CONTINUITY

THINGS TO REMEMBER :

1. A function f(x) is said to be continuous at $x = a$ if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$ i.e. L.H.L. = R.H.L. = value of the function at $x = a$ i.e. $\lim_{x\to a} f(x) = f(a)$

If f(x) is not continuous at $x = a$, we say that f(x) is discontinuous at $x = a$. f(x) will be discontinuous at $x = a$ in any of the following cases.

(i) $\lim_{x\to a} f(x)$ does not exist

(ii) $\lim_{x\to a} f(x) \neq f(a)$

(iii) f(a) is not defined.

2. Types of Discontinuities :

Type - 1: (Removable type of discontinuities)

In case $\underset{x\to c}{\text{Limit}}\, f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\underset{x\to c}{\text{Limit}}\, f(x) = f(c)$ & make it continuous at $x = c$. Removable type of discontinuity can be further classified as :

(A) MISSING POINT DISCONTINUITY : Where $\underset{x\to a}{\text{Limit}}\, f(x)$ exists finitely but f(a) is not defined.

e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at $x = 1$, and $f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at $x = 0$

(B) ISOLATED POINT DISCONTINUITY : Where $\underset{x\to a}{\text{Limit}}\, f(x)$ exists & f(a) also exists but ; $\underset{x\to a}{\text{Limit}} \neq f(a)$.

e.g. $f(x) = \frac{x^2-16}{x-4}, x \neq 4$ & $f(4) = 9$ has an isolated point discontinuity at $x = 4$.

Similarly $f(x) = [x] + [-x] = \begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ has an isolated point discontinuity at all $x \in I$.

Type-2: (Non - Removable type of discontinuities)

In case $\underset{x\to c}{\text{Limit}}\, f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non - removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(a) Finite discontinuity:

e.g. $f(x) = x - [x]$ at all integral x ; $f(x) = \tan^{-1}\frac{1}{x}$ at $x = 0$ and $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite discontinuity:

e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$ at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at $x = 0$.

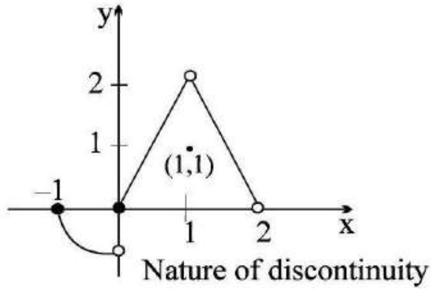
(c) Oscillatory discontinuity:

e.g. $f(x) = \sin\frac{1}{x}$ at $x = 0$.

In all these cases the value of f(a) of the function at $x = a$ (point of discontinuity) may or may not exist but $\lim_{x\to a} f(x)$ does not exist.

Note: From the adjacent graph note that

- f is continuous at $x = -1$
- f has isolated discontinuity at $x = 1$
- f has missing point discontinuity at $x = 2$
- f has non removable (finite type) discontinuity at the origin.



Nature of discontinuity

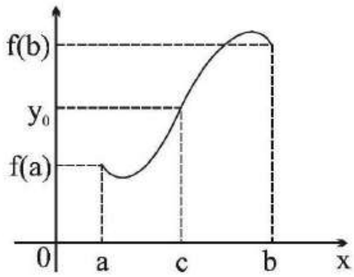
3. In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at $x = c$ & LHL at $x = c$ is called **THE JUMP OF DISCONTINUITY**. A function having a finite number of jumps in a given interval I is called a **PIECE WISE CONTINUOUS** or **SECTIONALLY CONTINUOUS** function in this interval.

4. All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

5. If f & g are two functions that are continuous at $x = c$ then the functions defined by :
$F_1(x) = f(x) \pm g(x)$; $F_2(x) = Kf(x)$, K any real number; $F_3(x) = f(x) \cdot g(x)$ are also continuous at $x = c$. Further, if $g(c)$ is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at $x = c$.

6. **The intermediate value theorem:**

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), their exists a number c between a and b such that $f(c) = y_0$.



