The Cyclic Cutwidth of a $P_2 \times P_2 \times P_n$ Mesh

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Abstract

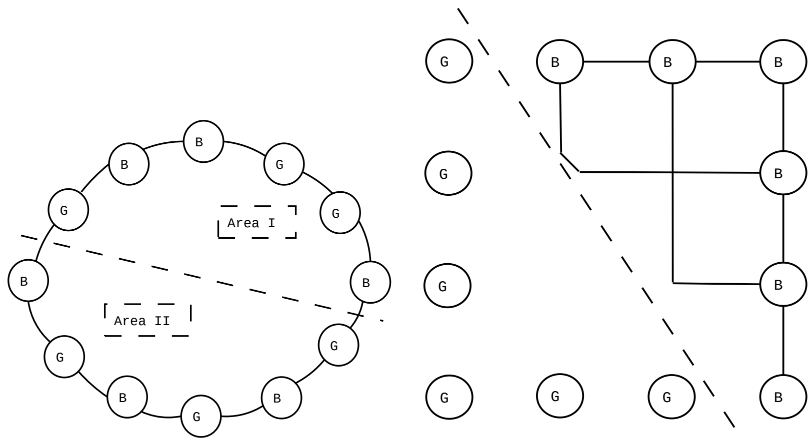
This is a proof for the cyclic cutwidth of a $P_2 \times P_2 \times P_n$ mesh. We will make use of a version of Schroder's lemma for two dimensional meshes and thereby give upper and lower bounds for the cyclic cutwidth of the mesh. We will also look at the possible extension into finding the general cyclic cutwidth of the three dimensional rectangular mesh.

1 Introduction

In Computer Science, there is much interest in the cutwidths of meshes since these seem to be useful in various networks such as computer and communication networks. It is for this reason that many people have worked on the problem of the cyclic cutwidth of meshes, where the vertices are embedded onto a cycle so as to minimize the maximum cutwidth between adjacent vertices. J. Rolim et. al [1] did work on the upper bounds of two dimensional meshes with H. Schroder doing later work on the lower bounds to solve the cyclic cutwidth problem for two dimensional meshes. There were problems in the original statement of Schroder's theorem so D.W. Clarke wrote a paper to correct a small portion of it and prove a case not originally stated in Schroder's paper. We will now look at three dimensional meshes by extending Schroder's work. We will adapt the lemma used by Schroder to establish lower bounds for meshes in three dimensions.

1.1 Schroder's Theorem for Two Dimensional Meshes

Schroder's Lemma [2] states that if you color the vertices of any graph two colors (i.e. half blue, half green) and position them on a circle, it is always possible to draw a diameter dividing them such that exactly half of each color type lie on either side of the diameter (See Figure 1a). This method was used by Schroder to cut a two dimensional mesh in half (See Figure 1b) and then pair vertices together to yield disjoint edge paths that must cross the diameter. This set a lower bound for the cutwidth equal to at least half the total number of disjoint edges crossing this diameter.



(a) (b)

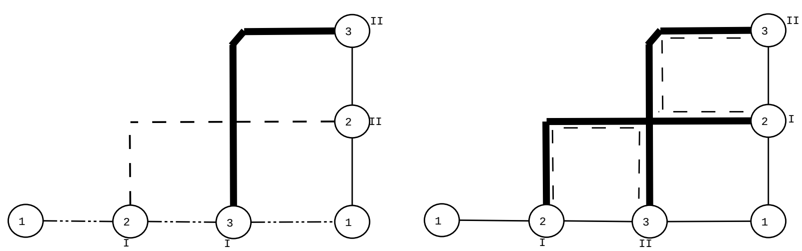
Figure 1

Schroder's Theorem (as corrected by D.W. Clarke [3]) states the following for a graph G which is a $P_m \times P_n$ mesh where $m \geq n \geq 3$:

$$ccw(G) = \begin{cases} n-1, & m = n \text{ is even} \\ n, & \text{where } m = n, n+1 \text{ and } n \text{ is odd or} \\ & m = n+1, n+2 \text{ and } n \text{ is even} \\ n+1, & \text{otherwise.} \end{cases}$$

Schroder's method uses disjoint edges and induction to prove the above theorem.

However, Schroder's method can be modified to include edges outside the two dimensional mesh, which corresponds to a three dimensional graph, and hence find the cyclic cutwidth of the three dimensional mesh.



(a) (b)

Figure 2

1.2 Applying Schroder's Method

Schroder's method pairs up vertices based on their location along the horizontal and the vertical axes (See Figure 2a). Two cases are possible depending on whether the vertices correspond to vertices in Area I or Area II. If they are in opposite areas, then a path is drawn from the side vertex to the point lying above the bottom vertex and horizontal from the side vertex. These edges are then removed from the graph, as well as the vertices they were connecting.

In the second case, the pair of vertices lie within the same area. This implies that there is a pair of vertices related in similar fashion lying in the opposite area. Considering the paths followed by the two pairs of vertices, we can connect the vertices from one pair to the vertices of the other pair and remove the two pairs together, just as we did in the first case. Removing them in this way uses no more than the same edges which would have been removed if the pairs would have connected to themselves instead (See dotted lines in Figure 2b). By induction, we can see this process can be repeated so that all the vertices can be paired in this manner.

Please note: In this induction proof, the edge between vertices 1 and 2 and the edge between vertices 2 and 3 along the vertical are never considered. These edges make up what we will define as the ***wall*** of the graph and will be referred to as such throughout this paper. There will always be a *wall* coming off the corner, either horizontally or vertically.

2 Theorem 1: Cyclic Cutwidth (ccw) of the $P_2 \times P_2 \times P_n$ Mesh

Theorem 1: Let the graph G be a $P_2 \times P_2 \times P_n$ mesh. The cyclic cutwidth of G is then given by:

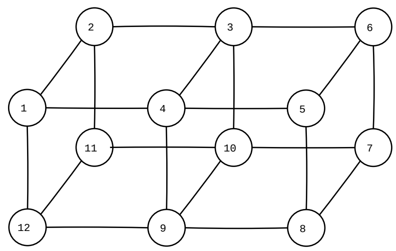
$$ccw(G) = \begin{cases} 1, & n = 1 \\ n+1, & 2 \leq n \leq 5 \\ 6, & n \geq 5. \end{cases}$$

2.1 Proof

For $n = 1$, this reduces to a two dimensional mesh, and by Schroder's Theorem the cyclic cutwidth is 1. A $P_2 \times P_2 \times P_2$ mesh is equivalent to a 3-cube (Q_3) and is already known to have a cyclic cutwidth of 3 [4]. For all other cases ($n \geq 4$) of the cyclic cutwidth of the $P_2 \times P_2 \times P_n$ mesh, we will consider $n = 3, 4, 5$ and the case when $n \geq 6$.

