61A Lecture 20

Monday, March 11

Announcements

- Project 3 due Thursday 3/12 @ 11:59pm
  - Project party on Tuesday 3/10 5pm-6:30pm in 2050 VLSB
  - Bonus point for early submission by Wednesday 3/11
- Guerrilla section this weekend on recursive data (linked lists and trees)
- Homework 6 due Monday 3/16 @ 11:59pm
- Midterm 2 is on Thursday 3/19 7pm-9pm
  - Fill out conflict form if you cannot attend due to a course conflict

Time

The Consumption of Time

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

```
def factors(n):
```

	Time (number of divisions)
Slow: Test each k from 1 through n	n
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	Greatest integer less than $\sqrt{n}$

(Demo)

Space

The Consumption of Space

Which environment frames do we need to keep during evaluation?

At any moment there is a set of active environments

Values and frames in active environments consume memory

Memory that is used for other values and frames can be recycled

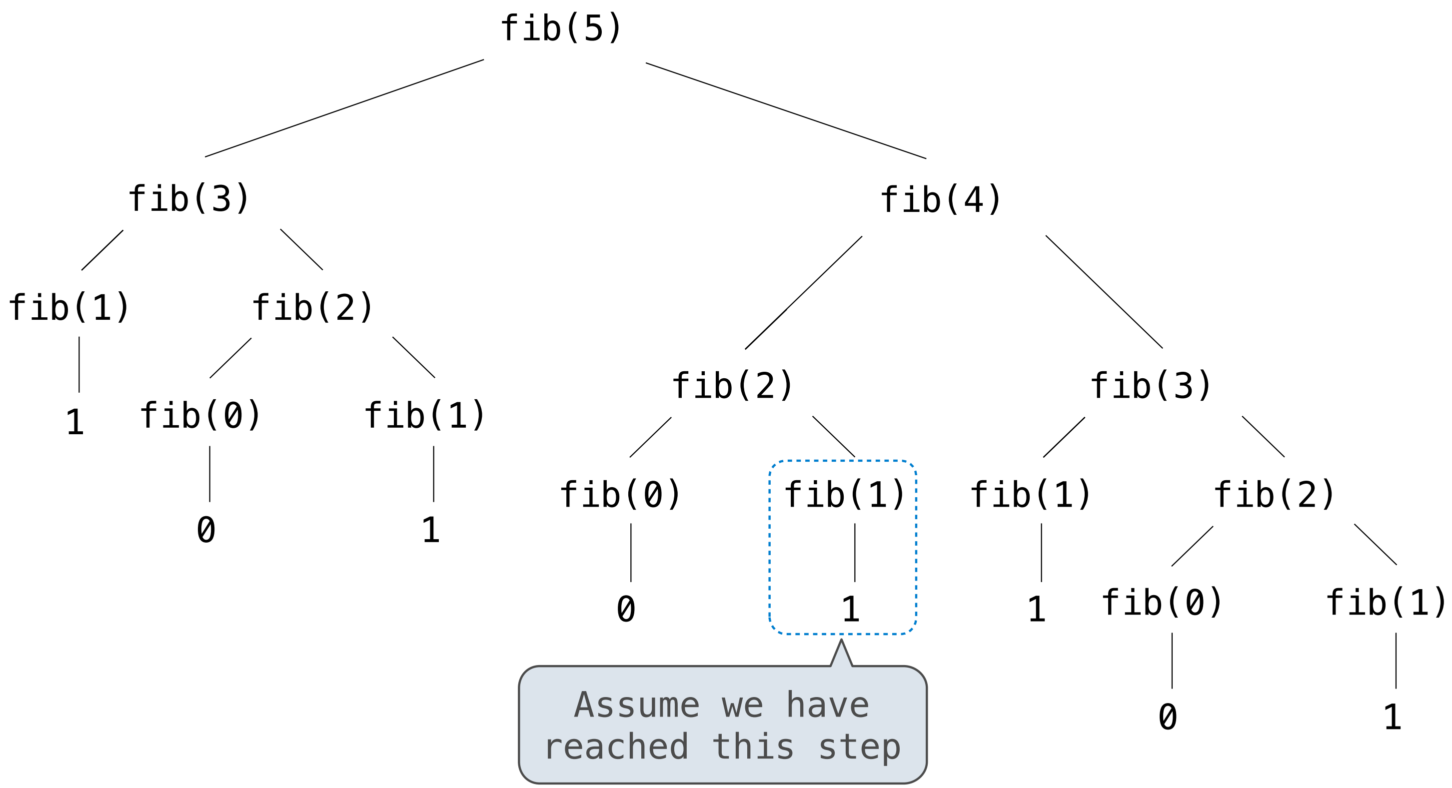
Active environments:

- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

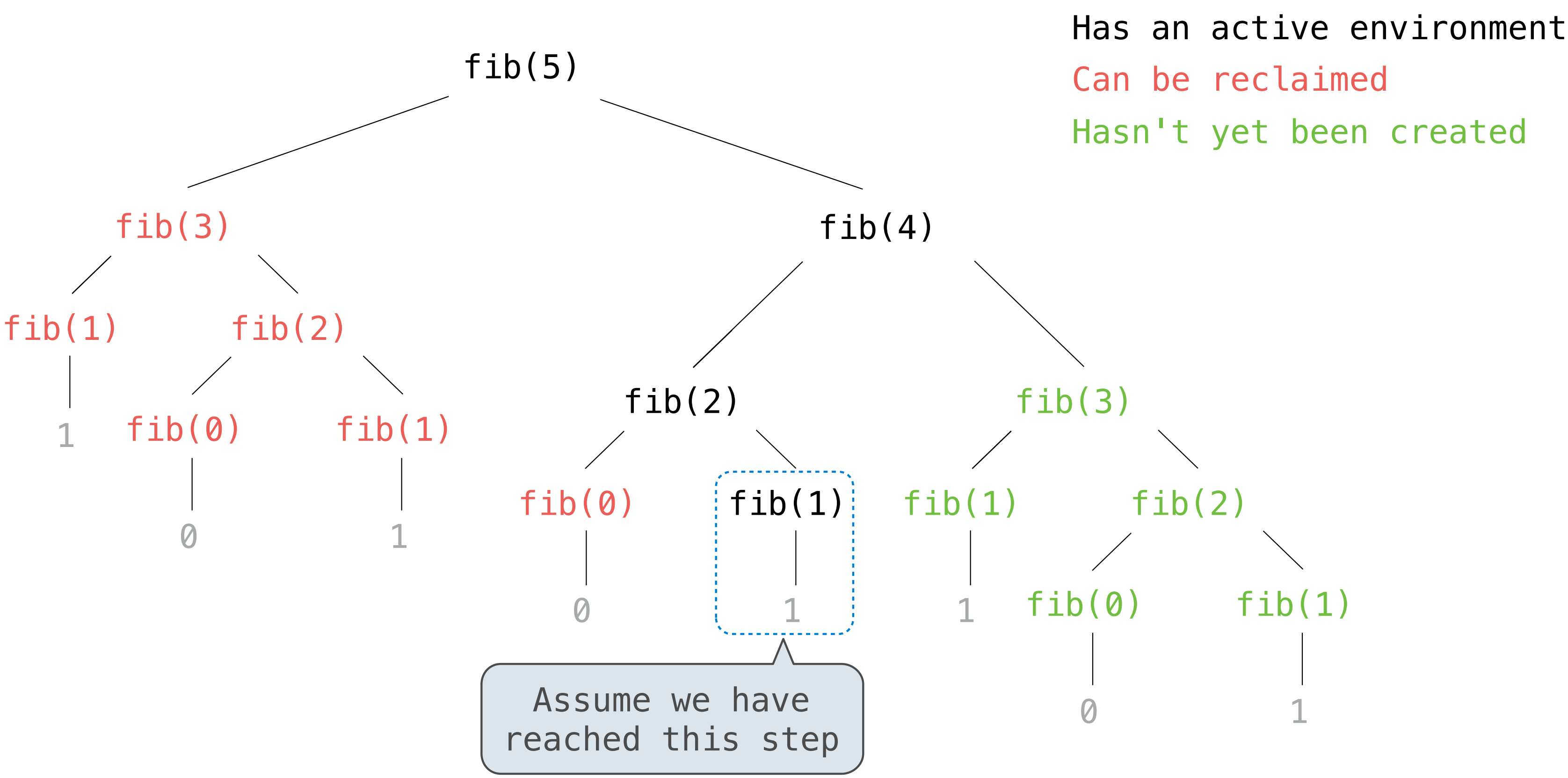
(Demo)

