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De Moverie's Theorem

$$(cis\theta)^n = (e^{i\theta})^n = \begin{cases} e^{in\theta} = cis\,n\theta & n \in I. \\ cin(n\theta) \text{ \& more values.} & n \in Q - \{I\} \end{cases}$$

$$(e^{i\theta})^5 = e^{i5\theta}.$$

$$(e^{i\theta})^{2/3} = \left(e^{i2\theta}\right)^{\frac{1}{3}} = z$$

$$z^3 = e^{i2\theta}$$

$$\sqrt{3+4i} = z..$$

$$3+4i = z^2.$$

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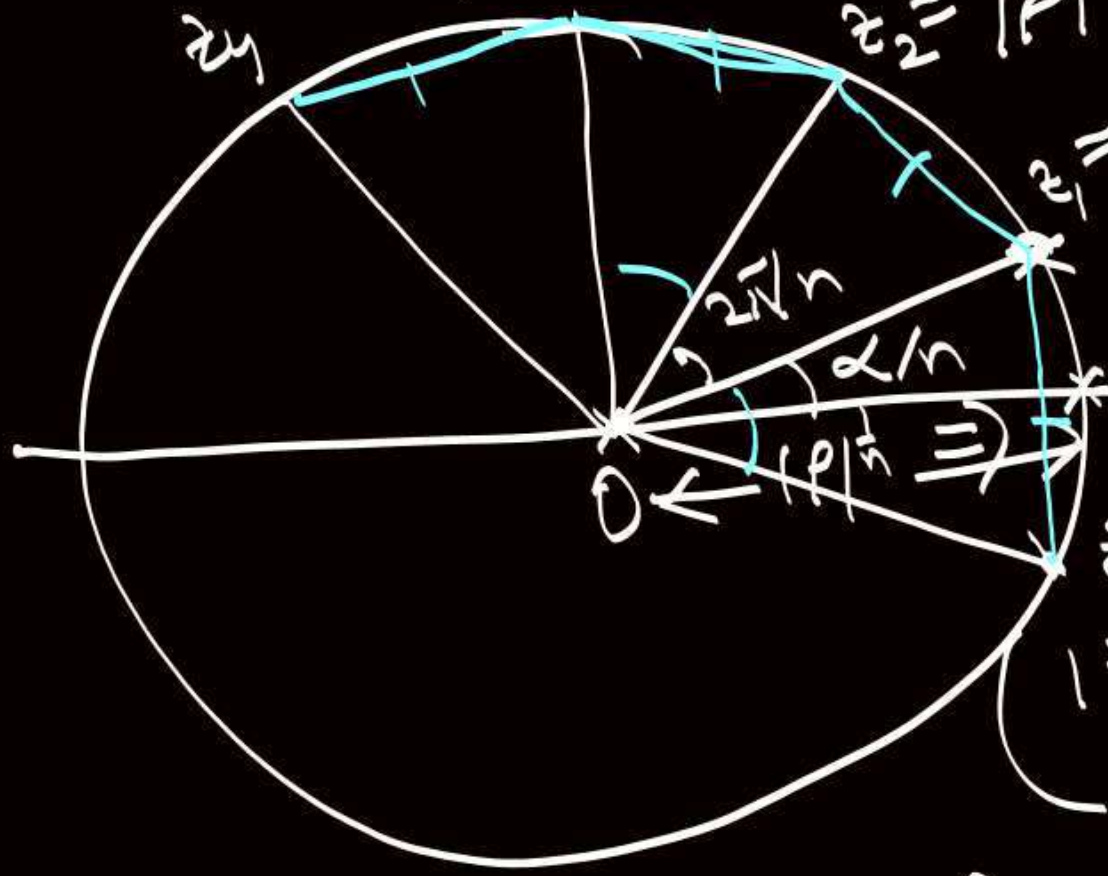
n^{th} root of a complex number



$z = \rho^{\frac{1}{n}}, \ n \in N.$

Note →

$z^n = \rho$

$\left(|z| e^{i\theta}\right)^n = |\rho| e^{i(\alpha+2k\pi)}$ $\quad k \in I.$

$|z|^n e^{in\theta} = |\rho| e^{i(\alpha+2k\pi)}$

$|z|^n = |\rho|, \ n\theta = \alpha+2k\pi \Rightarrow |z| = |\rho|^{\frac{1}{n}}, \ \theta = \frac{\alpha+2k\pi}{n}$

