CONTINUITY

DPP–01(CONTINUITY AT A POINT)

SOLUTION

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1. A function f(x) satisfies the following property: f(x + y) = f(x)f(y) Show that the function is continuous for all values of x if it is continuous at x = 1.

Sol. As the function is continuous at $x = 1$, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) \quad \text{or} \quad \lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$$

Or $\lim_{h\to 0} f(1)f(-h) = \lim_{h\to 0} f(1)f(h) = f(1)$ [Using $f(x+y) = f(x)f(y)$]

Or $\lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 1$ ... (i)

Now, consider any arbitrary point $x = \text{a}$.

LHL $= \lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a)f(-h) = f(a)\lim_{h\to 0} f(-h) = f(a)$ [As $\lim_{h\to 0} f(-h) = 1$, using (i)]

RHL $= \lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a)f(h) = f(a)\lim_{h\to 0} f(h) = f(a)$ [As $\lim_{h\to 0} f(h) = 1$, using (i)]

[As $\lim_{h\to 0} f(h) = 1$, using (i)]
Hence, at any arbitrary point $(x = a)$, LHL = RHL = f(a).
Therefore, the function is continuous for all values of x if it is continuous at 1 .

2. Find the points of discontinuity of the following functions.

(i) f(x) $= \frac{1}{2\sin x - 1}$

(ii) f(x) = [[x]] − [x − 1], where [.] represent the greatest integer function.

Ans. **(i)** $x = 2n\pi + \frac{\pi}{6}$ or $x = 2n\pi + \frac{5\pi}{6}, n \in Z$ **(ii)** continuous ∀ x ∈ R.

Sol. (i) $f(x) = \frac{1}{2\sin x - 1}$
$f(x)$ is discontinuous when $2\sin x - 1 = 0$
or

$$\sin x = \frac{1}{2}, \text{ i.e., } x = 2n\pi + \frac{\pi}{6} \text{ or } x = 2n\pi + \frac{5\pi}{6}, n \in Z$$

(ii) f(x) = [[x]] − [x − 1] = [x] − ([x] − 1) = 1.
Therefore, f(x) is continuous ∀x ∈ R.

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3. Let $f(x)=\begin{cases}\frac{\log_e \cos x}{\sqrt[4]{1+x^2}-1}, & x>0\\ \frac{e^{\sin 4x}-1}{\log_e(1+\tan 2x)}, & x<0\end{cases}$. Find the value of f(0) which makes the function continuous at x = 0,

Ans. f(0) cannot be defined.

Sol. $\text{LHL} = \lim\limits_{x\to 0^-}\frac{e^{\sin 4x}-1}{\log_e(1+\tan 2x)} = \lim\limits_{x\to 0^-}\frac{\frac{e^{\sin 4x}-1}{\sin 4x}\sin 4x}{\frac{\log_e(1+\tan 2x)}{\tan 2x}\tan 2x} \Rightarrow \lim\limits_{x\to 0^-}\frac{\sin 4x}{\tan 2x}=2$

$\text{RHL} = \lim\limits_{x\to 0^+}\left(\frac{\log_e \cos x}{\sqrt[4]{(1+x^2)}-1}\right) = \lim\limits_{x\to 0^+}\left(\frac{-\tan x}{\frac{1}{4}(1+x^2)^{-\frac{3}{4}}2x}\right) = -2$ [Using L'HR]

[Using L'HR]
Here $f(0^-) \neq f(0^+)$
Hence $f(x)$ cannot be defined.
Hence, $f(x)$ has non-removable type of discontinuity.

4. $f(x)=\begin{cases}\cos^{-1}\{\cot x\} & x<\frac{\pi}{2}\\ \pi[x]-1 & x\geq\frac{\pi}{2}\end{cases}$; find jump of discontinuity , where [ ] denotes greatest integer & { } denotes fractional part function.

Ans. $\frac{\pi}{2}-1$

Sol. $f(x)=\begin{cases}\cos^{-1}\{\cot x\}\ x<\frac{\pi}{2}\\ \pi[x]-1\ x\geq\frac{\pi}{2}\end{cases}$

$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}\cos^{-1}\{\cot x\} = \lim_{h\to 0}\cos^{-1}\left\{\cot\left(\frac{\pi}{2}-h\right)\right\} = \lim_{h\to 0}\cos^{-1}\{\tan h\} = \frac{\pi}{2}$

$= \lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\pi[x]-1 = \lim_{h\to 0}\pi\left[\frac{\pi}{2}+h\right]-1 = \pi-1$

$\therefore$ jump of discontinuity $=(\pi-1)-\frac{\pi}{2}=\frac{\pi}{2}-1$

5. $f(x)=\begin{cases}|x+1|; & x\leq 0\\ x; & x>0\end{cases}$ and $g(x)=\begin{cases}|x|+1; & x\leq 1\\ -|x-2|; & x>1\end{cases}$

Draw its graph and discuss the continuity of f(x) + g(x).

