

KEY CONCEPTS

I. **Sine Formula :** In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

II. **Cosine Formula :**

$$(i) \cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ or } a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$(ii) \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$(iii) \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

III. **Projection Formula :**

$$(i) a = b \cos C + c \cos B$$

$$(ii) b = c \cos A + a \cos C$$

$$(iii) c = a \cos B + b \cos A$$

IV. **NAPIER'S ANALOGY - TANGENT RULE :** (i) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$

$$(ii) \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$(iii) \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

V. **Trigonometric Functions of Half Angles :**

$$(i) \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; \sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}; \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$(ii) \cos \frac{A}{2} = \sqrt{\frac{s(s-b)}{bc}}; \cos \frac{B}{2} = \sqrt{\frac{s(s-c)}{ca}}; \cos \frac{C}{2} = \sqrt{\frac{s(s-a)}{ab}}$$

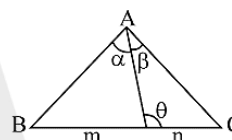
$$(iii) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} \text{ where } s = \frac{a+b+c}{2} \text{ \& } \Delta = \text{area of triangle.}$$

$$(iv) \text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

VI. **M – N Rule :** In any triangle,

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta$$

$$= n \cot B - m \cot C$$



VII. $\frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \text{area of triangle ABC}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Note that $R = \frac{abc}{4\Delta}$; Where R is the radius of circumcircle & Δ is area of triangle

VIII. Radius of the incircle 'r' is given by:

$$(a) r = \frac{\Delta}{s} \text{ where } s = \frac{a+b+c}{2}$$

$$(b) r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

$$(c) r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \text{ \& so on}$$

$$(d) r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

IX. Radius of the Ex-circles r_1, r_2 & r_3 are given by:

$$(a) r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c}$$

$$(b) r_1 = s \tan \frac{A}{2}; r_2 = s \tan \frac{B}{2}; r_3 = s \tan \frac{C}{2}$$

$$(c) r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} \text{ \& so on}$$

$$(d) r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$$

$$r_2 = 4R \sin \frac{B}{2} \cdot \cos \frac{A}{2} \cdot \cos \frac{C}{2}; r_3 = 4R \sin \frac{C}{2} \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2}$$

$$(d) r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2};$$

X. **Length Of Angle Bisector & Medians :**

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

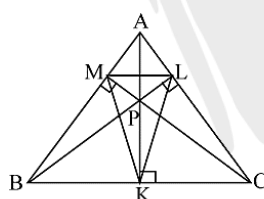
$$m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2} \text{ and } \beta_a = \frac{2bc \cos \frac{A}{2}}{b+c}$$

$$\text{Note that } m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

XI. **Orthocentre and Pedal Triangle :**

The triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle. the distances of the orthocentre from the angular points of the $\triangle ABC$ are $2R \cos A, 2R \cos B$ and $2R \cos C$ the distances of P from sides are $2R \cos B \cos C, 2R \cos C \cos A$ and $2R \cos A \cos B$ the sides of the pedal triangle are $a \cos A (= R \sin 2A), b \cos B (= R \sin 2B)$ and $c \cos C (= R \sin 2C)$ and its angles are $\pi - 2A, \pi - 2B$ and $\pi - 2C$.

circumradii of the triangles PBC, PCA, PAB and ABC are equal .



XII. **Excentral Triangle:**

The triangle formed by joining the three excentres I_1, I_2 and I_3 of $\triangle ABC$ is called the excentral or excentric triangle.

Note that :

Incentre I of $\triangle ABC$ is the orthocentre of the excentral $\triangle I_1 I_2 I_3$.

