CIRCLE

SINGLE CORRECT ANSWER TYPE

1. S1: The locus of the centre of a circle which cuts a given circle orthogonally and also touches a given straight line is a parabola.

S2: Two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touches iff $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

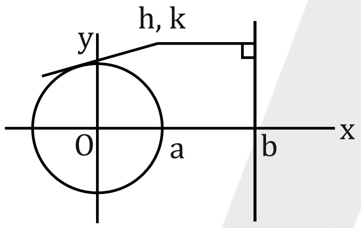
S3: The two circles which passes through $(0, a)$ and $(0, -a)$ and touch the straight line $y = mx + c$, will cut orthogonally if $c^2 = a^2(2 + m^2)$.

S4: The length of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\frac{ab}{\sqrt{a^2-b^2}}$.

(A) TFTF (B) TTFF (C) TFTT (D) FFTT

Ans. **(A)**

Hint. S1: Assume circle as $x^2 + y^2 = a^2$ and the given line as $x - b = 0$ Let (h, k) be the centre of the required circle. Then length of tangent from (h, k) to the circle and distance of (h, k) from the line should be equal.



Hence $\sqrt{h^2 + k^2 - a^2} = |h - b|$

S2: Apply $c_1c_2 = |r_1 \mp r_2|$

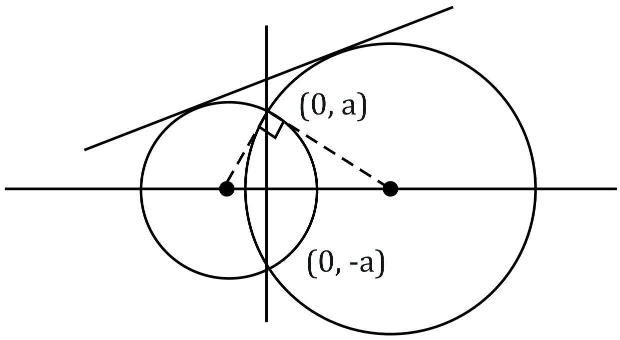
S3: Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle passing through the points $(0, a) \& (0, -a)$

$\therefore\ a^2 + 2af + c = 0 \ \& \ a^2 - 2af + c = 0$

$\therefore f = 0$ and $c = -a^2$

$\therefore$ Equation of the circle is

$x^2 + y^2 + 2gx - a^2 = 0$



Since it touches the line $y = mx + c$ then use P=r

it gives two values of g, therefore, there are two circles further these two circles cut each other orthogonally

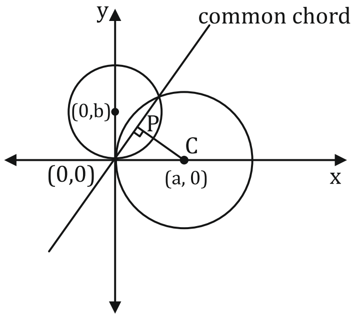
then use $2\,g_1\,g_2 = f_1 + f_2$

S4: Equation of common chord is $S_1 - S_2 = 0$

$\Rightarrow\ ax - by = 0$

from the figure, $CP = P_1$(let)

$\therefore$ length of common chord $= 2\sqrt{{r_1}^2 - {P_1}^2}$



2. P is a variable point on the line $L = 0$. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch it at Q and R. The parallelogram PQSR is completed.

If $L = 2x + y - 6 = 0$, then the locus of circumcentre of $\triangle$ PQR is

(A) $2x - y - 4$ (B) $2x + y = 3$ (C) $x - 2y = 4$ (D) $x + 2y = 3$

Ans. **(B)**

Sol. $\because$ $PQ = PR$ i.e. parallelogram PQRS is a rhombus

$\therefore$ Mid point of QR = Midpoint of PS and $QR \perp PS$

$\therefore$ S is the mirror image of P w.r.t. QR

$\therefore$ S is the mirror image

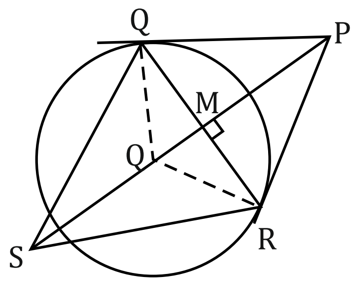
$\because L \equiv 2x + y = 6$ Let $P \equiv (k, 6 - 2k)$

$\because \angle PQO = \angle PRO = \frac{\pi}{2}$

$\therefore$ OP is diameter of circumcircle PQR, then centre is $\left(\frac{k}{2}, 3 - k\right)$

$\therefore x = \frac{k}{2} \Rightarrow k = 2x$

$y = 3 - k \therefore 2x + y = 3$



PARABOLA

SINGLE CORRECT ANSWER TYPE

3. A circle is described whose centre is the vertex and whose diameter is three-quarters of the latus rectum of the parabola $y^2 = 4ax$. If PQ is the common chord of the circle and the parabola and L_1L_2 is the latus rectum, then the area of the trapezium $PL_1\,L_2Q$ is

(A) $3\sqrt{2}a^2$ (B) $2\sqrt{2}a^2$ (C) $4a^2$ (D) $\left(\frac{2+\sqrt{2}}{2}\right)a^2$

Ans. **(D)**

Sol. Centre (0,0), radius $= \frac{1}{2}\cdot\frac{3}{4}\cdot 4a = \frac{3a}{2}$

Equation of the circle is $4(x^2 + y^2) = 9a^2$ ...(i)

Equation of the parabola is $y^2 = 4ax$ ...(ii)

Solving (i) and (ii) $x^2 + 4ax - \frac{9a^2}{4} = 0$

$$x = \frac{-4a \pm \sqrt{16a^2 + 9a^2}}{2} = \frac{-4a \pm 5a}{2}$$

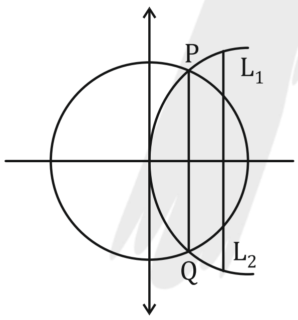
$\therefore\ x = a/2$

For $x = a/2$ $y^2 = 4ax = 4aa/2 = 2a^2$

$\Rightarrow\ y = \pm\sqrt{2}a$

$\therefore$ The double ordinate $= 2\sqrt{2}\,a$

$\therefore$ area of the trapezium $PL_1L_2Q = \frac{1}{2}\left(a - \frac{a}{2}\right)(4a + 2\sqrt{2}a) = \left(\frac{2+\sqrt{2}}{2}\right)a^2$



MATRIX - MATCH TYPE

4. **Column-I** **Column-II**

(A) Area of a triangle formed by the tangents drawn from a point $(-2,2)$ to the parabola $y^2 = 4(x+y)$ and their corresponding chord of contact is — (P) 8

(B) Length of the latus rectum of the conic $25\{(x-2)^2 + (y-3)^2\} = (3x + 4y - 6)^2$ is — (Q) $4\sqrt{3}$

(C) If focal distance of a point on the parabola $y = x^2 - 4$ is $25/4$ and points are of the form $(\pm\sqrt{a}, b)$ then value of $a + b$ is

