Sayyed Faisal
Fall 2024

MAP 2302 Homework 7.2

Problem 1. Find the Laplace transform of $f(t) = 6t^2$.

Solution.

$$\begin{aligned}
F(s) &= \int_0^\infty 6t^2 e^{-st} \\
&= 6\int_0^\infty t^2 e^{-st}\,dt \\
&= 6\lim_{N\to\infty}\left[\frac{-t^2e^{-st}}{s} - \int -\frac{2te^{-st}}{s}\,dt\right]_0^N \\
&= 6\lim_{N\to\infty}\left[-\frac{t^2e^{-st}}{s} + \frac{2}{s}\left(-\frac{te^{-st}}{s} - \int -\frac{e^{-st}}{s}\,dt\right)\right]_0^N \\
&= 6\lim_{N\to\infty}\left[-\frac{t^2e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^N \\
&= 6\lim_{N\to\infty}\left[-\frac{N^2e^{-sN}}{s} - \frac{2Ne^{-sN}}{s^2} - \frac{2e^{-sN}}{s^3} + \frac{2}{s^3}\right] \\
&= \frac{12}{s^3}, \quad s > 0
\end{aligned}$$

Problem 2. Find the Laplace transform of $f(t) = te^{5t}$.

Solution.

$$\begin{aligned}
F(s) &= \int_0^\infty te^{(5-s)t}\,dt \\
&= \lim_{N\to\infty}\left[\frac{te^{(5-s)t}}{5-s} - \int \frac{e^{(5-s)t}}{5-s}\,dt\right]_0^N \\
&= \lim_{N\to\infty}\left[\frac{te^{(5-s)t}}{5-s} - \frac{e^{(5-s)t}}{(5-s)^2}\right]_0^N \\
&= \lim_{N\to\infty}\left[\frac{Ne^{(5-s)N}}{5-s} - \frac{e^{(5-s)N}}{(5-s)^2} + \frac{1}{(5-s)^2}\right] \\
&= \frac{1}{(5-s)^2}, \quad s > 5
\end{aligned}$$

Problem 3. Find the Laplace transform of $f(t) = -\sin bt, b$ a constant.

Solution.

$$\begin{aligned}
F(s) &= \int_0^\infty -e^{-st}\sin bt\,dt \\
&= \lim_{N\to\infty}\left[\frac{e^{-st}\sin bt}{s} - \int \frac{be^{-st}}{s}\,dt\right]_0^N \\
&= \lim_{N\to\infty}\left[\frac{e^{-st}\sin bt}{s} - \frac{b}{s}\left(-\frac{e^{-st}\cos bt}{s} - \int \frac{be^{-st}\sin bt}{s}\,dt\right)\right]_0^N \\
&= \lim_{N\to\infty}\left[\frac{e^{-st}\sin bt}{s} + \frac{be^{-st}\cos bt}{s^2}\right]_0^N + \frac{b^2}{s^2}\int_0^\infty e^{-st}\sin bt\,dt \\
\left(1+\frac{b^2}{s^2}\right)F(s) &= \lim_{N\to\infty}\left[\frac{e^{-sN}\sin bN}{s} + \frac{be^{-sN}\cos bN}{s^2} - \frac{b}{s^2}\right] \\
&= -\frac{b}{s^2} \\
F(s) &= -\frac{b}{s^2+b^2}, \quad s > 0
\end{aligned}$$

Problem 4. Find the Laplace transform of $f(t) = e^{-5t}\sin 2t$.

