KEY CONCEPTS (METHOD OF DIFFERENTIATION)

1. DEFINITION:

If x and $x + h$ belong to the domain of a function f defined by $y = f(x)$, then

$\text{Limit}_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if it exists, is called the Derivative of f at x & is denoted by

$f'(x)$ or $\frac{dy}{dx}$. We have therefore, $f'(x) = \text{Limit}_{h\to 0} \frac{f(x+h)-f(x)}{h}$

2. The derivative of a given function f at a point $x = a$ of its domain is defined as :

$\text{Limit}_{h\to 0} \frac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define $f'(a) = \text{Limit}_{x\to a} \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

3. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE /ab INITIO METHOD:

If f(x) is a derivable function then, $\underset{\delta x\to 0}{\text{Limit}} \frac{\delta y}{\delta x} = \text{Limit}_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$

4. THEOREMS ON DERIVATIVES:

If u and v are derivable function of x, then,

(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(Ku) = K\frac{du}{dx}$,where K is any constant

(iii) $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} \pm v\frac{du}{dx}$ known as **"Product Rule"**

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$ where $v \neq 0$ known as **"QUOTIENT RULE"**

(v) If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ **"CHAIN RULE"**

5. DERIVATIVE OF STANDARDS FUNCTIONS:

(i) $D(x^n) = n \cdot x^{n-1}; x \in R, n \in R, x > 0$

(ii) $D(e^x) = e^x$

(iii) $D(a^x) = a^x. \text{Ina}$, $a > 0$

(iv) $D(\ln x) = \frac{1}{x}$

(v) $D(\log_a x) = \frac{1}{x}\log_a e$

(vi) $D(\sin x) = \cos x$ (vii) $D(\cos x) = -\sin x$ (viii) $D(\tan x) = \sec^2 x$

(ix) $D(\sec x) = \sec x \cdot \tan x$ (x) $D(\text{cosec} x) = -\text{cosec} x \cdot \cot x$

(xii) $D(\cot x) = -\text{cosec}^2 x$ (xii) D (constant) $= 0$ where $\mathbf{D} = \frac{\mathbf{d}}{\mathbf{dx}}$

6. INVERSE FUNCTIONS AND THEIR DERIVATIVES:

(a) **Theorem**: If the inverse functions f & g are defined by $y = f(x)$ & $x = g(y)$ & if $f'(x)$ exists & $f'(x) \neq 0$ then $g'(y) = \frac{1}{f'(x)}$. This result can also be written as,

if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then $\frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right)$ or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy} \neq 0\right]$

(b) **Results** :

(i) $D(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(ii) $(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(iii) $D(\tan^{-1}x) = \frac{1}{1+x^2}, x \in R$

(iv) $D(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(v) $D(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

(vi) $D(\cot^{-1}x) = \frac{-1}{1+x^2}, x \in R$

Note: In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$.

7. LOGARITHMIC DIFFERENTIATION: To find the derivative of:

(i) a function which is the product or quotient of a number of functions **OR**

(ii) a function of the form $[f(x)]^{[(x)}$ where f& g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate. This is called **LOGARITHMIC DIFFERENTIATION.**

8. IMPLICIT DIFFERENTIATION: $\phi(x, y) = 0$

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x & y are present.

9. PARAMETRIC DIFFERENTIATION:

If $y = f(\theta) \& x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

10. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION:

Let $y = f(x);\ z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

11. DERIVATIVES OF ORDER TWO & THREE:

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] & is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''. Similarly, the 3rd order derivative of y w. r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ It is also denoted by $f'''(x)$ or y'''.

12. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

13. L' HOSPITAL'S RULE :

If f(x) & g(x) are functions of x such that :

(i) $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$ OR $\lim\limits_{x\to a} f(x) = \infty = \lim\limits_{x\to a} g(x)$ and

(ii) Both f(x) & g(x) are continuous at $x = a$ &

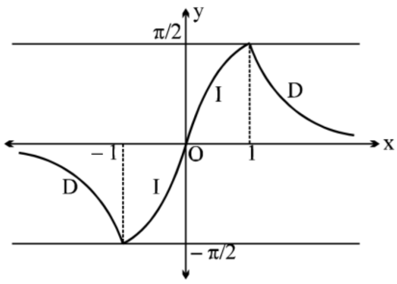
(iii) Both f(x) & g(x) are differentiable at $x = a$ &

(iv) Both $f'(x)$ & $g'(x)$ are continuous at $x = a$, Then

$$\underset{x\to a}{\text{Limit}} \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Limit}} \frac{f'(x)}{g'(x)} = \underset{x\to a}{\text{Limit}} \frac{f''(x)}{g''(x)}$$ & so on till indeterminant form vanishes.

