The Singularity Theory Problem List

A list of problems related to Singularity Theory by and for its community

This is a (very) preliminary version, visit https://rgimenezconejero.github.io/list.html

Edited by R. Giménez Conejero

Chapter 1

Singular varieties

1.1 Classification of algebraic curves and surfaces

Meirav Amram meiravt@sce.ac.il

Classification of algebraic curves and surfaces is a big mathematical problem over the years. Here are two problems to study, the first one is related to the classification of algebraic surfaces, and the second one relates to the classification of algebraic curves.

Problem 1.1. We take a deformation of an algebraic surface to a union of planes and project it onto the projective plane. We compute the fundamental group of the complement of the branch curve in the projective plane, and also other related fundamental groups. Those fundamental groups are invariants of the surfaces and help us to classify the surfaces in the moduli space. Surfaces in the same connected component in the moduli space will have the same related fundamental group.

Moreover, we have singularities of many interesting types in those deformations, it is interesting and challenging to find their regenerations and determine fundamental groups. The study of non-planar deformations can supply appearance of special and complicated singularities.

Studying non-planar deformations of degree 8 and higher than this, check the fundamental groups of the complements of the branch curves.

The study of deformations to unions of planes involves also the study of Zappatic singularities. But if we study much more complicated singularities than Zappatic ones, or Zappatic singularities with higher multiplicity, then additional tools like homotopy techniques, special softwares, or specific results in the theory of singularities could be considered. In particular, non-planar deformations (degenerations) are intriguing because these might reveal more intricate or exotic singularities.

This study can direct us to study Galois covers of algebraic surfaces, because the fundamental group of the Galois cover of a surface is also an invariant of the classification. It is a special quotient of the fundamental group of the complement of the branch curve of the surface. It is interesting to study this group because there is a use of Coxeter or Artin groups to determine fundamental groups.

One of the promising goals is the work on how these fundamental groups change under deformations and what they tell us about moduli spaces of surfaces.

References: [Amr25, AGR$^+$24, AGM23, AGM24a, AGM24b, ALV12, Lib85, Lib07, Lib14, CCFM08, CCFM07, CLM93]

Key words: Deformation of surfaces, regeneration of singularities, Zariski pairs, fundamental groups

Problem 1.2. Concerning algebraic curves, we can study line arrangements and conic line arrangements, and find Zariski pairs. Much of the work is focused on conic-line arrangements of degree equal or higher than 7.

Also deformations of plane curves can be very interesting. So we take the above arrangements and we study the possible deformations and the differences in the groups.

The study of algebraic curve classification has a connection with line arrangements, stick curves and Zariski pairs; this could yield additional insights into how the topology of the curves relates to their algebraic classification.

Exploring the fundamental groups of the complements of algebraic curves through deformation theory could provide further connections to other areas, such as the classification of singularities or the study of moduli spaces for algebraic curves.

References: [ASS+23, ABS+23, ABST20, BT20, ABCAMM19, ABCT08]

Chapter 2

Singular mappings

2.1 Algebra-computing spectral sequences

R. Giménez Conejero *roberto.gimenez@uam.es*

In the theory of singularities of maps, also called *Thom-Mather theory*, one of the commonly studied topics is the topology of the discriminants of maps and how they change after small perturbations. In close connection with this, the algebraic properties of the singularities and their deformations are often expressed in terms of certain *codimensions*, *versal unfoldings*, *bifurcations sets*, etc. Some modern references for this are [MNB20] and [**?**, Chapter 2].

In the case that the dimension of the target space of a map germ, say p, is greater than the dimension of the source, say n, the discriminant coincides with the image. There is a tool to compute the homology of the image of a map that works in great generality, called the *Image-Computing Spectral Sequence* (ICSS). However, there is no such tool to help us to compute the algebra of the singularity of a map, something similar to an *Algebra-Computing Spectral Sequence* (ACSS).

The ICSS of a map $f : N \to P$ uses the *homological properties* of the multiple point spaces of f $D^k(f)$ to compute (to some extent) the homology of $f(N)$ and, for this reason, one hopes that there is an ACSS that uses the *algebraic information* of the multiple point spaces to compute the algebraic invariants we want. More precisely:

Problem 2.1. Find a spectral sequence that, for any map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, computes the $\mathscr{A}_e$-codimension of f using algebraic information of the multiple point spaces $D^k(f)$.

