$ $\quad$ $I=\int_0^{\frac{x}{2}}\frac{\sec^2\frac{x}{2}dx}{2\cos^2\frac{\theta}{2}+2\sin^2\frac{\theta}{2}\tan^2\frac{x}{2}}$

$\int_0^{\frac{x}{2}}\frac{\sec^2\frac{x}{2}dx}{2\sin^2\frac{\theta}{2}\left(\cot^2\frac{\theta}{2}+\tan^2\frac{x}{2}\right)}$ $\quad$ Put $\tan\frac{x}{2}=t$

$\sec^2 \frac{x}{2}\frac{1}{2}dx = dt$ $\qquad \sec^2 \frac{x}{2}dx = 2dt$

$I = \int_0^1 \frac{2dt}{2\sin^2\frac{\theta}{2}\left(\cot^2\frac{\theta}{2}+t^2\right)}$ $\qquad I = \frac{1}{\sin^2\frac{\theta}{2}}\int_0^1 \frac{dt}{\cot^2\frac{\theta}{2}+t^2}$

$I = \frac{1}{\sin^2\frac{\theta}{2}} \cdot \frac{1}{\cot\frac{\theta}{2}}\tan^{-1}\left.\frac{t}{\cot\frac{\theta}{2}}\right]_0^1$ $\qquad I = \frac{2}{\sin\theta}\left(\frac{\theta}{2}\right) = \frac{\theta}{\sin\theta}$

17. $\int_0^{\frac{\ln 3}{2}} \frac{e^x+1}{e^{2x}+1}dx$

Ans. $\frac{1}{2}\left(\frac{\pi}{6} + \ln 3 - \ln 2\right)$

Sol. $I = \int_0^{\frac{\ln 3}{2}} \frac{e^x+1}{e^{2x}+1}dx$ $\qquad I = \int_0^{\frac{\ln 3}{2}} \frac{e^x}{e^{2x}+1}dx + \int_0^{\frac{\ln 3}{2}} \frac{1}{e^{2x}+1}dx$

for I_1 $\qquad I_1\ I_2$ $\qquad$ put $e^x = t$

$e^x dx = dt$ $\qquad I_1 = \int_1^{\sqrt{3}} \frac{dt}{t^2+1} = \tan^{-1} t\Big]_1^{\sqrt{3}}$

$I_1 = \frac{x}{3} - \frac{x}{4} = \frac{x}{12}$

for I_2

Put $e^x = t$

$e^x dx = dt$

$dx = \frac{dt}{t}$

$I_2 = \int_1^{\sqrt{3}} \frac{dt}{t(t^2+1)}$

$I_2 = \int_1^{\sqrt{3}} \left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt$

$I_2 = \ln t\Big]_1^{\sqrt{3}} - \frac{1}{2}\ln(t^2+1)\Big]_1^{\sqrt{3}}$

$I_2 = \ln\sqrt{3} - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 2$

$\therefore I = I_1 + I_2$

$= \frac{x}{12} + \frac{1}{2}\ln 3 - \frac{1}{2}\ln 2$

$= \frac{1}{2}\left(\frac{x}{6} + \ln 3 - \ln 2\right)$

18. $\int_0^{\pi/4} \cos 2x\sqrt{1-\sin 2x}\,dx$

Ans. $\frac{1}{3}$

Sol. $I=\int_0^{\frac{\pi}{4}}\cos\ 2x\sqrt{1-\sin\ 2x}dx$

$I=\int_0^{\frac{\pi}{4}}\cos\ 2x\sqrt{(\cos\ x-\sin\ x)^2}dx$

$I=\int_0^{\frac{\pi}{4}}\cos\ 2x(\cos\ x-\sin\ x)dx$

$I=\frac{1}{2}\int_0^{\frac{\pi}{4}}2\cos\ 2x(\cos\ x-\sin\ x)dx$

$I=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos\ 3x+\cos\ x-\sin\ 3x+\sin\ x)dx$

$I=\frac{1}{2}\left[\frac{\sin\ 3x}{3}+\sin\ x+\frac{\cos\ 3x}{3}-\cos\ x\right]_0^{\frac{\pi}{4}}$

$I=\frac{1}{2}\left[\frac{1}{3\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{3\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{2}{3}\right]$