Note Very Carefully That :

(a) If f(x) is continuous & g(x) is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$$f(x) = x \;\&\; g(x) = \begin{cases} \sin\dfrac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) If f(x) and g(x) both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(c) A Continuous function whose domain is closed must have a range also in closed interval.

(d) If f is continuous at $x = c$ & g is continuous at $x = f(c)$ then the composite $g[f(x)]$ is continuous at $x = c$. eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at $x = 0$, hence the composite $(gof)\ (x) = \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at $x = 0$.

7. CONTINUITY IN AN INTERVAL :

(a) A function f is said to be continuous in (a,b) if f is continuous at each & every point $\in (a,b)$.

(b) A function f is said to be continuous in a closed interval $[a,b]$ if :

(i) f is continuous in the open interval (a,b)&

(ii) f is right continuous at ' a ' i.e. $\text{Limit}_{x\to a^+}\ f(x) = f(a) =$ a finite quantity.

(iii) f is left continuous at ' b ' i.e. $\text{Limit}_{x\to b^-}\ f(x) = f(b) = a$ finite quantity.

Note that a function f which is continuous in $[a,b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a,b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation $f(x) = K$ in the open inetrval (a,b).

PROFICIENCY TEST

1. If the function $f(x)=\frac{3x^2+ax+a+3}{x^2+x-2}$ is continuous at $x=-2$. Find $f(-2)$.

2. Find all possible values of a and b so that f(x) is continuous for all $x \in R$ if

$$f(x)=\begin{cases}|ax+3| & \text{if } x\le -1\\ |3x+a| & \text{if } -1<x\le 0\\ \frac{b\sin 2x}{x}-2b & \text{if } 0<x<\pi\\ \cos^2 x-3 & \text{if } x\ge \pi\end{cases}$$

3. $f(x)=\begin{cases}ax^2+bx+c, & |x|>1\\ x+3, & |x|\le 1\end{cases}$. Find the values of a, b, c so that f(x) is continuous for all values of x.

4. Draw the graph of the function $f(x)=x-|x-x^2|, -1\le x\le 1$ & discuss the continuity or discontinuity of f in the interval $-1\le x\le 1$

5. Let $f(x)=\begin{cases}\frac{1-\sin \pi x}{1+\cos 2\pi x}, & x<\frac{1}{2}\\ p, & x=\frac{1}{2}\\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x>\frac{1}{2}\end{cases}$. Determine the value of p, if possible, so that the function is continuous at $x=1/2$

6. Given the function $g(x)=\sqrt{6-2x}$ and $h(x)=2x^2-3x+a$. Then

(a) evaluate $h(g(2))$ **(b)** If $f(x)=\begin{cases}g(x), & x\le 1\\ h(x), & x>1\end{cases}$, find 'a' so that f is continuous.

7. Let $f(x)=\begin{cases}1+x & ,0\le x\le 2\\ 3-x & ,2<x\le 3\end{cases}$. Determine the form of $g(x)=f[f(x)]$ & hence find the point of discontinuity of g, if any.

8. Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x)=\begin{cases}\frac{(\exp\{(x+2)\ell n4\})^{\frac{[x+1]}{4}}-16}{4^x-16} & ,x<2\\ A\frac{1-\cos(x-2)}{(x-2)\tan(x-2)} & ,x>2\end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

9. Discuss the continuity of the function ' f ' defined as follows: $f(x)=\begin{cases}\frac{1}{x-1} & \text{for } 0\le x\le 2\\ \frac{3}{x+1} & \text{for } 2<x\le 4\\ \frac{x+1}{x-5} & \text{for } 4<x\le 6\end{cases}$ and

draw the graph of the function for $x\in[0,6]$. Also indicate the nature of discontinuities if any.

10. State whether True or False.

(i) $f(x) = \text{Limit}_{n\to\infty} \frac{1}{1+n\sin^2 \pi x}$ is continuous at $x = 1$.

(ii) The function defined by $f(x) = \frac{x}{|x|+2x^2}$ for $x \neq 0$ & $f(0) = 1$ is continuous at $x = 0$.