14. ANALYSIS AND GRAPHS OF SOME USEFUL FUNCTIONS:

(i) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$



HIGHLIGHTS:

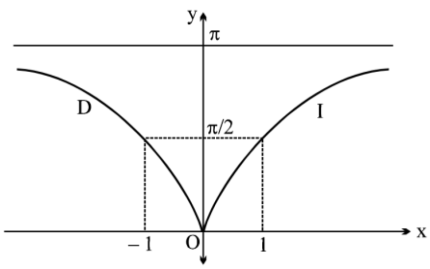
(a) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) f is continuous for all x but not diff. at $x = 1, -1$

(c) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{non existent} & \text{for } |x| = 1 \\ -\frac{2}{1+x^2} & \text{for } |x| > 1 \end{cases}$

(d) I in $(-1,1)$ & D in $(-\infty,-1) \cup (1,\infty)$

(ii) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$



HIGHLIGHTS :

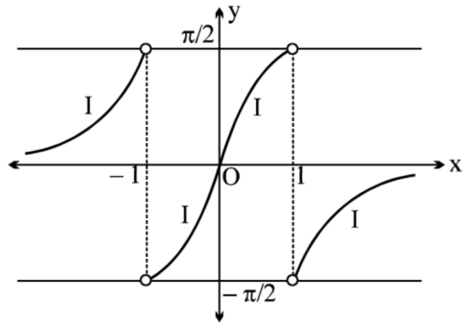
(a) Domain is $x \in R$ & range is $[0, \pi)$

(b) Continuous for all x but not diff. at $x = 0$

(c) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } x > 0 \\ \text{non existent} & \text{for } x = 0 \\ -\frac{2}{1+x^2} & \text{for } x < 0 \end{cases}$

(d) I in $(0,\infty)$ & D in $(-\infty, 0)$

(iii) $y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -(\pi - 2\tan^{-1}x) & x > 1 \end{cases}$



HIGHLIGHTS:

(a) Domain is $R - \{1, -1\}$ & range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(b) f is neither continuous nor diff. at $x = 1, -1$

(c) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(d) I ∀ x in its domain

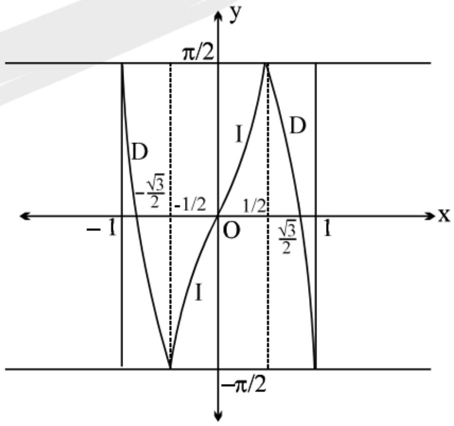
(e) It is bound for all x

(iv) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

HIGHLIGHTS:

(a) Domain is $x \in [-1,1]$ &

range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(b) Not derivable at $|x| = \frac{1}{2}$

(c) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(d) Continuous everywhere in its domain

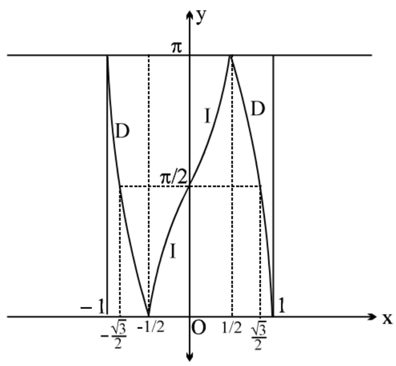
(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

HIGHLIGHTS:

(a) Domain is $x \in [-1,1]$ & range is $[0, \pi]$

(b) Continuous everywhere in its domain

but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$



(c) I in $\left(-\frac{1}{2},\frac{1}{2}\right)$ & D in $\left(\frac{1}{2},1\right] \cup \left[-1,-\frac{1}{2}\right)$

(d) $\frac{dy}{dx} = \begin{bmatrix} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2},\frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1,-\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right) \end{bmatrix}$

GENERAL NOTE:

Concavity in each case is decided by the sign of 2^{nd} derivative as:

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards; $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

D = DECREASING; ; I = INCREASING

PROFICIENCY TEST-01

1. If $f(x) = \frac{2x-4}{x^2-1}$ and $f'(x) = \frac{p}{(x^2-1)^2}$, then p equals-

(A) $x^2 - 8x - 2$ (B) $-2x^2 + 8x + 2$ (C) $4x + 2$ (D) $-2x^2 + 8x - 2$

2. If $y = \sqrt{\frac{1-\cos x}{1+\cos x}}, x \in (0, \pi)$ then $\frac{dy}{dx}$ equals-

(A) $\frac{1}{2}\sec^2 x/2$ (B) $\frac{1}{2}\operatorname{cosec}^2 x/2$ (C) $\sec^2 x/2$ (D) $\operatorname{cosec}^2 x/2$

3. $\frac{d}{d\theta}\left\{\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\right\}$ equals-

(A) 1/2 (B) 1 (C) $\sec\theta$ (D) $\operatorname{cosec}\theta$

4. $d/dx(\sec x^\circ)$ equals -

(A) $\sec x \tan x$ (B) $\sec x^\circ \tan x^\circ$

(C) $\left(\frac{\pi}{180}\right)\sec x^\circ \tan x^\circ$ (D) $\left(\frac{\pi}{180}\right)\sec x \tan x$

5. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \ldots}}}$, then dy/dx equals-