(D) Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$ whose one angular point is vertex of the parabola, is

(R) $\frac{12}{5}$

(T) $\frac{24}{5}$

Ans. ((A) → (r), (B) → (t), (C) → (p), (D) → (q))

Sol. (A) Point of contact of tangent drawn from $(-2,2)$ on $y^2 = 4(x + y)$ are $(0,4)$ and $(0,0)$

$\therefore$ Area $= 4$

(B) The conic is a parabola having focus is $(2,3)$ & Directrix

$3x + 4y - 6 = 0]$

$\therefore$ Latus rectum $= 2$ ($\perp$ distance of focus from the directrix)

$$= 2\left(\frac{6 + 12 - 6}{5}\right) = \frac{24}{5}$$

(C) $y + 4 = x^2$

$$x^2 = 4 \cdot \frac{1}{4}(y + 4)$$

focal distance $= \frac{25}{4}$

$\therefore$ distance from directrix $\left(y = \frac{-17}{4}\right)$ = ordinate of points on the parabola whose focal distance is $\frac{25}{4}$

$$= \frac{-17}{4} + \frac{25}{4}$$

$$= 2$$

$\Rightarrow$ Points are $(\pm\sqrt{6}, 2)$

$a + b = 8$

(D) Length of side $= 8\sqrt{3}a = 8\sqrt{3}\frac{1}{2} = 4\sqrt{3}$

ELLIPSE

MULTIPLE CORRECT ANSWER TYPE

5. If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose focii are S and S′. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then

(A) $PS + PS' = 2a$, if $a > b$

(B) $PS + PS' = 2b$, if $a < b$

(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$

(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{a^2-b^2}}{b^2}\left[a-\sqrt{a^2-b^2}\right]$ when $a > b$

Ans. **(ABC)**

Sol. Focal property of ellipse

$PS + PS' = 2a;$ if $a > b$

$PS + PS' = 2b$; if $a < b$

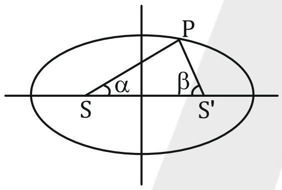
$PS\cos\alpha + PS'\cos\beta = 2ae$ ...(i)

$PS\sin\alpha - PS'\sin\beta = 0$ ...(ii)

$PS + PS' = 2a$ ...(iii)

from (i) and (ii), we get $PS = \frac{2ae\sin\beta}{\sin(\alpha+\beta)}$, $P' = \frac{2ae\sin\alpha}{\sin(\alpha+\beta)}$

from (iii) and (iv)



$e(\sin\alpha + \sin\beta) = \sin(\alpha+\beta)$

$\therefore$ $e.\, 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$

$\therefore$ $e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$\therefore$ $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e} = \frac{2a\left(a-\sqrt{a^2-b^2}\right)-b^2}{b^2}$

$\therefore$ C is correct option & D is incorrect

INTEGER TYPE

6. Origin O is the centre of two concentric circles whose radii are a & b respectively, $a < b$. A line OPQ is drawn to cut the inner circle in P& the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. The locus of R is an ellipse touching the two circles. If the focii of this ellipse lie on the inner circle, if eccentricity is $\sqrt{2}\lambda$, then find λ

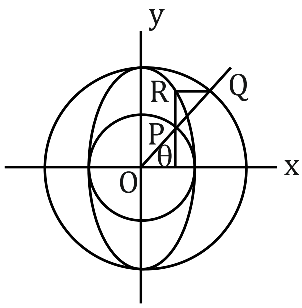
Ans. (1)

Sol. Let line OPQ makes angle θ with x-axis so $P \equiv (a\cos\theta, a\sin\theta), Q(b\cos\theta, b\sin\theta)$

and Let $R(x, y)$

So $X = a\cos\theta$ $Y = b\sin\theta$

eliminating θ, we get



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, locus of R is an ellipse.

Also $a < b$ so vertices are $(0, b)$ and $(0, -b)$

and extremities of minor axis are $(\pm a, 0)$.

So ellipse touches both inner circle and outer circle

if focii are $(0, \pm a)$

$\Rightarrow$ $a = be$ i.e $e = a/b$

Also $e = \sqrt{1 - e^2}$ $\Rightarrow$ $e^2 = 1 - e^2$

$\Rightarrow$ $e = 1/\sqrt{2}$

and ratio of radii, is $\frac{a}{b} = e = \frac{1}{\sqrt{2}}$.

HYPERBOLA

COMPREHENSION TYPE (7-8)

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the normal at P meets the transverse axis AA′ in G and the conjugate axis BB′ in g and CF be perpendicular to the normal from the centre.

7. PF . PG = K CB^2 , then K =

(A) 2 (B) 1 (C) $\frac{1}{2}$ (D) 4

Ans. (B)

Sol. Equation of normal at $P(a\sec\theta, b\tan\theta)$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$

$G\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$, $g\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$ equation of CF is $bx\cot\theta - ay\cos\theta = 0$

$\therefore PF = \frac{ab}{\sqrt{b^2\sec^2\phi + a^2\tan^2\phi}}$ and $PG^2 = \frac{b^2}{a^2}(b^2\sec^2\phi + a^2\tan^2\phi)$

$\therefore PF.PG = b^2$.

8. PF. Pg equals to

(A) CA^2 (B) CF^2 (C) CB^2 (D) $CA \cdot CB$

Ans. **(A)**

Sol. $Pg^2 = \frac{a^2}{b^2}(b^2\sec^2\theta + a^2\tan^2\theta) \Rightarrow PF.Pg = a^2 = CA^2$

FUNCTION

SINGLE CORRECT ANSWER TYPE

9. Let $f:\{x, y, z\} \to \{1,2,3\}$ be a one-one mapping such that only one of the following three statements is true and remaining two are false: $f(x) \neq 2, f(y) = 2, f(z) \neq 1$, then

(A) $f(x) > f(y) > f(z)$ (B) $f(x) < f(y) < f(z)$

(C) $f(y) < f(x) < f(z)$ (D) $f(y) < f(z) < f(x)$

Ans. **(C)**

Sol. Case - I $f(x) \neq 2$ is true, $f(y) = 2$ and $f(z) \neq 1$ are false, then

$f(x) = 1$ or $3, f(y) = 1$ or 3 and $f(z) = 1$

$\Rightarrow$ f is not one-one

Case - II $f(x) \neq 2$ is false, $f(y) = 2$ is true, $f(z) \neq 1$ is false, then

$f(x) = 2, \quad f(y) = 2, \quad f(z) = 1$

$\Rightarrow$ not possible

Case - III $f(x) \neq 2$ is false, $f(y) = 2$ is false, $f(z) \neq 1$ is true, then

$f(x) = 2,\ f(y) = 1$ or $3,\ f(z) = 2$ or 3

$\Rightarrow f(x) = 2, \quad f(z) = 3, \quad f(y) = 1$

INTEGER TYPE

10. If $f(x) + f(y) + f(xy) = 2 + f(x).f(y)$, for all real values of x & y and f(x) is a polynomial function with $f(4) = 17$, then find the value of $f(5)/14$, where $f(1) \neq 1$.