$I=\frac{1}{3}$

19. $\int_0^3\sqrt{\frac{x}{3-x}}dx$

Ans. $\frac{3\pi}{2}$

Sol. $I=\int_0^3\sqrt{\frac{x}{3-x}}dx$

put $3-x=t^2$

- $dx=2tdt$

$I=-\int_{\sqrt{3}}^{0}\frac{\sqrt{3-t^2}}{t}\cdot 2tdt$

$I=2\int_0^{\sqrt{3}}\sqrt{3-t^2}dt$

$I=2\cdot\left[\frac{t}{2}\sqrt{3-t^2}+\frac{3}{2}\sin^{-1}\frac{t}{\sqrt{3}}\right]_0^{\sqrt{3}}$

$I=2\left(\frac{3}{2}\cdot\frac{\pi}{2}-0\right)=\frac{3\pi}{2}$

20. $\int_0^{1/2}\frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

Ans. $\frac{1}{2}\ln(2+\sqrt{3})$

Sol. $I=\int_0^{\frac{1}{2}}\frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

$x=\sin\ \theta$

$dx=\cos\ \theta d\theta$

$I=\int_0^{\frac{\pi}{6}}\frac{\cos\ \theta d\theta}{\cos\ 2\theta\cdot\cos\ \theta}$

$I=\int_0^{\frac{\pi}{6}}\sec\ 2\theta d\theta$

$I = \frac{1}{2}[\ln(\sec 2\theta + \tan 2\theta)]_0^{\frac{\pi}{6}}$

$I = \frac{1}{2}(\ln(2+\sqrt{3}))$

21. $\int_1^2 \frac{dx}{x(x^4+1)}$

Ans. $\frac{1}{4}\ln\frac{32}{17}$

Sol. $I = \int_1^2 \frac{dx}{x(x^4+1)}$

$$I = \int_1^2 \frac{x^3dx}{x^4(x^4+1)}$$

put $x^4 = t$

$4x^3dx = dt$

$$I = \frac{1}{4}\int_1^{16}\frac{dt}{t(t+1)}$$

$$I = \frac{1}{4}\int_1^{16}\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$I = \frac{1}{4}[\ln t - \ln t+1]_1^{16}$$

$$I = \frac{1}{4}\left[\ln\frac{t}{t+1}\right]_1^{16}$$

$$I = \frac{1}{4}\left[\ln\frac{16}{17} - \ln\frac{1}{2}\right] = \frac{1}{4}\ln\left(\frac{32}{17}\right)$$

22. $\int_0^{\pi/2}\sin\phi\cos\phi\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}d\phi a \neq b\ (a > 0, b > 0)$

Ans. $\frac{1}{3}\frac{a^3-b^3}{a^2-b^2}$

Sol. $I = \int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta}d\theta$

$a \neq b(a > 0, b > 0)$

put $a^2\sin^2\theta + b^2\cos^2\theta = t$

$(a^2 2\sin\theta\cos\theta - b^2\cos\theta\sin\theta)d\theta = dt$

$$\sin\theta\cos\theta d\theta = \frac{dt}{2(a^2-b^2)}$$

$$I = \int_{b^2}^{a^2}\frac{t^{\frac{1}{2}}dt}{2(a^2-b^2)}$$

$$I = \frac{1}{2(a^2-b^2)}\frac{2}{3}\left[t^{\frac{3}{2}}\right]_{b^2}^{a^2}$$

$$I = \frac{1}{3}\left(\frac{a^3-b^3}{a^2-b^2}\right)$$

23. (a) $\int_0^{3\pi/4}((1+x)\sin x+(1-x)\cos x)dx$

(b) $\int_{\pi/2}^{\pi}x^{\sin x}(1+x\cos x\cdot\ln x+\sin x)dx$

Ans. **(a)** $2(\sqrt{2}+1)$;**(b)** $\left(\pi-\frac{\pi^2}{4}\right)$

Sol. $I=\int_0^{\frac{3\pi}{4}}((1+x)\sin x-(1-x)\cos x)dx$

$I=\int_0^{\frac{3\pi}{4}}(1+x)\sin xdx+\int_0^{\frac{3\pi}{4}}(1-x)\cos xdx$

$I=(1+x)(-\cos x)|_0^{\frac{3\pi}{4}}+\int_0^{\frac{3\pi}{4}}\cos xdx+(1-x)\sin x|_0^{\frac{3\pi}{4}}+\int_0^{\frac{3\pi}{4}}\sin xdx$

$I=-\left(\left(1+\frac{3\pi}{4}\right)\left(\frac{-1}{\sqrt{2}}\right)-1\right)+\left(\frac{1}{\sqrt{2}}\right)+\left(1-\frac{3\pi}{4}\right)\frac{1}{\sqrt{2}}-\left(\frac{-1}{\sqrt{2}}-1\right)$

