ROUND-01

FUNCTION

1. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ = greatest integer $\leq x$, then

(A) $f(\pi/2) = -1$ (B) $f(\pi) = 1$ (C) $f(-\pi) = -1$ (D) $f(\pi/4) = 2$

Ans. **(A)**

Sol. $\because [\pi^2] = 9$ and $[-\pi^2] = -10$, so

$f(x) = \cos 9x + \cos(-10)x = \cos 9x + \cos 10x$

$\Rightarrow f(\pi/2) = \cos 9\pi/2 + \cos 5\pi = -1$

$f(\pi) = \cos(9\pi) + \cos 10\pi = 0$

$f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = 0$

$f(\pi/4) = \cos 9\pi/4 + \cos 5\pi/2 = 1/\sqrt{2}$.

2. $f: (-\pi/2, \pi/2) \to (-\infty, \infty), f(x) = \tan x$ is

(A) onto but not one-one

(B) one-one but not onto

(C) one-one onto

(D) neither one-onr nor onto

Ans. **(C)**

Sol. Domain of f lies in first and fourth quadrant where corresponding to two different values, tanx assumes different values. Hence f is one-one.

Further in these two quadrants tanx takes all real values. Hence range $f = (-\infty, \infty)$ which shows that it is onto.

3. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(A) [1,2] (B) [2,3) (C) (1,2) (D) [2,3)

Ans. **(B)**

Sol. $\sin^{-1}(x-3)$ is defined when

$-1 \leq x - 3 \leq 1 \Rightarrow 2 \leq x \leq 4$

$\therefore$ domain of $\sin^{-1}(x-3) = [2,4]$

Also $\sqrt{9-x^2}$ is defined when $9 - x^2 \geq 0$

$\Rightarrow x^2 \leq 9$

$\Rightarrow |x| \leq 3$

$\therefore$ domain of $\frac{1}{\sqrt{9-x^2}} = (-3,3)$

Hence $D_f = [2,4] \cap (-3,3) = [2,3)$

4. If x be real then the range of the function $f(x) = \frac{x}{1+x^2}$ is

(A) $[-1/2, 1/2]$ (B) $(-2, 2)$ (C) $(-1, 1)$ (D) $(-1/2, 1/2)$

Ans. **(A)**

Sol. Let $\frac{x}{1+x^2} = y$

$\Rightarrow yx^2 - x + y = 0$

$\because x \in \mathbf{R}$,

$\therefore B^2 - 4AC \geq 0$

$\Rightarrow 1 - 4y^2 \geq 0$

$\Rightarrow (1-2y)(1+2y) \geq 0$

$\Rightarrow (y - 1/2)(y + 1/2) \leq 0$

$\Rightarrow -1/2 \leq y \leq 1/2$

$\therefore$ range $= [-1/2, 1/2]$

5. If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(g \circ f)(x) = 2x^2 - 5x + 2$, then f(x) is equal to

(A) $2x - 3$ (B) $2x + 3$ (C) $2x^2 + 3x + 1$ (D) $2x^2 - 3x - 1$

Ans. **(A)**

Sol. $g(x) = x^2 + x - 2$

$\Rightarrow (g \circ f)(x) = g[f(x)] = [f(x)]^2 + f(x) - 2$

Given $\frac{1}{2}(g \circ f)(x) = 2x^2 - 5x + 2$

$\Rightarrow [f(x)]^2 + f(x) = 4x^2 - 10x + 6$

$\Rightarrow f(x)[f(x) + 1] = (2x - 3)[(2x - 3) + 1]$

$\Rightarrow f(x) = 2x - 3$

LIMITS

6. $\lim_{x \to \pi/4} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \pi^2/16}$ is equal to

(A) $\frac{8}{\pi} f(2)$ (B) $\frac{2}{\pi} f(2)$ (C) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (D) $4f(2)$

Ans. **(A)**

Sol. $\because$ when $x = \pi/4$, then $\int_2^{\sec^2 x} f(t)dt = \int_2^2 f(t)dt = 0$

$\therefore$ limit is in $0/0$ form. So, by Hospital rule

$$\text{Limit} = \lim_{x \to \pi/4} \frac{f(\sec^2 x) \cdot (2\sec^2 x \tan x) - 0}{2x}$$

$$= \frac{f(2) \cdot 2(2)(1)}{\pi/2} = \frac{8}{\pi} f(2)$$

7. For $a \in R, a \neq -1$,

$$\lim_{n\to\infty} \frac{1^a + 2^a + \cdots + n^a}{(n+1)^{a-1}[(na+1) + (na+2) + \cdots + (na+n)]} = \frac{1}{60}.$$

Then a is equal to

(A) 5 (B) 7 (C) $-15/2$ (D) $-17/2$

Ans. **(B)**

Sol. Given

$$\text{limit} = \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left\{n(na)+\frac{n(n+1)}{2}\right\}}$$

$$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{n^{a+1}\left(1+\frac{1}{n}\right)^{a-1}\left(a+\frac{1}{2}+\frac{1}{2n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{a-1}\left(\frac{r}{n}\right)^{a}\left(a+\frac{1}{2}+\frac{1}{2n}\right)}$$

$$= \frac{\int_0^1 x^a dx}{1\cdot(a+1/2)}\left[\because \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx\right]$$

$$= \frac{\left(\frac{1}{(a+1)}\right)}{(a+1/2)} = \frac{2}{(2a+1)(a+1)}$$

$\therefore$ From given relation, we have

$$\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$$

$(2a+1)(a+1) = 120$

$\Rightarrow 2a^2 + 3a - 119 = 0$

$\Rightarrow 2a^2 - 14a + 17a - 119 = 0$

$\Rightarrow 2a(a-7) + 17(a-7) = 0$

$\Rightarrow (2a+17)(a-7) = 0$

$\Rightarrow a = 7, -17/2.$

But $a \neq -17/2$ because $\int_0^1 x^a dx$ converges if $a > -1$.

$\therefore\ a = 7.$

8. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$ is equal to

(A) 1/2 (B) -1 (C) 2 (D) -2

Ans. **(A)**

Sol. Using expansion of tanx and sinx, we have

$$\text{Limit} = \lim_{x\to 0} \frac{\left(1+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots\right)-\left(x-\frac{x^3}{13}+\frac{x^5}{[5}-\cdots\right)}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{2}+\frac{1}{8}x^2+\cdots\right) = \frac{1}{2}$$

9. $\lim_{x\to 0} \frac{[\sin(x+a)+\sin(a-x)-2\sin a]}{x\sin x}$ is equal to

(A) $\sin a$ (B) $-\sin a$ (C) 1 (D) 0

Ans **(B)**

Sol. Limit $= \lim\limits_{x\to 0} \frac{2\sin a\cos x-2\sin a}{x\sin x}$

$= \lim\limits_{x\to 0} 2\sin a\left[\frac{\cos x-1}{x\sin x}\right]$

$= 2\sin a \cdot \lim\limits_{x\to 0}\left[\frac{-2\sin^2 x/2}{2x\sin x/2\cos x/2}\right]$

$= -\sin a \cdot \lim\limits_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)\cdot\frac{1}{\cos x/2}$

$= -\sin a \cdot (1)(1)$

$= -\sin a$

10. If f(x) is differentiable and strictly increasing function, then $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is equal to

(A) 1 (B) 0 (C) 2 (D) -1

Ans. **(D)**

Sol. Limit is in 0/0 form, so by Hospital rule

Limit $= \lim\limits_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)}$

$= \lim\limits_{x\to 0}\left[\frac{2xf'(x^2)}{f'(x)}-1\right] = 0-1\ [\because f(x^2)>0]$

$= -1$

11. $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$ is equal to

(A) e^3 (B) $e^{1/3}$ (C) 1 (D) e

Ans. **(D)**

Sol. Limit $= \lim_{x\to 0}\left(\frac{x+x^3/3+\cdots}{x}\right)^{1/x^2}$

$= \lim_{x\to 0}\left(1+\frac{x^2}{3}\right)^{1/x^2}$

[$\because x\to 0$, so neglecting higher powers of x]

[1^∞ form]

$= e^{\lim_{x\to 0}\left(\frac{x^2}{3}\right)\frac{1}{x^2}} = e^{1/3}$

CONTINUITY

12. If $f(x) = \begin{cases} \frac{\log(1+ax)-\log(1-bx)}{x} & ,x \neq 0 \\ k & ,x = 0 \end{cases}$

is continuous at $x = 0$, then the value of k is

(A) 0 (B) $a + b$ (C) $a - b$ (D)

Ans. **(B)**

Sol. $\because$ f(x) is continuous at $x = 0$, so

$f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow k = \lim\limits_{x\to0} \frac{\log(1+ax)-\log(1-bx)}{x}\ \left(\frac{0}{0} \text{ form}\right)$

$= \lim\limits_{x\to0}\left(\frac{a}{1+ax} + \frac{b}{1-bx}\right) = a + b.$

13. If $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & ,x \neq 0 \\ k & ,x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k will be

(A) 1 (B) -1 (C) 0 (D) none of these

Ans. **(C)**

Sol. Since f(x) is continuous at $x = 0$, so

$f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow\ k = \lim\limits_{x\to0} x\sin\frac{1}{x}$

$= 0 \times (\text{ a finite quantity }) = 0$

14. If $f(x) = \begin{cases} x^{\alpha}\cos\left(\frac{1}{x}\right) & ,x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$, then

(A) $\alpha < 0$ (B) $\alpha > 0$ (C) $\alpha = 0$ (D) $\alpha \geq 0$

Ans. **(B)**

Sol. Since f(x) is continuous at $x = 0$, so

$f(0 - 0) = f(0 + 0) = f(0) = 0$

But $f(0 - 0) = \lim\limits_{h\to0} (-h)^{\alpha}\cos(-1/h) = 0$, if $\alpha > 0$

$f(0 + 0) = \lim\limits_{h\to0} (h)^{\alpha}\cos 1/h = 0$, if $\alpha > 0$

Hence f(x) is continuous at $x = 0$ when $\alpha > 0$.

15. If $f(x) = \begin{cases} -2\sin x & , \quad x \leq -\pi/2 \\ a\sin x + b & , \quad -\pi/2 < x < \pi/2 \\ \cos x & , \quad x \geq \pi/2 \end{cases}$

is a continuous function, then

(A) $a = 1, b = 1$ (B) $a = -1, b = 1$ (C) $a = 1, b = -1$ (D) $a = b = -1$

Ans. **(B)**

Sol. f(x) is a continuous function, so it is continuous at $x = \pi/2$ and $x = -\pi/2$.

Now f(x) is continuous at $x = \pi/2$

$\Rightarrow f(\pi/2) = f(\pi/2 - 0)$

$\Rightarrow \cos\pi/2 = \lim_{h\to 0}\left[a\sin\left(\frac{\pi}{2} - h\right) + b\right]$

$\Rightarrow 0 = a + b$

Also f(x) is continuous at $x = -\pi/2$

$\Rightarrow f(-\pi/2) = f(-\pi/2 + 0)$

$\Rightarrow -2\sin(-\pi/2) = \lim_{h\to 0}\left[a\sin\left(-\frac{\pi}{2} + h\right) + b\right]$

$2 = -a + b$

(1) and (2) $a = -1, b = 1$.

