G-MODELING & WARPED PRODUCT CURVATURE MODELS

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Abstract. In this paper, we construct a warped product manifold of dimension n that is G-modeled up to order 1 by an $(n-1)$-dimensional Lie group. We then define what it is to be a warped product curvature model and prove a result concerning the kernel of a warped product curvature tensor.

1. Introduction

The theory of G-modeling generalizes the notion of curvature homogeneity. In [1], it was shown that there exists a 3-dimensional Riemannian warped product manifold that is G-modeled up to order 1 by a 2-dimensional Lie group. In Section 2, background material and preliminary definitions are given. In Section 3, we generalize the construction in [1] to an n-dimensional Riemannian warped product manifold G-modeled up to order 1 by an $(n-1)$-dimensional Lie group. In Section 4, we then define what it is to be a warped product curvature model and prove that under certain dimensionality and degeneracy assumptions, the kernel of a warped product curvature tensor takes on a nice form. We conclude the paper with some open questions and further avenues for exploration.

2. Preliminaries

We commence by defining an object central to our discussion.

Definition 2.1. *Let V be a finite-dimensional vector space. An* **algebraic curvature tensor** *R is a $(4,0)$ tensor with the following three properties: for vectors $x, y, z, w \in V$, we have*

$$\begin{aligned} &R(x, y, z, w) = -R(y, x, z, w),\\ &R(x, y, z, w) = R(z, w, x, y),\\ &R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \end{aligned}$$

The space of algebraic curvature tensors on V is denoted $\mathcal{A}(V)$.

Algebraic curvature tensors are tensors that mimic the symmetry properties of the standard Riemann curvature tensor (see [2] for a detailed overview of Riemannian curvature tensors). The following tool allows us to view curvature tensors and tangent spaces to manifolds from a purely algebraic lens.

Definition 2.2. *Let V be a finite-dimensional vector space. A* **k-model** *(or, less descriptively,* **model space***) $\mathcal{M}$ is a $(k+3)$-tuple consisting of a vector space V, an inner product $\langle .,. \rangle$ on V, an algebraic curvature tensor $R_0 \in \mathcal{A}(V)$, and k covariant derivative algebraic curvature tensors, $(R_i)_{i=1}^k$ where R_i satisfies the same algebraic properties as the ith covariant derivative of the Riemann curvature tensor, $\nabla^i R$; that is, $\mathcal{M} = (V, \langle .,. \rangle, R_0, R_1, \cdots, R_n)$.*

Model spaces are essentially algebraic portraits of tangent spaces to manifolds and the Riemann curvature tensor associated to the manifold. There is a natural notion of equivalence between model spaces.

Definition 2.3. *Let $\mathcal{M} = (V, \langle .,. \rangle_1, R_0, \cdots R_\ell)$ and $\tilde{\mathcal{M}} = (W, \langle .,. \rangle_2, \tilde{R}_0, \cdots \tilde{R}_\ell)$ be two ℓ-models. The model spaces $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are* **isomorphic** *if there exists an invertible linear map $A : V \to W$ with $A^*\mathcal{M} = \tilde{\mathcal{M}}$, where*

$$A^*\mathcal{M} = (AV, A^*\langle .,. \rangle_1, A^*R_0, \cdots, A^*R_\ell),$$

and A^* acts on tensors by by precomposition.

The notion of curvature homogeneity has many equivalent definitions, but for our purposes, the following from [1] suffices.

Definition 2.4. *Let (M, g) be a pseudo-Riemannian manifold. Letting T_pM denote the tangent space to the manifold at p as usual, let $\mathcal{M}_p^k := (T_pM, g_p, R_p, \cdots, \nabla^k R_p)$, and let $\mathcal{M}^k = (V, \langle .,. \rangle, R_0, R_1, \cdots, R_k)$ be a k-model. The manifold (M, g) is* **curvature homogeneous up to order** k *with model $\mathcal{M}^k$ (or* $\mathbf{CH_k}$*) if for each $p \in M$, we have that $\mathcal{M}_p^k \cong \mathcal{M}^k$. The manifold (M, g) is said to be* **properly** $\mathbf{CH_k}$ *if it is CH_k but not CH_{k+1}.*

There is a weakening of this definition in which the isomorphism is taken up to a 1-paramater scaling.

