HOMEWORK–3

SOLUTION

Link to View Video Solution: Click Here

DIFFRENTIATION OF A FUNCTION RESPECT TO ANOTHER FUNCTION

1. If $x = e^{\sin^{-1}t}, y = \tan^{-1}t$, then $\frac{dy}{dx} =$

(A) $\frac{1}{1+t^2}e^{-\sin^{-1}t}\sqrt{1-t^2}$ (B) $\frac{1}{1+t^2}e^{-\sin^{-1}t}$

(C) $(1+t^2)e^{-\sin^{-1}t}\sqrt{1-t^2}$ (D) None of these

Ans. **(A)**

Sol. $\frac{dx}{dt} = \frac{e^{\sin^{-1}t}}{\sqrt{1-t^2}}$

$$\frac{dy}{dt} = \frac{1}{1+t^2}$$

$$\frac{1}{1+t^2}e^{-\sin^{-1}t}\sqrt{1-t^2}$$

2. Find derivative of f(tanx) w.r.t. g(secx) at $x = \frac{\pi}{4}$

where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$ is

(A) 3 (B) -4 (C) $\frac{2}{19}$ (D) $\frac{1}{\sqrt{2}}$

Ans. **(D)**

Sol. $y = f(\tan x)$ and $z = g(\sec x)$

$\frac{dy}{dx} = f'(\tan x)\cdot\sec^2x$ and $\frac{dz}{dx} = g'(\sec x)\cdot\sec x\tan x$

$\therefore \frac{dy}{dz} = \frac{f'(\tan x)\cdot\sec^2x}{g'(\sec x)\cdot\sec x\tan x}$

$\left.\frac{dy}{dz}\right|_{x=\frac{\pi}{4}} = \frac{f'(1)\cdot(\sqrt{2})^2}{g'(\sqrt{2})\cdot(\sqrt{2})} = \frac{1}{\sqrt{2}}$

3. Differential coefficient of $\sin^{-1}x$ with respect to $\text{Sin}^{-1}(3x-4x^3)$ is

(A) $\frac{1}{3}$ if $\frac{-\pi}{8} < x < \frac{\pi}{8}$ (B) 3 if $\frac{-\pi}{8} < x < \frac{\pi}{8}$

(C) $\frac{1}{3}$ if $\frac{-\pi}{9} < x < \frac{\pi}{9}$ (D) $\frac{1}{3}$ if $\frac{-1}{2} < x < \frac{1}{2}$

Ans. **(A, C, D)**

Sol. let $u = \sin^{-1}x \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$v = \sin^{-1}(3x-4x^3)$

$\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Link to View Video Solution: Click Here

$$v = \sin^{-1}(3x - 4x^3) = \begin{cases} -\pi - 3\sin^{-1} x; -1 \le x < -\frac{1}{2} \\ 3\sin^{-1} x; \frac{-1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x; \frac{1}{2} < x < 1 \end{cases}$$

$$\frac{dv}{dx} = \begin{cases} \frac{-3}{\sqrt{1-x^2}}; & -1 \le x < -\frac{1}{2} \\ \frac{3}{\sqrt{1-x^2}}; & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{-3}{\sqrt{1-x^2}}; & \frac{1}{2} < x < 1 \end{cases}$$

$-1 \le x < -\frac{1}{2}$ $\qquad$ $-1 \le x \le \frac{1}{2}$ $\qquad$ $\frac{1}{2} < x \le 1$

$\frac{du}{dv} = \frac{1/\sqrt{1-x^2}}{-3/\sqrt{1-x^2}}$ $\qquad$ $\frac{du}{dv} = \frac{1/\sqrt{1-x^2}}{3/\sqrt{1-x^2}}$ $\qquad$ $\frac{du}{dv} = \frac{1/\sqrt{1-x^2}}{-3/\sqrt{1-x^2}}$

$\frac{du}{dv} = -\frac{1}{3}$ $\qquad$ $\frac{du}{dv} = \frac{1}{3}$ $\qquad$ $\frac{du}{dv} = -\frac{1}{3}$

4. The differential coefficient of $\sin^{-1}\frac{t}{\sqrt{1+t^2}}$ w.r.t. $\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ is

(A) $1\ \forall\ t > 0$ $\qquad$ (B) $-1\ \forall\ t < 0$ $\qquad$ (C) $1\ \forall\ t \in R$ $\qquad$ (D) $2\ \forall\ t > 0$

