FUNCTIONS

DEFINITION

A relation R from a set A to a set B is called a function if each element of A has unique image in B. It is denoted by the symbol.

$$f: A \to B \text{ or } A \xrightarrow{f} B$$

which reads ' f ' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a ' or 'image of a under f ' 'or' 'the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

$$b = f(A) \text{ or } f: a \to b \text{ or } f: (a, b)$$

Thus a function ' f ' from a set A to a set B is a subset of $A \times B$ in which each ' a ' belonging to A appears in one and only one ordered pair belonging to f.

REPRESENTATION OF FUNCTION

(A) **Ordered pair :** Every function from $A \to B$ satisfies the following conditions :

(i) $f \subset AxB$

(ii) $\forall a \in A$ there exist $b \in B$ and

(iii) $(a, b) \in f \;\&\; (a, c) \in f \Rightarrow b = c$

(B) **Formula based** (uniformly/non-uniformly) :

e.g. **(i)** $f: R \to R, y = f(x) = 4x, f(x) = x^2$

(uniformly defined)

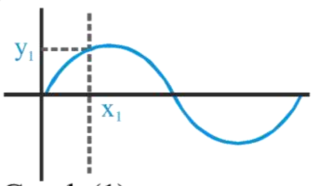
(ii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$

(non-uniformly defined)

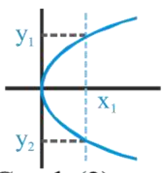
(iii) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$

(non-uniformly defined)

(C) **Graphical representation:**



Graph (1)



Graph (2)

Graph (1) represent a function but graph (2) does not represent a function.

Domain, Co-domain & Range Of a Function

Let $f: A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Thus: Domain of $f = \{a \mid a \in A, (a, f(A)) \in f\}$

Range of $f = \{f(A) \mid a \in A, f(A) \in B\}$

KEY POINTS

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

(ii) Every function is a relation but every relation is not necessarily a function.

(iii) It should be noted that range is a subset of co-domain.

(iv) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

METHODS OF DETERMINING RANGE

(i) Representing x in terms of y

If $y = f(x)$, try to express as $x = g(y)$, then domain of $g(y)$ represents possible values of y, which is range of $f(x)$.

Ex. Find the range of $f(x) = \frac{x^2+x+1}{x^2+x-1}$

Sol. $f(x) = \frac{x^2+x+1}{x^2+x-1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor $\}$ $y = \frac{x^2+x+1}{x^2+x-1}$

➢ $y^2 + yx - y = x^2 + x + 1$

➢ $(y - 1)x^2 + (y - 1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. Which is not true.

Further if $y \neq 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if

$(y - 1)^2 - 4(y - 1)(-y - 1) \geq 0$

i.e. if $y \leq -3/5$ or $y \geq 1$ but $y \neq 1$

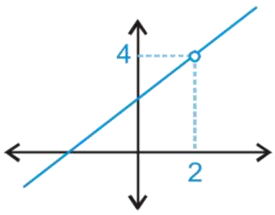
Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

(ii) Graphical Method

The set of y-coordinates of the graph of a function is the range.

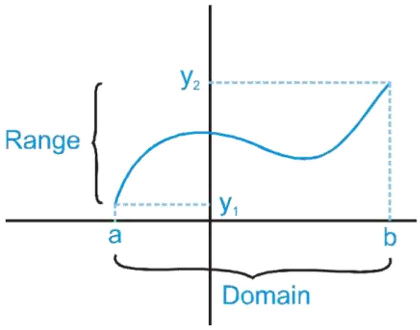
Ex. Find the range of $f(x) = \frac{x^2-4}{x-2}$

Sol. $f(x) = \frac{x^2-4}{x-2} = x + 2; x \neq 2$



➢ According to the graph of f(x), the range of f(x) is $R - \{4\}$

Further if f(x) happens to be continuous in its domain then range of f(x) is [min. f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min. f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Ex. Find the Domain of the following function:

(i) $f(x) = \sqrt{\sin\ x} + \sqrt{16 - x^2}$

(ii) $f(x) = \log_2 \left(-\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

Sol. **(i)** $\sin\ x \geq 0$ and $16 - x^2 \geq 0$

$2n\pi \leq x \leq (2n+1)\pi$ and $-4 \leq x \leq 4$

Domain is $[-4, -\pi] \cup [0, \pi]$

(ii) We have $f(x) = \log_2 \left(-\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$.

f(x) is defined if $-\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1 > 0$

or if $\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}}\right) < -1$

or if $\left(1 + \frac{1}{\sqrt[4]{x}}\right) > (1/2)^{-1}$ or if $1 + \frac{1}{\sqrt[4]{x}} > 2$

or if $\frac{1}{\sqrt[4]{x}} > 1$ or if $x^{1/4} < 1$ or if $0 < x < 1$

$D(F) = (0,1)$

Ex. Find the range of following functions :

(i) $f(x) = \log_2 \left(\frac{\sin\ x - \cos\ x + 3\sqrt{2}}{\sqrt{2}}\right)$

(ii) $f(x) = \log_{\sqrt{2}} \left(2 - \log_2 (16\sin^2\ x + 1)\right)$

Sol. **(i)** Let $y = \log_2 \left(\frac{\sin\ x - \cos\ x + 3\sqrt{2}}{\sqrt{2}}\right)$

$2^y = \sin\left(x - \frac{\pi}{4}\right) + 3 \Rightarrow -1 \leq 2^y - 3 \leq 1$

$2 \leq 2^y \leq 4 \Rightarrow y \Rightarrow [1,2]$

(ii) $f(x) = \log_{\sqrt{2}} \left(2 - \log_2 (16\sin^2\ x + 1)\right)$

$1 \leq 16\sin^2\ x + 1 \leq 17$

$0 \le \log_2 (16\sin^2 x + 1) \le \log_2 17$

$2 - \log_2 17 \le 2 - \log_2 (16\sin^2 x + 1) \le 2$

Now consider $0 < 2 - \log_2(16\sin^2 x + 1) \le 2$

$-\infty < \log_{\sqrt{2}}[2 - \log_2(16\sin^2 x + 1)] \le \log_{\sqrt{2}} 2$

The range is $(-\infty, 2]$

NUMBER OF FUNCTION

Let A and B be two finite sets having m and n elements respectively. Then, each element of set A can be associated to any one of n elements of set A. So, total number of functions from set A to set B is equal to the number of ways of doing m jobs where each job can be done in n ways.

The total number of such ways is $n \times n \times n \ldots . . \times n$ (m-times) $= n^m$.

Hence, the total number of functions from A to B is n^m For example, the total number of functions from a set $A = \{a, b, c, d\}$ to a set $B = \{1,2,3\}$ is $3^4 = 81$.

The total number of relations from a set A having m elements to a set n having n elements is 2^{mn}.

So, the number of relations from A to B which are not functions is $2^{mn} - n^m$.

(i) Polynomial Function

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

KEY POINTS

(A) A polynomial of degree one with no constant term is called an odd linear function. i.e.

$f(x) = ax, a \neq 0$

(B) There are two polynomial functions, satisfying the relation; $f(x) \cdot f(1/x) = f(x) + f(1/x)$.

They are :

(a) $f(x) = x^n + 1$ &

(b) $f(x) = 1 - x^n$, where n is a positive integer.

(ii) Algebric Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form

$P_0(x)y^n + P_1(x)y^{n-1} + \cdots \ldots + P_{n-1}(x)y + P_n(x) = 0$

Where n is a positive integer and $P_0(x), P_1(x) \ldots \ldots \ldots\ldots$ are Polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

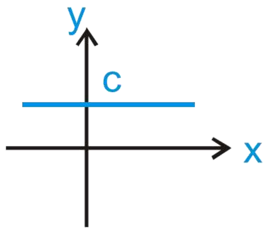
(iii) Rational Function

A rational function is a function of form. $y = f(x) = \frac{g(x)}{h(x)}$, where $g(x)$&$h(x)$ are polynomials & $h(x) \neq 0$.

(iv) Constant function

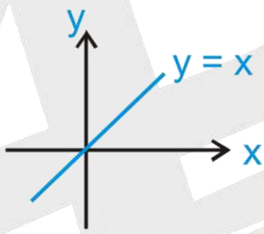
A function f: A → B is said to be a constant function, if every element of A has the same f image in B. Thus f: A → B;

$f(x) = c, \forall x \in A, c \in B$ is a constant function.



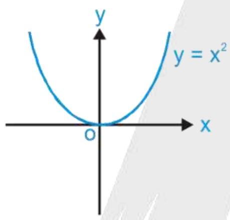
(v) Identity function

The function f: A → A defined by, $f(x) = x, \forall x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.



BASIC ALGEBRAIC FUNCTION

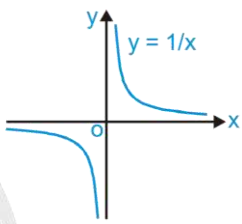
(i) $y = x^2$



Domain : R

Range : $R^+ \cup \{0\}$ or $[0, \infty)$

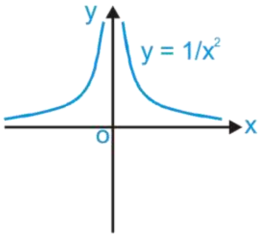
(ii) $y = \frac{1}{x}$



Domain : $R - \{0\}$ or R_0

Range : $R - \{0\}$

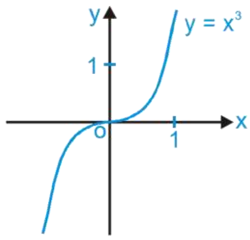
(iii) $y = \frac{1}{x^2}$



Domain : R_0

Range : R^+or $(0, \infty)$

(iv) $y = x^3$



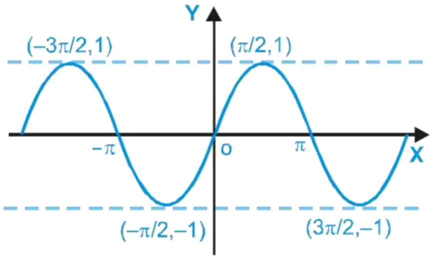
Domain : R

Range : R

TRIGONOMETRIC FUNCTIONS

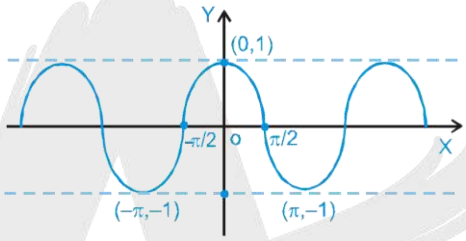
(I) **Sine function** $f(x) = \sin x$

Domain: R **Range** : $[-1,1]$, Period 2π



(ii) **Cosine function** $f(x) = \cos x$

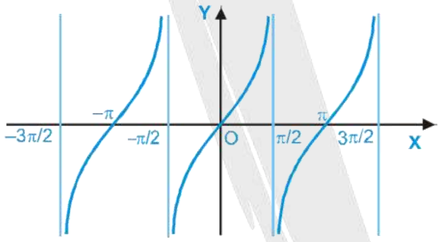
Domain : R **Range** : $[-1,1]$, period 2π



(iii) **Tangent function** $f(x) = \tan x$

Domain: $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

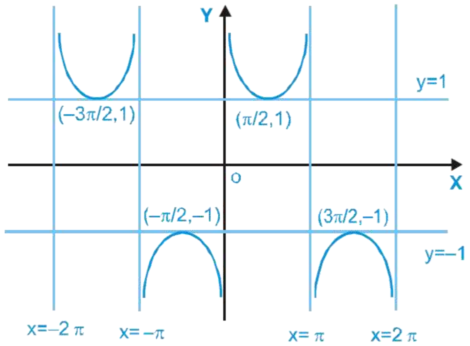
Range: R, period π



(iv) **Cosecant function** $f(x) = \operatorname{cosec} x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

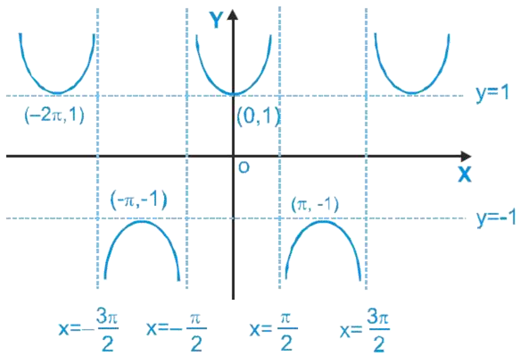
Range: R $- (-1,1)$, period 2π



(v) **Secant function** $f(x) = \sec\ x$

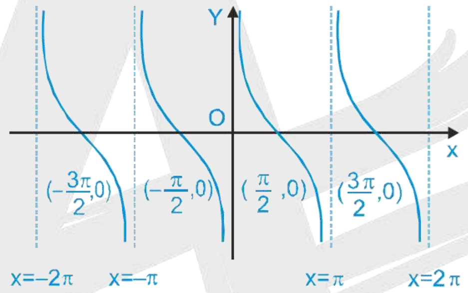
Domain: $R - \{x \mid x = (2n+1)\pi/2 : n \in I\}$

Range : $R - (-1,1)$, period 2π



(vi) **Cotangent function** $f(x) = \cot\ x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$ Range : R, period π

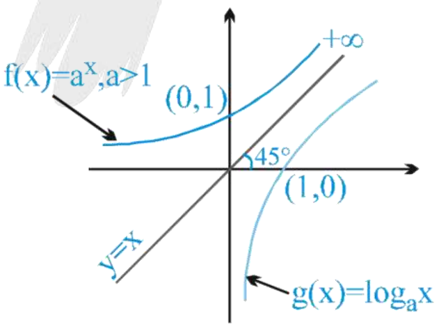


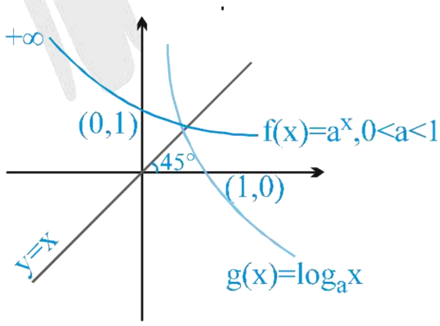
EXPONENTIAL FUNCTION

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function.

i.e. $g(x) = \log_a\ x$.

Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown.





ABSOLUTE VALUE FUNCTION

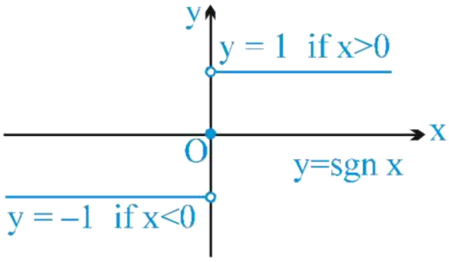
A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined

as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

SIGNUM FUNCTION

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

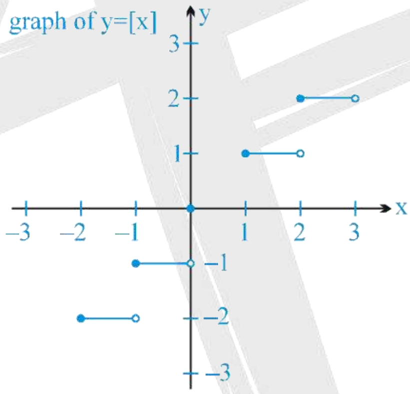


It is also written as $\text{Sgn } x = |x|/x; x \neq 0; f(0) = 0$

GREATEST INTEGER OR STEP UP FUNCTION

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

$$-1 \leq x < 0; [x] = -1 \quad 0 \leq x < 1; [x] = 0$$
$$1 \leq x < 2; [x] = 1 \quad 2 \leq x < 3; [x] = 2 \text{ and so on}$$



Properties of greatest integer function

(A) $[x] \leq x < [x] + 1$ and

$x - 1 < [x] \leq x, 0 \leq x - [x] < 1$

(B) $[x + m] = [x] + m$ if m is an integer .

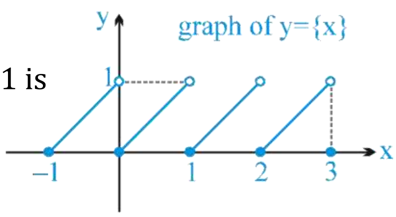
(C) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(D) $[x] + [-x] = \begin{cases} 0 & \text{if x is an int eger} \\ -1 & \text{otherwise.} \end{cases}$

FRACTIONAL PART FUNCTION

It is defined as : $g(x) = \{x\} = x - [x]$ e.g. the fractional part of the no. 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3.

The period of this function is 1 and graph of this function is as shown.



Ex. Determine the values of x satisfying the equality $|x^4 - x^2 - 6| = |x^4 - 4| - |x^2 + 2|$.

Sol. The equality $|a - b| = |a| - |b|$ holds true if and only if a and b have the same sign and $|a| \geq |b|$. In our case the equality will hold true for the value of x at which $x^4 - 4 \geq x^2 + 2$. Hence $x^2 - 2 \geq 1; |x| \geq \sqrt{3}$.

Ex. If $y = 2[x] + 3 \& y = 3[x - 2] + 5$ then find $[x + y]$ where [. . denotes greatest integer function.

Sol. $y = 3[x - 2] + 5 = 3[x] - 1$ So $3[x] - 1 = 2[x] + 3$ $[x] = 4 \Rightarrow 4 \leq x < 5$ Then $y = 11$

So $x + y$ will lie in the interval $[15,16) \therefore [x + y] = 15$

Ex. Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and $\{\cdot\}$ denotes fractional part function.

Sol. We are given that, $|2x - 1| = 3[x] + 2\{x\}$

Let, $2x - 1 \leq 0$ i.e. $x \leq \frac{1}{2}$.

The given equation yields. $1 - 2x = 3[x] + 2\{x\}$

- $1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \Rightarrow 1 - 5[x] = 4\{x\}$
- $\{x\} = \frac{1-5[x]}{4} \Rightarrow 0 \leq \frac{1-5[x]}{4} < 1$
- $0 \leq 1 - 5[x] < 4 \Rightarrow -\frac{3}{5} < [x] \leq \frac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

- $\{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$,

Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \frac{1}{2}$

- $2x - 1 = 3[x] + 2\{x\}$
- $2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$
- $[x] = -1$
- $-1 \leq x < 0$ which is not a solution as $x > \frac{1}{2}+$
- $x = \frac{1}{4}$ is the only solution.

ALGEBRAIC OPERATIONS ON FUNCTIONS

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f.g)$ and (f/g) as follows :

$$\left.\begin{array}{ll}\textbf{(i)} & (f \pm g)(x) = f(x) \pm g(x) \\ \textbf{(ii)} & (f.g)(x) = f(x) \cdot g(x)\end{array}\right] - \text{domain in each case is } A \cap B$$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is **(iv)** $(kf)(x) = kf(x)$ where k is a scalar.

Equal or Identical Function

Two functions f&g are said to be equal if:

(i) The domain of f = the domain of g

(ii) The range of f = the range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain.

eg. $f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions

Ex. The functions $f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$ are identical when x lies in the interval

Sol. Since $f(x) = \log(x-1) - \log(x-2)$.

Domain of f(x) is $x > 2$ or $x \in (2, \infty)$

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

From (i) and (ii), $x \in (2, \infty)$.

CLASSIFICATION OF FUNCTIONS

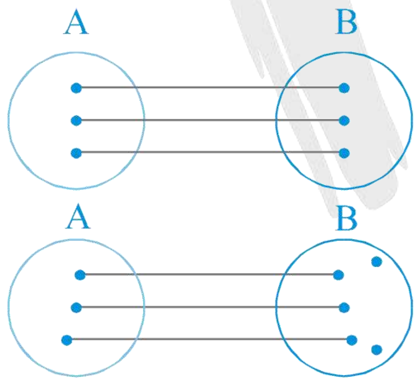
One - One Function (Injective mapping)

A function $f: A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Examples : $R \to R f(x) = x^3 + 1; f(x) = e^{-x}; f(x) = \ln x$

Diagramatically an injective mapping can be shown as



KEY POINTS

(i) A continuous function which is always increasing or decreasing in whole domain, then f(x) is one-one.

