HOME WORK–2

(PROBLEMS BASED ON FUNDAMENTALS)

A trigonometric equation is of the form, a $\cos\theta \pm$ b $\sin\theta = c$

Q. Solve for $\boldsymbol{\theta}$:

1. $\sin(\theta) + \cos(\theta) = 1$

Ans. $\left(n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right)$

Sol. We have $\sin(\theta) + \cos(\theta) = 1$

$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)\right) = 1$ $\quad\Rightarrow \left(\frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ $\quad\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \left(\sin\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \left(\theta + \frac{\pi}{4}\right) = \left(n\pi + (-1)^n\left(\frac{\pi}{4}\right)\right)$ $\quad\Rightarrow \theta = \left(n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right), n \in I$

2. $\sqrt{3}\sin(\theta) + \cos(\theta) = 2$

Ans. $n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$

Sol. We have $\sqrt{3}\sin(\theta) + \cos(\theta) = 2$

$\Rightarrow \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = 1$ $\quad\Rightarrow \sin\left(\theta + \frac{\pi}{6}\right) = 1$

$\Rightarrow \sin\left(\theta + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$ $\quad\Rightarrow \left(\theta + \frac{\pi}{6}\right) = n\pi + (-1)^n\left(\frac{\pi}{2}\right)$

$\Rightarrow \theta = n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$

3. $\sin(2\theta) + \cos(2\theta) + \sin(\theta) + \cos(\theta) + 1 = 0$

Ans. $n\pi - \frac{\pi}{4}$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right)$

Sol. We have

$\sin(2\theta) + \cos(2\theta) + \sin(\theta) + \cos(\theta) + 1 = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta)) + (1 + \sin(2\theta)) + \cos(2\theta) = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta)) + (\sin(\theta) + \cos(\theta))^2 + (\cos^2\theta - \sin^2\theta) = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta)) + (\sin(\theta) + \cos(\theta))^2 + (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta))$

$(1 + (\sin(\theta) + \cos(\theta)) + (\cos\theta - \sin\theta)) = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta))(1 + 2\cos\theta) = 0$

$\Rightarrow (\sin(\theta) + \cos(\theta)) = 0$ and $(1 + 2\cos\theta) = 0$

$\Rightarrow \left(\sin\left(\frac{\pi}{4} + \theta\right)\right) = 0$ and $\cos\theta = -\frac{1}{2}$

$\Rightarrow \left(\frac{\pi}{4} + \theta\right) = n\pi$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right)$

$\Rightarrow \theta = n\pi - \frac{\pi}{4}$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right), n \in I$

4. $\sin^3\theta + \sin\theta\cos\theta + \cos^3\theta = 1$

Ans. $\frac{1}{2}\left(n\pi + (-1)^n\left(\frac{\pi}{6}\right)\right)$

Sol. We have $\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta = 1$

$\Rightarrow (\sin^3\ \theta + \cos^3\ \theta) + \sin\ \theta\cos\ \theta = 1$

$\Rightarrow (\sin\ \theta + \cos\ \theta)(1 - \sin\ \theta\cos\ \theta) + \sin\ \theta\cos\ \theta = 1$

$\Rightarrow (\sin\ \theta + \cos\ \theta)(1 - \sin\ \theta\cos\ \theta) = (1 - \sin\ \theta\cos\ \theta)$

$\Rightarrow (\sin\ \theta + \cos\ \theta - 1)(1 - \sin\ \theta\cos\ \theta) = 0$

$\Rightarrow (\sin\ \theta + \cos\ \theta - 1) = 0$ and $(1 - \sin\ \theta\cos\ \theta) = 0$

$\Rightarrow (\sin\ \theta + \cos\ \theta) = 1$ and $\sin\ (2\theta) = \frac{1}{2}$

$\Rightarrow \left(\sin\ \left(\theta + \frac{\pi}{4}\right)\right) = \sin\ \left(\frac{\pi}{2}\right)$

and $\sin\ (2\theta) = \sin\ \left(\frac{\pi}{6}\right)$

$\Rightarrow\ \theta = n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{4}\right)$

and $\theta = \frac{1}{2}\left(n\pi + (-1)^n\left(\frac{\pi}{6}\right)\right)$, where $n \in I$

5. $\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$

Ans. $n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{3}$

Sol. Given equation is

$\Rightarrow \sin\ \theta + \sqrt{3}\cos\ \theta = \sqrt{2}$

$\Rightarrow \frac{1}{2}\sin\ \theta + \frac{\sqrt{3}}{2}\cos\ \theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\ \left(\theta + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \left(\theta + \frac{\pi}{3}\right) = n\pi + (-1)^n\left(\frac{\pi}{4}\right)$

$\Rightarrow \theta = n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{3}, n \in I$

6. $\sqrt{2}\sec\theta + \tan\theta = 1$

Sol. $\sqrt{2}\sec\ \theta + \tan\ \theta = 1$

$\Rightarrow \frac{\sqrt{2}}{\cos\ \theta} + \frac{\sin\ \theta}{\cos\ \theta} = 1$

$\Rightarrow \sqrt{2} + \sin\ \theta = \cos\ \theta$

$\Rightarrow \cos\ \theta - \sin\ \theta = \sqrt{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\ \theta - \frac{1}{\sqrt{2}}\sin\ \theta = 1$

$\Rightarrow \cos\ \left(\theta + \frac{\pi}{4}\right) = \cos\ 0$

$\Rightarrow \theta + \frac{\pi}{4} = 2n\pi \pm 0$

$\Rightarrow \theta = 2n\pi - \frac{\pi}{4}, n \in I$

7. $\cot\theta + \mathrm{cosec}\theta = \sqrt{3}$

Sol. Given expression:

$\Rightarrow \cot\ \theta + \mathrm{cosec}\ \theta = \sqrt{3}$

$\frac{\cos\ \theta}{\sin\ \theta} + \frac{1}{\sin\ \theta} = \sqrt{3}$

$\Rightarrow$ Taking square both side,

$\cos^2\ \theta + 1 + 2\cos\ \theta = 3 - 3\cos^2\ \theta$ [\ $\sin\ cesin^2\ \theta = 1 - \cos^2\ \theta$]

$4\cos^2\ \theta + 2\cos\ \theta - 2 = 0$

$4\cos^2\ \theta + 4\cos\ \theta - 2\cos\ \theta - 2 = 0$

We get

$(\cos\ \theta + 1)(4\cos\ \theta - 2) = 0$

When,

$(\cos\ \theta + 1) = 0$

$\cos\ \theta = -1$

$\cos\ \pi = \cos\ (2n + 1)\pi = -1$

Hence $\theta = (2n + 1)\pi$.