Ans. **(A, B)**

Sol. $\sin^{-1}\frac{t}{\sqrt{1-t^2}}$ w.r.t. $\cos^{-1}\frac{t}{\sqrt{1+t^2}}$

Put $t = \tan\theta\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

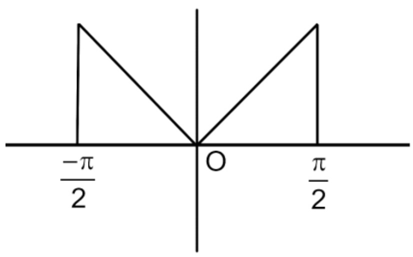
$u = \sin^{-1}\left(\frac{\tan\theta}{|\sec\theta|}\right)$

$\Rightarrow u = \sin^{-1}(\sin\theta)$

$u = \theta \Rightarrow \tan^{-1}t$

$\frac{du}{dt} = \frac{1}{(1+t^2)}$

$V = \cos^{-1}\frac{1}{|\sec\theta|}$



Link to View Video Solution: Click Here

$\Rightarrow \cos^{-1}(\cos\theta)$

$\theta \quad \theta > 0$ | $-\theta,\ \theta < 0$

$\tan^{-1} > 0$ | $t < 0$

$t > 0$

$V = \tan^{-1}t$

$$\frac{dv}{dt} = \frac{1}{1+t^2} \qquad t > 0 \qquad \text{for } t > 0 \quad 1$$

$$\frac{dv}{dt} = \frac{1}{1+t^2}\ , \qquad t < 0 \qquad \&\ \text{for } t < 0 \quad -1$$

LOGARITHMIC FUNCTION/TRIGONOMETRIC SUBSTITUTIONS

5. $y = \cos^{-1}\sqrt{\frac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$ then $\frac{dy}{dx}$ is

(A) $\frac{1}{2(1+x^2)}, x \in R$

(B) $\frac{1}{2(1+x^2)}, x > 0$

(C) $\frac{-1}{2(1+x^2)}, x < 0$

(D) $\frac{1}{2(1+x^2)} < 0$

Ans. **(B, C)**

Sol. $y = \cos^{1}\sqrt{\frac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$

put $x = \tan\theta\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$y = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right)$$

$-\frac{\theta}{2}, x < 0$ | $\frac{\theta}{2}, x > 0$

$$\frac{dy}{dx} = -\frac{1}{2(1+x)}, x < 0, \frac{1}{2(1+x)}, x > 0$$

Link to View Video Solution: Click Here

INFINITE SERIES

6. If $y=\sqrt{x+\sqrt{y+\sqrt{x+\cdots}}}$, then $\frac{dy}{dx}=$

(A) $\frac{x-y^2}{2y^3-2xy+1}$ (B) $\frac{x-y^2}{2y^3-2xy-1}$

(C) $\frac{x+y^2}{2y^3-2xy-1}$ (D) $\frac{y^2-x}{(y^2-x)^3-1}$

Ans. **(D)**

Sol. $y=\sqrt{x+\sqrt{y+y}}$

$(y^2)^2=(\sqrt{x+\sqrt{2y}})^2$

$y^2=x+\sqrt{2y}$

$\left((y^2-x)\right)^2=(\sqrt{2y})^2$ ....(1)

