

**KEY CONCEPTS (DIFFERENTIAL EQUATION)****DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE****DEFINITIONS :**

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION**.
2. A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be **PARTIAL** if there are two or more independent variables. We are concerned with ordinary differential equations only. eg.  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$  is a partial differential equation.
3. Finding the unknown function is called **SOLVING OR INTEGRATING** the differential equation. The solution of the differential equation is also called its **PRIMITIVE**, because the differential equation can be regarded as a relation derived from it.
4. The order of a differential equation is the order of the highest differential coefficient occurring in it.
5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$$f(x, y) \left[ \frac{d^m y}{dx^m} \right]^p + \phi(x, y) \left[ \frac{d^{m-1} y}{dx^{m-1}} \right]^q + \dots = 0 \text{ is order } m \text{ & degree } p.$$

Note that in the differential equation  $e^{y''} - xy'' + y = 0$  order is three but degree doesn't apply.

**FORMATION OF A DIFFERENTIAL EQUATION :**

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

- ☞ Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it .
- ☞ Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for  $y^2 = 4a(x + b)$  where a and b are arbitrary constant .

**Note :** A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

**GENERAL AND PARTICULAR SOLUTIONS :**

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE)** . A solution obtainable from the general solution by giving particular values to the constants is called a **PARTICULAR SOLUTION**.

Note that the general solution of a differential equation of the  $n^{\text{th}}$  order contains 'n' & only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation  $y = A e^{x+B}$  are not independent since the equation can be written as  $y = A e^B \cdot e^x$ . Similarly the solution  $y = A \sin x + B \cos(x + C)$  appears to contain three arbitrary constants, but they are really equivalent to two only.

**8. Elementary Types Of First Order & First Degree Differential Equations .****TYPE-1. VARIABLES SEPARABLE :** If the differential equation can be expressed as ;

$f(x)dx + g(y)dy = 0$  then this is said to be variable – separable type.

A general solution of this is given by  $\int f(x) dx + \int g(y) dy = c$  ;

where c is the arbitrary constant . consider the example  $(dy/dx) = e^{x-y} + x^2 \cdot e^{-y}$ .

**Note :** Sometimes transformation to the polar co-ordinates facilitates separation of variables.

In this connection it is convenient to remember the following differentials.

If  $x = r \cos \theta$  ;  $y = r \sin \theta$  then,

$$(i) x dx + y dy = r dr \quad (ii) dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \quad (iii) x dy - y dx = r^2 d\theta$$

If  $x = r \sec \theta$  &  $y = r \tan \theta$  then  $x dx - y dy = r dr$  and  $x dy - y dx = r^2 \sec \theta d\theta$  .



**TYPE-2:**  $\frac{dy}{dx} = f(ax + by + c), b \neq 0.$

To solve this, substitute  $t = ax + by + c$ . Then the equation reduces to separable type in the variable  $t$  and  $x$  which can be solved.

Consider the example  $(x + y)^2 \frac{dy}{dx} = a^2$ .

**TYPE-3. HOMOGENEOUS EQUATIONS :**

A differential equation of the form  $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$

where  $f(x, y) & \phi(x, y)$  are homogeneous functions of  $x & y$ , and of the same degree, is called

**HOMOGENEOUS**. This equation may also be reduced to the form  $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$  & is solved by putting

$y = vx$  so that the dependent variable  $y$  is changed to another variable  $v$ , where  $v$  is some unknown function, the differential equation is transformed to an equation with variables separable. Consider

$$\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0.$$

**TYPE-4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :**

If  $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$  ; where  $a_1b_2 - a_2b_1 \neq 0$ , i.e.  $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution  $x = u + h$ ,  $y = v + k$  transform this equation to a homogeneous type in the new variables  $u$  and  $v$  where  $h$  and  $k$  are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

- (i)  $a_1b_2 - a_2b_1 = 0$ , then a substitution  $u = a_1x + b_1y$  transforms the differential equation to an equation with variables separable. and
- (ii)  $b_1 + a_2 = 0$ , then a simple cross multiplication and substituting  $d(xy)$  for  $x dy + y dx$  & integrating term by term yields the result easily.

Consider  $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$  ;  $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$  &  $\frac{dy}{dx} = \frac{2x-y+1}{6x-5y+4}$

- (iii) In an equation of the form :  $yf(xy) dx + xg(xy)dy = 0$  the variables can be separated by the substitution  $xy = v$ .

**IMPORTANT NOTE :**

- (a) The function  $f(x, y)$  is said to be a homogeneous function of degree  $n$  if for any real number  $t (\neq 0)$ , we have  $f(tx, ty) = t^n f(x, y)$ .

For e.g.  $f(x, y) = ax^{2/3} + bx^{1/3} \cdot y^{1/3} + by^{2/3}$  is a homogeneous function of degree  $2/3$ .

- (b) A differential equation of the form  $\frac{dy}{dx} = f(x, y)$  is homogeneous if  $f(x, y)$  is a homogeneous function of degree zero i.e.  $f(tx, ty) = t^0 f(x, y) = f(x, y)$ . The function  $f$  does not depend on  $x & y$  separately but only on their ratio  $\frac{y}{x}$  or  $\frac{x}{y}$ .

**LINEAR DIFFERENTIAL EQUATIONS :**

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The  $n$ th order linear differential equation is of the form ;

$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x) \cdot y = \phi(x)$  . Where  $a_0(x), a_1(x) \dots a_n(x)$  are called the coefficients of the differential equation.



Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation  $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$  is not linear, though its degree is 1.

#### TYPE-5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is  $\frac{dy}{dx} + Py = Q$ , where P & Q are functions of x .

To solve such an equation multiply both sides by  $e^{\int P dx}$ .

#### NOTE:

- (1) The factor  $e^{\int P dx}$  on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y , is called integrating factor of the differential equation popularly abbreviated as I. F.
- (2) It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F.
- (3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1) \frac{dy}{dx} = y^2 + 3$  can be written as  $(y^2 + 3) \frac{dx}{dy} = x + y + 1$  which is a linear differential equation.

#### TYPE-6. EQUATIONS REDUCIBLE TO LINEAR FORM :

The equation  $\frac{dy}{dx} + py = Q \cdot y^n$  where P & Q functions of x , is reducible to the linear form by dividing it by  $y^n$  & then substituting  $y^{-n+1} = Z$ . Its solution can be obtained as in Type-5. Consider the example  $(x^3 y^2 + xy) dx = dy$ .

The equation  $\frac{dy}{dx} + Py = Q \cdot y^n$  is called **BERNOULI'S EQUATION**.

#### 9. TRAJECTORIES :

Suppose we are given the family of plane curves.

$$\Phi(x, y, a) = 0$$

depending on a single parameter a.