16. Let $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$ where [x] denotes the greatest integer function. The set of points of discontinuity of f in its domain is

(A) Z (B) Z_0 (C) N (D) none of these

Ans. **(B)**

Sol. $\because\ [x + 1] = 0$ if $0 \le x + 1 < 1$

$\Rightarrow\ [x + 1] = 0$ if $-1 \le x < 0$

$\Rightarrow$ domain of $f = R - [-1,0]$

Now [x] is continuous on $\mathbf{R} - \mathbf{Z}$ and $\frac{\pi}{[x+1]}$ is defined at all points of $\mathbf{R} - [-1,0)$. Thus points where f can possibly be discontinuous are $\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$

If n be any non-zero integer then at $x = n$.

$\text{LHL} = \lim_{h\to 0}[n - h]\sin\frac{\pi}{[n+1-h]}$

$= (n - 1)\sin\frac{\pi}{n}$

$\text{RHL} = \lim_{h\to 0}[n + h]\sin\frac{\pi}{[n+1+h]}$

$= n\sin\frac{\pi}{n+1}$

$\because$ LHL $\neq$ RHL, so f is discontinuous at every non-zero integer n. Now at $x = 0, f(0) = 0$.

$\text{RHL} = \lim_{h\to 0}[h]\sin\frac{\pi}{[h+1]} = 0.$

Also f is not defined for $x < 0$, so f is right continuous i.e., continuous at $x = 0$. Hence f is discontinuous at every non-zero integer in its domain.

17. Function $f: R/\{0\} \to R, f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining f(0) as

(A) 1 (B) 2 (C) -1 (D) 0

Ans. **(A)**

Sol. If f(x) is continuous at x = 0, then

$$f(0)=\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)$$

$$=\lim_{x\to 0}\frac{e^{2x}-2x-1}{x(e^{2x}-1)}\ \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{1+2x+\frac{(2x)^2}{\lfloor 2}+\cdots-2x-1}{x\left(1+2x+\frac{(2x)^2}{\lfloor 2}+\cdots-1\right)}$$

$$=2/2=1.$$

DIFFERENTIATION

18. Let $f:(-1,1)\to R$ be a differentiable function with $f(0)=-1$ and $f'(0)=1$. Let $g(x)=\{f(2f(x)+2)\}^2$. Then $g'(0)$ is equal to

(A) -2 (B) 4 (C) -4 (D) 0

Ans. **(C)**

Sol. By derivative of the function of the function rule, we have

$$g'(x)=2\{f(2f(x)+2)\}\cdot f'(2f(x)+2)\cdot 2f'(x)$$

$$\Rightarrow\ g'(0)=2\{f(2f(0)+2)\}\cdot f'(2f(0)+2)\cdot 2f'(0)$$

$$=2\{f(-2+2)\}\cdot f'(2(-1)+2)\cdot 2(1)$$

$$=2f(0)\cdot f'(0)\cdot 2$$

$$=2\cdot(-1)\cdot 1\cdot 2$$

$$=-4$$

19. If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & ,x\neq 0 \\ 0, & x=0\end{cases}$, then f(x) is

(A) discontinuous everywhere

(B) continuous as well as differentiable for all x

(C) continuous for all x but not differentiable at x = 0

(D) neither differentiable nor continuous at x = 0

Ans. **(C)**

Sol. Obviously f is continuous at every $x\in R, x\neq 0$.

At x = 0, we observe

$$f(0-0)=\lim_{h\to 0}(-h)e^{-\left(\frac{1}{|-h|}-\frac{1}{h}\right)}=\lim_{h\to 0}(-h)e^{0}=0$$

$$f(0+0)=\lim_{h\to 0} he^{-\left(\frac{1}{|h|}+\frac{1}{h}\right)}=\lim_{h\to 0}(h)e^{-(2/h)}=0\times 0=0$$

$$\because\ f(0-0)=f(0+0)=f(0)$$

$\therefore$ f is continuous at $x = 0$ and as such f is continuous for all x. For differentiability at $x = 0$, we find that

$$f'(0-0) = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-he^{0}-0}{-h} = 1$$

$$f'(0+0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{he^{-2/h}-0}{h} = 0$$

$\because\ f'(0-0) \neq f'(0+0)$

$\therefore$ f is not differentiable at $x = 0$.

Hence option (3) is correct.

20. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$ equals

(A) ½ (B) $-1/2$ (C) 1 (D) -1

Ans. **(B)**

Sol. $\because\ \frac{\cos x}{1+\sin x} = \frac{\sin(\pi/2-x)}{1+\cos(\pi/2-x)} = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

$\therefore \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$

$\Rightarrow$ derivative $= -1/2$.

21. If $x^p y^q = (x+y)^{p+q}$, then $\frac{dy}{dx}$ equals

(A) y/x (B) x/y

(C) $(x+y)/x$ (D) $(x+y)/y$

Ans. **(A)**

Sol. Taking log on both sides, we get

$\Rightarrow\ p\log x + q\log y = (p+q)\log(x+y)$

$\therefore\ p\log x + q\log y - (p+q)\log(x+y) = 0$

$\therefore\ \frac{dy}{dx} = -\frac{p/x-(p+q)/(x+y)}{q/y-(p+q)/(x+y)}$

$= -\frac{y}{x}\cdot\frac{p(x+y)-x(p+q)}{q(x+y)-y(p+q)}$

$= \frac{y}{x}\cdot\frac{qx-py}{qx-py} = \frac{y}{x}$

22. If $x^2+y^2 = t-\frac{1}{t}, x^4+y^4 = t^2+\frac{1}{t^2}$, then $\frac{dy}{dx}$ equals

(A) $\frac{1}{x^2y}$ (B) $\frac{1}{xy^3}$

(C) $-\frac{1}{x^3y}$ (D) $-\frac{1}{xy^3}$

Ans. **(A)**

Sol. Squaring the first equation, we have

$$x^4+y^4+2x^2y^2 = t^2+\frac{1}{t^2}-2$$

$\Rightarrow t^2 + \frac{1}{t^2} + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$ [from second equation]

$\Rightarrow x^2y^2 = -1$

$\Rightarrow y^2 = -\frac{1}{x^2}$

$\therefore 2y\frac{dy}{dx} = \frac{2}{x^3}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^3y}$

TANGENTS AND NORMALS

23. The equation of the tangent to the curve $y = be^{-xa}$ at the point where it meets y-axis is

(A) $\frac{x}{b} + \frac{y}{a} = 1$ (B) $\frac{x}{a} + \frac{y}{b} = 1$ (C) $\frac{x}{b} + \frac{y}{a} = 2$ (D) $\frac{x}{a} + \frac{y}{b} = 2$

Ans. **(B)**

Sol. The point of intersection of the curve with y-axis is $(0, b)$. Differentiating $y = be^{-xa}$, we have

$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b}{a}$

$\therefore$ the equation of the required tangent is

$y - b = -\frac{b}{a}(x - 0)$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

24. The sum of the squares of intercepts on axes made by a tangent at any point on the curve $x^{2n} + y^{2/3} = a^{2/3}$ is

(A) a (B) 2a (C) a^2 (D) $2a^2$

Ans. **(C)**

Sol. Here at any point (x_1, y_1)

$\frac{dy}{dx} = -\frac{y_1^{1/3}}{x_1^{1/3}}$ and $x_1^{2/3} + y_1^{2/3} = a^{2/3}$.

Now equation of the tangent at (x_1, y_1) is

$y - y_1 = -\frac{y_1^{1/3}}{x_1^{1/3}}(x - x_1)$

$\Rightarrow \frac{y}{y_1^{\frac{1}{3}}} + \frac{x}{x_1^{\frac{1}{3}}} = x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = a^{\frac{2}{3}}$

$\therefore$ sum of squares of intercepts

$= \left(x_1^{1/3}a^{1/3}\right)^2 + \left(y_1^{1/3}a^{1/3}\right)^2$

$= a^{4/3}\left(x_1^{2/3} + y_1^{2/3}\right)$

$= a^{4/3} \cdot a^{2/3} = a^2.$

25. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point ' θ '

(A) makes angle $(\pi/2 + \theta)$ with x-axis.

(B) passes through the origin.

(C) is at a constant distance from origin.

(D) passes through the point $(a\pi/2, -a)$.

Ans. **(AC)**

Sol. $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a(\cos\theta - \cos\theta + \theta\sin\theta)}{a(-\sin\theta + \sin\theta + \theta\cos\theta)} = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow$ slope of the normal at any point 'θ' $= \frac{-\cos\theta}{\sin\theta} = \tan(\pi/2 + \theta)$

So, it makes angle $\pi/2 + \theta$ with x-axis.

Also, equation of the normal at any point 'θ' is

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$\Rightarrow$ $x\cos\theta + y\sin\theta = a$.

Its distance form the origin $= a$ (constant).

26. The angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$ is

(A) $\pi/2$ (B) $\tan^{-1}(4/3)$ (C) $\tan^{-1}(4\sqrt{2}/7)$ (D) none of these

Ans. **(C)**

Sol. Solving the given equations, we get

$x = \pm\sqrt{2}, y = 2$

$\therefore$ point of intersection $= (\pm\sqrt{2}, 2)$. Now

$y = 4 - x^2 \Rightarrow \frac{dy}{dx} = -2x$

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$\therefore$ at $(\pm\sqrt{2}, 2)$ $\left(\frac{dy}{dx}\right)_1 = \mp 2\sqrt{2}$, $\left(\frac{dy}{dx}\right)_2 = \pm 2\sqrt{2}$

So required angle $= \tan^{-1}\left|\frac{\mp 2\sqrt{2} \mp 2\sqrt{2}}{1 + (\mp 2\sqrt{2})(\pm 2\sqrt{2})}\right|$

$= \tan^{-1}(4\sqrt{2}/7)$.

27. If the line $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at point (a, b), then n is equal to

(A) 1 (B) 2 (C) 3 (D) all non-zero values

Ans. **(D)**

Sol. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\Rightarrow \frac{dy}{dx} = -\frac{n}{a}\left(\frac{x}{a}\right)^{n-1} \Big/ \frac{n}{b}\left(\frac{y}{b}\right)^{n-1} = -\left(\frac{b}{a}\right)^n \left(\frac{x}{y}\right)^{n-1}$

$\therefore \left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{b}{a}$

So, tangent to the curve at (a, b) is

$y - b = -\frac{b}{a}(x - a) \Rightarrow \frac{x}{a} + \frac{y}{b} = 2.$

which is the given line and it is independent of n. Hence for every non-zero value of n given line touches the given curve.

28. The normal to the curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a

(A) circle (B) parabola (C) ellipse (D) hyperbola

Ans. **(D)**

Sol. If PM is the perpendicular from P on x-axis, then as given

$OG = 2OM$

$\Rightarrow OM = MG$

But $MG =$ subnormal at P

$= y\frac{dy}{dx}$

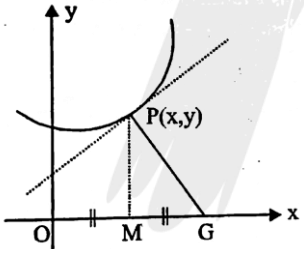
$\therefore$ From (1), $x = y\frac{dy}{dx}$

$\Rightarrow xdx = ydy$

$\Rightarrow x^2 = y^2 + c$

[on integration]

$\Rightarrow x^2 - y^2 = c$, which is a hyperbola.