$y^4+x^2-2xy^2=2y$

$4y^3\cdot y'+2x-2(x(2yy')+y^2(1))=2y'$

$(2y^3-2xy-1)y'=y^2-x$

$y'=\frac{y^2-x}{2y^3-2xy-1}$

$y'=\frac{(y^2-x)}{2y(y^2-x)-1}$ {from(1) $(y^2-x)^2=2y$}

$y'=\frac{(y^2-x)}{(y^2-x)^2(y^2-x)-1}$

$y'=\frac{(y^2-x)}{(y^2-x)^3-1}$

DIFFERENTION OF PARAMETRIC EQUATIONS

7. If $x=\frac{1+t}{t^3}, y=\frac{3}{2t^2}+\frac{2}{t}$ then, $x\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=$

(A) 0 (B) -1 (C) 1 (D) 2

Ans. **(C)**

Sol. $x=\frac{1+t}{t^3}, y=\frac{3}{2t^2}+\frac{2}{t}$

$\frac{dx}{dt}=-3t^{-4}-2t^{-3}$

$\frac{dy}{dt}=-3t^{-3}-2t^{-2}$

Link to View Video Solution: Click Here

$$\frac{dy}{dx}=\frac{-t^2(3t^{-1}+2)}{-t^{-2}(3t^{-1}+2)}$$

$$\left(\frac{dy}{dx}\right)=t\left(\frac{3+2t}{3+2t}\right)=t$$

$$x\cdot t^3-t=0$$

$$\frac{1+t}{t^3}\times t^3-t=1$$

8. If $\sin x=\frac{2t}{1+t^2}$ and $\cot y=\frac{1-t^2}{2t}$. Then value of $\frac{d^2x}{dy^2}$ is equal to

(A) 0 (B) 1 (C) -1 (D) $\frac{1}{2}$

Ans. **(A)**

Sol. put $t=\tan\theta$

$\sin x=\sin 2\theta$ $\qquad$ $\cot y=\cot 2\theta$

$x=n\pi+(-1)^n 2\theta$ $\qquad$ $y=m\pi+2\theta$

$\frac{dx}{d\theta}=(-1)^n 2$ $\qquad$ $\frac{dy}{d\theta}=2$

$\frac{dx}{dy}=1 \text{ or } -1$ $\qquad$ $\Rightarrow \frac{d^2x}{dy^2}=0$

MIXED PROBLEMS

9. If $y=\tan^{-1}\left(\frac{\ln\frac{e}{x^2}}{\ln ex^2}\right)+\tan^{-1}\frac{3+2\ln x}{1-6\ln x}$ then

(A) $\frac{dy}{dx}=0$ (B) $\frac{d^2y}{dx^2}=0$ (C) $\frac{dy}{dx}=\frac{2}{x(1+\ln^2x)}$ (D) $\frac{dy}{dx}=1$

Ans. **(A, B)**

Sol. $y=\tan^{-1}\left(\frac{\ln e-\ln(x^2)}{\ln e+\ln(x^2)}\right)+\tan^{-1}\left\{\frac{3+2\ln x}{1-(3)(2\ln x)}\right\}$

$y=\tan^{-1}\left(\frac{1-2\ln x}{1+2\ln x}\right)+\tan^{-1}(3)+\tan^{-1}(2\ln x)$

$y=\tan^{-1}(1)-\tan^{-1}(2\ln x)+\tan^{-1}(3)+\tan^{-1}(2\ln x)$

$$y=\frac{\pi}{4}+\tan^{-1}(3)$$

$$\frac{dy}{dx}=0\Rightarrow\frac{d^2y}{dx^2}=0$$

Link to View Video Solution: Click Here

JEE MAIN

10. If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is - **[AIEEE-2006]**

(A) $\frac{x+y}{xy}$ (B) xy (C) $\frac{x}{y}$ (D) $\frac{y}{x}$

Ans. **(D)**

Sol. $x^m \times y^n = (x+y)^{m+n}$

Taking log both sides we get

$m\log x + n\log y = (m+n)\log(x+y)$

Differentiating w.r.t. x we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{nx-my}{nx-my}\right)\frac{y}{x} = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

11. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. then $y'(1)$ equals :

(A) log2 (B) $-\log 2$ (C) -1 (D) 1 **[AIEEE-2009]**

Ans. **(C)**

Sol. $x^{2x} - 2x^x \cot y - 1 = 0$ ......(i)

at $x = 1$ we have

$1 - 2\cot y - 1 = 0$

$\Rightarrow \cot y = 0 \;\therefore\; y = \pi/2$

Differentiating (i) w.r.t. x, we have

$$2x^{2x}(1 + \ln x) - 2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx}\right.$$

$$\left. + \cot y \cdot x^x(1 + \ln x)\right] = 0$$

At $P(1, \pi/2)$ we have

$$2(1 + \ln 1) - 2\left[1(-1)\left(\frac{dy}{dx}\right)_P + 0\right] = 0$$

$$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_P = 0 \;\therefore\; \left(\frac{dy}{dx}\right)_P = -1$$

Link to View Video Solution: Click Here

12. Let f: $(-1,1) \rightarrow R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$.

Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0)$: – **[AIEEE-2010]**

(A) 4 (B) -4 (C) 0 (D) -2

Ans. **(B)**

Sol. Given, $f(0) = -1, f'(0) = 1$

Also given $g(x) = f(2f(x) + 2)^2$

Thus $g'(x) = 2f[(2f(x) + 2)]2f'(x)$

$\Rightarrow g'(0) = 2f[2f(0) + 2]2f'(0)$

$\Rightarrow g'(0) = 2[-1]2(1) = -4$

13. $\frac{d^2x}{dy}$ equals :- **[AIEEE-2011]**

(A) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (B) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (D) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

