

MAP 2302 Homework 7.2

Problem 1. Find the Laplace transform of $f(t) = 6t^2$.

Solution.

$$\begin{aligned} F(s) &= \int_0^{\infty} 6t^2 e^{-st} dt \\ &= 6 \int_0^{\infty} t^2 e^{-st} dt \\ &= 6 \lim_{N \rightarrow \infty} \left[\frac{-t^2 e^{-st}}{s} - \int -\frac{2te^{-st}}{s} dt \right]_0^N \\ &= 6 \lim_{N \rightarrow \infty} \left[-\frac{t^2 e^{-st}}{s} + \frac{2}{s} \left(-\frac{te^{-st}}{s} - \int -\frac{e^{-st}}{s} dt \right) \right]_0^N \\ &= 6 \lim_{N \rightarrow \infty} \left[-\frac{t^2 e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^N \\ &= 6 \lim_{N \rightarrow \infty} \left[-\frac{N^2 e^{-sN}}{s} - \frac{2Ne^{-sN}}{s^2} - \frac{2e^{-sN}}{s^3} + \frac{2}{s^3} \right] \\ &= \frac{12}{s^3}, \quad s > 0 \end{aligned}$$

Problem 2. Find the Laplace transform of $f(t) = te^{5t}$.

Solution.

$$\begin{aligned} F(s) &= \int_0^{\infty} te^{(5-s)t} dt \\ &= \lim_{N \rightarrow \infty} \left[\frac{te^{(5-s)t}}{5-s} - \int \frac{e^{(5-s)t}}{5-s} dt \right]_0^N \\ &= \lim_{N \rightarrow \infty} \left[\frac{te^{(5-s)t}}{5-s} - \frac{e^{(5-s)t}}{(5-s)^2} \right]_0^N \\ &= \lim_{N \rightarrow \infty} \left[\frac{Ne^{(5-s)N}}{5-s} - \frac{e^{(5-s)N}}{(5-s)^2} + \frac{1}{(5-s)^2} \right] \\ &= \frac{1}{(5-s)^2}, \quad s > 5 \end{aligned}$$

Problem 3. Find the Laplace transform of $f(t) = -\sin bt$, b a constant.

Solution.

$$\begin{aligned}
F(s) &= \int_0^\infty -e^{-st} \sin bt \, dt \\
&= \lim_{N \rightarrow \infty} \left[\frac{e^{-st} \sin bt}{s} - \int \frac{be^{-st}}{s} dt \right]_0^N \\
&= \lim_{N \rightarrow \infty} \left[\frac{e^{-st} \sin bt}{s} - \frac{b}{s} \left(-\frac{e^{-st} \cos bt}{s} - \int \frac{be^{-st} \sin bt}{s} dt \right) \right]_0^N \\
&= \lim_{N \rightarrow \infty} \left[\frac{e^{-st} \sin bt}{s} + \frac{be^{-st} \cos bt}{s^2} \right]_0^N + \frac{b^2}{s^2} \int_0^\infty e^{-st} \sin bt \, dt \\
\left(1 + \frac{b^2}{s^2}\right) F(s) &= \lim_{N \rightarrow \infty} \left[\frac{e^{-sN} \sin bN}{s} + \frac{be^{-sN} \cos bN}{s^2} - \frac{b}{s^2} \right] \\
&= -\frac{b}{s^2} \\
F(s) &= -\frac{b}{s^2 + b^2}, \quad s > 0
\end{aligned}$$

