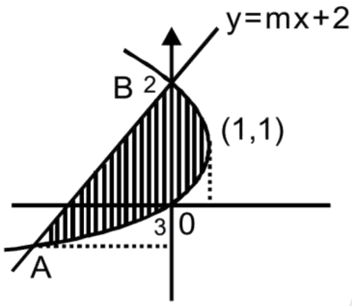
DPP–02 (AREA UNDER THE CURVE)

SUBJECTIVE

1. Find the values of $m(m > 0)$ for which the area bounded by the line $y = mx + 2$ and the curve $x = 2y - y^2$ is, (i) $\frac{9}{2}$ square units and (ii) minimum. Also find the minimum area.

Ans. (i) $m = 1$, (ii) $m = \infty; A_{min} = 4/3$

Sol. (a) Area $= \frac{1}{2}.2 \times \left(\frac{-1-2m}{m^2}\right) + \int_0^1 (2y - y^2)dy = \frac{3}{2}$



(b) If area is minimum $\frac{dA}{dm} = 0$

$\Rightarrow m = \infty \& Amin = \frac{4}{3}$

2. For what value of ' a ' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line $y = 0, x = 0$ and $x = 1$ the least?

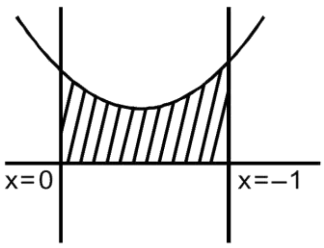
Ans. $a = -\frac{3}{4}$

Sol. $a^2x^2 + ax + 1$ is clearly positive

$$A = \int_0^1 (a^2x^2 + ax + 1)dx$$

$$= \frac{1}{6}\left(2\left(a + \frac{3}{4}\right)^2 + \frac{39}{8}\right)$$

minimum for $a = -3/4$.



3. Let ' c ' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point $(1, c)$ with gradient ' m ' and the parabola $y = x^2$ is 36 sq, units find the value of $(c^2 + m^2)$.

Ans. **104**

Sol. Equation of line

$y - c = m(x - 1)$

$\Rightarrow y = mx + c - m$

solving line & parabola

$$x = \frac{m \pm \sqrt{m^2 - 4m + 4c}}{2} <^{x_1}_{x_2} \text{ (let)}$$

$$A = \left| \int_{x_1}^{x_2} (x^2 - (mx + c - m))dx \right| = 36$$

4. If $f(x)$ is monotonic in (a, b) then prove that the area bounded by the ordinates at $x = a; x = b$; $y = f(x)$ and $y = f(c), c \in (a, b)$ is minimum when $c = \frac{a+b}{2}$. Hence if the area bounded by the graph of $f(x) = \frac{x^3}{3} - x^2 + a$, the straight lines $x = 0, x = 2$ and the x-axis is minimum then find the value of ' a ',

Ans. $a = \frac{2}{3}$

Sol. $A = \int_a^c (f(c) - f(x))dx + \int_c^b f(x) - f(c)\big) dx$

$= f(c)[c - a + c - b] - \int_a^c f(x)dx + \int_c^b f(x)dx$

differentiating w.r.t. 'c'

$\frac{dA}{dc} = f(c) \cdot 2 + (2c - a - b) \cdot f'(c) - f(c) - f(b) \cdot 0 - f(c)$

$\Rightarrow (2c - a - b)(f'(c)) = 0 \Rightarrow 2c - a - b = 0$

$c = \frac{a+b}{2}$

$f'(x) = x^2 - 2x + a = 0$

$\frac{2+a}{2} = -a \Rightarrow a = \frac{2}{3}$

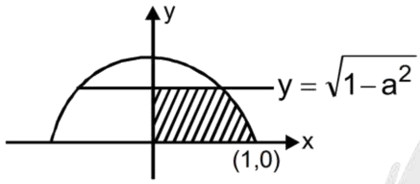
5. For what values of $a \in [0,1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0, x = 1$ and $y = f(a)$ is at a minimum and for what values it is at a maximum if $f(x) = \sqrt{1 - x^2}$. Find also the maximum and the minimum areas.

Ans. $a=\frac{1}{2}$ gives minima, $A\left(\frac{1}{2}\right)=\frac{3\sqrt{3}-\pi}{12}$; $a=0$ gives local maxima $A(0)=1-\frac{\pi}{4}$; $a=1$ gives maximum value, $A(1)=\frac{\pi}{4}$

Sol. $A=\int_0^{\sqrt{1-a^2}}(\sqrt{1-y})dy$

$\frac{dA}{da}=0, a=\frac{1}{2}$

so $A_{min}=\frac{3\sqrt{3}-\pi}{12}$ & maxima at $a=0$



6. A figure is bounded by the curves $y=\left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, $y=0, x=2$ and $x=4$. At what angles to the positive x-axis straight lines must be drawn through (4,0) so that these lines partition the figure into three parts of the same size.