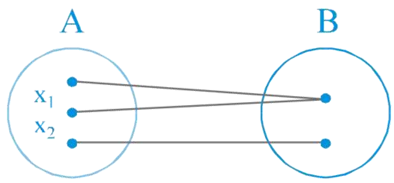
(ii) If any line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

Many-one function (Not injective)

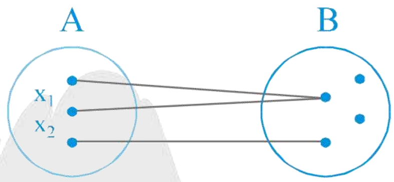
A function $f: A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f: A \rightarrow B$ is many one if for; $x_1, x_2 \in A, f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Ex. $R \rightarrow R$ $f(x) = [x]; f(x) = |x|; f(x) = ax^2 + bx + c; f(x) = \sin x$

Diagramatically a many one mapping can be shown as



OR



KEY POINTS

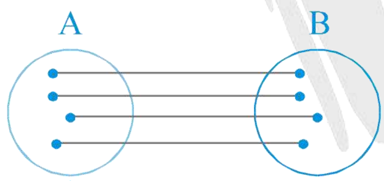
(i) Any continuous function which has atleast one local maximum or local minimum, then f(x) is many-one . In other words, if a line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(ii) If a function is one-one, it cannot be many-one and vice versa.

One One + Many One = Total number of mappings.

Onto function (Surjective mapping)

If the function $f: A \rightarrow B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \rightarrow B$ is surjective iff $\forall b \in B, \exists$ some $a \in A$ such that $f(A) = b$.



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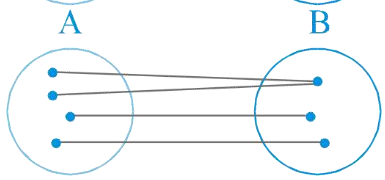
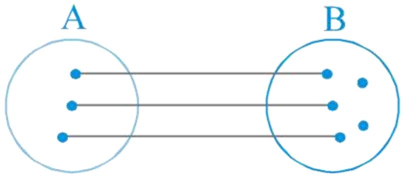


If range = co-domain, then f(x) is onto.

Into function

If $f: A \rightarrow B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.



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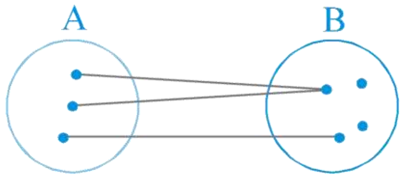


KEY POINTS

If a function is onto, it cannot be into and vice versa. A polynomial of degree even define from $R \to R$ will always be into & a polynomial of degree odd defined from $R \to R$ will always be onto.

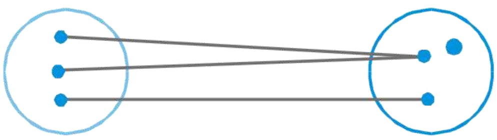
Thus a function can be one of these four types :

(i) one-one onto (injective & surjective) $(I \cap S)$

(ii) one-one into (injective but not surjective) $(I \cap \bar{S})$

(iii) many-one onto (surjective but not injective) $(S \cap \bar{I})$

(iv) many-one into (neither surjective nor injective) $(\bar{I} \cap \bar{S})$



(v) If f is both injective & surjective, then it is called a Bijective mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(vi) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one one.

Ex Let $f: R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then which type of function f is ?

Sol. We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \neq -2$

Also, f is not onto as $f(x) \geq 0 \forall x \in R$

$\therefore$ range of $f = [0, \infty) \subset R$

Ex Let $f(x) = \frac{x^2+3x+a}{x^2+x+1}$, where $f: R \to R$. Find the value of parameter ' a ' so that the given function is one-one.

Sol. $f(x) = \frac{x^2+3x+a}{x^2+x+1}$;

$$f'(x) = \frac{(x^2+x+1)(2x+3) - (x^2+3x+a)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{-2x^2 + 2x(1-a) + (3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x(1-a) + (3-a)$, g(x) will be negative if $4(1-a)^2 + 8(3-a) < 0$

➢ $1 + a^2 - 2a + 6 - 2a < 0$

➢ $(a-2)^2 + 3 < 0$ (which is not possible)

Therefore function is not monotonic.

Hence, no value of a is possible.

COMPOSITE OF UNIFORMLY & NON-UNIF ORMLY DEFINED FUNCTIONS

Let $f: A \to B$ & $g: B \to C$ be two functions. Then the function $gof : A \to C$ defined by

$(gof)(x) = g(f(x)) \forall x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} f \xrightarrow{f(x)} g \longrightarrow g(f(x))$. Thus the image of every $x \in A$ under the function gof is the g-image of the f-image of x.

Note that gof is defined only if $\forall x \in A$, f(x) is an element of the domain of g so that we can take its g image. Hence for the product gof of two functions f & g, the range of f must be a subset of the domain of g.

Properties of Composite Functions

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then $fo(goh) = (fog)oh$.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables. For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y. Symbolically if $f(tx, ty) = t^n$. Then $f(x, y)$ is homogeneous function of degree n.

BOUNDED FUNCTION

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

IMPLICIT & EXPLICIT FUNCTION

A function defined by an equation not solved for the dependent variable is called an **Implicit Function.**

For eg. the equation $\sin(x + e^y) = 3y$ defines y as an implicit function.

If y has been expressed in terms of x alone then it is called an **Explicit Function.**

e.g. $x + 2y = 0$

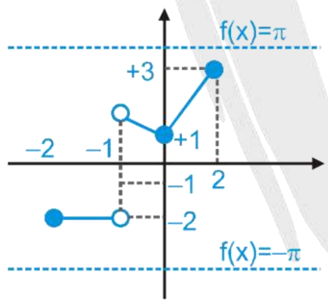
Ex. Find the domain and range of $h(x) = g(f(x))$,

where $f(x) = \begin{cases} [x], & -2 \leq x \leq -1 \\ |x| + 1, & -1 < x \leq 2 \end{cases}$ and $g(x) = \begin{cases} [x], & -\pi \leq x \leq 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$,

[.] denotes the greatest integer function.

Sol. $h(x) = g(f(x)) = \begin{cases} [f(x)], & -\pi \leq f(x) < 0 \\ \sin(f(x)), & 0 \leq f(x) \leq \pi \end{cases}$ From graph of f(x), we get

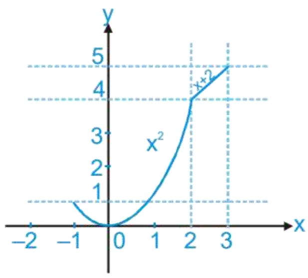
$h(x) = \begin{cases} [[x]], & -2 \leq x \leq -1 \\ \sin(|x| + 1), & -1 < x \leq 2 \end{cases}$



➢ Domain of h (x) is $[-2,2]$ and Range of $h(x)$ is $\{-2,1\} \cup [\sin 3, 1]$

Ex Let $f(x) = \begin{cases} x + 1, & x \leq 1 \\ 2x + 1, & 1 < x \leq 2 \end{cases}$ and

$g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x + 2, & 2 \leq x \leq 3 \end{cases}$, find $(fog)(x)$



Sol. $f(g(x)) = \begin{cases} g(x) + 1, & g(x) \leq 1 \\ 2g(x) + 1, & 1 < g(x) \leq 2 \end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.

It is clear that $g(x) \leq 1;\ \forall x \in [-1,1]$ and $1 < g(x) \leq 2;\ \forall x \in (1, \sqrt{2}]$

➢ $f(g(x)) = \begin{cases} x^2 + 1, & -1 \le x \le 1 \\ 2x^2 + 1, & 1 < x \le \sqrt{2} \end{cases}$

Ex Which of the following function is not homogeneous?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$

(C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Sol. It is clear that (D) does not have the same degree in each term.

Ex Which of the following function is implicit function?

(A) $y = \frac{x^2+e^x+5}{\sqrt{1-\cos^{-1} x}}$ (B) $y = x^2$

(C) $xy - \sin(x + y) = 0$ (D) $y = \frac{x^2 \log x}{\sin x}$

Sol. It is clear that in (C) y is not clearly expressed in x.

INVERSE OF A FUNCTION

Let $f: A \to B$ be a one-one & onto function, then their exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall x \in A$ & $y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of Inverse Function

(i) The inverse of a bijection is unique.

(ii) If $f: A \to B$ is a bijection & $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. Note that the graphs of f&g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2$ $(x \ge 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

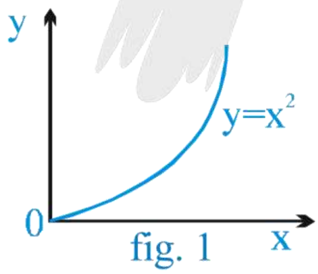


fig. 1

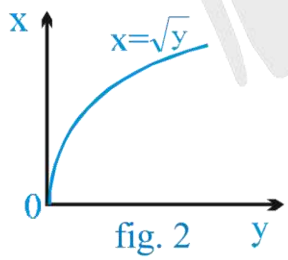


fig. 2

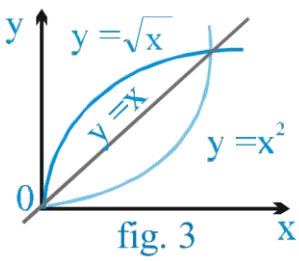


fig. 3

(iii) The inverse of a bijection is also a bijection.

(iv) If f & g are two bijections $f: A \to B$, $g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

Ex Find the inverse of the function $f(x) = \begin{cases} x; x < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x}; x > 4 \end{cases}$

Sol. Given $f(x) = \begin{cases} x; x < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x}; x > 4 \end{cases}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\therefore x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ \frac{y^2}{64}, & \frac{y^2}{64} > 4 \end{cases} \Rightarrow f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ \frac{y^2}{64}, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x; < 1 \\ \sqrt{x}; 1 \le x \le 16 \\ \frac{x^2}{64}; x > 16 \end{cases}$

Ex. **(i)** Determine whether $f(x) = \frac{2x+3}{4}$ for $f: R \to R$, is invertible or not? If so find it.

(ii) Is the function $f(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ invertible ?

Sol. **(i)** Given function is one-one and onto, therefore it is invertible $y = \frac{2x+3}{4}$

➢ $x = \frac{4y-3}{2} \quad \therefore f^{-1}(x) = \frac{4x-3}{2}$

(ii) Domain of f is $[-1,1]$ $f(0) = 0 = f(1)$

$\Rightarrow$ f is not one - one $\Rightarrow$ f is not invertible

ODD & EVEN FUNCTIONS

If a function is such that whenever ' x ' is in it's domain ' $-x$ ' is also in it's domain & it satisfies $f(-x) = f(x)$, it is an even function, $f(-x) = -f(x)$, it is an odd function

KEY POINTS

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many - one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '- x ' in it's domain whenever ' x ' is in it's domain, can be expressed as the sum of an even & an odd function.

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$

(v) The only function which is defined on the entire number line & even and odd at the same time is $f(x) = 0$.

$f(x)$	$g(x)$	$f(x)+g(x)$	$f(x)-g(x)$	$f(x)\cdot g(g)$	$f(x)/g(x)$	$(gof)(x)$	$(fog)(x)$
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Ex. Show that $a^x + a^{-x}$ is an even function.

Sol. Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.

Hence f(x) is an even function

Ex Identify the given functions as odd, even or neither :

(i) $f(x) = \frac{x}{e^x-1} + \frac{x}{2} + 1$

(ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

Sol. (i) Clearly domain of f(x) is $R \sim \{0\}$.

We have,

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1$$

$$= \frac{-e^x \cdot x}{1-e^x} - \frac{x}{2} + 1$$

$$= \frac{(e^x-1+1)x}{(e^x-1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x-1} - \frac{x}{2} + 1 = \frac{x}{e^x-1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) Replacing x, y by zero, we get $f(0) = 2f(0)$

➢ $f(0) = 0$

Replacing y by $-x$, we get $f(x) + f(-x) = f(0) = 0$

➢ $f(x) = -f(-x)$

Hence f(x) is an odd function.

PERIODIC FUNCTIONS

A function f(x) is called periodic with a period T if there exists a real number $T > 0$ such that for each x in the domain of f the numbers $x - T$ and $x + T$ are also in the domain of f and $f(x) = f(x + T)$ for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of ' T '.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π. The least positive period is called the Principal or Fundamental period of f(x) or simply the period of the function.

⋆Inverse of a periodic function does not exist.

Properties of Periodic Functions

(A) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(B) If f(x) has a period T, then $f(ax + b)$ has a period $\frac{T}{|a|}$.

(C) Every constant function defined for all real x, is always periodic, with no fundamental period.

(D) If f(x) has a period T_1 and g(x) also has a period T_2 then period of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ or $\frac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $= \left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M. }(a,p,\ell)}{\text{H.C.F. }(b,q,m)}$

e.g. $|\sin x|$ has the period π, $|\cos x|$ also has the period π $\therefore$ $|\sin x| + |\cos x|$ also has a period π. But the fundamental period of $|\sin x| + |\cos x|$ is $\frac{\pi}{2}$.

(E) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Ex. Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) $f(x) = \{x\} + \sin x$, where $\{.\}$ denotes fractional part function

(iii) $f(x) = \cos x \cdot \cos 3x$

(iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Sol. **(i)** Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π. Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π { L.C.M. of 4 and 6 is 12}

(ii) Period of $\sin x = 2\pi$ Period of $\{x\} = 1$ but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

➢ It is aperiodic.

(iii) $f(x) = \cos x \cdot \cos 3x$ period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \frac{2\pi}{n}$, where $n \in N$.

Hence cross-checking for $n = 1,2,3, \ldots$ we find π to be fundamental period

$f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of

$\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3} =$ L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2} = 12\pi$

KEY POINTS

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y)$

➢ $f(x) = k\ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y)$

➢ $f(x) = x^n, n \in R$

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y)$

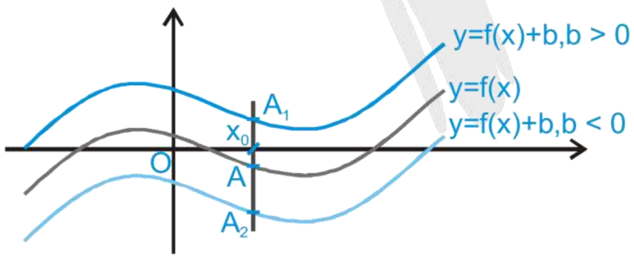
➢ $f(x) = kx$, where k is a constant.

Ex. If f(x) is a polynomial function satisfying $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall x \in R - \{0\}$ and $f(2) = 9$ then find f(3)

Sol. $f(x) = 1 \pm x^n$ As $f(2) = 9, n = 3\ \therefore\ f(x) = 1 + x^3$ Hence $f(3) = 1 + 3^3 = 28$

BASIC TRANSFORMATIONS ON GRAPHS

(i) Drawing the graph of $y = f(x) + b, b \in R$, from the known graph of $y = f(x)$

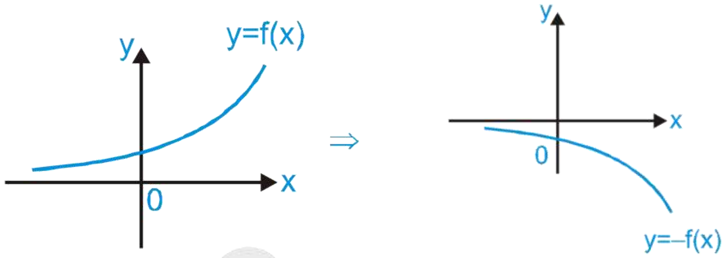


It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of $f(x). y|_{x=x_0} = f(x_0)$ The corresponding point on f(x) +b would be $f(x_0) + b$.
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on $f(x) + b$ would be lying at a distance ' b ' units above the point on f(x).
For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on $f(x) + b$ would be lying at a distance ' b ' units below the point on f(x).

Accordingly the graph of $f(x)+b$ can be obtained by translating the graph of $f(x)$ either in the positive y-axis direction (if $b>0$) or in the negative y-axis direction (if $b<0$), through a distance $|b|$ units.

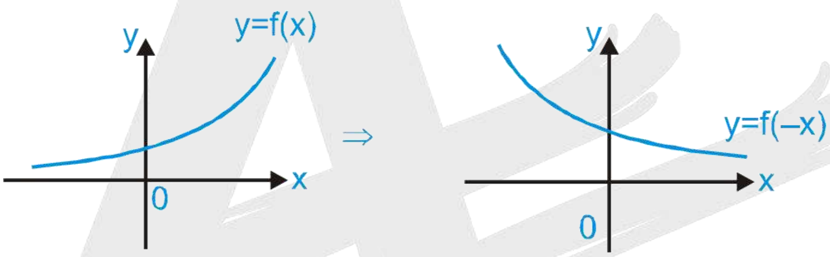
(ii) Drawing the graph of $y=-f(x)$ from the known graph of $y=f(x)$

To draw $y=-f(x)$, take the image of the curve $y=f(x)$ in the x-axis as plane mirror.



(iii) Drawing the graph of $y=f(-x)$ from the known graph of $y=f(x)$

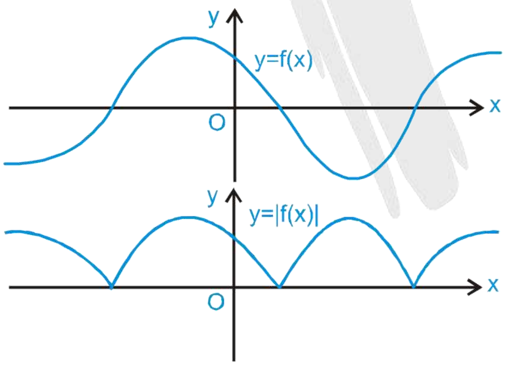
To draw $y=f(-x)$, take the image of the curve $y=f(x)$ in the y-axis as plane mirror.



(iv) Drawing the graph of $y=|f(x)|$ from the known graph of $y=f(x)$

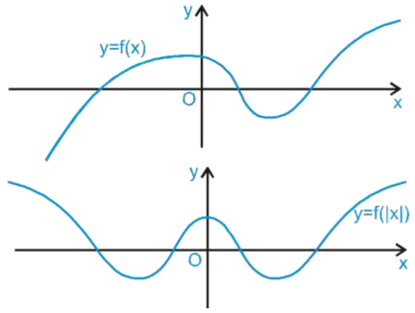
$|f(x)|=f(x)$ if $f(x)\geq 0$ and $|f(x)|=-f(x)$ if $f(x)<0$.

It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x)\geq 0$ and for the portions where $f(x)<0$ graph of $|f(x)|$ would be image of $y=f(x)$ in x-axis.



(v) Drawing the graph of $y=f(|x|)$ from the known graph of $y=f(x)$

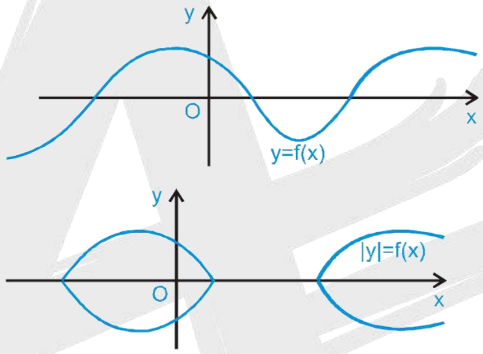
It is clear that, $f(|x|)=\begin{cases} f(x), & x\geq 0 \\ f(-x), & x<0 \end{cases}$. Thus $f(|x|)$ would be a even function, graph of $f(|x|)$ and $f(x)$ would be identical in the first and the fourth quadrants (ax $x\geq 0$) and as such the graph of $f(|x|)$ would be symmetric about the y-axis (as ($|\mathbf{x}|$) is even).



