LIMIT

THINGS TO REMEMBER :

1. Limit of a function f(x) is said to exist as, $x \to a$ when

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) =$ finite quantity.

2. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exists then :

(i) $\lim\limits_{x\to a} f(x) \pm g(x) = l \pm m$

(ii) $\lim\limits_{x\to a} f(x) \cdot g(x) = l.m$

(iii) $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$, provided $m \neq 0$

(iv) $\lim\limits_{x\to a} k\,f(x) = k \lim\limits_{x\to a} f(x)$; where k is a constant.

(v) $\lim\limits_{x\to a} f[g(x)] = f\left(\lim\limits_{x\to a} g(x)\right)$; provided f is continuous at $x = \lim\limits_{x\to a} g(x)$

3. STANDARD LIMITS :

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \frac{\tan^{-1} x}{x} = \lim\limits_{x\to 0} \frac{\sin^{-1} x}{x}$

[Where x is measured in radians]

(b) $\lim\limits_{x\to 0} (1+x)^{1/x} = e = \lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ note however there $\lim\limits_{\substack{h\to 0\\ n\to\infty}} (1-h^2)^n = 0$

and $\lim\limits_{\substack{h\to 0\\ n\to\infty}} (1+h^2)^n \to \infty$

(c) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} \phi(x) = \infty$, then ;

$\lim\limits_{x\to a} [f(x)]^{\phi(x)} = e^{\lim\limits_{x\to a} \phi(x)[f(x)-1]}$

(d) If $\lim\limits_{x\to a} f(x) = A > 0$ & $\lim\limits_{x\to a} \phi(x) = B$ (a finite quantity) then ;

$\lim\limits_{x\to a} [f(x)]^{\phi(x)} = A^B$

(e) $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \ln a\ (a > 0)$. In particular $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

(f) $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

4. SQUEEZE PLAY THEOREM :

If $f(x) \le g(x) \le h(x)\ \forall\ x$ & $\lim\limits_{x\to a} f(x) = l = \text{Limit}_{x\to a}\ h(x)$ then $\text{Limit}_{x\to a}\ g(x) = l$.

5. INDETERMINANT FORMS :

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^\circ, \infty^\circ, \infty - \infty$ and 1^∞

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementry algebra.

(ii) $\infty + \infty = \infty$ (iii) $\infty \times \infty = \infty$ (iv) $(a/\infty) = 0$ if a is finite

(v) $\frac{a}{0}$ is not defined, if $a \neq 0$. (vi) $ab = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation

(b) Rationalisation or double rationalisation

(c) Use of trigonometric transformation ;

appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx, cos x, tan x should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \cdots \ldots \ldots .\, a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \ldots \ldots$

(iii) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots$ for $-1 < x \leq 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \cdots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \cdots$

(vii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \cdots$

(viii) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \cdots \ldots$

(ix) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots \ldots$

PROFICIENCY TEST-01

1. $\lim\limits_{x\to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

2. $\lim\limits_{x\to 1} \frac{x^2-x\cdot \ln x+\ln x-1}{x-1}$

3. $\lim\limits_{x\to 1} \frac{\sqrt[13]{x}-\sqrt[7]{x}}{\sqrt[5]{x}-\sqrt[3]{x}}$

4. $\lim\limits_{x\to 0} [\ln(1+\sin^2 x)\cdot \cot(\ln^2(1+x))]$

5. (a) $\lim\limits_{x\to 0} \tan^{-1}\frac{a}{x^2}$, where $a \in R$;

(b) Plot the graph of the function $f(x) = \lim\limits_{t\to 0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

6. $\lim\limits_{x\to 1} \frac{\left[\sum_{k=1}^{100} x^k\right]-100}{x-1}$

7. $\lim\limits_{x\to \infty} \frac{2\sqrt{x}+3x^{1/3}+5x^{1/5}}{\sqrt{3x-2}+(2x-3)^{1/3}}$

8. $\lim\limits_{x\to 0} \frac{\sec 4x-\sec 2x}{\sec 3x-\sec x}$

9. $\lim\limits_{x\to 0} \frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

10. $\lim\limits_{\theta\to \frac{\pi}{4}} \frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

