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Brief paper

Target-point formation control[☆]

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ABSTRACT

In this paper a new distributed feedback strategy is proposed for controlling a rigid, acyclic formation of kinematic point-modeled mobile autonomous agents in the plane. The strategy makes use of a new concept called a "target point" and is applicable to any two-dimensional, acyclic formation whose underlying directed graph can be generated by a sequence of Henneberg vertex additions. It is shown that the method can cause a group of agents starting in any given initial positions in the plane to move into a prescribed formation exponentially fast provided the formation's designated leader and first follower start in different positions.

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1. Introduction

Multi-agent formations have been employed to perform tasks such as surveillance (Diehl, Satharishi, Hampshire, & Khosla, 1999), exploration (Burgard, Moors, Fox, Simmons, & Thrun, 2000), search and rescue (Shiroma, Chiu, Sato, & Matsuno, 2005), ocean sampling (Leonard et al., 2007) and space missions (Krieger, Hajnsek, Papathanssiou, Younis, & Moreira, 2010). By a *multi-agent formation* is meant a collection of autonomous agents in which the distance between every pair of the agents is a prescribed constant as time evolves. Assuming the kinematic model of each agent is a single integrator (Krick, Broucke, & Francis, 2009), double-integrator (Olfati-Saber & Murray, 2002) or nonholonomic (Desai, Ostrowski, & Kumar, 2001), the problem of distributed formation control is to maintain a multi-agent formation by choosing a control input for each agent using the agent's local sensed information about its neighbors. The local measurement of each agent can be range only (Cao, Yu, & Anderson, 2011), bearing only (Basiri, Bishop, & Jensfelt, 2010) or relative positions (Dorfler & Francis, 2010). When a multi-agent formation is rigid (Anderson, Yu, Fidan, & Hendrickx, 2008; Asimow & Roth, 1979), the formation can be achieved by maintaining the desired distances between some chosen pairs of agents. If each such distance in a formation is maintained by both associated agents, the formation is *undirected*; otherwise, it is *directed*, in which the agent assigned with the task of maintaining the desired distance is called a *follower* and the other agent is correspondingly called its *leader*.

In the research line of controlling undirected formations, perhaps the most comprehensive distributed method based on rigidity is the gradient control proposed in Krick et al. (2009). It has been shown by center manifold theory that the gradient control locally stabilizes a large class of rigid undirected formations. Recent studies in Belabbas, Mou, Morse, and Anderson (2012), Helmke, Mou, Sun, and Anderson (2014) and Mou, Morse, Belabbas, Sun, and Anderson (in press) have revealed that undirected formations are problematic in the sense that they will rotate in the plane under the gradient control if there exists inconsistency in neighboring agents' distance measurements. Although progress to fix such an issue has recently been made in Mou, Morse, and Anderson (2014), Mou and Morse (2014) and Marina, Cao, and Jayawardhana (2015), how to achieve robustness in controlling rigid undirected formations under inconsistent measurements between neighboring agents is still an open problem. Growing interests have then been given to directed formations because of their known additional advantage of less usage of sensing

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and communication capabilities. Along this direction, sufficient and necessary graphical conditions have been derived for driving directed formations to a rendezvous point or a line (Lin, Francis, & Maggiore, 2005); switching has been introduced in the control design to stabilize a class of minimal persistent directed formations in Fidan, Gazi, Zhai, Cen, and Karatas (2013) and Sandeep, Fidan, and Yu (2006); and a virtual leader strategy has been employed in Ogren, Egerstedt, and Hu (2002). However, controlling directed formations in a distributed manner is challenging even for a four-agent formation consisting of two cycles (Belabbas, 2013). Instead of attacking the problem of controlling directed formations in general, researchers have recently focused on the class of minimally rigid acyclic formations (Baillieul & Suri, 2003; Ding, Yan, & Lin, 2010; Fidan et al., 2013; Mou et al., 2011; Sandeep et al., 2006).

The authors of Cao, Morse, Yu, Anderson, and Dasgupta (2011) have shown that the gradient control is able to stabilize an acyclic triangular formation if the three agents are not initialized collinear. Otherwise, the formation will drift to infinity with agents remaining in collinear positions. One reason for the difficulty in the global analysis of gradient control is the possibility of the formation converging to some equilibria determined by the local minima of the associated potential functions calculating the gradient. What this implies in terms of the evolution of the formation shape dynamics is that there are initial agent positions that may lead to incorrect formation shapes when agents are under gradient control. For example, a four-agent minimally rigid acyclic formation will fail to converge to its desired shape if its three-agent sub-formation starts with a wrong orientation. Thus there is an urgent need to look at other types of control to overcome these problems and complements the existing gradient control. And this is exactly the aim of this paper.

