61A Lecture 20

Monday, March 11

Announcements

- Project 3 due Thursday 3/12 @ 11:59pm
  - Project party on Tuesday 3/10 5pm-6:30pm in 2050 VLSB
  - Bonus point for early submission by Wednesday 3/11
- Guerrilla section this weekend on recursive data (linked lists and trees)
- Homework 6 due Monday 3/16 @ 11:59pm
- Midterm 2 is on Thursday 3/19 7pm-9pm
  - Fill out conflict form if you cannot attend due to a course conflict

Time

The Consumption of Time

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

`def factors(n):`	**Time (number of divisions)**
Slow: Test each k from 1 through n	n
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	Greatest integer less than $\sqrt{n}$

(Demo)

Space

The Consumption of Space

Which environment frames do we need to keep during evaluation?

At any moment there is a set of active environments

Values and frames in active environments consume memory

Memory that is used for other values and frames can be recycled

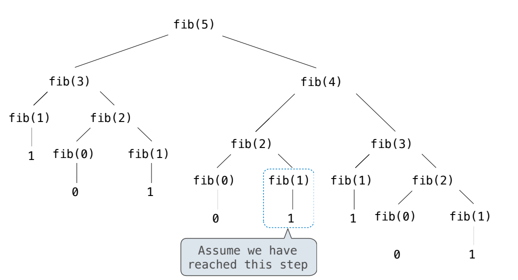
Active environments:

- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

(Demo)

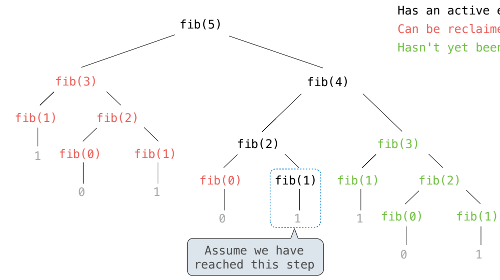
Interactive Diagram

Fibonacci Space Consumption



Fibonacci Space Consumption

Has an active environment
Can be reclaimed
Hasn't yet been created



Order of Growth

Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

n: size of the problem

R(n): measurement of some resource used (time or space)

$$R(n) = \Theta(f(n))$$

means that there are positive constants **k₁** and **k₂** such that

$$k_1 \cdot f(n) \leq \boxed{R(n)} \leq k_2 \cdot f(n)$$

for all **n** larger than some minimum **m**

Counting Factors

Number of operations required to count the factors of n using factors_fast is $\Theta(\sqrt{n})$

```
def factors_fast(n):
    sqrt_n = sqrt(n)
    k, total = 1, 0
    while k < sqrt_n:
        if divides(k, n):
            total += 2
        k += 1
    if k * k == n:
        total += 1
    return total
```

To check the *lower bound*, we choose k_1 = 1:

- Statements outside the while: 4 or 5
- Statements within the while (including header): 3 or 4
- while statement iterations: between $\sqrt{n}-1$ and $\sqrt{n}$
- Total number of statements executed: at least $4+3(\sqrt{n}-1)$

To check the *upper bound*

- Maximum statements executed: $5+4\sqrt{n}$
- Maximum operations required per statement: some **p**
- We choose k_2 = 5**p** and **m** = 25

Assumption: every statement, such as addition-then-assignment using the += operator, takes some fixed number of operations to execute

Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

`def factors(n):`

	Time	Space
Slow: Test each k from 1 through n	$\Theta(n)$	$\Theta(1)$
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	$\Theta(\sqrt{n})$	$\Theta(1)$

Assumption: integers occupy a fixed amount of space

Exponentiation

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{n-1} & \text{otherwise} \end{cases}$$

```
def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)

def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

	Time	Space
exp	$\Theta(n)$	$\Theta(n)$
exp_fast	$\Theta(\log n)$	$\Theta(\log n)$

Comparing Orders of Growth

Properties of Orders of Growth

Constants: Constant terms do not affect the order of growth of a process

$\Theta(n)$ $\Theta(500 \cdot n)$ $\Theta(\frac{1}{500} \cdot n)$

Logarithms: The base of a logarithm does not affect the order of growth of a process

$\Theta(\log_2 n)$ $\Theta(\log_{10} n)$ $\Theta(\ln n)$

Nesting: When an inner process is repeated for each step in an outer process, multiply the steps in the outer and inner processes to find the total number of steps

```
def overlap(a, b):
    count = 0
    for item in a:          # Outer: length of a
        if item in b:       # Inner: length of b
            count += 1
    return count
```

If a and b are both length **n**, then overlap takes $\Theta(n^2)$ steps

Lower-order terms: The fastest-growing part of the computation dominates the total

$\Theta(n^2)$ $\Theta(n^2 + n)$ $\Theta(n^2 + 500 \cdot n + \log_2 n + 1000)$

Comparing orders of growth (n is the problem size)

$\Theta(b^n)$ Exponential growth. Recursive fib takes $\Theta(\phi^n)$ steps, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61828$
Incrementing the problem scales R(n) by a factor

$\Theta(n^2)$ Quadratic growth. E.g., overlap
Incrementing n increases R(n) by the problem size n

$\Theta(n)$ Linear growth. E.g., slow factors or exp

$\Theta(\sqrt{n})$ Square root growth. E.g., factors_fast

$\Theta(\log n)$ Logarithmic growth. E.g., exp_fast
Doubling the problem only increments R(n).

$\Theta(1)$ Constant. The problem size doesn't matter