

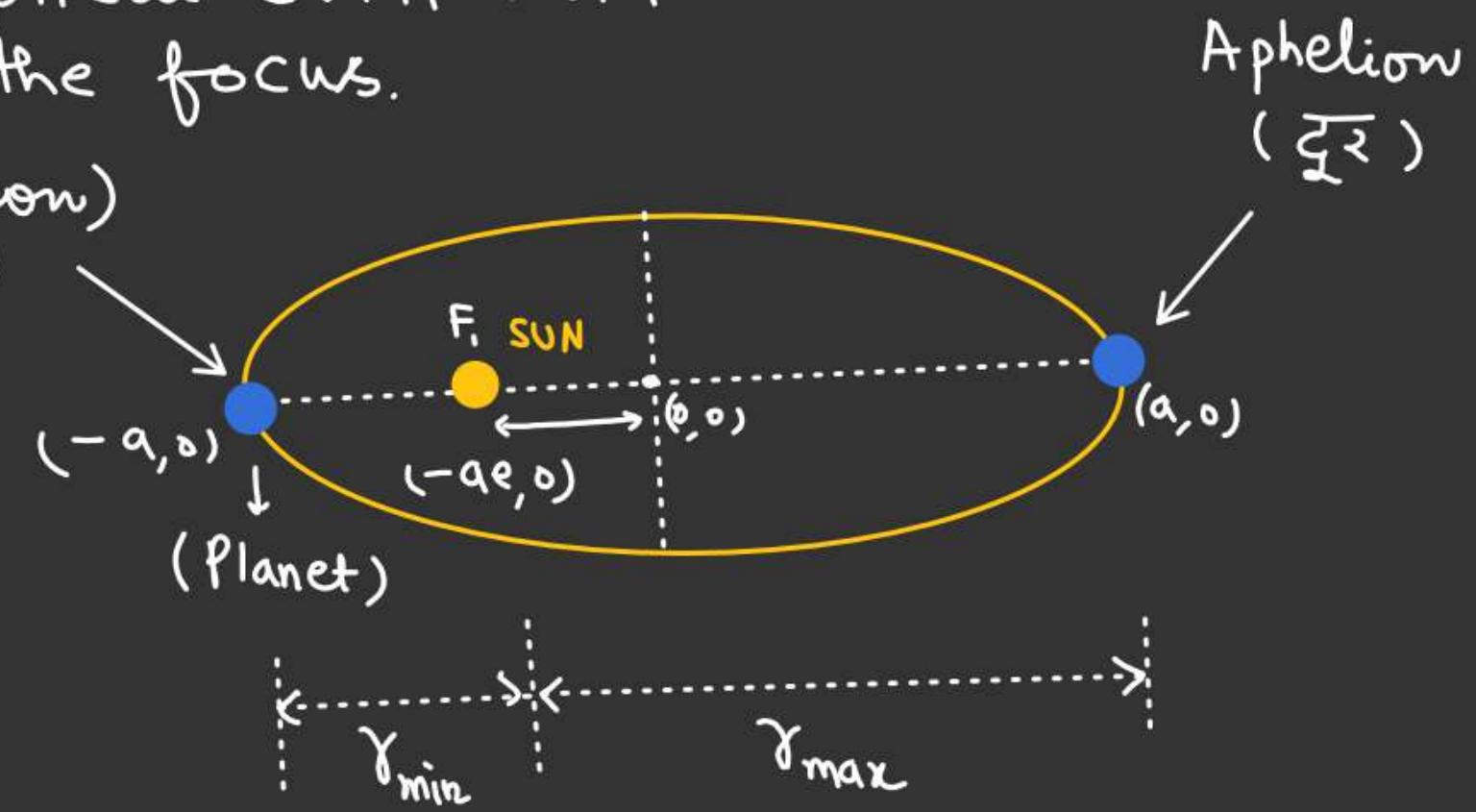


PLANETARY MOTION

KEPLER'S LAW

First law :- [All the planets revolve around the Sun in an elliptical Orbit with Sun at one of the focus.