(A) 1 (B) $1/2y$ (C) $1/y - 2$ (D) $1/2y - 1$

6. If $a\cos^2(x+y) = b$, then dy/dx equals-

(A) 2 (B) -2 (C) 1 (D) -1

7. If $y = \log\sqrt{\frac{1-\sin x}{1+\sin x}}$, then dy/dx equals-

(A) $\sec x$ (B) $-\sec x$ (C) $\operatorname{cosec} x$ (D) $\sec x \tan x$

8. If $y = \log_{10}(\sin x)$, then dy/dx equals-

(A) $\sin x \log_{10} e$ (B) $\cos x \log_{10} e$ (C) $\cot x \log_{10} e$ (D) $\cot x$

9. The derivative of $x|x|$ is-

(A) $2x$ (B) $-2x$ (C) $2|x|$ (D) Does not exist

10. The differential coefficient of $\tan^{-1} mx$ with respect to m is-

(A) $\frac{x}{m\sqrt{1+x^2}}$ (B) $\frac{x}{1+m^2x^2}$ (C) $\frac{m^2x}{1+x^2}$ (D) $\frac{m^2x}{1-mx^2}$

PROFICIENCY TEST-02

1. Derivative of $\cos^{-1}\sqrt{x}$ with respect to $\sqrt{1-x}$ is-

(A) $\sqrt{x}$ (B) $-1/\sqrt{x}$ (C) $-\sqrt{x}$ (D) $1/\sqrt{x}$

2. If $x = a\sin^3 t, y = a\cos^3 t$ then dy/dx equals-

(A) $\tan t$ (B) $\cot t$ (C) $-\tan t$ (D) $-\cot t$

3. If $x = t^2 + \frac{1}{t^2}$, $y = t^4 + \frac{1}{t^4}$, then dy/dx equals-

(A) $2x$ (B) x (C) x^2 (D) None of these

4. If $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, then dy/dx equals-

(A) $(y/x)^{\frac{1}{3}}$ (B) $-(y/x)^{\frac{1}{3}}$ (C) $(x/y)^{1/3}$ (D) $-(x/y)^{1/3}$

5. If $y = a\,\sin mx + b\cos mx$, then the value of $\frac{d^2y}{dx^2}$ equals –

(A) m^2y (B) $-m^2y$

(C) $-am^2\sin x + bm^2\cos x$ (D) None of these

6. If $x + y = x^y$, then dy/dx equals-

(A) $\frac{yx^{y-1}-1}{1-x^y\log x}$ (B) $\frac{yx^{y-1}-1}{x^y\log x-1}$ (C) $\frac{yx^{y-1}+1}{x^y\log x+1}$ (D) None of these

7. If $y = \log(x^x)$, then dy/dx equals-

(A) $\log(ex)$ (B) $\log\left(\frac{e}{x}\right)$ (C) $\log\left(\frac{x}{e}\right)$ (D) 1

8. If $y = e^{ax+b}$, then $(y_2)_0$ is equal to –

(A) ae^b (B) e^b (C) a^2e^a (D) a^2e^b

9. The differential coefficient of the function, $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t. x^2 is-

(A) $\frac{1}{1+x^2}$ (B) $\frac{1}{1-x^2}$ (C) $\frac{1}{x(1+x^2)}$ (D) $\frac{x}{1+x^2}$

10. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots}}}$ ..., then dy/dx equals-

(A) $\frac{x}{2y+1}$ (B) $\frac{1}{x(2y-1)}$ (C) $\frac{2y-1}{x}$ (D) $x(2y-1)$

PROFICIENCY TEST-03

Evaluate the following limits using L'Hospital's Rule or otherwise:

1. $\text{Lim}_{x\to 0}\left[\frac{1}{x\sin^{-1}x}-\frac{1-x^2}{x^2}\right]$

2. $\text{Lim}_{x\to 0}\frac{x+\ln\left(\sqrt{x^2+1}-x\right)}{x^3}$

3. $\text{Lim}_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$

4. $\text{Lim}_{x\to 0^+}\ X^{(X^X-1)}$

5. $\text{Lim}_{x\to 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x\cdot\tan^2 x}$

6. Determine the values of a, b and c so that $\text{Lim}_{x\to 0}\frac{(a+b\cos x)x-c\sin x}{x^5}=1$

7. $\text{Lim}_{x\to\frac{\pi}{2}}\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}$

8. $\text{Lim}_{x\to 0}\frac{3x\ln\left(\frac{\sin x}{x}\right)^2+x^3}{(x-\sin x)(1-\cos x)}$

9. Find the value of f(0) so that the function $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$, $x\neq 0$ is continuous at $x=0$ & examine the differentiability of f(x) at $x=0$.

10. $\text{Lim}_{x\to 0}\frac{\sin(3x^2)}{\ln\cdot\cos(2x^2-x)}$

11. If $\text{Lim}_{x\to 0}\frac{a\sin x-bx+cx^2+x^3}{2x^2\cdot\ln(1+x)-2x^3+x^4}$ exists & is finite, find the values of a, b, c & the limit.

12. Evaluate: $\text{Lim}_{x\to 0}\frac{x^{6000}-(\sin x)^{6000}}{x^2\cdot(\sin x)^{6000}}$

13. If $\text{Lim}_{x\to 0}\frac{1-\cos x\cdot\cos 2x\cdot\cos 3x\ldots\ldots\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n\in N$).