MONOTONICITY

29. Function $f(x) = 2x^3 - 9x^2 + 12x + 29$ is monotonic decreasing when

(A) $x < 2$ (B) $x > 2$ (C) $x > 3$ (D) $1 < x < 2$

Ans. **(D)**

Sol. f(x) is monotonic decreasing if

$\Rightarrow f'(x) < 0 \Rightarrow 6x^2 - 18x + 12 < 0$

$\Rightarrow (x-1)(x-2) < 0 \Rightarrow 1 < x < 2$

30. $f(x) = 2x^2 - \log|x|$ $(x \neq 0)$ is monotonic increasing in the interval

(A) $(1/2, \infty)$ (B) $(-\infty, -1/2) \cup (1/2, \infty)$

(C) $(-\infty, -1/2) \cup (0, 1/2)$ (D) $(-1/2, 0) \cup (1/2, \infty)$

Ans. **(D)**

Sol. $f'(x) = 4x - 1/x$

$f(x)$ is monotonic increasing when $f'(x) > 0$

$\Rightarrow 4x - \frac{1}{x} > 0$

$\Rightarrow \frac{4x^2-1}{x} > 0$

$\Rightarrow \begin{cases} 4x^2 - 1 > 0 & \text{when } x > 0 \\ 4x^2 - 1 < 0 & \text{when } x < 0 \end{cases}$

But $x > 0, 4x^2 - 1 > 0 \Rightarrow x^2 > 1/4 \Rightarrow |x| > 1/2$

$\Rightarrow x \in (1/2, \infty)$

and $x < 0, 4x^2 - 1 < 0 \Rightarrow x^2 < 1/4 \Rightarrow |x| < 1/2$

$\Rightarrow x \in (-1/2, 0)$

$\therefore$ $x \in (-1/2, 0) \cup (1/2, \infty)$

31. Function $f(x) = x^2 e^{-x}$ is monotonic increasing when

(A) $x \in R - [0,2]$ (B) $0 < x < 2$ (C) $2 < x < \infty$ (D) $x < 0$

Ans. **(B)**

Sol. $f'(x) = xe^{-x}(2 - x)$

$f(x)$ is monotonic increasing when $f'(x) > 0$

$\Rightarrow xe^{-x}(2 - x) > 0 \Rightarrow x(2 - x) > 0 [\because e^{-x} > 0]$

$\Rightarrow x(x - 2) < 0$

$\Rightarrow 0 < x < 2.$

32. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is

(A) increasing in $[-1/2, 1]$ (B) decreasing in R

(C) increasing in R (F) decreasing in $[-1/2, 1]$

Ans. **(A)**

Sol. $f(x) = xe^{x(1-x)}$

$f'(x) = e^{x(1-x)} + xe^{x(1-x)}(1 - 2x)$

$= e^{x(1-x)}[1 + x - 2x^2]$

$= -e^{x(1-x)}(x - 1)(2x + 1).$

Now $f(x)$ is increasing when $f'(x) > 0$

$\Rightarrow -e^{x(1-x)}(x - 1)(2x + 1) > 0$

$\Rightarrow (x - 1)(2x + 1) < 0$

$\Rightarrow -1/2 < x < 1$

$[\because e^{x(1-x)} > 0]$

Also f(x) is decreasing when $f'(x) < 0$

$\Rightarrow (x-1)(2x+1) > 0$

$\Rightarrow x < -1/2$ or $x > 1$

Hence (1) is correct.

33. Function $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$ is increasing in

(A) (2,2) (B) $(0, \infty)$ (C) $(-\infty, 0)$ (D) no where

Ans. **(C)**

Sol. $f'(x) = e^{-(x^2+1)^2} \cdot 2x - e^{-x^4} \cdot 2x$

$= 2xe^{-x^4}\left(e^{-(2x^2+1)} - 1\right)$

f is increasing when $f'(x) > 0$

$\Rightarrow \; xe^{-x^4}\left[e^{-(2x^2+1)} - 1\right] > 0$

This is true for all negative values of x, because $e^{-(2x^2+1)} < 1.\ \forall x \in \mathbf{R}^{-}$.

Hence f is increasing when $x \in (-\infty, 0)$.

MAXIMA AND MINIMA

34. $f(x) = 2x^3 - 21x^2 + 36x + 7$ has

(A) a local maxima at $x = 1$ and minima at $x = 6$.

(B) a local maxima at $x = 6$ and minima at $x = 1$.

(C) a local maxima at $x = 1$ and no local minima.

(D) a local minima at $x = 6$ and no local maxima.

Ans. **(A)**

Sol. $f'(x) = 6x^2 - 42x + 36$

$f''(x) = 12x - 42$

Now $f'(x) = 0 \Rightarrow 6(x^2 - 7x + 6) = 0 \Rightarrow x = 1, 6$

Also $f''(1) = 12 - 42 = -30 < 0$ and $f''(6) = 30 > 0$

$\therefore$ f(x) has a local maxima at $x = 1$ and a local minima at $x = 6$

35. The maximum value of $\sin x + \cos x$ is

(A) 1 (B) 2 (C) $\sqrt{2}$ (D) none of these

Ans. **(C)**

Sol. Let

$\Rightarrow\ f(x) = \sin x + \cos x$

$\Rightarrow\ f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

Now $f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \tan x = 1 \Rightarrow x = n\pi + \pi/4,\ n = 0,1,2, \ldots$

$\Rightarrow x = \pi/4, 5\pi/4, \ldots$

Also $f''(\pi/4) = -1/\sqrt{2} - 1/\sqrt{2} = -\sqrt{2} < 0$

$\therefore$ f(x) is maximum at $x = \pi/4$ and

So, its maximum value $= 1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2}$.

36. The maximum and minimum values of $\sin x - x$ are

(A) $1, -1$

(B) $\frac{3\sqrt{3}-\pi}{6}, \frac{\pi-3\sqrt{3}}{6}$

(C) $\frac{\pi-3\sqrt{3}}{6}, \frac{3\sqrt{3}-\pi}{6}$

(D) do not exist

Ans. **(B)**

Sol. Let

$f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f''(x) = -4\sin 2x$

Now $f''(x) = -4\sin 2x$

$f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0$

$\Rightarrow x = m\pi \pm \pi/6,\ n = 0,1,2, \ldots$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, -\frac{\pi}{6}, \ldots..$

But $f''(\pi/6) = -2/\sqrt{3} < 0$

$\Rightarrow x = \pi/6$ is a point of maxima

Also $f''(5\pi/6) = 2\sqrt{3} > 0$

$\Rightarrow x = 5\pi/6$ is a point of minima

Hence one max value $= f(\pi/6) = \frac{3\sqrt{3}-\pi}{6}$

one min. value $= f(5\pi/6) = -\frac{3\sqrt{3}+5\pi}{6}$

But it is not there in given alternative. Hence by alternative position another point of minima is $-\pi/6$ so

One minimum value $= f\left(-\frac{\pi}{6}\right) = \frac{\pi-3\sqrt{3}}{\mathbf{6}}$.

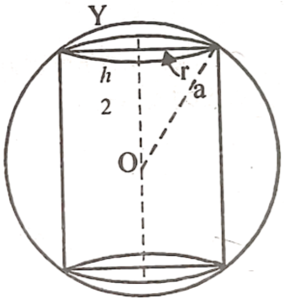
37. The height of a cylinder of maximum volume inscribed in a sphere of radius a is

(A) $a/\sqrt{3}$ (B) $2a/\sqrt{3}$ (C) $\sqrt{3}a$ (D) $2\sqrt{3}a$

Ans. **(B)**

Sol. Let radius of the base = r,

height = h and volume = V. Then



$V = \pi r^2 h$

$= \pi\left(a^2 - \frac{h^2}{4}\right)h$

$= \pi a^2 h - \frac{\pi h^2}{4}$

$\Rightarrow \frac{dV}{dh} = \pi a^2 - \frac{3\pi h^2}{4}, \frac{d^2V}{dh^2} = -\frac{2\pi h}{2}$

$\therefore \frac{dV}{dh} = 0 \Rightarrow a^2 - \frac{3h^2}{4} = 0 \Rightarrow h = \frac{2a}{\sqrt{3}}$

Also $\frac{d^2V}{dh^2}$ is - ve for $h = \frac{2a}{\sqrt{3}}$

So, V is maximum when

$h = 2a/\sqrt{3}$.

38. The height of a right circular cone of maximum volume inscribed in a sphere of diameter a is

(A) $(2/3)a$ (B) $(3/4)a$ (C) $(1/3)a$ (D) $(1/4)a$

Ans. **(A)**

Sol. Let r be the radius of the base and x be the height of the inscribed cone. Then

$r^2 = \frac{a^2}{4} - \left(x - \frac{a}{2}\right)^2 = ax - x^2$

If V be the volume of the cone, then

$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi(ax - x^2)x$

$= \frac{\pi}{3}(ax^2 - x^3)$

$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(2ax - 3x^2), \frac{d^2V}{dx^2} = \frac{2\pi a}{3} - 2\pi x$

Now $\frac{dV}{dx} = 0 \Rightarrow x = 0$ or $x = \frac{2a}{3}$.

But $x \neq 0$ and at $x = \frac{2a}{3}, \frac{d^2V}{dx^2} = -\frac{2\pi a}{3} < 0$.

So, V is maximum when height of cone = $(2/3)a$.

39. The semi-vertical angle of a right circular cone of given slant height and maximum volume is

(A) $\tan^{-1}2$ (B) $\tan^{-1}\sqrt{2}$

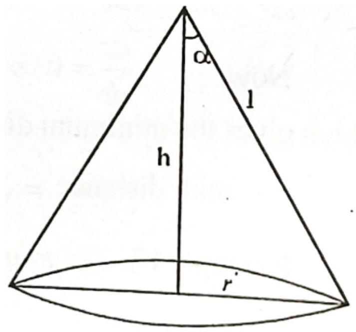
(C) $\tan^{-1}1/2$ (D) $\tan^{-1}1/\sqrt{2}$

Ans. **(B)**

Sol. Let l be given slant height and α be semi-vertical angle of the cone. Then its height

$h = l\cos\alpha$,

radius of the base $r = l\sin\alpha$



Volume, $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha$

$\Rightarrow \frac{dV}{d\alpha} = \frac{1}{3}\pi l^3[2\sin\alpha\cos^2\alpha - \sin^3\alpha]$

$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3[2\cos^3\alpha - 7\sin^2\alpha\cos\alpha]$

Now $\frac{dV}{d\alpha} = 0 \Rightarrow \sin\alpha = 0$ or $\tan\alpha = \sqrt{2}$

But $\sin \neq 0$ so $\alpha = \tan^{-1}\sqrt{2}$.

Also then $\frac{d^2V}{d\alpha^2} < 0$, hence when $\alpha = \tan^{-1}\sqrt{2}$, volume is maximum.

40. A rectangular sheet of fixed perimeter with side having their lengths in the ratio 8: 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume, then the lengths of sides of the rectangular sheet are

(A) 24 (B) 32 (C) 45 (D) 60

Ans. **(AC)**

Sol. Let x be the side of each removed square. Then $4x^2 = 100 \Rightarrow x = 5$. Let 8a and 15a be lengths of sides of the rectangular sheet. Then the volume of the box V is given by

$V = (15a - 2x)(8a - 2x)x$

$= 2(2x^3 - 23ax^2 + 60a^2x) = f(x)$

$\Rightarrow \frac{dV}{dx} = 4(3x^2 - 23ax + 30a^2)$

But as given when $x = 5$, V is maximum, so

$\left(\frac{dV}{dx}\right)_{x=5} = 0 \Rightarrow 4(75 - 115a + 30a^2) = 0$

$\Rightarrow 6a^2 - 23a + 15 = 0$

$\Rightarrow (6a - 5)(a - 3) = 0 \Rightarrow a = 3, 5/6$

Also $\frac{d^2V}{dx^2} = 4(6x - 23a)$

$\Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=5,a=3} < 0$ and $\left(\frac{d^2V}{dx^2}\right)_{x=5,a=5/6} > 0.$

Thus V is maximum when $a = 3$.

So, the sides are 24 and 45.