A curve making at each of its points a fixed angle  $\alpha$  with the curve of the family passing through that point is called an **isogonal trajectory** of that family ; if in particular  $\alpha = \pi/2$ , then it is called an **orthogonal trajectory**.

**Orthogonal trajectories** : We set up the differential equation of the given family of curves. Let it be of the form

$$F(x, y, y') = 0$$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x, y, -\frac{1}{y'}\right) = 0$$

The general integral of this equation

$$\Phi_1(x, y, C) = 0$$

gives the family of orthogonal trajectories.



Note : Following exact differentials must be remembered :

$$(i) \quad xdy + y dx = d(xy)$$

$$(ii) \quad \frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$

$$(iii) \quad \frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$$

$$(iv) \quad \frac{xdy + ydx}{xy} = d(\ln xy)$$

$$(v) \quad \frac{dx + dy}{x + y} = d(\ln(x + y))$$

$$(vi) \quad \frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$$

$$(vii) \quad \frac{ydx - xdy}{xy} = d\left(\ln\frac{x}{y}\right)$$

$$(viii) \quad \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$(ix) \quad \frac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$$

$$(x) \quad \frac{xdx + ydy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$$

$$(xi) \quad d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2 y^2}$$

$$(xii) \quad d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$$

$$(xiii) \quad d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$$

## EXERCISE-I

### (FORMATION & VARIABLES SEPARABLE)

1. State the order and degree of the following differential equations:

$$(i) \quad \left[ \frac{d^2x}{dt^2} \right]^3 + \left[ \frac{dx}{dt} \right]^4 - xt = 0$$

$$(ii) \quad \frac{d^2y}{dx^2} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}$$

2. Obtain the differential equation of the family of circles  $x^2 + y^2 + 2gx + 2fy + c = 0$ ; where g, f & c are arbitrary constants.

3. Form the differential equation of the family of curves represented by,  $c(y + c)^2 = x^3$ ; where c is any arbitrary constant.

$$4. \quad \frac{\ln(\sec x + \tan x)}{\cos x} dx = \frac{\ln(\sec y + \tan y)}{\cos y} dy$$

$$5. \quad \frac{dy}{dx} + \frac{\sqrt{(x^2 - 1)(y^2 - 1)}}{xy} = 0$$

$$6. \quad y - x \frac{dy}{dx} = a \left( y^2 + \frac{dy}{dx} \right)$$

$$7. \quad \frac{dy}{dx} = \sin(x + y) + \cos(x + y)$$

$$8. \quad \frac{dy}{dx} = \frac{x(2\ln x + 1)}{\sin y + y \cos y}$$

9. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at  $t = 0$ , the mass of the radium was  $m_0$  and during time  $t_0$   $\alpha$  % of the original mass of radium decay.



10. The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being  $k > 0$ . Find  
 (a) Population at any time  $t$ , given initial population of the town being 2500.  
 (b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.  
 (c) Predict about the population of the town in the long run.
11.  $e^{(dy/dx)} = x + 1$  given that when  $x = 0$ ,  $y = 3$
12. A normal is drawn at a point  $P(x, y)$  of a curve. It meets the  $x$ -axis at  $Q$ . If  $PQ$  is of constant length  $k$ , then show that the differential equation describing such curves is,  $y \frac{dy}{dx} = \pm \sqrt{k^2 - y^2}$ . Find the equation of such a curve passing through  $(0, k)$ .
13. Obtain the differential equation associated with the primitive,  

$$y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x, \quad \text{where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$
14. Find the curve  $y = f(x)$  where  $f(x) \geq 0$ ,  $f(0) = 0$ , bounding a curvilinear trapezoid with the base  $[0, x]$  whose area is proportional to  $(n+1)^{\text{th}}$  power of  $f(x)$ . It is known that  $f(1) = 1$ .
15. 
$$\frac{x dx - y dy}{x dy - y dx} = \sqrt{\frac{1+x^2-y^2}{x^2-y^2}}$$
16. A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point. Form the differential equation and solve it to find the equation of the curve.

## EXERCISE-II

### (HOMOGENEOUS)

1. (a)  $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$       (b)  $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$
2. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.
3. The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the  $x$ -axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.
4. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through  $(1, 1)$ .
5. Show that the equation of the curve intersecting with the  $x$ -axis at the point  $x = 1$  and for which the length of the subnormal at any point of the curve is equal to the arithmetic mean of the co-ordinates of this point is  $(y-x)^2(x+2y) = 1$ .
6. Use the substitution  $y^2 = a - x$  to reduce the equation  $y^3 \cdot \frac{dy}{dx} + x + y^2 = 0$  to homogeneous form and hence solve it.
7.  $\left[ x \cos \frac{y}{x} + y \sin \frac{y}{x} \right] y = \left[ y \sin \frac{y}{x} - x \cos \frac{y}{x} \right] x \frac{dy}{dx}$
8. Find the curve for which any tangent intersects the  $y$ -axis at the point equidistant from the point of tangency and the origin.



9.  $(x - y) dy = (x + y + 1) dx$

10.  $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

11.  $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

12.  $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

13.  $\frac{dy}{dx} = \frac{2(y + 2)^2}{(x + y - 1)^2}$

14.  $\frac{dy}{dx} + \frac{\cos x (3 \cos y - 7 \sin x - 3)}{\sin y (3 \sin x - 7 \cos y + 7)} = 0$

15. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point,

$$\sqrt{x^2 + y^2} = c e^{\pm \tan^{-1} \frac{y}{x}}$$

### EXERCISE-III

#### (LINEAR)

1. Show that the equation of the curve whose slope at any point is equal to  $y + 2x$  and which pass through the origin is  $y = 2(e^x - x - 1)$ .

2. Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa .

3.  $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

4.  $(1 + y + x^2y) dx + (x + x^3) dy = 0$

5. Find the curve possessing the property that the intercept , the tangent at any point of a curve cuts off on the  $y$ -axis is equal to the square of the abscissa of the point of tangency.

6.  $\sin x \frac{dy}{dx} + 3y = \cos x$

7.  $x \frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$

8.  $(1 + y^2) dx = (\tan^{-1} y - x) dy$

9. Find the curve such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is equal to  $a^2$ . (Initial ordinate means  $y$  intercept of the tangent).

10. Let the function  $\ln f(x)$  is defined where  $f(x)$  exists for  $x \geq 2$  &  $k$  is fixed positive real number, prove that if  $\frac{d}{dx}(x \cdot f(x)) \leq -k f(x)$  then  $f(x) \leq A x^{-1-k}$  where  $A$  is independent of  $x$ .

