Computer Graphics

CGT 520

Computer Graphics Technology Dept.
Purdue University

February 3, 2015


Ceci n'est pas une pipe.


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Abstract 2D types and operations

- **Scalar** - a one-component value
  - scalar + scalar = scalar
  - scalar * scalar = scalar
  - We will consider the real numbers.
- **Vector** - a quantity with magnitude and direction
  - vector + vector = vector
  - scalar * vector = vector
  - Represented as a 2-element array.
- **Point** - a position in 2D (for now) space
  - point + vector = point
  - point - point = vector
  - Also represented as a 2-element array.

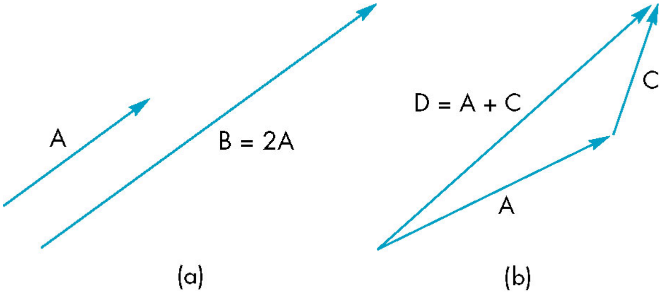
Concrete 2D types

- Scalar : We will consider the real numbers.
- Vector : Represented as a 2-element ordered set of real numbers.
- Point : Also represented as a 2-element ordered set of real numbers.

Specifying **vector** components $\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ implies a choice of basis vectors $\mathbf{v_1}, \mathbf{v_2}$ such that $\mathbf{a} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2}$.

For convenience, we most often choose basis vectors $\mathbf{e_1}, \mathbf{e_2}$ with representations $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Vector operations



Scalar-vector multiplication : $\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix}$

Vector-vector addition : $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

Concrete 2D types

Points:

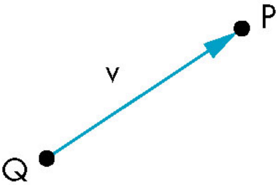
Specifying point coordinates $\mathbf{p} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ implies a choice of basis vectors $\mathbf{v_1}, \mathbf{v_2}$ and a reference point, $\mathbf{p_0}$, such that $\mathbf{p} = \mathbf{p_0} + \beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2}$.

For convenience, we most often choose the reference point to have the representation $\mathbf{p_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Frame:

a reference point,$\mathbf{p_0}$, and basis vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$

Point operations



Point-vector addition: $\mathbf{Q} + \mathbf{v} = \mathbf{P}$
Point-point subtraction: $\mathbf{P} - \mathbf{Q} = \mathbf{v}$

Point operations

Although it is mathematically incorrect to ”scale a point” ($\mathbf{p}$), we can scale the vector $\mathbf{p} - \mathbf{p_0}$.
(Recall the point is defined as $\mathbf{p} = \mathbf{p_0} + \beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2}$)

So scaling $\mathbf{p} - \mathbf{p_0}$ by α:
$\alpha(\mathbf{p} - \mathbf{p_0}) = \alpha(\beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2})$

Adding the result to $\mathbf{p_0}$ yields a new point $\mathbf{p}'$ in the same frame as $\mathbf{p}$:
$\mathbf{p}' = \mathbf{p_0} + \alpha(\beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2})$ which has the representation $\begin{bmatrix} \alpha\beta_1 \\ \alpha\beta_2 \end{bmatrix}$.

- So, in the end, the result is the same as if we had considered $\mathbf{p}$ to be a vector.
- It is still important to recognize the difference between points and vectors.
- Later we will use a representation (homogeneous coordinates) that will allow us to distinguish between points and vectors.

Spaces

Definitions:

Space : A **set** with some **operations** and **properties**.

- Vector space:
  - set of vectors
  - with addition and scalar multiplication operations
  - Properties : Addition is commutative and associative, etc...
- Affine space:
  - set of vectors and points
  - with vector space operators, and the point-vector operations.
  - Properties : Addition is associative, etc...
- Euclidean space:
  - An affine space with the addition of an inner product operation
  - This allows us to measure distance between points and do geometry.

Affine space : Lines

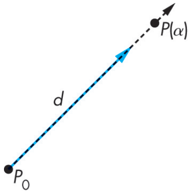
Using the elements and operations of affine space, it is possible to define a line:

$$P(\alpha) = P_0 + \alpha\mathbf{d}$$

These are all valid operation in affine space:

- $\alpha\mathbf{d}$: scalar $\times$ vector $\rightarrow$ vector
- $P_0 + (\alpha\mathbf{d})$: point $+$ vector $\rightarrow$ point
- This is called the **parametric form** of a line.
- Different values of the parameter (α) give different point on the line.

Affine space : Lines



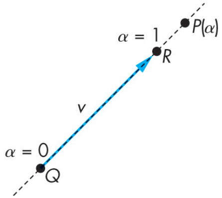
$$P(\alpha) = P_0 + \alpha \mathbf{d}$$

- Line (no endpoints): $\alpha \in (-\infty, \infty)$
- Ray (one endpoint): $\alpha \in [0, \infty)$
- Line segment (two endpoints) : $\alpha \in [\alpha_0, \alpha_1]$

Affine space : Lines

We may also define a line in terms of 2 points on the line

$$P(\alpha) = Q + \alpha(R - Q)$$



When computing points on the line, the equation can be rewritten as

$$P(\alpha) = \alpha R + (1 - \alpha)Q$$

Note that the operations involved are no longer defined in affine space. This is a useful operation to have, so we define it as a valid operation called **affine addition**.

