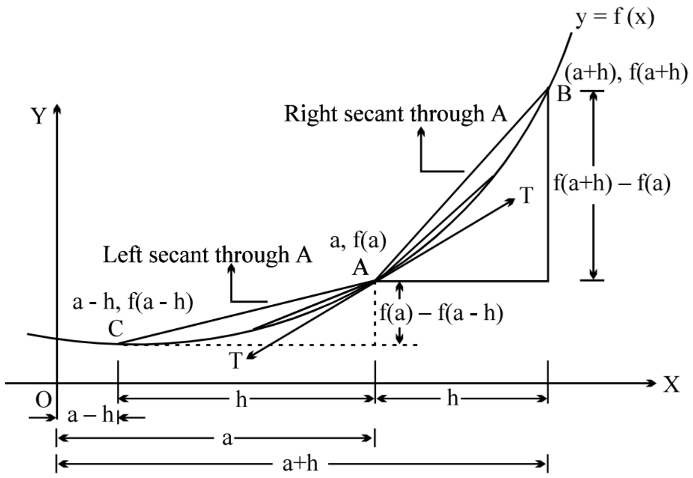
KEY CONCEPTS

THINGS TO REMEMBER:

1. Right hand & Left hand Derivatives;

By definition : $f'(a) = \underset{x \to a}{\text{Limit}} \frac{f(x)-f(a)}{x-a}$ if it exist



(i) The right hand derivative of f' at $x = a$ denoted by $f'(a^+)$is defined by:

$$f'(a^+) = \text{Limit}_{h \to 0^+} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists & is finite.

(ii) The left hand derivative : of f at $x = a$ denoted by $f'(a^-)$is defined by:

$$f'(a^-) = \text{Limit}_{h \to 0^+} \frac{f(a-h)-f(a)}{-h},$$

Provided the limit exists & is finite.

We also write $f'(a^+) = f'_+(a) \& f'(a^-) = f'_-(a)$.

This geometrically means that a unique tangent with finite slope can be drawn at $x = a$ as shown in the figure.

(iii) **Derivability & Continuity:**

(a) If $f'(a)$ exists then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$.

(b) If a function f is derivable at x then f is continuous at x.

For: $f'(x) = \text{Limit}_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h) - f(x) = \frac{f(x+h)-f(x)}{h} \cdot h [h \neq 0]$

Therefore:

$$\text{Limit}_{h \to 0} [f(x+h) - f(x)] = \text{Limit}_{h \to 0} \frac{f(x+h)-f(x)}{h} \cdot h = f'(x) \cdot 0 = 0$$

Therefore $\underset{h \to 0}{\text{Limit}} [f(x+h) - f(x)] = 0 \Rightarrow \text{Limit}_{h \to 0} \; f(x+h) = f(x) \Rightarrow f$ is continuous at x.

Note: If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain. The Converse of the above result is not true:

"IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x| \& g(x) = x \sin\frac{1}{x}; x \neq 0 \& g(0) = 0$ are continuous at $x = 0$ but not derivable at $x = 0$.

Note Carefully:

(a) Let $f'_{+}(a) = p$ & $f'_{-}(a) = q$ where p & q are finite then:

(i) $p = q \Rightarrow$ f is derivable at $x = a \Rightarrow$ f is continuous at $x = a$.

(ii) $p \neq q \Rightarrow$ f is not derivable at $x = a \Rightarrow$ f is continuous at $x = a$.

In short, for a function f :

Differentiability $\Rightarrow$ Continuity ; Continuity $\not\Rightarrow$ derivability;

Non derivibality $\not\Rightarrow$ discontinuous; But discontinuity $\Rightarrow$ Non derivability

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

3. DERIVABILITY OVER AN INTERVAL:

f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval

f(x) is said to be derivable over the closed interval [a, b] if:

(i) for the points a and b , $f'(a^+)$ & $f'(b^-)$ exist &

(ii) for any point c such that $a < c < b$, $f'(c+)$&$f'(c-)$ exist & are equal.

Note:

1. If f(x) & g(x) are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$ will also be derivable at $x = a$ & if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

2. If f(x) is differentiable at $x = a$ & g(x) is not differentiable at $x = a$, then the product function $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = x$ & $g(x) = |x|$.

3. If f(x) & g(x) both are not differentiable at $x = a$ then the product function $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$ e.g. $f(x) = |x|$ & $g(x) = |x|$.