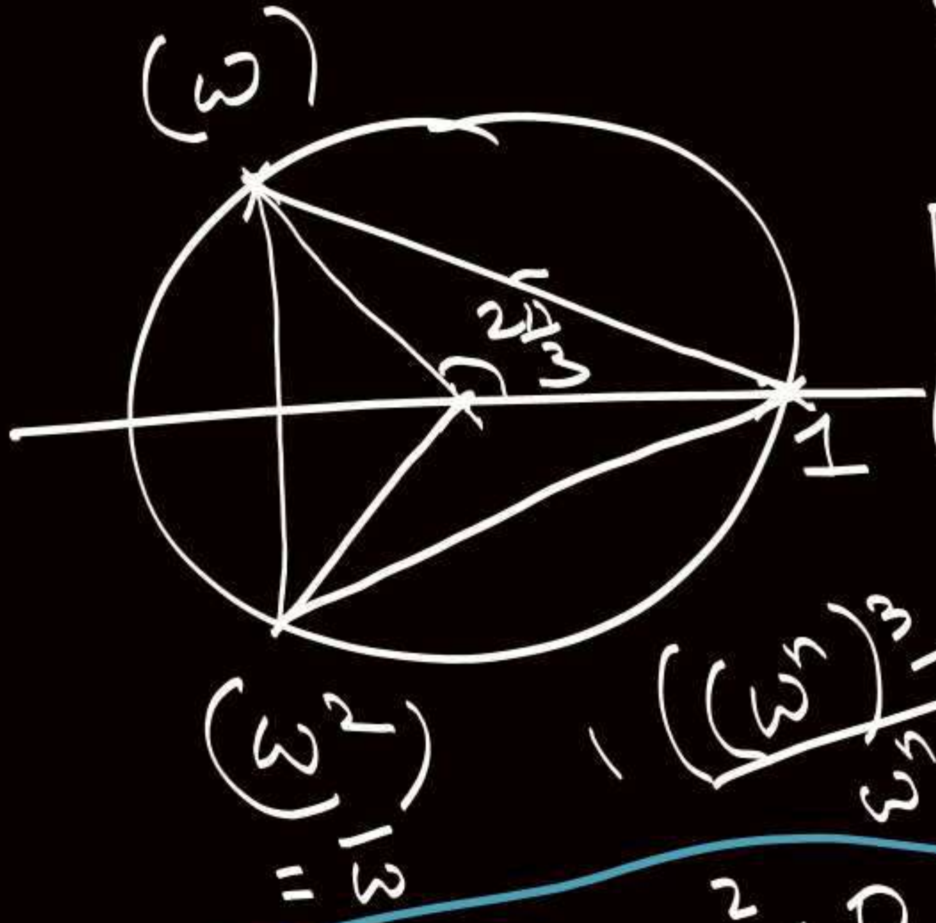
$$\boxed{z = |\rho|^{\frac{1}{n}} e^{i\frac{\alpha+2k\pi}{n}}, \ k=0,1,2,\ldots,(n-1)}$$

$k = 0,1,2,3,\ldots,n-1$

Cube root of Unity

$z^3 - 1 = 0 <^{1}_{\omega}{}_{\omega^2}$

$z = (1)^{\frac{1}{3}}$



$$\boxed{z = 1^{1/3} e^{i\frac{0+2k\pi}{3}} \quad k = 0, 1, 2}$$

$$= 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$$

$$e^{i\frac{2\pi}{3}} = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$e^{i\frac{4\pi}{3}} = \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\frac{(\omega^n)^3 - 1}{\omega^n - 1} = 0$$

* $1 + \omega + \omega^2 = 0$
* $\omega^3 = 1$
* $n \in I,\ 1 + \omega^n + \omega^{2n} = \begin{cases} 3 & n = 3k,\ k \in I \\ 0 & n \neq 3k \end{cases}$

$$a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$$

$$a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

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Solve for z

1. $2\sqrt{2}\, z^4 = (\sqrt{3}-1) + i(\sqrt{3}+1)$

$z^4 = 1 \cdot e^{i\frac{5\pi}{12}}$

$$\boxed{z = 1^{\frac{1}{4}}\, e^{i\left(\frac{\frac{5\pi}{12}+2k\pi}{4}\right)} \qquad k=0,1,2,3}$$

2. $(2-3i)z^6 + (1+5i) = 0$

$z^6 = 1-i = \sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$

$z = 2^{\frac{1}{12}}\, e^{i\left(\frac{-\frac{\pi}{4}+2k\pi}{6}\right)} \qquad k=0,1,2,3,4,5$