11. Find the differentiable function which satisfies the equation  $f(x) = - \int_0^x f(t) \tan t dt + \int_0^x \tan(t-x) dt$

where  $x \in (-\pi/2, \pi/2)$

12. Find all functions  $f(x)$  defined on  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  with real values and has a primitive  $F(x)$  such that

$$f(x) + \cos x \cdot F(x) = \frac{\sin 2x}{(1 + \sin x)^2} . \text{ Find } f(x).$$



13. If  $y_1$  &  $y_2$  be solutions of the differential equation  $\frac{dy}{dx} + Py = Q$ , where  $P$  &  $Q$  are functions of  $x$  alone, and  $y_2 = y_1 z$ , then prove that

$$z = 1 + a e^{-\int \frac{Q}{y_1} dx}, \text{ 'a' being an arbitrary constant.}$$

14.  $\frac{dy}{dx} + \frac{y}{x} \ln y = \frac{y}{x^2} (\ln y)^2$

15.  $\frac{dy}{dx} + xy = y^2 e^{x^2/2} \cdot \sin x$

16.  $2 \frac{dy}{dx} - y \sec x = y^3 \tan x$

17.  $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$

18.  $y(2xy + e^x) dx - e^x dy = 0$

19. Find the curve for which the area of the triangle formed by the  $x$ -axis, the tangent line and radius vector of the point of tangency is equal to  $a^2$ .

20. A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min, and the mixture is pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

21.  $(x^3 + y^2 + 2) dx + 2y dy = 0$

22.  $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$

23.  $\left(\frac{dy}{dx}\right)^2 - (x+y) \frac{dy}{dx} + xy = 0$

24.  $(1 - xy + x^2 y^2) dx = x^2 dy$

25.  $y y' \sin x = \cos x (\sin x - y^2)$

## EXERCISE-IV

### (MISCELLANEOUS)

1.  $\frac{dy}{dx} - y \ln 2 = 2^{\sin x} \cdot (\cos x - 1) \ln 2$ ,  $y$  being bounded when  $x \rightarrow +\infty$ .

2.  $\frac{dy}{dx} = y + \int_0^1 y dx$  given  $y = 1$ , where  $x = 0$

3. Given two curves  $y = f(x)$  passing through the points  $(0, 1)$  &  $y = \int_{-\infty}^x f(t) dt$  passing through the points  $(0, 1/2)$ . The tangents drawn to both curves at the points with equal abscissas intersect on the  $x$ -axis. Find the curve  $f(x)$ .

4. Consider the differential equation,  $\frac{dy}{dx} + P(x)y = Q(x)$

- (i) If two particular solutions of given equation  $u(x)$  and  $v(x)$  are known, find the general solution of the same equation in terms of  $u(x)$  and  $v(x)$ .
- (ii) If  $\alpha$  and  $\beta$  are constants such that the linear combinations  $\alpha \cdot u(x) + \beta \cdot v(x)$  is a solution of the given equation, find the relation between  $\alpha$  and  $\beta$ .

- (iii) If  $w(x)$  is the third particular solution different from  $u(x)$  and  $v(x)$  then find the ratio  $\frac{v(x) - u(x)}{w(x) - u(x)}$ .



5. Find the curve which passes through the point  $(2, 0)$  such that the segment of the tangent between the point of tangency & the  $y$ -axis has a constant length equal to 2.
6.  $x \frac{dy}{dx} + y = \frac{x dy - y dx}{x^2 + y^2} = 0$
7.  $\frac{y dx - x dy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$ , given that  $y=2$  when  $x=1$
8. Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the  $x$ -axis lies on the parabola  $2y^2 = x$ .
9. Find the continuous function which satisfies the relation,  $\int_0^x t f(x-t) dt = \int_0^x f(t) dt + \sin x + \cos x - x - 1$ , for all real number  $x$ .
10.  $(1-x^2)^2 dy + \left( y \sqrt{1-x^2} - x - \sqrt{1-x^2} \right) dx = 0$ .
11.  $3x^2 y^2 + \cos(xy) - xy \sin(xy) + \frac{dy}{dx} \{2x^3 y - x^2 \sin(xy)\} = 0$ .
12. Find the integral curve of the differential equation,  $x(1-x/n)y \cdot \frac{dy}{dx} + y = 0$  which passes through  $\left(1, \frac{1}{e}\right)$ .
13. Find all the curves possessing the following property; the segment of the tangent between the point of tangency & the  $x$ -axis is bisected at the point of intersection with the  $y$ -axis.
14. A perpendicular drawn from any point  $P$  of the curve on the  $x$ -axis meets the  $x$ -axis at  $A$ . Length of the perpendicular from  $A$  on the tangent line at  $P$  is equal to ' $a$ '. If this curve cuts the  $y$ -axis orthogonally, find the equation to all possible curves, expressing the answer explicitly.
15. A curve passing through  $(1, 0)$  such that the ratio of the square of the intercept cut by any tangent off the  $y$ -axis to the subnormal is equal to the ratio of the product of the co-ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.
16. A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water?
17. A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If ' $m$ ' grams of salt are present in the tank after  $t$  minute, express ' $m$ ' in terms of  $t$  and find the amount of salt present after 10 minutes.
18. Let  $f(x, y, c_1) = 0$  and  $f(x, y, c_2) = 0$  define two integral curves of a homogeneous first order differential equation. If  $P_1$  and  $P_2$  are respectively the points of intersection of these curves with an arbitrary line,  $y = mx$  then prove that the slopes of these two curves at  $P_1$  and  $P_2$  are equal.
19. Find the curve for which the portion of  $y$ -axis cut-off between the origin and the tangent varies as cube of the abscissa of the point of contact.



20. Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.  
 (i)  $y = ax^2$       (ii)  $\cos y = a e^{-x}$       (iii)  $x^k + y^k = a^k$   
 (iv) Find the isogonal trajectories for the family of rectangular hyperbolas  $x^2 - y^2 = a^2$  which makes with it an angle of  $45^\circ$ .