(Perihelion)
(पृथिवी)



$$\begin{aligned} r_{max} &= a + ae \\ &= a(1+e) \end{aligned}$$

$$\begin{aligned} r_{min} &= a - ae \\ &= a(1-e) \end{aligned}$$

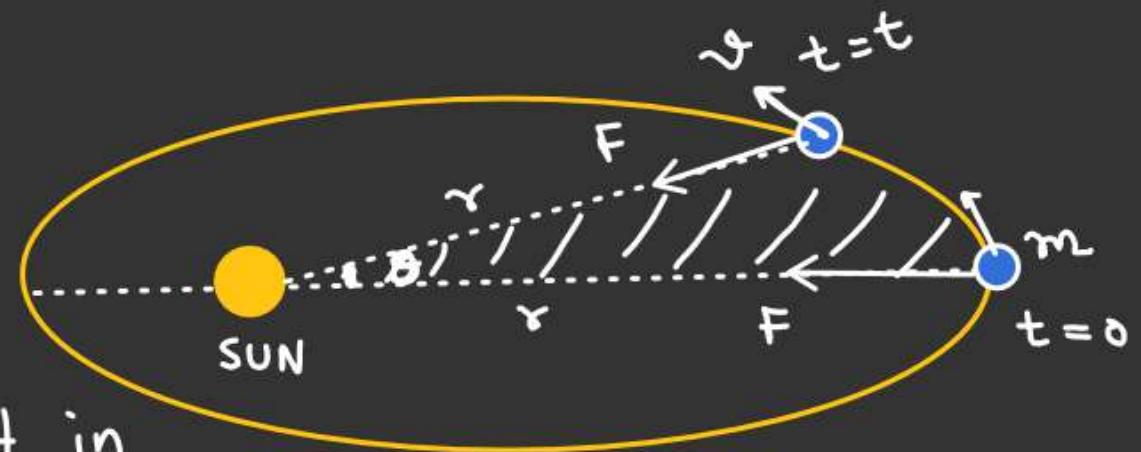
2nd Law :-

Planet Swept Equal area in equal interval of time.

OR

Rate of Change of areal velocity is constant and is equal to $\frac{L}{2m}$.

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$



L = Angular Momentum of the planet about Sun

Area Swept in $t=0$ to $t=t$ = Area of sector

$$L = I\omega$$

$$A = \frac{r^2 \theta}{2}$$

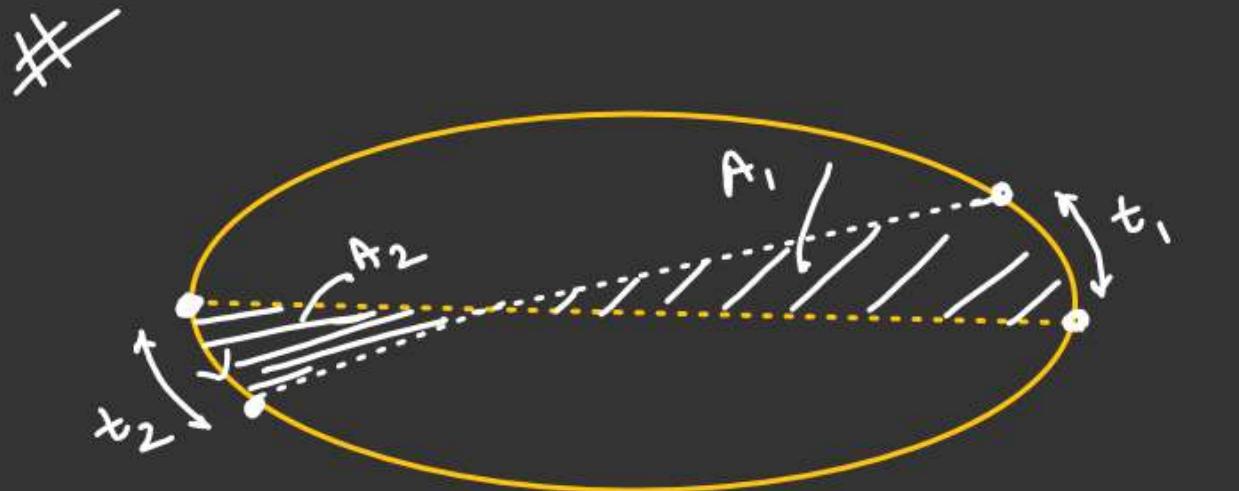
$$L = mr^2 \omega$$

$$\frac{dA}{dt} = \frac{r^2}{2} \left(\frac{d\theta}{dt} \right) = \left(\frac{r^2 \omega}{2} \right)$$

$$L = m r^2 \frac{2}{\pi} \left(\frac{dA}{dt} \right)$$

$$\omega = \frac{r}{r^2} \left(\frac{dA}{dt} \right)$$

$$\frac{dA}{dt} = \frac{L}{2m}$$



$$\frac{A_1}{t_1} = \frac{A_2}{t_2}$$

Kepler's 3rd Law

"Square of the time period of the planet is directly proportional to Cube of Semi major axis."