2.2 Lower Bounds for Cyclic Cutwidth

Proposition 1: When $n = 3$, the ccw of G is at least 4.



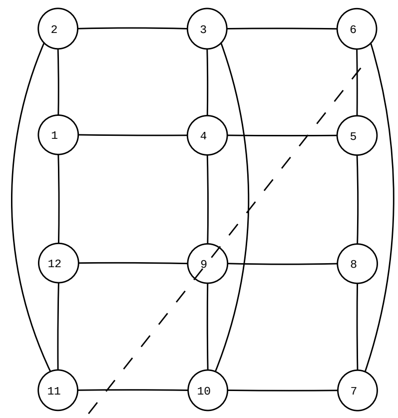


Figure 3: Two embeddings of $P_2 \times P_2 \times P_3$

Proof: Figure 3a shows the graph of the $P_2 \times P_2 \times P_3$ mesh, with an equivalent graph in Figure 3b. If we remove the edge connecting vertices 2 and 11, the edge connecting vertices 3 and 10, and the edge connecting vertices 6 and 7, we have a $P_3 \times P_4$ mesh which according to Schroder's Theorem has a cyclic cutwidth of 3 before adding these edges. Using Schroder's method as previously described, we see this two dimensional mesh contributes five disjoint edges across the diameter between the colored vertices: three due to the section with six vertices, and two due to the four vertices of the smaller section. By examining the edge connecting vertices 2 and 11, we can show that this adds one to the cyclic cutwidth across the diameter yielding a lower bound of 4 for this graph (see Figure 4).

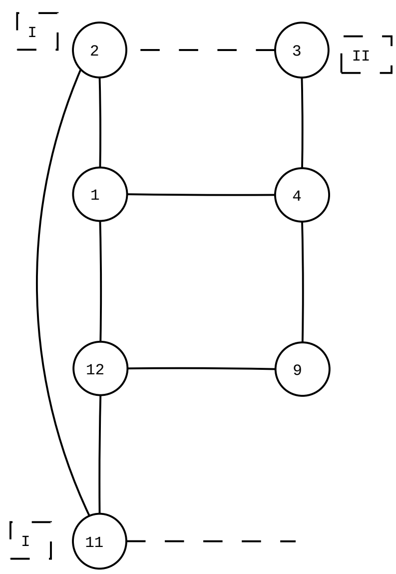


Figure 4

In Figure 4, if vertex 2 lies in Area I and vertex 11 lies in Area II or vice versa, then the edge must cross the diameter adding one to the cutwidth and we're done. Recall from Figure 2a that in using Schroder's Lemma, we never used any edges running between the side vertices. If any of these vertices along the *wall* are in an area opposite the other *wall* vertices, then Schroder's Lemma automatically gives us at least one more path disjoint from the other paths, adding one to the total edges crossing the diameter and hence adds one to the lower bound of the cyclic cutwidth. This happens from both sections adding a total of two to the edges crossing the diameter giving a total of seven crossings and therefore at least a cutwidth of 4. Therefore:

$$ccw(P_2 \times P_2 \times P_3) \geq 4.$$

Proposition 2: When $n = 4$, the ccw of G is at least 5.

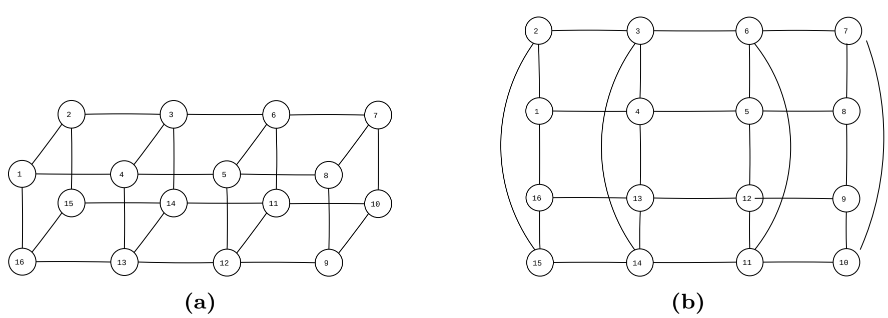


Figure 5

Proof: In Figure 5b, there is an edge connecting vertices 2 and 15. The graph is a $P_4 \times P_4$ mesh after removing the additional loopy edges. Schroder's theorem gives us a lower bound of 3 for the cutwidth due to the two dimensional mesh alone. This additional edge adds one to the cutwidth across the diameter as we saw in the proof of Proposition 1. However, by examining also the edges between vertices 3 and 14 and vertices 6 and 11 together, we can show that these also add one to the cutwidth across the diameter yielding a lower bound of 5 for this case (see Figure 6).

