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MAP 2302 Homework 7.5

Problem 1. Solve the initial value problem below using the method of Laplace transforms.

$$y" + 5y' - 6y = 0,\ y(0) = -3,\ y'(0) = 39$$

Solution.

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) - 6Y(s) = 0,\ Y(s) = \mathscr{L}\{y\} \\
s^2Y + 3s - 39 + 5sY + 15 - 6Y &= 0 \\
Y(s) &= -\frac{3s-24}{s^2+5s-6} \\
-\frac{3s-24}{(s+6)(s-1)} &= \frac{A}{s+6} + \frac{B}{s-1} \\
-3s + 24 &= A(s-1) + B(s+6) \\
27 &= 7B, \quad B = 3; \quad 42 = -7A, \quad A = -6 \\
Y(s) &= -\frac{6}{s+6} + \frac{3}{s-1} \\
y(t) &= -6e^{-6t} + 3e^{t}
\end{aligned}$$

Problem 2. Solve the initial value problem below using the method of Laplace transforms.

$$y" + 8y' + 12y = 462e^{5t},\ y(0) = 2,\ y'(0) = 42$$

Solution.

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 8(sY(s) - y(0)) + 12Y(s) &= \frac{462}{s-5} \\
s^2Y - 2s - 42 + 8sY - 16 + 12Y &= \frac{462}{s-5} \\
Y(s) &= \frac{1}{s^2+8s+12}\left(\frac{462}{s-5} + 2s + 58\right) \\
&= \frac{462}{(s+6)(s+2)(s-5)} + \frac{2s+58}{(s+6)(s+2)} \\
&= \frac{462 + (2s+58)(s-5)}{(s+6)(s+2)(s-5)} = \frac{A}{s+6} + \frac{B}{s+2} + \frac{C}{s-5} \\
462 + (2s+58)(s-5) &= A(s+2)(s-5) + B(s+6)(s-5) + C(s+6)(s+2) \\
-28B = 84, \quad B = -3; &\quad 44A = -44, \quad A = -1; \\
10 + 90 + 12C &= 172, \quad C = 6 \\
Y(s) &= -\frac{1}{s+6} - \frac{3}{s+2} + \frac{6}{(s-5)} \\
y(t) &= -e^{-6t} - 3e^{-2t} + 6e^{5t}
\end{aligned}
$$

Problem 3. Solve the initial value problem below using the method of Laplace transforms.

$$y" - 12y' + 37y = 130e^t,\ y(0) = 5,\ y'(0) = 6$$

Solution.

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - 12(sY(s) - y(0)) + 37Y(s) &= \frac{130}{s-1} \\
s^2Y - 5s - 6 - 12sY + 60 + 37Y &= \frac{130}{s-1} \\
Y(s) &= \frac{1}{s^2 - 12s + 37}\left(\frac{130}{s-1} + 5s - 54\right) \\
&= \frac{130 + (s-1)(5s-54)}{(s-1)(s^2-12s+37)} = \frac{A(s-6)+B}{(s-6)^2+1^2} + \frac{C}{s-1} \\
130 + (s-1)(5s-54) &= (s-1)[A(s-6)+B] + C(s^2 - 12s + 37) \\
130 &= 26C,\ C = 5; \quad 5B + C = 10,\ B = 1; \\
6A - B + 37C &= 184,\ A = 0 \\
Y(s) &= \frac{1}{(s-6)^2 + 1^2} + \frac{5}{s-1} \\
y(t) &= e^{6t}\sin t + 5e^t
\end{aligned}$$

Problem 4. Solve the initial value problem below using the method of Laplace transforms.

$$y" + 36y = 108t^2 - 36t + 78,\ y(0) = 0,\ y'(0) = 11$$

Solution.

