Ramsey Numbers and Other Knot Invariants

Maribeth Johnson
Research Experience for Undergraduates
California State University, San Bernardino
San Bernardino, California 92407

August 22, 2012

Abstract

We make use of a particular linear spatial embedding, the cyclic polytope, in an exploration of bounds on the Ramsey number of knots. Using arc presentations to simplify knotted cycles of this embedding, we examine the relationships between the Ramsey number, bridge number, crossing number, stick number and arc index of knots. In particular we show the Ramsey number is at least as large as the sum of the bridge number and the arc index, and at least as large as the sum of the crossing number and the bridge number plus 2. We also show that for a particular class of torus knots, $T_{p-1,p}$, the difference between the Ramsey number and stick number grows without bound.

1 Introduction

A **knot** is a closed curve embedded in three dimensions that is not self-intersecting. Of particular interest to us is the family of knots, $T_{p-1,p}$, or the $(p-1,p)$ torus knots. In general, **torus knots** are knots that lie on an unknotted torus without any self-crossings. Torus knots are identified by a pair of relatively prime integers, p and q, and denoted either $T_{p,q}$ or as a (p,q) torus knot. A (p, q) torus knot is the knot consisting of a closed curve on the torus that winds around the axis of revolution p times and wraps around the core of the torus q times (See Figure 1). For example, the trefoil knot is a $(2,3)$ torus knot.

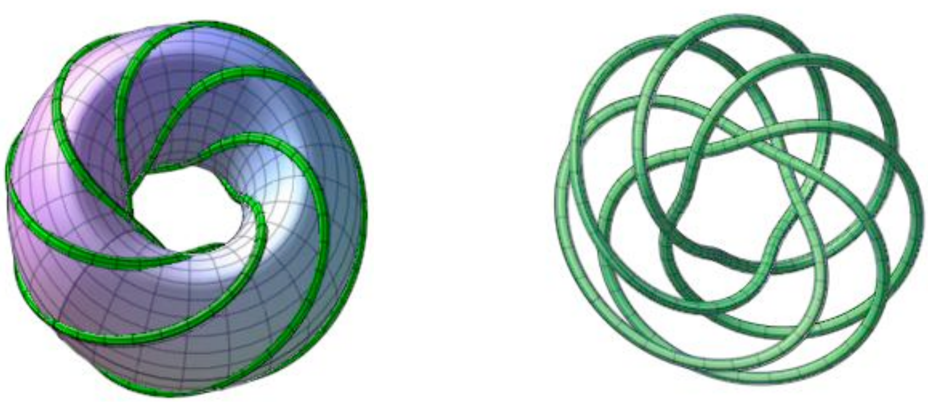

Figure 1: A (5, 6) torus knot.

A knot can be represented in many ways, one of which is the stick representation, which constructs the knot out of straight line segments in 3-space. The **stick number** of a knot, denoted $s(K)$, is the least number of sticks needed to create K in 3-space. For example, $s(unknot) = 3$ since it can be constructed by a triangle. The stick numbers of many knots are known, such as $s(T_{2,3}) = 6$. A knot can also be represented by an arc presentation. In an **arc presentation** of a knot, K is constructed by finitely many arcs, each on a half-plane arranged in an open-book such that each arc meets the binding at exactly two vertices, and each vertex is contained by exactly two arcs. The **arc index** of a link, $\alpha(K)$ is the least number of half-planes needed to construct the knot K. For example, it is known that $\alpha(unknot) = 2$ and $\alpha(T_{2,3}) = 5$ (See Figure 2).