Definition 2.5. *Let (M, g) be a pseudo-Riemannian manifold, and let $\mathcal{M}_p^k$ and let $\mathcal{M}^k$ be a as in the previous definition. The manifold (M, g) is* **homothety curvature homogeneous up to order** k *with model $\mathcal{M}^k$ (or* $\mathbf{HCH_k}$*) if for each $p \in M$, we have that $\mathcal{M}_p^k \cong (V, \lambda(p)\langle .,. \rangle, \lambda(p)R_0, \lambda(p)^{\frac{3}{2}}R_1, \cdots, \lambda(p)^{\frac{k+2}{2}}R_k)$. The manifold (M, g) is said to be* **properly** $\mathbf{HCH_k}$ *if it is HCH_k but not HCH_{k+1}.*

The following generalization of curvature homogeneity is due to C. Dunn, A. Luna, and S. Sbiti.

Definition 2.6. *Let (M, g) be a pseudo-Riemannian manifold, let $\mathcal{M}^k$ be a k-model as in (1.4), let $G \leq GL(V)$ be a Lie group with an associated action $A \mapsto A \cdot \mathcal{M}^k$, $A \in G$, on the set of model spaces over V. The manifold (M, g) is* **G-modeled up to order** k *if (i) for each $p \in M$ there exists $A \in G$ such that $\mathcal{M}_p^k \cong A \cdot \mathcal{M}^k$, and (ii) for each $A \in G$ there exists $p \in M$ with $\mathcal{M}_p^k \cong A \cdot \mathcal{M}^k$.*

Proving that a space is not CH_k or not HCH_k is quite difficult to do directly from the definitions, so we employ *invariants*, which are simply properties that must hold In the event a manifold is either CH_k or HCH_k. In order to define some (homothety) curvature homogeneity invariants, we will need the following notions.

Definition 2.7. *Let (M, g) be a pseudo-Riemannian manifold. The quantity*

$$\tau_R = \sum g^{i\ell} g^{jk} R_{ijk\ell}$$

is the **sectional curvature of** M *(taken at some implicit $p \in M$). The quantities*

$$||R||^2 = \sum g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g^{i_4 j_4} R_{i_1 i_2 i_3 i_4} R_{j_1 j_2 j_3 j_4},$$
$$||\nabla R||^2 = \sum g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g^{i_4 j_4} g^{i_5 j_5} \nabla R_{i_1 i_2 i_3 i_4 ; i_5} \nabla R_{j_1 j_2 j_3 j_4 ; i_5},$$

are the **square norm** *of R and ∇R (taken at some implicit $p \in M$), respectively.*

The following lemma from [1] establishes some basic (homothety) curvature homogeneity invariants.

Lemma 2.8. *Let (M, g) be a connected pseudo-Riemannian manifold. If (M, g) is HCH_0, then $\frac{\tau_R^2}{||R||^2}$ is constant wherever it is defined. If (M, g) is HCH_1, then $\frac{(||R||^2)^3}{(||\nabla R||^2)^2}$ is constant wherever it is defined.*

3. An n-dimensional manifold G-modeled by an $(n-1)$-dimensional Lie Group

In this section, we generalize the scaling construction of [1] to dimension n. Before doing so, we define the notion of warping a flat manifold onto a flat manifold, as we will investigate suitable curvature models for these types of spaces later. Such spaces can be thought of as "almost" cartesian products of manifolds.

Definition 3.1. *Given pseudo-Riemannian manifolds (M, g),(N, g'), and a smooth function $f : M \to \mathbb{R}$, the* **warped product** *of M onto N by f, denoted $M \times_f N$ is given by $(M \times N, \tilde{g})$, with*

$$\tilde{g}|_{(p,q)} = \pi_1^*(g|_p) + f(p)\pi_2^*(g'|_q),$$

where π_1 and π_2 are projections onto M and N, respectively.