14. Given a real valued function f(x) as follows:

$f(x)=\frac{x^2+2\cos x-2}{x^4}$ for $x<0$; $f(0)=\frac{1}{12}$ & $f(x)=\frac{\sin x-\ln(e^x\cos x)}{6x^2}$ for $x>0$. Test the continuity and differentiability of f(x) at $x=0$.

15. If $\text{Lim}_{x\to 0}\frac{1-\cos 3x\cdot\cos 9x\cdot\cos 27x\ldots\ldots\cos 3^n x}{1-\cos\frac{1}{3}x\cdot\cos\frac{1}{9}x\cdot\cos\frac{1}{27}x\ldots\ldots\cos\frac{1}{3^n}x}=3^{10}$, find the value of n.

EXERCISE-I

1. **(a)** If $y = (\cos x)^{\ln x} + (\ln x)^x$ find $\frac{dy}{dx}$.

(b) If $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$. Find $\frac{dy}{dx}$.

2. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ln y'$. Where ' denotes the derivative.

3. If $x = \operatorname{cosec}\theta - \sin\theta$; $y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$.

4. If $y = \sec 4x$ and $x = \tan^{-1}(t)$, prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$.

5. If $x = \frac{1+\ln t}{t^2}$ and $y = \frac{3+2\ln t}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

6. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w.r.t. $\sqrt{1-x^4}$.

7. Find the derivative with respect to x of the function:

$(\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \arcsin\frac{2x}{1+x^2}$ at $x = \frac{\pi}{4}$.

8. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3 \cdot (x^3 - y^3)$, prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

9. If $y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots\ldots\ldots}}}$, prove that $\frac{dy}{dx} = \cfrac{1}{2-\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots\ldots\ldots}}}}$

10. Suppose $f(x) = \tan(\sin^{-1}(2x))$

(a) Find the domain and range of f.

(b) Express f(x) as an algebraic function of x.

(c) Find $f'(1/4)$

11. Prove that the curves $y_1 = f(x)$ $(f(x) > 0)$ and $y_2 = f(x)\sin ax$, where f(x) is a differentiable function, are tangent to each other at the common points.

12. If $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$, find $\frac{dy}{dx}$ if $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

13. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1,1)$.

14. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + \cdots$ ... to n terms. Find dy/dx, expressing your answer in 2 terms.

15. If $y = \ln\left(x^{e^x \cdot a^y}\right)^{y^x}$ find $\frac{dy}{dx}$.

16. If $x = \tan\frac{y}{2} - \ln\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$. Show that $\frac{dy}{dx} = \frac{1}{2}\sin y(1+\sin y+\cos y)$.

17. If $y = \frac{2}{\sqrt{a^2 - b^2}}\left(\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right)$, then show that $\frac{d^2y}{dx^2} = \frac{b\sin x}{(a + b\cos x)^2}$

18. If f: R → R is a function such that $f(x) = x^3 + x^2f'(1) + xf''(2) + f'''(3)$ for all $x \in R$, then prove that $f(2) = f(1) - f(0)$.

19. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Find the continuous function f(x) satisfying $f'(1) = f(-1)$

20.

	Column-1		Column-II
(A)	$f(x) = \begin{cases} \ln(1+x^3)\cdot\sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(P)	continuous everywhere but not differentiable at $x = 0$
(B)	$g(x) = \begin{cases} \ln^2(1+x)\cdot\sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(Q)	differentiable at $x = 0$ but derivative is discontinuous at $x = 0$
(C)	$u(x) = \begin{cases} \ln\left(1+\frac{\sin x}{2}\right), & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(R)	differentiable and has Continuous derivative
(D)	$v(x) = \text{Lim}_{t\to0}\frac{2x}{\pi}\tan^{-1}\left(\frac{2}{t^2}\right)$	(S)	continuous and differentiable at $x = 0$

EXERCISE-II

1. If $\sin y = x\sin(a+y)$, show that $\frac{dy}{dx} = \frac{\sin a}{1-2x\cos a+x^2}$.

2. Find a polynomial function f(x) such that $f(2x) = f'(x)f''(x)$.

3. If $y = \arccos\sqrt{\frac{\cos 3x}{\cos^3 x}}$ then show that $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x+\cos 4x}}$, $\sin x > 0$.

4. Let $y = x\sin kx$. Find the possible value of k for which the differential equation

$\frac{d^2y}{d^2} + y = 2k\cos kx$ holds true for all $x \in R$. true for all $x \in R$.

5. Prove that if $|a_1\sin x + a_2\sin 2x + \cdots \ldots . + a_n\sin nx| \le |\sin x|$ for $x \in R$, then

$|a_1 + 2a_2 + 3a_3 + \cdots \ldots + na_n| \le 1$

6. The function $f: R \to R$ satisfies $f(x^2)\cdot f''(x) = f'(x)\cdot f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

7.(a) Show that the substitution $z = \ln\left(\tan\frac{x}{2}\right)$ changes the equation $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\operatorname{cosec}^2 x = 0$ to $(d^2y/dz^2) + 4y = 0$

(b) If the dependent variable y is changed to ' z ' by the substitution $y = \tan z$ then the differential equation $\frac{d^2y}{dx} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$ is changed to $\frac{d^2z}{dx^2} = \cos^2 z + k\left(\frac{dz}{dx}\right)^2$, then find the value of k.