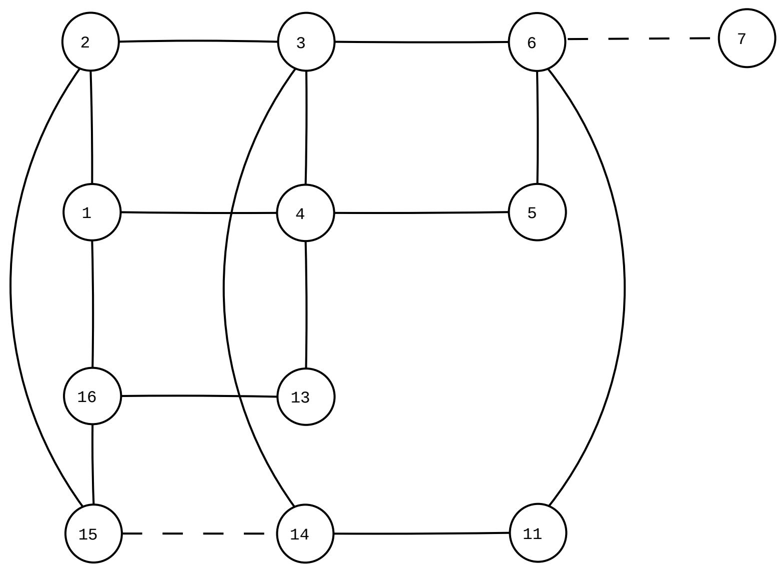


Figure 6

In Figure 6, we have already used the $P_4 \times P_4$ mesh, the edge between vertices 2 and 15, and the edge between vertices 7 and 10 to give us eight disjoint edge paths across the diamter. We have not used the edge between vertices 3 and 14, or the edge between 6 and 11. Schroder's theorem also never uses the edge between vertices 14 and 15 (dotted line) or between vertices 6 and 7. We need only one more edge across the diameter to give a total of nine diameter crossings and therefore to claim a lower bound of 5 for the cyclic cutwidth. This only *doesn't* happen if the four pairs of vertices connected by these edges lie pairwise in the same area. This implies that vertices 3, 14, and 15 are in the same area, as are vertices 6, 7, and 11. Now only consider the two cases for the vertices 2 and 15. Since only three vertices from vertices 1, 2, 3, 6, 15, and 16 can be in each area, if vertices 2 and 15 are in the same area, this forces all 12 colored vertices into specific areas forming a specific graph with at least ten total edges crossing the diameter (see Figure 7). In the second case, vertices 2 and 15 are opposites and so the external edge between vertices 2 and 15 adds one without affecting the lower bound of 3 from Schroder's Lemma, but also the edge between vertices 2 and 3 adds one since this is in the *wall* that is never used in Schroder's Lemma.

Therefore:

$$ccw(P_2 \times P_2 \times P_4) \geq 5.$$

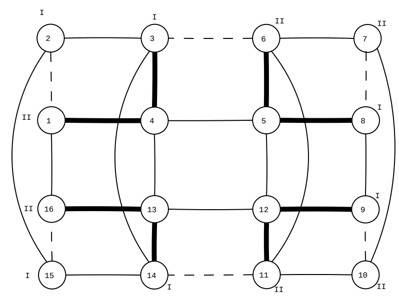
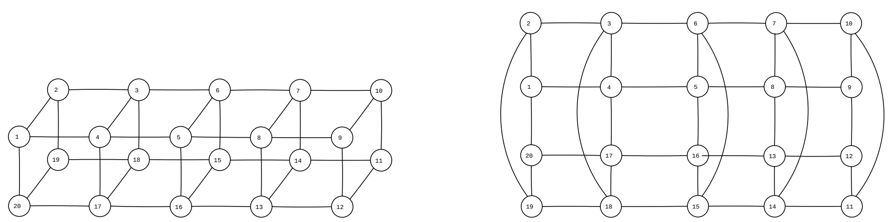


Figure 7

Proposition 3: When $n = 5$, the cyclic cutwidth (ccw) is at least 6.



(a) **(b)**

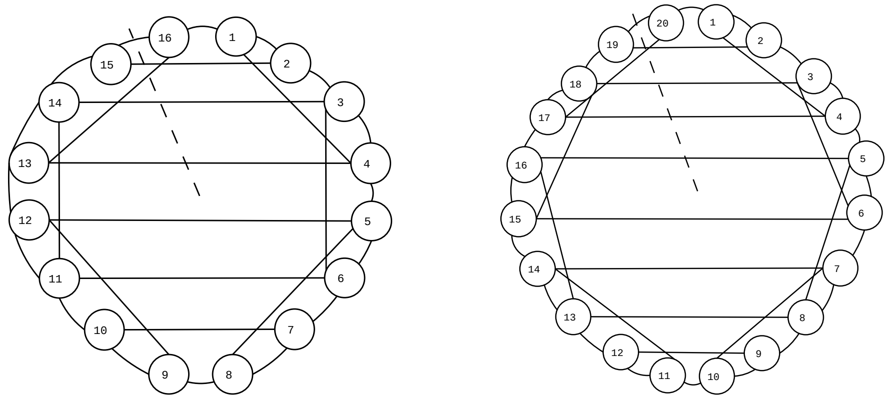
Figure 8

Proof: This is a $P_5 \times P_4$ graph after removing the additional loopy edges, and so has at least a cyclic cutwidth of 4 by Schroder's Theorem due to the two dimensional mesh alone. In Figure 8, we have a graph similar to the proof of Proposition 2. We will get a cutwidth of 4 from the two dimensional mesh and two additional cuts from the external edges using the same techniques as from Proposition 2. This yields a lower bound of 6.

$$ccw(P_2 \times P_2 \times P_5) \geq 6.$$

2.3 Upper Bounds for Cyclic Cutwidth

Using the algorithm of "snaking" through planes (refer to Figure 10), dropping to the next plane, and repeating to form a Hamiltonian cycle, we can generate a cyclic graph with a cyclic cutwidth equal to the lower bounds previously established (see Figures 3,5, and 7 and follow the numbered vertices to find this cycle). The cycles formed are shown below:

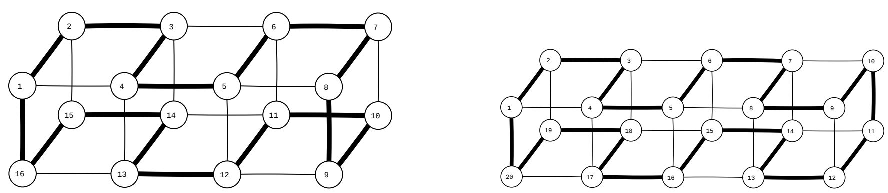


(a) **(b)**

Figure 9

In Figure 9a, the cyclic cutwidth is 5 and in Figure 9b, the cyclic cutwidth is 6. This sets an upper bound on these graphs equal to the lower bound. However, when we get to a cyclic cutwidth of six, this algorithm no longer yields the optimal graph for the cyclic cutwidth. It is at this threshold that the cyclic cutwidth equals the linear cutwidth and therefore we take the linear cutwidth and put it in a circle to yield (see Figure 11):