In preparation for the following proposition, define a 1-model $\mathcal{M}_4^1$ by $\mathcal{M}^1 = (V, \langle .,. \rangle, R, \nabla R)$ where $V = \text{span}\{X, Y, Z, W\}$ is an orthonormal basis (here we have abused notation and written ∇R in place of R_1). Define R and ∇R to have the following nonzero entries up to symmetry,

$$\begin{aligned}
&R_{XYXY} = 2,\\
&R_{XYXZ} = -2,\\
&R_{XZXZ} = 2,\\
&R_{XWXW} = 2,\\
&R_{XWXZ} = 2,\\
&R_{XWXY} = 2,\\
&\nabla R_{XZZX;Z} = 4,\\
&\nabla R_{ZXWX;W} = -1,\\
&\nabla R_{YWYX;X} = -1,\\
&\nabla R_{WYXW;X} = 2,\\
&\nabla R_{XZZW;X} = 1,\\
&\nabla R_{YXXZ;Y} = -1,\\
&\nabla R_{YXZX;Z} = -2,\\
&\nabla R_{XWXW;W} = -4,\\
&\nabla R_{XYXY;Y} = 4.
\end{aligned}$$

Proposition 3.2. *Let $M = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ with coordinates $y, z,$ and w, let $N = \mathbb{R}$ with coordinates x, and let $f(y, z, w) = \frac{y^4}{2z^2w^2}$. Equip M and N with the Euclidean metrics. The space $M \times_f N$ is G-modeled up to order 1. Moreover, this space is not HCH_0.*

Proof. The warped product metric on M is given by

$$g = \frac{y^4}{2z^2w^2} dx \otimes dx + dy \otimes dy + dz \otimes dz + dw \otimes dw.$$

Now, let G be the 3-dimensional Lie group

$$G = \{\mathrm{diag}(1, t, s, u) : t, s \in \mathbb{R}^+\} \cong (\mathbb{R}^+, \cdot) \oplus (\mathbb{R}^+, \cdot) \oplus (\mathbb{R}^+, \cdot).$$

This action is characterized by $X \mapsto X, Y \mapsto tY, Z \mapsto sZ,$ and $W \mapsto uW$.

Action by precomposition, i.e. $A \cdot \mathcal{M}^1 = (V, \langle .,. \rangle, A^*R, A^*\nabla R)$ for $A \in G$, yields

$$\begin{aligned}
A^*R_{XYXY} &= 2t^2, \\
A^*R_{XYXZ} &= -2ts, \\
A^*R_{XZXZ} &= 2s^2, \\
A^*R_{XWXW} &= 2u^2, \\
A^*R_{XWXZ} &= 2su, \\
A^*R_{XWXY} &= 2tu, \\
A^*\nabla R_{XZZX;Z} &= 4s^3, \\
A^*\nabla R_{ZXWX;W} &= -su^2, \\
A^*\nabla R_{YWYX;X} &= -t^2u, \\
A^*\nabla R_{WYXW;X} &= 2tu^2, \\
A^*\nabla R_{XZZW;X} &= s^2u, \\
A^*\nabla R_{YXXZ;Y} &= -st^2, \\
A^*\nabla R_{YXZX;Z} &= -2ts^2, \\
A^*\nabla R_{XWXW;W} &= -4u^3, \\
A^*\nabla R_{XYXY;Y} &= 4t^3.
\end{aligned}$$

The components of ${}^gR$ up to symmetry are

$$\begin{aligned}
{}^gR_{\partial_x\partial_y\partial_x\partial_y} &= \frac{y^2}{z^2w^2}, \\
{}^gR_{\partial_x\partial_y\partial_x\partial_z} &= -\frac{y^3}{z^3w^2}, \\
{}^gR_{\partial_x\partial_z\partial_x\partial_z} &= \frac{y^4}{z^4w^2}, \\
{}^gR_{\partial_x\partial_w\partial_x\partial_w} &= \frac{y^4}{z^2w^4}, \\
{}^gR_{\partial_x\partial_w\partial_x\partial_z} &= \frac{y^4}{2z^3w^3}, \\
{}^gR_{\partial_x\partial_w\partial_x\partial_y} &= \frac{y^3}{z^2w^3}.
\end{aligned}$$

Take the orthonormal frame $X = \frac{\sqrt{2}zw}{y^2}\partial_x, Y = \partial_y, Z = \partial_z$ and $W = \partial_w$. In our orthonormal coordinate frame, the tensor ${}^gR$ has components