(iii) The function $f(x) = 2^{-2^{1/(1-x)}}$ if $x \neq 1$ & $f(1) = 1$ is not continuous at $x = 1$.

(iv) There exists a continuous function $f : [0,1]$ onto $[0,10]$, but there exists no continuous function $g : [0,1]$ onto $(0,10)$

(v) If $f(x)$ is continuous in $[0,1]$ & $f(x) = 1$ for all rational numbers in $[0,1]$ then $f(1/\sqrt{2})$ equal to 1.

(vi) If $f(x) = \begin{cases} \frac{\cos \pi x+\sin (\pi x/2)}{(x-1)(3x^2-2x-1)} & \text{if } x \neq 1 \\ k & \text{if } k = 1 \end{cases}$ is continuous, then the value of k is $\frac{3\pi^2}{32}$.

EXERCISE-I

1. Let $f(x)=\begin{cases}\dfrac{\ln\cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x>0\\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x<0\end{cases}$

Is it possible to define f(0) to make the function continuous at $x=0$. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

2. Let $y_n(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\cdots\ldots\ldots+\frac{x^2}{(1+x^2)^{n-1}}$ and $y(x)=\text{Lim}_{n\to\infty}\, y_n(x)$

Discuss the continuity of $y_n(x)(n\in N)$ and $y(x)$ at $x=0$

3. The function $f(x)=\begin{cases}\left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } \quad 0<x<\frac{\pi}{2}\\ b+2 & \text{if } \quad x=\frac{\pi}{2}\\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \quad \frac{\pi}{2}<x<\pi\end{cases}$

Determine the values of ' a ' & ' b ', if f is continuous at $x=\pi/2$.

4. $f(x)=\begin{cases}ax+b & \text{if } \quad -1\le x<0\\ \left[\frac{e^x-1}{x}\right] & \text{if } \quad 0<x\le 1\\ bx+a & \text{if } \quad 1<x\le 2\end{cases}$ Where [.] denotes the greatest integer function.

Find values of 'a' and 'b' so that f(x) is continuous for all values of $x\in[-1,2]$

5. Let $f(x)=\begin{cases}1+x^3, & x<0\\ x^2-1, & x\ge 0\end{cases}$; $g(x)=\begin{cases}(x-1)^{1/3}, & x<0\\ (x+1)^{1/2}, & x\ge 0\end{cases}$. Discuss the continuity of g(f(x)).

6. Determine a & b so that f is continuous at $x=\frac{\pi}{2}\cdot f(x)=\begin{cases}\frac{1-\sin^3 x}{3\cos^2 x} & \text{if } \quad x<\frac{\pi}{2}\\ a & \text{if } \quad x=\frac{\pi}{2}\\ \frac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } \quad x>\frac{\pi}{2}\end{cases}$

7. Determine the values of a, b & c for which the function $f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } \ x>0\end{cases}$ is continuous at $x=0$.

8. If $f(x)=\frac{\sin 3x+A\sin 2x+B\sin x}{x^5}$ $(x\neq 0)$ is cont. at $x=0$. Find A & B. Also find f(0).

Select the correct alternative : (More than one are correct)

9. A function f is defined on an interval [a,b]. Which of the following statement(s) is/are INCORRECT ?

(A) If f(a) and f(b), have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

(B) If f is continuous on [a, b], $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a,b)$ such that $f(c) = 0$

(C) If f is continuous on [a, b] and there is a point c in (a, b) such that $f(c) = 0$, then f(a) and f(b) have opposite sign.

(D) If f has no zeroes on [a, b], then f(a) and f(b) have the same sign.