Problem 4. Find the Laplace transform of $f(t) = e^{-5t} \sin 2t$.**Solution.**

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-(s+5)t} \sin 2t \, dt \\
&= \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s+5)t} \sin 2t}{s+5} - \int -\frac{2e^{-(s+5)t} \cos 2t}{s+5} dt \right]_0^N \\
&= \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s+5)t} \sin 2t}{s+5} + \frac{2}{s+5} \left(-\frac{e^{-(s+5)t} \cos 2t}{s+5} - \int \frac{2e^{-(s+5)t} \sin 2t}{s+5} dt \right) \right]_0^N \\
&= \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s+5)t} \sin 2t}{s+5} - \frac{2e^{-(s+5)t}}{(s+5)^2} \right]_0^N - \frac{4}{(s+5)^2} \int_0^\infty e^{-(s+5)t} \sin 2t \, dt \\
\left(1 + \frac{4}{(s+5)^2}\right) F(s) &= \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s+5)N} \sin 2N}{s+5} - \frac{2e^{-(s+5)N} \cos 2N}{(s+5)^2} + \frac{2}{(s+5)^2} \right] \\
&= \frac{2}{(s+5)^2} \\
F(s) &= \frac{2}{(s+5)^2 + 4}, \quad s > -5
\end{aligned}$$

Problem 5. Find the Laplace transform of

$$\begin{cases} 3-t, & 0 < t < 3 \\ 0, & 3 < t \end{cases}.$$

Solution.

$$\begin{aligned} F(s) &= \int_0^3 (3-t)e^{-st} dt + \int_3^\infty 0e^{-st} dt \\ &= \left[-\frac{e^{-st}(3-t)}{s} - \int \frac{e^{-st}}{s} dt \right]_0^3 \\ &= \left[-\frac{e^{-st}(3-t)}{s} + \frac{e^{-st}}{s^2} \right]_0^3 \\ &= \frac{e^{-3s}}{s^2} + \frac{3}{s} - \frac{1}{s^2} \\ &= \begin{cases} \frac{e^{-3s} + 3s - 1}{s^2}, & s \neq 0, \\ \frac{9}{2}, & s = 0 \end{cases} \end{aligned}$$

Problem 6. Find the Laplace transform of

$$\begin{cases} e^{4t}, & 0 < t < 2 \\ 3, & 2 < t \end{cases}.$$

Solution.

$$\begin{aligned} F(s) &= \int_0^2 e^{(4-s)t} dt + \int_2^\infty 3e^{-st} dt \\ &= \left[\frac{e^{(4-s)t}}{4-s} \right]_0^2 + 3 \lim_{N \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_2^N \\ &= \frac{e^{8-2s}}{4-s} - \frac{1}{4-s} + 3 \lim_{N \rightarrow \infty} \left[-\frac{e^{-sN}}{s} + \frac{e^{-2s}}{s} \right] \\ &= \begin{cases} \frac{e^{8-2s} - 1}{4-s} + \frac{3e^{-2s}}{s}, & s \neq 4 \\ 2 + \frac{3e^{-8}}{4}, & s = 4 \end{cases} \end{aligned}$$

Problem 7. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathcal{L} \{4e^{-9t} - t^3 + 3t - 2\}.$$

Solution.

$$\mathcal{L} \{4e^{-9t} - t^3 + 3t - 2\} = \frac{4}{s+9} - \frac{6}{s^4} + \frac{3}{s^2} - \frac{2}{s}, \quad s > 0$$

Problem 8. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathcal{L} \{e^{7t} \sin 9t - t^5 + e^{6t}\}.$$

Solution.

$$\mathcal{L} \{e^{7t} \sin 9t - t^5 + e^{6t}\} = \frac{9}{(s-7)^2 + 81} - \frac{120}{s^6} + \frac{1}{s-6}, \quad s > 7$$

Problem 9. Use the Laplace transform table and the linearity of the Laplace transform to determine

$$\mathcal{L} \{2t^2 e^{-3t} - e^{7t} \cos \sqrt{6}t\}.$$

Solution.

$$\mathcal{L} \{2t^2 e^{-3t} - e^{7t} \cos \sqrt{6}t\} = \frac{4}{(s+3)^3} - \frac{s-7}{(s-7)^2 + 6}, \quad s > 7$$