DPP-04

SOLUTION

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1. Solve the following inequalities

(i) $\text{arc}\ \cot^2\ x - 5\ \text{arc}\cot\ x + 6 > 0$

Sol. Substitute $y = \cot - 1x$

$\therefore\ y2 - 5y + 6 > 0$

$\therefore\ y2 - 3y - 2y + 6 > 0$

$\therefore\ (y-3)(y-2) > 0$

$\therefore\ y < 2Uy > 3$

But, $y = \cot - 1x$

$\therefore\ y \in (0,2) \cup (3,\pi)$

$\therefore\ x \in (-\infty, \cot\ 3) \cup (\cot\ 2, \infty)$

(ii) $\arcsin\ x > \arccos\ x$

(iii) $\tan^2\ (\arcsin\ x) > 1$

Sol. $\tan^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) > 1$

$\Rightarrow \left\{\tan\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\right\}^2 > 1$

$\Rightarrow \frac{x^2}{(1-x^2)} > 1$

$\Rightarrow \frac{x^2}{(1-x^2)} - 1 > 0$

$\Rightarrow \frac{x^2-1+x^2}{(1-x^2)} > 0$

$\Rightarrow \frac{2x^2-1}{(x^2-1)} < 0$

$\Rightarrow \frac{(\sqrt{2}x+1)(\sqrt{2}x-1)}{(x+1)(x-1)} < 0$

$\Rightarrow x \in (-1, -1/\sqrt{2}) \cup (1, 1/\sqrt{2})$

Ans. (i) $x \in (-\infty, \cot\ 3) \cup (\cot\ 2, \infty)$

(ii) $1 \geq x > \frac{1}{\sqrt{2}}$

(iii) $x \in \left(-1, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$

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2. Solve the following inequalities

$4\arctan\ \tan^2\ x - 8\arctan\ x + 3 < 0$ &

$4\ \text{arc}\cot x - \text{arc}\ \cot^2\ x - 3 \geq 0$

Ans. $\tan\ x \in \left(\frac{1}{2}, \frac{3}{2}\right)$

Sol. $(2\tan^{-1}\ x - 3)(2\tan^{-1} - 1) < 0$

$+\quad 1/2 \quad - \quad 3/2 \quad +$

$\frac{1}{2} < \tan^{-1}\ x < 3/2$

$\tan^{1/2} \leq x \leq \tan^3\ /2$ ii)

$4(\cot^{-1}\ x)^- - (\cot^{-1}\ x)^2 - 3 \geq 0$

$(\cot^{-1}\ x)^2 - 4\cot^{-1}\ x + 3 \leq 0$

$(\cot^{-1}\ x - 1)(\cot^{-1}\ x - 3) \leq 0$

$+\quad 1 \quad - \quad 3 \quad +$

$\therefore 1 \leq \cot^{-1}\ x \leq 3$

$\cot\ 1 \leqslant x \leqslant 6\cot\ 3$ (ii) fical answer is indersection of (i) & (i) ie $x \in [\ \text{ton}\ /2, \cot\ 1]$

$(2\tan^{-1}\ x - 3)(2\tan^{-1} - 1) < 0$

$+\quad 1/2 \quad - \quad 3/2 \quad +$

3. find the sum of series

Ans. (i) $\frac{\pi}{2}$.

(ii) $\text{Tan}^{-1}\ 2^n - \frac{\pi}{4}$

(iii) $\frac{\pi}{4}$

(iv) $\text{Tan}^{-1}\ (x+n) - \text{Tan}^{-1}\ x$

(v) $\frac{\pi}{4}$

(i) $\sin^{-1}\ \frac{1}{\sqrt{2}} + \sin^{-1}\ \frac{\sqrt{2}-1}{\sqrt{6}} + \cdots + \sin^{-1}\ \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \cdots \infty$

Sol. Let $S = \sin^{-1}\ \frac{1}{\sqrt{2}} + \sin^{-1}\ \frac{\sqrt{2}-1}{\sqrt{6}} + \sin\ \frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}} + \cdots + \sin^{-1}\ \left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

Now $T_n = \sin^{-1}\ \left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

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$= \sin^{-1}\left[\frac{1}{\sqrt{n}}\sqrt{1-\left(\frac{1}{\sqrt{n+1}}\right)^2}-\frac{1}{\sqrt{n+1}}\sqrt{1-\left(\frac{1}{\sqrt{n}}\right)^2}\right]$

$= \sin^{-1}\frac{1}{\sqrt{n}}-\sin^{-1}\frac{1}{\sqrt{n+1}}$

$\therefore S = \sin^{-1}\frac{1}{\sqrt{2}}+\left(\sin^{-1}\frac{1}{\sqrt{2}}-\sin\frac{1}{\sqrt{3}}\right)+\left(\sin^{-1}\frac{1}{\sqrt{3}}-\sin\frac{1}{\sqrt{4}}\right)+\cdots+\infty$

$= 2\sin^{-1}\frac{1}{\sqrt{2}}=2\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$

