Let $\hat{\vec{\beta}} \sim N(0, \Sigma)$, K be a positive definite matrix and $S = \hat{\vec{\beta}}' K \hat{\vec{\beta}}$. Set $b, s > 0$ and define the following sets for some fixed index j

$$A := \{\hat{\vec{\beta}} : \ S < s\}, \qquad B := \{\hat{\vec{\beta}} : \hat{\beta}_j^2 < b\}$$

The sets A and B are both convex and symmetric about the origin. By the Gaussian Correlation Inequality we have:

$$Pr(A, B) \geq Pr(A)Pr(B) \tag{1}$$

The left-hand side of equation (1) can be written as

$$Pr(A, B) = 1 - Pr(\hat{\beta}_j^2 > b) - Pr(S > s) + Pr(\hat{\beta}_j^2 > b, S > s),$$

and similarly the right-hand side can be written as

$$Pr(A)Pr(B) = 1 - Pr(\hat{\beta}_j^2 > b) - Pr(S > s) + Pr(\hat{\beta}_j^2 > b)Pr(S > s).$$

Subtracting $1 - Pr(\hat{\beta}_j^2 > b) - Pr(S > s)$ from both sides of (1) yields:

$$Pr(\hat{\beta}_j^2 > b, S > s) \geq Pr(\hat{\beta}_j^2 > b)Pr(S > s).$$

Finally,

$$Pr(\hat{\beta}_j^2 > b | S > s) = \frac{Pr(\hat{\beta}_j^2 > b, S > s)}{Pr(S > s)} \geq \frac{Pr(\hat{\beta}_j^2 > b)Pr(S > s)}{Pr(S > s)} = Pr(\hat{\beta}_j^2 > b)$$

The left-hand side is $p'_{j,GN}$ and the right-hand side is p'_j and thus the result follows.