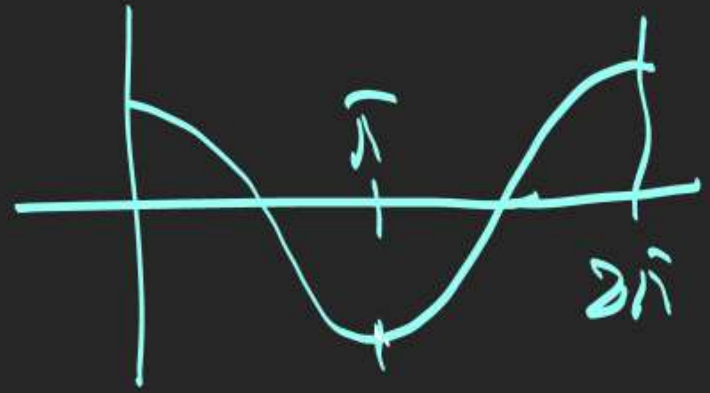
Nishant Jindal

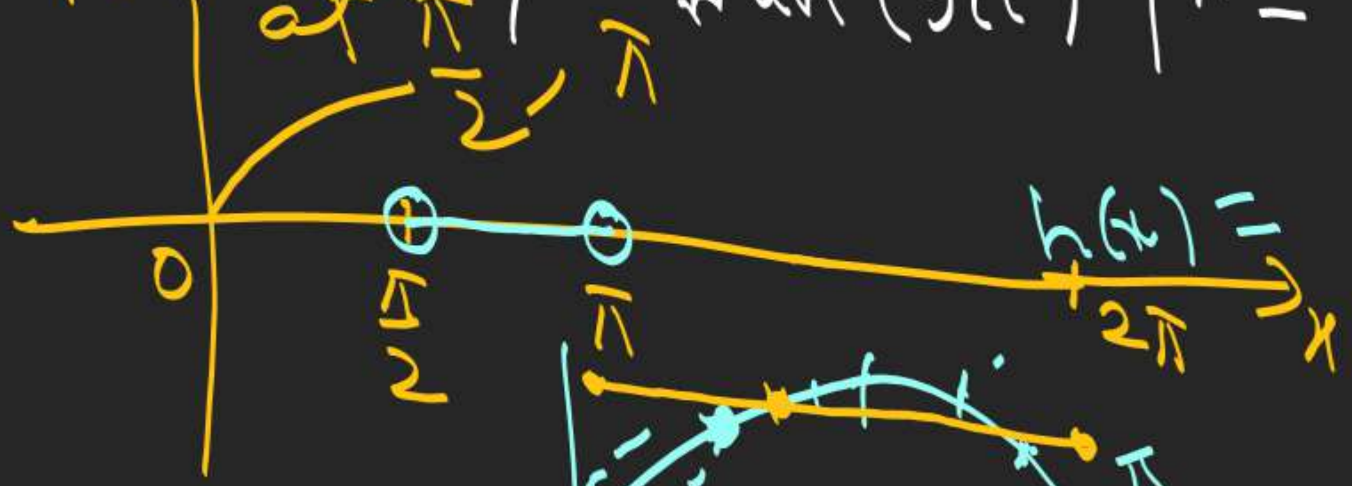
$[0, 2\pi]$

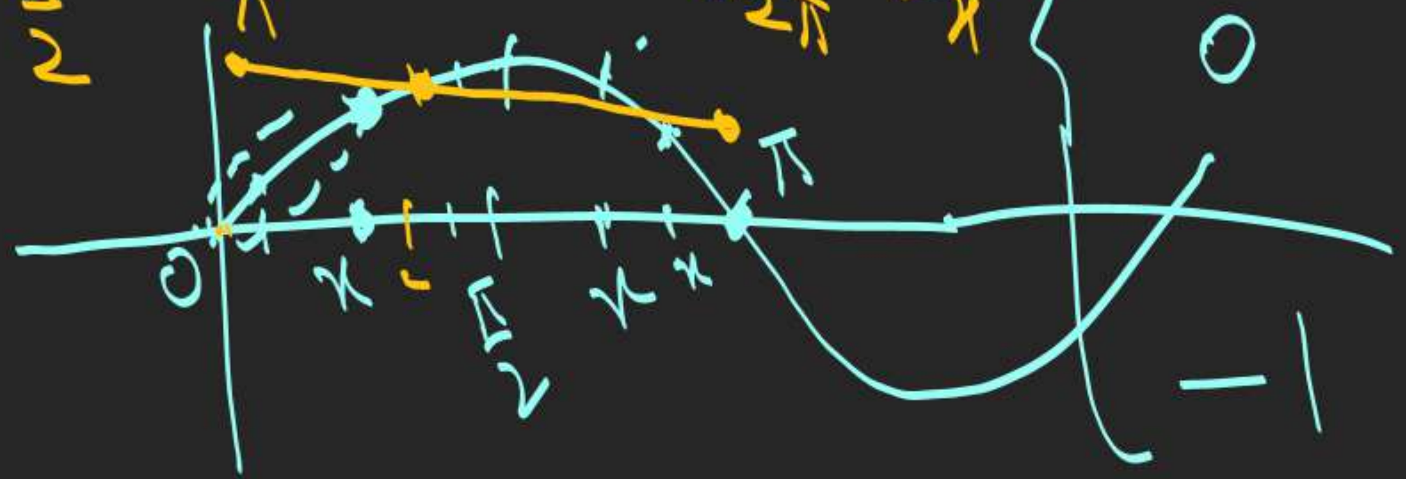
$f(x) = \sin x,\ g(x) = \cos x$

$$h(x) = \begin{cases} \max\left(f(t) \mid 0 \le t \le x\right), & x \in \left[0, \frac{\pi}{2}\right] \\ \min\left(f(t) \mid x \le t \le \pi\right), & x \in \left(\frac{\pi}{2}, \pi\right) \\ \min\left(g(t) \mid \pi \le t \le 2\pi\right), & x \in \left[\pi, 2\pi\right] \end{cases}$$



Discont & Non. diff. at $\frac{\pi}{2}$





$$h(x) = \begin{cases} \sin x, & x \in \left[0, \frac{\pi}{2}\right] \\ 0, & x \in \left(\frac{\pi}{2}, \pi\right) \\ -1, & x \in \left[\pi, 2\pi\right] \end{cases}$$

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Functional Equations

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$f(x+y) = f(x) + f(y)$ Obtain $f'(x)$ in terms of x.

Integrate & get $f(x)$

$$LHD = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \quad h>0$$

$$RHD = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Let f be a differentiable function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$; $\forall\, x, y > 0$. If $f'(1) = 1$, find $f(x)$