### EXERCISE-V

1. The order and degree of the differential equation  $\left(1 + 3 \frac{dy}{dx}\right)^{2/3} = 4 \frac{d^3y}{dx^3}$  are [AIEEE 2002]  
 (A)  $\left(1, \frac{2}{3}\right)$       (B) (3, 1)      (C) (3, 3)      (D) (1, 2)
2. The solution to the equation  $\frac{d^2y}{dx^2} = e^{-2x}$  is : [AIEEE 2002]  
 (A)  $\frac{e^{-2x}}{4}$       (B)  $\frac{e^{-2x}}{4} + cx + d$       (C)  $\frac{1}{4} e^{-2x} + cx^2 + d$       (D)  $\frac{1}{4} e^{-4x} + cx + d$
3. The degree and order of the differential equation of the family of all parabolas whose axis is x - axis, are respectively [AIEEE 2003]  
 (A) 2, 3      (B) 2, 1      (C) 1, 2      (D) 3, 2
4. The solution of the differential equation  $(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$ , is [AIEEE 2003]  
 (A)  $x e^{2\tan^{-1}y} = e^{\tan^{-1}y} + k$       (B)  $(x - 2) = k e^{-\tan^{-1}y}$   
 (C)  $2x e^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$       (D)  $x e^{\tan^{-1}y} = \tan^{-1}y + k$
5. The differential equation for the family of curves  $x^2 + y^2 - 2ay = 0$ , where a is an arbitrary constant is [AIEEE 2004]  
 (A)  $2(x^2 - y^2) y' = xy$       (B)  $2(x^2 + y^2) y' = xy$       (C)  $(x^2 - y^2) y' = 2xy$       (D)  $(x^2 + y^2) y' = 2xy$
6. The solution of the differential equation,  $y dx + (x + x^2 y) dy = 0$  is [AIEEE 2004]  
 (A)  $\frac{1}{xy} = C$       (B)  $-\frac{1}{xy} + \log y = C$       (C)  $\frac{1}{xy} + \log y = C$       (D)  $\log y = Cx$
7. The differential equation representing the family of curves  $y^2 = 2c(x + \sqrt{c})$ , where  $c > 0$  is a parameter, is of order and degree as follows [AIEEE- 2005]  
 (A) order 1, degree 2      (B) order 1, degree 1      (C) order 1, degree 3      (D) order 2, degree 2
8. If  $x \frac{dy}{dx} = y(\log y - \log x + 1)$ , then the solution of the equation is [AIEEE-2005]  
 (A)  $y \log\left(\frac{x}{y}\right) = cx$       (B)  $x \log\left(\frac{y}{x}\right) = cy$       (C)  $\log\left(\frac{y}{x}\right) = cx$       (D)  $\log\left(\frac{x}{y}\right) = cy$
9. The differential equation whose solution is  $Ax^2 + By^2 = 1$ , where A and B are arbitrary constants, is of  
 (A) first order and second degree      (B) first order and first degree [AIEEE 2006]  
 (C) second order and first degree      (D) second order and second degree



10. The differential equation of all circles passing through the origin and having their centres on the x-axis is  
 (A)  $x^2 = y^2 + xy \frac{dy}{dx}$       (B)  $x^2 = y^2 + 3xy \frac{dy}{dx}$       [AIEEE 2007]  
 (C)  $y^2 = x^2 + 2xy \frac{dy}{dx}$       (D)  $y^2 = x^2 - 2xy \frac{dy}{dx}$
11. The differential equation of the family of circles with fixed radius 5 units and centre on the line  $y = 2$  is  
 (A)  $(y - 2)^2 y'^2 = 25 - (y - 2)^2$       (B)  $(y - 2)^2 y'^2 = 25 - (y - 2)^2$       [AIEEE 2008]  
 (C)  $(x - 2)^2 y'^2 = 25 - (y - 2)^2$       (D)  $(x - 2)^2 y'^2 = 25 - (y - 2)^2$
12. The solution of the differential equation  $\frac{dy}{dx} = \frac{x+y}{x}$  satisfying the condition  $y(1) = 1$  is      [AIEEE 2008]  
 (A)  $y = x \ln x + x^2$       (B)  $y = xe^{(x-1)}$       (C)  $y = x \ln x + x$       (D)  $y = \ln x + x$
13. The differential equation which represents the family of curves  $y = c_1 e^{c_2 x}$  where  $c_1$  and  $c_2$  are arbitrary constants, is      [AIEEE 2009]  
 (A)  $y' = y^2$       (B)  $y'' = y' y$       (C)  $yy'' = y'$       (D)  $yy'' = (y')^2$
14. The solution to the differential equation  $\cos x \frac{dy}{dx} = y(\sin x - y)$ ,  $0 < x < \frac{\pi}{2}$ , is      [AIEEE 2010]  
 (A)  $\tan x = (\sec x + c)y$       (B)  $\sec x = (\tan x + c)y$   
 (C)  $y \sec x = \tan x + c$       (D)  $y \tan x = \sec x + c$
15. Let  $I$  be the purchase value of an equipment and  $V(t)$  be the value after it has been used for  $t$  years. The value  $V(t)$  depreciates at a rate given by differential equation  $\frac{dV(t)}{dt} = -k(T-t)$ , where  $k > 0$  is a constant and  $T$  is the total life in years of the equipment. Then the scrap value  $V(T)$  of the equipment is :      [AIEEE 2011]  
 (A)  $e^{-kT}$       (B)  $T^2 - \frac{I}{k}$       (C)  $I - \frac{kT^2}{2}$       (D)  $I - \frac{k(T-t)^2}{2}$
16. If  $\frac{dy}{dx} = y + 3 > 0$  and  $y(0) = 2$ , then  $y(\ln 2)$  is equal to      [AIEEE 2011]  
 (A) -2      (B) 7      (C) 5      (D) 13
17. The population  $p(t)$  at time  $t$  of a certain mouse species satisfies the differential equation      [AIEEE 2012]  

$$\frac{dp(t)}{dt} = 0.5 p(t) - 450.$$
 If  $p(0) = 850$ , then the time at which the population becomes zero is :  
 (A)  $2 \ln 18$       (B)  $\ln 9$       (C)  $\frac{1}{2} \ln 18$       (D)  $\ln 18$
18. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production  $P$  w.r.t. additional number of workers  $x$  is given by  $\frac{dP}{dx} = 100 - 12\sqrt{x}$ . If the firm employs 25 more workers, then the new level of production of items is :      [JEE Main 2013]  
 (A) 4500      (B) 2500      (C) 3000      (D) 3500



19. Let the population of rabbits surviving at a time  $t$  be governed by the differential equation  $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$ .

If  $p(0) = 100$ , then  $p(t)$  equals

[JEE Main 2014]

- (A)  $400 - 300 e^{-t/2}$       (B)  $400 - 300 e^{t/2}$       (C)  $300 - 200 e^{-t/2}$       (D)  $600 - 500 e^{t/2}$

20. Let  $y(x)$  be the solution of the differential equation  $(x \log x) \frac{dy}{dx} + y = 2x \log x$ , ( $x \geq 1$ ). Then  $y(e)$  is equal to:

[JEE Main 2015]

- (A)  $2e$       (B)  $e$       (C)  $0$       (D)  $2$

21. If a curve  $y = f(x)$  passes through the point  $(1, -1)$  and satisfies the differential equation,  $y(1 + xy) dx = x dy$ ,

then  $f\left(-\frac{1}{2}\right)$  is equal to :

[JEE Main 2016]

- (A)  $-\frac{2}{5}$       (B)  $-\frac{4}{5}$       (C)  $\frac{2}{5}$       (D)  $\frac{4}{5}$

