1. INTRODUCTION :

The algebraic expression of the form $ax^2 + bx + c, a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2 + bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS &COEFFICIENT :

(a) the form of quadratic equation is $ax^2 + bx + c = 0, a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminate of the quadratic equation.

(c) If $\alpha \& \beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) A quadratic equation whose roots are $\alpha \& \beta$ is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 - (\text{ sum of roots })x + \text{product of roots} = 0$.

Illustration 1 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$

(C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore \alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7) \& (\beta^2 - 3\beta + 7)$

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$x^2 - 4x + 4 = 0$

Illustration 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\frac{b}{a} \& \alpha\beta = \frac{c}{a}$

$(2\alpha+b)^{-2}+(2\alpha+b)^{-2}=\frac{1}{(2\alpha+b)^2}+\frac{1}{(a\beta+b)^2}$

$=\frac{a^2\beta^2+b^2+2ab\beta+a^2\alpha^2+b^2+2ab\alpha}{(a^2\alpha\beta+ba\beta+ba\alpha+b^2)^2}=\frac{a^2(\alpha^2+\beta^2)+2ab(\alpha+\beta)+2b^2}{(a^2\alpha\beta+ab(\alpha+\beta)+b^2)^2}$

$(\alpha^2+\beta^2$ can always be written as $(\alpha+\beta)^2-2\alpha\beta)$

$\frac{a^2[(\alpha+\beta)^2-2\alpha\beta]+2ab(\alpha+\beta)+2b^2}{(a^2\alpha\beta+ab(\alpha+\beta)+b^2)^2}=\frac{a^2\left[\frac{b^2-2ac}{a^2}\right]+2ab\left(-\frac{b}{a}\right)+2b^2}{\left(a^2\frac{c}{a}+ab\left(-\frac{b}{a}\right)+b^2\right)}=\frac{b^2-2ac}{a^2c^2}$

Alternatively :

If $\alpha\&\beta$ are roots of $ax^2+bx+c=0$ then, $a\alpha^2+b\alpha+c=0$

$\Rightarrow\ a\alpha+b=-\frac{c}{\alpha}$

same as $a\beta+b=-\frac{c}{\beta}$

$\therefore\ (2\alpha+b)^{-2}=(\alpha\beta+b)^{-2}=\frac{\alpha^2}{c^2}+\frac{\beta^2}{c^2}$

$=\frac{(\alpha+\beta)^2-2\alpha\beta}{c^2}=\frac{(-b/a)^2-2(c/a)}{c^2}=\frac{b^2-2ac}{a^2c^2}$

Do yourself - 1 :

(i) Find the roots of following equations :

(a) $x^2+3x+2=0$

(b) $x^2-8x+16=0$

(c) $x^2-2x-1=0$

(ii) Find the roots of the equation $a(x^2+1)-(a^2+1)x=0$, where $a\neq 0$.

(iii) Solve : $\frac{6-x}{x^2-4}=2+\frac{x}{x+2}$

(iv) If the roots of $4x^2+5k=(5k+1)x$ differ by unity, then find the values of k.

3. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2+bx+c=0$ where $a, b, c\in R\&a\neq 0$ then ;

$x=\frac{-b\pm\sqrt{D}}{2a}$

(i) $D>0\Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D=0\Leftrightarrow$ roots are real & coincident (equal)

(iii) $D<0\Leftrightarrow$ roots are imaginary.

(iv) If $p+iq$ is one root of a quadratic equation, then the other root must be the conjugate $p-iq\&$ vice versa. $(p,q\in R\&i=\sqrt{-1})$.

(b) Consider the quadratic equation $ax^2+bx+c=0$ where $a,b,c\in Q\&a\neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

Illustration 3: If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1 , then find the equation one of whose roots is $\tan\frac{\pi}{8}$.

Solution : We know that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$(x - (-1 + \sqrt{2}))(x - (-1 - \sqrt{2})) = 0 \Rightarrow x2 + 2x - 1 = 0$

Illustration 4 : Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow D = (a - 10)^2 - 4 \Rightarrow 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow (a - 10) = \pm 2 \Rightarrow a = 12,8$ Ans.

Do yourself - 2 :

(i) If $2 + \sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in Q$, find b, c.

(ii) For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2 - 6x + 10 = 0$ (b) $x^2 - (7 + \sqrt{3})x + 6(1 + \sqrt{3}) = 0$ (c) $4x^2 + 28x + 49 = 0$

(iii) If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

4. ROOTS UNDER PARTICULAR CASES :

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0$ $\Rightarrow$ roots are equal in magnitude but opposite in sign

(b) If $c = 0$ $\Rightarrow$ one root is zero other is $-b/a$

(c) If $a = c$ $\Rightarrow$ roots are reciprocal to each other

(d) If $\begin{matrix} a > 0 & c < 0 \\ a < 0 & c > 0 \end{matrix}$ $\Rightarrow$ roots are of opposite signs

(e) If $\begin{matrix} a > 0, & b > 0, c > 0 \\ a < 0, & c > 0, c < 0 \end{matrix}$ $\Rightarrow$ both roots are negative.

(f) If $\begin{matrix} a > 0, & b < 0, c > 0 \\ a < 0, & b > 0, c < 0 \end{matrix}$ $\Rightarrow$ both roots are positive.

(g) If sign of a a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0$ $\Rightarrow$ one root is 1 and second root is c/a or $(-b-a)/a$.

Illustration 5 : If equation $\frac{x^2-bx}{ax-c} = \frac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Solution : Let the roots are $\alpha \& -\alpha$. given equation is

$(x^2 - bx)(k+1) = (k-1)(ax-c)\{$ Considering, $x \neq c/a \& k \neq -1\}$

$\Rightarrow x^2(k+1) - bx(k+1) = ax(k-1) - c(k-1)$

$\Rightarrow x^2(k+1) - bx(k+1) - ax(k-1) + c(k-1) = 0$

Now sum of roots $= 0\ (\because \alpha - \alpha = 0)$

$\therefore b(k+1) + a(k-1) = 0 \Rightarrow k = \frac{a-b}{a+b}$

Ans. **(B)**

Illustration 6: If roots of the equation $(a-b)x^2 + (c-a)^x + (b-c) = 0$ are equal, then a, b, care in

(A) A.P. (B) H.P. (C) G.P. (D) none of these

Solution : $(a-b)x^2 + (c-a)x + (b-c) = 0$

As roots are equal so

$B^2 - 4AC = 0$

$\Rightarrow (c-a)^2 - 4(a-b)(b-c) = 0$

$\Rightarrow (c-a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow (c-a)^2 + 4ac - 4b(c+a) + 4b^2 = 0$

$\Rightarrow (c+a)^2 - 2\cdot(2b)(c+a) + (2b)^2 = 0$

$\Rightarrow [c+a-2b]^2 = 0 \Rightarrow c+a-2b = 0$

$\Rightarrow c + a = 2b$

Hence a, b, c are in A. P.

Alternative method :

$\because$ Sum of the coefficients $= 0$

Illustration 6: If roots of the equation $(a-b)x^2 + (c-a)^x + (b-c) = 0$ are equal, then a, b, care in

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Solution : $(a-b)x^2 + (c-a)x + (b-c) = 0$

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$\Rightarrow (c+a)^2 - 2\cdot(2b)(c+a) + (2b)^2 = 0$

$\Rightarrow [c+a-2b]^2 = 0 \Rightarrow c+a-2b = 0$

$\Rightarrow c+a = 2b$

Hence a, b, c are in A. P.

Alternative method :

$\because$ Sum of the coefficients $= 0$

Hence one root is 1 and other root is $\frac{b-c}{a-b}$.

Given that both roots are equal, so

$$1 = \frac{b-c}{a-b} \Rightarrow a-b = b-c \;\Rightarrow\; 2b = a+c$$

Hence a, b, c are in A.P.

Ans. **(A)**

Do yourself - 3 :

(i) Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \frac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b&c, if $\alpha < 0 < \beta \,\&\, |\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

5. IDENTITY :

An equation which is true for every value of the variable within the domain is called an identity, for example : $5(a-3) = 5a - 15$, $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note :A quadratic equation cannot have three or more roots & if it has, it becomes an identity. If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

Illustration 7 : If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

Solution : As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = 2,3$

and $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \quad \lambda = 1,2$

and $\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda = 2, -2$

So $\lambda =$ Ans. $= 2$

7. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Only one common root.

Let α be the common root of $ax^2+bx+c=0 \& a'x^2+b'x+c'=0$ then $a\alpha^2+b\alpha+c=0 \& a'\alpha^2+b'\alpha+c'=0$.

By Cramer's Rule $\frac{\alpha^2}{bc'-b'c}=\frac{\alpha^2}{a'c-ac'}=\frac{1}{ab'-a'b}$

Therefore, $\alpha=\frac{ca'-c'a}{ab'-a'b}-\frac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is $(ca'-c'a)^2=(ab'-a'b)(bc'-b'c)$.