$y = \ln x + C \Rightarrow f(x) = \ln x + C$ Put $x = 1$ $f(1) = \ln 1 + C = 0 \Rightarrow \boxed{C = 0}$

$\int dy = \int \frac{dx}{x}$

$\frac{dy}{dx} = \frac{1}{x}$

$\boxed{f(x) = \ln x}$

$x = y = 1$

$f(1) = f(1) - f(1) = 0$

$\boxed{f(1) = 0}$

$\boxed{f'(x) = \frac{1}{x}}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}} = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{x \cdot \frac{h}{x}} = \frac{1}{x} f'(1)$$

$$= \frac{1}{x}$$

$\lim_{x \to 0} 2 \frac{\sin x}{x}$

$2 \lim_{x \to 0} \frac{\sin x}{x}$

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$$f\left(\frac{x}{y}\right) = f(x) - f(y) \quad \forall\ x>0, y>0$$

Diff. w.r.t x

$$\frac{1}{y} f'\left(\frac{x}{y}\right) = f'(x) - 0$$

Put $x = y$

$$\frac{1}{x} f'(1) = f'(x) = \frac{1}{x}$$

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Q: A differentiable function satisfies $f'(x+y) = f'(x) + 2y$, put $x=0$ $\boxed{f'(y) = f'(0) + 2y}$

$\boxed{f(x+y) = f(x) + f(y) + 2xy - 1}$ $\forall x, y \in R$.

