

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$$

$$p\alpha^4 + q\beta^4 = p(3\alpha + 2) + q(3\beta + 2) = p\left(\frac{3+3\sqrt{5}}{2} + 2\right) + q\left(\frac{3-3\sqrt{5}}{2} + 2\right)$$

$$= \frac{7}{2}(p+q) + \frac{3\sqrt{5}}{2}(p-q) = \underline{28}$$

$$\alpha^4 = (\alpha^2 - \alpha - 1)(\alpha^2 + \alpha + 2) + 3\alpha + 2 = 3\alpha + 2$$

$$p - q = 0$$

$$7p = 28$$

$$p = q = 4$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha^2 = \alpha + 1$$

$$\alpha^3 = \alpha^2 + \alpha = 2\alpha + 1$$

$$\alpha^4 = 3\alpha + 2$$

$$\alpha^5 = 4\alpha + 3$$

$$\alpha^6 = 7\alpha + 4$$

$$\alpha^7 = 11\alpha + 7$$

$$\alpha^8 = 18\alpha + 11$$

$$\alpha^9 = 28\alpha + 18$$

$$\alpha^{10} = 44\alpha + 28$$

$$\alpha^{11} = 69\alpha + 44$$

$$\alpha^{12} = 107\alpha + 69$$

$$\alpha^{13} = 166\alpha + 107$$

$$\alpha^{14} = 255\alpha + 166$$

$$\alpha^{15} = 394\alpha + 255$$

$$\alpha^{16} = 599\alpha + 394$$

$$\alpha^{17} = 898\alpha + 599$$

$$\alpha^{18} = 1367\alpha + 898$$

$$\alpha^{19} = 2066\alpha + 1367$$

$$\alpha^{20} = 3095\alpha + 2066$$

$$\alpha^{21} = 4612\alpha + 3095$$

$$\alpha^{22} = 6889\alpha + 4612$$

$$\alpha^{23} = 10287\alpha + 6889$$

$$\alpha^{24} = 15382\alpha + 10287$$

$$\alpha^{25} = 22769\alpha + 15382$$

$$\alpha^{26} = 34151\alpha + 22769$$

$$\alpha^{27} = 50840\alpha + 34151$$

$$\alpha^{28} = 75491\alpha + 50840$$

$$\alpha^{29} = 111932\alpha + 75491$$

$$\alpha^{30} = 167423\alpha + 111932$$

$$\alpha^{31} = 249355\alpha + 167423$$

$$\alpha^{32} = 367778\alpha + 249355$$

$$\alpha^{33} = 546103\alpha + 367778$$

$$\alpha^{34} = 813881\alpha + 546103$$

$$\alpha^{35} = 1211234\alpha + 813881$$

$$\alpha^{36} = 1805115\alpha + 1211234$$

$$\alpha^{37} = 2686349\alpha + 1805115$$

$$\alpha^{38} = 3991564\alpha + 2686349$$

$$\alpha^{39} = 5902813\alpha + 3991564$$

$$\alpha^{40} = 8714377\alpha + 5902813$$

$$\alpha^{41} = 12926190\alpha + 8714377$$

$$\alpha^{42} = 19140567\alpha + 12926190$$

$$\alpha^{43} = 28256767\alpha + 19140567$$

$$\alpha^{44} = 41897334\alpha + 28256767$$

$$\alpha^{45} = 62154101\alpha + 41897334$$

$$\alpha^{46} = 91410868\alpha + 62154101$$

$$\alpha^{47} = 135568169\alpha + 91410868$$

$$\alpha^{48} = 200978987\alpha + 135568169$$

$$\alpha^{49} = 297547156\alpha + 200978987$$

$$\alpha^{50} = 438525143\alpha + 297547156$$

$$\alpha^{51} = 652073130\alpha + 438525143$$

$$\alpha^{52} = 968100273\alpha + 652073130$$

$$\alpha^{53} = 1426175403\alpha + 968100273$$

$$\alpha^{54} = 2124275576\alpha + 1426175403$$

$$\alpha^{55} = 3162375749\alpha + 2124275576$$

$$\alpha^{56} = 4680551325\alpha + 3162375749$$

$$\alpha^{57} = 6912927074\alpha + 4680551325$$

$$\alpha^{58} = 10248478401\alpha + 6912927074$$

$$\alpha^{59} = 15161405475\alpha + 10248478401$$

$$\alpha^{60} = 22409883549\alpha + 15161405475$$

$$\alpha^{61} = 33481309024\alpha + 22409883549$$

$$\alpha^{62} = 49791192573\alpha + 33481309024$$

$$\alpha^{63} = 73440574122\alpha + 49791192573$$

$$\alpha^{64} = 108431766695\alpha + 73440574122$$

$$\alpha^{65} = 160872240819\alpha + 108431766695$$