Ans. **(B)**

Sol. Here, $\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^{-1}$

Differentiating both sides w.r.t. y, we get

$$\frac{d^2x}{dy^2} = \left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d}{dy} \cdot \left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d}{dy}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dx}$$

$$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d}{dx} \cdot \left(\frac{dy}{dx}\right) \cdot \frac{dx}{dy}$$

$$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx}\right)^{-1}$$

$$= -\left(\frac{dy}{dx}\right)^{-3} \cdot \left(\frac{d^2y}{dx^2}\right)$$

14. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to : **[JEE-MAIN-2013]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\sqrt{2}$

Ans. **(A)**

Sol. $\frac{1}{\sqrt{2}}$

$$\frac{dy}{dx} = \sec(\tan^{-1}x) \cdot \tan(\tan^{-1}x) \times \frac{1}{1 + x^2}$$

Link to View Video Solution: Click Here

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x=1} = \sec(\tan^{-1}1) \times 1 \times \frac{1}{1+1^2}$$

$$= \sec\frac{\pi}{4} \times \frac{1}{2} = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}$$

15. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to : **[JEE-MAIN-2014]**

(A) $1 + x^5$ (B) $5x^4$ (C) $\frac{1}{1+\{g(x)\}^5}$ (D) $1 + \{g(x)\}^5$

Ans. **(D)**

Sol. Here, g is the inverse of f(x).

$\Rightarrow$ $fog(x) = x$

On differentiating w.r.t. x, we get

$f'\{g(x)\} \times g'(x) = 1$

$$g'(x) = \frac{1}{f'\{g(x)\}} = 1 + \{g(x)\}^5 \quad \left[\because f'(x) = \frac{1}{1+x^5}\right]$$

$$g'(x) = \frac{1}{f'\{g(x)\}} = 1 + \{g(x)\}^5 \quad \left[\because f'(x) = \frac{1}{1+x^5}\right]$$

$\Rightarrow g'(x) = 1 + \{g(x)\}^5$

16. If for $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **[JEE(Main)-2016]**

(A) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

(B) g is not differentiable at $x = 0$

(C) $g'(0) = \cos(\log 2)$

(D) $g'(0) = -\cos(\log 2)$

Ans. **(C)**

Sol. We have, $f(x) = |\log 2 - \sin x|$

and $g(x) = f(f(x)), x \in R$

Note that, for $x \to 0, \log 2 > \sin x$

$\therefore f(x) = \log 2 - \sin x$

$\Rightarrow g(x) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is differentiable.

Now, $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$= \cos x \cdot \cos(\log 2 - \sin x)$

$\Rightarrow g'(0) = 1 \cdot \cos(\log 2)$

Link to View Video Solution: Click Here

17. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}.g(x)$ then g(x) equals : **[JEE (Main)2017]**

(A) $\frac{3}{1+9x^3}$ (B) $\frac{9}{1+9x^3}$ (C) $\frac{3x\sqrt{x}}{1-9x^3}$ (D) $\frac{3x}{1-9x^3}$

Ans. **(B)**

Sol. Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left[\frac{2\cdot(3x^{3/2})}{1-(3x^{3/2})^2}\right]$

$= 2\tan^{-1}(3x^{3/2})\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{1+(3x^{3/2})^2}\cdot 3\times\frac{3}{2}(x)^{1/2}$

$= \frac{9}{1+9x^3}\cdot\sqrt{x}$

$\therefore g(x) = \frac{9}{1+9x^3}$

18. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is : **[JEE (Main)2019]**

(A) $\frac{3}{2\sqrt{2}}$ (B) $\frac{1}{6\sqrt{2}}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{6}$

Ans. **(B)**

Sol. $x = 3\tan t, y = 3\sec t$

$\frac{dx}{dt} = 3\sec^2 t, \frac{dy}{dt} = 3\sec t\tan t$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{3\sec t\tan t}{3\sec^2 t}$

$\Rightarrow \frac{dy}{dx} = \sin t \Rightarrow \frac{d^2y}{dx^2} = \cos t\cdot\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \cos t\cdot\frac{1}{3\sec^2 t} \Rightarrow \frac{d^2y}{dx^2} = \frac{\cos^3 t}{3}$

$\frac{d^2y}{dx^2} x = \pi/4 = \frac{1}{6\sqrt{2}}$

19. Let $f: R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3), x \in \mathbf{R}$. Then f(2) equals: **[JEE (Main)2019]**