8. Let $f(x) = \frac{\sin x}{x}$ if $x \ne 0$ and $f(0) = 1$. Define the function $f'(x)$ for all x and find $f''(0)$ if it exist.

9. Suppose f and g are two functions such that $f, g: R \to R$,

$f(x) = \ln\left(1+\sqrt{1+x^2}\right)$ and $g(x) = \ln\left(x+\sqrt{1+x^2}\right)$

then find the value of $xe^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x)$ at $x = 1$.

10. Let $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then prove that

(a) f is continuous and differentiable for all x.

(b) f' is continuous and differentiable for all x.

11. $f: [0,1] \to R$ is defined as $f(x) = \begin{cases} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$, then prove that

(a) f is differentiable in [0,1] (b) f is bounded in [0,1] (c) f' is bounded in [0,1]

12. Let $f(x)$ be a derivable function at $x = 0$ & $f\left(\frac{x+y}{k}\right) = \frac{f(x)+f(y)}{k}$ $(k \in R, k \neq 0,2)$. Show that $f(x)$ is either a zero or an odd linear function.

13. Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-a}{2} + xy$ for all real x and y. If $f(x)$ is differentiable and $f'(0)$ exists for all real permissible values of ' a ' and is equal to $\sqrt{5a-1-a^2}$. Prove that $f(x)$ is positive for all real x.

14. If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ then find $f'(x)$.

15. If α be a repeated root of a quadratic equation $f(x) = 0$ & $A(x), B(x), C(x)$ be the polynomials of degree 3 , 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

16. If $Y = sX$ and $Z = tX$, where all the letters denotes the functions of X and suffixes denotes the differentiation w.r.t. x then prove that

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

17. If $y = 1 + \frac{x_1}{x-x_1} + \frac{x_2 \cdot x}{(x-x_1)(x-x_2)} + \frac{x_3 \cdot x^2}{(x-x_1)(x-x_2)(x-x_3)} + \ldots\ldots.$ upto $(n+1)$ terms then prove that

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \frac{x_3}{x_3-x} + \cdots + \frac{x_n}{x_n-x}\right]$$

18. If $\sqrt{x^2+y^2} = e^{\arcsin\frac{y}{\sqrt{x^2+y^2}}}$. Prove that $\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}, x > 0$

EXERCISE-III

1. If $y = x - x^2$, then the derivative of y^2 w.r.t. x^2 is-

(A) $2x^2 + 3x - 1$ (B) $2x^2 - 3x + 1$ (C) $2x^2 + 3x + 1$ (D) None of these

2. If $y = \log\sqrt{\left(\frac{1+\cos x}{1-\cos x}\right)}$, then the value of $\frac{dy}{dx}$ is-

(A) $\sec x$ (B) $\operatorname{cosec} x$ (C) $-\sec x$ (D) $-\operatorname{cosec} x$

3. If $y = \frac{x}{(x+5)}$, then $\frac{dx}{dy}$ equals-

(A) $\frac{5}{(1-y)^2}$ (B) $\frac{5}{(1+y)^2}$ (C) $\frac{1}{(1-y)^2}$ (D) None of these

4. If $y = e^{3x}\sin 4x$, then the value of dy/dx is-

(A) $e^{3x}\sin\left(4x + \tan^{-1}\frac{4}{3}\right)$ (B) $e^{3x}\cos\left(4x + \tan^{-1}\frac{4}{3}\right)$

(C) $5e^{3x}\sin\left(4x + \tan^{-1}\frac{4}{3}\right)$ (D) $5e^{3x}\cos\left(4x + \tan^{-1}\frac{4}{3}\right)$

5. If $x = t + \frac{1}{t}, y = t - \frac{1}{t}$, then $\frac{d^2y}{dx^2}$ equals –

(A) $-4t(t^2 - 1)^{-2}$ (B) $-4t^3(t^2 - 1)^{-3}$

(C) $(t^2 + 1)(t^2 - 1)^{-1}$ (D) $-4t^2(t^2 - 1)^{-2}$

6. If $e^x + e^y = e^{x+y}$, then dy/dx equals-

(A) e^{x-y} (B) e^{y-x} (C) $-e^{x-y}$ (D) $-e^{y-x}$

7. If $y = x^x + a^x + x^a$, then $\frac{dy}{dx}$ at $x = 1$ equals-

(A) $1 + a\log a$ (B) $1 + a + \log a$ (C) $a + a\log(ea)$ (D) $1 + a\log(ea)$

8. If $(\cos x)^y = (\sin y)^x$, then dy/dx equals-

(A) $\frac{\log\sin y - y\tan x}{\log\cos x + x\cot y}$ (B) $\frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$ (C) $\frac{\log\sin y + y\tan x}{\log\cos x + x\cot y}$ (D) None of these

9. If $y = \sqrt{x}^{\sqrt{x}^{\sqrt{x}^{\sqrt{x}\ldots\ldots\infty}}}$, then the value of dy/dx is-

(A) $\frac{xy^2}{2 - y\log x}$ (B) $\frac{x^2}{y(2 - y\log x)}$ (C) $\frac{y^2}{x(2 - y\log x)}$ (D) $\frac{y^2}{x(2 + y\log x)}$

10. If $y = (\tan x)^{\tan x^{\tan x \ldots\ldots\infty}}$, then $\tan x\left(\frac{dy}{dx}\right)$ equals-

(A) $\frac{y^2\sec^2 x}{1-y\log\tan x}$ (B) $\frac{y^2\sec^2 x}{1+y\log\tan x}$ (C) $\frac{y\sec^2 x}{1-y\log\tan x}$ (D) None of these

11. Let f, g and h are differentiable functions. If $f(0) = 1; g(0) = 2; h(0) = 3$ and the derivatives of their pair wise products at $x = 0$ are

$(fg)'(0) = 6; (gh)'(0) = 4$ and $(hf)'(0) = 5$

then compute the value of $(fgh)'(0)$.