$$
\begin{aligned}
{}^gR_{XYXY} &= \frac{2z^2w^2}{y^4}\cdot\frac{y^2}{z^2w^2} = 2\frac{1}{y^2} = \frac{1}{t^2} = R_{XYXY},\\
{}^gR_{XYXZ} &= \frac{2z^2w^2}{y^4}\cdot -\frac{y^3}{z^3w^2} = \frac{2}{yz} = \frac{2}{ts} = R_{XYXZ},\\
{}^gR_{XZXZ} &= \frac{2z^2w^2}{y^4}\cdot\frac{y^4}{z^4w^2} = \frac{2}{z^2} = \frac{2}{s^2} = R_{XZXZ},\\
{}^gR_{XWXW} &= \frac{2z^2w^2}{y^4}\cdot\frac{y^4}{z^2w^4} = \frac{2}{w^2} = \frac{2}{u^2} = R_{XWXW},\\
{}^gR_{XWXZ} &= \frac{2z^2w^2}{y^4}\cdot\frac{y^4}{2z^3w^3} = \frac{2}{zw} = \frac{2}{su} = R_{XWXZ},\\
{}^gR_{XWXY} &= \frac{2z^2w^2}{y^4}\cdot\frac{y^3}{z^2w^3} = \frac{2}{yw} = \frac{2}{tu} = R_{XWXY},
\end{aligned}
$$

where we take $t = \frac{1}{y}, s = \frac{1}{z},$ and $u = \frac{1}{w}$.

This means that (M, g) is G-modeled up to order 0 by $\mathcal{M}^0 := (V, \langle .,. \rangle, R)$.

Now, we verify that (M, g) is G-modeled up to order 1. There are 9 components of ${}^g\nabla R$ up to symmetry:

$$
\begin{aligned}
{}^g\nabla R_{XZZX;Z} &= \frac{2y^4}{z^5w^2},\\
{}^g\nabla R_{ZXWX;W} &= -\frac{y^4}{2w^4z^3},\\
{}^g\nabla R_{YWYX;X} &= -\frac{y^2}{w^3z^2},\\
{}^g\nabla R_{WYXW;X} &= \frac{y^3}{w^4z^2},\\
{}^g\nabla R_{XZZW;X} &= -\frac{y^4}{2w^3z^4},\\
{}^g\nabla R_{YXXZ;Y} &= -\frac{y^2}{z^3w^2},\\
{}^g\nabla R_{YXZX;Z} &= -\frac{y^3}{z^4w^2},\\
{}^g\nabla R_{XWXW;W} &= -\frac{2y^4}{w^5z^2},\\
{}^g\nabla R_{XYXY;Y} &= \frac{2y}{z^2w^2}.
\end{aligned}
$$

Using the coordinate system from before and once again taking $t = \frac{1}{y}, s = \frac{1}{z},$ and $u = \frac{1}{w}$, we have that

$$\begin{aligned}
{}^g\nabla R_{XZZX;Z} &= \frac{2z^2w^2}{y^4} \cdot \frac{2y^4}{z^5w^2} = \frac{4}{z^3} = 4s^3 = A^*\nabla R_{XZZX;Z}, \\
{}^g\nabla R_{ZXWX;W} &= -\frac{2z^2w^2}{y^4} \cdot \frac{y^4}{2w^4z^3} = -\frac{1}{w^2z} = -su^2 = A^*\nabla R_{ZXWX;W}, \\
{}^g\nabla R_{YWYX;X} &= -\frac{2z^2w^2}{y^4} \cdot \frac{y^2}{w^3z^2} = \frac{1}{y^2w} = -t^2u = A^*\nabla R_{YWYX;X}, \\
{}^g\nabla R_{WYXW;X} &= \frac{2z^2w^2}{y^4} \cdot \frac{y^3}{w^4z^2} = \frac{2}{w^2y} = 2tu^2 = A^*\nabla R_{WYXW;X}, \\
{}^g\nabla R_{XZZW;X} &= -\frac{2z^2w^2}{y^4} \cdot \frac{y^4}{2w^3z^4} = \frac{1}{wz^2} = s^2u = A^*\nabla R_{XZZW;X}, \\
{}^g\nabla R_{YXXZ;Y} &= -\frac{2z^2w^2}{y^4} \cdot \frac{y^2}{z^3w^2} = \frac{1}{y^2z} = -t^2s = A^*\nabla R_{YXXZ;Y}, \\
{}^g\nabla R_{YXZX;Z} &= -\frac{2z^2w^2}{y^4} \cdot \frac{y^3}{z^4w^2} = \frac{2}{yz^2} = -2ts^2 = A^*\nabla R_{YXZX;Z}, \\
{}^g\nabla R_{XWXW;W} &= -\frac{2z^2w^2}{y^4} \cdot \frac{2y^4}{w^5z^2} = \frac{4}{w^3} = -4u^3 = A^*\nabla R_{XWXW;W}, \\
{}^g\nabla R_{XYXY;Y} &= \frac{2z^2w^2}{y^4} \cdot \frac{2y}{z^2w^2} = \frac{4}{y^3} = 4t^3 = A^*\nabla R_{XYXY;Y}.
\end{aligned}$$

This means that (M, g) is G-modeled up to order 1 by $\mathcal{M}^1 := (V, \langle ., . \rangle, R, \nabla R)$.

