Nishant Jindal

Transpose of matrix

A^T

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

$$\boxed{(A^T)_{ij} = A_{ji}}$$

Nishant Jindal

Properties

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(KA)^T = KA^T$
- $(A^n)^T = (A^T)^n, n \in N$

$$(A_1 \underbrace{A_2 A_3 \cdots A_n}) ^T = (A_2 \underbrace{A_3 \cdots A_n})^T A_1^T$$

$$= (A_3 A_4 \cdots A_n)^T A_2^T A_1^T$$

$$\vdots$$

$$= A_n^T A_{n-1}^T \cdots A_3^T A_2^T A_1^T$$

$A_{m \times n}, B_{n \times p}$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{r=1}^{n} A_{jr} B_{ri}$$

$$(B^T A^T)_{ij} = \sum_{r=1}^{n} A^T_{rj} B^T_{ir}$$

Nishant Jindal

Symmetric & Skew Symmetric matrices

$A_{ij} = A_{ji} \quad \forall i,j \Rightarrow A$ is symmetric matrix

$A_{ij} = A^T_{ij}$

$\Rightarrow \boxed{A^T = A}$

$$\begin{bmatrix} a_{11} & \alpha & \beta \\ \alpha & a_{22} & \gamma \\ \beta & \gamma & a_{33} \end{bmatrix}$$

$$\boxed{A^T = -A}$$

A is skew symmetric $\Leftarrow$ $A_{ij} = -A_{ji} \quad \forall i,j$

$i=j$: $A_{ii} = -A_{ii} \Rightarrow A_{ii} = 0$

$$\begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}$$

$A_{ij} = -A^T_{ij} = (-A^T)_{ij} \Rightarrow A = -A^T$

Nishant Jindal

If A is skew symm.

$$A^T = -A$$

$$\Rightarrow |A^T| = |-A|$$

$$|A| = (-1)^n |A|$$

$$\begin{vmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{vmatrix} = 0.$$

$$\begin{vmatrix} 0 & \alpha \\ -\alpha & 0 \end{vmatrix} = \alpha^2$$

If <u>n is odd</u> $|A| = -|A|$

$$|A| = 0$$

Nishant Jindal

Note → A Square matrix can always be expressed as a sum of symmetric and skew symm. matrix in a unique way.

$$A = \underbrace{P}_{\text{symm}} + \underbrace{Q}_{\text{skew symm.}} \quad -(1)$$

$$A^T = (P+Q)^T = P^T + Q^T$$

$$A^T = P - Q \quad -(2)$$

$$\boxed{P = \frac{A+A^T}{2} \qquad Q = \frac{A-A^T}{2}}$$

Nishant Jindal

Orthogonal matrix

$$AA^T = A^T A = I \quad \Rightarrow A \text{ is orthogonal matrix.}$$

$$AA^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} \sum a_1^2 & \sum a_1 b_1 & \sum a_1 c_1 \\ \sum a_1 b_1 & \sum b_1^2 & \sum b_1 c_1 \\ \sum a_1 c_1 & \sum b_1 c_1 & \sum c_1^2 \end{bmatrix}$$

$$\vec{V_1} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{V_2} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{V_3} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

$$\sum a_1^2 = \sum b_1^2 = \sum c_1^2 = 1$$

$$\sum a_1 b_1 = \sum a_1 c_1 = \sum b_1 c_1 = 0.$$

Nishant Jindal

1. Show that $B^T A B$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

$$(B^T A B)^T = B^T A^T B = \begin{cases} B^T A B & A \text{ is symm} \\ -B^T A B & A \text{ is skew} \end{cases}$$

$B^T(-A)B$

2. Comment upon $A^n, n \in N$ symm, skew symm or none if A is (i) symm. (ii) skew symm.

$$(A^n)^T = (A^T)^n = \begin{cases} A^n \Rightarrow \text{symm} & A \text{ is symm} \\ (-A)^n = (-1)^n A^n & A \text{ is skew} \end{cases}$$

$\longrightarrow -A^n$ n odd

$\longrightarrow A^n$ n even

Nishant Jindal

3. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & y-x & -1 \\ x & 1 & 2 \end{bmatrix}$, such that

AB is symmetric matrix, find x, y.

$$AB = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & y-x & -1 \\ x & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1-2x & y-x-2 & -5 \\ 3+x & 3y-3x+1 & -1 \\ -1+2x & x-y+2 & 5 \end{bmatrix}$$

$-5 = -1+2x$

$\boxed{x = -2}$

$y-x-2 = 3+x$

$\boxed{y = 1}$

4. Express $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as sum of symmetric and skew symmetric matrix

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ -5 & +1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Nishant Jindal

Adjoint of matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ C_{13} & C_{23} & & C_{n3} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

Nishant Jindal

Properties

- $A(\text{adj } A) = (\text{adj} A)A = |A| I$

- $|\text{adj} A| = |A|^{n-1}$, if A is non singular of order n.

$\boxed{\text{PT-1}}$ matrix

$A \text{ adj} A = |A| I$

$|A \text{ adj} A| = \big| |A| I \big|$

$\Rightarrow |A| |\text{adj} A| = |A|^n |I| = |A|^n$

$|AB| = |A| |B|$ A, B are square of same order

$\Rightarrow |\text{adj} A| = |A|^{n-1}$

Nishant Jindal

$$A \operatorname{adj} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & & C_{n2} \\ C_{13} & C_{23} & \vdots & & \vdots \\ \vdots & \vdots & & & \\ C_{1n} & C_{2n} & C_{3n} & & C_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & |A| \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$= |A| I$$