Ans. f(x) + g(x) is discontinuous at x = 0,1

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Sol. $f(1^+) = \lim\limits_{x\to 1^+} \frac{x^2-1}{x^2-2|x-1|-1} = \lim\limits_{x\to 1^+} \frac{x^2-1}{x^2-2(x-1)-1}$

$= \lim\limits_{x\to 1^+} \frac{(x+1)(x-1)}{(x+1)^{-2}} = \lim\limits_{x\to 1^+} \frac{x+1}{x-1} = \lim\limits_{x\to 1^+} \frac{(1+h)+1}{(1+h)-1} = \frac{2+h}{h} = \frac{2}{0} = \infty$

$$= \lim_{x\to 1^-} \frac{(x+1)(x-1)}{(x-1)(x+3)} = \lim_{x\to 1^+} \frac{x^2-1}{x+3} = \lim_{x\to 1^-} \frac{x^2-1}{x^2-2(1-x)-1} \frac{(1-h)+1}{(1-h)+3} = \lim_{x\to 1^+} \frac{2-h}{4-h} = \frac{1}{2}$$

$= \lim\limits_{x\to 1^-} \frac{(x+1)}{(x+1)+2} = \frac{1}{2}$. Hence, $f(x)$ is discontinuous at $x = 1$.

6. Draw the graph and discuss continuity of $\mathrm{f(x)} = [\sin\ \mathrm{x} + \cos\ \mathrm{x}], \mathrm{x} \in [0, 2\pi]$, where [.] represents the greatest integer function.

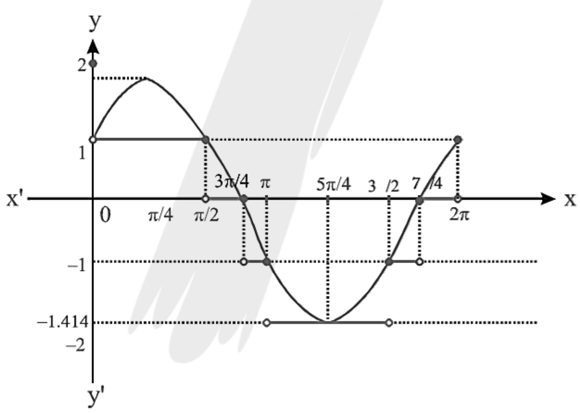
Ans. discontinuous at $\mathrm{x} = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

Sol. Sol. $f(x) = [\sin\ x + \cos\ x] = [g(x)]$, where $\mathrm{g(x)} = \sin\ \mathrm{x} + \cos\ \mathrm{x}$,

$\mathrm{g}(0) = 1, \mathrm{g}\left(\frac{\pi}{4}\right) = \sqrt{2}, \mathrm{g}\left(\frac{\pi}{2}\right) = 1$

$\mathrm{g}\left(\frac{3\pi}{4}\right) = 0, \mathrm{g}(\pi) = -1, \mathrm{g}\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

Clearly, from the graph given in fig. $f(x)$ is discontinuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$



7. Let $\mathrm{f(x)} = \mathrm{Lim}_{n\to\infty} \frac{1}{1+\mathrm{nsin^2x}}$, then find $\mathrm{f}\left(\frac{\pi}{4}\right)$ and also comment on the continuity at $\mathrm{x} = 0$

Ans. $\mathrm{f}\left(\frac{\pi}{4}\right) = 0$, f(x) is discontinuous at x = 0

Sol. Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x} \Rightarrow f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n\cdot\sin^2\frac{\pi}{4}} = \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$

Now $f(0) = \lim_{n\to\infty} \frac{1}{n\cdot\sin^2(0)+1} = \frac{1}{1+0} = 1 \Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty} \frac{1}{1+n\sin^2 x}\right] = 0$
{here $\sin^2$ x is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞} $\therefore$ f(x) is not continuous at x = 0

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8. Discuss the continuity of $f(x)=\begin{cases} x\{x\}+1, & 0\le x<1 \\ 2-\{x\}, & 1\le x\le 2 \end{cases}$. where $\{x\}$ denotes the fractional part function.

Ans. discontinuous at $x = 2$

Sol. $f(0)=f(0^+)=1$ $f(2)=2$ and $f(2^-)=1$
Hence, $f(x)$ is discontinuous at $x=2$. Also, $f(1^+)=2, f(1^-)=1+1=2$, and $f(1)=2$ Hence, $f(x)$ is continuous at $x=1$.