(A) 12 (B) 15 (C) 16 (D) 20

12. Let $f(x) = x + \frac{1}{2x+}\frac{1}{2x+}\frac{1}{2x+}\ldots\ldots\infty$. Compute the value of $f(100)\cdot f'(100)$.

(A) 1 (B) 10 (C) 50 (D) 100

13. Let $f(x) = x^2 - 4x - 3, x > 2$ and let g be the inverse of f. Find the value of g' where $f(x) = 2$.

(A) 0 (B) 1/6 (C) 6 (D) does not exist

14. If $x = 2\cos t - \cos 2t$ & $y = 2\sin t - \sin 2t$, find the value of (d^2y/dx^2) when $t = (\pi/2)$.

(A) $-3/2$ (B) $3/2$ (C) $-1/2$ (D) $1/2$

15. If $f(x) = \begin{vmatrix}(x-a)^4 & (x-a)^3 & 1\\ (x-b)^4 & (x-b)^3 & 1\\ (x-c)^4 & (x-c)^3 & 1\end{vmatrix}$ then $f'(x) = \lambda\cdot\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\ (x-b)^4 & (x-b)^2 & 1\\ (x-c)^4 & (x-c)^2 & 1\end{vmatrix}$. Find the value of λ.

(A) 0 (B) 1 (C) 3 (D) 4

EXERCISE-IV

1. If $y = \log_y x$, then $\frac{dy}{dx} =$ **[AIEEE 2002]**

(A) $\frac{1}{x+\log y}$ (B) $\frac{1}{\log x(1+y)}$ (C) $\frac{1}{x(1+\log y)}$ (D) $\frac{1}{y+\log x}$

2. If $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$, then $\frac{dy}{dx}$ **[AIEEE 2002]**

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$ (D) $\cot\theta$

3. If $y = \left(x + \sqrt{1+x^2}\right)^n$ then $(1+x^2)y_2 + xy_1 =$ **[AIEEE 2002]**

(A) ny^2 (B) n^2y (C) n^2y^2 (D) None of these

4. If $f(x) = x^n$, then the value of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \cdots \ldots + \frac{(-1)^n f^n(1)}{n!}$ is: **[AIEEE 2003]**

(A) 1 (B) 2^n (C) 2^{n-1} (D) 0

5. Let $f(x) = ax^2 + bx + c$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A.P., then $f'(a), f'(b)$ and $f'(c)$ are in: **[AIEEE 2003]**

(A) A.P. (B) G.P. (C) H.P. (D) None of these

6. If $x = e^{y+e^{y+\ldots\ldots\text{to }\infty}}$, $x > 0$, then $\frac{dy}{dx}$ is: **[AIEEE 2004]**

(A) $\frac{x}{1+x}$ (B) $\frac{1}{x}$ (C) $\frac{1-x}{x}$ (D) $\frac{1+x}{x}$

7. If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is: **[AIEEE 2007]**

(A) $\frac{x+y}{xy}$ (B) xy (C) $\frac{x}{y}$ (D) $\frac{y}{x}$

8. Let y be an implicit function of x defined by $x^{2x} - 2x^x\cot y - 1 = 0$. Then $y'(1)$ equals

(A) –1 (B) 1 (C) $\log 2$ (D) $-\log 2$ **[AIEEE 2009]**

9. Let $f: (-1,1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x)+2)]^2$. Then $g'(0) =$ **[AIEEE 2010]**

(A) -2 (B) 4 (C) -4 (D) 0

10. $\frac{d^2x}{dy^2}$ equals : **[AIEEE 2011]**

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

11. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to **[IIT Main-2014]**

(A) $1 + \{g(x)\}^5$ (B) $1 + x^5$ (C) $5x^4$ (D) $\frac{1}{1+\{g(x)\}^5}$

12. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at $(1,1)$: **[IIT Main-2015]**

(A) meets the curve again in the fourth quadrant

(B) does not meet the curve again

(C) meets the curve again in the second quadrant

(D) meets the curve again in the third quadrant

13. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **[IIT Main-2016]**

(A) g is not differentiable at $x = 0$

(B) $g'(0) = \cos(\log 2)$

(C) $g'(0) = -\cos(\log 2)$

(D) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

14. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals: **[IIT Main-2017]**

(A) $\frac{3x}{1-9x^3}$ (B) $\frac{3}{1+9x^3}$ (C) $\frac{9}{1+9x^3}$ (D) $\frac{3x\sqrt{x}}{1-9x^3}$