10. Which of the following functions f has/have a removable discontinuity at the indicated point ?

(A) $f(x) = \frac{x^2-2x-8}{x+2}$ at $x = -2$

(B) $f(x) = \frac{x-7}{|x-7|}$ at $x = 7$

(C) $f(x) = \frac{x^3+64}{x+4}$ at $x = -4$

(D) $f(x) = \frac{3-\sqrt{x}}{9-x}$ at $x = 9$

11. Let 'f ' be a continuous function on R . If $f(1/4^n) = (\sin e^n)e^{-n^2} + \frac{n^2}{n^2+1}$ then f(0) is :

(A) not unique

(B) 1

(C) data sufficient to find f(0)

(D) data insufficient to find f(0)

12. Indicate all correct alternatives if, $f(x) = \frac{x}{2} - 1$, then on the interval $[0, \pi]$

(A) $\tan(f(x))$ & $\frac{1}{f(x)}$ are both continuous

(B) $\tan(f(x))$ & $\frac{1}{f(x)}$ are both discontinuous

(C) $\tan(f(x))$ & $f^{-1}(x)$ are both continuous

(D) $\tan(f(x))$ is continuous but $\frac{1}{f(x)}$ is not

13. Let $f(x) = \begin{cases} \sin\left(\frac{5}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$, then $(fg)(x)$ is continuous at $x = 0$, if g(x) is equal to

(A) $(x^2 + 1), x \in R$

(B) $\ell n(1 + x), x \in (-1, \infty)$

(C) $\cos x, x \in R$

(D) $e^x, x \in R$

14. If $f(x) = \frac{1}{2-x}$ and $g(x) = \frac{f(f(x))}{3f(f(x))-1}$, then the value(s) of x at which g(x) is discontinuous is/are

(A) 1

(B) 3/2

(C) 2

(D) 3

15. Let $f(x) = \begin{bmatrix} x^2 + px + 1 ; x \in Q \\ px^2 + 2x + r ; x \notin Q \end{bmatrix}$ and f(x) is continuous at x = 1 and x = 2, then

(A) $p = 2$

(B) $r = 0$

(C) $p = 1/2$

(D) $r = 2$

EXERCISE-II

1. If $f(x)=x+\{-x\}+[x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x. Discuss the continuity of f in $[-2,2]$

2. Find the locus of (a,b) for which the function $f(x)=\begin{cases} ax-b & \text{for} \quad x\le 1 \\ 3x & \text{for} \quad 1<x<2 \\ bx^2-a & \text{for} \quad x\ge 2 \end{cases}$ is continuous at $x=1$ but discontinuous at $x=2$.

3. Let $g(x)=\text{Lim}_{n\to\infty}\frac{x^n f(x)+h(x)+1}{2x^n+3x+3}$, $x\ne 1$ and $g(1)=\text{Lim}_{x\to 1}\frac{\sin^2(\pi\cdot 2^x)}{\ln(\sec(\pi\cdot 2^x))}$ be a continuous function at $x=1$, find the value of $4g(1)+2f(1)-h(1)$. Assume that $f(x)$ and $h(x)$ are continuous at $x=1$.

4. If $g:[a,b]$ onto $[a,b]$ is continous show that there is some $c\in[a,b]$ such that $g(c)=c$.

5. The function $f(x)=\left(\frac{2+\cos x}{x^3\sin x}-\frac{3}{x^4}\right)$ is not defined at $x=0$. How should the function be defined at $x=0$ to make it continuous at $x=0$.

6. $f(x)=\frac{a^{\sin x}-a^{\tan x}}{\tan x-\sin x}$ for $x>0=\frac{\ln(1+x+x^2)+\ln(1-x+x^2)}{\sec x-\cos x}$ for $x<0$, if f is continuous at $x=0$, find 'a'.

Now if $g(x)=\ln\left(2-\frac{x}{a}\right)\cdot\cot(x-a)$ for $x\ne a, a\ne 0, a>0$. If g is continuous at $x=a$ then show that $g(e^{-1})=-e$

7. (a) Let $f(x+y)=f(x)+f(y)$ for all x ,y & if the function $f(x)$ is continuous at $x=0$, then show that $f(x)$ is continuous at all x.

(b) If $f(x\cdot y)=f(x)\cdot f(y)$ for all x ,y and $f(x)$ is continuous at $x=1$. Prove that $f(x)$ is continuous for all x except at $x=0$. Given $f(1)\ne 0$.