9. If $f(x)=\begin{cases} x+2, & \text{when } x<1 \\ 4x-1, & \text{when } 1\le x\le 3 \\ x^2+5, & \text{when } x>3 \end{cases}$, then correct statement is -

(A) $\lim_{x\to 1} f(x)=\lim_{x\to 3} f(x)$ (B) $f(x)$ is continuous at $x=3$
(C) $f(x)$ is continuous at $x=1$ (D) $f(x)$ is continuous at $x=1$ and 3

Ans. C

10. If $f(x)=\frac{x-e^x+\cos 2x}{x^2}, x\neq 0$ is continuous at $x=0$, then

(A) $f(0)=\frac{5}{2}$ (B) $[f(0)]=-2$
(C) $\{f(0)\}=-0.5$ (D) $[f(0)]\cdot\{f(0)\}=-1.5$

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function

Ans. D

Sol. $\text{Lim}_{x\to 0}\frac{x-e^x+1-(1-\cos 2x)}{x^2}=-\frac{1}{2}-2=-\frac{5}{2}$;
Hence for continuity $f(0)=-\frac{5}{2}$

$\therefore [f(0)]=-3; \{f(0)\}=\left\{-\frac{5}{2}\right\}=\frac{1}{2}$;

Hence $[f(0)]\{f(0)\}=-\frac{3}{2}=-1.5$

11. A function $f(x)$ is defined as below $f(x)=\frac{\cos(\sin x)-\cos x}{x^2}, x\neq 0$ and $f(0)=a$
$f(x)$ is continuous at $x=0$ if 'a' equals
(A) 0 (B) 4 (C) 5 (D) 6

Ans. A

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Sol. Correct option is A)
By L'Hospital's Rule,

$$= \lim_{x\to 0} \frac{-\sin(\sin x)\cdot\cos x + \sin x}{2x}$$

$$= \lim_{x\to 0} \frac{\sin(\sin x)\sin x - \cos^2 x\cdot\cos(\sin x) + \cos x}{2}$$

$$= \frac{-\cos^2 0\cdot\cos(0) + \cos 0}{2} \qquad = \frac{0}{2} \qquad = 0 = a$$

12. Consider the function $f(x) = \begin{cases} x\{x\} + 1 & 0 \le x < 1 \\ 2 - \{x\} & 1 \le x \le 2 \end{cases}$ where $\{x\}$ denotes the fractional part function.
Which one of the following statements is NOT correct?

(A) $\text{Lim}_{x\to 1} f(x)$ exists
(B) $f(0) \ne f(2)$
(C) $f(x)$ is continuous in $[0,2]$
(D) Rolle's theorem is not applicable to $f(x)$ in $[0,2]$

Ans. **C**

Sol. $f(1^+) = f(1^-) = f(1) = 2 \; f(0) = 1,\; f(2) = 2$

$f(2^-) = 1;\; f(2) = 2$

$\Rightarrow$ f is not continuous at $x = 2$

13. Given $f(x) = \frac{e^x - \cos 2x - x}{x^2}$ for $x \in R - \{0\}$

$$g(x) = \begin{cases} f(\{x\}) & \text{for } n < x < n + \frac{1}{2} \\ f(1 - \{x\}) & \text{for } n + \frac{1}{2} \le x < n + 1, n \in I \\ \frac{5}{2} & \text{otherwise} \end{cases} \left\{ \begin{array}{l} \text{where } \{x\} \text{ denotes} \\ \text{fractional part function} \end{array} \right.$$

then g (x) is

(A) discontinuous at all integral values of x only
(B) continuous everywhere except for $x = 0$
(C) discontinuous at $x = n + \frac{1}{2}$; $n \in I$ and at some $x \in I$
(D) continuous everywhere

Ans. **D**

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Sol. $\text{Limit}_{h\to 0}\ g(n+h) = \text{Limit}_{h\to 0} \frac{e^h - \cos\ 2h - h}{h^2}$

$$= \text{Limit}_{h\to 0} \frac{e^h - h - 1}{h^2} + \text{Limit}_{h\to 0} \frac{(1-\cos\ 2h)}{4h^2} \cdot 4 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\text{Limit}_{h\to 0}\ g(n-h) = \frac{e^{1-\{n-h\}} - \cos\ 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2}$$

$$= \text{Lim}_{h\to 0} \frac{e^h - \cos\ 2h - h}{h^2} = \frac{5}{2} (\{n-h\} = \{-h\} = 1 - h)$$

$g(n) = \frac{5}{2}$. Hence $g(x)$ is continuous at $\forall x \in I$.

