DPP -01

SOLUTION

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1. Simplify the expression $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$.

Ans. $= (\mathbf{x + 2a})^5$

Sol. $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$

$= {}^5C_0x^5 + {}^5C_1x^4(2a) + {}^5C_2x^3(2a)^2 + {}^5C_3x^2(2a)^3 + {}^5C_4x(2a)^4 + {}^5C_5(2a)^5$

$= (x + 2a)^5$

2. Find n, if the ratio of the fifth term form the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}: 1$.

Ans. $\mathbf{n = 10}$

Sol. In the expansion $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, the fifth term from the beginning is ${}^nC_4(\sqrt[4]{2})^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$.

According to the question, $\dfrac{{}^nC_4(\sqrt[4]{2})^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4}{{}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}} = \sqrt{6}$ or $\dfrac{{}^nC_{n-4}(\sqrt[4]{2})^{n-8}}{{}^nC_{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-8}} = \sqrt{6}$

or $(\sqrt[4]{2}\sqrt[4]{3})^{n-8} = \sqrt{6}$

or $6^{\frac{n-8}{4}} = 6^{\frac{1}{2}}$

or $n = 10$

3. Find the term in $\left(\sqrt[3]{\frac{a}{\sqrt{b}}} + \sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ which has the same power of a and b.

Ans. **9**

Sol. We have, $T_{r+1} = {}^{21}C_r\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^r = {}^{21}C_r a^{7-(r/2)} b^{(2/3)r-(7/2)}$

Since the powers of a and b are same, we have

$7 - \frac{r}{2} = \frac{2}{3}r - \frac{7}{2}$ or $r = 9$

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4. Prove that $\sqrt{10}[(\sqrt{10}+1)^{100}-(\sqrt{10}-1)^{100}]$ is an even integer.

Sol. $\sqrt{10}[(\sqrt{10}+1)^{100}-(\sqrt{10}-1)^{100}]$

$= 2\sqrt{10}[\,{}^{100}C_1(\sqrt{10})^{99} + {}^{100}C_3(\sqrt{10})^{97} + {}^{100}C_5(\sqrt{10})^{95} + \cdots]$

$= 2[\,{}^{100}C_1(\sqrt{10})^{100} + {}^{100}C_3(\sqrt{10})^{98} + {}^{100}C_5(\sqrt{10})^{96} + \cdots]$

$= 2[\,{}^{100}C_1(\sqrt{10})^{50} + {}^{100}C_3(\sqrt{10})^{49} + {}^{100}C_5(\sqrt{10})^{48} + \cdots]$

which is an even number.

5. If 9^7+7^9 is divisible by 2^n, then find the greatest value of n, where $n \in N$.

Ans. **6**

Sol. We have, $9^7+7^9=(1+8)^7-(1-8)^9$

$=(1+{}^7C_1 8^1+{}^7C_2 8^2+\cdots+{}^7C_7 8^7)-(1-{}^9C_1 8^1+{}^9C_2 8^2-\cdots-{}^9C_9 8^9)$

$=16\times 8+64[({}^7C_2+\cdots+{}^7C_7 8^5)-({}^9C_2-\cdots-{}^9C_9 8^7)]$

$=64k$ (where k is odd integer)

Therefore, 9^7+7^9 is divisible by 2^6.

6. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=0$, where $k=3n/2$ and n is an even integer.

Sol. $S=\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}$, $k=\frac{3n}{2}$ and n is even

Let $k=\frac{3(2m)}{2}=3m$

Then $S=\sum_{r=1}^{3m}(-3)^{r-1}\times{}^{6m}C_{2r-1}$

$={}^{6m}C_1-3\,{}^{6m}C_3+3^2\,{}^{6m}C_5-\cdots(-3)^{3m-1}\,{}^{6m}C_{6m-1}$

$=\frac{1}{\sqrt{3}}[\sqrt{3}\,{}^{6m}C_1-(\sqrt{3})^3\,{}^{6m}C_3+(\sqrt{3})^5\,{}^{6m}C_5-\cdots+(-1)^{3m-1}(\sqrt{3})^{6m-1}\,{}^{6m}C_{6m-1}]$

There is an alternate sign series with odd binomial coefficients.