(A) 8 (B) 30 (C) -4 (D) -2

Ans. **(D)**

Sol. $f(x) = x^3 + x^2 f'(1) + xf''(2) + f''(3), x \in R$

$f'(x) = 3x^2 + 2xf'(1) + f''(2)$

$\Rightarrow f'(1) = 3 + 2f'(1) + f''(2)$

Link to View Video Solution: Click Here

$\Rightarrow f'(1) + f''(2) + 3 = 0$

$f''(x) = 6x + 2f'(1) \Rightarrow f''(2) = 12 + 2f'(1)$

$\Rightarrow -f'(1) - 3 = 12 + 2f'(1)$

$\Rightarrow f'(1) = -5; f''(2) = 2$

$f''(x) = 6 \Rightarrow f''(3) = 6$

$f(x) = x^3 - 5x^2 + 2x + 6$

$f(2) = 8 - 20 + 4 + 6 = -2$

20. If $x\log_e(\log_e x) - x^2 + y^2 = 4 (y > 0)$, then $\frac{dy}{dx}$ at $x = e$ is equal to : **[JEE (Main)2019]**

(A) $\frac{e}{\sqrt{4+e^2}}$ (B) $\frac{(2e-1)}{2\sqrt{4+e^2}}$ (C) $\frac{(1+2e)}{2\sqrt{4+e^2}}$ (D) $\frac{(1+2e)}{\sqrt{4+e^2}}$

Ans. (B)

Sol. Diff. w.r.t. x

$$x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log_e(\log_e x) - 2x + 2y' = 0$$

y at $x = e, y^2 = 4 + e^2$ is $y = \sqrt{4 + e^2}$

y' at $x = e, 1 + 0 - 2e + 2\sqrt{4 + e^2}y'$

$$y' = \frac{2e - 1}{2\sqrt{4 + e^2}}$$

21. For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then $(1 + \log_e 2x)^2 \frac{dy}{dx}$ is equal to :**[JEE (Main)2019]**

(A) $\frac{x\log_e 2x - \log_e 2}{x}$ (B) $x\log_e 2x$

(C) $\log_e 2x$ (D) $\frac{x\log_e 2x + \log_e 2}{x}$

Ans. (A)

Sol. $(2x)^{2y} = 4e^{2x-2y}$

$2y \cdot \ln(2x) = \ln(4) + (2x - 2y)$

$\Rightarrow y = \frac{x+\ln 2}{1+\ln 2x}$

On differentating w.r.t. x

$$\frac{dy}{dx} = \frac{(1 + \ln 2x) \cdot (1 + 0) - (x + \ln 2)\left(\frac{1}{2x}\right) \cdot 2}{(1 + \ln 2x)^2}$$

$$\Rightarrow (1 + \ln 2x)^2 \cdot \frac{dy}{dx} = 1 + \ln 2x - \frac{x + \ln 2}{x}$$

$$\Rightarrow (1 + \ln 2x)^2 \cdot \frac{dy}{dx} = \frac{x\ln 2x - \ln 2}{x}$$

Link to View Video Solution: Click Here

22. If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x+\sin x}{\cos x-\sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0,\frac{\pi}{2}\right)$, then $\frac{dy}{dx}$ is equal to **[JEE (Main)2019]**

(A) $\frac{\pi}{6}-x$ (B) $x-\frac{\pi}{6}$ (C) $2x-\frac{\pi}{3}$ (D) $\frac{\pi}{3}-x$

Ans. **(B)**

Sol. $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x+\sin x}{\cos x-\sqrt{3}\sin x}\right)\right)^2$

$$2y = \left(\cot^{-1}\left(\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}\right)\right)^2$$

$$2y = \left(\cot^{-1}\left(\tan\left(\frac{\pi}{3}+x\right)\right)\right)^2$$

$$2y = \left(\frac{\pi}{2}-\tan^{-1}\left(\tan\left(\frac{\pi}{3}+x\right)\right)\right)^2 = \left(\frac{\pi}{2}-\left(\frac{\pi}{3}+x\right)\right)^2$$

$$2y = \left(x-\frac{\pi}{6}\right)^2$$

$$2y = x^2-\frac{\pi}{3}x+\frac{\pi^2}{36}$$

$$y' = x-\frac{\pi}{6}$$

23. The derivative of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x \in \left(0,\frac{\pi}{2}\right)\right)$ is :