3. $z^{10} - z^5 - 992 = 0$

4. $z^4 - z^3 + z^2 - z + 1 = 0 \longrightarrow z^5 + 1 = 0$

$$\boxed{z = e^{i\frac{\pi+2k\pi}{5}} \qquad k=0,1,3,4}$$

$z^5 = 32, -31 = 32e^{i0}, 31e^{i\pi}$

$z = 2e^{i\frac{2k\pi}{5}}, (31)^{\frac{1}{5}} e^{i\left(\frac{\pi+2k\pi}{5}\right)}$

$k=0,1,2,3,4.$

5. Find the cube root of complex number $2+11i$ having the least positive argument.

$$= \sqrt{125}\, e^{i\alpha}$$

$$\tan\alpha = \frac{11}{2}, \quad \alpha \in \left(0, \frac{\pi}{2}\right)$$

$$z = \sqrt{5}\, e^{i\frac{\alpha+2k\pi}{3}} \qquad k=0,1,2$$

$$\boxed{z = \sqrt{5}\, e^{i\frac{\alpha}{3}}}$$

$$= \sqrt{5}\left(\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right)$$

$$\boxed{z = 2+i} \qquad \boxed{\tan\frac{\alpha}{3} = \frac{1}{2}}$$

$$\frac{11}{2} = \frac{3t-t^3}{1-3t^2}$$

$$2t^3 - 33t^2 - 6t + 11 = 0$$

$$(2t-1)(t^2-16t-11) = 0$$

$$\frac{16 \pm \sqrt{300}}{2}$$

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6. If $z = \cos(60^\circ) + i\sin(60^\circ)$, find.

$\frac{1}{2}+i\frac{\sqrt{3}}{2} = -\omega^2$

$z + 2z^2 + 3z^3 + 4z^4 + 5z^5$

$e^{i\pi/3} + 2e^{i\frac{2\pi}{3}} + 3e^{i\pi} + 4e^{i\frac{4\pi}{3}} + 5e^{i\frac{5\pi}{3}} = \left(\frac{-1-i\sqrt{3}}{2}\right)^3$

$-\omega^2 + 2\omega^4 - 3\omega^6 + 4\omega^8 - 5\omega^{10} = -\omega^2 + 2\omega - 3 + 4\omega - 5\omega$

