Twist Numbers of Links from the Jones Polynomial

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Abstract

A theorem of Dasbach and Lin's states that the twist number of any alternating knot is the sum of the absolute value of the second coefficient and the absolute value of the second to last coefficient of the Jones Polynomial. We extend this result to links, finding that the Jones Polynomial doesn't detect Hopf link factors. This yields a formula which gives the twist number of any link: each region which is not part of a twist adds one, then each Hopf link factor minuses one, then minus two to get the Jones Polynomial twist number. Furthermore, we show what the individual numbers represent in terms of the link diagram.

1   Background

For the purposes of this paper, a *knot* is a simple closed curve in $\mathbb{R}^3$, whereas *link* is defined to have any number of components. The Reidemeister moves are the most basic operations on a link diagram which leave the link unchanged. The type I move adds a loop, the type II moves lays a portion of a strand over or under another strand, and the type III move translates a strand from one side of a crossing to the other side. For illustration of these three moves, see figure 1. For more information, refer to [1].

In [2], Kauffman details many results and terminology which are needed for this paper. The Kauffman Bracket polynomial was introduced and a connection to the Jones polynomial was found. Using the Kauffman Bracket
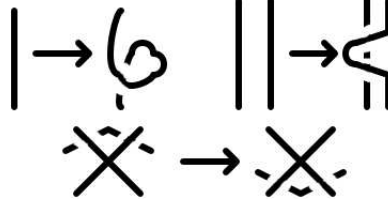
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Figure 1: Reidemeister moves type I, II, and III

polynomial, he was able to prove that any two connected, reduced, alternating projections of a link L have the same number of crossings, which was a long-standing conjecture of Tait's.

Definition 1. Let L be an unoriented link diagram. Let $< L >$ be the element of the ring $\mathbb{Z}\left[A, A^{-1}, -A^2 - A^{-2}\right]$ defined by the rules:

1. $< O >= 1$,

2. $< O \cup K > = (-A^2 - A^{-2}) < K >$ (where $K \neq \emptyset$),

3. $< \diagdown\!\!\!\!\diagup >= A < \smile\!\!\frown > + A^{-1} <)(>$, or
   $< \diagup\!\!\!\!\diagdown >= A <)(> + A^{-1} < \smile\!\!\frown >.$

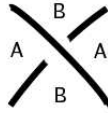Figure 2: A crossing

Remark To see *3* easier, use the right-hand rule on each undercrossing strand with your thumb pointed in the direction of the crossing. Label the area that your fingers point to as A, then label the other two areas as B (see figure 2). Now, we simply smooth out the crossing so that the two A areas are connected (figure 3). This action corresponds to $A < \smile\!\!\frown >$. Doing the smoothing to connect the B areas corresponds to $B <)(>$ (figure 3). These
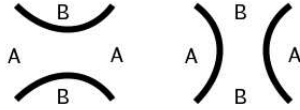
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Figure 3: A-smoothing and B-smoothing

smoothings are called an *A-smoothing* and a *B-smoothing*, respectively. Also denote the A and B areas as *A-regions* and *B-regions*, respectively.
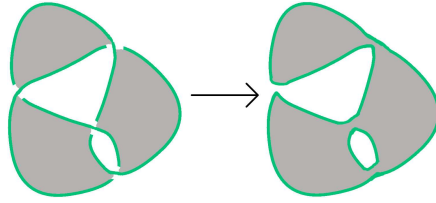

Figure 4: A given state

For any given link diagram L, performing all smoothings in any combination such that no crossings are left in L is called a *state S* of the diagram L. Since performing all smoothings yields a collection of circles, we see that the state contributes

$$A^i A^{-j} (-A^2 - A^{-2})^{|S|-1},$$

where i is the number of A-smoothings, j is the number of B-smoothings, and $|S|$ is the number of circles in the state diagram S. In figure 4, three A-smoothings are performed and 1 B-smoothing is performed. The sum of all the state contributions yields the bracket polynomial. In Kauffman's original paper, he shows that the bracket polynomial is invariant under Reidemeister moves of type II and III, but not under Reidemeister move I.