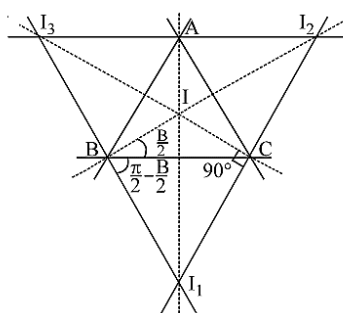
$\triangle ABC$ is the pedal triangle of the $\triangle I_1 I_2 I_3$.

the sides of the excentral triangle are

$$4R \cos \frac{A}{2}, 4R \cos \frac{B}{2} \text{ and } 4R \cos \frac{C}{2}$$

and its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.

$$- II_1 = 4R \sin \frac{A}{2}; II_2 = 4R \sin \frac{B}{2}; II_3 = 4R \sin \frac{C}{2}.$$



XIII. The Distances Between The Special Points :

(a) The distance between circumcentre and orthocentre is $= R \cdot \sqrt{1 - 8\cos A \cos B \cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$

(c) The distance between incentre and orthocentre is $= \sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$

XIV. Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are

given by $P = 2nr \sin \frac{\pi}{n}$ and $A = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}$

Perimeter and area of a regular polygon of n sides circumscribed about a given circle of radius r

is given by $P = 2nr \tan \frac{\pi}{n}$ and $A = nr^2 \tan \frac{\pi}{n}$

PROFICIENCY TEST-01

- In a triangle ABC, $a = 5, b = 7$ and $\sin A = \frac{3}{4}$ how many such triangles are possible.
- In a $\triangle ABC$, if $c^2 + a^2 - b^2 = ac$, then $\angle B =$
- In $\triangle ABC$, $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) =$
- In $\triangle ABC$, if $(a + b + c)(a - b + c) = 3ac$, then $\angle B =$
- In $\triangle ABC$, $\operatorname{cosec} A (\sin B \cos C + \cos B \sin C) =$
- If the angles of a triangle be in the ratio 1: 2: 7, then the ratio of its greatest side to the least side is :
- In $\triangle ABC$, if $a = 3, b = 4, c = 5$, then $\sin 2B =$
- If the sides of a triangle are in the ratio $2: \sqrt{6}: (\sqrt{3} + 1)$, then the largest angle of the triangle will be :
- If the lengths of the sides of a triangle be $7, 4\sqrt{3}$ and $\sqrt{13}$ cm, then the smallest angle is :
- In $\triangle ABC$, if $\angle C = 90^\circ, \angle A = 30^\circ, c = 20$, then the values of a and b are
- If $a = 9, b = 8$ and $c = x$ satisfies $3\cos C = 2$, then find x
- If the sides of a triangle are p, q and $\sqrt{p^2 + pq + q^2}$, then the biggest angle is :

(MATHEMATICS)

SOLUTIONS OF TRIANGLE

13. In a $\triangle ABC$, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the side $a = 2$, then area of the triangle is:
14. The perimeter of an acute $\triangle ABC$ is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then the angle A is :
15. In a triangle ABC , if $a = 2$, $B = 60^\circ$ and $C = 75^\circ$, then $b =$
16. In a $\triangle ABC$, $b = 2$, $C = 60^\circ$, $c = \sqrt{6}$, then $a =$
17. If $A = 30^\circ$, $c = 7\sqrt{3}$ and $\angle C = 90^\circ$ in $\triangle ABC$, then $a =$
18. If angles of a triangle are in the ratio of 2: 3: 7, then the sides are in the ratio of :
19. In $\triangle ABC$, if $b = 6$, $c = 8$ and $\angle A = 90^\circ$, then $R =$
20. In an equilateral triangle of side $2\sqrt{3}$ cm, the circumradius is :