Ans. **(9)**

Sol. Let $x = y = 1$

$f(x) + f(y) + f(xy) = 2 + f(x) \cdot f(y)$

$3f(1) = 2 + (f(1))^2 \Rightarrow f(1) = 1,2$. But given that $f(1) \neq 1$ so $f(1) = 2$

Now put $y = \frac{1}{x}$

$f(x) + f\left(\frac{1}{x}\right) + f(1) = 2 + f(x) \cdot f\left(\frac{1}{x}\right)$

so $f(x) = \pm x^n + 1$ Now $f(4) = 17 \Rightarrow \pm(4)^n + 1 = 17 \Rightarrow n = 2$

$f(x) = +(x)^2 + 1 \Rightarrow f(5) = 126$

LIMIT OF FUNCTION

ASSERTION AND REASON TYPE

11. Statement 1: $\lim_{x\to\infty}\left(\frac{1^2}{x^3}+\frac{2^2}{x^3}+\frac{3^2}{x^3}+\cdots\ldots\ldots+\frac{x^2}{x^3}\right)=\frac{1}{3}$

Statement 2: $\lim_{x\to a}(f_1(x)+f_2(x)+\cdots\ldots.+f_n(x))=\lim_{x\to a}f_1(x)+\lim_{x\to a}f_2(x)\ldots\ldots.+\lim_{x\to a}f_n(x)$ where $n\in\mathbb{N}$

(A) Statement 1 is true, statement 2 is true, statement 2 is a correct explanation for statement 1

(B) Statement 1 is true, statement 2 is true, statement 2 is NOT correct explanation for statement 1

(C) Statement 1 is true, statement 2 is false

(D) Statement 1 is false, statement 2 is true

Ans. **(C)**

Sol. $\lim\limits_{x\to\infty}\left(\frac{1^2}{x^3}+\frac{2^2}{x^3}+\frac{3^2}{x^3}+\cdots\ldots+\frac{x^2}{x^3}\right)$

$S_n=\frac{1^2+2^2+3^2+\cdots\ldots+x^2}{x^3}$

$S_n=\lim\limits_{x\to\infty}\frac{x(x+1)(2x+1)}{x^3}$

$S_n=\frac{\left(1+\frac{1}{x}\right)\left(2+\frac{1}{x}\right)}{6}$

$S_n=\frac{1}{3}$

Statement-2 is correct only when R.H.S. does not take any indeterminate form.

INTEGER TYPE

12. Let $P=\frac{\left(1^4+\frac{1}{4}\right)\left(3^4+\frac{1}{4}\right)\left(5^4+\frac{1}{4}\right)\ldots\left((2n-1)^4+\frac{1}{4}\right)}{\left(2^4+\frac{1}{4}\right)\left(4^4+\frac{1}{4}\right)\left(6^4+\frac{1}{4}\right)\ldots\left((2n)^4+\frac{1}{4}\right)}$ and $\lim_{n\to\infty}(n^aP)$ exists, then find a

Ans. **(2)**

Sol. $P=\frac{\prod_{r=1}^{n}\left((2r-1)^4+\frac{1}{4}\right)}{\prod_{r=1}^{n}\left((2r)^4+\frac{1}{4}\right)}$

$\because\ (2r-1)^4+\frac{1}{4}=(2r-1)^4+\frac{1}{4}+(2r-1)^2-(2r-1)^2$

$=\left((2r-1)^2+\frac{1}{2}\right)^2-(2r-1)^2=\left((2r-1)^2+(2r-1)+\frac{1}{2}\right)\left((2r-1)^2-(2r-1)+\frac{1}{2}\right)$

$\Rightarrow\ (2r-1)^4+\frac{1}{4}=\left((2r-1)2r+\frac{1}{2}\right)\left((2r-1)(2r-2)+\frac{1}{2}\right)$ ...(1)

Now $(2r)^4+\frac{1}{4}=(2r)^4+\frac{1}{4}+(2r)^2-(2r)^2=\left((2r)^2+\frac{1}{2}\right)^2-(2r)^2$

$=\left((2r)^2+2r+\frac{1}{2}\right)\left((2r)^2-2r+\frac{1}{2}\right)$

$\Rightarrow\ (2r)^4+\frac{1}{4}=\left(2r(2r+1)+\frac{1}{2}\right)\left(2r(2r-1)+\frac{1}{2}\right)$ ...(2)

Now by equation (1) and (2)

$$\therefore\ P = \frac{\prod_{r=1}^{n}\left((2r-1)(2r-2)+\frac{1}{2}\right)}{\prod_{r=1}^{n}\left(2r(2r+1)+\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}\cdot\left(3.2+\frac{1}{2}\right)\left(5\cdot 4+\frac{1}{2}\right)\ldots\ldots\left((2n-1)(2n-2)+\frac{1}{2}\right)}{\left(2.3+\frac{1}{2}\right)\left(4.5+\frac{1}{2}\right)\ldots\ldots\left((2n-2)(2n-1)+\frac{1}{2}\right)\left(2n(2n+1)+\frac{1}{2}\right)}$$

$$\Rightarrow\ P = \frac{1}{2\cdot\left(2n(2n+1)+\frac{1}{2}\right)} \qquad \ldots(3)$$

as $\lim_{n\to\infty} n^a P$ exists

$$\Rightarrow\ \lim_{n\to\infty} n^a \frac{1}{2\cdot\left(2n(2n+1)+\frac{1}{2}\right)} \text{ exists}$$

$$\Rightarrow\ \alpha = 2 \text{ and } \lim_{n\to\infty} n^a P = \frac{1}{8}$$

CONTINUITY & DERIVABILITY

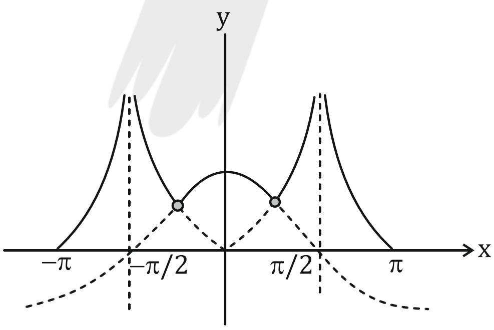
SINGLE CORRECT ANSWER TYPE

13. Number of points where the function $f(x) = \max(|\tan x|, \cos|x|)$ is non differentiable in the interval $(-\pi, \pi)$ is

(A) 4 (B) 6 (C) 3 (D) 2

Ans. **(A)**

Sol. The functions is not differentiable at two points between $x = -\pi/2 \,\&\, x = \pi/2$ also function is not continuous at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ hence at four points function is not differentiable.



