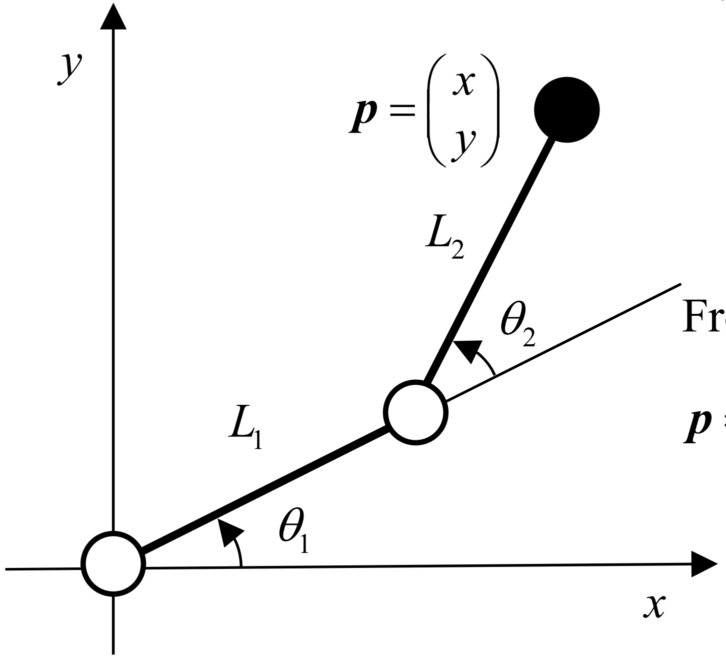
Robot Arm Kinematics
（ロボットアームの運動学）

Keitaro Naruse, UoA
（成瀬継太郎，会津大）

Forward Kinematics（順運動学）

Forward kinematics problem（順運動学問題）
Given: Controllable variables（制御値）, e.g. joint angles（関節角度）
Find: Controlled states（状態値）, e.g. a hand position（手先位置）
We can find a series robot arm forward kinematics solution easily
（直列ロボットアームでは簡単に計算することができる）

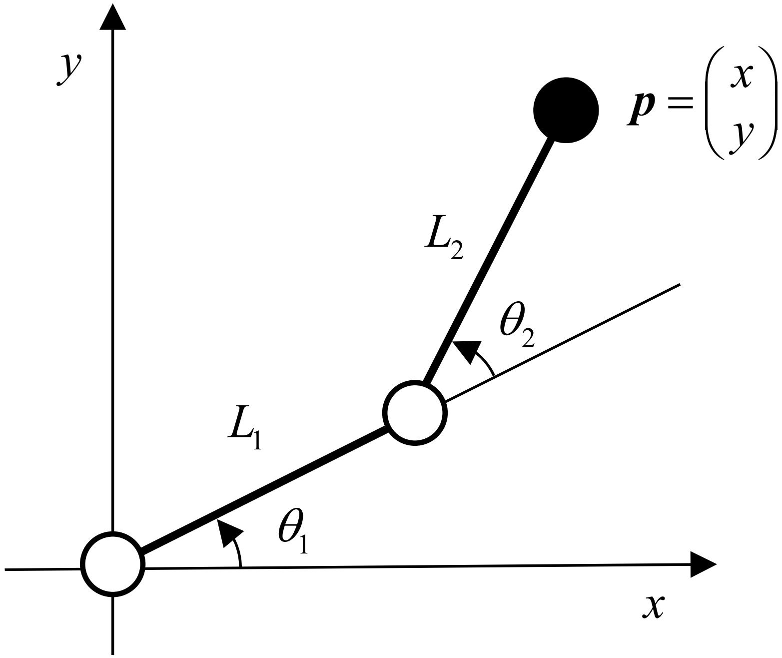


From θ_1, θ_2 to x, y

$$\boldsymbol{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_x(\theta_1, \theta_2) \\ f_y(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