$I=\frac{1}{\sqrt{2}}\left(1+\frac{3\pi}{4}\right)+1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(1-\frac{3\pi}{4}\right)+\frac{1}{\sqrt{2}}+1$

$I=\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+1$

$I=\frac{4}{\sqrt{2}}+2=2\sqrt{2}+2=2(\sqrt{2}+1)$

24. $\int_0^1 x(\tan^{-1}x)^2dx$

Ans. $\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\ln 2$

Sol. $I=\int_0^1 x(\tan^{-1}x)^2dx$

$I=(\tan^{-1}x)^2|xdx|_0^1-\int_0^1\frac{x^2}{2}\cdot\frac{2\tan^{-1}x}{1+x^2}dx$

$I=\left[(\tan^{-1}x)^2\frac{x^2}{2}\right]_0^1-\int_0^1\frac{x^2}{2}\cdot\frac{2\tan^{-1}x}{1+x^2}dx$

$I=\left[\left(\frac{x}{4}\right)^2\cdot\frac{1}{2}-0\right]-\int_0^1\tan^{-1}x+\int_0^1\frac{\tan^{-1}x}{1+x^2}dx$

$I=\frac{x^2}{3^2}-[\tan^{-1}x\cdot x]_0^1-\int_0^1\frac{x^0}{1+x^2}dx\Bigg]+\frac{1}{2}[(\tan^{-1}x)^2]_0^1$

$I=\frac{\pi^2}{3^2}-\left(\frac{\pi}{4}-0\right)+\frac{1}{2}\ln 2+\frac{\pi^2}{3^2}=\frac{\pi^2}{6}-\frac{\pi}{4}+\frac{1}{2}\ln 2=\frac{x}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\ln 2$

25. Suppose that f,f' and f'' are continuous on $[0,\ln 2]$ and that $f(0)=0,f'(0)=3,f(\ln 2)=6$, $f'(\ln 2)=4$ and $\int_0^{\ln 2}e^{-2x}\cdot f(x)dx=3$. Find the value of $\int_0^{\ln 2}e^{-2x}\cdot f''(x)dx$.

Ans. **13**

Sol. Given

f,f' and f'' are continuous on $[0,\ln 2]$

$f(0)=0 f'(0)=3, f(\ln 2)=6 f'(\ln 2)=4$

$\int_0^{\ln 2}e^{-2x}f(x)dx=3$

$I=\int_0^{\ln 2} e^{-2x}f''(x)dx$

$I=e^{-2x}\int f''(x)dx|_0^{\ln 2}+2\int_0^{\ln 2} f'(x)e^{-2x}$

$I=e^{-2x}f'(x)|_0^{\ln 2}+2\left[e^{-2x}\int f'(x)dx|_0^{\ln 2}+\int_0^{\ln 2} e^{-2x}f'(x)dx\right]$

$I=e^{-2\ln 2}f'(\ln 2)-f'(0)+2$

$I=\frac{1}{4}\cdot 4-3+2\left[\frac{1}{4}\times 6-0+6\right]$

$I=1-3+3+12=13$

26. $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$ where $-\pi<\alpha<\pi$

Ans. $\frac{\alpha}{2\sin\alpha}$ **if** $a\neq 0;\frac{1}{2}$ **if** $\alpha=0$

Sol. $I=\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$

$I=\int_0^1 \frac{dx}{(x+\cos\alpha)^2+1-\cos^2\alpha}$

$I=\int_0^1 \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$

put $x+\cos\alpha=t$

$dx=dt$

$I=\int \frac{dt}{t^2+\sin^2\alpha}=\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{1}{\sin\alpha}\right]_{\cos\alpha}^{1+\cos\alpha}$

$I=\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$

$\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\tan^{\frac{\pi}{2}}-\frac{d}{2}\right)-\tan^{-1}\left(\tan^2\left(\frac{\pi}{2}-\alpha\right)\right)\right]$

$I=\frac{1}{\sin\alpha}\left[\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right]$

$I=\frac{1}{2}\cdot\frac{\alpha}{\sin\alpha}$

$\alpha\neq 0\ I=\frac{1}{2}\cdot\frac{\alpha}{\sin\alpha}$

$\alpha=0\frac{\alpha}{\sin\alpha}\to I$

$\therefore I=\frac{1}{2}$

27. $\int_a^b \frac{dx}{\sqrt{1+x^2}}$ where $a=\frac{e-e^{-1}}{2}$ &$b=\frac{e^2-e^{-2}}{2}$