$$T^2 \propto a^3$$

ORBITAL VELOCITY

$$F = \frac{GMm}{R^2}$$

$$F = \frac{mv^2}{R}$$

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

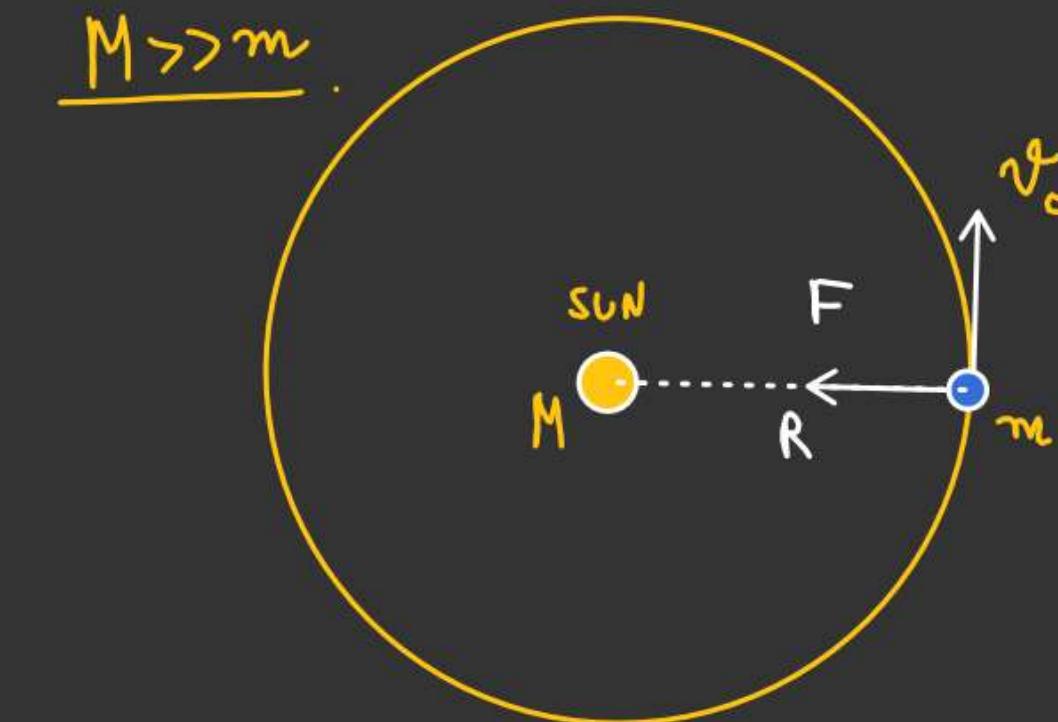
$$(v_0 = \sqrt{\frac{GM}{R}})$$

R = (Orbital radius)

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{GM}} \times \text{Constant}$$

$$T = \frac{2\pi}{\sqrt{GM}} R^{3/2}$$

$$T^2 = \left(\frac{4\pi^2}{GM} \right) R^3$$



$$\Rightarrow r_{\max} = a(1+e)$$

$$\Rightarrow r_{\min} = a(1-e)$$

Angular Momentum Conservation
b/c A to P.

$$\gamma \cdot v_{\min} \cdot r_{\max} = \gamma \cdot v_{\max} \cdot r_{\min}$$

$$\frac{v_{\min}}{v_{\max}} = \frac{r_{\min}}{r_{\max}} = \left(\frac{1-e}{1+e}\right)$$