$$\alpha^{66} = 238303907514\alpha + 160872240819$$

$$\alpha^{67} = 352135668333\alpha + 238303907514$$

$$\alpha^{68} = 520438575852\alpha + 352135668333$$

$$\alpha^{69} = 768574143366\alpha + 520438575852$$

$$\alpha^{70} = 1128412719218\alpha + 768574143366$$

$$\alpha^{71} = 1670950862584\alpha + 1128412719218$$

$$\alpha^{72} = 2469363581800\alpha + 1670950862584$$

$$\alpha^{73} = 3630314344384\alpha + 2469363581800$$

$$\alpha^{74} = 5364678206184\alpha + 3630314344384$$

$$\alpha^{75} = 7953042068000\alpha + 5364678206184$$

$$\alpha^{76} = 11706710274184\alpha + 7953042068000$$

$$\alpha^{77} = 17230752332168\alpha + 11706710274184$$

$$\alpha^{78} = 25537462606352\alpha + 17230752332168$$

$$\alpha^{79} = 37744214938536\alpha + 25537462606352$$

$$\alpha^{80} = 55550927270720\alpha + 37744214938536$$

$$\alpha^{81} = 82087640002904\alpha + 55550927270720$$

$$\alpha^{82} = 120638567273624\alpha + 82087640002904$$

$$\alpha^{83} = 177184107446528\alpha + 120638567273624$$

$$\alpha^{84} = 261821674720152\alpha + 177184107446528$$

$$\alpha^{85} = 388959242000000\alpha + 261821674720152$$

$$\alpha^{86} = 573596809280000\alpha + 388959242000000$$

$$\alpha^{87} = 850743611500000\alpha + 573596809280000$$

$$\alpha^{88} = 1250390420800000\alpha + 850743611500000$$

$$\alpha^{89} = 1831137232000000\alpha + 1250390420800000$$

$$\alpha^{90} = 2701884043200000\alpha + 1831137232000000$$

$$\alpha^{91} = 3992630854400000\alpha + 2701884043200000$$

$$\alpha^{92} = 5853377665600000\alpha + 3992630854400000$$

$$\alpha^{93} = 8624024476800000\alpha + 5853377665600000$$

$$\alpha^{94} = 12620492608000000\alpha + 8624024476800000$$

$$\alpha^{95} = 18526960720000000\alpha + 12620492608000000$$

$$\alpha^{96} = 27233428832000000\alpha + 18526960720000000$$

$$\alpha^{97} = 40140896944000000\alpha + 27233428832000000$$

$$\alpha^{98} = 58947365056000000\alpha + 40140896944000000$$

$$\alpha^{99} = 86753833168000000\alpha + 58947365056000000$$

$$\alpha^{100} = 127060301280000000\alpha + 86753833168000000$$

23. x, \dots, A_r, \dots, y
 $\quad \quad \quad n$

$2x, \dots, A'_r, \dots, y$

$5-3\beta, 5-\beta, 5+\beta, 5+3\beta$

$\frac{25-9\beta^2}{25-\beta^2} = \frac{2}{3}$

$\beta = 1$

$$A_r = x + r \left(\frac{2y-x}{n+1} \right) = \frac{(n+1-r)x + 2ry}{n+1}$$

$$A'_r = 2x + r \left(\frac{y-2x}{n+1} \right) = \frac{(2(n+1)-2r)x + ry}{n+1}$$

$$\frac{p}{2} (2a + (p-1)d) = \frac{q}{2} (2a + (q-1)d)$$

$$a(p-q) + d \left(\frac{p^2 - q^2}{2} - \frac{(p-q)}{2} \right) = 0$$

$$2a + d(p+q-1) = 0$$

$$\frac{\sum_{r=1}^m (2a + (m-1)d)}{\sum_{r=1}^n (2a + (n-1)d)} =$$

$$\frac{\sum_{r=1}^m}{\sum_{r=1}^n} \Rightarrow$$

$$\frac{a + \left(\frac{m-1}{2}\right)d}{a + \left(\frac{n-1}{2}\right)d} = \frac{m}{n}$$

$$\frac{m-1}{2} = M-1 \Rightarrow m = 2M-1$$

$$\frac{n-1}{2} = N-1 \Rightarrow n = 2N-1$$

$$\frac{a + (m-1)d}{a + (n-1)d} = ?$$

$$\frac{a + (M-1)d}{a + (N-1)d} = \frac{2M-1}{2N-1}$$

$$T_1, T_2, T_3, T_4, \dots, T_n$$

$$\begin{cases} T_1 + T_3 + T_5 + \dots + T_{n-1} = 24 \\ T_2 + T_4 + T_6 + \dots + T_n = 30 \end{cases}$$

