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Reduction Formula

$$I_n = \int \sec^n x\,dx = \int \sec^2 x \sec^{n-2} x\,dx$$

$$= \tan x \sec^{n-2} x - (n-2)\int \tan^2 x \sec^{n-2} x\,dx$$

$\tan^2 x = (\sec^2 x - 1)$

$$I_n = \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n = \int \tan^n x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$$

(I: $\sin^{n-1} x$, II: $\sin x$)

$$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x\,dx$$

$\cos^2 x \to (1 - \sin^2 x)$

$$-\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$n I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$$

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$$I_{n,m} = \int \frac{1 \cdot dx}{(a^m + x^m)^n} = \frac{x}{(a^m + x^m)^n} + nm \int \frac{x^m + a^m - a^m \, dx}{(a^m + x^m)^{n+1}}$$

$$I_{n,m} = \frac{x}{(a^m + x^m)^n} + nm \, I_{n,m} - nm \, a^m \, I_{n+1,m}$$

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1. If $u_n = \int_0^{\frac{\pi}{2}} x(\sin x)^n\,dx$, $n>0$, then

P.T. $u_n = \frac{(n-1)}{n} u_{n-2} + \frac{1}{n^2}$.

$$u_n = \int_0^{\frac{\pi}{2}} x\underbrace{(\sin x)}_{\text{II}}(\sin^{n-1} x)\,dx = (-\cos x)\,x(\sin^{n-1} x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\pi/2} \cos x\left(\sin^{n-1} x + x(n-1)\sin^{n-2} x\cos x\right)dx$$

$$u_n = \frac{1}{n} + (n-1)\int_0^{\frac{\pi}{2}} x\sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= \frac{1}{n} + (n-1)u_{n-2} - (n-1)u_n$$

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Q. If $u_n = \int_0^1 x^n \tan^{-1}x\,dx$, then P.T.

$$(n+1)u_n + (n-1)u_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$

$$(n+1)u_n + (n-1)u_{n-2} = \frac{\pi}{2} - \int_0^1 x^{n-1}\,dx$$

$$= \frac{\pi}{2} - \frac{1}{n}$$

$$(n+1)u_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,dx$$

$$(n-1)u_{n-2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n-1}\,dx}{1+x^2}$$

$$u_n = \frac{x^{n+1}}{n+1}\tan^{-1}x\Big|_0^1 - \frac{1}{(n+1)}\int_0^1 \frac{x^{n+1}}{1+x^2}\,dx = \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 \frac{(x^2+1-1)x^{n-1}}{1+x^2}\,dx$$

$$= \frac{\pi}{4(n+1)} - \frac{1}{n(n+1)} + \frac{1}{n+1}\int_0^1 \frac{x^{n-1}\,dx}{1+x^2}$$

$$u_n = \frac{\pi}{4(n+1)} - \frac{1}{n(n+1)} + \frac{1}{(n+1)}\left(\tan^{-1}x\right)x^{n-1}\Big|_0^1 - \frac{(n-1)}{n+1}\int_0^1 x^{n-2}\tan^{-1}x\,dx$$

u_{n-2}

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3. Find (i) $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x}\,dx$ (ii) $\int_0^{\pi/2} \frac{\sin(nx)}{\sin x}\,dx$, $n \in \mathbb{N}$.

$$I_n - I_{n-2} = \int_0^{\pi/2} \frac{\sin(nx) - \sin(n-2)x}{\sin x}\,dx = 2\int_0^{\pi/2} \cos(n-1)x\,dx$$

$$= \frac{2\sin(n-1)x}{(n-1)}\Big|_0^{\pi/2} = \frac{2\sin(n-1)\frac{\pi}{2}}{(n-1)}$$

<u>n is odd</u> $I_n = I_{n-2} = I_{n-4} = \cdots = I_1 = \frac{\pi}{2}$.

<u>n is even</u>

$I_2 - I_0 = 2\left(1\right)$

$I_4 - I_2 = 2\left(-\frac{1}{3}\right)$

$I_6 - I_4 = 2\left(\frac{1}{5}\right)$

$I_n - I_{n-2} = 2\frac{(-1)^{\frac{n}{2}+1}}{(n-1)}$

$I_n = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{\frac{n}{2}+1}}{n-1}\right)$

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4. $\frac{{}^nC_0}{1} - \frac{{}^nC_1}{4} + \frac{{}^nC_2}{7} - \frac{{}^nC_3}{10} + \cdots + \frac{(-1)^n\,{}^nC_n}{(3n+1)} = ?$

$= \sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{(3r+1)}$

$(1-x^3)^n = \sum_{r=0}^{n} {}^nC_r (-1)^r x^{3r}$

$I_n = \int_0^1 (1-x^3)^n\,dx = \int_0^1 \sum_{r=0}^{n} {}^nC_r (-1)^r x^{3r}\,dx = \sum_{r=0}^{n} \frac{{}^nC_r(-1)^r}{(3r+1)}$

$I_n = \int_0^1 1\,(1-x^3)^n\,dx = x(1-x^3)^n\Big|_0^1 + 3n\int_0^1 (1-x^3)^{n-1} x^3\,dx = 3n\,I_{n-1} - 3n\,I_n$

$x^3 \to (1-(1-x^3))$

$$\boxed{I_n = \frac{3n}{(3n+1)} I_{n-1}}$$

$$\boxed{I_n = \frac{3n}{(3n+1)} I_{n-1}} = \frac{3n}{3n+1}\,\frac{3(n-1)}{(3n-2)} I_{n-2}$$

$$= \frac{3^2\,n(n-1)}{(3n+1)(3n-2)}\,\frac{3(n-2)}{(3n-5)} I_{n-3}$$

$\vdots$

$$= \frac{3^n\,(n(n-1)\cdots 1)\,I_{n-n}}{(3n+1)(3n-2)(3n-5)\cdots 4}$$

$$\boxed{I_0 = 1}$$

$$I_n = \frac{3^n\,n!}{4\cdot 7\cdot 10\cdots(3n-2)(3n+1)}$$

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5. Find $\left[\sum_{r=1}^{10^6} \frac{1}{\sqrt{r}}\right]$ $[\cdot] = G.I.F.$

$\boxed{\begin{array}{l} \text{Ex-II (rem)} \\ \text{Ex-3 (1-6)} \end{array}}$