In this paper we utilize the idea of having the follower agents tracking their "target points", which are the desired relative positions for the follower agent to move to, to develop another type of formation control. Note that in a minimally rigid acyclic formation there is one agent called the *global leader*, which has no leader to follow, one agent called the *first follower*, which only follows the global leader, and each of all the other agents has exactly two leaders (Hendrickx, Anderson, Delvenne, & Blondel, 2007). In the following we will consider the case when an agent has one leader and the case when an agent has two leaders, separately, and propose a distributed control based on target points for both cases. Such analytical results will then be further used to prove the main result of this paper, namely, the proposed target-point control is able to drive minimally rigid acyclic formations to converge to their desired shapes as long as the global leader and the follower are not coincident.

2. Problem formulation

Consider the class of minimally rigid acyclic formations in the plane consisting of $n \geq 2$ mobile autonomous agents. It has been shown in Hendrickx et al. (2007) that any such an n-agent formation can be generated by the Henneberg vertex addition operations. And in addition that the agents can be labeled such that starting from a two-agent directed formation in which agent 2 follows agent 1, one adds in sequence an agent i, $3 \leq i \leq n$, which chooses two agents $i, j \in \{1, 2, \ldots, i-1\}$ as its leaders. We further suppose the distance between a follower and any of its leaders and the distance between the two leaders of agent i, $i = 3, 4, \ldots, n$ are positive. We call this class of formations *vertex-addition formations*. For example, the formations in Fig. 1(a)–(c) are vertex-addition formations while the one in Fig. 1(d) is not.

Assume each agent's motion in the plane is described by a simple kinematic point model

$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

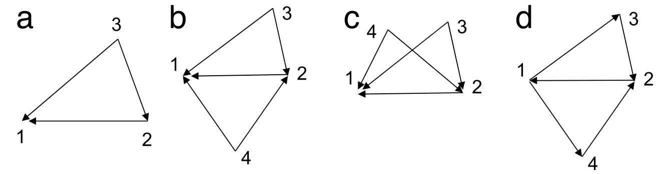

Fig. 1. Minimally rigid directed formations.

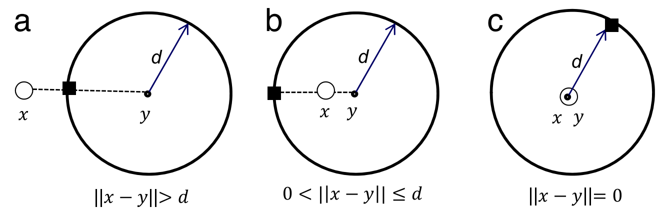

Fig. 2. Target points in the one-leader case.

where $x_i \in \mathbb{R}^2$ denotes the position of agent i and u_i denotes agent i's control input. Suppose each agent i's local measurements are $x_i - x_j$, the relative position away from each of its leader j. The goal of distributed formation control is to design u_i using agent i's local measurements such that the formation converges to its *desired shape*, that is, the distances between agent i, $i = 2, \ldots, n$ and each of its leader converge to the prescribed constants respectively, and each agent i, $i = 3, 4, \ldots, n$ and its two leaders are in desired clockwise or counter-clockwise orientation.

3. Target-point control

In a vertex-addition formation, a follower agent has at most two leaders. In the following we will devise target-point controls for the follower for both the one-leader case and the two-leader case.

3.1. One-leader case

Consider a two-agent directed formation in the plane with one follower and one leader, whose positions are denoted by $x, y \in \mathbb{R}^2$, respectively. Suppose the follower's motion in the plane is modeled by $\dot{x} = u$. The goal in this case is to choose u in terms of the follower's local measurement $x - y$ such that $\|x - y\|$ converges to the desired constant d. To accomplish this we define the following *target point*

$$\tau(x, y) = \begin{cases} x + \dfrac{\|y - x\| - d}{\|y - x\|}(y - x), & x \neq y; \\ y + \begin{bmatrix} d & 0 \end{bmatrix}', & x = y, \end{cases}$$

which is indicated by a black square in Fig. 2.

Note that the target point $\tau(x, y)$ is such that $\|\tau(x, y) - y\| = d$. One way to move the follower to keep d distance away from its leader is to drive x to converge to $\tau(x, y)$. Inspired by this observation, we choose

$$u = -\lambda(x - \tau(x, y)),$$

where λ is a non-negative parameter to be designed such that the control u is continuous. One choice for λ is

$$\lambda = \|y - x\|(\|y - x\| + d),$$

which leads to the following *target-point control* in one-leader case

$$u = -(\|x - y\|^2 - d^2)(x - y). \tag{2}$$

Note that the target-point control (2) is distributed in the sense that its implementation only requires the follower's local measurement $x - y$, and nothing else.