It remains to show that the space is not HCH_0. We compute the invariant of Lemma 1.8 and show that it is not constant. A maple computation yields

$$\begin{aligned}
\frac{\|R\|^2}{\tau_R^2} &= \frac{2w^4y^4 + 4w^4y^2z^2 + 2w^4z^4 + w^2y^4z^2 + 4w^2y^2z^4 + 2y^4z^4}{2(w^2y^2 + z^2w^2 + y^2z^2)^2} \\
&= 1 + \frac{3w^2y^4z^2}{2(w^2y^2 + z^2w^2 + y^2z^2)^2}.
\end{aligned}$$

Since this last expression is non-constant, the space is not HCH_0. $\square$

This construction is easily modified so as to produce an n-dimensional manifold G-modeled by an $(n-1)$-dimensional Lie group. Let G be the $(n-1)$-dimensional Lie group

$$G = \{\mathrm{diag}(1, a_1, \cdots, a_n) : a_i \in \mathbb{R}\},$$

and let it act on $\mathcal{M}(V)$ by precomposition.

Proposition 3.3. *Let $B = (\mathbb{R}^+)^n$ with coordinates $x_1, \cdots, x_n$, let $F = \mathbb{R}$ with coordinates y, and let $f(x_1, \cdots, x_n) = \frac{x_1^4}{2x_2^2 \cdots x_n^2}$. Equip B and F with the Euclidean metrics. The space $B \times_f F$ is G-modeled up to order 1.*

Proof. The metric tensor for $B \times_f F$ is given by

$$g = \frac{x_1^4}{2x_2^2 \cdots x_n^2} dy \otimes dy + dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n.$$

We compute the nonzero entries of the curvature tensor. In what follows, we abuse notation and make the identification $x \approx \partial_{x_i}$. A simple calculation shows that the only nonzero entries up to symmetry are of the form $R(y,\alpha,y,\beta)$, where $\alpha,\beta \in \{x_i\}_i$. First, notice that

$$^gR(y,\alpha)y = \nabla_y\nabla_\alpha y - \nabla_\alpha\nabla_y y = \nabla_y\nabla_\alpha y - 0 = \Gamma^y_{\alpha y}\nabla_y y.$$

Now,

$$\begin{aligned} ^gR(y,\alpha,y,\beta) = g(R(y,\alpha)y,\beta) = g(\Gamma^y_{\alpha y}\nabla_y y,\beta) &= \Gamma^y_{\alpha y} g(\nabla_y y,\beta) \\ &= \Gamma^y_{\alpha y} g(\Gamma_{x_1 yy} y + \cdots + \Gamma_{x_n yy} x_n, \beta) \\ &= \Gamma^y_{\alpha y}\Gamma_{\beta yy} \\ &= g^{yy}\cdot\Gamma_{y\alpha y}\cdot\Gamma_{\beta yy} \\ &= -\frac{1}{4}\cdot g^{yy}\cdot g_{yy/\alpha}\cdot g_{yy/\beta}. \end{aligned}$$

where we used that

$$\Gamma^\beta_{yy} = \sum_m g^{\beta m}\Gamma_{myy} = g^{\beta\beta}\Gamma_{\beta yy} = \Gamma_{\beta yy},$$

and

$$\Gamma^y_{\alpha y} = \sum_m g^{ym}\Gamma_{m\alpha y} = g^{yy}\cdot\Gamma_{y\alpha y},$$

where $m \in \{y, x_1, \cdots, x_n\}$.