Ans. $\pi-\tan^{-1}\frac{2\sqrt{2}}{3\pi}$; $\pi-\tan^{-1}\frac{4\sqrt{2}}{3\pi}$

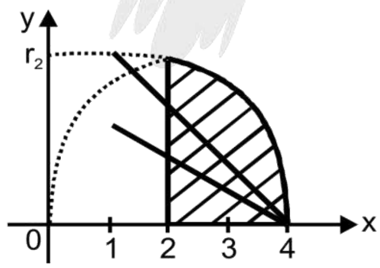
Sol. $A=\left|\int_2^4\sqrt{2}\sin\frac{\pi x}{4}dx\right|$

$=\frac{4\sqrt{2}}{\pi}$ sq. units.

let the line is $y=mx-4m$

$$A=\left(\frac{4\sqrt{2}}{\pi}\right)\times\frac{1}{3}=\int_2^4\left(\sqrt{2}\sin\frac{\pi x}{4}-mx+4m\right)dx$$

similarly for m_2



7. The line $3x+2y=13$ divides the area enclosed by the curve, $9x^2+4y^2-18x-16y-11=0$ into two parts. Find the ratio of the larger area to the smaller area.

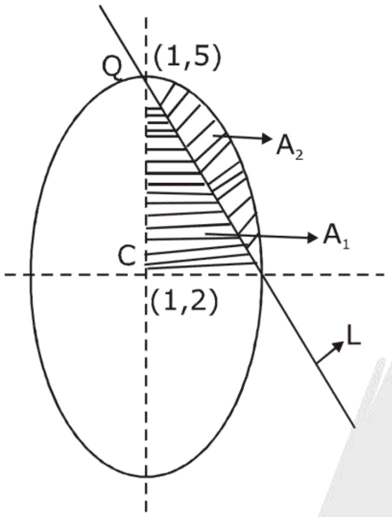
Ans. $\frac{3\pi+2}{\pi-2}$

Sol. $9x^2-18x+4y^2-16y-11=0$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1 \Rightarrow \text{ (Ellipse)}$$

L: $3x + 2y = 13$

Point of intersection of line & Ellipse is P(3,2), Q(1,5)



Area of ellipse (A) $= \pi ab == \pi \times 3 \times 2 = 6\pi$

$A_1 + A_2 = \frac{A}{4}$

$A_1 = \frac{1}{2} \times CP \times CQ = \frac{1}{2} \times 2 \times 3 = 3$

$A_2 = \frac{A}{4} = 3 = \frac{3\pi}{2} - 3$

Smaller Area $= \frac{3\pi}{2} - 3$

8. Find the area bounded by the curve $y = xe^{-x^2}$, the x-axis and the line $x = c$ where $y(c)$ is maximum.

Ans. $\frac{1}{2}\left(1 - e^{-1/2}\right)$

Sol. $y = x\bar{e}^{x^2}$

$y' = \bar{e}^{x^2} - 2x^2e^{-x^2} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

y is maximum at $x = \frac{1}{\sqrt{2}}$

Area $= \int_0^{1/\sqrt{2}} x\bar{e}^{x^2} dx = \frac{1}{2}\left(1 - e^{-1/2}\right)$

9. A polynomial function $f(x)$ satisfies the condition $f(x+1) = f(x) + 2x + 1$. Find $f(x)$ if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

Ans. $f(x) = x^2 + 1; y = \pm 2x; A = \frac{2}{3}$ sq. units

Sol. $f(x+1) = f(x) + 2x + 1; f(0) = 1$

put $x = 0$

$f(1) = f(0) + 1 = 2$

put $x = 1$

$f(2) = f(1) + 3 = 5$

put $x = 2$

$f(3) = f(2) + 5 = 10$

$\Rightarrow f(x) = 1 + x^2$

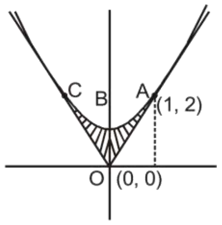
Let the pair of tangents $y = mx$

$mx = 1 + x^2 \Rightarrow x^2 - mx + 1 = 0$

$D = 0 \Rightarrow m = \pm 2$

pair of tangent $y = \pm 2x$

Area $= 2\int_0^1 (1 + x^2 - 2x)dx = \frac{2}{3}$



10. Find the equation of the line passing through the origin and dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ and $x = 1$ into two parts of equal area.

