1 GLMER E-Step

$$
\begin{aligned}
P(Z_i = 1|y,\theta) &= \frac{\int \prod_j f(y_{ij}|X,\beta_1) g(v) dv}{\int \prod_j f(y_{ij}|X,\beta_1) f(v) dv + \int \prod_j f(y_{ij}|X,\beta_0) g(v) dv} \\
&\approx \frac{\sum_{m=1}^{M} \frac{g(v)}{\tilde{g}_1(v)} \prod_j f(y_{ij}|X,\beta_1,v_m)}{\sum_{m=1}^{M} \frac{g(v)}{\tilde{g}_1(v)} \prod_j f(y_{ij}|X,\beta_1,v_m) + \frac{g(v)}{\tilde{g}_0(v)} \prod_j f(y_{ij}|X,\beta_0,v_m)} \\
&= \left(1 + \frac{\sum_{m=1}^{M} \frac{g(v)}{\tilde{g}_0(v)} \prod_j f(y_{ij}|X,\beta_0,v_m)}{\sum_{m=1}^{M} \frac{g(v)}{\tilde{g}_1(v)} \prod_j f(y_{ij}|X,\beta_1,v_m)}\right)^{-1}
\end{aligned}
$$

2 Some Computation

$$\mathcal{L}(\theta) = \int \prod_{k=1}^{K} f_\theta(x_k, v)\pi(v)dv$$

$$l(\theta) = \log \int \prod_{k=1}^{K} f_\theta(x_k, v)\pi(v)dv$$

$$\frac{\partial}{\partial\theta} l(\theta) = \frac{1}{\mathcal{L}(\theta)} \int \frac{\partial}{\partial\theta} \prod_{k=1}^{K} f_\theta(x_k)\pi(v)dv$$

$$= \frac{1}{\mathcal{L}(\theta)} \int \prod_{k=1}^{K} f_\theta(x_k, v) \left(\sum_{k=1}^{K} \frac{\partial}{\partial\theta} \log f_\theta(x_k, v) \right) \pi(v)dv$$

$$\approx \left(\sum_{j=1}^{J} \prod_{k=1}^{K} f_\theta(x_k, v_j) \right)^{-1} \left(\sum_{j=1}^{J} \prod_{k=1}^{K} f_\theta(x_k, v_j) \left(\sum_{k=1}^{K} \frac{\partial}{\partial\theta} \log f_\theta(x_k, v_j) \right) \right)$$

3 Constrained Random Effect

$$\min \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{j=1}^{J} w_{ijk} (x_{ijk} - \hat{\mu}_{ik})^2$$

$$w_{ik} := \sum_j w_{ijk}$$

$$s.t. \sum_i w_{ik} \mu_{ik} = 0 \;\; \forall k$$

For a set k:

$$L_k = \frac{1}{2} \sum_i \sum_j w_{ij} (x_{ij} - \hat{\mu}_i)^2 + \lambda \sum_i w_i \hat{\mu}_i$$

Solve a system of equations:

$$-\sum_j w_{ij} (x_{ij} - \hat{\mu}_i) + \lambda w_i = 0 \qquad \forall i$$

$$\sum_i w_i \hat{\mu}_i = 0$$

$$\Rightarrow \lambda = \frac{\sum_{ij} w_{ij} x_{ij}}{\sum_i w_i}$$

$$\Rightarrow \hat{\mu}_i = \frac{\sum_j w_{ij} x_{ij} - \lambda w_i}{w_i} = \frac{\sum_j w_{ij} x_{ij}}{\sum_j w_{ij}} - \lambda := \bar{x}_i - \bar{\mu}$$