(b) If both roots are same then $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

Illustration 8: Find p and q such that $px^2+5x+2=0$ and $3x^2+10x+q=0$ have both roots in common.

Solution : $a_1=p, b_1=5, c_1=2$

$a_2=3,\ b_2=10, c_2=q$

We know that :

$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \Rightarrow \frac{p}{3}=\frac{5}{10}=\frac{2}{q}$

$\Rightarrow \pi=\frac{3}{2}; q=4$

Illustration 9: Find the possible value(s) of a for which the equations $x^2+ax+1=0$ and $x^2+x+a=0$ have atleast one common root.

Solution : Let α is a common root

then $\alpha^2+a\alpha+1=0$

$\&\alpha^2+\alpha+a=0$

by Cramer's rule

$\frac{\alpha^2}{a^2-1}=\frac{\alpha}{1-a}=\frac{1}{1-a}$

$\Rightarrow (1-a)2=(a-1)(1-a)$

$\Rightarrow a=1,-2$

Do yourself - 4 :

(i) If $x^2+bx+c=0 \& 2x^2+9x+10=0$ have both roots in common, find b&c.

(ii) If $x^2-7x+10=0 \& x^2-5x+c=0$ have a common root, find c.

(iii) Show that $x^2+(a^2-2)x-2a=0$ and $x^2-3x+2=0$ have exactly one common root for all $a \in R$.

7. REMAINDER THEOREM :

If we divide a polynomial $f(x)$ by $(x-\alpha)$ the remainder obtained is $f(\alpha)$. If $f(\alpha)$ is 0 then $(x-\alpha)$ is a factor of $f(x)$.

Consider $f(x)=x^3-9x^2+23x-15$

If $f(1)=0 \Rightarrow (x-1)$ is a factor of $f(x)$.

If $f(x)=(x-2)(x^2-7x+9)+3$. Hence $f(2)=3$ is remainder when $f(x)$ is divided by $(x-2)$.

Illustrations 10 : A polynomial in x of degree greater than three, leaves remainders 2,1 and -1 when divided, respectively, by $(x-1)$, $(x+2)$ and $(x+1)$. What will be the remainder when it is divided by $(x-1)(x+2)(x+1)$.

Solution : Let required polynomial be $f(x)=p(x)(x-1)(x+2)(x+1)+a_0x^2+a_1x+a_2$ By remainder theorem, $f(1)=2, f(-2)=1, f(-1)=-1$.

$\Rightarrow a_0+a_1+a_2=2$

$4a_0-2a_1+a_2=1$

$a_0-a_1+a_2=-1$

Solving we get, $a_0=\frac{7}{6}, a_1=\frac{3}{2}, a_2=\frac{2}{3}$

Remainder when $f(x)$ is divided by $(x-1)(x+2)(x+1)$ will be $\frac{7}{6}x^2+\frac{3}{2}x+\frac{2}{3}$

8. SOLUTION OF RATIONAL INEQUALITIES :

Let $y=\frac{f(x)}{g(x)}$ be an expression in x where $f(x)\&g(x)$ are polynomials in x. Now, if it is given that $y>0$ (or <0 or ≥ 0 or ≤ 0), this calls for all the values of x for which y satisfies the constraint. This solution set can be found by following steps :

Step I : Factorize $f(x)\&$ $g(x)$ and generate the form :

$$y=\frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\ldots\ldots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots\ldots(x-b_p)^{m_p}}$$

where $n_1n_2\ldots\ldots n_k, m_1, m_2\ldots\ldots$ mp are natural numbers and $a_1, a_2\ldots. a_k, b_1, b_2\ldots. b_p$ are real numbers. Clearly, here $a_1, a_2\ldots\ldots$ ak are roots of $f(x)=0\&b_1, b_2, \ldots. b_p$ are roots of $g(x)=0$.

Step II : Here y vanishes (becomes zero) for $a_1, a_2, \ldots\ldots\ldots a_k$. These points are marked on the number line with a black dot. They are solution of $y=0$.

If $g(x)=0, y=\frac{f(x)}{g(x)}$ attains an undefined form, hence $b_1, b_2\ldots. b_k$ are excluded from the solution. These points are marked with white dots.

e.g. $f(x)=\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$

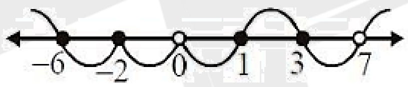
Step-III : Check the value of y for any real number greater than the right most marked number on the number line. If it is positive, then y is positive for all the real numbers greater than the right most marked number and vice versa.

Step-IV : If the exponent of a factor is odd, then the point is called simple point and if the exponent of a factor is even, then the point is called double point

$$\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$

Here 1,3, -6 and 7 are simple points and -2&0 are double points. From right to left, beginning above the number line (if y is positive in step 3 otherwise from below the line), a wavy curve should be drawn which passes through all the marked points so that when passing through a simple point, the curve intersects the number line and when passing through a double point, the curve remains on the same side of number line.

$$f(x) = \frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$



As exponents of $(x+2)$ and x are even, the curve does not cross the number line. This method is called wavy curve method.

Step-V : The intervals where the curve is above number line, y will be positive and the intervals where the curve is below the number line, y will be negative. The appropriate intervals are chosen in accordance with the sign of inequality & their union represents the solution of inequality.

Note :

(i) Points where denominator is zero will never be included in the answer.

(ii) If you are asked to find the intervals where f(x) is non-negative or non-positive then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

(iii) Normally we cannot cross-multiply in inequalities. But we cross multiply if we are sure that quantity in denominator is always positive.

(iv) Normally we cannot square in inequalities. But we can square if we are sure that both sides are non negative.

(v) We can multiply both sides with a negative number by changing the sign of inequality.

(vi) We can add or subtract equal quantity to both sides of inequalities without changing the sign of inequality.

Illustration 11: Find x such that $3x2 - 7x + 6 < 0$

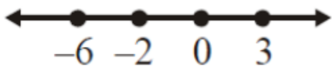
Solution : $D = 49 - 72 < 0$

As $D < 0, 3x^2 - 7x + 6$ will always be positive. Hence $x \in \phi$.

Illustration 12 : $(x^2 - x - 6)(x^2 + 6x) \geq 0$

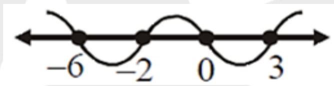
Solution : $(x-3)(x+2)(x)(x+6) \geq 0$

Consider $E = x(x-3)(x+2)(x+6), E = 0 \Rightarrow x = 0, 3, -2, -6$ (all are simple points)



For $x \geq 3 E = \underbrace{x}_{+ve}\ \underbrace{}_{+ve}\ (x-3)\ \underbrace{(x+2)}_{+ve}\ \underbrace{(x+6)}_{+ve}$

$=$ positive



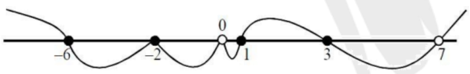
Hence for $x(x-3)(x+2)(x+6) \geq 0$ $x \in (-\infty, -6] \cup [-2, 0] \cup [3, \infty)$

Illustration 13 : Let $f(x) = \frac{(x-1)^3 + (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3}$. Solve the following inequality

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution: We mark on the number line zeros of the function : $1, -2, 3$ and -6 (with black and circles)and the points of discontinuity 0 and 7 (with white circles), isolate the double points : -2 and 0

draw the wavy curve :



From graph, we get

(i) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(ii) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$

(iii) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(iv) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Do yourself - 5 :

(i) Find range of x such that

(a) $(x-2)(x+3) \geq 0$ **(b)** $\frac{x}{x+1} > 2$

(c) $\frac{3x-1}{4x+1} \leq 0$ **(d)** $\frac{(2x-1)(x+3)(2-x)(1-x)^2}{x^4(x+6)(x-9)(2x^2+4x+9)} < 0$

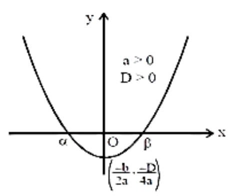
(e) $\frac{7x-17}{x^2-3x+4} \geq 1$ **(f)** $x^2+2 \leq 3x \leq 2x^2-5$

9. QUADRATIC EXPRESSION AND IT'S GRAPHS :

Consider the quadratic expression, $y = ax^2+bx+c, a \neq 0 \& a, b, c \in R$ then ;

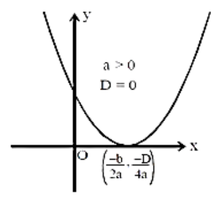
(a) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

Fig. 1



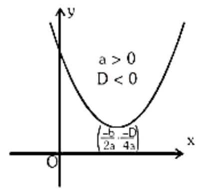
Roots are real & distinct
$ax^2+bx+c>0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$
$ax^2+bx+c<0 \ \forall \ x \in (\alpha, \beta)$

Fig. 2



Roots are coincident
$ax^2+bx+c>0 \forall x \in R-\{\alpha\}$
$ax^2+bx+c=0$ for $x = \alpha = \beta$

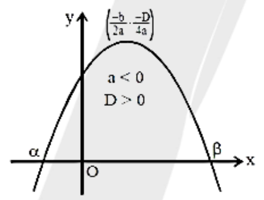
Fig.3



Roots are complex conjugate
$ax^2+bx+c>0 \ \forall x \in R$

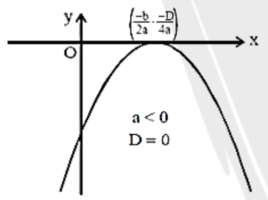
(b) The graph of $y = ax^2+bx+c$ can be divided in 6 broad categories which are as follows: (Let the real roots of quadratic equation $ax^2+bx+c=0$ be $\alpha \& \beta$ where $\alpha \geq \beta$).