PROFICIENCY TEST-02

1. If in a triangle ABC , $(s - a)(s - b) = s(s - c)$, then angle C is equal to :
2. In a $\triangle ABC$, if $2s = a + b + c$ and $(s - b)(s - c) = x \sin^2 \frac{A}{2}$, then $x =$
3. In $\triangle ABC$, if $a = 16$, $b = 24$ and $c = 20$, then $\cos \frac{B}{2} =$
4. In triangle ABC if a, b, c are in A.P., then the value of $\frac{\sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{B}{2}} =$
5. In $\triangle ABC$, $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) =$
6. If in a triangle ABC , $b = \sqrt{3}$, $c = 1$ and $B - C = 90^\circ$ then $\angle A$ is :
7. If in a triangle the angles A, B, C are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then $\angle A$ is equal to :
8. In $\triangle ABC$, $(b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2}$ is equal to :
9. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60° . If the third side is 3, the remaining fourth side is :
10. If in the $\triangle ABC$, $AB = 2BC$, then $\tan \frac{B}{2} : \cot \left(\frac{C-A}{2} \right) =$
11. In a $\triangle ABC$, if $A = 30^\circ$, $b = 2$, $c = \sqrt{3} + 1$, then $\frac{C-B}{2} =$
12. If in a triangle ABC side $a = (\sqrt{3} + 1)$ cms and $\angle B = 30^\circ$, $\angle C = 45^\circ$, then the area of the triangle is :
13. In an acute $\triangle ABC$, if $b = 20$, $c = 21$ and $\sin A = \frac{3}{5}$, then $a =$
14. The area of triangle ABC , in which $a = 1$, $b = 2$, $\angle C = 60^\circ$ is :
15. In a $\triangle ABC$ if the sides are $a = 3$, $b = 5$ and $c = 4$, then $\sin \frac{B}{2} + \cos \frac{B}{2}$ is equal to :
16. Find the radius of incircle if the sides of triangle are
(i) 13,14,15 (ii) 3,5,6 (iii) 18,24,30

17. If the radius of the circumcircle of an isosceles triangle PQR is equal to $PQ (= PR)$, then the angle P is :
18. In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is :
19. If the sides of the triangle are $5K, 6K, 5K$ and radius of incircle is 6 then value of K is equal to :
20. In a triangle ABC, if $b = 2, B = 30^\circ$ then the area of circumcircle of triangle ABC in square units is :



EXERCISE-I

With usual notations, prove that in a triangle ABC :

1. $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$
2. $a \cot A + b \cot B + c \cot C = 2(R + r)$
3. $\frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)} = \frac{3}{r}$
4. $\frac{r_1-r}{a} + \frac{r_2-r}{b} = \frac{c}{r_3}$
5. $\frac{abc}{s} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \Delta$
6. $(r_1 + r_2) \tan \frac{C}{2} = (r_3 - r) \cot \frac{C}{2} = c$
7. $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$
8. $(r + r_1) \tan \frac{B-C}{2} + (r + r_2) \tan \frac{C-A}{2} + (r + r_3) \tan \frac{A-B}{2} = 0$
9. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$
10. $(r_3 + r_1)(r_3 + r_2) \sin C = 2r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$
11. $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$
12. $\left(\frac{1}{r} - \frac{1}{r_1}\right) \left(\frac{1}{r} - \frac{1}{r_2}\right) \left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{4R}{r^2 s^2}$
13. $\frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$
14. $\left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^2 = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$
15. $Rr (\sin A + \sin B + \sin C) = \Delta$
16. $2R \cos A = 2R + r - r_1$
17. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$
18. $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}$
19. Given a triangle ABC with sides $a = 7$, $b = 8$ and $c = 5$. If the value of the expression $(\sum \sin A) \left(\sum \cot \frac{A}{2} \right)$ can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\frac{p}{q}$ is in its lowest form find the value of $(p + q)$.
20. If $r_1 = r + r_2 + r_3$ then prove that the triangle is a right angled triangle.
21. If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.
22. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

23. Given a right triangle with $\angle A = 90^\circ$. Let M be the mid-point of BC. If the inradii of the triangle ABM and ACM are r_1 and r_2 then find the range of r_1/r_2 .
24. If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.
25. Prove that in a triangle $\frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3} = 2R \left[\left(\frac{a}{b} + \frac{b}{a} \right) + \left(\frac{b}{c} + \frac{c}{b} \right) + \left(\frac{c}{a} + \frac{a}{c} \right) - 3 \right]$.



