

1. GENERAL DEFINITION:

If to every value (Considered as real unless other-wise stated) of a variable x , which belongs to some collection (Set) E , there corresponds one and only one finite value of the quantity y , then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable.

If to every value of x belonging to some set E there corresponds one or several values of the variable y , then y is called a multiple valued function of x defined on E . Conventionally the word "FUNCTION" is used only as the meaning of a single valued function, if not otherwise stated.

Pictorially : $\begin{matrix} x \\ \text{input} \end{matrix} \xrightarrow{\boxed{f}} \begin{matrix} f(x)=y \\ \text{output} \end{matrix}$ is called the image of x & x is the pre-image of y under f .

Every function from $A \rightarrow B$ satisfies the following conditions.

- (i) $f \subset A \times B$
- (ii) $\forall a \in A \Rightarrow (a, f(a)) \in f$ and
- (iii) $(a, b) \in f \& (a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f: A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of all f images of elements of A is known as the range of f .

Thus : Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

3. IMPORTANT TYPES OF FUNCTIONS :

(i) POLYNOMIAL FUNCTION :

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .

Note:

(a) A polynomial of degree one with no constant term is called an odd linear function.

i.e. $f(x) = ax, a \neq 0$

(b) There are two polynomial functions, satisfying the relation; $f(x) \cdot f(1/x) = f(x) + f(1/x)$.

They are :

(i) $f(x) = x^n + 1$ &

(ii) $f(x) = 1 - x^n$, where n is a positive integer.

(ii) **ALGEBRAIC FUNCTION :**

y is an algebraic function of x , if it is a function that satisfies an algebraic equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0$ Where n is a positive integer and $P_0(x), P_1(x) \dots$ are Polynomials in x .

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called **Transcendental Function**.

(iii) **FRACTIONAL RATIONAL FUNCTION :**

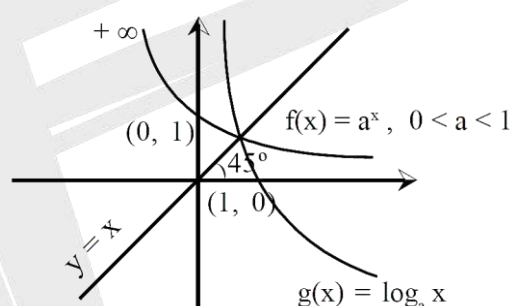
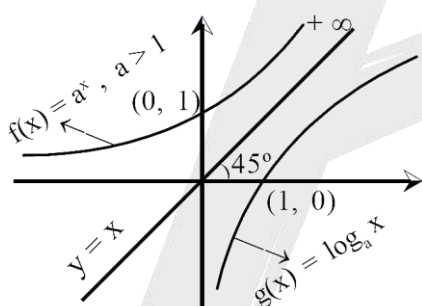
A rational function is a function of the form. $y = f(x) = \frac{g(x)}{h(x)}$,

where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$.

(iv) **EXPONENTIAL FUNCTION :**

A function $f(x) = a^x = e^{x \ln a}$ ($a > 0, a \neq 1, x \in \mathbb{R}$) is called an exponential function. The inverse of the exponential function is called the logarithmic function. i.e. $g(x) = \log_a x$.

Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown.



(v) **ABSOLUTE VALUE FUNCTION :**

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :

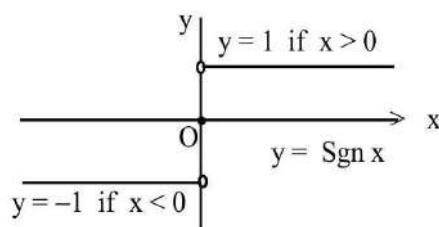
$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(vi) **SIGNUM FUNCTION :**

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{Sgn } x = |x|/x; x \neq 0; f(0) = 0$



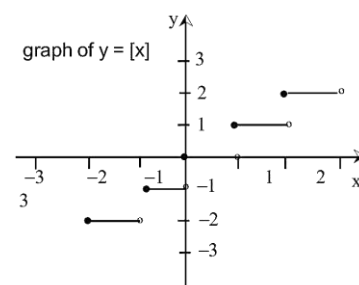
(vii) **GREATEST INTEGER OR STEP UP FUNCTION :**

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x . Note that for :

$$\begin{array}{llll} -1 \leq x < 0 & ; & [x] = -1 & \quad 0 \leq x < 1 & ; & [x] = 0 \\ 1 \leq x < 2 & ; & [x] = 1 & \quad 2 \leq x < 3 & ; & [x] = 2 \text{ and so on.} \end{array}$$

Properties of greatest integer function :

- (a) $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$
- (b) $[x + m] = [x] + m$ if m is an integer.
- (c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$
- (d) $[x] + [-x] = 0$ if x is an integer
 $= -1$ otherwise.

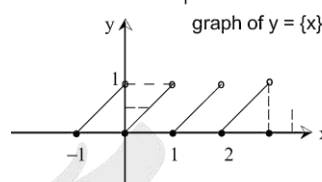


(viii) **FRACTIONAL PART FUNCTION :**

It is defined as : $g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is

$2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3 . The period of this function is 1 and graph of this function is as shown.



4. **DOMAINS AND RANGES OF COMMON FUNCTION :**

Function	Domain	Range
$(y = f(x))$	(i.e. values taken by x)	(i.e. values taken by $f(x)$)

A. Algebraic Functions

(i) $x^n, (n \in \mathbb{N})$	$\mathbb{R} = (\text{set of real numbers}) \mathbb{R},$	if n is odd $\mathbb{R}^+ \cup \{0\}$, if n is even
(ii) $\frac{1}{x^n}, (n \in \mathbb{N})$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$, if n is odd \mathbb{R}^+ , if n is even
(iii) $x^{1/n}, (n \in \mathbb{N})$	\mathbb{R} , if n is odd $\mathbb{R}^+ \cup \{0\}$, if n is even	\mathbb{R} , if n is odd $\mathbb{R}^+ \cup \{0\}$, if n is even
(iv) $\frac{1}{x^{1/n}}, (n \in \mathbb{N})$	$\mathbb{R} - \{0\}$, if n is odd \mathbb{R}^+ , if n is even	$\mathbb{R} - \{0\}$, if n is odd \mathbb{R}^+ , if n is even

B. Trigonometric Functions

(i) $\sin x$	\mathbb{R}	$[-1, +1]$
(ii) $\cos x$	\mathbb{R}	$[-1, +1]$
(iii) $\tan x$	$\mathbb{R} - (2k + 1)\frac{\pi}{2}, k \in \mathbb{I}$	\mathbb{R}

(MATHEMATICS)

(iv) $\sec x$	$\mathbb{R} - (2k + 1)\frac{\pi}{2}, k \in \mathbb{I}$	$(-\infty, -1] \cup [1, \infty)$
(v) $\operatorname{cosec} x$	$\mathbb{R} - k\pi, k \in \mathbb{I}$	$(-\infty, -1] \cup [1, \infty)$
(vi) $\cot x$	$\mathbb{R} - k\pi, k \in \mathbb{I}$	\mathbb{R}