Let $z = x - y$ and $v = \dot{y}$, where v is assumed to be at least piecewise continuous. From (2) and $\dot{x} = u$, one has

$$\dot{z} = (d^2 - \|z\|^2)z - v. \tag{3}$$

Since system (3) is locally Lipschitz continuous in z, for each $z(0) \in \mathbb{R}^2$, there exists a maximal interval $[0, T)$ and a unique continuous solution to (3) on $[0, T)$ that starts at $z(0)$ at $t = 0$. Actually the maximal interval of existence is $[0, \infty)$. To see why this is so, we suppose T is finite and let $z^* = \lim_{t\to T} z(t)$. Then there exists an interval $[T, \bar{T})$ of positive length, on which there were a continuous solution $\bar{z}(t)$ to (3) passing through z^* at $t = T$. This in turn would imply that the concatenated function θ defined by $\theta(t) = z(t), t \in [0, T)$ and $\theta(t) = \bar{z}(t), t \in [T, \bar{T})$ would be a continuous function to (3) passing through z_0 at $t = 0$. Then $\bar{T} > T$, which contradicts to the assumption that $[0, T)$ is the maximal interval of existence.

Note that if $v = 0$, system (3) has two equilibrium sets, namely the point $z = 0$ and the set of z for which $\|z\| = d$. The former is repulsive and the latter is attractive. Thus, if $v = 0$, all trajectories starting outside the set for $z = 0$ remain bounded away from $z = 0$. The assumption $v = 0$ will typically lead to easier analysis, which is also often made in relevant references (Cao et al., 2011). Perturbing (3) with nonzero v will change everything. However, if v is “small” in some sense one might still expect trajectories starting outside $z = 0$ to be bounded away from $z = 0$ for all t as shown in the following lemma, the proof of which will be given in the Appendix.

Lemma 1. *Suppose $z(0)$ is nonzero, v converges to 0 exponentially fast as t goes to infinity and $\int_0^\infty \|v\|dt < \min\{\|z(0)\|, d\}$. Then y converges exponentially fast to a constant vector y^* and x converges exponentially fast to a constant vector x^* such that $\|x^* - y^*\| = d$.*

Although in the one-leader case the target-point control (2) is the same as the gradient control used in Krick et al. (2009), it has been derived in a different way. The paper (Cao et al., 2011) has shown that the gradient control drives $\|x - y\|$ to converge to d exponentially fast assuming that the leader keeps still, that is, $\dot{y} = 0$. We have relaxed this assumption in Lemma 1 by showing that if the leader moves sufficiently slowly, the target-point control (2) still causes $\|x - y\|$ to converge exponentially fast to d.

3.2. Two-leader case

Consider a three-agent formation in the plane with one follower agent and two leaders labeled by 1 and 2, whose positions are denoted by $x, y_1, y_2 \in \mathbb{R}^2$ respectively. The motion of the follower is modeled by $\dot{x} = u$. Suppose the follower is able to measure $x - y_1$ and $x - y_2$. Let d_1 and d_2 denote the desired distances for the follower to keep away from its leaders, in which without loss of generality we assume that $d_1 \geq d_2$. The aim in this case is to choose u in terms of $x - y_1$ and $x - y_2$ such that x converges to a *desired position* x^*, at which $\|x^* - y_1\| = d_1$, $\|x - y_2\| = d_2$, and x^*, y_1 and y_2 are in desired clockwise or counter-clockwise orientation. To accomplish this goal we introduce the *target point* $\tau(y_1, y_2)$ in this two-leader case to be the point indicated by a black square in Fig. 3 when $y_1 \neq y_2$.

When $y_1 = y_2$, we simply let $\tau(y_1, y_2) = y_1$ to guarantee that the target point is well-defined. Then

$$\tau(y_1, y_2) = \begin{cases} y_1 + \dfrac{d_1}{\|y_1 - y_2\|}R(s)(y_2 - y_1), & y_1 \neq y_2; \\ y_1, & \text{otherwise}, \end{cases} \tag{4}$$

where

$$R(s) = \begin{bmatrix} s & \gamma\sqrt{1-s^2} \\ -\gamma\sqrt{1-s^2} & s \end{bmatrix}$$

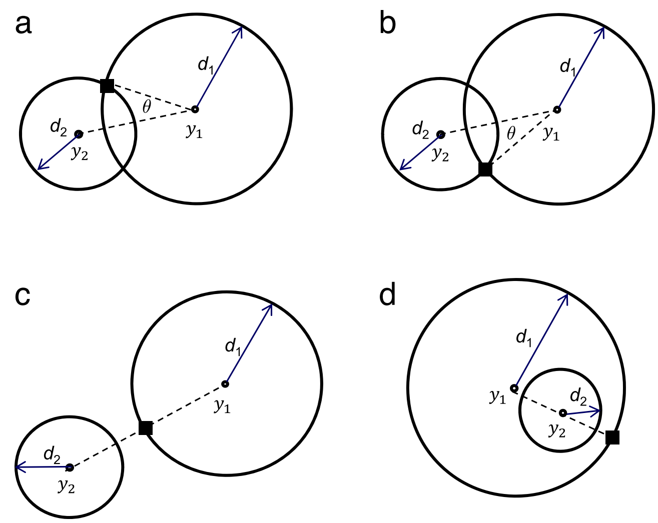

Fig. 3. (a) $d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2$ and the follower and its leaders 1 and 2 are clockwisely oriented; (b) $d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2$ and the follower and its two leaders 1 and 2 are counter-clockwisely oriented; (c) $\|y_1 - y_2\| \geq d_1 + d_2$; (d) $0 < \|y_1 - y_2\| \leq d_1 - d_2$.