There are three cases (without loss of generality) to consider when attempting to get explicit expressions: α,β both unequal to x_1, $\alpha = x_1$ and $\beta \neq x_1$, and $\alpha = x_1$ and $\beta = x_1$. In the event that α,β are both unequal to x_1, we have

$$^gR(y,\alpha,y,\beta) = -\frac{1}{4}\cdot\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{-x_1^4}{x_2^2\cdots\alpha^3\cdots x_n^2}\cdot\frac{-x_1^4}{x_2^2\cdots\beta^3\cdots x_n^2} = -\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2};$$

in the event that $\alpha = x_1$ and $\beta \neq x_1$, we have

$$^gR(y,\alpha,y,\beta) = -\frac{1}{4}\cdot\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{4x_1^3}{2x_2^2\cdots x_n^2}\cdot\frac{-x_1^4}{x_2^2\cdots\beta^3\cdots x_n^2} = \frac{1}{\beta}\cdot\frac{x_1^3}{x_2^2\cdots x_n^2};$$

in the event that $\alpha = x_1$ and $\beta = x_1$, we have

$$^gR(y,\alpha,y,\beta) = -\frac{1}{4}\cdot\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{4x_1^3}{2x_2^2\cdots x_n^2}\cdot\frac{4x_1^3}{2x_2^2\cdots x_n^2} = -\frac{2x_1^2}{x_2^2\cdots x_n^2}.$$

Next we compute the total covariant derivative of ^gR. The only nonzero entries up to symmetry are $^g\nabla R(y,\alpha,y,\beta;\gamma)$ where $\gamma = \alpha$ or $\gamma = \beta$. First notice that

$$\begin{aligned} ^g\nabla R(y,\alpha,y,\beta;\alpha) &= \frac{\partial}{\partial\alpha}R(y,\alpha,y,\beta) - R(\nabla_\alpha y,\alpha,y,\beta) - R(y,\nabla_\alpha\alpha,y,\beta) \\ &\quad - R(y,\alpha,\nabla_\alpha y,\beta) - R(y,\alpha,y,\nabla_\alpha\beta) \\ &= \frac{\partial}{\partial\alpha}R(y,\alpha,y,\beta) - 2R(\nabla_\alpha y,\alpha,y,\beta), \end{aligned}$$

and, analogously,

$$\begin{aligned} ^g\nabla R(y,\alpha,y,\beta;\beta) &= \frac{\partial}{\partial\beta}R(y,\alpha,y,\beta) - R(\nabla_\beta y,\alpha,y,\beta) - R(y,\nabla_\beta\alpha,y,\beta) \\ &\quad - R(y,\alpha,\nabla_\beta y,\beta) - R(y,\alpha,y,\nabla_\beta\beta) \\ &= \frac{\partial}{\partial\beta}R(y,\alpha,y,\beta) - 2R(\nabla_\beta y,\alpha,y,\beta). \end{aligned}$$

We will only consider the first case, as the calculations in the second are nearly identical. Once again, we can get explicit expressions in terms of our coordinate system by considering the three cases: α, β both unequal to x_1, $\alpha = x_1$ and $\beta \neq x_1$, and $\alpha = x_1$ and $\beta = x_1$. In the first case,

$$\begin{aligned}
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= \frac{\partial}{\partial\alpha}R(y,\alpha,y,\beta) - 2R(\nabla_\alpha y,\alpha,y,\beta) \\
&= -\frac{\partial}{\partial\alpha}\left(\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) - 0 \\
&= \frac{1}{2\alpha^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2};
\end{aligned}$$

In the second case,

$$\begin{aligned}
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= \frac{\partial}{\partial\alpha}R(y,\alpha,y,\beta) - 2R(\nabla_\alpha y,\alpha,y,\beta) \\
&= -\frac{\partial}{\partial\alpha}\left(\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) - 2\cdot\frac{2}{\alpha}R(y,\alpha,y,\beta) \\
&= -\frac{\partial}{\partial\alpha}\left(\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) - 2\cdot\frac{2}{\alpha}\left(\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) \\
&= \frac{1}{2\alpha^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} - \frac{2}{\alpha^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} \\
&= -\frac{3}{2x_1^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2};
\end{aligned}$$

In the third case,

$$\begin{aligned}
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= \frac{\partial}{\partial\alpha}R(y,\alpha,y,\beta) - 2R(\nabla_\alpha y,\alpha,y,\beta) \\
&= -\frac{\partial}{\partial\alpha}\left(\frac{1}{2\alpha^2}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) - 2\cdot\frac{2}{\alpha}R(y,\alpha,y,\beta) \\
&= -\frac{\partial}{\partial\alpha}\left(\frac{1}{2\alpha^2}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) - 2\cdot\frac{2}{\alpha}\left(\frac{1}{2\alpha^2}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2}\right) \\
&= \frac{1}{\alpha^3}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} - \frac{2}{\alpha^3}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} \\
&= -\frac{1}{x_1^3}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} \\
&= -\frac{x_1}{x_2^2\cdots x_n^2}.
\end{aligned}$$