INTEGER TYPE

14. Let $f(x)$ is differentiable function & $f(0) = 1$. Also if $f(x)$ satisfies

$f(x+y+1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2\ \forall\ x, y \in R$ and $f(x) = a(x+1)^b$, then find $(a+b)$

Ans. **(3)**

Sol. $f(x+y+1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$

$f(1) = 4$

$\frac{\partial}{\partial x} f'(x+y+1) = 2(\sqrt{f(x)} + \sqrt{f(y)}) \cdot \frac{f'(x)}{2\sqrt{f(x)}}$

$\frac{\partial}{\partial y} f'(x+y+1) = (\sqrt{f(x)} + \sqrt{f(y)}) \cdot \frac{f'(y)}{\sqrt{f(y)}} \Rightarrow \ \Rightarrow \frac{f'(x)}{\sqrt{f(x)}} = \frac{f'(y)}{\sqrt{f(y)}} = k$

$\therefore \int \frac{f'(x)}{\sqrt{f(x)}dx = \int kdx \Rightarrow 2\sqrt{f(x)} = kx + c}$

put $x = 0$

$\Rightarrow 4 = k + 2 \Rightarrow k = 2$

$\therefore 2\sqrt{f(x)} = 2(x+1) \ \Rightarrow \ f(x) = (x+1)^2$

METHOD OF DIFFERENTIANTION

MULTIPLE CORRECT ANSWER TYPE

15. Given $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5a - x\sin a \cdot \sin 2a - 5\sin^{-1}(a^2 - 8a + 17)$ then:

(A) $f'(x) = -x^2 + 2x \sin 6 - \sin 4 \sin 8$

(B) $f'(\sin 8) > 0$

(C) $f'(x)$ is not defined at $x = \sin 8$

(D) $f'(\sin 8) < 0$

Ans. **(AD)**

Sol. $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5a - x\sin a \cdot \sin 2a - 5\sin^{-1}(a^2 - 8a + 17)$

$f(x)$ is defined when $\ -1 \le a^2 - 8a + 17 \le 1.$

$-1 \le (a-4)^2 + 1 \le 1 \ \Rightarrow \ a = 4$

$\therefore \ f(x) = -\frac{x^3}{3} + x^2 \sin 6 - x\sin 4 \sin 8 - \frac{5\pi}{2}$

$\therefore \ f'(x) = -x^2 + 2x\sin 6 - \sin 4 \sin 8$

$f'(\sin 8) = -\sin^2 8 + 2\sin 8 \sin 6 - \sin 4 \sin 8 = \sin 8[2\sin 6 - (\sin 8 + \sin 4)]$

$= \sin 8[2\sin 6 - 2\sin 6 \cos 2] = 2\sin 6 \sin 8(1 - \cos 2)$

$\sin 6 < 0, \sin 8 > 0, 1 - \cos 2 > 0$

$\therefore \ f'(\sin 8) < 0$

APPLICATION OF DERIVATIVES

SINGLE CORRECT ANSWER TYPE

16. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection is

(A) 0 (B) $\frac{4\sqrt{2}}{3}$ (C) $\frac{4\sqrt{2}}{7}$ (D) $2\sqrt{2}$

Ans. **(C)**

Sol. Points of intersection of curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ are $(\pm\sqrt{2}, 1)$

$y = |x^2 - 1|$ and $y = |x^2 - 3|$

$\frac{dy}{dx} = 2x$ and $\frac{dy}{dx} = -2x$

$m_1 = \left.\frac{dy}{dx}\right|_{(\sqrt{2},1)} = 2\sqrt{2}$ and $m_2 = \left.\frac{dy}{dx}\right|_{(\sqrt{2},1)} = -2\sqrt{2}$

$\tan\theta = \left|\frac{4\sqrt{2}}{7}\right|$

SINGLE CORRECT ANSWER TYPE

17. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots \ldots \ldots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \ldots . + a_{n-1}x + a_n = 0$ has

(A) exactly one root in (0,1) (B) at least one root in (0,1)

(C) no root in (0,1) (D) at the most one root in (0,1)

Ans. **(B)**

Sol. Consider the function $f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1} + \cdots \ldots \ldots + \frac{a_{n-1}x^2}{2} + a_nx$. Then $f(0) = 0$ and $f(1) = 0$ hence $f'(x) = 0$ has at least one solution in (0,1)

INTEGER TYPE

18. A cubic f(x) vanishes at $x = -2$ and has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)dx = \frac{14}{3}$, the cubic $f(x) = \lambda_1x^3 + \lambda_2x^2 - x + 2$, then find $(\lambda_1 + \lambda_2)$

Ans. **(2)**

Sol. $x = -1$ and $x = \frac{1}{3}$ are roots of $f'(x) = 0$

$\Rightarrow f'(x) = a(3x - 1)(x + 1) = a(3x^2 + 2x - 1)$

$\Rightarrow f(x) = a(x^3 + x^2 - x + b)$

$f(-2) = 0 \Rightarrow b = 2$

$\Rightarrow f(x) = a(x^3 + x^2 - x + 2)$

$\int_{-1}^{1} f(x)dx = \frac{14}{3}$

$\Rightarrow \int_{-1}^{1} a(x^3 + x^2 - x + 2) = \frac{14}{3}$

$\Rightarrow a\int_{-1}^{1} x^2 + 2 = \frac{14}{3} \Rightarrow 2a\left(\frac{1}{3} + 2\right) = \frac{14}{3} \Rightarrow a = 1$

$\therefore f(x) = x^3 + x^2 - x + 2$

INDEFINITE INTEGRATION

SINGLE CORRECT ANSWER TYPE

19. Evaluate $\int x^2 \log(1 - x^2) \cdot dx$ and hence find the value of:

$\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \cdots \ldots \ldots . =$

(A) $\frac{1}{4}\log 2$ (B) $\frac{2}{7} - \frac{2}{3}\log 2$ (C) $\frac{8}{9} - \frac{2}{3}\log 2$ (D) $-\frac{2}{3}\log 2$

Ans. **(C)**

Sol. we know

$$\log(1 - x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \ldots \infty\right\}$$

Put x^2 instead of x

$$\log(1 - x^2) = -\left\{x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \cdots \ldots \infty\right\}$$

$$\Rightarrow x^2\log(1 - x^2) = -\left\{x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \frac{x^{10}}{4} + \cdots \ldots \infty\right\}$$

Integrating both sides, we get

$$\int x^2\log(1 - x^2) \cdot dx = -\left\{\frac{x^5}{1.5} + \frac{x^7}{2.7} + \frac{x^9}{3.9} + \cdots \ldots \infty\right\}$$

Now to find constant, put $x = 0$

$0 = 0 + c \Rightarrow c = 0$

Apply integration by parts & taking limit 0 to 1

$$\Rightarrow \left[\frac{x^3}{3} \cdot \log(1 - x^2)\right]_0^1 - \int_0^1 \frac{x^3}{3} \cdot \frac{(-2x)}{1 - x^2} \cdot dx$$

$$\Rightarrow \left[\frac{x^3}{3}\log(1 - x^2)\right]_0^1 + \frac{2}{3}\left[\frac{-x^3}{3} + \frac{1}{2}\log\left|\frac{1 + x}{1 - x}\right|\right]_0^1$$