Ans. $y = \frac{2x}{3}$

Sol. Area bounded by

$y = 2x - x^2, y = 0 \& x = 1$

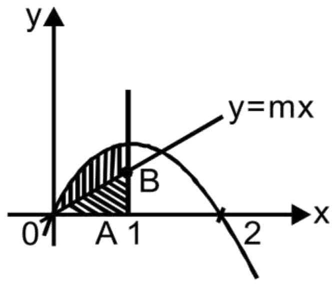
$$A = \int_0^1 (2x - x^2)dx$$

$$= \frac{2}{3}$$

Let line is $y = mx$ which divides the area into two equal parts so area of $\triangle$ OAB $= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

$$\Rightarrow \frac{1}{2} \times 1 \times m = \frac{1}{3} \Rightarrow m = \frac{2}{3}$$

so line is $y = \frac{2}{3}x$



11. Consider the curve $y = x^n$ where $n > 1$ in the quadrant. If the area bounded by the curve, the x axis and the tangent line to the graph of $y = x^n$ at the point (1,1) is maximum then find the value of n

Ans. $\sqrt{2} + 1$

Sol. Equation of tangent at (1,1)

$y - 1 = n(x - 1) \Rightarrow y = nx - n + 1$

area bounded by the curve, the tangent & the x axis is

$$A = \left| \int_0^1 \left((nx - n + 1) - x^n\right) \cdot dx \right| = \left| \frac{n^2 - n}{2(n + 1)} \right|$$

Now A to be maximum, $\frac{dA}{dn} = 0 \Rightarrow$ at $n = \sqrt{2} + 1$

12. In the adjacent figure, graphs of two functions $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at $A(a, f(a))$; $B(\pi, 0)$ and $C(2\pi, 0)$. $A_i (i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$ between $x = 0$ and $x = a$; $i = 1$, between $x = a$ and $x = \pi$; $i = 2$, between $x = \pi$ and $x = 2\pi$, $i = 3$. If $A_1 = 1 - \sin a + (a - 1)\cos a$, determine the furction $f(x)$. Hence determine ' a ' and A_1. Also calcu. late A_2 and A_3.

Ans. $f(x) = x\sin x, a = 1; A_1 = 1 - \sin 1; A_2 = \pi - 1 - \sin 1; A_3 = (3\pi - 2)$ sq. units

Sol. From the figure it is clear that

$\int_0^a (\sin x - f(x))dx = 1 - \sin a + (a - 1)\cos a$

differentiate w.r.t.

$\sin a - f(a) = -\cos a + \cos a - (a - 1)\sin a$

$\Rightarrow \sin a - f(a) = -a\sin a + \sin a$

$\Rightarrow f(a) = a\sin a \Rightarrow f(x) = x\sin x$

The points where $f(x)$&$\sin x$ intersect are $x\sin x = \sin x \Rightarrow \sin x = 0$ or $x = 1$.

We can say that $a = 1$

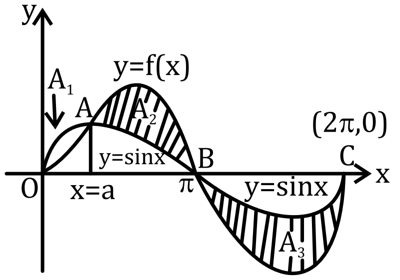
$A_1 = \int_0^1 (\sin x - x\sin x)dx = (1 - \sin 1)$

$A_2 = \int_1^\pi (f(x) - \sin x)dx = \int_1^\pi (x\sin x - \sin x)dx = (\pi - 1 - \sin 1)$

$A_3 = \left|\int_{\pi}^{2\pi}(\sin x - x\sin x)dx\right| = (3\pi - 2)$

13. Let A_n be the area bounded by the curve $y = (\tan x)^2$ and the lines $x = 0, y = 0$ and $x = \frac{\pi}{4}$.

Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{(n-1)}$ and deduce that $\frac{1}{(2n+2)} < A_n < \frac{1}{(2n-2)}$



Sol. $A_n = \int_0^{\pi/4}(\tan x)^n dx$

$0 < \tan x < 1$ when $0 < x < \pi/4$

$0 < (\tan x)^{n+1} < (\tan x)^n, n \in N$

$\Rightarrow \int_0^{\pi/4}(\tan x)^{n+1}dx < \int_0^{\pi/4}(\tan x)^n dx$

$\Rightarrow A_{n+1} < A_n$

for $n < 2$

$A_n + A_{n+2} = \int_0^{\pi/4}[(\tan x)^n + (\tan x)^{n+2}]dx$

$= \int_0^{\pi/4}(\tan x)^n(1 + \tan^2 x)dx$

$= \frac{1}{n+1}$

since $A_{n+2} < A_{n+1} < A_n$

so $A_n + A_{n+2} < 2A_n$

$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n$

Also for $n > 2$; $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

$\Rightarrow 2A_n < \frac{1}{n-1} \Rightarrow A_n < \frac{1}{2n-2}$

combining (1) & (2) use get

$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$

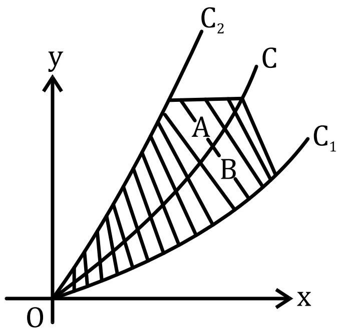
14. Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes (asympt totes are the lines which meet the curve at infinity).

