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An Extension of System F with Subtyping

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System F is a well-known typed λ-calculus with polymorphic types, which provides a basis for polymorphic programming languages. We study an extension of F, called $F_{<:}$ (pronounced *ef-sub*), that combines parametric polymorphism with subtyping. The main focus of the paper is the equational theory of $F_{<:}$, which is related to PER models and the notion of parametricity. We study some categorical properties of the theory when restricted to closed terms, including interesting categorical isomorphisms. We also investigate proof-theoretical properties, such as the conservativity of typing judgments with respect to F. We demonstrate by a set of examples how a range of constructs may be encoded in $F_{<:}$. These include record operations and subtyping hierarchies that are related to features of object-oriented languages.

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1. INTRODUCTION

System F [Gir 71, Rey 74] is a well-known typed λ-calculus with polymorphic types that provides a basis for polymorphic programming languages. We study an extension of F that combines parametric polymorphism [Str 67] with subtyping. We call this language $F_{<:}$, where $<:$ is our symbol for the subtype relation. $F_{<:}$ is closely related to the language $F_{\leqslant}$ identified by Curien, and used by Curien and Ghelli primarily as a test case for certain mathematical techniques [Ghe 90, CG 91]. $F_{\leqslant}$ is, in turn, a fragment of the language *Fun* [CW 85]. In spite of $F_{<:}$'s apparent minimality, it has become apparent that a range of constructs may be encoded in it (or in $F_{\leqslant}$); these include many of the record operations and subtyping features of [Car 88, CM 91] and related work that are connected to operations used in object-oriented programming. We illustrate some of the power of $F_{<:}$ in Section 3; see also [Car 93].

We have also found that the study of $F_{<:}$ raises semantic questions of independent interest. A major concern in this paper is an equational theory for $F_{<:}$ terms. The equational axioms for most systems of typed λ-calculi arise naturally as a consequence of characterizing type connectives by adjoint situations (for example). In addition, it is often the case that provable equality may be captured by a reduction system obtained by orienting the equational axioms in a straightforward way. However, both of these properties appear to fail for $F_{<:}$.

A simple example illustrates some of the basic issues.

Consider the polymorphic type $\forall(A)A \to A \to A$. This type is commonly referred to as *Bool*, since in system F and related systems there are two definable elements of this type. These elements are written as the following normal forms:

$$\mathit{true} \triangleq \lambda(A)\,\lambda(x{:}A)\,\lambda(y{:}A)x$$

$$\mathit{false} \triangleq \lambda(A)\,\lambda(x{:}A)\,\lambda(y{:}A)\,y.$$

If $F_{<:}$, however, there are two additional normal forms of type *Bool*. These arise because we have a maximal type *Top*, which has all other types as its subtypes. The main idea behind the additional terms is that we can change the type of any argument not used in the body of a term to *Top*, and still have a term of the same type (by antimonotonicity of the left operand of $\rightarrow$ with respect to $<:$). This gives us the following two normal forms of type *Bool*:

$$true' \triangleq \lambda(A)\,\lambda(x{:}A)\,\lambda(y{:}Top)x$$

$$false' \triangleq \lambda(A)\,\lambda(x{:}Top)\,\lambda(y{:}A)\,y.$$

However, *true* and *true'* are completely equivalent terms when considered at type *Bool*. Specifically, for any type A, the terms $true(A)$ and $true'(A)$ define extensionally equal functions of type $A \rightarrow A \rightarrow A$. Put proof-theoretically, if we take any term a containing *true* with the property that when reducing a to normal form we apply each occurrence of *true* to two arguments, then we may replace any or all occurrences of *true* by *true'* and obtain a provably equal term. For this reason, it seems natural to consider that $true = true'$, and similarly $false = false'$, even though these terms have different normal forms. When we add these two equations to our theory, we restore the pleasing property that *Bool* contains precisely two equivalence classes of normal forms.

While our initial examination of the equational theory of $F_{<:}$ was motivated by a vague intuition about observable properties of normal forms, our primary guide is the PER semantics of polymorphic λ-calculus with subtyping [BL 88, CL 91, Ghe 90, Sce 90]. One relevant characteristic of PER models is the *parametric* behavior of polymorphic functions. Specifically, since polymorphic functions operate independently of their type parameter, they may be considered equivalent at all their type instances. In $F_{<:}$ we can state a consequence of this notion of parametricity, namely that whenever the two type instances have a common supertype, they will be equal when considered as elements of that supertype (see the rule (*Eq appl2*) in Section 2.2). Hence the syntax of $F_{<:}$ can state, at least to some extent, the semantic notion of parametricity investigated in [Rey 83, Fre 93, MS 93]. A general principle we have followed is to adopt axioms that express parametricity properties satisfied by PER models, but not to capture explicitly the exact theory of PER models [Mit 90]. This leads us to a new angle on parametricity which may prove useful in further study, and also gives us a set of axioms that are sufficient to prove $true = true'$, and other expected equations, without appearing contrived to fit these particular examples.

While $F_{<:}$ differs from each of the λ-calculi mentioned above, several properties of $F_{<:}$ transfer easily from related work; in particular, $F_{<:}$ differs from $F_{\leqslant}$ [CG 91] only in the equational theory. For syntactic properties

we have strong normalization [Ghe 90]; canonical type derivations, coherence, minimum typing [CG 91]; and confluence of the β–η–*Top*-*Collapse* equational theory [CG 91a]. The PER semantics follows easily from the work in [BL 88, CL 91, Ghe 90, Sce 90]. While an alternative semantics could perhaps be developed in the style of [BFSS 90, Fre 93], we do not explore that possibility here.

The main results of this paper are an equational theory for $F_{<:}$, some proof-theoretic properties developed in Section 2 including conservativity of $F_{<:}$ typing over F, a set of examples in Section 3 demonstrating the expressiveness of $F_{<:}$ (some reported earlier in [CL 91], and in [Ghe 90] with attribution), and in Section 4 some categorical properties of the theory when it is restricted to closed terms.

2. System $F_{<:}$

$F_{<:}$ is obtained by extending F [Gir 71, Rey 74] (see Appendix) with a notion of subtyping ($<:$). This extension allows us to remain within a pure calculus. That is, we introduce neither the basic types, nor the structured types, normally associated with subtyping in programming languages. Instead, we show that these programming types can be obtained via encodings within the pure calculus. In particular, we can encode record types with their subtyping relations [Car 88].

2.1. *Syntax*

Subtyping is reflected in the syntax of types by a new type constant *Top* (the supertype of all types), and by a subtype bound on second-order quantifiers: $\forall(X<:A)A'$ (*bounded quantifiers* [CW 85]). Ordinary second-order quantifiers are recovered by setting the quantifier bound to *Top*; we use $\forall(X)A$ for $\forall(X<:Top)A$. The syntax of values is extended by a constant *top* of type *Top* (mostly a convenience), and by a subtype bound on polymorphic functions, $\lambda(X<:A)a$. We use $\lambda(X)a$ for $\lambda(X<:Top)a$.

Syntax.

$A, B ::=$	Types
X	type variables
Top	the supertype of all types
$A \to B$	function spaces
$\forall(X<:A)B$	bounded quantifications

$a, b ::=$	Values
x	value variables
top	the canonical value of type Top
$\lambda(x:A)b$	functions
$b(a)$	applications
$\lambda(X<:A)b$	bounded type functions
$b(A)$	type applications

The $\rightarrow$ operator associates to the right. The scoping of λ and $\forall$ extends to the right as far as possible. Types and terms can be parenthesized.

A subtyping judgment is added to F's judgments. Moreover, the equality judgment on values is made relative to a type; this is important since values in $F_{<:}$ can have many types, and two values may or may not be equivalent depending on the type that those values are considered as possessing (see, for example, the rule (*Eq collapse*) in Section 2.2).

Judgments.

$\vdash E\ env$	E is a well-formed environment
$E \vdash A\ type$	A is a type
$E \vdash A <: B$	A is a subtype of B
$E \vdash a:A$	a has type A
$E \vdash a \leftrightarrow b:A$	a and b are equal members of type A

We use $dom(E)$ for the set of variables defined by an environment E.

As usual, we identify terms up to renaming of bound variables; that is, using $B\{X \leftarrow C\}$ for the substitution of C for X in B, and $FV(-)$ for sets of free variables,

$$\forall(X<:A)B \equiv \forall(Y<:A)\,B\{X \leftarrow Y\} \quad \text{where} \quad Y \notin FV(B)$$
$$\lambda(x:A)b \equiv \lambda(y:A)\,b\{x \leftarrow y\} \quad \text{where} \quad y \notin FV(b)$$
$$\lambda(X<:A)b \equiv \lambda(Y<:A)\,b\{X \leftarrow Y\} \quad \text{where} \quad Y \notin FV(b).$$

These identifications can be made directly on the syntax; that is, without knowing whether the terms involved are the product of formal derivations in the system. By adopting these identifications, we avoid the need of a type equivalence judgment for quantifier renaming.

Environments, however, are not identified up to renaming of variables in their domains; environment variables are kept distinct by construction.

A more formal approach would use de Bruijn indices for free and bound variables [deB 72].

2.2. *Rules*

The inference rules of $F_{<:}$ are listed below; the only essential difference between these and the ones of $F_{\leqslant}$ [Ghe 90, CG 91] is in the more general (*Eq appl2*) rule. We now comment on the most interesting aspects of the rules. (See also the discussion about (*Eq appl2*) in Section 2.4.)

The subtyping judgment, $E \vdash A <: B$, is, for any E, a reflexive and transitive relation on types with a *subsumption* property; that is, a member of a type is also a member of any supertype of that type. Every type is a subtype of *Top*. The function space operator $\rightarrow$ is antimonotonic in its first argument and monotonic in its second. A bounded quantifier is antimonotonic in its bound and monotonic in its body under an assumption about the free variable.

The rules for the typing judgment, $E \vdash a : A$, are the same as the corresponding rules in F, except for the extension to bounded quantifiers. However, additional typing power is hidden in the subsumption rule, which allows a function to take an argument of a subtype of its input type.

Most of the equivalence rules, $E \vdash a \leftrightarrow b : A$, are unremarkable. They provide symmetry, transitivity, congruence on the syntax, and β and η equivalences. Two rules, however, stand out. The first, (*Eq collapse*) (also called the *Top-collapse* rule), states that any two terms are equivalent when "seen" at type *Top*; since no operations are available on members of *Top*, all values are indistinguishable at that type. The second, (*Eq appl2*), is the congruence rule for polymorphic type application, giving general conditions under which two expressions $b'(A')$ and $b''(A'')$ are equivalent at a type C. This rule has many intriguing consequences, which will be amply explored throughout this work. (We occasionally write $E \vdash A, B <: C$ for $E \vdash A <: C \wedge E \vdash B <: C$, and so on.)

Environments.

$$\frac{}{\vdash \phi \; env}\ (Env\ \phi) \qquad \frac{E \vdash A\ type \quad x \notin dom(E)}{\vdash E, x : A\ env}\ (Env\ x) \qquad \frac{E \vdash A\ type \quad X \notin dom(E)}{\vdash E, X <: A\ env}\ (Env\ X)$$

Types.

$$\frac{\vdash E, X <: A, E'\ env}{E, X <: A, E' \vdash X\ type}\ (Type\ X) \qquad \frac{\vdash E\ env}{E \vdash Top\ type}\ (Type\ Top)$$

$$\frac{E \vdash A\ type \quad E \vdash B\ type}{E \vdash A \rightarrow B\ type}\ (Type \rightarrow) \qquad \frac{E, X <: A \vdash B\ type}{E \vdash \forall (X <: A) B\ type}\ (Type \forall)$$

Subtypes.

(*Sub refl*)

$$\frac{E \vdash A\ type}{E \vdash A <: A}$$

(*Sub trans*)

$$\frac{E \vdash A <: B \quad E \vdash B <: C}{E \vdash A <: C}$$

(*Sub X*)

$$\frac{\vdash E, X <: A, E'\ env}{E, X <: A, E' \vdash X <: A}$$

(*Sub Top*)

$$\frac{E \vdash A\ type}{E \vdash A <: Top}$$

(*Sub* $\rightarrow$)

$$\frac{E \vdash A' <: A \quad E \vdash B <: B'}{E \vdash A \rightarrow B <: A' \rightarrow B'}$$

(*Sub* $\forall$)

$$\frac{E \vdash A' <: A \quad E, X <: A' \vdash B <: B'}{E \vdash \forall(X <: A)B <: \forall(X <: A')B'}$$

Values.

(*Subsumption*)

$$\frac{E \vdash a : A \quad E \vdash A <: B}{E \vdash a : B}$$

(*Val x*)

$$\frac{\vdash E, x : A, E'\ env}{E, x : A, E' \vdash x : A}$$

(*Val top*)

$$\frac{\vdash E\ env}{E \vdash top : Top}$$

(*Val fun*)

$$\frac{E, x : A \vdash b : B}{E \vdash \lambda(x : A)b : A \rightarrow B}$$

(*Val appl*)

$$\frac{E \vdash b : A \rightarrow B \quad E \vdash a : A}{E \vdash b(a) : B}$$

(*Val fun2*)

$$\frac{E, X <: A \vdash b : B}{E \vdash \lambda(X <: A)b : \forall(X <: A)B}$$

(*Val appl2*)

$$\frac{E \vdash b : \forall(X <: A)B \quad E \vdash A' <: A}{E \vdash b(A') : B\{X \leftarrow A'\}}$$

Equivalence.

(*Eq symm*)

$$\frac{E \vdash a \leftrightarrow b : A}{E \vdash b \leftrightarrow a : A}$$

(*Eq trans*)

$$\frac{E \vdash a \leftrightarrow b : A \quad E \vdash b \leftrightarrow c : A}{E \vdash a \leftrightarrow c : A}$$

(*Eq x*)

$$\frac{E \vdash x : A}{E \vdash x \leftrightarrow x : A}$$

(*Eq collapse*)

$$\frac{E \vdash a : Top \quad E \vdash b : Top}{E \vdash a \leftrightarrow b : Top}$$

(*Eq fun*)

$$\frac{E, x : A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(x : A)b \leftrightarrow \lambda(x : A)b' : A \rightarrow B}$$

(*Eq appl*)

$$\frac{E \vdash b \leftrightarrow b' : A \to B \quad E \vdash a \leftrightarrow a' : A}{E \vdash b(a) \leftrightarrow b'(a') : B}$$

(*Eq fun2*)

$$\frac{E, X <: A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(X <: A)b \leftrightarrow \lambda(X <: A)b' : \forall(X <: A)B}$$

(*Eq appl2*)

$$\frac{E \vdash b' \leftrightarrow b'' : \forall(X <: A)B \quad E \vdash A', A'' <: A \quad E \vdash B\{X \leftarrow A'\}, B\{X \leftarrow A''\} <: C}{E \vdash b'(A') \leftrightarrow b''(A'') : C}$$

(*Eq eta*)

$$\frac{E \vdash b \leftrightarrow b' : A \to B \quad y \notin dom(E)}{E \vdash \lambda(y:A)\, b(y) \leftrightarrow b' : A \to B}$$

(*Eq eta2*)

$$\frac{E \vdash b \leftrightarrow b' : \forall(X <: A)B \quad Y \notin dom(E)}{E \vdash \lambda(Y <: A)\, b(Y) \leftrightarrow b' : \forall(X <: A)B}$$

(*Eq beta*)

$$\frac{E, x:A \vdash b \leftrightarrow b' : B \quad E \vdash a \leftrightarrow a' : A}{E \vdash (\lambda(x:A)b)(a) \leftrightarrow b'\{x \leftarrow a'\} : B}$$

(*Eq beta2*)

$$\frac{E, X <: A \vdash b \leftrightarrow b' : B \quad E \vdash A' <: A}{E \vdash (\lambda(X <: A)b)(A') \leftrightarrow b'\{X \leftarrow A'\} : B\{X \leftarrow A'\}}$$

2.3. *Basic Properties*

We now state some basic lemmas about $F_{<:}$ derivations. Most of these are proven by (simultaneous) induction on the size of the derivations; the proofs are long, but straightforward if carried out in the order indicated. We conclude the section with an application of these lemmas, showing that typing is preserved under β–η-reductions.

Notation. Let $\mathcal{J}$ stand for either C *type*, $C <: C'$, $c:C$, or $c \leftrightarrow c' : C$.