PROFICIENCY TEST-02

1. If $l = \lim\limits_{n\to\infty} \sum_{r=2}^{n} \left((r+1)\sin\frac{\pi}{r+1} - r\sin\frac{\pi}{r}\right)$ then find $\{l\}$. (where {} denotes the fractional part function)

2. $\lim\limits_{x\to-\infty} \frac{(3x^4+2x^2)\sin\frac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$

3. Let $S_n = a_1 + a_2 + a_3 + \cdots . + a_n$ where $\lim\limits_{n\to\infty} a_n = 2$, then find the value of $\lim\limits_{n\to\infty} \frac{S_{n+1}-S_n}{\sum_{k=1}^{n} k}$

4. Calculate $\lim\limits_{x\to 0} \left(1 + xe^{-\frac{1}{x^2}}\sin\frac{1}{x^4}\right)$

5. $\lim\limits_{x\to\infty} \left(\frac{x+c}{x-c}\right)^x = 4$ then find c

6. $\lim\limits_{x\to\infty} \left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$

7. $\lim\limits_{n\to\infty} (\cos(\ln(n-1) - \ln(n+1)))^{(n+1)^2}$

8. Evaluate, $\lim\limits_{x\to 1} \frac{1-x+\ln x}{1+\cos\pi x}$

9. $\lim\limits_{y\to 0} \left[\lim\limits_{x\to\infty} \frac{\exp\left(x\ln\left(1+\frac{ay}{x}\right)\right)-\exp\left(x\ln\left(1+\frac{by}{x}\right)\right)}{y}\right]$

10. $\lim\limits_{x\to 1} \frac{ax^2+bx+c}{(x-1)^2} = 2$, then find the value of $\lim\limits_{x\to 1} \frac{(x-a)(x-b)(x-c)}{(x+1)}$

EXERCISE- I

1. $\lim\limits_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$ p, q $\in$ N

2. If $L=\lim\limits_{x\to 1}\frac{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{2n})}{[(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)]^2}$ then show that L can be equal to

(a) $\prod_{r=1}^{n}\frac{n+r}{r}$ (b) $\frac{1}{n!}\prod_{r=1}^{n}(4r-2)$

(c) The sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.

(d) The coefficient of x^n in the expansion of $(1+x)^{2n}$.

3. $\lim\limits_{x\to\infty}(x-\ln\cosh x)$ where $\cosh t=\frac{e^t+e^{-t}}{2}$.

4. $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{\cos^{-1}\left(2x\sqrt{1-x^2}\right)}{x-\frac{1}{\sqrt{2}}}$

5. $\lim\limits_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$

6. If $\lim\limits_{x\to 0}\frac{a\sin x-\sin 2x}{\tan^3 x}$ is finite then find the value of ' a ' & the limit.

7. If $f_1(x)=\frac{x}{2}+10\ \forall\ x\in R$ and $f_n(x)=f_1(f_{n-1}(x)),\forall\ n\geq 2, n\in N$, then evaluate $\lim\limits_{n\to\infty}f_n(x)$

8. Let $f(x)=\frac{x}{\sin x}, x>0$ and $g(x)=x+3,\ x<1$

$=2-x, x\leq 0$ $\quad$ $=x^2-2x-2, 1\leq x<2$

$=x-5.\quad x\geq 2$

find LHL and RHL of $g(f(x))$ at $x=0$ and hence find $\lim\limits_{x\to 0}g(f(x))$.

9. Let $P_n=a^{P_{n-1}}-1,\forall\ n=2,3$ and Let $P_1=a^x-1$ where $a\in R^+$ then evaluate $\lim\limits_{x\to 0}\frac{P_n}{x}$.

10. Determine $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left(\sin\left(\frac{\pi/2}{k}\right)-\cos\left(\frac{\pi/2}{k}\right)-\sin\left(\frac{\pi/2}{k+2}\right)+\cos\left(\frac{\pi/2}{k+2}\right)\right)$.