In order to make the bracket polynomial invariant under Reidemeister move type I, we need to introduce the *writhe* $w(L)$ of a link L. It is simply the sum of all crossings where each crossing is given a value of either $+1$ or -1 using the right-hand rule on the overstrand (figure 5). Now we define a Laurent Polynomial $f[L]$ by the formula
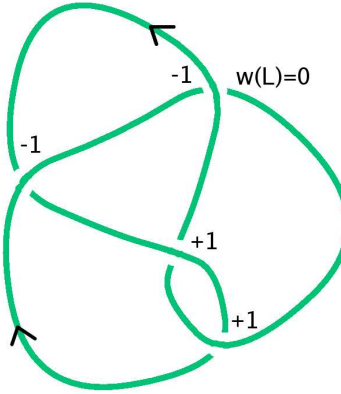
$$f[L] = (-A)^{-3w(L)} < L >,$$

Figure 5: Writhe of a Link

where $w(L)$ is the writhe of a link L. This formula is invariant under Reidemeister move I. To obtain the Jones Polynomial, we substitute $t^{-1/4}$ in for A. For the rest of this paper, the direct relation to the Jones Polynomial will rarely be discussed.

Lemma 1.1 *If a link L has an n-crossing twist formed by n type I Reidemeister moves, then*

$$< \text{\raisebox{-2pt}{\includegraphics}} >= (-1)^n A^{\pm 3n} < L > .$$

Proof The proof of this is left for the reader.
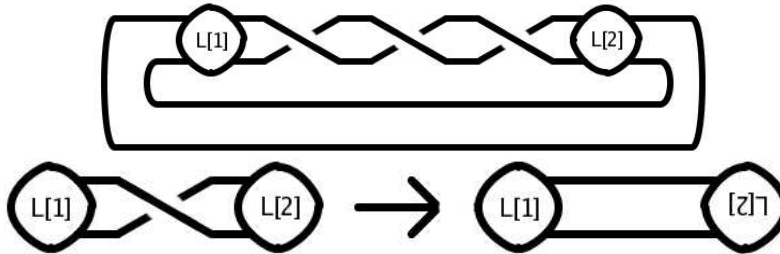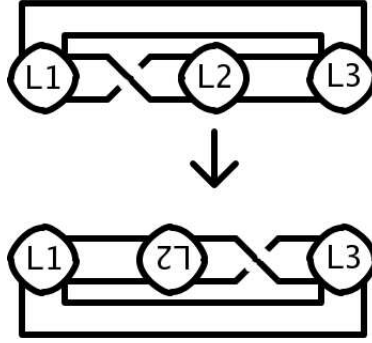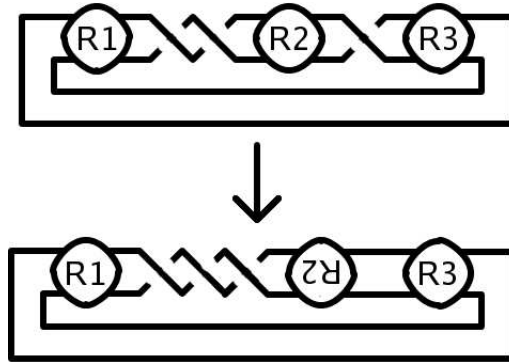

Figure 6: Twist and an Isthmus

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Figure 7: Flype

Figure 8: Reducing twists by flyping

Before proving any results, more definitions are needed. A *twist* in a diagram D is two strands which cross each other (figure 6). A *flype* is a rotation of some portion of the link which translates a twist from one area to another area (figure 7). Note that flypes can reduce the number of twists in a link (figure 8), and for every link L, there exists a diagram which has the minimum number of twists for that link. Thus, the diagram is said to be *twist-minimal*. The number of twists in this twist-minimal diagram is the twist number $T(L)$ of the link L. An *isthmus* is a crossing in a diagram D so that two of the four local regions at the crossing are part of the same region in the overall diagram, and a diagram is *reduced* if it does not contain an isthmus (figure 6).