Fig. 4



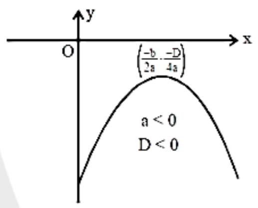
Roots are real & distinct
$ax^2+bx+c>0 \ \forall \ x \in (\alpha, \beta)$
$ax^2+bx+c<0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$

Fig. 5



Roots are coincident
$ax^2+bx+c<0 \ \forall \ x \in R-\{\alpha\}$
$ax^2+bx+c=0$ for $x = \alpha = \beta$

Fig.6



Roots are complex conjugate
$ax^2+bx+c<0 \ \forall \ x \in R$

Important Note :

(i) The quadratic expression $ax^2+bx+c>0$ for each $x \in R \Rightarrow a > 0, D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2+bx+c<0$ for each $x \in R \Rightarrow a < 0, D < 0$ & vice-versa (Fig. 6)

10. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS :

$y = ax^2 + bx + c$:

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2-4ac)}{4a^2}\right]$, which is a parabola.

$\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex ; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression $ax^2 + bx + c$ is a perfect square, then $a > 0$ and $D = 0$

Illustration 14 : The value of the expression $x^2 + 2bx + c$ will be positive for all real x if -

(A) $b^2 - 4c > 0$ (B) $b^2 - 4c < 0$ (C) $c^2 < b$ (D) $b^2 < c$

Solution : As $a > 0$, so this expression will be positive if $D < 0$ so $4b^2 - 4c < 0$

$b^2 < c$ Ans. (D)

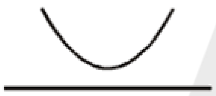


Illustration 15: The minimum value of the expression $4x^2 + 2x + 1$ is -

(A) 1/4 (B) ½ (C) ¾ (D) 1

Solution : Since $a = 4 > 0$

therefore its minimum value $= -\frac{D}{4a} = \frac{4(4)(1)-(2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$ Ans.(C)

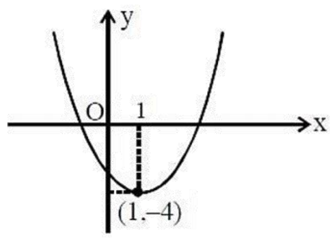
Illustration 16: If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

Solution : We know that minimum value of y will occur at

$x = -\frac{b}{2a} = -\frac{(-2)}{2\times 1} = 1$

$y_{min} = -\frac{D}{4a} = \frac{-(4+3\times 4)}{4} = -4$



(i) $x \in R$;

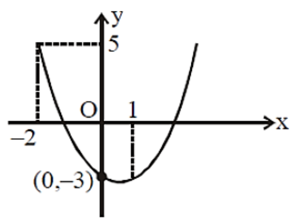
$y \in [-4, \infty)$ Ans.

(ii) $x \in [0,3]$

$f(0) = -3, f(1) = -4,\ f(3) = 0$

$\because f(3) > f(0)$

$\therefore$ y will take all the values from minimum to f(3).



$y \in [-4,0]$ Ans.

(iii) $x \in [-2,0]$

This interval does not contain the minimum x value of y for $x \in R$.

y will take values from $f(0)$ to $f(-2)$

$f(0) = -3$

$f(-2) = 5$

$y \in [-3,5]$ Ans.

Illustration 17: If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of $5a - b$.

Solution : Either $f(x) \geq 0 \forall x \in R$ or $f(x) \leq 0 \forall x \in R$

$\because f(0) = 10 > 0 \Rightarrow f(x) \geq x \in R$

$\Rightarrow f(-5) = 25a - 5b + 10 \geq 0$

$\Rightarrow 5a - b \geq -2$

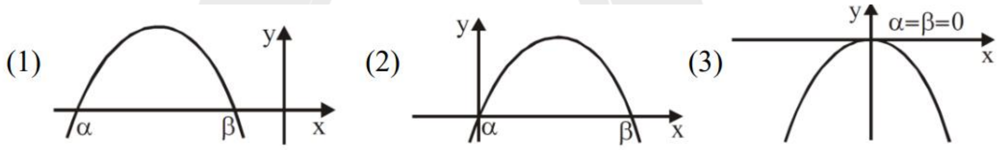
Do yourself - 6

(i) Find the minimum value of :

(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

(ii) For following graphs of $y = ax^2 + bx + c$ with $a, b, c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$



(iii) Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a, b, c \in R^+$, then the vertex of the graph will lie in which quadrant.

(iv) Find the range of 'a' for which:

(a) $ax^2 + 3x + 4 > 0 \forall x \in R$ (b) $ax^2 + 4x - 2 < 0 \forall x \in R$

11. MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS:

$$y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}, \frac{1}{ax^2 + bx + c}, \frac{a_1x + b_1}{a_2x^2 + b_2x + c_2}, \frac{a_1x^2 + b_1x + c_1}{a_2x + b_2};$$

Sometime we have to find range of expression of form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$.

The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.

$$(a_1 - a_2y)x^2 + (b1 - b_2y)x + (c_1 - c_2y) = 0$$

Step 3 : Put Discriminate ≥ 0 and solve the inequality for possible set of values of y.

Illustration 18 : For $x \in R$, find the set of values attainable by $\frac{x^2-3x+4}{x^2+3x+4}$

Solution : Let $y = \frac{x^2-3x+4}{x^2+3x+4}$

$x^2(y-1) + 3x(y+1) + 4(y-1) = 0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R, D \geq 0$

$\Rightarrow 9(y+1)^2 - 16(y-1)^2 \geq 0 \Rightarrow 7y^2 - 50y + 7 \leq 0$

$\Rightarrow (7y-1)(y-7) \leq 0 \ \Rightarrow y \in \left[\frac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$$\Rightarrow 1 = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

$\Rightarrow x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

Illustration 19: Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

Solution : Let $y = \frac{ax^2+3x-4}{3x-4x^2+a}$

$x^2(a+4y) + 3(1-y)x - (4+ay) = 0$

If $x \in R, D \geq 0$

$\Rightarrow 9(1-y)^2 + 4(a+4y)(4+ay) \geq 0$

$\Rightarrow (9+16a)y^2 + (4a^2+46)y + (9+16a) \geq 0$

for all $y \in R, (9+16a) > 0 \& D \leq 0$

$\Rightarrow (4a^2+46)^2 - 4(9+16a)(9+16a) \leq 0 \Rightarrow 4(a^2-8a+7)(a^2+8a+16) \leq 0$

$\Rightarrow a^2 - 8a + 7 \leq 0 \ \Rightarrow \ 1 \leq a \leq 7$

$9 + 16a > 0 \& 1 \leq a \leq 7$

Taking intersection, $a \in [1,7]$

Now, checking the boundary values of a

For $a = 1$

$$y = \frac{x^2 + 3x - 4}{3x - 4x^2 + 1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because x \neq 1 \;\Rightarrow\; y \neq -1$

$\Rightarrow a = 1$ is not possible.

if $a = 73$

$$y = \frac{7x^2 + 3x - 4}{3x - 4x^2 + 7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \quad \because \quad \because\; x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $a \in (1,7)$

Do yourself - 7 :

(i) Prove that the expression $\frac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4 , in its domain

(ii) Find the range of $\frac{x^2+2x+1}{x^2+2x+7}$, where x is real

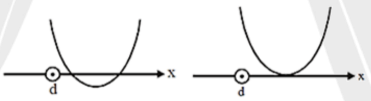
12. LOCATION OF ROOTS :

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation $ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$

Type-1 : Both roots of the quadratic equation are greater than a specific number (say d). The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\frac{b}{2a} > d$

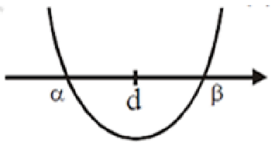


Note : When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-b\frac{b}{2a} < d$

Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than ' d ' or ' d ' lies between the roots of the given equation. The necessary and sufficient condition for this are : $f(d) < 0$

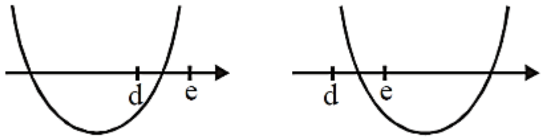


Note: Consideration of discriminate is not needed.