11. $\lim\limits_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+a_3^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}{n}\right)^{nx}$ where $a_1,a_2,a_3,\ldots\ldots a_n>0$

12. Let $L=\prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right)$; $M=\prod_{n=2}^{\infty}\left(\frac{n^3-1}{n^3+1}\right)$ and $N=\prod_{n=1}^{\infty}\frac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1}+M^{-1}+N^{-1}$

13. If $f(n,\theta)=\prod_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$, then compute $\lim\limits_{n\to\infty}f(n,\theta)$

14. Using Sandwich theorem, evaluate

(a) $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots\ldots\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$

(b) $\lim\limits_{n\to\infty}\frac{1}{1+n^2}+\frac{2}{2+n^2}+\cdots\ldots\ldots+\frac{n}{n+n^2}$

(c) $\lim\limits_{n\to\infty}(\sqrt{2}-2^{1/3})(\sqrt{2}-2^{1/5})\ldots\ldots(\sqrt{2}-2^{1/(2n+1)})$

(d) $\lim\limits_{n\to\infty}\sum_{k=0}^{n}\frac{1}{{}^nC_k}$

15. If $L=\lim\limits_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\ln\left(x+\sqrt{1+x^2}\right)}\right)$ then find the value of $\frac{L+153}{L}$.

EXERCISE- II

1. $\lim\limits_{x\to 1} \dfrac{(\ln(1+x)-\ln 2)(3\cdot 4^{x-1}-3x)}{\left[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right]\cdot\sin(x-1)}$

2. (i) $\lim\limits_{x\to 2} \dfrac{(\cos\alpha)^x+(\sin\alpha)^x-1}{x-2}$

(ii) $\lim\limits_{x\to 4} \dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$

3. If the $\lim\limits_{x\to 0} \frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$ exists and has the value equal to l, then find the value of $\frac{1}{a}-\frac{2}{l}+\frac{3}{b}$.

4. Find a & b if :

(i) $\lim\limits_{x\to\infty}\left[\frac{x^2+1}{x+1}-ax-b\right]=0$

(ii) $\lim\limits_{x\to\infty}\left[\sqrt{x^2-x+1}-ax-b\right]=0$

5. $\lim\limits_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

6. $\lim\limits_{x\to\infty}\left[\cos\left(2\pi\left(\frac{x}{1+x}\right)^a\right)\right]^{x^2}$ $a\in R$

7. Let $f(x)=\frac{\sin^{-1}(1-\{x\})\cdot\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\cdot(1-\{x\})}$ then find $\lim\limits_{x\to 0^+}f(x)$ and $\lim\limits_{x\to 0^-}f(x)$, where $\{x\}$ denotes the fractional part function.

8. Find the values of a, b & c so that $\lim\limits_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\cdot\sin x}=2$

9. If a sequence of numbers $\{x_n\}$, determine by the equality $x_n=\frac{x_{n-1}+x_{n-2}}{2}$ and the values x_0 & x_1,

Prove that $\lim\limits_{n\to\infty}x_n=\frac{x_0+2x_1}{3}$

10. If $\lim\limits_{x\to 1}(1+ax+bx^2)^{\frac{a}{x-1}}=e^{-3}$, find the values of ordered pair (a,b).

11. $\lim\limits_{n\to\infty}\frac{[1.x]+[2.x]+[3.x]+\cdots+[n.x]}{n^2}$, Where [.] denotes the greatest integer function.

12. Let $f(x)=\lim\limits_{n\to\infty}\sum_{n=1}^{n}3^{n-1}\sin^3\frac{x}{3^n}$ and $g(x)=x-4f(x)$. Evaluate $\lim\limits_{x\to 0}(1+g(x))^{\cot x}$.

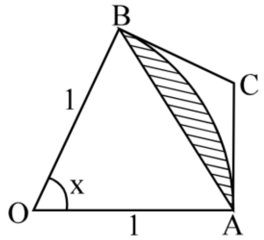
13. $L=\lim\limits_{x\to 0}\dfrac{\sqrt{\frac{\cos 2x+(1+3x)^{1/3}}{2}}-\sqrt[3]{\frac{4\cos^3 x-\ln(1+x)^4}{4}}}{x}$

If $L=a/b$ where ' a ' and ' b ' are relatively primes find $(a+b)$.

14. $\lim\limits_{x\to\infty}\left(\frac{\cosh(\pi/x)}{\cos(\pi/x)}\right)^{x^2}$ where $\cosh t=\frac{e^t+e^{-t}}{2}$

15. At the end-points and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arcAB decreases indefinitely.

16. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:



(a) T(x)

(b) S(x) &

(c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$.

17. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

EXERCISE- III

1. $\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x}$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

2. Find the sum of an infinite geometric series whose first term is the limit of the function $f(x) = \frac{\tan x-\sin x}{\sin^3 x}$ as $x \to 0$ and whose common ratio is the limit of the function $g(x) = \frac{1-\sqrt{x}}{(\cos^{-1} x)^2}$ as $x \to 1$

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) $\frac{3}{5}$ (D) $\frac{5}{3}$

3. $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{4}{\pi}$ (D) does not exist

4. $\lim_{x\to 0} \frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

(A) $4\sqrt{2}(\ln\ 3)^2$ (B) $8\sqrt{2}(\ln\ 3)^2$ (C) $4\sqrt{2}(\ln\ 3)$ (D) $8\sqrt{2}(\ln\ 3)$

5. $\lim_{x\to 0} \left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$

(A) e (B) $\frac{1}{e}$ (C) $e^{-1/2}$ (D) $\sqrt{e}$

6. $\lim_{x\to 0} \left[\frac{x^2}{\sin x \tan x}\right]$, where [.] denotes greatest integer function.

(A) 0 (B) -1 (C) 1 (D) does not exist

7. $\lim_{x\to 1} \left(\tan \frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

(A) e (B) $\frac{1}{e}$ (C) $e^{-1/2}$ (D) $\sqrt{e}$

8. Let $x_0 = 2\cos \frac{\pi}{6}$ and $x_n = \sqrt{2+x_{n-1}}$, $n = 1,2,3, \ldots \ldots \ldots$, find $\lim_{n\to\infty} 2^{(n+1)} \cdot \sqrt{2-x_n}$

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$

9. $\lim_{x\to 0} \frac{2(\tan x-\sin x)-x^3}{x^5}$

(A) $-\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{4}$ (D) $\frac{5}{6}$

10. Calculate $\lim_{x\to 0} \left(1+e^{-\frac{1}{x^2}}\arctan \frac{1}{x^2}+xe^{-\frac{1}{x^2}}\sin \frac{1}{x^4}\right)^{e^{\frac{1}{x^2}}}$

(A) e (B) e^2 (C) $e^{\pi/2}$ (D) $e^{-\pi/2}$

11. If $f(x) = \begin{cases} \sin x, & x \neq n\pi, n \in I \\ 2, & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x^2+1, & x \neq 0 \\ 4, & x = 0 \\ 6, & x = 2 \end{cases}$, then $\lim\limits_{x\to 0} g\{f(x)\}$ is :

(A) 1 (B) 4 (C) 5 (D) 6

12. $\lim\limits_{n\to\infty}\left(1+\frac{1}{5}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{5^4}\right)\ldots\ldots\left(1+\frac{1}{5^{2^n}}\right)$ is equal to :

(A) 0 (B) 5/4 (C) 4/5 (D) 1/5

13. The value of the limit $\lim\limits_{n\to\infty} n^2\left(a^{1/n}-a^{1/(n+1)}\right)(a>0)$ is :

(A) In a (B) e^a (C) e^{-a} (D) 1

14. $\lim\limits_{n\to\infty}\frac{1^6+2^6+3^6+\cdots\ldots+n^6}{(1^2+2^2+3^2+\cdots\ldots+n^2)(1^3+2^3+3^3+\cdots\ldots+n^3)} =$

(A) 2 (B) $\frac{21}{8}$ (C) 1 (D) $\frac{12}{7}$

15. If a, b, c are non zero constant numbers then $\lim\limits_{r\to\infty}\frac{\cos\frac{a}{r}-\cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}}$ equals :

(A) $\frac{a^2+b^2-c^2}{2bc}$ (B) $\frac{c^2+a^2-b^2}{2bc}$ (C) $\frac{b^2+c^2-a^2}{2bc}$ (D) Independent of a, b, c

EXERCISE- IV(Mains)

1. If $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then the values of a and b are **[AIEEE 2004]**

(A) $a\in R, b\in R$ (B) $a=1, b\in R$ (C) $a\in R, b=2$ (D) $a=1, b=2$

2. Let α and β be the distinct roots of $ax^2+bx+c=0$, then $\lim\limits_{x\to a}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to **[AIEEE 2005]**

(A) $\frac{a^2}{2}(\alpha-\beta)^2$ (B) 0 (C) $\frac{a}{2}(\alpha-\beta)^2$ (D) $\frac{1}{2}(\alpha-\beta)^2$