INTEGRATION

41. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$ equals

(A) $2\sqrt{\sec x} + c$ (B) $2\sqrt{\tan x} + c$ (C) $\frac{2}{\sqrt{\tan x}} + c$ (D) $\frac{2}{\sqrt{\sec x}} + c$

Ans. **(B)**

Sol. $I = \int \frac{\sqrt{\tan x}}{\tan x} \sec^2 x dx$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = 2\sqrt{\tan x} + c$$

42. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$ equals

(A) $\tan(xe^x) + c$ (B) $\cot(xe^x) + c$ (C) $\tan(e^x) + c$ (D) $\cot(e^x) + c$

Ans. **(A)**

Sol. Put $xe^x = t \Rightarrow e^x(1 + e^x)dx = dt$

$$\therefore\ I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t dt = \tan(xe^x) + c$$

43. $\int \frac{x}{1+\cos x} dx$ is equal to

(A) $x\tan\frac{x}{2} + \log\cos\frac{x}{2} + c$ (B) $x\tan\frac{x}{2} + 2\log\cos\frac{x}{2} + c$

(C) $x\tan\frac{x}{2} - 2\log\cos\frac{x}{2} + c$ (D) none of these

Ans. **(B)**

Sol. $I = \int \frac{x}{2\cos^2 x/2} dx$

$= \int x\left(\frac{1}{2}\sec^2\frac{x}{2}\right) dx$

$= x\tan\frac{x}{2} - \int \tan\frac{x}{2} dx$

$= x\tan\frac{x}{2} - 2\log\sec\frac{x}{2} + c$

$= x\tan\frac{x}{2} + 2\log\cos\frac{x}{2} + c$

44. $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right) dx$ is equal to

(A) $e^x\tan\frac{x}{2} + c$ (B) $e^x\cot\frac{x}{2} + c$ (C) $e^x\tan x + c$ (D) $e^x\cot x + c$

Ans. **(A)**

Sol. $I = \int e^x\left(\frac{1+2\sin x/2\cos x/2}{2\cos^2 x/2}\right) dx$

$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$

$= \int e^x[f'(x) + f(x)]dx$

$= e^x\tan\frac{x}{2} + c$

45. $\int \frac{dx}{x(x^n+1)}$ is equal to

(A) $\log\left(\frac{x^n}{x^n+1}\right)+c$

(B) $\log\left(\frac{x^n+1}{x^n}\right)+c$

(C) $\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+c$

(D) $\frac{1}{n}\log\left(\frac{x^n+1}{x^n}\right)+c$

Ans. **(C)**

Sol. $I=\int \frac{x^{n-1}}{x^n(x^n+1)}dx$

$=\frac{1}{n}\int \frac{dt}{t(t+1)}$, where $t=x^n$

$$=\frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+c$$

46. $\int\left\{\frac{\log x-1}{1+(\log x)^2}\right\}^2 dx$ is equal to

(A) $\frac{x}{(\log x)^2+1}+c$ (B) $\frac{xe^x}{1+x^2}+c$ (C) $\frac{x}{x^2+1}+c$ (D) $\frac{\log x}{(\log x)^2+1}+c$

Ans. **(A)**

Sol. Let $\log x=t$, then $x=e^t\Rightarrow dx=e^t dt$

$$\therefore\ I=\int\left(\frac{t-1}{1+t^2}\right)^2 e^t dt=\int e^t\cdot\frac{t^2+1-2t}{(1+t^2)^2}dt$$

$=\int e^t\cdot\left\{\frac{1}{1+t^2}-\frac{2t}{(1+t^2)^2}\right\}dt$

$=\int e^t\{f(t)+f'(t)\}dt$, where $f(t)=\frac{1}{1+t^2}$

$=e^t f(t)+c$

$=\frac{x}{1+(\log x)^2}+c$

47. $\int \sec^{2/3}x\operatorname{cosec}^{4/3}x\,dx$ is equal to

(A) $-3(\cot x)^{1/3}+c$

(B) $-3(\tan x)^{-2/3}+c$

(C) $3(\operatorname{cosec}x)^{1/3}+c$

(D) $3(\cos 2x)^{1/3}+c$

Ans. **(A)**

Sol. $I=\int \frac{1}{\cos^{2/3}x\sin^{4/3}x}dx$

$$=\int\frac{\sin^{2/3}x}{\cos^{2/3}x}\cdot\operatorname{cosec}^2x\,dx$$

$=\int(\cot x)^{-2/3}\operatorname{cosec}^2x\,dx$

$=-3(\cot x)^{1/3}+c$

DEFINITE INTEGRAL

48. The value of the integral $\int_{-4}^{4}(ax^3 + bx + c)dx$ depends on

(A) b and c (B) a, b and c (C) only c (D) a and c

Ans. **(C)**

Sol. $\because I = \int_{-4}^{4} (ax^3 + bx)dx + \int_{-4}^{4} cdx$

$= 0 + 2\int_{0}^{4} cdx$ [by P_7]

$= 2c[x]_0^4 = 8c.$

Hence the value of I depends only on c.

49. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ equals

(A) 0 (B) $\pi/4$ (C) $\pi^2/4$ (D) $\pi^2/2$

Ans. **(C)**

Sol. $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$ [by P_4]

$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$

$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$= -\pi[\tan^{-1}(\cos x)]_0^{\pi}$

$= -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2}$

$\therefore I = \pi^2/4.$

50. $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$ equals

(A) 3/2 (B) ½ (C) ¼ (D) 1

Ans. **(B)**

Sol. $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$

$\Rightarrow I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}} dx$

$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}} dx$

$\therefore 2I = \int_2^3 1dx = [x]_2^3 = 1$

$\Rightarrow I = 1/2.$

51. $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ ($n \in \mathbf{N}$) is equal to

(A) π^2 (B) $2\pi^2$ (C) π (D) 2π

Ans. **(A)**

Sol. $I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\Rightarrow\ I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)}dx$ [by P_4]

$= \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\therefore\ 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\Rightarrow\ I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ [by P_6]

$= 4\pi\left(\frac{\pi}{4}\right) = \pi^2.$

52. If $f(x) = \begin{cases} e^{\cos x}\sin x & , |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)dx$ is equal to

(A) 0 (B) 1 (C) 2 (D) 3

Ans. **(C)**

Sol. $f(x) = \begin{cases} 2 & , \quad x < -2 \\ e^{\cos x}\sin x & , \quad -2 \le x \le 2 \\ 2, & \quad x > 2 \end{cases}$

$\therefore\ \int_{-2}^{3} f(x)dx = \int_{-2}^{2} e^{\cos x}\sin x dx + \int_{2}^{3} 2dx$

$= 0 + 2[x]_2^3$ [$\because$ $e^{\cos x}\sin x$ is odd function]

$= 2.$

53. Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}, x > 0$. If $\int_1^4 \frac{3}{x}e^{\sin x^3}dx = F(k) - F(1)$, then one of the possible volume of k, is

(A) 15 (B) 16 (C) 63 (D) 64

Ans. **(D)**

Sol. $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}, x > 0$

$\Rightarrow\ F(x) = \int \frac{e^{\sin x}}{x}dx$

Also as given

$\int_1^4 \frac{3}{x}e^{\sin x^3}dx = F(k) - F(1)$

$\Rightarrow\ \int_1^4 \frac{3x^2}{x^3}e^{\sin x^3}dx = F(k) - F(1)$

$\Rightarrow\ \int_1^{64} \frac{e^{\sin t}}{t}dt = F(k) - F(1)$, where $t = x^3$

$\Rightarrow\ [F(t)]_1^{64} = F(k) - F(1)$ [using (1)]

$\Rightarrow F(64) - F(1) = F(k) - F(1)$

$k = 64.$

54. $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\cdots.+\frac{1}{n}\sec^2 1\right]$ equals

(A) $\frac{1}{2}\tan 1$ (B) $\tan 1$ (C) $\frac{1}{2}\operatorname{cosec} 1$ (D) $\frac{1}{2}\sec 1$

Ans. **(A)**

Sol. Limit $=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\sec^2\frac{r^2}{n^2}$

$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r}{n}\right)\sec^2\left(\frac{r}{n}\right)^2\cdot\frac{1}{n}$

$=\int_0^1 x\sec^2 x^2 dx=\frac{1}{2}[\tan x^2]_0^1=\frac{1}{2}\tan 1$

Area under Curve

55. The area between the parabola $y^2=4ax$ and its latus rectum is

(A) $(8/3)a$ (B) $(8/3)a^2$ (C) $(4/3)a$ (D) $(4/3)a^2$

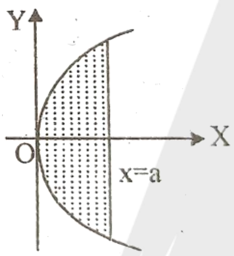
Ans. **(B)**

Sol. Required area is symmetrical about x-axis and it lies between $x=0$ and $x=a$ (latus rectum). Hence

area $=2\int_0^a y dx=2\int_0^a 2\sqrt{a}\sqrt{x}dx$

$=4\sqrt{a}\frac{2}{3}\left[x^{3/2}\right]_0^a$

$=\frac{8}{3}\sqrt{a}(a\sqrt{a}-0)=\frac{8}{3}a^2$



56. The area bounded by the circle $x^2+y^2=4$, the line $x=\sqrt{3}y$ and x-axis lying in the first quadrant is

(A) π (B) $\pi/2$ (C) $\pi/3$ (D) $2\pi/3$

Ans. **(C)**

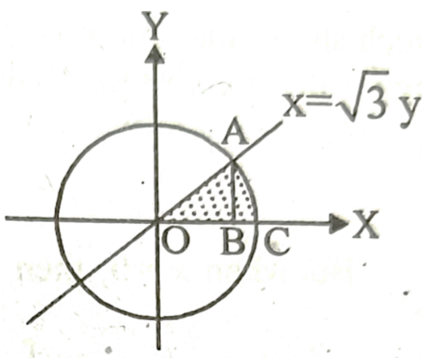
Sol. Solving the given equations for x, we have

$x^2+\frac{x^2}{3}=4$

$\Rightarrow x^2=3$

$\Rightarrow x=\pm\sqrt{3}$

$\therefore$ reqd. area $=\int_0^{\sqrt{3}}\frac{x}{\sqrt{3}}dx+\int_{\sqrt{3}}^{2}\sqrt{4-x^2}dx$



$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^2$

$= \frac{\sqrt{3}}{2} + \left[\pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right] = \frac{\pi}{3}.$

Alter.

Reqd. area $= \int_{y=0}^{1}(x_2 - x_1)dy = \int_0^1\left(\sqrt{4-y^2} - \sqrt{3}y\right)dy$

$= \left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} - \sqrt{3}\frac{y^2}{2}\right]_0^1$

$= \frac{1}{2}\left[\sqrt{3} + \frac{2\pi}{3} - \sqrt{3} - 0\right] = \frac{\pi}{3}.$

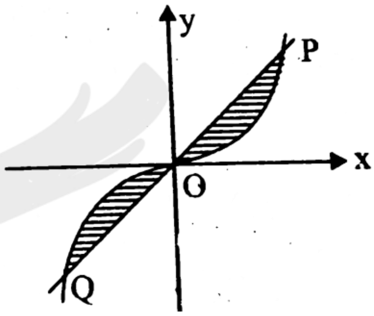
57. The area between the curves $y = x$ and $y = x^3$ is

(A) 1/4 (B) 1/2 (C) 1/3 (D) 1

Ans. **(B)**

Sol. Solving the equations of the given curves for x, we get $x = 0, 1, -1$.