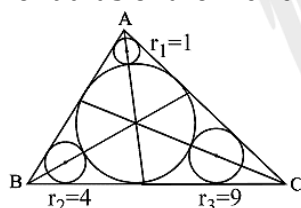
EXERCISE-II

1. With usual notation, if in a $\triangle ABC$, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$; then prove that, $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.
2. For any triangle ABC, if $B = 3C$, show that $\cos C = \sqrt{\frac{b+c}{4c}}$ & $\sin \frac{A}{2} = \frac{b-c}{2c}$.
3. In a triangle ABC, BD is a median. If $l(BD) = \frac{\sqrt{3}}{4} \cdot l(AB)$ and $\angle DBC = \frac{\pi}{2}$. Determine the $\angle ABC$.
4. ABCD is a trapezium such that AB, DC are parallel & BC is perpendicular to them.
If angle ADB = θ , BC = p & CD = q, show that $AB = \frac{(p^2+q^2)\sin \theta}{p\cos \theta + q\sin \theta}$
5. If sides a, b, c of the triangle ABC are in A.P., then prove that $\sin^2 \frac{A}{2} \operatorname{cosec} 2A$; $\sin^2 \frac{B}{2} \operatorname{cosec} 2B$; $\sin^2 \frac{C}{2} \operatorname{cosec} 2C$ are in H.P.
6. Find the angles of a triangle in which the altitude and a median drawn from the same vertex divide the angle at that vertex into 3 equal parts.
7. In a triangle ABC, if $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ are in AP. Show that $\cos A$, $\cos B$, $\cos C$ are in AP.
8. ABCD is a rhombus. The circumradii of $\triangle ABD$ & $\triangle ACD$ are 12.5 & 25 respectively. Find the area of rhombus.
9. In a triangle ABC if $a^2 + b^2 = 101c^2$ then find the value of $\frac{\cot C}{\cot A + \cot B}$.
10. The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is 60° . If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.
11. If I be the in-centre of the triangle ABC and x, y, z be the circum radii of the triangles IBC, ICA & IAB, show that $4R^3 - R(x^2 + y^2 + z^2) - xyz = 0$.
12. Sides a, b, c of the triangle ABC are in H.P., then prove that $\operatorname{cosec} A(\operatorname{cosec} A + \cot A)$; $\operatorname{cosec} B(\operatorname{cosec} B + \cot B)$ & $\operatorname{cosec} C(\operatorname{cosec} C + \cot C)$ are in A.P.
13. A point 'O' is situated on a circle of radius R and with centre O, another circle of radius $3R/2$ is described. Inside the smaller crescent shaped area intercepted between these circles, a circle of radius $R/8$ is placed. If the same circle moves in contact with the original circle of radius R, then find the length of the arc described by its centre in moving from one extreme position to the other.
14. ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that $\cos A \cdot \cos C = \frac{2(c^2 - a^2)}{3ac}$
15. In a $\triangle ABC$, (i) $\frac{a}{\cos A} = \frac{b}{\cos B}$
(ii) $2\sin A \cos B = \sin C$
(iii) $\tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} \tan \frac{C}{2} - 1 = 0$, prove that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).
16. The sequence a_1, a_2, a_3, \dots is a geometric sequence. The sequence b_1, b_2, b_3, \dots is a geometric sequence.

$$b_1 = 1; b_2 = \sqrt[4]{7} - \sqrt[4]{28} + 1; a_1 = \sqrt[4]{28} \text{ and } \sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} b_n$$

If the area of the triangle with sides lengths a_1, a_2 and a_3 can be expressed in the form of p/q where p and q are relatively prime, find $(p + q)$

17. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ denotes the area of the triangle, prove that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$.
18. If $a \tan A + b \tan B = (a + b) \tan \frac{A+B}{2}$, prove that triangle ABC is isosceles.
19. The triangle ABC (with side lengths a, b, c as usual) satisfies $\log a^2 = \log b^2 + \log c^2 - \log (2b \cos A)$. What can you say about this triangle?
20. With reference to a given circle, A_1 and B_1 are the areas of the inscribed and circumscribed regular polygons of n sides, A_2 and B_2 are corresponding quantities for regular polygons of $2n$ sides. Prove that
 - (1) A_2 is a geometric mean between A_1 and B_1 .
 - (2) B_2 is a harmonic mean between A_2 and B_1 .
21. The sides of a triangle are consecutive integers $n, n + 1$ and $n + 2$ and the largest angle is twice the smallest angle. Find n .
22. The triangle ABC is a right angled triangle, right angle at A . The ratio of the radius of the circle circumscribed to the radius of the circle escribed to the hypotenuse is, $\sqrt{2} : (\sqrt{3} + \sqrt{2})$. Find the acute angles B & C . Also find the ratio of the two sides of the triangle other than the hypotenuse.
23. ABC is a triangle. Circles with radii as shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle of the $\triangle ABC$.



24. Line l is a tangent to a unit circle S at a point P . Point A and the circle S are on the same side of l , and the distance from A to l is 3. Two tangents from point A intersect line l at the point B and C respectively. Find the value of $(PB)(PC)$.
25. In a scalene triangle ABC the altitudes AD & CF are dropped from the vertices A & C to the sides BC & AB . The area of $\triangle ABC$ is known to be equal to 18, the area of triangle BDF is equal to 2 and length of segment DF is equal to $2\sqrt{2}$. Find the radius of the circle circumscribed.

EXERCISE-III

1. In a $\triangle ABC$, if $b^2 + c^2 = 3a^2$, then $\cot B + \cot C - \cot A =$
 (A) 1 (B) 2 (C) 3 (D) 0
2. In an acute triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to :
 (A) $\frac{\pi}{8}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$
3. Area of the triangle is $10\sqrt{3}$ sq. cm, $\angle C = 60^\circ$ and its perimeter is 20 cm, then side c will be :
 (A) 3 (B) 5 (C) 6 (D) 7
4. Point D, E are taken on the side BC of a triangle ABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z}$
 (A) 2 (B) 3 (C) 4 (D) 6
5. In a $\triangle ABC$, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then side c is equal to :
 (A) 3 (B) 6 (C) 8 (D) 9
6. If the sides of a triangle are in ratio 3: 7: 8, then R: r is equal to :
 (A) 3: 1 (B) 5: 2 (C) 7: 2 (D) 5: 3
7. In triangle ABC, $\frac{b^2 \sin 2C + c^2 \sin 2B}{\Delta}$ is always equal to :
 (A) 1 (B) 2 (C) 3 (D) 4
8. In triangle ABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$ is equal to:
 (A) 2 (B) $\frac{1}{2}$ (C) 1 (D) None of these
9. In triangle ABC, $(r_1 + r_2 + r_3 - r)$ is equal to:
 (A) $2a \sin A$ (B) $2a \operatorname{cosec} A$ (C) $2a \sin \frac{A}{2}$ (D) $2a \operatorname{cosec} \frac{A}{2}$
10. In triangle ABC, $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$, then $\angle A$ is equal to :
 (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$
11. The expression $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$ is equal to :
 (A) $\cos^2 A$ (B) $\sin^2 A$ (C) $\cos A \cos B \cos C$ (D) none of these
12. If the area of a triangle ABC is given by $\Delta = a^2 - (b - c)^2$, then $\tan A$ is equal to:
 (A) $\frac{1}{4}$ (B) $\frac{8}{15}$ (C) $\frac{4}{15}$ (D) $\frac{3}{4}$
13. If in a triangle ABC, $\frac{\sin A}{4} = \frac{\sin B}{5} = \frac{\sin C}{6}$, the value of $\cos A + \cos B + \cos C$ is equal to :
 (A) $\frac{4}{15}$ (B) $\frac{1}{3}$ (C) $\frac{2}{5}$ (D) $\frac{23}{16}$
14. In triangle ABC, $\frac{a \cos A + b \cos B + c \cos C}{a+b+c}$ is equal to :
 (A) $r + R$ (B) R/r (C) r/R (D) rR
15. In triangle ABC, $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then the triangle is necessarily :
 (A) right-angled (B) obtuse-angled (C) isosceles (D) equilateral

