1. MEANING OF DERIVATIVE :

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let ' f ' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of ' f ' then

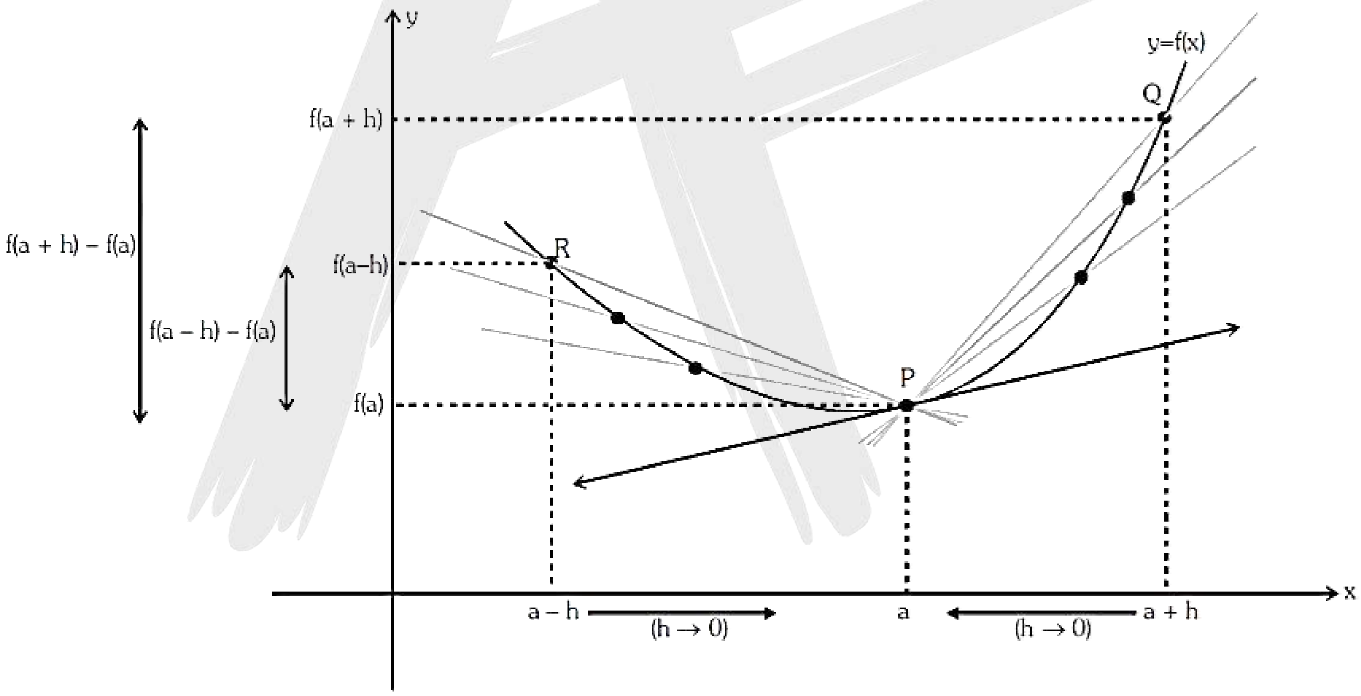
$$\Delta f = f(x+\Delta x) - f(x)$$
$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of ' f ' at the point x. The derivative of a function ' f ' is a function; this function is denoted by symbols such as

$$f'(x), \frac{df}{dx}, \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$
$$\Rightarrow \frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a), \left.\frac{df(x)}{dx}\right|_{x=a}, f'(x)_{x=a}$, etc.

2. EXISTENCE OF DERIVATIVE AT x = a :



(a) Right hand derivative :

The right hand derivative of f(x) at $x = a$ denoted by $f'(a^+)$ is defined as : $f'(a^+) = \text{Lim}_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists and is finite. $(h > 0)$

(b) Left hand derivative :

The left hand derivative of f(x) at $x = a$ denoted by $f'(a^-)$ is defined as :

$f'(a^-) = \text{Lim}_{h\to 0}\frac{f(a-h)-f(a)}{-h}$, provided the limit exists and is finite. $(h > 0)$

Hence f(x) is said to be derivable or differentiable at $x = a$. If $f'(a^+) = f'(a^-)$ =finite quantity and it is denoted by $f'(a)$; where $f'(a) = f'(a^-) = f'(a^+)$and it is called derivative or differential coefficient of f(x) at $x = a$.

3. DIFFERENTIABILITY AND CONTINUITY :

Theorem : If a function f(x) is derivable at $x = a$, then f(x) is continuous at $x = a$.

Proof : $f'(a) = \text{Lim}_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

Also $f(a+h) - f(a) = \frac{f(a+h)-f(a)}{h} \cdot h[h \neq 0]$

$\therefore \text{Lim}_{h\to 0} [f(a+h) - f(a)] = \text{Lim}_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a) \cdot 0 = 0$

$\Rightarrow \text{Lim}_{h\to 0} [f(a+h) - f(a)] = 0 \Rightarrow \text{Lim}_{h\to 0}\ f(a+h) = f(a) \Rightarrow f(x)$ is continuous at $x = a$.

Note :

(i) Differentiable ⇒ Continuous ; Continuity ⇒ Differentiable ; Not Differentiable ⇒ Not Continuous But Not Continuous ⇒ Not Differentiable

(ii) All polynomial, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If f(x) and g(x) are differentiable at $x = a$ then the function $f(x) + g(x), f(x) - g(x)$, $f(x) \cdot g(x)$ will also be differentiable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be differentiable at $x = a$.

Illustration 1 : Let $f(x) = \begin{cases} \text{sgn}\ (x) + x; & -\infty < x < 0 \\ -1 + \sin\ x; & 0 \le x < \frac{\pi}{2} \\ \cos\ x; & \frac{\pi}{2} \le x < \infty \end{cases}$.

Discuss the continuity and differentiability at $x = 0$ and $\frac{\pi}{2}$.

Solution : $f(x) = \begin{cases} -1 + x & ; -\infty < x < 0 \\ -1 + \sin\ x & ; 0 \le x < \frac{\pi}{2} \\ \cos\ x & ;\ \frac{\pi}{2} \le x < \infty \end{cases}$

To check the differentiability at $x = 0$

$\text{LHD} = \lim\limits_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0} \frac{-1+0-h-(-1)}{-h} = 1$

$\text{RHD} = \lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{-1+\sin h+1}{h} = 1$

$\because$ LHD = RHD

$\therefore$ Differentiable at $x = 0$.

$\Rightarrow$ Continuous at $x = 0$.

To check the continuity at $x = \frac{\pi}{2}$

LHL $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}(-1+\sin\ x) = 0$

RHL $\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\cos\ x = 0$

$\because$ LHL = RHL $= f\left(\frac{\pi}{2}\right) = 0 \therefore$ Continuous at $x = \frac{\pi}{2}$.