$$ccw(G) = 6, n \geq 5.$$



(a) **(b)**

Figure 10

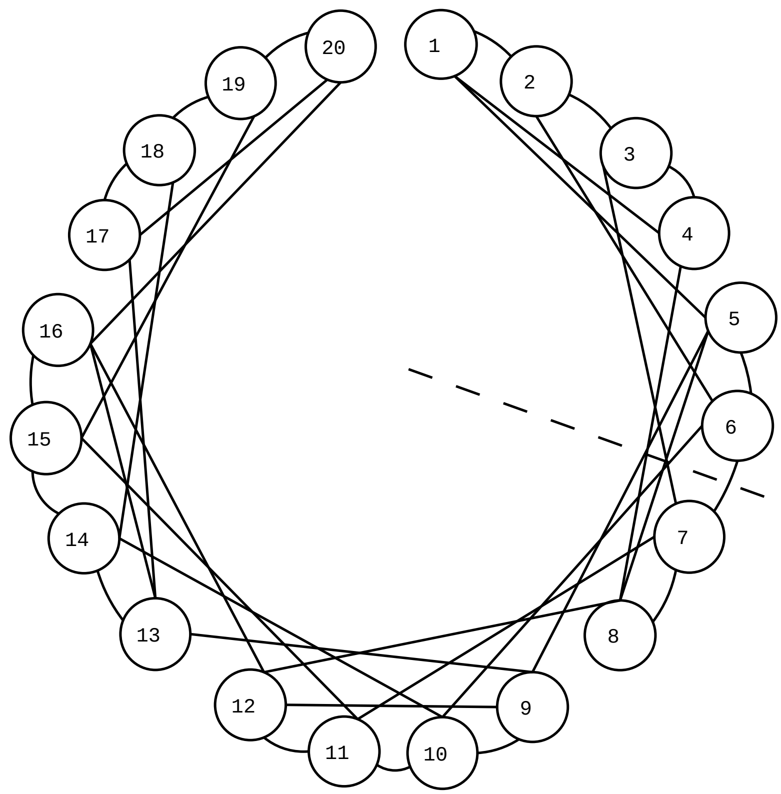


Figure 11

2.4 Completing the Proof

Since every graph of $P_2 \times P_2 \times P_n, n \geq 5$ has $P_2 \times P_2 \times P_5$ as a subgraph, then

$$ccw(P_2 \times P_2 \times P_n, n \geq 5) \geq 6$$

and since extending the algorithm for $P_2 \times P_2 \times P_5$ shows that for $P_2 \times P_2 \times P_n, n \geq 5$, there exists a graph for each $n \geq 5$ for which ccw $= 6$, then the cyclic cutwidth is 6 when $n \geq 5$.

End of proof.

3 The $P_m \times C_n$ Mesh

The $P_2 \times P_2 \times P_n$ mesh is the same as the cylinder $P_n \times C_4$. In Schroder's paper, the cyclic cutwidth of a cylinder is defined to be

$$ccw(P_m \times C_n) = min\{m + 1, n + 2\}.$$

This predicts our cyclic cutwidth of $min\{n+1, 6\}$. However, Schroder does not give the proof of this explicitly in the paper and we were unaware of this result when this work was done. However, after examining some graphs I was able to see how the upper bound of this could be determined.

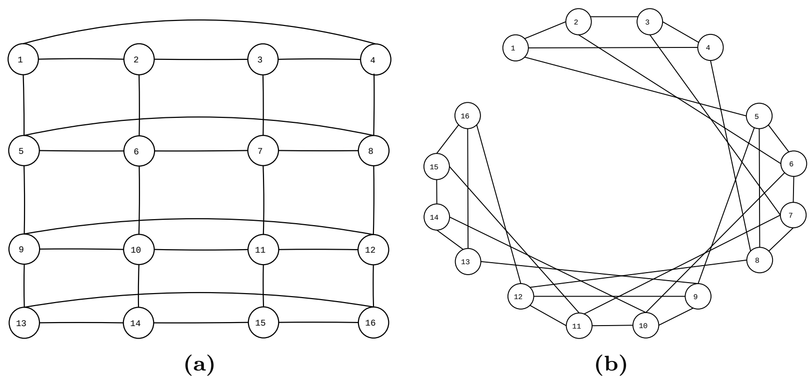


Figure 12: An embedding of $P_2 \times P_2 \times P_4$ onto a cycle

In Figure 12b, we see an embedding of a $P_2 \times P_2 \times P_4$ mesh (such as seen in Figure 5a) onto a cycle. This is the same as a $P_4 \times C_4$ cylinder. Considering the general case of a $P_m \times C_n$ mesh, Figure 12 shows an upper bound of $n + 2$ for this configuration. Referring to each crescent-shaped part of the graph as *islands*, we see there are n connecting edges from each *island* to another *island*, but then there are two additional edges due to the crescent shape itself. This gives us an upper bound of $n+2$ for the cyclic cutwidth. The number of islands (determined by m) are irrelevant to the cyclic cutwidth in this scenario.

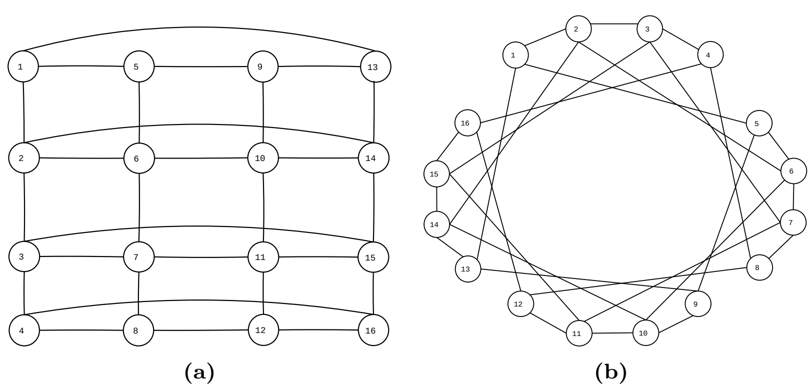


Figure 13: A different embedding of $P_2 \times P_2 \times P_4$ onto a cycle

In Figure 13b, we see a different embedding of the $P_m \times C_n$ mesh, but in this arrangement the *islands* are formed by the columns instead of the rows. There are m edges connecting the vertices of an *island* with those of another *island*, but also we get an additional edge from the edges connecting the vertices in the *island* yielding a cyclic cutwidth of $m + 1$ for the graph. Notice that in this arrangement the cutwidth is the same between every adjacent pair of vertices all around the cycle. This will be important when we consider concentric cylinders ($P_2 \times P_m \times P_n$ mesh).

4 The $P_2 \times P_m \times P_n$ Mesh

Just as the $P_2 \times P_2 \times P_n$ can be viewed as a $P_n \times C_4$ cylinder, the $P_2 \times P_m \times P_n$ mesh can be viewed as several concentric cylinders. We can then extend the upper bound method from the previous section to get an upper bound for the $P_2 \times P_m \times P_n$ mesh.