EXERCISE-IV

1. The sides of a triangle are $3x + 4y$, $4x + 3y$ and $5x + 5y$ where $x, y > 0$ then the triangle is
(A) Right angled (B) Obtuse angled (C) Equilateral (D) None of these
[AIEEE-2002]
2. In a triangle with sides a, b, c , $r_1 > r_2 > r_3$ (which are the exradii) then [AIEEE-2002]
(A) $a > b > c$ (B) $a < b < c$ (C) $a > b$ and $b < c$ (D) $a < b$ and $b > c$
3. In a triangle ABC, medians AD and BE are drawn. If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of the $\triangle ABC$ is :
(A) $\frac{64}{3}$ (B) $\frac{8}{3}$ (C) $\frac{16}{3}$ (D) $\frac{32}{3\sqrt{3}}$ [AIEEE-2003]
4. If in a $\triangle ABC$ $a \cos^2\left(\frac{C}{2}\right) + c \cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides a, b and c [AIEEE-2003]
(A) satisfy $a + b = c$ (B) are in A.P.
(C) are in G.P. (D) are in H.P.
5. The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1 + \sin\alpha \cos\alpha}$ for some $0 < \alpha < \frac{\pi}{2}$. Then the greatest angle of the triangle is [AIEEE-2004]
(A) 150° (B) 90° (C) 120° (D) 60°
6. In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle ABC, then $2(r + R)$ equals [AIEEE-2005]
(A) $b + c$ (B) $a + b$ (C) $a + b + c$ (D) $c + a$
7. If in a $\triangle ABC$, the altitudes from the vertices A, B, C on opposite sides are in H.P., then $\sin A, \sin B, \sin C$ are in [AIEEE-2005]
(A) G.P. (B) A.P. (C) A.P. - G.P. (D) H.P.
8. ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to : [JEE-Main 2013]
(A) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (B) $\frac{(p^2+q^2)\cos\theta}{p\cos\theta+q\sin\theta}$ (C) $\frac{p^2+q^2}{p\cos\theta+q\sin\theta}$ (D) None of these
9. Let the orthocentre and centroid of a triangle be A(-3,5) and B(3,3) respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter is: [JEE-Main 2018]
(A) $\frac{3\sqrt{5}}{2}$ (B) $\sqrt{10}$ (C) $2\sqrt{10}$ (D) $3\sqrt{\frac{5}{2}}$

EXERCISE-V

1. If in a $\triangle ABC$, $a = 6$, $b = 3$ and $\cos(A - B) = 4/5$ then find its area. [REE '97, 6]
2. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in A.P., then [JEE '98, 2]
 - (A) the altitudes are in A.P.
 - (B) the altitudes are in H.P.
 - (C) the medians are in G.P.
 - (D) the medians are in A.P.
3. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30° . How many such triangles are possible? Find the length of their third side and area. [REE '98, 6]
4. The radii r_1 , r_2 , r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides. [REE '99, 6]
5. (a) In a triangle ABC, Let $\angle C = \frac{\pi}{2}$. If 'r' is the inradius and 'R' is the circumradius of the triangle, then $2(r + R)$ is equal to:
 - (A) $a + b$
 - (B) $b + c$
 - (C) $c + a$
 - (D) $a + b + c$
 (b) In a triangle ABC, $2ac \sin \frac{1}{2}(A - B + C) =$ [JEE '2000 (Screening) 1+1]
 - (A) $a^2 + b^2 - c^2$
 - (B) $c^2 + a^2 - b^2$
 - (C) $b^2 - c^2 - a^2$
 - (D) $c^2 - a^2 - b^2$
6. Let ABC be a triangle with incentre 'I' and inradius 'r'. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA & AB respectively. If r_1 , r_2 & r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF & CEID respectively, prove that