If $f'(0) = \sqrt{3+a-a^2}$, find $f(x)$ and also P.T. $f(x) > 0 \quad \forall x \in R$

$= -\frac{3}{4} - \left(a - \frac{1}{2}\right)^2 < 0$

$D = 3 + a - a^2 - 4$

$= a - a^2 - 1$

Put $x = y = 0$

$0 = f(0) - 1 \Rightarrow \boxed{f(0) = 1}$

$$\boxed{f(x) = x^2 + \sqrt{3+a-a^2}\, x + 1}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(h) + 2xh - 1}{h} = \lim_{h \to 0} \left(2x + \frac{f(h) - 1}{h}\right)$$

$$f'(x) = 2x + f'(0)$$

$$\int dy = \int (2x + f'(0))\, dx \Rightarrow y = x^2 + f'(0)x + C$$

$$f(x) = x^2 + f'(0)x + C$$

$x = 0$, $f(0) = C = 1$

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JEE '95 3.

If $\boxed{f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}}$ $\forall\, x, y \in R$. If

$f'(0) = -1$ & $f(0) = 1$, find $f(2)$.

Put $y = 0$, $x \to 2x$

$\boxed{2f(x) = f(2x) + f(0)}$

$f(2x) - 2f(x) = -f(0)$

$$f'(x) = \lim_{h\to 0} \frac{f\left(\frac{(2x)+(2h)}{2}\right) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2} - f(x)}{h} = \lim_{h\to 0} \frac{f(2x)+f(2h)-2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{f(2h)-f(0)}{2h} = f'(0)$$

$\frac{1}{2} f'\left(\frac{x+y}{2}\right) = \frac{f'(x)}{2}$

$x = 0$

$f'\left(\frac{y}{2}\right) = f'(0) = -1$

$f'(x) = -1$

$\int dy = -\int dx$

$y = -x + C$

Put $x = 0$ $\quad 1 = 0 + C$

$\boxed{f(x) = 1 - x}$

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4. Determine all differentiable function on $(-1,1)$ satisfying $f\left(\frac{x+y}{1+xy}\right) = f(x)+f(y)$

$\boxed{f(x) = k\ln\left(\frac{1+x}{1-x}\right)}$

Put $x=y=0 \Rightarrow \boxed{f(0)=0}$

$$f'\left(\frac{x+y}{1+xy}\right)\left(\frac{(1+xy)(1)-(x+y)y}{(1+xy)^2}\right) = f'(x)$$

Put $x=0$

$$f'(y)\left(\frac{1-y^2}{1}\right) = f'(0) \Rightarrow \boxed{f'(x) = \frac{f'(0)}{1-x^2}}$$

$$f'(x) = \frac{f'(0)}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$$

$$\int dy = \frac{f'(0)}{2}\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx \Rightarrow y = \frac{f'(0)}{2}\left(\ln(1+x) - \ln(1-x)\right) + C$$