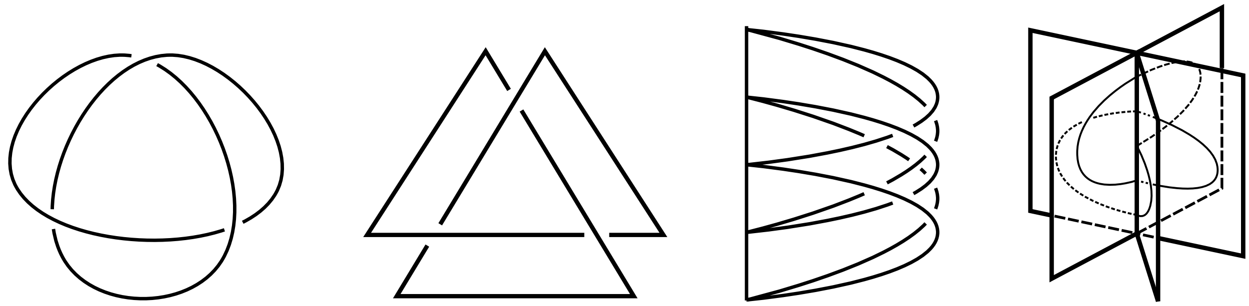

Figure 2: *The Trefoil Knot.* A general knot diagram (far left), stick diagram (center left), and arc presentations (center right, far right).

The **bridge number** of a knot is another invariant that we will make use of. Any knot that is not the unknot cannot be embedded in a plane. A knot can, however, be embedded in a plane with the exception of bridges.

Definition 1. *The **bridge number** of a knot, denoted br(K), is the minimum number of bridges needed for a projection of the knot into a plane.*

The bridge number can also be interpreted as the least number of maxima in any picture of the knot. So, any representation of the knot must have at least $br(K)$ maxima. See Figure 3.

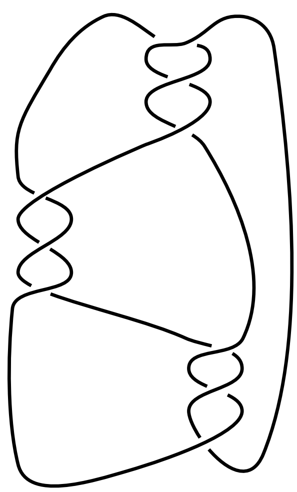

Figure 3: *A bridge diagram.* A 2-bridge knot

The **crossing number** of a knot, denoted $c(K)$, is the minimum number of crossings over all diagrams of that knot. So, given any diagram of K it will have at least $c(K)$ crossings. For example, the unknot clearly has crossing number 0, and $c(T_{2,3}) = 3$ (See Figure 2, far left).

A **graph** is a set of vertices connected by edges. A given vertex can be contained by multiple edges, or no edges. Each edge contains 2 vertices. The **complete graph** on n vertices, K_n, is the set of n vertices with $\binom{n}{2}$ edges such that each pair of vertices is connected by an edge. A **Hamiltonian cycle** of a graph is a cycle which visits every vertex exactly once.

Definition 2. *A **linear spatial embedding** of K_n is a copy of K_n in 3-space such that every edge is straight and no two edges intersect one another.*

In this paper we will be concerned with a certain embedding of K_n, the cyclic polytope.

Definition 3. *The **cyclic polytope** of three dimensions with $n \geq 4$ vertices, $C_n = C(n) = C(t_1, t_2, ..., t_n)$ is the linear spatial embedding of K_n whose vertices are the points $x(t_1), x(t_2), ..., x(t_n)$ of the moment curve $x : \mathbb{R} \to \mathbb{R}^3, t \to (t, t^2, t^3)$. See Figure 4 below.*

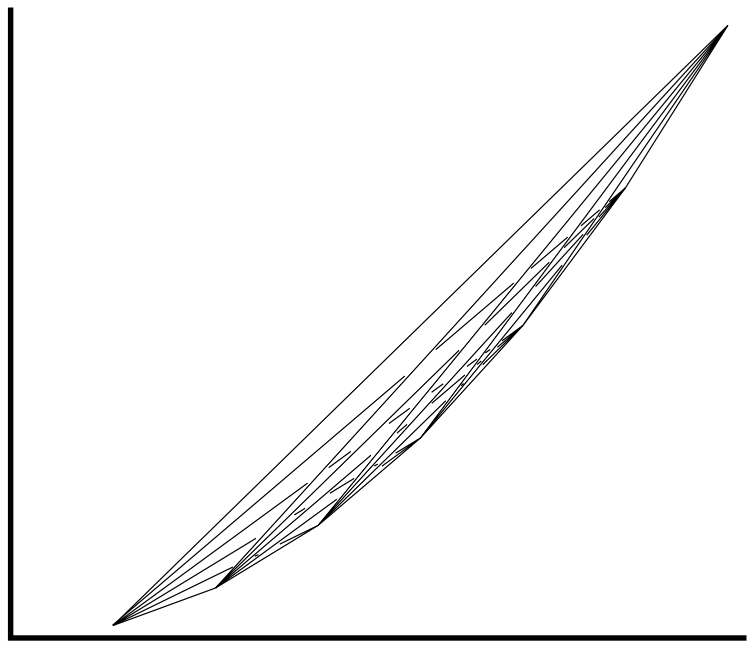

Figure 4: The cyclic polytope on seven vertices projected on the x,y plane.