LEMMA (Renaming). *Assume* $Y \notin dom(E, X <: D, E')$:

$$\vdash E, X <: D, E' \; env \Rightarrow \vdash E, Y <: D, E'\{X \leftarrow Y\} \; env$$

(*equal-size derivations*)

$$E, X <: D, E' \vdash \vartheta \Rightarrow E, Y <: D, E'\{X \leftarrow Y\} \vdash \vartheta\{X \leftarrow Y\}$$

(*equal-size derivations*).

Assume $y \notin dom(E, x:D, E')$:

$\vdash E, x:D, E' \; env \Rightarrow \vdash E, y:D, E' \; env$ (*equal-size derivations*)

$E, x:D, E' \vdash \vartheta \Rightarrow E, y:D, E' \vdash \vartheta\{x \leftarrow y\}$ (*equal-size derivations*).

LEMMA (Implied judgments).

(ϑ/env) $\vdash E, F \; env \Rightarrow \vdash E \; env$

$E, F \vdash \vartheta \Rightarrow \vdash E \; env.$

$(env/type)$ $\vdash E, X <: D, E' \; env \Rightarrow E \vdash D \; type$

$\vdash E, x:D, E' \; env \Rightarrow E \vdash D \; type.$

LEMMA (Bound change).

$$\vdash E, X <: D', E' \; env, \; E \vdash D \; type \Rightarrow \vdash E, X <: D, E' \; env$$

$$E, X <: D', E' \vdash C \; type, \; E \vdash D \; type \Rightarrow E, X <: D, E' \vdash C \; type.$$

LEMMA (Weakening). *Let* β *stand for either* $X <: D$ *or* $x:D$. *Assume* $\vdash E, \beta \; env$, *and* $X, x \notin dom(E')$:

$$\vdash E, E' \; env \Rightarrow \vdash E, \beta, E' \; env$$

$$E, E' \vdash \vartheta \Rightarrow E, \beta, E' \vdash \vartheta.$$

LEMMA (Multiple weakening). *Assume* $\vdash E, F \; env$ *and* $dom(F) \cap dom(E') = \phi$:

$$\vdash E, E' \; env \Rightarrow \vdash E, F, E' \; env$$

$$E, E' \vdash \vartheta \Rightarrow E, F, E' \vdash \vartheta.$$

Proof. Induction on the length of F. ∎

LEMMA (Implied judgments, continued).

$$(sub/type) \quad E \vdash C <: C' \Rightarrow E \vdash C\ type,\ E \vdash C'\ type.$$

LEMMA (Bound weakening). *Let* $\langle \beta, \beta' \rangle$ *stand for either* $\langle X <: D, X <: D' \rangle$ *or* $\langle x:D, x:D' \rangle$.
Assume $E \vdash D' <: D$:

$$\vdash E, \beta, E'\ env \Rightarrow \vdash E, \beta', E'\ env$$
$$E, \beta, E' \vdash \vartheta \Rightarrow E, \beta', E' \vdash \vartheta.$$

LEMMA (Type substitution). *Assume* $E \vdash D' <: D$; *then*

$$\vdash E, X <: D, E'\ env \Rightarrow \vdash E, E'\{X \leftarrow D'\}\ env$$
$$E, X <: D, E' \vdash \vartheta \Rightarrow E, E'\{X \leftarrow D'\} \vdash \vartheta\{X \leftarrow D'\}.$$

LEMMA (Value substitution). *Assume* $E \vdash d:D$; *then*

$$\vdash E, x:D, E'\ env \Rightarrow \vdash E, E'\ env$$
$$E, x:D, E' \vdash \vartheta \Rightarrow E, E' \vdash \vartheta\{x \leftarrow d\}.$$

LEMMA (Value strengthening). *Assume* $x \notin FV(\vartheta)$; *then, for* $\vartheta \neq c \leftrightarrow c':C$,

$$\vdash E, x:D, E'\ env \Rightarrow \vdash E, E'\ env$$
$$E, x:D, E' \vdash \vartheta \Rightarrow E, E' \vdash \vartheta.$$

LEMMA (Implied judgments, continued).

$$(val/type) \quad E \vdash c:C \Rightarrow E \vdash C\ type,$$
$$(eq/val) \quad E \vdash c \leftrightarrow c':C \Rightarrow E \vdash c:C,\ E \vdash c':C.$$

LEMMA (Eq subsumption).

$$E \vdash c \leftrightarrow c':C,\ E \vdash C <: D \Rightarrow E \vdash c \leftrightarrow c':D.$$

Proof. By sub/type lemma, $E \vdash C\ type$. Take $x \notin dom(E)$.
Then $\vdash E, x:C\ env$ and $E, x:C \vdash x:C$.
By weakening lemma $E, x:C \vdash C <: D$.
By (*Subsumption*) $E, x:C \vdash x:D$, and by (*Eq x*), $E, x:C \vdash x \leftrightarrow x:D$.
By (*Eq fun*), $E \vdash \lambda(x:C)x \leftrightarrow \lambda(x:C)x:C \to D$.
By hypothesis and (*Eq appl*), $E \vdash (\lambda(x:C)x)(c) \leftrightarrow (\lambda(x:C)x)(c'):D$.

By (*Eq beta*), $E \vdash (\lambda(x:C)x)(c) \leftrightarrow c':D$.
By (*Eq symm*)(*Eq beta*), $E \vdash (\lambda(x:C)x)(c') \leftrightarrow c:D$.
Hence by (*Eq symm*)(*Eq trans*), $E \vdash c \leftrightarrow c':D$. ∎

LEMMA (Implied judgments, continued).

$$(val/eq) \quad E \vdash c:C \Rightarrow E \vdash c \leftrightarrow c:C.$$

LEMMA (Congruence).

$$E \vdash d \leftrightarrow d':D \wedge E, x:D, E' \vdash c:C \Rightarrow E, E' \vdash c\{x \leftarrow d\} \leftrightarrow c\{x \leftarrow d'\}:C.$$

LEMMA (Exchange). *Let* β *stand for either* $X <: D$ *or* $x:D$.
Let β' *stand for either* $X' <: D'$ *or* $x':D'$.
Assume $\vdash E, \beta'$ *env*:

$$\vdash E, \beta, \beta', E' \text{ env} \Rightarrow \vdash E, \beta', \beta, E' \text{ env}$$

$$E, \beta, \beta', E' \vdash \mathcal{J} \Rightarrow E, \beta', \beta, E' \vdash \mathcal{J}.$$

LEMMA (Substitution exchange). *Let* β *stand for either* $x':D'$ *or* $X' <: D'$:

$$\vdash E, X <: D, \beta, E' \text{ env} \Rightarrow \vdash E, \beta\{X \leftarrow D\}, X <: D, E' \text{ env}$$

$$E, X <: D, \beta, E' \vdash C \text{ type} \Rightarrow E, \beta\{X \leftarrow D\}, X <: D, E' \vdash C \text{ type}.$$

The following two lemmas draw conclusions about the shape of terms and derivations from the fact that certain subtyping and typing judgments have been derived.

LEMMA (Subtyping decomposition).

- *If* $E \vdash A <: X$, *then* $A \equiv Y_1$ *for some type variable* Y_1 *and either* $Y_1 \equiv X$, *or for some* $n \geqslant 1$, $Y_1 <: Y_2 \in E \cdots Y_n <: X \in E$.
- *If* $E, X <: B, E' \vdash X <: A$, *then either* $A \equiv X$ *or* $E, X <: B, E' \vdash B <: A$.
- *If* $E \vdash Top <: A$, *then* $A \equiv Top$.
- *If* $E \vdash B' \to B'' <: A$, *then either* $A \equiv Top$ *or* $A \equiv A' \to A''$, $E \vdash A' <: B'$ *and* $E \vdash B'' <: A''$.
- *If* $E \vdash A <: B' \to B''$, *then*

  either $A \equiv A' \to A''$ *for some* A', A'', *with* $E \vdash B' <: A'$ *and* $E \vdash A'' <: B''$

or $A \equiv X_1$ *and for some* $A', A'', n \geqslant 1$: $X_1 <: X_2 \in E \cdots X_n <: A' \to A'' \in E$ *with* $E \vdash B' <: A'$ *and* $E \vdash A'' <: B''$.

- *If* $E \vdash \forall(X <: B')B'' <: A$, *then either* $A \equiv Top$ *or* $A \equiv \forall(X <: A')A''$, $E \vdash A' <: B'$ *and* $E, X <: A' \vdash B'' <: A''$.
- *If* $E \vdash A <: \forall(X <: B')B''$, *then*

either $A \equiv \forall(X <: A')A''$ *for some* A', A'' *with* $E \vdash B' <: A'$ *and* $E, X <: B' \vdash A'' <: B''$

or $A \equiv X_1$ *and for some* $A', A'', n \geqslant 1$: $X_1 <: X_2 \in E \cdots X_n <: \forall(X <: A')A'' \in E$ *with* $E \vdash B' <: A'$ *and* $E, X <: B' \vdash A'' <: B''$.

Proof (Sketch). All cases are proven by induction on the size of the derivations, in order to circumvent the (*Sub refl*) and (*Sub trans*) rules that do not follow the structure of terms. Otherwise the proofs are straightforward. ▮

LEMMA (Typing decomposition).

- *If* $E, x:D, E' \vdash x:C$, *then* $E \vdash D <: C$.
- *If* $E \vdash top:A$, *then* $A \equiv Top$.
- *If* $E \vdash \lambda(x:B')b:A$, *then either* $A \equiv Top$, *or, for some* A', A'', B'', $A \equiv A' \to A''$, *with* $E \vdash A' <: B'$, $E \vdash B'' <: A''$, *and* $E, x:B' \vdash b:B''$.
- *If* $E \vdash b(c):B''$ *then for some* B', $E \vdash b:B' \to B''$ *and* $E \vdash c:B'$.
- *If* $E \vdash \lambda(X <: B')b:A$, *then either* $A \equiv Top$, *or, for some* A', A'', B'', $A \equiv \forall(X <: A')A''$, *with* $E \vdash A' <: B'$, $E, X <: A' \vdash B'' <: A''$, *and* $E, X <: B' \vdash b:B''$.
- *If* $E \vdash b(C):D$ *then for some* B', B'', X, $E \vdash C <: B'$, $E \vdash B''\{X \leftarrow C\} <: D$, *and* $E \vdash b:\forall(X <: B')B''$.

Proof (Sketch). All cases are proven by induction on the size of the derivations, in order to circumvent the (*Subsumption*) rule that does not follow the structure of terms. Otherwise the proofs are straightforward. ▮

We conclude with a proposition about the preservation of typing under β and η reduction. The second-order η case is by far the hardest, and it requires the following lemma about the elimination of unused free variables (FV).

LEMMA (Non-occurring type variable). *If* $X \notin FV(c, E')$ *and* $E, X <: D, E' \vdash c:C$, *then for some* C_0 *with* $X \notin FV(C_0)$, $E, X <: D, E' \vdash c:C_0$ *and* $E, X <: D, E' \vdash C_0 <: C$.

Proof. By induction on the derivation of $E, X <: D, E' \vdash c : C$. The interesting cases are (*Val appl*) and (*Val appl2*), where we use the subtyping decomposition lemmas for $\rightarrow$ and $\forall$. We show the (*Val appl2*) *case*, where we have:

$$c \equiv b(A'),\ C \equiv B\{Y \leftarrow A'\} \qquad (\text{for } Y \notin dom(E, X <: D, E'))$$
$$E, X <: D, E' \vdash b : \forall(Y <: A)B, \qquad E, X <: D, E' \vdash A' <: A.$$

Since $X \notin FV(b)$, by induction there is a type AB_0 with $X \notin FV(AB_0)$, and

$$E, X <: D, E' \vdash b : AB_0, \qquad E, X <: D, E' \vdash AB_0 <: \forall(Y <: A)B.$$

By the (subtyping decomposition lemma) $AB_0 \equiv \forall(Y <: A_0)B_0$ with:

either $AB_0 \equiv \forall(Y <: A_0)B_0$ for some A_0, B_0, with $E, X <: D, E' \vdash A <: A_0$ and $E, X <: D, E', Y <: A_0 \vdash B_0 <: B$, hence, $X \notin FV(\forall(Y <: A_0)B_0)$, $E, X <: D, E' \vdash b : \forall(Y <: A_0)B_0$

or $AB_0 \equiv X_1$ and for some A_0, B_0, $n \geqslant 1$,

$$X_1 <: X_2 \in E, X <: D, E' \cdots X_n <: \forall(Y <: A_0)B_0 \in E, X <: D, E',$$

with $E, X <: D, E' \vdash A <: A_0$ and $E, X <: D, E', Y <: A_0 \vdash B_0 <: B$.

If $X_n <: \forall(Y <: A_0)B_0 \in E$; $X \notin FV(\forall(Y <: A_0)B_0)$ since X comes after E.
If $X_n <: \forall(Y <: A_0)B_0 \equiv X <: D$; $X \notin FV(D \equiv \forall(Y <: A_0)B_0)$.
If $X_n <: \forall(Y <: A_0)B_0 \in E'$; $X \notin FV(\forall(Y <: A_0)B_0)$ by the hyp. $X \notin FV(E')$.

By n uses of (*Sub X*) and (*Subsumption*), $E, X <: D, E' \vdash b : \forall(Y <: A_0)B_0$. Hence, in both cases, by (*Sub Trans*), $E, X <: D, E' \vdash A' <: A_0$, and $E, X <: D, E' \vdash b(A') : B_0\{Y \leftarrow A'\}$, with $X \notin FV(B_0\{Y \leftarrow A'\})$. Moreover, from $E, X <: D, E', Y <: A_0 \vdash B_0 <: B$ by (bound weakening lemma) $E, X <: D, E', Y <: A' \vdash B_0 <: B$ and by (type substitution lemma) $E, X <: D, E' \vdash B_0\{Y \leftarrow A'\} <: B\{Y \leftarrow A'\}$.

Hence we can take $C_0 \equiv B_0\{Y \leftarrow A'\}$. ∎

PROPOSITION (Preservation of typing under β-η-reductions).

($\beta 1$) $E \vdash (\lambda(x : B)b)(c) : A \Rightarrow E \vdash b\{x \leftarrow c\} : A$

($\eta 1$) $E \vdash \lambda(x : B)\, c(x) : A,\ x \notin FV(c) \Rightarrow E \vdash c : A$

($\beta 2$) $E \vdash (\lambda(X <: B)b)(C) : A \Rightarrow E \vdash b\{X \leftarrow C\} : A$

($\eta 2$) $E \vdash \lambda(X <: B)\, c(X) : A,\ X \notin FV(c) \Rightarrow E \vdash c : A$.

Proof. The first three cases are obtained easily by applying the appropriate decomposition lemmas, along with weakening, bound weakening, value and type substitution, and value strengthening.

The $(\eta 2)$ case goes as follows. From $E \vdash \lambda(X <: B)\, c(X) : A$ by the (typing decomposition lemma) for *fun2* and *appl2*, we obtain (omitting the easy case of $A \equiv Top$), for some A', A'', B'', Y, C', C'':

$$A \equiv \forall(X <: A')A'' \qquad \text{with} \quad E \vdash A' <: B',\ E, X <: A' \vdash B'' <: A'',$$
$$\text{and} \qquad E, X <: B' \vdash c(X) : B''$$
$$E, X <: B' \vdash c : \forall(Y <: C')C'' \qquad \text{with} \quad E, X <: B' \vdash X <: C'$$
$$\text{and} \qquad E, X <: B' \vdash C''\{Y \leftarrow X\} <: B''.$$

Since $X \notin FV(c)$, by the (nonoccurring type variable lemma) there is a D with

$$X \notin FV(D) \text{ and } E, X <: B' \vdash c : D,\ E, X <: B' \vdash D <: \forall(Y <: C')C''.$$

Using the (subtyping decomposition lemma) on D we obtain two subcases that, for some D', D'', both lead to

$$E, X <: B' \vdash c : \forall(Y <: D')D'', \qquad X \notin FV(\forall(Y <: D')D'')$$
$$\text{with } E, X <: B' \vdash C' <: D' \text{ and } E, X <: B', Y <: C' \vdash D'' <: C''.$$

By the (type strengthening lemma) from $E, X <: B' \vdash c : \forall(Y <: D')D''$:

$$E \vdash c : \forall(Y <: D')D''; \qquad \text{i.e.,} \quad E \vdash c : \forall(X <: D')\, D''\{Y \leftarrow X\}.$$

Now, to obtain the final goal $E \vdash c : \forall(X <: A')A''$ via subsumption, we need to show only that $E \vdash \forall(X <: D')\, D''\{Y \leftarrow X\} <: \forall(X <: A')A''$; i.e., that

(1) $E \vdash A' <: D'$

(2) $E, X <: A' \vdash D''\{Y \leftarrow X\} <: A''$.

