Nishant Jindal

P.T.

1. $\frac{\pi}{128} < \int_{\pi/4}^{\pi/2} (\sin x)^{10}\, dx < \frac{\pi}{4}$

$\cos x < \frac{\sin x}{x} < 1$

$\frac{1}{\sqrt{2}} \le \sin x \le 1$

$\frac{1}{32} \le (\sin x)^{10} \le 1$

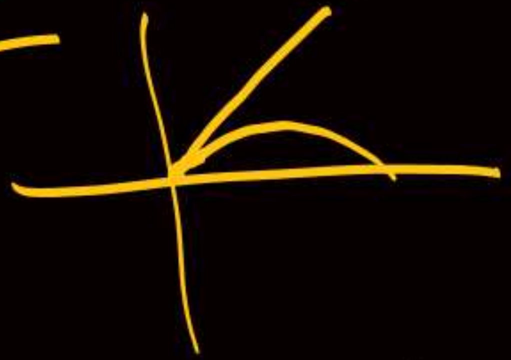
$\frac{1}{32}\int_{\pi/4}^{\pi/2} dx < \int_{\pi/4}^{\pi/2} \sin^{10} x\, dx < \int_{\pi/4}^{\pi/2} 1\, dx$

$(0, \frac{\pi}{2})$ $\tan x > x$

$\frac{\sin x}{x} > \cos x$

$1 = \int_0^{\pi/2} \frac{2}{\pi}\, dx < \int_0^{\pi/2} \frac{\sin x}{x}\, dx < \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$

2. $\frac{e-1}{3} < \int_1^e \frac{dx}{2+\ln x} < \frac{e-1}{2}$



$(0, \pi/2)$

$\sin x < x$

$2+0 \le 2+\ln x \le 2+1$

$\int_1^e \frac{1}{3}\, dx < \int_1^e \frac{1}{2+\ln x}\, dx < \int_1^e \frac{1}{2}\, dx$

3. $1 < \int_{0^+}^{\pi/2} \frac{\sin x}{x}\, dx < \frac{\pi}{2}$

$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x\,(x - \tan x)}{x^2} < 0$

$\frac{2}{\pi} = f\left(\frac{\pi}{2}\right) < f(x) < \lim_{x\to 0^+} \frac{\sin x}{x} = 1$

$f(x) = \frac{\sin x}{x}$

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4. $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^5}} < \frac{\pi}{6}$

$4-x^2 \le 4-(x^2-x^5) \le 4$

$\frac{1}{2} \le \frac{1}{\sqrt{4-x^2+x^5}} \le \frac{1}{\sqrt{4-x^2}}$

$\frac{1}{2} = \int_0^1 \frac{1}{2}dx < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^5}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

5. $1 < \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^3 x}\, dx < \frac{1}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right)$

$\sin^4 x \le \sin^3 x \le \sin^2 x$

$1 = \int_0^{\frac{\pi}{2}} \cos x\, dx < \int_0^{\pi/2} \sqrt{1-\sin^3 x}\, dx < \int_0^{\frac{\pi}{2}} \sqrt{1+\sin^2 x}\, \cos x\, dx$

$\frac{1}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right) = \left[\frac{\sin x}{2}\sqrt{1+\sin^2 x} + \frac{1}{2}\ln\left|\sin x+\sqrt{1+\sin^2 x}\right|\right]_0^{\frac{\pi}{2}}$