The required area is symmetrical about the origin as shown in the diagram. So



reqd. area $= 2\int_0^1 (x - x^3)dx$

$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

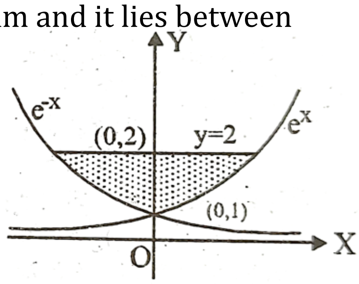
$= 2\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{2}$

58. The area bounded by the curves $y = e^x, y = e^{-x}$ and $y = 2$ is

(A) log(16/e) (B) log(4/e) (C) 2log(4/e) (D) none of these

Ans. **(C)**

Sol. The required area is symmetrical about y-axis as shown in the diagram and it lies between $y = 1$ and $y = 2$. So



reqd. area $= 2\int_1^2 x dy$

$= 2\int_1^2 \log y dy = 2[y\log y - y]_1^2$

$= 2[(2\log 2 - 2) - (0 - 1)]$

$= 2[\log 4 - 1] = 2\log(4/e)$

59. The area of the smaller portion between curves $x^2 + y^2 = 8$ and $y^2 = 2x$ is

(A) $\pi + 2/3$ (B) $2\pi + 2/3$ (C) $2\pi + 4/3$ (D) $\pi + 4/3$

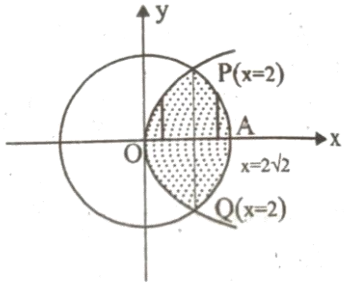
Ans. **(C)**

Sol. Two curves meet at P and Q where $x = 2$. Obviously, the required area lies between $x = 0$ and $x = 2\sqrt{2}$. It is symmetrical about x-axis and bounded by two given curves.

So, required area $= 2\left[\int_0^2 \sqrt{2}\sqrt{x}dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}dx\right]$

$$= 2\left[\left(\frac{2\sqrt{2}}{3}x^{\frac{3}{2}}\right)_0^2 + \left(\frac{x}{2}\sqrt{8-x^2} + 4\sin^{-1}\frac{x}{2\sqrt{2}}\right)_2^{2\sqrt{2}}\right]$$

$$= 2\left[\left(\frac{8}{3}-0\right) + (2\pi - 2 - \pi)\right] = 2\pi + \frac{4}{3}.$$



DIFFERENTIAL EQUATIONS

60. The order and degree of differential equation

$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$ are

(A) 2,2 (B) 2,3 (C) 2,1 (D) 1,6

Ans. **(A)**

Sol. Squaring both sides, given equation becomes

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

It contains $\frac{d^2y}{dx^2}$ as the greatest order derivative. Since its order is 2 , so order of given differential equation is 2 .

Further index of $\frac{d^2y}{dx^2}$ is also 2 , so degree of given equation is also 2 .

61. Solution of the differential equation

$\sec^2x \tan y\, dx + \sec^2y \tan x\, dy = 0$ is

(A) $\tan x \sec y = c$ (B) $\tan x \tan y = c$

(C) $\tan x = c\tan(x+y)$ (D) $\tan x = c\tan(x-y)$

Ans. **(B)**

Sol. Given equation can be written as

$\frac{\sec^2x}{\tan x}dx + \frac{\sec^2y}{\tan y}dy = 0$

$\Rightarrow \int \frac{\sec^2x}{\tan x}dx + \int \frac{\sec^2y}{\tan y}dy = 0$

$\Rightarrow \log(\tan x \tan y) = \log c$

$\Rightarrow \tan x \tan y = c$

62. Solution of differential equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is

(A) $e^y = e^x + \frac{1}{3}x^3 + c$ (B) $y = e^x + \frac{1}{3}x^3 + c$

(C) $e^{-y} = e^x + \frac{1}{3}x^3 + c$ (D) none of these

Ans. **(A)**

Sol. From given equation

$\frac{dy}{dx} = e^{-y}(e^x + x^2)$

$\Rightarrow e^y dy = (e^x + x^2)dx$

$\Rightarrow \int e^y dy = \int (e^x + x^2)dx$

$\Rightarrow e^y = e^x + \frac{1}{3}x^3 + c$

63. Solution of differential equation $\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$ is

(A) $y = \tan x - 1 + ce^{-\tan x}$ (B) $y^2 = \tan x - 1 + ce^{\tan x}$

(C) $ye^{\tan x} = \tan x - 1 + c$ (D) $ye^{-\tan x} = \tan x - 1 + c$

Ans. **(A)**

Sol. Given equation is linear equation in y. So its

I.F. $= e^{\int \sec^2 x dx} = e^{\tan x}$

$\therefore$ solution is $y \cdot e^{\tan x} = \int (\tan x \sec^2 x)e^{\tan x} dx + c$

$= \int te^t dt + c$ where $t = \tan x$

$= (t-1)e^t + c$

$= (\tan x - 1)e^{\tan x} + c$

$\Rightarrow y = \tan x - 1 + ce^{-\tan x}$

64. Solution of differential equation

$(1 + y^2)dx + \left(x - e^{\tan^{-1}y}\right)dy = 0°$ is

(A) $ye^{\tan^{-1}x} = \tan^{-1}x + c$ (B) $xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + c$

(C) $2x = e^{\tan^{-1}y} + c$ (D) $y = xe^{-\tan^{-1}x} + c$

Ans. **(B)**

Sol. From given equation

$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2}$

which is linear in x. So, its

I.F. $= e^{\int 1/(1+y^2)dy} = e^{\tan^{-1}y}$

$\therefore$ solution is $x \cdot e^{\tan^{-1}y} = \int \left(\frac{e^{\tan^{-1}y}}{1+y^2}\right)e^{\tan^{-1}y} dy + c$

$\Rightarrow x\tan^{-1}y = \frac{1}{2}e^{2\tan^{-1}y} + c$

65. The order and degree of the differential equation of all parabolas with their axes as x-axis will be

(A) 1,2 (B) 2,2 (C) 3,2 (D) 2,1

Ans. **(D)**

Sol. Let equation of the given family of parabolas be

$y^2 = Ax + B$

where A and B are parameters.

Differentiating (1), we get

$2y\frac{dy}{dx} = A \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

which is the differential equation of the given family.

Obviously, its order $= 2$, degree $= 1$.

66. Solution of differential equation $x\frac{dy}{dx} = y(\log y - \log x + 1)$ is

(A) $y + e^x = 0$ (B) $y = xe^{cx}$ (C) $y + xe^{cx} = 0$ (D) none of these

Ans. **(B)**

Sol. From given equation

$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

which is a homogeneous equation. So, putting $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$ and transformed equation will be

$v + x\frac{dv}{dx} = v(\log v + 1)$

$\Rightarrow \frac{dv}{dx} = \frac{v\log v}{x}$

$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$

$\Rightarrow \log(\log v) = \log x + \log c$

$\Rightarrow \log(y/x) = cx$

$\Rightarrow y = xe^{cx}$

VECTORS

67. If $2i - j + k, i - 3j - 5k$ and $3i - 4j - 4k$ are position vectors of the vertices of a triangle then this triangle is

(A) equilateral (B) isosceles

(C) right angled isosceles (D) right angled

Ans. **(D)**

Sol. Let the given points be A, B, C respectively. Then

$\overrightarrow{AB} = -i - 2j - 6k, \overrightarrow{BC} = 2i - j + k, \overrightarrow{AC} = i - 3j - 5k$

$\therefore AB = |\overrightarrow{AB}| = \sqrt{1 + 4 + 36} = \sqrt{41}$

$\therefore C = |\overrightarrow{BC}| = \sqrt{4+1+1} = \sqrt{6}$

$AC = |\overrightarrow{AC}| = \sqrt{1+9+25} = \sqrt{35}$

$\because\ BC^2 + AC^2 = AB^2$

$\therefore$ A, B, C are vertices of a right-angled triangle.

68. If two adjacent sides of a parallelogram are represented by vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, then its area is

(A) $5\sqrt{6}$ (B) $6\sqrt{2}$ (C) $6\sqrt{5}$ (D) 180

Ans. **(C)**

Sol. If given vectors be a and b, then

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -4i - 10j + 8k$$

$\therefore$ area of parallelogram $= |a \times b|$

$= \sqrt{16+100+64} = 6\sqrt{5}$.

69. $a \cdot [(b+c) \times (a+b+c)]$ equals

(A) 0

(B) $[a\ \ b\ \ c] + [bc\, a\,]$

(C) $[a\ \ b\ \ c]$

(D) none of these

Ans. **(A)**

Sol. $\text{Exp.} = a \cdot \{(b+c) \times (a+b+c)\}$

$= a \cdot \{(b+c) \times a + (b+c) \times b + (b+c) \times c\}$

$= a \cdot \{(b \times a) + (c \times a) + (b \times b) + (c \times b) + (b \times c)\}$

$= a \cdot \{(b \times a) + (c \times a) + (c \times b) - (c \times b)\}$

$= a \cdot \{(b \times a) + (c \times a)\}$

$= [aba] + [aca] = 0 + 0 = 0.$

70. If a, b, c are coplanar unit vectors, then $[2a-b\ \ 2b-c\ \ 2c-a]$ is equal to

(A) 0 (B) 1 (C) $-\sqrt{3}$ (D) $\sqrt{3}$

Ans. **(A)**

Sol. $[2a-b\ \ 2b-c\ \ 2c-a]$

$= (2a-b) \cdot [(2b-c) \times (2c-a)]$

$= (2a-b) \cdot [4b \times c - 2b \times a - 2c \times c + c \times a]$

$= (2a-b) \cdot (4b \times c - 2b \times a + c \times a)$

$= 8[abc] - 4[aba] + 2[aca] - 4[bbc] + 2[bba] - [bca]$

$= 8[abc] - 0 + 0 - 0 + 0 - [abc]$

$= 7[abc] = 0.$ [$\because$ a, b, c are coplanar]

71. Let $a = 2i + j - 2k$ and $b = i + j$. If c is a vector such that $a \cdot c = |c|, |c - a| = 2\sqrt{2}$ and the angle between $a \times b$ and c is 30°, then $|(a \times b) \times c|$ is equal to

(A) 2/3 (B) 3/2 (C) 2 (D) 3

Ans. **(B)**

Sol. $a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2i - 2j + k$

$\Rightarrow |a \times b| = \sqrt{4+4+1} = 3$

Also, $|c - a| = 2\sqrt{2}$

$\Rightarrow |c - a|^2 = 8$

$\Rightarrow |c|^2 + |a|^2 - 2c \cdot a = 8$

$\Rightarrow |c|^2 + 9 - 2|c| = 8$ [$\because c \cdot a = |c|$]

$\Rightarrow (|c| - 1)^2 = 0$

$\Rightarrow |c| = 1$

Now

$|(a \times b) \times c| = |a \times b||c| \sin 30°$

$= (3)(1)(1/2)$ [by (1), (2)]

$= 3/2$

72. Let $u = i + j, v = i - j$ and $w = i + 2j + 3k$. If $\hat{n}$ is a unit vector such that $u \cdot \hat{n} = 0$ and $v \cdot \hat{n} = 0$, then $|w \cdot \hat{n}|$ is equal to

(A) 3 (B) 0 (C) 1 (D) 2

Ans. **(A)**

Sol. $\because \hat{n} \perp u$ and $\hat{n} \perp v \Rightarrow \hat{n} = \frac{u \times v}{|u \times v|}$

$\therefore |w \cdot \hat{n}| = \left|\frac{w \cdot (u \times v)}{|u \times v|}\right| = \frac{|w \cdot (u \times v)|}{|u \times v|}$ .....(1)

But $w \cdot (u \times v) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -6$

$\Rightarrow |w_0(u \times v)| = 6$ .....(2)

and $u \times v = (i + j) \times (i - j) = -2k \Rightarrow |u \times w| = 2$ .....(3)

Hence (1), (2), (3) $\Rightarrow |w \cdot \hat{n}| = 6/2 = 3$.