(ii) $\mathrm{Tan}^{-1}\frac{1}{3}+\mathrm{Tan}^{-1}\frac{2}{9}+\cdots+\mathrm{Tan}^{-1}\frac{2^{n-1}}{1+2^{2n-1}}+\cdots$ n terms

Sol. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{4}{33}+\cdots+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$

$= \tan^{-1}\left(\frac{2-1}{1+2\times1}\right)+\tan^{-1}\left(\frac{4-2}{1+4\times2}\right)+\tan^{-1}\left(\frac{8-4}{1+8\times4}\right)+\cdots+$

$\tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n\cdot2^{n-1}}\right)$

$= (\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}4-\tan^{-1}2)+(\tan^{-1}8-\tan^{-1}4)+\cdots+$

$(\tan^{-1}2^{n-1}-\tan^{-1}2^{n-2})+(\tan^{-1}2^n-\tan^{-1}2^{n-1})$

$= \tan^{-1}2^n-\tan^{-1}1$

$= \tan^{-1}2^n-\frac{\pi}{4}$

(iii) $\cot^{-1}7+\cot^{-1}13+\cot^{-1}21+\cot^{-1}31+\cdots$ ... to ∞.

Sol. $S_n = \cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+\cot^{-1}(21)+\cdots$ ... n

$t_n = \cot^{-1}(1+n(n+1))$

$= \tan^{-1}\left(\frac{1}{1+n(n+1)}\right)$

$= \tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right)$

$= \tan^{-1}(n+1)-\tan^{-1}n$

$S_n = \sum_{r=1}^{n}\tan^{-1}(r+1)-\tan^{-1}r$

$S_n = \tan^{-1}(n+1)-\tan^{-1}1=\tan^{-1}\left(\frac{n}{n+2}\right)$

$S_{10} = \tan^{-1}\frac{5}{6}$

$s_\infty = \tan^{-1}(1)=\frac{\pi}{4}$

$S_{20} = \tan^{-1}\left(\frac{20}{22}\right)=\cot^{-1}\left(\frac{11}{10}\right)$

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(iv) $\mathrm{Tan}^{-1}\frac{1}{x^2+x+1}+\mathrm{Tan}^{-1}\frac{1}{x^2+3x+3}+\mathrm{Tan}^{-1}\frac{1}{x^2+5x+7}+\mathrm{Tan}^{-1}\frac{1}{x^2+7x+13}+\cdots$ to nterms.

Sol. $y=\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\cdots$. to n terms

$=\tan^{-1}\left\{\frac{1}{1+x(x+1)}\right\}+\tan^{-1}\left\{\frac{1}{1+(x+1)(x+2)}\right\}+$

$\tan^{-1}\left\{\frac{1}{1+(x+2)(x+3)}\right\}+\cdots..+\tan^{-1}\left\{\frac{1}{1+(x+n-1)\{x+n\}}\right\}$

$=\tan^{-1}\left\{\frac{(x+1)-x}{1+(x+1)x}\right\}+\tan^{-1}\left\{\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right\}+$

$\tan^{-1}\left\{\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right\}+\cdots..+\tan^{-1}\left\{\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right\}$

$\therefore y=\{\tan^{-1}(x+1)-\tan^{-1}(x)\}+\{\tan^{-1}(x+2)-\tan^{-1}(x+1)\}+$

$\{\tan^{-1}(x+3)-\tan^{-1}(x+2)\}+\cdots..$

$+\{\tan^{-1}(x+n)-\tan^{-1}(x+n-1)\}$

(v) $\mathrm{Tan}^{-1}\frac{1}{2}+\mathrm{Tan}^{-1}\frac{1}{8}+\mathrm{Tan}^{-1}\frac{1}{18}+\mathrm{Tan}^{-1}\frac{1}{32}+\cdots\infty$

Sol. Expand the given expression using trigonometric identities

Lets denote the given expression as ' S_n'

Let, $S_n=\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+\cdots\infty$

$S_n=\cot^{-1}2*(1)^2+\cot^{-1}2*(2)^2+\cot^{-1}2*(3)^2+\cdots\ldots\ldots\cot^{-1}2*(n)^2$

$S_n=\lim_{n\to\infty}\left(\sum_{r=1}^{n}\cot^{-1}(2r^2)\right)=\lim_{n\to\infty}\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)\right)$

$=\lim_{n\to\infty}\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)\right)$

$=\lim_{n\to\infty}\left(\sum_{r=1}^{n}\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right)$

$=(\tan^{-1}\infty-\tan^{-1}1)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

Hence the sum of given trigonometric series is $\frac{\pi}{4}$

4. Find the value of $\mathrm{Tan}\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$, if $x>y>1$.

Sol. Let $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$

$\therefore\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x$

Let $y=\tan\phi\Rightarrow\phi=\tan^{-1}y$

$\therefore\cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)=\cos^{-1}(\cos 2\phi)=2\phi=2\tan^{-1}y$

$\therefore\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

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$= \tan \frac{1}{2}[2\tan^{-1} x + 2\tan^{-1} y]$

$= \tan [\tan^{-1} x + \tan^{-1} y]$

$= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy}\right)\right] = \frac{x+y}{1-xy}$

5. If $\alpha = 2\tan^{-1} \left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1} \left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then Prove that $\alpha + \beta = \pi$. What the value of $\alpha + \beta$ will be if $x > 1$?