Hence, we should replace x by $\sqrt{3}i$ in $(1+x)^{6m}$. Therefore,

$(1+\sqrt{3}i)^{6m}={}^{6m}C_0+{}^{6m}C_1(\sqrt{3}i)+{}^{6m}C_2(\sqrt{3}i)^2+{}^{6m}C_3(\sqrt{3}i)^3+\cdots+{}^{6m}C_{6m}(\sqrt{3}i)^{6m}$

$\Rightarrow \sqrt{3}\times{}^{6m}C_1-(\sqrt{3})^3\,{}^{6m}C_3+(\sqrt{3})^5\,{}^{6m}C_5+\cdots$

$=$ Imaginary part in $(1+\sqrt{3}i)^{6m}$

$$=\text{Im}\left[2^{6m}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{6m}\right]$$

$$=\text{Im}\left[2^{6m}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^{6m}$$

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$= \text{Im}\,[2^{6m}(\cos\ 2m\pi + i\sin\ 2m\pi)]$

$= \text{Im}\,[2^{6m}] = 0$

$\Rightarrow\ S = 0$

7. Find the value of $\frac{1}{81^n} - \frac{10}{81^n}\ {}^{2n}C_1 + \frac{10^2}{81^n}\ {}^{2n}C_2 - \frac{10^3}{81^n}\ {}^{2n}C_3 + \cdots + \frac{10^{2n}}{81^n}$

Ans. **1**

Sol. We have $\frac{1}{81^n} - \frac{10}{81^n}\ {}^{2n}C_1 + \frac{10^2}{81^n}\ {}^{2n}C_2 - \frac{10^3}{81^n}\ {}^{2n}C_3 + \cdots + \frac{10^{2n}}{81^n}$

$= \frac{1}{81^n}[\ {}^{2n}C_0 - {}^{2n}C_1 10^1 + {}^{2n}C_2 10^2 - {}^{2n}C_3 10^3 + \cdots + {}^{2n}C_{2n}10^{2n}]$

$= \frac{1}{81^n}[1-10]^{2n} = \frac{(-9)^{2n}}{81^n} = \frac{81^n}{81^n} = 1$

8. Find the number of nonzero terms in the expansion of $(1+3\sqrt{2}x)^9 + (1-3\sqrt{2}x)^9$

Ans. **5**

Sol. Given expression is $2\left[1 + {}^9C_2(3\sqrt{2}x)^2 + {}^9C_4(3\sqrt{2}x)^4 + {}^9C_6(3\sqrt{2}x)^6 + {}^9C_8(3\sqrt{2}x)^8\right]$

Therefore, the number of nonzero terms is 5 .

9. Find

(i) the last digit,

(ii) the last two digits, and

(iii) the last three digits of 17^{256}

Ans. **(i) 1,(ii) 8,1(iii) 6,8,1**

Sol. We have,

$17^{256} = (17^2)^{128} = (289)^{128} = (290-1)^{128}$

$\therefore 17^{256} = {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - \cdots$

$- {}^{128}C_{125}(290)^3 + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= [\ {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - \cdots$

$= {}^{128}C_{125}(290)^3] + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= 1000m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1\ (m \in I)$

$= 1000m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$

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$= 1000m + (128)(127)(290)(145) - 128 \times 290 + 1$

$= 1000m + (128)(290)(127 \times 145 - 1) + 1$

$= 1000m + (128)(290)(18414) + 1$

$= 1000m + 683527680 + 1$

$= 1000m + 683527000 + 680 + 1$

$= 1000(m + 683527) + 681$

Hence, the last three digits of 17^{256} are 6,8,1. As a result, the last two digits of 17^{256} are 8,1 and the last digit of 17^{256} is 1 .

10. Find the remainder when $6^n - 5n$ is divided by 25 .

Ans. **1**

Sol. $6^n - 5n = (1+5)^n - 5n$

$= (1 + 5n + {}^nC_2 \cdot 5^2 + {}^nC_3 5^3 + \cdots) - 5n$

$= 25({}^nC_2 + {}^nC_3 + \cdots) + 1$

Hence, $6^n - 5n$ when divided by 25 leaves 1 as remainder.

11. Using binomial theorem, show that $2^{3x} - 7n - 1$ is divisible by 49. Hence, show that $2^{3n+3} - 7n - 8$ is divisible by 49, $n \in N$.

Sol. $2^{3n} - 7n - 1 = (2^3)^n - 7n - 1$

$= (1+7)^n - 7n - 1 = 1 + 7n + {}^nC_2 7^2 + {}^nC_3 7^3 + \cdots + {}^nC_n 7^n - 7n - 1$

$= 7^2[{}^nC_2 + {}^nC_3 7 + \cdots + {}^nC_n 7^{n-2}] = 49K(1)$

where K is an integer.