C. Inverse Circular Functions

(Refer after Inverse is taught)

(i) $\sin^{-1} x$	$[-1, +1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $\cos^{-1} x$	$[-1, +1]$	$[0, \pi]$
(iii) $\tan^{-1} x$	\mathbb{R}	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(v) $\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi) $\cot^{-1} x$	\mathbb{R}	$(0, \pi)$

Function

Domain

Range

$(y = f(x))$

(i.e. values taken by x)

(i.e. values taken by $f(x)$)

D. Exponential Functions

(i) e^x	\mathbb{R}	\mathbb{R}^+
(ii) $e^{1/x}$	$\mathbb{R} - \{0\}$	$\mathbb{R}^+ - \{1\}$
(iii) $a^x, a > 0$	\mathbb{R}	\mathbb{R}^+
(iv) $a^{1/x}, a > 0$	$\mathbb{R} - \{0\}$	$\mathbb{R}^+ - \{1\}$

E. Logarithmic Functions

(i) $\log_a x, (a > 0)(a \neq 1)$	\mathbb{R}^+	\mathbb{R}
(ii) $\log_x a = \frac{1}{\log_a x}$ ($a > 0$)($a \neq 1$)	$\mathbb{R}^+ - \{1\}$	$\mathbb{R} - \{0\}$

F. Integral Part Functions

(i) $[x]$	\mathbb{R}	\mathbb{I}
(ii) $\frac{1}{[x]}$	$\mathbb{R} - [0, 1)$	$\left\{\frac{1}{n}, n \in \mathbb{I} - \{0\}\right\}$

G. Fractional Part Functions

(i) $\{x\}$	\mathbb{R}	$[0, 1)$
(ii) $\frac{1}{\{x\}}$	$\mathbb{R} - \mathbb{I}$	$(1, \infty)$

H. Modulus Functions

(i) $ x $	\mathbb{R}	$\mathbb{R}^+ \cup \{0\}$
(ii) $\frac{1}{ x }$	$\mathbb{R} - \{0\}$	\mathbb{R}^+

(MATHEMATICS)

I. Signum Function

$$\begin{aligned} \operatorname{sgn}(x) &= \frac{|x|}{x}, x \neq 0 \\ &= 0, x = 0 \end{aligned} \quad \begin{array}{l} \mathbb{R} \\ \{-1, 0, 1\} \end{array}$$

J. Constant Function

$$\text{say } f(x) = c \quad \begin{array}{l} \mathbb{R} \\ \{c\} \end{array}$$

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if :

- (i) The domain of f = the domain of g .
- (ii) The range of f = the range of g and
- (iii) $f(x) = g(x)$, for every x belonging to their common domain. e.g.
 $f(x) = 1/x$ & $g(x) = x/x^2$ are identical functions.

6. CLASSIFICATION OF FUNCTIONS :

One-One Function (Injective mapping) :

A function $f: A \rightarrow B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B . Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



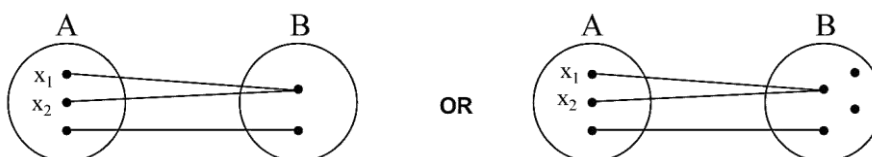
Note:

- (i) Any function which is entirely increasing or decreasing in whole domain, then $f(x)$ is one-one.
- (ii) If any line parallel to x -axis cuts the graph of the function at most at one point, then the function is one-one.

Many-one function :

A function $f: A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B . Thus $f: A \rightarrow B$ is many one if for ; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



Note:

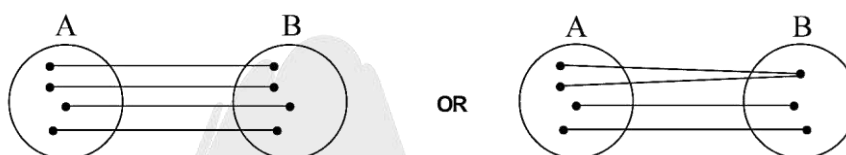
(i) Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many-one. In other words, if a line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(ii) If a function is one-one, it cannot be many-one and vice versa.

Onto function (Surjective mapping) :

If the function $f: A \rightarrow B$ is such that each element in B (co-domain) is the f image of atleast one element in A , then we say that f is a function of A 'onto' B . Thus $f: A \rightarrow B$ is surjective iff $\forall b \in B, \exists$ some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as



Note that : If range = co-domain, then $f(x)$ is onto.

Into function :

If $f: A \rightarrow B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then $f(x)$ is into.

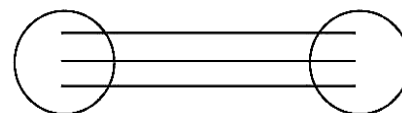
Diagrammatically into function can be shown as



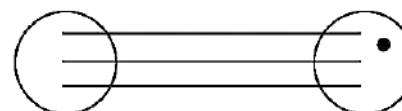
Note that : If a function is onto, it cannot be into and vice versa. A polynomial of degree even will always be into.

Thus a function can be one of these four types :

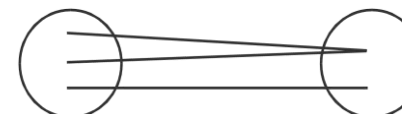
(A) one-one onto (injective & surjective)



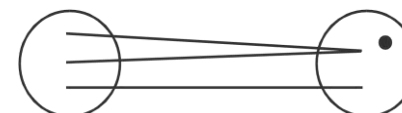
(B) one-one into (injective but not surjective)



(C) many-one onto (surjective but not injective)



(D) many-one into (neither surjective nor injective)



Note:

- (i) If f is both injective & surjective, then it is called a **Bijjective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.
- (ii) If a set A contains n distinct elements then the number of different functions defined from $A \rightarrow A$ is n^n & out of it $n!$ are one one.

Identity function :

The function $f: A \rightarrow A$ defined by $f(x) = x \forall x \in A$ is called the identity of A and is denoted by I_A . It is easy to observe that identity function is a bijection.

Constant function:

A function $f: A \rightarrow B$ is said to be a constant function if every element of A has the same f image in B . Thus $f: A \rightarrow B; f(x) = c, \forall x \in A, c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g, f - g, (f \cdot g)$ & (f/g) as follows :

- (i) $(f \pm g)(x) = f(x) \pm g(x)$
- (ii) $(f \cdot g)(x) = f(x) \cdot g(x)$
- (iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
- domain in each case is $A \cap B$
- domain is $\{x \mid x \in A \cap B \text{ s.t } g(x) \neq 0\}$.

8. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f: A \rightarrow B$ & $g: B \rightarrow C$ be two functions.

Then the function $g \circ f: A \rightarrow C$ defined by $(g \circ f)(x) = g(f(x)) \forall x \in A$ is called the composite of the two functions f & g .

Diagrammatically $x \xrightarrow{\quad} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x)).$

Thus the image of every $x \in A$ under the function $g \circ f$ is the g -image of the f -image of x .

Note that $g \circ f$ is defined only if $\forall x \in A, f(x)$ is an element of the domain of g so that we can take its g -image. Hence for the product $g \circ f$ of two functions f & g , the range of f must be a subset of the domain of g .

Properties Of Composite Functions :

- (i) The composite of functions is not commutative i.e. $g \circ f \neq f \circ g$.
- (ii) The composite of functions is associative i.e. if f, g, h are three functions such that $f \circ (g \circ h)$ & $(f \circ g) \circ h$ are defined, then $f \circ (g \circ h) = (f \circ g) \circ h$.
- (iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that $g \circ f$ is defined, then $g \circ f$ is also a bijection.

9. HOMOGENEOUS FUNCTIONS:

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y .

Symbolically if, $f(tx, ty) = t^n \cdot f(x, y)$ then $f(x, y)$ is homogeneous function of degree n .

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

11. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an IMPLICIT FUNCTION. For e.g. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an EXPLICIT FUNCTION.

12. INVERSE OF A FUNCTION :

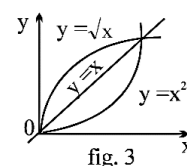
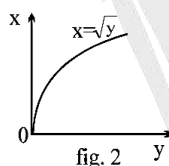
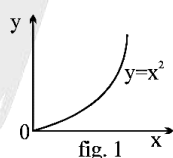
Let $f: A \rightarrow B$ be a one-one & onto function, then there exists a unique function $g: B \rightarrow A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall x \in A \text{ \& } y \in B$.

Then g is said to be inverse of f .

Thus $g = f^{-1}: B \rightarrow A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties Of Inverse Function :

- (i) The inverse of a bijection is unique.
- (ii) If $f: A \rightarrow B$ is a bijection & $g: B \rightarrow A$ is the inverse of f , then $f \circ g = I_B$ and $g \circ f = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = x$ the changed form of $x = \sqrt{y}$.



- (iii) The inverse of a bijection is also a bijection.
- (iv) If f & g are two bijections $f: A \rightarrow B, g: B \rightarrow C$ then the inverse of $g \circ f$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

13. ODD & EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of ' f ' then f is said to be an even function.

e.g. $f(x) = \cos x; g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of ' f ' then f is said to be an odd function.

e.g. $f(x) = \sin x; g(x) = x^3 + x$.

- (a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

- (b) A function may neither be odd nor even.
- (c) Inverse of an even function is not defined.
- (d) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.
- (e) Every function can be expressed as the sum of an even & an odd function.

$$\text{e.g. } f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

- (f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.
- (g) If f and g both are even or both are odd then the function $f \cdot g$ will be even but if any one of them is odd then $f \cdot g$ will be odd.

14. PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number $T (T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x .

e.g. The function $\sin x$ & $\cos x$ both are periodic over 2π & $\tan x$ is periodic over π .

Note:

- (a) $f(T) = f(0) = f(-T)$, where ' T ' is the period.
- (b) Inverse of a periodic function does not exist.
- (c) Every constant function is always periodic, with no fundamental period.
- (d) If $f(x)$ has a period T & $g(x)$ also has a period T then it does not mean that $f(x) + g(x)$ must have a period T . e.g. $f(x) = |\sin x| + |\cos x|$.
- (e) If $f(x)$ has a period p , then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p .
- (f) If $f(x)$ has a period T then $f(ax + b)$ has a period $\frac{T}{a}$ ($a > 0$).

15. GENERAL:

If x, y are independent variables, then:

- (i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \cdot \ln x$ or $f(x) = 0$.
- (ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in \mathbb{R}$
- (iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.
- (iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

PROFICIENCY TEST-01

- Which of the following is a function?
 (A) $\{(2,1), (2,2), (2,3), (2,4)\}$ (B) $\{(1,4), (2,5), (1,6), (3,9)\}$
 (C) $\{(1,2), (3,3), (2,3), (1,4)\}$ (D) $\{(1,2), (2,2), (3,2), (4,2)\}$
- Find the domains of definitions of the following functions :
 (Read the symbols $[*]$ and $\{*\}$ as greatest integer and fractional part functions respectively.)
 (i) $f(x) = \sqrt{\cos 2x} + \sqrt{16 - x^2}$ (ii) $f(x) = \log_7 \log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x)$
 (iii) $f(x) = \ln (\sqrt{x^2 - 5x - 24} - x - 2)$ (iv) $f(x) = \sqrt{\frac{1-5^x}{7^{-x}-7}}$
 (v) $y = \log_{10} \sin (x - 3) + \sqrt{16 - x^2}$ (vi) $f(x) = \log_{100x} \left(\frac{2\log_{10} x + 1}{-x} \right)$
 (vii) $f(x) = \frac{1}{\sqrt{4x^2 - 1}} + \ln x(x^2 - 1)$ (viii) $f(x) = \sqrt{\log_{\frac{1}{2}} \frac{x}{x^2 - 1}}$
 (ix) $f(x) = \sqrt{x^2 - |x|} + \frac{1}{\sqrt{9 - x^2}}$ (x) $f(x) = \sqrt{(x^2 - 3x - 10) \cdot \ln^2 (x - 3)}$
 (xi) $f(x) = \sqrt{\log_x (\cos 2\pi x)}$ (xii) $f(x) = \frac{\sqrt{\cos x - (1/2)}}{\sqrt{6 + 35x - 6x^2}}$
 (xiii) $f(x) = \sqrt{\log_{1/3} (\log_4 ([x]^2 - 5))}$ (xiv) $f(x) = \frac{[x]}{2x - [x]}$ (xv) $f(x) = \log_x \sin x$
- Find the domain & range of the following functions.
 (Read the symbols $[*]$ and $\{*\}$ as greatest integer and fractional part functions respectively.)
 (i) $f(x) = \log_{\sqrt{5}} (\sqrt{2}(\sin x - \cos x) + 3)$ (ii) $f(x) = \frac{2x}{1+x^2}$
 (iii) $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$ (iv) $f(x) = \frac{x}{1+|x|}$
 (v) $f(x) = 2 + x - [x - 3]$ (vi) $f(x) = \sin^{-1} \sqrt{x^2 + x + 1}$
 (vii) $f(x) = \log_3 (5 + 4x - x^2)$
- The range of the function $f(x) = |x - 1| + |x - 2|, -1 \leq x \leq 3$ is
 (A) $[1,3]$ (B) $[1,5]$ (C) $[3,5]$ (D) None of these
- The range of the function $f(x) = 2|\sin x| - 3|\cos x|$ is :
 (A) $[-2, \sqrt{13}]$ (B) $[-2,3]$ (C) $[3, \sqrt{13}]$ (D) $[-3,2]$
- (i) The function $f(x)$ is defined on the interval $[0,1]$. Find the domain of definition of the functions.
 (A) $f(\sin x)$ (B) $f(2x + 3)$
 (ii) Given that $y = f(x)$ is a function whose domain is $[4,7]$ and range is $[-1,9]$. Find the range and domain of
 (A) $g(x) = \frac{1}{3}f(x)$ (B) $h(x) = f(x - 7)$

PROFICIENCY TEST-02

1. Classify the following functions $f(x)$ defined in $\mathbb{R} \rightarrow \mathbb{R}$ as injective, surjective, both or none.