4. If f(x) & g(x) both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. e.g. $f(x) = |x|$ & $g(x) = -|x|$.

5. If f(x) is derivable at $x = a \Rightarrow f'(x)$ is continuous at $x = a$.

$$\text{e.g. } f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

PROFICIENCY TEST

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin |x|, x \in R$. Draw a rough sketch of the graph of f(x).

2. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2|, x \in R$.
Also draw the graph of f(x).

3. A function f is defined as follows: $f(x) = \begin{bmatrix} 1 & \text{for} & -\infty < x < 0 \\ 1 + |\sin x| & \text{for} & 0 \le x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{for} & \frac{\pi}{2} \le x < \infty \end{bmatrix}$

Discuss the continuity & differentiability at $x = 0$ & $x = \pi/2$.

4. Examine the origin for continuity & derivability in the case of the function f defined by $f(x) = x \tan^{-1}(1/x)$, $x \neq 0$ and $f(0) = 0$.

5. Discuss the continuity & the derivability of ' f ' where $f(x) =$ degree of $\left(u^{x^2} + u^2 + 2u - 3\right)$ at $x = \sqrt{2}$

Fill in the blanks:

6. If f(x) is derivable at $x = 3$ & $f'(3) = 2$, then $\text{Limit}_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2} =$ ________.

7. If $f(x) = |\sin x|$ & $g(x) = x^3$ then $f[g(x)]$ is ________ & ________ at $x = 0$. (State continuity and derivability)

8. Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then its value is ________.

9. For the function $f(x) = \begin{bmatrix} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{bmatrix}$, the derivative from the right, $f'(0^+) =$ ________ & the derivative from the left, $f'(0^-) =$ ________.

10. The number of points at which the function $f(x) = \max.\{a - x, a + x, b\}, -\infty < x < \infty, 0 < a < b$ cannot be differentiable is ________.

EXERCISE-I

1. Given a differentiable function f(x) defined for all real x, and is such that $f(x+h)-f(x) \leq 6h^2$ for all real h and x. Show that f(x) is constant.

2. Let $f(0)=0$ and $f'(0)=1$. For a positive integer k, show that

$$\operatorname{Lim}_{x\to 0} \frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+\cdots \ldots f\left(\frac{x}{k}\right)\right)=1+\frac{1}{2}+\frac{1}{3}+\cdots \ldots+\frac{1}{k}$$

3. Let $f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, x \neq 0 ; f(0)=0$. Test the continuity & differentiability at $x=0$.

4. If $f(x)=|x-1|.([x]-[-x])$, then find $f'(1^+)$ & $f'(1^-)$ where [x] denotes greatest integer function.

5. If $f(x)=\begin{cases} ax^2-b & \text{if } |x|<1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at $x=1$. Find the values of a&b.

6. Consider the functions $f(x)=x^2-2x$ and $g(x)=-|x|$

Statement-1 : The composite function $F(x)=f(g(x))$ is not derivable at $x=0$.

because

Statement-2 : $F'(0^+)=2$ and $F'(0^-)=-2$.

(A) Statement- 1 is true, statement- 2 is true and statement- 2 is correct explanation for statement- 1.

(B) Statement- 1 is true, statement- 2 is true and statement- 2 is NOT the correct explanation for statement- 1.

(C) Statement- 1 is true, statement- 2 is false.

(D) Statement- 1 is false, statement- 2 is true.

Select the correct alternative: (More than one are correct)

7. $f(x)=|x[x]|$ in $-1 \leq x \leq 2$, where [x] is greatest integer $\leq x$ then f(x) is:

(A) continuous at $x=0$ (B) discontinuous at $x=0$

(C) not differentiable at $x=2$ (D) differentiable at $x=2$

8. $f(x)=1+x \cdot [\cos x]$ in $0<x \leq \pi/2$, where [ ] denotes greatest integer function then,

(A) It is continuous in $0<x<\pi/2$ (B) It is differentiable in $0<x<\pi/2$

(C) Its maximum value is 2 (D) It is not differentiable in $0<x<\pi/2$

9. $f(x)=(\sin^{-1}x)^2 \cdot \cos(1/x)$ if $x \neq 0 ; f(0)=0$, f(x) is:

(A) continuous no where in $-1 \leq x \leq 1$

(B) continuous every where in $-1 \leq x \leq 1$

(C) differentiable no where in $-1 \leq x \leq 1$

(D) differentiable everywhere in $-1<x<1$

10. $f(x) = |x| + |\sin x|$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is:

(A) Continuous no where
(B) Continuous every where
(C) Differentiable no where
(D) Differentiable everywhere except at $x = 0$

11. If $f(x) = 2 + |\sin^{-1}x|$, it is:

(A) continuous no where
(B) continuous everywhere in its domain
(C) differentiable no where in its domain
(D) not differentiable at $x = 0$

12. If $f(x) = x^2 . \sin(1/x), x \neq 0$ and $f(0) = 0$ then,

(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is derivable at $x = 0$
(C) $f'(x)$ is continuous at $x = 0$
(D) $f''(x)$ is not derivable at $x = 0$

13. A function which is continuous & not differentiable at $x = 0$ is:

(A) $f(x) = x$ for $x < 0$ & $f(x) = x^2$ for $x \geq 0$
(B) $g(x) = x$ for $x < 0$ & $g(x) = 2x$ for $x \geq 0$
(C) $h(x) = x|x|, x \in R$
(D) $K(x) = 1 + |x|, x \in R$

14. If $\sin^{-1} x + |y| = 2y$ then y as a function of x is:

(A) defined for $-1 \leq x \leq 1$
(B) continuous at $x = 0$
(C) differentiable for all **x**
(D) such that $\frac{dy}{dx} = \frac{1}{3\sqrt{1-x^2}}$ for $-1 < x < 0$

15. Let $f(x) = \cos(x)$ & $H(x) = \begin{bmatrix} \text{Min}\,[f(t) \,/\, 0 \leq t \leq x] & \text{for } 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} < x \leq 3 \end{bmatrix}$, then

(A) $H(x)$ is continuous & derivable in $[0,3]$
(B) $H(x)$ is continuous but not derivable at $x = \pi/2$
(C) $H(x)$ is neither continuous nor derivable at $x = \pi/2$
(D) Maximum value of $H(x)$ in $[0, 3]$ is 1

EXERCISE-II

1. Let f(x) be defined in the interval $[-2,2]$ such that $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$ &

$g(x) = f(|x|) + |f(x)|$. Test the differentiability of g(x) in $(-2,2)$.

2. Given $f(x) = \cos^{-1}\left(\text{sgn}\left(\frac{2[x]}{3x-[x]}\right)\right)$ where sgn (.) denotes the signum function & [.] denotes the greatest integer function. Discuss the continuity & differentiability of f(x) at $x = \pm 1$.

3. Examine for continuity & differentiability at the points $x = 1$ & $x = 2$, the function f defined by

$f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)[x], & 2 \le x \le 3 \end{cases}$ where $[x]$ = greatest integer less than or equal to x.

4. $f(x) = x \cdot \left(\frac{e^{[x]+|x|} - 2}{[x]+|x|}\right), x \neq 0$ & $f(0) = -1$ where [x] denotes greatest integer less than or equal to x. Test the differentiability of f(x) at $x = 0$.

5. Discuss the continuity & the derivability in $[0,2]$ of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\frac{\pi x}{2} & \text{for } x < 1 \end{cases}$

where [ ] denote greatest integer function.

6. If $f(x) = -1 + |x-1|, -1 \le x \le 3$; $g(x) = 2 - |x+1|, -2 \le x \le 2$, then calculate (fog) (x) & (gof) (x). Draw their graph. Discuss the continuity of (fog) (x) at $x = -1$ & the differentiability of (gof) (x) at $x = 1$.

7. The function $f(x) = \begin{cases} ax(x-1)+b & \text{when } x < 1 \\ x-1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x

(ii) f ' (1) does not exist

(iii) $f'(x)$ is continuous at $x = 3$

8. Examine the function, $f(x) = x \cdot \frac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}, x \neq 0 (a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin.

9. $f(x) = \begin{cases} 1-x & , (0 \le x \le 1) \\ x+2 & , (1 < x < 2) \\ 4-x & , (2 \le x \le 4) \end{cases}$ Discuss the continuity & differentiability of

$y = f[f(x)]$ for $0 \le x \le 4$.

10. $f: R \to R$ be a function satisfying $f(x+2y) = f(x) + f(2y) + 4xy + e^x \cdot e^{2y} - e^x - e^{2y} \ \forall x, y \in R$. If $f'(0) = 1$, find f(x) in terms of x.