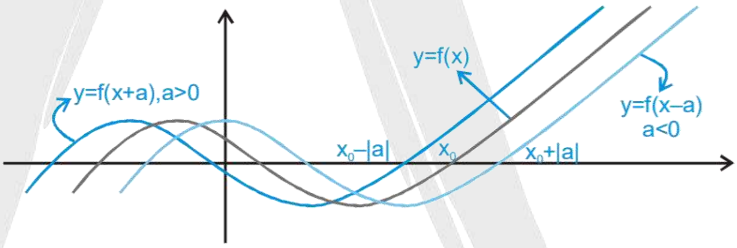
(vi) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$.

Hence graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about the x-axis only in those regions.



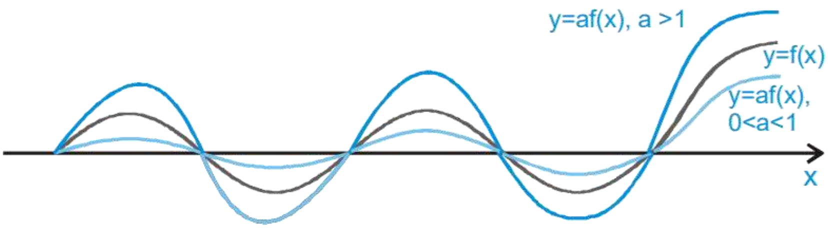
(vii) Drawing the graph of $y = f(x + a), a \in R$ from the known graph of $y = f(x)$



(i) If $a > 0$, shift the graph of $f(x)$ through ' a ' units towards left of $f(x)$.

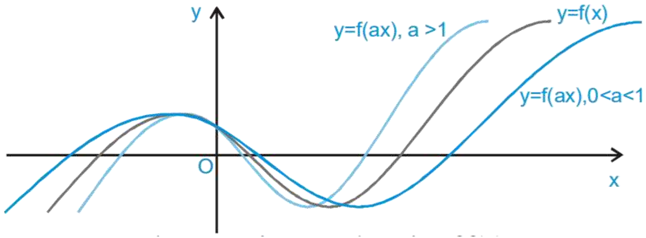
(ii) If $a < 0$, shift the graph of $f(x)$ through ' a ' units towards right of $f(x)$.

(viii) Drawing the graph of $y = af(x)$ from the known graph of $y = f(x)$



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of $1: a$.

(ix) Drawing the graph of $y = f(ax)$ from the known graph of $y = f(x)$



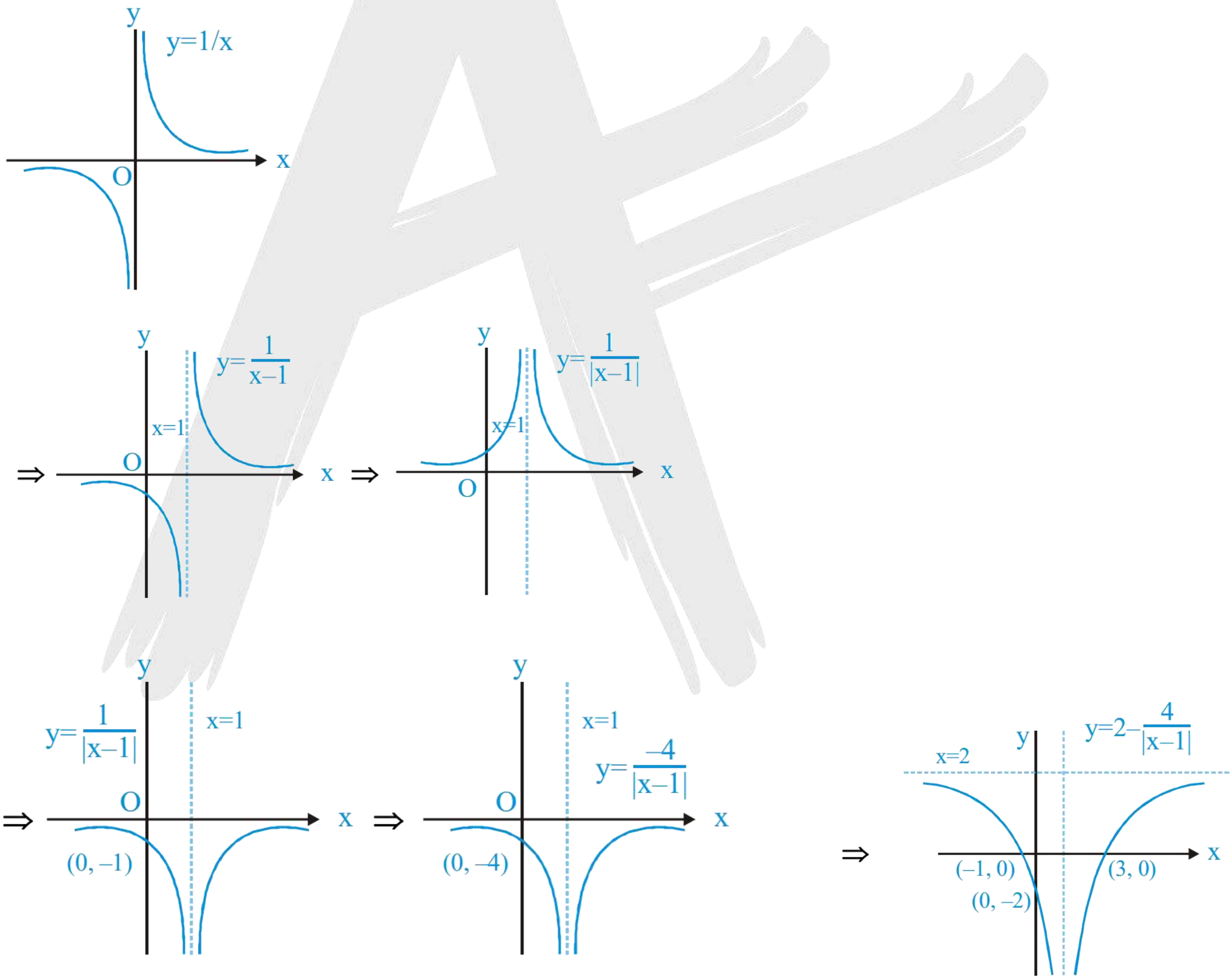
Let us take any point $x_0 \in$ domain of $f(x)$. Let $ax = x_0$ or

$$x = \frac{x_0}{a}$$

Clearly if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by $\frac{1}{a}$ units along the x-axis and if $a > 1, x < x_0$, then $f(x)$ will compress by ' a ' units along the x-axis.

Ex Draw the graph of $y = 2 - \frac{4}{|x-1|}$

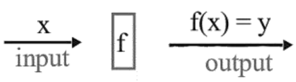
Sol.



Tips & Formulas

1. Definition

If to every value (considered as real unless other-wise stated) of a variable x, which belongs to a set A, there corresponds one and



only one finite value of the quantity y which belong to set B, then y is said to be a function of x and written as $f: A \rightarrow B, y = f(x)$, x is called argument or independent variable and y is called dependent variable.

Pictorially:

y is called the image of x & x is the pre-image of y, under f. Every function $f: A \rightarrow B$ satisfies the following conditions.

(i) $f \subset A \times B$

(ii) $\forall a \in A \Rightarrow (a, f(A)) \in f$

(iii) If $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

2. Domain, Co-Domain & Range of a Function

Let $f: A \rightarrow B$, then the set A is known as the domain of ' f ' & the set B is known as co-domain of ' f '. The set of all f images of elements of A is known as the range of ' f '. Thus

Domain of $f = \{x \mid x \in A, (x, f(x)) \in f\}$

Range of $f = \{f(x) \mid x \in A, f(x) \in B\}$

Range is a subset of co-domain.

3. Important Types of Function

(A) Polynomial function :

If a function ' f ' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \ldots .. + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots \ldots . a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, \neq 0$.

(ii) There are two polynomial functions, satisfying the relation;

$f(x). f(1/x) = f(x) + f(1/x)$ They are:

(A) $f(x) = x^n + 1$&

(B) $f(x) = 1 - x^n$, where n is a positive integer.

(iii) Domain of a polynomial function is R.

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(B) Algebric function :

A function ' f ' is called an algebric function if it can be constructed using algebric operations (such as addition, subtraction, multiplication, division and taking radicals) straight with polynomials.

(C) Rational function :

A rational function is a function of the form $y = f(x) = \frac{g(x)}{h(x)}$, where $g(x)$&$h(x)$ are polynomials & $h(x) \neq 0$,

Domain: $R - \{x \mid h(x) = 0\}$

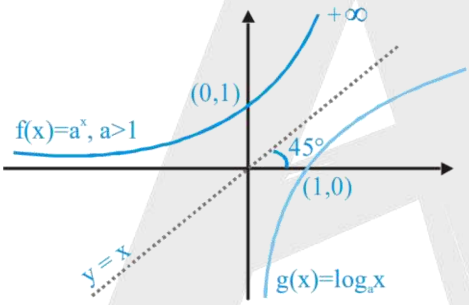
Any rational function is automatically an algebric function.

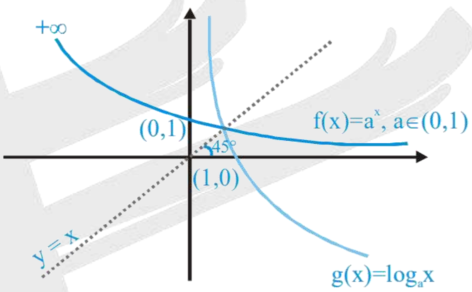
(D) Exponential and Logarithmic Function :

A function $f(x) = a^x (a > 0), a \neq 1, x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$. Note that $f(x)$&$g(x)$ are inverse of each other & their graphs are as shown. (Functions are mirror image of each other about the line $y = x$)

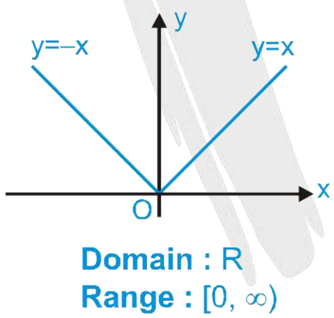
Domain of a^x is R **Range** R^+

Domain of $\log_a x$ is R^+ **Range** R





(E) Absolute value function



It is defined as: $y = |x|$

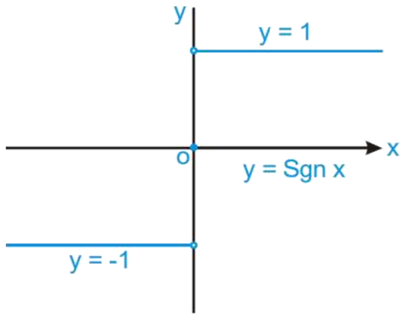
$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Also defined as max $\{x, -x\}$

Note : $f(x) = \frac{1}{|x|}$, Domain : $R - \{0\}$, Range : R^+

(F) Signum function

Signum function y = sgn (x) is defined as follows



$$y=\begin{cases}\frac{|x|}{x}, x\neq 0\\ 0, x=0\end{cases}=\begin{cases}1 & \text{for } x>0\\ 0 & \text{for } x=0\\ -1 & \text{for } x<0\end{cases}$$

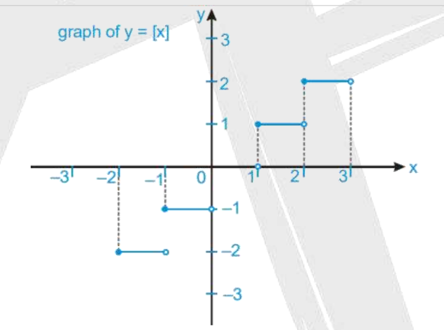
Domain : R

Range : $\{-1,0,1\}$

(G) Greatest integer or step up function

The function $y = f(x) = [x]$ is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
$[-2,-1)$	-2
$[-1,0)$	-1
$[0,1)$	0
$[1,2)$	1



Domain : R **Range : I**

Properties of greatest integer function :

(i) $[x]\leq x<[x]+1$ and $x-1<[x]\leq x,\ 0\leq x-[x]<1$

(ii) $[x+m]=[x]+m$ if m is an integer.

(iii) $[x]+[-x]=\begin{cases}0, & x\in I\\ -1, & x\notin I\end{cases}$

Note: $f(x)=\frac{1}{[x]}$ Domain : $R-[0,1)$

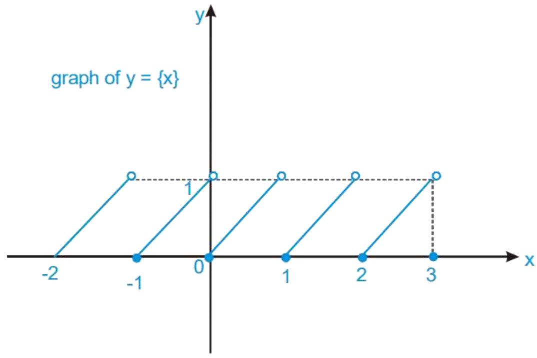
Range : $\left\{x \mid x=\frac{1}{n}, n\in I_0\right\}$

(H) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$

e.g.

x	{x}
$[-2,-1)$	$x+2$
$[-1,0)$	$x+1$
$[0,1)$	x
$[1,2)$	$x-1$



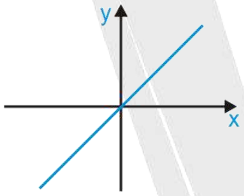
Domain : R **Range** : $[0,1)$ **Period** : 1

The fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.

Note : $f(x) = \frac{1}{\{x\}}$ Domain : R - I Range : $(1, \infty)$

(I) Identity function :

The function $f: A \to A$ defined by $f(x) = x \forall x \in A$ is called the identity function on A and is denoted by I_A.



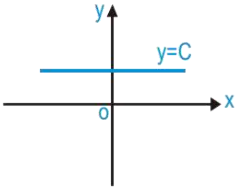
(J) Constant function :

$f: A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f: A \to B$;

$f(x) = c, \forall x \in A, c \in B$

is constant function.

Domain : R **Range** : $\{C\}$



(K) Trigonometric functions :

(i) Sine function $f(x) = \sin x$

Domain : R Range : $[-1,1]$, period 2π

(ii) **Cosine function** $f(x) = \cos x$

Domain : R **Range** : $[-1,1]$, period 2π

(iii) **Tangent function** $f(x) = \tan x$

Domain : $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R, period π

(iv) **Cosecant function** $f(x) = \operatorname{cosec} x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : $R - (-1,1)$, period 2π

(v) **Secant function** $f(x) = \sec x$

Domain : $R - \{x \mid x = (2n+1)\pi/2 : n \in I\}$

Range : $R - (-1,1)$, period 2π

(vi) **Cotangent function** $f(x) = \cot x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : R, period π

(L) **Inverse Trigonometric function :**

(i)	$f(x) = \sin^{-1} x$	Domain : $[-1,1]$	Range : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$f(x) = \cos^{-1} x$	Domain : $[-1,1]$	Range : $[0, \pi]$
(iii)	$f(x) = \tan^{-1} x$	Domain : R	Range : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(iv)	$f(x) = \cot^{-1} x$	Domain : R	Range : $[0, \pi]$
(v)	$f(x) = \operatorname{cosec}^{-1} x$	Domain : $R - (-1,1)$	Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(vi)	$f(x) = \sec^{-1} x$	Domain : $R - (-1,1)$	Range : $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

4. **Equal or Identical Function**

Two function f & g are said to be equal if :

(A) The domain of f = the domain of g

(B) The range of f = range of g and

(C) $f(x) = g(x)$, for every x belonging to their common domain (i.e. should have the same graph)

5. **Algebraic Operations on Functions**

If f & g are real valued functions of x with domain set A, B respectively, $f + g$, $f - g$, $(f.g)$ & are (f/g) as follows :

(A) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(B) $(f.g)(x) = f(x) \cdot g(x)$ domain is $A \cap B$

(C) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x \mid g(x) = 0\}$

6. Classification of Functions

(A) One-One function (Injective mapping) :

A function $f: A \rightarrow B$ is said to be a oneone function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A \& f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note (i) Any continuous function which is entirely increasing or decreasing in whole domain is oneone.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point.

(B) Many-one function :

A function $f: A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f: A \rightarrow B$ is many one if $\exists x_1, x_2 \in A, f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Note: If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

(C) Onto function (Surjective mapping) :

If range = co-domain, then f(x) is onto.

(D) Into function :

If $f: A \rightarrow B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Note:

(i) If ' f ' is both injective & surjective, then it is called a Bijective mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined f rom $A \rightarrow A$ is n^n & out of it n ! are one one and rest are many one.

(iii) $f: R \rightarrow R$ is a polynomial

(A) Of even degree, then it will neither be injective nor surjective.

(B) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

7. Composite of Uniformly & Non-Uniformly Defined Function

Let $f: A \rightarrow B$ & $g: B \rightarrow C$ be two functions. Then the function gof : $A \rightarrow C$ defined by $(gof)(x) = g(f(x)) \forall x \in A$ is called the composite of the two functions f& g. Hence in gof (x) the range of " f ' must be a subset of the domain of ' g '.

Properties of composite functions :

(A) In general composite of functions is not commutative i.e. gof $\neq$ fog.

(B) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then $fo(g \circ h) = ($ fog$)$).

(C) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

(D) If gof is one-one function then f is one-one but g may not be one-one.

8. Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

9. Bounded Function

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

10. Implicit & Explicit Function

A function defined by an equation not solved for the dependent variable is called an implicit function. e.g. the equations $x^3 + y^3 = 1 \& x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an Explicit function. e.g. $y = 5x^3 - 3$

11. Inverse of a Function

Let $f: A \rightarrow B$ be a one-one & onto function, then their exists a unique function $g: B \rightarrow A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall x \in A \& y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1}: B \rightarrow A = \{(f(x), x)) \mid (x, f(x)) \in f\}$.

Properties of inverse function :

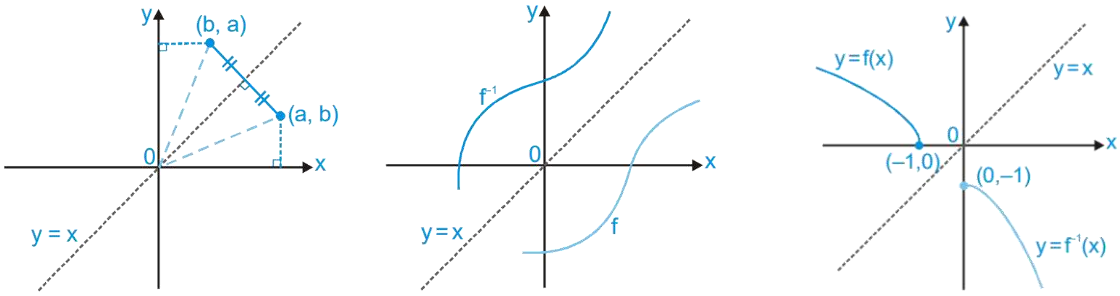
(A) The inverse of a bijection is unique.

(B) If $f: A \rightarrow B$ is a bijection & $g: B \rightarrow A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where $I_A \& I_B$ are identity functions on the sets A & B respectively. If fof $=$ I, then f is inverse of itself.

(C) The inverse of a bijection is also a bijection.

