

TANGENT & NORMAL

THINGS TO REMEMBER :

- I The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left. \frac{dy}{dx} \right|_{x_1 y_1} = \text{Slope of tangent at}$$

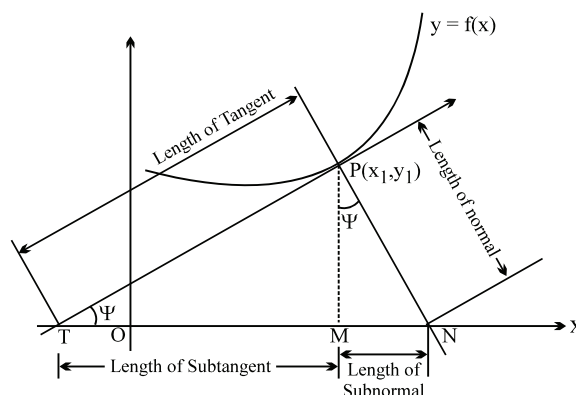
$$P(x_1, y_1) = m \text{ (say).}$$

- II Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left. \frac{dy}{dx} \right|_{x_1 y_1} (x - x_1).$$

- III Equation of normal at (x_1, y_1) is ;

$$y - y_1 = - \frac{1}{\left. \frac{dy}{dx} \right|_{x_1 y_1}} (x - x_1).$$



NOTE :

- The point $P(x_1, y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.
 - If the tangent at any point P on the curve is parallel to the axis of x then $dy/dx = 0$ at the point P.
 - If the tangent at any point on the curve is parallel to the axis of y, then $dy/dx = \infty$ or $dx/dy = 0$.
 - If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.
 - If the tangent at any point makes equal intercept on the coordinate axes then $dy/dx = -1$.
 - Tangent to a curve at the point $P(x_1, y_1)$ can be drawn even though dy/dx at P does not exist.
e.g. $x = 0$ is a tangent to $y = x^{2/3}$ at $(0, 0)$.
 - If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.
e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.
- IV Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **ORTHOGONAL** curves.

V (a) Length of the tangent (PT) = $\frac{y_1 \sqrt{1 + [f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\frac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1 \sqrt{1 + [f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

VI DIFFERENTIALS :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable.

Thus if, $y = \tan x$ then $dy = \sec^2 x dx$.

In general $dy = f'(x) dx$.

Note that : $d(c) = 0$ where 'c' is a constant.

$$d(u + v - w) = du + dv - dw$$

$$d(uv) = u dv + v du$$

Note :

- For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.
- The relation $dy = f'(x) dx$ can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

PROFICIENCY TEST-1

- If tangent to the curve $y = f(x)$ at any point is parallel to y - axis , then at that point dy/dx is
(A) 0 (B) 1 (C) -1 (D) Not defined
- If normal to the curve $y = f(x)$ at a point makes 135° angle with x - axis (taken counter-clockwise w.r.t positive x -axis) then at that point dy/dx equals-
(A) 1 (B) -1 (C) 0 (D) Not defined
- The slope of the curve $y = \sin x + \cos^2 x$ is zero at a point, whose x -coordinate can be
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) $\frac{\pi}{3}$
- The slope of the tangent to the curve $x^2 + 2y = 8x - 7$ at the point with $x = 5$ is -
(A) 1 (B) $\sqrt{3}$ (C) -1 (D) $-\sqrt{3}$
- The equation of the tangent to the curve $y = \cos x$ at $x = \pi/3$ is-
(A) $3x - 2\sqrt{3}y = \pi + \sqrt{3}$ (B) $3x + 2\sqrt{3}y = \pi + \sqrt{3}$
(C) $3x + 2\sqrt{3}y = \pi - \sqrt{3}$ (D) $3x - 2\sqrt{3}y = \pi - \sqrt{3}$
- At what point the tangent to the curve, $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is perpendicular to the x - axis-
(A) (0, 0) (B) (a, a) (C) (a,0) (D) (0,a)
- If the curve $y = x^2 + bx + c$, touches the line $y = x$ at the point (1,1), then values of b and c are (respectively)
(A) $-1, 2$ (B) $-1, 1$ (C) $2, 1$ (D) $-2, 1$
- If the slope of the tangent to the curve $xy + ax - 2y = 0$ at point (1,1) is 2, then a equals-
(A) 0 (B) 1 (C) 2 (D) 3
- The equation of the tangents to the curve $y = (x^3 - 1)(x - 2)$ at the points where it meets x - axis are-
(A) $y + 3x = 3, y + 7x - 14 = 0$ (B) $y - 3x = -3, y - 7x + 14 = 0$
(C) $y + 3x = 3, y - 7x + 14 = 0$ (D) $y - 3x = -3, y + 7x - 14 = 0$
- At what point on the curve $y = e^{-x}$, the tangent cuts intercept equal in length on coordinate axes-
(A) (0,1) (B) $(-1, e)$ (C) $(1, 1/e)$ (D) $(-1, 1/e)$
- The equation of the normal to the curve, $y^2 = 4ax$ at point (a, 2a) is-
(A) $x - y + a = 0$ (B) $x + y - 3a = 0$ (C) $x - 2y + 3a = 0$ (D) $2x + y - 4a = 0$
- The equation of normal to the curve, $x^{2/3} + y^{2/3} = a^{2/3}$ at the point (a, 0) is-
(A) $x = a$ (B) $y = 0$ (C) $x + y = a$ (D) $x - y = a$
- The equation of normal to the curve $y = e^x$ at the point (0, 1) is-
(A) $x + y = 1$ (B) $y - x = 1$ (C) $ey - x = e$ (D) $e(y-1) + x = 0$
- The equation of normal to the curve $y^2 = 16x$ at the point (1, 4) is-
(A) $2x + y = 6$ (B) $2x - y + 2 = 0$ (C) $x + 2y = 9$ (D) $x - 2y + 7 = 0$
- The equation of normal to the curve $y = \tan x$ at the point (0, 0) is-
(A) $x + y = 0$ (B) $x - y = 0$ (C) $x + 2y = 0$ (D) None of these

PROFICIENCY TEST-2

- The angle made by the tangent to the curve $x = e^t \cos t$, $y = e^t \sin t$ at point $t = \pi/4$ with x- axis is -
(A) 0 (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$
- The coordinates of any point P on a curve are represented by $x = \frac{1}{2}t^2$, $y = \frac{1}{3}t^3$, where t is a parameter, then equation of tangent to the curve at P is-
(A) $6tx - 6y = t^3$ (B) $4tx + 3y = t^3$ (C) $3tx + 2y = t^3$ (D) $3tx + y = t^3$
- The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at point (4,4) on coordinate axes is-
(A) $4\sqrt{2}$ (B) $6\sqrt{3}$ (C) $8\sqrt{2}$ (D) $\sqrt{256}$
- If $x = t^2$ and $y = 2t$, then equation of normal at $t = 1$ is-
(A) $x + y + 3 = 0$ (B) $x - y + 1 = 0$ (C) $x + y - 1 = 0$ (D) $x + y - 3 = 0$
- The points on the curve $9y^2 = x^3$ where the normal to the curve cuts equal non-zero intercepts from the axes are-
(A) (4, 8/3) (B) (4, -8/3)
(C) (0, 0) (D) None of these
- The angle of intersection between the curve $y = 4x^2$ and $y = x^2$ is -
(A) 0° (B) 30° (C) 45° (D) 90°
- The angle of intersection of curves $2y = x^3$ and $y^2 = 32x$ at the origin is-
(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/2$ (D) None of these
- The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16,8) is-
(A) 60° (B) 90° (C) $\tan^{-1}(3/5)$ (D) $\tan^{-1}(4/3)$
- At any point of a curve (subtangent) \times (subnormal) is equal to the square of the-
(A) slope of the tangent at that point (B) slope of the normal at that point
(C) abscissa of that point (D) ordinate of that point
- The length of subtangent to the curve, $x^2 + xy + y^2 = 7$ at the point (1,-3) is-
(A) 3 (B) 5 (C) 15 (D) 3/5
- The length of subnormal at any point to the parabola $y^2 = 4ax$ is ($a > 0$)
(A) 1 (B) 2 (C) 2a (D) 4a
- For a curve $\frac{\text{length of normal}}{\text{length of tangent}}$ equals-
(A) Length of subtangent (B) Length of subnormal
(C) slope of tangent (D) slope of normal
- The length of subtangent at any point of the curve $y = be^{x/a}$ is ($a, b > 0$)
(A) ab (B) a (C) b (D) b/a
- A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 3)$. The rate of change of volume with respect to x will be-
(A) $\frac{27\pi}{8} (2x - 3)^2$ (B) $\frac{27\pi}{8} (2x + 3)^2$ (C) $\frac{27\pi}{8} (3x + 2)^2$ (D) $\frac{8}{27\pi} (2x + 3)^2$

(Mathematics) APPLICATION OF DERIVATIVE

15. The rate of change of the volume of a cone with respect to the radius of its base is-
 (A) $\pi r^2 h$ (B) $\frac{4}{3} \pi r h$ (C) $\frac{4}{3} \pi r^2 h$ (D) $\frac{2}{3} \pi r h$
16. The radius of a circle is increasing at the rate of 0.7 cm/sec. The rate of increase of its circumference is -
 (A) 0.7π cm/sec (B) 2.1π cm/sec (C) 1.4π cm/sec (D) 2.8π cm/sec
17. The complete set of points where the tangent to the curve $y^3 - 3xy + 2 = 0$ is horizontal is-
 (A) $\{(1,1)\}$ (B) $\{(0, -\sqrt[3]{2})\}$ (C) $\{(1,1), (0, -\sqrt[3]{2})\}$ (D) ϕ
18. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then that angle is
 (A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$
19. The tangent to the curve $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$ at the points $\theta = (2k+1)\pi$, $k \in \mathbb{Z}$ are parallel to
 (A) $y = x$ (B) $y = -x$ (C) $y = 0$ (D) $x = 0$
20. A balloon is pumped at the rate of a cm³/minute. The rate of increase of its surface area when the radius is b cm, is -
 (A) $\frac{a}{b}$ cm²/min (B) $\frac{a}{2b}$ cm²/min (C) $\frac{2a}{b}$ cm²/min (D) none of these

(Mathematics) APPLICATION OF DERIVATIVE

EXERCISE-I

- Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.
- Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point $(1, 2)$.
- Find the point of intersection of the tangents drawn to the curve $x^2 y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$.
- Find all the lines that pass through the point $(1, 1)$ and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.
- The tangent to $y = ax^2 + bx + \frac{7}{2}$ at $(1, 2)$ is parallel to the normal at the point $(-2, 2)$ on the curve $y = x^2 + 6x + 10$. Find the value of a and b .
- A straight line is drawn through the origin and parallel to the tangent to a curve $\frac{x + \sqrt{a^2 - y^2}}{a} = \ln \left(\frac{a + \sqrt{a^2 - y^2}}{y} \right)$ at an arbitrary point M . Show that the locus of the point P of intersection of the straight line through the origin & the straight line parallel to the x -axis & passing through the point M is $x^2 + y^2 = a^2$.
- Prove that the segment of the tangent to the curve $y = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$ contained between the y -axis & the point of tangency has a constant length.
- A function is defined parametrically by the equations

$$f(t) = x = \begin{cases} 2t + t^2 \sin \frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and } g(t) = y = \begin{cases} \frac{1}{t} \sin t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for $t = 0$ if exist.
- Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \leq x \leq 2\pi$, that are parallel to the line $x + 2y = 0$.
- Prove that the segment of the normal to the curve $x = 2a \sin t + a \sin t \cos^2 t$; $y = -a \cos^3 t$ contained between the co-ordinate axes is equal to $2a$.
- Show that the normals to the curve $x = a(\cos t + t \sin t)$; $y = a(\sin t - t \cos t)$ are tangent lines to the circle $x^2 + y^2 = a^2$.
- The chord of the parabola $y = -a^2 x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point $x = 2$ and is bisected by that point. Find 'a'.
- If the tangent at the point (x_1, y_1) to the curve $x^3 + y^3 = a^3$ ($a \neq 0$) meets the curve again in (x_2, y_2) then show that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.
- Determine a differentiable function $y = f(x)$ which satisfies $f'(x) = [f(x)]^2$ and $f(0) = -\frac{1}{2}$. Find also the equation of the tangent at the point where the curve crosses the y -axis.