22. If  $(2 + \sin x) \frac{dy}{dx} + (y + 1) \cos x = 0$  and  $y(0) = 1$  then  $y\left(\frac{\pi}{2}\right)$  is equal to :      [JEE Main 2017]

- (A)  $\frac{4}{3}$       (B)  $\frac{1}{3}$       (C)  $-\frac{2}{3}$       (D)  $-\frac{1}{3}$

23. Let  $y = y(x)$  be the solution of the differential equation

$\sin x \frac{dy}{dx} + y \cos x = 4x$ ,  $x \in (0, \pi)$ . If  $y\left(\frac{\pi}{2}\right) = 0$ , then  $y\left(\frac{\pi}{6}\right)$  is equal to      [JEE Main 2018]

- (A)  $-\frac{4}{9}\pi^2$       (B)  $\frac{4}{9\sqrt{3}}\pi^2$       (C)  $\frac{-8}{9\sqrt{3}}\pi^2$       (D)  $-\frac{8}{9}\pi^2$

24. If  $y = y(x)$  is the solution of the differential equation,  $x \frac{dy}{dx} + 2y = x^2$  satisfying  $y(1) = 1$ , then  $y\left(\frac{1}{2}\right)$  is equal to      [IIT Main Online-2019]

- (A)  $\frac{49}{16}$       (B)  $\frac{13}{16}$       (C)  $\frac{7}{64}$       (D)  $\frac{1}{4}$

25. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be such that  $f(x y) = f(x).f(y)$ , for all  $x, y \in [0, 1]$ , and  $f(0) \neq 0$ . If  $y = y(x)$  satisfies the

differential equation,  $\frac{dy}{dx} = f(x)$  with  $y(0) = 1$ , then  $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$  is equal to :

[IIT Main Online-2019]

- (A) 3      (B) 5      (C) 2      (D) 4



26. If  $\frac{dy}{dx} + \frac{3}{\cos^2 x} y = \frac{1}{\cos^2 x}$ ,  $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ , and  $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$ , then  $y\left(-\frac{\pi}{4}\right)$  equals :

[IIT Main Online-2019]

- (A)  $-\frac{4}{3}$       (B)  $\frac{1}{3} + e^3$       (C)  $\frac{1}{3}$       (D)  $\frac{1}{3} + e^6$

27. Let  $f$  be a differentiable function such that  $f'(x) = 7 - \frac{3}{4} \frac{f(x)}{x}$ , ( $x > 0$ ) and  $f(1) \neq 4$ . Then  $\lim_{x \rightarrow 0^+} xf\left(\frac{1}{x}\right)$

[IIT Main Online-2019]



**IIT Main Online 2013**

- (A)  $y(\log_e 2) = \frac{\log_e 2}{4}$       (B)  $y(x)$  is decreasing in  $(0, 1)$

(C)  $y(x)$  is decreasing in  $\left(\frac{1}{2}, 1\right)$       (D)  $y(\log_e 2) = \log_e 4$

- 30.** The solution of the differential equation,  $\frac{dy}{dx} = (x - y)^2$ , when  $y(1) = 1$ , is : [IIT Main Online-2019]

- $$(A) \log_e \left| \frac{2-y}{2-x} \right| = 2(y-1) \quad (B) -\log_e \left| \frac{1+x-y}{1-x+y} \right| = x+y-2$$

- $$(C) \log_e \left| \frac{2-x}{2-y} \right| = x - y \quad (D) -\log_e \left| \frac{1-x+y}{1+x-y} \right| = 2(x-1)$$

31. Let  $y = y(x)$  be the solution of the differential equation,  $x \frac{dy}{dx} + y = x \log_e x$ , ( $x > 1$ ). If  $2y(2) = \log_e 4 - 1$ , then  $y(e)$  is equal to : [IIT Main Online-2019]



32. If a curve passes through the point  $(1, -2)$  and has slope of the tangent at any point  $(x, y)$  on it as  $\frac{x^2 - 2y}{x}$ , then the curve also passes through the point : [IIT Main Online-2019]
- (A)  $(\sqrt{3}, 0)$       (B)  $(3, 0)$       (C)  $(-\sqrt{2}, 1)$       (D)  $(-1, 2)$
33. If  $y = y(x)$  is the solution of the differential equation,  $e^y \left( \frac{dy}{dx} - 1 \right) = e^x$  such that  $y(0) = 0$ , then  $y(1)$  is equal to : [IIT Main Online-2020]
- (A)  $2 + \log_e 2$       (B)  $2e$       (C)  $\log_e 2$       (D)  $1 + \log_e 2$
34. Let  $y = y(x)$  be the solution curve of the differential equation,  $(y^2 - x) \frac{dy}{dx} = 1$ , satisfying  $y(0) = 1$ . This curve intersects the x-axis at a point whose abscissa is : [IIT Main Online-2020]
- (A)  $-e$       (B)  $2 + e$       (C)  $2$       (D)  $2 - e$
35. Let  $y = y(x)$  be a solution of the differential equation,  $\sqrt{1-x^2} \frac{dy}{dx} + \sqrt{1-y^2} = 0, |x| < 1$ . If  $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$ , then  $y\left(\frac{-1}{\sqrt{2}}\right)$  is equal to : [IIT Main Online-2020]
- (A)  $\frac{\sqrt{3}}{2}$       (B)  $-\frac{\sqrt{3}}{2}$       (C)  $-\frac{1}{\sqrt{2}}$       (D)  $\frac{1}{\sqrt{2}}$
36. The differential equation of the family of curves,  $x^2 = 4b(y+b)$ ,  $b \in \mathbb{R}$ , is: [IIT Main Online-2020]
- (A)  $x(y')^2 = x - 2yy'$       (B)  $x(y')^2 = 2yy' - x$       (C)  $x(y')^2 = x + 2yy'$       (D)  $xy'' = y'$
37. If  $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ ;  $y(1) = 1$ , then a value of  $x$  satisfying  $y(x) = e$  is [IIT Main Online-2020]
- (A)  $\sqrt{3}e$       (B)  $\frac{e}{\sqrt{2}}$       (C)  $\sqrt{2}e$       (D)  $\frac{1}{2}\sqrt{3}e$
38. If for  $x \geq 0$ ,  $y = y(x)$  is the solution of the differential equation,  $(x+1)dy = ((x+1)^2 + y - 3) dx$ ,  $y(2) = 0$ , then  $y(3)$  is equal to \_\_\_\_\_. [IIT Main Online-2020]



## **EXERCISE-VI**

## **(PROBLEMS ASKED IN JEE & REE)**

1. (a) A solution of the differential equation,  $\left(\frac{dy}{dx}\right)^2 - x \frac{dy}{dx} + y = 0$  is :

(A)  $y = 2$       (B)  $y = 2x$       (C)  $y = 2x - 4$       (D)  $y = 2x^2 - 4$