Ans. **1**

Sol. $\int_a^b \frac{dx}{\sqrt{1+x^2}}$ where

$$a=\frac{e-e^{-1}}{2}\&b=\frac{e^2-e^{-2}}{2}$$

$$I=\int_a^b \frac{dx}{\sqrt{1+x^2}}=\left[\ln\left(x+\sqrt{1+x^2}\right)\right]_a^b$$

$$I=\ln\left(b+\sqrt{1+b^2}\right)-\ln\left(a+\sqrt{1+a^2}\right)$$

$$I=\ln\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

$$I=\ln\left(\frac{e^2-e^{-2}}{2}+\sqrt{1+\left(\frac{e^2-e^{-2}}{2}\right)^2}\right)$$

$$I=\ln\left(\frac{e^2-e^{-2}+e^2+e^{-2}}{e-e^{-1}+e+e^{-1}}\right)\sqrt{1+\left(\frac{e-e^{-1}}{2}\right)^2}\Bigg)$$

$$I=\ln e=1$$

28. $\int_{0^+}^{1}\frac{x^x(x^{2x}+1)(\ln x+1)}{x^{4x}+1}dx$

Ans. **0**

Sol. $I=\int_0^1\frac{x^2(x^{2x}+1)(\ln x+1)}{x^{4x}+1}dx$

Put $x^x=t$

$x^x(\ln x+1)dx=dt$

$I=\int_0^1\frac{(t^2+1)}{(t^4+1)}dt=0$

29. $\int_0^1 x^5\sqrt{\frac{1+x^2}{1-x^2}}dx$

Ans. $\mathbf{\frac{3\pi+8}{24}}$

Sol. $I=\int_0^1 x^5\sqrt{\frac{1+x^2}{1-x^2}}dx$

$I=\int_0^1 x^4\cdot\sqrt{\frac{1+x^2}{1-x^2}}xdx$

put $x^2=\cos\theta$

$2xdx=-\sin\theta d\theta$

$I=-\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\cos^2\theta\sin\theta\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}d\theta$

$I=-\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\cos^2\theta\sin\theta\cot\frac{\theta}{2}d\theta$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}2\cos^2\theta\cos^2\frac{\theta}{2}d\theta$$

$$I=\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1+\cos\theta}{2}\right)d\theta$$

$$I=\frac{1}{4}\left[\theta+\frac{\sin\ \theta}{2}+\sin\ \theta\right]_0^{\frac{\pi}{2}}+\frac{1}{8}\int_0^{\frac{\pi}{2}}(1+\cos\ 3\theta+\cos\ \theta)d\theta$$

$$I=\frac{1}{4}\left[\frac{\pi}{2}+0+1-(0+0+0)\right]+\frac{1}{8}\left[\frac{\sin\ 3\theta}{3}+\sin\ \theta\right]_0^{\frac{\pi}{2}}$$

$$I=\frac{1}{4}\left(\frac{\pi}{2}+1\right)+\frac{1}{8}\left(-\frac{1}{3}+1\right)$$

$$=\frac{\pi}{8}+\frac{1}{4}+\frac{1}{12}=\frac{\pi}{8}+\frac{1}{3}=\frac{3\pi+8}{24}$$

30. Suppose that the function f, g, f' and g' are continuous over $[0,1], g(x)\neq 0$ for $x\in[0,1], f(0)=0,\ g(0)=\pi,\ f(1)=\frac{2009}{2}$ and $g(1)=1$. Find the value of the definite integral

$$\int_0^1 \frac{f(x)\cdot g'(x)\{g^2(x)-1\}+f'(x)\cdot g(x)\{g^2(x)+1\}}{g^2(x)}dx$$

Ans. **2009**

Sol. Given f, g, f' and g' are continuous in $[0,1]$

$g(x)\neq 0\forall x\in[0,1]$

$f(0)=0g(0)=x, f(1)=\dfrac{2009}{2}\&g(1)=1$

$$I=\int_0^1 \frac{(f(x)g'(x)\{g^2(x)-1\}+f'(x)\cdot g(x)\{g^2(x)+1\})}{g^2(x)}dx$$

$$I=\int_0^1 (f(x)g'(x)+f'(x)g(x))dx+\int_0^1\frac{(f'(x)g(x)-f(x)g'(x))}{g^2(x)}dx$$

$$I=\int_0^1 d[f(x)g(x)]+\int_0^1 d\left(\frac{f(x)}{g(x)}\right)$$

$$I=f(x)g(x)|_0^1+\left.\frac{f(x)}{g(x)}\right|_0^1$$

$$I=f(1)\cdot g(1)-f(0)g(0)+\frac{f(1)}{g(1)}-\frac{f(0)}{g(0)}$$

$$I=\frac{2009}{2}\cdot 1-0+\frac{2009}{2}-0$$

$$I=2009$$

31. $\displaystyle\int_0^{\pi/4}\frac{\sin\ \theta+\cos\ \theta}{9+16\sin\ 2\theta}d\theta$