We get

$(\cos\ \theta + 1)(4\cos\ \theta - 2) = 0$

When,

$(\cos\ \theta + 1) = 0$

$\cos\ \theta = -1$

$\cos\ \pi = \cos\ (2n+1)\pi = -1$
Hence $\theta = (2n+1)\pi$.
When,
$(4\cos\ \theta - 2) = 0$
$\cos\ \theta = \dfrac{1}{2}$
$\cos\ \dfrac{\pi}{3} = \cos\ \left(2n\pi \pm \dfrac{\pi}{3}\right) = \dfrac{1}{2}$

8. $\sin\theta + \cos\theta = \sqrt{2}$

Sol. Consider the given equation,
$\Rightarrow L\sin\ \theta + \cos\ \theta = \sqrt{2}$
Taking square both sides,
$\Rightarrow \sin^2\ \theta + \cos^2\ \theta + 2\sin\ \theta\cos\ \theta = 2$
$1 + 2\sin\ \theta\cos\ \theta = 2$
$\Rightarrow 2\sin\ \theta\cos\ \theta = 1$
$\sin\ \theta\cos\ \theta = \frac{1}{2}$ ……. (2)
Now, divided by $\cos\ \theta$ in equation 1^{st}, we get $\Rightarrow \tan\ \theta + 1 = \frac{\sqrt{2}}{\cos\ \theta}$
Again divided by $\sin\ \theta$ in equation 1^{st}, we get $\Rightarrow 1 + \cot\ \theta = \frac{\sqrt{2}}{\sin\ \theta}$
Add equation 1^{st} and 2^{nd}, we get
$\Rightarrow \tan\ \theta + \cot\ \theta + 2 = \frac{\sqrt{2}}{\cos\ \theta} + \frac{\sqrt{2}}{\sin\ \theta}$
$\tan\ \theta + \cot\ \theta = \sqrt{2}\cdot\left(\frac{\sin\ \theta + \cos\ \theta}{\sin\ \theta\cos\ \theta}\right) - 2$
$\Rightarrow$ Now, from equation 1^{st}, 2^{nd} and 5^{th}, we get
$\tan\ \theta + \cot\ \theta = \sqrt{2}\cdot\left(\frac{\sqrt{2}}{\frac{1}{2}}\right) - 2$
$\Rightarrow \tan\ \theta + \cot\ \theta = 2$

9. $\sqrt{3}\cos\theta + \sin\theta = 1$

Sol.

10. $\sin\theta + \cos\theta = 1$

Sol. Given, $\sin\ \theta + \cos\ \theta = 1$
$\Rightarrow$ Squaring both sides gives,
$\sin^2\ \theta + \cos^2\ \theta + 2\sin\ \theta\cos\ \theta = 1$
$\Rightarrow 1 + 2\sin\ \theta\cos\ \theta = 1$
$\Rightarrow 2\sin\ \theta\cos\ \theta = 0$
$\Rightarrow \sin\ \theta\cos\ \theta = 0$

11. $\mathrm{cosec}\theta = 1 + \cot\theta$

Sol.

12. $\tan\theta + \sec\theta = \sqrt{3}$

Sol.

13. $\cos\theta + \sqrt{3}\sin\theta = 2\cos2\theta$

Ans. $\frac{2n\pi}{3} + \frac{\pi}{9}$

Sol. Given equation is
$\Rightarrow \cos\ \theta + \sqrt{3}\sin\ \theta = 2\cos\ 2\theta$
$\Rightarrow \frac{1}{2}\cos\ \theta + \frac{\sqrt{3}}{2}\sin\ \theta = \cos\ 2\theta$
$\Rightarrow \cos\ \left(\theta - \frac{\pi}{3}\right) = \cos\ 2\theta$
$\Rightarrow \left(\theta - \frac{\pi}{3}\right) = 2n\pi \pm 2\theta$
$\Rightarrow$ Taking positive one, we get
$\theta = -\left(2n\pi + \frac{\pi}{3}\right)$
$\Rightarrow$ Taking negative one, we get,

$\Rightarrow \ \theta = \frac{2n\pi}{3} + \frac{\pi}{9}, n \in I$ $\Rightarrow$

14. $\sqrt{3}\left(\cos\theta - \sqrt{3}\sin\theta\right) = 4\sin 2\theta \cdot \cos 3\theta$

Ans. $(2k+1)\frac{\pi}{4} - \frac{\pi}{12}$

Sol. Given equation is

$\sqrt{3}(\cos\ \theta - \sqrt{3}\sin\ \theta) = 4\sin\ 2\theta \cdot \cos\ 3\theta$

$\Rightarrow \sqrt{3}\cos\ \theta - 3\sin\ \theta = 2(\sin\ 5\theta - \sin\ \theta)$ $\Rightarrow$ $\sqrt{3}\cos\ \theta - \sin\ \theta = 2(\sin\ 5\theta)$

$\Rightarrow \frac{\sqrt{3}}{2}\cos\ \theta - \frac{1}{2}\sin\ \theta = (\sin\ 5\theta)$ $\Rightarrow \sin\left(\frac{\pi}{3} - \theta\right) = \sin\ 5\theta$

$\Rightarrow 5\theta = n\pi + (-1)^n\left(\frac{\pi}{3} - \theta\right)$ $\Rightarrow$ when n is even

$5\theta = 2k\pi + \left(\frac{\pi}{3} - \theta\right)$ $\Rightarrow 6\theta = 2k\pi + \frac{\pi}{3}$

$\Rightarrow \theta = \frac{k\pi}{3} + \frac{\pi}{18}, k \in I$ $\Rightarrow$ when n is odd

$5\theta = (2k+1)\pi - \left(\frac{\pi}{3} - \theta\right)$ $\Rightarrow 4\theta = (2k+1)\pi - \frac{\pi}{3}$