15. Tangent at a point P_1 [other than $(0, 0)$] on the curve $y = x^3$ meets the curve again at P_2 . The tangent at P_2 meets the curve at P_3 & so on. Show that the abscissae of $P_1, P_2, P_3, \dots, P_n$, form a GP. Also find the ratio $\frac{\text{area}(P_1 P_2 P_3)}{\text{area}(P_2 P_3 P_4)}$.
16. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x -axis at $P(-2, 0)$ & cuts the y -axis at a point Q where its gradient is 3. Find a, b, c .
17. The tangent at a variable point P of the curve $y = x^2 - x^3$ meets it again at Q . Show that the locus of the middle point of PQ is $y = 1 - 9x + 28x^2 - 28x^3$.
18. Show that the distance from the origin of the normal at any point of the curve $x = ae^{\theta} \left(\sin \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \right)$ & $y = ae^{\theta} \left(\cos \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \right)$ is twice the distance of the tangent at the point from the origin.
19. Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $(x^2/a^2) + (y^2/b^2) = 1$ may touch if $c = a + b$.
20. The graph of a certain function f contains the point $(0, 2)$ and has the property that for each number ' p ' the line tangent to $y = f(x)$ at $(p, f(p))$ intersect the x -axis at $p + 2$. Find $f(x)$.
21. A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q . Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.
22. A and B are points of the parabola $y = x^2$. The tangents at A and B meet at C. The median of the triangle ABC from C has length ' m ' units. Find the area of the triangle in terms of ' m '.
23. (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.
(b) Show that in the curve $y = a \ln(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.
24. (a) Show that the curves $\frac{x^2}{a^2 + K_1} + \frac{y^2}{b^2 + K_1} = 1$ & $\frac{x^2}{a^2 + K_2} + \frac{y^2}{b^2 + K_2} = 1$ intersect orthogonally.
(b) If the two curves $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k .
25. Show that the angle between the tangent at any point 'A' of the curve $\ln(x^2 + y^2) = C \tan^{-1} \frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

EXERCISE-II

RATE MEASURE AND APPROXIMATIONS

1. Water is being poured on to a cylindrical vessel at the rate of $1 \text{ m}^3/\text{min}$. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.
2. A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.
 - (i) how fast is the farther end of the shadow moving on the pavement ?
 - (ii) how fast is his shadow lengthening ?
3. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.
4. An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of $1.5 \text{ cm}^3/\text{min}$. Find the rate at which level of water in the cone is rising, when the depth of water is 4 cm.
5. A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1 cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.
6. Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always $1/6$ th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.
7. An open Can of oil is accidentally dropped into a lake; assume the oil spreads over the surface as a circular disc of uniform thickness whose radius increases steadily at the rate of 10 cm/sec. At the moment when the radius is 1 meter, the thickness of the oil slick is decreasing at the rate of 4 mm/sec, how fast is it decreasing when the radius is 2 meters.
8. Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of $2 \text{ cm}^3/\text{sec}$ through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.
9. An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R Km, how fast the area of the earth, visible from the plane increasing at 3min after it started ascending. Take visible area $A = \frac{2\pi R^2 h}{R + h}$ Where h is the height of the plane in kms above the earth.
10. A variable $\triangle ABC$ in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time $t = 0$ and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec.
11. A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$.
12. Water is flowing out at the rate of $6 \text{ m}^3/\text{min}$ from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3} \cdot y^2 (3R - y)$ when the water is y meter deep. Find
 - (a) At what rate is the water level changing when the water is 8 m deep.
 - (b) At what rate is the radius of the water surface changing when the water is 8 m deep.

(Mathematics) APPLICATION OF DERIVATIVE

13. If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.
14. At time $t > 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 unit and at $t = 15$ the radius is 2 units.
- (a) Find the radius of the sphere as a function of time t .
- (b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$.
15. (i) Use differentials to approximate the values of ; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.
- (ii) If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

EXERCISE-III

1. The abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts off equal intercepts from the axes is-
- (A) 1 (B) $4a/3$ (C) 3 (D) $4a/9$
2. The distance of normal at any point on the curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ from origin is ($a > 0$)
- (A) a (B) $a/2$ (C) $2a$ (D) Not constant
3. The angle of intersection between the curve $y^2 = 16x$ and $2x^2 + y^2 = 4$ is-
- (A) 0° (B) 30° (C) 45° (D) 90°
4. The rate of change of the area of a circular disc with respect to its circumference when the radius is 3 units, is-
- (A) 1 (B) 4 (C) 3 (D) 2
5. The equation of normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec \theta, b \tan \theta)$ is-
- (A) $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$ (B) $\frac{ax}{\sec \theta} - \frac{by}{\tan \theta} = a^2 - b^2$
- (C) $\frac{ax}{\sec \theta} - \frac{by}{\tan \theta} = a - b$ (D) $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a + b$
6. If length of subnormal at any point of the curve $y^n = a^{n-1}x$ is constant, then n equals-
- (A) 1 (B) 3 (C) 2 (D) 0
7. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then-
- (A) $a, b \in \mathbb{R}$ (B) $a > 0, b > 0$
- (C) $a < 0, b > 0$ or $a > 0, b < 0$ (D) $a < 0, b < 0$
8. At what points the tangent line to the curve $y = \cos(x+y)$, $(-2\pi \leq x \leq 2\pi)$ is parallel to $x + 2y = 0$ -
- (A) $(\pi/2, 0)$ (B) $(-\pi/2, 0)$ (C) $(3\pi/2, 0)$ (D) $(-3\pi/2, 0)$
9. Tangents are drawn from origin to the curve $y = \sin x$, then point of contact lies on-
- (A) $x^2y^2 = y^2 - x^2$ (B) $x^2y^2 = x^2 - y^2$ (C) $x^2y^2 = x^2 + y^2$ (D) None of these

(Mathematics) APPLICATION OF DERIVATIVE

10. A man 2 metres tall, walks at a uniform speed of 6 metres per minute away from a 5 metres tall lamp post in a straight line. The rate at which the length of his shadow increases is –
(A) 1 metres/minute (B) 2 metres/minute (C) 4 metres/minute (D) 3 metres/minute
11. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall, at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?
(A) $\frac{4}{3}$ m/s (B) $\frac{8}{3}$ m/s (C) $\frac{10}{3}$ m/s (D) None of these
12. The curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ intersect orthogonally, then
(A) $\frac{1}{a} + \frac{1}{A} = \frac{1}{b} + \frac{1}{B}$ (B) $\frac{1}{a} - \frac{1}{A} = \frac{1}{b} - \frac{1}{B}$ (C) $\frac{1}{a} + \frac{1}{b} = \frac{1}{B} + \frac{1}{A}$ (D) none of these
13. For the parabola $y^2 = 4ax$, the ratio of the length of subtangent to the abscissa is -
(A) 1 : 1 (B) 2 : 1 (C) 1 : 2 (D) Not constant
14. Let the equation of a curve be defined by $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$. If θ changes at a constant rate k then the rate of change of the slope of the tangent to the curve at $\theta = \frac{\pi}{3}$ is -
(A) $\frac{2k}{\sqrt{3}}$ (B) $\frac{k}{\sqrt{3}}$ (C) $\sqrt{3} k$ (D) $\frac{2k}{3}$
15. On the curve $x^3 = 12y$ the abscissa changes at a faster rate than the ordinate. Then x belongs to the interval
(A) $(-2, 2)$ (B) $(-1, 1)$ (C) $(-\sqrt{2}, \sqrt{2})$ (D) none of these

EXERCISE-IV

1. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point $(2, 1)$ and at that point the tangent to the graph is $y = 3x - 5$, then the function, is- [AIEEE 2004]
(A) $(x - 1)^2$ (B) $(x - 1)^3$ (C) $(x + 1)^3$ (D) $(x + 1)^2$
2. The normal to the curve $x = a(1 + \cos \theta)$, $y = a \sin \theta$ at ' θ ' always passes through the fixed point- [AIEEE 2004]
(A) $(a, 0)$ (B) $(0, a)$ (C) $(0, 0)$ (D) (a, a)
3. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is [AIEEE-2004]
(A) $(2, 6)$ (B) $(2, -6)$ (C) $\left(\frac{9}{8}, \frac{-9}{2}\right)$ (D) $\left(\frac{9}{8}, \frac{9}{2}\right)$
4. The normal to the curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ at any point ' θ ' is such that -
(A) it passes through the origin (B) it makes angle $\frac{\pi}{2} - \theta$ with the x-axis
(C) it passes through $\left(a\frac{\pi}{2}, -a\right)$ (D) it is at a constant distance from the origin.

(Mathematics) APPLICATION OF DERIVATIVE

5. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 \text{ cm}^3/\text{min}$. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is - [AIEEE-2005]
 (A) $\frac{1}{36\pi} \text{ cm/min}$. (B) $\frac{1}{18\pi} \text{ cm/min}$. (C) $\frac{1}{54\pi} \text{ cm/min}$. (D) $\frac{5}{6\pi} \text{ cm/min}$.
6. Let f be differentiable for all x . If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then - [AIEEE-2005]
 (A) $f(6) \geq 8$ (B) $f(6) < 8$ (C) $f(6) < 5$ (D) $f(6) = 5$
7. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is - [AIEEE 2006]
 (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$
8. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, which is parallel to the x -axis, is [AIEEE-2010]
 (A) $y = 8$ (B) $y = 0$ (C) $y = 3$ (D) $y = 2$
9. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is [AIEEE-2012]
 (A) $\frac{9}{7}$ (B) $\frac{7}{9}$ (C) $\frac{2}{9}$ (D) $\frac{9}{2}$
10. The intercepts on x -axis made by tangents to the curve, $y = \int_0^x |t| dt, x \in \mathbb{R}$, which are parallel to the line $y = 2x$, are equal to : [JEE (Main)-2013]
 (A) ± 4 (B) ± 1 (C) ± 2 (D) ± 3
11. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1): [JEE (Main)-2015]
 (A) meets the curve again in the fourth quadrant (B) does not meet the curve again
 (C) meets the curve again in the second quadrant (D) meets the curve again in the third quadrant
12. Consider : $f(x) = \tan^{-1} \left(\sqrt{\frac{1+\sin x}{1-\sin x}} \right), x \in \left(0, \frac{\pi}{2} \right)$. A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point: [JEE (Main)-2016]
 (A) (0, 0) (B) $\left(0, \frac{2\pi}{3} \right)$ (C) $\left(\frac{\pi}{6}, 0 \right)$ (D) $\left(\frac{\pi}{4}, 0 \right)$
13. The normal to the curve $y(x-2)(x-3) = x+6$ at the point where the curve intersects the y -axis passes through the point : [JEE (Main)-2017]
 (A) $\left(\frac{1}{2}, -\frac{1}{3} \right)$ (B) $\left(\frac{1}{2}, \frac{1}{3} \right)$ (C) $\left(-\frac{1}{2}, -\frac{1}{2} \right)$ (D) $\left(\frac{1}{2}, \frac{1}{2} \right)$

14. If the curve $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :
[JEE (Main)-2018]
(A) $\frac{9}{2}$ (B) 6 (C) $\frac{7}{2}$ (D) 4
15. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is :
[JEE (Main)-2018]
(A) 95 (B) 195 (C) 185 (D) 85
16. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q . If these tangents intersect at the point $T(0, 3)$ then the area (in sq. units) of $\triangle PTQ$ is :
[JEE (Main)-2018]
(A) $36\sqrt{5}$ (B) $45\sqrt{5}$ (C) $54\sqrt{3}$ (D) $60\sqrt{3}$

EXERCISE-V

1. Find the equation of the straight line which is tangent at one point and normal at another point of the curve, $x = 3t^2$, $y = 2t^3$.
[REE 2000 (Mains) 5 out of 100]
2. If the normal to the curve, $y = f(x)$ at the point $(3, 4)$ makes an angle $\frac{3\pi}{4}$ with the positive x -axis. Then $f'(3) =$
(A) -1 (B) $-\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 1
[JEE 2000 (Scr.) 1 out of 35]
3. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are)
[JEE 2002 (Scr.), 3]
(A) $\left(\pm \frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm \sqrt{\frac{11}{3}}, 1\right)$ (C) $(0, 0)$ (D) $\left(\pm \frac{4}{\sqrt{3}}, 2\right)$
4. Tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q . Then the coordinates of Q are
(A) $(-6, -11)$ (B) $(-9, -13)$ (C) $(-10, -15)$ (D) $(-6, -7)$
[JEE 2005 (Scr.), 3]
5. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$
(A) on the left of $x = c$ (B) on the right of $x = c$
(C) at no point (D) at all points
[JEE 2007, 3]
6. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is
[JEE Advanced 2014]

MONOTONOCITY

(Significance of the sign of the first order derivative)

DEFINITIONS :

1. A function $f(x)$ is called an Increasing Function at a point $x = a$ if in a sufficiently small neighbourhood around

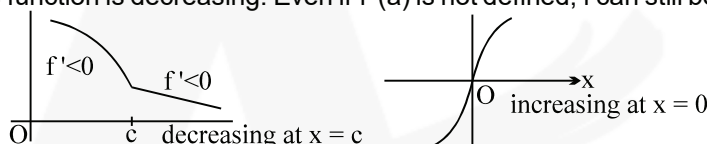
$$x = a \text{ we have } \left. \begin{array}{l} f(a+h) > f(a) \text{ and} \\ f(a-h) < f(a) \end{array} \right\} \text{ increasing;}$$

$$\text{Similarly decreasing if } \left. \begin{array}{l} f(a+h) < f(a) \text{ and} \\ f(a-h) > f(a) \end{array} \right\} \text{ decreasing.}$$

disregards whether f is non derivable or even discontinuous at $x = a$

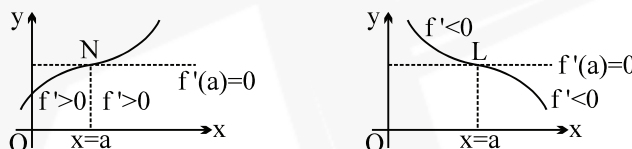
2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.
3. A function which in a given interval is increasing or decreasing is called “Monotonic” in that interval.
4. **Tests for increasing and decreasing of a function at a point :**

If the derivative $f'(x)$ is positive at a point $x = a$, then the function $f(x)$ at this point is increasing. If it is negative, then the function is decreasing. Even if $f'(a)$ is not defined, f can still be increasing or decreasing.