Hence g(x) is continuous $\forall x \in R$

14. If $f(x) = \begin{cases} \frac{\log(1+2ax) - \log(1-bx)}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$, is continuous at $x = 0$, then k is equal to -

(A) $2a + b$ (B) $2a - b$ (C) $b - 2a$ (D) $a + b$

Ans. **A**

Sol. Given, $f(x) = \begin{cases} \frac{\log(1+2ax) - \log(1-bx)}{x}, & x \equiv 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$.

$$\therefore f(0) = \lim_{x\to 0} \frac{\log(1+2ax) - \log(1-bx)}{x} \left(\frac{0}{0} \text{ form}\right)$$

$$\Rightarrow k = \lim_{x\to 0} \frac{\frac{1}{2ax+1}(2a) - \frac{1}{1-bx}(-b)}{+1} \text{ (by 'L' Hospital's rule)}$$

$$\Rightarrow k = \frac{2a}{0+1} + \frac{b}{1-0} = 2a + b$$

15. If $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & , x < 0 \\ a & , x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, & x > 0 \end{cases}$, then correct statement is -

(A) f(x) is discontinuous at $x = 0$ for any value of a

(B) f(x) is continuous at $x = 0$ when $a = 8$

(C) f(x) is continuous at $x = 0$ when $a = 0$

(D) none of these

Ans. **B**

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Sol. $\lim\limits_{x\to0^-}\frac{1-\cos 4x}{x^2}=8$

$\lim\limits_{x\to0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}=8 \because f(0)=8$

So $f(x)$ is continuous at $x=0$ when $a=8$

16. Let $f(x)=\begin{cases}\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}} & \text{if } x>2\\ \dfrac{x^2-4}{x-\sqrt{3x-2}} & \text{if } x<2\end{cases}$ then

(A) $f(2)=8 \Rightarrow f$ is continuous at $x=2$
(B) $f(2)=16 \Rightarrow f$ is continuous at $x=2$
(C) $f(2^-)\neq f(2^+) \Rightarrow f$ is discontinuous
(D) f has a removable discontinuity at $x=2$

Ans. C

Sol. $f(2^+)=8; f(2^-)=16$

More than one answer type

17. Let $f(x)=\frac{|x+\pi|}{\sin x}$, then

(A) $f(-\pi^+)=-1$ (B) $f(-\pi)=1$
(C) $\lim_{x\to-\pi}f(x)$ does not exist (D) $\lim_{x\to\pi}f(x)$ does not exist

Ans. ABCD

Sol. $f(x)=\frac{|x+\pi|}{\sin x}$

(A) $f(-\pi^+)=\lim_{h\to0}\frac{|-\pi+h+\pi|}{\sin(-\pi+h)}=\lim_{h\to0}\frac{|h|}{-\sin h}=-1$
(B) $f(-\pi^-)=\lim_{h\to0}\frac{|-\pi-h+\pi|}{\sin(-\pi-h)}=\lim_{h\to0}\frac{|h|}{\sin h}=1$
(C) $f(-\pi^+)\neq f(-\pi^-)$ So $\lim_{x\to-\pi}f(x)$ does not exist
(D) for $\lim_{x\to\pi}f(x)$

$$\text{LHL} = \lim_{x\to\pi^-}\frac{|x+\pi|}{\sin x}=\lim_{h\to0}\frac{2\pi-h}{\sin h}=\frac{2\pi}{0}=\infty$$

$$\text{RHL} = \lim_{x\to\pi^+}\frac{|x+\pi|}{\sin x}=\lim_{h\to0}\frac{2\pi+h}{-\sin h}=-\frac{2\pi}{0}=-\infty$$

LHL $\neq$ RHL

So $\lim_{x\to\pi}f(x)$ does not exist.

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18. On the interval $I=[-2,2]$, the function $f(x)=\begin{cases}(x+1)e^{-\left[\frac{1}{|x|}+\frac{1}{x}\right]} & (x\neq 0)\\ 0 & (x=0)\end{cases}$ then which one of the following hold good?