(D) If &g are two bijections $f: A \rightarrow B, g: B \rightarrow C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o g^{-1}$.

(E) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of ' f ' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.



The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

12. Odd & Even Functions

If a function is such that whenever ' x ' is in it's domain ' $-x$ ' is also in it's domain & it satisfies $f(-x) = f(x)$ it is an even function $f(-x) = -f(x)$ it is an odd function

Note

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many - one function.

(iii) Every even function is symmetric about the y axis & every odd function is symmetric about the origin.

(iv) Every function which has ' $-x$ ' in it's domain whenever ' x ' is in it's domain, can be expressed as the sum of an even & an odd function .

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$

(v) The only function which is defined on the entire number line & even and odd at the same time is $f(x) = 0$.

(vi) If f(x) and g(x) both are even or both are odd then the function $f(x) \cdot g(x)$ will be even but if any one of them is odd & other is even, then f.g will be odd.

13. Periodic Function

A function f(x) is called periodic if there exists a least positive number $T(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of f(x).

Note: (i) Inverse of a periodic function does not exist.

(ii) Every constant function is periodic, with no fundamental period.

(iii) If f(x) has a period T&g(x) also has a period T then it does not mean that $f(x) + g(x)$ must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.

(iv) If f(x) has period p and g(x) has period q, then period of $f(x) + g(x)$ will be LCM of p &

q provided f(x)&g(x) are not interchangeable. If f(x)&g(x) can be interchanged by adding a least positive number r, then smaller of LCM & r will be the period.

(v) If f(x) has period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then $f(ax+b)$ has a period $T/a(a>0)$.

(vii) $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x| \& |\operatorname{cosec} x|$ are periodic function with period π.

(viii) $\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$, are periodic function with period 2π when ' n ' is odd or π when n is even.

(ix) $\tan^n x, \cos^n x$ are periodic function with period π.

14. General

If x, y are independent variables, then :

(A) $f(xy)=f(x)+f(y) \Rightarrow f(x)=k\ell nx$

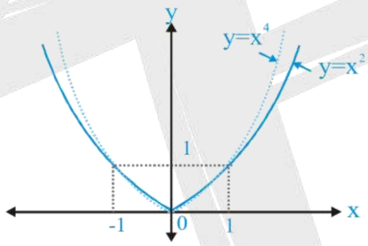
(B) $f(xy)=f(x), f(y) \Rightarrow f(x)=x^n, n \in R$ or $f(x)=0$

(C) $f(x+y)=f(x)\cdot f(y) \Rightarrow f(x)=a^{kx}$ or $f(x)=0$

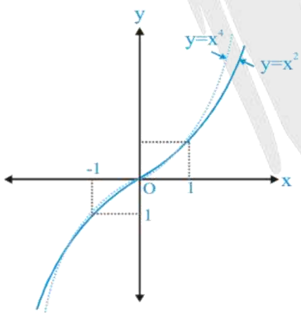
(D) $f(x+y)=f(x)+f(y) \Rightarrow f(x)=kx$, where k is a constant.

15. Some Basic Function & their Graph

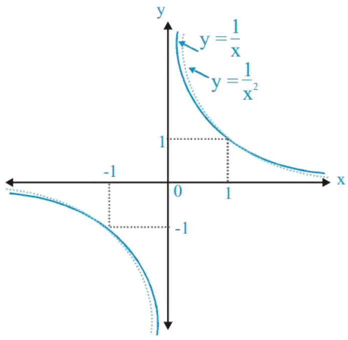
(A) $y=x^{2n}$, where $n \in N$



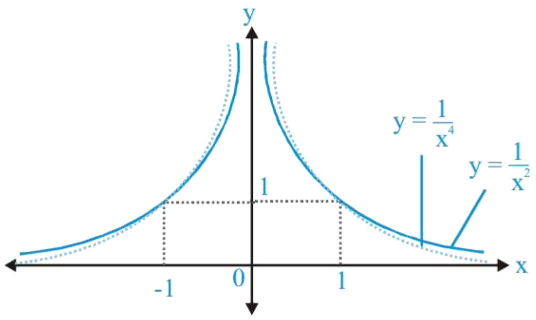
(B) $y=x^{2n+1}$, where $n \in N$



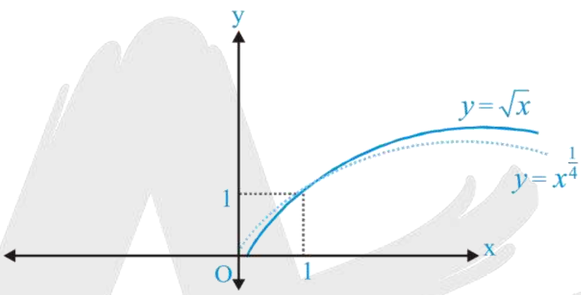
(C) $y=\frac{1}{x^{2n-1}}$, where $n \in N$



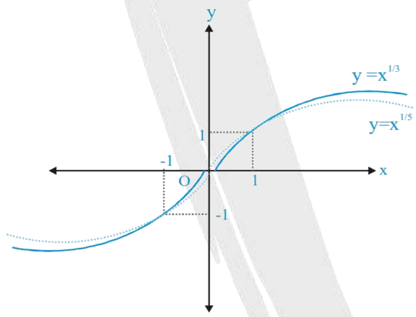
(D) $y = \frac{1}{x^{2n}}$, where $n \in N$



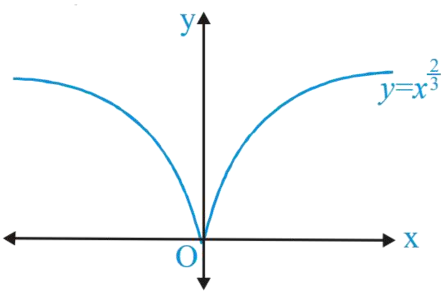
(E) $y = x^{\frac{1}{2n}}$, where $n \in N$



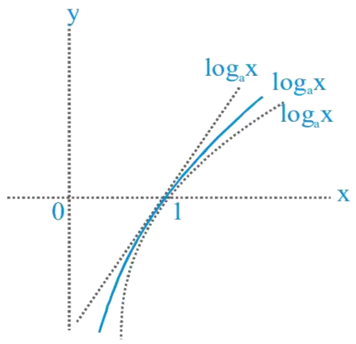
(F) $y = x^{\frac{1}{2n+1}}$, where $n \in N$



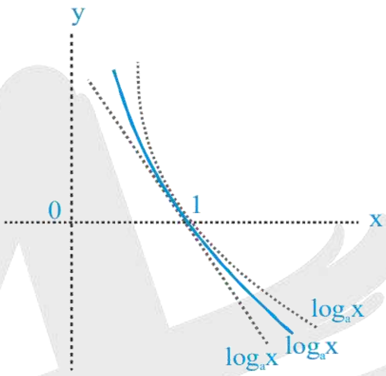
Note : $y = x^{2/3}$



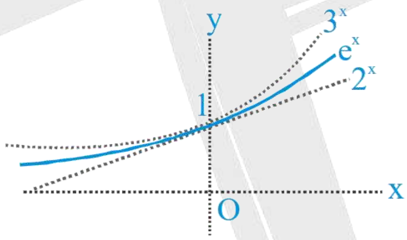
(G) $y = \log_a x$; when $a > 1$;



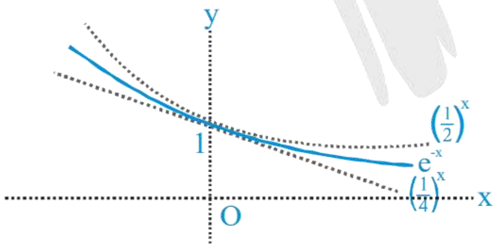
when $0 < a < 1$



(H) $y = a^x; a > 1$

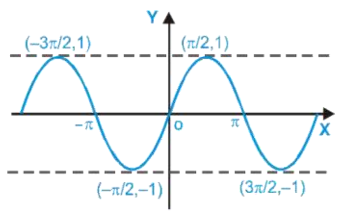


$0 < a < 1$

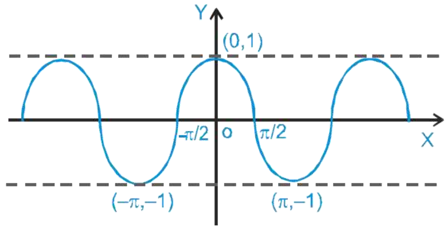


(I) Trigonometric functions

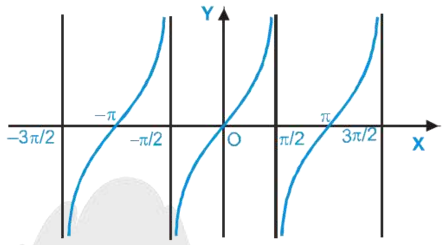
$y = \sin x$



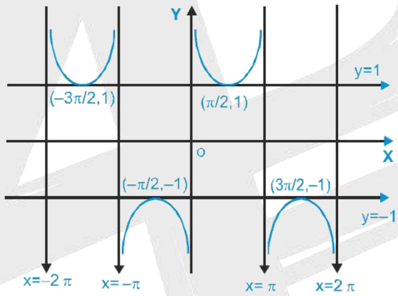
$y = \cos x$



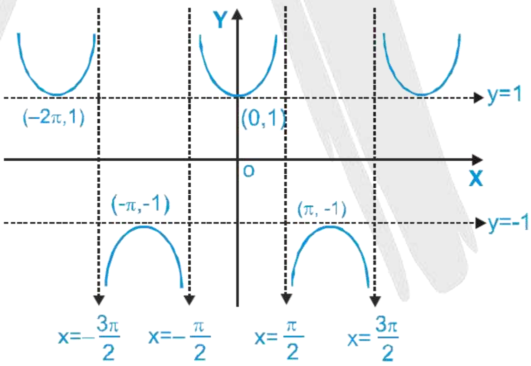
$y = \text{cosec } x$



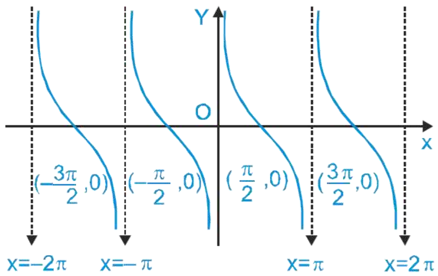
$y = \sec x$



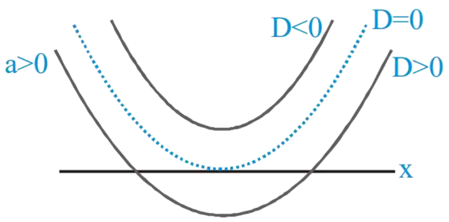
$y = \cot x$



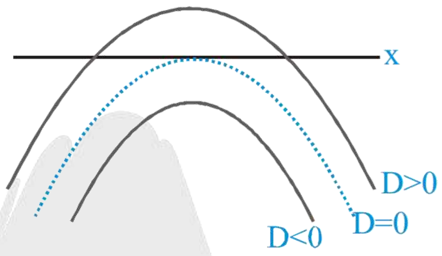
$y = \cot x$



(J) $y = ax^2 + bx + c$



vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ where $D = b^2 - 4ac$

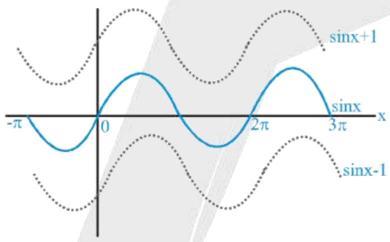


16. Transformation of Graph

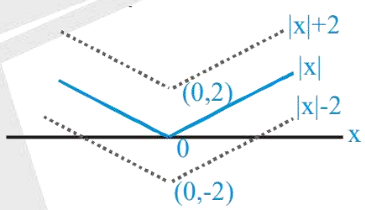
(A) when $f(x)$ transforms to $f(x) + k$ if $k > 0$ then shift graph of $f(x)$ upward through k if $k < 0$ then shift graph of $f(x)$ downward through k

Examples:

1.



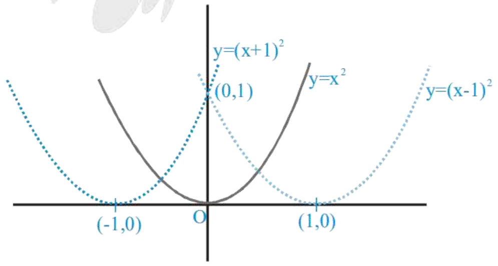
2.



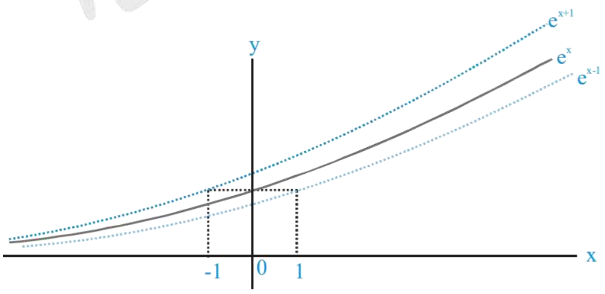
(B) $f(x)$ transforms to $f(x + k)$: if > 0 then shift graph of $f(x)$ through k towards left. if $k < 0$ then shift graph of $f(x)$ through k towards right.

Examples :

1.



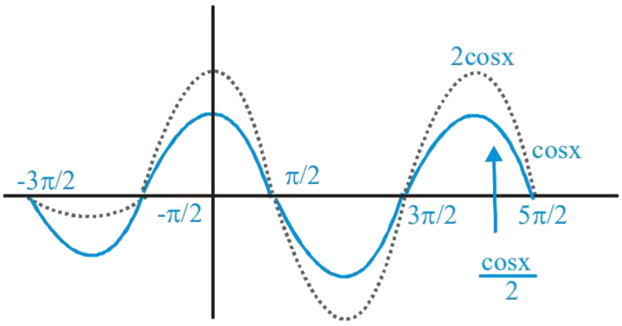
2.



(C) $f(x)$ transforms to $kf(x)$

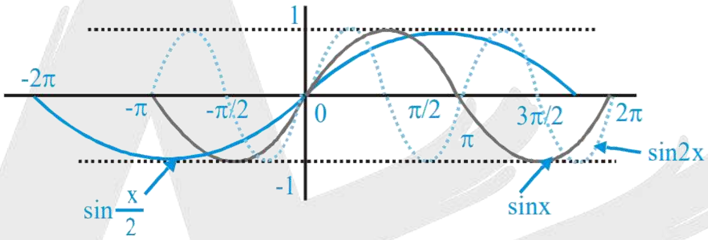
if $k > 1$ then strech graph of $k(x)k$ times along $y -$ axis if $0 < k < 1$ then shrink graph of $f(x)$, k times along y-axis

Examples:



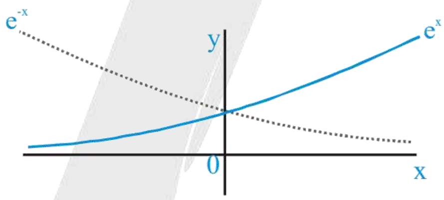
(D) f(x) transforms to f(kx) : if $k > 1$ then shrink graph of f(x), ' k ' times along x-axis. If $0 < k < 1$ then strech graph of f(x), ' k ' times along x-axis.

Examples:

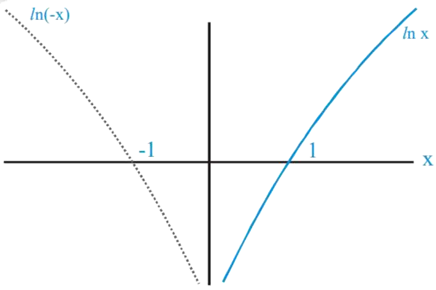


(E) f(x) transforms to f(−x) : Take mirror image of the curve y = f(x) in y-axis as plane mirror.

1.



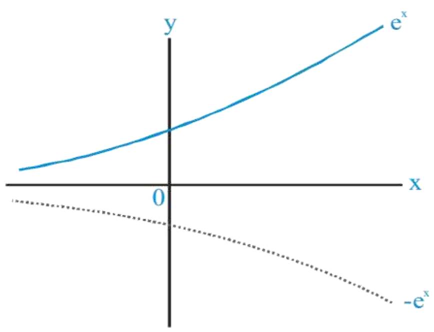
2.

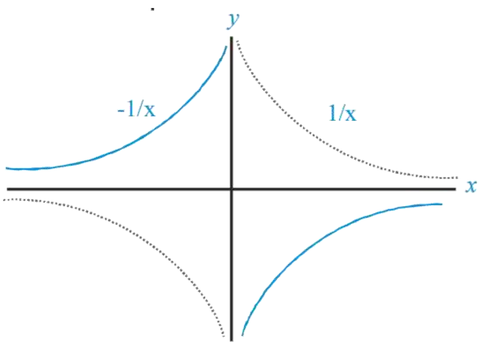


Examples:

(F) f(x) transforms to −f(x) : Take image of y = f(x) in the x axis as plane mirror.

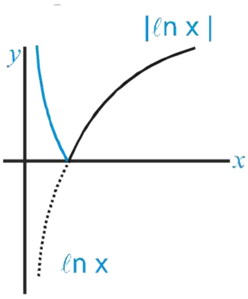
Examples :

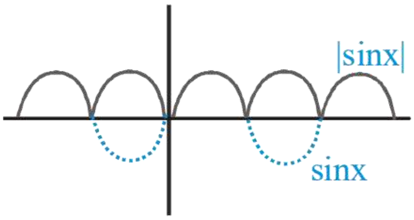




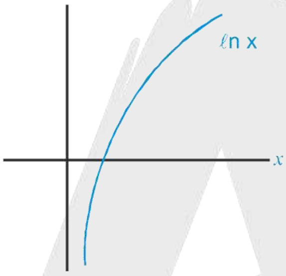
(G) f(x) tansforms to |f(x)| : Take mirror image (in x axis) of the portion of the graph of f(x) which lies below x-axis.

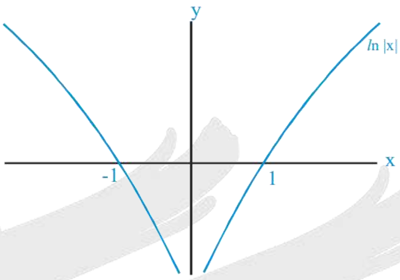
Examples:





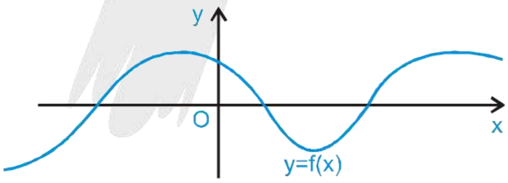
(H) f(x) transforms to f(|x|) : Neglect the curve for $x < 0$ and take the image of curve for $x \geq 0$ about y-axis.

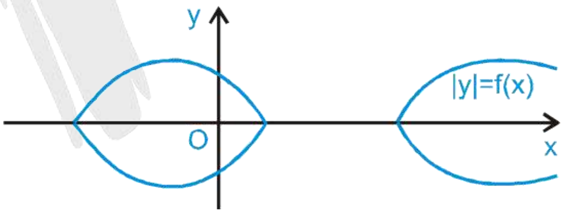




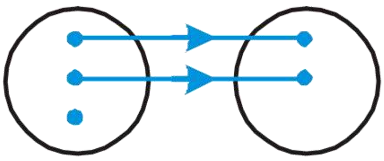
(I) y = f(x) transforms to |y| = f(x) :
Remove the portion of graph which lies below x − axis & then take mirror image (in x-axis) of remain ing portion of graph.