11. Let $f(x)$ be a function defined on $(-a, a)$ with $a > 0$. Assume that $f(x)$ is continuous at $x = 0$ and $\text{Lim}_{x\to 0} \frac{f(x)-f(kx)}{x} = \alpha$, where $k \in (0,1)$ then compute $f'(0^+)$and $f'(0^-)$, and comment upon the differentiability of f at $x = 0$.

12. Consider the function, $f(x) = \begin{cases} x^2 \left|\cos \frac{\pi}{2x}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Show that $f'(0)$ exists and find its value (b) Show that $f'(1/3)$ does not exist.

(c) For what values of x , $f'(x)$ fails to exist.

13. Let $f(x)$ be a real valued function not identically zero satisfies the equation,

$f(x + y^n) = f(x) + (f(y))^n$ for all real x&y and $f'(0) \geq 0$ where $n(> 1)$ is an odd natural number. Find $f(10)$.

14. A derivable function $f : R^+ \to R$ satisfies the condition $f(x) - f(y) \geq \ln\frac{x}{y} + x - y$ for every $x, y \in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$.

15. Suppose that f & g are non constant differentiable, real valued functions on R. If for every $x, y \in R$, $f(x + y) = f(x)f(y) - g(x)g(y)$; $g(x + y) = g(x)f(y) + f(x)g(y)$ & $f'(0) = 0$ then prove that $f^2(x) + g^2(x) = 1, \forall\, x \in R$.

EXERCISE-III

1. Let $f(x)=\frac{|x|}{\sin x}$ for $x\neq 0$ & $f(0)=1$ then,

(A) $f(x)$ is continuous & derivable at $x=0$

(B) $f(x)$ is continuous & not derivable at $x=0$

(C) $f(x)$ is discontinuous at $x=0$

(D) none

2. Given $=\begin{cases}\log_a\left(a|[x]+[-x]|\right)^x\left(\dfrac{a^{\frac{2}{\left(\frac{[x]+[-x]}{|x|}\right)}-5}}{3+a^{\frac{1}{|x|}}}\right) & \text{for } |x|\neq 0;\ a>1\\ 0 & \text{for } x=0\end{cases}$

where [ ] represents the integral part function, then:

(A) f is continuous but not differentiable at $x=0$

(B) f is cont. & diff. at $x=0$

(C) the differentiability of ' f ' at $x=0$ depends on the value of a

(D) f is cont. & diff. at $x=0$ and for $a=e$ only.

3. For what triplets of real numbers (a,b,c) with $a\neq 0$ the function

$f(x)=\begin{cases}x & x\leq 1\\ ax^2+bx+c & \text{otherwise}\end{cases}$ is differentiable for all real x?

(A) $\{(a,1-2a,a)\mid a\in R, a\neq 0\}$

(B) $\{(a,1-2a,c)\mid a,c\in R, a\neq 0\}$

(C) $\{(a,b,c)\mid a,b,c\in R, a+b+c=1\}$

(D) $\{(a,1-2a,0)\mid a\in R, a\neq 0\}$

4. A function f defined as $f(x)=x[x]$ for $-1\leq x\leq 3$ where $[x]$ defines the greatest integer $\leq x$ is:

(A) continuous at all points in the domain of f but non-derivable at a finite number of points

(B) discontinuous at all points & hence non-derivable at all points in the domain of $f(x)$

(C) discontinuous at a finite number of points but not derivable at all points in the domain of $f(x)$

(D) discontinuous & also non-derivable at a finite number of points of $f(x)$.

5. $[x]$ denotes the greatest integer less than or equal to x. If $f(x)=[x][\sin\ \pi x]$ in $(-1,1)$ then $f(x)$ is:

(A) continuous at $x=0$

(B) continuous in $(-1,0)$

(C) differentiable in $(-1,1)$

(D) none

6. The function $f(x)$ is defined as follows $f(x)=\begin{cases}-x & \text{if} & x<0\\ x^2 & \text{if} & 0\leq x\leq 1\\ x^3-x+1 & \text{if} & x>1\end{cases}$ then $f(x)$ is:

(A) derivable and continuous at $x=0$

(B) derivable at $x=1$ but not continuous at $x=1$

(C) neither derivable nor continuous at $x=1$

(D) not derivable at $x=0$ but continuous at $x=1$

7. If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ where $\{x\}$ denotes the fractional part function, then:

(A) ' f ' is continuous & diff. at $x = 0$ (B) ' f ' is continuous but not diff. at $x = 0$