Inverse Kinematics（逆運動学）

Inverse kinematics problem（逆運動学問題）
Given: Controlled states（状態値）, e.g. a hand position（手先位置）
Find: Controllable variables（制御値）, e.g. joint angles（関節角度）
We cannot find a direct solution in general case
（逆運動学の解法は一般的に困難）



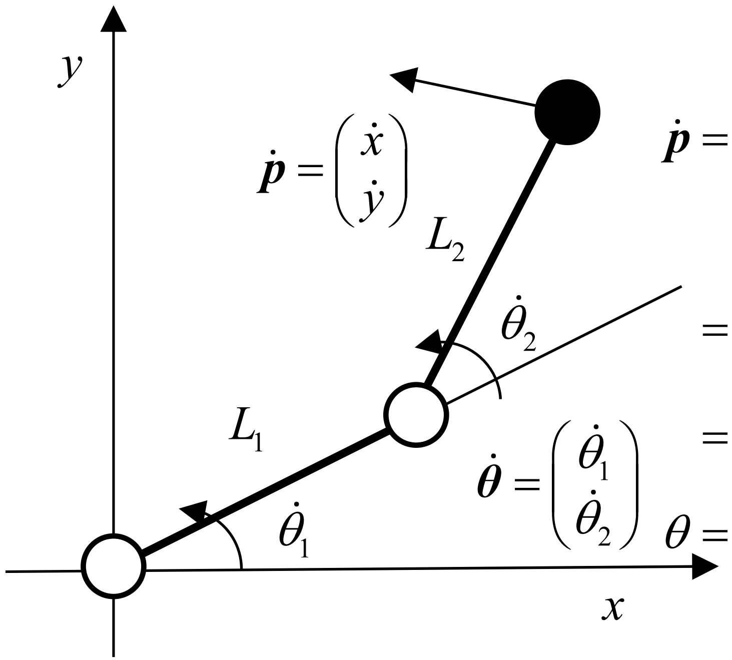
From x, y to θ_1, θ_2

$$\theta_1 = \pm\cos^{-1}\left(\frac{x^2 + y^2 + L_1^2 - L_2^2}{2L_1\sqrt{x^2 + y^2}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{y - L_1\sin(\theta_1)}{x - L_1\cos(\theta_1)}\right) - \theta_1$$

Consider Angular and Translational Velocity
（回転速度の並進速度を考えると）

Consider a time derivative of x and y, i.e, velocity
（xとyの時間微分，すなわち速度を考える）
We can have a relation between angular velocity of joint angles and translational velocity of a hand
（関節角速度と手先の並進速度の関係が得られる）



$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \left(-L_1 \sin\theta_1\right)\dot{\theta}_1 + \left(-L_2 \sin\left(\theta_1+\theta_2\right)\right)\left(\dot{\theta}_1+\dot{\theta}_2\right) \\ \left(L_1 \cos\theta_1\right)\dot{\theta}_1 + \left(L_2 \cos\left(\theta_1+\theta_2\right)\right)\left(\dot{\theta}_1+\dot{\theta}_2\right) \end{pmatrix}$$

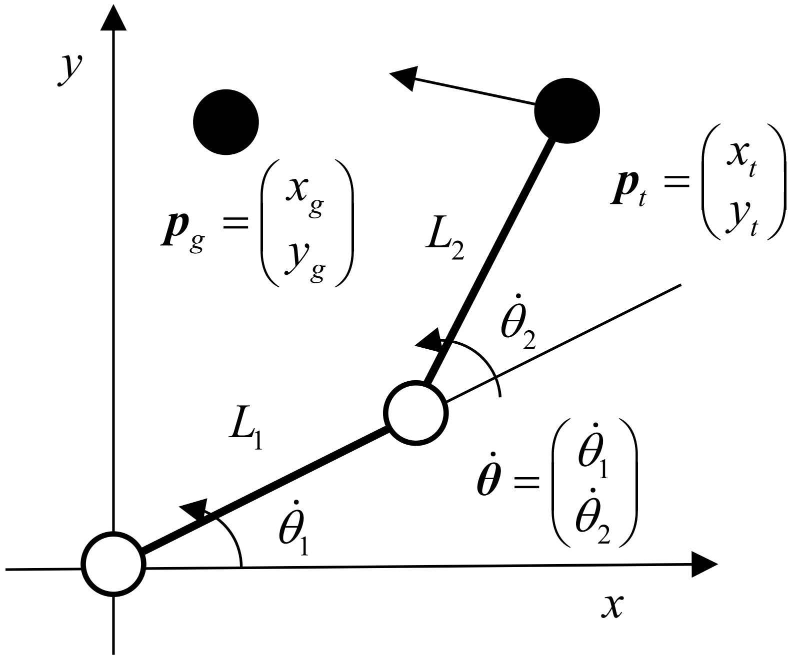
$$= \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin\left(\theta_1+\theta_2\right) & -L_2 \sin\left(\theta_1+\theta_2\right) \\ L_1 \cos\theta_1 + L_2 \cos\left(\theta_1+\theta_2\right) & L_2 \cos\left(\theta_1+\theta_2\right) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$= J\dot{\boldsymbol{\theta}}$$