Affine space : Affine addition

Affine addition can be defined for more than 2 points:

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots \alpha_n P_n$$

under the condition that

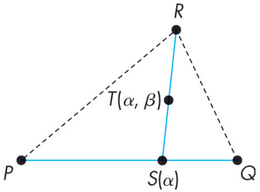
$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$$

Affine space : Planes

Recall from geometry that 3 **noncollinear** points define a plane. In affine space, a plane can be defined as a direct extension of the parametric line.

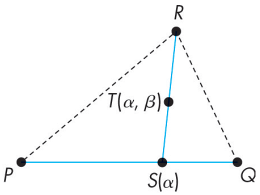
Let P, Q, R be 3 noncollinear points. Consider the line $S(\alpha)$ joining point P and Q.

$$S(\alpha) = \alpha P + (1 - \alpha)Q$$



Now construct a line, T, between any point S on that line and the point R.

Affine space : Planes



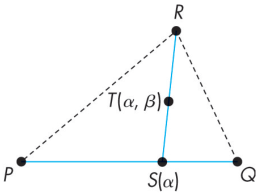
The line T can be written as : $T(\beta) = \beta S + (1 - \beta)R$
where : $S(\alpha) = \alpha P + (1 - \alpha)Q$.

Substituting the equation for S into the equation for T we have

$$T(\alpha, \beta) = \beta(\alpha P + (1 - \alpha)Q) + (1 - \beta)R$$

Plugging in values for α, β gives points on the plane defined by P, Q, R.

Affine space : Planes



$$T(\alpha, \beta) = \beta(\alpha P + (1 - \alpha)Q) + (1 - \beta)R$$

Can you show that this is an affine combination (i.e. the coefficients sum to one)?

Euclidean space : Dot product

Recall the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i = ||\mathbf{a}||||\mathbf{b}|| \cos\theta$$

Euclidean space adds the notions of length and distance to the affine space. We will define the **length** of a vector, u, to be

$$||u|| = \sqrt{u \cdot u}$$

and the **distance** between points P, Q as $||P - Q||$.

We may **normalize** a vector, **b** to obtain a new vector **a** whose length is 1.

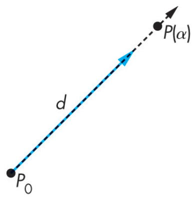
$$\mathbf{a} = \frac{\mathbf{b}}{||\mathbf{b}||}, ||\mathbf{a}|| = 1$$

Euclidean space : Dot product

The dot product can be used to compute the angle between 2 vectors, $\mathbf{u}$, $\mathbf{v}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

We can use the concepts of Euclidean space to obtain different forms of the line and plane definitions.

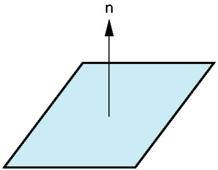


Note that for all points, $P(\alpha)$, on the line defined by P_0 and $\mathbf{d}$ we have

$$(P - P_0) \cdot \mathbf{d}_{\perp} = 0$$

Euclidean space : Dot product

We can specify a plane by specifying one point P_0 on the plane, and a vector, **n**, perpendicular to the plane.



All points, P, on the plane defined by P_0 and **n** we have

$$(P - P_0) \cdot \mathbf{n} = 0$$

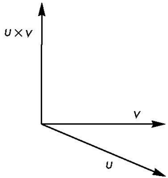
These representations of points and lines are called **implicit**.
They don't allow us to directly generate points on the line/plane, they do allow us to test whether a given point is on the line/plane.

Euclidean space : Cross product

For vectors $\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ the cross product is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

Euclidean space : Cross product



If **u** and **v** are not parallel, then the vector $\mathbf{u} \times \mathbf{v}$ is **orthogonal** to **u** and **v**.

The length of the resulting vector is related to the angle between **u** and **v** by

$$|\sin\theta| = \frac{||\mathbf{u} \times \mathbf{v}||}{||\mathbf{u}||\ ||\mathbf{v}||}$$

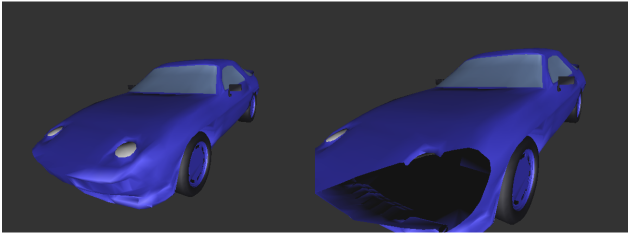
Three-dimensional primitives

We have seen how to represent simple 3D objects : lines and planes. One challenge of computer graphics is representing complex real objects. We will use the following approach:

- Represent objects as **surfaces**, not solids.
- Store the coordinates of points on the surface.
- Approximate the surface using planar polygons.

Represent objects as **surfaces**, not solids.

If the camera does not pass through objects, and if objects never break apart, we will only ever see the surface.

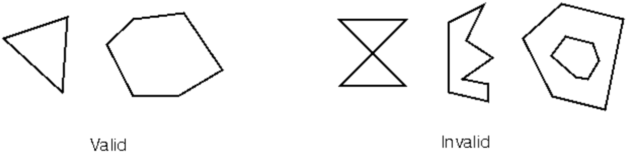


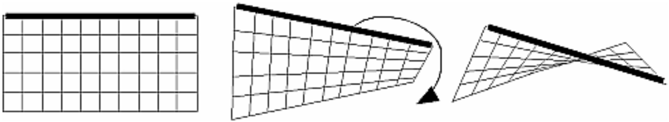
This is only a visual representation, the application programmer must devise techniques to prevent objects from intersecting.

Three-dimensional primitives

Why triangles?

- Always planar.
- Always **simple** (edges do not intersect).
- Always **convex** (line between any 2 points is in triangle).
- It is easy to optimize the hardware for triangles.





Nonplanar skew quadrilateral.