We are going to denote the points $x(t_1), x(t_2), ..., x(t_n)$, as $v_1, v_2, ..., v_n$. Ramírez Alfonsín [2] also uses the cyclic polytope in his exploration of Ramsey numbers of knots. For our purposes we can think of the cyclic polytope as a linear spatial embedding with vertices $v_1, v_2, ..., v_n$ such any edge containing v_1 crosses over all other edges, the edges containing v_2 cross over all edges except those containing v_1, etc. for all n. See Figure 5.

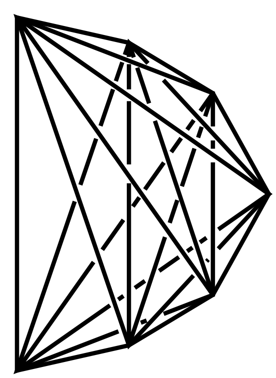

Figure 5: A linear embedding isotopic to the cyclic polytope, C_7.

Note that we will say $v_i < v_j$ if and only if $i < j$. We will use this linear spatial embedding to explore the Ramsey number of knots.

Definition 4. *Given a knot K, the* ***Ramsey number*** *of K, denoted $R(K)$, is the smallest n such that any linear spatial embedding of K_n contains the knot K.*

The Ramsey number is unknown for all but two knots. It is known that $R(unknot) = 3$, since any three vertices can be connected to make a triangle, the stick diagram of an unknot. Ramírez Alfonsín [1] showed that $R(T_{2,3}) = 7$. The Ramsey numbers of all other knots are unknown, although various upper and lower bounds have been found. Negami [7] proved that $R(K)$ is finite for any knot K.

2 Arc Presentations

We will construct arc presentations of Hamiltonian cycles of C_n. We will then provide an algorithm for reducing our arc presentations, which will lead to interesting results on the arc index of cycles in C_n.

2.1 Construction given a Hamiltonian Cycle

We will first construct the complete graph as a pseudo-arc presentation, an open-book decomposition of K_n on $n-1$ half-planes which is isotopic to the cyclic polytope, C_n.

Definition 5. *Let $\mathbf{A_n}$ be a pseudo arc presentation of all posible $\binom{n}{2}$ arcs on $n-1$ half-planes such that all arcs with v_1 as the least vertex are on the first half-plane, followed by a half-plane of all arcs with v_2 as least vertex, continuing this pattern to the half-plane of arcs with v_{n-1} as least vertex. See Figure 6.*

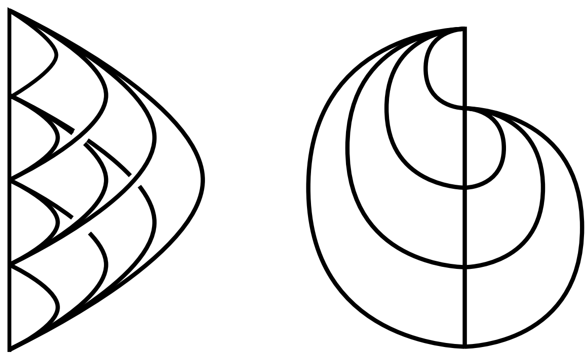

Figure 6: $\mathbf{A_5}$, the pseudo arc presentation of C_5 in 4 half-planes (left) and detail of what the first two half-planes look like (right).