73. If C is the middle point of AB and P is any point outside AB, then

(A) $\overrightarrow{PA} + \overrightarrow{PB} = \overrightarrow{PC}$ (B) $\overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$

(C) $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \vec{0}$ (D) $\overrightarrow{PA} + \overrightarrow{PB} + 2\overrightarrow{PC} = \vec{0}$

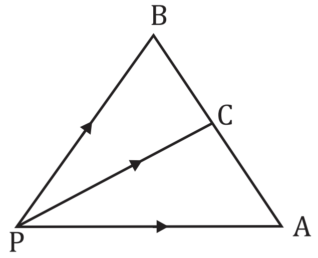
Ans. **(B)**

Sol. $\because \frac{AC}{CB} = \frac{1}{1}$

$\Rightarrow \overrightarrow{AC} = \overrightarrow{CB}$

$\Rightarrow \overrightarrow{AP} + \overrightarrow{PC} = \overrightarrow{CP} + \overrightarrow{PB}$

$\Rightarrow \overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$



3D-Geometry

75. If a line makes α, β, γ angles with coordinate axes, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to

(A) -2 (B) -1 (C) 1 (D) 2

Ans. **(B)**

Sol. Dc's of the line $= \cos\alpha, \cos\beta, \cos\gamma \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \frac{1}{2}(1 + \cos 2\alpha) + \frac{1}{2}(1 + \cos 2\beta) + \frac{1}{2}(1 + \cos 2\gamma) = 1$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

76. If dc's of two lines satisfy $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$, then angle between these lines is

(A) $2\pi/3$ (B) $\pi/6$ (C) $\pi/2$ (D) none of these

Ans. **(A)**

Sol. $l + m + n = 0$ .....(1)

$l^2 + m^2 - n^2 = 0$ .....(2)

Eliminating n, we have $l + m^2 - (-l - m)^2 = 0$

$\Rightarrow lm = 0 \Rightarrow l = 0$ or $m = 0$

When $l = 0$, then $l + 0 \cdot m + 0 \cdot n = 0$ .....(3)

$(1), (3) \Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{1}$

When $m = 0$, then $0 \cdot l + m + 0 \cdot n = 0$ .....(4)

$(1), (4) \Rightarrow \frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$

Hence if θ be the angle between given lines, then

$$\cos\theta = \frac{0(1) + (-1)0 + 1(-1)}{\sqrt{2}\sqrt{2}} = -\frac{1}{2}$$

$\Rightarrow \theta = 120^\circ$

77. If a line makes $\alpha, \beta, \gamma, \delta$ angles with four diagonats of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(A) 1 (B) 2/3 (C) 4/3 (D) none of these

Ans. **(C)**

Sol. In the diagram of Ex.15, OG, BF, AE and CD are four diagonals of the cube. Dr's of these diagonals are $1,1,1; 1,-1,1; -1,1,1$ and $1,1,-1$. If l, m, n be dc's of the given line, then

$\cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{l-m+n}{\sqrt{3}}$

$\cos\gamma = \frac{-l+m+n}{\sqrt{3}}, \cos\delta = \frac{l+m-n}{\sqrt{3}}$

$\Sigma\cos^2\alpha = \frac{1}{3}[4l^2 + 4m^2 + 4n^2] = \frac{4}{3}$

PLANE STRAIGHT LINE & SPHERE

78. Equation of the plane through point (4,2,4) and perpendicular to planes $2x + 5y + 4z + 1 = 0$ and $4x + 7y + 6z + 2 = 0$ is

(A) $x + 2y - 3z + 4 = 0$ (B) $x + 2y - 3z - 4 = 0$

(C) $x - 2y + 3z + 4 = 0$ (D) none of these

Ans. **(A)**

Sol. Let equation of the plane be

$a(x-4) + b(y-2) + c(z-4) = 0$ .....(1)

It is perpendicular to given two planes. So, we have

and

$2a + 5b + 4c = 0$ .....(2)

$4a + 7b + 6c = 0$ .....(3)

$(2), (3) \Rightarrow$

$\frac{a}{2} = \frac{b}{4} = \frac{c}{-6} \Rightarrow \frac{a}{1} = \frac{b}{2} = \frac{c}{-3}.$

Putting these proportional values of a, b, c in (1), required equation is

$x + 2y - 3z + 4 = 0.$

79. Equation of the plane passing through the point (4,3,7) and through the line $\frac{x-1}{5} = \frac{y+2}{6} = \frac{z-3}{4}$ will be

(A) $4x + 8y + 7z = 41$ (B) $4x - 8y + 7z = 41$

(C) $4x - 8y - 7z = 41$ (D) $4x - 8y + 7z = 39$

Ans. **(B)**

Sol. Let equation of the plane be

$a(x-1)+b(y+2)+c(z-3)=0$

$5a+6b+4c=0$

where

$\because$ it passes through point (4,3,7), so

$3a+5b+4c=0$

(2), (3) $\Rightarrow$ $\frac{a}{4}=\frac{b}{-8}=\frac{c}{7}$.

Putting these proportional values of a, b, c in (1), equation of the required plane will be

$4(x-1)-8(y+2)+7(z-3)=0$

$\Rightarrow$ $4x-8y+7z=41$.

80. Equation of the plane containing lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ is

(A) $17x-47y-24z+172=0$ (B) $17x+47y-24z+172=0$

(C) $17x+47y+24z+172=0$ (D) $17x-47y+24z+172=0$

Ans. **(A)**

Sol. Equation of the required plane is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$

$\Rightarrow (x-5)(12+5)+(y-7)(-35-12)+(z+3)(4-28)=0$

$\Rightarrow 17x-47y-24z+172=0$

81. The distance between point $(-1,-5,-10)$ and the point of intersection of line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$ is

(A) 10 (B) 8 (C) 21 (D) 13

Ans. **(D)**

Sol. Let point of intersection of the line and plane be

$(3r+2, 4r-1, 12r+2)$

It lies in the plane, so

$3r+2-(4r-1)+12r+2=5 \Rightarrow r=0$.

$\Rightarrow$ point of intersection $=(2,-1,2)$

$\therefore$ required distance $=\sqrt{9+16+144}=13$

82. The vector equation of the plane $r=(1+s-t)i+(2-s)j+(3-2s+2t)k$ in scalar product form is

(A) $r\cdot(i+2j)=5$ (C) $r\cdot(2i+k)=5$

(B) $r\cdot(i+2k)=5$ (D) none of these

Ans. **(C)**

Sol. Equation of the given plane is

$r = (1+s-t)i + (2-s)j + (3-2s+2t)k$

$\Rightarrow r = (i+2j+3k) + s(i-j-2k) + t(-i+2k)$

$\Rightarrow r = a + sb + tc$ (say).

So, given plane passes through the point a and parallel to vectors b and c. As such $b \times c$ will be vector perpendicular to this plane. But

$$b \times c = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 0 & 2 \end{vmatrix} = -2i - k$$

Hence equation of the given plane in scalar product form is given by

$(r-a)\cdot(b\times c) = 0 \Rightarrow \{r-(i+2j+3k)\}\cdot(-2i-k) = 0$

$\Rightarrow r\cdot(2i+k) = 5$

83. Two lines

$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect at a point, then k is equal to

(A) 3/2 (B) 2 (C) 9/2 (D) 2/9

Ans. **(C)**

Sol. Lines intersect at a point ⇒ lines are coplanar. So, using condition of coplanarity, we have

$$\begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(3-8) + (k+1)(4-2) - 1(4-3) = 0$

$\Rightarrow -10 + 2k + 2 - 1 = 0$

$\Rightarrow k = 9/2.$

PROBABILITY

84. A coin is tossed 4 times. The probability of showing tail at least once will be

(A) 15/16 (B) 1/16 (C) 1/4 (D) ¾

Ans. **(A)**

Sol. Here total number of exhaustive cases $= 2^4 = 16$.

Favourable cases for showing tail atleast once

$= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

$\therefore$ reqd. prob. $= 15/16$.

Aliter. Use binomial distribution: $n = 4, p = 1/2, q = 1/2$.

probability $= \sum_{r=1}^{4} {}^nC_r q^{n-r} p^r = \frac{15}{16}$

85. If A, B, C can hit a target 4 times in 5 shots, 3 times in 4 shots and 2 times in 3 shots respectively, then the probability that exactly two of them will hit the target is

(A) 13/30 (B) 5/6 (C) 17/30 (D) none of these

Ans. **(A)**

Sol. The given event has following three disjoint cases:

$AB\bar{C}, A\bar{B}C, \bar{A}BC$.

Since A, B, C are independent events, so

$$P(AB\bar{C}) = P(A)\,P(B)P(\bar{C}) = \frac{4}{5}\cdot\frac{3}{4}\cdot\left(1-\frac{2}{3}\right) = \frac{12}{60}.$$

$$\text{Similarly } P(A\bar{B}C) = \frac{4}{5}\cdot\frac{1}{4}\cdot\frac{2}{3} = \frac{8}{60}$$

$$P(\bar{A}BC) = \frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} = \frac{6}{60}$$

$\therefore$ reqd. prob. $= P(AB\bar{C}) + P(A\bar{B}C) + P(\bar{A}BC)$

$= \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{26}{60} = \frac{13}{30}.$

86. A speaks truth in 75% cases and B in 80% cases. What is the probability that they contradict each other in stating the same fact?

(A) 7/20 (B) 13/20 (C) 3/20 (D) 1/5

Ans. **(A)**

Sol. There are two mutually exclusive cases in which they contradict each other i.e., $\bar{A}B$ and $A\bar{B}$.

Hence

reqd. prob. $= P(A\bar{B} + \bar{A}B) = P(A\bar{B}) + P(\bar{A}B)$

$= P(A)P(\bar{B}) + P(\bar{A})P(B)$

$= \frac{3}{4}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{4}{5} = \frac{7}{20}.$

87. The probability that a leap year has 53 Sundays is

(A) 2/7 (B) 3/5 (C) 2/3 (D) 1/7

Ans. **(A)**

Sol. There are 366 days in a leap year. As such there will be 52 weeks +2 remaining days. Hence the required probability is the probability of coming Sunday in these two remaining days. Now remaining two days may occur in the following ways:

MT, TW, WTh, ThF, FS, SSu, SuM.

Two of them are favourable. So

$\therefore$ reqd. prob. $= 2/7$.

88. For three events A, B and C, if

P (Happening of exactly A or B) $= p$

P (Happening of exactly B or C) $= p$

P (Happening of exactly C or A) $= p$

P (happening A, B, C together) $= p^2$,

where $0 < p < 1/2$; then probability of happening of atleast one of A, B, C is

(A) $\frac{3p+2p^2}{2}$ (B) $\frac{p+3p^2}{4}$

(C) $\frac{p+3p^2}{2}$ (D) $\frac{3p+2p^2}{4}$

Ans. (A)

Sol.