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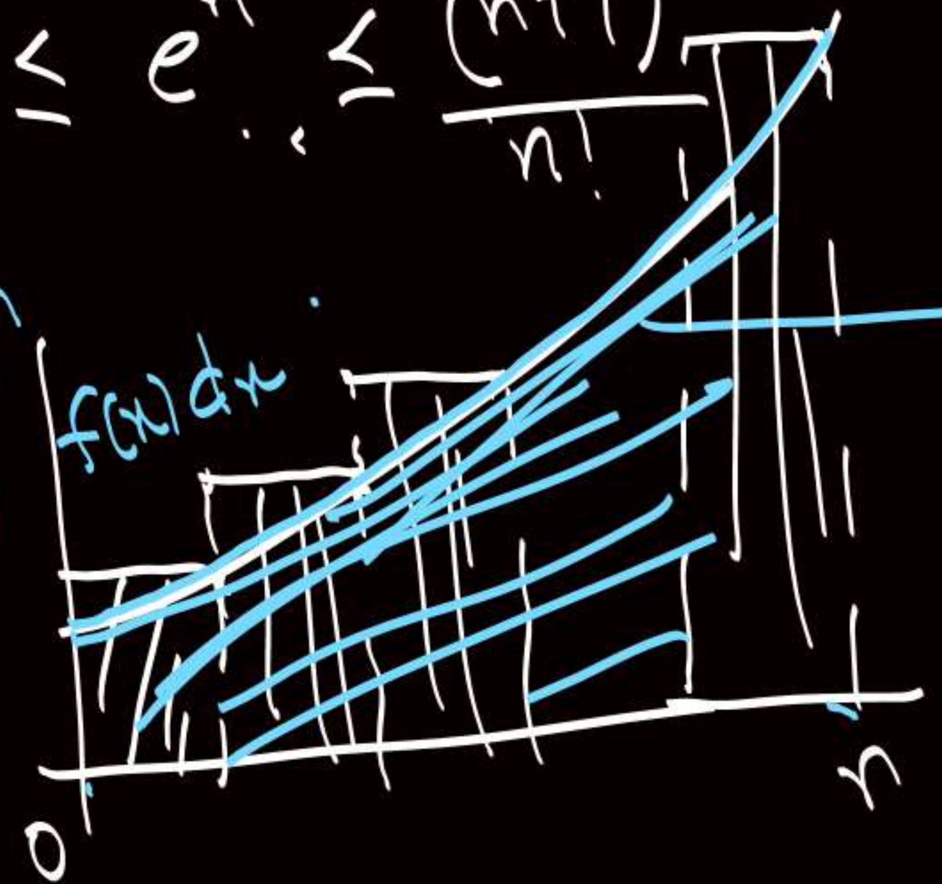
6. If f is continuous and strictly increasing on $[0,\infty)$. P.T.

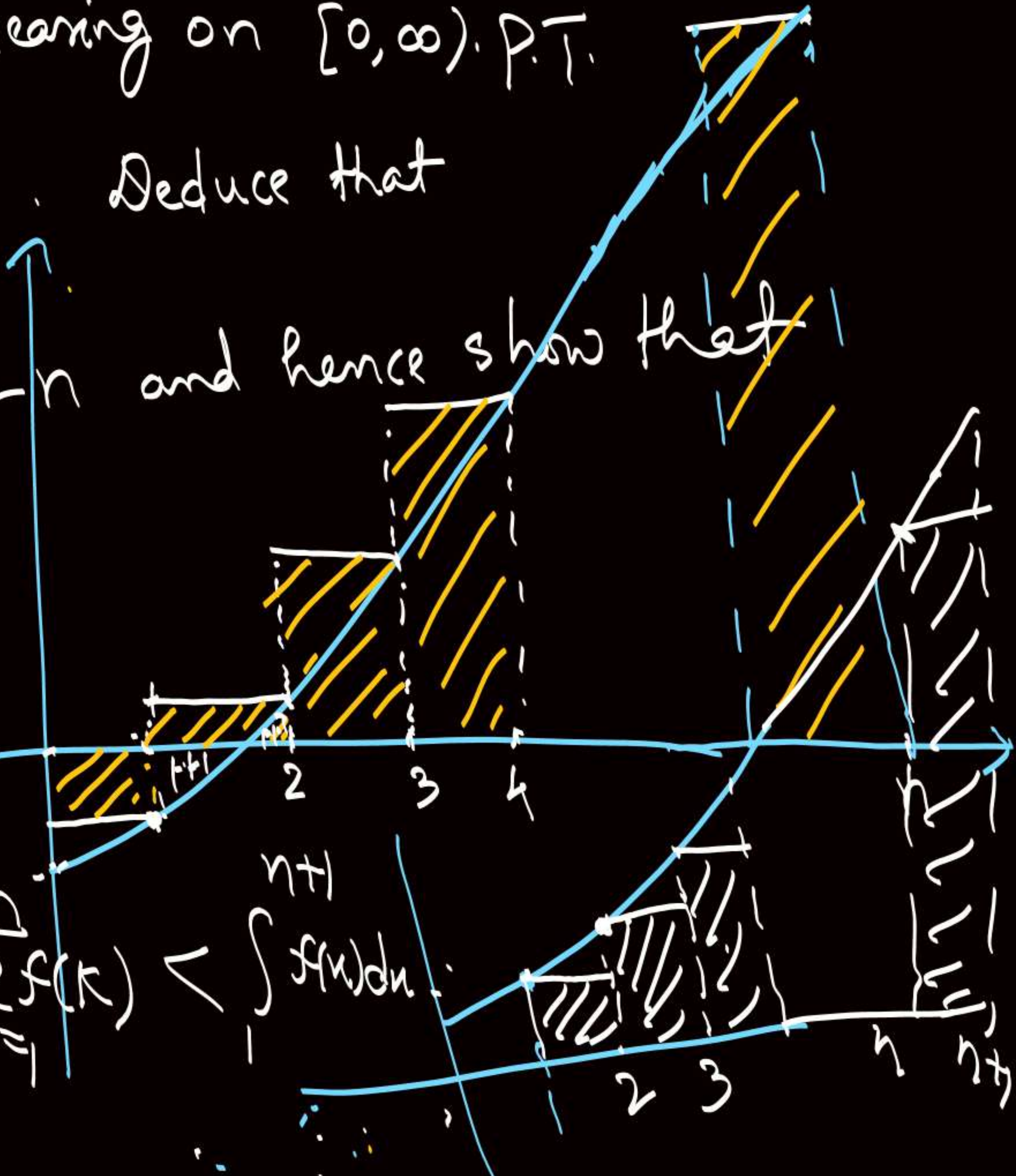
$$\int_0^n f(x)\,dx \le \sum_{k=1}^{n} f(k) \le \int_1^{n+1} f(x)\,dx \,.$$ Deduce that