Taking $\log(1 - x^2) = \log(1 + x) + \log(1 - x)$ & $\log\left(\frac{1 + x}{1 - x}\right) = \log(1 + x) - \log(1 - x)$

$$\Rightarrow \frac{1}{3}\log 2+\frac{1}{3}\log 2-\frac{2}{3}-\frac{2}{9}+\lim_{x\to 1}\left(\frac{x^3}{3}-\frac{1}{3}\right)\log(1-x)$$

$$\Rightarrow \frac{2}{3}\log 2-\frac{8}{9}=\text{ R.H.S.}$$

$$-\left\{\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\cdots\right\}=\frac{2}{3}\log(2)-\frac{8}{9}$$

INTEGER TYPE

20. $\int\left(\frac{x-1}{x+1}\right)\frac{dx}{\sqrt{x^3+x^2+x}}=\lambda\tan^{-1}\sqrt{\left(x+\frac{1}{x}+1\right)}+c$, then find λ

Ans. **(2)**

Sol. $I=\int\left(\frac{x-1}{x+1}\right)\frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$; So, $I=\int\frac{(x-1)dx}{(x+1)x\sqrt{x+1+\frac{1}{x}}}$

$$=\int\frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)dx}{(x+1)\left(1+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Put $x+1+\frac{1}{x}=t^2$

$$\left(1-\frac{1}{x^2}\right)dx=2tdt\ =\int\frac{2tdt}{(t^2+1)t}=2\tan^{-1}t+c=2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)+c$$

$$\left(x+\frac{1}{x}+1\right)$$

DEFINITE INTEGRATION

SINGLE CORRECT ANSWER TYPE

21. The absolute value of $\int_{10}^{18}\frac{\sin x}{1+x^8}dx$, is

(A) less than 10^{-7} (B) more than 10^{-7}

(C) less than 10^{-6} (D) more than 10^{-6}

Ans. **(A)**

Sol. $\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|\le\int_{10}^{19}\frac{|\sin x|}{1+x^8}dx$

$\because\ \left|\int f(x)dx\right|\le\int|f(x)|dx\ \sin x\le 1$ and $1+x^6>x^8$

$\therefore\ \frac{1}{1+x^8}<\frac{1}{x^8}\ \ \therefore\ \frac{|\sin x|}{1+x^8}<\frac{1}{x^8}$

So, $\int_{10}^{19}\frac{\sin x}{1+x^8}dx\ |\le\int_{10}^{18}\frac{|\sin x|}{1+x^8}dx<\int_{10}^{19}\frac{1}{x^8}dx<\int_{10}^{19}\frac{1}{10^8}dx=9\times10^{-8}<10\times10^{-8}<10^{-7}$

MULTIPLE CORRECT ANSWER TYPE

22. If the value of the definite integral $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$ is $\frac{\pi^2}{\sqrt{n}}$ where $n \in N$, then the value of ' n ' is divisible by prime number

(A) 2 (B) 3 (C) 5 (D) 9

Ans. **(A, B)**

Sol. $I = \int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx \ldots\ldots (i)$

$$I = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2-x+1}dx = \int_0^1 \frac{\cos^{-1}\sqrt{x}}{x^2-x+1}dx \ldots\ldots (ii)$$

(Applying $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$)

$\therefore$ On adding (1) and (2), we get

$$2I = \int_0^1 \frac{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}{x^2-x+1}dx = \frac{\pi}{2}\int_0^1 \frac{dx}{x^2-x+1}dx = \frac{\pi}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$2I = \frac{\pi}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = \frac{\pi^2}{3\sqrt{3}}$$

$$\text{Hence } I = \frac{\pi^2}{6\sqrt{3}} = \frac{\pi^2}{\sqrt{108}} \equiv \frac{\pi^2}{\sqrt{n}} \Rightarrow n = 108$$

INTEGER TYPE

23. A function $f: R \to R$ satisfies the equation $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$ and is continuous throughout the domain. If $I_1 + I_2 + I_3 + I_4 + I_5 = 450$ where $I_n = n\int_0^n f(x)dx$ and $f(x) = \lambda x$, then find λ

Ans. **(4)**

Sol. Given the $f(x+y) = f(x) + f(y)$

Putting $x = 0$ and $y = 0$ in (i), we get

$f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$

Now $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0} f'(h) = f'(0)$

$$\Rightarrow f(x) = \int f'(0)dx = xf'(0) + c$$

Putting $x = 0$ in (ii), we get

$f(0) = 0 + c \Rightarrow c = 0\ [\because f(0) = 0 \text{ from (ii) }]$

$\Rightarrow f(x) = xf'(0)$

Thus $I_n = n\int_0^n f(x)dx = n\int_0^n xf'(0)dx \Rightarrow I_n = \frac{n^3 \cdot f'(0)}{2}$

therefore $I_1 + I_2 + I_3 + I_4 + I_5 = \frac{f'(0)}{2}(1^3 + 2^3 + 3^3 + 4^3 + 5^3)$

$\Rightarrow 450 = \frac{f'(0)}{2} \cdot \left\{\frac{5 \cdot (5+1)}{2}\right\}^2 \Rightarrow f'(0) = 4 \therefore f(x) = 4x$ (from equation (iii))

AREA UNDER CURVE

SINGLE CORRECT ANSWER TYPE

24. If $f(x) = \sin x \ \forall\, x \in \left[0, \frac{\pi}{2}\right], f(x) + f(\pi - x) = 2 \ \forall\, x \in \left(\frac{\pi}{2}, \pi\right]$ and $f(x) = f(2\pi - x) \ \forall\, x \in (\pi, 2\pi]$, then the area enclosed by $y = f(x)$ and x - axis is

(A) π (B) 2π (C) 2 (D) 4

Ans. **(B)**

Sol. $f(x) = \sin x$

$f(x) + f(\pi - x) = 2$

$f(x) = 2 - f(\pi - x) = 2 - \sin(\pi - x) = 2 - \sin x \ x \in \left(\frac{\pi}{2}, \pi\right]$

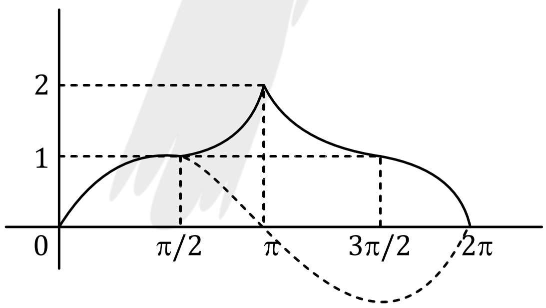
$f(x) = f(2\pi - x)$

$\therefore\ f(x + \pi) = f(\pi - x)$

so curve is symmetric w.r.t. line $x = \pi$ for $(\pi, 2\pi]$

$f(x) = f(2\pi - x) = -\sin x$

Area $= 2\left(\int_0^{\pi/2} \sin x dx \int_{\pi/2}^{\pi} (2 - \sin x)dx\right) = 2\left(1 + 2x\frac{\pi}{2} - 1\right) = 2\pi$



DIFFERENTIAL EQUATION

SINGLE CORRECT ANSWER TYPE

25. Solution of differential equation $x^2 = 1 + \left(\frac{x}{y}\right)^{-1}\frac{dy}{dx} + \frac{\left(\frac{x}{y}\right)^{-2}\left(\frac{dy}{dx}\right)^2}{2!} + \frac{\left(\frac{x}{y}\right)^{-3}\left(\frac{dy}{dx}\right)^3}{3!} + \cdots$. Is