$\Rightarrow \theta = (2k+1)\frac{\pi}{4} - \frac{\pi}{12}, k \in I$

Q. Solve for x:

15. $\sin x + \cos x - 2\sqrt{2}\sin x\cos x = 0$

Ans. $x = n\pi - (-1)^n\frac{\pi}{6} - \frac{\pi}{4}, n \in Z$

Sol. Let $\sin\ x + \cos\ x = t$ $\Rightarrow \ \sin\ x \cdot \cos\ x = \frac{t^2-1}{2}$

So, the given equation can be reduces to

$t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right) = 0$ $\Rightarrow \sqrt{2}t^2 - t - \sqrt{2} = 0$

$\Rightarrow \ (\sqrt{2}t + 1)(t - \sqrt{2}) = 0$ $\Rightarrow \ \ t = \sqrt{2}, -\frac{1}{\sqrt{2}}$

When $\sin\ x + \cos\ x = \sqrt{2}$ $\Rightarrow \ \frac{1}{\sqrt{2}}\sin\ x + \frac{1}{\sqrt{2}}\cos\ x = 1$

$\Rightarrow \ \sin\left(x + \frac{\pi}{4}\right) = 1 = \sin\left(\frac{\pi}{2}\right)$ $\Rightarrow \ x + \frac{n\pi}{4} = n\pi + (-1)^n\frac{\pi}{2}, n \in Z$

When $\sin\ x + \cos\ x = -\frac{1}{\sqrt{2}}$ $\Rightarrow$ When $\sin\ x + \cos\ x = -\frac{1}{\sqrt{2}}$

$\Rightarrow \ \frac{1}{\sqrt{2}}\sin\ x + \frac{1}{\sqrt{2}}\cos\ \frac{1}{2}$ $\Rightarrow \ \sin\left(x + \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \ \left(x + \frac{\pi}{4}\right) = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ $\Rightarrow \ x = n\pi - (-1)^n\frac{\pi}{6} - \frac{\pi}{4}, n \in Z$

16. $\sin^3 x + \sin x\cos x + \cos^3 x = 1$

Sol. The correct options are $\Rightarrow$ A $2n\pi, n \in Z$

C $2n\pi + \frac{\pi}{2}, n \in Z$ $\Rightarrow$ Given equation can be written as

$(\sin\ x + \cos\ x)(\sin^2\ x - \sin\ x\cos\ x + \cos^2\ x) - (1 - \sin\ x\cos\ x) = 0$

$\Rightarrow (\sin\ x + \cos\ x)(1 - \sin\ x\cos\ x) - (1 - \sin\ x\cos\ x) = 0$

$\Rightarrow (\sin\ x + \cos\ x - 1)(1 - \sin\ x\cos\ x) = 0$

$\Rightarrow \sin\ x + \cos\ x = 1$

or

$1 - \sin\ x\cos\ x = 0$

This gives

$$\cos\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

or

$\sin 2x = 2$

But $\sin 2x$ can't be equal to 2 .

Therefore,

$$\cos\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$\Rightarrow x-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$ $\Rightarrow x=2n\pi+\frac{\pi}{2}$

or

$x=2n\pi, n\in Z$

17. $\sin x+\cos x=1-\sin x\cos x$

Ans. $x=n\pi+(-1)^n\left(\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in I$

Sol. Given equation is

$\sin x+\cos x=1-\sin x\cos x$ ....(i)

Put $\sin x+\cos x=t$

$$\Rightarrow \sin x\cdot\cos x=\frac{t^2-1}{2}$$

Now, equation (i) becomes

$$t=1-\frac{t^2-1}{2}$$

$\Rightarrow 2t=2-t^2+1$ $\Rightarrow t^2+2t-3=0$

$\Rightarrow (t+3)(t-1)=0$ $\Rightarrow (t+3)=0, (t-1)=0$

$\Rightarrow \sin x+\cos x=1, \sin x+\cos x=-3$ $\Rightarrow \sin x+\cos x=1$

$\Rightarrow \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$ $\Rightarrow \sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow x=n\pi+(-1)^n\left(\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in I$

18. $1+\sin^3 x+\cos^3 x=\frac{3}{2}\sin 2x$

Ans. $x=n\pi+(-1)^n\left(-\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in I$

Sol. Given equation is

$1+\sin^3 x+\cos^3 x=\frac{3}{2}\sin 2x$ ....(i)

$\Rightarrow 1+(\sin x+\cos x)(1-\sin x\cos x)=3\sin x\cos x$

Put $\sin x+\cos x=t$

$$\Rightarrow \sin x\cdot\cos x=\frac{t^2-1}{2}$$

Now, equation (i) becomes

$$1+t\left(1-\frac{t^2-1}{2}\right)=\frac{3}{2}(t^2-1)$$

$\Rightarrow 2+t(3-t^2)=3(t^2-1)$ $\Rightarrow 2+3t-t^3-3t^2+3=0$

$\Rightarrow 3t-t^3-3t^2+5=0$ $\Rightarrow t^3+3t^2-3t-5=0$

$\Rightarrow t^3+t^2+2t^2+2t-5t-5=0$ $\Rightarrow t^2(t+1)+2t(t+1)-5(t+1)=0$

$\Rightarrow (t^2 + 2t - 5)(t + 1) = 0$ $\Rightarrow t = -1, t = \frac{-2\pm\sqrt{24}}{2}$

$\Rightarrow \sin x + \cos x = -1, \frac{-2\pm\sqrt{24}}{2}$ $\Rightarrow \sin x + \cos x = -1$

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$ $\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

19. $\sin 2x - 12(\sin x - \cos x) + 12 = 0$, where $0 \le x \le 2\pi$

Ans. $x = 0, \frac{\pi}{2}, 2\pi$ 20. $(A + B) = n\pi + \left(\frac{\pi}{4}\right)$

Sol. Given equation is

$\sin 2x - 12(\sin x - \cos x) + 12 = 0$ $\Rightarrow \sin x + \cos x = -1$

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$ $\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

SOLUTIONS IN CASE IF TWO EQUATIONS ARE GIVEN:

20. If $(1 + \tan A)(1 + \tan B) = 2$, then find all the values of $A + B$

Sol. We have, $(1 + \tan A)(1 + \text{tab } B) = 2$

$\Rightarrow 1 + \tan A + \tan B + \tan A \cdot \tan B = 2$ $\Rightarrow \tan A + \tan B = 1 - \tan A \cdot \tan B$

$\Rightarrow \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right) = 1$ $\Rightarrow \tan(A + B) = 1$

$\Rightarrow \tan(A + B) = \tan\left(\frac{\pi}{4}\right)$ $\Rightarrow (A + B) = n\pi + \left(\frac{\pi}{4}\right)$, where $n \in I$

21. If $\tan(A - B) = 1$ and $\sec(A + B) = \frac{2}{\sqrt{3}}$, then find the smallest +ve values of A and B and their most general values.