Theorem 2: For a $P_2 \times P_m \times P_n$ mesh, where $2 \leq m \leq n$:

$$ccw(P_2 \times P_m \times P_n) \leq 3m.$$

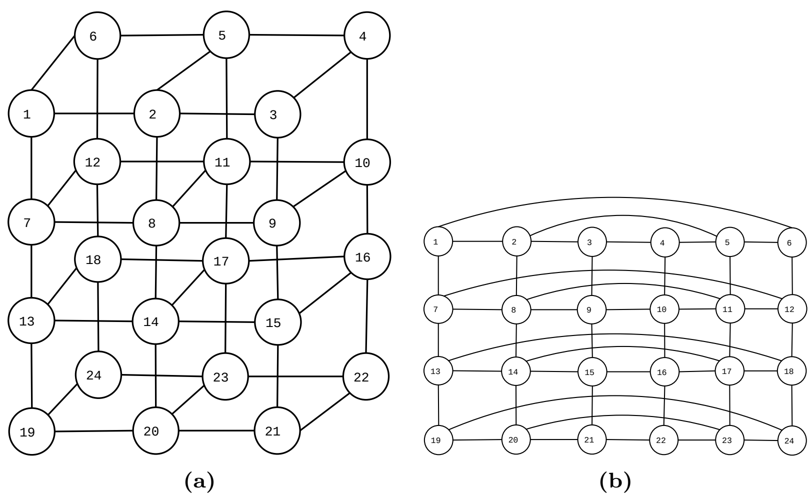


Figure 14: Two embeddings of a $P_2 \times P_3 \times P_4$ mesh

Proof: In Figure 14, we see that $P_2 \times P_m \times P_n$ meshes can be embedded as concentric cylinders. Using the same technique that was used to prove a cyclic cutwidth of $n+2$ in the $P_m \times C_n$ mesh, we see that the new *islands* formed are made up of $2m$ vertices (where $m \leq n$) and there are n *islands*. By the same reasoning as before, there are $2m$ edges connecting an *island* to another *island*, but also vertices within an *island* are connected pairwise as seen in Figure 15 (i.e. the arcs between vertices 1 and 6, 2 and 5, and vertices 3 and 4) giving additional m edges. This gives a cyclic cutwidth of $3m$ irregardless of the value of n.

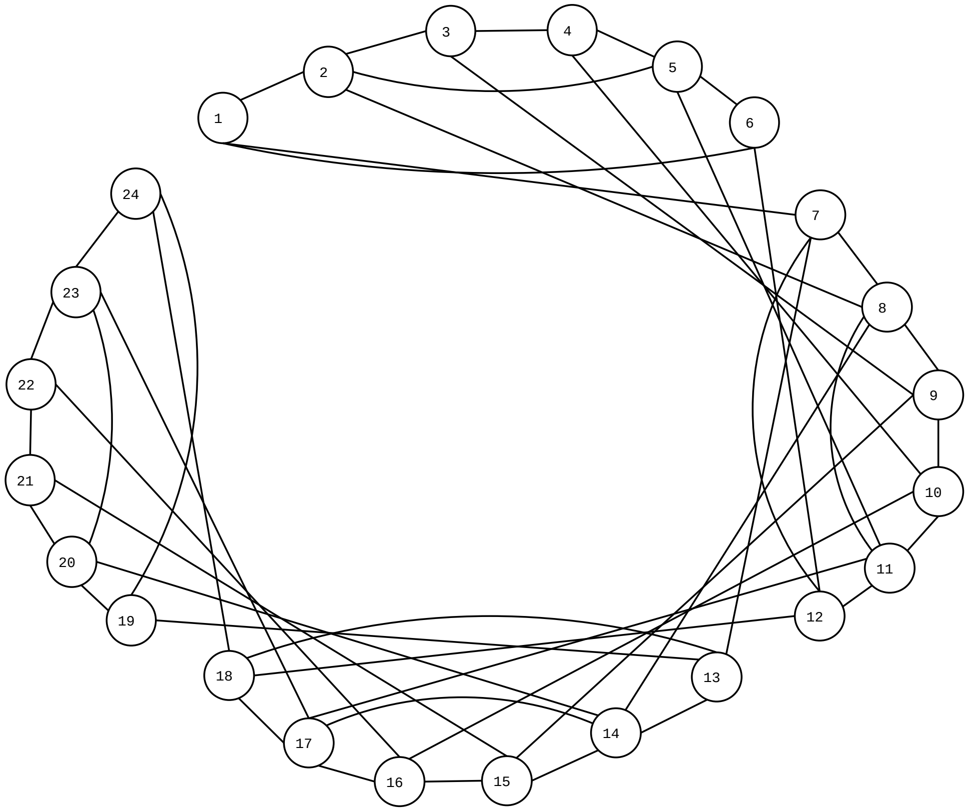


Figure 15: A cyclic embedding of a $P_2 \times P_3 \times P_4$ mesh

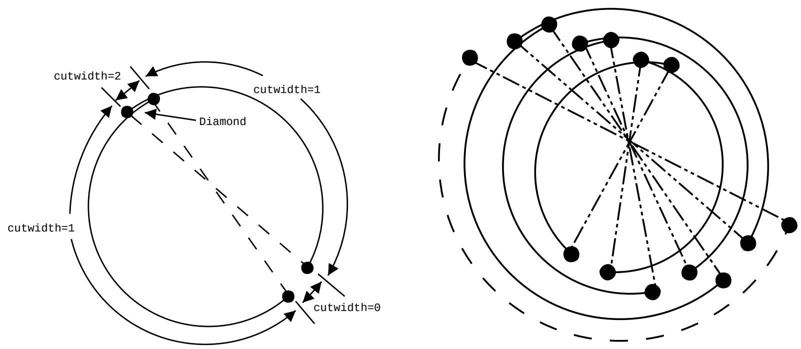
Theorem 3: For a $P_2 \times P_m \times P_n$ mesh, where $2 \leq m \leq n$:

$$ccw(P_2 \times P_m \times P_n) \leq \begin{cases} \frac{mn+2}{2}, & n \text{ is even} \\ \frac{mn+3}{2}, & m,\, n \text{ odd} \\ \frac{mn+4}{2}, & m \text{ even, } n \text{ odd.} \end{cases}$$

Proof: In order to prove this theorem, we will need to make use of the following proposition:

Proposition 4: Given a graph G whose optimal embedding has equal cutwidth between all pairs of adjacent vertices on a cycle, then the change in the cyclic cutwidth due to the addition of n pairwise disjoint diameters (where each diameter connects vertices adjacent to other vertices also connected by a diameter) is given by:

$$\triangle ccw(G) = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{n+2}{2}, & n \text{ is even.} \end{cases}$$



(a) Diamond **(b) Diamond Rings**

Figure 16: Diamonds and Diamond Rings

Proof of Proposition 4: Define a *diameter* as an edge on a cycle such that it divides the cycle in half (i.e. it passes through the same number of vertices irregardless of which way it goes around the cycle). Define *adjacent diameters* as diameters with one vertex from each diameter forming a pair of adjacent vertices on the cycle. Also define a *diamond ring* as a pair of *adjacent diameters* running in opposite directions and a *diamond* is the section between adjacent vertices on a *diamond ring* having cutwidth 2. Hence there's only one *diamond* per *diamond ring*. We run all *adjacent diameters* in opposite directions around the cycle forming concentric *diamond rings*. We can't help but get a cutwidth of 2 from the *diamond* in the first *diamond ring*. This method will keep the graph optimal always since it distributes the edges as evenly as possible. This also ensures that every disjoint *diamond ring* added afterwards contributes only one more cut to cyclic cutwidth. The first diamond ring gives a cutwidth of 2, but every disjoint diamond ring added afterward adds just one more to the cutwidth. It also distributes all the *diamonds* onto one half of the cycle. For n diameters, if n is even we have $\frac{n}{2}$ *diamond rings* and therefore $\frac{n}{2}$ cuts plus an additional cut for the first *diamond ring*. Thus we have $\frac{n}{2}+1$ or $\frac{n+2}{2}$ cuts proving the upper bound for the even case.

For the n odd case (see dotted line in Figure 16b), it will have the same number of cuts as for the $n-1$ even case since we run the extra diameter in the direction opposite the half of the cycle containing all the *diamonds*. Thus the cyclic cutwidth is $\frac{(n-1)+2}{2}$ or $\frac{n+1}{2}$ proving the upper bound for the odd case.

To prove the lower bounds, let m represent the number of vertices on the cycle. Since the wirelength of each diameter is $\frac{m}{2}$, this mean the total wirelength for the diameters is $n \cdot \frac{m}{2}$ or $\frac{mn}{2}$. Dividing by m gives an average cutwidth of $\frac{n}{2}$ between each pair of adjacent vertices on the cycle. This proves the odd case. To prove the even case, we make notice that the only way to make a cutwidth of $\frac{n}{2}$ for the even case is if every pair of adjacent vertices has the same cutwidth between them. This is impossible with disjoint diameters, so the minimum cutwidth for the graph must be $\frac{n}{2}+1$ proving the even case.

End of Proof for Proposition 4.

Proposition 5: An upper bound for the cyclic cutwidth of a $P_2 \times P_m \times P_n$ mesh where n is even is:

$$ccw(P_2 \times P_m \times P_n) \leq \frac{mn+2}{2}.$$

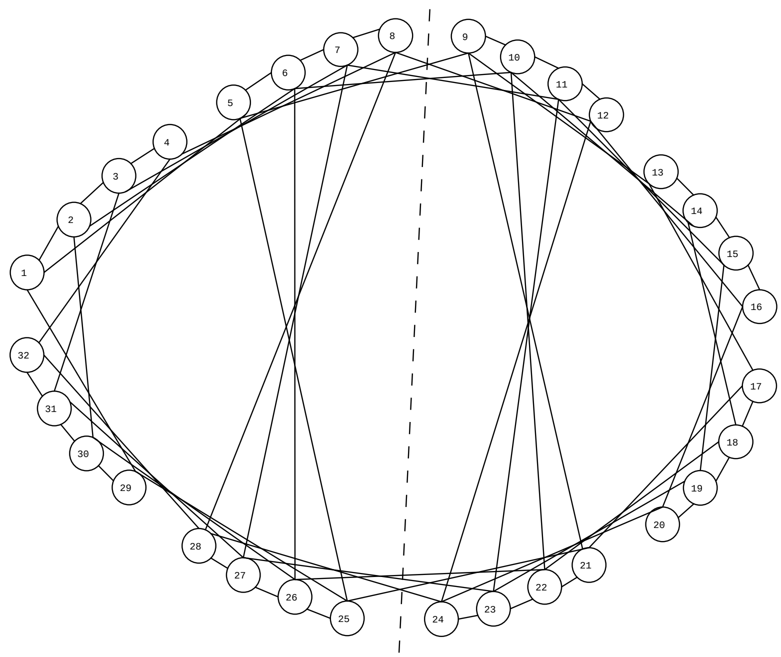


Figure 17: Cyclic Embedding of $P_2 \times P_4 \times P_4$ Mesh

Proof for Proposition 5: Extending the techniques outlined in proving the $m+1$ upper bound for the $P_m \times C_n$ mesh (see Figure 17), we have $2n$ *islands* with m vertices on each *island*. As before we get the cutwidth of $m+1$ between all pairs of adjacent vertices. However, we also have additional edges that we can run pairwise in opposite directions (Note that none cross the dotted line). Since the lines run from one half of the *islands* to the other half, we need only consider the top half n *islands*. The end islands aren't used, leaving only $n-2$ *islands*, of which half the additional edges run either direction, leaving $\frac{n-2}{2}$ islands with m edges giving total additional cuts of $\frac{m(n-2)}{2}$. Added to the $m+1$ original cuts between all pairs of vertices, this yields an upper bound of $\frac{mn}{2} - m + m + 1$ or $\frac{mn+2}{2}$.

End of Proof for Proposition 5.