8. Given $f(x)=\sum_{r=1}^{n}\tan\left(\frac{x}{2^r}\right)\sec\left(\frac{x}{2^{r-1}}\right)$; $r,n\in N$

$g(x)=\text{Limit}_{n\to\infty}\frac{\ell n\left(f(x)+\tan\frac{x}{2^n}\right)-\left(f(x)+\tan\frac{x}{2^n}\right)^n\cdot\left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+\left(f(x)+\tan\frac{x}{2^n}\right)^n}$ for $x\ne\frac{\pi}{4}$ $=k$ for $x=\frac{\pi}{4}$ and the domain of $g(x)$ is $(0,\pi/2)$. where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that $g(x)$ is continuous at $x=\pi/4$. Also state the points of discontinuity of $g(x)$ in $(0,\pi/4)$, if any.

9. Let $f(x)=x^3-x^2-3x-1$ and $h(x)=\frac{f(x)}{g(x)}$ where h is a rational function such that

(a) It is continuous every where except when $x=-1$,

(b) $\text{Lim}_{x\to\infty} h(x)=\infty$ and

(c) $\text{Lim}_{x\to -1} h(x)=\frac{1}{2}$.

Find $\text{Lim}_{x\to 0}(3h(x)+f(x)-2g(x))$

10. Let f be continuous on the interval $[0,1]$ to R such that $f(0)=f(1)$. Prove that there exists a point c in $\left[0,\frac{1}{2}\right]$ such that $f(c)=f\left(c+\frac{1}{2}\right)$

11. Consider the function $g(x) = \begin{cases} \frac{1-a^x+xa^x \ell na}{a^x x^2} & \text{for } x < 0 \\ \frac{2^x a^x - x\ell n2 - x\ell na - 1}{x^2} & \text{for } x > 0 \end{cases}$ where $a > 0$.

find the value of 'a' & 'g(0)' so that the function g(x) is continuous at $x = 0$.

12. Let $f(x) = \begin{cases} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)} & \text{for } x \neq 0 \\ \frac{\pi}{2} & \text{for } x = 0 \end{cases}$ where $\{x\}$ is the fractional part of x.

Consider another function g(x); such that $g(x) = f(x)$ for $x \geq 0 = 2\sqrt{2}\, f(x)$ for $x < 0$

Discuss the continuity of the functions f(x) & g(x) at $x = 0$.

13. Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \leq 1 \end{cases}$; where [x] is the greatest integer not greater than x. Also draw the graph.

14. Let $g(x) = \tan^{-1}|x| - \cot^{-1}|x|, f(x) = \frac{[x]}{[x+1]}\{x\}$, $h(x) = |g(f(x))|$ where $\{x\}$ denotes fractional part and [x] denotes the integral part then which of the following holds good?

(A) h is continuous at $x = 0$

(B) h is discontinuous at $x = 0$

(C) $h(0^-) = \pi/2$

(D) $h(0^+) = -\pi/2$

15. $f(x) = \begin{cases} \sin\left(\frac{a-x}{2}\right)\tan\left[\frac{\pi x}{2a}\right] & \text{for } x > a \\ \frac{\left[\cos\left(\frac{\pi x}{2a}\right)\right]}{a-x} & \text{for } x < a \end{cases}$

where [x] is the greatest integer function of x , and $a > 0$, then

(A) $f(a^-) < 0$

(B) f has a removable discontinuity at $x = a$

(C) f has an irremovable discontinuity at $x = a$

(D) $f(a^+) < 0$

EXERCISE-III

1. f is a continuous function on the real line. Given that for all real values of x,

$x^2 + (f(x) - 2)x - \sqrt{3} \cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined (B) is $2(1 - \sqrt{3})$

(C) is zero (D) is $\frac{2(\sqrt{3}-2)}{\sqrt{3}}$

2. If $f(x) = \text{sgn}\,(\cos 2x - 2\sin\ x + 3)$, where sgn ( ) is the signum function, then f(x)

(A) is continuous over its domain

(B) has a missing point discontinuity

(C) has isolated point discontinuity

(D) has irremovable discontinuity.