(a) $f(x) = \frac{x^2+4x+30}{x^2-8x+18}$

(b) $f(x) = x^3 - 6x^2 + 11x - 6$

(c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$

(d) $f(x) = \frac{x^2}{1+x^2}$

(e) $f(x) = x + |x|$

(f) $f(x) = e^x - e^{-x}$

(g) $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$

2. If $f(x) = |x|$ and $g(x) = [x]$, then value of $\text{fog}\left(-\frac{1}{4}\right) + \text{gof}\left(-\frac{1}{4}\right)$ is, ($[x]$ denotes greatest integer function)

(A) 0

(B) 1

(C) -1

(D) $\frac{1}{4}$

3. If $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3 + 3$, and $g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = 2x + 1$, then $f^{-1} \circ g^{-1}(23)$ equals :

(A) 2

(B) 3

(C) $(14)^{1/3}$

(D) $(15)^{1/3}$

4. Which of the following functions has its inverse:

(A) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = a^x$

(B) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x| + |x - 1|$

(C) $f: \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = |x|$

(D) $f: [\pi, 2\pi] \rightarrow [-1, 1], f(x) = \cos x$

5. If function $f(x) = \begin{cases} \sqrt{2}, & \text{when } x \in \mathbb{Q} \\ 0, & \text{when } x \notin \mathbb{Q} \end{cases}$, $(\text{fof}) \sqrt{4}$ the value will be :

(A) 0

(B) 2

(C) $\sqrt{2}$

(D) None of these

6. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \left(\frac{3x+x^3}{1+3x^2}\right)$, then $f[g(x)]$ is equal to :

(A) $-f(x)$

(B) $3f(x)$

(C) $[f(x)]^3$

(D) None of these

7. If $f: \mathbb{R} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 3x + 4$ and $(\text{gof})(x) = 2x - 1$, then the value of $g(x)$ is :

(A) $2x - 1$

(B) $2x - 11$

(C) $\frac{1}{3}(2x - 11)$

(D) None of these

8. If $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + 2x - 3$ and $g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = 3x - 4$, then the value of $\text{fog}(x)$ is :

(A) $3x^2 + 6x - 13$

(B) $9x^2 - 18x + 5$

(C) $(3x - 4)^2 + 2x - 3$

(D) None of these

9. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cdot \cos\left(x + \frac{\pi}{3}\right)$ and $g(x)$ is a one-one function defined in $\mathbb{R} \rightarrow \mathbb{R}$, then $(\text{gof})(x)$ is

(A) One-one

(B) Onto

(C) Constant function

(D) Periodic with fundamental period π

10. Compute the inverse of the functions:

(a) $f(x) = \ln(x + \sqrt{x^2 + 1})$

(b) $f(x) = 2^{\frac{x}{x-1}}$

(c) $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

PROFICIENCY TEST-03

1. Find whether the following functions are even or odd or none

(a) $f(x) = \log(x + \sqrt{1+x^2})$

(c) $f(x) = \sin x + \cos x$

(e) $f(x) = \sin x - \cos x$

(g) $f(x) = \left\{ \frac{x^{2n}}{(x^{2n} \operatorname{sgn} x)^{2n+1}} \left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \right) \right\}, x \neq 0 \text{ and } n \in \mathbb{N}$

(h) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

(b) $f(x) = \frac{x(a^x+1)}{a^x-1}$

(d) $f(x) = x \sin^2 x - x^3$

(f) $f(x) = \frac{(1+2^x)^2}{2^x}$
2. Let $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ ($x \neq 0$), then $f(x)$ equals :

(A) $x^2 - 2$

(B) $x^2 - 1$

(C) x^2

(D) None of these
3. Find the period of following function:

(i) $f(x) = |\sin 2x|$ is :

(A) $\pi/4$

(B) $\pi/2$

(C) π

(D) 2π

(ii) $f(x) = \sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right)$ is

(A) 4

(B) 6

(C) 12

(D) 24

(iii) $f(x) = \log \cos 2x + \tan 4x$ is

(A) $\pi/2$

(B) π

(C) 2π

(D) $2\pi/5$
4. In the following which function is not periodic

(A) $\tan 4x$

(B) $\cos 2\pi x$

(C) $\cos x^2$

(D) $\cos^2 x$
5. Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of $f(21)$.
6. Let ' f ' be a function defined from $\mathbb{R}^+ \rightarrow \mathbb{R}^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and $f(2) = 6$, find the value of $f(50)$.
7. Let $f(x)$ be a function with two properties

(i) for any two real number x and y , $f(x+y) = x + f(y)$ and

(ii) $f(0) = 2$. Find the value of $f(100)$.
8. The period of $\cos(x + 4x + 9x + \dots + n^2x)$ is $\pi/7$, then $n \in \mathbb{N}$ is equal to :

(A) 2

(B) 3

(C) 4

(D) 5
9. Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :

(a) $10^x + 10^y = 10$

(b) $x + |y| = 2y$

10. Function f & g are defined by $f(x) = \sin x, x \in \mathbb{R}; g(x) = \tan x, x \in \mathbb{R} - \left(K + \frac{1}{2}\right)\pi$ where $K \in \mathbb{I}$. Find

(i) periods of $fo g$ & $go f$. **(ii)** range of the function $fo g$ & $go f$.