Note this is not a true arc presentation because there are multiple arcs on a single half-plane. C_n is clearly isotopic to A_n. Any cycle in C_n inherits an arc presentation by first considering the corresponding cycle in A_n. This is still not an arc presentation because arcs with the same least vertex will still be on the same half-plane (but now there are at most two arcs per half-plane). In order to make the construction a true arc presentation, we must separate the arcs that are on the same half-plane. We note that we can arbitrarily choose an arc to be on the half-plane above the other and either way we will have an arc presentation representative of our cycle. For the sake of consistency we will let the arc with the least second vertex be on a half-plane above the arc with the greater second vertex. Thus, we have defined an arc presentation that represents a knotted cycle in C_n.

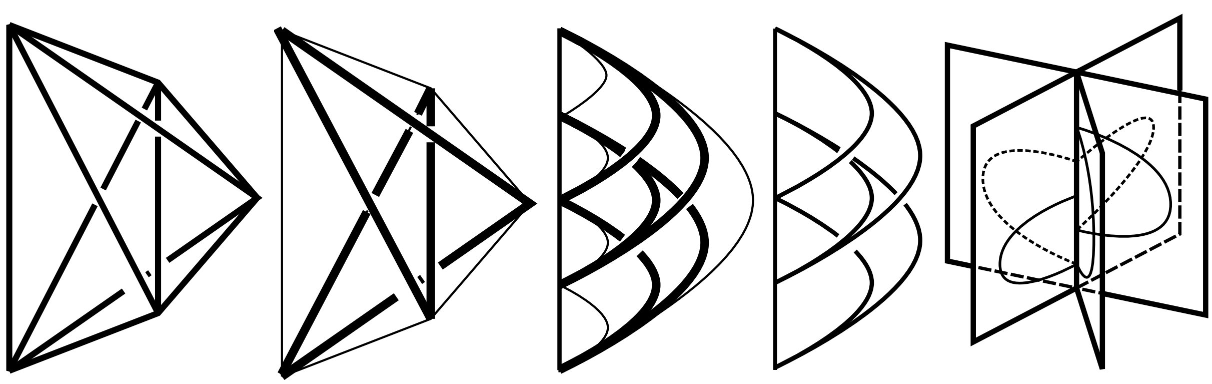

Figure 7: Creating an arc presentation from a knotted cycle in C_5. First we take the cycle in C_5 and transfer it to A_5. Then we eliminate arcs not in the given cycle and separate the arcs so each is on a separate half-plane.

Note this arc presentation is not necessarily minimal, so we present an algorithm for reducing the arc presentation below.

2.2 Notation and Reduction

Cromwell [4] defines moves for reducing arc presentations. The arc merge and the vertex merge are two of these moves that we will make use of (Figures 8 and 9). The **vertex merge** allows arcs with two adjacent vertices to be shrunk until the two endpoints come together and combine, resulting in the arc disappearing. The **arc merge** allows two arcs that meet at a vertex and are on consecutive half-planes to be merged, and their common vertex to be deleted.

First, we need to define a useful notation for arc presentations. We will represent each arc as an ordered pair. The ordered pair will consist of the two vertices contained by the arc. We will use the convention that the least vertex is first, followed by the greater vertex. The ordered pairs will be listed in the order of their half-planes, starting with the top half-plane. Remember that the arc presentation is circular so the first half-plane also comes after the last half-plane.

For example, in a three half-plane arc presentation, say the first half-plane contains an arc connecting v_1 and v_2, the second half-plane contains an arc connecting v_1 and v_3 and the third half-plane contains an arc connecting v_2 and v_3. Using our notation, this arc presentation would be denoted $(1,2)(1,3)(2,3)$.

We will define an algorithm for reducing arc presentations constructed in Section 2.1 while maintaining the knot type of the cycle. We will now define three moves in terms of the notation above. The first two are exactly Cromwell's moves, translated into our notation. The third is a renumbering process that allows us to keep consistent notation and preserve the knot.

1. **Arc Merge:** If two consecutive ordered pairs have the same first number, delete the vertex in common by replacing those two ordered pairs with an ordered pair consisting of the two second numbers. Remember to always have the least number as the first digit in the ordered pair. So, $(X,Y)(X,Z) \to (Y,Z)$ if $(Y < Z)$. See Figure 8.