$3\omega^2 - 3\omega - 3$

$= 3\omega^2 + 3\omega^2 = 6\omega^2$

7. Let z_1, z_2, z_3 be complex numbers s.t. $z_1 + z_2 + z_3 = 0$ & $z_1z_2 + z_2z_3 + z_3z_1 = 0$.

then P.T. $|z_1| = |z_2| = |z_3|$ ✓

$\sum z_1^2 + 2\sum z_1 z_2 = 0$

$\Rightarrow \sum z_1^2 = 0$

$\sum z_1^2 = \sum z_1 z_2$

$\boxed{\text{Ex-II, IV}}$

$z^3 - 0z^2 + 0z - p = 0$ z_1, z_2, z_3

$|z|^3 = |p|$

$|z_1| = |z_2| = |p|^{1/3} = |z_3|$

n^{th} root of Unity

$$z = (-1)^{1/n} \checkmark \quad e^{i\frac{\pi+2k\pi}{n}}$$

$$z^n+1$$

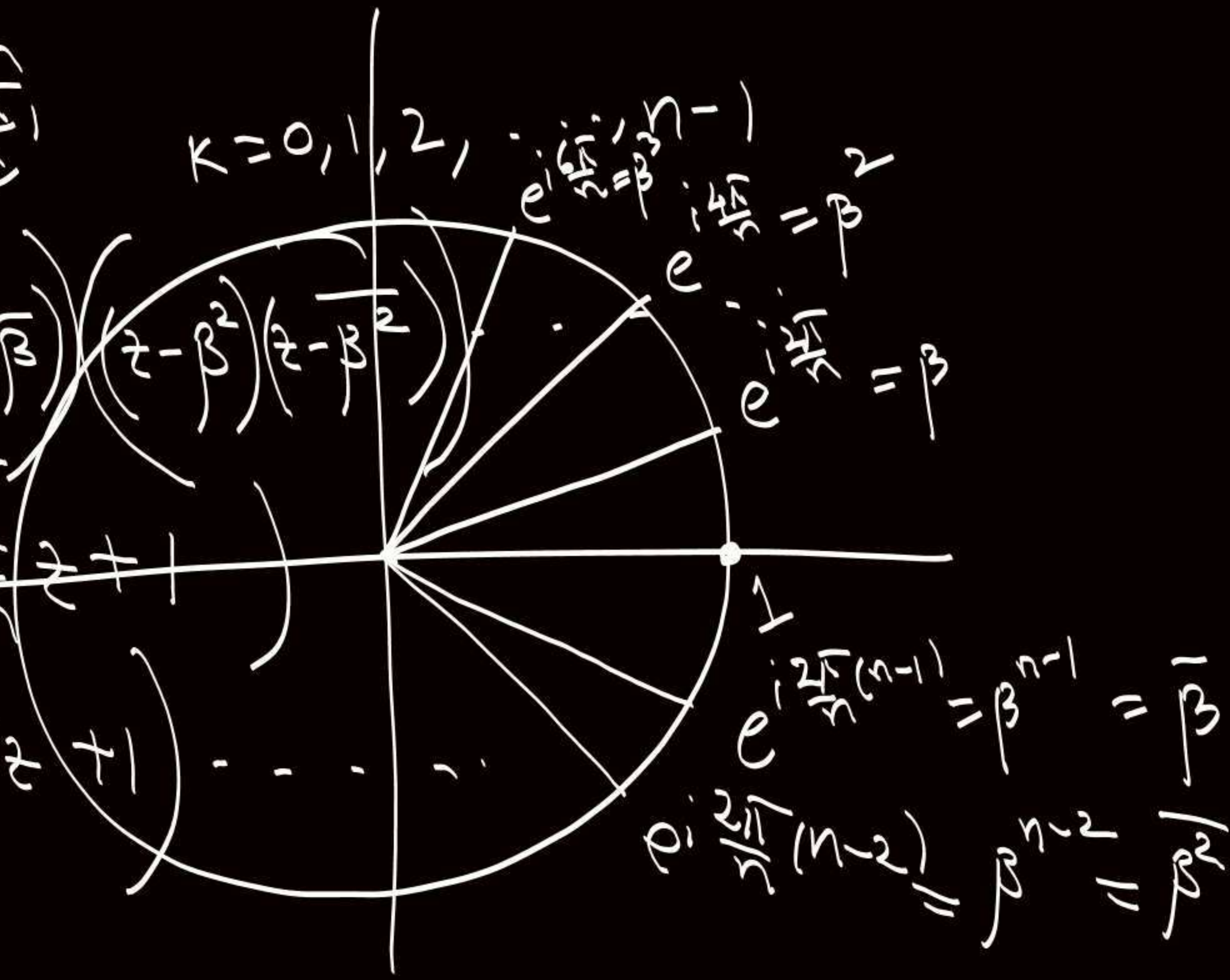
$$z = (1)^{\frac{1}{n}}$$

$$z = e^{i\left(\frac{2k\pi}{n}\right)}$$

$$k = 0, 1, 2, \ldots n-1$$

$$z^n - 1 = (z-1)\underbrace{(z-\beta)(z-\bar{\beta})}(z-\beta^2)(z-\bar{\beta}^2)$$

$$1+z+z^2+\cdots+z^{n-1} = \left(z^2 - 2\cos\frac{2\pi}{n}z+1\right)\left(z^2-2\cos\frac{4\pi}{n}z+1\right)\cdots\cdots$$



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1. (i) $\prod_{r=1}^{5} \sin\frac{r\pi}{11}$ ✓ (ii) $\prod_{r=1}^{5} \cos\frac{r\pi}{11}$ ✓ (iii) $\prod_{r=1}^{5} \cos\left(\frac{2r\pi}{11}\right)$ ✓

$-\cos\frac{16\pi}{11}$

(iv) $\prod_{r=1}^{5} \sin\frac{2r\pi}{11} = (i)$ 2^5 (i) x (2) $= \boxed{\frac{\sqrt{11}}{32}}$

(iii) $z = i$ $\frac{1+i}{i-i} = \frac{-i-1}{i-1} = \frac{i^{11}-1}{i-1} = -32i\prod_{r=1}^{5}\cos\frac{2r\pi}{11} \Rightarrow \boxed{\prod_{r=1}^{5}\cos\frac{2r\pi}{11} = -\frac{1}{32}}$