To check the differentiability at $x = \frac{\pi}{2}$

$$\text{LHD} = \lim_{h\to0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to0}\frac{-1+\cos h-0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to0}\frac{-\sin h-0}{h} = -1$$

$\because$ LHD $\neq$ RHD $\therefore$ not differentiable at $x = \frac{\pi}{2}$.

Illustration 2: If $f(x) = \begin{cases} A+Bx^2 & ; x<1 \\ 3Ax-B+2 & ; x\geq 1 \end{cases}$

then find A and B so that f(x) become differentiable at x = 1.

Solution : $f'(1^+) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{3A(1+h)-B+2-3A+B-2}{h} = \lim_{h\to0}\frac{3Ah}{h} = 3A$

$f'(1^-) = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{A+B(1-h)^2-3A+B-2}{-h} = \lim_{h\to0}\frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$

hence for this limit to be defined

$-2A+2B-2=0$

$B = A+1$

$f'(1^-) = \lim_{h\to0} -(Bh-2B) = 2B$ $\qquad\therefore f'(1^-) = f'(1^+)$

$3A = 2B = 2(A+1)$

$A = 2, B = 3$

Illustration 3: $f(x) = \begin{cases} [\cos\ \pi x] & x\leq 1 \\ 2\{x\}-1 & x>1 \end{cases}$ comment on the derivability at x = 1, where [ ] denotes greatest integer function and {} denotes fractional part function.

Solution : $f'(1^-) = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{[\cos(\pi-\pi h)]+1}{-h} = \lim_{h\to0}\frac{-1+1}{-h} = 0$

$f'(1^+) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{2\{1+h\}-1+1}{h} = \lim_{h\to0}\frac{2h}{h} = 2$

Hence f(x) is not differentiable at x = 1.

Do yourself - 1: (i) A function is defined as follows :

(ii) If $f(x) = \begin{cases} ax^3+b, & \text{for } 0\leq x\leq 1 \\ 2\cos\ \pi x+\tan^{-1}\ x, & \text{for } 1<x\leq 2 \end{cases}$ be the differentiable function in [0,2], then find a and b. (where [.] denotes the greatest integer function)

4. IMPORTANT NOTE :

(a) Let $f'(a^+) = p$ and $f'(a^-) = q$ where p and q are finite then :

(i) $p = q \Rightarrow$ f is differentiable at $x = a \Rightarrow$ f is continuous at $x = a$

(ii) $p \neq q \Rightarrow$ f is not differentiable at $x = a$, but f is continuous at $x = a$.

Illustration 4 : Determine the values of x for which the following functions fails to be continuous or differentiable $f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \leq x \leq 2 \\ (3-x), & x > 2 \end{cases}$, Justify your answer.

Solution: By the given definition it is clear that the function f is continuous and differentiable at all points except possibly at $x = 1$ and $x = 2$.

Check the differentiability at $x = 1$

$$q = \text{LHD} = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{1-(1-h)-0}{-h} = -1$$

$$p = \text{RHD} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

$\because q = p \therefore$ Differentiable at $x = 1. \Rightarrow$ Continuous at $x = 1$.

Check the differentiability at $x = 2$

$$q = \text{LHD} = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h\to 0} \frac{(1-2+h)(2-2+h)-0}{-h} = 1 = \text{ finite}$$

$$p = \text{RHD} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(3-2-h)-0}{h} \to \infty \text{ (not finite)}$$

$\because q \neq p \therefore$ not differentiable at $x = 2$.

Now we have to check the continuity at $x = 2$

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3-x) = \lim_{h\to 0}(3-(2+h)) = 1$$

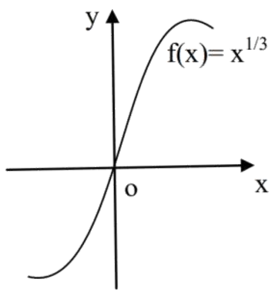
$\therefore$ LHL $\neq$ RHL

$\Rightarrow$ not continuous at $x = 2$.

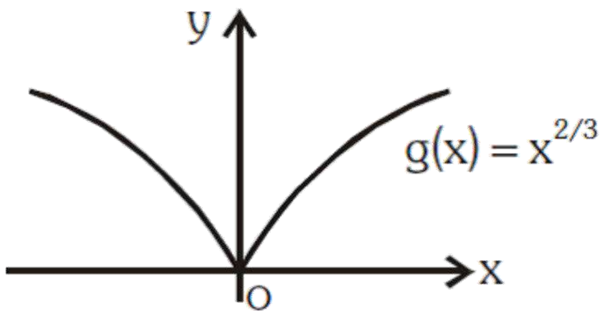
Do yourself - 2: (i) Let $f(x) = (x-1)|x-1|$. Discuss the continuity and differentiability of $f(x)$ at $x = 1$.

(b) Vertical tangent :

(i) If for $y = f(x)$; $f'(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a) \to -\infty$ then at $x = a$, $y = f(x)$ has vertical tangent.



e.g. (i) $f(x)=x^{1/3}$ has vertical tangent at $x=0$ since $f'(0^+)\to\infty$ and $f'(0^-)\to\infty$ hence $f(x)$ is not differentiable at $x=0$



(2) $g(x)=x^{2/3}$ doesn't have vertical tangent at $x=0$

since $g'(0^+)\to\infty$ and $g'(0^-)\to-\infty$ hence $g(x)$ is not differentiable at $x=0$.

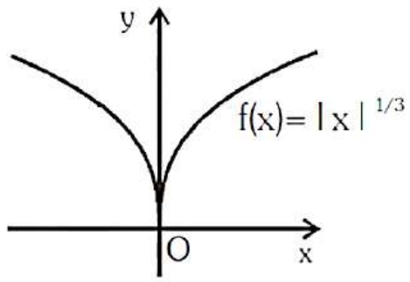
(ii) If a function has vertical tangent at $x=a$ then it is non differentiable at $x=a$.

(c) Geometrical interpretation of differentiability :

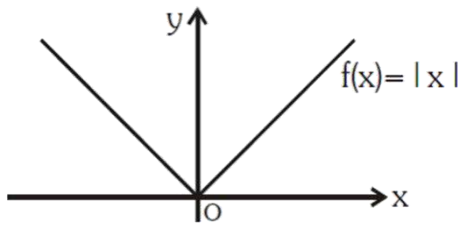
(i) If the function $y=f(x)$ is differentiable at $x=0$, then a unique tangent can be drawn to the curve $y=f(x)$ at the point $P(a,f(a))$ and $f'(a)$ represent the slope of the tangent at point P.

(ii) If the function $f(x)$ does not have a unique tangent $(p\neq q)$ but is continuous at $x=a$. it geometrically implies a sharp corner at $x=a$. Note that p and q may not be finite, where $p=f'(a^+)$and $q=f'(a^-)$

e.g. (1) $f(x)=|x|$ and $|x|^{1/3}$ is continuous but not differentiable at $x=0$ and there is sharp corner at $x=0$.