Solution.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-(s+5)t}\sin 2t\,dt \\
&= \lim_{N\to\infty}\left[-\frac{e^{-(s+5)t}\sin 2t}{s+5} - \int -\frac{2e^{-(s+5)t}\cos 2t}{s+5}\,dt\right]_0^N \\
&= \lim_{N\to\infty}\left[-\frac{e^{-(s+5)t}\sin 2t}{s+5} + \frac{2}{s+5}\left(-\frac{e^{-(s+5)t}\cos 2t}{s+5} - \int \frac{2e^{-(s+5)t}\sin 2t}{s+5}\,dt\right)\right]_0^N \\
&= \lim_{N\to\infty}\left[-\frac{e^{-(s+5)t}\sin 2t}{s+5} - \frac{2e^{-(s+5)t}}{(s+5)^2}\right]_0^N - \frac{4}{(s+5)^2}\int_0^\infty e^{-(s+5)t}\sin 2t\,dt \\
\left(1+\frac{4}{(s+5)^2}\right)F(s) &= \lim_{N\to\infty}\left[-\frac{e^{-(s+5)N}\sin 2N}{s+5} - \frac{2e^{-(s+5)N}\cos 2N}{(s+5)^2} + \frac{2}{(s+5)^2}\right] \\
&= \frac{2}{(s+5)^2} \\
F(s) &= \frac{2}{(s+5)^2+4}, \quad s > -5
\end{aligned}$$

Problem 5. Find the Laplace transform of

$$\begin{cases} 3-t, & 0<t<3 \\ 0, & 3<t \end{cases}.$$

Solution.

$$\begin{aligned}
F(s) &= \int_0^3 (3-t)e^{-5t}\,dt + \int_3^\infty 0e^{-st}\,dt \\
&= \left[-\frac{e^{-st}(3-t)}{5} - \int \frac{e^{-st}}{s}\,dt\right]_0^3 \\
&= \left[-\frac{e^{-st}(3-t)}{5} + \frac{e^{-st}}{s^2}\right]_0^3 \\
&= \frac{e^{-3s}}{s^2} + \frac{3}{s} - \frac{1}{s^2} \\
&= \begin{cases} \dfrac{e^{-3s}+3s-1}{s^2}, & s \neq 0, \\ \dfrac{9}{2}, & s = 0 \end{cases}
\end{aligned}$$

Problem 6. Find the Laplace transform of

$$\begin{cases} e^{4t}, & 0< \ t<2 \\ 3, & 2<t \end{cases}.$$

Solution.

$$\begin{aligned}
F(s) &= \int_0^2 e^{(4-s)t}\,dt + \int_2^\infty 3e^{-st}\,dt \\
&= \left[\frac{e^{(4-s)t}}{4-s}\right]_0^2 + 3\lim_{N\to\infty}\left[-\frac{e^{-st}}{s}\right]_2^N \\
&= \frac{e^{8-2s}}{4-s} - \frac{1}{4-s} + 3\lim_{N\to\infty}\left[-\frac{e^{-sN}}{s} + \frac{e^{-2s}}{s}\right] \\
&= \begin{cases} \dfrac{e^{8-2s}-1}{4-s} + \dfrac{3e^{-2s}}{s}, & s \neq 4 \\ 2 + \dfrac{3e^{-8}}{4}, & s = 4 \end{cases}
\end{aligned}$$

Problem 7. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathscr{L}\left\{4e^{-9t} - t^3 + 3t - 2\right\}.$$

Solution.

$$\mathscr{L}\left\{4e^{-9t} - t^3 + 3t - 2\right\} = \frac{4}{s+9} - \frac{6}{s^4} + \frac{3}{s^2} - \frac{2}{s}, \quad s > 0$$

Problem 8. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathscr{L}\left\{e^{7t}\sin 9t - t^5 + e^{6t}\right\}.$$

Solution.

$$\mathscr{L}\left\{e^{7t}\sin 9t - t^5 + e^{6t}\right\} = \frac{9}{(s-7)^2+81} - \frac{120}{s^6} + \frac{1}{s-6}, \quad s > 7$$

Problem 9. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathscr{L}\left\{2t^2e^{-3t} - e^{7t}\cos\sqrt{6}t\right\}.$$

Solution.

$$\mathscr{L}\left\{2t^2e^{-3t} - e^{7t}\cos\sqrt{6}t\right\} = \frac{4}{(s+3)^3} - \frac{s-7}{(s-7)^2+6}, \quad s > 7$$