3. Consider $f(x) = \text{Limit}_{n\to\infty} \frac{x^n - \sin\ x^n}{x^n + \sin\ x^n}$ for $x > 0, x \neq 1, f(1) = 0$ then

(A) f is continuous at $x = 1$

(B) f has an infinite or oscillatory discontinuity at $x = 1$.

(C) f has a finite discontinuity at $x = 1$

(D) f has a removable type of discontinuity at $x = 1$.

4. Given $f(x) = \frac{[\{|x|\}]e^{x^2}\{[x+\{x\}]\}}{\left(e^{1/x^2}-1\right)\text{sg n}(\text{si n}\,x)}$ for $x \neq 0$

$= 0$ for $x = 0$

where {x} is the fractional part function ; [x] is the step up function and sgn (x) is the signum function of x then, f(x)

(A) is continuous at $x = 0$

(B) is discontinuous at $x = 0$

(C) has a removable discontinuity at $x = 0$

(D) has an irremovable discontinuity at $x = 0$

5. Consider $f(x) = \begin{cases} x[x]^2\log_{(1+x)}2 & \text{for } -1 < x < 0 \\ \dfrac{\ln\left(e^{x^2}+2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases}$

where [*] & { *} are the greatest integer function & fractional part function respectively, then

(A) $f(0) = \ln 2 \Rightarrow$ f is continuous at $x = 0$

(B) $f(0) = 2 \Rightarrow$ f is continuous at $x = 0$

(C) $f(0) = e^2 \Rightarrow$ f is continuous at $x = 0$

(D) f has an irremovable discontinuity at $x = 0$

6. Consider $f(x)=\frac{\sqrt{1+x}-\sqrt{1-x}}{\{x\}}, x\neq 0$; $g(x)=\cos\ 2x, -\frac{\pi}{4}<x<0$, $h(x)=\begin{cases}\frac{1}{\sqrt{2}}f(g(x)) & \text{for } x<0\\ 1 & \text{for } x=0\\ f(x) & \text{for } x>0\end{cases}$

then, which of the following holds good. where $\{x\}$ denotes fractional part function.

(A) 'h' is continuous at $x=0$

(B) 'h' is discontinuous at $x=0$

(C) $f(g(x))$ is an even function

(D) $f(x)$ is an even function

7. The function $f(x)=[x]\cdot\cos\frac{2x-1}{2}\pi$, where $[\cdot]$ denotes the greatest integer function, is discontinuous at

(A) all x

(B) all integer points

(C) no x

(D) x which is not an integer

8. Consider the function defined on $[0,1]\to R$, $f(x)=\frac{\sin\ x-x\cos\ x}{x^2}$ if $x\neq 0$ and $f(0)=0$, then the function $f(x)$

(A) has a removable discontinuity at $x=0$

(B) has a non removable finite discontinuity at $x=0$

(C) has a non removable infinite discontinuity at $x=0$

(D) is continuous at $x=0$

9. Consider the function $f(x)=\mathrm{Lim}_{n\to\infty}\frac{\sin\ \pi x-x^{2n}\sin\ (x-1)}{1+x^{2n+1}-x^{2n}}$, where $n\in N$

Statement-1: $f(x)$ is discontinuous at $x=1$.

because

Statement-2: $f(1)=0$.

(A) Statement-1 is true, statement- 2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement- 1 is true, statement- 2 is false.

(D) Statement- 1 is false, statement- 2 is true.

10. Consider the functions, $f(x) = \operatorname{sgn}(x-1)$ and $g(x) = \cot^{-1}[x-1]$

where [ ] denotes the greatest integer function.

Statement-1: The function $F(x) = f(x) \cdot g(x)$ is discontinuous at $x = 1$.

because

Statement-2: If $f(x)$ is discontinuous at $x = a$ and $g(x)$ is also discontinuous at $x = a$ then the product function $f(x) \cdot g(x)$ is discontinuous at $x = a$.

(A) Statement- 1 is true, statement- 2 is true and statement- 2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement- 1 is true, statement- 2 is false.

(D) Statement-1 is false, statement- 2 is true.