(A) 2 (B) $\frac{2}{3}$ (C) 1 (D) $\frac{1}{2}$ **[JEE (Main)2019]**

Ans. **(A)**

Sol. $u = \tan^{-1}\left\{-\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\right\}$

$$u = -\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$u = -\tan^{-1}\left(\tan\left(\frac{\pi}{4}-x\right)\right)$$

$$u = x-\pi/4$$

$$\frac{du}{dx} = 1$$

$v = \frac{x}{2}$

$\frac{dv}{dx} = \frac{1}{2}$

Link to View Video Solution: Click Here

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{(1/2)} = 2$$

$\frac{du}{dv} = 2$

24. Let $(x)^k + (y)^k = (a)^k$ where $a, k > 0$ and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then find k - **[JEE (Main)2020]**

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) 2

Ans. (B)

Sol. $x^k + y^k = a^k$

Differentiating w.r.t. x

$$kx^{k-1} + k \cdot y^{k-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{kx^{k-1}}{k \cdot y^{k-1}}$$

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1}$$

$$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$$

$$k - 1 = \frac{-1}{3}$$

$k = \frac{2}{3}$

25. If $y^{1/4} + y^{-1/4} = 2x$, and $(x^2 - 1)\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0$ then $|\alpha - \beta|$ is equal to .

[JEE (Main)2021]

Ans. 17

Sol. $y^{1/4} + \frac{1}{y^{1/4}} = 2x$

$$\left(y^{1/4}\right)^2 + 1 = 2x\left(y^{1/4}\right)$$

$$\left(y^{1/4}\right)^2 - 2x\left(y^{1/4}\right) + 1 = 0$$

$$y^{1/4} = x + \sqrt{x^2 - 1} \text{ or } x - \sqrt{x^2 - 1}$$

$$y^{1/4} = x + \sqrt{x^2 - 1}$$

$$\frac{1}{4}y^{-3/4}y' = 1 + \frac{2x}{2\sqrt{x^2 - 1}}$$

$$\frac{y'}{4y^{3/4}} = \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{y^{1/4}}{\sqrt{x^2 - 1}}$$

Link to View Video Solution: Click Here

$y' = \frac{4y}{\sqrt{x^2-1}} \Rightarrow \sqrt{x^2-1} \cdot y' = 4y$

$\sqrt{x^2-1} \cdot y'' + (y')\frac{1}{2\sqrt{x^2-1}}(2x) = 4y'\left(x\sqrt{x^2-1}\right)$

$(x^2-1) \cdot y'' + xy' = 4\left(y'\sqrt{x^2-1}\right)$ $\qquad \because y' = \frac{4y}{\sqrt{x^2-1}}$

$(x^2-1) \cdot y'' + xy' = 16y$ $\qquad \therefore y' \cdot \sqrt{x^2-1} = 4y$

$(x^2-1) \cdot y'' + xy' - 16y = 0$

$\alpha = 1,\ \beta = -16$

$|\alpha - \beta| = |1 + 16| = 17$

26. If $y(x) = (x^x)^x, x > 0$ then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to: **[JEE (Main)2022]**

Ans. **16**

Sol. $y = (x) = (x^x)^x$

$\ln y(x) = x^2 \cdot \ln x$

$$\frac{1}{y(x)} \cdot y'(x) = \frac{x^2}{x} + 2x \cdot \ln x$$

$y'(x) = y(x)[x + 2x\ln x]$

$y(1) = 1; y'(1) = 1$

$y''(x) = y'(x)[x + 2x \cdot \ln(x)]$

$+y(x)[1 + 2(1 + \ln x)]$

$y''(1) = 1[1 + 0] + 1(1 + 2) = 4$

$$\frac{d^2y}{d^2} = -\left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2x}{dy^2} \Rightarrow 4 = -\frac{d^2x}{dy^2}$$

$$\frac{d^2x}{d^2} = -4$$

$-4 + 20 = 16$

27. Let $y(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$ **[JEE (Main)2023]**

Then $y' - y''$ at $x = -1$ is equal to

(A) 976 (B) 464 (C) 496 (D) 944

Ans. **(C)**

Sol. $y = \frac{1-x^{32}}{1-x} \Rightarrow y - xy = 1 - x^{32}$

$y' - xy' - y = -32x^{31}$

$y'' - xy'' - y' - y' = -(32)(31)x^{30}$

at $x = -1 \Rightarrow y' - y'' = 496$

Link to View Video Solution: Click Here

JEE ADVANCED

SECTION-1

28. For $x > 0, \lim_{x\to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is :- **[JEE 2006, 3]**

(A) 0 (B) -1 (C) 1 (D) 2

Ans. **(C)**

Sol. Limit $= \lim_{x\to 0}\left((\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right)$

$= e^{\lim_{x\to 0}\left(\frac{\log\sin x}{x}\right)} + e^{\lim_{x\to 0}\left(\frac{-\log x}{\operatorname{cosec} x}\right)}$