15. If $x^2 + y^2 + \sin y = 4$, then the value of $\frac{d^2y}{dx^2}$ at the point $(-2, 0)$ is : **[IIT Main-2018]**

(A) –34 (B) –32 (C) 4 (D) –2

EXERCISE-V

1. If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, then find the domain and the range of f. Show that f is one-one. Also find the function $\frac{df^{-1}(x)}{dx}$ and its domain. **[REE '99, 6]**

2.(a) If $x^2 + y^2 = 1$, then : **[JEE 2000, Screening, 1 out of 35]**

(A) $yy'' - 2(y')^2 + 1 = 0$
(B) $yy'' + (y')^2 + 1 = 0$
(C) $yy'' - (y')^2 - 1 = 0$
(D) $yy'' + 2(y')^2 + 1 = 0$

(b) Suppose $p(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$ prove that

$| a_1 + 2a_2 + \cdots\ldots + na_n | \le 1$ **[JEE 2000 (Mains) 5 out of 100]**

3.(a) If $\ln(x + y) = 2xy$, then $y'(0) =$ **[JEE 2004 (Scr.)]**

(A) 1 (B) –1 (C) 2 (D) 0

(b) $f(x) = \begin{cases} b\sin^{-1}\left(\frac{x+c}{2}\right), & -\frac{1}{2} < x < 0 \\ \frac{1}{2} & \text{at } x = 0 \\ \frac{e^{ax/2} - 1}{x}, & 0 < x < \frac{1}{2} \end{cases}$.

If f(x) is differentiable at $x = 0$ and $|c| < 1/2$ then find the value of ' a ' and prove that

$64b^2 = 4 - c^2$. **[JEE 2004, 4 out of 60]**

4.(a) If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$, then $y''(0)$

(A) 1 (B) –1 (C) π (D) $-\pi$

(b) If P(x) is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(1) = 1, P(0) = 0$ and $P'(x) > 0\ \forall\ x \in [0,1]$, then

(A) $S = \phi$
(B) $S = \{(1 - a)x^2 + ax, 0 < a < 2\}$
(C) $(1 - a)x^2 + ax, a \in (0, \infty)$
(D) $S = \{(1 - a)x^2 + ax, 0 < a < 1\}$

(c) If f(x) is a continuous and differentiable function and $f(1/n) = 0, \forall\ n \ge 1$ and $n \in I$, then

(A) $f(x) = 0, x \in (0,1]$
(B) $f(0) = 0, f'(0) = 0$ **[JEE 2005 (Scr.)]**
(C) $f'(x) = 0 = f''(x), x \in (0,1]$
(D) $f(0) = 0$ and $f'(0)$ need not to be zero

(d) If $f(x - y) = f(x) \cdot g(y) - f(y) \cdot g(x)$ and $g(x - y) = g(x) \cdot g(y) + f(x) \cdot f(y)$ for all $x, y \in R$. If right hand derivative at $x = 0$ exists for f(x). Find derivative of g(x) at $x = 0$.

[JEE 2005 (Mains), 4]

5. For $x > 0$, $\text{Lim}_{x\to 0}\left((\sin x)^{\frac{1}{x}} + (1/x)^{\sin x}\right)$ is **[JEE 2006, 3]**

(A) 0 (B) -1 (C) 1 (D) 2

6. $\frac{d^2x}{dy^2}$ equals **[JEE 2007, 3]**

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

7.(a) Let $g(x) = \ln f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that

$f(x+1) = xf(x)$. Then for $N = 1, 2, 3$

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots.+\frac{1}{(2N-1)^2}\right\}$ (B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots\ldots+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots.+\frac{1}{(2N+1)^2}\right\}$ (D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots.+\frac{1}{(2N+1)^2}\right\}$

(b) Let f and g be real valued functions defined on interval $(-1, 1)$ such that g " (x) is continuous, $g(0) \neq 0, g'(0) = 0, g''(0) \neq 0$, and $f(x) = g(x)\sin x$ **[JEE 2008, 3+3]**

STATEMENT-1: $\text{Lim}_{x\to 0}\,[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

and

STATEMENT-2 : $f'(0) = g(0)$

(A) Statement-1 is True, Statement-2 is True ; statement-2 is a correct explanation for statement-1

(B) Statement-1 is True, Statement-2 is True ; statement-2 is NOT a correct explanation for statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

8. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is. **[JEE 2009]**

9. Let $f: (0,1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on $(0, 1)$

(B) $f \neq f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$ **[JEE 2011]**

(C) $f = f^{-1}$ on $(0, 1)$ and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on $(0, 1)$

10. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}\left(f(\theta)\right)$ is

[JEE 2011]

11. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is : **[JEE Advance 2013]**

(A) 6 (B) 4 (C) 2 (D) 0

12. Let $f(x) = x\sin\pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at **[JEE Advance 2013]**

(A) a unique point in the interval $\left(n, n + \frac{1}{2}\right)$

(B) a unique point in the interval $\left(n + \frac{1}{2}, n + 1\right)$

(C) a unique point in the interval $(n, n + 1)$

(D) two points in the interval $(n, n + 1)$

13. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1,3)$ is

[JEE Advance-2014]

14. Let $f, g: [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table: **[JEE Advanced-2015]**

	$x = -1$	$x = 0$	$x = 2$
f(x)	3	6	0
g(x)	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

Answer the following by appropriately matching the lists based on the information given in the paragraph (Q.15 to Q.16) **[IIT Advanced 2019]**

Let $f(x) = \sin(\pi\cos x)$ and $g(x) = \cos(2\pi\sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$X = \{x : f(x) = 0\}$, $Y = \{x : f'(x) = 0\}$,

$Z = \{x : g(x) = 0\}$, $W = \{x : g'(x) = 0\}$.

