Задание 1

$$A = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = E = I$$

$$\det(A - \lambda I) = 0$$

$$\begin{pmatrix} \cos\theta - \lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta - \lambda \end{pmatrix} = (A - \lambda I)$$

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$-(\cos\theta - \lambda)(\cos\theta + \lambda) + i^2\sin^2\theta = 0$$

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = 0$$

$$\lambda^2 = \sin^2\theta + \cos^2\theta$$

$$\lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

$\lambda = 1$:
$$\begin{pmatrix} \cos\theta - 1 & -i\sin\theta \\ i\sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(\cos\theta - 1)x_1 - x_2 i\sin\theta = 0$$
$$i x_1 \sin\theta - x_2(\cos\theta + 1) = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i\sin\theta \\ \cos\theta - 1 \end{pmatrix} \frac{1}{(\cos^2\theta - \sin^2\theta - 2\cos\theta + 1)^{1/2}}$$

λ_2:
$$\begin{pmatrix} \cos\theta + 1 & -i\sin\theta \\ i\sin\theta & 1 - \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{cases} (\cos\theta + 1)x_1 - x_2 i\sin\theta = 0 \\ x_1 i\sin\theta - x_2(1 - \cos\theta) = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i\sin\theta \\ \cos\theta + 1 \end{pmatrix} \frac{1}{(\cos^2\theta - \sin^2\theta - 2\cos\theta + 1)^{1/2}}$$

Проверим собственные вектора на ортогональность:

$$\begin{pmatrix} i\sin\theta \\ \cos\theta - 1 \end{pmatrix} \begin{pmatrix} i\sin\theta \\ \cos\theta + 1 \end{pmatrix} = -\sin^2\theta - \cos^2\theta + 1 = 0 = 0$$

Ответ: $\lambda = \pm 1$ $\quad X(\lambda = 1) = \begin{pmatrix} i\sin\theta \\ \cos\theta - 1 \end{pmatrix} \frac{1}{(\cos^2\theta - \sin^2\theta - 2\cos\theta + 1)^{1/2}}$

$$X(\lambda = -1) = \begin{pmatrix} i\sin\theta \\ \cos\theta + 1 \end{pmatrix} \frac{1}{(\cos^2\theta - \sin^2\theta - 2\cos\theta + 1)^{1/2}}$$

$e^{i\alpha A}$ - ?

α - действительный оператор

$\alpha = const$

A - матрица

№2

$\hat{S}$; $e^{i\alpha A} = e^{\hat{S}} = 1 + \frac{\hat{S}}{1!} + \frac{\hat{S}^2}{2!} + \dots = 1 + \sum_i \frac{\hat{S}^i}{i!}$

e

$$e^{i\alpha A} = 1 + \frac{i\alpha A}{1} + \frac{(i\alpha A)^2}{2} + \frac{(i\alpha A)^3}{6} \dots = 1 + \frac{\alpha^2 A^2}{2} + \frac{-i\alpha^3 A^3}{6} + \frac{\alpha^4 A^4}{24} + \frac{-i\alpha^5 A^5}{120}$$

$$A^2 = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = E$$

$$e^{i\alpha A} = 1 + i\alpha A - \frac{\alpha^2 E}{2} - \frac{i\alpha^3 A}{6} + \frac{\alpha^4 E}{24} + \frac{i\alpha^5 A}{120} \dots$$

$$e^{i\alpha A} = \left(1 - \frac{\alpha^2 E}{2!} + \frac{\alpha^4 E}{4!} \dots\right) + \left(i\alpha A - \frac{i\alpha^3 A}{3!} + \frac{i\alpha^5 A}{5!}\right)$$

$$e^{i\alpha A} = E\cos\alpha + iA\sin\alpha = e^{iA\alpha}$$

$$E\cos\alpha + iA\sin\alpha = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha \end{pmatrix} + \begin{pmatrix} +i\sin\alpha\cos\theta & \sin\alpha\sin\theta \\ -\sin\alpha\sin\theta & -i\sin\alpha\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} i\sin\alpha\cos\theta + \cos\alpha & \sin\theta\sin\alpha \\ -\sin\alpha\sin\theta & \cos\alpha - i\sin\alpha\cos\theta \end{pmatrix}$$