Ans. $14a^2$

Sol. $x^2y^2 = a^2y^2 - a^2x^2$

$y^2 = \frac{a^2x^2}{a^2-x^2}$ Area $= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}dx = 14a^2$

15. Let C_1 and C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 and C_2, if for each point P of C, the two shaded regions A and B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ and that the lower curve C_1 has the equation $y = \frac{x^2}{2}$.



Ans. $\left(\frac{16}{9}\right)x^2$

Sol. Let a point $P(t, t^2)$ is on the curve c.
y cordinate of Q is also t^2&$R = (t, t^2/2)$

& $Q = (x, t^2)$

Area of OPQ = Area of OPR

$$\Rightarrow \int_0^{t^2} \left(\sqrt{y} - f(x)dy = \int_0^t (x^2 - x^2/2)dx\right.$$

differentiating both sides w.r.t. t

$$f(t) = \frac{16}{9}t^2 \Rightarrow f(x) = \left(\frac{16}{9}\right)x^2$$

PREVIOUS YEAR(JEE ADVANCED)

16. (a) The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is **[JEE Adv. 2008]**

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

(B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(C) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(D) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

Ans. **(B)**

Sol. (a) $I = \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right) dx$

$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1+\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right) dx$

$= \int_0^{\pi/4} \frac{\left(1+\tan\frac{x}{2}\right)-\left(1-\tan\frac{x}{2}\right)}{\sqrt{1-\tan^2\frac{x}{2}}} dx$

$= \int_0^{\pi/4} \frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}} dx = \int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}} dt$

as $\tan\frac{x}{2} = t$

(b) Comprehension (3 questions together) : Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2,2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$

(i) If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$

(B) $-\frac{4\sqrt{2}}{7^3 3^2}$

(C) $\frac{4\sqrt{2}}{7^3 3}$

(D) $-\frac{4\sqrt{2}}{7^3 3}$

Ans. **(B)**

Sol. Differentiating the given equation, use get $3y^2y'$

$-3y' + 1 = 0$

$y'(-10\sqrt{2}) = -\frac{1}{21}$

Differentiation again use get

$6y(y')^2 + 3y^2y'' - 3y'' = 0$

$f''(-10\sqrt{2}) = -\frac{6\cdot 2\sqrt{2}}{(21)^4} = -\frac{4\sqrt{2}}{7^3 3^2}$

(ii) The area of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(A) $\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) = af(a)$

(B) $-\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

(D) $-\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

Ans. **(A)**

Sol. The reqd. Area

$= \int_a^b f(x)dx = xf(x)|_a^b - \int_a^b xf'(x)dx$

$= bf(b) - af(a) + \int_a^b \frac{x}{[f(x)^2-1]}dx$

(iii) $\int_{-1}^{1} g'(x)dx$ equals

(A) $2\, g(-1)$

(B) 0

(C) $-2g(1)$

(D) $2\, g(1)$

Ans. **(D)**

Sol. We have $y' = \frac{1}{3[1-[f(x)]^2]}$ which is even Hence $\int_{-1}^{1} g'(x) = g(1) - g(-1) = 2g(1)$

17. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **[JEE Adv. 2009]**

(A) $e - 1$

(B) $\int_1^c \ln(e + 1 \cdot y)dt$

(C) $e - \int_0^1 e^x dx$

(D) $\int_1^e \text{In}\, ydy$

Ans. **(BCD)**

Sol. Reqd. area

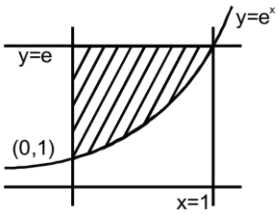
$$= \int_1^e \ln\, ydy$$

$= (y\ln\, y - y)_1^e$

$= e - e - \{-1\} = 1$

Also $\int_1^e \ell n ydy = \int_1^e \ell n(e + 1 - y)dy$

Further the reqd. area $= (e \times 1) - \int_0^1 e^x dx$



18. Let the straight line $x = b$ divide the area enclosed by $y = (1-x)^2, y = 0$, and $x = 0$ into two parts R_1 $(0 \le x \le b)$ and $R_2 (b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **[JEE Adv. 2011]**