For (1) we use the (type substitution lemma) to get

$$E \vdash B' <: C'\{X \leftarrow B'\} \qquad \text{(from } E, X <: B' \vdash X <: C')$$
$$E \vdash C'\{X \leftarrow B'\} <: D'\{X \leftarrow B'\} \equiv D' \qquad \text{(from } E, X <: B' \vdash C' <: D').$$

Hence $E \vdash A' <: B' <: C'\{X \leftarrow B'\} <: D'$.

For (2) we use the (bound weakening lemma) twice to get:

$$E, X <: A', Y <: X \vdash D'' <: C''$$
$$\text{(from } E, X <: B', Y <: C' \vdash D'' <: C'',$$
$$E, X <: B' \vdash X <: C',\ E \vdash A' <: B');$$

from this by the (type substitution lemma)

$$E, X <: A' \vdash D''\{Y \leftarrow X\} <: C''\{Y \leftarrow X\}.$$

We also have, by the (bound weakening lemma):

$$E, X <: A' \vdash C''\{Y \leftarrow X\} <: B''$$

(from $E, X <: B' \vdash C''\{Y \leftarrow X\} <: B''$, $E \vdash A' <: B'$).

Finally, $E, X <: A' \vdash D''\{Y \leftarrow X\} <: C''\{Y \leftarrow X\} <: B'' <: A''$. ∎

Note that this proposition is nontrivial; for example, the ($\beta 1$) case does not follow simply from the (*Eq beta*) rule and the eq/val lemma. Moreover, the derivation of $E \vdash b\{x \leftarrow c\} : A$ will have, in general, quite a different shape than the derivation of $E \vdash (\lambda(x:B)b)(c) : A$.

2.4. *Derived Rules*

Most of the lemmas in the previous section can be written down as derived inference rules. Here we discuss some derived rules of special significance.

First, the eq-subsumption lemma in the previous section gives us a very interesting rule that lifts subsumption to the equality judgment. We remark that this is proven via the (*Eq beta*) rule:

(*Eq subsumption*)

$$\frac{E \vdash a \leftrightarrow a' : A \quad E \vdash A <: B}{E \vdash a \leftrightarrow a' : B}$$

Note that, in general, it is not true that $E \vdash a \leftrightarrow a' : B$ and $E \vdash A <: B$ imply $E \vdash a \leftrightarrow a' : A$.

The following two lemmas concern the equivalence of functions modulo domain restriction; the first one will find a useful application in Section 3.1.

LEMMA (Domain restriction). *If $f : A \to B$, then f is equivalent to its restriction $f|_{A'}$ to a smaller domain $A' <: A$, when they are both seen at type $A' \to B$. That is,*

(*Eq fun'*)

$$\frac{E \vdash A' <: A \quad E \vdash B <: B' \quad E, x:A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(x:A)b \leftrightarrow \lambda(x:A')b' : A' \to B'}$$

Proof (Sketch). First derive $E \vdash \lambda(y:A')(\lambda(x:A)b)(y) \leftrightarrow \lambda(x:A')b' : A' \to B'$ via (*Eq-subsumption*) and (*Eq beta*). Then pass from $E \vdash \lambda(x:A)b \leftrightarrow \lambda(x:A)b : A \to B$ to $E \vdash \lambda(x:A)b \leftrightarrow \lambda(x:A)b : A' \to B'$ by (*Eq subsumption*), and to $E \vdash \lambda(y:A')(\lambda(x:A)b)(y) \leftrightarrow \lambda(x:A)b : A' \to B'$ by (*Eq eta*). Conclude by transitivity. ∎

LEMMA (Bound restriction). *If* $f: \forall(X <: A)B$, *then* f *is equivalent to its restriction* $f|_{A'}$ *to a smaller bound* $A' <: A$, *when they are both seen at type* $\forall(X <: A')B$. *That is*:

(*Eq fun2'*)

$$\frac{E \vdash A' <: A \quad E, X <: A' \vdash B <: B' \quad E, X <: A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(X <: A)b \leftrightarrow \lambda(X <: A')b' : \forall(X <: A')B'}$$

Proof. Similar to the previous lemma, using (*Eq beta2*) and (*Eq eta2*). ∎

We now turn to the (*Eq appl2*) rule. This rule asserts that if a polymorphic function $b:\forall(X <: A)B$ is instantiated at two types $A' <: A$ and $A'' <: A$, then both instantiations evaluate to the same value with respect to any result type that is an upper bound of $B\{X \leftarrow A'\}$ and $B\{X \leftarrow A''\}$:

(*Eq appl2*)

$$\frac{\begin{array}{c} E \vdash b' \leftrightarrow b'' : \forall(X <: A)B \quad E \vdash A' <: A \quad E \vdash A'' <: A \\ E \vdash B\{X \leftarrow A'\} <: C \quad E \vdash B\{X \leftarrow A''\} <: C \end{array}}{E \vdash b'(A') \leftrightarrow b''(A'') : C}$$

Note that this rule asserts that the result of $b(A)$ is independent of A, in the proper result type.

A simpler derived rule (used in $F_{\leqslant}$ [CG 91]) is obtained by setting $A' = A''$:

(*Eq appl2* $A' = A''$)

$$\frac{E \vdash b' \leftrightarrow b'' : \forall(X <: A)B \quad E \vdash A' <: A}{E \vdash b'(A') \leftrightarrow b''(A') : B\{X \leftarrow A'\}}$$

However, the (*Eq appl2*) rule is most useful when $A' \neq A''$ and we can find a nontrivial upper bound to $B\{X \leftarrow A'\}$ and $B\{X \leftarrow A''\}$. This fact motivates the following derived rule, which is often used in practice.

Denote by $B\{X \leftarrow C, X^+ \leftarrow D\}$ the substitution of C for the negative occurrences of X in B, and of D for the positive ones. Take $A' <: A''$ $(<: A)$; then we have

$$B\{X \leftarrow A'\} \equiv B\{X \leftarrow A', X^+ \leftarrow A'\} <: B\{X \leftarrow A', X^+ \leftarrow A''\}$$

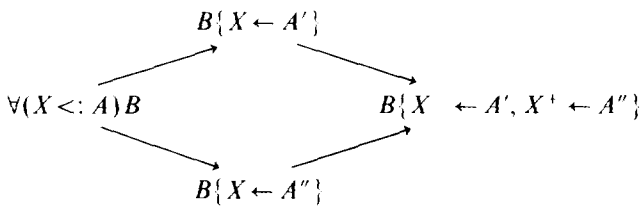
$$B\{X \leftarrow A''\} \equiv B\{X \leftarrow A'', X^+ \leftarrow A''\} <: B\{X \leftarrow A', X^+ \leftarrow A''\}$$

(A proof of this may be found in [Ghe 90, Section 14.3].) Hence, for $A' <: A'' <: A$ we have a (nontrivial) common supertype for $B\{X \leftarrow A'\}$ and $B\{X \leftarrow A''\}$. This fact then justifies the rule

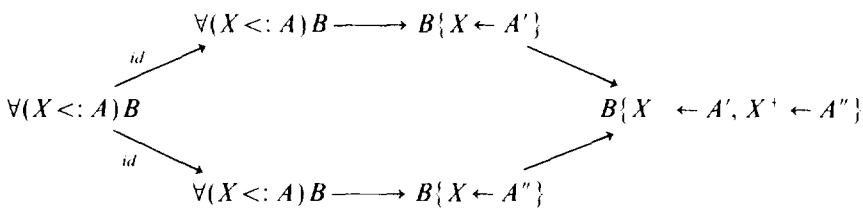
$$(Eq\ appl2\ ')$$

$$\frac{E \vdash b' \leftrightarrow b'' : \forall(X <: A)B \quad E \vdash A' <: A'' <: A}{E \vdash b'(A') \leftrightarrow b''(A'') : B\{X \leftarrow A', X^+ \leftarrow A''\}}$$

This rule is in fact a special case of *dinaturality* of type application [BFSS 90], where the dinaturality is required only with respect to coercions $A' <: A''$, for all A', A'' subtypes of A. We have the diagram



The two arrows on the left are the A' and A'' instances of generic type application $x(X)$, where x is a variable of type $\forall(X <: A)B$, and B might have the type variable X free. The two arrows on the right are coercions induced by $A' <: A''$. Here $\forall(X <: A)B$ is constant in X, so the coercion $A' <: A''$ has no effect on this type. Hence the diagram above is just a brief version of



where now the two horizontal arrows are the A' and A'' instances of $x(X)$. In the terminology of [BFSS 90, p. 42], the family given by $\{x(X) | X <: A\}$ is dinatural in the coercions.

We conclude this section with an application of *(Eq appl2)*, which is used in Sections 3.6 and 4.1.

PROPOSITION (Eq-substitution). *Assume* $E, X <: A, x:S \vdash b:B$ *and* X *positive in* S *and* B.

If $E \vdash A_1, A_2 <: A$, $E \vdash s_1 : S\{X \leftarrow A_1\}$, $E \vdash s_2 : S\{X \leftarrow A_2\}$, $E \vdash s_1 \leftrightarrow s_2 : S\{X \leftarrow A\}$ *then* $E \vdash b\{X \leftarrow A_1, x \leftarrow s_1\} \leftrightarrow b\{X \leftarrow A_2, x \leftarrow s_2\} : B\{X \leftarrow A\}$.

Proof. Let $M \triangleq \lambda(X <: A)\, \lambda(x:S) b$. Then $E \vdash M : \forall(X <: A) S \to B$. Now prove:

(1) $E \vdash M(A_1)(s_1) \leftrightarrow M(A)(s_1) : B\{X \leftarrow A\}$, by *(Eq appl2)* and *(Eq appl)*, since X is positive in S and B.

(2) $E \vdash M(A_2)(s_2) \leftrightarrow M(A)(s_2) : B\{X \leftarrow A\}$ similarly to (1).

(3) $E \vdash M(A)(s_1) \leftrightarrow M(A)(s_2) : B\{X \leftarrow A\}$ by *(Eq appl2)* and *(Eq appl)*, since $E \vdash s_1 \leftrightarrow s_2 : S\{X \leftarrow A\}$.

Conclude by *(Eq trans)*, *(Beta2)*, and *(Beta)*. ∎

The proposition can be easily generalized to the case where there are several variables $x_1:S_1, ..., x_n:S_n$ (X positive in all of them) and terms $E \vdash s_1 : S\{X \leftarrow A_1\}, ..., E \vdash s_n : S\{X \leftarrow A_n\}$, with $E \vdash A_1, ..., A_n <: A$ and $E \vdash s_1 \leftrightarrow \cdots \leftrightarrow s_n : S\{X \leftarrow A\}$.

2.5. *PER Semantics*

For the PER semantics, the reader can consult [BL 88, CL 91, Ghe 90, Sce 90]. The interpretation of $F_{<:}$ in PER is explained in those papers, except that the rule *(Eq appl2)* must be shown to be sound. The proof rests on the fact that, given types $\forall(X <: A) B$ and $A' <: A$ and denoting by $[\![-]\!]$ the interpretation function for types, we have $[\![\forall(X <: A) B]\!] \subseteq [\![B\{X \leftarrow A'\}]\!]$. From this, and the observation that the interpretation for terms is given by erasing the type information, the conclusion is straightforward.

2.6. *Conservativity of Typing*

Besides the presence of subtypes, the main new feature of $F_{<:}$ with respect to F lies in its equational theory, which extends the standard β–η equality in two directions, by adding a terminal type *Top* and introducing the rule *(Eq appl2)*. Besides nonessential syntactic variants, the language of

F is included in $F_{<:}$'s language and thus it makes sense to investigate whether $F_{<:}$ is conservative over F. We may, however, consider also an intermediate system between F and $F_{<:}$, with the property that the language inclusion of F into $F_{<:}$ "splits."

The system we are interested in is F_1, obtained by adding to F the type constant *Top*, together with rule (*Eq collapse*) for making *Top* a terminal type. If we want to compare $F_{<:}$ with its underlying subtype-free systems, we need a system such as F_1, and not F, since it is well known that the terminal type is not definable in F. Moreover, the conservativity result we prove with respect to F holds because $F_{<:}$ proves only trivial subtype judgments between F types, while the situation for F_1 is more complex and its analysis sheds some more light on the structure of subtype proofs.

First of all, the equational theory ($\leftrightarrow$) of $F_{<:}$ is not conservative over F, because of the rule (*Eq appl2*). Consider, for example:

PROPOSITION.

$$E \vdash B\ type, \quad E \vdash c : \forall(X) X \to B, \quad E \vdash a : A$$
$$\Rightarrow E \vdash c(Top)(top) \leftrightarrow c(A)(a) : B.$$

Proof.

$E \vdash c(Top)(top) \leftrightarrow c(Top)(a) : B$	val/eq lemma (*Eq appl2*)(*Eq collapse*)(*Eq appl*)
$E \vdash c(Top)(a) \leftrightarrow c(A)(a) : B$	val/eq lemma (*Eq appl2*)(*Eq appl*)
$E \vdash c(Top)(top) \leftrightarrow c(A)(a) : B$	(*Eq trans*). ∎

By applying this fact twice via (*Eq trans*) we can show that

$$y : \forall(X) X \to Bool \vdash y(Bool)(true) \leftrightarrow y(Bool)(false) : Bool,$$

which is an F-judgment equating two different $\beta\eta$-normal forms. It is well known that no such judgment is derivable in F. A further application of (*Eq fun*) produces two closed terms with the same property.

As for the *typing* theory, however, $F_{<:}$'s rules are designed to maintain and carefully generalize those of its subsystems. Writing $\vdash_F$ for derivations in F, $\vdash_1$ for derivations in F_1, and $\vdash_{<:}$ for derivations in $F_{<:}$, we can prove the following result.

THEOREM. (i) *If* $E \vdash_{<:} a : A$, *where* E, a, *and* A *are in the language of* F, *then* $E \vdash_F a : A$.

(ii) *If* $E \vdash_{<:} a : A$, *where* E, a, *and* A *are in the language of* F_1, *then there exists an* F_1*-term*, $a1$, *such that* $E \vdash_1 a1 : A$ *and* $E \vdash_{<:} a \leftrightarrow a1 : A$.

The proof of these statements (inspired by some results in [Ghe 90]) requires a detour on *normal form proofs* in $F_{<:}$. These normal forms are studied in [CG 91] for a slightly different system, but which shares with $F_{<:}$ the same typing judgments. The reason for the detour is that trivial proofs by induction on the derivation of $E \vdash_{<:} a:A$ do not work, since $F_{<:}$ has "cut rules" (e.g., (*Subsumption*), (*Sub trans*), or (*Val appl*)) that may introduce non-F (or non-F_1) types.

2.6.1. *Normal and minimal proofs in* $F_{<:}$

In $F_{<:}$ a single typing judgment may have many proofs. The nondeterminism of the proof search arises from the freedom in the order in which the rules (*Subsumption*) and (*Sub trans*) can be applied. However, as shown in [CG 91], this freedom does not provide additional proving power. In subtype proofs we can do without (*Sub trans*) except for the uses where the first (i.e., smallest) type is a variable appearing in the environment. In type proofs, we can restrict the use of (*Subsumption*) so as to derive only the *least* type for a given term, which may be then given a larger type with a single, last application of (*Subsumption*). These ideas are the inspiration for the notions of normal and minimal normal proofs.

Subtype Proofs. A *normal form proof* of $\vdash_{<:} A <: B$ is a proof of $E \vdash_{\mathrm{nf}} A <: B$ obtained in the formal system $\vdash_{\mathrm{nf}}$ consisting of the rules (*Sub Top*), (*Sub* $\to$), (*Sub* $\forall$) (where $\vdash_{<:}$ is replaced by $\vdash_{\mathrm{nf}}$), plus the following rules:

$$\text{(Sub Refl-X)} \qquad \frac{E \vdash_{\mathrm{nf}} X \text{ type}}{E \vdash_{\mathrm{nf}} X <: X}$$

$$\text{(Sub Trans-X)} \qquad \frac{E', X <: B, E'' \vdash_{\mathrm{nf}} B <: A}{E', X <: B, E'' \vdash_{\mathrm{nf}} X <: A} \quad A \not\equiv Top$$

Type Proofs. Normal form proofs and minimal normal form proofs of $E \vdash_{<:} a:A$ are simultaneously defined as follows.