Note : If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a separate rule. e.g. $f(x) = x^3$ is increasing at every point.

Note that, $dy/dx = 3x^2$.



5. **Tests for Increasing & Decreasing of a function in an interval :**

SUFFICIENCY TEST : If the derivative function $f'(x)$ in an interval (a, b) is every where positive, then the function $f(x)$ in this interval is Increasing ;

If $f'(x)$ is every where negative, then $f(x)$ is Decreasing.

General Note :

- (1) If a continuous function is invertible it has to be either increasing or decreasing.
- (2) If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
- (3) If f is increasing in $[a, b]$ and is continuous then $f(b)$ is the greatest and $f(a)$ is the least value of f in $[a, b]$. Similarly if f is decreasing in $[a, b]$ then $f(a)$ is the greatest value and $f(b)$ is the least value.

6.

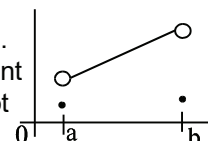
(a) **ROLLE'S THEOREM :**

Let $f(x)$ be a function of x subject to the following conditions :

- (i) $f(x)$ is a continuous function of x in the closed interval of $a \leq x \leq b$.
- (ii) $f'(x)$ exists for every point in the open interval $a < x < b$.
- (iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

Note that if f is not continuous in closed $[a, b]$ then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in (a, b) .



(b) **LMVT THEOREM :**

Let $f(x)$ be a function of x subject to the following conditions :

- (i) $f(x)$ is a continuous function of x in the closed interval of $a \leq x \leq b$.
- (ii) $f'(x)$ exists for every point in the open interval $a < x < b$.
- (iii) $f(a) \neq f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at $x = a$ & $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$. Note the following :

✎ Rolle's theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

Note : Now $[f(b) - f(a)]$ is the change in the function f as x changes from a to b so that $[f(b) - f(a)] / (b - a)$ is the *average rate of change* of the function over the interval $[a, b]$. Also $f'(c)$ is the actual rate of change of the function for $x = c$. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) APPLICATION OF ROLLES THEOREM FOR ISOLATING THE REAL ROOTS OF AN EQUATION $f(x)=0$

Suppose a & b are two real numbers such that ;

- (i) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \leq x \leq b$.
- (ii) $f(a)$ & $f(b)$ have opposite signs.
- (iii) $f'(x)$ is different from zero for all values of x between a & b .

Then there is one & only one real root of the equation $f(x) = 0$ between a & b .

PROFICIENCY TEST

- If $f(x) = x^5 - 20x^3 + 240x$, then $f(x)$ is
(A) Monotonic increasing everywhere (B) Monotonic decreasing only in $(0, \infty)$
(C) Monotonic decreasing everywhere (D) Monotonic increasing only in $(-\infty, 0)$
- The function $y = \frac{x}{\log x}$ increases in the interval
(A) $(-\infty, 0)$ (B) (e, ∞) (C) $(0, 1)$ (D) $(1, e)$
- For $x > 0$, which of the following statement is true
(A) $x < \log(1+x)$ (B) $x > \log(1+x)$ (C) $x \leq \log(1+x)$ (D) None of these
- The function $f(x) = 2\log(x-2) - x^2 + 4x + 1$ increases in the interval
(A) $(1, 2)$ (B) $(2, 3)$ (C) $(-\infty, 1)$ (D) $(3, \infty)$
- If $a < 0$ then function $(e^{ax} + e^{-ax})$ is monotonic decreasing when
(A) $x < 0$ (B) $x > 0$ (C) $x > 1$ (D) $x < 1$
- Function $f(x) = x^{100} + \sin x - 1$ is increasing in the interval
(A) $(0, 1)$ (B) $(-\pi/2, \pi/2)$ (C) $(-1, 1)$ (D) None of these
- In which interval $f(x) = 2x^2 - \log|x|$, ($x \neq 0$) is monotonically decreasing
(A) $(-1/2, 0) \cup (1/2, \infty)$ (B) $(-\infty, 0)$
(C) $(-\infty, -1/2) \cup (0, 1/2)$ (D) $(-\infty, -1/2) \cup (1/2, \infty)$
- If the domain of $f(x) = \sin x$ is $D = \{x : 0 \leq x \leq \pi\}$, then $f(x)$ is
(A) Increasing in D (B) Decreasing in D
(C) Decreasing in $[0, \pi/2]$ and increasing in $[\pi/2, \pi]$ (D) None of these
- Function $f(x) = \log(1+x) - \frac{2x}{1+x}$ is monotonic increasing when
(A) $-1 < x < 0$ (B) $x > 1$ (C) $x \in \mathbb{R}$ (D) $x \in (0, 1)$
- $f(x) = 2x - \tan^{-1} x - \log(x + \sqrt{1+x^2})$ is monotonic increasing for
(A) $x > 0$ (B) $x < 0$ (C) $x \in \mathbb{R}$ (D) $x \in \mathbb{R} \sim \{0\}$
- If $f(x) = g(x)(x - \lambda)^2$ where $g(\lambda) \neq 0$ and $g(x)$ is continuous at $x = \lambda$ then function $f(x)$
(A) increasing at x close to λ if $g(\lambda) > 0$ (B) decreasing at x close to λ if $g(\lambda) > 0$
(C) increasing at x close to λ if $g(\lambda) < 0$ (D) None of these
- Function $\cos^2 x + \cos^2(\pi/3 + x) - \cos x \cos(\pi/3 + x)$ for all real values of x will be
(A) Increasing (B) Constant (C) Decreasing (D) None of these
- The interval where $f(x) = x - e^x + \tan(2\pi/7)$ is increasing is
(A) $(0, \infty)$ (B) $(-\infty, 0)$ (C) $(-\infty, \infty)$ (D) None of these
- Let $y = x^2 e^{-x}$, then the interval in which y increases with respect to x is -
(A) $(-\infty, -2)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$
- The function $y = x^3 - 3x^2 + 6x - 17$
(A) Increases everywhere
(B) Decreases everywhere
(C) Increases for positive x and decreases for negative x
(D) Increases for negative x and decreases for positive x

EXERCISE-I

- Find the intervals of monotonicity for the following functions & represent your solution set on the number line.
 (a) $f(x) = 2e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$
 Also plot the graphs in each case & state their range.
- Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$
- Find the intervals of monotonicity of the functions in $[0, 2\pi]$
 (a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ (b) $g(x) = 2 \sin x + \cos 2x$ in $(0 \leq x \leq 2\pi)$.
 (c) $f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$
- Let $f(x)$ be a increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a .
- Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) : 0 \leq t \leq x\} & , 0 \leq x \leq 1 \\ 3 - x & , 1 < x \leq 2 \end{cases}$
 Discuss the conti. & differentiability of $g(x)$ in the interval $(0, 2)$.
- Find the set of all values of the parameter 'a' for which the function,
 $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in \mathbb{R}$ and has no critical points for all $x \in \mathbb{R}$.
- Find the greatest & the least values of the following functions in the given interval if they exist.
 (a) $f(x) = \sin^{-1} \frac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$ (b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$
 (c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$
- Find the values of 'a' for which the function $f(x) = \sin x - a \sin 2x - \frac{1}{3} \sin 3x + 2ax$ increases throughout the number line.
- If $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$ is monotonic increasing for every $x \in \mathbb{R}$ then find the range of values of 'a'.
- Find the set of values of 'a' for which the function,
 $f(x) = \left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \sqrt{7}$ is increasing at every point of its domain.
- Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x , prove that
 $\int_0^a g(x) dx + \int_0^b g(x) dx$ increases as $(b - a)$ increases.
- Find the range of values of 'a' for which the function $f(x) = x^3 + (2a + 3)x^2 + 3(2a + 1)x + 5$ is monotonic in \mathbb{R} . Hence find the set of values of 'a' for which $f(x)$ is invertible.
- Find the value of $x > 1$ for which the function
 $F(x) = \int_x^{x^2} \frac{1}{t} \ln\left(\frac{t-1}{32}\right) dt$ is increasing and decreasing.
- Find all the values of the parameter 'a' for which the function;
 $f(x) = 8ax - a \sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in \mathbb{R}$.

15. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for every $x \in \mathbb{R}$ then find the range of values of 'a'.
16. Prove that, $x^2 - 1 > 2x \ln x > 4(x - 1) - 2 \ln x$ for $x > 1$.
17. Prove that $\tan^2 x + 6 \ln \sec x + 2 \cos x + 4 > 6 \sec x$ for $x \in \left(\frac{3\pi}{2}, 2\pi\right)$.
18. If $0 < x < 1$ prove that $y = x \ln x - (x^2/2) + (1/2)$ is a function such that $d^2y/dx^2 > 0$. Deduce that $x \ln x > (x^2/2) - (1/2)$.
19. Find the set of values of x for which the inequality $\ln(1+x) > x/(1+x)$ is valid.
20. If $b > a$, find the minimum value of $|(x-a)^3| + |(x-b)^3|$, $x \in \mathbb{R}$.
21. Discuss the monotonicity of function $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2) \forall x \in \mathbb{R}$ where $f''(x) > 0 \forall x \in \mathbb{R}$.
22. Find a for $f(x) = \sin^3 x - a \sin^2 x$ should not have any critical point in $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.
23. Let $f(x) = (b^2 + (a-1)b + 2)x + \int (\sin^2 x + \cos^4 x) dx$. If $f(x)$ be strictly increasing function $\forall x \in \mathbb{R}$ and for all real values of b , find a .
24. Let $f(x) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \ln(x^2 + x + 1) + (b^2 - 5b + 3)x + c$ is strictly decreasing for all real values of x , find b .
25. Find a for which $f(x) = \log_a(4ax - x^2)$ is strictly increasing $\forall x \in \left[\frac{3}{2}, 2\right]$.
26. If $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, $A > 0$ (constant). Find the smallest A such that $f(x) \geq 24 \forall x > 0$.
27. If $ax^2 + \frac{b}{x} \geq c \forall x > 0$, where $a, b > 0$, then show that $27ab^2 \geq 4c^3$.

EXERCISE-II

1. Verify Rolles thorem for $f(x) = (x-a)^m(x-b)^n$ on $[a, b]$; m, n being positive integer.
2. Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c , $0 < c < 1$ such that $f(c) = 0$.
3. Using LMVT prove that : (a) $\tan x > x$ in $\left(0, \frac{\pi}{2}\right)$, (b) $\sin x < x$ for $x > 0$
4. Let f be continuous on $[a, b]$ and assume the second derivative f'' exists on (a, b) . Suppose that the graph of f and the line segment joining the point $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.
5. Prove that if f is differentiable on $[a, b]$ and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

(Mathematics) APPLICATION OF DERIVATIVE

6. For what value of a , m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \leq x \leq 2 \end{cases}$ satisfy the hypothesis of the mean value theorem for the interval $[0, 2]$.
7. Assume that f is continuous on $[a, b]$, $a > 0$ and differentiable on an open interval (a, b) . Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.
8. Let f, g be differentiable on \mathbb{R} and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.
9. Let f be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.
10. Let f defined on $[0, 1]$ be a twice differentiable function such that, $|f''(x)| \leq 1$ for all $x \in [0, 1]$. If $f(0) = f(1)$, then show that, $|f'(x)| < 1$ for all $x \in [0, 1]$.
11. $f(x)$ and $g(x)$ are differentiable functions for $0 \leq x \leq 2$ such that $f(0) = 5, g(0) = 0, f(2) = 8, g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.
12. If f, ϕ, ψ are continuous in $[a, b]$ and derivable in $]a, b[$ then show that there is a value of c lying between a & b such that,
- $$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \Psi(a) & \Psi(b) & \Psi'(c) \end{vmatrix} = 0$$
13. Assume $|f''(x)| \leq m$ for each x in interval $[0, a]$ and assume f takes on its largest value at an interior point of this interval. Show that $|f'(0)| + |f'(a)| \leq a m$. Assume $f''(x)$ is continuous in $[0, a]$.
14. Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.
15. Prove the inequality $e^x > (1+x)$ using LMVT for all $x \in \mathbb{R}_0$ and use it to determine which of the two numbers e^π and π^e is greater.
16. If $f'''(x)$ exists $\forall x \in \mathbb{R}$ such that $f(x) = f(6-x), f'(0) = 0 = f'(2) = f'(5)$. Find the minimum number of roots of equation $(f''(x))^2 + f'(x)f'''(x) = 0$ in interval $[0, 6]$.
17. Let $f: [0, 1] \rightarrow \mathbb{R}$ be continuous with $f(0) = f(1) = 0$. Assume that $f''(x)$ exists on $0 < x < 1$, with $f''(x) + 2f'(x) + f(x) \geq 0$. Show that $f(x) \leq 0$ for all $x \in [0, 1]$.