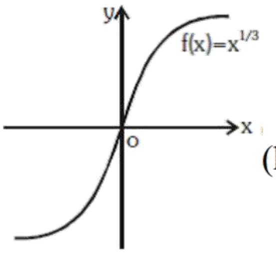


(does not have unique tangent) $\begin{cases} p\to+\infty \\ q\to-\infty \end{cases}$

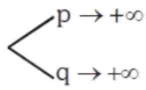


(does not have unique tangent) $\begin{cases} p=1 \\ q=-1 \end{cases}$

(2) $f(x)=x^{1/3}$ is continuous but not differentiable at $x=0$ because $f'(0^+)\to\infty$ and $f'(0^-)\to\infty$.



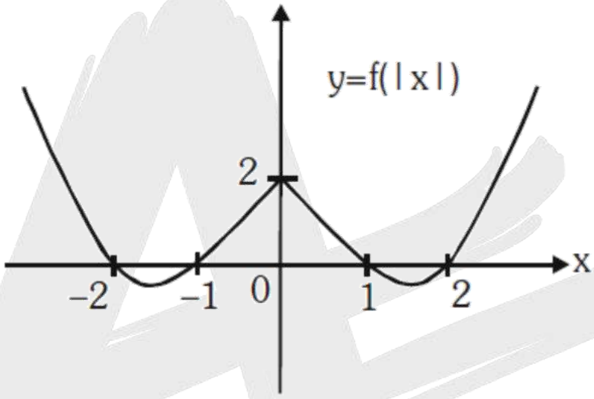
(have a unique tangent but does not have sharp corner)



Note : sharp corner ⇒ non differentiable non differentiable ⇒ sharp corner

Illustration 5: If $f(x) = \begin{cases} x-3 & x<0 \\ x^2-3x+2 & x \geq 0 \end{cases}$. Draw the graph of the function and discuss the continuity and differentiability of f(|**x**|) and |f(x)|.

Solution :



$$f(|x|) = \begin{cases} |x|-3; & |x|<0 \rightarrow \text{ not possible} \\ |x|^2-3|x|+2; & |x| \geq 0 \end{cases}$$

$$f(|x|) = \begin{cases} x^2+3x+2, & x<0 \\ x^2-3x+2, & x \geq 0 \end{cases}$$

at $x = 0$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{-h} = 3$$

$$p = \text{ RHD } = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

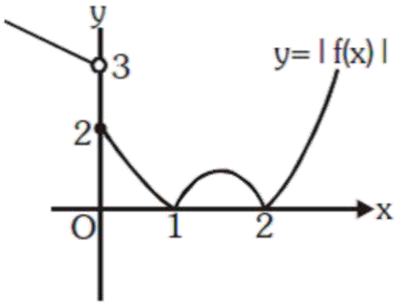
$\because q \neq p$

$\therefore$ not differentiable at $x = 0$. but p and q both are finite

$\Rightarrow$ continuous at $x = 0$

$$\text{Now, } |f(x)| = \begin{cases} 3-x & , & x<0 \\ (x^2-3x+2) & , & 0 \leq x < 1 \\ -(x^2-3x+2) & , & 1 \leq x \leq 2 \\ (x^2-3x+2) & , & x>2 \end{cases}$$

To check differentiability at $x = 0$



To check differentiability at x = 0

$$q = \quad LHD = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{3+h-2}{-h} = \lim_{h\to 0}\frac{(1+h)}{-h} \to -\infty$$

$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\Rightarrow$ not differentiable at x = 0.

Now to check continuity at x = 0

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3-x = 3$$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-3x+2 = 2$$

$\Rightarrow$ not continuous at x = 0.

To check differentiability at x = 1

$$q = LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h^2)-3(1-h)+2-0}{-h} = \lim_{h\to 0}\frac{h^2+h}{-h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim_{h\to 0}\frac{-(h^2-h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 1.

but |f(x)| is continuous at x = 1, because p ≠ q and both are finite.

To check differentiability at x = 2

$$q = LHD = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{-(4+h^2-4h-6+3h+2)-0}{-h} = \lim_{h\to 0}\frac{h^2-h}{h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(h^2+4h+4-6-3h+2)-0}{h} = \lim_{h\to 0}\frac{(h^2+h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 2.

but |f(x)| is continuous at x = 2, because p ≠ q and both are finite.

but |f(x)| is continuous at x = 2, because p ≠ q and both are finite.

Do yourself - 3: (i) Let $f(x) = \begin{cases} -4; & -4 < x < 0 \\ x^2-4; & 0 \le x < 4 \end{cases}$

Discuss the continuity and differentiability of g(x) = |f(x)|.

(ii) Let f(x) = min{|x − 1|, |x + 1|,1}. Find the number of points where it is not differentiable.

5. DIFFERENTIABILITY OVER AN INTERVAL :

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each and every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if :

(i) f(x) is differentiable in (a, b) and

(ii) for the points a and b, $f'(a^+)$ and $f'(b^-)$ exist.

Illustration 6: If $f(x)=\begin{cases} e^{-|x|}, & -5<x<0 \\ -e^{-|x-1|}+e^{-1}+1, & 0\le x<2 \\ e^{-|x-2|}, & 2\le x<4 \end{cases}$

Discuss the continuity and differentiability of f(x) in the interval (−5,4).

Solution: $f(x)=\begin{cases} e^{+x} & -5<x<0 \\ -e^{x-1}+e^{-1}+1 & 0\le x\le 1 \\ -e^{-x+1}e^{-1}+1 & 1<x<2 \\ e^{-x+2} & 2\le x<4 \end{cases}$

Check the differentiability at x = 0

$$\text{LHD}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{-e^{h-1}+e^{-1}+1-1}{h}=-e^{-1}$$

∵ LHD ≠ RHD

∴ Not differentiable at x = 0, but continuous at x = 0 since LHD and RHD both are finite. Check the differentiability at x = 1

$$\text{LHD}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{-e^{1-h-1}+e^{-1}+1-e^{-1}}{-h}=-1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{-e^{1-h-1}+e^{-1}+1-e^{-1}}{h}=1$$

∵ LHD ≠ RHD

∴ Not differentiable at x = 1, but continuous at x = 1 since LHD and RHD both are finite.

Check the differentiability at x = 2

$$\text{LHD}=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}=\lim_{h\to 0}\frac{-e^{-2+h+1}+e^{-1}+1-1}{-h}=\lim_{h\to 0}\frac{-e^{-1}(e^{h}-1)}{-h}=e^{-1}$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{e^{-h}-1}{h}=-1$$

∵ LHD ≠ RHD

∴ Not differentiable at x = 2, but continuous at x = 2 since LHD and RHD both are finite.

Note :

(i) If f(x) is differentiable at x = a and g(x) is not differentiable at x = a, then the product function F(x) = f(x) · g(x) can still be differentiable at x = a.

e.g. Consider $f(x) = x$ and $g(x) = |x|$, f is differentiable at $x = 0$ and g is non-differentiable at $x = 0$, but $f(x) \cdot g(x)$ is still differentiable at $x = 0$.