Therefore, $2^{3n} - 7n - 1$ is divisible by 49 . Now,

$2^{3n+3} - 7n - 8 = 2^3 2^{3n} - 7n - 8$

$= 8(2^{3n} - 7n - 1) + 49n$

$= 8 \times 49K + 49n$

$= 49(8K + n)$ [From (1)]

Therefore, $2^{3n+3} - 7n - 8$ is divisible by 49.

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12. If $(2+\sqrt{3})^n = I + f$, where I and n are positive integers and $0 < f < 1$, show that I is an odd integer and $(1-f) \times (I+f) = 1$.

Sol. $(2+\sqrt{3})^n = I + f$

or $I + f = 2^n + {}^nC_1 2^{n-1}\sqrt{3} + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_3 2^{n-3}(\sqrt{3})^3 + \cdots$ (1)

Now, $0 < 2-\sqrt{3} < 1$

$\Rightarrow 0 < (2-\sqrt{3})^n < 1$

Let $(2-\sqrt{3})^n = f'$, where $0 < f' < 1$.

$\therefore\ f = 2^n - {}^nC_1 2^{n-1}\sqrt{3} + {}^nC_2 \quad 2^{n-2}(\sqrt{3})^2 - {}^nC_3 2^{n-3}(\sqrt{3})^3 + \cdots$ (2)

Adding (1) and (2), we get

$I + f + f' = 2\left[2^n + {}^nC_2 2^{n-2} \times \sqrt{3} + \cdots\right]$

or $I + f + f' =$ even integer (3)

Now, $0 < f < 1$ and $0 < f' < 1$.

$\therefore\ 0 < f + f' < 2$

Hence, from (3), we conclude that $f + f'$ is an integer between 0 and 2. Therefore,

$f + f' = 1$ or $f' = 1 - f$ (4)

From (3) and (4), we get $I + 1$ is an even integer. Therefore, I is an odd integer. Now,

$I + f = (2+\sqrt{3})^n, f' = 1 - f = (2-\sqrt{3})^n$

$\therefore (I+f)(1-f) = [(2+\sqrt{3})(2-\sqrt{3})]^n = (4-3)^n = 1$

$\therefore (I+f)(1-f) = 1$

13. Show that $9^{n+1} - 8n - 9$ is divisible by 64 , whenever n is apositive integer.

Sol. In order to show that $9^{n+1} - 8n - 9$ is divisible by 64 , it has to be proved that

$9^{n+1} - 8n - 9 = 64k$, where k is some natural number

$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \cdots + {}^{n+1}C_{n+1}(8)^{n+1}$

or $9^{n+1} = 1 + (n+1)(8) + 8^2[\,{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \cdots + {}^{n+1}C_{n+1}(8)^{n-1}]$

or $9^{n+1} = 9 + 8n + 64[\,{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \cdots C_{n+1}(8)^{n-1}]$

or $9^{n+1} - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \cdots {}^{n+1}C_{n+1}(8)^{n-1}$ is a natural number

Thus, $9^{n+1} - 8n - 9$ is divisible by 64 , whenever n is a positive integer.

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14. Show that $2^{4n+4}-15n-15$, where $n \in N$ is divisible by 225.

Sol. We have $2^{4n+4}-15n-16$

$= 2^{4(n+1)}-15n-16$

$= 16^{n+1}-15n-16$

$= (1+15)^{n+1}-15n-16$

$= {}^{n+1}C_0 15^0 + {}^{n+1}C_1 15^1 + {}^{n+1}C_2 15^2 + {}^{n+1}C_3 15 + \cdots + {}^{n+1}C_{n+1}(15)^{n+1} - 15n - 16$

$= 1 + (n+1)15 + {}^{n+1}C_2 15^2 + {}^{n+1}C_3 15^3 + \cdots + {}^{n+1}C_{n+1}(15)^{n+1} - 15n - 16$

$= 15^2[\,{}^{n+1}C_2 + {}^{n+1}C_3 15 + \cdots]$

Thus, $2^{4n+4}-15n-16$ is divisible by 225 .

15. find the remainder when 7^{103} is divided by 25 .

Ans. **18**

Sol. $\frac{7^{103}}{25} = \frac{7(49)^{51}}{25} = \frac{7(50-1)^{51}}{25} = \frac{7(25k-1)}{25} = \frac{175(k)-25+25-7}{25}$
$= \frac{25(7k-1)+18}{25}$

Therefore, the remainder is 18 .