A *normal form proof* $E \vdash_{\mathrm{nf}} a:A$ is either (1) a minimal normal form proof $E \vdash_{\mathrm{mnf}} a:A$, or (2) a minimal normal form proof followed by a single nontrivial use of subsumption; in this case the final step has the form

$$\frac{E \vdash_{\mathrm{mnf}} a:A' \quad E \vdash_{\mathrm{nf}} A' <: A}{E \vdash_{\mathrm{nf}} a:A} \qquad \text{where} \quad A' \not\equiv A.$$

A *minimal normal form proof* $E \vdash_{\mathrm{mnf}} a:A$ is a proof using only the rules (*Val x*), (*Val top*), (*Val fun*), (*Val fun2*) (where $\vdash_{<:}$ is replaced by $\vdash_{\mathrm{mnf}}$), or one of the two rules below, which use the following notation:

- $E(X) \equiv A$ if $E \equiv E_1, X <: A, E_2$.
- $E^*(C) \equiv C$ if C is not a variable;

  $E^*(X) \equiv E(X)$ if $E(X)$ is not a variable,

  $E^*(X) \equiv E1^*(E(X))$ if $E(X)$ is a variable and $E \equiv E1, X <: A, E2$.

(*Val appl-min*)

$$\frac{E \vdash_{\text{mnf}} b : C \quad E \vdash_{\text{nf}} a : A}{E \vdash_{\text{mnf}} b(a) : B} \qquad E^*(C) \equiv A \to B$$

(*Val appl2-min*)

$$\frac{E \vdash_{\text{mnf}} b : C \quad E \vdash_{\text{nf}} A' <: A}{E \vdash_{\text{mnf}} b(A') : B\{X \leftarrow A'\}} \qquad E^*(C) \equiv \forall(X <: A)B.$$

PROPOSITION. *For any provable judgment* $E \vdash_{<:} a : A$, *there exists a* unique *derivation of* $E \vdash_{\text{nf}} a : A$.

Proof. [CG 91]. ∎

2.6.2. $F_{<:}$ *Typing is Conservative over F Typing*

It is not difficult to see F as a subsystem of $F_{<:}$. We can define a translation function τ over the language of F so that

$$\tau(\forall X.A) \equiv \forall(X <: Top)\,\tau(A)$$
$$\tau(\Lambda X.M) \equiv \lambda(X <: Top)\,\tau(M)$$

and which is trivially defined on all the other constructs. A well-formed environment E in F consists of a collection $E1 \equiv X_1, ..., X_h$ of type variables and a list $E2 \equiv x_1 : S_1, ..., x_h : S_h$ of type assumptions, where at most the type variables in $E1$ can appear free. Then

$$\tau(E) \equiv X_1 <: Top, ..., X_h <: Top, x_1 : \tau(S_1), ..., x_h : \tau(S_h).$$

From this, it is almost obvious that F-derivations $E \vdash_F a : A$ and $E \vdash_F a \leftrightarrow a' : A$ are mapped to $F_{<:}$-derivations $\tau(E) \vdash \tau(a) : \tau(A)$ and $\tau(E) \vdash \tau(a) \leftrightarrow \tau(a') : \tau(A)$ with the following properties. The resulting derivations never use (*Subsumption*) (and thus subtyping rules) or *Top* rules, and (*Eq appl2*) is always applied in its special case when $A' \equiv A''$ and $C \equiv B\{X \leftarrow A'\}$. In the following we argue directly in the language of $F_{<:}$ (thus dispensing with τ).

LEMMA. *Let* E *be an F-environment, and let* A *and* B *be F-types.* $E \vdash_{<:} A <: B$ *iff* $A \equiv B$.

Proof. The "if" direction is a routine induction. For the other direction, take the normal form proof of $E \vdash_{<:} A <: B$. Then, (*Sub* $\rightarrow$) and (*Sub* $\forall$) proceed by induction, and (*Sub Refl-X*) is trivial. For (*Sub Trans-X*), $E \vdash_{\mathrm{nf}} X <: A$ must have been derived from $E', X <: Top, E'' \vdash_{\mathrm{nf}} Top <: A$, but the latter implies $A \equiv Top$ by the subtyping decomposition lemma, which is absurd since A is an F-type. ∎

LEMMA. *Let E be an F-environment, a be an F-term, and let $E \vdash_{\mathrm{mnf}} a : A$. Then A is an F-type and $E \vdash_F a : A$.*

Proof. By induction on the derivation $E \vdash_{\mathrm{mnf}} a : A$.

(*Val x*) $E', x:A, E'' \vdash_{\mathrm{mnf}} x : A$. Then A is an F-type, since E is an F-environment.

(*Val fun*) The last rule is

$$\frac{E, x:A \vdash_{\mathrm{mnf}} b : B}{E \vdash_{\mathrm{mnf}} \lambda(x:A)b : A \rightarrow B}.$$

By hypothesis, $\lambda(x:A)b$ is an F-term and therefore A is an F-type.
By induction hypothesis, B is an F-type and $E, x:A \vdash_F b : B$.

(*Val fun2*) is analogous to (*Val fun*).

(*Val appl-min*) The last rule is

$$\frac{E \vdash_{\mathrm{mnf}} b : C \quad E \vdash_{\mathrm{nf}} a : A}{E \vdash_{\mathrm{mnf}} b(a) : B} \qquad E^*(C) \equiv A \rightarrow B.$$

Consider first the premise $E \vdash_{\mathrm{mnf}} b : C$.
We show that C cannot be a variable. Indeed, if it were the case that $C \equiv X$, then $E^*(C) \equiv E(X) \equiv Top$, since E is an F-environment, contrary to the side-condition that $E^*(C)$ has to be a function type.
Therefore C is not a variable, and $E^*(C) \equiv C \equiv A \rightarrow B$.
By the induction hypothesis, $A \rightarrow B$ is an F-type and $E \vdash_F b : A \rightarrow B$.
Consider now the proof $E \vdash_{\mathrm{nf}} a : A$. We claim it is actually a minimal normal form proof. In fact, we already proved that $A \rightarrow B$ is an F-type; hence A is an F-type. It is were the case that the last step of the proof $E \vdash_{\mathrm{nf}} a : A$ is

$$\frac{E \vdash_{\mathrm{mnf}} a : A' \quad E \vdash_{\mathrm{nf}} A' <: A}{E \vdash_{\mathrm{nf}} a : A}$$

with $A' \not\equiv A$, then, by the induction hypothesis, A' would be an F-type and $A' \equiv A$ by the previous lemma. Hence the proof $E \vdash_{\mathrm{nf}} a : A$ is a minimal normal proof $E \vdash_{\mathrm{mnf}} a : A$ and, by the induction hypothesis, $E \vdash_F a : A$.

(*Val appl2-min*) The last rule is

$$\frac{E \vdash_{\text{mnf}} b : C \quad E \vdash_{\text{nf}} A' <: A}{E \vdash_{\text{mnf}} b(A') : B\{X \leftarrow A'\}} \qquad E^*(C) \equiv \forall(X <: A)B$$

Note first that since $b(A')$ is an F-term, A' is an F-type. As in the previous case, C cannot be a variable, and $C \equiv \forall(X <: A)B$.

By the induction hypothesis, $\forall(X <: A)B$ is an F-type (thus $A \equiv Top$, making trivial the other premise $E \vdash_{\text{nf}} A' <: Top$) and $E \vdash_F b : \forall(X <: Top)B$.

Then $E \vdash_F b(A') : B\{X \leftarrow A'\}$. ∎

THEOREM (Conservativity of typing over F). *Let E be an F-environment, a be an F-term, and A be an F-type. Then*

$$E \vdash_{<:} a : A \Rightarrow E \vdash_F a : A.$$

Proof. Consider the unique normal form proof $E \vdash_{\text{nf}} a : A$. If its last step is:

$$\frac{E \vdash_{\text{mnf}} a : A' \quad E \vdash_{\text{nf}} A' <: A}{E \vdash_{\text{nf}} a : A}$$

with $A' \not\equiv A$, then, by the previous lemma, A' would be an F-type and $A' \equiv A$ by the other lemma. The proof $E \vdash_{\text{nf}} a : A$ is then a proof $E \vdash_{\text{mnf}} a : A$; the previous lemma allows us to obtain the conclusion. ∎

2.6.3. *$F_{<:}$ Typing Is Conservative "Modulo an Equality" over F_1 Typing*

As in the case of F, system F_1 can easily be viewed as a subsystem of $F_{<:}$. Consider the subsystem of $F_{<:}$ obtained by restricting (*Env X*) to the case where $A \equiv Top$, dropping all the subtyping rules but (*Sub Top*), removing (*Subsumption*), and restricting (*Eq appl2*) to the case where $A' \equiv A''$ and $C \equiv B\{X \leftarrow A'\}$. We will identify F_1 with this subsystem and write $\vdash_1$ for F_1-derivations.

The reason the typing theory of $F_{<:}$ is conservative over that of F (expressed in the first lemma of the previous subsection) is that only trivial subtype judgments $E \vdash_{<:} A <: B$ with $A \equiv B$ can be proved when A and B are F-types. The situation for F_1-types is more interesting, since, due to (*Sub Top*), nontrivial inclusions can be proved.

A first remark is that the typing of $F_{<:}$ is *not* conservative over that of F_1,

$$X <: Top, x : X \vdash_{<:} x : Top,$$

but, of course,

$$\neg(X <: Top,\ x:X \vdash_1 x:Top).$$

This failure is, indeed, one of the pragmatic reasons (from the programming language design viewpoint) for introducing (*Subsumption*), since this is the mechanism by which a program (method, function, ...) can be inherited in other types.

We can look, however, for conservativity modulo an $F_{<:}$-equality. If $E \vdash_{<:} a:A$, where E, a, and A are in the language of F_1, then there exists an F_1-term, $a1$ say, such that $E \vdash_1 a1:A$ and $E \vdash_{<:} a \leftrightarrow a1:A$. In the example above, it is obvious that $X <: Top, x:X \vdash_1 top:Top$ and $X <: Top, x:X \vdash_1 x \leftrightarrow top:Top$, by (*Eq Top*).

We start with some preliminary lemmas. Let

$$id \equiv \lambda(X <: Top)\,\lambda(x:X)x.$$

LEMMA (Identity coercions). *Let E be an F_1-environment, A and B be F_1-types, and $E \vdash_{<:} A <: B$. Then there exists an F_1-term $k_{A,B}$ such that*

$$E \vdash_1 k_{A,B}:A \to B \qquad \textit{and} \qquad E \vdash_{<:} k_{A,B} \leftrightarrow id(A):A \to B.$$

Proof. By induction on the normal form proof $E \vdash_{\mathrm{nf}} A <: B$.

Note first that (*Sub Trans-X*) cannot be the last rule of such a proof, because its premise would be that $E', X <: Top, E'' \vdash_{\mathrm{nf}} Top <: A$ (since E is an F_1-environment), which would imply $A \equiv Top$ by subtyping decomposition lemma, which is impossible because of the side condition requiring that $A \not\equiv Top$. In the other cases, we take $k_{A,B}$ as the (inductively defined) explicit coercion between A and B. Details are as follows.

(*Sub Refl-X*) is trivial.

(*Sub Top*) $E \vdash_{<:} A <: Top$. Take then $k_{A,Top} \equiv \lambda(x:A)top$. Rules (*Eq collapse*) and (*Eq fun*) give $E \vdash_{<:} k_{A,Top} \leftrightarrow id(A):A \to Top$.

(*Sub* $\to$) Define $k_{A \to B,\, A' \to B'} \equiv \lambda(f:A \to B)\, k_{B,B'} \circ f \circ k_{A',A}$. From $E \vdash_{\mathrm{nf}} A \to B <: A' \to B'$, by the induction hypothesis and an easy argument,

$$E,\ f:A \to B \vdash_{<:} \lambda(x:A')\, k_{B,B'}(f(k_{A',A}(x))) \leftrightarrow \lambda(x:A')\, f(x):A' \to B';$$

by (*Eq eta*) and transitivity,

$$E,\ f:A \to B \vdash_{<:} \lambda(x:A')\, k_{B,B'}(f(k_{A',A}(x))) \leftrightarrow f:A' \to B';$$

by (*Eq fun*),

$$E \vdash_{<:} \lambda(f:A \to B)\,\lambda(x:A')\, k_{B,B'}(f(k_{A',A}(x))) \leftrightarrow \lambda(f:A \to B) f:(A \to B) \to (A' \to B').$$

(*Sub* $\forall$) $E \vdash_{\text{nf}} \forall(X <: A)B <: \forall(X <: A')B'$ where $A \equiv A' \equiv Top$ because both $\forall(X <: A)B$ and $\forall(X <: A')B'$ are F_1-types. Let

$$C \equiv \forall(X <: Top)B \qquad \text{and} \qquad C' \equiv \forall(X <: Top)B'$$

and define

$$k_{C,C'} \equiv \lambda(x:C)\,\lambda(X <: Top)\,k_{B,B'}(x(X)).$$

From $E \vdash_{\text{nf}} C <: C'$, by induction and an easy argument,

$$E, x:C \vdash_{<:} \lambda(X <: Top)\,k_{B,B'}(x(X)) \leftrightarrow \lambda(X <: Top)\,x(X):C';$$

by (*Eq eta2*) and transitivity,

$$E, x:C \vdash_{<:} \lambda(X <: Top)\,k_{B,B'}(x(X)) \leftrightarrow x:C';$$

and hence the thesis, by (*Eq fun*). ∎

LEMMA. *Let E be an F_1-environment, a an F_1-term and $E \vdash_{\text{mnf}} a:A$. Then*

(i) *A is an F_1-type*

(ii) *there exists an F_1-term $a1$ such that $E \vdash_1 a1:A$ and $E \vdash_{<:} a \leftrightarrow a1:A$.*

Proof. By induction on $E \vdash_{\text{mnf}} a:A$.

(*Val x*) $E', x:A, E'' \vdash_{\text{mnf}} x:A$. Then A is an F_1-type, since E is an F_1-environment and $a1 \equiv x$; the conclusion (ii) follows by (*Eq x*).

(*Val top*) $E \vdash_{\text{mnf}} top:Top$. Then also $E \vdash_1 top:Top$ and we can take $a1 \equiv top$.

(*Val fun*) The last rule is

$$\frac{E, x:A \vdash_{\text{mnf}} b:B}{E \vdash_{\text{mnf}} \lambda(x:A)b:A \to B}.$$

By hypothesis, $\lambda(x:A)b$ is an F_1-term and therefore A is an F_1-type.

By the induction hypothesis, B is an F_1-type and there exists a term $b1$ such that $E, x:A \vdash_1 b1:B$ and $E, x:A \vdash_{<:} b \leftrightarrow b1:B$.

The thesis follows by (*Eq fun*).

(*Val fun2*) is analogous to (*Val fun*).

(*Val appl-min*) The last rule is

$$\frac{E \vdash_{\text{mnf}} b:C \quad E \vdash_{\text{nf}} a:A}{E \vdash_{\text{mnf}} b(a):B} \qquad E^*(C) \equiv A \to B.$$

Consider first the left premise, $E \vdash_{\text{mnf}} b : C$. We observe that C cannot be a variable X. If it were, since E is an F_1-environment, we would have $E^*(C) \equiv E(X) \equiv Top$, contradicting the assumption that $E^*(C) \equiv A \to B$. Thus, $C \equiv A \to B$, induction applies, $A \to B$ is an F_1-type, and we obtain an F_1-term $b1$ such that

$$E \vdash_1 b1 : A \to B \qquad \text{and} \qquad E \vdash_{<:} b \leftrightarrow b1 : A \to B.$$

Consider now the other premise, $E \vdash_{\text{nf}} a : A$. If it happens to be a minimal normal form proof $E \vdash_{\text{nf}} a : A$ then by the induction hypothesis we have a term $a1$ such that

$$E \vdash_1 a1 : A \qquad \text{and} \qquad E \vdash_{<:} a \leftrightarrow a1 : A.$$

Otherwise, the last step of $E \vdash_{\text{nf}} a : A$ is

$$\frac{E \vdash_{\text{mnf}} a : A' \quad E \vdash_{\text{nf}} A' <: A}{E \vdash_{\text{nf}} a : A}.$$

By the induction hypothesis, A' is an F_1-type and we get an F_1-term a' such that $E \vdash_1 a' : A$ and $E \vdash_{<:} a \leftrightarrow a' : A'$.