$$\boxed{f(x) = \frac{f'(0)}{2}\ln\left(\frac{1+x}{1-x}\right)+C}$$

$x=0$, $\boxed{C=0}$

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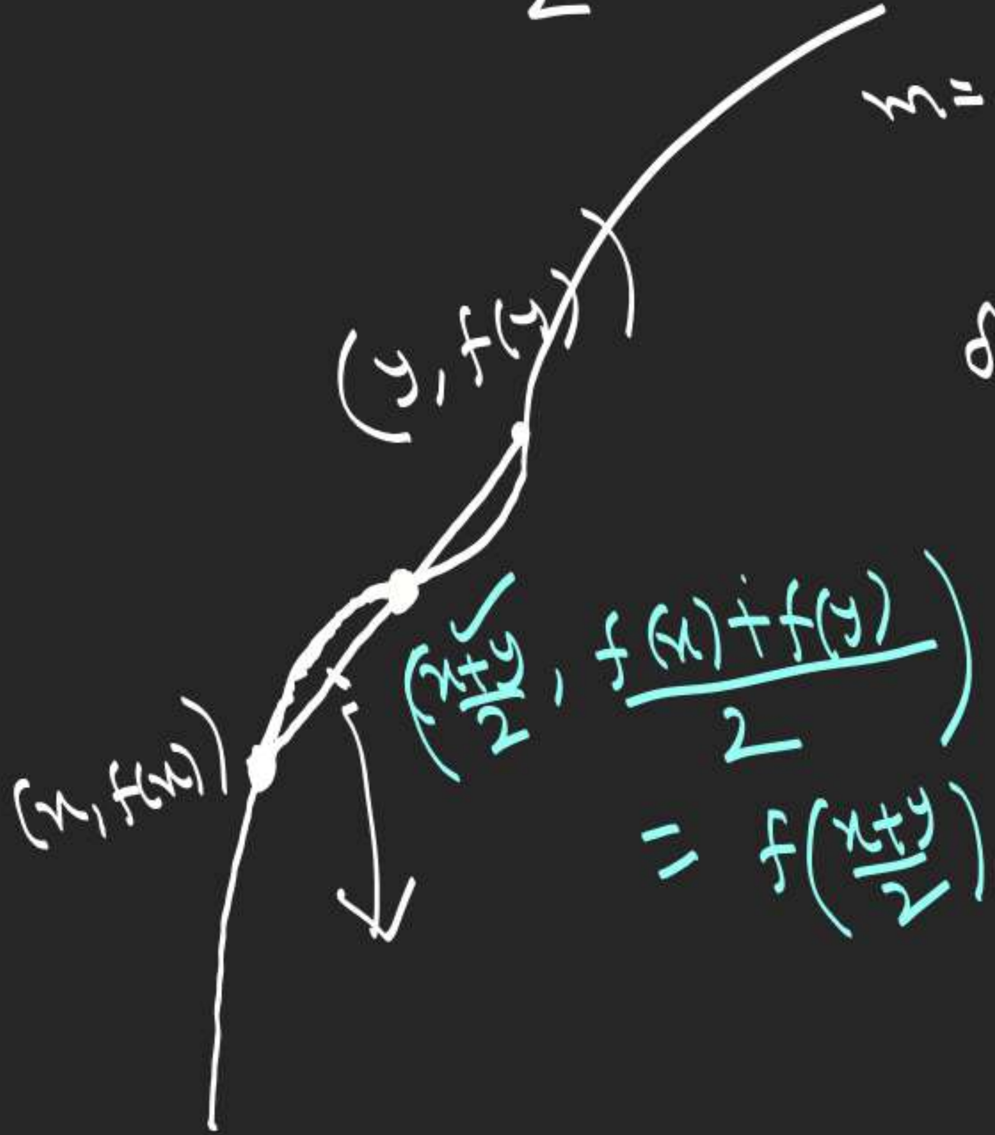
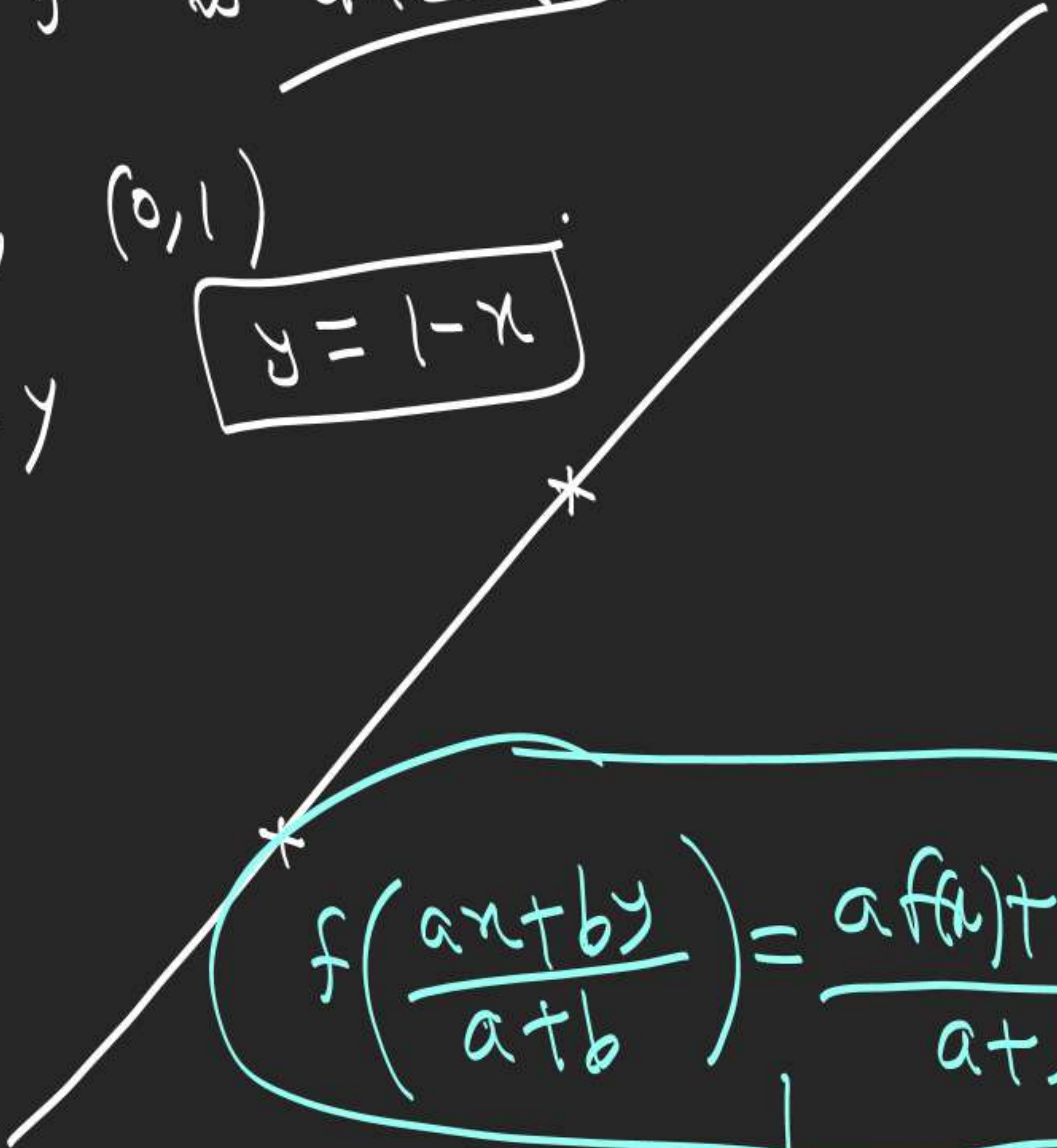
$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \longrightarrow f \text{ is linear}$$

$m = -1, \quad (0,1).$

$$\boxed{y = 1-x}$$

$\delta(x) = y$





$$f\left(\frac{ax+by}{a+b}\right) = \frac{af(x)+bf(y)}{a+b}$$

f is linear.

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$$f\left(\frac{x+y}{1+xy}\right) = f(x)+f(y)$$

$x=y=0$

$$\boxed{f(0)=0}$$

$$x+h = \frac{x+y}{1+xy}$$

$x \to x$ ✓

$$y = \frac{h}{1-x^2-xh}$$

$$h = \frac{y-x^2y}{1+xy}$$

$$h+xyh = y-x^2y$$

$$\frac{h}{1-x^2-xh} = y$$

$y=-x$

$$f(0) = f(x)+f(-x)$$

$$\boxed{-f(x) = f(-x)}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)+f(-x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{h}{1-x(x+h)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{h}{1-x(x+h)}\right) - f(0)}{(1-x(x+h))\left(\frac{h}{1-x(x+h)}\right)}$$

$$= \frac{f'(0)}{1-x^2}$$