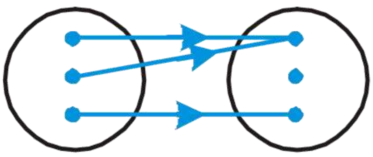
Examples:

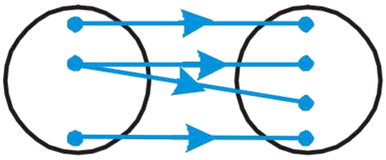


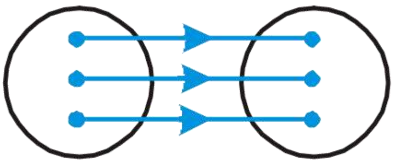


Ex. 1 Which of the following pictorial diagrams represent the function

(A) 

(B) 

(C) 

(D) 

Sol. B and D. In (A) one element of domain has no image, while in (**C**) one element of 1st set has two images in 2nd set.

Ex. 2 Find the Domain of the following function :

(i) $y = \log_{(x-4)}(x^2 - 11x + 24)$

(ii) $f(x) = \sqrt{x^2 - 5}$

(iii) $\sin^{-1}(2x - 1)$

(iv) $f(x) = \sqrt{\sin x} - \sqrt{16 - x^2}$

Sol. (i) $y = \log_{(x-4)}(x^2 - 11x + 24)$

Here ' y ' would assume real value if,

$x - 4 > 0$ and $\neq 1, x^2 - 11x + 24 > 0$

$\Rightarrow x > 4$ and $\neq 5, (x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5, x < 3$ or $x > 8$

$\Rightarrow x > 8$

$\Rightarrow$ Domain $(y) = (8, \infty)$

(ii) $f(x) = \sqrt{x^2 - 5}$ is real iff $x^2 - 5 \geq 0$

$\Rightarrow |x| \geq \sqrt{5}$

$\Rightarrow x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$

(iii) $\sin^{-1}(2x - 1)$ is real iff $-1 \leq 2x - 1 \leq +1$

$\therefore$ domain is $x \in [0,1]$

(iv) $\sqrt{\sin x}$ is real iff $\sin x \geq 0$

$\Leftrightarrow x \in [2n\pi, 2n\pi + \pi], n \in I.$

$\sqrt{16 - x^2}$ is real iff $16 - x^2 \geq 0 \Leftrightarrow -4 \leq x \leq 4.$

Thus the domain of the given function is $\{x : x \in [2n\pi, 2n\pi + \pi], n \in I\} \cap [-4,4] = [-4, -\pi] \cup [0, \pi].$

Ex. 3 Find the range of following functions :

(i) $f(x) = \frac{1}{8 - 3\sin x}$

(ii) $f(x) = \frac{x^2 - 4}{x - 2}$

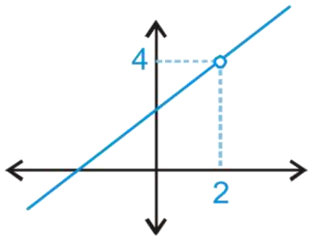
Sol. (i) $f(x) = \frac{1}{8 - 3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of $f = \left[\frac{1}{11}, \frac{1}{5}\right]$

(ii) $f(x) = \frac{x^2-4}{x-2}$

$= x + 2; x \neq 2$



$\therefore$ graph of f(x) would be

Thus the range of f(x) is $R - \{4\}$

Ex. 4 Find the range of following functions :

(i) $y = \ln(2x - x^2)$

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Sol. (i) Step-1: We have $2x - x^2 \in (-\infty, 1]$

Step-2 Let $t = 2x - x^2$

For ℓnt to be defined accepted values are (0,1] Now, using monotonocity of ℓnt,

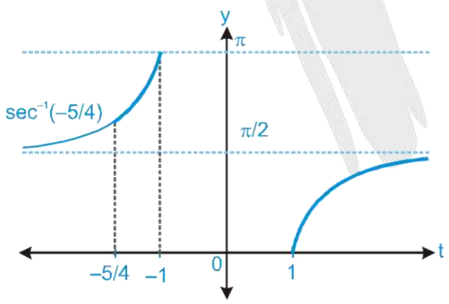
$\ell n(2x - x^2) \in (-\infty, 0]$

$\therefore$ range is $(-\infty, 0]$

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Let $t = x^2 + 3x + 1$ for $x \in R$, then $t \in \left[-\frac{5}{4}, \infty\right)$ but $y = \sec^{-1}(t)$

$\Rightarrow t \in \left[-\frac{5}{4}, -1\right] \cup [1, \infty)$



From graph the range is

$\left[0, \frac{\pi}{2}\right) \cup \left[\sec^{-1}\left(-\frac{5}{4}\right), \pi\right]$

Ex. 5 (i) Let {x} and [x] denote the fractional and integral part of a real number x respectively.

Solve $4\{x\} = x + [x]$

(ii) Draw graph of $f(x) = \text{sgn}(\ln x)$

Sol.

(i) As $x = [x] + \{x\}$

$\therefore$ Given equation

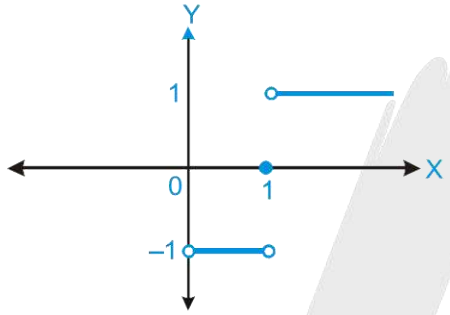
$$\Rightarrow 4\{x\} = [x] + \{x\} + [x] \Rightarrow \{x\} = \frac{2[x]}{3}$$

As $[x]$ is always an integer and $\{x\} \in [0,1)$, possible values are

$[x]$	$\{x\}$	$x = [x] + \{x\}$
0	0	0
1	$\frac{2}{3}$	$\frac{5}{3}$

$\therefore$ There are two Solution of given equation $x = 0$ and $x = \frac{5}{3}$

(ii)



Ex. 6 Find the domain $f(x) = \frac{1}{\sqrt{|[|x|-5]|-11}}$ where [.] denotes greatest integer function.

Sol. $|[|x| - 5]| > 11$

So $[|x| - 5] > 11$ or $[|x| - 5] < -11$

$[|x|] > 16$ or $[|x|] < -6$

$|x| \geq 17$ or $[|x|] < -6$

(Not Possible)

$\Rightarrow x \leq -17$ or $x \geq 17$

So $x \in (-\infty, -17] \cup [17, \infty)$

Ex. 7 Examine whether following pair of functions are identical or not?

(i) $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2 x + \cos^2 x$ and $(x) = \sec^2 x - \tan^2 x$

Sol. (i) No, as domain of $f(x)$ is $R - \{1\}$, while domain of $g(x)$ is R

(ii) No, as domain are not same. Domain of $f(x)$ is R, while that of $g(x)$ is

$$R - \left\{(2n+1)\frac{\pi}{2}; n \in I\right\}$$

Ex. 8 Find the value of

$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\cdots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

where [.] denotes greatest integer function.

Sol.

$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\cdots\ldots\left[\frac{1}{2}+\frac{499}{1000}\right]+$$

$$\left[\frac{1}{2}+\frac{500}{1000}\right]+\cdots\ldots\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+\cdots\ldots$$

$$+\left[\frac{1}{2}+\frac{2499}{1000}\right]+\left[\frac{1}{2}+\frac{2500}{1000}\right]+\cdots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

$$=0+1\times1000+2\times1000+3\times447=3000+1341=4341$$

Ex. 9 Find the range of $f(x)=\frac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Sol. $y=\frac{x-[x]}{1+x-[x]}=\frac{\{x\}}{1+\{x\}}$

$\therefore \frac{1}{y}=\frac{1}{\{x\}}+1 \Rightarrow \left[\frac{1}{\{x\}}=\frac{1-y}{y}\right]$

$\Rightarrow \{x\}=\frac{y}{1-y}$ $[\because 0\le\{x\}<1]$

$\Rightarrow 0\le\frac{y}{1-y}<1$

Range $=[0,1/2)$

Ex. 10 Let $f(x)=e^x; R^+\to R$ and $g(x)=\sin^{-1}x; [-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Find domain and range of fog(x)

Sol. Domain of $f(x): (0,\infty)$

Range of $g(x): \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

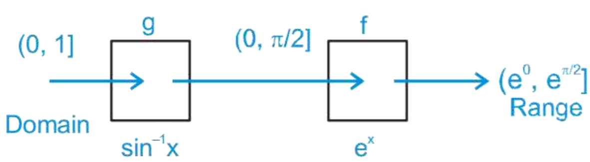
values in range of g(x) which are accepted by f(x)

are $\left(0,\frac{\pi}{2}\right]$

$\Rightarrow 0<g(x)\le\frac{\pi}{2}\Rightarrow 0<\sin^{-1}x\le\frac{\pi}{2}$

$\Rightarrow 0<x\le1$

Hence domain of $f\circ g(x)$ is $x\in(0,1]$



Therefore Domain : $(0,1]$

Range : $\left(1,e^{\pi/2}\right]$

Ex. 11 Let $A = \{x: -1 \leq x \leq 1\} = B$ be a mapping $f: A \to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(A) $f(x) = |x|$

(B) $f(x) = x|x|$

(C) $f(x) = x^3$

(D) $f(x) = [x]$

(E) $f(x) = \sin\frac{\pi x}{2}$

Sol. (A) $f(x) = |x|$

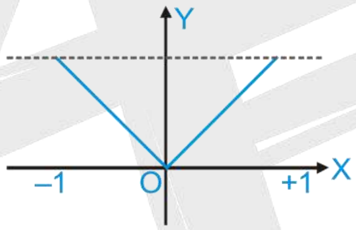
Graphically

It shows many one, as the straight line parallel to $x -$ axis cuts at two points. Here range for $f(x) \in [0,1]$

Which is clearly subset of co-domain i.e. $[0,1] \subseteq [-1,1]$ Thus, into.

Hence, function is many-one-into

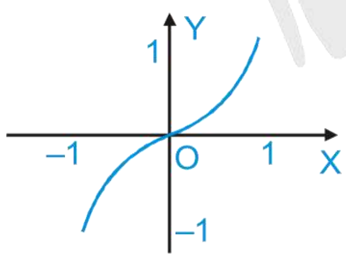
$\therefore$ Neither injective nor surjective.



(B) $f(x) = x|x| = \begin{Bmatrix} -x^2, & -1 < x < 0 \\ x^2 & , 0 < x < 1 \end{Bmatrix},$

Graphically

The graph shows f(x) one-one, as the straight line parallel to x-axis cuts the graph only at one point.



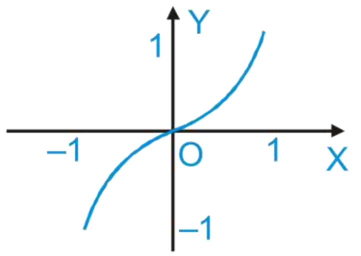
Here, range $f(x) \in [-1,1]$

Thus, range = co-domain

Hence, onto.

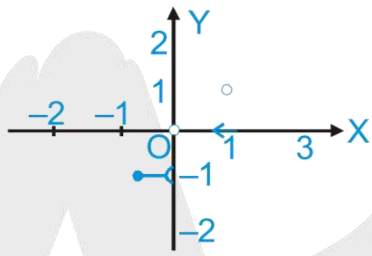
Therefore, f(x) is one-one, onto or (Bijective).

(C) $f(x) = x^3$, Graphically; Graph shows f(x) is one-one onto (i.e. Bijective)



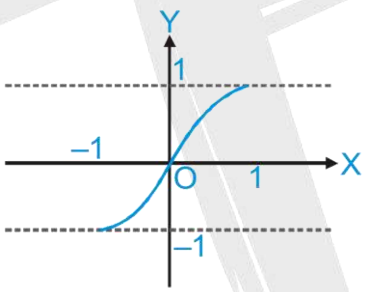
(D) $f(x) = [x]$, Graphically; Which shows f(x) is manyone, as the straight line parallel to x-axis meets at more than one point.

Here, range of $f(x) \in \{-1,0,1\}$ which shows into as range $\subseteq$ co-domain



Hence, many-one-into.

(E) $f(x) = \sin\ x$ Graphically; Which shows f(x) is oneone and onto as range = co-domain.



Therefore, f(x) is bijective.

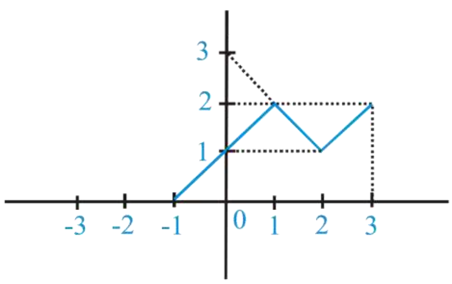
Ex. 12 Composition of piecewise defined functions :

If $f(x) = ||x-3|-2|;\ 0 \le x \le 4$

$g(x) = 4 - |2-x|\ ;\ -1 \le x \le 3$

then find fog(x) and draw rough sketch of fog(x).

Sol. $f(x) = \begin{cases} |x-1| & 0 \le x < 3 \\ |x-5| & 3 \le x \le 4 \end{cases} = \begin{cases} 1-x & 0 \le x < 1 \\ x-1 & 1 \le x < 3 \\ 5-x & 3 \le x \le 4 \end{cases}$



$g(x) = \begin{cases} 4-(2-x) & -1 \le x < 2 \\ 4-(x-2) & 2 \le x \le 3 \end{cases} = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$

$$\therefore f(x)=\begin{cases}1-g(x) & 0\le g(x)<1\\ g(x)-1 & 1\le g(x)<3\\ 5-g(x) & 3\le g(x)\le 4\end{cases}=\begin{cases}1-(2+x) & 0\le 2+x<1 & \text{and} & -1\le x<2\\ 2+x-1 & 1\le 2+x<3 & \text{and} & -1\le x<2\\ 5-(2+x) & 3\le 2+x\le 4 & \text{and} & -1\le x<2\\ 1-6+x & 0\le 6-x<1 & \text{and} & 2\le x\le 3\\ 6-x-1 & 1\le 6-x<3 & \text{and} & 2\le x\le 3\\ 5-6+x & 3\le 6-x\le 4 & \text{and} & 2\le x\le 3\end{cases}$$

$$=\begin{cases}-1-x & -2\le x<-1 & \text{and} & -1\le x<2\\ 1+x & -1\le x<1 & \text{and} & -1\le x<2\\ 3-x & 1\le x\le 2 & \text{and} & -1\le x<2\\ x-5 & -6\le -x<-5 & \text{and} & 2\le x\le 3\\ 5-x & -5\le -x<-3 & \text{and} & 2\le x\le 3\\ x-1 & -3\le -x\le -2 & \text{and} & 2\le x\le 3\end{cases}$$

$$=\begin{cases}-1-x & -2\le x<-1 & \text{and} & -1\le x<2\\ 1+x & -1\le x<1 & \text{and} & -1\le x<2\\ 3-x & 1\le x\le 2 & \text{and} & -1\le x<2\\ x-5 & 5<x\le 6 & \text{and} & 2\le x\le 3\\ 5-x & 3<x\le 5 & \text{and} & 2\le x\le 3\\ x-1 & 2\le x\le 3 & \text{and} & 2\le x\le 3\end{cases}=\begin{cases}1+x & -1\le x<1\\ 3-x & 1\le x<2\\ x-1 & 2\le x\le 3\end{cases}$$

Ex. 13 (i) Find whether $f(x)=x+\cos\,x$ is one-one.

(ii) Identify whether the function $f(x)=-x^3+3x^2-2x+4$ for $f: R\to R$ is ONTO or INTO

(iii) $f(x)=x^2-2x+3$; $[0,3]\to A$. Find whether $f(x)$ is injective or not. Also find the set A, if $f(x)$ is surjective.

Sol. (i) The domain of $f(x)$ is R.

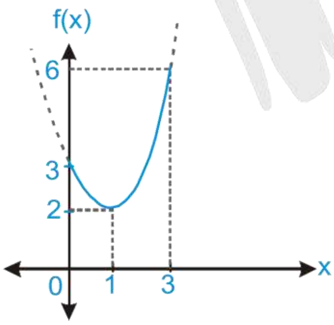
$f'(x)=1-\sin\,x$.

$\therefore f'(x)\ge 0\,\forall x\in$ complete domain and equality holds at discrete points only

$\therefore f(x)$ is strictly increasing on R. Hence $f(x)$ is one-one.

(ii) As range $\equiv$ codomain, therefore given function is ONTO.

(iii) $f'(x)=2(x-1);\ 0\le x\le 3$



$$\therefore f'(x)=\begin{cases}-\text{ve} & ;\ 0\le x<1\\ +\text{ve} & ;\ 1<x<3\end{cases}$$

$\therefore f(x)$ is non monotonic. Hence it is not injective. For $f(x)$ to be surjective, A should be equal to its range. By graph range is [2,6]

$\therefore A\equiv[2,6]$

Ex. 14 If f be the greatest integer function and g be the modulus function, then

$(gof)\left(-\frac{5}{3}\right)-(fog)\left(-\frac{5}{3}\right)=$

(A) 1 (B) -1 (C) 2 (D) 4

Sol. Given $(gof)\left(\frac{-5}{3}\right)-(fog)\left(\frac{-5}{3}\right)$

$= g\left\{f\left(\frac{-5}{3}\right)\right\}-f\left\{g\left(\frac{-5}{3}\right)\right\}=g(-2)-f\left(\frac{5}{3}\right)=2-1=1$

Ex. 15 Show that $\log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

Sol. Let $f(x)=\log\left(x+\sqrt{x^2+1}\right)$.

Then $f(-x)=\log\left(-x+\sqrt{(-x)^2+1}\right)$

$=\log\left(\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}\right)$

$=\log\frac{1}{\sqrt{x^2+1}+x}=-\log\left(x+\sqrt{x^2+1}\right)=-f(x)$ or $f(x)+f(-x)=0$.

Hence f(x) is an odd function.

Ex. 16 Show that $\cos^{-1}x$ is neither odd nor even.

Sol. Let $f(x)=\cos^{-1}x$. Then $f(-x)=\cos^{-1}(-x)=\pi-\cos^{-1}x$ which is neither equal to f(x) nor equal to $-f(x)$.

Hence $\cos^{-1}x$ is neither odd nor even.

Ex.17 Which of the following functions is (are) even, odd or neither :

(i) $f(x)=x^2\sin x$

(ii) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$

(iii) $f(x)=\log\left(\frac{1-x}{1+x}\right)$

(iv) $f(x)=\sin x-\cos x$

(v) $f(x)=\frac{e^x+e^{-x}}{2}$

Sol. (i) $f(-x)=(-x)^2\sin(-x)=-x^2\sin x=-f(x)$. Hence f(x) is odd.

(ii) $f(-x)=\sqrt{1+(-x)+(-x)^2}-\sqrt{1-(-x)+(-x)^2}$

$=\sqrt{1-x+x^2}-\sqrt{1+x+x^2}=-f(x)$.

Hence f(x) is odd

(iii) $f(-x)=\log\left(\frac{1-(-x)}{1+(-x)}\right)=\log\left(\frac{1+x}{1-x}\right)=-f(x)$.

Hence f(x) is odd

(iv) $f(-x)=\sin(-x)-\cos(-x)=-\sin x-\cos x$. Hence f(x) is neither even nor odd.

(v) $f(-x)=\frac{e^{-x}+e^{-(-x)}}{2}=\frac{e^{-x}+e^{x}}{2}=f(x)$. Hence f(x) is even

Ex.18 Let f: R → R be defined by $f(x) = (e^x - e^{-x})/2$. Is f(x) invertible? If so, find its inverse.