2   Standard Twist Numbers

Lemma 2.1 *The reduced, prime link L is a $T(2,n)$ link if $T(L) = 1$.*

Proof (Figure 9) Suppose that L has one twist. There is only one way for the twist to be connected to itself, and this is obviously a $T(2,n)$ link. ∎
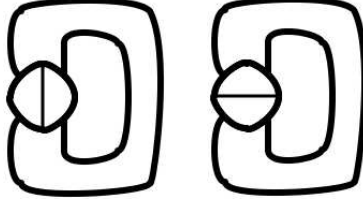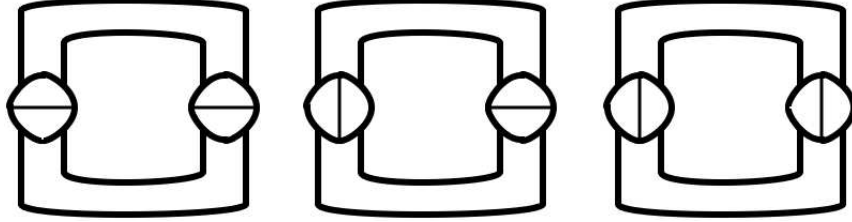

Figure 9: $T(2,n)$ link and $T(L) = 1$

Figure 10: Possible two-twist combinations

Theorem 2.2 *The reduced, prime link L is a two-twist 2-bridge link if $T(L) = 2$.*

Proof (Figure 10) Assume $T(L) = 2$. This means there is only one way of connecting the two twists, but there are three different ways to orient them: both twists pointing to the inside, one twist pointing to the inside, and no twist pointing to the inside. If no twists point to the inside of the diagram, then L is not reduced and the two twists are actually one twist which is separated. Similarly, if both twists point to the inside, then L is in the form of a two-twist pretzel knot, but two-twist pretzel links are not reduced (again

there is one twist separated into two twists). Finally, if only one twist points inside, then L forms a two-twist 2-bridge link, which is reduced. Thus, this is the only two-twist reduced link. ∎

Theorem 2.3 *The reduced, prime link L is a three-twist pretzel link or a three-twist 2-bridge link if $T(L) = 3$.*

Proof There are two parts to this proof. For the first part, we will prove that there is only one way to arrange three twists. After that, we will show that the twist orientation determines what kind of link it is.

First, each twist can be thought of a vertex in a graph where the following rules are obeyed: 1) each vertex has degree four (since a twist must have four lines exiting it), 2) there can be no crossings (i.e. there are no crossings outside of the twists), and 3) no vertex can have an edge to itself (i.e. the overall link is prime). Given those rules, there is only one possibility: each vertex is connected to every other vertex by two edges. If two vertices have three edges between them, then there will be one edge from each of those vertices connecting the last vertex. For the last vertex to have degree four, it would need to have an edge to itself which is not allowed. Thus, there is only the one possibility.

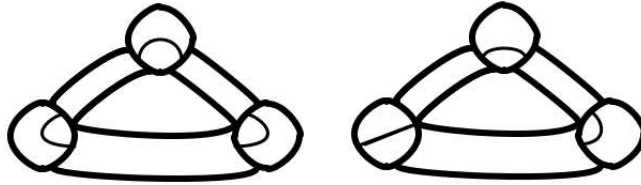Next, we will consider the twist orientation of each twist using cases.

Figure 11: Case 1

Case 1: (Figure 11) If none of the twists point towards the interior of the graph, then all three twists point towards each other which means the diagram is not twist reduced. Similarly, if one of the twists points towards the interior, then the other two twists point toward each other, so they are not twist reduced either.