(b) The differential equation representing the family of curves,  $y^2 = 2c(x + \sqrt{c})$ , where  $c$  is a positive parameter, is of  
 (A) order 1      (B) order 2      (C) degree 3      (D) degree 4

(c) A curve passing through the point  $(1, 1)$  has the property that the perpendicular distance of the origin from the normal at any point  $P$  of the curve is equal to the distance of  $P$  from the  $x$ -axis. Determine the equation of the curve.  
**[JEE '99, 2 + 3 + 10, out of 200]**

2. Solve the differential equation,  $(x^2 + 4y^2 + 4xy) dy = (2x + 4y + 1) dx$ .      **[REE '99, 6]**

3. A country has a food deficit of 10%. Its population grows continuously at a rate of 3%. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after ' $n$ ' years, where ' $n$ ' is the smallest integer bigger than or equal to,  $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$ .      **[JEE '2000 (Main) 10]**

4. A hemispherical tank of radius 2 metres is initially full of water and has an outlet of  $12 \text{ cm}^2$  cross sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law  $V(t) = 0.6\sqrt{2gh(t)}$ , where  $V(t)$  and  $h(t)$  are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time  $t$ , and  $g$  is the acceleration due to gravity. Find the time it takes to empty the tank.      **[JEE '2001 (Main) 10]**

5. Find the equation of the curve which passes through the origin and the tangent to which at every point  $(x, y)$  has slope equal to  $\frac{x^4 + 2xy - 1}{1 + x^2}$ .      **[REE '2001 (Main) 3]**

6. Let  $f(x)$ ,  $x \geq 0$ , be a nonnegative continuous function, and let  $F(x) = \int_0^x f(t)dt$ ,  $x \geq 0$ . If for some  $c > 0$ ,  $f(x) \leq cF(x)$  for all  $x \geq 0$ , then show that  $f(x) = 0$  for all  $x \geq 0$ .      **[JEE 2001 (Main) 5 out of 100]**

7. (a) A right circular cone with radius  $R$  and height  $H$  contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant =  $k > 0$ ). Find the time after which the cone is empty.  
 (b) If  $P(1) = 0$  and  $\frac{dP(x)}{dx} > P(x)$  for all  $x \geq 1$  then prove that  $P(x) > 0$  for all  $x > 1$ .  
**[JEE 2003, (Main) 4 + 4]**

8. (a) If  $\left(\frac{2 + \sin x}{1 + y}\right) \frac{dy}{dx} = -\cos x$ ,  $y(0) = 1$ , then  $y\left(\frac{\pi}{2}\right) =$   
 (A) 1      (B) 1/2      (C) 1/3      (D) 1/4  
**[JEE 2004 (Scr.)]**



- (b) A curve passes through (2, 0) and the slope of tangent at point P (x, y) equals  $\frac{(x+1)^2 + y - 3}{(x+1)}$ . Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant.

[JEE 2004 (Main)]

9. (a) The solution of primitive integral equation  $(x^2 + y^2)dy = xy \, dx$ , is  $y = y(x)$ . If  $y(1) = 1$  and  $y(x_0) = e$ , then  $x_0$  is

(A)  $\sqrt{2(e^2 - 1)}$       (B)  $\sqrt{2(e^2 + 1)}$       (C)  $\sqrt{3} e$       (D)  $\sqrt{\frac{e^2 + 1}{2}}$

- (b) For the primitive integral equation  $ydx + y^2dy = xdy$ ;  $x \in \mathbb{R}$ ,  $y > 0$ ,  $y = y(x)$ ,  $y(1) = 1$ , then  $y(-3)$  is  
 (A) 3      (B) 2      (C) 1      (D) 5

[JEE 2005 (Scr.)]

- (c) If length of tangent at any point on the curve  $y = f(x)$  intercepted between the point and the x-axis is of length 1. Find the equation of the curve.

[JEE 2005 (Main)]

10. A tangent drawn to the curve,  $y = f(x)$  at P(x, y) cuts the x-axis and y-axis at A and B respectively such that  $BP : AP = 3 : 1$ , given that  $f(1) = 1$ , then

[JEE 2006, 5]

(A) equation of the curve is  $x \frac{dy}{dx} - 3y = 0$       (B) equation of curve is  $x \frac{dy}{dx} + 3y = 0$   
 (C) curve passes through  $(2, 1/8)$       (D) normal at  $(1, 1)$  is  $x + 3y = 4$

11. (a) Let  $f(x)$  be differentiable on the interval  $(0, \infty)$  such that  $f(1) = 1$  and  $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$  for each  $x > 0$ . Then  $f(x)$  is

(A)  $\frac{1}{3x} + \frac{2x^2}{3}$       (B)  $\frac{-1}{3x} + \frac{4x^2}{3}$       (C)  $\frac{-1}{x} + \frac{2}{x^2}$       (D)  $\frac{1}{x}$

- (b) The differential equation  $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$  determines a family of circles with

(A) variable radii and a fixed centre at  $(0, 1)$   
 (B) variable radii and a fixed centre at  $(0, -1)$   
 (C) fixed radius 1 and variable centres along the x-axis.  
 (D) fixed radius 1 and variable centres along the y-axis.

[JEE 2007, 3+3]

12. Let a solution  $y = y(x)$  of the differential equation,  $x\sqrt{x^2 - 1}dy - y\sqrt{y^2 - 1}dx = 0$  satisfy  $y(2) = \frac{2}{\sqrt{3}}$ .

**STATEMENT-1 :**  $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$   
**and**

**STATEMENT-2 :**  $y(x)$  is given by  $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

- (A) Statement-1 is True, Statement-2 is True; statement-2 is a correct explanation for statement-1  
 (B) Statement-1 is True, Statement-2 is True; statement-2 is NOT a correct explanation for statement-1  
 (C) Statement-1 is True, Statement-2 is False  
 (D) Statement-1 is False, Statement-2 is True

[JEE 2008, 3]