(A) $\frac{3}{4}$

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{1}{4}$

Ans. **(B)**

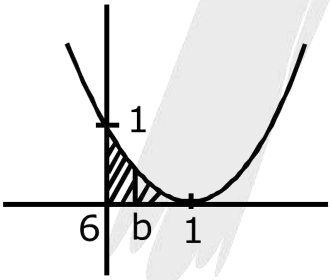
Sol. $R_1 + R_2 = \int_0^1 (1-x)^2 \cdot dx = 1/3 \ldots (1)$

$R_1 - R_2 = 1/4 \ldots (2)$

Now $2R_1 = 7/12$

$\Rightarrow R_1 = 7/24$

$\Rightarrow \int_0^b (1-x)^2 dx = 7/24 \Rightarrow b = 1/2$



19. Let $f: [-1,2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let $R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x), x = -1, x = 2$, and the x axis. Then **[JEE Adv. 2011]**

(A) $R_1 = 2R_2$

(B) $R_1 = 3R_2$

(C) $2R_1 = R_2$

(D) $3R_1 = R_2$

Ans. **(C)**

Sol. $R_1 = \frac{1}{2}\int_{-1}^{2} f(x)dx; R_2$

$= \int_{-1}^{2} f(x)dx \Rightarrow R_2 = 2R_1$

20. Let S be the area of the region enclosed by $y = e^{-x^2}, y = 0, x = 0$, and $x = 1$. Then

[JEE Adv. 2012]

(A) $s \geq \frac{1}{e}$

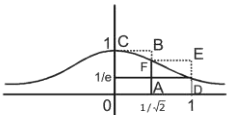
(B) $s \geq 1 - \frac{1}{e}$

(C) $s \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$

(D) $s \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 \cdot \frac{1}{\sqrt{2}}\right)$

Ans. **(ABD)**

Sol. (B) $x \geq x^2 \int_0^1 e^{-x}dx \leq \int_0^1 e^{-x^2}dx$



(D) $S \leq$ area OABC + area AFDE

$$s \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$$

21. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is **[JEE Adv. 2013]**

(A) $4(\sqrt{2} - 1)$

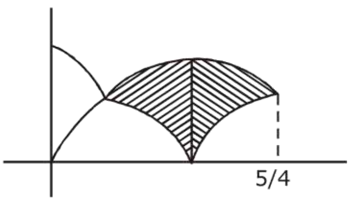
(B) $2\sqrt{2}(\sqrt{2} - 1)$

(C) $2(\sqrt{2} + 1)$

(D) $2\sqrt{2}(\sqrt{2} + 1)$

Ans. **(B)**

Sol. $A = 2\int_0^{5/4} (\sin x + \cos x - (\cos x - \sin x)) \cdot dx$



$A = 2 \times 2(-\cos x)_0^{\pi/4}$

$A = 2 \times 2\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) = \frac{4(\sqrt{2}-1)}{\sqrt{2}}$

$2\sqrt{2}(\sqrt{2} - 1)$

22. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2\, t\,dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \rightarrow [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0, y = 0, y = f(x)$ and $x = a$, then $f(0)$ is. **[JEE Adv. 2015]**

(A) 1

(B) 2

(C) 3

(D) 4

Ans. **(C)**

Sol. $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2\, t\,dt$

$F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right) \cdot 2x - 2\cos^2\, x \cdot 1$

$F''(x) = -4\cos\left(x^2 + \frac{\pi}{6}\right) \cdot \sin\left(x^2 + \frac{\pi}{6}\right) \cdot 2x \cdot 2x$

$+2\cos^2\left(x^2 + \frac{\pi}{6}\right) \cdot 2$

$+4\cos\, x \cdot \sin\, x$

$F''(0) = 0$

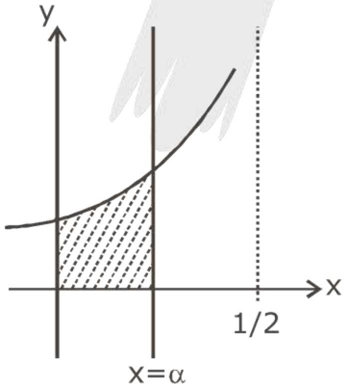
$+2\cos^2\left(\frac{\pi}{6}\right) \cdot 2$

$= 3$

$\int_0^{\alpha} f(x)dx = F'(\alpha) + 2$

$f(\alpha) = F''(\alpha)$

$f(0) = F''(0) = 3$



23. Area of the region $\left\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq x + 9 \leq 15\right\}$ is equal to **[JEE Adv. 2016]**