Type-3 :

Exactly one root lies in the interval (d, e). The necessary and sufficient condition for this are :

$f(d) \cdot f(e) < 0$



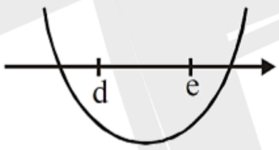
Type-4 :

When both roots are confined between the number d and e $(d < e)$. The necessary and sufficient condition for this are:

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$ (iv) $d < -\frac{b}{2a} < e$

Type-5 :

One root is greater than e and the other roots is less than $d(d < e)$. The necessary and sufficient condition for this are : $f(d) < 0$ and $f(e) < 0$



Note : If $a < 0$ in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by ' a '. Now assume $x^2 + \frac{b}{a}x + \frac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustration 20 : Find the values of the parameter ' a ' for which the roots of the quadratic equation $x^2 + 2(a-1)x + a + 5 = 0$ are

(i) real and distinct (ii) equal

(iii) opposite in sign (iv) equal in magnitude but opposite in sign

(v) positive (vi) negative

(vii) greater than 3 (viii) smaller than 3

(ix) such that both the roots lie in the interval (1,3)

Solution : Let $f(x) = x^2 + 2(a-1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow A = 1, B = 2(a-1), C = a + 5.$

Also $D = B^2 - 4AC = 4(a-1)^2 - 4(a+5) = 4(a+1)(a-4)$

(i) $D > 0 \Rightarrow (a+1)(a-4) > 0 \Rightarrow a \in (-\infty, -1) \cup (4, \infty)$.

(ii) $D=0 \Rightarrow (a+1)(a-4)=0 \Rightarrow a=-1,4$.

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow\ f(0)<0$ and $D>0$ i.e. $a \in (-\infty,-1) \cup (4,\infty)$

$\Rightarrow\ a+5<0 \Rightarrow a<-5 \Rightarrow a \in (-\infty,-5)$.

(iv) This means that the sum of the roots is zero $\Rightarrow\ -2(a-1)=0$ and $D>0$ i.e.

$a \in -(-\infty,-1) \cup (4,\infty) a=1$ which does not belong to $(-\infty,-1) \cup (4,\infty) \Rightarrow a \in \phi$.

(v) This implies that both the roots are greater than zero

$\Rightarrow\ -\frac{B}{A}>0, \frac{C}{A}>D \geq 0 \Rightarrow -(a-1)>0, a+5>0, a \in (-\infty,-1] \cup [4,\infty)$

$\Rightarrow\ a<1,-5<a, a \in (-\infty,-1] \cup [4,\infty) \Rightarrow\ a \in (-5,-1]$.

(vi) This implies that both the roots are less than zero

$\Rightarrow\ \frac{B}{A}<0, \frac{C}{A}>0, D \geq 0 \Rightarrow -(a-1)<0, a+5>0, a \in (-\infty,-1] \cup [4,\infty)$

$\Rightarrow\ a>1, a>-5, a \in (-\infty,-1] \cup [4,\infty)\ \Rightarrow\ a \in [4,\infty)$.

(vii) In this case $-\frac{B}{2a}>$, A.f(3) >0 and $D \geq 0$.

$\Rightarrow\ -(a-1)>3, 7a+8>0$ and $a \in (-\infty,-1] \cup [4,\infty)$

$\Rightarrow\ a<-2, a>-8/7$ and $a \in (-\infty,-1] \cup [4,\infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3 .

(viii) In this case

$-\frac{B}{2a}<3$, A.f (3) >0 and $D \geq 0$

$\Rightarrow\ a>-2, a>-8/7$ and $a \in (-\infty,-1] \cup [4,\infty) \Rightarrow a \in (-8/7,-1] \cup [4,\infty)$

(ix) In this case

$1<-\frac{B}{2\,A}<3$, A.f (1) >0, A.f (3) >0, $D \geq 0$

$\Rightarrow\ 1<-1(a-1)<3, 3a+4>0, 7a+8>0, a \in (-\infty,-1] \cup [4,\infty)$

$\Rightarrow\ -2<a<0, a>-4/3, a>-8/7, a \in (-\infty,-1] \cup [4,\infty) \Rightarrow a \in \left(-\frac{8}{7},-1\right)$

Illustration 21 : Find value of k for which one root of equation $x^2-(k+1)x+k^2+k-8=0$ exceeds

2& other is less than 2.

Solution : $4-2(k+1)+k^2+k-8<0 \Rightarrow k^2-k-6<0$

$(k-3)(k+2)<0 \Rightarrow -2<k<3$

Taking intersection, $k \in (-2,3)$.

Illustration 22 : Find all possible values of a for which exactly one root of $x^2 - (a+1)x + 2a = 0$ lies In interval (0,3).

Solution : $f(0).f(3) < 0$

$\Rightarrow 2a(9 - 3(a+1) + 2a) < 0 \Rightarrow 2a(-a+6) < 0$

$\Rightarrow a(a-6) > 0 \Rightarrow a < 0$ or $a > 6$

Checking the extremes.

If $a = 0,\ x^2 - x = 0$

$x = 0,1\ ,\ 1 \in (0,3)$

If $a = 6$,

$x^2 - 7x + 12 = 0, x = 3,4$ But $4 \notin (0,3)$

Hence solution set is

$a \in (-\infty, 0] \cup (6, \infty)$

Do yourself - 8 :

(i) If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:

(a) -3 (b) 0 (c) 1

(ii) If $a > 1$, roots of the equation $(1-a)x^2 + 3ax - 1 = 0$ are -

(A) one positive one negative (B) both negative

(C) both positive (D) both non-real

(iii) Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3 .

(iv) If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

(v) If α, β are roots of $4x^2 - 16x + \lambda = 0, \lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

13. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Illustration 23 : If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

(A) 6,2 (B) $-6,2$ (C) $6,-2$ (D) $-6,-2$

Solution : Here $a = 1, h = 1, b = 0, g = 1, f = m/2, c = -3$

$$\text{So } \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0 \Rightarrow -\frac{m^2}{4} + m + 3 = 0$$

$\Rightarrow m^2 - 4m - 12 = 0 \Rightarrow m = -2,6$

Ans. **(C)**

Do yourself - 9 :

(i) Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.

14. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots . a_{n-1}x +$ an $= 0$, where $a_0, a_1, \ldots \ldots . . a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - a_3) \ldots \ldots . . (x - \alpha_n)$

$\therefore\ a_0x_n + a_1x^{n-1} + \cdots \ldots \ldots . a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots . . (x - \alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = \frac{a_1}{a_0} = S_1 \text{ (say)}$$

or $S_1 = \frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$.

$S_2 = \sum_{i \neq j} \alpha_i\alpha_j = (-1)^2 \frac{a_2}{a_0}$

$S3 = \sum_{j \neq j \neq k} \alpha_i\alpha_j\alpha_k(-1)^3 \frac{a_3}{a_0}$

$$S_n = \alpha_1\alpha_2 \ldots \ldots \ldots . . \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constan term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

Quadratic equation :If α. β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$b\alpha + \beta = \frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Cubic equation :If α. β, γare roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \text{ and } \alpha\beta\gamma = \frac{d}{a}$$

Note :

(i) If α is a root of the equation $f(x) = 0$, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

(ii) Every equation of nth degree $(n \geq 1)$ has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha + i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) If the coefficients in the equation are all rational &$\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$&β is not a perfect square.

(v) If there be any two real numbers ' a ' & 'b' such that f(a)&f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between ' a ' and ' b '.

(vi) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

Illustration 24: If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$\therefore \alpha \cdot \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \Rightarrow \alpha^2 + 2\alpha\beta = -\frac{23}{4} \& \alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \Rightarrow \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4}$

$\Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \alpha = 1/2, -\frac{23}{6}$

when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5 - 1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6}$

$$\Rightarrow \alpha^2\beta = \frac{23 \times 23}{36}\left(-5 - 2x\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2} \Rightarrow \alpha = \frac{1}{2}\beta = -6$$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$

Illustration 25: If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum a^3$ (ii) $\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)$

Solution : We know that $\alpha + \beta + \gamma = p,\ \alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = r$

(i) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) = \alpha^2(p - \alpha) + \beta^2(p - \beta) + \gamma^2(p - \gamma)$

$= p(\alpha^2 + \beta^2 + \gamma 2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Illustration 26: If $b^2 < 2ac$ and $a, b, c, d \in R$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

$\Rightarrow \alpha^2+\beta^2+\gamma^2<0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two nonreal and one real roots.

Do yourself - 10 :

(i) Let α, β be two of the roots of the equation $x^3-px^2+qx-r=0$. If $\alpha+\beta=0$, then show that $pq=r$

(ii) If two roots of $x^3+3x^2-9x+c=0$ are equal, then find the value of c.