3. Let $f: R\to R$ be a positive, increasing function with $\lim\limits_{x\to\infty}\frac{f(3x)}{f(x)}=1$. **[AIEEE 2010]**

Then $\lim\limits_{x\to\infty}\frac{f(2x)}{f(x)}$ is equal to

(A) 3 (B) 1 (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

4. $\lim\limits_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[AIEEE 2011]**

(A) Equals $\frac{1}{\sqrt{2}}$ (B) Does not exist (C) Equals $\sqrt{2}$ (D) Equals $-\sqrt{2}$

5. $\lim\limits_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to **[IIT Mains - 2014]**

(A) π (B) $\frac{\pi}{2}$ (C) 1 (D) $-\pi$

6. Let f(x) be a polynomial of degree four having extreme values at $x=1$ and $x=2$. If

$\lim\limits_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then f(2) is equal to: **[IIT Mains - 2015]**

(A) 4 (B) -8 (C) -4 (D) 0

7. $\lim\limits_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[IIT Mains - 2015]**

(A) $\frac{1}{2}$ (B) 4 (C) 3 (D) 2

8. Let $p=\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to : **[IIT Mains - 2016]**

(A) 2 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

9. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals : **[IIT Mains - 2017]**

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{24}$ (D) $\frac{1}{16}$

10. For each $t\in R$, let [t] be the greatest integer less than or equal to t. **[IIT Mains - 2018]**

Then $\lim\limits_{x\to 0}x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\cdots\ldots\ldots+\left[\frac{15}{x}\right]\right)$

(A) does not exist (in R) (B) is equal to 0

(C) is equal to 15 (D) is equal to 120

EXERCI SE- V (Advanced)

1. $\lim_{x\to0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ is : **[JEE '99, 2 (out of 200)]**

(A) 2 (B) -2 (C) ½ (D) $-1/2$

2. For $x \in R, \lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x =$ **[JEE 2000, Screening]**

(A) e (B) e^{-1} (C) e^{-5} (D) e^5

3. Evaluate $\lim_{x\to0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a > 0.$ **[REE 2001, 3 out of 100]**

4. The integer n for which $\lim_{x\to0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is

(A) 1 (B) 2 (C) 3 (D) 4

[JEE 2002 (screening), 3]

5. If $\lim_{x\to0}\frac{\sin (nx)[(a-n)nx-\tan x]}{x^2} = 0 (n > 0)$ then the value of 'a' is equal to

[JEE 2003 (screening)]

(A) $\frac{1}{n}$ (B) $n^2 + 1$ (C) $\frac{n^2+1}{n}$ (D) None

6. Find the value of $\lim_{n\to\infty}\left[\frac{2}{\pi}(n + 1)\cos^{-1}\left(\frac{1}{n}\right) - n\right]$. **[JEE ' 2004, 2 out of 60]**

(A) $a = 2$ (B) $a = 1$ (C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

7. Let $L = \lim_{x\to0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}, a > 0$. If L is finite, then **[JEE ' 2009, 3]**

(A) $a = 2$ (B) $a = 1$ (C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

8. If $\lim_{x\to0}[1 + x\ln (1 + b^2)]^{1/x} = 2b\sin^2\theta, b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is

[JEE ' 2011]

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

9. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation **[JEE · 2012]**

$(\sqrt[3]{1 + a} - 1)x^2 + (\sqrt{1 + a} - 1)x + (\sqrt[6]{1 + a} - 1) = 0$ where $a > -1$.

Then $\lim_{a\to0^+}\alpha(a)$ and $\lim_{a\to0^+}\beta(a)$ are

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and -1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

10. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then **[JEE · 2012]**

(A) $a = 1, b = 4$ (B) $a = 1, b = -4$

(C) $a = 2, b = -3$ (D) $a = 2, b = 3$

11. The largest value of the non-negative integer a for which $\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is

[IIT Advance-2014]

12. Let m and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$ then the value of $\frac{m}{n}$ is **[IIT Advance - 2015]**

13. Let $\alpha,\beta\in\mathbf{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$. Then $6(\alpha+\beta)$ equals : **[IIT Advance - 2016]**

14. Let $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x\neq 1$. Then **[IIT Advance - 2017]**

(A) $\lim_{x\to 1^+} f(x)$ does not exist (B) $\lim_{x\to 1^-} f(x)$ does not exist

(C) $\lim_{x\to 1^-} f(x)=0$ (D) $\lim_{x\to 1^+} f(x)=0$

15. For any positive integer n, define $f_n:(0,\infty)\to R$ as **[IIT Advance - 2018]**

$$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)\text{ for all } x\in(0,\infty).$$

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE?