Sol. Let us check for invertibility of f(x) :

(A) One-One :

Let $x_1, x_2 \in R$ and $x_1 < x_2$

$\Rightarrow e^{x_1} < e^{x_2}$ (Because base e > 1)

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$\Rightarrow e^{-x_2} < e^{-x_1}$ (Because base e > 1)

(ii)

(i) + (ii) $\Rightarrow e^{x_1} + e^{-x_2} < e^{x_2} + e^{-x_1}$

$\Rightarrow \frac{1}{2}(e^{x_1} - e^{-x_1}) < \frac{1}{2}(e^{x_2} - e^{-x_2})$

$\Rightarrow f(x_1) < f(x_2)$ i.e. f is one-one.

(B) Onto :

As x tends to larger and larger values so does f(x) and when $x \to \infty, f(x) \to \infty$.

long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

(C) To find f^{-1} :

Let f^{-1} be the inverse function of f, then by rule of identity $fof^{-1}(x) = x$

$$\frac{e^{f^{-1}(x)} - e^{-f^{-1}(x)}}{2} = x$$

$$\Rightarrow e^{2f^{-1}(x)} - 2xe^{f^{-1}(x)} - 1 = 0$$

$$\Rightarrow e^{f^{-1}(x)} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$\Rightarrow e^{f^{-1}(x)} = x \pm \sqrt{1 + x^2}$$

Since $e^{f^{-1}(x)} > 0$, hence negative sign is ruled out and Hence $e^{f^{-1}(x)} = x + \sqrt{1 + x^2}$

Taking logarithm, we have $f^1(x) = \ln\left(x + \sqrt{1 + x^2}\right)$.

Ex.19 Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{tn(\sin x)} + \tan^3 x - \text{cosec}\,(3x - 5)$

(ii) $f(x) = x - [x - b], b \in R$

(iii) $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

(iv) $f(x) = \tan \frac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$

(vi) $f(x) = \frac{(1 + \sin x)(1 + \sec x)}{(1 + \cos x)(1 + \text{cosec } x)}$

(vii) $f(x) = e^{x - [x] + |\cos \pi x| + |\cos 2\pi x| + \cdots \ldots + |\cos n\pi x|}$

Sol. (i) $f(x) = e^{\ln(\sin x)} + \tan^3 x - \operatorname{cosec}(3x-5)$

Period of $e^{\ell n \sin x} = 2\pi, \tan^3 x = \pi$

$\operatorname{cosec}(3x-5) = \dfrac{2\pi}{3}$

$\therefore$ Period $= 2\pi$

(ii) $f(x) = x - [x-b] = b + \{x-b\}$

$\therefore$ Period $= 1$

(iii) $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\frac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan\frac{\pi}{2}[x]$

$\tan\frac{\pi}{2}[x+T] = \tan\frac{\pi}{2}[x] \Rightarrow \frac{\pi}{2}[x+T] = n\pi + \frac{\pi}{2}[x]$

$\therefore T = 2$

$\therefore$ Period $= 2$

(v) Let f(x) is periodic then $f(x+T) = f(x)$

$\Rightarrow \cos(\sin(x+T)) + \cos(\cos(x+T)) = \cos(\sin x) + \cos(\cos x)$

If $x = 0$ then $\cos(\sin T) + \cos(\cos T) = \cos(0) + \cos(1)$

$= \cos\left(\cos\frac{\pi}{2}\right) + \cos\left(\sin\frac{\pi}{2}\right)$

On comparing $T = \frac{\pi}{2}$

(vi) $f(x) = \frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$

$= \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)} \Rightarrow f(x) = \tan x$

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+\cdots\ldots+|\cos n\pi x|}$

Period of $x - [x] = 1$

Period of $|\cos \pi x| = 1$

Period of $|\cos 2\pi x| = \frac{1}{2}$

Period of $|\cos n\pi x| = \frac{1}{2n}$

So period of f(x) will be L.C.M. of all period = 1

Ex.20 Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$

(ii) $f(x) = \sin\frac{\pi x}{\sqrt{2}} + \cos\frac{\pi x}{\sqrt{3}}$

(iii) $f(x) = \sin\frac{\pi x}{\sqrt{3}} + \cos\frac{\pi x}{2\sqrt{3}}$

Sol. (i) Period of $e^{x-[x]} = 1$ Period of $\sin x = 2\pi$

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $= \sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $= \cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

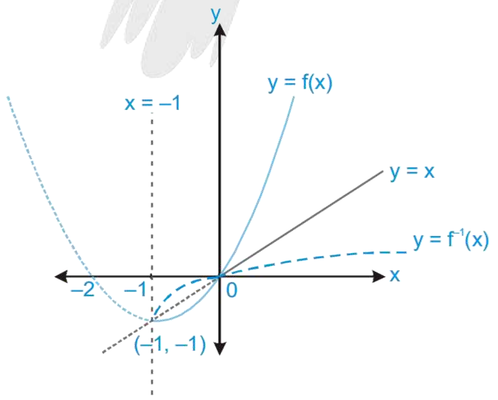
$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period $= 4\sqrt{3}$

Ex. 21.(i) Let $f(x) = x^2 + 2x; x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

(ii) If $y = f(x) = x^2 - 3x + 1, x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Sol.



(i)

$f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$

$\Rightarrow x^2 + 2x = x \Rightarrow x(x+1) = 0 \Rightarrow x = 0, -1$

Hence two solution for $f(x) = f^{-1}(x)$

(ii) $y = 1$

$\Rightarrow x^2 - 3x + 1 = 1 \Rightarrow x(x-3) = 0$

$\Rightarrow x = 0,3 \Rightarrow$ But $x \geq 2$ $\therefore x = 3$

Now $g(f(x)) = x$

Differentiating both sides w.r.t. x

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow g'(f(3)) = \frac{1}{f'(3)} \Rightarrow g'(1) = \frac{1}{6-3} = \frac{1}{3}$$

(As $f'(x) = 2x - 3$)

EXERCISE-1

OBJECTIVE TYPE QUESTION

Daily Work Sheet-1

SINGLE CORRECT TYPE

1. The range of the function $y=\frac{8}{9-x^2}$ is

(A) $(-\infty,\infty)-\{\pm 3\}$ (B) $\left[\frac{8}{9},\infty\right)$

(C) $\left(0,\frac{8}{9}\right)$ (D) $(-\infty,0)\cup\left[\frac{8}{9},\infty\right)$

2. For the function $f(x)=\frac{e^x+1}{e^x-1}$, if $n(d)$ denotes the number of integers which are not in its domain and $n(r)$ denotes the number of integers which are not in its range, then $n(d)+n(r)$ is equal to

(A) 2 (B) 3 (C) 4 (D) Infinite

3. Number of integral values of x in the domain of function $f(x)=\sqrt{\ln|\ln|x||}+\sqrt{7|x|-|x|^2-10}$ is equal to

(A) 4 (B) 5 (C) 6 (D) 7

4. If a polynomial function ' f ' satisfies the relation $\log_2(f(x))=\log_2\left(2+\frac{2}{3}+\frac{2}{9}+\cdots\infty\right)\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$ and $f(10)=1001$ then the value of $f(20)$ is

(A) 2002 (B) 7999 (C) 8001 (D) 16001

5. If the range of function $f(x)=\frac{x^2+x+c}{x^2+2x+c}, x\in R$ is $\left[\frac{5}{6},\frac{3}{2}\right]$ then c is equal to

(A) -4 (B) 3 (C) 4 (D) 5

6. If $x=\frac{4l}{1+l^2}$ and $y=\frac{2-2l^2}{1+l^2}$ where ' l ' is a parameter and range of $f(x,y)=x^2-xy+y^2$ is $[a,b]$ then $(a+b)$ is equal to

(A) 4 (B) 6 (C) 8 (D) 12

7. If minimum and maximum values of $f(x)=2|x-1|+|x+3|-3|x-4|$ are m and M respectively then $(m+M)$ equals

(A) 0 (B) 1 (C) 2 (D) 3

8. If the domain of $g(x)$ is $[3,4]$, then the domain of $g(\log_2(x^2+3x-2))$ is

(A) $[-4,-1]\cup[2,7]$ (B) $[-3,2]$

(C) $[-6,-5]\cup[2,3]$ (D) $\left[\frac{3}{2},5\right]$

MULTIPLE CORRECTTYPE

9. Consider the function $f(x)=x+\sqrt{1-x^2}$, then which of the following is/are CORRECT?

(A) Range of $f(x)$ is $[-1,\sqrt{2}]$.

(B) f is many one.

(C) f is either even or odd.

(D) Range of $f(x)$ is identical to range of $g(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$.

10. Consider, $f(x)=\{x+[\log_2(2+x)]\}+\{x+[\log_2(2+x^2)]\}+\cdots..+\{x+[\log_2(2+x^{10})]\}$

Identify the correct statement(s)

(A) $[f(e)]=7$.

(B) $f(\pi)=20\pi-60$.

(C) the number of solutions of the equation $f(x)=x$ is 9 .

(D) the number of solutions of the equation $f(x)=x$ is 10 .

[Note : {y} and [y] denotes the fractional part function and greatest integer function respectively.]

INTEGERTYPE

11. Find the number of integer in the range of the function,

$$f(x)=\sqrt{\sin\frac{\pi x}{2}}+\sqrt{16-x^2}+\sqrt{x}+\log_2(x(x-2))$$

Daily Work Sheet-2

SINGLE CORRECT TYPE

1. Which of the following statements are incorrect? I. If $f(x)$ and $g(x)$ are one to one then $f(x)+g(x)$ is also one to one.

II. If $f(x)$ and $g(x)$ are one-one then $f(x)\cdot g(x)$ is also one-one.

III. If $f(x)$ is odd then it is necessarily one to one.

(A) I and II only
(B) II and III only
(C) III and I only
(D) I, II and III

2. Let $f:[0,2]\rightarrow[2,5]$ be defined as $f(x)=3x^2-6x+5$, then $f(x)$ is

(A) injective but not surjective
(B) surjective but not injective
(C) injective as well as surjective
(D) neither injective nor surjective

3. If the functions f(x) and g(x) are defined on $R \to R$ such that

$f(x) = \begin{cases} x+3, & x \in \text{rational} \\ 4x, & x \in \text{irrational} \end{cases}$ and $g(x) = \begin{cases} x+\sqrt{5}, & x \in \text{irrational} \\ -x, & x \in \text{rational} \end{cases}$ then $(f-g)(x)$ is

(A) one - one and onto
(B) neither one-one nor onto
(C) one-one but not onto
(D) onto but not one-one

4. Let $f: R \to [1, \infty)$ be defined as

$f(x) = \log_{10}\left(\sqrt{3x^2 - 4x + k + 1} + 10\right)$. If f(x) is surjective, then

(A) $k = \frac{1}{3}$
(B) $k < \frac{1}{3}$
(C) $k > \frac{1}{3}$
(D) $k = 1$

5. Which one of the following function is surjective but not injective?

(A) $f: R \to R, f(x) = x^3 + x + 1$
(B) $f: [0, \infty) \to (0,1]; f(x) = e^{-|x|}$.
(C) $f: R \to R, f(x) = x^3 + 2x^2 - x + 1$
(D) $f: R \to R^+, f(x) = \sqrt{1+x^2}$

6. Let $A = \{1,2,3,4\}$ and $B = \{1,2,3,4\}$. If $f: A \to B$ is an one-one function and $f(x) \neq x$ for all $x \in A$, then the number of such possible functions, is

(A) 6
(B) 9
(C) 24
(D) 44

7. Let $p, q, r \in R$ such that $3q > p^2$. Then the function $g: R \to R$ given by $g(x) = x^3 + px^2 + qx + r$, is

(A) one-one and onto
(B) onto but not one-one
(C) one-one but not onto
(D) neither one-one nor onto

INTEGERTYPE

8. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$. Suppose $P(0) = 6, P(1) = 7, P(2) = 8$ and $P(3) = 9$, then find the value of P(4).

9. Let a function f defined from $R \to R$ as

$f(x) = \begin{cases} x + p^2, & \text{for } x \leq 2 \\ px + 5, & \text{for } x > 2 \end{cases}$

If the function is surjective, then find the sum of all possible integral values of p in $[-100,100]$.

10. Let f be a function satisfying the functional equation $f(x) + 2f\left(\frac{2x+1}{x-2}\right) = 3x, x \neq 2$, then find the value of $\left|\frac{f(3)}{f(7)}\right|$.

Daily Work Sheet-3

SINGLE CORRECTTYPE

1. Let

$$F(x)=\begin{cases} x|x| & \text{if} \quad x\le -1\\ [1+x]+[1-x] & \text{if} \quad -1<x<1\\ -x|x| & \text{if} \quad x\ge 1\end{cases}$$

where [x] denotes the greatest integer function then F(x) is

(A) even (B) odd

(C) neither odd nor even (D) even as well as odd

2. If $g(x)=\left(4\cos^4 x-2\cos 2x-\frac{1}{2}\cos 4x-x^7\right)^{\frac{1}{7}}$ then the value of $g(g(100))$ is equal to

(A) -1 (B) 0 (C) 1 (D) 100

3. If f(x) is defined on (0,1), then the domain of definition of $f(e^x)+f(\ln|x|)$ is

(A) $(-e,-1)$ (B) $(-e,-1)\cup(1,e)$

(C) $(-\infty,-1)\cup(1,\infty)$ (D) $(-e,e)$

4. Let $f(x)=\begin{cases}2+x, & x\ge 0\\ 4-x, & x<0\end{cases}$

If $f(f(x))=k$ has atleast one solution, then smallest value of k is

(A) 2 (B) 3 (C) 4 (D) 6

5. If $f(g(x))=g(f(x))=x$ for all real numbers x, and $f(2)=5$ and $f(5)=3$, then the value of $g(3)+g(f(2))$ is

(A) 7 (B) 5 (C) 3 (D) 2

6. Let f: A → B and g: B → C be two functions and gof : A → C is defined. Then which of the following statement(s) is true?

(A) If gof is onto then f must be onto.

(B) If f is into and g is onto then gof must be onto function.

(C) If gof is one-one then g is not necessarily one-one.

(D) If f is injective and g is surjective then gof must be bijective mapping.

MULTIPLE CORRECT TYPE

7. Let $f(x)=\begin{cases}x^2-4, & \text{if } |x|\le 3\\ 5\,\mathrm{sgn}|x-3|, & \text{if } |x|>3\end{cases}$ and $g(x)=2\tan^{-1}(e^x)-\frac{\pi}{2}$ for all $x\in R$, then which of the following is(are) correct?

[Note: sgn(k) denotes the signum function of k.]

(A) fog(x) is an even function (B) gof (x) is an even function.

(C) gog (x) is an odd function (D) fof (x) is an odd function.

MATCHTHE COLUMN

8. Let $f(x) = \ln\, x$ and $g(x) = x^2 - 1$

Column-I contains composite functions and column-II contains their domain. Match the entries of column-I with their corresponding answer is column-II.

Column-I	**Column-II**
(A) fog	(P) $(1, \infty)$
(B) gof	(Q) $(-\infty, \infty)$
(C) fof	(R) $(-\infty, -1) \cup (1, \infty)$
(D) gog	(S) $(0, \infty)$

INTEGER TYPE

9. Let $f(x) = \left[\frac{1}{\cos\{x\}}\right]$ where $[y]$ and $\{y\}$ denote greatest integer and fractional part functions respectively and $g(x) = 2x^2 - 3x(k+1) + k(3k+1)$. If $g(f(x)) < 0\, \forall x \in R$ then find the number of integral values of k.

10. If $h(x) = Ax^5 + B\sin\, x + C\ln\left(\frac{1+x}{1-x}\right) + 7$, where A, B, C are non-zero real constants and $h\left(\frac{-1}{2}\right) = 6$, then find the vale of $h\left(\frac{\operatorname{sgn}(e^{-x})}{2}\right)$.

Daily Work Sheet-4

SINGLE CORRECTTYPE

1. Let $f: R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map $g: R - \left\{\frac{4}{3}\right\} \to R - \left\{\frac{-4}{3}\right\}$ is given by

(A) $g(y) = \frac{3y}{3-4y}$

(B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$

(D) $g(y) = \frac{3y}{4-3y}$

2. The function f(x) is defined by $f(x) = \cos^4 x + K\cos^2 2x + \sin^4 x$, where K is a constant. If the function f(x) is a constant function, the value of k is

(A) -1 (B) $-1/2$ (C) 0 (D) $1/2$

3. If $f(x) = \sqrt[3]{\frac{9}{\log_2(3-2x)}} - 1$ then the value of 'a' which satisfies $f^{-1}(2a-4) = \frac{1}{2}$, is

(A) 4 (B) 3 (C) 2 (D) 1

4. Let $f: [0, a] \to S$ be a function defined by $f(x) = 3\cos\frac{x}{2}$. If the largest value of a for which f(x) has an inverse function $f^{-1}(x)$ is $k\pi$, then the value of k is

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2

5. Let $f: (-\infty, 2] \rightarrow [6, \infty)$ be defined as $f(x) = 4x^2 - 16x + 22$ and $g(x)$ is a function such that graphs of $f(x)$ and $g(x)$ are mirror image of each other with respect to line $x - y = 0$, then $g(10)$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

6. Let $f(x) = \frac{3}{2} + \sqrt{x - \frac{7}{4}}$ and $g(x)$ be the inverse function of $f(x)$ then the value of $(f^{-1}og^{-1})(17)$ is equal to

(A) $\frac{3+\sqrt{61}}{2}$ (B) 242 (C) 17 (D) $\frac{3-\sqrt{61}}{2}$

7. Let f be a function defined as $f: \left(0, e^{\frac{-3}{2}}\right] \rightarrow \left[\frac{-1}{4}, \infty\right)$, $f(x) = (\ln\ x)^2 + 3\ln\ x + 2$ then $f^{-1}(x)$ equals

(A) $\log\left(\frac{-3+\sqrt{4x+1}}{2}\right)$ (B) $\log\left(\frac{-3-\sqrt{4x+1}}{2}\right)$

(C) $e^{\frac{-3+\sqrt{4x+1}}{2}}$ (D) $e^{\frac{-3-\sqrt{4x+1}}{2}}$

8. Let $g: R \rightarrow R$ defined by $g(x) = \{e^x\}$, where $\{x\}$ denotes fractional part function.

Statement-1 : $g(x)$ is periodic function.

Statement-2 : $\{x\}$ is periodic function.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true

MULTIPLE CORRECTTYPE

9. Let $f: I \rightarrow I$, defined as $f(x) = 2\sin(2\pi x) - 10\tan(5\pi x) + 7\cos(4\pi x) + 3$, then which of the following statement(s) is/are TRUE?

(A) $f(x)$ is periodic function. (B) $f(x)$ is an even function.

(C) $f(x)$ is an odd function and its inverse exists. (D) $f(f(f(x))) = f(f(x))$ for all $x \in I$.

[Note : I denote the set of all integers.]

INTEGER TYPE

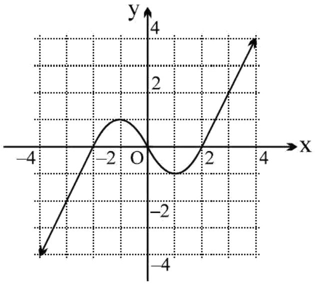
10. If $f: [2, \infty) \rightarrow [8, \infty)$ is a surjective function defined by $f(x) = x^2 - (p-2)x + 3p - 2$, $p \in R$ then sum of values of p is $m + \sqrt{n}$, where $m, n \in N$. Find the value of $\frac{n}{m}$.

Daily Work Sheet-5

SINGLE CORRECTTYPE

1. The graph of the function $y = g(x)$ is shown.

The number of solutions of the equation $||g(x)| - 1| = \frac{1}{2}$, is



(A) 4 (B) 5 (C) 6 (D) 8

2. Which of the following equations have the same graphs?

I. $y = x - 2$

II. $y = \frac{(x^2-4)}{(x+2)}$

III. $(x + 2)y = x^2 - 4$

(A) I and II only.