with

$$s = \begin{cases} 1 + \dfrac{(\|y_1 - y_2\| - (d_1 + d_2))(\|y_1 - y_2\| - (d_1 - d_2))}{2d_1\|y_1 - y_2\|}, \\ \quad d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2; \\ 1, \quad \text{otherwise}, \end{cases}$$

and

$$\gamma = \begin{cases} 1 & \text{if the clockwise orientation is desired;} \\ -1 & \text{otherwise.} \end{cases}$$

When $d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2$, note that $s = \cos\theta$, where θ is indicated in Fig. 3, the matrix $R(s)$ is just a rotation matrix, and thus $\tau(y_1, y_2)$ is the desired position for x. One choice for u to move x to the target point is

$$u = -\lambda(x - \tau(y_1, y_2)),$$

where λ is a non-negative parameter. Note that $R(s)$ is continuous for $y_1 \neq y_2$, so is $\tau(y_1, y_2)$. In order to get a continuous u, one could choose $\lambda = \|y_1 - y_2\|^2$, which leads to the following target-point control

$$u = -\|y_1 - y_2\|^2(x - y_1) - d_1\|y_1 - y_2\|R(s)(y_1 - y_2). \tag{5}$$

Since $y_1 - y_2 = (x - y_2) - (x - y_1)$ and the follower is able to measure $x - y_1$ and $x - y_2$, then $y_1 - y_2$ is available to the follower. So is $R(s)$. Thus the target-point control (5) can be implemented using the follower's local measurements.

Under the target-point control one has

$$\dot{x} = -\|y_1 - y_2\|^2 x + \|y_1 - y_2\|^2 \tau(y_1, y_2) \tag{6}$$

which is a forced linear differential equation. When y_1, y_2 are bounded and continuous, there is a unique solution to (6) for $t \in [0, \infty)$ for each $x(0)$. Convergence of the follower is shown in the following lemma, the proof of which will be given in the Appendix.

Lemma 2. *Suppose y_1 and y_2 converge exponentially fast to the constant vectors y_1^*, y_2^*, respectively, where $d_1 - d_2 < \|y_1^* - y_2^*\| < d_1 + d_2$. Then under the target-point control (5), x converges exponentially fast to a constant vector x^* such that $\|x^* - y_1^*\| = d_1$, $\|x^* - y_2^*\| = d_2$, and the three-agent formation is correctly oriented.*

The gradient control proposed in Krick et al. (2009) in the two-leader case will lead to the following dynamics for x:

$$\dot{x} = (\|x - y_1\|^2 - d_1^2)(y_1 - x) + (\|x - y_2\|^2 - d_2^2)(y_2 - x). \quad (7)$$

Note that the equilibrium set of (7) is a large set, which contains the points for which x, y_1, y_2 are collinear or in wrong orientations. This is perhaps the major reason why the gradient control fails in controlling a three-agent triangular formation if their initial positions are collinear or controlling a four-agent vertex-addition formation if its sub-formation starts with the wrong orientation. In contrast, system (6) has only one equilibrium point $x = \tau(y_1, y_2)$ when $y_1 \neq y_2$. If further $d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2$, this equilibrium point is the desired position for x. Thus no matter whether x is collinear with y_1 and y_2 or they are in a wrong orientation, the target-point control (5) drives x to its desired position as long as $d_1 - d_2 < \|y_1 - y_2\| < d_1 + d_2$ as claimed in Lemma 2.

3.3. Main result

The main result of this paper is as follows:

Theorem 1. *Given an n-agent vertex-addition formation with $n \geq 2$ which is generated and labeled as stated in Section 2. Assume $\dot{x}_1$ converges exponentially fast to 0, $x_1(0) \neq x_2(0)$ and $\int_0^\infty \|\dot{x}_1\| dt < \min\{\|x_1(0) - x_2(0)\|, d\}$, where d is the desired distance for agent 2 to keep away from agent 1. By Applying the target-point control (2) to agent 2 and (5) to agent i, $i = 3, \ldots, n$ when $n \geq 3$, one has the n-agent vertex-addition formation converges exponentially fast to its desired shape.*

Proof of Theorem 1. We prove Theorem 1 by induction. When $n = 2$, Lemma 1 implies that x_1 and x_2 converge exponentially fast to the constant vectors $x_1^*, x_2^* \in \mathbb{R}^2$, respectively, such that $\|x_2^* - x_1^*\| = d$. Then Theorem 1 holds for $n = 2$.

