DPP 01

SOLUTION

1. $\rho(r)=\rho_0\left(1-\frac{r^2}{R^2}\right)$

Let an elementary portion of radius r and thickness dr

$dm=\rho\cdot 4\pi r^2 dr$

$$dm=\rho_0\left(1-\frac{r^2}{R^2}\right)\cdot 4\pi r^2 dr$$

$$m=4\pi\rho_0\int_0^r\left[r^2-\frac{r^4}{R^2}\right]dr$$

$$m=4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5R^2}\right]_0^r;\ m=4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5R^2}\right]$$

$$E=\frac{Gm}{r^2}=\frac{G}{r^2}\cdot 4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5R^2}\right]$$

$$E=4\pi G\rho_0\left[\frac{r}{3}-\frac{r^3}{5R^2}\right]$$

For maximum value, $\frac{dE}{dr}=0\Rightarrow\frac{1}{3}-\frac{3r^2}{5R^2}=0;\ r=\sqrt{\frac{5}{9}}R$

2. Here $\rho=\frac{K}{r^2}\Rightarrow M=\int\rho dV;$

$$M=\int\frac{K}{r^2}(4\pi r^2)dr=4\pi Kr$$

Now the gravitational force on the particle,

$$\frac{GMm}{r^2}=\frac{mv^2}{r};\ v^2=\frac{GM}{r}$$

$$\frac{v^2}{(2\pi r)^2}=\frac{GM}{(2\pi r)^2 r}\Rightarrow\frac{1}{T^2}=C\times\frac{1}{r^2}\ [M=4\pi kr]$$

$\Rightarrow\frac{T}{R}=$ constant. [where $r=R$, radius of orbit]

3. $dF=\frac{Gm(\mu dx)}{x^2}\Rightarrow F=Gm\int_{x=a}^{x=(a+L)}\frac{(A+Bx^2)}{x^2}dx$

$$F=Gm\left(A\int_{x=a}^{x=a+L}x^{-2}dx+\int_{x=a}^{x=a+L}B\cdot dx\right)$$

$$F=Gm\left(A\left[\frac{-1}{x}\right]_a^{a+L}+B[x]_a^{a+L}\right)$$

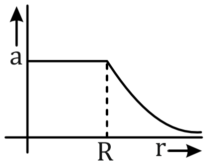
$$F=Gm\left(-A\left[\frac{1}{a+L}-\frac{1}{a}\right]+B[a+L-a]\right)$$

$$F = Gm\left(A\left[\frac{1}{a} - \frac{1}{a+L}\right] + BL\right)$$

4. $\rho(r) = \frac{k}{r}$ $\quad r \le R, \rho(r) = 0 \quad r > R$

$$g = \frac{G \times \frac{4}{3}\pi r^3 \times \frac{k}{r}}{r^2}, r \le R$$

$$= \frac{4G\pi k}{3} = \text{ const And}, g = \frac{G\frac{4}{3}\pi R^3}{r^2} = \frac{4G\pi R^3}{3} \times \frac{1}{r^2} \text{ for } r > R$$



5. For motion of a planet in circular orbit,

Centripetal force = Gravitational force

$\therefore\ mR\omega^2 = \frac{GMm}{R^n}$ or $\omega = \sqrt{\frac{GM}{R^{n+1}}}$

$\therefore\ T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^{n+1}}{GM}} = \frac{2\pi}{\sqrt{GM}}R^{\left(\frac{n+1}{2}\right)}$

$\therefore$ T is proportional to $R^{\left(\frac{n+1}{2}\right)}$.

6. As per question,

Weight of body at height h above earth surface

= Weight of body at depth h below the earth surface

$\Rightarrow mg_{above} = mg_{below}$

$$\frac{GM}{(R+h)^2} = \frac{GM}{R^3}(R-h)$$

$\Rightarrow R^3 = (R-h)(R+h)^2 = (R^2-h^2)(R+h)$

$\Rightarrow R^3 = R^3 - Rh^2 + R^2h - h^3 \Rightarrow Rh^2 - R^2h + h^3 = 0$

$\Rightarrow h(h^2 - R^2 + Rh) \Rightarrow h = 0$ or $h^2 + Rh - R^2 = 0$

$\therefore\ h = \frac{-R \pm \sqrt{R^2 - 4(1)(-R^2)}}{2} \Rightarrow h = \frac{-R \pm R\sqrt{5}}{2}$

Negative value of h is not possible.