$= e^{-\infty} + e^{\lim_{x\to 0}\left(\frac{-\log x}{\operatorname{cosec} x}\right)}$

$(\because x \to 0^+, \log(\sin x) \to -\infty)$

$= e^{-\infty} + e^{\lim_{x\to 0}\left(-\frac{\frac{1}{x}}{-\operatorname{cosec} x\cot x}\right)}$

$= e^{-\infty} + e^{\lim_{x\to 0}\left(\frac{\tan x}{x}\times\sin x\right)}$

$= e^{-\infty} + e^{0}$

$= 0 + 1$

$= 1$

29. $\frac{d^2x}{dy^2}$ equals :-

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Ans. **(D)**

Sol. Here, $\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^{-1}$

Differentiating both sides w.r.t. y, we get

$\frac{d^2x}{dy^2} = \left(\frac{dy}{dx}\right)^{-2}\cdot\frac{d}{dy}\cdot\left(\frac{dy}{dx}\right)$

$= -\left(\frac{dy}{dx}\right)^{-2}\cdot\frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dx}$

$= -\left(\frac{dy}{dx}\right)^{-2}\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dy}$

$= -\left(\frac{dy}{dx}\right)^{-2}\cdot\frac{d^2y}{dx^2}\cdot\left(\frac{dy}{dx}\right)^{-1}$

$= -\left(\frac{dy}{dx}\right)^{-3}\cdot\left(\frac{d^2y}{dx^2}\right)$

Link to View Video Solution: Click Here

30. (a) Let $g(x) = \ln f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. then for $N = 1,2,3, \ldots\ldots\ldots, g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots\ldots\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots\ldots\cdot\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots\ldots\cdot\frac{1}{(2N+1)^2}\right\}$

Ans. **(A)**

Sol. Since, $f(x) = e^{g(x)}$

$\Rightarrow\ e^{g(x+1)} = f(x+1) = xf(x) = xe^{g(x)}$

and $g(x+1) = \log x + g(x)$

ie, $g(x+1) - g(x) = \log x$ ....(ii)

Replacing x by $x - \frac{1}{2}$, we get

$$g\left(x+\frac{1}{2}\right) - g\left(x-\frac{1}{2}\right) = \log\left(x-\frac{1}{2}\right)$$

$= \log(2x-1) - \log 2$

$\therefore g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = \frac{-4}{(2x-1)^2} \ldots$ (iI)

On substituting, $x = 1,2,3, \ldots, N$ in Eq. (ii) and adding, we get

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N-1)^2}\right\}$$

(b) Let f and g be real valued functions defined on interval $(-1,1)$ such that $g(x)$ is continuous, $g(0) \neq 0,\ g'(0) = 0, g''(0) \neq 0$, and $f(x) = g(x)\sin x$.

Statement-1 : $\lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

And

Statement-2 : $f'(0) = g(0)$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation of statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. **[JEE 2008, 3 + 3]**

Ans. **(A)**

Sol. $f(x) = g(x) \cdot \sin x$

$f'(x) = g(x) \cdot \cos x + g'(x)\sin x$

Link to View Video Solution: Click Here

Put $x = 0 \Rightarrow f'(0) = g(0)$ $\Rightarrow$ Statement "2" is true

Now

$f''(x) = y(x)(-\sin x) + g'(x)\cos x + g'(x)\cos x + g''(x)\sin x$

Put $x = 0 \Rightarrow f''(0) = g'(0) + g'(0) = 2g'(0) = 0$

$f''(0) = 0$

Now,

$\lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x]$

$\lim_{x\to 0}\left(\frac{g(x)\cos x - g(0)}{\sin x}\right)\left(\frac{0}{0}\right)$

$\lim_{x\to 0}\frac{g'(x)\cos x - g(x)\sin x}{\cos x}$

$= g'(0) = 0 = f''(0) \Rightarrow$ Statement "1" is also true

31. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[JEE 2009, 4]**

Ans. **2**

Sol. $f(x) = x^3 + e^{x/2}$

$f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

Given g is inverse of f $\Rightarrow g(f(x)) = x$

Differentiating both sides w.r.t x

$g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

Clearly $f'(0) = 1 \therefore g'(1) = g'(f(0)) = \frac{1}{f'(0)} = 2$

32. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

Ans. **1**

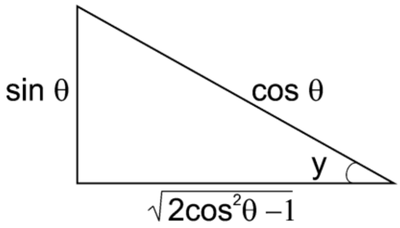
Sol. Given $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