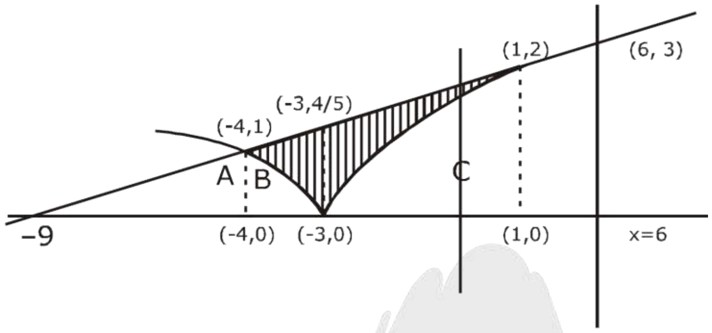
(A) $\frac{1}{6}$

(B) $\frac{4}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{3}$

Ans. **(C)**



Sol.

$$\text{Area } = \frac{1}{2}\cdot 10 \cdot 2 - \left[\frac{1}{2}\cdot 5 \cdot 1\right] -$$

$$\int_{-4}^{-3}\sqrt{-(x+3)}dx - \int_{-3}^{1}\sqrt{(x+3)}$$

$$= 10 - \frac{5}{2} + \frac{2}{3}(-(x+3))^{3/2}\Big|_{-4}^{-3} - \frac{2}{3}(x+3)^{3/2}\Big|_{-3}^{1}$$

$$= 10 - \frac{5}{2} - \frac{2}{3}[0-[1]] - \frac{2}{3}[8] - 0\Big]$$

$$= \frac{15}{2} - \frac{2}{3} - \frac{16}{3}$$

$$= \frac{45-4-32}{6} = \frac{5}{2}$$

$$\frac{45-36}{6} = \frac{9}{6} = \frac{3}{2}$$

24. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then **[JEE Adv. 2017]**

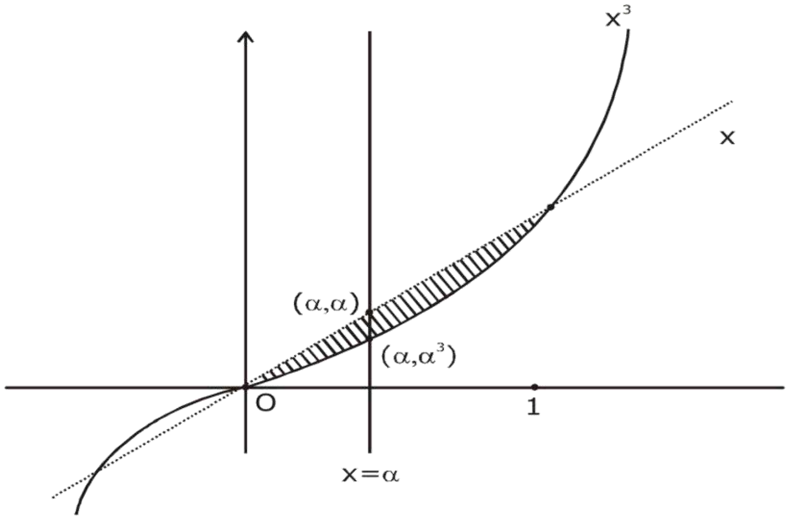
(A) $\alpha^4 + 4\alpha^2 - 1 = 0$

(B) $0 < a \le \frac{1}{2}$

(C) $2\alpha^4 - 4\alpha^2 + 1 = 0$

(D) $\frac{1}{2} < \alpha < 1$

Ans. **(CD)**



Sol.

$\int_0^\alpha (x - x^3)dx = \frac{1}{2}\int_0^1 (x - x^3)dx$

$\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^\alpha = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1$

$\left(\frac{2\alpha^2 - \alpha^4}{4}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}\cdot\frac{1}{2}$

$2\alpha^2 - \alpha^4 = \frac{1}{2}$

$\boxed{4\alpha^2 - 2\alpha^4 = 1}$ $\qquad t = \frac{4 \pm 2\sqrt{2}}{4}$

Check 'D' U $\qquad t = 1 \pm \frac{1}{\sqrt{2}}$

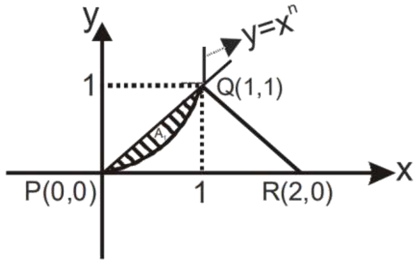
$\alpha^2 = t$ $\qquad \alpha^2 = 1 + \frac{1}{\sqrt{2}}$

$\alpha^2 = 1 - \frac{1}{\sqrt{2}}$

$2t^2 - 4t + 1 = 0$ $\qquad 1 > \alpha > \frac{1}{2}$

25. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0,0)$, $Q(1,1)$ and $R(2,0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of $\triangle$ PQR, then the value of n is **[JEE Adv. 2018]**

Ans. **(4)**



Sol.