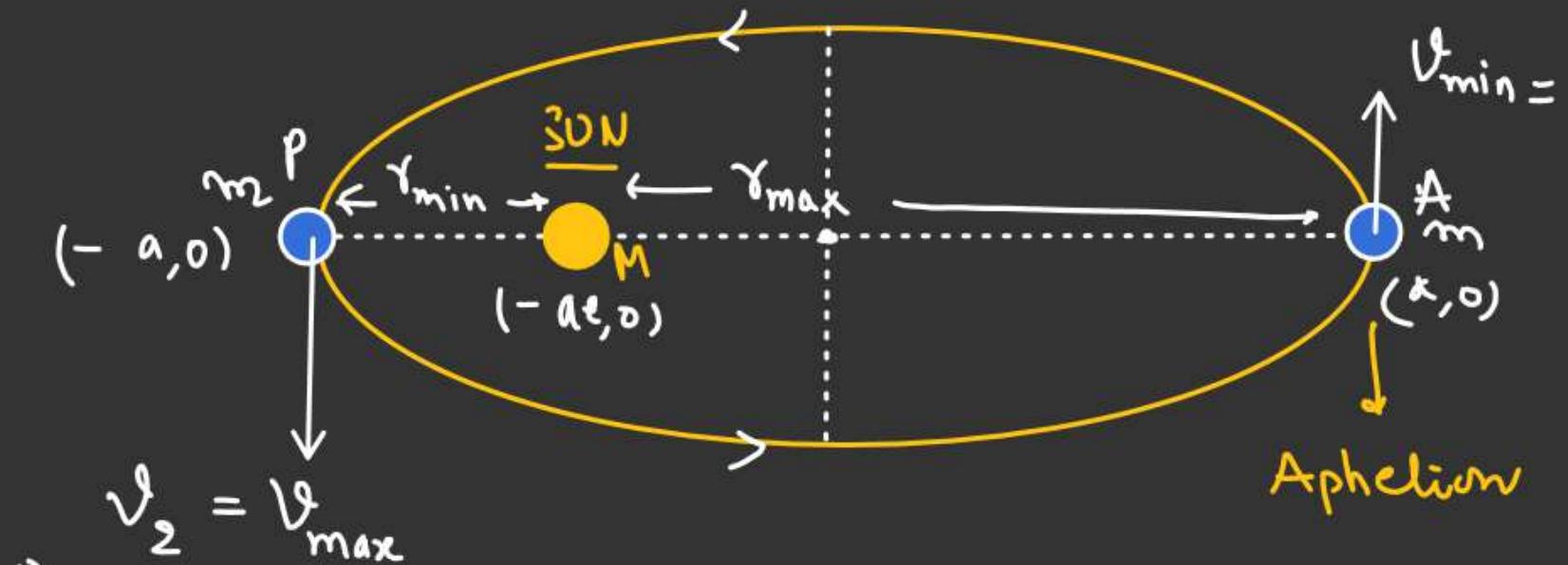
Energy conservation from A to P

$$-\frac{GMm}{r_{\max}} + \frac{1}{2}mv_{\min}^2 = -\frac{GMm}{r_{\min}} + \frac{1}{2}mv_{\max}^2$$

$$\frac{GMm}{r_{\min}} \left(\frac{1}{r_{\min}} - \frac{1}{r_{\max}} \right) = \frac{1}{2}mv(v_{\max}^2 - v_{\min}^2)$$

$$\frac{GM}{r_{\min}} \left(1 - \frac{r_{\min}}{r_{\max}} \right) = \frac{1}{2}v_{\max}^2 \left(1 - \frac{v_{\min}^2}{v_{\max}^2} \right) = \frac{1}{2}v_{\max}^2 \left(1 - \frac{r_{\min}^2}{r_{\max}^2} \right)$$

$$L = mvr = \text{constant}$$



$$U = mv$$

$$\frac{GM}{r_{min}} \left(1 - \frac{r_{min}}{r_{max}} \right) = \frac{1}{2} v_{max}^2 \left(1 - \frac{v_{min}^2}{v_{max}^2} \right) = \frac{1}{2} v_{max}^2 \left(1 - \frac{r_{min}^2}{r_{max}^2} \right)$$

$$\frac{GM}{r_{min}} \left(1 - \frac{r_{min}}{r_{max}} \right) = \frac{1}{2} v_{max}^2 \left(1 - \frac{r_{min}}{r_{max}} \right) \left(1 + \frac{r_{min}}{r_{max}} \right)$$

$$\frac{GM}{r_{min}} = \frac{1}{2} v_{max}^2 \left(\frac{r_{max} + r_{min}}{r_{max}} \right)$$

$$v_{max} = \sqrt{\frac{r_{max}}{r_{min}} \left(\frac{GM}{a} \right)}$$

$$= \sqrt{\frac{\alpha(1+\epsilon)}{\alpha(1-\epsilon)} \left(\frac{GM}{a} \right)}$$

$$r_{max} + r_{min} = 2a$$

~~xx~~.