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 36Y(s) &= \frac{216}{s^3} - \frac{36}{s^2} + \frac{78}{s} \\
s^2Y - 11 + 36Y &= \frac{216}{s^3} - \frac{36}{s^2} + \frac{78}{s} \\
Y(s) &= \frac{1}{s^2+36}\left(\frac{216}{s^3} - \frac{36}{s^2} + \frac{78}{s} + 11\right) \\
&= \frac{78s^2 - 36s + 216}{s^3(s^2+36)} + \frac{11}{s^2+36} \\
\frac{78s^2 - 36s + 216}{s^3(s^2+36)} &= \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+36} \\
78s^2 - 36s + 216 &= A(s^2+36) + Bs(s^2+36) + Cs^2(s^2+36) + s^3(Ds+E) \\
&= As^2 + 36A + Bs^3 + 36Bs + Cs^4 + 36Cs^2 + Ds^4 + Es^3 \\
&= (C+D)s^4 + (B+E)s^3 + (A+36C)s^2 + 36Bs + 36A
\end{aligned}$$

$$C + D = 0; \quad B + E = 0; \quad A + 36C = 78; \quad 36B = -36; \quad 36A = 216$$

$$\begin{aligned}
A = 6; \quad B = -1; \quad C &= 2; \quad D = -2; \quad E = 1 \\
Y(s) &= \frac{6}{s^3} - \frac{1}{s^2} + \frac{2}{s} - \frac{2s-1}{s^2+36} + \frac{11}{s^2+36} \\
y(t) &= 3t^2 - t - 2\cos 6t + 2\sin 6t + 2
\end{aligned}$$

Problem 5. Solve the initial value problem below using the method of Laplace transforms.

$$y\text{''} - 16y = 32t - 24e^{-4t},\ y(0) = 0,\ y'(0) = 41$$

Solution.

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - 16Y(s) &= \frac{32}{s^2} - \frac{24}{s+4} \\
s^2Y - 41 - 16Y &= \frac{32}{s^2} - \frac{24}{s+4} \\
Y(s) &= \frac{1}{s^2-16}\left(\frac{32}{s^2} - \frac{24}{s+4} + 41\right) \\
&= \frac{1}{s^2-16}\left(\frac{32(s+4) - 24s^2 + 41s^2(s+4)}{s^2(s+4)}\right) \\
&= \frac{32s + 128 - 24s^2 + 41s^3 + 164s^2}{s^2(s-4)(s+4)^2} \\
&= \frac{41s^3 + 140s^2 + 32s + 128}{s^2(s-4)(s+4)^2} \\
&= \frac{A}{(s+4)^2} + \frac{B}{s^2} + \frac{C}{s+4} + \frac{D}{s} + \frac{E}{s-4} \\
41s^3 + 140s^2 + 32s + 128 &= As^2(s-4) + B(s-4)(s+4)^2 + \\
&\quad Cs^2(s-4)(s+4) + Ds(s-4)(s+4)^2 + Es^2(s+4)^2 \\
&= As^3 - 4As^2 + Bs^3 + 4Bs^2 - 16Bs - 64B + Cs^4 - 16Cs^2 + Ds^4 + \\
&\quad 4Ds^3 - 16Ds^2 - 64Ds + Es^4 + 8Es^3 + 16Es^2 \\
&(C+D+E)s^4 + (A+B+4D+8E)s^3 + (-4A+4B-16C-16D+16E)s^2 + \\
&\quad (-16B-64D)s - 64B \\
C+D+E = 0; \quad A+B+4D+8E &= 41; \\
-4A+4B-16C-16D+16E &= 140; \quad -16B-64D = 32; \quad -64B = 128 \\
B = -2; \quad D = 0; \quad E &= 5; \quad A = 3; \quad C = -5 \\
Y(s) &= \frac{3}{(s+4)^2} - \frac{2}{s^2} - \frac{5}{s+4} + \frac{5}{s-4} \\
y(t) &= 3te^{-4t} - 2t - 5e^{-4t} + 5e^{4t}
\end{aligned}
$$

Problem 6. Solve for Y(s), the Laplace transform of the solution y(t) to the initial value problem below.

$$y" + 9y = 6t^2 - 5,\ y(0) = 0,\ y'(0) = -5$$

Solution.

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{12}{s^3} - \frac{5}{s} \\
s^2Y + 5 + 9Y &= \frac{12}{s^3} - \frac{5}{s} \\
Y(s) &= \frac{1}{s^2+9}\left(\frac{12}{s^3} - \frac{5}{s} - 5\right) \\
&= \frac{1}{s^2+9}\left(\frac{12s - 5s^2 - 5s^3}{s^3}\right) \\
&= -\frac{5s^3 + 5s^2 - 12s}{s^3(s^2+9)}
\end{aligned}
$$