Case 2: When two of the twists point towards the interior, the knot diagram is that of a three twist 2-bridge link (figure 12).
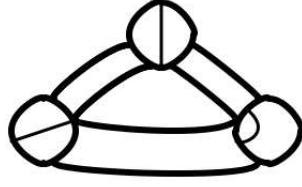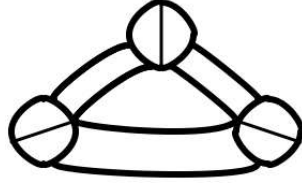
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Figure 12: Case 2

Figure 13: Case 3

Case 3: If all three twists point towards the center, then the knot diagram is that of a three twist pretzel link (figure 13).

Thus, any twist reduced diagram with three twists is either a three twist 2-bridge link or a three twist pretzel link. ▮
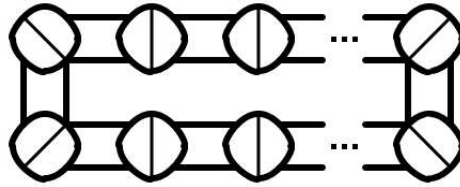

Figure 14: Pretzel diagram

Theorem 2.4 *All twist-reduced, prime pretzel links $L = (p_1, p_2, ..., p_n)$, where $|p_i| > 1$ for $i = 1, ..., n$, have isomorphic diagrams given by (figure 14). Furthermore, $T(D) \leq n$.*
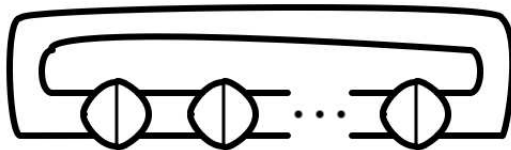
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Figure 15: Normal pretzel diagram

Remark This box can be rearranged to show the familiar pretzel link structure by putting all twists onto a line (figure 15).

Proof By induction, suppose there are three twists, each with more than 1 crossing. It was shown in Theorem 2.3 that three-twist pretzel links have all their twists pointing towards the interior, so the first case is done.

Now assume there are n twists, each with greater than 1 crossings connected in a box with all twists pointing towards the interior like figure 15. Adding another twist with crossing greater than 1 yields a diagram which is isomorphic to the n twist diagram. This means the diagram is twist-reduced: flypes will only rearrange the twists, so the twist number $T(L)$ doesn't change.

Thus, by the principle of mathematical induction, all twist-reduced pretzel links L have isomorphic diagrams, so $T(D) \leq n$. ∎

3   Relation of Twist Numbers to Jones Polynomial

Let n be the number of crossings in a reduced, alternating link L. Since L is alternating, note that the labels of each region are all the same. If L was non-alternating, the region labels might not always be the same. Shade the A-regions and denote the number of them by s. Similarly, do not shade the B-regions, and denote the amount of them by w (figure 16).

A *twist decomposition* is given to mean a twist, along with its interior regions. If the region inside of the twist is an A-region, then the twist is defined to be an A-*twist* (figure 18). A similar definition holds for B-regions and B-*twists*. Furthermore, if a region is not a part of any twist, it will be called an *exterior A-region* (or *B-region*) and the total amount of them will be denoted by r_s (or r_w). For example, in figure 17, and assuming the link
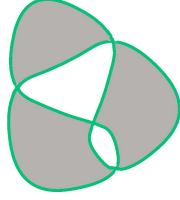
Figure 16: Shadings

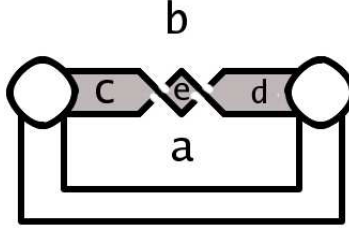

Figure 17: Example

is alternating and the diagram is twist-minimal, then the a and e regions
contribute 1 each to r_w, but only the b and d regions contribute 1 to r_s. This
is because the c region is an A-twist region so it is not included in r_s. Finally,
if T is a 2-crossing A-twist, then T is a *Hopf twist* if both exterior A-regions
are the same region in the overall diagram (again, a similar definition holds
for B-twists and exterior B-regions). The number of Hopf twists will be
denoted by h_w and h_s for B-region Hopf twists and A-region Hopf twists,
respectively. In figure 18, the Hopf twist is a B-Hopf twist, so $h_w \geq 1$, since
there could be more Hopf twists inside the tangles.