(C) ' f ' is continuous & diff. at $x = 2$ (D) none of these

8. The set of all points where the function $f(x) = \frac{x}{1+|x|}$ is differentiable is:

(A) $(-\infty, \infty)$ (B) $[0, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

9. Let f be an injective and differentiable function such that $f(x) \cdot f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x and y with $f'(0) = 0, f'(1) = 2 \neq f(0)$, then

(A) $x\,f'(x) - 2\,f(x) + 2 = 0$ (B) $x\,f'(x) + 2\,f(x) - 2 = 0$

(C) $x\,f'(x) - f(x) + 1 = 0$ (D) $2\,f(x) = f'(x) + 2$

10. Let $f(x) = [n + p\sin x], x \in (0, \pi), n \in I$ and p is a prime number. The number of points where f(x) is not differentiable is

(A) $p - 1$ (B) $p + 1$ (C) $2p + 1$ (D) $2p - 1$.

Here [x] denotes greatest integer function.

11. The function $f(x) = \sin^{-1}(\cos x)$ is:

(A) discontinuous at $x = 2\pi$ (B) differentiable at $x = \frac{3\pi}{2}$

(C) not differentiable at $x = \frac{\pi}{2}$ (D) differentiable at $x = 2\pi$

12. $f_1(x) = (x^2 - 4)|(x-2)(x-3)|; f_2(x) = \sin(|x-2|) - |x-2|, f_3(x) = \tan(|x-2|) + |x-2|$

How many of the above functions $\{f_1(x), f_2(x), f_3(x)\}$ are differentiable at $x = 2$?

(A) 0 (B) 1 (C) 2 (D) 3

13. Consider the function $f(x) = \max\{||x-2| - 1|, \alpha\}$, where, α is some real number. Find the number of values of $x \in R$ at which f(x) is non-differentiable if $\alpha = \frac{1}{2}$

(A) 2 (B) 3 (C) 4 (D) 5

14. The function $f(x) = \frac{|x| - x(3^{1/x}+1)}{3^{1/x}-1}, x \neq 0, f(0) = 0$ is:

(A) discontinuous at $x = 0$

(B) continuous at $x = 0$ but not differentiable there

(C) both continuous and differentiable at $x = 0$

(D) differentiable but not continuous at $x = 0$

15. If $f(x) = |1 - x|$, then the points where $\sin^{-1}(f|x|)$ is non-differentiable are:

(A) $\{0,1\}$ (B) $\{0,-1\}$ (C) $\{0,1,-1\}$ (D) None of these

EXERCISE-IV

1. If $f(x) = \begin{cases} x\, e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ then f(x) is **[AIEEE 2003]**

(A) discontinuous everywhere

(B) continuous as well as differentiable for all x

(C) continuous for all x but not differentiable at $x = 0$

(D) neither differentiable nor continuous at $x = 0$

2. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \leq (x-y)^2, x, y \in R$ and $f(0) = 0$, then f(1) equals **[AIEEE-2005]**

(A) -1 (B) 0 (C) 2 (D) 1

3. Suppose f(x) is differentiable at $x = 1$ and $\lim_{h\to 0} \frac{1}{h} f(1+h) = 5$, then $f'(1)$ equals-

[AIEEE-2005]

(A) 3 (B) 4 (C) 5 (D) 6

4. The set of points where $f(x) = \frac{x}{1+|x|}$ is differentiable is– **[AIEEE 2006]**

(A) $(-\infty, -1) \cup (-1, \infty)$ (B) $(-\infty, \infty)$

(C) $(0, \infty)$ (D) $(-\infty, 0) \cup (0, \infty)$

5. Let $f: R \to R$ be a function defined by $f(x) = \min\{x+1, |x|+1\}$. Then which of the following is true? **[AIEEE 2007]**

(A) $f(x) \geq 1$ for all $x \in R$ (B) f(x) is not differentiable at $x = 1$

(C) f(x) is differentiable everywhere (D) f(x) is not differentiable at $x = 0$

6. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Then which one of the following is true? **[AIEEE 2008]**

(A) f is differentiable at $x = 0$ and at $x = 1$

(B) f is differentiable at $x = 0$ but not at $x = 1$

(C) f is differentiable at $x = 1$ but not at $x = 0$

(D) f is neither differentiable at $x = 0$ nor at $x = 1$

7. If the function, $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx+2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is:

[IIT Mains - 2015]

(A) 4 (B) 2 (C) $\frac{16}{5}$ (D) $\frac{10}{3}$

8. Let $S = \{t \in R : f(x) = |x-\pi| \cdot (e^{|x|}-1)\sin|x| \text{ is not differentiable at } t\}$. Then the set S is equal to **[IIT Mains - 2018]**

(A) $\{0, \pi\}$ (B) ϕ (an empty set) (C) $\{0\}$ (D) $\{\pi\}$

EXERCISE-V

1. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at:

[JEE'99, 2(out of 200)]

(A) -1 (B) 0 (C) 1 (D) 2

2. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is

[JEE 2000, Screening, 1 out of 35]

(A) onto if f is onto (B) one one if f is one one

(C) continuous if f is continuous (D) differentiable if f is differentiable.

3. Discuss the continuity and differentiability of the function,

$$f(x) = \begin{cases} \frac{x}{1+|x|} & , |x| \geq 1 \\ \frac{x}{1-|x|} & , |x| < 1 \end{cases}.$$

[REE,2000,3]

4. **[JEE 2001 (Screening)]**

(a) Let $f : R \to R$ be a function defined by , $f(x) = \max[x, x^3]$. The set of all points where f(x) is NOT differentiable is:

(A) $\{-1,1\}$ (B) $\{-1,0\}$ (C) $\{0,1\}$ (D) $\{-1,0,1\}$

(b) The left hand derivative of , $f(x) = [x]\sin(\pi x)$ at $x = k$, k an integer is :

where [ ] denotes the greatest function.

(A) $(-1)^k(k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$

(C) $(-1)^k k\pi$ (D) $(-1)^{k-1}k\pi$

(c) Which of the following functions is differentiable at $x = 0$?

(A) $\cos(|x|) + |x|$ (B) $\cos(|x|) - |x|$

(C) $\sin(|x|) + |x|$ (D) $\sin(|x|) - |x|$

5. The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1}x & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x| - 1) & \text{if } |x| > 1 \end{cases}$

(A) $R - \{0\}$ (B) $R - \{1\}$ (C) $R - \{-1\}$ (D) $R - \{-1,1\}$ **[JEE 2002]**

6. Let $f : R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. The $\text{Limit}_{x\to 0} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals

(A) 1 (B) $e^{1/2}$ (C) e^2 (D) e^3 **[JEE 2002]**

7. $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x - 1| & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \geq 0 \end{cases}$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x , determine the values of a and b. Further, for these values of a and b, is gof differentiable at $x = 0$? Justify your answer. **[JEE 2002, 5 out of 60]**

8. If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at $x = a$ is 0 then find the left hand derivative at $x = -a$.

[JEE 2003(Mains) 2 out of 60]

9. (a) The function given by $y = ||x| - 1|$ is differentiable for all real numbers except the points **[JEE 2005 (Screening), 3]**

(A) $\{0, 1, -1\}$ (B) ± 1 (C) 1 (D) -1

(b) If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1,2). **[JEE 2005 (Mains), 2]**

10. If $f(x) = \min.(1, x^2, x^3)$, then **[JEE 2006, 5]**

(A) $f(x)$ is continuous $\forall\, x \in R$

(B) $f'(x) > 0, \forall\, x > 1$

(C) $f(x)$ is not differentiable but continuous $\forall\, x \in R$

(D) $f(x)$ is not differentiable for two values of x

11. Let $g(x) = \frac{(x-1)^n}{\ell n \cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0, n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then **[JEE 2008, 3]**

(A) $n = 1, m = 1$ (B) $n = 1, m = -1$ (C) $n = 2, m = 2$ (D) $n > 2, m = n$

12. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y), \forall\, x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then **[JEE 2011]**

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall\, x \in R$

(C) $f'(x)$ is constant $\forall\, x \in R$

(D) $f(x)$ is differentiable except at finitely many points

13. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$, then **[JEE 2011]**

(A) $f(x)$ is continuous at $x = -\pi/2$ (B) $f(x)$ is not differentiable at $x = 0$

(C) $f(x)$ is differentiable at $x = 1$ (D) $f(x)$ is differentiable at $x = -3/2$

14. Let $f(x) = \begin{cases} x^2 \left|\cos \frac{\pi}{x}\right|, & x \neq 0 \\ 0 & x = 0 \end{cases}, x \in R$, then f is **[JEE 2012]**