Now, using the coordinate system $y = \frac{\sqrt{2}x_2\cdots x_n}{x_1^2}\partial_y, x_i = \partial_{x_i}$, we have that the curvature tensor components are

$$\begin{aligned}
{}^gR(y,\alpha,y,\beta) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{1}{2\alpha\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} = -\frac{1}{\alpha\beta}, && \text{(Case 1)}, \\
{}^gR(y,\alpha,y,\beta) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{1}{\beta}\cdot\frac{x_1^3}{x_2^2\cdots x_n^2} = -\frac{2}{x_1\beta}, && \text{(Case 2)}, \\
{}^gR(y,\alpha,y,\beta) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot-\frac{2x_1^2}{x_2^2\cdots x_n^2} = \frac{4}{x_1^2}, && \text{(Case 3)},
\end{aligned}$$

and we have that the covariant derivative curvature tensor components are

$$
\begin{aligned}
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot\frac{1}{2\alpha^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} = -\frac{1}{\alpha^2\beta}, && \text{(Case 1)},\\
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot -\frac{3}{2x_1^2\beta}\cdot\frac{x_1^4}{x_2^2\cdots x_n^2} = \frac{3}{x_1^2\beta}, && \text{(Case 2)},\\
{}^g\nabla R(y,\alpha,y,\beta;\alpha) &= -\frac{2x_2^2\cdots x_n^2}{x_1^4}\cdot -\frac{x_1}{x_2^2\cdots x_n^2} = \frac{2}{x_1^3}, && \text{(Case 3)}.
\end{aligned}
$$

Now, choose the model space $\mathcal{M}_n^1$ to be the triple $(V,\langle.,.\rangle,R,\nabla R)$, where V is an n-dimensional vector space with basis $\{Y,X_1,\cdots,X_n\}$. Let $x_1=\frac{1}{a_1},\cdots,x_{n-1}=\frac{1}{a_{n-1}}$. Choose components of R and ∇R so as to ensure action by some $A\in G$ will yield $R(y,\alpha,y,\beta,y)={}^g R(y,\alpha,y,\beta,y)$ (as was implicitly done in the $n=4$ case). For instance,

$$
{}^gR(y,x_2,y,x_2) = -\frac{1}{a_2^2} \implies R(Y,X_2,Y,X_2)=-1.
$$

Constructing the 1-model in this way brings us to an analogous situation as the $n=4$ case, so we have our result. □

4. Warped Product Curvature Models

In this section, we define warped product curvature models. We do this by looking at the curvature components of the Riemann curvature tensor of warped product spaces, given in [4], and replacing any mathematical structures involving the manifold directly with algebraic analogues.

Let $V=V_1\perp V_2$. Let $x_\alpha\in V_1$ and let $y_\beta\in V_2$, where α and β are arbitrary elements of distinct indexing sets. By [4],

$$
\begin{aligned}
R(y_i,x_j,x_k,y_\ell) &= g(R(y_i,x_j)x_k,y_\ell)\\
&= g(H^f(x_j,x_k)y_i,y_\ell)\\
&= H^f(x_j,x_k)\cdot g(y_i,y_\ell)\\
&\approx H(x_j,x_k)\cdot\langle y_i,y_\ell\rangle,
\end{aligned}
$$

where H^f is the Hessian of a smooth positive function f, and H is a symmetric bilinear form. We also have

$$
\begin{aligned}
R(y_i,y_j,y_k,y_\ell) &= g(R(y_i,y_j)y_k,y_\ell)\\
&= g\left({}^FR(y_i,y_j)y_k - \frac{\langle \mathrm{grad} f,\mathrm{grad} f\rangle(g(y_i,y_k)y_j - g(y_j,y_k)y_i)}{f^2}, y_\ell\right)\\
&= g({}^FR(y_i,y_j)y_k,y_\ell) - \frac{g(\mathrm{grad} f,\mathrm{grad} f)}{f^2}[g(y_i,y_k)g(y_j,y_\ell)-g(y_j,y_k)g(y_i,y_\ell)].\\
&\approx c\cdot[\langle y_i,y_k\rangle\langle y_j,y_\ell\rangle - \langle y_j,y_k\rangle\langle y_i,y_\ell\rangle]\\
&= c\cdot R_{\langle.,.\rangle},\ \text{where } c\in\mathbb{R},
\end{aligned}
$$