13. Match the statements/expression in **Column I** with the open intervals in **Column II**. [JEE 2009]
- | <b>Column I</b>  | <b>Column II</b>                                 |
|--|--|
| (A) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x - 3)^2 y' + y = 0$ | (P) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |
| (B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$                                    | (Q) $\left(0, \frac{\pi}{2}\right)$              |
| (C) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies                                | (R) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$ |
| (D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing   | (T) $(-\pi, \pi)$                                |
14. Let  $f$  be a real-valued differentiable function on  $\mathbb{R}$  (the set of all real numbers) such that  $f(1) = 1$ . If the  $y$ -intercept of the tangent at any point  $P(x, y)$  on the curve  $y = f(x)$  is equal to the cube of the abscissa of  $P$ , then the value of  $f(-3)$  is equal to [JEE 2010]
15. Let  $y'(x) + y(x) g'(x) = g(x) g'(x)$ ,  $y(0) = 0$ ,  $x \in \mathbb{R}$ , where  $f'(x)$  denotes  $\frac{df(x)}{dx}$  and  $g(x)$  is a given non-constant differentiable function on  $\mathbb{R}$  with  $g(0) = g(2) = 0$ . Then the value of  $y(2)$  is [JEE 2011]
16. If  $y(x)$  satisfies the differential equation  $y' - y \tan x = 2x \sec x$  and  $y(0) = 0$ , then [JEE 2012]
- (A)  $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$       (B)  $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$       (C)  $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$       (D)  $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$
17. A curve passes through the point  $\left(1, \frac{\pi}{6}\right)$ . Let the slope of the curve at each point  $(x, y)$  be  $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$ ,  $x > 0$ . Then the equation of the curve is : [JEE (Adv.) 2013]
- (A)  $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$       (B)  $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x - 2$   
 (C)  $\sec\left(\frac{2y}{x}\right) = \log x + 2$       (D)  $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$
18. The function  $y = f(x)$  is the solution of the differential equation [JEE (Adv.) 2014]
- $$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1-x^2}} \quad \text{in } (-1, 1) \text{ satisfying } f(0) = 0. \quad \text{Then } \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x) dx \text{ is :}$$
- (A)  $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$       (B)  $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$       (C)  $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$       (D)  $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$
19. Let  $y(x)$  be a solution of the differential equation  $(1 + e^x) y' + y e^x = 1$ . If  $y(0) = 2$ , then which of the following statements is (are) true? [JEE (Adv.) 2015]
- (A)  $y(-4) = 0$       (B)  $y(-2) = 0$   
 (C)  $y(x)$  has a critical point in the interval  $(-1, 0)$       (D)  $y(x)$  has no critical point in the interval  $(-1, 0)$



20. Consider the family of all circles whose centers lie on the straight line  $y = x$ . If this family of circles is represented by the differential equation  $Py'' + Qy' + 1 = 0$ , where P, Q are functions of x, y and  $y'$  (here

$y' = \frac{dy}{dx}$ ,  $y'' = \frac{d^2y}{dx^2}$ ), then which of the following statements is (are) true?

[JEE Advanced 2015]

(A)  $P = y + x$

(B)  $P = y - x$

(C)  $P + Q = 1 - x + y + y' + (y')^2$

(D)  $P - Q = x + y - y' - (y')^2$

21. A solution curve of the differential equation  $(x^2 + xy + 4x + 2y + 4) \frac{dy}{dx} - y^2 = 0$ ,  $x > 0$ , passes through the point  $(1, 3)$ . Then the solution curve [JEE Advanced 2016]
- (A) intersects  $y = x + 2$  exactly at one point  
 (B) intersects  $y = x + 2$  exactly at two points  
 (C) intersects  $y = (x + 2)^2$   
 (D) does NOT intersect  $y = (x + 3)^2$

22. If  $y = y(x)$  satisfies the differential equation

[JEE Advanced 2017]

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, \quad x > 0$$

and  $y(0) = \sqrt{7}$ , then  $y(256) =$

(A) 80

(B) 3

(C) 16

(D) 9

23. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be a differentiable function with  $f(0) = 0$ . If  $y = f(x)$  satisfies the differential equation

$$\frac{dy}{dx} = (2 + 5y)(5y - 2), \text{ then the value of } \lim_{x \rightarrow -\infty} f(x) \text{ is } \underline{\hspace{2cm}}.$$

[JEE Advanced 2018]

24. Let  $\Gamma$  denote a curve  $y = y(x)$  which is in the first quadrant and let the point  $(1, 0)$  lie on it. Let the tangent to  $\Gamma$  at a point P intersect the y-axis at  $Y_P$ . If  $PY_P$  has length 1 for each point P on  $\Gamma$ , then which of the following options is/are correct? [JEE Advanced 2019]

(A)  $xy' - \sqrt{1-x^2} = 0$

(B)  $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(C)  $xy' + \sqrt{1-x^2} = 0$

(D)  $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

25. Let  $b$  be a nonzero real number. Suppose  $f : \mathbf{R} \rightarrow \mathbf{R}$  is a differentiable function such that  $f(0) = 1$ . If the derivative  $f'$  of  $f$  satisfies the equation

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

for all  $x \in \mathbf{R}$ , then which of the following statements is/are TRUE?

[JEE Advanced 2020]

- (A) If  $b > 0$ , then  $f$  is an increasing function  
 (B) If  $b < 0$ , then  $f$  is a decreasing function  
 (C)  $f(x)f(-x) = 1$  for all  $x \in \mathbf{R}$   
 (D)  $f(x) - f(-x) = 0$  for all  $x \in \mathbf{R}$



26. For any real numbers  $\alpha$  and  $\beta$ , let  $y_{\alpha,\beta}(x)$ ,  $x \in \mathbb{R}$ , be the solution of the differential equation

$$\frac{dy}{dx} + ay = xe^{\beta x}, y(1) = 1$$

Let  $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$ . Then which of the following functions belong(s) to the set  $S$ ?

[JEE Advanced 2021]

(A)  $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B)  $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

(C)  $f(x) = \frac{e^x}{2} \left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D)  $f(x) = \frac{e^x}{2} \left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$



## ANSWER KEY

DIFFERENTIAL EQUATION  
EXERCISE-I

1. (i) order 2 & degree 3 (ii) order 2 & degree 2
2.  $[1 + (y')^2] \cdot y''' - 3y'(y'')^2 = 0$
3.  $12y(y')^2 = x[8(y')^3 - 27]$
4.  $\ln^2(\sec x + \tan x) - \ln^2(\sec y + \tan y) = c$
5.  $\sqrt{x^2 - 1} - \sec^{-1} x + \sqrt{y^2 - 1} = c$
6.  $y = c(1 - ay)(x + a)$
7.  $\ln\left[1 + \tan\frac{x+y}{2}\right] = x + c$
8.  $y \sin y = x^2 \ln x + c$
9.  $m = m_0 e^{-kt}$  where  $k = -\frac{1}{t_0} \ln\left(1 - \frac{\alpha}{100}\right)$
10. (a)  $P = 1000 + 1500e^{-kt}$  where  $k = \frac{1}{10} \ln\left(\frac{5}{3}\right)$ ; (b)  $T = 10 \log_{5/3}(3)$ ; (c)  $P = 1000$  as  $t \rightarrow \infty$
11.  $y = (x+1) \cdot \ln(x+1) - x + 3$
12.  $x^2 + y^2 = k^2$
13.  $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$
14.  $y = x^{1/n}$
15.  $\sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = \frac{c(x+y)}{\sqrt{x^2 - y^2}}$
16.  $y = kx$  or  $xy = c$