EXERCISE-I

1. Find the domains of definitions of the following functions :

(Read the symbols $[*]$ and $\{*\}$ as greatest integers and fractional part functions respectively.)

$$(i) \quad f(x) = \log_2 \left(-\log_{1/2} \left(1 + \frac{1}{\sin \left(\frac{x^\circ}{100} \right)} \right) \right) + \sqrt{\log_{10} (\log_{10} x) - \log_{10} (4 - \log_{10} x) - \log_{10} 3}$$

$$(ii) \quad f(x) = \frac{1}{[x]} + \log_{1-\{x\}} (x^2 - 3x + 10) + \frac{1}{\sqrt{2 - |x|}} + \frac{1}{\sqrt{\sec (\sin x)}}$$

$$(iii) \quad f(x) = \sqrt{(5x - 6 - x^2)[\ln \{x\}]} + \sqrt{(7x - 5 - 2x^2)} + \left(\ln \left(\frac{7}{2} - x \right) \right)^{-1}$$

$$(iv) \quad f(x) = \log_{\left[\frac{x+1}{x} \right]} |x^2 - x - 6| + {}^{16-x}C_{2x-1} + {}^{20-3x}P_{2x-5}$$

$$(v) \quad f(x) = \log \left\{ \log_{|\sin x|} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} \right\}.$$

2. Find the domain & range of the following functions.

(Read the symbols $[*]$ and $\{*\}$ as greatest integers and fractional part functions respectively.)

$$(i) \quad y = \sqrt{2 - x} + \sqrt{1 + x}$$

$$(ii) \quad f(x) = \log_{(\operatorname{cosec} x - 1)} (2 - [\sin x] - [\sin x]^2)$$

$$(iii) \quad f(x) = \frac{\sqrt{x+4}-3}{x-5}$$

3. **Column I**

Column II

$$(A) f(x) = \cos \left(\frac{\pi}{\sqrt{3}} \sin x + \sqrt{\frac{2}{3}} \pi \cos x \right)$$

(P) Domain of $f(x)$ is $(-\infty, \infty)$

$$(B) f(x) = \log_2 (|\sin x| + 1)$$

(Q) Range of $f(x)$ contains only one positive integer

$$(C) f(x) = \cos \{[x] + [-x]\}$$

(R) $f(x)$ is many-one function

$$(D) f(x) = [\{e^x\}]$$

(S) $f(x)$ is constant function

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function.

4. (a) The function $f(x)$ defined on the real numbers has the property that

$f(f(x)) \cdot (1 + f(x)) = -f(x)$ for all x in the domain of f . If the number 3 is in the domain and range of f , compute the value of $f(3)$.

(b) Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x . Then find the value of $f(300)$.

5. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is such that $f\left(\frac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Prove the following.

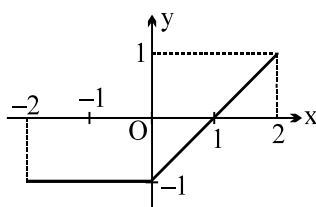
$$(a) f(f(x)) = x$$

$$(b) f(1/x) = -f(x), x \neq 0$$

$$(c) f(-x - 2) = -f(x) - 2$$

(MATHEMATICS)

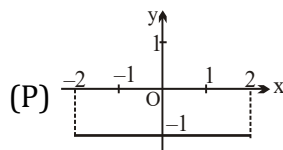
6. $f(x) = \begin{cases} x+1, & -1 \leq x \leq 2 \\ 4-x, & 2 < x \leq 5 \end{cases}$. Find domain and range of $f(f(x))$.
7. Let $f(x) = \sqrt{ax^2 + bx}$. Find the set of real values of 'a' for which there is at least one positive real value of 'b' for which the domain of f and the range of f are the same set.
8. $f(x) = \begin{cases} 1-x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} -x & \text{if } x < 1 \\ 1-x & \text{if } x \geq 1 \end{cases}$ find $(f \circ g)(x)$ and $(g \circ f)(x)$
9. Find the inverse of $f(x) = 2^{\log_{10} x} + 8$ and hence solve the equation $f(x) = f^{-1}(x)$.
10. Suppose $p(x)$ is a polynomial with integer coefficients. The remainder when $p(x)$ is divided by $x - 1$ is 1 and the remainder when $p(x)$ is divided by $x - 4$ is 10. If $r(x)$ is the remainder when $p(x)$ is divided by $(x - 1)(x - 4)$, find the value of $r(2006)$.
11. (i) Prove that the function defined as, $f(x) = \begin{cases} e^{-\sqrt{|n\{x\}|}} - \{x\} & \text{where ever it exists} \\ \{x\} & \text{otherwise, then} \end{cases}$
 $f(x)$ is odd as well as even.
 (where $\{x\}$ denotes the fractional part function)
- (ii) If $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$, then $f(x)$ is
 (A) an odd function (B) an even function
 (C) neither even nor odd (D) both even and odd
12. In a function $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2} \sin\left(\pi\left(x + \frac{1}{4}\right)\right)\right|\right) = 4\cos^2 \frac{\pi x}{2} + x\cos \frac{\pi}{x}$ Prove that
 (i) $f(2) + f(1/2) = 1$ and
 (ii) $f(2) + f(1) = 0$
13. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If $f(x)$ is divided by $x^3 - x$ then the remainder is some function of x say $g(x)$. Find the value of $g(10)$.
14. Let $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5$ where $x \in [-6, 6]$. If the range of the function is $[a, b]$ where $a, b \in \mathbb{N}$ then find the value of $(a + b)$.
15. If $a, b \in \mathbb{R}$ be fixed positive numbers such that
 $f(a + x) = b + [b^3 + 1 - 3b^2 \cdot f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$ for all $x \in \mathbb{R}$ then
 prove that $f(x)$ is a periodic function.
16. The graph of the function $y = f(x)$ is as follows.
 Match the function mentioned in **Column-I** with the respective graph given in **Column-II**.



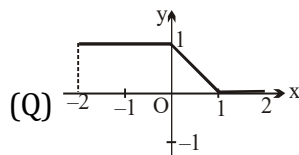
Column-I

Column-II

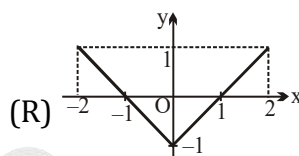
(A) $y = |f(x)|$



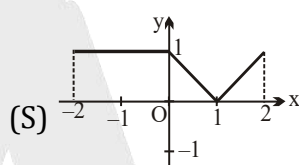
(B) $y = f(|x|)$



(C) $y = f(-|x|)$



(D) $y = \frac{1}{2}(|f(x)| - f(x))$



17. **Column I** contains functions and **Column II** contains their natural domains. Exactly one entry of **column II** matches with exactly one entry of **column I**.

