Nishant Jindal

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n$$

$$\int_0^x (1+x)^n\,dx = \int_0^x \left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n\right) dx$$

$$\frac{(1+x)^{n+1} - 1}{n+1} = {}^nC_0 x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + {}^nC_3 \frac{x^4}{4} + \cdots + \frac{{}^nC_n}{n+1} x^{n+1}$$

$x=1$, $$\frac{2^{n+1} - 1}{n+1} = \frac{{}^nC_0}{1} + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} + \cdots + \frac{{}^nC_n}{n+1}$$

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1. $2\,\frac{{}^nC_0}{1} + 2^2\,\frac{{}^nC_1}{2} + 2^3\,\frac{{}^nC_2}{3} + 2^4\,\frac{{}^nC_3}{4} + \cdots + 2^{n+1}\,\frac{{}^nC_n}{n+1} = ?$

$= \sum_{r=0}^{n} 2^{r+1}\frac{{}^nC_r}{r+1} = \frac{3^{n+1}-1}{n+1}$ $\quad (1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

2. $\frac{2^2\,{}^nC_0}{1\cdot 2} + \frac{2^3\,{}^nC_1}{2\cdot 3} + \frac{2^4\,{}^nC_2}{3\cdot 4} + \cdots + \frac{2^{n+2}\,{}^nC_n}{(n+1)(n+2)} = \sum_{r=0}^{n} \frac{{}^nC_r\,2^{r+2}}{(r+1)(r+2)}$

$= \frac{3^{n+2}-1}{(n+1)(n+2)} - \frac{2}{n+1}$

$\int_0^x (1+x)^n\,dx = \int_0^x \left(\sum_{r=0}^{n} {}^nC_r\, x^r\right) dx \Rightarrow \frac{(1+x)^{n+1}-1}{n+1} = \sum_{r=0}^{n} \frac{{}^nC_r\, x^{r+1}}{r+1}$

$$\int_0^x \left(\frac{(1+x)^{n+1}-1}{n+1}\right) dx = \int_0^x \left(\sum_{r=0}^{n} \frac{{}^nC_r\, x^{r+1}}{r+1}\right) dx$$

$$\left.\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1}\right|_0^x = \frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \sum_{r=0}^{n} \frac{{}^nC_r\, x^{r+2}}{(r+1)(r+2)}$$

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1.

$$\sum_{r=0}^{n} \frac{{}^{n}C_r 2^{r+1}}{(r+1)} = \frac{1}{(n+1)} \sum_{r=0}^{n} {}^{n+1}C_{r+1} 2^{r+1}$$

$$= \frac{(1+2)^{n+1} - {}^{n+1}C_0 2^0}{(n+1)}$$

2.

$$\sum_{r=0}^{n} \frac{{}^{n}C_r 2^{r+2}}{(r+1)(r+2)} = \frac{1}{(n+1)} \sum_{r=0}^{n} \frac{{}^{n+1}C_{r+1} 2^{r+2}}{(r+2)} = \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} {}^{n+2}C_{r+2} 2^{r+2}$$

$$\frac{3^{n+2} - (1+2(n+2))}{(n+1)(n+2)} = \frac{(1+2)^{n+2} - \left({}^{n+2}C_0 2^0 + {}^{n+2}C_1 2^1\right)}{(n+1)(n+2)}$$

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① ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + {}^nC_3^2 + \cdots + {}^nC_n^2 = ?$

② ${}^nC_0\,{}^nC_4 + {}^nC_1\,{}^nC_5 + {}^nC_2\,{}^nC_6 + {}^nC_3\,{}^nC_7 + \cdots + {}^nC_{n-4}\,{}^nC_n = {}^{2n}C_{n-4}$

$=$ Coeff. of x^{n-4} in $(1+x)^{2n}$ $= {}^{2n}C_{n-4}$

${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + {}^nC_3\,{}^nC_{n-3} + \cdots + {}^nC_n\,{}^nC_0 = {}^{2n}C_n$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + \cdots + {}^nC_n x^n$ —— ①

$(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + {}^nC_3 x^{n-3} + {}^nC_4 x^{n-4} + {}^nC_5 x^{n-5} + \cdots + {}^nC_n$

① × ② $(1+x)^{2n} = {}^nC_0\,{}^nC_n + ({}^nC_1\,{}^nC_n + {}^nC_0\,{}^nC_{n-1})x + \cdots + \underbrace{({}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \cdots {}^nC_n^2)}x^n + \cdots + {}^nC_0\,{}^nC_n x^{2n}$

Equate Coeff. of x^n ${}^{2n}C_n = {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \cdots + {}^nC_n^2$.

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$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

$$f\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots \frac{a_n}{x^n}$$

$$= \frac{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n}{x^n}$$

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3. ${}^nC_0^2 - {}^nC_1^2 + {}^nC_2^2 - {}^nC_3^2 + \dots + (-1)^n\, {}^nC_n^2$

4. ${}^nC_0\,{}^nC_1 - {}^nC_1\,{}^nC_2 + {}^nC_2\,{}^nC_3 - {}^nC_3\,{}^nC_4 + \dots + (-1)^{n-1}\,{}^nC_{n-1}\,{}^nC_n$

$= \text{coeff of } x^{n-1} \text{ in } (1-x^2)^n = \begin{cases} 0 & n \text{ is even} \\ (-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}} & n \text{ is odd} \end{cases}$

$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + (-1)^n\,{}^nC_n x^n$

${}^nC_r (-1)^r x^{2r}$

$(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + {}^nC_3 x^{n-3} + \dots + {}^nC_n$

${}^nC_r (-1)^r x^{2r}$

${}^nC_0^2 - {}^nC_1^2 + {}^nC_2^2 - {}^nC_3^2 + \dots + (-1)^n\,{}^nC_n^2 = \text{coeff. of } x^n \text{ in } (1-x^2)^n$

$= \begin{cases} 0 & , n \text{ is odd} \\ {}^nC_{\frac{n}{2}} (-1)^{\frac{n}{2}} & , n \text{ is even} \end{cases}$

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$$\text{(1)}\ \sum_{0\le i<j\le n}\sum {}^nC_i\,{}^nC_j = {}^nC_0\,{}^nC_1 + {}^nC_0\,{}^nC_2 + {}^nC_1\,{}^nC_3 + \cdots + {}^nC_{n-1}\,{}^nC_n$$

$$= \frac{1}{2}\left[\left({}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots {}^nC_n\right)^2 - \left({}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \cdots + {}^nC_n^2\right)\right]$$

$$= \frac{1}{2}\left(2^{2n} - {}^{2n}C_n\right)$$

$$\sum_{1\le i<j\le n}\sum 1 = 1+1+1+\cdots+1 = \boxed{{}^nC_2}$$

$(i,j) = (1,2),(1,3),\ldots(1,n)$

$(2,3),(2,4),\ldots(2,n)$

$\vdots$

$(n-1,n)$

Black book → SOT, QE

Binomial Theorem

P&C → remaining sheet