Ans. $\mathbf{\frac{1}{20}ln3}$

Sol. $I=\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin\ \theta+\cos\ \theta}{9+16\sin\ 2\theta}d\theta$

put $\sin\ \theta-\cos\ \theta=t$

$(\cos\ \theta+\sin\ \theta)d\theta=dt$

$(\sin\ \theta-\cos\ \theta)^2=t^2$

$1-\sin\ 2\theta=t^2$

$\sin 2\theta = 1 - t^2$

$I = \int_{-1}^{\theta} \frac{dt}{9+16(1-t^2)}$

$I = \int_{-1}^{0} \frac{dt}{28-16t^2} = \frac{1}{6}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$I = \frac{1}{16} \cdot \frac{1}{2\left(\frac{5}{4}\right)}\left[\ln \left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$

$I = \frac{1}{40}\left[\ln 1 - \ln \frac{1}{9}\right]$

$I = \frac{1}{40}(0 + 2\ln 3)$

$I = \frac{1}{20}\ln 3$

32. $\int_0^{\pi} \theta\sin^2\theta\cos\theta d\theta$

Ans. $-\frac{4}{9}$

Sol. $I = \int_0^{\pi} \theta\sin^2\theta\cos\theta d\theta$

$I = \int_0^{\pi} \theta(1-\cos^2\theta)\cos\theta d\theta$

$I = \int_0^{\pi} \theta(\cos\theta - \theta\cos^3\theta)d\theta$

$I = \int_0^{\pi} \theta\cos\theta d\theta - \int_0^{\pi} \theta\cos^3\theta d\theta$

$I = \theta\int \cos\theta d\theta|_0^{\theta} - \int_0^{\pi} \sin\theta d\theta - \int_0^{\pi} \theta\left(\frac{\cos 3\theta + 3\cos\theta}{4}\right)d\theta$

$I = \theta\sin\theta|_0^{\pi} + \cos\theta|_0^{\pi} - \frac{1}{4}\int_0^{\pi} \theta\cos 3\theta d\theta - \frac{3}{4}\int_0^{\pi} \theta\cos\theta d\theta$

$I = 0 - (-1,-1) - \frac{1}{4}\left[\theta\int \cos 3\theta d\theta|_0^{\pi} - \frac{1}{3}\int_0^{\pi} \sin 3\theta d\theta\right]$

$-\frac{3}{4}\left(\theta\int\cos\theta\Big|_0^{\pi} - \int_0^{\pi}\sin\theta d\theta\right)$

$I = 0 - 2 - \frac{1}{4}\left[\theta \cdot \frac{\sin 3\theta}{3}\Big|_0^{\pi} + \frac{1}{9}\cos 3\theta\Big|_0^{\pi}\right] - \frac{3}{4}(\theta\sin\theta|_0^{\pi} + \cos\theta|_0^{\pi})$

$I = 0 - 2 - \frac{1}{4}(0) - \frac{1}{36}(-1,-1) - \frac{3}{4}(0) - \frac{3}{4}(-1,-1)$

$I = -2 + \frac{1}{18} + \frac{3}{2} = \frac{1}{18} - \frac{1}{2} = \frac{-8}{18} = \frac{-4}{9}$

33. $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}dx$

Ans. $\frac{1}{2}$

Sol. $I = \int_0^{\frac{\pi}{2}} \frac{1+2\cos x}{(2+\cos x)^2}dx$

divide N^r&DN^r by $\operatorname{cosec}^2 X$

$$I = \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec} e^2x + 2\cot x\operatorname{cosec} x}{(2\operatorname{cosec} x + \cot x)^2}$$

put $2\operatorname{cosec} x+\cot x=t$

$(-2\operatorname{cosec} x\cot x-\operatorname{cosec}^2 x)dx=dt$

$I=\int_{\infty}^{2}\frac{dt}{t^2}$

$I=\int_{2}^{\infty}\frac{dt}{t^2}=-\left[\frac{1}{t}\right]_2^{\infty}$

$I=-\left(0-\frac{1}{2}\right)=\frac{1}{2}$

34. $\int_0^{\pi/2}\frac{x+\sin x}{1+\cos x}dx$

Ans. $\frac{\pi}{2}$

Sol. $I=\int_0^{\frac{\pi}{2}}\frac{x+\sin x}{1+\cos x}$

$I=\int_0^{\frac{\pi}{2}}\frac{x}{2\cos^2\frac{x}{2}}dx+\int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$

