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Polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_2 A^2 + a_1 A + a_0 I$$

if $f(A) = O$

then A is root/zero of matrix polynomial.

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Cayley Hamilton Theorem

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

$$|A - \lambda I| = 0 = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$f(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)\left((a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}\right)$$
$$- a_{12}\left(a_{21}(a_{33}-\lambda) - a_{23}a_{31}\right)$$
$$+ a_{13}\left(a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right)$$

$$-\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 + (\quad)\lambda + (\quad) = 0$$

$$\Rightarrow \boxed{-\lambda^3 + \mathrm{Tr}(A)\lambda^2 + (\quad)\lambda + |A| = 0}$$

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$A_{n\times n}$

$$\boxed{|A-\lambda I| = 0 \qquad (-1)^n \lambda^n + (-1)^{n-1} Tr(A)\, \lambda^{n-1} + \cdots + |A| = 0}$$

characteristic eqn.

Cayley Hamilton theorem

$|A-\lambda I| = 0$

$$\Rightarrow \quad \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_1\lambda + \alpha_0 = 0$$

$$\boxed{\alpha_n A^n + \alpha_{n-1}A^{n-1} + \alpha_{n-2}A^{n-2} + \cdots + \alpha_1 A + \alpha_0 I = O}$$

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$$AB = 0$$

$$\not\Rightarrow \quad A = 0 \text{ or } B = 0$$

$$IA = A$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AI = A.$$

$$A = 0$$

$$AB = 0$$

$$\underset{3\times 2}{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} a_1 & a_2 \\ & \\ & \end{bmatrix}$$

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1. Find all matrices which commute with matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

B

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ \gamma & \delta \end{pmatrix}$$

$$\shortparallel$$

$$BA = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha+\beta \\ \gamma & \gamma+\delta \end{pmatrix}$$

$$\alpha+\gamma = \alpha \Rightarrow \gamma = 0$$

$$\beta+\delta = \alpha+\beta \Rightarrow \alpha = \delta$$

$$B = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

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2: Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$

If $(A+B)^2 = A^2 + B^2$, find a, b.

$$(A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + B^2$$

$$\Rightarrow AB + BA = O \checkmark$$

$$\boxed{(a,b) = (1,4)}$$

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} = \begin{pmatrix} a-b & 2 \\ 2a-b & 3 \end{pmatrix}$$

$$BA = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{pmatrix}$$

$a-b = -a-2$

$a+1 = 2$

$2-b = 2a-b$

$b-1 = 3$

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3: Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ and $f(x) = x^2 - 4x + 7$, then P.T. $f(A) = O$. Use this result to find A^5

$$A^5 = \begin{pmatrix} -118 & -93 \\ 31 & -118 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 3 \\ -1 & 2-\lambda \end{vmatrix} = 0 = 4 + \lambda^2 - 4\lambda + 3 \Rightarrow \boxed{A^2 - 4A + 7I = 0}$$

$$x^5 = (x^2 - 4x + 7)(x^3 + 4x^2 + 9x + 8) - 31x - 56$$

$$A^5 = (A^2 - 4A + 7I)(A^3 + 4A^2 + 9A + 8I) - 31A - 56I$$

$$= -31A - 56I.$$

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4. Find an upper triangular matrix A such that $A^3 = \begin{pmatrix} 8 & -57 \\ 0 & 27 \end{pmatrix}$.

$$\boxed{a = 2, \quad c = 3 \quad b = -3}$$

$$A^3 = \underbrace{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab+bc \\ 0 & c^2 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & a^2b + (ab+bc)c \\ 0 & c^3 \end{pmatrix}$$

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5. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$. Use induction to P.T.

$$A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix} \text{ for } n \in N.$$

Mathematical Induction

① P.T. $n=1$ is true

② Let $n=K$ to be true

③ Prove $n=K+1$ is true

① $A^1 = \begin{pmatrix} 1+2 & -4(1) \\ 1 & 1-2(1) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is true.

② Let $A^K = \begin{pmatrix} 1+2K & -4K \\ K & 1-2K \end{pmatrix}$

③ $A^{K+1} = A^K A = \begin{pmatrix} 1+2K & -4K \\ K & 1-2K \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3+2K & -4-4K \\ K+1 & -2K-1 \end{pmatrix}$

$\begin{pmatrix} 1+2(K+1) & -4(K+1) \\ K+1 & 1-2(K+1) \end{pmatrix} =$

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$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$$

$$A^n = (I+B)^n = {}^nC_0 I + {}^nC_1 B + {}^nC_2 B^2 + {}^nC_3 B^3 + \cdots + {}^nC_n B^n$$

$$B^2 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(I+B)^2 = (I+B)(I+B) = I + B^2 + 2B$$

$$IB = BI = B$$

$$A^n = I + nB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2n & -4n \\ n & -2n \end{pmatrix}$$

$$= \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$$

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Q. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$A^n = ?$

$A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A$

$A^3 = 3A^2 = 3(3A) = 3^2 A$

$A^4 = 3^2 A^2 = 3^3 A$

$A^n = 3^{n-1} A = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$

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7. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$. Use induction

to P.T. $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, $n \in N$.

$n \in N$.