Column I

Column II

(A) $f(x) = \sin^{-1} \left(\frac{x+1}{x} \right)$

(P) $(1, 3) \cup (3, \infty)$

(B) $g(x) = \sqrt{\ln \left(\frac{x^2+3x-2}{x+1} \right)}$

(Q) $(-\infty, 2)$

(C) $h(x) = \frac{1}{\ln \left(\frac{x-1}{2} \right)}$

(R) $\left(-\infty, -\frac{1}{2} \right]$

(D) $\phi(x) = \ln (\sqrt{x^2+12} - 2x)$

(S) $[-3, -1) \cup [1, \infty)$

EXERCISE-II

1. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false.
 $f(x) = 1; f(y) \neq 1; f(z) \neq 2$. Determine $f^{-1}(1)$
2. Let $x = \log_4 9 + \log_9 28$
 show that $[x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x .
3. (a) A function f is defined for all positive integers and satisfies $f(1) = 2005$ and
 $f(1) + f(2) + \dots + f(n) = n^2 f(n)$ for all $n > 1$. Find the value of $f(2004)$.
 (b) If a, b are positive real numbers such that $a - b = 2$, then find the smallest value of the constant L for which $\sqrt{x^2 + ax} - \sqrt{x^2 + bx} < L$ for all $x > 0$
 (c) Let $f(x) = x^2 + kx$; k is a real number. The set of values of k for which the equation $f(x) = 0$ and $f(f(x)) = 0$ have same real solution set.
 (d) If $f(2x + 1) = 4x^2 + 14x$, then find the sum of the roots of the equation $f(x) = 0$.
 (e) Let a and b be real numbers and let $f(x) = a \sin x + b \sqrt[3]{x} + 4, \forall x \in \mathbb{R}$.
 If $f(\log_{10}(\log_3 10)) = 5$ then find the value of $f(\log_{10}(\log_{10} 3))$.
4. Let $[x]$ = the greatest integer less than or equal to x . If all the values of x such that the product $\left[x - \frac{1}{2}\right] \left[x + \frac{1}{2}\right]$ is prime, belongs to the set $[x_1, x_2) \cup [x_3, x_4)$, find the value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$.
5. Let $f: \mathbb{R} \rightarrow \mathbb{R} - \{3\}$ be a function with the property that there exist $T > 0$ such that
 $f(x + T) = \frac{f(x) - 5}{f(x) - 3}$ for every $x \in \mathbb{R}$. Prove that $f(x)$ is periodic.
6. If $f(x) = -1 + |x - 2|, 0 \leq x \leq 4$ and $g(x) = 2 - |x|, -1 \leq x \leq 3$
 Then find $f \circ g(x), g \circ f(x), f \circ f(x)$ & $g \circ g(x)$. Draw rough sketch of the graphs of $f \circ g(x)$ & $g \circ f(x)$.
7. Let $\{x\}$ & $[x]$ denote the fractional and integral part of a real number x respectively.
 Solve $4\{x\} = x + [x]$
8. Let $f(x) = \frac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\frac{1}{2006}\right) + f\left(\frac{2}{2006}\right) + f\left(\frac{3}{2006}\right) + \dots + f\left(\frac{2005}{2006}\right)$
9. The set of real values of ' x ' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where $[]$ denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in \mathbb{N}$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.
10. $f(x)$ and $g(x)$ are linear function such that for all $x, f(g(x))$ and $g(f(x))$ are Identity functions.
11. If for all real values of u & $v, 2f(u)\cos v = f(u + v) + f(u - v)$, prove that, for all real values of x .
 (i) $f(x) + f(-x) = 2a \cos x$ (ii) $f(\pi - x) + f(-x) = 0$
 (iii) $f(\pi - x) + f(x) = -2b \sin x$. Deduce that $f(x) = a \cos x - b \sin x, a, b$ are arbitrary constants.

(MATHEMATICS)

12. Find out for what integral values of n the number 3π is a period of the function :
 $f(x) = \cos nx \cdot \sin (5/n)x$
13. If $f(x) = \frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$, then the fundamental period of $f(x)$ is :
 (A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) None of these
14. If $f(x) = -\frac{x|x|}{1+x^2}$ then $f^{-1}(x)$ equals.
 (A) $\sqrt{\frac{|x|}{1-|x|}}$ (B) $(\operatorname{sgn}(-x))\sqrt{\frac{|x|}{1-|x|}}$ (C) $-\sqrt{\frac{x}{1-x}}$ (D) None of these
15. If $f(x)$ satisfies $x + |f(x)| = 2f(x)$ then $f^{-1}(x)$ satisfies.
 (A) $3x + |f^{-1}(x)| = 2f^{-1}(x)$ (B) $x + |f^{-1}(x)| = 2f^{-1}(x)$
 (C) $f^{-1}(x) - |x| = 2x$ (D) $3x - |f^{-1}(x)| = 2f^{-1}(x)$
16. **Column I** **Column II**
- (A) The integral values of x for which
 $f(x) = \cos^{-1} \left(\frac{2[|\sin x| + |\cos x|]}{\sin^2 x + 2\sin x + \frac{11}{4}} \right)$, is defined
 (where $[.]$ denotes greatest integer function) is/are (P) -1
- (B) The possible value(s) of $\tan a$, such that
 $[\cos a] + [\sin a + 1] = 0$,
 where $[.]$ denote greatest integer function is/are (Q) 0
- (C) The integers in the domain of function
 $f(x) = \log_{10} (ax^3 + (b+a)x^2 + (b+c)x + c)$
 if $b^2 < 2ac, a > 0$ is/are (R) 2
- (D) The number of integers in the domain of
 function, $f(x) = \operatorname{sgn} \left(\log_{(4-|x|)} (x^2 + 4x + 4) \right)$ is (S) 4

Comprehension (Q.18 to Q. 20)

Consider the function $f(x) = \begin{cases} x^2 - 1, & -1 \leq x \leq 1 \\ \ell n x, & 1 < x \leq e \end{cases}$

Let $f_1(x) = f(|x|)$ and $f_2(x) = |f(|x|)|$

$f_3(x) = f(-x)$

Now answer the following questions.

17. Number of positive solution of the equation $2f_2(x) - 1 = 0$ is (are)
 (A) 4 (B) 3 (C) 2 (D) 1
18. Number of integral solution of the equation $f_1(x) = f_2(x)$ is (are)
 (A) 1 (B) 2 (C) 3 (D) 4
19. If $f_4(x) = \log_{27} (f_3(x) + 2)$, then range of $f_4(x)$ is
 (A) $[1, 9]$ (B) $\left[\frac{1}{3}, \infty\right)$ (C) $\left[0, \frac{1}{3}\right]$ (D) $[1, 27]$

EXERCISE-III

1. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is : [JEE Main 2014]
 (A) $(-\infty, \infty)$ (B) $(0, \infty)$ (C) $(-\infty, 0)$ (D) $(-\infty, \infty) - \{0\}$
2. If $a \in \mathbb{R}$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval [JEE Main 2014]
 (A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-1, 0) \cup (0, 1)$
 (C) $(1, 2)$ (D) $(-2, -1)$
3. If $X = \{4^n - 3n - 1 : n \in \mathbb{N}\}$ and $Y = \{9(n - 1) : n \in \mathbb{N}\}$, where \mathbb{N} is the set of natural numbers, then $X \cup Y$ is equal to : [JEE Main 2014]
 (A) Y (B) \mathbb{N} (C) $Y - X$ (D) X
4. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$, and $S = \{x \in \mathbb{R} : f(x) = f(-x)\}$; then S : [JEE Main 2016]
 (A) is an empty set (B) contains exactly one element
 (C) contains exactly two elements (D) contains more than two elements
5. Let $a, b, c \in \mathbb{R}$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy, \forall x, y \in \mathbb{R}$, then $\sum_{n=1}^{10} f(n)$ is equal to : [JEE Main 2017]
 (A) 190 (B) 255 (C) 330 (D) 165
6. The function $f: \mathbb{R} \rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is : [JEE Main 2017]
 (A) surjective but not injective (B) neither injective nor surjective
 (C) invertible (D) injective but not surjective
7. If $g(x) = x^2 + x - 1$ and $g(f(x)) = 4x^2 - 10x + 5$, then find $f\left(\frac{5}{4}\right)$. [JEE Main 2020]
 (A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$
8. Find the number of solution of $\log_{1/2} |\sin x| = 2 - \log_{1/2} |\cos x|, x \in [0, 2\pi]$ [JEE Main 2020]
 (A) 2 (B) 4 (C) 6 (D) 8
9. The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}, x \in (-1, 1)$, is [JEE Main 2020]
 (A) $\frac{1}{4}(\log_8 e) \log_e \left(\frac{1+x}{1-x}\right)$ (B) $\frac{1}{2} \log_8 \left(\frac{1-x}{1+x}\right)$
 (C) $\frac{1}{4}(\log_8 e) \log_e \left(\frac{1-x}{1+x}\right)$ (D) $\frac{1}{2} \log_8 \left(\frac{1+x}{1-x}\right)$