(ii) If $f(x)$ and $g(x)$ both are not differentiable at $x = a$ then the product function ; $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$.

e.g. Consider $f(x) = |x|$ & $g(x) = -|x|$. f& g are both non differentiable at $x = 0$, but $f(x) \cdot g(x)$ still differentiable at $x = 0$.

(iii) If $f(x)$& $g(x)$ both are non-differentiable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

e.g. $f(x) = |x|$& $g(x) = -|x| \cdot$ f& g are both non differentiable at $x = 0$, but $(f + g)(x)$ still differentiable at $x = 0$.

(iv) If $f(x)$ is differentiable at $x = a f'(x)$ is continuous at $x = a$.

e.g. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Do yourself - 4 :

(i) Let $f(x) = \max\{\sin x, 1/2\}$, where $0 \leq x \leq \frac{5\pi}{2}$. Find the number of points where it is not differentiable.

(ii) Let $f(x) = \begin{cases} [x] & ; 0 < x \leq 2 \\ 2x - 2 & ; 2 < x < 3 \end{cases}$, where [.] denotes greatest integer function.

(a) Find that points at which continuity and differentiability should be checked.

(b) Discuss the continuity and differentiability of $f(x)$ in the interval $(0,3)$.

6. DETERMINATION OF FUNCTION WHICH SATISFYING THE GIVEN FUNCTIONAL RULE :

Illustration 7: Let $f(x + y) = f(x) + f(y) - 2xy - 1$ for all x and y. If $f'(0)$ exists and $f'(0) = -\sin \alpha$, then find $f\{f'(0)\}$.

Solution : $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{\{f(x) + f(h) - 2xy - 1\} - f(x)}{h}$$

$$= \lim_{h \to 0} - 2x + \lim_{h \to 0} \frac{f(h) - 1}{h} = -2x + \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

(Using the given relation)

[Putting $x = 0 = y$ in the given relation we find $f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$]

$\therefore f'(x) = -2x + f'(0) = -2x - \sin \alpha$

$\Rightarrow f(x) = -x^2 - (\sin \alpha) \cdot x + c$

$f(0) = -0 - 0 + c \Rightarrow c = 1$

$\therefore f(0) = -x^2 - (\sin \alpha) \cdot x + 1$

So, $f\{f'(0)\} = f(-\sin \alpha) = -\sin^2 \alpha + \sin^2 \alpha + 1 \;\; \therefore f\{f'(0)\} = 1$.

Do yourself - 5:

(i) A function $f: R \to R$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R, f(x) \neq 0$. suppose that the function is differentiable everywhere and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Miscellaneous Illustration :

Illustration 8: Discuss the continuity and differentiability of the function $y = f(x)$ defined parametrically ; $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Solution : Here $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Now when $t < 0$; $x = 2t - \{-(t-1)\} = 3t - 1$ and $y = 2t^2 - t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2$

when $0 \leq t < 1$

$x = 2t - (-(t-1)) = 3t - 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$

when $t \geq 1$; $x = 2t - (t-1) = t + 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$

$$\text{Thus, } y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \leq x < 2 \\ 3(x-1)^2, & x \geq 2 \end{cases}$$

We have to check differentiability at $x = -1$ and 2 . Differentiability at $x = -1$;

$$\text{LHD} = f'(-1^-) = \lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h\to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = f'(-1^+) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{-h} = 0$$

Hence $f(x)$ is differentiable at $x = -1. \Rightarrow$ continuous at $x = -1$.

To check differentiability at $x = 2$;

$$\text{LHD} = f'(2^-) = \lim_{h\to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2 \text{ and RHD} = f'(2^+) = \lim_{h\to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

Hence $f(x)$ is not differentiable at $x = 2$.

But continuous at $x = 2$, because LHD and RHD both are finite.

$\therefore$ $f(x)$ is continuous for all x and differentiable for all x, except $x = 2$.

ANSWERS FOR DO YOURSELF

1. (i) Continuous but not differentiable at $x = 1$

(ii) $a = \frac{1}{6}, b = \frac{\pi}{4} - \frac{13}{6}$

2. (i) Continuous and differentiable at $x = 1$

3. (i) Continuous everywhere but not differentiable at $x = 2$ only

(ii) 5

4. (i) 3

(ii) (a) 1 and 2

(b) Not continuous at $x = 1$ and 2 and not differentiable at $x = 1$ and 2 .

EXERCISE – 1

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x) = [\tan^2 x]$, (where [.] denotes greatest integer function). Then -

(A) $\lim_{x\to 0} f(x)$ does not exist
(B) $f(x)$ is continuous at $x = 0$.
(C) $f(x)$ is not differentiable at $x = 0$
(D) $f'(0) = 1$

2. The number of points where $f(x) = [\sin x + \cos x]$ (where [-] denotes the greatest integer function), $x \in (0,2\pi)$ is not continuous is -

(A) 3 (B) 4 (C) 5 (D) 6

3. If 6,8 and 12 are ℓ^{th}, m^{th} and n^{th} terms of an A.P. and $f(x) = nx^2 + 2\ell x - 2m$, then the equation $f(x) = 0$ has -

(A) a root between 0 and 1
(B) both roots imaginary.
(C) both roots negative.
(D) both roots greater than 1 .

4. Let f be differentiable at $x = 0$ and $f'(0) = 1$. Then $\lim_{h\to 0} \frac{f(h)-f(-2h)}{h} =$

(A) 3 (B) 2 (C) 1 (D) -1

5. Let $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for} \quad x < 1 \\ ax + b & \text{for} \quad x \geq 1 \end{cases}$

If g(x) is continuous and differentiable for all numbers in its domain then -

(A) $a = b = 4$
(B) $a = b = -4$
(C) $a = 4$ and $b = -4$
(D) $a = -4$ and $b = 4$

6. If $f(x)f(y) + 2 = f(x) + f(y) + f(xy)$ and $f(1) = 2, f'(1) = 2$ then sgn $f(x)$ is equal to (where sgn denotes signum function) -

(A) 0 (B) 1 (C) -1 (D) 4

7. The function $g(x) = \begin{cases} x + b, & x < 0 \\ \cos x & x \geq 0 \end{cases}$ can be made differentiable at $x = 0$ -

(A) if b is equal to zero
(B) if b is not equal to zero
(C) if b takes any real value
(D) for no value of b

8. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable?