Assume that result holds for all k-agent vertex-addition formations with $k \geq 2$. Then in any $(k+1)$-agent vertex-addition formation one has agent $k+1$ follows two other agents $1 \leq i, j \leq k$ of a k-agent formation. Let $x, y_1, y_2 \in \mathbb{R}^2$ denote the positions of agent $k+1$, agent i and agent j, respectively. Let d_1 and d_2 denote the prescribed distances for agent $k+1$ to keep from agents i and j respectively. Without loss of generality, we assume $d_1 \geq d_2$. From the induction assumption, one has that y_1 and y_2 converge to constants y_1^* and y_2^* respectively exponentially fast and $d_1 - d_2 < \|y_1^* - y_2^*\| < d_1 + d_2$. From Lemma 2, one has x converges exponentially fast to a constant $x^* \in \mathbb{R}^2$ such that $\|x^* - y_1^*\| = d_1$, $\|x^* - y_2^*\| = d_2$ and the three-agent sub-formation consisting of agents i, j, and $k+1$ are correctly oriented. Then Theorem 1 holds for $n = k+1$. We complete the proof. ■

4. Simulations

Example 1. Consider the three-agent formation in Fig. 1(a) with the prescribed distances $d_1 = 3$, $d_2 = 4$, and $d_3 = 5$. Take agent 1's velocity to be $v = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$. The three agents are initially positioned collinearly. The simulation result shown in Fig. 4 implies that the formation still converges to its desired shape.

Example 2. Consider the four-agent formation in Fig. 1(b) with the same d_1, d_2, d_3 as in Example 1, and in addition $d_4 = 5$ and $d_5 = 4$. Take $v = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$. The agents' initial positions are shown in Fig. 1(c), where the sub-formation of agents 1, 2 and 4 starts with the wrong orientation. In this case the formation still converges to its desired shape as shown in Fig. 5.

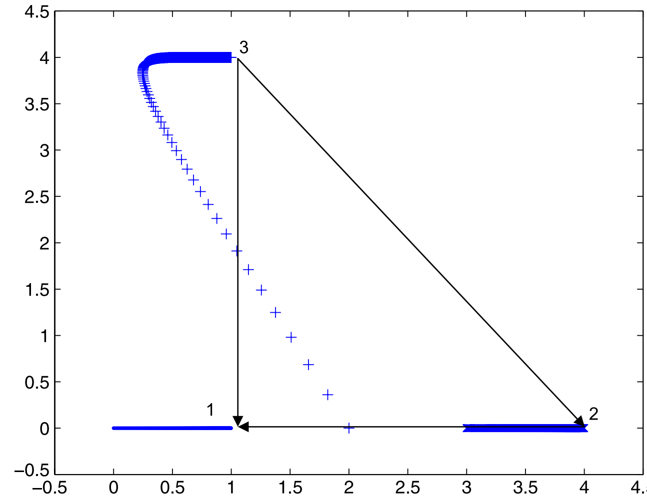

Fig. 4. Convergence of the three-agent formation.

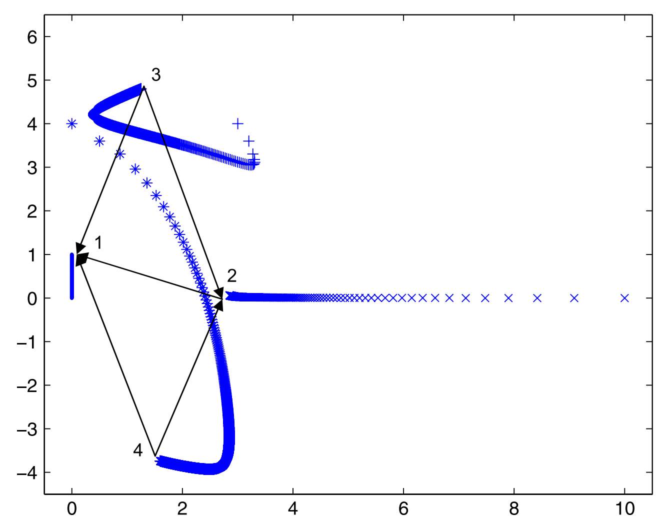

Fig. 5. Convergence of the four-agent formation.

5. Concluding remarks

This paper has proposed a target-point control which globally stabilizes the class of vertex-addition formations. We are currently working on implementing the proposed control on a mobile robotic testbed, for which practical issues, such as measurement noise, actuation errors and collision avoidance, will be considered.

Appendix

Proof of Lemma 1. Since $\dot{y} = v$ converges to 0 exponentially fast, one has y converges to a constant $y^* \in \mathbb{R}^2$ exponentially fast. To prove that x converges exponentially fast to a constant vector $x^* \in \mathbb{R}^2$ such that $\|x^* - y^*\| = d$, it is sufficient to show that $V = e^2$ converges to 0 exponentially fast, where $e = \|z\|^2 - d^2$.