EXERCISE-III

1. Let $f'(x) > 0$ and $g'(x) < 0$ for all $x \in \mathbb{R}$, then comparing $f(g(x))$ with $f(g(x+1))$ we have
 (A) $f(g(x)) > f(g(x+1)) \forall x \in \mathbb{R}$ (B) $f(g(x)) < f(g(x+1)) \forall x \in \mathbb{R}$
 (C) $f(g(x)) > f(g(x+1)) \forall x > 0$ only (D) $f(g(x)) < f(g(x+1)) \forall x > 0$ only
2. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is an increasing function $\forall x \in \mathbb{R}$ if a and b satisfy the condition
 (A) $a^2 - 3b - 15 \geq 0$ (B) $a^2 - 3b + 15 \geq 0$ (C) $a^2 - 3b + 15 \leq 0$ (D) $a^2 - 3b - 15 \leq 0$

3. In which of the following intervals is function $f(x) = \cos(\pi/x)$ decreasing ?
- (A) $(2n, 2n+1), n \in \mathbb{N}$ (B) $\left(\frac{1}{2n+1}, \frac{1}{2n}\right), n \in \mathbb{N}$
- (C) $\left(\frac{1}{2n}, \frac{1}{2n-1}\right), n \in \mathbb{N}$ (D) none of these
4. The function $f(x) = x^x$ decreases in the interval -
- (A) $(0, e)$ (B) $(0, 1)$ (C) $(0, 1/e)$ (D) none of these
5. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ has, in the interval $(0, 1)$,
- (A) Exactly one root (B) Atleast one root
- (C) Atmost one root (D) No root.
6. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval -
- (A) $(0, 1)$ (B) $(1, 2)$ (C) $(2, 3)$ (D) none of these
7. If $f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}$, where $0 < a < b < \pi/2$, then the equation $f'(x) = 0$ has, in the interval (a, b) -
- (A) Atleast one root (B) Atmost one root (C) No root (D) None of these
8. If $\tan(\pi \cos \theta) = \cot(\pi \sin \theta)$, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$; and $f(x) = (\sin \theta + \cos \theta)^x$, then $f(x)$ is
- (A) increasing $\forall x \in \mathbb{R}$ (B) decreasing $\forall x \in \mathbb{R}$
- (C) increasing $\forall x > 0$ only (D) decreasing $\forall x > 0$ only
9. Let $f(x) = ax + \sin 2x + b$, then $f(x) = 0$ has
- (A) exactly one positive real root, if $a > 2$ and $b < 0$
- (B) exactly one positive real root, if $a > 2$ and $b > 0$
- (C) Both (A) & (B)
- (D) None of these
10. If $f(x)$ is a twice differentiable real valued function such that $f''(x) - f'(x) > 1 \forall x \geq 0$ and $f'(0) = -1$, then $\forall x > 0$, $g(x) = f(x) + x$, is
- (A) decreasing function of x (B) increasing function of x
- (C) constant function (D) None of these

EXERCISE-IV

- A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched ? [AIEEE-2005]

interval	function
(A) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(B) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(C) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(D) $(-\infty, -4]$	$x^3 + 6x^2 + 6$
- If the equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x = 0$; $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + a_1 = 0$ has a positive root, which is - [AIEEE-2005]

(A) greater than α (B) smaller than α
 (C) greater than or equal to α (D) equal to α
- The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- [AIEEE 2007]

(A) $(\pi/4, \pi/2)$ (B) $(-\pi/2, \pi/4)$ (C) $(0, \pi/2)$ (D) $(-\pi/2, \pi/2)$
- A value of c for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is- [AIEEE 2007]

(A) $2\log_3 e$ (B) $\frac{1}{2} \log_e 3$ (C) $\log_3 e$ (D) $\log_e 3$
- How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? [AIEEE - 2008]

(A) 7 (B) 1 (C) 3 (D) 5
- Consider the function, $f(x) = |x - 2| + |x - 5|$, $x \in \mathbb{R}$. [AIEEE-2012]

Statement-1: $f'(4) = 0$
Statement-2: f is continuous in $[2, 5]$, differentiable in $(2, 5)$ and $f(2) = f(5)$.

(A) Statement-1 is false, Statement-2 is true.
 (B) Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1
 (C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for statement-1.
 (D) Statement-1 is true, Statement-2 is false.
- If f and g are differentiable functions in $[0, 1]$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in (0, 1)$ [JEE Main 2014]

(A) $f'(c) = 2g'(c)$ (B) $2f'(c) = g'(c)$ (C) $2f'(c) = 3g'(c)$ (D) $f'(c) = g'(c)$

EXERCISE-V

- 1.(a) For all $x \in (0, 1)$:
 (A) $e^x < 1 + x$ (B) $\log_e(1 + x) < x$ (C) $\sin x > x$ (D) $\log_e x > x$
- (b) Consider the following statements S and R :
 S : Both $\sin x$ & $\cos x$ are decreasing functions in the interval $(\pi/2, \pi)$.
 R : If a differentiable function decreases in an interval (a, b) , then its derivative also decreases in (a, b) .
 Which of the following is true ?
 (A) both S and R are wrong
 (B) both S and R are correct, but R is not the correct explanation for S
 (C) S is correct and R is the correct explanation for S
 (D) S is correct and R is wrong.
- (c) Let $f(x) = \int e^x (x - 1)(x - 2) dx$ then f decreases in the interval :
 (A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, +\infty)$
[JEE 2000 (Scr.) 1+1+1 out of 35]
- 2.(a) If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
 (A) increasing on $\left(-\frac{1}{2}, 1\right)$ (B) decreasing on $\left[-\frac{1}{2}, 1\right]$ (C) increasing on \mathbb{R} (D) decreasing on \mathbb{R}
- (b) Let $-1 \leq p \leq 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.
[JEE 2001, 1 + 5]
3. The length of a longest interval in which the function $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, is
 (A) $\pi/3$ (B) $\pi/2$ (C) $3\pi/2$ (D) π **[JEE 2002]**
4. (a) Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) \geq x$, $\forall x \in \left[0, \frac{\pi}{4}\right]$.
 (b) Let $f : [0, 4] \rightarrow \mathbb{R}$ be a differentiable function.
 (i) Show that there exist $a, b \in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$
 (ii) Show that there exist α, β with $0 < \alpha < \beta < 2$ such that

$$\int_0^4 f(t) dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$$
 [JEE 2003 (Mains), 4 + 4 out of 60]
5. (a) Let $f(x) = \begin{cases} x^\alpha / \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$
 (A) -2 (B) -1 (C) 0 (D) 1/2
- (b) If f is a strictly increasing function, then $\lim_{x \rightarrow 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to
 (A) 0 (B) 1 (C) -1 (D) 2 **[JEE 2004 (Scr)]**
6. If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem, prove that at least one root of $p(x)$ lies between $(45^{1/100}, 46)$. **[JEE 2004, 2 out of 60]**
7. If $f(x)$ is a twice differentiable function and given that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then **[JEE 2005 (Scr), 3]**
 (A) $f''(x) = 2$, for $\forall x \in (1, 3)$ (B) $f''(x) = f'(x) = 2$, for some $x \in (2, 3)$
 (C) $f''(x) = 3$, for $\forall x \in (2, 3)$ (D) $f''(x) = 2$, for some $x \in (1, 3)$

(Mathematics) APPLICATION OF DERIVATIVE

8.(a) Let $f(x) = 2 + \cos x$ for all real x .

Statement-1: For each real t , there exists a point ' c ' in $[t, t + \pi]$ such that $f'(c) = 0$.

because

Statement-2: $f(t) = f(t + 2\pi)$ for each real t .

(A) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

[JEE 2007, 3]

Paragraph :

[JEE 2007, 4+4+4]

8.(b) If a continuous function f defined on the real line \mathbf{R} , assumes positive and negative values in \mathbf{R} then the equation $f(x) = 0$ has a root in \mathbf{R} . For example, if it is known that a continuous function f on \mathbf{R} is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in \mathbf{R} .

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

(i) The line $y = x$ meets $y = ke^x$ for $k \leq 0$ at

(A) no point (B) one point (C) two points (D) more than two points

(ii) The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $1/e$ (B) 1 (C) e (D) $\log_e 2$

(iii) For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is

(A) $(0, 1/e)$ (B) $(1/e, 1)$ (C) $(1/e, \infty)$ (D) $(0, 1)$

Match the column.

[JEE 2007, 6]

8.(c) In the following $[x]$ denotes the greatest integer less than or equal to x .

Match the functions in **Column I** with the properties in **Column II**.

Column I

Column II

(A) $x | x |$

(P) continuous in $(-1, 1)$

(B) $\sqrt{|x|}$

(Q) differentiable in $(-1, 1)$

(C) $x + [x]$

(R) strictly increasing in $(-1, 1)$

(D) $|x - 1| + |x + 1|$

(S) non differentiable at least at one point in $(-1, 1)$

9. (a) Let the function $g : (-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2 \tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

9. (b) Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'(1/4) = 0$. Then

(A) $f''(x)$ vanishes at least twice on $[0, 1]$ (B) $f'(1/2) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(D) $\int_0^{1/2} f(t) e^{\sin \pi t} \, dt = \int_{1/2}^1 f(1-t) e^{\sin \pi t} \, dt$ [JEE 2008, 3 + 4]

10. For the function $f(x) = x \cos \frac{1}{x}$, $x \geq 1$,

[JEE 2009]

(A) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$ (B) $\lim_{x \rightarrow \infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$ (D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

11. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \rightarrow 0} \left(1 + \frac{p(x)}{x^2}\right) = 2$. Then the value of $p(2)$ is

[JEE 2009]

(Mathematics) APPLICATION OF DERIVATIVE

Paragraph for questions 12 to 14

[JEE 2010]

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

12. The real number s lies in the interval

- (A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

13. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

- (A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) $(9, 10)$ (D) $\left(0, \frac{21}{64}\right)$

14. The function $f'(x)$ is

(A) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

15. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is

[JEE 2011]

Paragraph for Question Nos. 16 to 17

[JEE 2012]

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t \right) f(t) dt$ for all $x \in (1, \infty)$

16. Which of the following is true ?

- (A) g is increasing on $(1, \infty)$ (B) g is decreasing on $(1, \infty)$
(C) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$ (D) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

17. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then

- (A) both P and Q are true (B) P is true and Q is false
(C) P is false and Q is true (D) both P and Q are false

18. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = \int_{\frac{1}{x}}^x e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$.

[JEE Advance 2014]

Then :

(A) $f(x)$ is monotonically increasing on $[1, \infty)$ (B) $f(x)$ is monotonically decreasing on $(0, 1)$

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$ (D) $f(2^x)$ is an odd function of x on \mathbb{R} .

19. For every pair of continuous functions $f, g : [0, 1] \rightarrow \mathbb{R}$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is(are) :

[JEE Advanced 2014]

- (A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$ (B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$ (D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

20. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is :

[JEE Advanced 2016]

- (A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

(Mathematics) APPLICATION OF DERIVATIVE

Answer Q. 21, Q. 22 and Q. 23 by appropriately matching the information given in the three columns of the following table. [JEE Advanced 2017]

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0, \infty)$

- Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
- Column 2 contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
- Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column-1	Column-2	Column-3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x \rightarrow \infty} f(x) = 0$	(P) f is increasing in $(0, 1)$
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x \rightarrow \infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x \rightarrow \infty} f'(x) = -\infty$	(R) f' is increasing in $(0, 1)$
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x \rightarrow \infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

21. Which of the following options is the only CORRECT combination?

- (A) (III) (iv) (P) (B) (I) (ii) (R) (C) (II) (iii) (S) (D) (IV) (i) (S)

22. Which of the following options is the only INCORRECT combination?

- (A) (II) (iv) (Q) (B) (III) (i) (R) (C) (I) (iii) (P) (D) (II) (iii) (P)

23. Which of the following options is the only CORRECT combination?

- (A) (I) (i) (P) (B) (II) (ii) (Q) (C) (IV) (iv) (S) (D) (III) (iii) (R)

24. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbf{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then :

[JEE Advanced 2017]

- (A) $0 < f'(1) \leq \frac{1}{2}$ (B) $f'(1) \leq 0$ (C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \leq 1$

25. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbf{R}$, and $f(0) = 1$, then

[JEE Advanced 2017]

- (A) $f(x) > e^{2x}$ in $(0, \infty)$ (B) $f(x)$ is decreasing in $(0, \infty)$
(C) $f(x)$ is increasing in $(0, \infty)$ (D) $f(x) < e^{2x}$ in $(0, \infty)$

26. For every twice differentiable function $f : \mathbf{R} \rightarrow [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is(are) TRUE? [JEE Advanced 2018]

- (A) There exist $r, s \in \mathbf{R}$, where $r < s$, such that f is one-one on the open interval (r, s)
(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$
(C) $\lim_{x \rightarrow \infty} f(x) = 1$
(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

27. Let the function $f : (0, \pi) \rightarrow \mathbf{R}$ be defined by

$$f(\theta) = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^4.$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1 \pi, \dots, \lambda_r \pi\}$,

where $0 < \lambda_1 < \dots < \lambda_r < 1$. Then the value of $\lambda_1 + \dots + \lambda_r$ is _____

[JEE Advanced 2020]

28. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

[JEE Advanced 2021]

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE** ?

