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1.

$$\int_0^{2\pi} x \sin^4 x \cos^6 x \, dx$$

$$= 2\pi \int_0^{\pi} \sin^4 x \cos^6 x \, dx$$

$$= 4\pi \int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$$

$$= 4\pi \int_0^{\pi/4} \sin^4 x \cos^4 x \, dx = \frac{\pi}{4} \int_0^{\pi/4} \sin^4 2x \, dx$$

$$= \frac{\pi}{8} \int_0^{\pi/2} \sin^4 x \, dx = \frac{\pi}{8} \frac{3\pi}{16} = \boxed{\frac{3\pi^2}{128}}$$

$$\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/4} \left(1 - \frac{1}{2}\sin^2 2x\right) dx$$

$$= \int_0^{\pi/8} \left(2 - \frac{1}{2}\left(s^2 2x + c^2 2x\right)\right) dx$$

$$= \int_0^{\pi/8} \frac{3}{2} \, dx = \boxed{\frac{3\pi}{16}}$$

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2. $I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin 2x}{2}\right) dx$.

$$I = \frac{1}{2}\int_0^{\pi/2} \ln\left(\frac{\sin x}{2}\right) dx = \frac{1}{2} I - \frac{1}{2}\int_0^{\pi/2} \ln 2\, dx.$$

$$\frac{I}{2} = -\frac{1}{2}(\ln 2)\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \ln \sin x\, dx = -\frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln \cos x\, dx \qquad I = -\frac{\pi}{2}\ln 2$$

$$\int_0^{\pi/2} \ln(\tan x)\, dx = 0 = \int_0^{\frac{\pi}{2}} \ln(\cot x)\, dx$$

$$\int_0^{\pi/2} \ln(\operatorname{cosec} x)\, dx = \frac{\pi}{2}\ln 2 = \int_0^{\pi/2} \ln \sec x\, dx.$$

Nishant Jindal

3. $$\int_0^{\pi} \ln(1-\cos x)\,dx = \int_0^{\pi/2} \ln \sin^2 x\,dx = \boxed{-\pi \ln 2}$$

4. $$\int_0^1 \frac{\sin^{-1}x}{x}\,dx$$

$\sin^{-1}x = \theta$

$x = \sin\theta$

$$= \int_0^{\pi/2} \underbrace{\theta}_{\text{I}} \cot\theta\,d\theta = \theta \ln\sin\theta \Big|_{0^+}^{\pi/2} - \int_0^{\pi/2} \ln\sin\theta\,d\theta$$

$$= 0 - \lim_{\theta\to 0^+} \frac{\ln\sin\theta}{1/\theta} + \frac{\pi}{2}\ln 2$$

$$= \boxed{\frac{\pi}{2}\ln 2}$$

$$\frac{\cot\theta}{-\frac{1}{\theta^2}} = \frac{\theta}{\tan\theta}\,\theta = 0$$

Nishant Jindal

5. $$\int_0^\infty \frac{\ln x\, dx}{(ax^2+bx+a)} \qquad a \neq 0 .$$

$$= \int_0^1 \frac{\ln x\, dx}{ax^2+bx+a} + \underbrace{\int_1^\infty \frac{\ln x\, dx}{ax^2+bx+a}}_{x=\frac{1}{t}} = \int_0^1 + \int_1^0 \frac{(-\ln t)\left(-\frac{dt}{t^2}\right)}{\frac{a}{t^2}+\frac{b}{t}+a}$$

$$= \cancel{\int_0^1 \frac{\ln x\, dx}{ax^2+bx+a}} - \cancel{\int_0^1 \frac{\ln t\, dt}{a+bt+at^2}}$$

$$= 0$$

6. $$\int_0^\infty \frac{\ln x\, dx}{(x^2+2x+4)} \qquad \boxed{\frac{\pi}{3}\frac{\ln 2}{\sqrt{3}}}$$

$$= 2\int_0^\infty \frac{\ln(2x)\, dx}{4x^2+4x+4} = \frac{2}{4}\underbrace{\int_0^\infty \frac{\ln x\, dx}{x^2+x+1}}_{=0} + \frac{1}{2}\ln 2\int_0^\infty \frac{dx}{(x^2+x+1)}$$

$$\frac{\ln 2}{2}\,\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)\Bigg|_0^\infty$$