First, We show that z is upper bounded. Along the solution to (3) on the maximal interval $[0, \infty)$, one has

$$\begin{aligned} \dot{V} &= -4\|z\|^2 V - 4ez'v \\ &\leq -4\|z\|^2 V + 4\|ez\|\|v\| \quad (8) \\ &\leq 4\|ez\|\|v\|. \quad (9) \end{aligned}$$

Since $\|ez\|^2 \le e^2(e+d^2) \le (\frac{e^2+e+d^2}{2})^2 \le (\frac{3}{4}e^2+\frac{1}{4}+\frac{1}{2}d^2)^2$, it must be true that

$$\|ez\| \le \frac{3}{4}e^2+\frac{1}{4}+\frac{1}{2}d^2. \quad (10)$$

From (9) and (10), one has

$$\dot{V} \le 3\|v\|V + (1+2d^2)\|v\|,$$

which implies that

$$V(t) \le V(0) + (1+2d^2)\int_0^t \|v(s)\|ds + 3\int_0^t \|v(s)\|V(s)ds.$$

It follows from Bellman–Gronwall Lemma (Bellman, 1943) that

$$V(t) \le (V(0)+(1+2d^2)a)e^{3a},$$

where $a = \min\{\|z(0)\|, d\}$. Then V is bounded on $[0,\infty)$, and so are e and z.

Second, we prove by contradiction that z is lower bounded away from 0 by showing

$$\|z(t)\| \ge b, \quad (11)$$

where $b = a - \int_0^\infty \|v\|d\tau > 0$. Assume that (11) is false. Then there exists a finite t_2 such that

$$\|z(t_2)\| < a - \int_0^\infty \|v\|dt. \quad (12)$$

Furthermore, since $\|z(0)\| > 0$, one can choose t_2 in such a way that $\|z(t)\| > 0$, $t \in [0,t_2]$. Since $\|z(0)\| \ge a$ and $\|z(t_2)\| < a$, there must exist a time $t_1 \in [0,t_2)$ such that $\|z(t_1)\| = a$ and $0 < \|z(t)\| \le a$ for $t \in [t_1,t_2]$. From (3), one has

$$\begin{aligned}\frac{d}{dt}\|z\| &= (d^2-\|z\|^2)\|z\| - v'\frac{z}{\|z\|}\\ &\ge (d^2-\|z\|^2)\|z\| - \|v\|, \quad t\in[t_1,t_2]\end{aligned}$$

from which, $\|z\| \le a$ for $t \in [t_1,t_2]$ and $a \le d$, one has

$$\frac{d}{dt}\|z\| \ge -\|v\|, \quad t \in [t_1,t_2].$$

Then

$$\|z(t_2)\| \ge \|z(t_1)\| - \int_{t_1}^{t_2}\|v\|dt \ge a - \int_0^\infty \|v\|dt$$

which contradicts to (12). Thus (11) is true.

Finally, we show $V(t)$ converges to 0 exponentially fast. Since e and z are bounded, and v converges to 0 exponentially fast, there exists a positive constant μ such that

$$V(0) + 4\int_0^\infty \|ez\|\|v\|dt \le \mu. \quad (13)$$

From (8), (11) and (13), one has

$$V(t) \le -4b\int_0^t V(s)ds + \mu,$$

which together with Bellman–Gronwall Lemma (Bellman, 1943) leads to

$$V(t) \le \mu e^{-4bt}.$$

We complete the proof. ■

Proof of Lemma 2. First, we show that there exists a finite time T such that x converges exponentially fast to $\tau(y_1^*,y_2^*)$ for $t \in [T,\infty)$. Since y_1 and y_2 converge exponentially fast to y_1^* and y_2^* respectively, where $0 \le d_1-d_2 < \|y_1^*-y_2^*\| < d_1+d_2$. Then there exists a finite time T and a positive constant γ such that

$$d_1-d_2 < \gamma \le \|y_1-y_2\| < d_1+d_2, \quad t\in[T,\infty). \quad (14)$$

From (14) and (6), one has

$$\dot{x} \le -\gamma(x-\tau(y_1,y_2)), \quad t\in[T,\infty). \quad (15)$$

Let $\epsilon = x-\tau(y_1^*,y_2^*)$ and $w = \tau(y_1,y_2)-\tau(y_1^*,y_2^*)$. Then $\dot{\epsilon} = \dot{x}$, which together with (15) implies

$$\dot{\epsilon} \le -\gamma\epsilon + \gamma w, \quad t\in[T,\infty). \quad (16)$$

From (14) and the definition of $\tau(y_1,y_2)$ in (4), one has $\tau(y_1,y_2)$ is continuously differentiable with respect to y_1 and y_2 for $t \in [T,\infty)$. Note in addition that y_1 and y_2 converge exponentially fast to y_1^* and y_2^*, respectively, for $t\in[T,\infty)$. Then $\tau(y_1,y_2)$ converges to $\tau(y_1^*,y_2^*)$ exponentially fast for $t\in[T,\infty)$. Then w converges to 0 exponentially fast for $t\in[T,\infty)$, from which, γ is a positive constant and (16), one has ϵ converges to 0 for $t\in[T,\infty)$. So does $\|x-\tau(y_1^*,y_2^*)\|$.