(A) $f(x) = x^{1/3}$
(B) $f(x) = \frac{|x|}{x}$
(C) $f(x) = e^{-x}$
(D) $f(x) = \tan x$

9. If the right hand derivative of $f(x) = [x]\tan \pi x$ at $x = 7$ is $k\pi$, then k is equal to ([y] denotes greatest integer $\leq$ y)

(A) 6 (B) 7 (C) -7 (D) 49

10. Let $f: R \rightarrow R$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall x \in R$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5,5]$, then the equation $f(x) = 0$ has-

(A) exactly three real roots (B) exactly two real roots

(C) atleast five real roots (D) atleast three real roots

11. Let $f(x) = \begin{cases} \lim_{n\to\infty} \dfrac{ax(x-1)\left(\cot\frac{\pi x}{4}\right)^n + (px^2+2)}{\left(\cot\frac{\pi x}{4}\right)^n + 1}, & x \in (0,1) \cup (1,2) \\ 0 & , x = 1 \end{cases}$

If $f(x)$ is differentiable for all $x \in (0,2)$ then $(a^2 + p^2)$ equals -

(A) 18 (B) 20 (C) 22 (D) 24

12. If $2x + 3|y| = 4y$, then y as a function of x i.e. $y = f(x)$, is -

(A) discontinuous at one point

(B) non differentiable at one point

(C) discontinuous& non differentiable at same point

(D) continuous& differentiable everywhere

13. If $f(x) = (x^5 + 1)|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then $f(x)$ is not differentiable at -

(A) 2 points (B) 3 points

(C) 4 points (D) zero points

14. Let $f(x) = \begin{cases} x^3 + 2x^2 & x \in Q \\ -x^3 + 2x^2 + ax & x \notin Q \end{cases}$, then the integral value of ' a ' so that $f(x)$ is differentiable at $x = 1$, is

(A) 2 (B) 6 (C) 7 (D) not possible

15. Let R be the set of real numbers and $f: R \rightarrow R$, be a differentiable function such that $|f(x) - f(y)| \le |x - y|^3 \forall x, y \in R$. If $f(10) = 100$, then the value of $f(20)$ is equal to -

(A) 0 (B) 10 (C) 20 (D) 100

16. For what triplets of real numbers (a, b, c) with $a \neq 0$ the function $f(x) = \begin{cases} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) $\{(a, 1 - 2a, a) \mid a \in R, a \neq 0\}$ (B) $\{(a, 1 - 2a, c) \mid a, c \in R, a \neq 0\}$

(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$ (D) $\{(a, 1 - 2a, 0) \mid a \in R, a \neq 0\}$

17. Number of points of non-differentiability of the function

$g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2\sin^2 4x + [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$ in $(-50,50)$ where $[x]$ and $\{x\}$ denotes the greatest integer function and fractional part function of x respectively, is equal to:-

(A) 98 (B) 99 (C) 100 (D) 0

18. Let $f(x) = [n + p\sin x], x \in (0, \pi), n \in I$ and p is a prime number. The number of points where $f(x)$ is not differentiable is :-

(A) $p - 1$ (B) $p + 1$ (C) $2p + 1$ (D) $2p - 1$

Here $[x]$ denotes greatest integer function.

19. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at :

(A) -1 (B) 0 (C) 1 (D) 2

20. Let $g(x) = \begin{cases} 2x + \tan^{-1} x + a, & -\infty < x \le 0 \\ x^3 + x^2 + bx, & 0 < x < \infty \end{cases}$.

If $g(x)$ is differentiable for all $x \in (-\infty, \infty)$ then $(a^2 + b^2)$ is equal to

(A) 20 (B) 13 (C) 9 (D) 4

21. Number of points in $[-2\pi, 2\pi]$ where $f(x) = |\cos^{-1}(\cos x)|$ is non-derivable is

(A) 0 (B) 2 (C) 3 (D) 5

22. Let $f(x) = \min.(|\mathbf{x}|, x^2)$ and $g(x) = \max\left\{|\sin^{-1}(\sin x)|, \frac{x^2}{4}\right\}$. Then total number of points where $f(x)$ and $g(x)$ are non-derivable is

(A) 4 (B) 5 (C) 6 (D) 7

23. If the function $f(x) = \begin{cases} ax + b, & -\infty < x \le 2 \\ x^2 - 5x + 6, & 2 < x < 3 \\ px^2 + qx + 1, & 3 \le x < \infty \end{cases}$ is differentiable in $(-\infty, \infty)$, then

(A) $a = -1, p = \frac{-4}{9}$ (B) $b = 2, q = \frac{5}{3}$ (C) $a = 1, b = 2$ (D) $a = -1, q = \frac{-5}{3}$

24. Let $f(x) = [x]$ and $g(x) = \begin{cases} x, & x \in [0,1) \\ x - 1, & x \in [1,2) \\ x - 2, & x \in [2,3) \\ 0, & x = 3 \end{cases}$.

Then $f(x) + g(x)$ is

(A) discontinuous at $x = 1$ and $x = 2$.

(B) continuous in $[0,3]$ but non derivable in $[0,3]$.

(C) not twice differentiable in $[0,3]$.

(D) twice differentiable in $[0,3]$

[Note: $[k]$ denotes the greatest integer function less than or equal to k.]

25. Let $f(x) = \left| \begin{matrix} x + 2, & x < 0 \\ -(2 + x^2), & 0 \le x < 1 \\ x, & x \ge 1 \end{matrix} \right.$

Then the number of points where $|f(x)|$ is non-derivable is

(A) 3 (B) 2 (C) 1 (D) 0

26. Let $g(x) = \min.(x, x^2)$ where $x \in R$, then $\lim_{x\to 0} \frac{g(1+x)-g(1)}{x}$ equals

(A) 0 (B) 1 (C) 2 (D) does not exist

EXERCISE – 2

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin |x|, x \in R$. Draw a rough sketch of the graph of f(x).

2. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2| x \in R$. Also draw the graph of f(x).

3. If the function f(x) defined as $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\frac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at x = 0 then find the range of n.

4. A function f is defined as follows: $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1 + |\sin x| & \text{for } 0 \le x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at $x = 0 \& x = \pi/2$.

5. Examine the origin for continuity & derivability in the case of the function f defined by $f(x) = x\tan^{-1}(1/x), x \ne 0$ and $f(0) = 0$.

6. Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that

$$\lim_{x\to 0}\frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \cdots \ldots + f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \ldots + \frac{1}{k}$$

7. Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}; x \ne 0, f(0) = 0$, test the continuity & differentiability at $x = 0$

8. If $f(x) = |x-1| \cdot ([x] - [-x])$, then find $f'(1^+) \& f'(1^-)$ where [x] denotes greatest integer function.

9. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \ge 1 \end{cases}$ is derivable at $x = 1$. Find the values of a&b.

10. Let $g(x) = \begin{cases} a\sqrt{x+2}, & 0 < x < 2 \\ bx + 2, & 2 \le x < 5 \end{cases}$. If g(x) is derivable on (0,5), then find (2a + b).

EXERCISE – 3 (JM)

1. The set of points where $f(x)=\frac{x}{1+|x|}$ is differentiable- **[AIEEE-2006]**

(A) $(-\infty,-1)\cup(-1,\infty)$ (B) $(-\infty,\infty)$

(C) $(0,\infty)$ (D) $(-\infty,0)\cup(0,\infty)$

2. Let $f(x)=x|x|$ and $g(x)=\sin\ x$. **[AIEEE-2009]**

Statement-1 : gof is differentiable at $x=0$ and its derivative is continuous at that point.

Statement-2 : gof is twice differentiable at $x=0$.

(A) Statement -1 is true, Statement -2 is false.

(B) Statement -1 is false, Statement -2 is true.

(C) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1.