(A) $\Sigma_{j=1}^{5}\tan^2\left(f_j(0)\right)=55$

(B) $\Sigma_{j=1}^{10}\left(1+f_j'(0)\right)\sec^2\left(f_j(0)\right)=10$

(C) For any fixed positive integer n, $\lim_{x\to\infty}\tan\left(f_n(x)\right)=\frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty}\sec^2\left(f_n(x)\right)=1$

16. For any positive integer n, let $S_n:(0,\infty)\to R$ be defined by

$$S_n(x)=\sum_{k=1}^{n}\cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$$

where for any $x\in R$, $\cot^{-1}(x)\in(0,\pi)$ and $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then which of the following statements is (are) TRUE? **[IIT Advance - 2021]**

(A) $S_{10}(x)=\frac{\pi}{2}-\tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x>0$

(B) $\lim_{x\to\infty}\cot\left(S_n(x)\right)=x$, for all $x>0$

(C) The equation $S_3(x)=\frac{\pi}{4}$ has a root in $(0,\infty)$

(D) $\tan\left(S_n(x)\right)\leq\frac{1}{2}$, for all $n\geq 1$ and $x>0$

Answer Key

PROFICIENCY TEST-01

1. 3 **2.** 2 **3.** $\frac{45}{91}$ **4.** 1

5. (a) $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$ (b) $f(x) = |x|$

6. 5050 **7.** $\frac{2}{\sqrt{3}}$ **8.** $\frac{3}{2}$ **9.** $\frac{1}{32}$ **10.** $\frac{1}{16\sqrt{2}}$

PROFICIENCY TEST-02

1. $\pi - 3$ **2.** -2 **3.** 0 **4.** 1 **5.** $c = \ln 2$ **6.** e^{-8}

7. e^{-2} **8.** $-\frac{1}{\pi^2}$ **9.** $a - b$ **10.** $5/2$

EXERCISE-I

1. $\frac{p-q}{2}$ **3.** $\ln 2$ **4.** does not exist **5.** $\frac{2\ln 2}{\pi}$

6. $a = 2;$ limit $= 1$ **7.** 20 **8.** $-3, -3, -3$

9. $(\ln a)^n$ **10.** 3 **11.** $(a_1 \cdot a_2 \cdot a_3 \dots a_n)$

12. 8 **13.** $\frac{\theta}{\tan\theta}$ **14.** (a) 2 (b) 1/2 (c) 0 (d) 2

15. 307

EXERCISE-II

1. $-\frac{9}{4}\ln\frac{4}{e}$

2. (i) $\cos^2\alpha\ \ln\cos\alpha + \sin^2\alpha\ \ln\sin\alpha,$

(ii) $\cos^4\alpha \ln\cos\alpha - \sin^4\alpha \ln\sin\alpha$

3. 72 **4.** (i) $a = 1, b = -1$ (ii) $a = -1, b = 1/2$

5. e^{-1} **6.** $e^{-2\pi^2 a^2}$ **7.** $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$

8. $a = c = 1, b = 2$ **10.** $(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$

11. $\frac{x}{2}$ **12.** $g(x) = \sin x$ and limit $= e$

13. 19 **14.** e^{π^2}

16. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin\ x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}, S(x) = \frac{1}{2}x - \frac{1}{2}\sin\ x$, limit $= \frac{3}{2}$

EXERCISE-III

1. D **2.** A **3.** D **4.** B **5.** C **6.** A **7.** B

8. D **9.** C **10.** C **11.** A **12.** B **13.** A **14.** D

15. C

EXERCISE-IV

1. B **2.** A **3.** B **4.** B **5.** A **6.** D **7.** D

8. C **9.** D **10.** D

EXERCISE-V

1.	C	**2.**	C	**3.**	lna	**4.**	C	**5.**	C	**6.**	$1-\frac{2}{\pi}$	**7.**	A,C
8.	D	**9.**	B	**10.**	B	**11.**	0	**12.**	2	**13.**	7	**14.**	A,C
15.	D	**16.**	A,B										