Lemma 3.1 *Let L be an alternating, reduced, and twist-minimal link dia-
gram. If an A-twist (or B-twist) with two crossings is not a Hopf twist,
then performing a single A-smoothing (or B-smoothing) inside the A-twist
(or B-twist) does not result in a Hopf twist being created in the resulting
diagram.*

Proof This is trivial since the A-smoothing connects two A-regions, but a
Hopf twist will not result since the original twist was not a Hopf twist. ∎
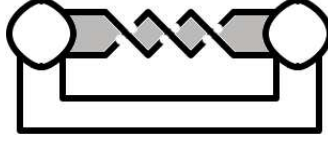
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Figure 18: *B*-Hopf Twist

Figure 19: *A*-twist

Lemma 3.2 *Let L be an alternating, reduced, and twist-minimal link diagram. If a twist has a single crossing, then performing an A-smoothing on that crossing will not result in a B-Hopf twist, and similarly, performing a B-smoothing will not result in an A-Hopf twist.*

Proof Without loss of generality, an *A*-smoothing on a 1-twist connects two *A*-regions, and leaves the white regions as they were, there can be no white Hopf twists created (figure 20). ∎

Let the Jones Polynomial of any link L be given by

$$J(L) = a_n t^n + a_{n+1} t^{n+1} + ... + a_{m-1} t^{m-1} + a_m t^m,$$

where $a_i \in \mathbb{Z}$ for $n \leq i \leq m$ and $n, m \in \mathbb{Z}$. Dasbach and Lin [3] proved that the Jones twist number of a knot K can be realized in the following way:

$$T_J(L) = |a_{n+1}| + |a_{m-1}|.$$

However, a stronger result is shown in the next theorem. To prove the theorem, four coefficients will need to be tracked. This means four degrees of A need to be tracked: the highest, second highest, second lowest, and lowest degree. The maximum, second highest, second lowest, and minimum degrees are given by

$$max \ term < L >= (-1)^{w-1} A^{n+2w-2}, \tag{1}$$

$$next \ highest \ term < L >= (-1)^{w-2} T_1 A^{n+2w-6}, \tag{2}$$

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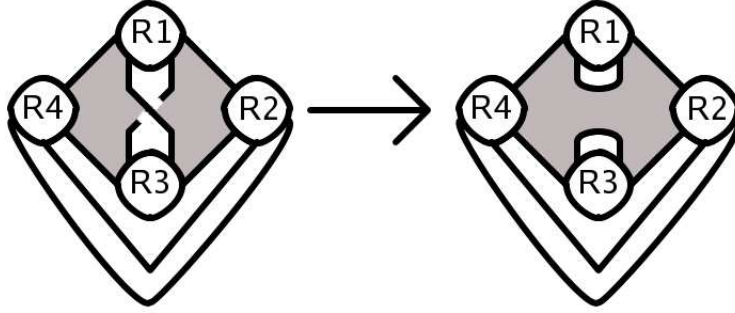
Figure 20: No white Hopf twists

$$next\ lowest\ term < L > = (-1)^{s-2}t_1A^{-n-2s+6}, \tag{3}$$

$$min\ term < L > = (-1)^{s-1}A^{-n-2s+2}, \tag{4}$$

where T_1 and t_1 are the twist numbers given in the theorem. These equations have seven variables to keep track of, so to decrease explanation, we will call them the *tracked variables*. Similarly, we will call equations (1) through (4) the *main equations*.