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \dot{\theta} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

Numerical Inverse Kinematics Solution（数値的な逆運動学解法）

Consider a time derivative of x and y, i.e, velocity
（xとyの時間微分，すなわち速度を考える）



$$\dot{\boldsymbol{\theta}} = J^{-1}\dot{\boldsymbol{p}}$$

Repeat until $\boldsymbol{p}_g \approx \boldsymbol{p}_t$

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + kJ^{-1}\left(\boldsymbol{p}_g - \boldsymbol{p}_t\right)$$

$$\boldsymbol{p}_{t+1} = \begin{pmatrix} x_{t+1} \\ y_{t;1} \end{pmatrix} = \begin{pmatrix} f_x\left(\boldsymbol{\theta}_{t+1}\right) \\ f_y\left(\boldsymbol{\theta}_{t+1}\right) \end{pmatrix}$$

Two methods to Find Jacobian
（ヤコビアンを求める二つの方法）

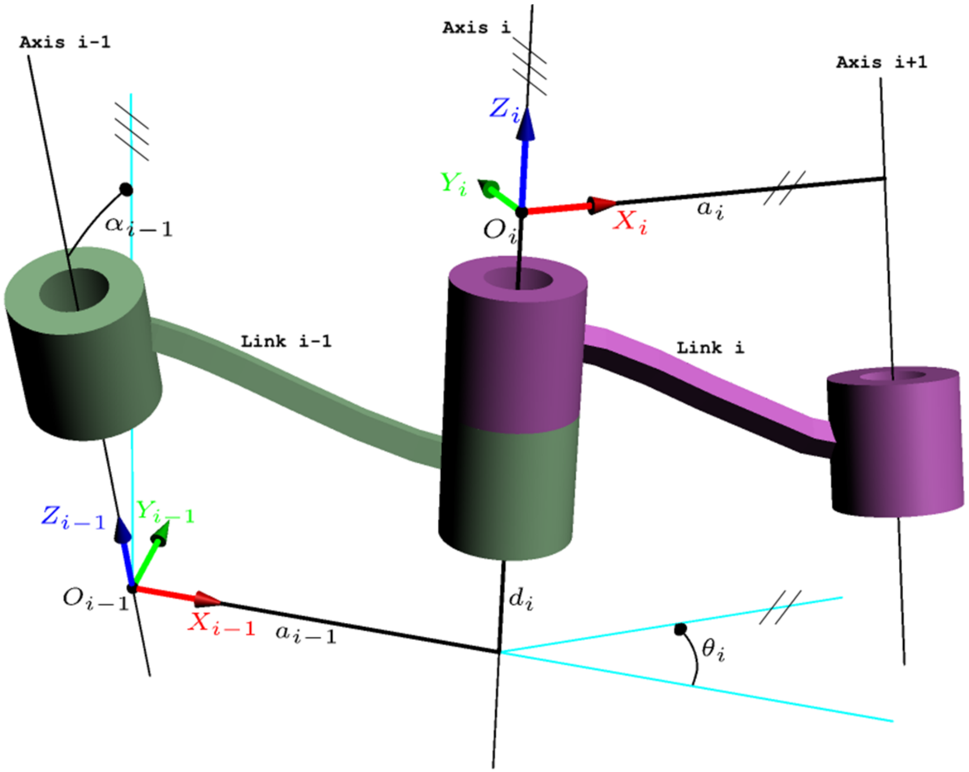
Differential Jacobian（微分的ヤコビアン）
Can be found by taking partial derivative of forward kinematics function by angles
（順運動学関数を関節角度で偏微分することによって求められる）