Let $\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = y$

$\Rightarrow \tan y = \frac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}$

Link to View Video Solution: Click Here

Refer to the figure below,



$\Rightarrow \sin y = \dfrac{\sin\theta}{\cos\theta}$

$f(\theta) = \sin y = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$\Rightarrow \dfrac{d}{d(\tan\theta)}(f(\theta)) = 1$

33. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1,3)$ is

[JEE(Advanced)-2014, 3]

Ans. **8**

Sol. Given curve is $(y - x^5)^2 = x(1 + x^2)^2$

$\Rightarrow 2(y - x^5)(d/dx - 5x^4) = (1 + x^2)^2 + 4x(1 + x^2)$

$\Rightarrow 2(y - x5)(dy/dx - 5x^4) = (1 + x^2)(1 + 5x^2)$

when $x = 1, y = 3$, then

$2(3 - 1)(dy/dx - 5) = (1 + 1)(1 + 5) = 12$

$\Rightarrow (dy/dx - 5) = 3$

$\Rightarrow dy/dx = 5 + 3 = 8$

SECTION-2

34. Let $f: R \to R, g: R \to R$ and $h: R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2, g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then-

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$

(C) $h(0) = 16$ (D) $h(g(3)) = 36$ **[JEE(Advanced)-2016, 4(-2)]**

Ans. **(B, C)**

Sol. $g'(f(x)) \cdot f'(x) = 1$

$g'(2) \cdot f'(0) = 1$

$g'(2) = \dfrac{1}{f'(0)}$

$f'(x) = 3x^2 + 3$

$g'(2) = \dfrac{1}{3}$

$h(g(g(x))) = x$

$h(g(g(f(x))) = f(x)$

Link to View Video Solution: Click Here

$h(g(x)) = f(x)$

$h(g(3)) = f(3) = 38$

$h(g(f(x))) = f(f(x))$

$h(x) = f(f(x))$

$h(x) = f'(f(x)) \cdot f'(x)$

$h'(1) = f'(f(1)) \cdot f'(1)$

$= 111 \times 6 = 666$

$h(0) = f(f(0)) = f(2) = 16$

35. For any positive integer n, define $f_n: (0, \infty) \to R$ as

[JEE(Advanced)-2018, 4(0)]

$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ for all $x \in (0, \infty)$

(Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1 + f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

Ans. **(D)**

Sol. The correct option is D For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$= \sum_{j=1}^{n} \left[\tan^{-1}(x+j) - \tan^{-1}(x+j-1)\right]$

$= \tan^{-1}(x+n) - \tan^{-1}x$

$f_n(x) = \tan^{-1}\left[\frac{n}{1+(x+n)x}\right]$

$\Rightarrow \tan[f_n(x)] = \left[\frac{n}{1+(x+n)x}\right]$

$\lim\limits_{x\to\infty} \tan(f_n(x)) = 0$

$\therefore$ (C) is wrong.

$\lim\limits_{x\to\infty} \sec^2(f_n(x)) = \lim\limits_{x\to\infty} (1 + \tan^2(f_n(x))$

Link to View Video Solution: Click Here

$$= 1 + \lim_{x\to\infty}\left[\frac{n}{1+(x+n)x}\right]^2 = 1$$

$\therefore$ (D) is correct.

ANSWER KEY

DIFFRENTIATION OF A FUNCTION RESPECT TO ANOTHER FUNCTION

1. (A) **2.** (D) **3.** (A,C,D) **4.** (A,B)

LOGARITHMIC FUNCTION/TRIGONOMETRIC SUBSTITUTIONS

5. (B,C)

INFINITE SERIES

6. (D)

DIFFERENTION OF PARAMETRIC EQUATIONS

7. (C) **8.** (A)

MIXED PROBLEMS

9. (A,B)

JEE MAIN

10. (D) **11.** (C) **12.** (B) **13.** (B) **14.** (A) **15.** (D) **16.** (C)
17. (B) **18.** (B) **19.** (D) **20.** (B) **21.** (A) **22.** (B) **23.** (A)
24. (B) **25.** (17) **26.** (16) **27.** (C)

JEE ADVANCED

SECTION-1

28. (C) **29.** (D) **30.** **(a)** A; **(b)** A **31.** (2) **32.** (1) **33.** (8)

SECTION-2

34. (B,C) **35.** (D)