(A) f is decreasing in the interval $(-2, -1)$ (B) f is increasing in the interval $(1, 2)$

(C) f is onto (D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Paragraph for Question Nos. 29 to 30

[JEE Advanced 2021]

Let $\psi_1 : [0, \infty) \rightarrow \mathbb{R}$, $\psi_2 : [0, \infty) \rightarrow \mathbb{R}$, $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, 8) \rightarrow \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x, x \geq 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, x \geq 0,$$

$$f(x) = \int_{-x}^x (|t| - t^2) e^{-t^2} dt, x > 0$$

and

$$g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, x > 0.$$

29. Which of the following statements is **TRUE** ?

(A) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

30. Which of the following statements is **TRUE** ?

(A) $\psi_1(x) \leq 1$, for all $x > 0$

(B) $\psi_2(x) \leq 1$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

MAXIMA - MINIMA

FUNCTIONS OF A SINGLE VARIABLE

HOW MAXIMA & MINIMA ARE CLASSIFIED

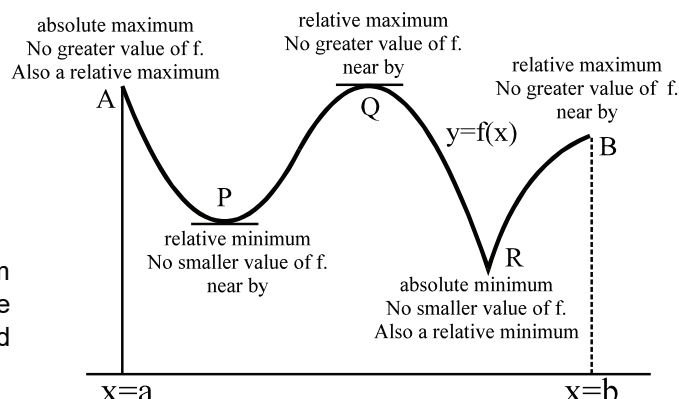
1. A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$. Symbolically

$$\left[\begin{array}{l} f(a) > f(a+h) \\ f(a) > f(a-h) \end{array} \right] \Rightarrow x = a \text{ gives maxima for}$$

a sufficiently small positive h .

Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$. Symbolically if

$$\left[\begin{array}{l} f(b) < f(b+h) \\ f(b) < f(b-h) \end{array} \right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive } h.$$



Note that :

- (i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.
- (ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.
- (iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
- (iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.
- (v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

2. A NECESSARY CONDITION FOR MAXIMUM & MINIMUM :

If $f(x)$ is a maximum or minimum at $x = c$ & if $f'(c)$ exists then $f'(c) = 0$.

Note :

- (i) The set of values of x for which $f'(x) = 0$ are often called as stationary points or critical points. The rate of change of function is zero at a stationary point.
- (ii) In case $f'(c)$ does not exist $f(c)$ may be a maximum or a minimum & in this case left hand and right hand derivatives are of opposite signs.
- (iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval $[a, b]$ are $f(a)$ or $f(b)$ or are given by the values of x for which $f'(x) = 0$.
- (iv) Critical points are those where $\frac{dy}{dx} = 0$, if it exists, or it fails to exist either by virtue of a vertical tangent or by virtue of a geometrical sharp corner but not because of discontinuity of function.

3. SUFFICIENT CONDITION FOR EXTREME VALUES :

$$\left[\begin{array}{l} f'(c-h) > 0 \\ f'(c+h) < 0 \end{array} \right] \Rightarrow x = c \text{ is a point of local maxima, where } f'(c) = 0.$$

$$\left[\begin{array}{l} f'(c-h) < 0 \\ f'(c+h) > 0 \end{array} \right] \Rightarrow x = c \text{ is a point of local minima, where } f'(c) = 0.$$

h is a sufficiently small positive quantity

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c , then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f .

4. USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA:

- (a) $f(c)$ is a minimum value of the function f , if $f'(c) = 0$ & $f''(c) > 0$.
- (b) $f(c)$ is a maximum value of the function f , $f'(c) = 0$ & $f''(c) < 0$.

Note : if $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaining the maxima or minima.

(Mathematics) APPLICATION OF DERIVATIVE

5. SUMMARY–WORKING RULE :

FIRST :

When possible , draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND :

Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

THIRD :

If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

FOURTH :

Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

- (a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow y$ is minimum.
 If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow y$ is maximum.
 If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.
- (b) If $\frac{dy}{dx}$ is $\left. \begin{array}{l} \text{positive for } x < x_0 \\ \text{zero for } x = x_0 \\ \text{negative for } x > x_0 \end{array} \right\} \Rightarrow \text{a maximum occurs at } x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called **INFLECTION POINTS**.

FIFTH :

If the function $y = f(x)$ is defined for only a limited range of values $a \leq x \leq b$ then examine $x = a$ & $x = b$ for possible extreme values.

SIXTH :

If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Important Note :

- Given a fixed point $A(x_1, y_1)$ and a moving point $P(x, f(x))$ on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.
- If the sum of two positive numbers x and y is constant then their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

$$xy = \frac{1}{4} [(x + y)^2 - (x - y)^2]$$
- If the product of two positive numbers is constant then their sum is least if they are equal. i.e. $(x + y)^2 = (x - y)^2 + 4xy$

6. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

- ☞ Volume of a cuboid = lbh .
- ☞ Surface area of a cuboid = $2(lb + bh + hl)$.
- ☞ Volume of a prism = area of the base x height.
- ☞ Lateral surface of a prism = perimeter of the base x height.
- ☞ Total surface of a prism = lateral surface + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).
- ☞ Volume of a pyramid = $\frac{1}{3}$ area of the base x height.
- ☞ Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).
- ☞ Volume of a cone = $\frac{1}{3} \pi r^2 h$.
- ☞ Curved surface of a cylinder = $2 \pi rh$.
- ☞ Total surface of a cylinder = $2 \pi rh + 2 \pi r^2$.
- ☞ Volume of a sphere = $\frac{4}{3} \pi r^3$.
- ☞ Surface area of a sphere = $4 \pi r^2$.
- ☞ Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

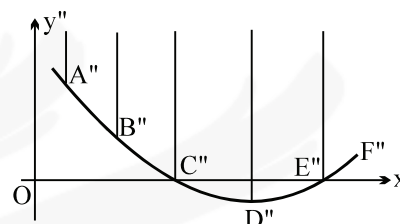
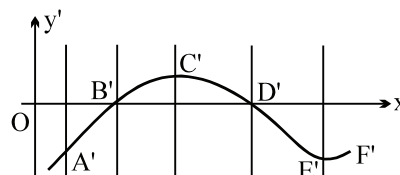
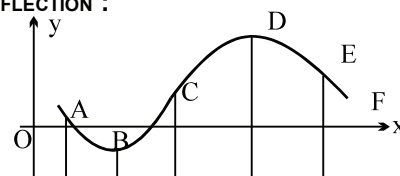
The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:

(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ &

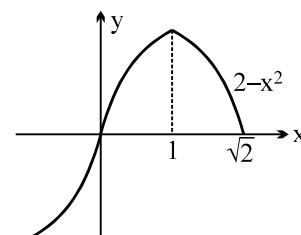
$\frac{d^2y}{dx^2}$ changes sign.



Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist. For example, consider the graph of the function defined as,

$$f(x) = \begin{cases} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{cases}$$

Note that the graph exhibits two critical points one is a point of local maximum & the other a point of inflection.



PROFICIENCY TEST-1

1. A double differentiable function $f(x)$ has a local maximum at $x = c$ if
 (A) $f'(c) = 0, f''(c) > 0$ (B) $f''(c) = 0, f''(c) < 0$
 (C) $f'(c) \neq 0, f''(c) = 0$ (D) None of these
2. $f(x)$ has a local maximum at $x = c$ if about $x = c$
 (A) $f'(x)$ changes sign from +ve to -ve (B) $f'(x)$ changes sign from -ve to +ve
 (C) $f'(x)$ does not change sign (D) None of these
3. Which of the following function has maximum value at $x = 0$
 (A) x^2 (B) $-x^2$ (C) $|x|$ (D) $[x]$
4. The abscissa of a local maximum of $\sec x$ is
 (A) $x = 0$ (B) $x = \pi/2$ (C) $x = \pi$ (D) $x = 3\pi/2$
5. $x^3 - 3x + 4$ has a local minimum at
 (A) $x = 1$ (B) $x = -1$ (C) $x = 0$ (D) No where
6. The value of $2x^3 - 9x^2 + 100$ at a local maximum is
 (A) 0 (B) 100 (C) 3 (D) 30
7. If $f(x) = x^3 - kx + 7$ has a local maximum at $x = -1$, then the value of k is
 (A) 3 (B) 6 (C) -3 (D) -6
8. Which of the following function has no extremum
 (A) 2^x (B) $[x]$ (C) $\log_{10} x$ (D) All these functions
9. Let $f(x) = |x|$, then
 (A) $f'(0) = 0$ (B) $f(x)$ has a maximum at $x = 0$
 (C) $f(x)$ has a minimum at $x = 0$ (D) $f(x)$ has no maximum and no minimum
10. The function $f(x) = \sum_{k=1}^5 (x-K)^2$ assumes minimum value for x equal to
 (A) 5 (B) 3 (C) $5/2$ (D) 2
11. In $x \in [0, 2]$, a local maximum of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ occurs at
 (A) $x = 0$ (B) $x = 1$ (C) $x = 1/2$ (D) None of these
12. If $f(x)$ is a differentiable function and $f'(x)$ changes sign from negative to positive as x is varied from $(c - h)$ to $(c + h)$, where h is a small positive quantity, then
 (A) $f(c)$ is neither a local maximum nor a local minimum value of $f(x)$
 (B) $f(c)$ is a local maximum value of $f(x)$
 (C) $f(c)$ is a local minimum value of $f(x)$
 (D) None of these
13. If $f'(c) < 0$ and $f''(c) > 0$, then at $x = c$, $f(x)$ has
 (A) local maximum (B) local minimum
 (C) neither local maximum nor local minimum (D) None of these

14. For what value of k , the function : $f(x) = kx^2 + \frac{2k^2 - 81}{2}x - 12$, is maximum at $x = 9/4$ -
 (A) $9/2$ (B) -9 (C) $-9/2$ (D) 9
15. The function $f(x) = a \sin x + \frac{1}{3} \sin 3x$ has a maximum at $x = \pi/3$, then a equals
 (A) -2 (B) 2 (C) -1 (D) 1
16. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at $x = 3$ and maximum at $x = -1$, then
 (A) $a = -3, b = -9, c = 0$ (B) $a = 3, b = 9, c = 0$
 (C) $a = -3, b = -9, c \in \mathbb{R}$ (D) None of these
17. The minimum value of function $f(x) = x^2 \log x$ is
 (A) e^2 (B) $\frac{e}{2}$ (C) $-\frac{1}{e^2}$ (D) $-\frac{1}{2e}$
18. The sum of two positive numbers is 6. The minimum value of the sum of their reciprocals is
 (A) 3 (B) 6 (C) $2/3$ (D) $6/5$
19. Divide 10 into two non-negative parts such that sum of double of one part and square of the other part is minimum, then the non-negative difference between the two parts is
 (A) 8 (B) 6 (C) 4 (D) 2
20. If $xy = 4$ and $x < 0$ then maximum value of $x + 16y$ is
 (A) -4 (B) -8 (C) -12 (D) -16
21. The point on the line $y = x$ such that the sum of the squares of its distance from the point $(a, 0)$, $(-a, 0)$ and $(0, b)$ is minimum will be
 (A) $(a/6, a/6)$ (B) (a, a) (C) (b, b) (D) $(b/6, b/6)$