$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\sec^2\frac{x}{2}dx+\int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$

$I=\frac{1}{2}x\int\sec^2\frac{x}{2}dx\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}2\cdot\tan\frac{x}{2}dx+\int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$

$I=\frac{1}{2}\left(\frac{\pi}{2}\cdot 2\cdot 1-0\right]$

$I=\frac{\pi}{2}$

35. Let $A=\int_{3/4}^{4/3}\frac{2x^2+x+1}{x^3+x^2+x+1}dx$ then find the value of e^A.

Ans. $\frac{16}{9}$

Sol. $A=\int_{\frac{3}{4}}^{\frac{4}{3}}\frac{2x^2+x+1}{x^3+x^2+x+1}dx$

$A=\int_{\frac{3}{4}}^{\frac{4}{3}}\frac{2x^2+x+1}{(x^2+1)(x+1)}$

$A=\int_{\frac{3}{4}}^{\frac{4}{3}}\left(\frac{x}{x^2+1}+\frac{x}{x+1}\right)dx$

$A=\int_{\frac{3}{4}}^{\frac{4}{3}}\frac{x}{x^2+1}dx+\int_{\frac{3}{4}}^{\frac{4}{3}}\frac{dx}{x+1}$

$A=\frac{1}{2}\ln(x^2+1)_{\frac{3}{4}}^{\frac{4}{3}}+\ln(x+1)|_{\frac{3}{4}}^{\frac{4}{3}}$

$A=\frac{1}{2}\left(\ln\left(\frac{16}{9}+1\right)-\ln\left(\frac{9}{16}+1\right)\right)+A=\frac{1}{2}\left(\ln\frac{25}{9}+1\right)-\ln\frac{28}{16}\right)+\left(\frac{3}{4}+1\right)\right)\frac{7}{3}-\ln\frac{7}{4}\right)$

$A=\frac{1}{2}\ln\left(\frac{16}{9}\right)+\ln\frac{4}{3}$

$A=\ln\frac{4}{3}+\ln\frac{4}{3}$

$A = 2\ln\frac{4}{3}$

$A = \ln\left(\frac{4}{3}\right)^2$

$\therefore e^A = e^{\ln\left(\frac{4}{3}\right)^2} = \frac{16}{9}$

36. $\int_0^1 \frac{2-x^2}{(x+1)\sqrt{1-x^2}}dx$

Ans. $\frac{\pi}{2}$

Sol. $I = \int_0^1 \frac{2-x^2}{(x+1)\sqrt{1-x^2}}dx$

put $x = \sin\ \theta$

$dx = \cos\ \theta d\theta$

$I = \int_0^{\frac{\pi}{2}} \frac{(2-\sin^2\ \theta)\cos\ \theta}{(\sin\ \theta+1)\sqrt{1-\sin^2\ \theta}}d\theta$

$I = \int_0^{\frac{\pi}{2}} \frac{(1+\cos^2\ \theta)\cos\ \theta}{(\sin\ \theta+1)\cos\ \theta}d\theta$

$I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\ \theta+1}d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos^2\ \theta d\theta}{1+\sin\ \theta}$

$I = \int_0^{\frac{\pi}{2}} \frac{1-\sin\ \theta}{1-\sin^2\ \theta}d\theta + \int_0^{\frac{\pi}{2}} \frac{(1-\sin\ \theta)(1+\sin\ \theta)}{(1+\sin\ \theta)}d\theta$

$I = \int_0^{\frac{\pi}{2}} (\sec^2\ \theta - \sec\ \theta\tan\ \theta)d\theta + \int_0^{\frac{\pi}{2}} (1-\sin\ \theta)d\theta$

$I = [\tan\ \theta - \sec\ \theta]_0^{\frac{\pi}{2}} + [\theta + \cos\ \theta]_0^{\frac{\pi}{2}}$

$I = \left[\frac{\sin\ \theta - 1}{\cos\ \theta}\right]_0^{\frac{\pi}{2}} + \left(\frac{\pi}{2} + 0 - (0+1)\right)$

$I = \left[0 - (-1) + \frac{\pi}{2} - 1 = 1 + \frac{\pi}{2} - 1 = \frac{\pi}{2}\right.$