$$6 = \frac{n}{2} d$$

$$nd = 12$$

$$n = ?$$

$$T_n - T_1 = \frac{21}{2}$$

$$\cancel{a} + (n-1)d - \cancel{a} = \frac{21}{2}$$

$$12 - d = \frac{21}{2}$$

$$d = ?$$

$$S_n - S_{n-1} = T_n$$

$$\begin{aligned} n(S_{n-1}) - (n-1)(S_n - 8) \\ = T_n \end{aligned}$$

$35, 2^{35}$

$$\textcircled{1} S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \dots + (a+(n-1)d)r^{n-1}$$

$$\textcircled{2} \times S_n = ar + (a+d)r^2 + (a+2d)r^3 + \dots + (a+(n-2)d)r^{n-1}$$

$$\textcircled{1} - \textcircled{2} \quad (1-r)S_n = a + dr + dr^2 + dr^3 + \dots + (a+(n-1)d)r^n$$

$$= a + \underbrace{\frac{dr(1-r^{n-1})}{(1-r)}}_{\substack{\text{as } n \rightarrow \infty \\ \rightarrow 0}} - \underbrace{(a+(n-1)d)r^n}_{\rightarrow 0}$$

Handwritten notes on a blackboard showing the derivation of the formula for the number of nodes in a full binary tree. The notes include:

- n nodes
- 2^n nodes
- $2^n - 1$ nodes
- $2^n - 1$ nodes

$$S = \lim_{n \rightarrow \infty} \left(\frac{a}{1-r} + \frac{dr - dr^n}{(1-r)^2} - \frac{ar^n + (n-1)r^n d}{1-r} \right)$$

$|r| < 1$ ✓

$$S = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \dots \infty$$

$$rS = ar + (a+d)r^2 + (a+2d)r^3 + \dots \infty$$

$$(1-r)S = a + \underbrace{(dr + dr^2 + dr^3 + \dots)}$$

$$S(1-r) = a + \frac{dr}{1-r}$$

$$S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \quad |r| < 1$$

1. 2) $|x| < 1$, find

(i) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots \infty =$

$$\sum_{r=1}^{\infty} r x^{r-1}$$

$$S = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots \infty$$

$$xS = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$\boxed{x^r}$$

$$(x+1)^2 = x^2 + 2x + 1$$

$$2(x+1) = 2x + 2$$

$$(1-x)S = 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}$$

$$S = \frac{1}{(1-x)^2}$$

$$1 + x + x^2 + x^3 + x^4 + \dots \infty = \frac{1}{1-x}$$

$$1 + 2x + 3x^2 + 4x^3 + \dots \infty = \frac{-1(-1)}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

$$(ii) \quad 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots - \infty$$

$$S = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots - \infty$$

$$xS = x + 3x^2 + 6x^3 + 10x^4 + \dots - \infty$$

$$(1-x)S = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots - \infty$$

$$x(1-x)S = x + 2x^2 + 3x^3 + 4x^4 + \dots - \infty$$

$$(1-x)(1-x)S = 1 + x + x^2 + x^3 + x^4 + \dots - \infty$$

$$= \frac{1}{1-x}$$

$$S = \frac{1}{(1-x)^3}$$

2. Find the sum upto n terms & infinite terms if exist.

$$\frac{4}{5}S_n = 1 + \frac{\frac{3}{5}(1 - \frac{1}{5}n-1)}{(1 - \frac{1}{5})} - \frac{3n-2}{5^n}$$

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \frac{13}{5^4} + \dots$$

$$\frac{4}{5}S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \dots$$

$$= 1 + \frac{\frac{3}{5}}{1 - \frac{1}{5}} = 1 + \frac{3}{4}$$

$$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \frac{13}{5^4} + \dots + \frac{3n-2}{5^{n-1}}$$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \frac{10}{5^4} + \dots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$$

$$\frac{4}{5}S_n = 1 + \left(\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \dots + \frac{3}{5^{n-1}} \right) - \frac{3n-2}{5^n}$$

$$\frac{3n-2}{5^n}$$

$$S = \frac{35}{18}$$

$$\underline{\sum x - \bar{I}(a)}$$

$$21 - 28 \cdot \checkmark$$

$$\underline{\sum x - \bar{I}(b)}$$

$$\rightarrow 3, 4, 7, 9, 10,$$

$$15, \boxed{17}, 18, 19, 20,$$

$$21, 22, 23.$$