List-I contains the sets X, Y, Z and W. List-II contains some information regarding these sets.

	List-I		**List-II**
(I)	X	**(P)**	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	**(Q)**	An arithmetic progression
(III)	Z	**(R)**	NOT an arithmetic progression
(IV)	W	**(S)**	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

15. Which of the following is the only CORRECT combination?

(A) (II), (Q), (T) (B) (I), (P), (R) (C) (I), (Q), (U) (D) (II), (R), (S)

16. Which of the following is the only CORRECT combination?

(A) (III), (P), (Q), (U) (B) (IV), (P), (R), (S) (C) (III), (R), (U) (D) (IV), (Q), (T)

17. For a polynomial g(x) with real coefficients, let m_g denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficients defined by

$$S = \{(x^2-1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}$$

For a polynomial f, let f ' and f " denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$ where $f \in S$, is **[IIT Advanced 2020]**

ANSWER KEY

PROFICIENCY TEST-01

1. D **2.** A **3.** A **4.** C **5.** D **6.** D **7.** B

8. C **9.** C **10.** B

PROFICIENCY TEST-02

1. D **2.** D **3.** A **4.** B **5.** B **6.** A **7.** A

8. D **9.** C **10.** B

PROFICIENCY TEST-03

1. $\frac{5}{6}$ **2.** $\frac{1}{6}$ **3.** $-\frac{1}{3}$ **4.** 1 **5.** $-\frac{1}{2}$

6. $a = 120; b = 60; c = 180$ **7.** 2 **8.** $-2/5$

9. $f(0) = 1$; differentiable at $x = 0, f'(0^+) = -(1/3); f'(0^-) = -(1/3)$

10. -6 **11.** $a = 6, b = 6, c = 0; \frac{3}{40}$ **12.** 1000

13. $n = 11$ **14.** f is cont. but not derivable at $x = 0$

15. $n = 4$

EXERCISE-I

1. **(a)** $Dy = (\cos x)^{\ln x}\left[\frac{\ln(\cos x)}{x} - \tan x \ln x\right] + (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]$;

(b) $\frac{dy}{dx} = e^{x^{e^x}} \cdot x^{e^x}\left[\frac{e^x}{x} + e^x \ln x\right] + e^{x^{x^e}} x^{e-1} x^{x^e}[1 + e\ln x] + x^{e^{e^x}} e^{e^x}\left[\frac{1}{x} + e^x \ln x\right]$

6. $\frac{1+\sqrt{1-x^4}}{x^6}$ 7. $\frac{32}{16+\pi^2} - \frac{8}{\ln 2}$

10. **(a)** $\left(-\frac{1}{2}, \frac{1}{2}\right), (-\infty, \infty)$; **(b)** $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; **(c)** $\frac{16\sqrt{3}}{9}$ **12.** $\frac{1}{2}$ or $-\frac{1}{2}$

13. $\frac{1-2x}{2\sqrt{1-x^2}}$ 14. $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ **15.** $\frac{y}{x} \cdot \frac{x\ln x + x\ln x \cdot \ln y + 1}{\ln x(1-x-y\ln a)}$

19. $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ln\frac{3}{2}\right]x & \text{if } x \leq 0 \\ -\left(\frac{1+x}{2+x}\right)^{1/x} & \text{if } x > 0 \end{cases}$ **20.** (A) R, S; (B) Q, S; (C) P; (D) R, S

EXERCISE-II

2. $\frac{4x^3}{9}$ **4.** $k = 1, -1$ or 0 **6.** 6 **7.** (b) $k = 2$

8. $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}; f''(0) = -\frac{1}{3}$ **9.** zero

14. $2(1 + 2x) \cdot \cos 2(x + x^2)$

EXERCISE-III

1.	B	**2.**	D	**3.**	A	**4.**	C	**5.**	B	**6.**	D	7.	D
8.	B	**9.**	C	**10.**	A	**11.**	C	**12.**	D	**13.**	B	**14.**	A
15.	C												

EXERCISE-IV

1.	C	**2.**	D	**3.**	B	**4.**	D	**5.**	A	**6.**	C	**7.**	D
8.	A	**9.**	C	**10.**	D	**11.**	A	**12.**	A	**13.**	B	**14.**	C
15.	A												

EXERCISE-V

1. Domain of $f(x) = R - \{-2,0\}$; Range of $f(x) = R - \{-1/2,1\}$; $\frac{d}{dx}[f^{-1}(x)] = \frac{3}{(1-x)^2}$

Domain of $f^{-1}(x) = R - \{-1/2,1\}$

2. **(a)** B **3.** **(a)** A; **(b)** $a = 1$ **4.** **(a)** C; **(b)** B; **(c)** B, **(d)** $g'(0) = 0$

5. C **6.** D **7.** **(a)** A, **(b)** A **8.** 2 **9.** A **10.** 1

11. C **12.** BC **13.** 8 **14.** B, C **15.** A **16.** B

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