We already proved that $A \to B$ is an F_1-type; hence A is an F_1-type.

From $E \vdash_{\text{nf}} A' <: A$, the identity coercions lemma gives an F_1 term $k_{A',A}$ such that $E \vdash_1 k_{A',A} : A' \to A$ and $E \vdash_{<:} k_{A',A} \leftrightarrow id(A') : A' \to A$. Take then $a1 \equiv k_{A',A}(a')$. Simple computations give

$$E \vdash_1 a1 : A \qquad \text{and} \qquad E \vdash_{<:} a \leftrightarrow a1 : A.$$

Finally, by (*Eq appl*),

$$E \vdash_1 b1(a1) : B \qquad \text{and} \qquad E \vdash_{<:} b1(a1) \leftrightarrow b(a) : B.$$

(*Val appl2-min*) The last rule is

$$\frac{E \vdash_{\text{mnf}} b : C \quad E \vdash_{\text{nf}} A' <: A}{E \vdash_{\text{mnf}} b(A') : B\{X \leftarrow A'\}} \qquad E^*(C) \equiv \forall(X <: A)B.$$

Note, first, that since $b(A')$ is an F_1-term, A' is an F_1-type. As in the previous case, in $E \vdash_{\text{mnf}} b : C$, C cannot be a variable. Therefore, the left premise is $E \vdash_{\text{mnf}} b : \forall(X <: A)B$. By the induction hypothesis, $\forall(X <: A)B$ is an F_1-type (thus $A \equiv Top$ and the second premise is trivial) and we have an F_1-term $b1$ such that

$$E \vdash_1 b1 : \forall(X <: Top)B \qquad \text{and} \qquad E \vdash_{<:} b \leftrightarrow b1 : \forall(X <: Top)B.$$

Then $E \vdash_1 b1(A') : B\{X \leftarrow A'\}$ and $E \vdash_{<:} b(A') \leftrightarrow b1(A') : B\{X \leftarrow A'\}$. ∎

We can finally prove our conservativity result:

THEOREM (Conservativity of typing over F_1). *If* $E \vdash_{<:} a:A$, *where* E, a, *and* A *are in the language of* F_1, *then there exists an* F_1*-term*, $a1$, *such that* $E \vdash_1 a1:A$ *and* $E \vdash_{<:} a \leftrightarrow a1:A$.

Proof. Take the normal form proof $E \vdash_{nf} a:A$. If it is a minimal normal form proof, then the thesis follows by the previous lemma. If, on the other hand, it consists of a minimal normal form proof $E \vdash_{mnf} a:A'$ followed by subsumption with premise $E \vdash_{nf} A' <: A$, then, by the previous lemma, A' is an F_1-type and we have an F_1-term, a', such that $E \vdash_1 a':A'$ and $E \vdash_{<:} a \leftrightarrow a':A'$. The thesis then follows by the identity coercions lemma and (*Eq appl*). ∎

3. EXPRESSIVENESS

Since $F_{<:}$ is an extension of F, one can already carry out all the standard encodings of algebraic data types that are possible in F [BB 85]. However, it is not clear that anything of further interest can be obtained from the subtyping rules of $F_{<:}$, which involve only an apparently useless type *Top* and the simple rules for $\rightarrow$ and $\forall$. In this section we begin to show that we can in fact construct rich subtyping relations on familiar data structures.

3.1. *Booleans*

In the rest of Section 3 we concentrate on inclusion of structured types, but for this to make sense we need to show that there are some nontrivial inclusions already at the level of basic types. We investigate here the type of booleans, illustrating some consequences of the $F_{<:}$ rules.

Starting from the encoding of Church's booleans in F, we can define three subtypes of *Bool* (cf. [Fai 89]),

$$Bool \triangleq \forall(A)\, A \rightarrow A \rightarrow A$$

$$True \triangleq \forall(A)\, A \rightarrow Top \rightarrow A$$

$$False \triangleq \forall(A)\, Top \rightarrow A \rightarrow A$$

$$None \triangleq \forall(A)\, Top \rightarrow Top \rightarrow A,$$

where

$$None <: True,\ None <: False,\ True <: Bool,\ False <: Bool.$$

Looking at all the closed normal forms (that is, the *elements*) of these types, we have

$$true_{Bool}: Bool \triangleq \lambda(A)\,\lambda(x:A)\,\lambda(y:A)\,x$$

$$false_{Bool}: Bool \triangleq \lambda(A)\,\lambda(x:A)\,\lambda(y:A)\,y$$

$$true_{True}: True \triangleq \lambda(A)\,\lambda(x:A)\,\lambda(y:Top)\,x$$

$$false_{False}: False \triangleq \lambda(A)\,\lambda(x:Top)\,\lambda(y:A)\,y.$$

We obtain four elements of type *Bool*; in addition to the usual two, $true_{Bool}$ and $false_{Bool}$, the extra $true_{True}$ and $false_{False}$ have type *Bool* by subsumption. This is somewhat surprising because computationally there are only two booleans. Intuitively, if two arguments of an arbitrary type are given, there are only two ways of providing a result of that type. This coincides with the fact that by removing all the type information in the terms above, we obtain only two distinct untyped terms. Fortunately, we can show that $true_{Bool}$ and $true_{True}$ are provably equivalent at type *Bool*, by using the domain restriction lemma (*Eq fun'*) from Section 2.4:

$$\dfrac{\dfrac{\dfrac{\dfrac{E, A<:Top, x:A, y:Top \vdash x \leftrightarrow x:A \quad E \vdash A<:Top}{E, A<:Top, x:A \vdash \lambda(y:Top)x \leftrightarrow \lambda(y:A)x:A \to A}\,(Eq\,fun')}{E, A<:Top \vdash \lambda(x:A)\,\lambda(y:Top)x \leftrightarrow \lambda(x:A)\,\lambda(y:A)x:A \to A \to A}}{E \vdash \lambda(A)\,\lambda(x:A)\,\lambda(y:Top)x \leftrightarrow \lambda(A)\,\lambda(x:A)\,\lambda(y:A)x:\forall(A)A \to A \to A}}{E \vdash true_{True} \leftrightarrow true_{Bool}: Bool}$$

Similarly, we can show that $E \vdash false_{False} \leftrightarrow false_{Bool}: Bool$. Hence, there really are only two different values in *Bool*; one value each in *True* and *False*, and none in *None*.

3.2. *Naturals*

The encoding of booleans in the previous section does not seem to generalize to other algebraic types. A different style of encoding (which can also be applied to booleans) works better for naturals. In the following encoding, *Nat* stands for the type of naturals, Nat_z for the type of zero naturals (the singleton zero), and Nat_s for the type of nonzero naturals:

$$Nat \triangleq \forall(N)\,\forall(N_z<:N)\,\forall(N_s<:N)\,N_z \to (N \to N_s) \to N$$

$$Nat_z \triangleq \forall(N)\,\forall(N_z<:N)\,\forall(N_s<:N)\,N_z \to (N \to N_s) \to N_z$$

$$Nat_s \triangleq \forall(N)\,\forall(N_z<:N)\,\forall(N_s<:N)\,N_z \to (N \to N_s) \to N_s.$$

The closed normal forms of minimal type for *Nat* are the usual Church numerals; for Nat_z we have only the zero natural, and for Nat_s the non-zero naturals. We obtain

$$Nat_z <: Nat, \qquad Nat_s <: Nat$$

$$zero : Nat_z$$
$$\triangleq \lambda(N)\,\lambda(N_z <: N)\,\lambda(N_s <: N)\,\lambda(z : N_z)\,\lambda(s : N \to N_s)\,z$$

$$succ : Nat \to Nat_s$$
$$\triangleq \lambda(n : Nat)$$
$$\lambda(N)\,\lambda(N_z <: N)\,\lambda(N_s <: N)\,\lambda(z : N_z)\,\lambda(s : N \to N_s)$$
$$s(n(N)(N_z)(N_s)(z)(s)).$$

3.3. *Products*

The standard encoding for pairs in F, shown below, already exhibits useful subtyping properties:

$$A \times B \triangleq \forall(C)(A \to B \to C) \to C.$$

Both A and B occur in monotonic positions in $A \times B$, being placed on the left of an $\to$ which is on the left of another $\to$. Hence we obtain the expected monotonic inclusion of products as a derived rule:

$$\frac{E \vdash A <: A' \quad E \vdash B <: B'}{E \vdash A \times B <: A' \times B'}.$$

The operations on pairs are defined, as usual, as

$$pair: \quad \forall(A)\,\forall(B)\,A \to B \to A \times B$$
$$\triangleq \lambda(A)\,\lambda(B)\,\lambda(a : A)\,\lambda(b : B)\,\lambda(C)\,\lambda(f : A \to B \to C)\,f(a)(b)$$

$$fst: \quad \forall(A)\,\forall(B)\,A \times B \to A$$
$$\triangleq \lambda(A)\,\lambda(B)\,\lambda(c : A \times B)\,c(A)(\lambda(x : A)\,\lambda(y : B)\,x)$$

$$snd: \quad \forall(A)\,\forall(B)\,A \times B \to B$$
$$\triangleq \lambda(A)\,\lambda(B)\,\lambda(c : A \times B)\,c(B)(\lambda(x : A)\,\lambda(y : B)\,y).$$

We often use the following abbreviations, disambiguated by context:

$$a, b \equiv a,_{A \times B} b \qquad \equiv pair(A)(B)(a)(b)$$
$$fst(c) \equiv fst_{A \times B}(c) \equiv fst(A)(B)(c)$$
$$snd(c) \equiv snd_{A \times B}(c) \equiv snd(A)(B)(c).$$

3.4. *Simple Tuples*

A *tuple type* is an iterated product type. When the last factor of this iterated product is a type variable, we have an *extensible tuple type.* When it is *Top*, we have a *simple tuple type.* In this paper we discuss only simple tuple types,

$$Tuple(Top) \triangleq Top$$

$$Tuple(A_1, ..., A_n, Top) \triangleq A_1 \times (\cdots \times (A_n \times Top) \cdots) \qquad n \geqslant 1,$$

with derived rule

$$\frac{E \vdash A_1 <: B_1 \cdots E \vdash A_n <: B_n \quad E \vdash A_{n+1}\ type \cdots E \vdash A_m\ type}{E \vdash Tuple(A_1, ..., A_n, ..., A_m, Top) <: Tuple(B_1, ..., B_n, Top)}.$$

For example,

$$Tuple(A, B, Top) <: Tuple(A, Top)$$

because $A <: A$, $B \times Top <: Top$, and $\times$ is monotonic.

We note here that the type *Top* assumes a very useful role, in allowing a longer tuple type to be a subtype of a shorter tuple type. The intuition is that a longer tuple value can always be regarded as a shorter tuple value, by "forgetting" the additional components, and this is possible since everything is forgotten in *Top.*

For tuple values we have

$$tuple(top) \triangleq top$$

$$tuple(a_1, ..., a_n, top) \triangleq a_1, (..., (a_n, top)...) \qquad n \geqslant 1,$$

with derived rules

$$\frac{E \vdash a_1 : A_1 \cdots E \vdash a_n : A_n}{E \vdash tuple(a_1, ..., a_n, top) : Tuple(A_1, ..., A_n, Top)}$$

$$\frac{E \vdash a_1 \leftrightarrow b_1 : A_1 \cdots E \vdash a_n \leftrightarrow b_n : A_n}{E \vdash tuple(a_1, ..., a_n, top) \leftrightarrow tuple(b_1, ..., b_n, top) : Tuple(A_1, ..., A_n, Top)}.$$

The basic tuple operations are $a \lfloor i$, dropping the first i components of tuple a; and $a.i$, selecting the ith component of a. These are defined by iterating product operations; again, we omit some typing information:

$$a \lfloor i \equiv snd^i(a)$$

$$a.i \equiv fst(a \lfloor i).$$

We obtain the derived rules

$$\frac{E \vdash a : Tuple(A_0, ..., A_n, Top)}{E \vdash a \lfloor i : Tuple(A_i, ..., A_n, Top)} \qquad n \geqslant 0, \quad i \in 0 \cdots n+1$$

$$\frac{E \vdash a : Tuple(A_0, ..., A_n, Top)}{E \vdash a.i : A_i} \qquad n \geqslant 0, \quad i \in 0 \cdots n$$

$$\frac{E \vdash a_0 : A_0 \cdots E \vdash a_n : A_n}{\left(\begin{array}{c} E \vdash tuple(a_0, ..., a_n, top) \lfloor i \\ \leftrightarrow tuple(a_i, ..., a_n, top) : Tuple(A_i, ..., A_n, Top) \end{array}\right)} \qquad n \geqslant 0, \quad i \in 0 \cdots n+1$$

$$\frac{E \vdash a_0 : A_0 \cdots E \vdash a_n : A_n}{E \vdash tuple(a_0, ..., a_n, top).i \leftrightarrow a_i : A_i} \qquad n \geqslant 0, \quad i \in 0 \cdots n.$$

3.5. *Simple Records*

We restrict ourselves to the encoding of *simple records* (the ones with a fixed number of components [CL 91]); *extensible records* are treated in [Car 93].

Let $\mathscr{L}$ be a countable set of *labels*, enumerated by a bijection $\iota \in \mathscr{L} \to Nat$. We indicate by l^i, with a superscript, the ith label in this enumeration. Often we need to refer to a list of n distinct labels out of this enumeration; we then use subscripts, as in $l_1 \cdots l_n$. So we may have, for example, $l_1, l_2, l_3 = l^5, l^1, l^{17}$. More precisely, $l_1 \cdots l_n$ stands for $l^{\sigma(1)}, ..., l^{\sigma(n)}$ for some injective $\sigma \in 1 \cdots n \to Nat$.

A record type has the form $Rcd(l_1 : A_1, ..., l_n : A_n, C)$; in this presentation C will always be Top. Once the enumeration of labels is fixed, a record type is encoded as a tuple type where the record components are allocated to tuple slots as determined by the index of their labels. The component of label l^i is allocated into the ith tuple slot; the remaining slots are filled with Top "padding." For example:

$$Rcd(l^2 : C, l^0 : A, Top) \triangleq Tuple(A, Top, C, Top).$$

Since record type components are canonically sorted under the encoding, two record types that differ only in the order of their components will be equal under the encoding. Hence we can consider record components as unordered.

From the encoding, we derive the familiar rule for simple records [Car 88]:

$$\frac{E \vdash A_1 <: B_1 \cdots E \vdash A_n <: B_n \quad E \vdash A_{n+1}\ type \cdots E \vdash A_m\ type}{E \vdash Rcd(l_1 : A_1, ..., l_n : A_n, ..., l_m : A_m, Top) <: Rcd(l_1 : B_1, ..., l_n : B_n, Top)}$$

This holds because any additional field $l_k:A_k$ $(n<k\leqslant m)$ on the left is absorbed either by the *Top* padding on the right, if $\iota(l_k)<max(\iota(l_1)\cdots\iota(l_n))$, or by the final *Top*, otherwise. For example,

$$Rcd(l^0:A, l^1:B, l^2:C, Top) \equiv Tuple(A, B, C, Top)$$
$$<: Tuple(Top, B, Top) \equiv Rcd(l^1:B, Top).$$

Record values are similarly encoded; for example,

$$rcd(l^2=c, l^0=a, top) \triangleq tuple(a, top, c, top),$$

from which we obtain the rules

$$\frac{E\vdash a_1:A_1\cdots E\vdash a_n:A_n}{E\vdash rcd(l_1=a_1, ..., l_n=a_n, top):Rcd(l_1:A_1, ..., l_n:A_n, Top)}$$

$$\frac{E\vdash a_1\leftrightarrow a_1':A_1\cdots E\vdash a_n\leftrightarrow a_n':A_n}{\left(\begin{array}{c}E\vdash rcd(l_1=a_1, ..., l_n=a_n, top)\leftrightarrow rcd(l_1=a_1', ..., l_n=a_n', top):\\ Rcd(l_1:A_1, ..., l_n:A_n, Top)\end{array}\right)}$$

Record selection is encoded as follows:

$$r.l_i \triangleq r.\iota(l_i)$$

$$\frac{E\vdash r:Rcd(l:A, Top)}{E\vdash r.l:A}$$

Note that, by subsumption, we have the following as (further) derived rules:

$$\frac{E\vdash a_1:A_1\cdots E\vdash a_n:A_n\cdots E\vdash a_m:A_m}{E\vdash rcd(l_1=a_1, ..., l_n=a_n, ..., l_m=a_m, top):Rcd(l_1:A_1, ..., l_n:A_n, Top)}$$

$$\frac{\begin{array}{c}E\vdash a_1\leftrightarrow b_1:A_1\cdots E\vdash a_n\leftrightarrow b_n:A_n\\ E\vdash a_{n+1}:B_{n+1}\cdots E\vdash a_p:B_p \quad E\vdash b_{n+1}:C_{n+1}\cdots E\vdash b_q:C_q\end{array}}{\left(\begin{array}{c}E\vdash rcd(l_1=a_1, ..., l_n=a_n, ..., l_p=a_p, top)\leftrightarrow rcd(l_1=b_1, ..., l_n=b_n, ..., l_q=b_q, top):\\ Rcd(l_1:A_1, ..., l_n:A_n, Top)\end{array}\right)}$$

$$\frac{E\vdash r:Rcd(l_1:A_1, ..., l_n:A_n, Top)}{E\vdash r.l_i:A_i} \qquad i\in 1\cdots n$$

The second rule above is particularly interesting. It expresses a form of observational equivalence: two records are equivalent if they coincide on the components that are observable at a given type. This holds ultimately because any two values are equivalent at type *Top*.