$$v_{max} = \sqrt{\frac{GM}{a} \left(\frac{1+\epsilon}{1-\epsilon} \right)}$$

$$v_{min} = \sqrt{\frac{GM}{a} \left(\frac{1-\epsilon}{1+\epsilon} \right)}$$

Nishant Jindal
At Aphelion.

$$E_T = -\frac{GMm}{r_{max}} + \frac{1}{2}mv_{min}^2$$

$$E_T = \left[-\frac{GMm}{a(1+e)} \right] + \frac{1}{2}m \left[\frac{GM}{a} \left(\frac{1-e}{1+e} \right) \right]$$

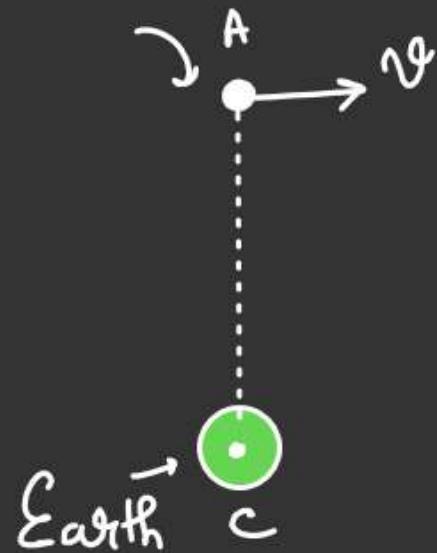
$$E_T = \frac{GMm}{a} \left[\frac{-1}{(1+e)} + \frac{(1-e)}{2(1+e)} \right]$$

$$E_T = \frac{GMm}{a} \left[\frac{-2+1-e}{2(1+e)} \right]$$

$$E_T = -\frac{GMm}{2a}$$

Trajectory of a Satellite When projected horizontally from a vertical height

Satellite



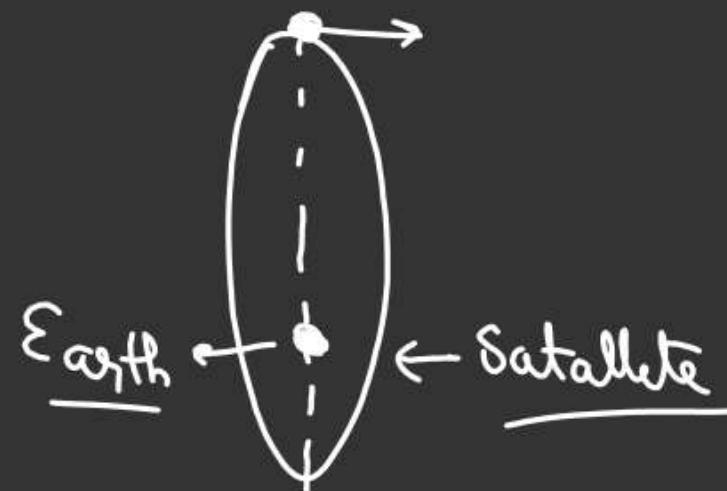
① $v = 0 \Rightarrow$ Motion \rightarrow St-line along AC

② $0 < v < v_0 \Rightarrow$ Motion is elliptical

③ $v = v_0 \Rightarrow$ Motion is circular



④ $v_0 < v < v_e \Rightarrow$ Motion is elliptical.



⑤ $v = v_e \Rightarrow$ Motion is parabolic.

(vi) $v > v_e \Rightarrow$ Motion is hyperbola



Case of Binary Stars. (Concept of Reduced mass)