$$\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$
 [JEE '2000, 7]
7. If Δ is the area of a triangle with side lengths a, b, c , then show that: $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$. Also show that equality occurs in the above inequality if and only if $a = b = c$. [JEE' 2001]
8. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)? [JEE '2002 (Scr), 3]
 - (A) $a, \sin A, \sin B$
 - (B) a, b, c
 - (C) $a, \sin B, R$
 - (D) $a, \sin A, R$
9. If I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that [JEE 2003, Mains, 4 out of 60]

$$I_n = \frac{O_n}{2} \left(1 + \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} \right)$$
10. The ratio of the sides of a triangle ABC is $1 : \sqrt{3} : 2$. The ratio A : B : C is [JEE 2004 (Screening)]
 - (A) 3 : 5 : 2
 - (B) $1 : \sqrt{3} : 2$
 - (C) 3 : 2 : 1
 - (D) 1 : 2 : 3
11. (a) In $\triangle ABC$, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is [JEE 2005 (Screening)]
 - (A) $(b - c) \sin \left(\frac{B-C}{2} \right) = a \cos \left(\frac{A}{2} \right)$
 - (B) $(b - c) \cos \left(\frac{A}{2} \right) = a \sin \left(\frac{B-C}{2} \right)$
 - (C) $(b + c) \sin \left(\frac{B+C}{2} \right) = a \cos \left(\frac{A}{2} \right)$
 - (D) $(b - c) \cos \left(\frac{A}{2} \right) = 2a \sin \left(\frac{B+C}{2} \right)$

(MATHEMATICS)

SOLUTIONS OF TRIANGLE

(b) Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact. [JEE 2005 (Mains), 2]

12. (a) Given an isosceles triangle, whose one angle is 120° and radius of its incircle is $\sqrt{3}$. Then the area of triangle in sq. units is :

(A) $7 + 12\sqrt{3}$ (B) $12 - 7\sqrt{3}$ (C) $12 + 7\sqrt{3}$ (D) 4π [JEE 2006, 3]

(b) Internal bisector of $\angle A$ of a triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of $\triangle ABC$ then [JEE 2006, 5]

(A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$
(C) $EF = \frac{4bc}{b+c} \sin \frac{A}{2}$ (D) the triangle AEF is isosceles

13. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression

$\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is [JEE 2010]

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

14. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) [JEE 2010]

(A) $-(2 + \sqrt{3})$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

15. Consider a triangle ABC and let, a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to [JEE 2010]

16. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. [JEE 2012]

Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

17. In a triangle PQR, P is the largest angle and $\cos P = 1/3$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is/are

[JEE (Adv.)2013]

(A) 16 (B) 18 (C) 24 (D) 22

(MATHEMATICS)

SOLUTIONS OF TRIANGLE

18. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z , respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

[JEE Advanced-2016]

- (A) area of the triangle XYZ is $6\sqrt{6}$
 (B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
 (C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
 (D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$
19. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE? [JEE Advanced-2018]
- (A) $\angle QPR = 45^\circ$
 (B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
 (C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$
 (D) The area of the circumcircle of the triangle PQR is 100π
20. In a non-right-angled $\triangle PQR$, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$ and the radius of the circumcircle of the $\triangle PQR$ equals 1, then which of the following options is/are correct?