So, $h = \frac{\sqrt{5}R - R}{2}$

7. Acceleration due to gravity at equator,

$g_A = g - \omega^2 R$

Acceleration due to gravity at a height h above the poles,

$$g_B = g\left(1 - \frac{2h}{R}\right)$$

Given that, $g_A = g_B$

$$\therefore\ g - \omega^2 R = g\left(1 - \frac{2h}{R}\right) \text{ or } h = \frac{R^2\omega^2}{2g}$$

8. Mass of object remains same.

Weight of object ∝ acceleration due to gravity.

$$\frac{W_{(earth)}}{W_{(planet)}} = \frac{9}{4} = \frac{g_{(earth)}}{g_{(planet)}}$$

$$\therefore\ \frac{9}{4} = \frac{GM_{(earth)}}{GM_{(planet)}} \times \frac{R^2_{(planet)}}{R^2_{(earth)}} = \frac{M_{(earth)}}{M_{(planet)}} \times \frac{R^2_{(planet)}}{R^2_{(earth)}}$$

$$= 9 \times \frac{R^2_{(planet)}}{R^2_{(earth)}} \therefore R_{(planet)} = \frac{R_{(earth)}}{2} = \frac{R}{2}$$

9. Let the area of the ellipse be A.

As per Kepler's 2^{nd} law, areal velocity of a planet around the sun is constant, i.e.,

$$\frac{dA}{dt} = \text{constant.}$$

$$\therefore \frac{t_1}{t_2} = \frac{\text{Area of abcsa}}{\text{Area of adcsa}} = \frac{\frac{A}{2} + \frac{A}{4}}{\frac{A}{2} - \frac{A}{4}} = \frac{\frac{3A}{4}}{\frac{A}{4}} = 3 \Rightarrow t_1 = 3t_2$$

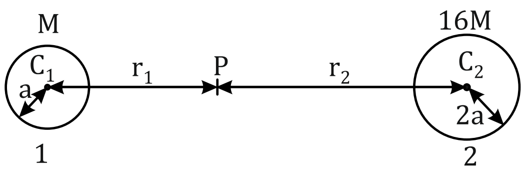
Note: Here db is the major axis of the ellipse, not semi-major axis and ca is the minor axis of the ellipse, not semi-minor axis.

10. Given, $\vec{E}_G = \frac{Ax}{(x^2+a^2)^{3/2}}$

∴ Gravitational potential on the x-axis at a distance x,

$$\vec{V}_G = \int E_G \cdot dx = \int_{\infty}^{x} \frac{Ax}{(x^2 + a^2)^{3/2}} dx = \frac{A}{(x^2 + a^2)^{1/2}}$$

11. Let there are two planets 1 and 2 as shown here.



Let P is a point between C_1 and C_2, where gravitational field strength is zero or at P, field strength due to planet 1 is equal and opposite to the field strength due to planet 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G(16M)}{r_2^2} \text{ or } \frac{r_2}{r_1} = 4. \text{ Also, } r_1 + r_2 = 10a$$

$$\therefore\ r_2 = \left(\frac{4}{4+1}\right)(10a) = 8a \text{ and } r_1 = 2a$$

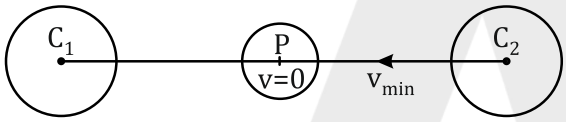
Now, the body of mass m is projected from the surface of larger planet towards the smaller one. Between C_2 and P, it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the body is attracted towards the smaller planet itself.

From conservation of mechanical energy,

$\frac{1}{2}mv_{min}^2$ = Potential energy of the body at P

Potential energy at the surface of the larger planet

$$\therefore \frac{1}{2}mv_{min}^2 = \left[-\frac{GMm}{r_1} - \frac{16GMm}{r_2}\right] - \left[-\frac{GMm}{10a-2a} - \frac{16GMm}{2a}\right]$$



$$= \left[-\frac{GMm}{2a} - \frac{16GMm}{8a}\right] - \left[-\frac{GMm}{8a} - \frac{8GMm}{a}\right]$$

$$\text{or } \frac{1}{2}mv_{min}^2 = \left(\frac{45}{8}\right)\frac{GMm}{a} \therefore v_{min} = \frac{3}{2}\left(\sqrt{\frac{5GM}{a}}\right)$$

12. Potential V(r) due to a large planet of radius R is given by

$$V_o(r) = -\frac{GM}{r};\ r > R$$

$$V(r) = \frac{-GM}{R};\ r = R$$

$$V_{in} = -\frac{3}{2}\frac{GM}{R}\left[1 - \frac{r^2}{3R^2}\right];\ r < R$$