Ans. $\begin{cases} A = (2n + m)\frac{\pi}{2} + \frac{\pi}{24} \\ B = (2n - m)\frac{\pi}{2} - \frac{5\pi}{24} \end{cases}$

Sol. Given, $\tan(A - B) = 1$ $\Rightarrow (A - B) = \frac{\pi}{4}, \frac{5\pi}{4}$

Also, $\sec(A + B) = \frac{2}{\sqrt{3}}$

$\Rightarrow \cos(A + B) = \frac{\sqrt{3}}{2}$ $\Rightarrow (A + B) = \frac{\pi \cdot 11\pi}{6!6}$

Here, we observe that $A - B$ is positive

So, $A > B$

$\begin{cases} A + B = \frac{11\pi}{6} \\ A - B = \frac{\pi}{4} \end{cases}$ or $\begin{cases} A + B = \frac{11\pi}{6} \\ A - B = \frac{6}{4} \end{cases}$

On solving, we get,

$\begin{cases} A = \frac{25\pi}{24} \\ B = \frac{19\pi}{24} \end{cases}$ or $\begin{cases} A = \frac{19\pi}{24} \\ B = \frac{7\pi}{24} \end{cases}$

General values of $\tan$ $\tan(A-B)=1$

is $(A-B)=n\pi+\frac{\pi}{4}, n\in I$.........(i)

General values of $\sec(A+B)=\frac{2}{\sqrt{3}}$

is $(A+B)=2n\pi+\frac{\pi}{6}, n\in I$......(ii)

On solving (i) and (ii), we get

$$\begin{cases} A=(2n+m)\dfrac{\pi}{2}+\dfrac{\pi}{24} \\ B=(2n-m)\dfrac{\pi}{2}-\dfrac{5\pi}{24} \end{cases}$$

22. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then prove that, $\cos\left(\theta\pm\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$.

Sol. We have $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$

$\Rightarrow \sin(\pi\cos\theta)=\sin\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$\Rightarrow (\pi\cos\theta)=\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$\Rightarrow \cos\theta=\left(\frac{1}{2}-\sin\theta\right)$

$\Rightarrow \cos\theta+\sin\theta=\frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta+\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

Similarly, we can prove that,

$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

23. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, then prove that $\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

Sol. We have $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta)=\tan\left(\frac{\pi}{2}-\pi\sin\theta\right)$ $\Rightarrow (\pi\cos\theta)=\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$\Rightarrow \cos(\theta)+\sin(\theta)=\frac{1}{2}$ $\Rightarrow \frac{1}{\sqrt{2}}\cos(\theta)+\frac{1}{\sqrt{2}}\sin(\theta)=\frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

24. If $\sin A=\sin B$ and $\cos A=\cos B$, then find the values of A in terms of B.

Ans. $A=n\pi+B$

Sol. Given, $\sin A=\sin B$ ......(i)

and $\cos A=\cos B$ ......(ii)

Dividing (i) and (ii), we get,

$\frac{\sin A}{\cos A}=\frac{\sin B}{\cos B}$

$\Rightarrow \tan A=\tan B$

$\Rightarrow A=n\pi+B$, where $n\in I$

25. If **A** and **B** are acute +ve angles satisfying the equations $\mathbf{3\sin^2 A + 2\sin^2 B = 1}$ and $\mathbf{3\sin 2A - 2\sin 2B = 0}$, then find $\mathbf{A + 2B}$.

Ans. $(A + 2B) = \frac{\pi}{2}$

Sol. Given equations are

$3\sin^2 A + 2\sin^2 B = 1$ .....(i)

and $3\sin 2A - 2\sin 2B = 0$....(ii)

From (ii), we get,

$3\sin 2A = 2\sin 2B$

$\Rightarrow \frac{\sin 2A}{2} = \frac{\sin 2B}{3}$

$\Rightarrow \frac{\sin 2B}{\sin 2A} = \frac{3}{2}$

From (i), we get

$\frac{3}{2}(2\sin^2 A) + (2\sin^2 B) = 1$

$\Rightarrow \frac{3}{2}(1 - \cos 2A) + (1 - \cos 2B) = 1$

$\Rightarrow \frac{3}{2}\cos 2A + \cos 2B = \frac{3}{2}$

$\Rightarrow \frac{\sin 2B}{\sin 2A}\cos 2A + \cos 2B = \frac{\sin 2B}{\sin 2A}$

$\Rightarrow \sin 2B\cos 2A + \sin 2A\cos 2B = \sin 2B$

$\Rightarrow \sin(2A + 2B) = \sin 2B$

$\Rightarrow \sin(2A + 2B) = \sin(\pi - 2B)$

$\Rightarrow (2A + 2) = (\pi = 2B)$

$\Rightarrow (2A + 4B) = \pi$

$\Rightarrow (A + 2B) = \frac{\pi}{2}$

DIFFERENT TYPES OF TRIGNOMETRIC EQUATION

TYPE – 4 (Product to Sum)

Q. Solve:

26. $\mathbf{4\sin x \cdot \sin 2x \cdot \sin 4x = \sin 3x}$

Ans. $x = n\pi, x = (3n \pm 1)\frac{\pi}{9}, n \in Z$

Sol. The given equation can be written as

$(2\sin 2x \cdot \sin x)2\sin 4x - \sin 3x = 0$

$\Rightarrow 2(\cos x - \cos 3x)\sin 4x - \sin 3x = 0$

$\Rightarrow 2\sin 4x\cos x - 2\sin 4x\cos 3x - \sin 3x = 0$

$\Rightarrow (\sin 5x + \sin 3x) - (\sin 7x + \sin x) - \sin 3x = 0$

$\Rightarrow (\sin 7x - \sin 5x) + \sin x = 0$

$\Rightarrow \sin x(2\cos 6x + 1) = 0$

$\Rightarrow \sin x = 0, \cos 6x = -1/2$

$\Rightarrow \sin x = 0, \cos 6x = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow x = n\pi, 6x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow x = n\pi, x = (3n \pm 1)\frac{\pi}{9}, n \in Z$

27. $\cos x \cdot \cos 2x \cdot \cos 3x = \frac{1}{4,0} \le x \le 2\pi$

Sol. Do yourself.