$A_1 = \int_0^1 (x - x^n)dx = \frac{3}{10}\left(\frac{1}{2} \times 2 \times 1\right)$

$\Rightarrow \left(\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right)_0^1 = \frac{3}{10}$

$\Rightarrow \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$

$\Rightarrow \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10}$

$= \boxed{n = 4}$

26. The area of region $\{(x, y): xy \leq 8, 1 \leq y \leq x^2\}$ is **[JEE Adv. 2019]**

(A) $16\log_c 2 - \frac{14}{3}$

(B) $8\log_e 2 - \frac{7}{3}$

(C) $8\log_c 2 - \frac{14}{3}$

(D) $16\log_c 2 - 6$

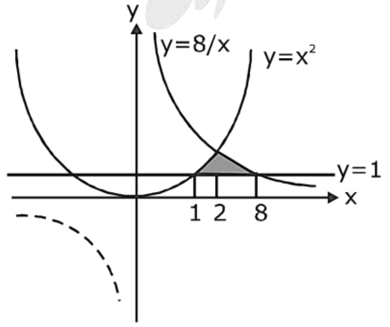
Ans. **(A)**

Sol. $xy \leq 8 \;\&\; 1 \leq y \leq x^2$

$A = \int_1^2 (x^2 - 1)dx + \int_2^8 \left(\frac{8}{x} - 1\right) dx$

$A = \frac{x^3}{3}\Big|_1^2 + 8\ln x|_2^8 - 1 - 6$

$A = \left(\frac{8}{3} - \frac{1}{3}\right) + 8(\ln 8 - \ln 2) - 7$



$A = \frac{7}{3} - 7 + 16\ln 2$

$\Rightarrow A = 16\ln 2 - \frac{14}{3}$

27. Let the functions $f: R \to R$ and $g: R \to R$ be defined by $f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$. Then the area of the region in the first quadrant bounded by the curves $y = f(x), y = g(x)$ and $x = 0$ is. **[JEE Adv. 2020]**

(A) $(2-\sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(B) $(2+\sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(C) $(2-\sqrt{3}) + \frac{1}{2}(e + e^{-1})$

(D) $(2+\sqrt{3}) + \frac{1}{2}(e + e^{-1})$

Ans **(A)**

Sol. $f(x) = 0 \quad ; x < 1$

$= e^{x-1} - e^{-(x-1)} \quad ; x \geq 1$

while $g(x) \geq 1$

so they will intersect in the region $x > 1$

solve $f(x) = g(x)$

$$e^{x-1} - e^{-(x-1)} = \frac{1}{2}(e^{x-1} + e^{1-x})$$

$$\frac{1}{2}e^{x-1} = \frac{3}{2}e^{1-x}$$

$$e^{2x} = 3e^2$$

$$2x = 2 + \ln 3$$

$$x = 1 + \ln\sqrt{3}$$

$$A = \int_0^1 g(x)dx + \int_1^{1+\ln\sqrt{3}} (g(x) - f(x))dx$$

$$= \frac{1}{2}\int_0^1 (e^{x-1} + e^{1-x})dx + \int_1^{1+\ln\sqrt{3}} \frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x})dx$$

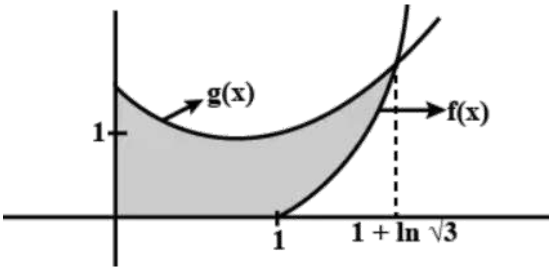
$$= \frac{1}{2}\int_1^{1+\ln\sqrt{3}} (e^{x-1} + e^{1-x})dx - \int_1^{1+\ln\sqrt{3}} (e^{x-1} - e^{1-x})dx$$

$$= \frac{1}{2}[e^{x-1} - e^{1-x}]_0^{1+\ln\sqrt{3}} - (e^{x-1} + e^{1-x})_1^{1+\ln\sqrt{3}}$$

$$= \frac{1}{3}\left[\sqrt{3} - \frac{1}{\sqrt{3}} - \frac{1}{e} + e\right] - \left[\sqrt{3} + \frac{1}{\sqrt{3}} - 1 - 1\right]$$

$$= 2 + \frac{1}{2}\left(e - \frac{1}{e}\right) - \frac{\sqrt{3}}{2} - \frac{3}{2\sqrt{3}}$$

$$= (2-\sqrt{3}) + \frac{e - \frac{1}{e}}{2}$$



28. The area of the region $\left\{(x,y): 0 \le x \le \frac{9}{4},\ 0 \le y \le 1,\ x \ge 3y,\ x+y \ge 2\right\}$ is **[JEE Adv. 2021]**