$$n\ln n - n \le \ln(n!) \le (n+1)\ln(n+1) - n$$ and hence show that

$$\frac{n^n}{n!} \le e^n \le \frac{(n+1)^n}{n!}$$



$$\sum_{k=1}^{n} f(k) > \int_0^n f(x)\,dx$$



$$\sum_{k=1}^{n} f(k) < \int_1^{n+1} f(x)\,dx$$

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7. Find a function 'f', continuous for all $x \in R$ (and not identically zero), such that

$$f^2(x) = \int_0^x \frac{f(t)\sin t\,dt}{(2+\cos t)}$$

$\hookrightarrow f(0) = 0$

$$2f(x)f'(x) = \frac{f(x)\sin x}{2+\cos x} \Rightarrow 2f'(x) = \frac{\sin x}{2+\cos x}$$

$$2f(x) = -\ln(2+\cos x) + C$$

$$f(0)=0 \Rightarrow 0 = -\ln 3 + C$$

$$\boxed{f(x) = \frac{1}{2}\ln\left(\frac{3}{2+\cos x}\right)}$$

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$$x \in [k, k+1] \qquad f(x) \geq f(k) \implies \int_k^{k+1} f(x)\,dx \geq \int_k^{k+1} f(k)\,dx = f(k)$$

$$x \in [k-1, k] \qquad f(x) \leq f(k) \implies \int_{k-1}^{k} f(x)\,dx \leq \int_{k-1}^{k} f(k)\,dx = f(k)$$

$$\int_{k-1}^{k} f(x)\,dx \leq f(k) \leq \int_{k}^{k+1} f(x)\,dx$$

$$\int_0^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq \int_1^{n+1} f(x)\,dx$$

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8. If $f(x) = x + \int_0^{\frac{\pi}{2}} \sin(x+y) f(y)\,dy$, find $f(x)$.

$$f(x) = x + \sin x \int_0^{\pi/2} \cos y\, f(y)\,dy + \cos x \int_0^{\pi/2} \sin y\, f(y)\,dy$$

$$f(x) = x + c_1 \sin x + c_2 \cos x$$

Monday

DI $\boxed{Ex-2}$

$$c_1 = \int_0^{\frac{\pi}{2}} \cos y\,(y + c_1 \sin y + c_2 \cos y)\,dy = \left(\frac{\pi}{2} - 1\right) + \frac{c_1}{2} + c_2 \frac{\pi}{4}$$

$$c_2 = \int_0^{\pi/2} \sin y\,(y + c_1 \sin y + c_2 \cos y)\,dy = 1 + c_1 \frac{\pi}{4} + \frac{c_2}{2}$$