(iii) If α, β, γbe the roots of $ax3+bx^2+cx+d=0$, then find the value of

(a) $\Sigma\alpha^2$ (b) $\sum\frac{1}{\alpha}$ (c) $\sum\alpha^2(\beta+\gamma)$

15. TRANSFORMATION OF THE EQUATION :

Let $ax^2+bx+c=0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in $\alpha\&\beta$, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$y=f(\alpha)$

By transformation, $\alpha=g(y)$

$a(g(y))^2+b(g(y))+c=0$

This is the required equation in y.

Illustration 27 : If the roots of $ax^2+bx+c=0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution : (a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ put, $y=\frac{-2}{\alpha} \Rightarrow \alpha=\frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2+b\left(-\frac{2}{y}\right)+c=0 \Rightarrow cy^2-2by+4a=0$

Required equation is $cx^2-2bx+4a=0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

$$\text{put, } y=\frac{\alpha}{\alpha+1} \Rightarrow \alpha=\frac{y}{1-y}$$

$\Rightarrow a\left(\frac{y}{1-y}\right)^2+b\left(\frac{y}{1-y}\right)+c=0$

$\Rightarrow (a+c-b)y^2+(-2c+b)y+c=0$

Required equation is $(a+c-b)x^2+(b-2c)x+c=0$

(c) α^2, β^2

put $y=\alpha^2$

$\Rightarrow \alpha=y$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

Illustration 28 : If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$

Solution : Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d}\left(\because \alpha\beta\gamma = -\frac{d}{a}\right)$

Put $x = - -\frac{dy}{a}$

$\Rightarrow a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$

Required equation is $d^2x^3 - bbx^2 + acx - a^2 = 0$

Do yourself - 11 :

(i) If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (B) $\frac{1}{2\alpha+b}, \frac{1}{a\beta+b}$ (c) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

(ii) If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^2\beta^3 + \alpha^3\beta^2$.

Miscellaneous Illustrations :

Illustrations 29: If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution : α, β are the roots of $x^2 + px + q = 0$

$\therefore \alpha + \beta = -p, \alpha\beta = q$ ....(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \gamma + \delta = -r, \gamma\delta = s \ldots\ldots\ldots (2)$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s\left((\alpha + \beta)^2 - 2\alpha\beta\right) + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq)$

For a common root (Let $\alpha = \gamma$ or $\beta = \delta$) ....(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ ....(4)

from (3) and (4), we get

$(q - s)^2 + (p - r)(sp - rq) = 0$

$\Rightarrow (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

Illustrations 30: If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution : $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \forall x \in R$

$\Rightarrow y^2 - 5y + 3 < \frac{2x}{x^2+x+1}$

Let $\frac{2x}{x^2+x+1} = P$

$\Rightarrow px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow -2 \leq p \leqq \frac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is -2 .

So, $y^2 - 5y + 3 < -2 \Rightarrow y^2 - 5y + 5 < 0$

$\Rightarrow y \in \left(\frac{5 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right)$

ANSWERS FOR DO YOURSELF

1. (i) (a) $-1, -2$; (b) 4; (c) $1 \pm \sqrt{2}$; (ii) $a, \frac{1}{a}$; (iii) $\frac{7}{3}$ (iv) $3, -\frac{1}{5}$

2. (i) $b = -4, c = 1$ (ii) (a)imaginary; (b) real & distinct ; (c) real & coincident

3. (i) (a) $c = 0$; (b) $c = 1$ (c) $b \to$ negative, $c \to$ negative

4. (i) $b\frac{9}{2}, c = 5$; (ii) $c = 0,6$

5. (i) (a) $x \in (-\infty, -3] \cup [2, \infty)$; (b) $x \in (-2, -1)$; (c) $\left(-\frac{1}{4}, \frac{1}{3}\right)$;

(d) $x \in (-6, -3) \cup \left(\frac{1}{2}, 2\right) - \{1\} \cup (9, \infty)$ (e) $[3,7]$ (f) ϕ

6. (i) (a) 1(b) -1 (ii) (1) (i) $a < 0$(ii) $b < 0$(iii) $c < 0$ (iv) $D > 0$ (v) $\alpha + \beta < 0$ (vi) $\alpha\beta > 0$

(2) (i) $a < 0$(ii) $b > 0$ (iii) $c = 0$ (iv) $D > 0$ (v) $\alpha + \beta > 0$ (vi) $\alpha\beta = 0$

(3) (i) $a < 0$ (ii) $b = 0$ (iii) $c = 0$ (iv) $D = 0$ (v) $\alpha + \beta = 0$ (vi) $\alpha\beta = 0$

(iii) Third quadrant

(iv) (a) $a > 9/16$ (b) $a < -2$

7. (ii) least value $= 0$, greatest value $= 1$.

8. (i) $-3 < \alpha < 0 < \beta < 1$;(ii) C; (iii) $a < 2$; (iv) $a < 2$ (v) $12 < \lambda < 16$

9. (i) 0,2

10. (ii) $-27,5$ (iii) (a) $\frac{1}{a^2}(b^2 - ac)$ (b) $-\frac{c}{d}$ (c) $\frac{1}{a^2}(3ad - bc)$

11. (i) (a) $c^2y^2 + y(2ac - b^2) + a^2 = 0$; (b) $acx^2 - bx + 1 = 0$; (c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

(ii) $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

EXERCISE – 1

1. If the roots of the equation $6x^2 - 7x + k = 0$ are rational, then k is equal to-

(A) -1 (B) $-1, -2$ (C) -2 (D) 1,2

2. If the equation $x^2 - m(2x - 8) - 15 = 0$ has equal roots, then $m =$

(A) $3, -5$ (B) $-3, 5$ (C) $3, 5$ (D) $-3, -5$

3. If α, β are roots of the equation $ax^2 - bx - c = 0$, then $\alpha^2 - \alpha\beta + \beta^2$ is equal to-

(A) $\frac{b^2+3ac}{a^2}$ (B) $\frac{b^2-3ac}{a^2}$ (C) $\frac{b^2+2ac}{a^2}$ (D) $\frac{b^2-2ac}{a^2}$

4. The roots of the equation $ax^2 + bx + c = 0$ will be imaginary if-

(A) $a > 0, b = 0, c < 0$ (B) $a > 0, b = 0, c > 0$ '

(C) $a = 0, b > 0, c > 0$ (D) $a > 0, b > 0, c = 0$

5. If α, β are roots of the equation $x^2 + px + q = 0$, then the equation whose roots are $\frac{q}{\alpha}, \frac{q}{\beta}$ will be

(A) $x^2 - qx + p = 0$ (B) $x^2 + px + q = 0$

(C) $x^2 - px - q = 0$ (D) $qx^2 + px + q = 0$

6. If p, q are the roots of equation $x^2 + px + q = 0$, then value of p must be equal to-

(A) 0,1 (B) 1 (C) 2 (D) $0, -1$

7. If α, β are the roots of the equation $a^2 + bx + c = 0$, then $\frac{\alpha}{a\beta+b} + \frac{\beta}{a\alpha+b}$ is equal to -

(A) $\frac{2}{2}$ (B) $\frac{2}{b}$ (C) $\frac{2}{c}$ (D) $-\frac{2}{a}$

8. If the roots of the equation $ax^2 + x + b = 0$ be real and different, then the roots of the equation $x^2 - 4\sqrt{abx} + 1 = 0$ will be-

(A) Rational (B) Irrational (C) Real (D) Imaginary

9. If α and β are roots of $x^2 - 2x + 3 = 0$, then the equation whose roots are $\frac{\alpha-1}{\alpha+1}$ and $\frac{\beta-1}{\beta+1}$ will be-

(A) $3x^2 - 2x - 1 = 0$ (B) $3x^2 + 2x + 1 = 0$

(C) $3x^2 - 2x + 1 = 0$ (D) $x^2 - 3x + 1 = 0$

10. If α, β are the roots of the equation $x^2 - 3x + 1 = 0$, then the equation with roots $\frac{1}{\alpha-2}, \frac{1}{\beta-2}$ will be-

(A) $x^2 - x - 1 = 0$ (B) $x^2 + x - 1 = 0$

(C) $x^2 + x + 2 = 0$ (D) None of these

11. The least integral value α of x such that $\frac{x-5}{x^2+5x-14} > 0$, satisfies

(A) $\alpha^2 - 7\alpha + 6 = 0$ (B) $\alpha^2 + 3\alpha - 4 = 0$

(C) $\alpha^2 + 5\alpha - 6 = 0$ (D) $\alpha^2 - 5\alpha + 4 = 0$

12. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ is 7 , then the roots of the equation are real if k equals-

(A) 1 (B) 2 (C) -2 (D) pm 2

13. If α and β are roots of the equation $x^2 + px + \frac{3p}{4} = 0$, such that $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set :-