$$J = \begin{pmatrix} \dfrac{\partial f_x}{\partial \theta_1} & \dfrac{\partial f_x}{\partial \theta_2} \\ \dfrac{\partial f_y}{\partial \theta_1} & \dfrac{\partial f_y}{\partial \theta_2} \end{pmatrix}$$

$$J = \begin{pmatrix} \dfrac{f_x\left(\theta_1 + \Delta\theta_1, \theta_2\right) - f_x\left(\theta_1, \theta_2\right)}{\left(\theta_1 + \Delta\theta_1\right) - \theta_1} & \dfrac{f_x\left(\theta_1, \theta_2 + \Delta\theta_2\right) - f_x\left(\theta_1, \theta_2\right)}{\left(\theta_2 + \Delta\theta_2\right) - \theta_2} \\ \dfrac{f_y\left(\theta_1 + \Delta\theta_1, \theta_2\right) - f_y\left(\theta_1, \theta_2\right)}{\left(\theta_1 + \Delta\theta_1\right) - \theta_1} & \dfrac{f_y\left(\theta_1, \theta_2 + \Delta\theta_2\right) - f_y\left(\theta_1, \theta_2\right)}{\left(\theta_2 + \Delta\theta_2\right) - \theta_2} \end{pmatrix}$$

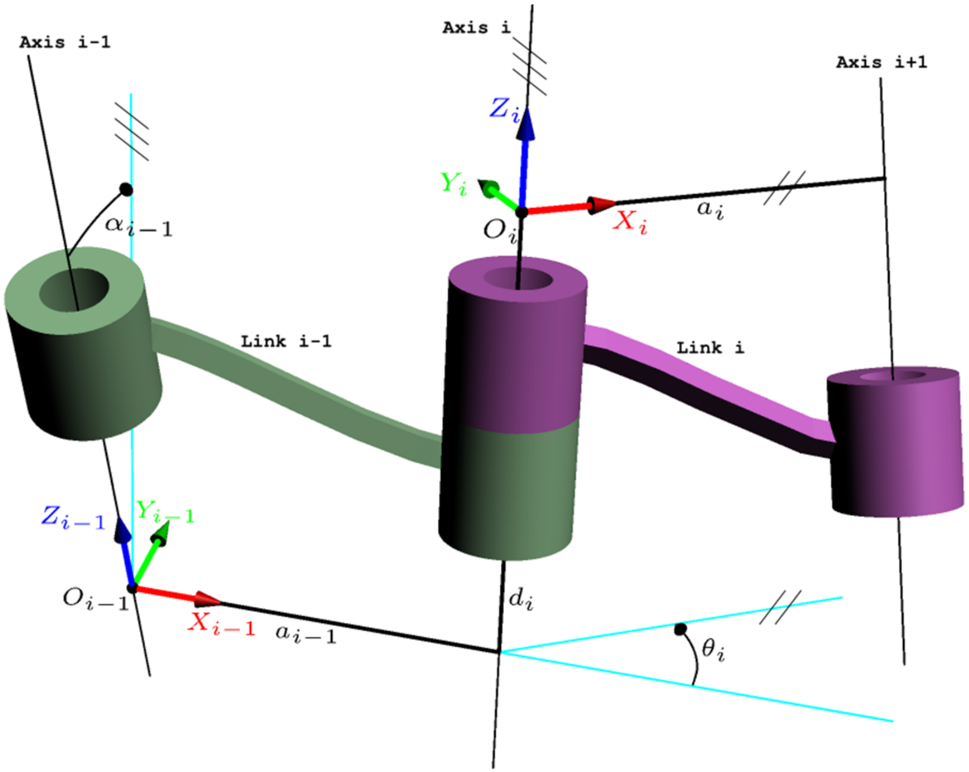
Numerical Jacobian（数値的ヤコビアン）
Can be found from forward kinematics function numerically
(順運動学関数から数値的に求める)