$$F_{\text{net}} = 0.$$

$$r_1 + r_2 = d. \quad \text{---} \textcircled{2}$$

$$\Delta r_{\text{com}} = 0.$$

$$r_1 = \left(\frac{m_2 d}{m_1 + m_2} \right).$$

$$-\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0$$

$$r_2 = \left(\frac{m_1 d}{m_1 + m_2} \right)$$

$$m_1 r_1 = m_2 r_2 \quad \text{---} \textcircled{1}$$

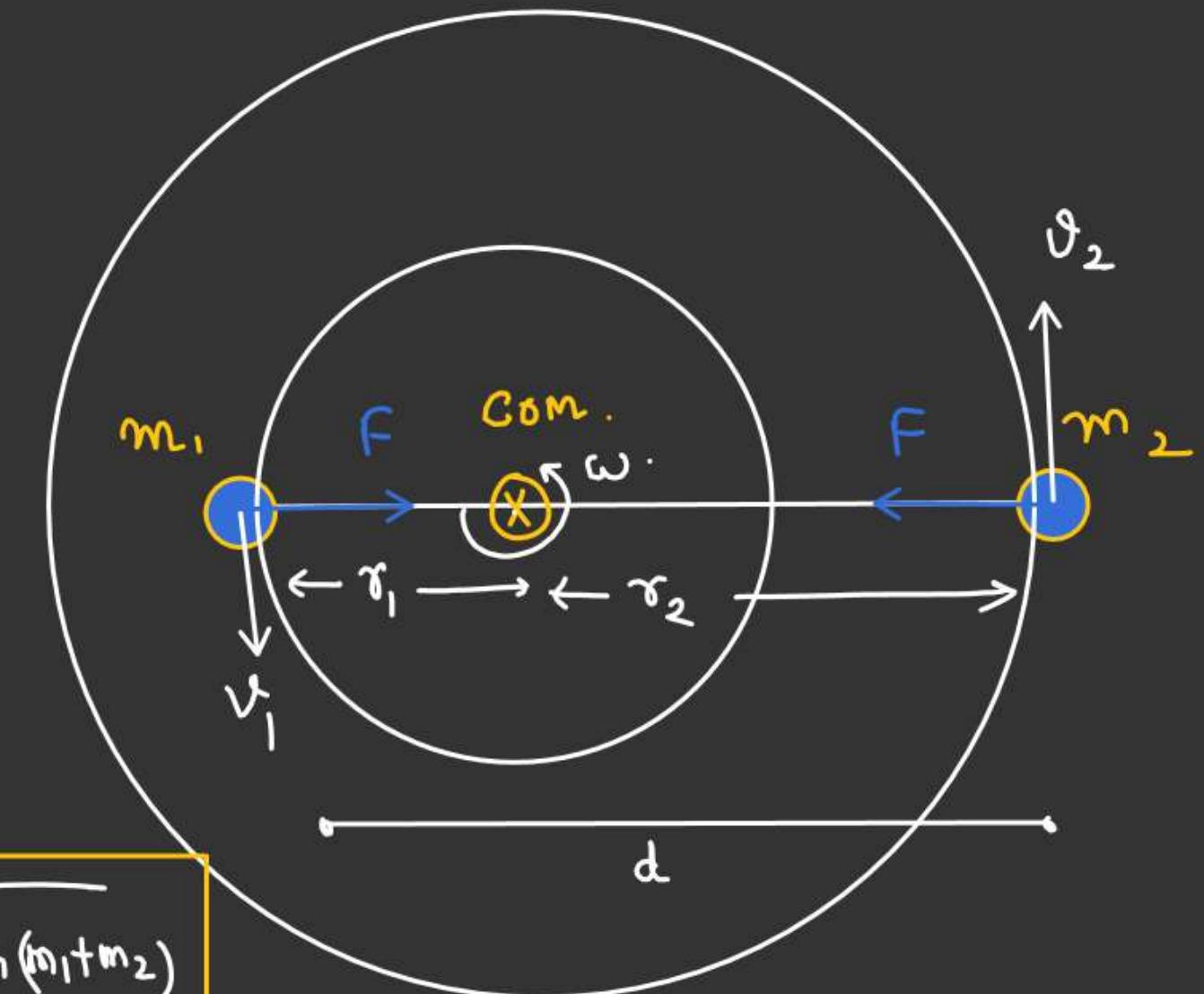
$$F = \frac{G m_1 m_2}{d^2} = m_1 \omega^2 r_1 = m_2 \omega^2 r_2$$