Задание №2

$$A=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

$\det(A-\lambda I)=0 \qquad (A-\lambda I)=\frac{1}{\sqrt{2}}\begin{pmatrix}-\lambda & -i & 0\\ i & -\lambda & -i\\ 0 & i & -\lambda\end{pmatrix}$

$$\begin{vmatrix}-\lambda & -i & 0\\ i & -\lambda & -i\\ 0 & i & -\lambda\end{vmatrix}=-\lambda^3+2\lambda=0 \Rightarrow \begin{cases}\lambda=0\\ \lambda=\sqrt{2}\\ \lambda=-\sqrt{2}\end{cases}$$

$\lambda: \lambda=\sqrt{2}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-\sqrt{2} & -i & 0\\ i & -\sqrt{2} & -i\\ 0 & i & -\sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=0$$

, фундаментальная система решений

ФСР: $\left\{x_3\cdot\begin{pmatrix}-1\\ -i\sqrt{2}\\ 1\end{pmatrix}\right\}$

$$x_3=i \Rightarrow \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}-i\\ +\sqrt{2}\\ i\end{pmatrix}\cdot\frac{1}{2}$$

$\lambda=-\sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2} & -i & 0\\ i & \sqrt{2} & -i\\ 0 & i & \sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=0$ ФСР: $\left\{x_3\begin{pmatrix}-1\\ i\sqrt{2}\\ 1\end{pmatrix}\right\}$, $x_3=1$

если $x_3=i \Rightarrow \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}-i\\ \sqrt{2}\\ i\end{pmatrix}\cdot\frac{1}{\sqrt{2+2}}=\frac{1}{2}\begin{pmatrix}-i\\ \sqrt{2}\\ i\end{pmatrix}$

$\lambda=0 \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=0 \qquad \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}$

Значит собственные вектора: $\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}$; $\frac{1}{2}\begin{pmatrix}-i\\ +\sqrt{2}\\ i\end{pmatrix}$; $\frac{1}{2}\begin{pmatrix}+i\\ \sqrt{2}\\ -i\end{pmatrix}$

Проверим их ортогональность:

$$\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\begin{pmatrix}-i\\ -\sqrt{2}\\ i\end{pmatrix}=\frac{1}{2\sqrt{2}}(-i+i)=0$$

$$\begin{pmatrix}-i\\ +\sqrt{2}\\ i\end{pmatrix}\begin{pmatrix}+i\\ \sqrt{2}\\ -i\end{pmatrix}=1+1-2=0$$

$$e^{i\theta A} = 1 + \frac{i\theta A}{1!} + \frac{(i\theta A)^2}{2!} + \frac{(i\theta A)^3}{3!}$$

$$A^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = B \qquad iA = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$A^4 = \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = B$$

$$A^3 = \frac{1}{2\sqrt{2}} \cdot 2A = \frac{1}{\sqrt{2}}A \qquad A^5 = \frac{1}{\sqrt{2}}A$$

$$e^{i\theta A} = \left(1 - \frac{(\theta A)^2}{2!} + \frac{(\theta A)^4}{4!} \dots\right) + i\left(\frac{\theta A}{1!} - \frac{(\theta A)^3}{3!} \dots\right)$$