PROFICIENCY TEST-2

1. Function $x - \sin x$ has
 (A) A maximum and no minimum (B) A minimum and no maximum
 (C) A maximum and a minimum (D) No maximum and no minimum
2. If $f(x) = x^3 - 3x^2 + 3x + 7$, then
 (A) $f(x)$ has a maximum at $x = 1$ (B) $f(x)$ has a minimum at $x = 1$
 (C) $f(x)$ has a point of inflection at $x = 1$ (D) None of these
3. The minimum value of $\frac{x}{\log x}$, ($x > 1$) is
 (A) e (B) $1/e$ (C) 0 (D) Does not exist
4. For what value of x , $x^2 \log(1/x)$ is maximum
 (A) $e^{-1/2}$ (B) $e^{1/2}$ (C) e (D) e^{-1}
5. For $f(x) = \sqrt{3} \sin x + 3 \cos x$, a local maximum occurs at
 (A) $x = \pi/6$ (B) $x = \pi/3$ (C) $x = \pi/4$ (D) $x = \pi/12$
6. The minimum value of $y = x(\log x)^2$ is -
 (A) 0 (B) 1 (C) 2 (D) None of these
7. For $x \in (-2, 2)$, the minimum value of $x^3 - 3x + 4$ is -
 (A) 0 (B) 1 (C) 2 (D) 3
8. The least value of $f(x) = x^3 - 12x^2 + 45x$ in $x \in [0, 7]$ is -
 (A) 0 (B) 50 (C) 45 (D) 54
9. If $0 \leq x \leq \pi$, then maximum value of $y = (1 + \sin x) \cos x$ is -
 (A) $\frac{2 + \sqrt{3}}{4}$ (B) $\frac{2 + \sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{4}$ (D) None of these
10. The maximum and minimum values of $y = xe^{-x}$ are (respectively)
 (A) $1/e$, Not Defined (B) $1/e$, 0 (C) 1 , $1/e$ (D) None of these
11. The point of inflection on the curve $y = x^{5/2}$ is
 (A) $(1, 1)$ (B) $(0, 0)$ (C) $(4, 32)$ (D) None of these
12. The area of a rectangle of maximum area inscribed in a circle of radius a is
 (A) $3a^2$ (B) a^2 (C) $2a^2$ (D) $4a^2$

EXERCISE-I

1. A cubic $f(x)$ vanishes at $x = -2$ & has relative minimum/maximum at $x = -1$ and $x = 1/3$.

If $\int_{-1}^1 f(x) dx = \frac{14}{3}$, find the cubic $f(x)$.

2. Investigate for maxima & minima for the function, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2] dt$

3. Find the greatest & least value for the function ;

(a) $y = x + \sin 2x$, $0 \leq x \leq 2\pi$

(b) $y = 2 \cos 2x - \cos 4x$, $0 \leq x \leq \pi$

4. Suppose $f(x)$ is real valued polynomial function of degree 6 satisfying the following conditions ;

(a) f has minimum value at $x = 0$ and 2

(b) f has maximum value at $x = 1$

(c) for all x , $\lim_{x \rightarrow 0} \frac{1}{x} \ln \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$.

Determine $f(x)$.

5. Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α .
6. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.
7. The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length ' $2r$ ' and width ' s '. If the surface area A of the pool is fixed, for what value of ' r ' and ' s ' the perimeter ' P ' of the pool is minimum.
8. For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1} \frac{1}{\sqrt{3}}$.
9. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2 + 3}$, find the equations of the tangent lines of minimum and maximum slope.
10. A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to have the most favourable view of statue.
11. By the post office regulations, the combined length & girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.
12. A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end. If the area of the rectangular portion is to be maximum, find the length of its sides.
Use : $\pi \approx 22/7$.
13. A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semicircle. The semicircular portion is fitted with coloured glass while the rectangular part is fitted with clean glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?
14. A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

(Mathematics) APPLICATION OF DERIVATIVE

15. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.
16. A trapezium ABCD is inscribed into a semicircle of radius r so that the base AD of the trapezium is a diameter and the vertices B & C lie on the circumference. Find the base angle θ of the trapezium ABCD which has the greatest perimeter.
17. If $y = \frac{ax + b}{(x-1)(x-4)}$ has a turning value at $(2, -1)$ find a & b and show that the turning value is a maximum.
18. If r is a real number then find the smallest possible distance from the origin $(0, 0)$ to the vertex of the parabola whose equation is $y = x^2 + rx + 1$.
19. A sheet of poster has its area 18 m^2 . The margin at the top & bottom are 75 cms and at the sides 50 cms. What are the dimensions of the poster if the area of the printed space is maximum?
20. A perpendicular is drawn from the centre to a tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.
21. Consider the function, $F(x) = \int_{-1}^x (t^2 - t) dt$, $x \in \mathbb{R}$.
 - (a) Find the x and y intercept of F if they exist.
 - (b) Derivatives $F'(x)$ and $F''(x)$.
 - (c) The intervals on which F is an increasing and the intervals on which F is decreasing.
 - (d) Relative maximum and minimum points.
 - (e) Any inflection point.
22. A beam of rectangular cross section must be sawn from a round log of diameter d . What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.
23. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.
24. The flower bed is to be in the shape of a circular sector of radius r & central angle θ . If the area is fixed & perimeter is minimum, find r and θ .
25. The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

EXERCISE-II

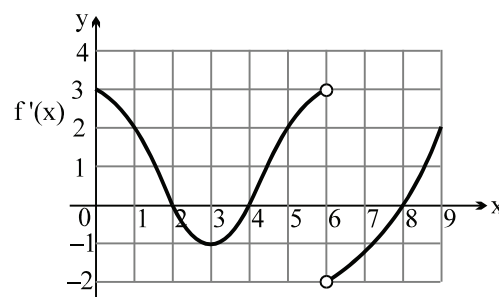
1. The mass of a cell culture at time t is given by, $M(t) = \frac{3}{1+4e^{-t}}$
 - (a) Find $\lim_{t \rightarrow -\infty} M(t)$ and $\lim_{t \rightarrow \infty} M(t)$
 - (b) Show that $\frac{dM}{dt} = \frac{1}{3}M(3-M)$
 - (c) Find the maximum rate of growth of M and also the value of t at which occurs.
2. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length l of the median drawn to its lateral side.
3. From a fixed point A on the circumference of a circle of radius ' a ', let the perpendicular AY fall on the tangent at a point P on the circle, prove that the greatest area which the $\triangle APY$ can have is $3\sqrt{3} \frac{a^2}{8}$ sq. units.
4. Given two points $A(-2, 0)$ & $B(0, 4)$ and a line $y = x$. Find the co-ordinates of a point M on this line so that the perimeter of the $\triangle AMB$ is least.
5. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum, the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.
6. Let α, β be real numbers with $0 \leq \alpha \leq \beta$ and $f(x) = x^2 - (\alpha + \beta)x + \alpha\beta$ such that $\int_{-1}^1 f(x) dx = 1$. Find the maximum value of $\int_0^\alpha f(x) dx$.
7. Let $f(x) = x^3 - 3x + a$, $a \in (0, 2)$ has 3 distinct real roots x_1, x_2, x_3 , find $\{x_1\} + \{x_2\} + \{x_3\}$, where $\{.\}$ denote fraction part function.
8. For $a > 0$, find the minimum value of the integral $\int_0^{1/a} (a^3 + 4x - a^5 x^2) e^{ax} dx$.
9. Consider the function $f(x) = \begin{cases} \sqrt{x} \ln x & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$
 - (a) Find whether f is continuous at $x = 0$ or not.
 - (b) Find the minima and maxima if they exist.
 - (c) Does $f'(0)$? Find $\lim_{x \rightarrow 0} f'(x)$.
 - (d) Find the inflection points of the graph of $y = f(x)$.
10. Consider the function $y = f(x) = \ln(1 + \sin x)$ with $-2\pi \leq x \leq 2\pi$. Find
 - (a) the zeroes of $f(x)$
 - (b) inflection points if any on the graph
 - (c) local maxima and minima of $f(x)$
 - (d) asymptotes of the graph
 - (e) sketch the graph of $f(x)$ and compute the value of the definite integral $\int_{-\pi/2}^{\pi/2} f(x) dx$.

(Mathematics) APPLICATION OF DERIVATIVE

11. The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$. If

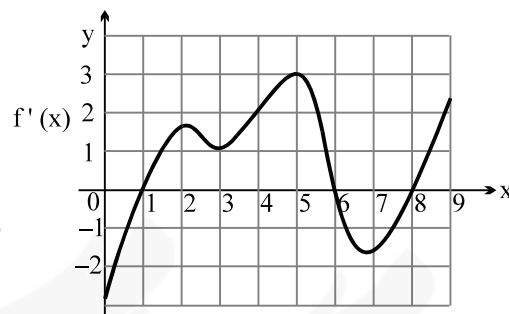
- f is monotonic increasing in the interval $[a, b) \cup (c, d) \cup (e, f]$ and decreasing in $(p, q) \cup (r, s)$.
- f has a local minima at $x = x_1$ and $x = x_2$.
- f is concave up in $(l, m) \cup (n, t]$
- f has inflection point at $x = k$
- number of critical points of $y = f(x)$ is 'w'.

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (l + m + n) + (x_1 + x_2) + (k + w)$.



12. The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$

- On what intervals is f increasing or decreasing?
- At what values of x does f have a local maximum or minimum?
- On what intervals is f concave upward or downward?
- State the x -coordinate(s) of the point(s) of inflection.
- Assuming that $f(0) = 0$, sketch a graph of f .



13. Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

14. Find the positive value of k for the value of the definite integral $\int_0^{\pi/2} |\cos x - kx| dx$ is minimised.

15. A cylinder is obtained by revolving a rectangle about the x -axis, the base of the rectangle lying on the x -axis and the entire rectangle lying in the region between the curve

$$y = \frac{x}{x^2 + 1} \text{ \& the } x\text{-axis. Find the maximum possible volume of the cylinder.}$$

16. The value of 'a' for which $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

17. What is the radius of the smallest circular disk large enough to cover every acute isosceles triangle of a given perimeter L ?

18. Find the magnitude of the vertex angle ' α ' of an isosceles triangle of the given area 'A' such that the radius 'r' of the circle inscribed into the triangle is the maximum.

19. The function $f(x)$ defined for all real numbers x has the following properties

- $f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find
 - the intervals on which f is increasing and decreasing and any local maximum or minimum values.
 - the intervals on which the graph f is concave down and concave up.
 - the function $f(x)$ and plot its graph.

20. Use calculus to prove the inequality, $\sin x \geq 2x/\pi$ in $0 \leq x \leq \pi/2$.

Use this inequality to prove that, $\cos x \leq 1 - x^2/\pi$ in $0 \leq x \leq \pi/2$.

EXERCISE-III

- The correct statement is
(A) $f(c)$ is an extremum of $f(x)$ if $f'(c) = 0$
(B) If $f(c)$ is an extremum of $f(x)$ then $f'(c) = 0$
(C) Both (A) and (B)
(D) None of these
- If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals
(A) $1/2$ (B) 3 (C) 1 (D) 2
- The minimum value of $a \sec x + b \operatorname{cosec} x$; $a, b > 0$; $0 < x < \pi/2$, is =
(A) $(a^2 + b^2)^{1/2}$ (B) $(a^3 + b^3)^{1/3}$ (C) $(a^{2/3} + b^{2/3})^{3/2}$ (D) $(a^{3/2} + b^{3/2})^{2/3}$
- If $xy = c^2$ then the minimum value of $ax + by$, ($x > 0, a > 0, b > 0, c > 0$), is
(A) $c\sqrt{ab}$ (B) $c\sqrt{ab}/2$ (C) $2c\sqrt{ab}$ (D) None of these
- The difference between two numbers is a ($a > 0$). If their product is minimum, then their sum is
(A) 0 (B) $a/2$ (C) a (D) $\sqrt{2}a$
- Sum of two positive numbers is 20 , such that product of cube of one number and square of the other is maximum, then non-negative difference between the two is
(A) 1 (B) 2 (C) 3 (D) 4
- A point on the curve $x^2 = 2y$ which is nearest to $(0, 5)$ is
(A) $(1, 0.5)$ (B) $(0, 0)$ (C) $(2, 2)$ (D) None of these
- The maximum distance of the point $(a, 0)$ from the curve $2x^2 + y^2 - 2x = 0$ is [$a \in (0, 1)$]
(A) $\sqrt{(1-2a+a^2)}$ (B) $\sqrt{(1+2a+2a^2)}$ (C) $\sqrt{(1+2a-a^2)}$ (D) $\sqrt{(1-2a+2a^2)}$
- The ratio between the height of a right circular cone of maximum volume inscribed in a sphere and the diameter of the sphere is
(A) $2 : 3$ (B) $3 : 4$ (C) $4 : 3$ (D) None of these
- The maximum distance from the origin of a point on the curve $x = a \sin t - b \sin\left(\frac{at}{b}\right)$, $y = a \cos t - b \cos\left(\frac{at}{b}\right)$, both $a, b > 0$, is
(A) $a - b$ (B) $a + b$ (C) $\sqrt{a^2 + b^2}$ (D) $\sqrt{a^2 - b^2}$