Second, we show that $\|x(t)-\tau(y_1^*,y_2^*)\|$ is bounded for $t \in [0,T]$. Since y_1 and y_2 converge to y_1^* and y_2^* exponentially fast, we know that $\|y_1-y_2\| \le a$ for $t\ge 0$ and $a \triangleq \max\{d_1+d_2, \|y_1(0)-y_2(0)\|\}$. Then it must be true that there exists a positive constant b such that

$$\begin{aligned}b \ge{}& \|x(0)\| + a^2\int_0^t \|y_1(s)\|ds\\ &+ ad_1\int_0^t \|y_1(s)-y_2(s)\|ds + \|\tau(y_1^*,y_2^*)\|.\end{aligned}$$

From (5) and $\dot{x} = u$, one has

$$\dot{x} = -\|y_1-y_2\|^2(x-y_1) - d_1\|y_1-y_2\|R(s)(y_1-y_2).$$

Then

$$\begin{aligned}\|x(t)-\tau(y_1^*,y_2^*)\| &\le \|x(t)\| + \|\tau(y_1^*,y_2^*)\|\\ &\le b + a^2\int_0^t \|x(s)\|ds.\end{aligned}$$

From Bellman–Gronwall Lemma (Bellman, 1943), one has

$$\|x(t)-\tau(y_1^*,y_2^*)\| \le be^{a^2T}, \quad t\in[0,T]. \quad (17)$$

Third, we prove x converges to $\tau(y_1^*,y_2^*)$ exponentially fast for $t\in[0,\infty)$ by showing the convergence of x to $\tau(y_1^*,y_2^*)$ is bounded by an exponentially decaying signal. Note that x converges to $\tau(y_1^*,y_2^*)$ exponentially fast for $t\in[T,\infty)$, one has there exist positive constants c and λ such that

$$\|x(t)-\tau(y_1^*,y_2^*)\| \le ce^{-\lambda t}, \quad t\in[T,\infty)$$

and

$$\|x(T)-\tau(y_1^*,y_2^*)\| = ce^{-\lambda T}. \quad (18)$$

Let $\bar{c} = be^{(a^2+\lambda)T}$. From (17) and (18), one has $ce^{-\lambda T} \le be^{a^2T}$. It follows that $c \le \bar{c}$. Then one has

$$\|x(t)-\tau(y_1^*,y_2^*)\| \le \bar{c}e^{-\lambda t}, \quad t\in[T,\infty). \quad (19)$$

In view of (17) and the fact that $be^{a^2T} \le \bar{c}e^{-\lambda t}$ for $t\in[0,T]$, one has

$$\|x(t)-\tau(y_1^*,y_2^*)\| \le \bar{c}e^{-\lambda t}, \quad t\in[0,T]. \quad (20)$$

From (19) and (20), one has

$$\|x(t)-\tau(y_1^*,y_2^*)\| \le \bar{c}e^{-\lambda t} \quad t\in[0,\infty).$$

Thus $x(t)$ converges to $\tau(y_1^*, y_2^*)$ exponentially fast. Note that $\tau(y_1^*, y_2^*)$ is the desired position for x since $d_1 - d_2 < \|y_1^* - y_2^*\| < d_1 + d_2$. Therefore Lemma 2 is true. ■

References

Anderson, B. D. O., Yu, C., Fidan, B., & Hendrickx, J. M. (2008). Rigid graph control architectures for autonomous formations. *IEEE Control Systems*, *28*(6), 48–63.

Asimow, L., & Roth, B. (1979). The rigidity of graphs ii. *Journal of Mathematical Analysis and Applications*, *68*, 171–190.

Baillieul, J., & Suri, A. (2003). Information patterns and hedging brockett's theorem in controlling vehicle formations. In: *Proc. of the 42th conference on decision and control* (pp. 556–563).

Basiri, M., Bishop, A. N., & Jensfelt, P. (2010). Distributed control of triangular formations with angle-only constraints. *Systems & Control Letters*, *59*, 147–154.

Belabbas, M. A. (2013). Decentralized design with localized objective in formation control. In: *Proc. of the 2013 American control conference* (pp. 1310–1315).

Belabbas, M. A., Mou, S., Morse, A. S., & Anderson, B. D. O. (2012). Robustness issues with undirected formations. In; *Proc. of the 51st conference on decision and control* (pp. 1445–1450).