(A) $y^2 = x^2(\ln x^2 - 1) + c$ (B) $y = x^2(\ln x - 1) + c$

(C) $y^2 = x(\ln x - 1) + c$ (D) $y = x^2e^{x^2} + c$

Ans. **(A)**

Sol. $x^2 = e^{\left(\frac{x}{y}\right)^{-1}\left(\frac{dy}{dx}\right)} \Rightarrow x^2 = e^{\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)} \Rightarrow \ln x^2 = \frac{y}{x}\frac{dy}{dx} \Rightarrow \int x\ln x^2 dx = \int ydy$

Put $x^2 = t \Rightarrow 2xdx = dt \Rightarrow \frac{1}{2}\int \ln t dt = \frac{y^2}{2}$

$\Rightarrow c + t\ln t - t = y^2 \Rightarrow y^2 = x^2\ln x^2 - x^2 + c$

INTEGER TYPE

26. Let the curve $y = f(x)$ passes through $(4, -2)$ satisfy the differential equation,

$y(x + y^3)dx = x(y^3 - x)dy$ & $y = g(x) = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}dt, 0 \le x \le \frac{\pi}{2}$.

The area of the region bounded by curves, $y = f(x), y = g(x)$ and $x = 0$ is $\frac{\lambda}{8}\left(\frac{3\pi}{16}\right)^4$,

then find the value of λ.

Ans. **(1)**

Sol. Since given differential equation is

$y(x + y^3)dx = x(y^3 - x)dy$

$\Rightarrow (xydx + x^2dy) + y^4dx - y^3xdy = 0$

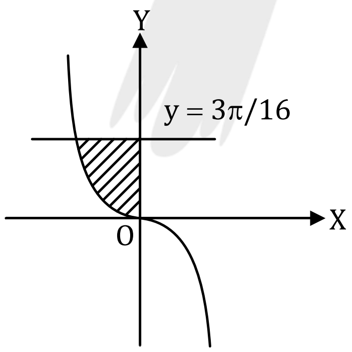
$\Rightarrow x(ydx + xdy) + y^3(ydx - xdy) = 0$

$\Rightarrow xd(xy) = y^3(xdy - ydx)$

$\Rightarrow xd(xy) = x^2y^3 d\left(\frac{y}{x}\right)$

$\Rightarrow \frac{d(xy)}{(xy)^2} = \left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)$

On integrating $-\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + c$ at $(4, -2)$



PROBABILITY

SINGLE CORRECT ANSWER TYPE

27. S_1 : Two persons each make a single throw with a die. The probability they get equal values is P_1. Four persons each make a single throw and probability of exactly three being equal is P_2. Then P_1 greater than P_2.

S_2 : Each of A & B throw 2 dice, if A throws 9 , then B 's probability of throwing a higher number is $\frac{1}{6}$

S_3: If $P(A_1 \cup A_2) = 1 - P(A_1^C) \cdot P(A_2^C)$, then A_1 and A_2 are independent

S_4 : If the events A, B, C are independent, then A, B, $\bar{C}$ are independent

(A) T T T T (B) TTFT

(C) TFTF (D) F TTF

Ans. **(A)**

Sol. S_1: $P_1 = \frac{1}{6}$

$P_2 = {}^4C_3 \times {}^6C_1 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{54}$

$\therefore P_1 > P_2$

$\therefore S_1$ is true.

S_2 : B throws 10,11, or 12. The cases are

$6+4, 5+5, 4+6$

$6+5, 5+6, 6+6$

$\therefore$ prob. $= \frac{6}{36} = \frac{1}{6}$

$\therefore S_2$ is true

S_3: $P(A_1 \cup A_2) = 1 - P(A_1 \cup A_2)' = 1 - P(A_1{}' \cap A_2{}')$

also $P(A_1 \cup A_2) = 1 - P(A_1')P(A_2')$

$\therefore P(A_1' \cap A_2') = P(A_1{}')P(A_2{}')$

$\therefore A_1'$ and A_2' are independent

$\therefore A_1$ and A_2 are independent

S_A: $P(A \cap B \cap C) = P(A)P(B)P(C)$

$P(A \cap B \cap \bar{C}) = P(A \cap B) - C) = P((A \cap B) - (A \cap B \cap C))$

$= P(A)P(B) - P(A) \cdot P(B)P(C) = P(A)P(B)(1 - P(C)) = P(A)P(B)P(\bar{C})$

$\therefore$ A, B, $\bar{C}$ are independent.

MULTIPLE CORRECT ANSWER TYPE

28. A bag initially contains one red & two blue balls. An experiment consisting of selecting a ball at random, noting its colour & replacing it together with an additional ball of the same colour. If three such trials are made, then:

(A) probability that atleast one blue ball is drawn is 0.9

(B) probability that exactly one blue ball is drawn is 0.2

(C) probability that all the drawn balls are red given that all the drawn balls are of same colour is 0.2

(D) probability that atleast one red ball is drawn is 0.6 .

Ans. **(A,B,C,D)**

Sol. (i) $P(E_1) = 1 - P(RRR)$

$$= 1 - \left[\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5}\right] = 0.9$$

(ii) $P(E_2) = 3P(BRR) = 3 \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5} = 0.2$

(iii) $P(E_3) = P\left(\frac{RRR}{RRR} \cup BBB = \frac{P(RRR)}{P(RRR)+P(BBB)}\right.$ but $P(B\,B\,B) = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{8}{20}$

$$\Rightarrow P(E_3) = \frac{0.1}{0.1 + 0.4} = 0.2$$

(iv) $P(E_4) = 1 - P(B\,B\,B) = 1 - \frac{2}{5} = 0.6$

MATRICES & DETERMINANTS

SINGLE CORRECT ANSWER TYPE

29. If a determinant of order 3×3 is formed by using the numbers 1 or -1 then minimum value of determinant is

(A) -2 (B) -4 (C) 0 (D) -8

Ans. **(B)**

Sol. Let $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$C_2 \to c_2 - \frac{a_{12}}{a_{11}} c_1 \quad C_3 \to C_3 - \frac{a_{13}}{a_{11}} c_1$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$ so minimum value $= -4$

30. **Match the following**

Column – I	**Column - II**
(A) Let $|A| = |a_i|_{3\times3} \neq 0$. Each element a_{ij} is multiplied by k^{i-1}. Let $|B|$ the resulting determinant, where $k_1|A| + k_2|B| = 0$. Then $k_1 + k_2 =$	(p) 0
(B) The maximum value of a third order determinant each of its entries are \pm 1 equals	(q) 4
(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$	(r) 1
(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$ where A and B	(s) 2
are determinants of order3. Then $A + 2\,B =$	(t) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

Ans. (A) − (p, t), (B) − (q), (C) − (r), (D) − (p, t)

Sol. (A) $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|B| = \begin{vmatrix} a_{11} & k^{-1}a_{12} & k^{-2}a_{13} \\ ka_{21} & a_{22} & k^{-1}a_{23} \\ k^2a_{31} & ka_{32} & a_{33} \end{vmatrix} = \frac{1}{k^3}\left[\begin{matrix} k^2a_{11} & ka_{12} & a_{13} \\ k^2a_{21} & ka_{22} & a_{23} \\ k^2a_{31} & ka_{32} & a_{33} \end{matrix}\right|$$