Interactive Diagram

Fibonacci Space Consumption



Fibonacci Space Consumption



Order of Growth

Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

n: size of the problem

R(n): measurement of some resource used (time or space)

$$R(n) = \Theta(f(n))$$

means that there are positive constants $\mathbf{k_1}$ and $\mathbf{k_2}$ such that

$$k_1 \cdot f(n) \leq R(n) \leq k_2 \cdot f(n)$$

for all **n** larger than some minimum **m**

Counting Factors

Number of operations required to count the factors of n using factors_fast is $\Theta(\sqrt{n})$

To check the *lower bound*, we choose $k_1 = 1$:

- Statements outside the while: 4 or 5
- Statements within the while (including header): 3 or 4
- while statement iterations: between $\sqrt{n} - 1$ and $\sqrt{n}$
- Total number of statements executed: at least $4 + 3(\sqrt{n} - 1)$

```
def factors_fast(n):
    sqrt_n = sqrt(n)
    k, total = 1, 0
    while k < sqrt_n:
        if divides(k, n):
            total += 2
        k += 1
    if k * k == n:
        total += 1
    return total
```

To check the *upper bound*

- Maximum statements executed: $5 + 4\sqrt{n}$
- Maximum operations required per statement: some **p**
- We choose $k_2 = 5p$ and $m = 25$

Assumption: every statement, such as addition-then-assignment using the += operator, takes some fixed number of operations to execute

Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

```
def factors(n):
```

	Time	**Space**
Slow: Test each k from 1 through n	$\Theta(n)$	$\Theta(1)$
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	$\Theta(\sqrt{n})$	$\Theta(1)$

Assumption: integers occupy a fixed amount of space

Exponentiation

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{n-1} & \text{otherwise} \end{cases}$$

```
def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)


def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

	Time	Space
`exp`	$\Theta(n)$	$\Theta(n)$
`exp_fast`	$\Theta(\log n)$	$\Theta(\log n)$

Comparing Orders of Growth

Properties of Orders of Growth

Constants: Constant terms do not affect the order of growth of a process

$$\Theta(n) \qquad \Theta(500 \cdot n) \qquad \Theta(\frac{1}{500} \cdot n)$$

Logarithms: The base of a logarithm does not affect the order of growth of a process

$$\Theta(\log_2 n) \qquad \Theta(\log_{10} n) \qquad \Theta(\ln n)$$

Nesting: When an inner process is repeated for each step in an outer process, multiply the steps in the outer and inner processes to find the total number of steps

```
def overlap(a, b):
    count = 0
    for item in a:          Outer: length of a
        if item in b:       Inner: length of b
            count += 1
    return count
```

If a and b are both length **n**, then overlap takes $\Theta(n^2)$ steps

Lower-order terms: The fastest-growing part of the computation dominates the total

$$\Theta(n^2) \qquad \Theta(n^2 + n) \qquad \Theta(n^2 + 500 \cdot n + \log_2 n + 1000)$$

Comparing orders of growth (n is the problem size)

$\Theta(b^n)$ Exponential growth. Recursive fib takes

$\Theta(\phi^n)$ steps, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61828$

Incrementing the problem scales R(n) by a factor

$\Theta(n^2)$ Quadratic growth. E.g., overlap

Incrementing n increases R(n) by the problem size n

$\Theta(n)$ Linear growth. E.g., slow factors or exp

$\Theta(\sqrt{n})$ Square root growth. E.g., factors_fast

$\Theta(\log n)$ Logarithmic growth. E.g., exp_fast

Doubling the problem only increments R(n).

$\Theta(1)$ Constant. The problem size doesn't matter