(D) Statement -1 is true, Statement - 2 is true ; Statement -2 is not a correct explanation for statement -1 .

3. If function f(x) is differentiable at $x=a$ then $\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a}$ **[AIEEE-2011]**

(A) $2af(a)+a^2f'(a)$ (B) $-a^2f'(a)$

(C) $a\,f(a)-a^2f'(a)$ (D) $2af(a)-a^2f'(a)$

4. Consider the function, $f(x)=|x-2|+|x-5|, x\in R$.

Statement - 1: $f'(4)=0$.

Statement - 2: f is continuous in [2,5], differentiable in (2,5) and $f(2)=f(5)$. **[AIEEE 2012]**

(A) Statement-1 is true, Statement -2 is false.

(B) Statement-1 is false, Statement-2 is true.

(C) Statement-1 is true, Statement-2 is true ; Statement-2 is a correct explanation for Statement 1 .

(D) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement 1 .

5. Let $f(x)=x|x|, g(x)=\sin\ x$ and $h(x)=(gof)(x)$. Then **[2014]**

(A) $h'(x)$ is differentiable at $x=0$

(B) h' (x) is continuous at $x=0$ but is not differentiable at $x=0$

(C) $h(x)$ is differentiable at $x=0$ but $h'(x)$ is not continuous at $x=0$

(D) $h(x)$ is not differentiable at $x=0$

6. Let $f, g: R\to R$ be two functions defined by $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$, and $g(x)=xf(x)$: –

Statement **I**: f is a continuous function at $x=0$. **[2014]**

Statement II : g is a differentiable function at $x=0$.

(A) Statement I is false and statement II is true

(B) Statement I is true and statement II is false

(C) Both statement I and II are true

(D) Both statements I and II are false

7. Let $f: R \to R$ be a function such that $|f(x)| \le x^2$, for all $x \in R$. Then, at $x = 0$, f is:

(A) Neither continuous nor differentiable **[2014]**

(B) Differentiable but not continuous

(C) Continuous as well as differentiable

(D) Continuous but not differentiable

8. Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2|x - y|^{3/2}$, for all $x, y \in R$. If $f(0) = 1$ then $\int_0^1 f^2(x)dx$ is equal to : **[JEE Mains -2019]**

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) 0

9. Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let S be the set of points in the interval $(-4,4)$ at which f is not differentiable. Then S :

[JEE Mains Online-2019]

(A) equals $(-2, -1, 1, 2)$ (B) equals $\{-2, 2\}$

(C) is an empty set (D) equals $\{-2, -1, 0, 1, 2\}$

10. Let $f: (-1,1) \to R$ be a function defined by $f(x) = \max\{-|x|, -\sqrt{1 - x^2}\}$. If K be the set of all points at which f is not differentiable, then K has exactly: **[JEE Mains -2019]**

(A) one element (B) two elements

(C) five elements (D) three elements

11. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1 & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2,2)$, g is :

[JEE Mains Online-2019]

(A) not continuous (B) not differentiable at one point

(C) differentiable at all points (D) not differentiable at two points

12. Let K be the set of all real values of x where the function $f(x) = \sin |x| - |x| + 2(x - \pi)\cos |x|$ is not differentiable. Then the set K is equal to: **[JEE Mains -2019]**

(A) $\{\pi\}$ (B) ϕ (an empty set)

(C) $\{0\}$ (D) $\{0, \pi\}$

13. Let S be the set of all points in $(-\pi, \pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following ? **[JEE Mains -2019]**

(A) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ (B) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

(C) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$ (D) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

14. Let $f(x) = 15 - |x - 10|; x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable is: **[JEE Mains -2019]**

(A) $\{10,15\}$ (B) $\{10\}$ (C) $\{5,10,15,20\}$ (D) $\{5,10,15\}$

15. Let $f: R \to R$ be differentiable at $c \in R$ and $f(c) = 0$, If $g(x) = |f(x)|$, then at $x = c$, g is : **[JEE Mains -2019]**

(A) differentiable if $f'(c) = 0$ (B) not differentiable

(C) not differentiable if $f'(c) = 0$ (D) differentiable if $f'(c) \neq 0$

16. Let $f: R \to R$ be a continuously differentiable function such that $f(2) = 6$ and $f'(2) = \frac{1}{48}$. If $\int_6^{f(x)} 4t^3 dt = (x - 2)g(x)$, then $\lim_{x\to 2} g(x)$ is equal to **[JEE Mains -2019]**

(A) 36 (B) 12 (C) 18 (D) 24

17. If $f(x) = |2 - |x - 3||$ is non differentiable in $x \in S$. Then value of $\sum_{x\in S}(f(f(x))$ is **[JEE Mains -2020]**

18. The function $f(x) = |x^2 - 2x - 3| \cdot e^{|9x^2 - 12x + 4|}$ is not differentiable at exactly: **[JEE Mains -2021]**

(A) four points (B) three points (C) two points (D) one point

19. Let $f(x) = \left\{\frac{\sin(x-[x])}{x-[x]}\right.$, $x \in (-2, -1)$ where $[t]$ denotes greatest integer $\leq t$. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is : **[JEE Mains -2022]**

(A) $(3,3)$ (B) $(2,4)$ (C) $(2,3)$ (D) $(3,4)$

20. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$; Then at $x = 0$ **[JEE Mains -2023]**

(A) f is continuous but not differentiable
(B) f is continuous but f' is not continuous
(C) f and f' both are continuous
(D) f' is continuous but not differentiable

EXERCISE – 4 (JA)

SECTION-1

1. Let $g(x) = \frac{(x-1)^n}{\ell n\cos^m (x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0, n > 0$ and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim_{x\to1^+} g(x) = p$, then :- **[JEE 2008, 3]**

(A) $n = 1, m = 1$ (B) $n = 1, m = -1$

(C) $n = 2, m = 2$ (D) $n > 2, m = n$

2. Let $f(x) = \begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x = 0, \\ 0 & , x \neq 0 \end{cases}$, then f is - **[JEE 2012, 3M, -1M]**

(A) Differentiable both at $x = 0$ and at $x = 2$

(B) Differentiable at $x = 0$ but not differentiable at $x = 2$

(C) Not differentiable at $x = 0$ but differentiable at $x = 2$

(D) Differentiable neither at $x = 0$ nor at $x = 2$

SECTION-2

3. If $f(x) = \min.(1, x^2, x^3)$, then **[JEE 2006, 5]**

(A) f(x) is continuous $\forall x \in R$

(B) $f'(x) > 0, \forall x > 1$

(C) f(x) is not differentiable but continuous $\forall x \in R$

(D) f(x) is not differentiable for two values of x

4. Let $f: R \to R$ be a function such that $f(x + y) = f(x) + f(y), \forall x, y \in R$.

If f(x) is differentiable at $x = 0$, then

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous $\forall x \in R$

(C) $f'(x)$ is constant $\forall x \in R$

(D) f(x) is differentiable except at finitely many points **[JEE 2011, 4M]**

5. If $f(x) = \begin{cases} -x-\frac{\pi}{2} & , \quad x \le -\frac{\pi}{2} \\ -\cos x & , \quad -\frac{\pi}{2} < x \le 0 \\ x-1 & , \quad 0 < x \le 1 \\ \ell nx & , \quad x > 1 \end{cases}$ then - **[JEE 2011, 4M]**