EXERCISE-II

1. (a)  $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6} = \exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$  where  $\exp x \equiv e^x$  (b)  $y^2 - x^2 = c(y^2 + x^2)^2$
2.  $\frac{y^2 \pm y\sqrt{y^2 - x^2}}{x^2} = \ell n \left| \left( y \pm \sqrt{y^2 - x^2} \right) \cdot \frac{c^2}{x^3} \right|$ , where same sign has to be taken.
4.  $x^2 + y^2 - 2x = 0$
6.  $\frac{1}{2} \ln|x^2 + a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c$ , where  $a = x + y^2$
7.  $xy \cos\frac{y}{x} = c$
8.  $x^2 + y^2 = cx$



9.  $\arctan \frac{2y+1}{2x+1} = \ln c \sqrt{x^2 + y^2 + x + y + \frac{1}{2}}$       10.  $(x+y-2) = c(y-x)^3$

11.  $\tan^{-1} \frac{y+3}{x+2} + \ln c \sqrt{(y+3)^2 + (x+2)^2} = 0$       12.  $x+y+\frac{4}{3} = ce^{3(x-2y)}$

13.  $e^{-2\tan^{-1} \frac{y+2}{x-3}} = c \cdot (y+2)$       14.  $(\cos y - \sin x - 1)^2 (\cos y + \sin x - 1)^5 = c$

### EXERCISE-III

1.  $y = 2(e^x - x - 1)$

2.  $y = cx^2 \pm x$

3.  $y(x-1) = x^2(x^2 - x + c)$

4.  $xy = c - \arctan x$

5.  $y = cx - x^2$

6.  $\left(\frac{1}{3} + y\right) \tan^3 \frac{x}{2} = c + 2 \tan \frac{x}{2} - x$

7.  $y = cx + x \ln \tan x$

8.  $x = ce^{-\arctan y} + \arctan y - 1$

9.  $y = cx \pm \frac{a^2}{2x}$

11.  $\cos x - 1$

12.  $f(x) = -\frac{2 \cos x}{(1+\sin x)^2} - Ce^{-\sin x} \cdot \cos x$

14.  $x = \ln y \left(cx^2 + \frac{1}{2}\right)$

15.  $e^{-x^2/2} = y(c + \cos x)$

16.  $\frac{1}{y^2} = -1 + (c+x) \cot \left(\frac{x}{2} + \frac{\pi}{4}\right)$

17.  $x^3 y^{-3} = 3 \sin x + c$

18.  $y^{-1} e^x = c - x^2$

19.  $x = cy \pm \frac{a^2}{y}$

20.  $27\frac{7}{9}$  minutes

21.  $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$

22.  $\sin y = (e^x + c)(1 + x)$

23.  $y = ce^x ; y = c + \frac{x^2}{2}$

24.  $y = \frac{1}{x} \tan(\ell \ln |cx|)$

25.  $y^2 = \frac{2}{3} \sin x + \frac{c}{\sin^2 x}$

EXERCISE-IV

1.  $y = 2^{\sin x}$

2.  $y = \frac{1}{3-e} (2e^x - e + 1)$

3.  $f(x) = e^{2x}$

4. (i)  $y = u(x) + K(u(x) - v(x))$  where  $K$  is any constant ; (ii)  $\alpha + \beta = 1$ ; (iii) constant

5.  $y = \pm \left[ \sqrt{4-x^2} + 2 \ln \frac{2-\sqrt{4-x^2}}{x} \right]$

6.  $xy + \tan^{-1} \frac{y}{x} = c$

7.  $\frac{\sin^{-1} x}{2} + \frac{y}{x-y} = \frac{\pi}{4} - 2$  8.  $y^2 = 2x + 1 - e^{2x}$  9.  $f(x) = e^x - \cos x$

10.  $y = \frac{x}{\sqrt{1-x^2}} + ce^{-\frac{x}{\sqrt{1-x^2}}}$  11.  $x(x^2 y^2 + \cos xy) = c$  12.  $x(ey + \ln y + 1) = 1$

13.  $y^2 = cx$  14.  $y = \pm a \frac{e^{x/a} + e^{-x/a}}{2}$  &  $y = \pm a$  15.  $x = e^{2\sqrt{y/x}}$  ;  $x = e^{-2\sqrt{y/x}}$

16.  $T = \log_{4/3} 2$  hrs from the start 17.  $y = 5t \left( 1 + \frac{50}{50+t} \right)$  gms ;  $91\frac{2}{3}$  gms 19.  $2y + Kx^3 = cx$

20. (i)  $x^2 + 2y^2 = c$ , (ii)  $\sin y = ce^{-x}$ , (iii)  $y = cx$  if  $k = 2$  and  $\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$  if  $k \neq 2$   
(iv)  $x^2 - y^2 + 2xy = c$  ;  $x^2 - y^2 - 2xy = c$

EXERCISE-V

- |       |       |       |       |           |       |       |       |          |       |
|-------|-------|-------|-------|-----------|-------|-------|-------|----------|-------|
| 1. C  | 2. B  | 3. C  | 4. C  | 5. C      | 6. B  | 7. C  | 8. C  | 9. C     | 10. C |
| 11. B | 12. C | 13. D | 14. B | 15. C     | 16. B | 17. A | 18. D | 19. B    | 20. D |
| 21. D | 22. B | 23. D | 24. A | 25. A     | 26. D | 27. C | 28. B | 29. C    | 30. D |
| 31. D | 32. A | 33. D | 34. D | 35. Bonus |       | 36. C | 37. A | 38. 3.00 |       |

EXERCISE-VI

1. (a) C, (b) A, C, (c)  $x^2 + y^2 - 2x = 0$

2.  $y = \ln((x+2y)^2 + 4(x+2y) + 2) - \frac{3}{2\sqrt{2}} \ln\left(\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right) + c$



4.  $\frac{7\pi \times 10^5}{135\sqrt{g}}$  sec.    5.  $y = (x - 2\tan^{-1}x)(1 + x^2)$     7. (a)  $T = H/k$
8. (a) C; (b)  $y = x^2 - 2x$ , area =  $4/3$  sq. units    9. (a) C; (b) A; (c)  $\sqrt{1-y^2} + \ln \left| \frac{1-\sqrt{1-y^2}}{y} \right| = \pm x + c$
10. A, B, C, D    11. (a) A, (b) C    12. C  
 13. (A)  $\rightarrow P, Q, S$ ; (B)  $\rightarrow P, T$ ; (C)  $\rightarrow P, Q, R, T$ ; (D)  $\rightarrow S$     14. 9    15. 0    16. A, D  
 17. A    18. B    19. A, C    20. B, C    21. A, D    22. B    23. 0.4  
 24. BC    25. AC    26. AC