11. $f(x) = \frac{a\cos x - \cos bx}{x^2}$, when $x \neq 0$ and $f(0) = 4$. If $f(x)$ is continuous at $x = 0$, then the value of $(n + \Sigma a_i + \Sigma b_i)$ is

[Here, $n =$ number of ordered pairs (a, b);

$\Sigma a_i =$ sum of all distinct possible values of a;

$\Sigma b_i =$ sum of all distinct possible values of b]

(A) 2 (B) 3 (C) 4 (D) 5

12. Statement-1: Function $f(x) = [\cos\ x]$ is discontinuous at $x = \pi$.

Statement-2 : Function $g(x) = [x]$ is discontinuous at integral values of x.

[Here, ['] denotes greatest integer function, i.e., [k] equals the greatest integer less than or equal to ' k ']

(A) Statement-1 is true, Statement-2 is true & Statement-2 is correct explanation for Statement-1

(B) Statement- 1 is true, Statement-2 is true & Statement-2 is NOT the correct explanation for Statement-1

(C) Statement- 1 is true, Statement- 2 is false

(D) Statement-1 is false, Statement- 2 is true

13. Consider the function $f(x) = \begin{cases} x^2 + ax + 2, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$, where ' a ' is a parameter that can take any real value. If $f(x)$ is discontinuous for all real values of x, then number of possible integral values of a is

(A) 3 (B) 2 (C) 1 (D) 0

14. If $f(x) = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$, then number of points of discontinuity of $f(x)$ in $[-1,1]$ is :

(A) 5 (B) 4 (C) 7 (D) 6

15. If $f(x) = \operatorname{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi), n \in N$, then

(A) the minimum possible value of n is 5

(B) the maximum possible value of n is 6

(C) there are exactly two possible values of n

(D) None of these

EXERCISE-IV

1. Let $f(x)=\frac{1-\tan x}{4x-\pi}, x \neq \frac{\pi}{4}, x \in \left[0,\frac{\pi}{2}\right]$. If f(x) is continuous in $\left[0,\frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is- **[AIEEE 2004]**

(A) 1 (B) 1/2 (C) $-1/2$ (D) -1

2. The function $f: R-\{0\} \to R$ given by $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$ can be made continuous at $x=0$ by defining f(0) as **[AIEEE 2007]**

(A) 2 (B) -1 (C) 0 (D) 1

3. The value of p and q for which the function $f(x)=\begin{cases}\frac{\sin(p+1)x+\sin x}{x} & x<0\\ q, & x=0\\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} & x>0\end{cases}$ is continuous for all $x \in R$, are : **[AIEEE 2011]**

(A) $p=\frac{1}{2}, q=\frac{3}{2}$ (B) $p=\frac{1}{2}, q=-\frac{3}{2}$

(C) $p=\frac{5}{2}, q=\frac{1}{2}$ (D) $p=-\frac{3}{2}, q=\frac{1}{2}$

4. If $f: R \to R$ is a function defined by $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is : **[AIEEE 2012]**

(A) continuous for every real x.

(B) discontinuous only at $x=0$

(C) discontinuous only at non-zero integral values of x.

(D) continuous only at $x=0$

5. Let $f(x)=\begin{cases}(x-1)^{\frac{1}{2-x}} & ,x>1, x\neq 2\\ k & ,x=2\end{cases}$ **[IIT Main 2018]**

The value of k for which f is continuous at $x=2$ is :

(A) 1 (B) e (C) e^{-1} (D) e^{-2}

6. If the function f defined as $f(x)=\frac{1}{x}-\frac{k-1}{e^{2x}-1}, x\neq 0$, is continuous at $x=0$, then the ordered pair (k, f(0)) is equal to **[IIT Main Online 2018]**

(A) (2,1) (B) (3,1) (C) $\left(\frac{1}{3},2\right)$ (D) (3,2)

EXERCISE-V

1. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at : **[JEE '99, 2 (out of 200)]**

(A) all integers

(B) all integers except 0 & 1

(C) all integers except 0

(D) all integers except 1

2. Determine the constants a, b & c for which the function

$$f(x) = \begin{cases} (1+ax)^{1/x} & \text{for } x < 0 \\ b & \text{for } x = 0 \\ \dfrac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1} & \text{for } x > 0 \end{cases}$$ is continuous at $x = 0$.