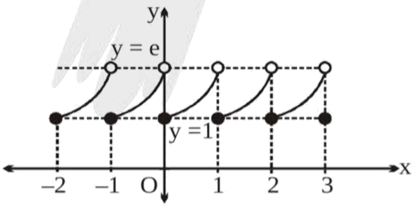
(B) I and III only.

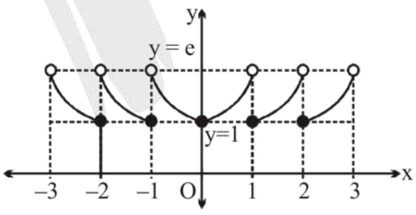
(C) II and III only.

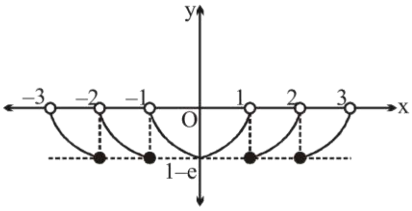
(D) All the equations have different graphs.

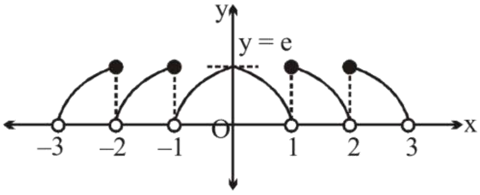
3. Which one of the following best represent the graph of function $f(x) = e^{\{|x|\}}$.

[Note: $\{\alpha\}$ denotes the fractional part of α.

(A) 

(B) 

(C) 

(D) 

4. If $f(x) = 2x - 1$ then number of solution(s) of the equation $|f(x)| = |f(|x| - 1)|$ is(are)

(A) 1 (B) 2 (C) 4 (D) 8

5. If $f(x) = |x+2| + |2x-p| + |x-2|$ attains its minimum value in the interval $(-1,1)$ then sum of all possible integral value of p is

(A) 0 (B) 1 (C) 3 (D) 4

PARAGRAPHBASED

Paragraph for question nos. 6&7

Let $f(x) = x^2 - 2x - 1 \forall x \in R$. Let $f: (-\infty, a] \to [b, \infty)$, where ' a ' is the largest real number for which f(x) is bijective.

6. The value of $(a+b)$ is equal to

(A) -2 (B) -1 (C) 0 (D) 1

7. Let $f: R \to R$, then range of values of k for which equation $f(|x|) = k$ has 4 distinct real roots is

(A) $(-2,-1)$ (B) $(-2,0)$ (C) $(-1,0)$ (D) $(0,1)$

MATHCH THE COLUMN

8. Let ' f ' be a function defined in $[-2,3]$ given as $f(x) = \begin{cases} 3(x+1)^{1/3}, & -2 \le x < 0 \\ -(x-1)^2, & 0 \le x < 1 \\ 2(x-1)^2, & 1 \le x < 2 \\ -x^2 + 4x - 3, & 2 \le x \le 3 \end{cases}$

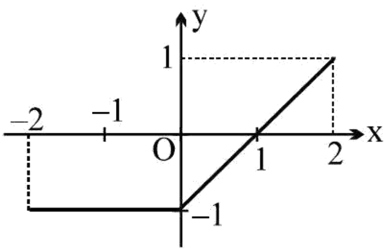
List-I	**List-II**
(P) The number of integers in the range of f(x) is	(1) 2
(Q) The number of integral values of x which are in the domain of $f(1-\lvert x\rvert)$, is	(2) 4
(R) The number of integers in the range of $\lvert f(-\lvert x\rvert)\rvert$, is	(3) 6
(S) The number of integral values of k for which the equation $f(\lvert x\rvert) = k$ has exactly four distinct solutions is	(4) 7

Code :

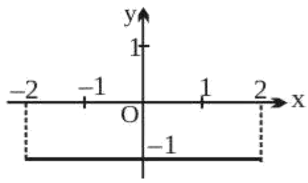
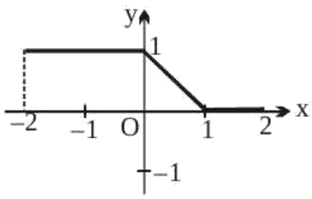
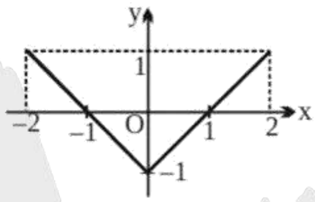
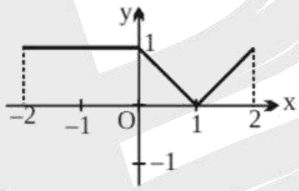
(A) P-3, Q-3, R-2, S-1 (B) P-4, Q-4, R-2, S-1

(C) P-3, Q-4, R-2, S-1 (D) P-3, Q-4, R-2, S-2

9. The graph of the function $y = f(x)$ is as follows.



Match the function mentioned in Column-I with the respective graph given in Column-II.

Column-I	Column-II
(A) $y = \lvert f(x) \rvert$	(P) 
(B) $y = f(\lvert x \rvert)$	(Q) 
(C) $y = f(-\lvert x \rvert)$	(R) 
(D) $y = \frac{1}{2}(\lvert f(x) \rvert - f(x))$	(S) 

Daily Work Sheet-6

SINGLE CORRECT TYPE

1. If the equation $(p^2-4)(p^2-9)x^3+\left[\frac{p-2}{2}\right]x^2+(p-4)(p^2-5p+6)x+\{2p-1\}=0$ is satisfied by all values of x in (0,3] then sum of all possible integral values of ' p ' is

(A) 0 (B) 5 (C) 9 (D) 10

2. The sum of all different values of λ for which the equation $4\lambda[x]^2=\lambda+12$ has a solution in $[1,\infty)$, is

[Note : [k] denotes greatest integer less than or equal to k.]

(A) 8 (B) 3 (C) 4 (D) 6

PARAGRAPH BASED

Paragraph for question nos. 3 & 4

Let f be an even function satisfying $f(x-2)=f\left(x+\left[\frac{6x^2+13}{x^2+2}\right]\right)\forall x\in R$

and $f(x)=\begin{cases}3x, & 0\le x<1\\ 4-x, & 1\le x\le 4\end{cases}$

[Note : [y] denotes greatest integer function of y.]

3. The area bounded by the graph of f(x) and the x-axis from $x=-1$ to $x=9$ is

(A) $\frac{31}{2}$ (B) 15 (C) 12 (D) $\frac{15}{2}$

4. The value of $f(-89)-f(-67)+f(46)$ is equal to

(A) 4 (B) 5 (C) 6 (D) 7

MULTIPLE CORRECT TYPE

5. Consider, $f(x)=(x^2-1)^{1/3}$ for $x<0$, $g(x)=-(x^3+1)^{1/2}$ for $x>-1$

Identify which of the following statement(s) is(are) correct.

(A) The range of f(f(x)) is (−1,0). (B) The domain of g(g(x)) is (−1,0).

(C) $f^{-1}og^{-1}(x)=x\forall x\in(-\infty,0)$. (D) $g^{-1}of^{-1}(x)=x\forall x\in(-1,\infty)$.

6. The maximum value of the function defined by $f(x)=\min(e^x,2+e^2-x,8)$ is α then integral value of x satisfying the inequality $\frac{x(x-[\alpha])}{x^2-[\alpha]x+12}<0$, is

[Note: [k] denotes greatest integer function less than or equal to k.

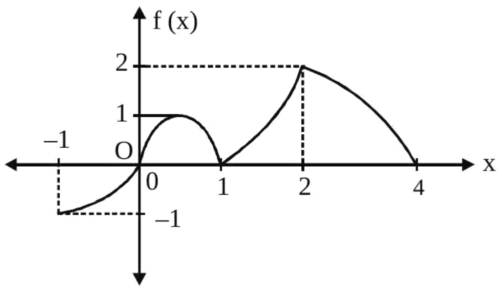
(A) 1 (B) 3 (C) 5 (D) 6

7. Let f be a constant function with domain R and g be a certain function with domain R. Two ordered pairs in f are $(4,a^2-5)$ and $(2,4a-9)$ for some real number a. Also domain of $\frac{f}{g}$ is $R-\{7\}$. Then

(A) $a = 2$ (B) $(f(10))^{100} = 1$

(C) $(100)^{g(7)} = 1$ (D) $\int_0^1 f(x)dx = 1$

8. If graph of a function f(x) which is defined in $[-1,4]$ is shown in the adjacent figure then identify the correct statement(s).



(A) domain of $f(|x|-1)$ is $[-5,5]$ (B) range of $f(|x|+1)$ is $[0,2]$

(C) range of $f(-|x|)$ is $[-1,0]$ (D) domain of $f(|x|)$ is $[-3,3]$

9. Find the sum of all the solutions of the equation $\cot\frac{\pi x}{2} = \log_2\{x\}$ in $x \in (0,100)$.

[Note: {k} denotes the fractional part function of k.]

INTEGERTYPE

10. Let $f(x) = \sin x - \cos^2 x$. If $f(x) = a$ has atleast one solution in $\left[0,\frac{\pi}{2}\right]$, then find the number of integral values of a. EXERCISE-2

INTERGER TYPE QUESTION

1. If $f(x) = 4x^3 - x^2 - 2x + 1$ and $g(x) = \begin{cases} \text{Min.}\{f(t): 0 \le t \le x\} & ; 0 \le x \le 1 \\ 3 - x & ; 1 < x \le 2 \end{cases}$ then find the value of λ if $2\lambda = g(1/4) + g(3/4) + g(5/4)$

2. Find the number of integral values of x in $[-\pi,\pi]$ which satisfies the domain of f(x)

$= \sqrt{\log_2\{4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} + 1\}}$

3. If $f(x) = x\ell n 2x - x \in \left[\frac{1}{2e},\frac{e}{2}\right]$ and rangeof f(x) is $\left[\frac{-1}{a}, b\right]$, then value of a + b is

4. Number of solutions of the equation

$$\sum_{n=0}^{\infty}(\sin^2 x)^n + \sum_{n=0}^{\infty}(\cos^2 x)^n = 4 \text{ in } (0,2\pi) \text{ will be}$$

5. Find the period of $f(x) = \sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2}$, where [.] denotes greatest integer function.

6. Suppose f and g both are linear function with $f(x) = -2x + 1$ and $f(g(x)) = 6x - 7$ and slope of $y = g(x)$ be m then 3 - m is equal to

7. If range of $f(x)=\frac{2\sin^2 x+2\sin x+3}{\sin^2 x+\sin x+1}$ is $[p, q]$ then $6p-3q$ equals

8. If $F(n+1)=\frac{2F(n)+1}{2}, n=2, \ldots \ldots .8F(1)=2$ then $\frac{F(101)}{26}$ equals

9. Number of solution for $|x-2|=[-2]$ is ($[-2\pi]$ denotes greatest integer)

10. The number of elements in the range of $f(x)=[x]+[2x]+\left[\frac{2}{3}x\right]+[3x]+[4x]+[5x]$ for $0 \leq x<3$ is

11. If $f(x)$ is a function such that $f(x-1)+f(x+1)=\sqrt{3}f(x)$ and $f(5)=8$, then $\sum_{r=0}^{250} f(5+12r)=\cdots$

12. Let $f(x)=([a]^2-5[a]+4)x^3-(6\{a\}2-5\{a\}+1)x-\operatorname{sgn} x.(\tan x)$ be an even function for $\forall x \in R$. If S be the sum of all possible values of 'a' then [S] is (Here [.] & {} represent greatest integer & fractional part functions respectively.)

13. Let $f(x)$ be a function such that $f(x-1)+f(x+1)=\sqrt{3}$ $f(x) \forall x \in R$. If $f(5)=100$, then $\sum_{r=0}^{49} f(5+12r)$

14. The period of the function

$$f(x)=\left(\sec^2\left(\frac{\pi x}{10}\right)-\tan^2\left(\frac{\pi x}{10}\right)\right)^{\cos^4 4\pi x+100\{x\}}$$

(where {.}denotesfractionalpartfunction)isλ, then $(\lambda/2)$ is equal to

15. Let $g(x)$ be a function such that $g(a+b)=g(a).g(b)\ \forall a, b \in R$. If zero is not an element in range of g, then $g(x).g(-x)$ is equal to

EXERCISE-3

SUBJECTIVE TYPE : S-1

1. Find the domains of definitions of the following functions :

(Read the symbols [*] and { *} as greatest integers and fractional part functions respectively.)

(i) $f(x)=\sqrt{\cos 2x}+\sqrt{16-x^2}$

(ii) $f(x)=\log_7 \log_5 \log_3 \log_2 (2x^3+5x^2-14x)$

(iii) $f(x)=\ln\left(\sqrt{x^2-5x-24}-x-2\right)$

(iv) $f(x)=\sqrt{\frac{1-5^x}{7^{-x}-7}}$

(v) $y=\log_{10} \sin(x-3)+\sqrt{16-x^2}$

(vi) $f(x)=\log_{100x}\left(\frac{2\log_{10} x+1}{-x}\right)$

(vii) $f(x)=\sqrt{x^2-|x|}+\frac{1}{\sqrt{9-x^2}}$

(viii) $f(x)=\sqrt{(x^2-3x-10)\cdot\ln^2(x-3)}$ (ix) $f(x)=\sqrt{(5x-6-x^2)[\{\ln\{x\}\}]}+$

$$\sqrt{(7x-5-2x^2)}+\left(\ln\left(\frac{7}{2}-x\right)\right)^{-1}$$

(x) $f(x)=\log_{\left[x+\frac{1}{x}\right]}|x^2-x-6|+{}^{16-x}C_{2x-1}+{}^{20-3x}P_{2x-5}$

2. Find the domain & range of the following functions. (Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $y=\log_{\sqrt{5}}(\sqrt{2}(\sin x-\cos x)+3)$

(ii) $y=\frac{2x}{1+x^2}$

(iii) $f(x)=\frac{x^2-3x+2}{x^2+x-6}$

(iv) $f(x)=\frac{x}{1+|x|}$

(v) $y=\sqrt{2-x}+\sqrt{1+x}$

(vi) $f(x)=\frac{\sqrt{x+4}-3}{x-5}$

3. (a) Draw graphs of the following function, where [ ] denotes the greatest integer function.

(i) $f(x)=x+[x]$

(ii) $y=(x)^{[x]}$ where $x=[x]+(x)$ & $x>0$ & $x\leq 3$

(iii) $y=\operatorname{sgn}[x]$

(iv) $\operatorname{sgn}(x-|x|)$

(b) Identify the pair(s) of functions which are identical? (where [x] denotes greatest integer and {x} denotes fractional part function)

(i) $f(x) = \text{sgn}\,(x^2 - 3x + 4)$ and $g(x) = e^{[\{x\}]}$

(ii) $f(x) = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ and $g(x) = \tan x$

(iii) $f(x) = \ln(1+x) + \ln(1-x)$ and $g(x) = \ln(1-x^2)$

(iv) $f(x) = \frac{\cos x}{1-\sin x}$ and $g(x) = \frac{1+\sin x}{\cos x}$

4. Classify the following functions f(x) definzed in $R \to R$ as injective, surjective, both or none .

(a) $f(x) = \frac{x^2+4x+30}{x^2-8x+18}$

(b) $f(x) = x^3 - 6x^2 + 11x - 6$

(c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$

5. Solve the following problems from (a) to (e) on functional equation.

(a) The function f(x) defined on the real numbers has the property that $f(f(x)) \cdot (1 + f(x)) = -f(x)$ for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of f(3).

(b) Suppose f is a real function satisfying $f(x + f(x)) = 4\,f(x)$ and $f(1) = 4$. Find the value of f(21).

(c) Let ' f ' be a function defined from $R^+ \to R^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and $f(2) = 6$, find the value of f(50).

(d) Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x. Then find the value of f(300).

6. Suppose $f(x) = \sin x$ and $g(x) = 1 - \sqrt{x}$. Then find the domain and range of the following functions.

(a) fog (b) gof (c) fof (d) gog

7. A function $f: R \to R$ is such that $f\left(\frac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Prove the following.

(a) $f(f(x)) = x$

(b) $f(1/x) = -f(x), x \neq 0$ (c) $f(-x-2) = -f(x) - 2$

8. Find the formula for the function fogoh, given $f(x) = \frac{x}{x+1}$; $g(x) = x^{10}$ and $h(x) = x + 3$. Find also the domain of this function. Also compute (fogoh)(-1).

9. Let f be a one-one function with domain {x, y, z} and range {1,2,3}. It is given that exactly one of the following statements is true and the remaining two are false.

$f(x) = 1$; $f(y) \neq 1$; $f(z) \neq 2$. Determine $f^{-1}(1)$

10. $f(x) = \begin{cases} 1-x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$ and

$g(x) = \begin{bmatrix} -x & \text{if} & x < 1 \\ 1-x & \text{if} & x \geq 1 \end{bmatrix}$ find $(fog)(x)$ and $(gof)(x)$

11. Find whether the following functions are even or odd or none

(a) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$

(b) $f(x) = \frac{x(a^x+1)}{a^x-1}$

(c) $f(x) = \sin x + \cos x$

(d) $f(x) = x\sin^2 x - x^3$

(e) $f(x) = \sin x - \cos x$

(f) $f(x) = \frac{(1+2^x)^2}{2^x}$

(g) $f(x) = \frac{x}{e^x-1} + \frac{x}{2} + 1$

(h) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

12. (i)Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :

(a) $10^x + 10^y = 10$

(b) $x + |y| = 2y$

(ii) The function f(x) is defined on the interval [0,1]. Find the domain of definition of the functions.

(a) $f(\sin x)$ (b) $f(2x+3)$

(iii) Given that $y = f(x)$ is a function whose domain is [4,7] and range is [−1,9]. Find the range and domain of

(a) $g(x) = \frac{1}{3} f(x)$

(b) $h(x) = f(x-7)$

13. Compute the inverse of the functions:

(a) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$

(b) $f(x) = 2^{\frac{x}{x-1}}$

(c) $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

14. Find the inverse of $f(x) = 2^{\log_{10} x} + 8$ and hence solve the equation $f(x) = f^{-1}(x)$.

13. (a) Function f & g are defined by $f(x) = \sin x, x \in R$;

$g(x) = \tan x, x \in R - \left(K + \frac{1}{2}\right)\pi$

where $K \in I$. Find

(i) periods of fog & gof.

(ii) range of the function fog & gof .

(b) Suppose that f is even, periodic function with period 2 , and that $f(x)=x$ for all x in interval $[0,1]$. Find the value of $f(3.14)$.

SUBJECTIVE TYPE : S-2

1. (a) Let $P(x)=x^6+ax^5+bx^4+cx^3+dx^2+ex+f$ be a polynomial such that

$P(1)=1; P(2)=2; P(3)=3; P(4)=4; P(5)=5$ and $P(6)=6$ then find the value of $P(7)$.

(b) Let a and b be real numbers and let $f(x)=a\sin x+b\sqrt[3]{x}+4, \forall x\in R$.

If $f(\log_{10}(\log_3 10))=5$ then find the value of $f(\log_{10}(\log_{10} 3))$.

2. Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by $x-1$ is 1 and the remainder when p(x) is divided by $x-4$ is 10 . If r(x) is the remainder when p(x) is divided by $(x-1)(x-4)$, find the value of $r(2006)$.

3. A function f, defined for all $x, y\in R$ is such that $f(1)=2; f(2)=8$

& $f(x+y)-kxy=f(x)+2y^2$, where k is some constant . Find f(x) & show that :

$f(x+y)f\left(\frac{1}{x+y}\right)=k$ for $x+y\neq 0$.