(A) Radius of incircle of $\triangle PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$ [JEE Advanced-2019]

(B) Area of $\triangle SOE = \frac{\sqrt{3}}{12}$

(C) Length of RS = $\frac{\sqrt{7}}{2}$

(D) Length of OE = $\frac{1}{6}$

21. Let x, y and z be positive real numbers, Suppose x, y and z are the length of the sides of a triangle opposite to its angles X, Y and Z , respectively. If $\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z}$, then which of the following statements is/are TRUE ? [JEE Advanced-2020]

(A) $2Y = X + Z$ (B) $Y = X + Z$ (C) $\tan \frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

22. Consider a triangle PQR having sides of length p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) True ? [JEE Advanced-2021]

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

(B) $\cos R \geq \left(\frac{q-r}{p+q} \right) \cos P + \left(\frac{p-r}{p+q} \right) \cos Q$

(C) $\frac{q+r}{p} < 2 \frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

23. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and $CA = 4$. Then the value of

$\frac{\cot A + \cot C}{\cot B}$ is

[JEE Advanced-2021]

ANSWER KEY

PROFICIENCY TEST-01

1. 0 2. $\frac{\pi}{3}$ 3. 0 4. 60° 5. 1 6. $(\sqrt{5} + 1):(\sqrt{5} - 1)$
 7. $24/25$ 8. 75° 9. 30° 10. $10, 10\sqrt{3}$ 11. $x = 7$ 12. $2\pi/3$
 13. $\sqrt{3}$ 14. $\pi/6$ 15. $\sqrt{6}$ 16. $\sqrt{3} + 1$ 17. $\frac{7\sqrt{3}}{2}$ 18. $\sqrt{2}:2:(\sqrt{3} + 1)$
 19. 5 20. 2 cm

PROFICIENCY TEST-02

1. 90° 2. bc 3. $\frac{3}{4}$ 4. $\frac{1}{2}$ 5. 0 6. 30° 7. 75°
 8. 0 9. 2 10. 1:3 11. 30° 12. $\frac{\sqrt{3}+1}{2} \text{ cm}^2$ 13. 13
 14. $\frac{\sqrt{3}}{2}$ 15. $\sqrt{2}$ 16. (i) 4 (ii) $\sqrt{8/7}$ (iii) 6 17. $\frac{2\pi}{3}$ 18. $\frac{16}{7}$ 19. 4
 20. 4π

EXERCISE-I

19. 107 23. $(\frac{1}{2}, 2)$

EXERCISE-II

3. 120° 6. $\pi/6, \pi/3, \pi/2$ 8. 400 9. 50 10. 3 cms & 2 cms
 13. $\frac{7\pi R}{12}$ 16. 9 19. triangle is isosceles 21. 4
 22. $B = \frac{5\pi}{12}; C = \frac{\pi}{12}; \frac{b}{c} = 2 + \sqrt{3}$ 23. $r = 11$ 24. 3
 25. $\frac{9}{2}$ units

EXERCISE-III

1. D 2. C 3. D 4. C 5. B 6. C 7. D
 8. B 9. B 10. B 11. B 12. B 13. D 14. C
 15. C

EXERCISE-IV

1. B 2. A 3. D 4. B 5. C 6. B 7. B
 8. A 9. D

EXERCISE-V

1. 9sq. Unit 2. B
 3. $2, (2\sqrt{3} - \sqrt{2}), (2\sqrt{3} + \sqrt{2}), (2\sqrt{3} - \sqrt{2})$ & $(2\sqrt{3} + \sqrt{2})$ sq. units
 4. 6,8,10cms 5. (a) A, (b) B 8. D 10. D 11. (a) B; (b) $\sqrt{5}$
 12. (a) C, (b) A, B, C, D 13. D 14. B 15. 3 16. C 17. B, D
 18. A, C, D 19. B, C, D 20. A, C, D 21. BC 22. A, B 23. 2.00