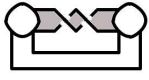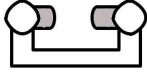**Theorem 3.3** *Let L be an alternating, twist-minimal link diagram. Then*

$$T_1 = r_s - h_w - 1, \ and\ t_1 = r_w - h_s - 1,$$

so the Jones twist number is

$$T_J(L) = r_s + r_w - h_w - h_s - 2.$$

Proof This proof will proceed by induction. First, suppose L_3 is a twist-minimal diagram with a 3-crossing twist. Then,

$$< \text{ } > = A < \text{ } > - A^{-7} < \text{ } > .$$

Denote by L_1, and by L_2. For L_1, the tracked variables are $n-1, w, s-1, r_w, r_s, h_w$, and h_s. Plugging these values into the four

equations, then multiplying each equation by A furnishes the distinct (from the main equations)

$$\textit{next lowest term } A < L_1 >= (-1)^{s-3}(r_w - h_s - 1)A^{-n-2s+10},$$

$$\textit{min term } A < L_1 >= (-1)^{s-2}A^{-n-2s+6}.$$

Note that the next lowest term doesn't contribute to the twist number.

Now, in the $A^{-1} < L_2 >$ portion, Lemma 1.1 was used to give $n-3$, $w-1$, $s-2$, r_w-1, r_s+x_0, h_w+x_1, and h_s, where x_0 and x_1 are positive integers. A problem arises here since $A^{-7} < L_2 >$ may or may not be twist-minimal. As seen in the following calculation, r_s and h_w don't contribute to the twist number so it doesn't matter if they change. So, assume that $A^{-7} < L_2 >$ is twist-minimal. Doing a similar plugin process as before yields

$$\textit{max term } A^{-7} < L_2 >= (-1)^{w+1}A^{n+2w-14},$$

$$\textit{next highest term } A^{-7} < L_2 >= (-1)^{w}(r_s - h_w - 1)A^{n+2w-18},$$

$$\textit{next lowest term } A^{-7} < L_2 >= (-1)^{s-2}(r_w - h_s - 2)A^{-n-2s+6}.$$
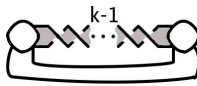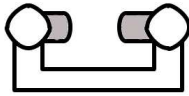
So, the maximum term of $A^{-7} < L_2 >$ and the next highest term of $A^{-7} < L_2 >$ don't contribute to the twist number. Finally,

$$\textit{next lowest term } A^{-7} < L_2 > +\textit{min term } A < L_1 >= (-1)^{s-2}(r_w-h_s-1)A^{-n-2s+6}.$$

Thus, all terms are exactly the terms shown in equations (1) through (4).

For the next step of the induction proof, suppose the equations (1) through (4) work for a twist-minimal link L_k with a k-crossing twist (for $k > 2$). Then,

Denote by L_{k-1}, and by L_0 for reference purposes.

The $A < L_{k-1} >$ portion has the same tracked variables as the case $k = 3$, and the only difference comes in the $A^{-3k+2} < L_0 >$ portion: $n-k, w-1, s-(k-1), r_w-1, r_s+y_0, h_w+y_1$, and h_s, where y_0 and y_1 are integers. Again, $A^{-3k+2} < L_0 >$ may or may not be twist-minimal. However, h_w and r_s

don't contribute to the twist number. So, multiplying all the equations by $(-1)^{k-1}A^{-3k+2}$ tenders the distinct (from the main equations) equations

$$max\ term\ A^{-3k+2} < L_0 >= (-1)^{w+k-2}A^{n+2w-4k-2},$$

$$next\ highest\ term\ A^{-3k+2} < L_0 >= (-1)^{w+k-3}(r_s - h_w - 1)A^{n+2w-4k-6},$$

$$next\ lowest\ term\ A^{-3k+2} < L_0 >= (-1)^{s-2}(r_w - h_s - 2)A^{-n-s+6}.$$

Since $k > 2$, the maximum term and the next highest term of $A^{-3k+2} < L_0 >$ do not contribute to the twist number. Adding together like terms gives

$$min\ term\ A < L_{k-1} > + next\ lowest\ term\ A^{-3k+2} < L_0 >= (-1)^{s-2}(r_w - h_s - 1)A^{-n-2s+6}$$

and this equals (3), which is the desired result.