P (happening of exactly A or B) $= P(A) + P(B) - 2P(AB)$

$\Rightarrow p = P(A) + P(B) - 2P(AB)$ .....(1)

Similarly, $p = P(B) + P(C) - 2P(BC)$ .....(2)

$p = P(C) + P(A) - 2P(CA)$ .....(3)

$(1) + (2) + (3) \Rightarrow P(A) + P(B) + P(C) - P(AB)$

$-P(BC) - P(CA) = 3p/2$ .....(4)

Also as given $P(ABC) = p^2$ .....(5)

$\therefore$ reqd. prob. $= P(A + B + C)$

$= P(A) + P(B) + P(C) + P(ABC) - P(AB) - P(BC) - P(CA)$

$= (3p/2) + p^2 = (3p + 2p^2)/2.$

89. Two dice are thrown together 4 times. The probability that both dice will show same numbers twice is

(A) 1/3 (B) 25/36

(C) 25/216 (D) none of these

Ans. (C)

Sol. The probability of showing same number by both dice

$p = 6/36 = 1/6.$

In binomial distribution here $n = 4, r = 2, p = 1/6, q = 5/6.$

$\therefore$ reqd. prob. $= {}^nC_r q^{n-r} p^r = {}^4C_2 (5/6)^2 (1/6)^2$

$= 6\left(\frac{25}{36}\right)\left(\frac{1}{36}\right) = \frac{25}{216}$

90. The probability distribution of a variate X is as follows:

X:	1	2	3	4	5	6	7	8
P(X):	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

If $E = \{X \text{ is a prime number }\}$ and $F = \{X < 4\}$, then $P(E \cup F)$ is equal to

(A) 0.35 (B) 0.77 (C) 0.87 (D) 0.50

Ans. **(B)**

Sol. $P(E) = P(2) + P(3) + P(5) + P(7)$

$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$

and

$P(F) = P(1) + P(2) + P(3)$

$= 0.15 + 0.23 + 0.12 = 0.50$

Also

$P(E \cap F) = 0.23 + 0.12 = 0.35$

$\therefore\ P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.62 + 0.50 - 0.35 = 0.77.$

MATHEMATICAL REASONING

91. Which one of the following statements is a tautology?

(A) $p \wedge q \equiv p \vee q$

(B) $(p \wedge q) \vee r \Leftrightarrow (p \vee q) \wedge r$

(C) $p \Rightarrow q \Leftrightarrow (\sim q) \Rightarrow (\sim p)$

(D) none of these

Ans. **(C)**

Sol. Truth table of the given statements are as follows:

(A)

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

(B)

p	q	r	$p \wedge q$	$p \vee q$	$(p \wedge q) \vee r$	$(p \vee q) \wedge r$	$(p \wedge q) \vee r \Leftrightarrow (p \vee q) \wedge r$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	T

(C)

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$(\sim q) \Rightarrow (\sim p)$	$p \Rightarrow q \Leftrightarrow (\sim q) \Rightarrow (\sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Observing to the above tables it is clear that statement (3) is a tautology.

92. The contrapositive statement of the statement $(\sim p \wedge q) \Rightarrow (q \wedge \sim r)$ is

(A) $(\sim p \vee r) \Rightarrow (\sim p \wedge \sim r)$ (B) $(\sim q \vee r) \Rightarrow (\sim p \vee q)$

(C) $(p \vee \sim q) \Rightarrow (\sim q \vee p)$ (D) $(\sim q \vee r) \Rightarrow (p \vee \sim q)$

Ans. **(D)**

Sol. The contrapositive statement of the given statement is

$\sim (q \wedge \sim r) \Rightarrow \sim (\sim p \wedge q)$

or $\sim q \vee \sim (\sim r) \Rightarrow \sim (\sim p) \vee (\sim q)$ [by De Morgan law]

or $\sim q \vee r \Rightarrow p \vee (\sim q)$ [by negation law]

93. The statement equivalent to $(\sim p \wedge q) \vee (\sim q)$ is

(A) $\sim (p \vee q)$ (B) $\sim (p \wedge q)$ (C) $p \vee q$ (D) $p \wedge q$

Ans. **(B)**

Sol. $(\sim p \wedge q) \vee (\sim q)$

$\equiv (\sim q) \vee (\sim p \wedge q)$ [by commutative law]

$\equiv (\sim q \vee \sim p) \wedge (\sim q \vee q)$ [by distributive law]

$\equiv \sim (q \wedge p) \wedge t$ [De Morgan law]

$\equiv \sim (q \wedge p)$ [by identity law]

$\equiv \sim (p \wedge q)$ [by commutative law]

94. The negation of $p \rightarrow (\sim p \vee q)$ is

(A) $p \vee (p \vee \sim q)$ (B) $p \rightarrow q$

(C) $p \wedge \sim q$ (D) $p \rightarrow \sim (p \vee q)$

Ans. **(C)**

Sol. $\because$ Negation of $p \rightarrow q$ is $p \wedge \sim q$,

So, negation of given statement is

$= p \wedge \sim (\sim p \vee q)$

$= p \wedge (p \wedge \sim q)$ $[\because \sim (p \vee q) = (\sim p) \wedge (\sim q)]$

$= (p \wedge p) \wedge (\sim q)$ [by associativity]

$= p \wedge \sim q$.

SETS & RELATIONS

95. If $A = \{0, \phi, \{\phi\}\}$, then

(A) $\phi \in P(A)$ (B) $\{\phi\} \in P(A)$

(C) $\{\{\phi\}\} \in P(A)$ (D) $0 \in P(A)$

Ans. **(A,B,C)**

Sol. Since subsets of A are as follows:

$A, \phi, \{0\}, \{\phi\}, \{\{\phi\}\}, \{0, \phi\}, \{0, \{\phi\}\}, \{\phi, \{\phi\}\}$.

Hence $\phi \in P(A), \{\phi\} \in P(A)$ and $\{\{\phi\}\} \in P(A)$.

So (1), (2), (3) are correct.

96. Two finite sets have m and n elements. If total number of subsets of first set is 56 more than that of second set, then (m, n) is equal to

(A) (7,6) (B) (6,3)

(C) (5,1) (D) (8,7)

Ans. **(B)**

Sol. As given

$2^m - 2^n = 56 \Rightarrow 2^n(2^{m-n} - 1) = 2^3(2^3 - 1)$

$\Rightarrow n = 3,\ m - n = 3$

$m = 6, n = 3$

97. Let R be a relation defined as $R = \{(a,b) \mid a \le b\}$ where a, b are real numbers. Then relation R is

(A) reflexive, symmetric and transitive.

(B) reflexive and transitive but not symmetric.

(C) symmetric and transitive but not reflexive.

(D) symmetric but neither reflexive nor transitive.

Ans. **(B)**

Sol. $\because$ For every real number $a, a = a$

$\Rightarrow (a,a) \in R \ \therefore$ R is reflexive.

Further if a, b are real numbers such that $(a,b) \in R$, then

$(a,b) \in R \not\Rightarrow a \le b \Rightarrow b \le a\ [\because a < b \Rightarrow b > a]$

$\therefore$ R is not symmetric.

Also if a, b, c are real numbers such that

$(a,b) \in R$ and $(b,c) \in R$, then

$a \le b$ and $b \le c \Rightarrow a \le c \Rightarrow (a,c) \in R$

$\therefore$ R is transitive.

Hence (2) is correct.

98. Let R be a relation defined on N as follows:

$'\, xRy \Leftrightarrow x + 2y = 10\, '$

Then range of R is

(A) N (B) $\{1,2,3,4\}$

(C) $\{2,3,4,6\}$ (D) $\{10,12,14,\ldots\}$

Ans. **(B)**

Sol. $R = \left\{\left(x, \frac{10-x}{2}\right) \mid x \in N, \frac{10-x}{2} \in N\right\}$

$= \{(2,4),(4,3),(6,2),(8,1)\}$

$\therefore$ range of $R = \{1,2,3,4\}$.

99. Let R be the real line. Consider the following subsets of the plane $R \times R$:

$S = \{(x,y): y = x + 1 \text{ and } 0 < x < 2\}$

$T = \{(x,y): x - y \text{ is an integer}\}$

Which one of the following is true?

(A) Both S and T are equivalence relations on R.

(B) S is an equivalence relation on R but T is not.

(C) T is an equivalence relation on R but S is not.

(D) Neither S nor T is an equivalence relation on R

Ans. **(C)**

Sol. S is not reflexive because $xSx \Rightarrow x = x + 1$ which is not possible for $0 < x < 2$.

$\therefore$ S is not an equivalence relation.

Further we note that for any $x \in R$

$x - x = 0$ (integer) $\Rightarrow xTx$

$\therefore$ T is reflexive.

If xTy, then $xTy \Rightarrow (x - y)$ is an integer

$\Rightarrow (y - x)$ is an integer

$\Rightarrow yTx$

$\therefore$ T is symmetric.

Also if xTy and yTz, then

xTy and yTz $\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow (x - y) - (y - z)$ is an integer

$\Rightarrow (x - z)$ is an integer

$\Rightarrow xTz$

$\therefore$ T is transitive.

Hence T is an equivalence relation. So, T is an equivalence relation but S is not.

DETERMINANTS

100. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative when

(A) a, b, c are positive

(B) a, b, c are negative

(C) a, b, c are positive and unequal

(D) never

Ans. **(C)**

Sol. Det. $= 3abc - a^3 - b^3 - c^3$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Now since $a^2 + b^2 + c^2 > ab + bc + ca$ when $a \neq b \neq c$.

$\therefore$ determinant will be negative when a, b, c are positive and unequal.

101. If x, y, z are positive numbers, then value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is equal to

(A) 0 (B) 3 (C) logxyz (D) none of these

Ans. (A)

Sol. $$\Delta = \begin{vmatrix} 1 & \log y/\log x & \log z/\log x \\ \log x/\log y & 1 & \log z/\log y \\ \log x/\log z & \log y/\log z & 1 \end{vmatrix}$$

$$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

[on multiplying R_1, R_2, R_3 by logx, logy, logz respectively]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

102. If for a fixed integer n, $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$, then $\left(\frac{\Delta}{(n!)^3} - 4\right)$ is divisible by

(A) $(n+1)$ (B) n (C) $(n+2)$ (D) none of these

Ans. (B)

Sol. $$\frac{\Delta}{(n!)^3} = \begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ n+1 & (n+2)(n+1) & (n+3)(n+2)(n+1) \\ (n+1)(n+2) & (n+3)(n+2)(n+1) & (n+4)(n+3)(n+2)(n+1) \end{vmatrix}$$

$$= (n+1)^2(n+2)\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & n+3 \\ n^2+3n+2 & n^2+5n+6 & n^2+7n+12 \end{vmatrix}$$

$$= (n+1)^2(n+2)\begin{vmatrix} 1 & 0 & 0 \\ n+1 & 1 & 1 \\ n^2+3n+2 & 2(n+2) & 2(n+3) \end{vmatrix}$$

$$= 2(n^3 + 4n^2 + 5n + 2)$$

$$\therefore \frac{\Delta}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

103. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$, then

(A) $\Delta_1 = 3\Delta_2{}^2$ (B) $\frac{d}{dx}(\Delta_1) = 3\Delta_2^2$

(C) $\frac{d}{dx}(\Delta_1) = 3\Delta_2$ (D) none of these

Ans. (C)

Sol. $\frac{d}{dx}(\Delta) = \begin{vmatrix} \frac{d}{dx}(R_1) \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ \frac{d}{dx}(R_2) \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ \frac{d}{dx}(R_3) \end{vmatrix}$

$\therefore \frac{d}{dx}(\Delta_1) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(b) & \frac{d}{dx}(b) \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ \frac{d}{dx}(a) & \frac{d}{dx}(x) & \frac{d}{dx}(b) \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ \frac{d}{dx}(a) & \frac{d}{dx}(a) & \frac{d}{dx}(x) \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\begin{vmatrix} x & b \\ a & x \end{vmatrix}$

$= 3\triangle_2$

104. If system of equations $x + 4ay + az = 0, x + 3by + bz = 0, x + 2cy + cz = 0$ has a ion-zero solution, then a, b, c are in

(A) AP (B) GP (C) HP (D) none of these

Ans. **(C)**

Sol. Given equations are homogeneous equations, so their system will have a non-zero solution, if $\Delta = 0$

$\Rightarrow \begin{vmatrix} 1 & 4a & a \\ 1 & 3b & b \\ 1 & 2c & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2a & a \\ 1 & b & b \\ 1 & 0 & c \end{vmatrix} = 0$ [by $C_2 - 2C_3$]

$\Rightarrow (bc + 2ab) - (ab + 2ac) = 0$

$\Rightarrow bc + ab = 2ac$

$\Rightarrow \frac{1}{a} + \frac{1}{c} = \frac{2}{b} \Rightarrow$ a, b, c are in HP.

105. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

Statement I: The system of equations has no solution for $k \neq 3$,

and

Statement II: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$ for $k \neq 3$.

In the following correct answer is

(A) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I.

(B) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(C) Statement I is true, Statement II is false.

(D) Statement I is false, Statement II is true.

Ans. **(A)**

Sol. For the given system

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4+9+4)-(3+6+8) = 0$$

$$\Delta_1 = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 3-k \Rightarrow \Delta_1 \neq 0 \text{ if } k \neq 3$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = k-3 \Rightarrow \Delta_2 \neq 0 \text{ if } k \neq 3$$

$$\Delta_3 = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & k \\ 1 & -3 & 1 \end{vmatrix} = k-3 \Rightarrow \Delta_3 \neq 0 \text{ if } k \neq 3$$

But system of equations has no solution when

$\Delta = 0$ and $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ or $\Delta_3 \neq 0$.

$\therefore$ system has no solution for $k \neq 3$.

So statement I is true.

Further $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} = 3-k$, which is not zero for $k \neq 3$.

So statement II is true and it is a correct explanation for I

106. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that and

$x = cy + bz$

$y = az + cx$

$z = bx + ay$

Then $a^2 + b^2 + c^2 + 2abc$ is equal to

(A) -1 (B) 0 (C) 1 (D) 2

Ans. **(C)**

Sol. Given equations are :

$-x + cy + bz = 0$

$cx - y + az = 0$

$bx + ay - z = 0$

$\because$ x, y, z $\in$ **R** and non-zero, so these equations have a solution

$\Rightarrow \Delta = 0$

$$\Rightarrow \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

MATRICES

107. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $A^2 - 4A - nI = 0$, then n is equal to

(A) 3 (B) -3

(C) 1/3 (D) $-1/3$

Ans. **(B)**

Sol. $A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, 4A = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix}, nI = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$

$\Rightarrow A^2 - 4A - nI = \begin{bmatrix} 5-8-n & -4+4-0 \\ -4+4-0 & 5-8-n \end{bmatrix} = \begin{bmatrix} -3-n & 0 \\ 0 & -3-n \end{bmatrix}$

$\therefore A^2 - 4A - nI = 0 \Rightarrow \begin{bmatrix} -3-n & 0 \\ 0 & -3-n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow -3 - n = 0 \Rightarrow n = -3$

108. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then $|A||\text{adj}A|$ is equal to

(A) a^3 (B) a^6

(C) a^9 (D) a^{27}

Ans. **(C)**

Sol. $\because |A| = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^3$ and $|\text{adj}A| = |A|^2 = a^6$

$\therefore |A||\text{adj}A| = a^3 \cdot a^6 = a^9$.

109. If A is a square matrix and B is a nonsingular matrix of the same order, then $\det(B^{-1}AB)$ equals

(A) $\det(A)$ (B) $\det(B)$

(C) $\det(A^{-1})$ (D) $\det(B^{-1})$

Ans. **(A)**

Sol. $\det(B^{-1}AB) = |B^{-1}||A||B| = |B^{-1}||B||A|$

$= \frac{1}{|B|}|B||A| = |A| = \det(A)$.

110. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then α is equal to

(A) 0 (B) ± 5

(C) ± 2 (D) ± 3

Ans. **(D)**

Sol. $|A^3| = 125 \Rightarrow |A|^3 = 125 \Rightarrow |A| = 5$

$\Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$.

111. If $P = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q = PAP^T$, $X = P^TQ^{2005}P$, then X is equal to

(A) $\begin{pmatrix} 1 & 2005 \\ 0 & 1 \end{pmatrix}$

(B) $\frac{1}{4}\begin{pmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 005\sqrt{3} \end{pmatrix}$

(C) $\frac{1}{4}\begin{pmatrix} 2005 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2005 \end{pmatrix}$

(D) none of these

Ans. **(A)**

Sol. $\because PP^T = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ P is orthogonal matrix.

$\Rightarrow P^TQ^{2005}P = Q^{2005}$

But $Q^{2005} = (PAP^T)^{2005} = A^{2005}$ [$\because$ P is orthogonal]

$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{2005} = \begin{pmatrix} 1 & 2005 \\ 0 & 1 \end{pmatrix}$

112. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

(A) 5 (B) 0 (C) 4 (D) 11

Ans. **(D)**

Sol. $\because |\text{adj}A| = |A|^{3-1}$

$\Rightarrow |P| = \begin{vmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{vmatrix} = 4^2.$

$\Rightarrow (12 + 12 + 6\alpha) - (18 + 12 + 4\alpha) = 16$

$\Rightarrow 2\alpha = 22$

$\therefore \alpha = 11$

113. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5?

(A) 198 (B) 126 (C) 135 (D) 162

Ans. **(A)**

Sol. Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

$tr(M^TM) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 5$

It is possible in two ways :

Case 1: When five entries are 1 and remaining four 0 .

Such total matrices $= {}^9C_5 \cdot 1 = 126$

Case II : When one entry is 2 , one entry is 1 and remaining 0 .

Such total matrices $= {}^9C_2 \cdot \lfloor 2 \cdot 1 = 72$

$\therefore$ Total number of required $M = 126 + 72 = 198$

MEASURES OF CENTRAL TENDENCY AND DISPERSION

114. If the mean of the series $x_1, x_2, \ldots, x_n$ is $\bar{x}$, then the mean of the series $x_i + 2i, i = 1,2, \ldots, n$ will be

(A) $\bar{x} + n$ (B) $\bar{x} + n + 1$ (C) $\bar{x} + 2$ (D) $\bar{x} + 2n$

Ans. **(B)**

Sol. As given $\bar{x} = \frac{x_1+x_2+\cdots+x_n}{n}$

If the mean of the series $x_i + 2i, i = 1,2, \ldots, n$ be $\bar{X}$, then

$$\bar{X} = \frac{(x_1 + 2) + (x_2 + 2 \cdot 2) + (x_3 + 2 \cdot 3) + \cdots + (x_n + 2 \cdot n)}{n}$$

$$= \frac{x_1 + x_2 + \cdots + x_n}{n} + \frac{2(1 + 2 + 3 + \cdots + n)}{n} = \bar{x} + \frac{2n(n+1)}{2n} = \bar{x} + n + 1$$

115. The variance of a distribution is σ^2. If each value of the distribution is increased by λ, then the variance of the new distribution is

(A) $\lambda^2\sigma^2$ (B) $\lambda^2 + \sigma^2$ (C) $\lambda + \sigma^2$ (D) σ^2

Ans. **(D)**

Sol. If $\bar{x}$ be the mean of the distribution, then $\bar{x} = \frac{\Sigma f_i x_i}{N}$

Let $u_i = x_i + \lambda$, then

$$\bar{u} = \frac{\sum f_i u_i}{N} = \frac{\sum f_i (x_i + \lambda)}{N}$$

$$= \frac{\sum f_i x_i}{N} + \frac{\lambda \Sigma f_i}{N} = \bar{x} + \lambda \qquad \ldots\ldots\ldots(1)$$

If σ_u^2 be the variance of the new distribution, then

$$\sigma_u^2 = \frac{\sum f_i (u_i - \bar{u})^2}{N}$$

$$= \frac{\sum f_i (x_i + \lambda - \bar{x} - \lambda)^2}{N} \text{ [from (1)]}$$

$$= \frac{\sum f_i (x_i - \bar{x})^2}{N} = \sigma^2$$

116. If frequencies of the values $0,1,2, \ldots, n$ of a variate are proportional to ${}^nc_0, {}^nc_1, {}^nc_2, \ldots, {}^nc_n$, then Var(X) is equal to

(A) $\frac{(n^2-1)}{12}$ (B) $\frac{n}{2}$ (C) $n/4$ (D) none of these

Ans. **(C)**

Sol. $\bar{x} = \frac{\sum_{r=0}^{n} r \cdot {}^{n}c_r}{\sum_{r=0}^{n} {}^{n}c_r} = \frac{n \cdot 2^{n-1}}{2^n} = \frac{n}{2}$

Also $\frac{1}{N}\sum f_i x_i^2 = \frac{1}{2^n}\sum_{r=0}^{n} r^2 \cdot {}^{n}c_r = \frac{1}{2^n}\sum_{r=0}^{n} [r(r-1)+r]^{n}c_r$

$= \frac{1}{2^n}[n(n+1)\cdot 2^{n-2}] = \frac{n(n+1)}{4}$

[$\because \Sigma r^2 c_r = n(n+1)\cdot 2^{n-2}$ See Binomial Theorem Chapter]

$\therefore \text{Var}(X) = \frac{1}{N}\sum f_i x_i^2 - \bar{x}^2$

$= \frac{n(n+1)}{4} - \frac{n^2}{4} = \frac{n}{4}$

117. The mean and variance of 5 observations of an experiment are 4 and 5.2 respectively. If from these observations three are 1,2 and 6, then the remaining will be

(A) 2,9 (B) 5,6 (C) 4,7 (D) 3,8

Ans. **(C)**

Sol. As given $\bar{x} = 4, n = 5$ and $\sigma^2 = 5 \cdot 2$. If the remaining observations are x_1, x_2, then $\sigma^2 = 5.2$

$\Rightarrow \frac{\Sigma|x_i - \bar{x}|^2}{n} = 5.2$

$\Rightarrow \frac{(x_1-4)^2 + (x_2-4)^2 + (1-4)^2 + (2-4)^2 + (6-4)^2}{5} = 5.2$ .... (1)

$\Rightarrow (x_1-4)^2 + (x_2-4)^2 = 9$

Also $\bar{x} = 4 \Rightarrow \frac{x_1+x_2+1+2+6}{5} = 4$

$\Rightarrow x_1 + x_2 = 11$ ......(2)

$\Rightarrow x_1, x_2 = 4,7$

118. If $a > 0$, then the minimum sum of the real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} will be

(A) 7 (B) 8 (C) 9 (D) 10

Ans. **(B)**

Sol. For eight +ve real numbers $a^{-5}, a^{-4}, a^{-3}, a^{-3}, a^{-3}, a^8, a^{10}, 1$, using

AM $\geq$ GM

$\Rightarrow \frac{a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1}{8} \geq (a^{-5}\cdot a^{-4}\cdot a^{-3}\cdot a^{-3}\cdot a^{-3}\cdot a^8\cdot a^{10}\cdot 1)^{1/8} \geq 1$

$\Rightarrow a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1 \geq 8$

$\therefore$ The minimum sum of the given numbers = 8.