(A) differentiable both at $x = 0$ and at $x = 2$

(B) differentiable at $x = 0$ but not differentiable at $x = 2$

(C) not differentiable at $x = 0$ but differentiable at $x = 2$

(D) differentiable neither at $x = 0$ nor at $x = 2$

15. Let $f_1 : R \to R$, $f_2 : [0, \infty) \to R$, $f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by

[JEE Advance 2014]

$f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases}$; $f_2(x) = x^2$; $f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$;

and $f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0 \end{cases}$

List-I	**List-II**
(P) f_4 is	(1) onto but not one-one
(Q) f_3 is	(2) neither continuous nor one-one
(R) $f_2 o f_1$ is	(3) differentiable but not one-one
(S) f_2 is	(4) continuous and one-one

Codes:

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

16. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$.

Define $h : R \to R$ by $h(x) = \begin{cases} \max \{f(x), g(x)\} & \text{if } x \leq 0 \\ \min \{f(x), g(x)\} & \text{if } x > 0 \end{cases}$

The number of points at which $h(x)$ is not differentiable is **[JEE Advance 2014]**

17. Let $g : R \to R$ be a differentiable function with $g(0) = 0, g'(0) = 0$ and $g'(1) \neq 0$.

Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let $(foh)(x)$ denote

$f(h(x))$ and $(hof)(x)$ denote $h(f(x))$.

Then which of the following is(are) true? **[IIT Advance - 2015]**

(A) f is differentiable at $x = 0$

(B) h is differentiable at $x = 0$

(C) foh is differentiable at $x = 0$

(D) hof is differentiable at $x = 0$

18. Let $a, b \in R$ and $f : R \to R$ be defined by $f(x) = a \cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is

(A) Differentiable at $x = 0$ if $a = 0$ and $b = 1$ **[IIT Advance - 2016]**

(B) Differentiable at $x = 1$ if $a = 1$ and $b = 0$

(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$

(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$

19. Let $f : \left[-\frac{1}{2}, 2\right] \to R$ and $g : \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by **[IIT Advance - 2016]**

$f(x) = [x^2 - 3]$ and $g(x) = |x|f(x) + |4x - 7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

20. Let $f: R \to R$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$f(x + y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in R$. **[JEE Advance 2018]**

Then, the value of $\log_e(f(4))$ is

21. Let $f: R \to R$ be a function. We say that f has **[JEE Advance 2019]**

PROPERTY 1 if $\lim_{h\to 0} \frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h\to 0} \frac{f(h)-f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct?

(A) $f(x) = x|x|$ has PROPERTY 2

(B) $f(x) = x^{2/3}$ has PROPERTY 1

(C) $f(x) = \sin x$ has PROPERTY 2

(D) $f(x) = |x|$ has PROPERTY 1

22. Let $f_1: R \to R$, $f_2: \left(-\frac{\pi}{3}, \frac{\pi}{2}\right) \to R$, $f_3: \left(-1, e^{\pi/2} - 2\right) \to R$ and $R_4: R \to R$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$, **[JEE Advnace 2019]**

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where, for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

List-I	**List-II**
(P) The function f_1 is	(1) NOT continuous at $x = 0$
(Q) The function f_2 is	(2) continuous at $x = 0$ and NOT differentiable at $x = 0$

(R) The function f_3 is (3) differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$

(S) The function f_4 is (4) differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is:

(A) $P \to 2$; $Q \to 3$; $R \to 1$; $S \to 4$ (B) $P \to 4$; $Q \to 1$; $R \to 2$; $S \to 3$

(C) $P \to 4$; $Q \to 2$; $R \to 1$; $S \to 3$ (D) $P \to 2$; $Q \to 1$; $R \to 4$; $S \to 3$

23. Let the function $f: R \to R$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and $g : R \to R$ be an arbitrary function $fg : R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE? **[JEE Advanced 2020]**

(A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

(B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

(C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

(D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

24. Let $f : R \to R$ and $g : R \to R$ be functions satisfying **[JEE Advanced 2020]**

$f(x+y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$

for all $x, y \in R$. If $\lim_{x\to0} g(x) = 1$, then which of the following statements is/are TRUE?