EXERCISE-IV

- Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is [AIEEE-2005]
 (A) $2ab$ (B) ab (C) \sqrt{ab} (D) a/b
- A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides, having fence, are of same length x . The maximum area enclosed by the park is [AIEEE-2006]
 (A) $\frac{3}{2}x^2$ (B) $\sqrt{\frac{x^3}{8}}$ (C) $\frac{1}{2}x^2$ (D) πx^2
- If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is- [AIEEE 2007]
 (A) 2 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$
- Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds ? [AIEEE 2008]
 (A) The cubic has minimum at $-\sqrt{\frac{p}{3}}$ and maximum at $\sqrt{\frac{p}{3}}$
 (B) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
 (C) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
 (D) The cubic has minimum at $\sqrt{\frac{p}{3}}$ and maximum at $-\sqrt{\frac{p}{3}}$
- Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$ - [AIEEE 2009]
 (A) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
 (B) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
 (C) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P
 (D) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P
- The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is - [AIEEE 2009]
 (A) $\frac{3\sqrt{2}}{8}$ (B) $\frac{2\sqrt{3}}{8}$ (C) $\frac{3\sqrt{2}}{5}$ (D) $\frac{\sqrt{3}}{4}$
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by [AIEEE - 2010]

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$
 If f has a local minimum at $x = -1$, then a possible value of k is
 (A) -1 (B) 1 (C) 0 (D) $-\frac{1}{2}$

(Mathematics) APPLICATION OF DERIVATIVE

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$. [AIEEE-2010]
- Statement-1:** $f(c) = 1/3$, for some $c \in \mathbb{R}$.
Statement-2: $0 < f(x) \leq 1/(2\sqrt{2})$, for all $x \in \mathbb{R}$.
 (A) Statement-1 is false, Statement-2 is true.
 (B) Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1
 (C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for statement-2.
 (D) Statement-1 is true, Statement-2 is false.
9. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has : [AIEEE - 2011]
- (A) local maximum at π and 2π . (B) local minimum at π and 2π
 (C) local minimum at π and local maximum at 2π . (D) local maximum at π and local minimum at 2π .
10. Let $a, b \in \mathbb{R}$ be such that the function f given by $f(x) = \ln |x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$. [AIEEE - 2012]
- Statement-1 :** f has local maximum at $x = -1$ and at $x = 2$.
Statement-2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$
 (A) Statement-1 is true, Statement-2 is true and Statement-2 is correct explanation for Statement-1
 (B) Statement-1 is true, Statement-2 is true and Statement-2 is **NOT** the correct explanation for Statement-1
 (C) Statement-1 is true, Statement-2 is false
 (D) Statement-1 is false, Statement-2 is true
11. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log |x| + \beta x^2 + x$ then [JEE Main 2014]
- (A) $\alpha = 2, \beta = \frac{1}{2}$ (B) $\alpha = -6, \beta = \frac{1}{2}$ (C) $\alpha = -6, \beta = -\frac{1}{2}$ (D) $\alpha = 2, \beta = -\frac{1}{2}$
12. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: [JEE Main 2016]
- (A) $2x = (\pi + 4)r$ (B) $(4 - \pi)x = \pi r$ (C) $x = 2r$ (D) $2x = r$
13. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is : [JEE Main 2017]
- (A) 25 (B) 30 (C) 12.5 (D) 10
14. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbb{R} - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is : [JEE Main 2018]
- (A) $2\sqrt{2}$ (B) 3 (C) -3 (D) $-2\sqrt{2}$

(Mathematics) APPLICATION OF DERIVATIVE

EXERCISE-V

- The function $f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$ has a local minimum at $x =$
(A) 0 (B) 1 (C) 2 (D) 3
[JEE '99 (Screening), 3]
- Find the co-ordinates of all the points P on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P.
[JEE '99, 10 out of 200]
- Find the normals to the ellipse $(x^2/9) + (y^2/4) = 1$ which are farthest from its centre.
[REE '99, 6]
- Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \leq 2 \\ 1 & \text{for } x = 0 \end{cases}$. Then at $x = 0$, 'f' has :
(A) a local maximum (B) no local maximum
(C) a local minimum (D) no extremum.
[JEE 2000 Screening, 1 out of 35]
- Find the area of the right angled triangle of least area that can be drawn so as to circumscribe a rectangle of sides 'a' and 'b', the right angle of the triangle coinciding with one of the angles of the rectangle.
[REE 2001 Mains, 5 out of 100]
- (a) Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is
(A) $[0, 1]$ (B) $\left(0, \frac{1}{2}\right]$ (C) $\left[\frac{1}{2}, 1\right]$ (D) $(0, 1]$
(b) The maximum value of $(\cos \alpha_1) \cdot (\cos \alpha_2) \cdots (\cos \alpha_n)$, under the restrictions
 $0 \leq \alpha_1, \alpha_2, \dots, \alpha_n \leq \frac{\pi}{2}$ and $\cot \alpha_1 \cdot \cot \alpha_2 \cdots \cot \alpha_n = 1$ is
(A) $\frac{1}{2^{n/2}}$ (B) $\frac{1}{2^n}$ (C) $\frac{1}{2n}$ (D) 1
[JEE 2001 Screening, 1 + 1 out of 35]
- If a_1, a_2, \dots, a_n are positive real numbers whose product is a fixed number e , the minimum value of $a_1 + a_2 + a_3 + \dots + a_{n-1} + 2a_n$ is
(A) $n(2e)^{1/n}$ (B) $(n+1)e^{1/n}$ (C) $2ne^{1/n}$ (D) $(n+1)(2e)^{1/n}$
[JEE 2002 Screening]
- (a) Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.
(b) For a circle $x^2 + y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum.
[JEE 2003, Mains, 2 + 2 out of 60]
- (a) Let $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$. Then f
(A) is bounded (B) has a local maxima
(C) has a local minima (D) is strictly increasing
[JEE 2004 (Scr.)]
(b) Prove that $\sin x + 2x \geq \frac{3x \cdot (x+1)}{\pi} \quad \forall x \in \left[0, \frac{\pi}{2}\right]$. (Justify the inequality, if any used).
[JEE 2004, 4 out of 60]
- If $P(x)$ be a polynomial of degree 3 satisfying $P(-1) = 10$, $P(1) = -6$ and $P(x)$ has maximum at $x = -1$ and $P'(x)$ has minima at $x = 1$. Find the distance between the local maximum and local minimum of the curve.
[JEE 2005 (Mains), 4 out of 60]

(Mathematics) APPLICATION OF DERIVATIVE

11.(a) If $f(x)$ is cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minima at $x = 0$, then

(A) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$.

(B) $f(x)$ is increasing for $x \in (1, 2\sqrt{5}]$

(C) $f(x)$ has local minima at $x = 1$

(D) the value of $f(0) = 5$

$$(b) f(x) = \begin{cases} e^x & 0 \leq x \leq 1 \\ 2 - e^{x-1} & 1 < x \leq 2 \\ x - e & 2 < x \leq 3 \end{cases} \text{ and } g(x) = \int_0^x f(t) dt, x \in [1, 3] \text{ then } g(x) \text{ has}$$

(A) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$

(B) local maxima at $x = 1$ and local minima at $x = 2$

(C) no local maxima

(D) no local minima

[JEE 2006, 5 marks each]

(c) If $f(x)$ is twice differentiable function such that $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$,

where $a < b < c < d < e$, then find the minimum number of zeros of $g(x) = (f'(x))^2 + f(x) \cdot f''(x)$ in the interval $[a, e]$.

[JEE 2006, 6]

12.(a) The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(A) 0

(B) 1

(C) 2

(D) 3

(b) **Comprehension:**

Consider the function $f: (-\infty, \infty) \rightarrow (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2$$

(i) Which of the following is true?

(A) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$

(B) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2-a)^2$

(D) $f'(1) f'(-1) = -(2+a)^2$

(ii) Which of the following is true?

(A) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$

(B) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x = 1$

(C) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum and nor a local minimum at $x = 1$.

(D) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum and nor a local minimum at $x = 1$.

(iii) Let $g(x) = \int_0^x \frac{f'(t)}{1+t^2} dt$

Which of the following is true?

(A) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(B) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(C) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(D) $g'(x)$ does not change sign on $(-\infty, 0)$ and $(0, \infty)$

[JEE 2008, 3 + 4 + 4 + 4]

13. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \leq 9x\}$ is

[JEE 2009]

(Mathematics) APPLICATION OF DERIVATIVE

14. Let f, g and h be real-valued functions defined on the interval $[0, 1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on $[0, 1]$, then
(A) $a = b$ and $c \neq b$ (B) $a = c$ and $a \neq b$ (C) $a \neq b$ and $c \neq b$ (D) $a = b = c$ [JEE 2010]
15. Let f be a function defined on \mathbf{R} (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$, for all $x \in \mathbf{R}$. If g is a function defined on \mathbf{R} with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in \mathbf{R}$, then the number of points in \mathbf{R} at which g has a local maximum is [JEE 2010]
16. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$ for all $x \in (0, \infty)$, then [JEE 2012]
(A) f has a local maximum at $x = 2$ (B) f is decreasing on $(2, 3)$
(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$ (D) f has a local minimum at $x = 3$
17. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is [JEE 2012]
18. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is [JEE 2012]

Comprehension (Q.19 to Q.20)

Let $f: [0, 1] \rightarrow \mathbf{R}$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$. [JEE (Adv.) 2013]

19. If the function $e^{-x}f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true ?
(A) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$ (C) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$ (D) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$
20. Which of the following is true for $0 < x < 1$?
(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2} < f(x) < \frac{1}{2}$ (C) $-\frac{1}{4} < f(x) < 1$ (D) $-\infty < f(x) < 0$
21. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at $x =$
(A) -2 (B) $-\frac{2}{3}$ (C) 2 (D) $\frac{2}{3}$ [JEE (Adv.) 2013]
22. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio $8 : 15$ is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100 , the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are : [JEE (Adv.) 2013]
(A) 24 (B) 32 (C) 45 (D) 60
23. A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of $V \text{ mm}^3$, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container. If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm , then the value of $\frac{V}{250\pi}$ is : [JEE Advance 2015]
24. Let $f: \mathbf{R} \rightarrow (0, \infty)$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on \mathbf{R} . Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$, and $g'(2) \neq 0$. If $\lim_{x \rightarrow 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then [JEE Advance 2016]
(A) f has a local minimum at $x = 2$ (B) f has a local maximum at $x = 2$
(C) $f''(2) > f(2)$ (D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbf{R}$

(Mathematics) APPLICATION OF DERIVATIVE

25. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \geq 3; \end{cases}$$

Then which of the following options is/are correct?

[JEE Advanced 2019]

- (A) f is increasing on $(-\infty, 0)$ (B) f is onto
(C) f' has a local maximum at $x = 1$ (D) f' is NOT differentiable at $x = 1$

26. Let $f(x) = \frac{\sin \pi x}{x^2}$, $x > 0$

Let $x_1 < x_2 < x_3 < \dots < x_n < \dots$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < \dots < y_n < \dots$ be all the points of local minimum of f .

Then which of the following options is/are correct?

[JEE Advanced 2019]

- (A) $x_1 < y_1$ (B) $x_{n+1} - x_n > 2$ for every n
(C) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n (D) $|x_n - y_n| > 1$ for every n

27. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = (x-1)(x-2)(x-5)$. Define $F(x) = \int_0^x f(t) dt$, $x > 0$.

Then which of the following options is/are correct?

[JEE Advanced 2019]

- (A) F has a local maximum at $x = 2$.
(B) F has two local maxima and one local minimum in $(0, \infty)$.
(C) $F(x) \neq 0$ for all $x \in (0, 5)$.
(D) F has a local minimum at $x = 1$.

Question Stem for Question No. 28 and 29

Question Stem

[JEE Advanced 2021]

Let $f_1: (0, \infty) \rightarrow \mathbb{R}$ and $f_2: (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j dt, \quad x > 0$$

$$\text{and } f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \quad x > 0,$$

Where for any positive integer n and real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, $\prod_{i=1}^n \alpha_i$ denotes the product of $\alpha_1, \alpha_2, \dots, \alpha_n$. Let m_i and n_i , respectively, denote the number of points of local minimum and the number of points of local maxima of function f_i , $i = 1, 2$, in the interval $(0, \infty)$.