4. If $f(x)=-1+|x-2|, 0\le x\le 4$ $g(x)=2-|x|, -1\le x\le 3$

Then find $f\circ g(x)$ & gof (x). Draw rough sketch of the graphs of fog(x) & gof(x).

5. Let $f(x)=x^{135}+x^{125}-x^{115}+x^5+1$. If f(x) is divided by x^3-x then the remainder is some function of x say g(x). Find the value of $g(10)$.

6. Let $\{x\}$&$[x]$ denote the fractional and integral part of a real number x respectively.

Solve $4\{x\}=x+[x]$

7. Let $f(x)=\frac{9^x}{9^x+3}$ then find the value of the sum $f\left(\frac{1}{2006}\right)+f\left(\frac{2}{2006}\right)+f\left(\frac{3}{2006}\right)+\cdots+f\left(\frac{2005}{2006}\right)$

8. Let $f(x)=(x+1)(x+2)(x+3)(x+4)+5$ where $x\in[-6,6]$. If the range of the function is $[a,b]$ where $a, b\in N$ then find the value of $(a+b)$.

9. The set of real values of ' x ' satisfying the equality $\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]=5$ (where [ ] denotes the greatest integer function) belongs to the interval (a, b/c] where $a, b, c\in N$ and b/c is in its lowest form. Find the value of a +b + c + abc.

10. f(x) and g(x) are linear function such that for all x, f(g(x)) and g(f(x)) are Identity functions. If $f(0)=4$ and $g(5)=17$, compute $f(2006)$.

EXERCISE-4

PREVIOUS YEAR QUESTION

PART - A : JEE-MAIN

1. The domain of the function $f(x)=\frac{1}{\sqrt{|x|-x}}$ is :- **[AIEEE 2011]**

(A) $(-\infty,0)$ (B) $(-\infty,\infty)-\{0\}$ (C) $(-\infty,\infty)$ (D) $(0,\infty)$

2. Let f be a function defined by $f(x)=(x-1)^2+1,(x\geq 1)$ **[AIEEE 2011]**

Statement - 1: The set $\{x: f(x)=f^{-1}(x)\}=\{1,2\}$

Statement - 2 : f is bijection and $f^{-1}(x)=1+\sqrt{x-1}, x>1$.

(A) Statement-1 is true, Statement-2 is false.

(B) Statement-1 is false, Statement-2 is true.

(C) Statement-1 is true, Statement-2 is true ; Statement2 is a correct explanation for Statement -1.

(D) Statement-1 is true, Statement-2 is true ; Statement2 is not a correct explanation for statement -1.

3. If $f: R\to R$ is a function defined by $f(x)=[x]\cos\pi\left(\frac{2x-1}{2}\right)$, where [x] denotes the greatest integer function, then f is : **[AIEEE 2012]**

(A) continuous only at $x=0$.

(B) continuous for every real x.

(C) discontinuous only at $x=0$.

(D) discontinuous only at non-zero integral values of x.

4. If $X=\{4^n-3n-1: n\in N\}$ and $Y=\{9(n-1): n\in N\}$, where N is the set of natural numbers, then $X\cup Y$ is equal to : **[JEE - Main 2014]**

(A) N (B) $Y-X$ (C) X (D) Y

5. If $f(x)+2f\left(\frac{1}{x}\right)=3x, x\neq 0$ and $S=\{x\in R: f(x)=f(-x)\}$; then S : **[JEE - Main 2016]**

(A) contains exactly one element.

(B) contains exactly two elements.

(C) contains more than two elements

(D) is an empty set.

6. For $x\in R, f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then: **[JEE - Main 2016]**

(A) $g'(0)=\cos(\log 2)$

(B) $g'(0)=-\cos(\log 2)$

(C) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

(D) g is not differentiable at $x = 0$

7. The function $f: R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is : **[JEE - Main 2017]**

(A) neither injective nor surjective. (B) invertible

(C) injective but not surjective (D) surjective but not injective.

8. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}, x \in R - \{-1,0,1\}$.

If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is **[JEE - Main 2018]**

(A) -3 (B) $-2\sqrt{2}$ (C) $2\sqrt{2}$ (D) 3

9. The domain of the definition of the function $f(x) = \frac{1}{4-x^2} + \log_{10}(x^3 - x)$ is[JEE - Main 2019]

(A) $(-1,0) \cup (1,2) \cup (3,\infty)$ (B) $(-2,-1) \cup (-1,0) \cup (2,\infty)$

(C) $(-1,0) \cup (1,2) \cup (2,\infty)$ (D) $(1,2) \cup (2,\infty)$

10. Let $f(x) = a^x (a > 0)$ be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. Then $f_1(x+y) + f_1(x-y)$ equals **[JEE - Main 2019]**

(A) $2f_1(x+y) \cdot f_2(x-y)$ (B) $2f_1(x+y) \cdot f_1(x-y)$

(C) $2f_1(x) \cdot f_2(y)$ (D) $2f_1(x) \cdot f_1(y)$

11. For $x \in \left(0, \frac{3}{2}\right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$ If $\phi(x) = (\,hof\,) \circ g)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to **[JEE - Main 2019]**

(A) $\tan\frac{\pi}{12}$ (B) $\tan\frac{11\pi}{12}$ (C) $\tan\frac{7\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

12. Let $f(x)x^2, x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R: f(x) \in A\}$. If $S = [0,4]$, then which one of the following statemetns is not true? **[JEE - Main 2019]**

(A) $f(g(S)) = S$ (B) $g(f(f)) \neq S$ (C) $g(f(S)) = g(S)$ (D) $f(g(S)) = f(S)$

13. Let $\sum_{k=1}^{10} f(a+k) = 16(2^{10} - 1)$, where the function f satisfies $f(x+y) = f(x)f(y)$ for all natural numbers x, y and $f(1) = 2$. Then, the natural number ' a ' is **[JEE - Main 2019]**

(A) 2 (B) 4 (C) 3 (D) 16

14. If $f(x) = \left(\frac{1-x}{1+x}\right)$, $|x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to **[JEE - Main 2019]**

(A) $2f(x)$ (B) $2f(x^2)$ (C) $(f(x))^2$ (D) $-2f(x)$

15. For $x \in R - \{0,1\}$, let $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$ be three given functions. If a function, $J(x)$ satisfies $(f_2.J.f_1)(x) = f_3(x)$, then $J(x)$ is equal to **[JEE - Main 2019]**

(A) $f_2(x)$ (B) $f_3(x)$ (C) $f_1(x)$ (D) $\frac{1}{x}f_3(x)$

16. If the function $f: \mathbf{R} - \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1-x^2}$, is surjective, then A is equal to **[JEE - Main 2019]**

(A) $\mathbf{R} - \{-1\}$ (B) $[0, \infty)$ (C) $\mathbf{R} - [-1,0)$ (D) $\mathbf{R} - (-1,0)$

17. Let a function $f: (0, \infty) \longrightarrow (0, \infty)$ be defined by $f(x) = \left|1 - \frac{1}{x}\right|$. Then, f is **[JEE - Main 2019]**

(A) injective only

(B) both injective as well as surjective

(C) not injective but it is surjective

(D) neither injective nor surjective

18. The number of functions f from $\{1,2,3, \ldots .20\}$ onto $\{1, 2,3, \ldots ,20\}$ such that f(k) is a multiple of 3, whenever k is a multiple of 4 , is **[JEE - Main 2019]**

(A) $(15)! \times 6!$ (B) $5^6 \times 15$ (C) $5! \times 6\,!$ (D) $6^5 \times (15)\,!$

19. Let $f: R \to R$ be defined by $f(x) = \frac{x}{1+x^2}$ $x \in R$. Then, the range of f is **[JEE - Main 2019]**

(A) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (B) $(-1,1) - \{0\}$ (C) $R - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $R - [-1,1]$

20. Let N be the set of natural numbers and two functions f and g be defined as $f, g: N \longrightarrow N$ such that $f(n) = \begin{cases} \frac{n+1}{2}; & \text{if n is odd} \\ \frac{n}{2}; & \text{if n is even} \end{cases}$ and $g(n) = n - (-1)^n$. Then, fog is **[JEE - Main 2019]**

(A) one-one but not onto

(B) onto but not one-one

(C) both one-one and onto

(D) neither one-one nor onto

21. Let $A = \{x \in R: x \text{ is not a positive integer }\}$. Define a function $f: A \to R$ as $f(x) = \frac{2x}{x-1}$, then f is **[JEE - Main 2019]**

(A) injective but not surjective

(B) not injective

(C) surjective but not injective

(D) neither injective nor surjective

PART - B : JEE ADAVANCE

1. Let $f(x) = x^2$ and $g(x) = \sin\, x$ for all $x \in R$. Then the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)\,(x) = f(g(x))$, is- **[JEE 2011]**

(A) $\pm\sqrt{n\pi}, n \in \{0,1,2, \ldots.$

(B) $\pm\sqrt{n\pi}, n \in \{1,2, \ldots\}$

(C) $\frac{\pi}{2} + 2n\pi, n \in \{\ldots., -2, -1,0,1,2, \ldots.$

(D) $2n\pi, n \in \{\ldots., -2, -1,0,1,2, \ldots..$

2. The function $f: [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[JEE 2012]**

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

3. Let $f: (-1,1) \to \mathbb{R}$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0,\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are)- **[JEE 2012]**

(A) $1-\sqrt{\frac{3}{2}}$ (B) $1+\sqrt{\frac{3}{2}}$ (C) $1-\sqrt{\frac{2}{3}}$ (D) $1+\sqrt{\frac{2}{3}}$

4. For every pair of continuous functions $f, g: [0,1] \to R$ such that **[JEE Ad. 2014]**

$\max\{f(x): x \in [0,1]\} = \max\{g(x): x \in [0,1]\}$, the correct statement(s) is (are) :

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3\,g(c)$ for some $c \in [0,1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3\,g(c)$ for some $c \in [0,1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0,1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0,1]$

5. Let $f: \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then, **[JEE Ad. 2014]**

(A) f(x) is an odd function (B) f(x) is a one-one function

(C) f(x) is an onto function (D) f(x) is an even function

6. Let $f_1: R \to R, f_2: [0,\infty) \to R, f_3: R \to R$ and $f_4: R \to [0,\infty)$ be defined by **[JEE Ad. 2014]**

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases};\quad f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0 \end{cases}$$

List – I	**List - II**
(p) f_4 is	(1) onto but not one-one
(q) f_3 is	(2) neither continuous nor one-one
(r) f_2 of f_1 is	(3) differentiable but not one-one
(s) f_2 is	(4) continuous and one-one

Codes :

	p	**q**	**r**	**s**
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

7. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$.

Let (fog)(x) denote f(g(x)) and (gof) (x) denote g(f(x)). Then which of the following is (are) true? **[JEE Ad. 2015]**

(A) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(B) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(C) $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$
(D) There is an $x \in R$ such that $(gof)(x) = 1$

8. Let $f: R \to R, g: R \to R$ and $h: R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2, g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then **[JEE Ad. 2016]**

(A) $g'(2) = \frac{1}{15}$
(B) $h'(1) = 666$
(C) $h(0) = 16$
(D) $h(g(3)) = 36$

9. Let $f: R \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval $(0,1)$? **[JEE Ad. 2017]**

(A) $e^x - \int_0^x f(t)\sin t\,dt$
(B) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$
(C) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$
(D) $x^9 - f(x)$

10. Let $f: R \to R$ and $g: R \to R$ be two non-constant differentiable functions.
If $f'(x) = \left(e^{(f(x))-g(x)}\right)g'(x)$ for all $x \in R$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ? **[JEE Ad. 2018]**

(A) $f(2) < 1 - \log_e 2$
(B) $f(2) > 1 - \log_e 2$
(C) $g(1) > 1 - \log_e 2$
(D) $g(1) < 1 - \log_e 2$

11. Let $E_1 = \left\{x \in R: x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and **[JEE Ad. 2018]**

$E_2 = \left\{x \in E_1: \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number }\right\}$

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.)

Let $f: E_1 \to R$ be the function defined by $f(x) = \log_e\left[\frac{x}{x-1}\right]$ and $g: E_2 \to R$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$

LIST-I	**LIST-II**
(P) The range of f is	(1) $\left[-\infty,\frac{1}{1-e}\right) \cup \left[\frac{e}{e-1},\infty\right)$
(Q) The range of g contains	(2) $(0,1)$
(R) The domain of f contains	(3) $\left[-\frac{1}{2},\frac{1}{2}\right]$
(S) The domain of g is	(4) $(-\infty,0) \cup (0,\infty)$
	(5) $\left(-\infty,\frac{e}{e-1}\right]$
	(6) $(-\infty,0) \cup \left(\frac{1}{2},\frac{e}{e-1}\right]$

(A) $P \to 4; Q \to 2; R \to 1; S \to 1$
(B) $P \to 3; Q \to 3; R \to 6; S \to 5$
(C) $P \to 4; Q \to 2; R \to 1; S \to 6$
(D) $P \to 4; Q \to 3; R \to 6; S \to 5$

ANSWER KEY

EXERCISE-1

Daily Work Sheet-1

1.	D	**2.**	C	**3.**	C	**4.**	C	**5.**	C	**6.**	C	**7.**	A
8.	C	**9.**	AB	**10.**	AC	**11.**	1						

Daily Work Sheet-2

1.	D	**2.**	B	**3.**	B	**4.**	A	**5.**	C	**6.**	B	**7.**	A
8.	9.	**10.**	11										

Daily Work Sheet-3

1.	A	**2.**	D	**3.**	A	**4.**	C	**5.**	A	**6.**	C
7.	ABC.	**8.**	(A) R; (B) S; (C) P; (D) Q					**9.**	1	**10.**	8

Daily Work Sheet-4

1.	B	**2.**	B	**3.**	B	4.	D	5.	A	6.	C	7.	D
8.	D	**9.**	ABD	**10.**	2								

Daily Work Sheet-5

1.	D	**2.**	D	**3.**	B	**4.**	B	**5.**	A	**6.**	B	**7.**	A
8.	C	**9.**	(A) S; (B) R; (C) P; (D) Q										

Daily Work Sheet-6

1.	B	**2.**	D	**3.**	B	**4.**	A	**5.**	ABCD	**6.**	ACD
7.	ABC	**8.**	ABC	**9.**	2525	**10.**	3				

EXERCISE-2

1.	5	**2.**	6	**3.**	2	**4.**	4	**5.**	8	**6.**	6	**7.**	4
8.	2	**9.**	0	**10.**	30	**11.**	2008	**12.**	11	**13.**	5000	**14.**	5
15.	1												

EXERCISE-3

Ss - 1

1. (i) $\left[-\frac{5\pi}{4},\frac{-3\pi}{4}\right]\cup\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$ (ii) $\left(-4,-\frac{1}{2}\right)\cup(2,\infty)$ (iii) $(-\infty,-3]$

(iv) $(-\infty,-1)\cup[0,\infty)$ (v) $(3-2\pi<x<3-\pi)U(3<x\leq 4)$

(vi) $\left(0,\frac{1}{100}\right)\cup\left(\frac{1}{100},\frac{1}{\sqrt{10}}\right)$ (vii) $(-3,-1]U\{0\}U[1,3)$

(viii) $\{4\}\cup[5,\infty)$ (ix) $(1,2)\cup(2,5/2)$; (x) $x\in\{4,5\}$

2. (i) D: $x\in R$ R: $[0,2]$ (ii) $D=R$; range $[-1,1]$

(iii) D: $\{x \mid x\in R; x\neq -3; x\neq 2\}$ R: $\{f(x)\mid f(x)\in R, f(x)\neq 1/5; f(x)\neq 1\}$

(iv) D: R; R: $(-1,1)$ (v) D: $-1\leq x\leq 2$ R: $[\sqrt{3},\sqrt{6}]$

(vi) D: $[-4,\infty)-\{5\}$; R: $\left(0,\frac{1}{6}\right)\cup\left(\frac{1}{6},\frac{1}{3}\right]$

3. (b) (i), (iii) are identical

4. (a) neither surjective nor injective

(b) surjective but not injective

(c) neither injective nor surjective

5. (a) $-3/4$; (b) 64; (c) 30, (d) 5050

6. (a) domain is $x\geq 0$; range $[-1,1]$;

(b) domain $2k\pi\leq x\leq 2k\pi+\pi$; range $[0,1]$

(c) Domain $x\in R$; range $[-\sin 1,\sin 1]$;

(d) domain is $0\leq x\leq 1$; range is $[0,1]$

8. (a) $\frac{(x+3)^{10}}{(x+3)^{10}+1}$, domain is R, $\frac{1024}{1025}$;

(b) $g(x)=\begin{cases}\frac{1}{x^2} & \text{if } 0<x\leq 1\\ x^2 & \text{if } x>1\end{cases}$

9. $f^{-1}(1)=y$

10. $(gof)(x)=\begin{cases}x & \text{if } x\leq 0\\ -x^2 & \text{if } 0<x<1\\ 1-x^2 & \text{if } x\geq 1\end{cases}$; $(fog)(x)=\begin{cases}x^2 & \text{if } x<0\\ 1+x & \text{if } 0\leq x<1\\ x & \text{if } x\geq 1\end{cases}$

11. (a) odd, (b) even, (c) neither odd nor even, (d) odd, (e) neither odd nor even, (f) even, (g) even, (h) even

12. (i)(a) $y=\log(10-10^x), -\infty<x<1$

(b) $y=x/3$ when $-\infty<x<0$ & $y=x$ when $0\leq x<+\infty$

(ii)(a) $2K\pi\leq x\leq 2K\pi+\pi$ where $K\in I$ (b) $[-3/2,-1]$

(iii)(a) Range : $[-1/3,3]$, Domain $=[4,7]$; (b) Range $[-1,9]$ and domain $[11,14]$

13. (a) $\frac{e^x - e^{-x}}{2}$;

(b) $\frac{\log_2 x}{\log_2 x - 1}$;

(c) $\frac{1}{2}\log\frac{1+x}{1-x}$

14. $x = 10; f^{-1}(x) = 10^{\log_2 (x-2)}$

15. (a) (i) period of fog is π, period of gof is 2π;

(ii) range of fog is $[-1,1]$, range of gof is $[-\tan 1, \tan 1]$

(b) 0.86

S – 2

1. (a) 727, (b) 3 **2.** 6016 **3.** $f(x) = 2x^2$

4. $\text{fog}(x) = \begin{cases} -(1+x) & , -1 \le x \le 0 \\ x-1 & , 0 < x \le 2 \end{cases}$; $\text{gof}(x) = \begin{cases} x+1 & , 0 \le x < 1 \\ 3-x, & 1 \le x \le 2 \\ x-1, & 2 < x \le 3 \\ 5-x, & 3 < x \le 4 \end{cases}$

$\text{fof}(x) = \begin{cases} x & 0 \le x \le 1 \\ 4-x & , 3 \le x \le 4 \end{cases}$; $\text{gog}(x) = \begin{cases} -x & , -1 \le x \le 0 \\ 4-x & , 0 < x \le 2 \\ 4 - 2 < x \le 3 \end{cases}$

5. 21 **6.** x = 0 or 5/3 **7.** 1002.5 **8.** 5049 **9.** 20

10. 122

EXERCISE- 4

PART – A

1. A **2.** C **3.** B **4.** D **5.** B **6.** A **7.** D

8. C **9.** C **10.** D **11.** B **12.** C **13.** C **14.** A

15. B **16.** C **17.** D **18.** A **19.** A **20.** B **21.** A

PART – B

1. A **2.** B **3.** AB **4.** AD **5.** ABC **6.** D

7. ABC **8.** BC **9.** C, D **10.** B, C **11.** A