(A) $\{2, -5\}$ (B) $\{-3,2\}$ (C) $\{3, -5\}$ (D) $\{-2,5\}$

14. If p and q are non-zero real numbers and $\alpha^3 + \beta^3 = -p, \alpha\beta = q$, then a quadratic equation whose roots are $\frac{\alpha^2}{\beta}, \frac{\beta^2}{\alpha}$ is -

(A) $qx^2 + px + q^2 = 0$ (B) $px^2 + qx + p^2 = 0$

(C) $qx^2 - px + q^2 = 0$ (D) $px^2 - qx + p^2 = 0$

15. If the equation $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ has roots equal in magnitude but opposite in sign, then m is equal to -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{b-a}{b+a}$ (D) None of these

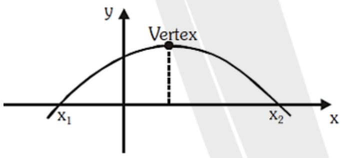
16. For what value of a the curve $y = x^2 + ax + 25$ touches the x-axis -

(A) 0 (B) pm 5 (C) pm 10 (D) none

17. The expression $a^2x^2 + bx + 1$ will be positive for all $x \in R$ if-

(A) $b^2 > 4a^2$ (B) $b^2 < 4a^2$ (C) $4b^2 > a^2$ (D) $4b^2 < a^2$

18. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then -



(A) $a > 0$ (B) $b > 0, c > 0$ (C) $c > 0, b < 0$ (D) $b^2 < 4ac$

19. If both the roots of the equations $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to-

(A) 1 (B) -1 (C) 2 (D) 0

20. All possible values of a, so that 6 lies between the roots of the equation $x^2 + 2(a - 3)x + 9 = 0$

(A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-\infty, -3/4)$

(C) $(2, \infty)$ (D)none of these

EXERCISE – 2

1. If α, β are roots of the equation $x^2 - px + q = 0$, then find the value of
(i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$ (ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$.

2. Find the value of a for which one root of the equation $x^2 + (2a-1)x + a^2 + 2 = 0$ is twice as large as the other.

3. Find a such that one of the roots of the equation $x^2 - \frac{15}{4}x + a = 0$ is the square of the other.

4. Find k in the equation $5x^2 - kx + 1 = 0$ such that the difference between the roots of the equation is unity.

5. Find b in the equation $5x^2 + bx - 28 = 0$ if the roots x_1 and x_2 of the equation are related as $5x_1 + 2x_2 = 1$ and b is an integer.

6. Find the values of the coefficient a for which the curve $y = x^2 + ax + 25$ touches the OX axis.

7. For what values of p does the vertex of the parabola $y = x^2 + 2px + 13$ lie at a distance of 5 from the origin?

8. If x_1, x_2 are the roots of $ax^2 + bx + c = 0$, then find the value of
(i) $(ax_1 + b)^{-2} + (ax_2 + b)^{-2}$ (ii) $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$.

9. α, β are the roots of the equation $K(x^2 - x) + x + 5 = 0$. If K_1& K_2 are the two values of K for which the roots α, β are connected by the relation $(\alpha/\beta) + (\beta/\alpha) = 4/5$.Find the value of $(K_1/K_2) + (K_2/K_1)$.

10. If α, β are the roots of $ax^2 + bx + c = 0, (a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0, (A \neq 0)$ for some constant δ, then prove that, $\frac{b^2-4ac}{a^2} = \frac{B^2-4ac}{A^2}$.

11. (a) If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then which of the following expressions in α, β will denote the symmetric functions of roots. Give proper reasoning.
(i) $f(\alpha, \beta) = \alpha^2 - \beta$ (ii) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$
(iii) $f(\alpha, \beta) = \ln \frac{\alpha}{\beta}$ (iv) $f(\alpha, \beta) = \cos(\alpha - \beta)$
(b) If α, β are the roots of the equation $x^2 - px + q = 0$, then find the quadratic equation the roots of which are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$&$\alpha^3\beta^2 + \alpha^2\beta^3$.

12. Solve following Inequalities over the set of real numbers -
(i) $\frac{x^2+2x-3}{x^2+1} < 0$ (ii) $\frac{(x-1)(x+2)^2}{-1-x} < 0$ (iii) $x^4 - 2x^2 - 63 \leq 0$ (iv) $\frac{x+1}{(x-1)^2} < 1$
(v) $\frac{x^2-7x+12}{2x^2+4x+5} > 0$ (vi) $\frac{x^2+6x-7}{x^2+1} \leq 2$ (vii) $\frac{x^4+x^2+1}{x^2-4x-5} < 0$ (viii) $\frac{x+7}{x-5} + \frac{3x+1}{2} \geq 0$
(ix) $\frac{1}{x+2} < \frac{3}{x-3}$ (x) $\frac{14x}{x+1} - \frac{9x-30}{x-4} < 0$ (xi) $\frac{x^2-5x+12}{x^2-4x+5} > 3$ (xii) $\frac{x^2+2}{x^2-1} < -2$
(xiii) $\frac{(2-x^2)(x-3)^3}{(x+1)(x^2-3x-4)} \geq 0$ (xiv) $\frac{5-4x}{3x^2-x-4} < 4$ (xv) $\frac{(x+2)(x^2-2x+1)}{4+3x-x^2} \geq 0$
(xvi) $\frac{x^4-3x^3+2x^2}{x^2-x-30} > \frac{1}{x}$ (xvii) $\frac{2x}{x^2-9} \leq \frac{1}{x+2}$ (xviii) $\frac{1}{x-2} + \frac{1}{x-1} > \frac{1}{x}$

(xix) $\frac{20}{(x-3)(x-4)}+\frac{10}{x-4}+1>0$ (xx) $\frac{(x-2)(x-4)(x-7)}{(x+2)(x+4)(x+7)}>1$

(xxi) $(x^2-2x)(2x-2)-9\frac{2x-2}{x^2-2x}\leq 0$

13. Find integral values of k for which the quadratic equation $(k-12)x^2+2(k-12)x+2=0$ possess no real roots?

14. For what values of k is the inequality $x^2-(k-3)x-k+6>0$ valid for all real x ?

15. Find all values of p for which the roots of the equation $(p-3)x^2-2px+5p=0$ are real and positive.

16. Find all values of a for which the inequality $(a+4)x^2-2ax+2a-6<0$ is satisfied for all $x\in R$

17. For what values of a do the graphs of the functions $y=2ax+1$ and $y=(a-6)x^2-2$ not intersect?

18. Find the range of values of a, such that $f(x)=\frac{ax^2+2(a+1)x+9a+4}{x^2-8x+32}$ is always negative.

19. Let the quadratic equation $x^2+3x-k=0$ has roots a, b and $x^2+3x-10=0$ has roots c, d such that modulus of difference of the roots of the first equation is equal to twice the modulus of the difference of the roots of the second equation. If the value of ' k ' can be expressed as rational number in the lowest form as mn then find the value of $(m+n)$.

20. Find the value of m for which the quadratic equations $x^2-11x+m=0$ and $x^2-14x+2m=0$ may have common root.

21. If the quadratic equations $x^2+bx+ca=0$ & $x^2+cx+ab=0$ (where $a\neq 0$) have a common root, prove that the equation containing their other root is $x^2+ax+bc=0$.

22. Consider the quadratic polynomial $f(x)=x^2-4ax+5a^2-6a$

(a) Find the smallest positive integral value of 'a' for which f(x) positive for every real x.

(b) Find the largest distance between the roots of the equation $f(x)=0$

(c) Find the set of values of 'a' for which range of f(x) is $[-8,\infty)$

23. We call ' p ' a good number if the inequality $\frac{2x^2+2x+3}{x^2+x+1}\leq p$ is satisfied for any real x. Find the smallest integral good number.

24. Let α, β and γ are the roots of the cubic $x^3-3x^2+1=0$. Find a cubic whose roots are $\frac{\alpha}{\alpha-2},\frac{\beta}{\beta-2}$ and $\frac{\gamma}{\gamma-2}$. Hence or otherwise find the value of $(\alpha-2)(\beta-2)(\gamma-2)$.

25. If x be real, then prove that $\frac{x}{x^2-5x+9}$ must lie between 1 and $-\frac{1}{11}$.

26. Find the greatest value of $\frac{x+2}{2x^2+3x+6}$ for real values of x.

27. For what values of m will the expression $y^2+2xy+2x+my-3$ be capable of resolution into two rational factors?

28. If x and y are two real quantities connected by the equation

$9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, then will x lie between 3 and 6 and y between 1 and 10 .

29. Find the complete set of real values of ' a ' for which both roots of the quadratic equation

$(a^2 - 6a + 5)x^2 - \sqrt{a^2 + 2a}x + (6a - a^2 - 8) = 0$ lie on either side of the origin.

30. Find all the values of the parameter 'a' for which both roots of the quadratic equation

$x^2 - ax + 2 = 0$ belong to the interval $(0,3)$.