3.6. *Lists*

Following the pattern used in the encoding of the naturals, we can define the algebra of parametric lists [BB 85]. $List[A]$ stands for the homogeneous lists of type A:

$$List[A] \triangleq \forall(L)\, L \to (A \to L \to L) \to L.$$

We have

$$A <: B \Rightarrow List[A] <: List[B]$$

$$nil : \forall(A)\, List[A]$$

$$\triangleq \lambda(A)\, \lambda(L)\, \lambda(n:L)\, \lambda(c:A \to L \to L)\, n$$

$$cons : \forall(A)\, A \to List[A] \to List[A]$$

$$\triangleq \lambda(A)\, \lambda(hd:A)\, \lambda(tl:List[A])$$

$$\lambda(L)\, \lambda(n:L)\, \lambda(c:A \to L \to L)$$

$$c(hd)(tl(L)(n)(c))$$

$$length : \forall(A)\, List[A] \to Nat$$

$$\triangleq \lambda(A)\, \lambda(l:List[A])$$

$$l(Nat)(zero)(\lambda(a:A)\, \lambda(n:Nat)\, succ(n)).$$

As an application of (*Eq appl2*) we can now show some interesting facts. Namely, any two null lists are equal in $List[Top]$, and have the same length in *Nat*. Similarly for two singleton lists, and so on. In the proof, we will use the Eq-substitution proposition of Section 2.4.

Take $b:B$ and $c:C$; then

$$\vdash nil(B) \leftrightarrow nil(C) : List[Top] \qquad (Eq\ appl2)$$

$$\vdash length(Top)(nil(B)) \leftrightarrow length(Top)(nil(C)) : Nat \qquad (Eq\ appl2,\ Eq\ appl)$$

$$\vdash cons(B)(b)(nil(B)) \leftrightarrow cons(C)(c)(nil(C)) : List[Top]$$

by Eq-substitution, starting from
$X <: Top,\ x:X,\ l:List[X]$
$\vdash cons(X)(x)(l) : List[X]$

$$\vdash length(B)(cons(B)(b)(nil(B))) \leftrightarrow length(C)(cons(C)(c)(nil(C))) : Nat$$

by Eq-substitution, starting from
$X <: Top,\ l:List[X] \vdash length(X)(l) : Nat.$

Note that we have proven an interesting property of the behavior of *length* uniquely from its type; any function $f:\forall(A)\ List[A] \to Nat$ has such a property. This fact is related to the theorems proved in [Wad 89] using only the types of terms. A difference is that our proof is carried out within $F_{<:}$, whereas Wadler uses semantic parametricity properties beyond the proof system of F.

4. The Category of Closed Terms

It is well known that the usual second-order encodings for products and coproducts, while logically sound, do not define true categorical constructions under β–η-equality. One can easily prove the existence of a term making a certain diagram commute, but its uniqueness does not follow from the standard equational rules.

As an example of the expressive power of (*Eq appl2*), we show that those encodings are really categorical constructions when the underlying equational theory is the one of $F_{<:}$. In the same vein, motivated by the semantic isomorphisms obtained in [BFSS 90] and [Fre 93] as consequences of parametricity, we investigate some provable isomorphisms in a suitable setting. The framework for our discussion is a category whose objects are the sets of closed terms of a closed type.

4.1. *Definitions and Basic Properties*

Recall that given a typed λ-calculus language and a λ-theory **T**, a category $Cl(\mathbf{T})$ is determined by taking as objects of $Cl(\mathbf{T})$ the (closed) types of **T** [LS 86, MS 93]. As for morphisms, choose first one variable for each type and define the morphisms from A to B to be equivalence classes of typing judgments $x:A \vdash t:B$, where x is the chosen variable of type A, and the equivalence relation is given by the equality judgments $x:A \vdash t \leftrightarrow t':B$ of **T**. We will write $[x:A \vdash t:B]$ for the morphism given by the judgment $x:A \vdash t:B$. Identity is given by $[x:A \vdash x:A]$ and composition is defined by substitution:

$$[y:B \vdash s:C] \circ [x:A \vdash t:B] = [x:A \vdash s\{y \leftarrow t\}:C].$$

The category $Cl(F_{<:})$, obtained by applying this construction to $F_{<:}$, has a terminal object, given by *Top*. For any object A, the canonical morphism from A to *Top* is $[x:A \vdash top:Top]$; uniqueness is guaranteed by (*Eq collapse*).

Now, given an arbitrary (small) category **C** with a terminal object 1, consider the canonical functor $\ulcorner \_ \urcorner$: $\mathbf{C} \to \mathbf{Sets}$ given by:

For any object A:

$$\ulcorner A \urcorner = \mathbf{C}(1, A) \qquad \text{(the set of all morphisms } 1 \to A\text{)}$$

For any morphism $f \in \mathbf{C}(A, B)$:

$\ulcorner f \urcorner$ is the mapping from $\ulcorner A \urcorner$ to $\ulcorner B \urcorner$ given by composing with f (that is, $\ulcorner f \urcorner(p) = f \circ p$ for $p \in \mathbf{C}(1, A)$).

Note that $\ulcorner \_ \urcorner$ is not faithful if $\mathbf{C}$ is not well-pointed (as defined in 4.2.5). Given $f, g \in \mathbf{C}(A, B)$, $\ulcorner f \urcorner$ and $\ulcorner g \urcorner$ are set-theoretical mappings and therefore, in order to have $\ulcorner f \urcorner = \ulcorner g \urcorner$, it is sufficient that $f \circ p = g \circ p$ for any $p \in \mathbf{C}(1, A)$. The values of the functor $\ulcorner \_ \urcorner : \mathbf{C} \to \mathbf{Sets}$ over all the objects and morphisms of $\mathbf{C}$ give a subcategory of **Sets** that can be denoted with $\ulcorner \mathbf{C} \urcorner$.

The category we are interested in is $\ulcorner Cl(F_{<:}) \urcorner$. We will prove, as consequences of (*Eq appl2*), that it has finite products and coproducts. For this, however, it is convenient to introduce the category **CL**, equivalent to $\ulcorner Cl(F_{<:}) \urcorner$, for which we can give a more explicit description.

Remark.

$\vdash A\ type$	reads "A is a closed type"
$\vdash a : A$	reads "a is a closed term of closed type A."

DEFINITION (cl-equality). For $\vdash f, f' : A \to B$, we say $\vdash f \leftrightarrow^{cl} f' : A \to B$ iff for all a, $\vdash a : A \Rightarrow \vdash f(a) \leftrightarrow f'(a) : B$.

The *objects* of $\ulcorner Cl(F_{<:}) \urcorner$ are, for any $\vdash A\ type$, the sets of morphisms $[z : Top \vdash t : A]$. By (*Eq collapse*) and congruence, $[z : Top \vdash t : A] = [z : Top \vdash t\{z \leftarrow top\} : A]$. The term $t\{z \leftarrow top\}$ is closed and $z : Top \vdash t\{z \leftarrow top\} : A$ iff $\vdash t\{z \leftarrow top\} : A$. Any object of $\ulcorner Cl(F_{<:}) \urcorner$ is therefore isomorphic to the set of equivalence classes $[\vdash a : A]$ of closed terms of a closed type; the equivalence relation is given by the equality judgments $\vdash a \leftrightarrow a' : A$. (Write $\vdash A\ type$ for such a set.) These sets are the objects of the category **CL**.

The *morphisms* of $\ulcorner Cl(F_{<:}) \urcorner$ are, for any morphism $f = [x : A \vdash t : B]$ of $Cl(F_{<:})$, the mappings from $\ulcorner A \urcorner$ to $\ulcorner B \urcorner$ given by $\ulcorner f \urcorner([z : Top \vdash a : A]) = [z : Top \vdash t\{x \leftarrow a\} : B]$ for any $[z : Top \vdash a : A]$. By β- and η-conversion one obtains a category equivalent to $\ulcorner Cl(F_{<:}) \urcorner$ by stipulating that a morphism of **CL** from $\vdash A\ type$ to $\vdash B\ type$ is an equivalence class of derivable term judgments,

$$\vdash f : A \to B,$$

where the morphism equivalence is

$$(\vdash f:A \to B) = (\vdash f':A \to B) \qquad \text{iff} \quad \vdash f \leftrightarrow^{cl} f':A \to B.$$

The *identity* judgment is

$$id_A \triangleq \vdash \lambda(x:A)x:A \to A$$

and the *composition* judgment is, for any $\vdash h:A \to B$ and $\vdash g:B \to C$,

$$g \circ h \triangleq \vdash \lambda(x:A)\ g(h(x)):A \to C.$$

(We also ambiguously use $g \circ h \triangleq \lambda(x:A)\ g(h(x))$.)

We remark that morphism equivalence is *not* provable equality. For two morphisms $\vdash f:A \to B$ and $\vdash f':A \to B$ to be equal it is sufficient that f and f' agree on the *closed* terms of type A. Similarly, the following two definitions correspond to isomorphism and uniqueness (for morphisms) in **CL**.

DEFINITION (cl-isomorphism). We say that $\vdash A \sim^{cl} B$ iff there exist $\vdash f:A \to B$, $\vdash g:B \to A$ such that

$$\vdash g \circ f \leftrightarrow^{cl} id_A:A \to A$$

$$\vdash f \circ g \leftrightarrow^{cl} id_B:B \to B.$$

DEFINITION (cl-uniqueness). We say that $\vdash f:A \to B$ is the cl-unique f satisfying $P(f)$ iff for any other $\vdash f':A \to B$ satisfying $P(f')$ we have $\vdash f \leftrightarrow^{cl} f':A \to B$.

In order to prove that **CL** has finite products and coproducts, we need some more lemmas in $F_{<:}$, and especially the crucial consequence of (*Eq appl2*) expressed in the eq-var-substitution lemma, below.

LEMMA (Type monotonicity). *Let* $E, X <: B \vdash C <: D <: B$ *and* $E, X <: B, E' \vdash S$ *type. Then*

(i) *X positive in* $S \Rightarrow E, X <: B, E' \vdash S\{X \leftarrow C\} <: S\{X \leftarrow D\}$

(ii) *X negative in* $S \Rightarrow E, X <: B, E' \vdash S\{X \leftarrow D\} <: S\{X \leftarrow C\}$.

Proof. By induction on the derivation $E, X <: B, E' \vdash S$ *type*. The only less trivial case is (*Type* $\forall$). Assume X positive in $\forall(Y <: S1)S2$. By the induction hypothesis,

$$E, X <: B, E' \vdash S1\{X \leftarrow D\} <: S1\{X \leftarrow C\}.$$

From $E, X <: B, E', Y <: S1 \vdash S2\ type$, by the bound change lemma,

$$E, X <: B, E', Y <: S1\{X \leftarrow D\} \vdash S2\ type.$$

Now conclude by induction and (*Sub* $\forall$). ∎

DEFINITION (Pointed on X). Given a type variable X, a type S is *pointed on* X iff X is positive in S and $S \equiv \forall(Y_1 <: B_1) \cdots \forall(Y_k <: B_k) T_1 \to (\cdots \to (T_h \to X) \cdots)$ for $k \geqslant 0$, $h \geqslant 0$.

LEMMA (Generalized collapse). *Let* $E, X <: Top \vdash S\ type$, *with* S *pointed on* X. *If* $E \vdash D\ type$ *and* $E \vdash s : S\{X \leftarrow D\}$, *then* $E, X <: Top, x : S \vdash x \leftrightarrow s : S\{X \leftarrow Top\}$.

Proof. Let $S \equiv \forall(Y_1 <: B_1) \cdots \forall(Y_k <: B_k) T_1 \to (\cdots \to (T_h \to X) \cdots)$. By the type monotonicity lemma,

$$E, X <: Top \vdash S <: S\{X \leftarrow Top\}$$

and

$$E, X <: Top \vdash S\{X \leftarrow D\} <: S\{X \leftarrow Top\}.$$

Let $F \equiv Y_1 <: B_1\{X \leftarrow Top\}, \ldots, Y_k <: B_k\{X \leftarrow Top\}, t_1 : T_1\{X \leftarrow Top\}, \ldots, t_h : T_h\{X \leftarrow Top\}$. By (*Val* x), weakening, and (*Subsumption*),

$$E, X <: Top, x : S, F \vdash x : S\{X \leftarrow Top\};$$

by (*Eq appl2*) and (*Eq appl*),

$$E, X <: Top, x : S, F \vdash x(Y_1) \cdots (Y_k)(t_1) \cdots (t_h) : Top.$$

Analogously, from $E \vdash s : S\{X \leftarrow D\}$ we obtain

$$E, X <: Top, x : S, F \vdash s : S\{X \leftarrow Top\}$$

and then

$$E, X <: Top, x : S, F \vdash s(Y_1) \cdots (Y_k)(t_1) \cdots (t_h) : Top.$$

By (*Eq collapse*),

$$E, X <: Top, x : S, F \vdash x(Y_1) \cdots (Y_k)(t_1) \cdots (t_h)$$
$$\leftrightarrow s(Y_1) \cdots (Y_k)(t_1) \cdots (t_h) : Top.$$

By (*Eq fun*), (*Eq fun2*), (*Eq eta*), and (*Eq eta2*),

$$E, X <: Top, x:S \vdash x \leftrightarrow s:S\{X \leftarrow Top\}. \quad \blacksquare$$

By generalized collapse and the eq-substitution property (Section 2.4) we obtain the following lemma, which expresses a parametricity property: a (possibly open) term a of a closed type A is provably equal to any term obtained by substituting specific types and terms for its free variables.

LEMMA (Eq-var-substitution). *Assume, for* $i = 1 \cdots n$, $E', X <: Top \vdash S_i$ *type and* S_i *pointed on* X. *Let* $E \equiv E', X <: Top, x_1:S_1, ..., x_n:S_n$. *If* $\vdash A$ *type,* $E \vdash a:A$, $E' \vdash D$ *type, and* $E' \vdash t_i:S_i\{X \leftarrow D\}$ *for* $i = 1 \cdots n$, *then* $E \vdash a \leftrightarrow a\{X \leftarrow D, x_1 \leftarrow t_1, ..., x_n \leftarrow t_n\}:A$.

Proof. By generalized collapse lemma, for $i = 1 \cdots n$:

$$E', X <: Top, x_i:S_i \vdash x_i \leftrightarrow t_i:S_i\{X \leftarrow Top\}.$$

The eq-substitution proposition (Section 2.4) allows us to conclude. ∎

4.2. **CL** *Finite Products and Coproducts; Well-Pointedness*

In this section we show that the equational theory of $F_{<:}$ is strong enough to entail some basic categorical properties of **CL**.

4.2.1. *Terminal Objects*

PROPOSITION. *For any object* $\vdash C$ *type, there is a unique morphism* $\vdash 1_C:C \to Top$.