10. Let $f: (1,3) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is: [Jee Main 2020]

(A) $\left(0, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{7}{5}\right]$ (B) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$
 (C) $\left(\frac{2}{5}, 1\right) \cup \left(1, \frac{4}{5}\right]$ (D) $\left(0, \frac{1}{3}\right) \cup \left(\frac{2}{5}, \frac{4}{5}\right]$

11. Let x denote the total number of one-one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one-one functions from the set A to the set $A \times B$. Then: [JEE Main 2021]

(A) $y = 273x$ (B) $2y = 91x$
 (C) $y = 91x$ (D) $2y = 273x$

12. A function $f(x)$ is given by $f(x) = \frac{5^x}{5^x + 5}$, then the sum of the series [JEE Main 2021]

$f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{3}{20}\right) + \dots + f\left(\frac{39}{20}\right)$ is equal to:

(A) $\frac{19}{2}$ (B) $\frac{49}{2}$ (C) $\frac{39}{2}$ (D) $\frac{29}{2}$

13. Let $R = \{P, Q\} \mid P \text{ and } Q \text{ are at the same distance from the origin}\}$ be a relation, then the equivalence class of $(1, -1)$ is the set: [JEE Main 2021]

(A) $S = \{(x, y) \mid x^2 + y^2 = 1\}$ (B) $S = \{(x, y) \mid x^2 + y^2 = 4\}$
 (C) $S = \{(x, y) \mid x^2 + y^2 = \sqrt{2}\}$ (D) $S = \{(x, y) \mid x^2 + y^2 = 2\}$

14. The number of roots of the equation, $(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$ in the interval $[0, \pi]$ is equal to: [JEE Main 2021]

(A) 3 (B) 4 (C) 8 (D) 2

15. Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ be defined by $f(x) = \frac{x-2}{x-3}$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be given as $g(x) = 2x - 3$. Then, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$ is equal to [JEE Main 2021]

(A) 7 (B) 2 (C) 5 (D) 3

16. The total number of functions, $f: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ such that $f(1) + f(2) = f(3)$, is equal to: [JEE Main 2022]

(A) 60 (B) 90 (C) 108 (D) 126

17. Let $f, g: \mathbb{N} - \{1\} \rightarrow \mathbb{N}$ be functions defined by $f(a) = \alpha$, where α is the maximum of the powers of those primes p such that p^α divides a , and $g(a) = a + 1$, for all $a \in \mathbb{N} - \{1\}$. Then, the function $f + g$ is [JEE Main 2022]

(A) one-one but not onto (B) onto but not one-one
 (C) both one-one and onto (D) neither one-one nor onto

(MATHEMATICS)

18. The domain of the function

$$f(x) = \sin^{-1} [2x^2 - 3] + \log_2 \left(\log_{\frac{1}{2}} (x^2 - 5x + 5) \right)$$

where $[t]$ is the greatest integer function, is :

[JEE Main 2022]

(A) $\left(-\sqrt{\frac{5}{2}}, \frac{5-\sqrt{5}}{2}\right)$

(B) $\left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$

(C) $\left(1, \frac{5-\sqrt{5}}{2}\right)$

(D) $\left[1, \frac{5+\sqrt{5}}{2}\right)$

19. The domain of the function

[JEE Main 2022]

$$f(x) = \frac{\cos^{-1} \left(\frac{x^2 - 5x + 6}{x^2 - 9} \right)}{\log_e (x^2 - 3x + 2)}$$
 is

(A) $(-\infty, 1) \cup (2, \infty)$

(B) $(2, \infty)$

(C) $\left[-\frac{1}{2}, 1\right) \cup (2, \infty)$

(D) $\left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{ \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2} \right\}$

20. Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function such that $f(x+y) = 2f(x)f(y)$ for natural numbers x and y . If $f(1) = 2$, then the value of α for which $\sum_{k=1}^{10} f(\alpha + k) = \frac{512}{3} (2^{20} - 1)$ holds, is

[JEE Main 2022]

(A) 2

(B) 3

(C) 4

(D) 6

EXERCISE-IV

1. Let $f: (0,1) \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

[JEE -2011]

- (A) f is not invertible on $(0,1)$ (B) $f \neq f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$
 (C) $f = f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$ (D) f^{-1} is differentiable on $(0,1)$

2. The function $f: [0,3] \rightarrow [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

[JEE -2012]

- (A) one-one and onto. (B) onto but not one-one.
 (C) one-one but not onto. (D) neither one-one nor onto.

3. Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ be given by

[JEE Advanced 2014]

Then $f(x) = (\log(\sec x + \tan x))^3$

- (A) $f(x)$ is an odd function (B) $f(x)$ is a one-one function
 (C) $f(x)$ is an onto function (D) $f(x)$ is an even function

4. Let $f_1: \mathbb{R} \rightarrow \mathbb{R}$, $f_2: [0, \infty) \rightarrow \mathbb{R}$, $f_3: \mathbb{R} \rightarrow \mathbb{R}$ and $f_4: \mathbb{R} \rightarrow [0, \infty)$ be defined by [JEE Advanced 2014]

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases}; f_2(x) = x^2; f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}; \text{ and } f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0 \end{cases}$$

List-I

- (A) f_4 is
 (B) f_3 is
 (C) f_2 Of f_1 is
 (D) f_2 is

List-II

- (P) onto but not one-one
 (Q) neither continuous nor one-one
 (R) differentiable but not one-one
 (S) continuous and one-one

Codes:

	P	Q	R	S
(A)	3	1	4	2
(C)	3	1	2	4

	P	Q	R	S
(B)	1	3	4	2
(D)	1	3	2	4

5. Let $f(x) = \sin\left(\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right)$ for all $x \in \mathbb{R}$ and $g(x) = \frac{\pi}{2} \sin x$ for all $x \in \mathbb{R}$. Let $(f \circ g)(x)$ denote $f(g(x))$ and $(g \circ f)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