$= |A|$

$k_1|A| + k_2|B| = 0$

$k_1 + k_2 = 0$

(B) $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$

(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

$\Rightarrow\ \sin^2\gamma - \cos\alpha(\cos\alpha - \cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma - \cos\beta)$

$= -\cos\alpha(-\cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma)$

$\Rightarrow\ \sin^2\gamma - \cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\beta = 2\cos\alpha\cos\beta\cos\gamma$

$\Rightarrow\ \sin^2\gamma = \cos^2\alpha + \cos^2\beta \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

$R_2 \to R_2 - (R_1 + R_3)$

$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = 4\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$

$= 4\begin{vmatrix} x+1 & -3 \\ 2x-1 & 0 \end{vmatrix} = (24x-12)$

$\therefore A = 24, B = -12$

$\therefore A + 2B = 0$

COMPLEX NUMBER

SINGLE CORRECT ANSWER TYPE

31. S1: If (z_1, z_2) and (z_3, z_4) are two pairs of non zero conjugate complex numbers then $\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right) = \pi/2$

S2: If ω is an imaginary fifth root of unity, then $\log_2\left|1 + \omega + \omega^2 + \omega^3 - \frac{1}{\omega}\right| = 1$

S3: If z_1 and z_2 are two of the 8th roots of unity, such that $\arg\left(\frac{z_1}{z_2}\right)$ is least positive, then $\frac{z_1}{z_2} = \frac{1+i}{\sqrt{2}}$

S4: The product of all the fifth roots of -1 is equal to -1

(A) TTFT (B) TFFT (C) FFTF (D)FTIT

Ans. **(D)**

Sol. S1. $\bar{z}_1 = z_2, \bar{z}_3 = z_4$

$\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right) = \arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = \arg\left(\frac{z_1 \bar{z}_1}{z_3 \bar{z}_3}\right) = 0$

S2. $\log_2|1 + \omega + \omega^2 + \omega^3 - \omega^4| = \log_2|-2\omega^4| = \log_2 2 = 1 \because |\omega^4| = 1$

S3. Since z_1, z_2 are 8^{th} roots of unity

$\therefore \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4}$ (least positive) $\therefore \frac{z_1}{z_2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}.$

S4. $z^5 + 1 = 0$

$\therefore$ product of all the roots $= (-1)^5 \cdot 1 = -1$

32. **Match the column :**

If z_1, z_2, z_3, z_4 are the roots of the equation $z^4 + z^3 + z^2 + z + 1 = 0$ then

Column-I	**Column - II**
(A) $\left|\sum_{i=1}^{4} z_i^4\right|$ is equal to	(p) 0
(B) $\sum_{i=1}^{4} z_i^5$ is equal to	(q) 4
(C) $\prod_{i=1}^{4} (z_i + 2)$ is equal to	(r) 1
(D) least value of $[|z_1 + z_2|]$ is	(s) 11
(Where [] represents greatest integer function)	(t) $\left|4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right|$

Ans. (A) → (r),(B) → (q, t),(C) → (s),(D) → (p)

Sol. The given equation is $\frac{z^5-1}{z-1}=0$ which means that z_1, z_2, z_3, z_4 are four out of five roots of unity except 1 .

(A) $z_1{}^4+z_2{}^4+z_3{}^4+z_4{}^4+1^4=0 \Rightarrow \left|\sum_{i=1}^{4} z_i{}^4\right|=1$

(B) $z_1^5+z_2^5+z_3^5+z_4^5+1^5=5 \Rightarrow \sum_{i=1}^{4} z_i^5=4$

(C) $z^4+z^3+z^2+z+1=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$.

Putting $z=-2$ both the sides and we get $\prod_{i=1}^{4}(z_i+2)=11$

(D) $|z_1+z_2|=\sqrt{2+2\cos 144^\circ}$ for minimum

$=2\cos 72^\circ=\frac{\sqrt{5}-1}{2}$ whose greatest integer is 0.

VECTORS

MULTIPLE CORRECT ANSWER TYPE

33. If in $\triangle ABC$, $\overrightarrow{AB}=\frac{\vec{u}}{|\vec{u}|}-\frac{\vec{v}}{|\vec{v}|}$ and $\overrightarrow{AC}=\frac{2\vec{u}}{|\vec{u}|}$, where $|\vec{u}|\neq|\vec{v}|$, then

(A) $1+\cos 2A+\cos 2B+\cos 2C=0$

(B) $\sin A=\cos C$

(C) projection of AC on BC equal to BC

(D) projection of AB on BC is equal to AB

Ans. **(A,B,C)**

Sol. $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\overrightarrow{BC}=\frac{2\vec{u}}{|\vec{u}|}-\frac{\vec{u}}{|\vec{u}|}+\frac{\vec{v}}{|\vec{v}|}=\frac{\vec{u}}{|\vec{u}|}+\frac{\vec{v}}{|\vec{v}|}$

$\overrightarrow{AB}\cdot\overrightarrow{BC}=\left(\frac{\vec{u}}{|\vec{u}|}-\frac{\vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{u}}{|\vec{u}|}+\frac{\vec{v}}{|\vec{v}|}\right)=(\hat{u}-\hat{v})\cdot(\hat{u}+\hat{v})=1-1=0$

$\Rightarrow B=90^\circ \Rightarrow 1+\cos 2A+\cos 2B+\cos 2C=0$

34. A vector ($\vec{d}$) is equally inclined to three vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{j}-2\hat{k}$. Let $\vec{x},\vec{y},\vec{z}$ be three vector in the plane of $\vec{a},\vec{b}$; $\vec{b},\vec{c}$; $\vec{c},\vec{a}$ respectively, then

(A) $\vec{x}\cdot\vec{d}=14$

(B) $\vec{y}\cdot\vec{d}=3$

(C) $\vec{z}\cdot\vec{d}=0$

(D) $\vec{r}\cdot\vec{d}=0$ where $\vec{r}=\lambda\vec{x}+\mu\vec{y}+\delta\vec{z}$

Ans. **(C,D)**

Sol. Since $[\vec{a}\vec{b}\vec{c}]=0$

$\therefore$ $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors.