Proof. Take $1_C \triangleq \lambda(x:C)top$. Take any other morphism $\vdash f:C \to Top$. We have

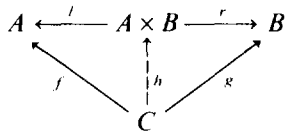
$x:C \vdash f:C \to Top$	(*weaken*)
$x:C \vdash f(x) \leftrightarrow top:Top$	(*Eq collapse*)
$\vdash \lambda(x:C)\, f(x) \leftrightarrow \lambda(x:C)top:C \to Top$	(*Eq fun*)
$\vdash f \leftrightarrow 1_C:C \to Top$	(*Eq eta*).

A fortiori, $\vdash f \leftrightarrow^{cl} 1_C:C \to Top$. ∎

4.2.2. *Binary Products*

DEFINITION. $A \times B \triangleq \forall(C)(A \to B \to C) \to C$.

PROPOSITION. *For any pair of objects* $\vdash A$ *type*, $\vdash B$ *type, the object* $\vdash A \times B$ *type is their categorical product. That is, there exist* $\vdash l: A \times B \to A$, $\vdash r: A \times B \to B$ *such that for any* $\vdash C$ *type, and for any* $\vdash f: C \to A$, $\vdash g: C \to B$, *there exists a unique (i.e., cl-unique)* $\vdash h: C \to A \times B$ *such that* $\vdash l \circ h \leftrightarrow^{cl} f: C \to A$ *and* $\vdash r \circ h \leftrightarrow^{cl} g: C \to B$:



Proof. Define:

$$px \triangleq \lambda(x:A)\,\lambda(y:B)\,x$$

$$py \triangleq \lambda(x:A)\,\lambda(y:B)\,y$$

$$l \triangleq \lambda(p:A \times B)\,p(A)(px) \qquad \text{then} \quad \vdash l: A \times B \to A$$

$$r \triangleq \lambda(p:A \times B)\,p(B)(py) \qquad \text{then} \quad \vdash r: A \times B \to B$$

$$pair \triangleq \lambda(a:A)\,\lambda(b:B)\,\lambda(C)\,\lambda(q:A \to B \to C)\,q(a)(b)$$

$$\text{then} \quad \vdash pair: A \to B \to A \times B$$

$$couple \triangleq \lambda(C)\,\lambda(f:C \to A)\,\lambda(g:C \to B)\,\lambda(c:C)\,pair(f(c))(g(c))$$

$$\text{then} \quad \vdash couple: \forall(C)(C \to A) \to (C \to B) \to C \to (A \times B).$$

Fix an object $\vdash C$ *type* and two morphisms $\vdash f: C \to A$ and $\vdash g: C \to B$.

(1) Existence. Take $h \triangleq couple(C)(f)(g) \leftrightarrow \lambda(c:C)\,pair(f(c))(g(c))$:

$$\vdash l \circ h \leftrightarrow \lambda(z:C)\,l(h(z)) \leftrightarrow \lambda(z:C)\,f(z) \leftrightarrow f: C \to A$$

$$\vdash r \circ h \leftrightarrow \lambda(z:C)\,r(h(z)) \leftrightarrow \lambda(z:C)\,g(z) \leftrightarrow g: C \to B.$$

(2) The morphism above is well defined. Just show that

$$\vdash f' \leftrightarrow^{cl} f: C \to A,\ \vdash g' \leftrightarrow^{cl} g: C \to B$$

$$\text{implies} \quad \vdash couple(C)(f)(g) \leftrightarrow^{cl} couple(C)(f')(g'): C \to A \times B.$$

(3) Uniqueness.

(3.1) Show, for $\vdash c:A \times B$, that $\vdash couple(A \times B)(l)(r)(c) \leftrightarrow c:A \times B$. The normal form of c must have the shape

$$c \equiv \lambda(C)\, \lambda(q:D)\, q(a)(b)$$

for some $C <: Top \vdash A \to B \to C <: D$, $C <: Top, q:D \vdash a:A$, and $C <: Top, q:D \vdash b:B$. By the bound weakening lemma,

$$C <: Top, q:A \to B \to C \vdash a:A \quad \text{and} \quad C <: Top, q:A \to B \to C \vdash b:B,$$

and by (*Eq fun'*), for $c' \triangleq \lambda(C)\, \lambda(q:A \to B \to C)\, q(a)(b)$,

$$\vdash c \leftrightarrow c':A \times B.$$

By β-conversion

$$\vdash l(c) \leftrightarrow c(A)(px) \leftrightarrow a\{C \leftarrow A, q \leftarrow px\}:A; \quad \text{let} \quad a1 \triangleq a\{C \leftarrow A, q \leftarrow px\}.$$
$$\vdash r(c) \leftrightarrow c(B)(py) \leftrightarrow b\{C \leftarrow B, q \leftarrow py\}:B; \quad \text{let} \quad b1 \triangleq b\{C \leftarrow B, q \leftarrow py\}.$$

By the eq-var-substitution lemma,

$$C <: Top, q:A \to B \to C \vdash a \leftrightarrow a1:A$$
$$C <: Top, q:A \to B \to C \vdash b \leftrightarrow b1:B$$
$$C <: Top, q:A \to B \to C \vdash q(a)(b) \leftrightarrow q(a1)(b1):C \qquad (Eq\text{-}appl)$$
$$\vdash \lambda(C)\, \lambda(q:A \to B \to C)\, q(a)(b) \leftrightarrow \lambda(C)\, \lambda(q:A \to B \to C)\, q(a1)(b1):A \times B$$
$$(Eq\ fun,\ Eq\ fun2).$$

Hence,

$$\vdash couple(A \times B)(l)(r)(c)$$
$$\leftrightarrow pair(l(c))(r(c))$$
$$\leftrightarrow \lambda(C)\, \lambda(q:A \to B \to C)\, q(a1)(b1) \leftrightarrow \lambda(C)\, \lambda(q:A \to B \to C)\, q(a)(b)$$
$$\leftrightarrow c' \leftrightarrow c:A \times B.$$

(3.2) Show, by β-conversion, that for any $\vdash D\ type$, $\vdash k:D \to C$, and $\vdash d:D$,

$$\vdash couple(D)(f \circ k)(g \circ k)(d) \leftrightarrow (couple(C)(f)(g) \circ k)(d):A \times B.$$

That h is *cl*-unique now follows by the usual argument. ∎

COROLLARY. $\vdash A \sim^{cl} A', \vdash B \sim^{cl} B' \Rightarrow \vdash A \times B \sim^{cl} A' \times B'$.

Proof. Standard diagram chasing, from the existence of products. ∎

4.2.3. *Initial Objects*

DEFINITION. $Bot \triangleq \forall(X)X$.

PROPOSITION. *For any object* $\vdash C$ *type, there is a unique morphism* $\vdash 0_C : Bot \to C$.

Proof. Take $0_C \triangleq \lambda(x:Bot)\, x(C)$.

Take any other morphism $\vdash f : Bot \to C$.

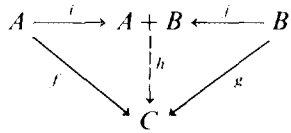
Since there are no terms c such that $\vdash c : Bot$, then it is vacuously true that for all $\vdash c : Bot$, $\vdash f(c) \leftrightarrow 0_C(c) : C$; that is, that $\vdash f \leftrightarrow^{cl} 0_C : Bot \to C$. ∎

Remark. $Bool \to Bot$ is also an initial object, by the same argument, since there are no terms of type $Bool \to Bot$. The unique map is the equivalence class of $\lambda(x:Bool \to Bot)\, x(true)(C)$, which includes $\lambda(x:Bool \to Bot)\, x(false)(C)$. More generally, any empty type V for which there exists a term $\vdash f : V \to Bot$ is initial. The canonical morphism is the equivalence class of $\lambda(x:V)\, f(x)(C)$, which is *cl*-unique since there are no closed terms $\vdash c : V$.

4.2.4. *Binary Coproducts*

DEFINITION. $A + B \triangleq \forall(C)(A \to C) \to (B \to C) \to C$.

PROPOSITION. *For any pair of objects* $\vdash A$ *type,* $\vdash B$ *type, the object* $\vdash A + B$ *type is their categorical coproduct. That is, there exist* $\vdash i : A \to A + B$, $\vdash j : B \to A + B$ *such that for any* $\vdash C$ *type, and for any* $\vdash f : A \to C$, $\vdash g : B \to C$, *there exists a unique* (*i.e.*, *cl-unique*) $\vdash h : A + B \to C$ *such that* $\vdash h \circ i \leftrightarrow^{cl} f : A \to C$ *and* $\vdash h \circ j \leftrightarrow^{cl} g : B \to C$:



Proof. Define:

$$i \triangleq \lambda(x:A)\, \lambda(C)\, \lambda(f:A \to C)\, \lambda(g:B \to C)\, f(x)$$

then $\vdash i : A \to A + B$

$$j \triangleq \lambda(y:B)\, \lambda(C)\, \lambda(f:A \to C)\, \lambda(g:B \to C)\, g(y)$$

then $\vdash j : B \to A + B$

$$case \triangleq \lambda(C)\, \lambda(f:A \to C)\, \lambda(g:B \to C)\, \lambda(c:A + B)\, c(C)(f)(g)$$

then $\vdash case : \forall(C)(A \to C) \to (B \to C) \to (A + B) \to C$.

(0) Let $\vdash c : A + B$; then the normal form of c must have one of the shapes

$c \equiv \lambda(C')\,\lambda(f' : D)\,\lambda(g' : G)\,f'(a)$

for some $C' <: Top \vdash A \to C <: D$, $C' <: Top \vdash B \to C' <: G$, and $C' <: Top, f' : D, g' : G \vdash a : A$

$c \equiv \lambda(C')\,\lambda(f' : D)\,\lambda(g' : G)\,g'(b)$

for some $C' <: Top \vdash A \to C <: D$, $C' <: Top \vdash B \to C' <: G$, and $C' <: Top, f' : D, g' : G \vdash b : B$.

By the bound weakening lemma,

$$C' <: Top, f' : A \to C', g' : B \to C' \vdash a : A,$$
$$C' <: Top, f' : A \to C', g' : B \to C' \vdash b : B,$$

and, by (*Eq fun'*),

either $\quad \vdash c \leftrightarrow \lambda(C')\,\lambda(f' : A \to C')\,\lambda(g' : B \to C')\,f'(a) : A + B$

or $\quad \vdash c \leftrightarrow \lambda(C')\,\lambda(f' : A \to C')\,\lambda(g' : B \to C')\,g'(b) : A + B.$

Fix an object $\vdash C\ type$ and two morphisms $\vdash f : A \to C$ and $\vdash g : B \to C$.

(1) Existence. Take $h \triangleq case(C)(f)(g)$:

$$\vdash h \circ i \leftrightarrow \lambda(x : A)\,h(i(x)) \leftrightarrow \lambda(x : A)\,f(x) \leftrightarrow f : A \to C$$
$$\vdash h \circ j \leftrightarrow \lambda(x : A)\,h(j(x)) \leftrightarrow \lambda(x : A)\,g(x) \leftrightarrow g : B \to C.$$

(2) The morphism above is well defined. Show that $\vdash f'' \leftrightarrow^{cl} f : A \to C$, $\vdash g'' \leftrightarrow^{cl} g : B \to C$ implies

$$\vdash case(C)(f)(g) \leftrightarrow^{cl} case(C)(f'')(g'') : A + B \to C.$$

That is, for $\vdash c : A + B$,

$$\vdash case(C)(f)(g)(c) \leftrightarrow case(C)(f'')(g'')(c) : C$$

By (0) and β-conversion, either

$$\vdash case(C)(f)(g)(c) \leftrightarrow f(a\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\}) : C$$

and

$$\vdash case(C)(f'')(g'')(c) \leftrightarrow f''(a\{C' \leftarrow C, f' \leftarrow f'', g' \leftarrow g''\}) : C$$

or

$$\vdash case(C)(f)(g)(c) \leftrightarrow g(b\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\}):C$$

and

$$\vdash case(C)(f'')(g'')(c) \leftrightarrow g''(b\{C' \leftarrow C, f' \leftarrow f'', g' \leftarrow g''\}):C$$

In the first case (the other one is similar), the eq-var-substitution lemma gives

$$C' <: Top, f':A \to C', g':B \to C' \vdash a \leftrightarrow a\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\}:A$$

and also

$$C' <: Top, f':A \to C', g':B \to C' \vdash a \leftrightarrow a\{C' \leftarrow C, f' \leftarrow f'', g' \leftarrow g''\}:A,$$

from which we infer that

$$\vdash a\{C' \leftarrow C, f' \leftarrow f'', g' \leftarrow g''\} \leftrightarrow a\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\}:A,$$

since both terms are closed. Now conclude by using $\vdash f'' \leftrightarrow^{cl} f:A \to C$.

(3) Uniqueness.

(3.1) Show, for $\vdash c:A + B$, that $\vdash case(A + B)(i)(j)(c) \leftrightarrow c:A + B$. By cases on the normal form of c, according to (0). In the first case,

$$\vdash case(A + B)(i)(j)(c) \leftrightarrow c(A + B)(i)(j)$$
$$\leftrightarrow i(a\{C' \leftarrow A + B, f' \leftarrow i, g' \leftarrow j\}):A + B.$$

Let $a1 \triangleq a\{C' \leftarrow A + B, f' \leftarrow i, g' \leftarrow j\}$. By the eq-var-substitution lemma,

$C' <: Top, f':A \to C', g':B \to C' \vdash a1 \leftrightarrow a:A$

$C' <: Top, f':A \to C', g':B \to C' \vdash f'(a1) \leftrightarrow f'(a):C'$ (*Eq appl*)

$\vdash \lambda(C')\,\lambda(f':A \to C')\,\lambda(g':B \to C')\,f'(a1)$
$\leftrightarrow \lambda(C')\,\lambda(f':A \to C')\,\lambda(g':B \to C')\,f'(a):A + B$ (*Eq fun, Eq fun2*)

$\vdash i(a1) \leftrightarrow c:A + B$ (def)

$\vdash case(A + B)(i)(j)(c) \leftrightarrow c:A + B$ (equation above).

The second case is similar.

(3.2) Show, for any $\vdash D\ type$, $\vdash k:C \to D$, and $\vdash c:A + B$,

$$\vdash case(D)(k \circ f)(k \circ g)(c) \leftrightarrow (k \circ case(C)(f)(g))(c):D.$$

By cases on the normal form of c, according to (0). In the first case we have

$$\vdash case(D)(k \circ f)(k \circ g)(c) \leftrightarrow c(D)(k \circ f)(k \circ g)$$
$$\leftrightarrow k(f(a\{C' \leftarrow D, f' \leftarrow k \circ f, g' \leftarrow k \circ g)\}\}) : D$$
$$\vdash (k \circ case(C)(f)(g))(c) \leftrightarrow k(f(a\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\})) : D.$$

From the eq-var-substitution lemma,

$$C' <: Top, f' : A \to C', g' : B \to C' \vdash a \leftrightarrow a\{C' \leftarrow D, f' \leftarrow k \circ f, g' \leftarrow k \circ g\} : A$$
$$C' <: Top, f' : A \to C', g' : B \to C' \vdash a \leftrightarrow a\{C' \leftarrow C, f' \leftarrow f, g' \leftarrow g\} : A$$

Conclude by transitivity and (*Eq appl*). The second case is similar.

(4) Uniqueness can now be shown by the standard argument. ∎

COROLLARY. $\vdash A \sim^{cl} A', \vdash B \sim^{cl} B' \Rightarrow \vdash A + B \sim^{cl} A' + B'$.

Proof. Standard diagram chasing, from the existence of coproducts. ∎

4.2.5. *Well-Pointedness*

A category $\mathbf{C}$ with a terminal object 1 is *well pointed* iff for any pair of objects A and B and any $f, g \in \mathbf{C}(A, B)$ we have

$$f = g \quad \text{iff for any} \quad h \in \mathbf{C}(1, A), f \circ h = g \circ h.$$

PROPOSITION. **CL** *is well pointed. That is, for any* $\vdash A$ *type,* $\vdash B$ *type, and any* $\vdash f, g : A \to B$, *we have*

$$\vdash f \leftrightarrow^{cl} g : A \to B \Leftrightarrow \text{for any } \vdash h : Top \to A, \vdash f \circ h \leftrightarrow^{cl} g \circ h : Top \to B.$$

Proof. ($\Rightarrow$)

$x : Top \vdash f(h(x)) \leftrightarrow f(h(top)) : B$	(*Eq collapse*) and (*Eq appl*)
$x : Top \vdash g(h(x)) \leftrightarrow g(h(top)) : B$	similarly
$x : Top \vdash f(h(top)) \leftrightarrow g(h(top)) : B$	hypothesis, weaken
$\vdash \lambda(x : Top)\, f(h(x)) \leftrightarrow \lambda(x : Top)\, g(h(x)) : Top \to B$	(*Eq trans*) and (*Eq fun*)

Hence $\vdash f \circ h \leftrightarrow g \circ h : Top \to B$.