28. $\sin 3\alpha = 4\sin\alpha \cdot \sin(x+\alpha) \cdot \sin(x-\alpha)$

Sol. Do yourself.

29. $\sin 2x \cdot \sin 4x + \cos 2x = \cos 6x$

Ans. $x = \frac{n\pi}{4}, x = \frac{n\pi}{2}, n \in I$

Sol. Given equation is

$\sin 4x\sin 2x = \cos 6x - \cos 2x$

$\Rightarrow \sin 4x\sin 2x = -2\sin 4x\sin 2x$ $\Rightarrow 3\sin 4x\sin 2x = 0$

$\Rightarrow \sin 4x = 0, \sin 2x = 0$ $\Rightarrow 4x = n\pi, 2x = n\pi, n \in I$

$\Rightarrow x = \frac{n\pi}{4}, x = \frac{n\pi}{2}, n \in I$

30. $\sec x \cdot \cos 5x + 1 = 0, 0 \le x \le 2\pi$

Ans. $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

Sol. We have, $\sec x\cos 5x + 1 = 0$

$\Rightarrow \sec x\cos 5x = -1$ $\Rightarrow \cos 5x = -\cos x$

$\Rightarrow 5x = 2n\pi \pm (\pi - x)$ $\Rightarrow x = \frac{(2n+1)\pi}{6}$ or $\frac{(2n-1)\pi}{4}$

Hence, $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{9\pi}{6}, \frac{11\pi}{6}$

31. $\cos x \cdot \cos 6x = -1$

Ans. $x = (2n+1)\frac{\pi}{7}, x = (2n+1)\frac{\pi}{5}, n \in I$

Sol. Given equation is

$\cos(6x)\cos x = 1$ $\Rightarrow 2\cos(6x)\cos x = -2$

$\Rightarrow \cos 7x + \cos 5x = -2$

It is possible only when

$\cos(7x) = -1, \cos(5x) = -1$

$\Rightarrow x = (2n+1)\frac{\pi}{7}, x = (2n+1)\frac{\pi}{5}, n \in I$

TYPE – 5 (Make one Variable)

Q. Solve:

32. $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$

Ans. $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), n \in Z$

Sol. The given equation can be written as

$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x$

$= -2(\sin^2 x + \cos^2 x)$ $\Rightarrow 2\tan^2 x - 5\tan x - 8 = -(\tan^2 x + 1)$

$\Rightarrow 4\tan^2 x - 5\tan x - 6 = 0$ $\Rightarrow (\tan x - 2)(4\tan x + 3) = 0$

$\Rightarrow \tan x = -2, \tan x = -\frac{3}{4}$ $\Rightarrow x = n\pi + \alpha, x = n\pi + \beta$, where

$\Rightarrow \alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), n \in Z.$

33. $5\sin^2 x - 7\sin x\cos x + 10\cos^2 x = 4$

Ans. $x = n\pi + \frac{\pi}{4}, x = n\pi + \alpha, \alpha = \tan^{-1}(5)$

Sol. Given equation is

$\Rightarrow 5\sin^2 x - 7\sin x\cos x + 10\cos^2 x = 4 \Rightarrow 5\tan^2 x - 7\tan x + 10 = 4\sec^2 x$

$\Rightarrow 5\tan^2 x - 7\tan x + 10 = 4 + 4\tan^2 x \Rightarrow \tan^2 x - 7\tan x + 6 = 0$

$\Rightarrow (\tan x - 1)(\tan x - 6) = 0$ $\Rightarrow \tan x = 1, 6$

34. $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3$

Ans. $x = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1 \pm \sqrt{5}}{2}\right)$

Sol. Given equation is

$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3 \Rightarrow 2\tan^2 x - 5\tan x - 8 = -3\sec^2 x$

$\Rightarrow 2\tan^2 x - 5\tan x - 8 = -3 - 3\tan^2 x$ $\Rightarrow 5\tan^2 x - 5\tan x - 5 = 0$

$\Rightarrow \tan^2 x - \tan x - 1 = 0 \Rightarrow \tan x = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow x = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1 \pm \sqrt{5}}{2}\right)$

35. $\sin^3 x\cos x + \sin^2 x\cos^2 x +$
$\sin x\cos^3 x = 1$

Ans. no solution

Sol. Given equation is

$\sin^2 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$

$\Rightarrow \sin x\cos x[\sin^2 x + \sin x\cos x + \cos^2 x] = 1$

$\Rightarrow \sin x\cos x[1 + \sin x\cos x] = 1 \Rightarrow 2\sin x\cos x[2 + 2\sin x\cos x] = 4$

$\Rightarrow \sin(2x)(2 + \sin(2x)) = 4 \Rightarrow \sin^2(2x) + 2\sin(2x) - 4 = 0$

$\Rightarrow \sin(2x) = \frac{-2\pm\sqrt{20}}{2} \Rightarrow \sin(2x) = \frac{-2\pm2\sqrt{5}}{2} = -1 \pm \sqrt{5}$

It is not possible.

So, it has no solution.

TYPE – 6 (Boundness)

Q. Solve for x:

36. $\sin 6x + \cos 4x + 2 = 0$

Ans. $x = m\pi + \frac{\pi}{4}, m \in Z$

Sol. The given equation can be written as

$\Rightarrow \sin 6x + \cos 4x = -2$

$\Rightarrow \sin 6x = -1$ and $\cos 4x = -1$

$\Rightarrow \sin 6x = \sin\frac{3\pi}{2}, \cos 4x = \cos\pi$

$\Rightarrow 6x = 2n\pi + \frac{3\pi}{2}, 4x = 2n\pi + \pi, n \in Z$

$\Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{4}, x = \frac{n\pi}{2} + \frac{\pi}{4}, n \in Z$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots \quad \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

Hence, the general solution will be,

$$\Rightarrow x = 2n\pi + \frac{\pi}{4}, 2n\pi + \frac{5\pi}{4}, n \in Z$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{4}, (2n+1)\pi + \frac{\pi}{4}, n \in Z \quad \Rightarrow x = m\pi + \frac{\pi}{4}, m \in Z$$

37. $\sin^6 x = 1 + \cos^4 3x$

Sol. do yourself

38. $\sin^4 x = 1 + \tan^8 x$

Ans. $x = n\pi \pm \frac{\pi}{2}, x = n\pi, n \in I$

Sol. Given equation is

$\sin^4 x = 1 + \tan^8 x$

It is possible only when

$\sin^4 x = 1, \tan^8 x = 0 \Rightarrow \sin^2 x = 1, \tan x = 0$

$\Rightarrow x = n\pi \pm \frac{\pi}{2}, x = n\pi, n \in I$

There is no common value which satisfies both the above equations. Hence, the equation has no solution.