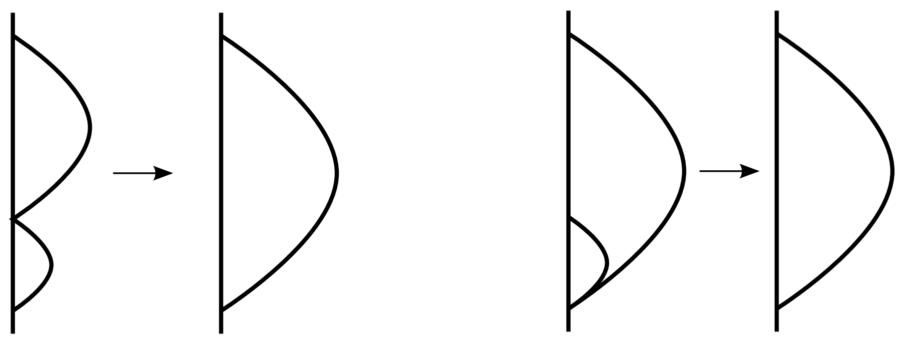

Figure 8: *The Arc Merge.* Two examples of the arc merge move on an arc presentation.

2. **Vertex Merge** If an ordered pair consists of two consecutive numbers, delete second vertex by combining the two vertices. To do this, delete the ordered pair with consecutive numbers and change the other occurrence of the second number (the higher of the two) to the first number (the lower of the two). So $(X, X+1)$ gets deleted and $X+1 \to X$ where it occurs elsewhere. See Figure 9.

3. **Renumbering Process:** Given vertex Z is deleted by an arc or vertex move,

$$f(X) = \begin{cases} X & : X < Z \\ X-1 & : X > Z \end{cases}$$

The algorithm consists of the vertex and arc merges. After each merge the renumbering process must occur. Do not continue to reduce a pair $(X,Y)(X,Y)$. Repeat until not possible. Note that this algorithm does not always give a minimal arc presentation, but it will simplify it.

Returning to our example above of the cycle $(1,2)(1,3)(2,3)$, we note both these ordered pairs begin with 1, so we can perform an arc merge, which leaves us with $(2,3)(2,3)$. Then we must renumber, which calls for subtracting one from all vertices above the one deleted (in this case, the vertex 1). We are left with $(1,2)(1,2)$, and cannot reduce further.

Lemma 1. *For a knot K, corresponding to a Hamiltonian cycle of C_n, $\alpha(K) \leq n-2$.*

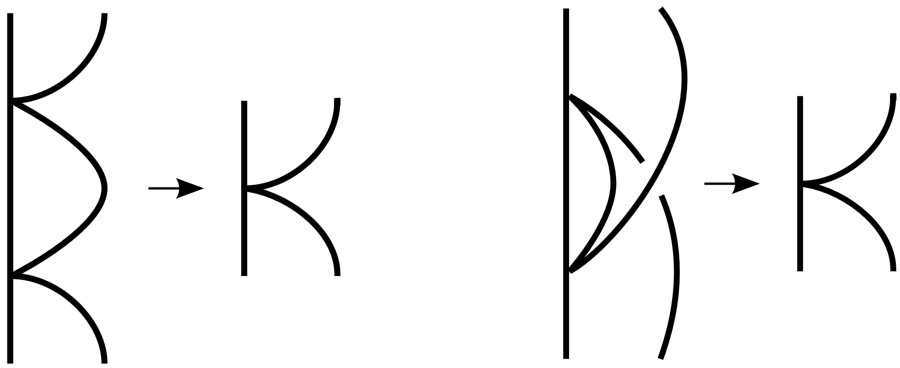

Figure 9: *The Vertex Merge.* Two examples of the vertex merge move on an arc presentation.

Proof. Let H be a Hamiltonian cycle of C_n. Construct the arc presentation of H as defined above. Now we claim that v_1 and v_2 can be eliminated from the arc presentation, decreasing the number of arcs/half-planes by 2.

Since v_1 is the least vertex in any arc containing it, there will be two consecutive arcs with v_1 as their least vertex, so by the arc merge move defined above, the arcs can be merged, deleting v_1 decreasing the presentation by one arc and therefore one half-plane.

We then have two cases to address the elimination of the v_2.