(A) $\frac{11}{32}$

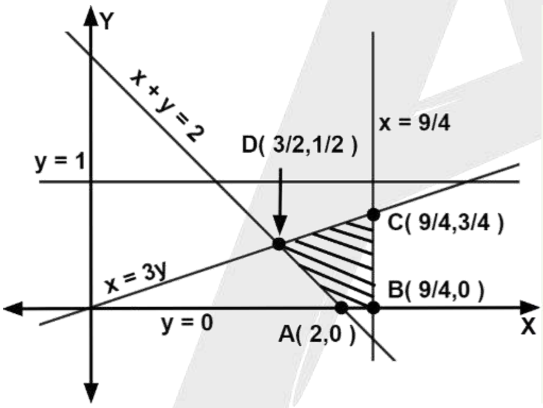
(B) $\frac{35}{96}$

(C) $\frac{37}{96}$

(D) $\frac{13}{32}$

Ans **(A)**

Sol. The correct option is **A** $\frac{11}{32}$ sq. units. Rough sketch of required region is



∴ Required area is

Area of △ ACD + Area of △ ABC

$$=\left|\frac{1}{2}\begin{vmatrix}2 & 0 & 1\\ 9/4 & 3/4 & 1\\ 3/2 & 1/2 & 1\end{vmatrix}\right|+\frac{1}{2}\times\left(\frac{9}{4}-2\right)\times\frac{3}{4}$$

$$=\frac{1}{2}\left|2\left(\frac{3}{4}-\frac{1}{2}\right)+1\left(\frac{9}{8}-\frac{9}{8}\right)\right|+\frac{3}{32}$$

$$=\frac{1}{4}+\frac{3}{32}=\frac{11}{32}\text{ sq. units.}$$

29. Consider the functions $f, g: R \to R$ defined by

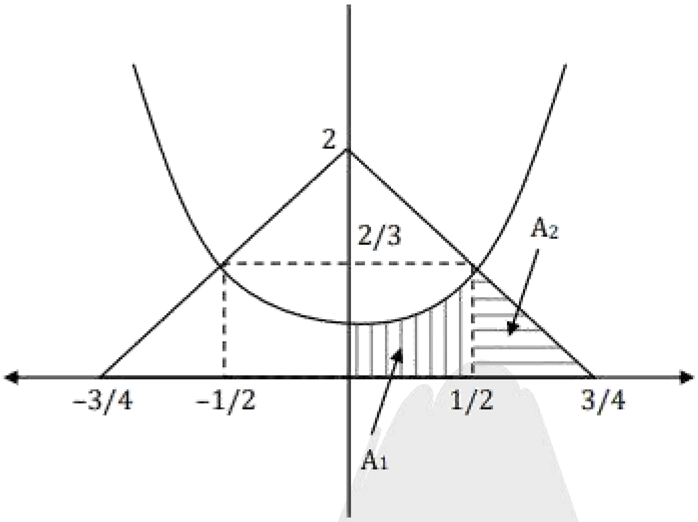
$f(x) = x^2 + \frac{5}{12}$ and $g(x) = \begin{cases} 2\left(1-4\frac{|x|}{3}\right), & |x| \le \frac{3}{4}, \\ 0, & |x| > \frac{3}{4} \end{cases}$. If α

is the area of the region $\left\{(x,y) \in R \times R: |x| \le \frac{3}{4}, 0 \le y \le \min\{f(x), g(x)\}\right\}$, then the value of 9α is

[JEE Adv. 2022]

Ans. (6)

Sol. Figure can be drawn as shown



Required area $= 2 \cdot (A_1 + A_2]$

$$\Rightarrow \alpha = \left(\int_0^{1/2}\left(x^2+\frac{5}{12}\right)\mathrm{dx}+\frac{1}{2}\left(\frac{3}{4}-\frac{1}{2}\right)\cdot\frac{2}{3}\right)\times 2$$

$$\Rightarrow \alpha = \left(\left(\frac{x^3}{3}+\frac{5x}{12}\right)\Bigg|_0^{\frac{1}{2}}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{3}\right)\times 2$$

$$\Rightarrow \alpha = \left(\frac{1}{4}+\frac{1}{12}\right)\times 2$$

$$\Rightarrow \alpha = \frac{2}{3}$$

$$\Rightarrow 9\alpha = 6$$