39. $\sin^2 x + \cos^2 y = 2\sec^2 z$

Ans. $x = (2n+1)\frac{\pi}{2}, y = m\pi, z = k\pi$

Sol. Given $\sin^2 x + \cos^2 y = 2\sec^2 z$

Here, LHS ≤ 2 and RHS ≥ 2

It is possible only when

$\sin^2 x = 1, \cos^2 y = 1, \sec^2 z = 1$

$\Rightarrow \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$

$\Rightarrow \cos^2 x = 0, \sin^2 y = 0, \sin^2 z = 1$

$\Rightarrow \cos x = 0, \sin y = 0, \sin z = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}, y = m\pi, z = k\pi$

where, $n, m, k \in I$

40. $\sin 3x + \cos 2x + 2 = 0$

Ans. $x = 2n\pi + \frac{\pi}{2}$

Sol. Given equation is

$\sin 3x + \cos 2x + 2 = 0$

It is possible only when

$\sin 3x = -1, \cos 2x = -1$

$\Rightarrow 3x = \frac{3\pi}{2}, 2x = \pi \quad \Rightarrow x = \frac{\pi}{2}, x = \frac{\pi}{2}$

Hence, the general solution is

$$x = 2n\pi + \frac{\pi}{2}, n \in I$$

41. $\cos 4x + \sin 5x = 2$

Ans. $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$

Sol. Given equation is $\cos 4x + \sin 5x = 2$

It is possible only when

$\cos 4x = 1, \sin 5x = 1 \qquad \Rightarrow 4x = 2n\pi, 5x = (4n+1)\frac{\pi}{2}$

$\Rightarrow x = \frac{2n\pi}{4}, x = (4n+1)\frac{\pi}{10}$

Thus, $x = \frac{\pi}{2}$ satisfies both

Hence, the solution is

$$x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$$

Type – 7 $\left(a^{f(x)} = b \text{ Type}\right)$

Q. Solve for x:

42. Find the values of x in $(-\pi, \pi)$ which satisfy the equation

$8^{1+|\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + |\cos x|^5 + \cdots \text{ to} \infty}$

$= 64$

Ans. $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

Sol. The given equation can be written as

$8^{1+|\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + |\cos x|^5 + \cdots \text{ to } \infty} = 8^2$

$\Rightarrow 1 + |\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + |\cos x|^5 +$

$.. \text{ to } \infty = 2$

$\Rightarrow \frac{1}{1-|\cos x|} = 2 \quad \Rightarrow \cos x = \pm\frac{1}{2}$

When $\cos x = \frac{1}{2}$

$\Rightarrow \cos x = \cos\left(\frac{\pi}{3}\right) \Rightarrow x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

When $\cos x = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$$\Rightarrow x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

Hence the values of x are $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

43. $2^{1+|\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + |\cos x|^5 + \cdots \text{ to } \infty} = 4$

Ans. $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

Sol. Given equation is

$2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\cdots \text{tos}} = 4$

$\Rightarrow 2^{\frac{1}{1-|\cos x|}} = 4 = 2^2 \quad \Rightarrow \frac{1}{1-|\cos x|} = 2$

$\Rightarrow 1 - |\cos x| = \frac{1}{2} \quad \Rightarrow |\cos x| = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos x = \pm\frac{1}{2}$

Hence the values of **x** are $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

44. $1 + \sin\theta + \sin^2\theta + \sin^3\theta + \cdots$ to ∞

$= 4 + 2\sqrt{3}$

Ans. $n\pi + (-1)^n\left(\frac{\pi}{3}\right)$

Sol. Given equation is

$1 + \sin\theta + \sin^2\theta + \cdots = 4 + 2\sqrt{3}$

$\Rightarrow \frac{1}{1-\sin\theta} = 4 + 2\sqrt{3} \quad \Rightarrow 1 - \sin\theta = \frac{1}{4+2\sqrt{3}}$

$\Rightarrow \sin\theta = 1 - \frac{1}{4+2\sqrt{3}} \quad \Rightarrow \sin\theta = 1 - \frac{1}{4+2\sqrt{3}} = 1 - \frac{2-\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = n\pi + (-1)^n\left(\frac{\pi}{3}\right), n \in I$

45. $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$

Ans. $n\pi + (-1)^n\left(\frac{\pi}{6}\right), 2n\pi, (2n+1)\pi$

Sol. Given equation is

$|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1 \quad \Rightarrow \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right)\log|\cos x| = 0$

$\Rightarrow (2\sin^2 x - 3\sin x + 1)\log|\cos x| = 0$

$\Rightarrow (\sin x - 1)(2\sin x - 1)\log|\cos x| = 0$

$\Rightarrow (\sin x - 1) = 0, (2\sin x - 1) = 0, \log|\cos x| = 0$

$\Rightarrow \sin x = 1, \sin x = \frac{1}{2}, \log|\cos x| = 0 \quad \Rightarrow \sin x = \frac{1}{2}, |\cos x| = 1$

$\Rightarrow \sin x = \frac{1}{2}, \cos = 1, \cos x = -1$

$\Rightarrow x = n\pi + (-1)^n\left(\frac{\pi}{6}\right), x = 2n\pi, x = (2n+1)\pi$

46. $e^{\sin x} - e^{-\sin x} - 4 = 0$

Ans. no solution

Sol. Given equation is

$e^{\sin x} - e^{-\sin x} - 4 = 0 \quad \Rightarrow t - \frac{1}{t} - 4 = 0, t = e^{\sin x}$

$\Rightarrow t^2 - 4t - 1 = 0 \quad \Rightarrow (t-2)^2 = 5$

$\Rightarrow t = 2 \pm \sqrt{5} \quad \Rightarrow e^{\sin x} = 2 \pm \sqrt{5}$

$\Rightarrow \sin x = \log_e (2 \pm \sqrt{5}) \quad \Rightarrow \sin x = \log_e (2 + \sqrt{5})$

$\Rightarrow \sin x = \log_e (2 + \sqrt{5}) > 1$

It is not possible

So, it has no solution.