(A) is continuous for all values of $x \in I$

(B) is continuous for $x \in I-\{0\}$

(C) assumes all intermediate values from $f(-2)\&f(2)$

(D) has a maximum value equal to $3/e$

Ans. **BCD**

Sol. $\lim\limits_{x\to 0^+}(x+1)e^{-[2/x]}=\lim\limits_{x\to 0^+}\frac{x+1}{e^{2/x}}=\frac{1}{e^{\infty}}=0$

$\lim\limits_{x\to 0^-}(x+1)e^{-\left(-\frac{1}{x}+\frac{1}{x}\right)}=1$

Hence continuous for $x \in I-\{0\}$

19. The function, $f(x)=[|x|]-|[x]|$ where $[x]$ denotes greatest integer function

(A) is continuous for all positive integers

(B) is discontinuous for all non positive integers

(C) has finite number of elements in its range

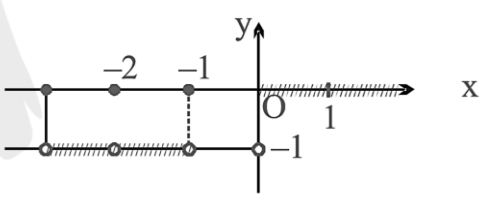
(D) is such that its graph does not lie above the x-axis.

Ans. **ABCD**

Sol. $[|x|]-|[x]|=\begin{bmatrix}0 & x=-1\\ -1 & -1<x<0\\ 0 & 0\le x\le 1\\ 0 & 1<x\le 2\end{bmatrix}$

$\Rightarrow$ range is $\{0,-1\}$

The graph is



20. f is a continous function in $[a,b]$; g is a continuous function in $[b,c]$

A function $h(x)$ is defined as $h(x)=\begin{cases}f(x) & \text{for } x\in[a,b)\\ g(x) & \text{for } x\in(b,c]\end{cases}$ if $f(B)=g(B)$, then

(A) $h(x)$ has a removable discontinuity at $x=b$.

(B) $h(x)$ may or may not be continuous in $[a,c]$

(C) $h(b^-)=g(b^+)$and $h(b^+)=f(b^-)$

(D) $h(b^+)=g(b^-)$and $h(b^-)=f(b^+)$

Ans. **AC**

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Sol. Given f is continuous in [a, b]
g is continuous in [b, c]
$f(b) = g(b)$

$$h(x) \left.\begin{aligned} &= f(x) \text{ for } x \in [a, b) \\ &= f(b) = g(b) \text{ for } x = b \\ &= g(x) \text{ for } x \in (b, c] \end{aligned}\right\}$$

h(x) is continuous in [a, b) ∪ (b, c] [using (i), (ii)]
also $f(b^-) = f(b); g(b^+) = g(b)$
$\therefore h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$
[using (iv), (v)]
now, verify each alternative. $g(b^-)$and $f(b^+)$are undefined.
$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$
and $h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$
hence $h(b^-) = h(b^+) = f(b) = g(b)$
and $h(b)$ is not defined ⇒ (A)

21. Function whose jump (non-negative difference of LHL & RHL) of discontinuity is greater than or equal to one, is/are -

(A) $f(x) = \begin{cases} \frac{(e^{1/x}+1)}{(e^{1/x}-1)}; & x < 0 \\ \frac{(1-\cos x)}{x}; & x > 0 \end{cases}$

(B) $g(x) = \begin{cases} \frac{x^{1/3}-1}{x^{1/2}-1}; & x > 1 \\ \frac{\ell nx}{(x-1)}; & \frac{1}{2} < x < 1 \end{cases}$

(C) $u(x) = \begin{cases} \frac{\sin^{-1}2x}{\tan^{-1}3x}; & x \in \left(0, \frac{1}{2}\right] \\ \frac{|\sin x|}{x}; & x < 0 \end{cases}$

(D) $v(x) = \begin{cases} \log_3(x+2); & x > 2 \\ \log_{1/2}(x^2+5); & x < 2 \end{cases}$

Ans. **ACD**

Sol. (A) LHL $= -1$ &RHL $= 0$
(B) LHL $= 1$ &RHL $= 2/3$
(C) LHL $= -1$ &RHL $= 2/3$
(D) LHL $= -2\log_2 3$&RHL $= 2\log_3 2$

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Answer Key

2. **(i)** $x = 2n\pi + \frac{\pi}{6}$ or $x = 2n\pi + \frac{5\pi}{6}, n \in Z$ **(ii)** continuous $\forall$ x $\in$ R.

3. f(0) cannot be defined. **4.** $\frac{\pi}{2} - 1$ **5.** f(x) + g(x) is discontinuous at x = 0,1

6. discontinuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ **7.** $f\left(\frac{\pi}{4}\right) = 0$, f(x) is discontinuous at x = 0

8. discontinuous at x = 2

9.	C	**10.**	D	**11.**	A	**12.**	C	**13.**	D	**14.**	A	**15.**	B	**16.**	C
17.	ABCD	**18.**	BCD	**19.**	ABCD	**20.**	AC	**21.**	ACD						