Nishant Jindal

1. Find 'a' for which $f(x) = \sin^3 x - a\sin^2 x$ has no critical point in $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

$$f'(x) = 3\sin^2 x\cos x - 2a\sin x\cos x$$

$$= \sin x\cos x\,(3\sin x - 2a)$$

$$a \neq \frac{3}{2}\sin x \quad x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \qquad \frac{3}{2}\sin x \in \left[\frac{3}{4}, \frac{3\sqrt{3}}{4}\right]$$

$$\boxed{a \in \left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3\sqrt{3}}{4}, \infty\right)}$$

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Q: Discuss the monotonocity of function $g(x)$,

$$g(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2), \quad \& \quad \underset{f' \uparrow}{f''(x) > 0} \quad \forall x \in R.$$

$$g'(x) = 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right) > 0$$

$x>0$ & $f'\left(\frac{x^2}{2}\right) > f'(6-x^2) \Rightarrow \frac{x^2}{2} > 6-x^2 \Rightarrow x^2 > 4 \Rightarrow \boxed{x \in (2,\infty)}$

OR

$x<0$ & $f'\left(\frac{x^2}{2}\right) < f'(6-x^2) \Rightarrow \frac{x^2}{2} < 6-x^2 \Rightarrow x^2 < 4 \Rightarrow \boxed{x \in (-2,0)}$

$g \uparrow \quad (-2,0) \cup (2,\infty)$

$\downarrow \quad (-\infty,-2) \cup (0,2)$.

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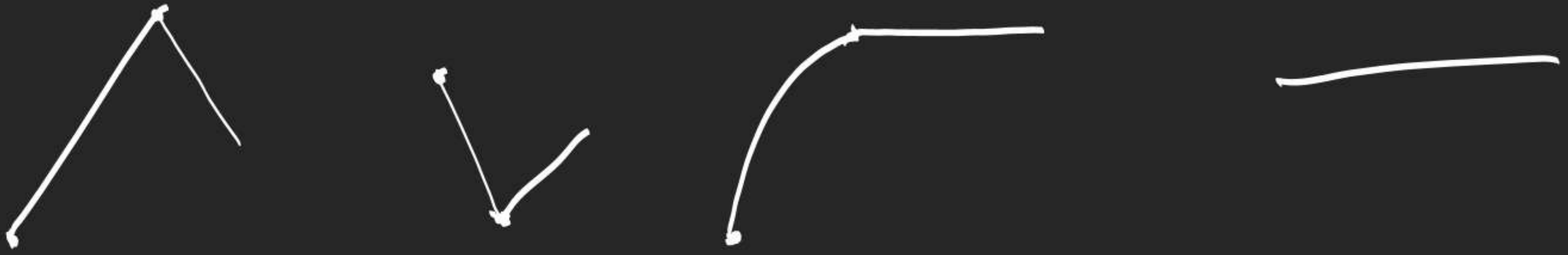
Global/Absolute Maximum/Minimum of a Continuous function in [a, b]

$$\boxed{f_{max} = \max\{f(a), f(b), f(c_1), f(c_2), \dots, f(c_n)\}}$$

$c_i \to$ critical points in (a, b)

$$\boxed{f_{min} = \min\{f(a), f(b), f(c_1), f(c_2), \dots, f(c_n)\}}$$



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Inequalities

① P.T. $f(x) \geq 0 \quad \forall x \geq a$.

$$f_{min} \geq 0$$

② P.T. $f(x) \leq 0 \quad \forall x \geq a$.

$$f_{max} \leq 0$$

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1. Find the highest & lowest values of

(i) $f(x) = e^{x^2-4x+3}$ in $[-5, 5]$

$R_f = \left[\frac{1}{e}, e^{48}\right]$ $\quad f_{min} = f(2) = e^{-1}$, $\quad f_{max} = f(-5) = e^{48}$

(ii) $f(x) = \cos 3x - 15\cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$ ✓

$$f'(x) = -3\sin 3x + 15\sin x = 12\sin^3 x + 6\sin x$$

$$= 6\sin x(2\sin^2 x + 1)$$

$\boxed{x = \pi}$

$$f\left(\frac{\pi}{3}\right) = -1 - \frac{15}{2} + 8 = -\frac{1}{2}$$

$$f(\pi) = -1 + 15 + 8 = 22$$

$$f\left(\frac{3\pi}{2}\right) = 8$$

$f_{max} = 22$

$f_{min} = -\frac{1}{2}$

$R_f = \left[-\frac{1}{2}, 22\right]$.

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2. Find the image of interval [-1, 3] under the mapping specified by the function