EXERCISE – 3

1. If value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ as sum the least value is- **[AIEEE-2005]**

(A) 2 (B) 3 (C) 0 (D) 1

2. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4$ cequals-

(A) 1 (B) 2 (C) 3 (D) -2 **[AIEEE-2005]**

3. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5 , then k lies in the interval- **[AIEEE-2005]**

(A) [4,5] (B) $(-\infty, 4)$ (C) $(6, \infty)$ (D) (5,6)

4. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than2 but less than 4 , lie in the interval- **[AIEEE-2006]**

(A) $-1 < m < 3$ (B) $1 < m < 4$ (C) $-2 < m < 0$ (D) $m > 3$

5. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30^\circ$ and $\tan 15^\circ$, respectively then the value of $2 + q - p$ is- **[AIEEE-2006]**

(A) 0 (B) 1 (C) 2 (D) 3

6. If x is real, then maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is- **[AIEEE-2006]**

(A) 1 (B) $\frac{17}{7}$ (C) $\frac{1}{4}$ (D) 41

7. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the setoff possible values of a is **[AIEEE-2007]**

(A) $(-3, \infty)$ (B)$(3, \infty)$ (C) $(-\infty, -3)$ (D) $(-3,3)$

8. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4: 3. Then the common root is **[AIEEE-2008]**

(A) 1 (B) 4 (C) 3 (D) 2

9. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is :- **[AIEEE-2009]**

(A) Greater than $-4ab$ (B) Less than $-4ab$

(C) Greater than 4ab (D) Less than 4ab

10. Let for $a \neq a_1 \neq 0, f(x) = ax^2 + bx + c, g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of p(B) is: **[AIEEE-2011]**

(A) 18 (B) 3 (C) 9 (D) 6

11. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4,3). Rahul made a mistake in writing down coefficient of x to get roots (3,2). The correct roots of equation are: **[AIEEE-2011]**

(A) $-4, -3$ (B) 6,1 (C) 4,3 (D) $-6, -1$

12. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has : **[AIEEE-2012]**

(A) exactly four real roots. (B) infinite number of real roots.

(C) no real roots. (D) exactly one real root.

13. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is : **[JEE-MAIN-2013]**

(A) $1:2:3$ (B) $3:2:1$ (C) $1:3:2$ (D) $3:1:2$

14. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to : **[JEE-MAIN-2015]**

(A) 3 (B) -3 (C) 6 (D) -6

15. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x24\times60} = 1$ is :- **[JEE-MAIN-2016]**

(A) 5 (B) 3 (C) -4 (D) 6

EXERCISE – 4

1. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **[JEE 2005(Mains), 2]**

2. (a) Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real, then **[JEE 2006, 3]**

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$ (C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

(b) If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then find the value of $a+b+c+d$. (a, b, c and d are distinct numbers) **[JEE 2006, 6]**

3. (a) Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of ' r ' is

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

MATCH THE COLUMN :

(b) Let $f(x) = \frac{x^2-6x+5}{x^2-5x+6}$

Match the expressions / statements in Column I with expressions / statements in Column II.

Column I	Column II
(A) If $-1 < x < 1$, then f(x) satisfies	(P) $0 < f(x) < 1$
(B) If $1 < x < 2$, the f(x) satisfies	(Q) $f(x) < 0$
(C) If $3 < x < 5$, then f(x) satisfies	(R) $f(x) > 0$
(D) If $x > 5$, then f(x) satisfies	(S) $f(x) < 1$

[JEE 2007, 3(+6)]

ASSERTION & REASON :

4. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT-1 : $(p^2 - q)(b^2 - ac) \geq 0$ and

STATEMENT-2 bb $\neq$ pa or c $\neq$ qa

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True **[JEE 2008, 3(−1)]**

5. The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4 , is [JEE 2009, 4(−1)]

6. Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 \neq -q$. If α and β are non zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[JEE 2010,3]**

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

7. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[JEE 2011]**

(A) 1 (B) 2 (C) 3 (D) 4

8. A value of b for which the equations

$x^2 + bx - 1 = 0$

$x^2 + x + b = 0$

have one root in common is -

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$ **[JEE 2011]**

9. Let S be the set of all non-zero numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ?

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

10. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals **[JEE 2016, 3,-1]**

(A) $2(\sec\theta - \tan\theta)$(B) $2\sec\theta$

(C) $-2\tan\theta$ (D) 0

Paragraph for Q.No. 11 to 12

Let p, q, be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0,1,2, \ldots$, let $a_n = p\alpha^n + q\beta^n$.

FACT :If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

11. If $a_4 = 28$, then $p + 2q =$ **[JEE 2017, 3M, 0]**

(A) 12 (B) 14 (C) 21 (D) 7

12. $a_{12} =$ **[JEE 2017, 3M, 0]**

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + 2a_{10}$ (D) $a_{11} + a_{10}$

EXERCISE – 5

1. The graph of curve $x^2 = 3x - y - 2$ is strictly below the line $y = k$, then -

(A) $-2 < k < 4$ (B) $k > \frac{1}{4}$ (C) $k = \frac{1}{4}$ (D) $k < -1$ or $k > 0$

2. If $a + b + c > \frac{9c}{4}$ and quadratic equation $ax^2 + 2bx - 5c = 0$ has non-real roots, then-

(A) $a > 0, c > 0$ (B) $a > 0, c < 0$ (C) $a < 0, c < 0$ (D) $a < 0, c > 0$

3. Ramesh and Mahesh solve a quadratic equation. Ramesh reads its constant term wrongly and finds its roots as 8 and 2 where as Mahesh reads the coefficient of x wrongly and finds its roots as-11 and 1 . The correct roots of the equation are

(A) 11,1 (B) $-11,1$ (C) $11,-1$ (D) None of these

4. The set of values of K for which both the roots of the equation $4x^2 - 20Kx + (25K^2 + 15K - 66) = 0$, are less than 2, is given by-

(A) $(2, \infty)$ (B) $(4/5, 2)$ (C) $(-\infty, -1)$ (D) None of these

5. Let $P(x) = kx^3 + 2k^2x^2 + k^3$. Find the sum of all real numbers k for which $x - 2$ is a factor of $P(x)$.

(A) 4 (B) 8 (C) -4 (D) -8

More than one correct :

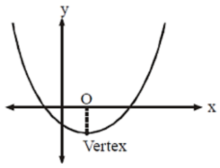
6. If $f(x) = x^2 + bx + c$ and $f(2 + t) = f(2 - t)$ for all real numbers t, then which of the following is true?

(A) $f(1) < f(2) < f(3)$ (B) $f(2) < f(1) < f(4)$
(C) $f(2) < f(4) < f(1)$ (D) $f(2.1) < f(1.5) < f(3)$

7. For $x \in [1,5]$, $y = x^2 - 5x + 3$ has -

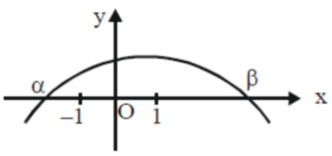
(A) least value $= -1.5$ (B) greatest value $= 3$
(C) least value $= -3.25$ (D) greatest value $= \frac{5+\sqrt{13}}{2}$

8. Graph of $y = ax^2 + bx + c$ is given adjacently. What conclusions can be drawn from this graph -



(A) $a > 0$ (B) $b < 0$ (C) $c < 0$ (D) $b^2 - 4ac > 0$

9. The graph of quadratic polynomial $f(x) = ax^2 + bx + c$ is shown below.



Which of the following are correct?

(A) $\frac{c}{a} < -1$ (B) $|\beta - \alpha| > 2$ (C) $f(x) > 0 \forall x \in (0, \beta)$ (D) $abc < 0$

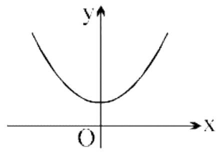
10. If S is the set of all real x such that $(2x-1)/(2x^3+3x^2+x)$ is positive, then S contains

(A) $(-\infty,-3/2)$ (B) $(-3/2,-1/4)$ (C) $(-1/4,1/2)$ (D) $(+1/2,3)$

11. If the quadratic equation $ax^2+bx+c=0(a>0)$hhas^2 θ and cosec^2 θ as its roots then which of the following must hold good?

(A) $b+c=0$ (B) $b^2-4ac\geq 0$ (C) $c\geq 4a$ (D) $4a+b\geq 0$

12. The graph of a quadratic polynomial $y=ax^2+bx+c(a,b,c\in R)$ with vertex on y-axis is as shown in the figure. Then which one of the following statement is CORRECT?



(A) Product of the roots of the corresponding quadratic equation is positive.

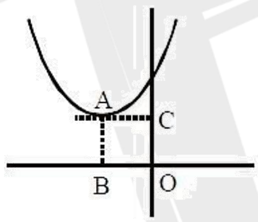
(B) Discriminate of the quadratic equation is negative.

(C) Nothing definite can be said about the sum of the roots, whether positive, negative or zero.

(D) Both roots of the quadratic equation $y=0$ are purely imaginary.

