Домашнее задание № 3. Федотов М.С
БМЗ-19-1

$$S=\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \quad \det(S-\lambda I)=0$$

$|\psi\rangle = \langle 1|\psi\rangle + \langle -1|\psi\rangle$

$$\begin{vmatrix} \cos\theta-\lambda & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta-\lambda \end{vmatrix} = -(\cos\theta-\lambda)(\cos\theta+\lambda)-(\sin\theta)^2\cdot 1 = \lambda^2-1=0$$

$\psi_1 \frac{1}{n} |1\rangle$

$$\lambda=\pm 1$$

$\lambda=1$: $\begin{pmatrix} \cos\theta-1 & \sin\theta e^{-i\varphi} \\ e^{i\varphi}\sin\theta & -\cos\theta-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta\cdot e^{-i\varphi} \\ \cos\theta-1 \end{pmatrix}\cdot\frac{1}{n}$

$$\begin{cases} (\cos\theta-1)x_1 + x_2\sin\theta\cdot e^{-i\varphi}=0 \\ x_1 e^{i\varphi}\sin\theta - x_2(\cos\theta+1)=0 \end{cases}$$

где $n=\frac{1}{(\sin^2\theta\cdot e^{-i\varphi\cdot 2}+1+\cos^2\theta-2\cos\theta)^{\frac{1}{2}}}$

$\lambda=-1$: $\begin{pmatrix} \cos\theta+1 & \sin\theta\cdot e^{i\varphi} \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{-i\varphi}\sin\theta \\ \cos\theta+1 \end{pmatrix}\cdot\frac{1}{\sqrt{\cos^2\theta+1+2\cos\theta+e^{-2i\varphi}\sin^2\theta}}$$

Проверим ортогональность:

$$\begin{pmatrix} -\sin\theta\cdot e^{-i\varphi} \\ \cos\theta-1 \end{pmatrix}\begin{pmatrix} e^{-i\varphi}\sin\theta \\ \cos\theta+1 \end{pmatrix}=0$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\alpha = \langle 1|\varphi\rangle =$

$|1\rangle = \begin{pmatrix} -e^{-i\varphi}\sin\theta \\ \cos\theta-1 \end{pmatrix} \Rightarrow \langle 1| = (\sin\theta\cdot e^{i\varphi} \;\; \cos\theta-1)$

$\alpha = (\sin\theta\cdot e^{i\varphi} \;\; \cos\theta-1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}$

Разложение $|\psi\rangle$ по собственным векторам $\hat{S}$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \lambda_1 C_1 S_1 + \lambda_2 C_2 S_2$$

$$S_2^2 = (\cos\theta - 1)^2 + \sin^2\theta = 2(1-\cos\theta)$$

$$S_2 = \begin{pmatrix} \frac{\cos\theta + 1}{\sqrt{-\cos\theta + 1}} \\ \frac{\sin\theta \cdot e^{i\varphi}}{\sqrt{2(-\cos\theta + 1)}} \end{pmatrix} \qquad S_1 = \begin{pmatrix} \frac{-\sin\theta \cdot e^{-i\varphi}}{\sqrt{2(\cos\theta + 1)}} \\ \frac{\cos\theta - 1}{\sqrt{2(-\cos\theta + 1)}} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \frac{-\sin\theta\cdot e^{-i\varphi}}{\sqrt{\cos\theta+1}} & \frac{\cos\theta - 1}{\sqrt{1-\cos\theta}} & 1 \\ \frac{\cos\theta - 1}{\sqrt{1-\cos\theta}} & \frac{\sin\theta\cdot e^{i\varphi}}{\sqrt{1\cancel{+}\cos\theta}} & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -\sin\theta\cdot e^{-i\varphi} & \cos\theta - 1 & \sqrt{1-\cos\theta} \\ \cos\theta - 1 & \sin\theta\cdot e^{i\varphi} & \sqrt{1-\cos\theta} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} -\sin^2\theta & (\cos\theta - 1)\sin\theta\cdot e^{i\varphi} & \sqrt{1-\cos\theta}\cdot\sin\theta\cdot e^{-i\varphi} \\ (\cos\theta-1)^2 & (\cos\theta - 1)\sin\theta\cdot e^{i\varphi} & \sqrt{1-\cos\theta}\cdot(\cos\theta - 1) \end{array}\right) \Rightarrow$$

$$\underbrace{((\cos\theta - 1)^2 + \sin^2\theta)}_{-2\cos\theta+2} C_1 = \sqrt{1-\cos\theta}\,(\cos\theta - 1 - \sin\theta\cdot e^{-i\varphi})$$

$$C_1 = \frac{\sqrt{1-\cos\theta}}{2(1-\cos\theta)}(\cos\theta - 1 - \sin\theta\cdot e^{-i\varphi}) = \frac{1}{2\sqrt{1-\cos\theta}}(\cos\theta - \sin\theta\cdot e^{-i\varphi} - 1)$$

$$C_2 = \frac{\sqrt{1-\cos\theta}\,(\cos\theta - \sin\theta - e^{i\varphi} + 1)}{\sin\theta\cdot e^{i\varphi}}$$

$$\langle S_\varphi\rangle = \langle\psi|\hat{A}_\varphi|\psi\rangle = \sum_{i=1}^{2}\lambda_i |\langle\psi|A_i\rangle|^2 = \frac{1}{2}\left(\left|\frac{-\sin\theta\cdot e^{-i\varphi}}{\sqrt{2(\cos\theta+1)}} + \frac{\cos\theta - 1}{\sqrt{2(-\cos\theta+1)}}\right|\right)^2 =$$

$$= \frac{(|\cos\theta - 1 - \sin\theta\cdot e^{-i\varphi}|)^2}{4(1-\cos\theta)} = C_1^2$$

$$\tilde{P}_2 = |\langle\psi|S_2\rangle|^2 = C_2^2 = \frac{1}{4(1-\cos\theta)}(|\cos\theta + 1 + \sin\theta\cdot e^{i\varphi}|)^2 = \tilde{P}_1$$

$$\begin{cases} \tilde{P}_2 + \tilde{P}_1 = 1 \\ \tilde{P}_2 = \tilde{P}_1 \end{cases} \Rightarrow P_1 = P_2 = 0{,}5$$