Modified Denavit–Hartenberg Convention （修正DH法）



- Method for finding a forward kinematics function（順運動学関数を求める方法）
- Describe link parameters in the way of DH method（DH法によりリンクパラメータを記述する）
- Forward kinematics function is found as a homogeneous transformation matrix（順運動学関数が同次変換行列として求められる）

Modified Denavit–Hartenberg Parameters （修正DHパラーメタ）



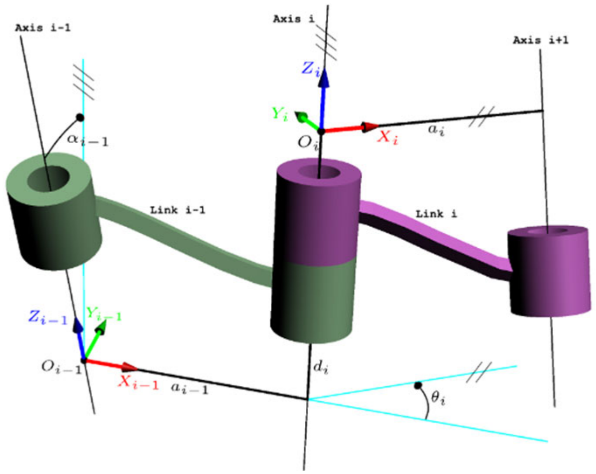
Define a local frame to each of joint（各関節に局所フレームを定義）

- Z-axis is set to a joint axis（関節軸はz軸）
- X-axis is set to a link direction（リンクはx軸）
- Y-axis is set to perpendicular to X and Z（Y軸はXとZに垂直になるように設定）

Describe mechanical link parameter by the following four（4つのリンクパラメータ）

- Length（リンク長）: a
- Offset（オフセット）: d
- Twist angle（ねじれ角）: α
- Joint angle（関節角）: θ

Homogeneous Transformation Matrix （同次変換行列）



Frame transformation matrix from n to n-1 frame
（nからn-1フレームへの座標変換行列）

$$ {}^{n-1}\boldsymbol{p} = {}^{n-1}T_n \; {}^{n}\boldsymbol{p} $$

${}^{n}\boldsymbol{p}$: point(coordinates) represented in n-th frame

${}^{n-1}\boldsymbol{p}$: one in n-1-th frame

$$ {}^{n}\boldsymbol{p} = \begin{pmatrix} {}^{n}x \\ {}^{n}y \\ {}^{n}z \\ \hline 1 \end{pmatrix}, \quad {}^{n-1}\boldsymbol{p} = \begin{pmatrix} {}^{n-1}x \\ {}^{n-1}y \\ {}^{n-1}z \\ \hline 1 \end{pmatrix}, $$

$$ \begin{aligned} {}^{n-1}T_n &= RotX_{n-1}(\alpha_{n-1})\, TransX(a_{n-1})\, RotZ_n(\theta_n)\, TransZ_n(d_n) \\ &= \left(\begin{array}{ccc|c} \cos(\theta_n) & -\sin(\theta_n) & 0 & a_{n-1} \\ \sin(\theta_n)\cos(a_{n-1}) & \cos(\theta_n)\cos(a_{n-1}) & -\sin(a_{n-1}) & -d_n \sin(\alpha_{n-1}) \\ \sin(\theta_n)\sin(a_{n-1}) & \cos(\theta_n)\sin(a_{n-1}) & \cos(a_{n-1}) & d_n \cos(\alpha_{n-1}) \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \end{aligned} $$