3. Discuss the continuity of the function $f(x) = \begin{cases} \frac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ at $x = 1$. **[REE '99, 6]**

[REE 2001 (Mains), 3 out of 100]

4. For every integer n, let a_n and b_n be real numbers. Let function $f : R \to R$ be given by **[JEE 2012]**

$$f(x) = \begin{cases} a_n + \sin\pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos\pi x, & \text{for } x \in (2n-1, 2n) \end{cases}$$, for all integers n.

If f is continuous, then which of the following hold(s) for all n ?

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$

(C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

5. For every pair of continuous functions $f, g : [0,1] \to R$, such that $\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\}$, the correct statement(s) is(are) : **[IIT Advance 2014]**

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0,1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0,1]$

6. Let $[x]$ be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous? **[IIT Advance - 2017]**

(A) $x = 1$ (B) $x = -1$ (C) $x = 0$ (D) $x = 2$

Answer Key

PROFICIENCY TEST

1. -1 **2.** $a = 0, b = 1$ **3.** $b = 1$ and $a, c \in R$ such that $a + c = 3$

4. f is cont. in $-1 \le x \le 1$ **5.** P not possible. **6.** (a) $4 - 3\sqrt{2} + a$ (b) $a = 3$

7. $g(x) = 2 + x$ for $0 \le x \le 1, 2 - x$ for $1 < x \le 2, 4 - x$ for $2 < x \le 3$,
g is discontinuous at $x = 1$ & $x = 2$ **8.** $A = 1; f(2) = 1/2$

9. discontinuous at x = 1, 4 & 5

10. (i) false ; (ii) false ; (iii) true ; (iv) true; (v) true; (vi) true

EXERCISE-I

1. $f(0^+) = -2; f(0^-) = 2$ hence f(0) not possible to define

2. $y_n(x)$ is continuous at $x = 0$ for all n and $y(x)$ is dicontinuous at $x = 0$

3. $a = 0; b = -1$ **4.** $a = 0, b = 1$ **5.** gof is dis-continuous at $x = 0, 1 \& -1$

6. $a = 1/2, b = 4$ **7.** $a = -3/2, b \neq 0, c = 1/2$ **8.** $A = -4, B = 5, f(0) = 1$

19. A, C, D **10.** A, C, D **11.** B, C **12.** C, D **13.** B **14.** B, C, D

15. B, C

EXERCISE-II

1. discontinuous at all integral values in $[-2,2]$

2. locus $(a, b) \to x, y$ is $y = x - 3$ excluding the points where $y = 3$ intersects it.

3. 5 **5.** $\frac{1}{60}$

8. $k = 0; g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

9. $g(x) = 4(x + 1)$ and limit $= -\frac{39}{4}$ **11.** $a = \frac{1}{\sqrt{2}}, g(0) = \frac{(\ell n\ 2)^2}{8}$

12. $f(0^+) = \frac{\pi}{2}; f(0^-) = \frac{\pi}{4\sqrt{2}} \Rightarrow$ f is discontinuous at $x = 0$
$g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is continuous at $x = 0$

13. The function f is continuous everywhere in $[0,2]$ except for $x = 0, \frac{1}{2}, 1$ & 2.

14. A **15.** B

EXERCISE-III

1.	B	**2.**	C	**3.**	C	**4.**	A	**5.**	D	**6.**	A	**7.**	C
8.	D	**9.**	B	**10.**	C	**11.**	B	**12.**	D	**13.**	A	**14.**	D
15.	C												

EXERCISE-IV

1.	C	**2.**	D	**3.**	D	**4.**	A	**5.**	C	**6.**	B

EXERCISE-V

1. D **2.** $a = \ln\frac{2}{3}; b = \frac{2}{3}; c = 1$

3. Discontinuous at $x = 1$; $f(1^+) = 1$ and $f(1^-) = -1$ **4.** BD **5.** AD **6.** ABD