Sol. $\alpha = 2\tan^{-1} ((1+x)/(1-x))$

$= 2(\tan^{-1} 1 + \tan^{-1} x)$

$= 2\left(\frac{\pi}{4} + \tan^{-1} x\right)$

$= \frac{\pi}{2} + 2\tan^{-1} x$

Also, $\beta = \sin^{-1} \left(\frac{1-x^2}{1+x^2}\right)$

$= \frac{\pi}{2} - \cos^{-1} \left(\frac{1-x^2}{1+x^2}\right)$

$= \frac{\pi}{2} - 2\tan^{-1} x$

Thus,

$\alpha + \beta = \frac{\pi}{2} + 2\tan^{-1} x + \frac{\pi}{2} - 2\tan^{-1} x$

$= \pi$

6. If $x \in \left[-1, \frac{-1}{2}\right]$ then express the functions $f(x) = \sin^{-1} (3x - 4x^3) + \cos^{-1} (4x^3 - 3x)$ in the form of $a\cos^{-1} x + b\pi$, where a and b are rational numbers.

7. Prove that the equations $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \alpha\pi^3$ has no roots for $\alpha < \frac{1}{32}$ and $\alpha > \frac{7}{8}$

Sol. The correct option is $C \left[\frac{1}{32} \frac{7}{8}\right]$

Let $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$

Put $\sin^{-1} x = t$

Then $f(t) = t^3 + \left(\frac{\pi}{2} - t\right)^3$

$f'(t) = 3\left[t^2 - \left(\frac{\pi}{2} - t\right)^2\right]$

$\Rightarrow f'(t) = 3 \cdot \frac{\pi}{2}\left(2t - \frac{\pi}{2}\right)$

Critical point: $f'(t) = 0 \Rightarrow t = \frac{\pi}{4}$

Also, $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

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$f\left(t=\frac{\pi}{4}\right)=\left(\frac{\pi}{4}\right)^3+\left(\frac{\pi}{4}\right)^3=\frac{\pi^3}{32}\rightarrow$ minimum

$f\left(t=\frac{-\pi}{2}\right)=\frac{7\pi^3}{8}\rightarrow$ maximum

$f\left(t=\frac{\pi}{2}\right)=\frac{\pi^3}{8}$

$\therefore$ Range of f is $\left[\frac{1}{32}\pi^3,\frac{7}{8}\pi^3\right]$

8. If $\cos^{-1}(2x^2-1)=2\pi-2\cos^{-1}x$, then

(A) $x\in[-1,0]$ (B) $x\in[0,1]$

(C) $x\in\left[0,\frac{1}{\sqrt{2}}\right]$ (D) $x\in\left[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

Ans. **(A)**

Sol. Now, $\cos^{-1}(2x^2-1)=2\pi-2\cos^{-1}x$ - (1)

Put $x=\cos\theta$ in (1), we get

$\cos^{-1}(2\cos^2\theta-1)=2\pi-2\cos^{-1}(\cos\theta)$

$cos^{-1}(cos\,2\theta)=2\pi-2\theta$

This is possible when $\pi<2\theta\leqslant 2\pi$

i.e, $\frac{\pi}{2}<\theta\leqslant\pi$

i.e, $\frac{\pi}{2}<\cos^{-1}x\leqslant\pi$

i.e if $-1\leqslant x<0$

$x\in[1,0)$

Option A is correct.

9. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is:

(A) $\frac{21}{19}$ (B) $\frac{19}{21}$ (C) $\frac{22}{23}$ (D) $\frac{23}{22}$

Ans. **(A)**

Sol. Given

$\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$

$=\cot\left(\sum_{n=1}^{19}\cot^{-1}(1+n(n+1))\right)$

$=\cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{1}{1+n(n+1)}\right)$

$=\cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{(n+1)-n}{1+n(n+1)}\right)$

$=\cot\left(\sum_{n=1}^{19}\tan^{-1}(n+1)-\tan^{-1}n\right)$

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$= \cot(\tan^{-1} 20 - \tan^{-1} 1)$

$= \cot\left(\tan^{-1}\frac{20-1}{1+20\cdot 1}\right)$

$= \cot\left(\tan^{-1}\frac{19}{21}\right)$

$= \cot\left(\cot^{-1}\frac{21}{19}\right)$

$= \frac{21}{19}$

10. The value of $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$, $|x| < \frac{1}{2}, x \neq 0$, is equal to

(A) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2$

(B) $\frac{\pi}{4}+\cos^{-1} x^2$

(C) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1} x^2$

(D) $\frac{\pi}{4}-\cos^{-1} x^2$

Ans. **(A)**

Sol. We have $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Let $x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1} x^2$

L.H.S.

$= \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$

$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$

$= \tan^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right]$

$= \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$

$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right]$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right) = \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2 =$ R.H.S.