where ${}^FR(y_i,y_j)$ denotes the pullback along the projection onto the second factor F,

and

$$
R(x_i,x_j,y_k,y_l) = R(x_i,x_j,y_k,x_l) = R(y_i,y_j,x_k,y_l) = 0.
$$

Definition 4.1. *Model spaces of the form* $(V, \langle .,. \rangle, R_{H,c})$ *with* $V = V_1 \perp V_2$ *and* $R_{H,c}$ *having the same nonzero entries as the tensor in the above considerations (up to symmetry) are* **warped product curvature models**.

When $c \neq 0$ and the dimension of V_2 is large enough, the kernel of a warped product curvature tensor has a nice form.

Proposition 4.2. *Let* $(V, \langle .,. \rangle, R_{H,c})$ *be the model space defined using the above and a non-degenerate inner product, and suppose that* $c \neq 0$ *and* $\dim V_2 \geq 2$. *Then,*

$$\ker(R_{H,c}) = \ker(H) \cap V_1.$$

Proof. Suppose $v \in \ker(H) \cap V_1$. Then, we have $R(w, v, w', \tilde{w}) = H(v, w_1)\langle w_2, \tilde{w}_2 \rangle \equiv 0$. Whence $v \in \ker(R_{H,c})$.

Conversely, suppose $v \in \ker(R_{H,c})$. Write $v = v_1 + v_2$ where $v_i \in V_i$. Then for $y_i, y_j, y_k \in V_2$, we have that

$$\begin{aligned} 0 = R_{H,c}(y_j, v_1 + v_2, y_j, y_k) &= R_{H,c}(y_i, v_1, y_j, y_k) + R_{H,c}(y_i, v_2, y_j, y_k) \\ &= 0 + R_{H,c}(y_i, v_2, y_j, y_k) \\ &= c \cdot R_{\langle .,. \rangle}(y_i, v_2, y_j, y_k) \\ &= c \cdot R_{\langle .,. \rangle |_{V_2}}(y_i, v_2, y_j, y_k). \end{aligned}$$

Since the restriction of a non-degenerate inner product to one factor in the orthogonal decomposition of a space is still a non-degenerate inner product, we must have $v_2 \in \ker(R_{\langle .,. \rangle |_{V_2}}) = \{0\}$ and since $rank(\langle .,. \rangle|_{V_2}) = dim V_1 \geq 2$, we have that $\ker(R_{\langle .,. \rangle |_{V_2}}) = \ker(\langle .,. \rangle|_{V_2})$. Whence $v \in V_1$. By non-degeneracy and $\dim V_2 \geq 2$, for nonzero $\tilde{v} \in V_2$, there exists a $\tilde{w} \in V_2$ with $\langle \tilde{v}, \tilde{w} \rangle \neq 0$. Now,

$$0 = R_{H,c}(\tilde{v}, v, x_i, \tilde{w}) = H(v, x_i)\langle \tilde{v}, \tilde{w} \rangle \implies H(v, x_i) = 0 \implies v \in \ker H,$$

so we have that $v \in \ker H \cap V_1$. □

5. Conclusion

We have established that in every dimension, a Riemannian manifold can be G-modeled by a Lie group of codimension 1, as well as established a curvature model for warped product spaces. There are a number of further considerations one can undertake from here.

- Extend Proposition 3.3 to a pseudo-Riemannian manifold of arbitrary signature.

- So far, the only examples of G-modeling have been by way of a scaling action. Construct a manifold that is G-modeled by a compact Lie group like $G = S^1$.

- Prove a recognition theorem for warped product spaces. That is, given curvature data, can one conclude that a space is diffeomorphic to a warped product space?

- In our considerations, we have not made use of the smooth structure of the Lie group G. Show that the smooth structure of G is vital to G-modeling; perhaps look at something in the realm of geodesics.
- Prove a more general result than Proposition 4.2.
- The space of warped product algebraic curvature tensors does not span the space of algebraic curvature tensors when the associated inner product is fixed since certain curvature entries are always zero. Make spanning considerations for tensors of the form $\{R_{H,c,\phi}\}$ where ϕ is an arbitrary symmetric bilinear form.
- Investigate the form that the structure group of an $R_{H,c}$ takes on.

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