Frame Transformation（座標変換）

Translation transformation
（並進変換）

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc:c} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ \hline 1 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{p}' \\ \hline 1 \end{pmatrix} = \left(\begin{array}{c:c} I & \boldsymbol{\Delta p} \\ \hdashline \boldsymbol{o}^T & 1 \end{array}\right) \begin{pmatrix} \boldsymbol{p} \\ \hline 1 \end{pmatrix}$$

Rotational transformation
（回転変換）

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a_x^X & a_x^Y & a_x^Z \\ a_y^X & a_y^Y & a_y^Z \\ a_z^X & a_z^Y & a_z^Z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc:c} a_x^X & a_x^Y & a_x^Z & 0 \\ a_y^X & a_y^Y & a_y^Z & 0 \\ a_z^X & a_z^Y & a_z^Z & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ \hline 1 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{p}' \\ \hline 1 \end{pmatrix} = \left(\begin{array}{c:c} R & \boldsymbol{o} \\ \hdashline \boldsymbol{o}^T & 1 \end{array}\right) \begin{pmatrix} \boldsymbol{p} \\ \hline 1 \end{pmatrix}$$

Homogeneous Transformation Matrix
（同次変換行列）

Translation and rotational transformation in a single step
（並進と回転変換を1ステップで）

$$\begin{pmatrix} X \\ Y \\ Z \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc|c} a_x^X & a_x^Y & a_x^Z & \Delta x \\ a_y^X & a_y^Y & a_y^Z & \Delta y \\ a_z^X & a_z^Y & a_z^Z & \Delta z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ \hline 1 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{p}' \\ \hline 1 \end{pmatrix} = \left(\begin{array}{c|c} R & \boldsymbol{\Delta p} \\ \hline \boldsymbol{o}^T & 1 \end{array}\right) \begin{pmatrix} \boldsymbol{p} \\ \hline 1 \end{pmatrix} = T \begin{pmatrix} \boldsymbol{p} \\ \hline 1 \end{pmatrix} = \begin{pmatrix} R\boldsymbol{p} + \boldsymbol{\Delta p} \\ 1 \end{pmatrix}$$

Figure in Craig, Introduction to Robotics

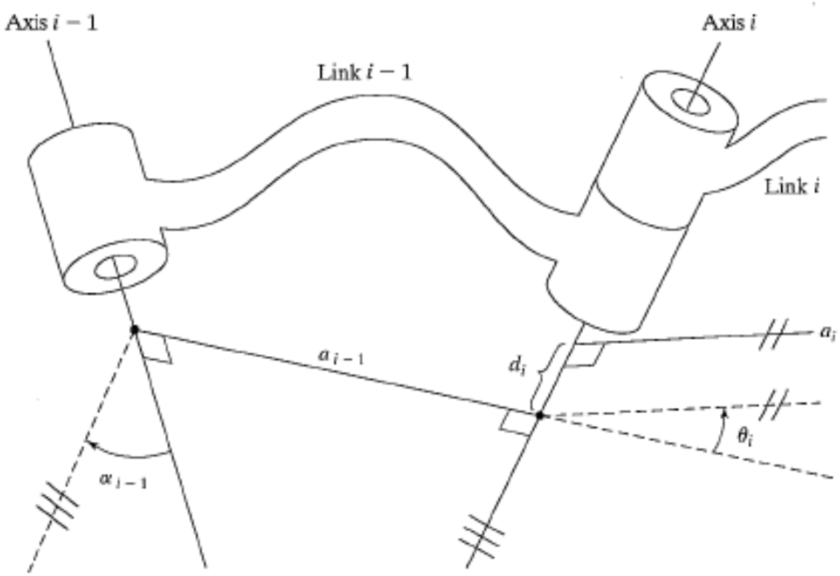


FIGURE 3.4: The link offset, d, and the joint angle, θ, are two parameters that may used to describe the nature of the connection between neighboring links.