$$e^{i\theta A} = B\cos\theta + (i\sin\theta)A = \frac{1}{2}\begin{pmatrix} \cos\theta & 0 & -\cos\theta \\ 0 & 2\cos\theta & 0 \\ -\cos\theta & 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$e^{i\theta A} = \begin{pmatrix} \frac{1}{2}\cos\theta & \frac{1}{\sqrt{2}}\sin\theta & -\frac{1}{2}\cos\theta \\ -\frac{1}{2}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ -\frac{1}{2}\cos\theta & -\frac{1}{2}\sin\theta & \frac{1}{2}\cos\theta \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\psi\rangle = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} + \frac{\beta}{2}\begin{pmatrix} +i \\ +\sqrt{2} \\ -i \end{pmatrix} + \frac{\gamma}{2}\begin{pmatrix} \frac{-i}{\sqrt{2}} \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \frac{\alpha}{\sqrt{2}} + \frac{i\beta}{2} - \frac{\gamma i}{2} = 1 \\ 0 - \frac{1}{\sqrt{2}}\beta + \frac{\gamma}{\sqrt{2}} = 0 \\ \frac{\alpha}{\sqrt{2}} - \frac{\beta i}{2} + \frac{\gamma i}{2} = 0 \end{cases} \iff \begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{i}{2} & -\frac{i}{2} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{2} & \frac{i}{2} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{-i}{2} \\ \frac{i}{2} \end{pmatrix}$$

Therefore: $|\psi\rangle = \underbrace{\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{-i}{4}\begin{pmatrix} +i \\ +\sqrt{2} \\ -i \end{pmatrix} + \frac{i}{4}\begin{pmatrix} \frac{-i}{\sqrt{2}} \\ +i \end{pmatrix}}$

Задание 3

Замена

① $\hat{T}_a \psi(x) = \psi(x+a)$

$$\langle \varphi(x) | \hat{T}_a \psi(x) \rangle = \int_{x_1}^{x_2} \varphi^*(x)\, \hat{T}_a \psi(x)\, dx = \int_{x_1}^{x_2} \varphi^*(x) \psi(x+a)\, dx = \left| \begin{cases} y = x+a \\ x = y-a \\ dx = dy \end{cases} \right| =$$

$$= \int_{y_1}^{y_2} \varphi^*(y-a) \psi(y)\, dy \neq \langle \hat{T}_a \varphi(y) | \psi(y) \rangle \Rightarrow$$ Оператор не эрмитов.

② $$\int_{y_1}^{y_2} \varphi^*(y-a) \psi(y)\, dy = \int_{y_1}^{y_2} (\hat{T}_{-a} \varphi(y))^* \psi(y)\, dy = \langle \hat{T}_a^+ \varphi(y) | \psi(y) \rangle$$

$$\hat{T}_a^+ = \hat{T}_{-a}$$

③ $\hat{T}_a^+ \hat{T}_a \psi(x) = \hat{T}_{-a} \psi(x+a) = \psi(x) \Rightarrow \hat{T}_a^+ \hat{T}_a = 1$

④ $\hat{T}_a \psi(x) = \psi(x+a) = \int \delta(x+a-x') \psi(x')\, dx' = \int T_a(x,x') \psi(x')\, dx'$

$$\hat{T}_a(x,x') = \delta(x+a-x')$$

Задание 4

$$\hat{S} = \alpha \left(x^2 \frac{d}{dx} - \frac{d}{dx} x^2 \right)$$

$$\frac{d(AB)}{dx} = A\frac{dB}{dx} + B\frac{dA}{dx} \Rightarrow A\frac{dB}{dx} - \frac{d(AB)}{dx} = -B\frac{dx}{dx} \Rightarrow \hat{S} = -2\alpha x$$

① $$\langle \varphi | \hat{S} \psi \rangle = -2\int \alpha x \varphi^* \psi\, dx = \int (-2\alpha^* x^* \varphi(x))^* \psi(x) = \int (-2\alpha^* x^* \varphi(x))^* \psi(x)\, dx$$

$$\hat{S}^+ = -2\alpha^* x^*$$

② Пусть $\hat{S}$ - эрмитов $\Rightarrow \hat{S}^+ = \hat{S} \Rightarrow -2\alpha^* x^* = -2\alpha x \quad \alpha^* x^* \neq \alpha x$

③ $$\hat{S}\psi(x) = -2\alpha x \psi(x) = \int (-2\alpha x) \delta(x-x') \psi(x')\, dx' = \int \delta(x,x') \psi(x')\, dx' \Rightarrow$$

$$\Rightarrow \delta(x,x') = -2\alpha x\, \delta(x-x')$$

① $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow BB^{-1} = I$

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab \\ dc-cd & -bc+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

② $O(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow O^{t}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

③ $O^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$