Case 1. There existed an arc containing v_1 and v_2 in the original cycle.
The new arc is on the top half-plane. The other arc containing v_2 is on the second half-plane, so these arcs are consecutive, and by the arc merge move, the arcs can be merged, deleting v_2 and eliminating one arc and therefore one half-plane.

Case 2. There did not exist an arc containing containing v_1 and v_2 in the original cycle.
Since there did not exist an arc containing v_1 and v_2, both arcs containing v_2 must have least vertex v_2 and therefore must be consecutive in their arc presentation. Then, by the arc merge move, the arcs can be merged to one, deleting v_2 and eliminating one arc and therefore one half-plane.

Thus, we have constructed an arc presentation using $(n-2)$ half-planes, which implies $\alpha(H) \leq n-2$. $\square$

So we have shown that the arc presentation obtained from any Hamiltonian cycle of C_n can be reduced by at least two arcs. We see that for C_7, a reduction of two gives us a 5-arc knot, which in the case of the trefoil knot cannot be reduced more, since $\alpha(T_{2,3}) = 5$. Therefore, we know that we cannot always remove a third half-plane, but we will show that in some cases we can.

Lemma 2. *For a knot K, corresponding to a Hamiltonian cycle of C_n for which there does not exist an edge containing v_1 and v_3, $\alpha(K) \leq n-3$.*

Proof. We will address this proof in two cases, depending on whether there exists an arc containing v_2 and v_3. Let H be a Hamiltonian cycle of C_n such that there does not exist an edge containing v_1 and v_3. Construct the arc presentation of H as defined above. Now we claim that v_1, v_2, and v_3 can be eliminated from the arc presentation, which in turn decreases the number of arcs/half-planes by 3.

Case 1: There exists an arc containing v_2 and v_3.
Perform the vertex merge move to combine v_2 and v_3. This reduces the number of half-planes by 1. Then two more half-planes can be eliminated by the process in Lemma 1.

Case 2: There does not exist an arc containing v_2 and v_3.
The first two vertices are eliminated by the process in Lemma 1. Then, since there are no arcs containing v_3 and v_1 or v_2, the arcs containing v_3 are on consecutive half-planes, and the arc merge move can be applied to them to reduce once more. So, three vertices have been eliminated, and therefore three arcs and three half-planes.

Therefore, in all cases we can eliminate three half-planes, reducing the arc index by 3. So, $\alpha(H) \leq n-3$. $\square$

3 Applications to Ramsey Numbers

Many bounds on different knot invariants, such as bridge number, arc index, crossing number, and stick number are known. Combining these bounds with our work with arc index of knotted cycles of the cyclic polytope, we find bounds on the Ramsey number of a knot. First, we will introduce a new concept, the bridge number of the diagram a cycle of C_n.

Definition 6. *Let H be a cycle in C_n. The bridge number of H, denoted $b(H)$ is the number of maxima of H in the 'y' direction when viewed as in Figure 5.*

For example, the cycle in Figure 7 has two maxima, v_1 and v_2. Negami [7] called these maxima 'plat tops' in his work. Note that by definition of bridge number, for a knot corresponding to the cycle H of C_n, $b(H) \geq br(K)$, since K has $b(H)$ bridges in this projection.

Lemma 3. *For a knot K, corresponding to a Hamiltonian cycle of C_n, $\alpha(K) + b(H) \leq n$*

Proof. Let H be a Hamiltonian cycle in C_n with $b(H)$ maxima in the 'y' direction.

Since in a Hamiltonian cycle of the cyclic polytope a vertex is maximum in the y direction if and only if both edges containing that vertex contain no vertices less than that vertex, we know any vertex that is maximum is the least vertex for both edges containing it. So, in the arc presentation of the corresponding knot K, the half-planes containing these arcs will be consecutive, and by the vertex merge move that vertex can be eliminated and the number of half-planes reduced by 1. This occurs for each of the $b(H)$ maxima, leaving a total of $(n - b(H))$ half-planes. Therefore, $\alpha(K) \leq n - b(H)$, and $\alpha(K) + b(H) \leq n$. ☐

Theorem 1. *For any knot, K, $\alpha(K) + br(K) \leq R(K)$.*

Proof. Let K be a knot. Note that since for any knotted cycle of C_n $b(H) \leq br(K)$, Lemma 3 implies that for K corresponding to H of C_n, $\alpha(K) + br(K) \leq n$. Since K lives on $C_{R(K)}$ by definition on Ramsey number, it follows that $\alpha(K) + br(K) \leq R(K)$. ☐

Note that this bound is sharp in the case of the unknot and the trefoil, the only two knots for which the Ramsey number is known. Now we can find the relationship between other knot invariants due to other known results.