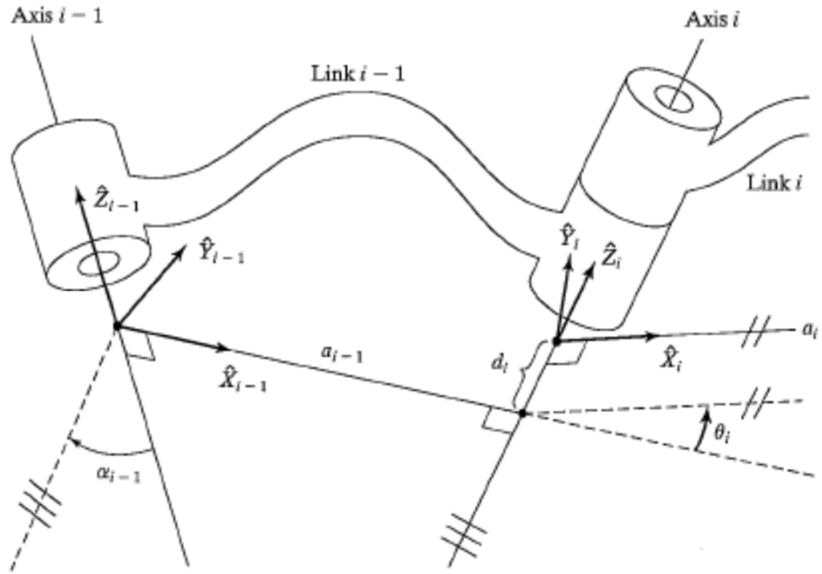
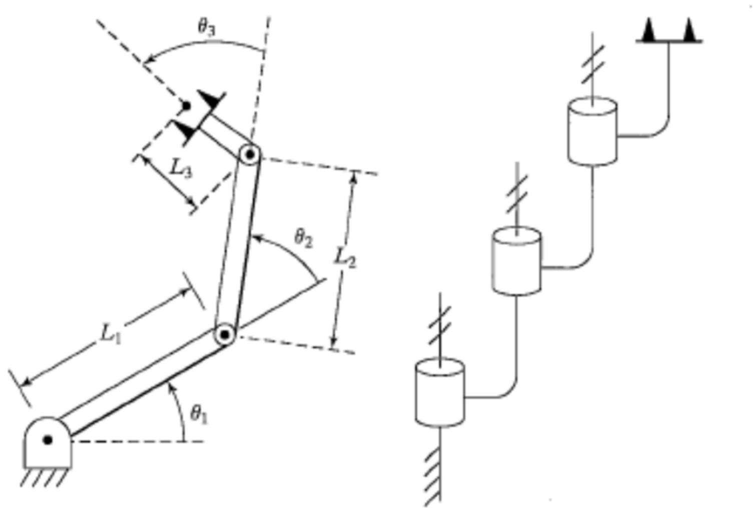


FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link i.

Figure in Craig, Introduction to Robotics



(a) (b)

FIGURE 3.6: A three-link planar arm. On the right, we show the same manipulator by means of a simple schematic notation. Hash marks on the axes indicate that they are mutually parallel.

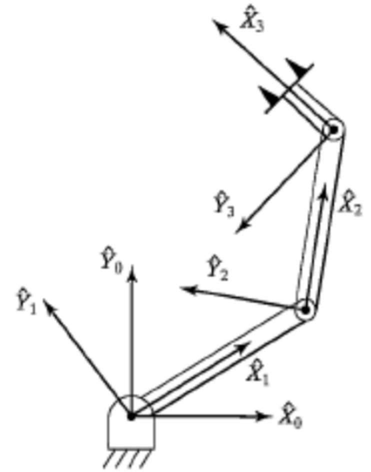


FIGURE 3.7: Link-frame assignments.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

FIGURE 3.8: Link parameters of the three-link planar manipulator.

Fiberscope Camera（内視鏡カメラのDH表）

	α	a	d	θ
1	0	0	0	θ1
2	90[deg]	L1	0	θ2
3	0	L2	0	θ3
4	0	L3	0	θ4
5	0	L4	0	θ5
6	0	L5	0	0