$z^{11}-1 = (z-1)\prod_{r=1}^{5}\left(z^2 - 2\cos\frac{2r\pi}{11}z+1\right)$

$(-2i)^5 = -2^5 i$

$1+z+z^2+\cdots z^{10} = \prod_{r=1}^{5}\left(z^2 - 2\cos\frac{2r\pi}{11}z+1\right)$

$z=1$ $\quad 11 = \prod_{r=1}^{5} 4\sin^2\frac{r\pi}{11} \Rightarrow \sqrt{\frac{11}{4^5}} = \prod_{r=1}^{5}\sin\frac{r\pi}{11} = \frac{\sqrt{11}}{32}$

$z=-1$ $\quad 1 = \prod_{r=1}^{5} 4\cos^2\frac{r\pi}{11} = 4^5\left(\prod_{r=1}^{5}\cos\frac{r\pi}{11}\right)^2 \Rightarrow \prod_{r=1}^{5}\cos\frac{r\pi}{11} = \frac{1}{32}$

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2. If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$ are n, n^{th} roots of unity.

Then P.T.

(i) $1^p + \alpha_1^p + \alpha_2^p + \alpha_3^p + \dots + \alpha_{n-1}^p = \begin{cases} 0, & p \neq nk \quad k \in I \\ n, & p = nk \end{cases}$ $p \in I$

$$\frac{1 \cdot \left(e^{i\frac{2\pi p}{n}\cancel{n}} - 1\right)}{e^{i\frac{2\pi p}{n}} - 1} = 0$$

$r = e^{i\frac{2\pi p}{n}} \neq 1, \quad p \neq nk$

(ii) $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3) \dots (1+\alpha_{n-1}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$z^n - 1 = (z-1)(z-\alpha_1)(z-\alpha_2) \dots (z-\alpha_{n-1}) \Rightarrow z = -1$

$$\frac{(-1)^n - 1}{-2} = (-1)^{n-1}(1+\alpha_1) \dots (1+\alpha_{n-1})$$

(iii) $(\omega-\alpha_1)(\omega-\alpha_2) \dots (\omega-\alpha_{n-1}) = \begin{cases} 0, & n = 3k \\ 1, & n = 3k+1 \\ 1+\omega, & n = 3k+2 \end{cases}$ $k \in I$

ω is usual cube root of unity.

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3. P.T. all roots of eqn. $\left(\frac{z+1}{z}\right)^n = 1$, $n \in N$ are collinear on complex plane. Also find sum of real parts of all roots & sum of imaginary parts of all roots. Also find the roots.

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$n \in N,\ (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n$

$x=i \quad (1+i)^n = \left({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots\right) + i\left({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \cdots\right)$

$= \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^n = 2^{n/2} e^{i\frac{n\pi}{4}} = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

✓ ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots = 2^{n/2}\cos\frac{n\pi}{4}$

${}^nC_0 - 2^2\,{}^nC_2 + 2^4\,{}^nC_4 - 2^6\,{}^nC_6 + \cdots$

${}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \cdots = 2^{n/2}\sin\frac{n\pi}{4}$

✓ ${}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \cdots = 2^{n-1}$

${}^nC_1 + {}^nC_3 + {}^nC_5 + {}^nC_7 + \cdots = 2^{n-1}$

${}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots = \frac{1}{2}\left(2^{n/2}\cos\frac{n\pi}{4} + 2^{n-1}\right)$

$C_3 + C_7 + C_{11} + \cdots = ?$

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$C_0 + C_4 + C_8 + \cdots$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + {}^nC_5 x^5 + {}^nC_6 x^6 + \cdots$$

$$(1+1)^n + \underbrace{(1-i)^n + (1+i)^n} + (1-1)^n = 4\left({}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots\right)$$

$(1-1)^n$

$(1+i)^n$

$(1-i)^n$

$C_3 + C_7 + C_{11} + C_{15} + \cdots$

$$e^{i\frac{\pi}{2}}\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^n = 2^{n/2}\, e^{i\left(\frac{n\pi}{4} + \frac{\pi}{2}\right)}$$

$x(1+x)^n$

$$1(1+1)^n - 1(1-1)^n + \underbrace{i(1+i)^n - i(1-i)^n}_{2\,\mathrm{Re}} = 4\left(C_3 + C_7 + C_{11} + \cdots\right)$$

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JEE

P.T. $\sum_{r=1}^{k} (-3)^{r-1} \, {}^{3n}C_{2r-1} = 0$, $k = \frac{3n}{2}$,

n is an even integer