($\Leftarrow$) Take $\vdash a : A$, consider $h = \lambda(x : Top)a$:

$\vdash (f \circ h)(top) \leftrightarrow (g \circ h)(top) : B$	hypothesis
$\vdash f(a) \leftrightarrow g(a) : B$	(*Eq beta*).

Hence $\vdash f \leftrightarrow^{cl} g : A \to B$. ∎

4.3. **CL** *Isomorphisms*

The following isomorphisms were inspired by [BFSS 90, Fre 91].

4.3.1. *Double Negation*

We prove that, for any $\vdash A\ type$ we have $A \sim \forall(C)(A \to C) \to C$. This is an isomorphism holding in the models studied in [BFSS 90], but which has no known proof in F. (See the remark at the end of this section.)

PROPOSITION.

$$\vdash A\ type \Rightarrow \vdash A \sim^{cl} \forall(C)(A \to C) \to C.$$

Proof. Define

$$f \triangleq \lambda(x{:}\forall(C)(A \to C) \to C)\, x(A)(id(A))$$
$$g \triangleq \lambda(y{:}A)\, \lambda(C)\, \lambda(z{:}A \to C)\, z(y).$$

Then

$$\vdash f{:}(\forall(C)(A \to C) \to C) \to A \qquad \text{and} \qquad \vdash g{:}A \to (\forall(C)(A \to C) \to C).$$

Take a such that $\vdash a{:}A$. Then, by β-conversion,

$$\begin{aligned} \vdash f(g(a)) &\leftrightarrow f(\lambda(C)\, \lambda(z{:}A \to C)\, z(a)) \\ &\leftrightarrow (\lambda(C)\, \lambda(z{:}A \to C)\, z(a))(A)(id(A)) \\ &\leftrightarrow id(A)(a) \leftrightarrow a{:}A. \end{aligned}$$

Take closed b such that $\vdash b{:}\forall(C)(A \to C) \to C$. Then b has a normal form of the shape

$$b = \lambda(C)\, \lambda(z{:}D)\, z(a1)$$

for some $C <: Top \vdash A \to C <: D$ and $C <: Top, z{:}D \vdash a1{:}A$. By the bound weakening lemma,

$$C <: Top, z{:}A \to C \vdash a1{:}A$$

and hence

$$\vdash b \leftrightarrow \lambda(C)\, \lambda(z{:}A \to C)\, z(a1).$$

Then

$$\vdash g(f(b)) \leftrightarrow \lambda(C)\, \lambda(z{:}A \to C)\, z(a1\{C \leftarrow A, z \leftarrow id(A)\}){:}\forall(C)(A \to C) \to C.$$

By the eq-var-substitution lemma,

$$C <: Top, z:A \to C \vdash a1 \leftrightarrow a1\{C \leftarrow A, z \leftarrow id(A)\}:A.$$

Hence,

$$C <: Top, z:A \to C \vdash z(a1) \leftrightarrow z(a1\{C \leftarrow A, z \leftarrow id(A)\}):C.$$

That is,

$$\vdash \lambda(C)\,\lambda(z:A \to C)\,z(a1)$$
$$\leftrightarrow \lambda(C)\,\lambda(z:A \to C)\,z(a1\{C \leftarrow A, z \leftarrow id(A)\}):\forall(C)(A \to C) \to C.$$

Combining the two equations above,

$$\vdash g(f(b)) \leftrightarrow \lambda(C)\,\lambda(z:A \to C)\,z(a1) \leftrightarrow b:\forall(C)(A \to C) \to C. \quad \blacksquare$$

Remark. Christine Paulin-Mohring has shown that, even for A closed, $A \sim \forall(C)(A \to C) \to C$ is not provable in F via the isomorphism we have used in the proof above. (It is not known whether some other isomorphism would work.) To see this, let T be $\forall(R)R \to R$; the term

$$\lambda(P)\,\lambda(x:(T \to T) \to P)\,x(\lambda(y:T)\,y(P \to T)(\lambda(u:P)\,y)(x(\lambda(v:T)v)))$$
$$:\forall(P)((T \to T \to P) \to P$$

is not convertible to any term of the form

$$\lambda(P)\,\lambda(x:(T \to T) \to P)\,x(c),$$

where c is a closed term of type $T \to T$.

Moreover, Roberto Di Cosmo [DiC 94] has shown that A is not isomorphic to $\forall(C)(A \to C) \to C$ in F in the usual sense of F-isomorphisms, as opposed to *cl*-isomorphisms.

4.3.2. *Existentials*

We prove in this section that the terminal type *Top* is isomorphic in **CL** to $\exists(X)X$. From the programming point of view this is consistent with the intuition that, although any value can be encapsulated as an object of type $\exists(X)X$, there is no way of using an object of this type. We prove, more generally, that $\exists(X <: A)X \sim A$ (i.e., $\vdash \exists(X <: A)X \sim^{cl} A$).

LEMMA 1.

$$E \vdash B\ type,\ E \vdash y:\forall(X <: A)X \to B,\ E \vdash A' <: A,\ E \vdash a':A',\ E \vdash a' \leftrightarrow a:A$$
$$\Rightarrow E \vdash y(A)(a) \leftrightarrow y(A')(a'):B.$$

Proof. First,

$$E \vdash y \leftrightarrow y : \forall(X <: A)\, X \to B \qquad \text{hypothesis, } (Eq\ x)$$
$$E \vdash y(A) \leftrightarrow y(A) : A \to B \qquad (Eq\ appl2), \text{ since } X \notin FV(B), \text{ by } E \vdash B\ type$$
$$E \vdash y(A)(a) \leftrightarrow y(A)(a') : B \qquad \text{hypothesis, } (Eq\ appl).$$

Then,

$$E \vdash y \leftrightarrow y : \forall(X <: A)\, X \to B \qquad \text{hypothesis, } (Eq\ x)$$
$$E \vdash y(A) \leftrightarrow y(A') : A' \to B \qquad (Eq\ appl2)$$
$$E \vdash y(A)(a') \leftrightarrow y(A')(a') : B \qquad \text{hypothesis, } (Eq\ appl).$$

Finally,

$$E \vdash y(A)(a) \leftrightarrow y(A')(a') : B. \quad \blacksquare$$

DEFINITION. Let $id : \forall(A)\, \forall(W <: A)\, W \to W \triangleq \lambda(A)\, \lambda(W <: A) \lambda(w : W) w$.

DEFINITION. $\exists(W <: A) B \triangleq \forall(V)(\forall(W <: A) B \to V) \to V$.

$$some : \forall(A)\, \forall(X <: A)\, X \to \exists(W <: A)\, W$$
$$\triangleq \lambda(A)\, \lambda(X <: A)\, \lambda(x : X)$$
$$\lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(X)(x).$$

PROPOSITION. $\vdash A\ type \Rightarrow \vdash A \sim^{cl} \exists(X <: A) X$.

Proof. Let $\vdash f : (\exists(W <: A)\, W) \to A$, where $f = \lambda(p : \exists(W <: A)\, W)\, p(A)(id(A))$.

Let $\vdash g : A \to (\exists(W <: A)\, W)$ where $g = \lambda(x : A)\, some(A)(A)(x)$.

Take a such that $\vdash a : A$. Then

$$\vdash f(g(a)) \leftrightarrow f(some(A)(A)(a))$$
$$\leftrightarrow f(\lambda(V)(\lambda(z : \forall(W <: A)\, W \to V)\, z(A)(a))$$
$$\leftrightarrow (\lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(A)(a))(A)(id(A))$$
$$\leftrightarrow id(A)(A)(a)$$
$$\leftrightarrow a : A.$$

Take closed b such that $\vdash b : \exists(W <: A)\, W$. Then b has a normal form of the shape

$$b = \lambda(V)\, \lambda(z : D)\, z(B1)(b1)$$

for some D, $B1$, $b1$ such that

$$V <: Top \vdash \forall(W <: A)\, W \to V <: D$$

$$V <: Top, z : D \vdash b1 : B1 <: A.$$

By the bound weakening lemma and (*Eq fun'*),

$$\vdash b \leftrightarrow \lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(B1)(b1).$$

Then

$$\begin{aligned}
\vdash g(f(b)) &\leftrightarrow g(b(A)(id(A))) \\
&\leftrightarrow g(id(A)(B1\{V \leftarrow A\})(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
&\leftrightarrow g(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
&\leftrightarrow some(A)(A)(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
&\leftrightarrow \lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(A)(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
&: \exists(W <: A)\, W.
\end{aligned}$$

By the eq-var-substitution lemma, since

$$\vdash id(A) : \forall(W <: A)\, W \to W <: \forall(W <: A)\, W \to A,$$

$$V <: Top, z : \forall(W <: A)\, W \to V \vdash b1 \leftrightarrow b1\{V \leftarrow A, z \leftarrow id(A)\} : A.$$

Hence by Lemma 1,

$$\begin{aligned}
V <: Top, z : \forall(W <: A)\, W \to V &\vdash z(A)(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
&\leftrightarrow z(B1)(b1) : V.
\end{aligned}$$

That is,

$$\begin{aligned}
\vdash \lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(A)(b1\{V \leftarrow A, z \leftarrow id(A)\}) \\
\leftrightarrow \lambda(V)\, \lambda(z : \forall(W <: A)\, W \to V)\, z(B1)(b1) \\
: \exists(W <: A)\, W.
\end{aligned}$$

Combining the two equations above,

$$\begin{aligned}
&\vdash g(f(b)) \\
&\quad \leftrightarrow \lambda(V)\,\lambda(z{:}\forall(W{<:}\,A)\,W \to V)\,z(B1)(b1) \\
&\quad \leftrightarrow b : \exists(W{<:}\,A)\,W. \quad \blacksquare
\end{aligned}$$

COROLLARY.

$$\vdash Top \sim^{cl} \exists(X)\,X.$$

4.3.3. *Other* cl-*isomorphisms*

Many other isomorphisms can be derived with the techniques developed in the previous sections. Among them we have the following.

Domain restriction.

$$C \sim \forall(X)\,X \to C$$

$$A \to C \sim \forall(X{<:}\,A)\,X \to C.$$

Categorical.

$$(A \times B) \times C \sim A \times (B \times C)$$

$$A \times Top \sim Top \times A \sim A$$

$$(A + B) + C \sim A + (B + C)$$

$$A + Bot \sim Bot + A \sim A.$$

Various.

$Top \to A \sim A$	(by simple top collapse)
$A \to Top \sim Top$	(by simple top collapse)
$Top \sim \forall(C)\,C \to C$	(by analyzing the normal forms)
$Bot \to A \sim Top$	(by analyzing the normal forms)
$A \to Bot \sim Bot$ for A nonempty	(by vacuous $f \circ g \leftrightarrow^{cl} id$ conditions since both types are empty)
$\forall(X)(A \to X) \sim A \to \forall(X)X$	(β–η suffices).

5. CONCLUSIONS

We study an extension of system F with subtyping and its equational theory. While the equational rules are not complete for PER models, the main inspirations for the most novel rules come from PER models and categorical notions of parametricity. Although our proof system is not a conservative extension of system F, we prove the conservativity of typing judgments with respect to F. We study some categorical properties of the theory when restricted to closed terms, including interesting categorical isomorphisms. These isomorphisms provide some confidence in the strength of the proof system. Additional evidence is given by a set of encodings; these include record operations and subtyping hierarchies that are related to features of object-oriented languages.

One important area we have not studied is an adequate computation system. Ideally we would like to have a notion of reduction such that any two provably equal terms reduce to a common term. If possible, we would like reductions to terminate as well. A standard approach is to orient each equational axiom in one direction. The two equational rules that lead to immediate problems are (*Eq collapse*) and (*Eq appl2*); for these it is not obvious how to produce an oriented reduction rule. Furthermore, in order to capture equivalence, a set of oriented rules would have to be proved confluent. If we had a computational characterization of equality, we would have decidability of the equational system; in its absence, decidability remains an open problem.

The final form of the (*Eq appl2*) rule is still under investigation. Some recent insights [ACC 93] seem to suggest that (*Eq appl2*$^{-+}$) should be taken instead. Specifically, formal systems considered in [BFSS 90] and [ACC 93] have the latter as a consequence, but not the former. The (*Eq appl2*) rule was adopted here because it is valid in PER and has a simpler syntactic form.

APPENDIX: SYSTEM F

Environments.

$$(Env\ \phi)\quad \frac{}{\vdash \phi\ env} \qquad (Env\ x)\quad \frac{E \vdash A\ type \quad x \notin dom(E)}{\vdash E, x{:}A\ env} \qquad (Env\ X)\quad \frac{\vdash E\ env \quad X \notin dom(E)}{\vdash E, X\ env}$$

Types.

$$(Type\ X)\quad \frac{\vdash E, X, E'\ env}{E, X, E' \vdash X\ type} \qquad (Type \rightarrow)\quad \frac{E \vdash A\ type \quad E \vdash B\ type}{E \vdash A \rightarrow B\ type} \qquad (Type \forall)\quad \frac{E, X \vdash B\ type}{E \vdash \forall(X)B\ type}$$

Values.

$$(Val\ x)\quad \frac{\vdash E, x:A, E'\ env}{E, x:A, E' \vdash x:A} \qquad (Val\ fun)\quad \frac{E, x:A \vdash b:B}{E \vdash \lambda(x:A)b:A \rightarrow B} \qquad (Val\ appl)\quad \frac{E \vdash b:A \rightarrow B \quad E \vdash a:A}{E \vdash b(a):B}$$

$$(Val\ fun2)\quad \frac{E, X \vdash b:B}{E \vdash \lambda(X)b:\forall(X)B} \qquad (Val\ appl2)\quad \frac{E \vdash b:\forall(X)B \quad E \vdash A\ type}{E \vdash b(A):B\{X \leftarrow A\}}$$

Equivalence.

$$(Eq\ symm)\quad \frac{E \vdash a \leftrightarrow b:A}{E \vdash b \leftrightarrow a:A} \qquad (Eq\ trans)\quad \frac{E \vdash a \leftrightarrow b:A \quad E \vdash b \leftrightarrow c:A}{E \vdash a \leftrightarrow c:A}$$

$$(Eq\ x)\quad \frac{E \vdash x:A}{E \vdash x \leftrightarrow x:A}$$

$$(Eq\ fun)\quad \frac{E, x:A \vdash b \leftrightarrow b':B}{E \vdash \lambda(x:A)b \leftrightarrow \lambda(x:A)b':A \rightarrow B} \qquad (Eq\ appl)\quad \frac{E \vdash b \leftrightarrow b':A \rightarrow B \quad E \vdash a \leftrightarrow a':A}{E \vdash b(a) \leftrightarrow b'(a'):B}$$

$$(Eq\ fun2)\quad \frac{E, X \vdash b \leftrightarrow b':B}{E \vdash \lambda(X)b \leftrightarrow \lambda(X)b':\forall(X)B} \qquad (Eq\ appl2)\quad \frac{E \vdash b \leftrightarrow b':\forall(X)B \quad E \vdash A\ type}{E \vdash b(A) \leftrightarrow b'(A):B\{X \leftarrow A\}}$$

$$(Eq\ eta)\quad \frac{E \vdash b \leftrightarrow b':A \rightarrow B \quad y \notin dom(E)}{E \vdash \lambda(y:A)\, b(y) \leftrightarrow b':A \rightarrow B} \qquad (Eq\ eta2)\quad \frac{E \vdash b \leftrightarrow b':\forall(X)B \quad Y \notin dom(E)}{E \vdash \lambda(Y)\, b(Y) \leftrightarrow b':\forall(X)B}$$

$$(Eq\ beta)\quad \frac{E, x:A \vdash b \leftrightarrow b':B \quad E \vdash a \leftrightarrow a':A}{E \vdash (\lambda(x:A)b)(a) \leftrightarrow b'\{x \leftarrow a'\}:B} \qquad (Eq\ beta2)\quad \frac{E, X \vdash b \leftrightarrow b':B \quad E \vdash A\ type}{E \vdash (\lambda(X)b)(A) \leftrightarrow b'\{X \leftarrow A\}:B\{X \leftarrow A\}}$$

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