Further since $\vec{d}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$

$\therefore\ \vec{d}\cdot\vec{a}=\vec{d}\cdot\vec{b}=\vec{d}\cdot\vec{c}=0 \therefore\ \vec{d}\cdot\vec{x}=\vec{d}\cdot\vec{y}=\vec{d}\cdot\vec{z}=0\ \therefore\ \vec{d}\cdot\vec{r}=0$

THREE-DIMENSIONAL GEOMETRY

SINGLE CORRECT ANSWER TYPE

35. Equation of the straight line in the plane $\vec{r}\cdot\vec{n}=d$ which is parallel to $\vec{r}=\vec{a}+\lambda\vec{b}$ and passes through the foot of perpendicular drawn from the point $P(\vec{a})$ to the plane $\vec{r}\cdot\vec{n}=d$ is (where $\vec{n}\cdot\vec{b}=0$)

(A) $\vec{r}=\vec{a}+\left(\frac{d-\vec{a}\cdot\vec{n}}{\vec{n}^2}\right)\vec{n}+\lambda\vec{b}$

(B) $\vec{r}=\vec{a}+\left(\frac{d-\vec{a}\cdot\vec{n}}{\vec{n}}\right)\vec{n}+\lambda\vec{b}$

(C) $\vec{r}=\vec{a}+\left(\frac{\vec{a}\cdot\vec{n}-d}{\vec{n}^2}\right)\vec{n}+\lambda\vec{b}$

(D) $\vec{r}=\vec{a}+\left(\frac{\vec{a}\cdot\vec{n}-d}{\vec{n}}\right)\vec{n}+\lambda\vec{b}$

Ans. **(A)**

Sol. Foot of perpendicular from point $A(\vec{a})$ on the plane $\vec{r}\cdot\vec{n}=d$ is $\vec{a}+\frac{(d-\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\vec{n}$

$\therefore$ Equation of line parallel to $\vec{r}=\vec{a}+\lambda\vec{b}$ in the plane $\vec{r}\cdot\vec{n}=d$ is given by

$$\vec{r}=\vec{a}+\frac{(d-\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\vec{n}+\lambda\vec{b}$$

36. If $P_1:\vec{r}\cdot\vec{n}_1-d_1=0, P_2:\vec{r}\cdot\vec{n}_2-d_2=0$ and $P_3:\vec{r}\cdot\vec{n}_3-d_3=0$ are three planes and $\vec{n}_1,\vec{n}_2$ and $\vec{n}_3$ are three non-coplanar vectors then, the three lines $P_1=0, P_2=0; P_2=0, P_3=0$ and $P_3=0, P_1=0$ are

(A) parallel lines

(B) coplanar lines

(C) coincident lines

(D) concurrent lines

Ans. **(D)**

Sol. $P_1=P_2=0, P_2=P_3=0$ and $P_3=P_1=0$ are lines of intersection of the three planes P_1, P_2 and P_3. As $\vec{n}_1,\vec{n}_2$ and $\vec{n}_3$ are non-coplanar, planes P_1, P_2 and P_3 will intersect at unique point. So the given lines will pass through a fixed point.

INVERSE TRIGONOMETRIC FUNCTION

MATRIX - MATCH TYPE

37. [.] represents greatest integer function in parts (A), (B) and (C)

Column -1	**Column - II**
(A) If $f(x) = \sin^{-1}x$ and $\lim\limits_{x\to\frac{1}{2}^+} f(3x-4x^3) = a - 3\lim\limits_{x\to\frac{1}{2}^+} f(x)$, then $[a] =$	(p) 2
(B) If $f(x) = \tan^{-1}g(x)$ where $g(x) = \frac{3x-x^3}{1-3x^2}$ and $\lim\limits_{h\to 0}\dfrac{f(a+3h)-f(a)}{3h} = \dfrac{3}{1+a^2}$, when $\dfrac{-1}{\sqrt{3}} < a < \dfrac{1}{\sqrt{3}}$, then find $\left[\lim\limits_{h\to 0}\frac{f\left(\frac{1}{2}+6h\right)-f\left(\frac{1}{2}\right)}{6h}\right] =$	(q) 3
(C) If $\cos^{-1}(4x^3-3x) = a + b\cos^{-1}x$ for $-1 < x < \frac{-1}{2}$, then $[a+b+2] =$	(r) 4
(D) If $f(x) = \cos^{-1}(4x^3-3x)$ and $\lim\limits_{x\to\frac{1}{2}^+} f'(x) = a$ and	(s) -2
$\lim\limits_{x\to\frac{1}{2}^-} f'(x) = b$, then $a+b-3 =$	(t) -3

Ans. (A) $\to$ (q) .(B) $\to$ (p),(C) $\to$ (s),(D) $\to$ (t)

Sol. (A) $\sin^{-1}(3x-4x^3) = \pi - 3\sin^{-1}x$ if $\frac{1}{2} < x < 1$

$\therefore \lim\limits_{x\to\frac{1}{2}^+} f(3x-4x^3) = \lim\limits_{x\to\frac{1}{2}^+}(\pi - 3\sin^{-1}x) = \pi - 3\lim\limits_{x\to\frac{1}{2}^+}\sin^{-1}x$

$\therefore a = \pi$

$\therefore [a] = 3$

(B) $f(x) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3\tan^{-1}x$, when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

if $-\frac{1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$, then $\lim_{h\to 0}\frac{f(a+3h)-f(a)}{3h} = \frac{3}{1+a^2} \Rightarrow f'(a) = \frac{3}{1+a^2}$

$$= \lim_{h\to 0}\frac{f\left(\frac{1}{2}+6h\right)-f\left(\frac{1}{2}\right)}{6h} = f'(1/2) == \frac{12}{5}$$

$\therefore$ required value $= 2$

(C) $\cos^{-1}(4x^3-3x) = \cos^{-1}(\cos 3\theta) = 3\theta - 2\pi \{\because\ 2\pi/3 < \theta < \pi\}$

$= -2\pi + 3\cos^{-1}x$

$\therefore [a+b+2] = [-2\pi+3+2] = -2$

(D) $f(x) = \cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta) = \begin{cases} 3\theta, & 0 < \theta < \frac{\pi}{3} \\ 2\pi - 3\theta, & \frac{\pi}{3} < \theta < \frac{\pi}{2} \end{cases}$

$$= \begin{cases} 3\cos^{-1}x, \frac{1}{2} < x < 1 \\ 2\pi - 3\cos^{-1}x, 0 < x < \frac{1}{2} \end{cases} \therefore\ f'(x) = \begin{cases} \frac{-3}{\sqrt{1-x^2}}, \frac{1}{2} < x < 1 \\ \frac{3}{\sqrt{1-x^2}}, 0 < x < \frac{1}{2} \end{cases}$$

$a = \lim_{x \to \frac{1}{2}^+} f'(x) = -2\sqrt{3}$

$b = \lim_{x \to \frac{1}{2}^-} f'(x) = 2\sqrt{3}$

$\therefore\ a + b - 3 = -3$

INTEGER TYPE

38. $\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right] + \tan^{-1}\left[\frac{\tan\alpha}{4}\right] = \lambda\alpha$, then find the value of λ, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Ans. **1**

Sol. $\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right] + \tan^{-1}\left[\frac{\tan\alpha}{4}\right] = \tan^{-1}\left(\frac{6\tan\alpha}{8+2\tan^2\alpha}\right) + \tan^{-1}\left(\frac{\tan\alpha}{4}\right)$

$$= \tan^{-1}\left(\frac{\frac{3\tan\alpha}{4+\tan^2\alpha} + \frac{\tan\alpha}{4}}{1 - \frac{3\tan^2\alpha}{16+4\tan^2\alpha}}\right) \left\{\because \frac{3\tan^2\alpha}{16+4\tan^2\alpha} < 1\right\}$$

$= \tan^{-1}(\tan\alpha) = \alpha$