$$f(x) = 4x^3 - 12x$$

$$f'(x) = 12(x^2 - 1)$$

$$f(-1) = 8$$

$$f(1) = -8$$

$$f(3) = 72$$

$$R_f = [-8, 72]$$

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3. Which is greater

(i) $\pi^e < e^\pi$

(ii) $(2023)^{2024} > (2024)^{2023}$

(iii) $(2.001)^{4.001} > (4.001)^{2.001}$

$(2023)^{\frac{1}{2023}} > (2024)^{\frac{1}{2024}}$

$a^b \quad b^a$

$a^{\frac{1}{a}} \,?\, b^{\frac{1}{b}}$

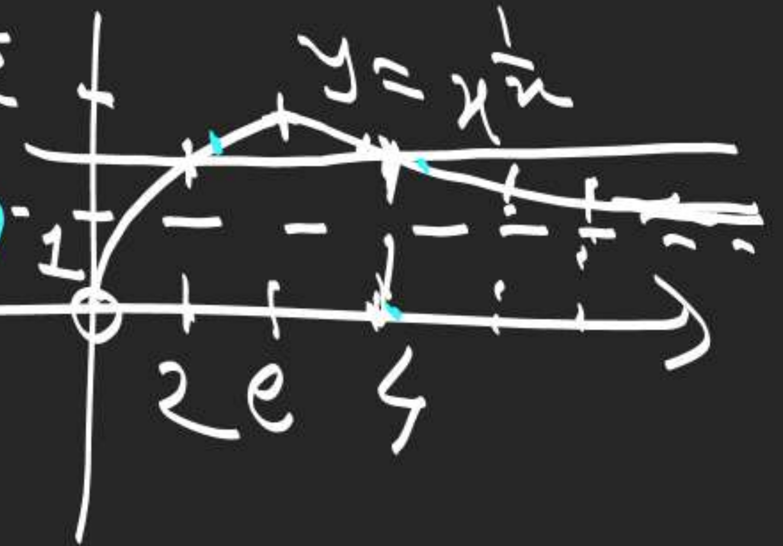
$\lim_{x\to\infty} \frac{\ln x}{x} \quad \frac{\frac{1}{x}}{1}$

$f(x) = x^{\frac{1}{x}}, x>0, \quad f'(x) = x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)$

$f \uparrow (0,e)$

$\downarrow (e,\infty)$

$e^{\frac{1}{e}}$



$(2.001)^{\frac{1}{2.001}} > (4.001)^{\frac{1}{4.001}}$

$e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}} \Rightarrow \boxed{e^\pi > \pi^e}$

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4. P.T. $2\sin x + \tan x \geq 3x \quad \forall\ 0 \leq x < \frac{\pi}{2}$.

$$f(x) = 2\sin x + \tan x - 3x$$

$$f'(x) = \left(2\cos x + \sec^2 x\right) - 3 \geq 0 \qquad \forall x \in \left[0, \frac{\pi}{2}\right)$$

$f \uparrow \qquad f_{min} = f(0) = 0$

$$\frac{\cos x + \cos x + \sec^2 x}{3} \geq \left(\cos^2 x \sec^2 x\right)^{1/3} = 1$$

$$f'(x) = \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$$f(x) \geq f(0) = 0.$$

$$= \frac{(\cos x - 1)\left(2\cos^2 x \overset{-2c+c}{-\cos x} - 1\right)}{\cos^2 x} = \frac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x} \geq 0.$$

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Q. P.T. $\dfrac{\tan x}{x} > \dfrac{x}{\sin x} \quad \forall\, x \in \left(0, \dfrac{\pi}{2}\right)$

$\tan x > x$

$f(x) = \tan x \sin x - x^2$

$f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$

$= \tan x\,(\underbrace{\sec x + \cos x}_{\geq 2}) - 2x \geq 2(\tan x - x) > 0$

$f \uparrow$

$f(x) > f(0) = 0 \quad \forall\, x \in \left(0, \dfrac{\pi}{2}\right)$

$1 < \dfrac{\overbrace{\left(\dfrac{\tan\frac{x}{2}}{x/2}\right)^2}^{>1}}{\underbrace{\left(1 - \tan^4\frac{x}{2}\right)}_{<1}} = \dfrac{\tan x \sin x}{x^2} > 1$

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3: P.T. $\frac{1}{x+\frac{1}{2}} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x} \quad \forall x > 0$.

$f(x) > \lim_{x\to\infty} f(x) = 0$.

$$f(x) = \ln\left(1+\frac{1}{x}\right) - \frac{2}{2x+1}$$

$$f'(x) = \frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x}\right)} + \frac{4}{(2x+1)^2}$$

$$= \frac{-1}{x(x+1)} + \frac{4}{(2x+1)^2}$$

$$= \frac{-1}{(2x+1)^2 x(x+1)} < 0$$

$f \downarrow$

$\ln(1+t) < t$
$t > 0$

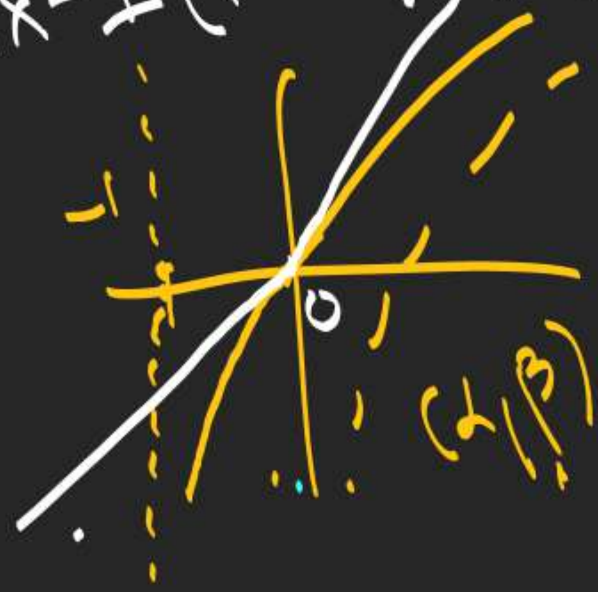
① Ex-IV (Differentiation)

② Ex-II (11-18) → Differentiation.

③ Ex-I (1-10) → Indefinite

$y = x$

$f(1+x) = (\alpha - 1, \beta)$



$(\alpha, \beta) \Leftarrow f(x)$