(A) f(x) is continuous at $x = -\frac{\pi}{2}$ (B) f(x) is not differentiable at $x = 0$

(C) f(x) is differentiable at $x = 1$ (D) f(x) is differentiable at $x = -\frac{3}{2}$

6. Let $f_1:\mathbb{R}\to\mathbb{R}$, $f_2:[0,\infty)\to\mathbb{R}$, $f_3:\mathbb{R}\to\mathbb{R}$ and $f_4:\mathbb{R}\to[0,\infty)$ be defined by

$$f_1(x)=\begin{cases}|x| & \text{if } x<0\\ e^x & \text{if } x\geq 0\end{cases}$$

$f_2(x)=x^2$;

$$f_3(x)=\begin{cases}\sin x & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases} \text{ and } f_4(x)=\begin{cases}f_2(f_1(x)) & \text{if } x<0,\\ f_2(f_1(x))-1 & \text{if } x\geq 0\end{cases}$$

List-I	**List-II**
P. f_4 is	1. onto but not one-one
Q. f_3 is	2. neither continuous nor one-one
R. $f_2\circ f_1$ is	3. differentiable but not one-one
S. f_2 is	4. continuous and one-one

Codes : **[JEE(Advanced)-2014, 3(-1)]**

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

7. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be respectively given by $f(x)=|x|+1$ and $g(x)=x^2+1$. Define

$$h:\mathbb{R}\to\mathbb{R} \text{ by } h(x)=\begin{cases}\max\{f(x),g(x)\} & \text{if } x\leq 0,\\ \min\{f(x),g(x)\} & \text{if } x>0.\end{cases}$$

The number of points at which $h(x)$ is not differentiable is **[JEE(Advanced)-2014, 3]**

8. Let $a,b\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=a\cos(|x^3-x|)+b|x|\sin(|x^3+x|)$. Then f is -

(A) Differentiable at $x=0$ if $a=0$ and $b=1$

(B) Differentiable at $x=1$ if $a=1$ and $b=0$

(C) NOT differentiable at $x=0$ if $a=1$ and $b=0$

(D) NOT differentiable at $x=1$ if $a=1$ and $b=1$ **[JEE(Advanced)-2016, 4(-2)]**

9. Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be function defined by $f(x)=[x^2-3]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y\in\mathbb{R}$. Then, **[JEE(Advanced)-2016, 4(-2)]**

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

10. Let $f:R\to R$ be a differentiable function with $f(0)=1$ and satisfying the equation $f(x+y)=f(x)f'(y)+f'(x)f(y)$ for all $x,y\in R$. Then, the value of $\log_e(f(4))$ is **[JEE Advanced-2018, 3(0)]**

11. Let the function $f: R \to R$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin\, x$ and let $g: R \to R$ be an arbitrary function. Let fg: $R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE? **[JEE Advanced-2020]**

(A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

(B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

(C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

(D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

12. Let $f: (0,1) \to \mathbb{R}$ be the function defined as $f(x) = [4x]\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Then which of the following is(are) true?

(A) The function f is discontinuous exactly at one point in $(0,1)$

(B) There is exactly one point in $(0,1)$ at which the function f is continuous but NOT differentiable

(C) The function f is NOT differentiable at more than three points in $(0,1)$

(D) The minimum value of the function f is $-\frac{1}{512}$ **[JEE Advanced-2023]**

EXERCISE – 5

[MULTIPLE CORRECT CHOICE TYPE]

1. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then-

(A) $Rf'(0)$ exist
(B) $Lf'(0)$ exist but $Rf'(0)$ does not exist
(C) $\lim_{x\to 0^+} f(x)$ exist
(D) $f(x)$ is differentiable at $x = 0$.

2. The function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right), & x < 1 \end{cases}$ is -

(A) continuous at $x = 1$
(B) differentiable at $x = 1$
(C) continuous at $x = 3$
(D) differentiable at $x = 3$

3. Select the correct statements -

(A) The function f defined by $f(x) = \begin{cases} 2x^2 + 3 & \text{for} \quad x \leq 1 \\ 3x + 2 & \text{for} \quad x > 1 \end{cases}$ is neither differentiable nor continuous at $x = 1$

(B) The function $f(x) = x^2|x|$ is twice differentiable at $x = 0$.

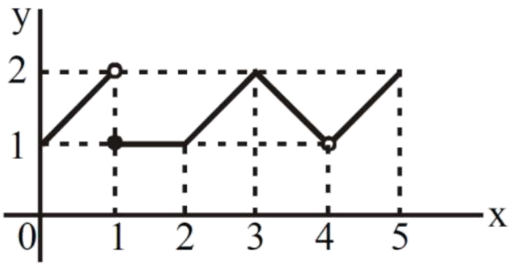
(C) If f is continuous at $x = 5$ and $f(5) = 2$ then $\lim_{x\to 2} f(4x^2 - 11)$ exists

(D) If $\lim_{x\to a}(f(x) + g(x)) = 2$ and $\lim_{x\to a}(f(x) - g(x)) = 1$ then $\lim_{x\to a} f(x)\cdot g(x)$ need not exist.

4. If $f(x) = \text{sgn}\,(x^5)$, then which of the following is/are false (where sgn denotes signum function) -

(A) $f'(0^+) = 1$
(B) $f'(0^-) = -1$
(C) f is continuous but not differentiable at $x = 0$
(D) f is discontinuous at $x = 0$

5. Graph of $f(x)$ is shown in adjacent figure, then in $[0,5]$



(A) $f(x)$ has non removable discontinuity at two points
(B) $f(x)$ is non differentiable at four points
(C) $\lim_{x\to 1} f(f(x)) = 1$
(D) Number of points of discontinuity = number of points of non-differentiability

6. Let S denotes the set of all points where $\sqrt[5]{x^2|x^3|} - \sqrt[3]{x^2|x|} - 1$ is not differentiable then S is a subset of -

(A) $\{0,1\}$
(B) $\{0,1,-1\}$
(C) $\{0,1\}$
(D) $\{0\}$

7. Which of the following statements is/are correct ?

(A) There exist a function $f:[0,1] \rightarrow R$ which is discontinuous at every point in $[0,1]$&$|f(x)|$ is continuous at every point in $[0,1]$

(B) Let $F(x) = f(x) \cdot g(x)$. If $f(x)$ is differentiable at $x = a$, $f(a) = 0$ and $g(x)$ is continuous at $x = a$ then $F(x)$ is always differentiable at $x = a$.