(A) f is differentiable at every $x \in R$

(B) If $g(0) = 1$, then g is differentiable at every $x \in R$

(C) The derivative $f'(1)$ is equal to 1

(D) The derivative $f'(0)$ is equal to 1

25. Let the functions $f : (-1,1) \to R$ and $g : (-1,1) \to (-1,1)$ be defined by **[JEE Advanced 2020]**

$f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,

where [x] denotes the greatest integer less than or equal to x. Let $fog : (-1,1) \to R$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1,1)$ at which fog is NOT continuous, and suppose d is the number of points in the interval $(-1,1)$ at which fog is NOT differentiable. Then the value of $c + d$ is ______.

ANSWER KEY

PROFICIENCY TEST

1. f(x) is continuous but not derivable at $x = 0$
2. continuous $\forall x \in R$, not derivable at $x = 0,1$ & 2
3. continuous but not derivable at $x = 0$; derivable & continuous at $x = \pi/2$
4. continuous but not derivable at $x = 0$
5. continuous but not derivable at $x = \sqrt{2}$
6. 2
7. continuous & derivable
8. 0
9. $f'(0^+) = 0, f'(0^-) = -1$
10. 2

EXERCISE-I

3. f is continuous but not derivable at $x = 0$
4. $f'(1^+) = 3, f'(1^-) = -1$
5. $a = 1/2, b = 3/2$
6. A
7. A, C
8. A, B
9. B, D
10. B, D
11. B, D
12. A, B, D
13. A, B, D
14. A, B, D
15. A, D

EXERCISE-II

1. not derivable at $x = 0$ & $x = 1$
2. f is continuous & derivable at $x = -1$ but f is neither continuous nor derivable at $x = 1$
3. discontinuous & not derivable at $x = 1$, continuous but not derivable at $x = 2$
4. not derivable at $x = 0$
5. f is continuous at $x = 1, 3/2$ & discontinuous at $x = 2$, f is not derivable at $x = 1, 3/2, 2$
6. $(fog)(x) = x + 1$ for $-2 \le x \le -1, -(x+1)$ for $-1 < x \le 0$ & $x - 1$ for $0 < x \le 2$. $(fog)(x)$ is continuous at $x = -1$, $(gof)(x) = x + 1$ for $-1 \le x \le 1$ & $3 - x$ for $1 < x \le 3$. $(gof)(x)$ is not differentiable at $x = 1$
7. $a \ne 1, b = 0, p = \frac{1}{3}$ and $q = -1$
8. If $a \in (0,1)$ $f'(0^+) = -1; f'(0^-) = 1 \Rightarrow$ continuous but not derivable
   $a = 1; f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable
   If $a > 1$ $f'(0^-) = -1; f'(0^+) = 1 \Rightarrow$ continuous but not derivable
9. f is continuous but not derivable at $x = 1$, discontinuous at $x = 2$ & $x = 3$. continuous & derivable at all other points
10. $f(x) = x^2 + e^x \ \forall x \in R$
11. $f'(0) = \frac{\alpha}{1-k}$
12. (a) $f'(0) = 0$,
    (b) $f'\left(\frac{1^-}{3}\right) = -\frac{\pi}{2}$ and $f'\left(\frac{1^+}{3}\right) = \frac{\pi}{2}$, (c) $x = \frac{1}{2n+1}$ $n \in 1$
13. $f(x) = x \Rightarrow f(10) = 10$
14. 5150

EXERCISE-III

1.	C	**2.**	B	**3.**	A	**4.**	D	**5.**	B	**6.**	D	**7.**	D
8.	A	**9.**	A	**10.**	D	**11.**	B	**12.**	C	**13.**	D	**14**	B
15.	C												

EXERCISE-IV

1.	C	**2.**	B	**3.**	C	**4.**	B	**5.**	C	**6.**	B	**7.**	B
8.	B												

EXERCISE-V

1. D **2.** C **3.** Discont. hence not deri. at $x = 1 \& -1$. Cont. & deri. at $x = 0$

4. (a) D, (b) A, (c) D **5.** D **6.** C **7.** $a = 1; b = 0$ $(gof)'(0) = 0$

8. $f'(a^-) = 0$ **9.** (a) A , (b) y – 2 = 0 **10.** A, C **11.** C **12.** B, C

13. A, B, C, D **14** B **15.** D **16.** 3 **17.** A, D **18.** A, B

19. B, C **20.** 2 **21.** B, D **22.** D **23.** A, C **24.** A, B, D

25. 4