↓

$$\omega^2 = \frac{G m_2}{d^2 r_1} = \frac{G m_2 (m_1 + m_2)}{d^2 (m_2 d)}$$

$$\omega^2 = \frac{G (m_1 + m_2)}{d^3}$$

$$\omega = \sqrt{\frac{G (m_1 + m_2)}{d^3}}$$



$$L_{\text{system}} = I_1 \omega_1 + I_2 \omega_2$$

$$\omega_1 = \omega_2 = \omega$$

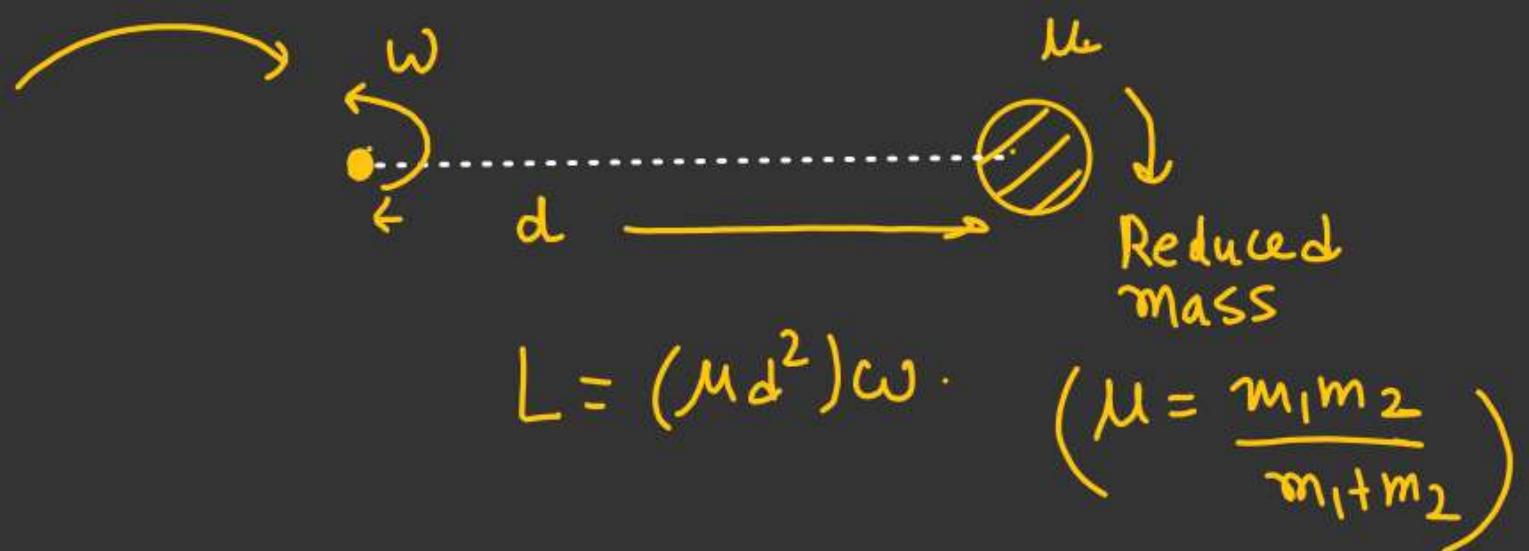
$$= (I_1 + I_2) \omega$$

$$= (m_1 r_1^2 + m_2 r_2^2) \omega$$

$$= \left[m_1 \left(\frac{m_2 d}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1 d}{m_1 + m_2} \right)^2 \right] \omega$$

$$= \left[\frac{m_1 m_2^2 d^2 + m_2 m_1^2 d^2}{(m_1 + m_2)^2} \right] \omega$$

$$L_{\text{system}} = \boxed{\frac{(m_1 m_2)}{(m_1 + m_2)} d \cdot \omega}$$



$$L = (\mu d^2) \omega$$

$$\left(\mu = \frac{m_1 m_2}{m_1 + m_2} \right)$$

$$\begin{aligned}
 (K \cdot E)_{\text{System}} &= \frac{1}{2} I_1 \omega^2 + \frac{1}{2} I_2 \omega^2 \\
 &= \frac{1}{2} (I_1 + I_2) \omega^2 \\
 &= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2 \\
 &= \frac{1}{2} \left[m_1 \frac{m_2 d^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2 d^2}{(m_1 + m_2)^2} \right] \omega^2
 \end{aligned}$$



$$K \cdot E = \frac{1}{2} (\mu d^2) \omega^2$$

$$K \cdot E_{\text{System}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) d^2 \omega^2$$

\Downarrow
 μ

$$= \frac{1}{2} (\mu d^2) \omega^2$$