Comprehension Type :

Graph of $f(x)=ax^2+bx+c$ is shown adjacently, for which $\lambda(AB)=2, \lambda(AC)=3$and $b^2-4ac=-4$.



On the basis of above information's, answer the following questions :

13. The value of $a+b+c$ is equal to -

(A) 7 (B) 8 (C) 9 (D) 10

14. The quadratic equation with rational coefficients whose one of the roots is $b+\sqrt{a+c}$, is -

(A) $x^2-6x+2=0$ (B) $x^2-6x-1=0$

(C) $x^2+6x+2=0$ (D) $x^2+6x-1=0$

15. Range of $g(x)=\left(a+\frac{1}{2}\right)x^2+(b+2)x-\left(c-\frac{1}{2}\right)$ when $x\in[-4,0]$ is -

(A) $[-10,-6]$ (B) $\left[-\frac{49}{4},-10\right]$ (C) $\left[-\frac{49}{4},-6\right]$ (D) $\left[-\frac{49}{4},-\infty\right]$

EXERCISE – 6

1. Let a, b be arbitrary real numbers. Find the smallest natural number 'b' for which the equation $x^2 + 2(a+b)x + (a-b+8) = 0$ has unequal real roots for all $a \in R$.

2. If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

3. Let $P(x) = 4x^2 + 6x + 4$ and $Q(y) = 4y^2 - 12y + 25$. Find the unique pair of real numbers (x, y) that satisfy $P(x) \cdot Q(y) = 28$.

4. Find the product of the real roots of the equation, $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

5. Find the values of ' a ' for which $-3 < [(x^2 + ax - 2)/(x^2 + x + 1)] < 2$ is valid for all real x.

6. If a & b are positive numbers, prove that the equation $\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x+b} = 0$ has two real roots, one between $a/3 \& 2a/3$ and the other between $-2b/3 \& -b/3$.

7. When $y^2 + my + 2$ is divided by $(y-1)$ then the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $(y+1)$ then quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$ then find the value of m.

8. If the roots of $x^2 - ax + b = 0$ are real & differ by a quantity which is less than $c(c > 0)$, prove that b lies between $(1/4)(a^2 - c^2) \& (1/4)a^2$.

9. Let $x^2 + y^2 + xy + 1 \geq a(x+y) \forall x, y \in R$. Find the possible integer(s) in the range of a.

10. If roots of the equation $(x-\alpha)(x-4+\beta) + (x-2+\alpha)(x+2-\beta) = 0$ are p and q then find the absolute value of the sum of the roots of the equations $2(x-p)(x-q) - (x-\alpha)(x-4+\beta) = 0$ and $2(x-p)(x-q) - (x-2+\alpha)(x+2-\beta) = 0$.

11. Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$). Find the sum of the squares of the roots of the cubic polynomial.

12. At what values of ' a ' do all the zeroes of the function, $f(x) = (a-2)x^2 + 2ax + a + 3$ lie on the interval $(-2,1)$?

13. Let α, β, γ be distinct real numbers such that

$a\alpha^2 + b\alpha + c = (\sin\theta)\alpha^2 + (\cos\theta)\alpha$

$a\beta^2 + b\beta + c = (\sin\theta)\beta^2 + (\cos\theta)\beta$

$a\gamma^2 + b\gamma + c = (\sin\theta)\gamma^2 + (\cos\theta)\gamma$

(where $a, b, c \in R$)

(a) Find the maximum value of the expression $\frac{a^2+b^2}{a^2+3ab+5b^2}$

(b) If $\vec{V}_1 = a\hat{i} + b\hat{j} + c\hat{k}$ makes an angle $\frac{\pi}{3}$ with $\vec{V}_2 = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$, then find the number of values of $\theta \in [0, 2\pi]$

14. If the range of the function $f(x)=\frac{x^2+ax+b}{x^2+2x+3}$ is $[-5,4]$, $a, b \in N$, then find the value of (a^2+b^2)

15. Fin the minimum value of $\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+x^3+\frac{1}{x^3}}$ for $x>0$.

16. Given $x, y \in R, x^2+y^2>0$. If the maximum and minimum value of the expression

$E=\frac{x^2+y^2}{x^2+xy+4y^2}$ are M and m, and A denotes the average value of M and m, compute (2007)A.

ANSWER KEY

EXERCISE – 1

1.	D	**2.**	C	**3.**	A	**4.**	B	**5.**	B	**6.**	A	**7.**	D
8.	D	**9.**	C	**10.**	A	**11.**	C	**12.**	B	**13.**	D	**14.**	A
15.	B	**16.**	C	**17.**	B	**18.**	B	**19.**	D	**20.**	B		

EXERCISE – 2

1. (i) $\frac{p(p^2-4p)(p^2-q)}{q}$ (ii) $\frac{p^4-4p^2q+2q^2}{q^4}$ **2.** $a=-4$ **3.** $-\frac{125}{8}, a_2=\frac{27}{8}$

4. $k=\pm 3\sqrt{5}$ **5.** $b=-13$ **6.** $a=\pm 10$

7. $\{-4,-3,3,4\}$ **8.** (i) $\frac{b^2-2ac}{a^2c^2}$ (ii) $\frac{b^2(b^2-3ac)}{a^3c^3}$

9. 254

11. (a) (ii) and (iv); (b) $x^2-p(p^4-5p^2q+5q^2)x+p^2q^2(p^2-4q)(p^2-q)=0$

12. Solve following Inequalities over the set of real numbers -

(i) $(-3,1)$ (ii) $(-\infty,-2)\cup(-2,-1)\cup(1,+\infty)$

(iii) $[-3,3]$ (iv) $(-\infty,0)\cup(3,+\infty)$

(v) $(-\infty,3)\cup(4,+\infty)$ (vi) $(-\infty,+\infty)$

(vii) $(-1,5)$ (viii) $[1,3]\cup(5,+\infty)$

(ix) $(-9/2,-2)\cup(3,+\infty)$ (x) $(-1,1)\cup(4,6)$

(xi) $(1/2,3)$ (xii) $(-1,0)\cup(0,1)$

(xiii) $[-\sqrt{2},-1)\cup(-1,\sqrt{2}]\cup[3,4)$ (xiv) $(-\infty,-\sqrt{7}/2)\cup(-1,\sqrt{7}/2)\cup(4/3,+\infty)$

(xv) $(-\infty,-2]\cup(-1,4)$ (xvi) $(-\infty,-5)\cup(1,2)\cup(6,+\infty)$

(xvii) $(-\infty,-3)\cup(-2,3)$ (xviii) $(-\sqrt{2},0)\cup(1,\sqrt{2})\cup(2,+\infty)$

(xix) $(-\infty,-2)\cup(-1,3)\cup(4,+\infty)$ (xx) $(-\infty,-7)\cup(-4,-2)$

(xxi) $(-\infty,-1]\cup(0,1]\cup(2,3]$

13. $k=13$

14. $(-3,5)$

15. For all $p\in\left[3,\frac{15}{4}\right]$

16. For all $a\in(-\infty,-6)$

17. For all $a\in(-6,3)$

18. $a\in\left(-\infty,-\frac{1}{2}\right)$

19. 19120.0 or 24

22. (a) 7, , (b) 6 , (c) 2 or 4

23. 4

24. $3y^3 - 9y^2 - 3y + 1 = 0; (\alpha - 2)(\beta - 2)(\gamma - 2) = 3$

26. $\frac{1}{3}$

27. -2

29. $(-\infty, -2] \cup [0,1) \cup (2,4) \cup (5, \infty)30.\ 2\sqrt{2} \le a < \frac{11}{3}$

30. $2\sqrt{2} \le a < \frac{11}{3}$

EXERCISE – 3

1.	D	**2.**	A	**3.**	B	**4.**	A	**5.**	D	**6.**	D	**7.**	D
8.	D	**9.**	A	**10.**	A	**11.**	B	**12.**	C	**13.**	A	**14.**	A
15.	B												

EXERCISE – 4

1. $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

2. (a) A; (b) 1210

3. (a) D; (b) (A) P,R,S; (B) Q,S; (C) Q,S; (D) P, R, S

4.	B	**5.**	2	**6.**	B	**7.**	C	**8.**	B	**9.**	A,D	**10.**	C
11.	A	**12.**	D										

EXERCISE – 5

1.	B	**2.**	B	**3.**	C	**4.**	C	**5.**	D	**6.**	BD	**7.**	BC
8.	ABCD	**9.**	ABCD	**10.**	AD	**11.**	ABC	**12.**	ABD	**13.**	D	**14.**	A
15.	C												

EXERCISE – 6

1. 5 **3.** $\left(-\frac{3}{4}, \frac{3}{2}\right)$ **4.** 20 **5.** $-2 < a < 1$ **7.** 0 **9.** $-1,0,1$

10. 4 **11.** 146 **12.** $\left(-\infty, -\frac{1}{4}\right) \cup \{2\} \cup (5,6]$

13. (a) 2 , (b) 3 **14.** 277 15. $y_{min} = 6$ **16.** 1338