Proposition 6: An upper bound for the cyclic cutwidth of a $P_2 \times P_m \times P_n$ mesh where m,n are odd is:

$$ccw(P_2 \times P_m \times P_n) \leq \frac{mn+3}{2}.$$

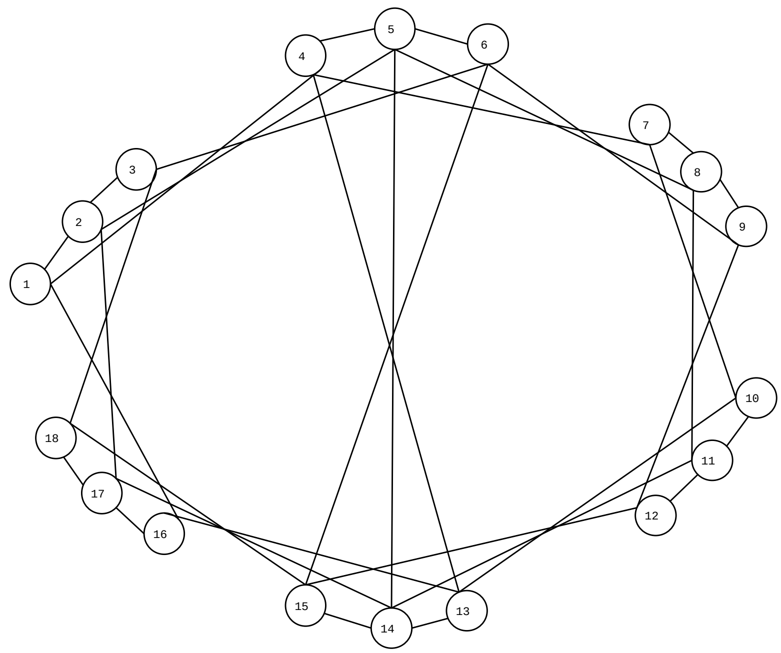


Figure 18: Cyclic Embedding of a $P_2 \times P_3 \times P_3$ Mesh

Proof for Proposition 6: This proof is similar to that of Proposition 5, but since n is now odd, we have an odd number of *islands* on the top half of the graph (see Figure 18). This implies that the middle *islands* on the top and bottom are connected by diameters, which require us to use Proposition 4. So we have $m+1$ cuts from the original graph, $\frac{m+1}{2}$ cuts from the diameters, and $\frac{m(n-3)}{2}$ cuts from the other *islands*, which didn't include the end or middle *islands*. Adding these gives us a total cut of:

$$m + 1 + \frac{mn}{2} - \frac{3m}{2} + \frac{m+1}{2} = \frac{mn+3}{2}.$$

End of Proof for Proposition 6.

Proposition 7: An upper bound for the cyclic cutwidth of a $P_2 \times P_m \times P_n$ mesh where m is even, n is odd, is:

$$ccw(P_2 \times P_m \times P_n) \leq \frac{mn+4}{2}.$$

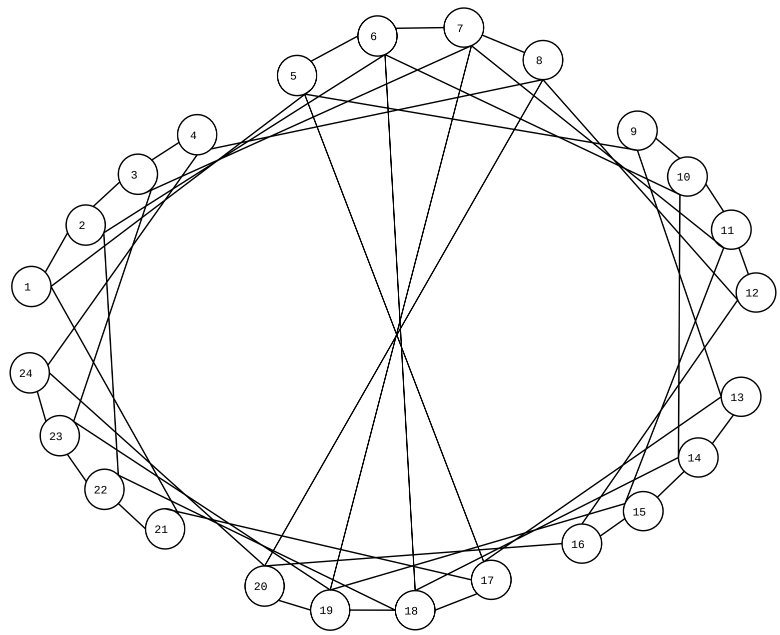


Figure 19: Cyclic Embedding of a $P_2 \times P_3 \times P_4$ Mesh

Proof for Proposition 7: Exactly the same as in Proposition 6, except that the number of diameters is now even (see Figure 19). So by Proposition 4, the contribution to the cut due to the diameters is $\frac{m+2}{2}$ and adding with the other contributors yields:

$$m + 1 + \frac{m+2}{2} + \frac{m(n-3)}{2} = \frac{mn}{2} + 2 = \frac{mn+4}{2}.$$

End of Proof for Proposition 7.

Propositions 5, 6, and 7 together complete the proof for Theorem 3.

End of Proof for Theorem 3.

Corollary: For a $P_2 \times P_m \times P_n$ mesh, where $2 \leq m \leq n$:

$$ccw(P_2 \times P_m \times P_n) \leq \begin{cases} \frac{mn+2}{2}, & n \text{ is even or } m \text{ is even} \\ \frac{mn+3}{2}, & m, n \text{ odd} \end{cases}$$

or

$$ccw(P_2 \times P_m \times P_n) \leq \lceil \frac{mn}{2} \rceil + 1.$$

Proof: Since in Theorem 3 we never made use of the fact that $m \leq n$, we can interchange their roles from the last case in Theorem 3 to make it the first case stated. These can then be combined to get the result stated.

5 The $P_l \times P_m \times P_n$ Mesh

Theorem 4: For a $P_l \times P_m \times P_n$ mesh, where $2 \leq m \leq n$:

$ccw(P_l \times P_m \times P_n) \leq \min\{l \cdot ccw(P_m \times P_n)+1, m \cdot ccw(P_l \times P_n)+1, n \cdot ccw(P_l \times P_m)+1\}$.