Now return to the case when $A^{-3k+2} < L_0 >$ is not twist-minimal. This situation (figure **??**) can be fixed by a flype, which changes only one value: $r_s - h_w - 2$ instead of $r_s - h_w - 1$. However, the term with this coefficient does not contribute to the twist number because the degree on A is $n + 2w - 4k - 6$. This same argument works for when $k = 3$.
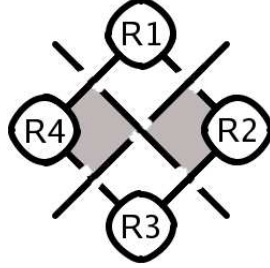

Figure 21: Setup of $k = 1$

Now for the case of $k = 1$, a little setup is required. Let $\circledR$ denote either the situation $\ominus$ or $\otimes$, and let $x, y = 0, 1, or\ 2$. A single crossing would then be pictured as in figure 21. After an A-smoothing on the main $k = 1$ crossing, the tracked variables are $n - 1, w, s - 1, r_w - y, r_s - 1, h_w$, and $h_s + j_s$ (figure 22), where j_s is a nonnegative integer. But, the situation in figure 23 could arise, which is where j_s (or j_w) originates.
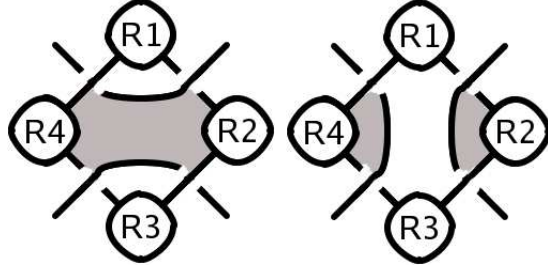
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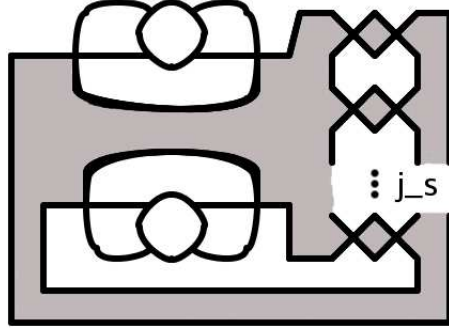Figure 22: A-smoothing and B-smoothing on $k = 1$, respectively

Figure 23: The case of j_s Hopf link factors

After a B-smoothing on the main crossing, the variables become $n-1, w-1, s, r_w - 1, r_s - x, h_w + j_w$, and h_s (figure 22). Again, j_w is a nonnegative integer. Again, j_w arises from a situation similar to the one in figure 23.

The term calculations are thus

$$max\ term\ A < \bowtie >= (-1)^{w-2}A^{n+2w-2},$$

$$next\ highest\ term\ A < \bowtie >= (-1)^{w-3}(r_s - h_w - 1)A^{n+2w-6},$$

$$next\ lowest\ term\ A < \bowtie >= (-1)^{s-2}(r_w - h_s - 2)A^{-n-2s+6},$$

$$min\ term\ A < \bowtie >= (-1)^{s-1}A^{-n-2s+2},$$

$$max\ term\ A < \bowtie >= (-1)^{w-2}A^{n+2w-2},$$

$$next\ highest\ term\ A^{-1} < \bowtie >= (-1)^{w-3}(r_s - h_w - 1)A^{n+2w-6},$$

$$next\ lowest\ term\ A^{-1} < \bowtie >= (-1)^{s-2}(r_w - h_s - 2)A^{-n-2s+6},$$

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$$min\ term\ A^{-1} < \bowtie >= (-1)^{s-1} A^{-n-2s+2}.$$

These calculations show that the induction hypothesis holds.

There are two main cases for the case when $k = 2$: either the twist is a Hopf twist or it isn't. Remember that the link starts as alternating, reduced, and twist-minimal.