Corollary 1. *For alternating knots, $c(K) + br(K) + 2 \leq R(K)$.*

Proof. By Theorem 1, $R(K) \geq \alpha(K) + br(K)$. Cromwell [3] proved that for any alternating knot, $\alpha(K) = c(K) + 2$. It follows that $R(K) \geq c(K) + br(K) + 2$. ☐

Now we concern ourselves with the $(p-1, p)$ torus knots, for which general values of other knot invariants are already known.

Corollary 2. $R(T_{p-1,p}) \geq 3p - 2$.

Proof. By Theorem 1, $R(T_{p-1,p}) \geq \alpha(T_{p,p-1}) + br(T_{p-1,p})$. Schubert [8] proved $br(T_{p,q}) = p$ for $2 \leq p < q$. Therefore, $br(T_{p-1,p}) = p - 1$. Matsuda [6] proved that $\alpha(T_{p,q}) = p + q$, so $\alpha(T_{p,p-1}) = 2p - 1$. It follows that $R(T_{p-1,p}) \geq p - 1 + 2p - 1 = 3p - 2$.

☐

Corollary 3. $R(T_{p-1,p}) - s(T_{p-1,p}) \geq p - 2$

Proof. By Corollary 2, $R(T_{p-1,p}) \geq 3p-2$. An application of a theorem proved by Jin [5] gives us that $s(T_{p-1,p}) = 2p$. It follows that $R(T_{p-1,p}) - s(T_{p-1,p}) \geq 3p - 2 - 2p = p - 2$. ☐

So, we have identified a class of knots, particularly the (p-1, p) torus knots, for which the difference between Ramsey number and stick number grows without bound.

4 Open Questions

1. Can we find other classes of knots for which the difference between the Ramsey number and stick number grows without bound?
2. We have shown that the difference between Ramsey number and stick number of $T_{p-1,p}$ is at least linear. Is it, in actuality, approximately linear? Quadratic? How can we best model this difference?
3. Do our reductions on arc presentations create a normal form from which we can gather more information about the knot?

ACKNOWLEDGEMENTS

I would like to thank Rollie Trapp for his insight and advice during this program, as well as Corey Dunn for his support throughout the REU. I would also like to thank California State University, San Bernardino and NSF grant DMS-1156608 for jointly funding this research program.

References

[1] J.L. Ramírez Alfonsín, *Spatial graphs and oriented matroids: the trefoil*, Discrete and Computational Geometry **22** (1999), no. 1, 149–158.

[2] ———, *Spatial graphs, knots and the cyclic polytope*, Beiträge Algebra Geometry **49** (2008), no. 2, 301–314.

[3] Peter Cromwell, *Arc presentations of knots and links*, Knot Theory (Warsaw) (Jósef H. Prztycki Pawel Traczyk Vladimir G. Turaev Vaughan F.R. Jones, Joanna Kania-Bartoszynska, ed.), vol. 42, Polish Academy of Sciences, Institute of Mathematics, Banach Center Publications, 1998, pp. 57–64.

[4] ———, *Knots and links*, Cambridge University Press, 2004.

[5] Gyo Taek Jin, *Polygon indices and superbridge indices of torus knots and links*, Journal of Knot Theory and Its Ramifications **6** (1997), no. 2, 281–289.

[6] Hiroshi Matsuda, *Links in an open book decomposition and the standard contact structure*, Preceedings of the American Mathematical Society **134** (2006), no. 12, 3697–3702.

[7] Seiya Negami, *Ramsey theorems for knots, links and spatial graphs*, Transactions of the American Mathematical Society **324** (1991), no. 2, 527–541.

[8] Horst Schubert, *Über eine numerische knoteninvariante*, Mathematische Zeitschrift **61** (1954), 245–288.