47. If $e^{[\sin^2 x + \sin^4 x + \sin^6 x + \cdots + \text{to}\infty]\log_e 2}$ satisfies the equations $x^2 - 9x + 8 = 0$, then find the value of $\frac{\cos x}{\cos x + \sin x}$, $0 < x < \frac{\pi}{2}$

Ans. $\frac{(\sqrt{3}-1)}{2}$

Sol. We have

$e^{[\sin^2 x + \sin^4 x + \sin^6 x + \cdots \text{toos}]\log_e 2}$

$= e^{\left(\frac{\sin^2 x}{1-\cos^2 x}\right)\log_e 2} = e^{\tan^2 x \log_e 2} = 2^{\tan^2 x}$

Let $a = 2^{\tan^2 x}$

Thus, $a^2 - 9a + 8 = 0$

$\Rightarrow (a-1)(a-8) = 0 \quad \Rightarrow a = 1, 8$

when $a = 1$, then $2^{\tan^2 x} = 1 = 2^0$

$\Rightarrow 2^{\tan^2 x} = 1 = 2^0 \quad \Rightarrow \tan^2 x = 0$

$\Rightarrow x = n\pi, n \in I$

when $a = 8$, then $2^{\tan^2 x} = 8 = 2^3 \Rightarrow \tan^2 x = 3$

$\Rightarrow \tan x = \sqrt{3}$

Now, $\frac{\cos x}{\cos x+\sin x}$ $\Rightarrow$ $=\frac{1}{1+\tan x}=\frac{1}{\sqrt{3}+1}$ $=\frac{(\sqrt{3}-1)}{2}$

TYPE – 8

Q. Solve for x:

48. $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}, 0 < x < \frac{\pi}{2}$

Ans. no solution

Sol. Given equation is

$$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$$

Here, LHS < 2 for $0 < x < \frac{\pi}{2}$ and RHS ≥ 2

So, it has no solutions.

49. $2\cos^2\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$

Sol. Do yourself

TYPE – 9 (Miscellaneous)

Q. Solve:

50. $1+2\operatorname{cosec} x = -\frac{\sec^2\left(\frac{x}{2}\right)}{2}$.

Ans. $x = 2n\pi - \frac{\pi}{2}, n \in I$

Sol. The given equation can be written as

$$1+\frac{2}{\sin x} = -\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow 2(\sin x+2) = -\left(1+\tan^2\left(\frac{x}{2}\right)\right)\sin x$$

$$\Rightarrow 2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+2\right) = -\left(1+\tan^2\frac{x}{2}\right)\cdot\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$$

$$\Rightarrow 2\left(\frac{2t}{1+t^2}+2\right) = -(1+t^2)\times\left(\frac{2t}{1+t^2}\right),$$

where $t = \tan(x/2)$

$$\Rightarrow t^3+2t^2+3t+2=0 \quad \Rightarrow t^3+t^2+t^2+t+2t+2=0$$

$$\Rightarrow (t+1)(t^2+t+2)=0 \quad \Rightarrow t+1=0, t^2+t+2\neq 0$$

$$\Rightarrow \tan\left(\frac{x}{2}\right) = -1 = \tan\left(\frac{-\pi}{4}\right) \qquad \Rightarrow \frac{x}{2} = n\pi-\frac{\pi}{4}, n\in Z$$

$$\Rightarrow x = 2n\pi-\frac{\pi}{2}, n \in Z.$$

51. $\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$

Ans. $x = 2n\pi$

Sol. Given equation is

$$\Rightarrow \cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$$

$$\frac{\cos\left(\frac{x}{2}\right) - 1}{\sin\left(\frac{x}{2}\right)} = \cot x \qquad \Rightarrow 2\sin^2\left(\frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right)\cot x$$

$$\Rightarrow \left(2\sin\left(\frac{x}{2}\right) + \cot x\right)\sin\left(\frac{x}{2}\right) = 0 \Rightarrow \left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0, \sin\left(\frac{x}{2}\right) = 0$$

when $\sin\left(\frac{x}{2}\right) = 0$

$$\Rightarrow x = 2n\pi, n \in 1$$

when $\left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0$

$$\Rightarrow 2\sin\left(\frac{x}{2}\right) = -\frac{\cos x}{\sin x} \qquad \Rightarrow 2\sin\left(\frac{x}{2}\right) = -\frac{\left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\Rightarrow 4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \cos^2\left(\frac{x}{2}\right) + 4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 0$$

For all real x,

$$16\sin^4\left(\frac{x}{2}\right) + 4\sin^2\left(\frac{x}{2}\right) \geq 0 \qquad \Rightarrow 4\sin^4\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \sin^2\left(\frac{x}{2}\right)\left(4\sin^2\left(\frac{x}{2}\right) + 1\right) = 0 \quad \Rightarrow \sin^2\left(\frac{x}{2}\right) = 0, \left(4\sin^2\left(\frac{x}{2}\right) + 1\right) = 0$$

$$\Rightarrow \sin^2\left(\frac{x}{2}\right) = 0 \qquad \Rightarrow \sin\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow x = 2n\pi, n \in I$$

52. If $\theta_1, \theta_2, \theta_3, \theta_4$ be the four roots of the equation $\sin(\theta + \alpha) = k\sin 2\theta$, no two of which differ by a multiple of 2π, then prove that

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\pi, (n \in Z)$$

Sol. Given equation is

$$\sin(\theta + \alpha) = k\sin(2\theta) \qquad \Rightarrow \sin\theta\cos\alpha + \cos\theta\sin\alpha = k\sin(2\theta)$$

$$\Rightarrow \left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right)\cos\alpha + \left(\frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right)\sin\alpha$$