For the first case, assume that the twist is a Hopf twist. Then, without loss of generality,

$$< \text{⊙⟋⊙} >= (-A^4 - A^{-4}) < \text{⊙—⊙} > .$$

The inside region cannot be a twist since a flype would have been admitted at the start (figure 24). Also, the outside region cannot be a twist either for the same reason. Thus, the only possibility is that both regions are not twists, so they must contribute to r_w and r_s. Also, there is nothing connecting the two tangles on either side of the Hopf link, so there are no added h_s values. This means the tracked values are $n - 2, w - 1, s - 1, r_s - 1, r_w - 1, h_s$, and $h_w - 1$.
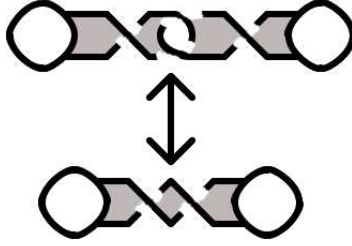

Figure 24: Both B-regions are twists

Now, assume the twist $k = 2$ in L is not a Hopf twist. Then

$$< \bowtie >= A < \bowtie > +(-1)A^{-4} < \text{)(} > .$$

Looking at $A < \bowtie >$, there are eight possibilities: all regions are twists, no regions are twists, one B-region (or A-region) is a twist, or two B-regions (or A-regions) are twists. Any case where an A-region is a twist is an impossibility since the starting link would have admitted flypes. Also, if both B-regions are twists, then there was a flypeable-twist in the starting link (figure 24).

Thus, the only possibilities are when one B-region is a twist or when no regions are a twist. Let y denote 0 or 1. Then the tracked variables are $n-1, w, s-1, r_w-y, r_s, h_w$, and h_s for $A < ⊃⊂ >$. Let x denote 0, 1, or 2. Then the tracked variables for $(-1)A^{-4} < ⊃⊂ >$ are $n-2, w-1, r_w-1, r_s-x, h_w+j$, and h_s. Thence, the distinct equations for $A < ⊃⊂ >$ are

$$\textit{next lowest term } A < \text{⋈} >= (-1)^{s-2}(r_w - h_s - y)A^{-n-2s+10},$$

$$\textit{min term } A < \text{⋈} >= (-1)^{s-1}A^{-n-2s+6},$$

and for $(-1)A^{-4} < ⊃⊂ >$,

$$\textit{max term } A^{-4} < \text{⋈} >= (-1)^{w-2}A^{n+2w-10},$$

$$\textit{next highest term } A^{-4} < \text{⋈} >= (-1)^{w-3}(r_s + j - x)A^{n+2w-14},$$

$$\textit{next lowest term } A^{-4} < \text{⋈} >= (-1)^{s-3}(r_w - h_s - 2)A^{-n-2s+6}.$$

Adding the two sets of equations together,

$$\textit{min term } A < \text{⋈} > +\textit{next lowest term } A^{-4} < \text{⋈} >= (-1)^{s-2}(r_w-h_s-1)A^{-n-2s+6},$$

and disregarding the non-contributing terms is precisely the induction conclusion, so the $k = 2$ case is determined.

Thus, this shows that the twist number can be calculated from counting regions outside of twists, and Hopf twists, and that

$$T_1 = |a_{n+1}| = r_s - h_w - 1, \text{ and } t_1 = |a_{m-1}| = r_w - h_s - 1.$$

However, the actual twist number of a link is

$$T(L) = r_s + r_w + h_s + h_w - 2. \quad \blacksquare$$

Corollary 3.4 *Let $T(L)$ be the actual twist number of the link L, $T_J(L)$ be the Jones Polynomial twist number of L, h_s be the number of shaded Hopf twists, and h_w be the number of unshaded Hopf twists. Then*

$$T(L) = T_J(L) + h_s + h_w.$$

Proof This follows from the Euler characteristic of the link L,

$$V - E + R = 2,$$

where V is the number of twists, E is 2 times the number of twists, and R is the total number of exterior regions. Then the formula becomes

$$-T + r_w + r_s - 2 = 0 \Rightarrow T = r_w + r_s - 2,$$

which is exactly what is needed. $\quad \blacksquare$

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