[JEE Advanced 2015]

- (A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (B) Range of $f \circ g$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$
 (C) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$ (D) There is an $x \in \mathbb{R}$ such that $(g \circ f)(x) = 1$

(MATHEMATICS)

6. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X , then the value of $\frac{1}{5!} (\beta - \alpha)$ is _____. **[JEE Advanced 2018]**
7. If the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is TRUE ? **[JEE Advanced 2020]**
- (A) f is one-one, but NOT onto (B) f is onto, but NOT one-one
(C) f is BOTH one-one and onto (D) f is NEITHER one-one NOR onto



ANSWER KEY FUNCTIONS

PROFICIENCY TEST-01

1. D
2. (i) $\left[-\frac{5\pi}{4}, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ (ii) $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$ (iii) $(-\infty, -3]$
 (iv) $(-\infty, -1) \cup [0, \infty)$ (v) $(3 - 2\pi < x < 3 - \pi) \cup (3 < x \leq 4)$
 (vi) $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{\sqrt{10}}\right)$ (vii) $\left(-1 < x < -\frac{1}{2}\right) \cup (x > 1)$
 (viii) $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$ (ix) $(-3, -1] \cup \{0\} \cup [1, 3)$
 (x) $\{4\} \cup [5, \infty)$ (xi) $\left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right) \cup \{x: x \in \mathbb{N}, x \geq 2\}$
 (xii) $\left(-\frac{1}{6}, \frac{\pi}{3}\right) \cup \left[\frac{5\pi}{3}, 6\right)$ (xiii) $[-3, -2) \cup [3, 4)$ (xiv) $\mathbb{R} - \left\{-\frac{1}{2}, 0\right\}$
 (xv) $2K\pi < x < (2K + 1)\pi$ but $x \neq 1$ where K is non-negative integer
3. (i) $D: x \in \mathbb{R}; R: [0, 2]$ (ii) $D: x \in \mathbb{R}; R: [-1, 1]$
 (iii) $D: \{x \mid x \in \mathbb{R}; x \neq -3; x \neq 2\}$ $R: \{f(x) \mid f(x) \in \mathbb{R}, f(x) \neq 1/5; f(x) \neq 1\}$
 (iv) $D: \mathbb{R}; R: (-1, 1)$ (v) $D: x \in \mathbb{R}; R: [5, 6)$
 (vi) $D: x \in [-1, 0]; R: [\pi/3, \pi/2]$ (vii) $D: x \in (-1, 5); R: (-\infty, 2]$
4. B 5. D
6. (i) (a) $2K\pi \leq x \leq 2K\pi + \pi$ where $K \in \mathbb{I}$; (b) $[-3/2, -1]$
 (ii) (a) Range: $[-1/3, 3]$, Domain = $[4, 7]$; (b) Range $[-1, 9]$ and domain $[11, 14]$

PROFICIENCY TEST-02

1. (a) neither surjective nor injective (b) surjective but not injective
 (c) neither injective nor surjective (d) neither injective nor surjective
 (e) neither injective nor surjective (f) Both injective and surjective
 (g) neither injective nor surjective
2. B 3. A 4. D 5. A 6. B
7. C 8. B 9. C
10. (a) $\frac{e^x - e^{-x}}{2}$ (b) $\frac{\log_2 x}{\log_2 (x-1)}$ (c) $\frac{1}{2} \log \frac{1+x}{1-x}$

PROFICIENCY TEST-03

1. (a) odd, (b) even, (c) neither odd nor even, (d) odd,
(e) neither odd nor even, (f) even, (g) even, (h) even
2. A 3. (i) B (ii) A (iii) B 4. C 5. 64 6. 30
7. 102 8. B
9. (a) $y = \log(10 - 10^x), -\infty < x < 1$
(b) $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \leq x < +\infty$
10. (i) period of fog is π , period of gof is 2π
(ii) range of fog is $[-1, 1]$, range of gof is $[-\tan 1, \tan 1]$

EXERCISE-I

1. (i) $\{x \mid 1000 \leq x < 10000\}$ (ii) $(-2, -1) \cup (-1, 0) \cup (1, 2)$ (iii) $(1, 2) \cup (2, 5/2)$;
(iv) $x \in \{4, 5\}$ (v) $x \in (3, 5) \sim \left\{\pi, \frac{3\pi}{2}\right\}$
2. (i) $D : -1 \leq x \leq 2$ R: $[\sqrt{3}, \sqrt{6}]$
(ii) $D: x \in (2n\pi, (2n+1)\pi) - \left\{2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{6}, n \in I\right\}$ and
R: $\log_a 2; a \in (0, \infty) - \{1\} \Rightarrow$ Range is $(-\infty, \infty) - \{0\}$
(iii) $D: [-4, \infty) - \{5\}; R: \left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{3}\right]$
3. (A) P, Q, R; (B) P, Q, R; (C) P, Q, R, S; (D) P, R, S
4. (a) $-3/4$; (b) 5050
5. Domain = $[-1, 5]$; Range = $[0, 3]$
7. $a \in \{0, -4\}$
8. $(\text{gof})(x) = \begin{cases} x & \text{if } x \leq 0 \\ -x^2 & \text{if } 0 < x < 1 \\ 1 - x^2 & \text{if } x \geq 1 \end{cases}; (\text{fog})(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 + x & \text{if } 0 \leq x < 1 \\ x & \text{if } x \geq 1 \end{cases}$
9. $f^{-1}(x) = 10^{\log_2(x-8)}; x = 10$ 10. 6016 11. (ii) B 13. 21
14. 5049 16. (A) S; (B) R; (C) P; (D) Q
17. (A) R; (B) S; (C) P; (D) Q

EXERCISE-II

1. $f^{-1}(1) = y$
3. (a) $\frac{1}{1002}$, (b) 1, (c) $[0, 4)$, (d) -5, (e) 3
4. 11
6. $f \circ g(x) = \begin{cases} -(1+x), & -1 \leq x \leq 0 \\ x-1, & x-1 \leq 2 \end{cases}; g \circ f(x) = \begin{cases} x+1, & 0 \leq x < 1 \\ 3-x, & 1 < x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases};$
 $f \circ f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 4-x, & 3 \leq x \leq 4 \end{cases}; g \circ g(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ x, & 0 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$
7. $x = 0$ or $5/3$
8. 1002.5
9. 20
10. 122
12. $\pm 1, \pm 3, \pm 5, \pm 15$
13. C
14. B
15. D
16. (A) - Q, R; (B) - P, Q; (C) - Q, R, S; (D) - S
17. C
18. D
19. C

EXERCISE-III

1. C
2. B
3. A
4. C
5. C
6. A
7. B
8. D
9. A
10. B
11. B
12. C
13. D
14. B
15. C
16. B
17. D
18. C
19. D
20. C

EXERCISE-IV

1. A
2. B
3. ABC
4. D
5. ABC
6. 119
7. A