$= 2k\left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\left(\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)$ $\Rightarrow \left(\frac{2t}{1+t^2}\right)\cos\ \alpha + \left(\frac{1-t^2}{1+t^2}\right)\sin\ \alpha$

$= 2k\cdot\left(-\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$ $\Rightarrow 2t(1+t^2)\cos\ \alpha + (1+t^4)\sin\ \alpha = 4kt(1-t^2)$

$\Rightarrow (\sin\ \alpha)t^4 - (4k+2\cos\ \alpha)t^3 + (4k-2\cos\ \alpha)t = \sin\ \alpha = 0$

Let t_1, t_2, t_3 and t_4 be its four roots

$$\Sigma t_1 = \frac{4k+2\cos\ \alpha}{\sin\ \alpha} = s_1$$

$$\Sigma t_1 t_2 = 0 = s_2$$

$$\Sigma t_1 t_2 t_3 = \frac{2\cos\ \alpha - 4k}{\sin\ \alpha} = s_3$$

$$\Sigma t_1 t_2 t_3 t_4 = -\frac{\sin\ \alpha}{\sin\ \alpha} = -1 = s_4$$

TYPE- 10 (Putting Value of y in Function)

Q. Solve for x & y:

53. Solve $x + y = \frac{\pi}{4}$ and $\tan\ x + \tan\ y = 1$.

Ans. no solutions

Sol. Given, $x + y = \frac{\pi}{4}$ and $\tan\ x + \tan\ y = 1$

$\Rightarrow \tan\ (x+y) = \tan\ \left(\frac{\pi}{4}\right)$

$\frac{\tan\ x+\tan\ y}{1-\tan\ x\cdot\tan\ y} = 1$ $\Rightarrow 1 - \tan\ x\cdot\tan\ y = 1$

$\Rightarrow \tan\ x\cdot\tan\ y = 0 \Rightarrow \tan\ x = 0 \& \tan\ y = 0$

$\Rightarrow x = n\pi = y$

Thus, no values of x and y satisfy the given equations. Therefore, the given equations have no solutions.

54. Solve $x + y = \frac{2\pi}{3}$ and $\sin x = 2\sin\ y$.

Ans. $\begin{cases} x = (2n+1)\frac{\pi}{2} \\ y = n\pi - \frac{\pi}{6} \end{cases}, n \in I$

Sol. Given, $\sin\ x = 2\sin\ y$

$\Rightarrow\ \sin\ x = 2\sin\ \left(\frac{2\pi}{3} - x\right)$

$\Rightarrow \sin x = 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)$

$\Rightarrow \sin x = \sqrt{3}\cos x + \sin x$

$\Rightarrow \sqrt{3}\cos x = 0$

$\Rightarrow \cos x = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$

when $x = (2n+1)\frac{\pi}{2}$, then $y = n\pi - \frac{\pi}{6}$

Hence, the solutions are

$$\begin{cases} x = (2n+1)\dfrac{\pi}{2} \\ y = n\pi - \dfrac{\pi}{6} \end{cases}, n \in I$$

55. Solve $x + y = \frac{2\pi}{3}$ and $\cos x + \cos y = \frac{3}{2}$

Ans. No Solutions.

Sol. Given, $x + y = \frac{2\pi}{3}$ and $\cos x + \cos y = \frac{3}{2}$

Now $\cos x + \cos y = \frac{3}{2}$

$$\Rightarrow \cos x + \cos\left(\frac{2\pi}{3} - x\right) = \frac{3}{2}$$

$$\Rightarrow \cos x - \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{3}{2}$$

$$\Rightarrow \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{3}{2}$$

$\Rightarrow \cos x + \sqrt{3}\sin x = 3$

It is not possible, since the maximum value of LHS is 2.

So, the given system of equations has no solutions.

Answer Key

1. $\left(n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right)$
2. $n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$
3. $n\pi - \frac{\pi}{4}$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right)$
4. $\frac{1}{2}\left(n\pi + (-1)^n\left(\frac{\pi}{6}\right)\right)$
5. $n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{3}$
13. $\frac{2n\pi}{3} + \frac{\pi}{9}$
14. $(2k+1)\frac{\pi}{4} - \frac{\pi}{12}$
15. $x = n\pi - (-1)^n\frac{\pi}{6} - \frac{\pi}{4}, n \in Z$
17. $x = n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$
18. $x = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$
19. $x = 0, \frac{\pi}{2}, 2\pi$ 20. $(A+B) = n\pi + \left(\frac{\pi}{4}\right)$
21. $\begin{cases} A = (2n+m)\frac{\pi}{2} + \frac{\pi}{24} \\ B = (2n-m)\frac{\pi}{2} - \frac{5\pi}{24} \end{cases}$
24. $A = n\pi + B$
25. $(A + 2B) = \frac{\pi}{2}$
26. $x = n\pi, x = (3n \pm 1)\frac{\pi}{9}, n \in Z$
29. $x = \frac{n\pi}{4}, x = \frac{n\pi}{2}, n \in I$
30. $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$
31. $x = (2n+1)\frac{\pi}{7}, x = (2n+1)\frac{\pi}{5}, n \in I$
32. $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), n \in Z$
33. $x = n\pi + \frac{\pi}{4}, x = n\pi + \alpha, \alpha = \tan^{-1}(5)$
34. $x = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1 \pm \sqrt{5}}{2}\right)$
35. no solution
36. $x = m\pi + \frac{\pi}{4}, m \in Z$
38. $x = n\pi \pm \frac{\pi}{2}, x = n\pi, n \in I$
39. $x = (2n+1)\frac{\pi}{2}, y = m\pi, z = k\pi$
40. $x = 2n\pi + \frac{\pi}{2}$
41. $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$
42. $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$
43. $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$
44. $n\pi + (-1)^n\left(\frac{\pi}{3}\right)$
45. $n\pi + (-1)^n\left(\frac{\pi}{6}\right), 2n\pi, (2n+1)\pi$

46. no solution

47. $\frac{(\sqrt{3}-1)}{2}$

48. no solution

50. $x = 2n\pi - \frac{\pi}{2}, n \in I$

51. $x = 2n\pi$

53. no solutions

54. $\begin{cases} x = (2n+1)\frac{\pi}{2} \\ y = n\pi - \frac{\pi}{6} \end{cases}, n \in I$

55. No Solutions.