37. $\int_{-1}^1 \left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx$

Ans. $\frac{2}{1+e}$

Sol. $I = \int_{-1}^1 \left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx$

function is not defined at $x = 0$

$I = \int_{-1}^0 \left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx + \int_{0^+}^1 \left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx$

$I = \left[\frac{1}{1+e^{1/x}}\right]_{-1}^{0^-} + \left[\frac{1}{1+e^{1/x}}\right]_{0^+}^{1}$

$$I = 1 - \frac{e}{1+e} + \frac{1}{1+e} = \frac{1+e-e+1}{1+e}$$

$$I = \frac{2}{1+e}$$

38. $\int_1^e \frac{dx}{\ln(x^x e^x)}$

Ans. **ln 2**

Sol. $I = \int_1^e \frac{dx}{\ln(x^x e^x)}$

$I = \int_1^e \frac{dx}{\ln((xe)^x)}$

$I = \int_1^e \frac{dx}{x\ln ex}$

put $\ln ex = t$

$\frac{e1}{ex} \cdot dx = dt$

$\frac{dx}{x} = dt$

$I = \int_1^2 \frac{dt}{t}$

$I = [\ln t]_1^2$

$I = \ln 2 - 0$

$I = \ln 2$

39. $\int_0^\pi \left[\cos^2\left(\frac{3\pi}{8} - \frac{x}{4}\right) - \cos^2\left(\frac{11\pi}{8} + \frac{x}{4}\right)\right] dx$

Ans. $\sqrt{2}$

Sol. $I = \int_0^\pi \left[\cos^2\left(\frac{3\pi}{8} - \frac{x}{4}\right) - \cos^2\left(\frac{11\pi}{8} + \frac{x}{4}\right)\right] dx$

$I = \int_0^\pi \left(\sin\left(\frac{3\pi}{8} - \frac{x}{4} + \frac{4x}{8} + \frac{x}{4}\right)\sin\left(\frac{11\pi}{8} + \frac{x}{4} - \frac{3\pi}{8} + \frac{x}{4}\right)\right) dx$

$I = \int_0^\pi \sin\left(\frac{14\pi}{8}\right)\sin\left(\pi + \frac{x}{2}\right) dx$

$I = \sin\left(\frac{7\pi}{4}\right)\int_0^\pi (-)\sin\frac{3}{2} dx$

$I = -\frac{1}{\sqrt{2}} 2 \cdot \left[\cos\frac{x}{2}\right]_0^\pi$

$I = -\sqrt{2}(0-1) = \sqrt{2}$

40. If $f(\pi) = 2 \& \int_0^\pi (f(x) + f''(x))\sin x dx = 5$, then find $f(0)$

Ans. **3**

Sol. Given $f(x) = 2$

$\int_0^\pi (f(x) + f''(x))\sin x dx = 5$

$\underbrace{\int_0^\pi f(x)\sin x dx}_{I_1} + \underbrace{\int_0^\pi f''(x)\sin x dx}_{I_2} = 5$

$I_1 = \int_0^{\pi} f(x)\sin xdx$

$I_1 = f(x)(-)\cos x|_0^{\pi} - \int_0^{\pi} (-)\cos xf'(x)dx$

$I_1 = -[f(x)\cos x]_0^{\pi} + \int_0^{\pi} f'(x)\cos xdx$

$I_1 = -[f(x)\cos x - f(0)\cos 0] + f'(x)\sin x|_0^{\pi} - \int_0^{\pi} f''(x)\sin xdx$

$I_1 = -(-2 - f(0)) + 0 - \int_0^{\pi} f''(x)\sin xdx$

$I_1 + I_2 = 5 + 2 + f(0) - \int_0^{\pi} f''(x)\sin xdx + \int_0^{\pi} f''(x)\sin xdx = 5$

$f(0) = 3$

41. $\int_a^b \frac{|x|}{x}dx$

Ans. $|\mathbf{b}| - |\mathbf{a}|$

Sol. $I = \int_a^b \frac{|x|}{x}dx$

case : 1

$0 \leq a \leq b$

$$\int_a^b \frac{x}{x}dx = |b| - |a|$$

case : 2

$$\int_a^b \frac{-x}{x}dx = -\int_a^b dx = -[b - a]$$

but a \& b both -e

$\therefore -[-b + a] = |b| - |a|$

but a&b both -e

$\therefore -[-b + a] = |b| - |a|$

case : 3

$$\int_a^b \frac{|x|}{x}dx = \int_a^{-0} \frac{-x}{x}dx + \int_0^b \frac{x}{x}dx$$