Bellman, R. (1943). The stability of solutions of linear differential equations. *Duke Math Journal*, *10*, 643–647.

Burgard, W., Moors, M., Fox, D., Simmons, R., & Thrun, S. (2000). Collaborative multi-robot exploration. In: *Proc. of the 2000 IEEE conference on robotics and automation* (pp. 476–481).

Cao, M., Morse, A. S., Yu, C., Anderson, B. D. O., & Dasgupta, S. (2011). Maintaining a directed, triangular formation of mobile autonomous agents. *Communications in Information and Systems*, *11*, 1–16.

Cao, M., Yu, C., & Anderson, B. D. O. (2011). Formation control using range-only measurements. *Automatica*, (4), 776–781.

Desai, J. P., Ostrowski, J. P., & Kumar, V. (2001). Modeling and control of formations of nonholonomic mobile robots. *IEEE Transactions on Robotics and Automation*, (6), 905–908.

Diehl, P., Satharishi, M., Hampshire, J., & Khosla, P. (1999). Collaborative surveillance using both fixed and mobile unattended groud sensor platforms. In: *Proc. of SPIE's 13th annual international symposium on aerospace defense sensing, simulation, and controls*.

Ding, W., Yan, G., & Lin, Z. (2010). Collective motions and formations under pursuit strategies on directed acyclic graphs. *Automatica*, *46*, 174–181.

Dorfler, F., & Francis, B. (2010). Geometric analysis of the formation problem for autonomous robots. *IEEE Transactions on Automatic Control*, *55*(10), 2379–2384.

Fidan, B., Gazi, V., Zhai, S., Cen, N., & Karatas, E. (2013). Single-view distance-estimation-based formation control of robotic swarms. *IEEE Transactions on Industrial Electronics*, (12), 5781–5791.

Helmke, U. R., Mou, S., Sun, Z., & Anderson, B. D. O. (2014). Geometrical methods for mismatched formation control. In: *Proc. of the 53rd conference on decision and control* (pp. 1341–1346).

Hendrickx, J. M., Anderson, B. D. O., Delvenne, J. C., & Blondel, V. D. (2007). Directed graphs for the analysis of rigidity and persistence in autonomous agent systems. *International Journal of Robust and Nonlinear Control*, *17*, 960–981.

Krick, L., Broucke, M., & Francis, B. (2009). Stabilization of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, *82*(3), 423–439.

Krieger, G., Hajnsek, I., Papathanssiou, K. P., Younis, M., & Moreira, A. (2010). Interferometric synthetic aperture radar (sar) missions employing formation flying. *Proceedings of IEEE*, *98*(5), 816–843.

Leonard, N. E., Paley, D. A., Lekien, F., Sepulchre, R., Fratantoni, D. M., & Davis, R. E. (2007). Collective motion, sensor networks, and ocean sampling. *Proceedings of IEEE*, *1*, 48–74.

Lin, Z., Francis, B., & Maggiore, M. (2005). Necessary and suficient conditions for formation control of unicyles. *IEEE Transactions on Automatic Control*, *50*(1), 121–127.

Marina, H., Cao, M., & Jayawardhana, B. (2015). Controlling rigid formations of mobile agents under inconsistent measurements. *IEEE Transactions on Robotics*, *31*(1), 31–39.

Mou, S., Cao, M., & Morse, A. S. (2011). A distributed control law for acyclic formations. In: *Proc. of the 18th IFAC congress* (pp. 7818–7823).

Mou, S., & Morse, A. S. (2014). Fix the non-robustness issue for a class of minimally rigid undirected formations. In: *American control conference* (pp. 2348–2352).

Mou, S., Morse, A. S., & Anderson, B. D. O. (2014). Toward robust control of minimally rigid undirected formations. In: *Proc. of the 53rd conference on decision and control* (pp. 643–647).

Mou, S., Morse, A. S., Belabbas, M. A., Sun, Z., & Anderson, B. D. O. (2015). Undirected formations are problematic. *IEEE Transactions on Automatic Control* (In press. http://dx.doi.org/10.1109/TAC.2015.2414771).

Ogren, P., Egerstedt, M., & Hu, X. (2002). A control lyapunov function approach to multiagent coordination. *IEEE Transactions on Robotics and Automation*, (5), 847–851.

Olfati-Saber, R., & Murray, R. M. (2002). Distributed cooperative control of multiple vehicle formations using structural potential functions. In: *Proc. of the 15th IFAC world congress* (pp. 346–352).

Sandeep, S., Fidan, B., & Yu, C. (2006). Decentralized cohensive motion control of multi-agent coordination. In: *Proc. of the 14th Mediterranean conference on control and automation*.

Shiroma, N., Chiu, Y., Sato, N., & Matsuno, F. (2005). Cooperative task excecution of a search and rescue mission by a multi-robot team. *Advanced Robotics*, (3), 311–329.

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