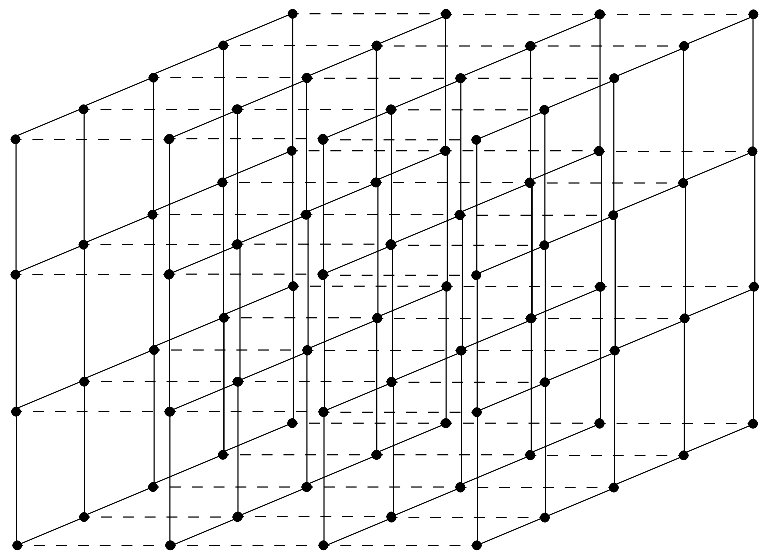
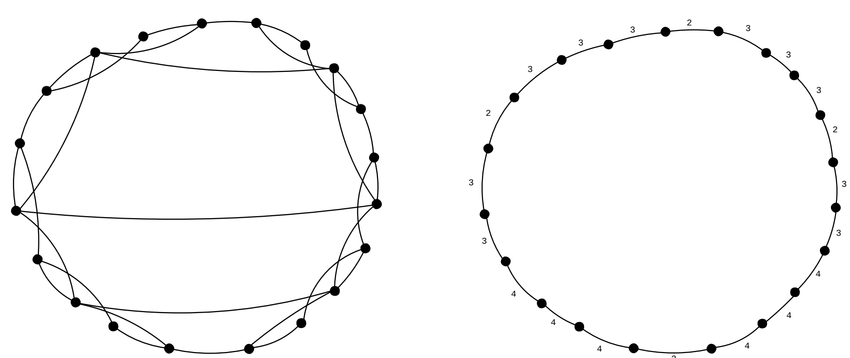


Figure 20: Parallel Two Dimensional Meshes with connecting edges



(a) Optimal Graph for $P_4 \times P_5$ Mesh **(b) Abbreviated Graph with Cutwidths**

Figure 21: The Optimal $P_4 \times P_5$ Mesh With Abbreviated Form

Proof: We will make use of the following Lemma:

Lemma 1: Every three dimensional mesh is made of parallel equivalent two dimensional meshes connected by edges (see Figure 20). In Figure 21a, the optimal graph of a cyclic embedding of a $P_4 \times P_5$ mesh is illustrated. We will represent the number of cuts between adjacent vertices on the cycle as numbers between vertices for clarity (see Figure 21b). Let the optimal cyclic embedding of each two dimensional graph be graph G_i where $i = 1..l, i = 1..m$, or $i = 1..n$ depending on which direction we go on the three dimensional mesh. Further, since each graph G_i doesn't share any vertices with any other graph $G_j, j \neq i$, we can embed the optimal graphs one on top of another (such as in Figure 22) and add the connecting edges in as seen by the dotted lines. Let $V_1, .., V_k$ represent the k vertices of each optimal graph G_i. Let G_1 be the outermost cycle, inserting concentric cycles inside one another until the innermost cycle is G_l (or G_m or G_n). We then align V_1 of G_2 just to the right of V_1 of G_1, V_1 of G_3 just to the right of V_1 of G_2, and so on until all the V_1's of all the graphs have been placed. We then begin with V_2 of G_1 and repeat the previous process until all vertices have been inserted.

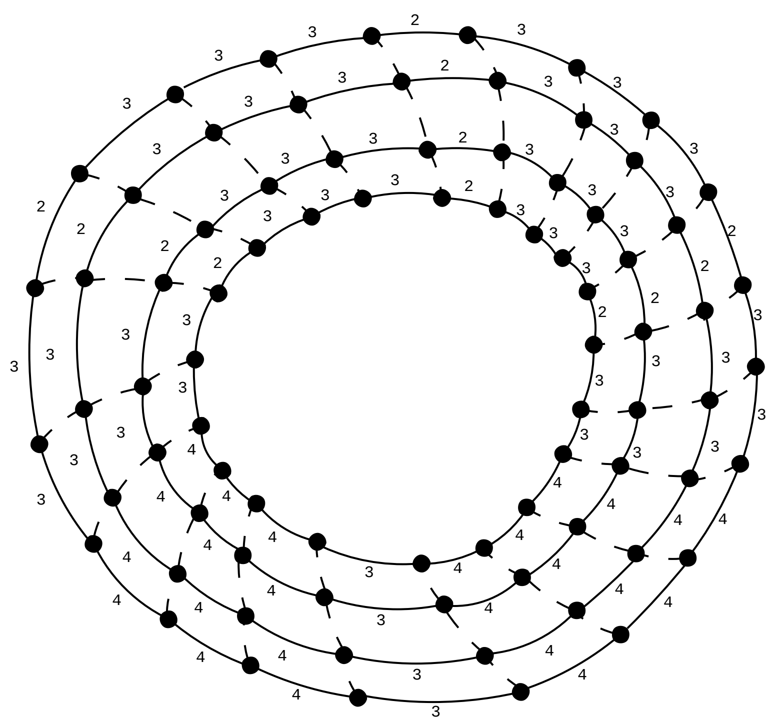


Figure 22: Concentric Optimal Graph Lemma

Using this lemma, we get an upper bound for each direction we use. The connecting edges add only one more to the cutwidth from this method. So for the graphs $G_i, i = 1..l$ we get l graphs, each with cyclic cutwidth represented by ccw$(P_m \times P_n)$ giving a total cutwidth of $l \cdot ccw(P_m \times P_n)$ plus one more for the connecting edges giving a total upper bound of $l \cdot ccw(P_m \times P_n) + 1$ for the cyclic cutwidth. The other two cases are proven similarly.

End of Proof.

6 Conclusions

We now have the cyclic cutwidth of the $P_2 \times P_2 \times P_n$ mesh as well as upper bounds for the cyclic cutwidths of the $P_2 \times P_m \times P_n$ mesh and $P_l \times P_m \times P_n$ mesh. There may be a limited extension of Schroder's method to finding the lower bounds of the $P_2 \times P_m \times P_n$ mesh, but it seems clear that it will *not* be extendable to finding lower bounds of the general case $P_l \times P_m \times P_n$. Our upper bounds seem promising for the general case in that they correspond exactly with the upper bounds found by D.W. Clarke for the cubic mesh by different means, and with a closer look at the algorithm to refine the upper bounds found by it, we can extend the algorithm into n dimensions and the upper bound it predicts exactly matches the cyclic cutwidth found by Dr. J. Chavez for the n-cube. We now need to find lower bounds that match these upper bounds and hopefully the lower bound technique will apply to any number of dimensions to solve not only the 3-dimensional general case, but the general cases in n dimensions.

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