$= [-x]_a^0 + [x]_0^b = -[-(-a)] + [b - 0] = b - a$

$\therefore \int_a^b \frac{|x|}{x}dx = |b| - |a|$

42. $\int_{\ln 2}^{\ln 3} f(x)dx$, where $f(x) = e^{-x} + 2e^{-2x} + 3e^{-3x} + \cdots . \infty$

Ans. $\frac{1}{2}$

Sol. $\int_{\ln 2}^{\ln 3} f(x)dx$

$f(x) = e^{-x} + 2e^{-2x} + 3 \cdot e^{-3x} + \cdots \ldots + \infty$

$e^{-x}f(x) = e^{-2x} + 2e^{-3x} + \cdots \ldots + \infty$

$f(x)(1-e^{-x}) = e^{-x} + e^{-2x} + e^{-3x} + \cdots \ldots \infty$

$f(x) = \frac{e^{-x}}{(1-e^{-x})^2}$

$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{(1-e^{-x})^2} dx$

$\text{put}^{-x}\ d - e^{-x} = t$

$\frac{2}{3} dt$

$\int_{\frac{1}{2}}^{\frac{d}{t^2}} = \left[\frac{-1}{t}\right]_{\frac{1}{2}}^{\frac{2}{3}}$

$= -\left[\frac{3}{2} - 2\right] = \frac{1}{2}$

43. $\int_0^{\pi/2} \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}} \frac{\operatorname{cosec} x}{\sqrt{1+2\operatorname{cosec} x}} dx$

Ans. $\pi/3$

Sol. $I = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}} \frac{\operatorname{cosec} x}{\sqrt{1+2\operatorname{cosec} x}} dx$

$I = \int_0^{\frac{\pi}{2}} \frac{1}{\sec x + \tan x} \cdot \frac{\operatorname{cosec} x}{\sqrt{1 + 2\operatorname{cosec} x}} dx$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)} \cdot \frac{1}{\sqrt{\sin x}\sqrt{\sin x + 2}} dx$

put $\sin x = t$

$\cos x dx = dt$

$I = \int_0^1 \frac{dt}{(1+t)\sqrt{t}\sqrt{t+2}}$

$I = \int_0^1 \frac{dt}{(1+t)\sqrt{t^2 + 2t}}$

$I = \int_0^1 \frac{dt}{(1+t)\sqrt{(t+1)^2 - 1}}$

$I = \sec^{-1}[1 + t1]_0^1$

$I = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

44. $\int_0^1 xf''(x)dx$, where $f(x) = \cos(\tan^{-1} x)$

Ans. $1 - \frac{3}{2\sqrt{2}}$

Sol. $I = \int_0^1 xf''(x)dx$

$I = x\left[f''(x)dx\right|_0^1 - \int_0^1 f'(x)dx$

$I = [xf'(x)]_0^1 - [f(x)]_0^1$

$f(x) = \cos \cdot \tan^{-1}(x)$

$f'(x) = \frac{-\sin(\tan^{-1} x)}{1+x^2}$

$f'(1) = -\frac{1}{2\sqrt{2}}$

$f'(0) = 0$

$f'(1) = \frac{1}{\sqrt{2}} \& f(0) = 1$

$I = [f'(1) - 0] - [f(1) - f(0)] \; -\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 = 1 - \frac{3}{2\sqrt{2}}$

45. (a) If $g(x)$ is the inverse of $f(x)$ and $f(x)$ has domain $x \in [1,5]$, where $f(1) = 2$ and $f(5) = 10$ then find the value of $\int_1^5 f(x)dx + \int_2^{10} g(y)dy$.

(b) Suppose f is continuous, $f(0) = 0, f(1) = 1, f'(x) > 0$ and $\int_0^1 f(x)dx = \frac{1}{3}$. Find the value of the definite integral $\int_0^1 f^{-1}(y)dy$.

Ans. **(a) 48 , (b) 2/3**

Sol. (a) gff(x) is inverse of g(x) then

$\int_a^b f(x)fx + \int_c^d g(y)dy = bd - ac$

$\int_1^5 f(x)dx + \int_2^{10} g(y)dy = 10 \times 5 - 2x1 = 48$

(b) $\int_0^1 f(x)dx + \int_0^1 f^{-1}(y)dy = 1 - 0$

$\frac{1}{3} + \int_0^1 f^{-1}(y)dy = 1$

$\int_0^1 f^{-1}(y)dy = 1 - \frac{1}{3} = \frac{2}{3}$