(C) If $f'(a^+) = 2$&$f'(a^-) = 3$, then $f(x)$ is non-differentiable at $x = a$ but will be always continuous at $x = a$

(D) If $f(a)$ and $f(b)$ possess opposite signs then there must exist at least one solution of the equation $f(x) = 0$ in (a, b) provided f is continuous on $[a, b]$

8. Let $f: R \rightarrow R$ be a function. Define $g: R \rightarrow R$ by $g(x) = |f(x)|$ for all x. Then which of the following is/are not always true-

(A) If f is continuous then g is also continuous

(B) If f is one-one then g is also one-one

(C) If f is onto then g is also onto

(D) If f is differentiable then g is also differentiable

9. If function defined by $f(x) = \begin{cases} \frac{(x-m)}{|x-m|}, & x \leq 0 \\ 2x^2 + 3ax + b, & 0 < x < 1, \\ m^2x + b - 2, & x \geq 1 \end{cases}$ is continuous & differentiable everywhere, then

(A) $b + m = -1$ (B) $b + m = 1$

(C) $b + m = -3$ (D) $m^2 + a + b = 3$

10. The function $\phi(x) = [|x| - \sin |x|]$ (where [.] denotes greater integer function) is -

(A) derivable at $x = 0$ (B) continuous at $x = 0$

(C) $\lim_{x\rightarrow 0} \phi(x)$ does not exists (D) continuous and derivable at $x = 0$

11. Let $f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & x < 0 \\ 0, & x = 0 \\ x^2 \sin\frac{1}{x}, & x > 0 \end{cases}$, then which of the following is (are) correct ?

(A) $f(x)$ is continuous but not differentiable at $x = 0$

(B) $f(x)$ is continuous and differentiable at $x = 0$

(C) $f'(x)$ is continuous but not differentiable at $x = 0$

(D) $f'(x)$ is discontinuous at $x = 0$

[MATCH THE COLUMN]

12. **Column – I** | **Column - II**

Column – I	Column - II
(A) If f(x) is derivable at $x = 3 \& f'(3) = 2$, then $\text{limit}_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2}$ equals	(P) 0
(B) Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then its value is equal to	(Q) 1
(C) For the function $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ the derivative from the left $f'(0-)$ equals	(R) 2
(D) The number of points at which the function $f(x) = \max.\{a - x, a + x, b\}, -\infty < x < \infty$, $0 < a < b$ cannot be differentiable is	(S) 3

EXERCISE – 6

1. Let f(x) be defined in the interval $[-2,2]$ such that

$f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$ & $g(x)=f(|x|)+|f(x)|$. Test the differentiability of g(x) in $(-2,2)$.

2. Discuss the continuity & the derivability in $[0,2]$ of $f(x)=\begin{cases}|2x-3|[x] & \text{for } x\ge 1\\ \sin\frac{\pi x}{2} & \text{for } x<1\end{cases}$ where [.] denotes the greatest integer function

3. Examine the function, $f(x)=x\cdot\frac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}, x\neq 0 (a>0)$ and $f(0)=0$ for continuity and existence of the derivative at the origin.

4. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-3,3]$ by $f(x)=\begin{cases}-x-[-x] & \text{if } [x] \text{ is even}\\ x-[x] & \text{if } [x] \text{ is odd}\end{cases}$

If L denotes the number of point of discontinuity and M denotes the number of points of non derivability of f(x), then find $(L+M)$.

5. $f(x)=\begin{cases}1-x, & (0\le x\le 1)\\ x+2, & (1<x<2)\\ 4-x, & (2\le x\le 4)\end{cases}$. Discuss the continuity & differentiability of $y=f[f(x)]$ for $0\le x\le 4$

6. A derivable function $f: R^+\to R$ satisfies the condition $f(x)-f(y)\ge \ln(x/y)+x-y$ for every $x,y\in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$.

7. If $\lim_{x\to 0}\frac{1-\cos\left(1-\cos\frac{x}{2}\right)}{2^m x^n}$ is equal to the left hand derivative of $e^{-|x|}$ at $x=0$, then find the value of $(n-10m)$

8. If f is a differentiable function such that $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)+f(0)}{3}, \forall x,y\in R$ and $f'(0)=2$, find f(x)

9. If $\lim_{x\to 0}\frac{f(3-\sin x)-f(3+x)}{x}=8$, then $|f'(3)|$ is

10. Let f(x) be a differentiable function such that $2f(x+y)+f(x-y)=3f(x)+3f(y)+2xy\,\forall x,y\in R$ & $f'(0)=0$, then $f(10)+f'(10)$ is equal to

ANSWER KEY

EXERCISE – 1

1.	B	**2.**	C	**3.**	A	**4.**	A	**5.**	C	**6.**	B	**7.**	D
8.	A	**9.**	B	**10.**	D	**11.**	B	**12.**	B	**13.**	A	**14.**	D
15.	D	**16.**	A	**17.**	D	**18.**	D	**19.**	D	**20.**	C	**21.**	C
22.	D	**23.**	D	**24.**	D	**25.**	A	**26.**	D				

EXERCISE – 2

1. f(x) is conti. but not derivable at $x = 0$ **2.** conti. $\forall x \in R$, not diff. at $x = 0,1\&2$

3. $0 < n \leq 1$

4. conti. but not diff. at $x = 0$; diff. & conti. at $x = \pi/2$

5. conti. but not diff. at $x = 0$ **7.** f is cont. but not diff. at $x = 0$

8. $f'(1^+) = 3, f'(1^-) = -1$ **9.** $a = 1/2, b = 3/2$

10. 3

EXERCISE – 3 (JM)

1.	**B**	**2.**	A	**3.**	D	**4.**	D	**5.**	B	**6.**	C	**7.**	C
8.	**A**	**9.**	**D**	**10.**	D	**11.**	B	**12.**	B	**13.**	C	**14.**	D
15.	**A**	**16.**	C	**17.**	3	**18.**	C	**19.**	C	**20.**	B		

EXERCISE # 4 (JA) SECTION-1

1. C **2.** B

SECTION-2

3.	AC	**4.**	BC	**5.**	ABCD	**6.**	D	**7.**	3	**8.**	AB	**9.**	BC
10.	2.00	**11.**	AC	**12.**	AB								

EXERCISE – 5

1.	ACD	**2.**	ABC	**3.**	BC	**4.**	ABC	**5.**	BC	**6.**	ABCD	**7.**	ABCD
8.	BCD	**9.**	BD	**10.**	ABD	**11.**	BD	**12.**	(A) R, (B) P, (C) Q, (D) R				

EXERCISE – 6

1. not derivable at $x = 0 \& x = 1$

2. f is conti. at $x = 1,3/2$& disconti. at $x = 2$, f is not diff. at $x = 1,3/2,2$

3. If $a \in (0,1) f'(0^+) = -1; f'(0^-) = 1 \Rightarrow$ continuous but not derivable $a = 1; f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable If $a > 1 f'(0^-) = -1; f'(0^+) = 1 \Rightarrow$ continuous but not derivable

4. 8

5. f is conti. but not diff. at $x = 1$, disconti. at $x = 2 \& x = 3$. cont. & diff. at all other points

6. 5150 **7.** 74 **8.** $f(x) = 2x + c$ **9.** 4 **10.** 120