28. The value of $2m_1 + 3n_1 + m_1n_1$ is _____

29. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____

ANSWER KEY

TANGENT & NORMAL

PROFICIENCY TEST-1

- Q.1 D Q.2 A Q.3 B Q.4 C Q.5 B Q.6 D Q.7 B
Q.8 B Q.9 C Q.10 A Q.11 B Q.12 A Q.13 A Q.14 C
Q.15 A

PROFICIENCY TEST-2

- Q.1 D Q.2 A Q.3 D Q.4 D Q.5 A Q.6 A Q.7 C Q.8 C
Q.9 D Q.10 C Q.11 C Q.12 C Q.13 B Q.14 B Q.15 D Q.16 C
Q.17 D Q.18 C Q.19 C Q.20 C

EXERCISE-I

- Q.1 $x + y - 1 = 0$ Q.2 $2\sqrt{3}x - y = 2(\sqrt{3} - 1)$ or $2\sqrt{3}x + y = 2(\sqrt{3} + 1)$
Q.3 $(0, 1)$ Q.4 $x = 1$ when $t = 1$, $m \rightarrow \infty$; $5x - 4y = 1$ if $t \neq 1$, $t = 1/3$
Q.5 $a = 1, b = \frac{-5}{2}$
Q.8 $T : x - 2y = 0$; $N : 2x + y = 0$ Q.9 $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$
Q.12 $a = 1$ Q.14 $-\frac{1}{x+2}$; $x - 4y = 2$ Q.15 $1/16$ Q.16 $a = -1/2$; $b = -3/4$; $c = 3$
Q.20 $2e^{-x/2}$ Q.22 $\frac{m\sqrt{m}}{\sqrt{2}}$ Q.23 (a) $n = -2$
Q.24 (b) $\pm \frac{1}{2\sqrt{2}}$ Q.25 $\theta = \tan^{-1} \frac{2}{C}$

EXERCISE-II

- Q.1 $1/9\pi$ m/min Q.2 (i) 6 km/h ; (ii) 2 km/hr Q.3 $(4, 11)$ & $(-4, -31/3)$
Q.4 $3/8\pi$ cm/min Q.5 $1 + 36\pi$ cu. cm/sec Q.6 $1/48\pi$ cm/s
Q.7 0.05 cm/sec Q.8 $\frac{\sqrt{2}}{4\pi}$ cm/s Q.9 $200\pi r^3 / (r+5)^2$ km²/h
Q.10 $\frac{66}{7}$ Q.11 $\frac{1}{4}$ cm/sec. Q.12 (a) $-\frac{1}{24\pi}$ m/min., (b) $-\frac{5}{288\pi}$ m/min.
Q.14 (a) $r = (1+t)^{1/4}$, (b) $t = 80$ Q.15 (i) (a) 6.05, (b) $\frac{80}{27}$; (ii) 9.72π cm³

EXERCISE-III

- Q.1 D Q.2 A Q.3 D Q.4 C Q.5 A Q.6 C Q.7 C Q.8 A Q.9 B Q.10 C
Q.11 B Q.12 B Q.13 B Q.14 D Q.15 A

EXERCISE-IV

- Q.1 B Q.2 A Q.3 D Q.4 D Q.5 B Q.6 A Q.7 A Q.8 C Q.9 C Q.10 B
Q.11 A Q.12 B Q.13 D Q.14 A Q.15 A Q.16 B

EXERCISE-V

- Q.1 $\sqrt{2}x + y - 2\sqrt{2} = 0$ or $\sqrt{2}x - y - 2\sqrt{2} = 0$ Q.2 D Q.3 D Q.4 D
Q.5 A Q.6 8

MONOTONOCITY

PROFICIENCY TEST

- Q.1 A Q.2 B Q.3 B Q.4 B Q.5 A Q.6 A Q.7 C
Q.8 D Q.9 B Q.10 C Q.11 A Q.12 B Q.13 B Q.14 D
Q.15 A

EXERCISE-I

- Q.1 (a) I in $(2, \infty)$ & D in $(-\infty, 2)$ (b) I in $(1, \infty)$ & D in $(-\infty, 0) \cup (0, 1)$
(c) I in $(0, 2)$ & D in $(-\infty, 0) \cup (2, \infty)$
(d) I for $x > \frac{1}{2}$ or $-\frac{1}{2} < x < 0$ & D for $x < -\frac{1}{2}$ or $0 < x < \frac{1}{2}$
Q.2 $(-2, 0) \cup (2, \infty)$
Q.3 (a) I in $[0, 3\pi/4) \cup (7\pi/4, 2\pi]$ & D in $(3\pi/4, 7\pi/4)$
(b) I in $[0, \pi/6) \cup (\pi/2, 5\pi/6) \cup (3\pi/2, 2\pi]$ & D in $(\pi/6, \pi/2) \cup (5\pi/6, 3\pi/2)$
(c) I in $[0, \pi/2) \cup (3\pi/2, 2\pi]$ and D in $(\pi/2, 3\pi/2)$
Q.4 $(0, 1/3) \cup (1, 5)$ Q.5 continuous but not diff. at $x = 1$ Q.6 $a < -(2 + \sqrt{5})$ or $a > \sqrt{5}$
Q.7 (a) $(\pi/6) + (1/2)\ln 3, (\pi/3) - (1/2)\ln 3$
(b) Maximum at $x = 1$ and $f(-1) = 18$; Minimum at $x = 1/8$ and $f(1/8) = -9/4$
(c) 2 & -10
Q.8 $[1, \infty)$ Q.9 $a \in (-\infty, -3] \cup [1, \infty)$ Q.10 $[-7, -1) \cup [2, 3]$
Q.12 $0 \leq a \leq \frac{3}{2}$ Q.13 \uparrow in $(3, \infty)$ and \downarrow in $(1, 3)$ Q.14 $(6, \infty)$ Q.15 $a \geq 0$

(Mathematics) APPLICATION OF DERIVATIVE

Q.19 $(-1, 0) \cup (0, \infty)$

Q.20 $(b - a)^{3/4}$

Q.21 Decreasing in $(-\infty, -2) \cup (0, 2)$, increasing in $(-2, 0) \cup (2, \infty)$

Q.22 $\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3\sqrt{3}}{4}, \infty\right)$

Q.23 $[1 - \sqrt{11}, 1 + \sqrt{11}]$

Q.24 $\left[\frac{5 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right]$

Q.25 $\left(\frac{1}{2}, \frac{3}{4}\right] \cup (1, \infty)$

Q.26 $2\left(\frac{24}{7}\right)^{7/2}$

EXERCISE-II

Q.1 $c = \frac{mb + na}{m + n}$ which lies between a & b

Q.6 $a = 3, b = 4$ and $m = 1$ Q.16 12

EXERCISE-III

- Q.1 A Q.2 C Q.3 C Q.4 C Q.5 B Q.6 A Q.7 A
Q.8 B Q.9 A Q.10 B

EXERCISE-IV

- Q.1 C Q.2 B Q.3 B Q.4 A Q.5 B Q.6 C Q.7 A

EXERCISE-V

- Q.1 (a) B ; (b) D ; (c) C Q.2 (a) A, (b) $\cos\left(\frac{1}{3}\cos^{-1}p\right)$ Q.3 A Q.5 (a) D; (b) C
Q.7 D Q.8 (a) B; (b) (i) B, (ii) A, (iii) A; (c) (A) P,Q,R; (B) P,S; (C) R,S; (D) P, Q
Q.9 (a) C, (b) A, B, C, D Q.10 B,C,D Q.11. $p(2) = 0$
Q.12. C Q.13. A Q.14. B Q.15. 2 Q.16. B Q.17. C
Q.18. A, C, D Q.19. A, D Q.20. C Q.21 C Q.22 B
Q.23 B Q.24 C Q.25 AC Q.26. ABD Q.27 0.50 Q.28. A,B
Q.29. C Q.30. D

MAXIMA - MINIMA

PROFICIENCY TEST-1

- Q.1. B Q.2. A Q.3. B Q.4. C Q.5. A Q.6. B Q.7. A
Q.8. D Q.9. C Q.10.B Q.11. C Q.12. C Q.13. C Q.14. B
Q.15. B Q.16. C Q.17. D Q.18. C Q.19. A Q.20. D Q.21. D

PROFICIENCY TEST-2

- Q.1. D Q.2. C Q.3. A Q.4. A Q.5. A Q.6. A Q.7. C
Q.8. A Q.9. C Q.10. A Q.11. D Q.12. C

EXERCISE-I

- Q.1 $f(x) = x^3 + x^2 - x + 2$ Q.2 max. at $x = 1$; $f(1) = 0$, min. at $x = 7/5$; $f(7/5) = -108/3125$
Q.3 (a) Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$
(b) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and
Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$
Q.4 $f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$ Q.5 $P_{\max} = a\left(1 + \operatorname{cosec} \frac{\alpha}{2}\right)$ Q.6 $75\sqrt{3}$ sq. units
Q.7 $r = \sqrt{\frac{2A}{\pi+4}}$, $s = \sqrt{\frac{2A}{\pi+4}}$ Q.9 $3x + 4y - 9 = 0$; $3x - 4y + 9 = 0$ Q.10 $4\sqrt{2}$ m Q.11 $1/\pi$ cu m
Q.12 $110'$, $70'$ Q.13 $6/(6+\pi)$ Q.14 side $10'$, height $10'$ Q.15 32 sq. units
Q.16 $\theta = 60^\circ$ Q.17 $a = 1$, $b = 0$ Q.18 $d_{\min} = \frac{\sqrt{3}}{2}$ when $r = \sqrt{2}$ or $-\sqrt{2}$
Q.19 width $2\sqrt{3}$ m, length $3\sqrt{3}$ m Q.20 $|a - b|$
Q.21 (a) $(-1, 0)$, $(0, 5/6)$;
(b) $F'(x) = (x^2 - x)$, $F''(x) = 2x - 1$, (c) increasing $(-\infty, 0) \cup (1, \infty)$, decreasing $(0, 1)$; (d) $(0, 5/6)$; $(1, 2/3)$;
(e) $x = 1/2$
Q.22 (a) $x = y = \frac{d}{\sqrt{2}}$, (b) $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}d$ Q.23 $6' \times 18'$
Q.24 $r = \sqrt{A}$, $\theta = 2$ radians Q.25 $\frac{4}{3\sqrt{3}}$

EXERCISE-II

- Q.1 (a) 0 , 3 , (c) $\frac{3}{4}$, $t = \ln 4$ Q.2 $\cos A = 0.8$ Q.4 $(0, 0)$
Q.6 $\frac{\sqrt{6}}{108}$ Q.7 1 Q.8 4 when $a = \sqrt{2}$
Q.9 (a) f is continuous at $x = 0$; (b) $-2/e$; (c) does not exist, does not exist; (d) pt. of inflection $x = 1$
Q.10 (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \pi/2$ and $-3\pi/2$ and no minima,

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(d) $x = 3\pi/2$ and $x = -\pi/2$, (e) $-\pi \ln 2$

Q.11 74

Q.12 (i) I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$; (ii) L.Min. at $x = 1$ and $x = 8$; L.Max. $x = 6$
(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$; (iv) $x = 2, 3, 5, 7$

Q.13 $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$

Q.14 $k = \frac{2\sqrt{2}}{\pi} \cos\left(\frac{\pi}{2\sqrt{2}}\right)$

Q.15 $\pi/4$

Q.16 320

Q.17 L/4

Q.18 $\pi/3$

Q.19 (a) increasing in $(0, 2)$ and decreasing in $(-\infty, 0) \cup (2, \infty)$, local min. value = 0 and local max. value = 2

(b) concave up for $(-\infty, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, \infty)$ and concave down in $(2 - \sqrt{2}, 2 + \sqrt{2})$

(c) $f(x) = \frac{1}{2}e^{2-x} \cdot x^2$

EXERCISE-III

Q.1 D

Q.2 D

Q.3 C

Q.4 C

Q.5 A

Q.6 D

Q.7 D

Q.8 D

Q.9 A

Q.10 B

EXERCISE-IV

Q.1. A

Q.2. C

Q.3. D

Q.4. D

Q.5. B

Q.6. A

Q.7. A

Q.8. B

Q.9. D

Q.10. A

Q.11. D

Q.12. C

Q.13. A

Q.14. A

EXERCISE-V

Q.1 B, D

Q.2 $\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}}$

Q.3 $\pm \sqrt{3}x \pm \sqrt{2}y = \sqrt{5}$

Q.4 A

Q.5 2ab

Q.6 (a) D; (b) A

Q.7 A

Q.8 (a) (2, 1); (b) 5

Q.9 (a) D

Q.10 $4\sqrt{65}$

Q.11 (a) B, C; (b) A (c) 6 solutions

Q.12 (a) C; (b) (i) A, (ii) A, (iii) B

Q.13 7

Q.14 D

Q.15 1

Q.16 A, B, C, D

Q.17 5

Q.18 9

Q.19 C

Q.20 D

Q.21 A, B

Q.22 A, C

Q.23 4

Q.24 A, D

Q.25 B, C, D

Q.26 B, C, D

Q.27 A, C, D

Q.28 57.00

Q.29 6.00