The case of Problem 2.1 for corank one map germs should be simpler, since the multiple point spaces are isolated complete intersection singularities when the germ has finite $\mathscr{A}_e$-codimension. See [MM89, Proposition 2.14].

Problem 2.2. For corank one map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, there is a spectral sequence that uses the Tjurina modules of the multiple point spaces $D^k(f)$ and computes the $\mathscr{A}_e$-codimension of f.

Some recommended references for the ICSS are [GM93, Hou07, CMM22] and [GC21, Chapter 2]. This problem was first stated in the Remark 7.7 of the related work [GC22].

Key words: Singularities of maps, Thom-Mather theory, codimension, spectral sequence, computation

2.2 A Kind of Non-isolated singularities: Flag Projections

Alexey Remizov alexey-remizov@yandex.ru

Introduction. Many problems arising in analysis, geometry, and physics lead us to equations

$$\Delta(x,y)\frac{dp}{dx} = M(x,y,p), \quad p = dy/dx, \tag{2.1}$$

where Δ and M are smooth or analytic functions, $\Delta(x,y)$ vanishes on a smooth regular curve Σ on the (x,y)-plane. Points of Σ are called singular points of Equation (2.1), the curve Σ is called the singular curve. The existence and behaviour of solutions of Equation (2.1) at its generic singular points are analyzed in [Rem20, SS21]. It is proved that a generic Equation (2.1) has no infinitely oscillating solutions, but solutions enter a singular point q_0 only in so-called *admissible* directions p defined by the condition $M(q_0, p)$.

In other words, admissible directions of Equation (2.1) correspond to singular (equilibrium) points of the vector field

$$\dot{x} = \Delta(x,y), \quad \dot{y} = p\Delta(x,y), \quad \dot{p} = M(x,y,p) \tag{2.2}$$

in the (x,y,p)-space. The projections of this field to the (x,y)-plane along the p-axis are the graphs of solutions of Equation (2.1).

Regarding Equation (2.1), a great interest is paid to equations whose right-hand side is a cubic polynomial in p, that is, $M = \sum \mu_i(x,y)p^i$, where $i = 0,\dots,3$. Equations of this type describes, for example, non-parametrized geodesics in 2D-metrics: $ds^2 = adx^2 + 2bdxdy + cdy^2$, where the coefficients a,b,c smoothly or analytically depend on x,y. Then $\Delta = ac - b^2$ is the determinant of the metric, it vanishes at points where the metric degenerates (generically, at points where the metric changes its signature). Generically, the number of admissible directions for geodesic equations at almost all singular points $q \in \Sigma$ is one or three, it is the number of real roots p of the cubic polynomial $M(q,p)$. Singularities of Equation (2.1), especially, equations of geodesics in pseudo-Riemannian metrics were studied in the series of papers [Rem09, Rem15, RT16]. The main tool is this study is the theory of normal forms (going back to H. Poincaré) applied to the germs of vector field of Equation (2.2) at its singular points (q, p_i), where $q \in \Sigma$ and $M(q, p_i) = 0$.

Under some general assumptions, the spectrum of the linear part of the germ of Equation (2.2) at its singular point (q_0, p_i) is $\{\lambda_1, \lambda_2, 0\}$, where

$$\lambda_1 = \Delta_x(q_0) + p_i\Delta_y(q_0) \neq 0, \quad \lambda_2 = M_p(q_0, p_i) \neq 0. \tag{2.3}$$

Then the center manifold of Equation (2.2) consists of its singular points, and the reduction principle [HPS77] yields that the germ of the field of Equation (2.2) at (q_0, p_i) is topologically equivalent to the germ

$$\dot{\xi} = \xi, \quad \dot{\eta} = \sigma\eta, \quad \dot{\zeta} = 0 \tag{2.4}$$

at the origin, where $\sigma = \pm 1$ is the signum of $\lambda(q_0, p_i) = \lambda_2(q_0, p_i)/\lambda_1(q_0, p_i)$.

The topological normal form of Equation (2.4) gives an idea of the phase portrait of the field of Equation (2.2) near its singular points, but for our purposes it is not enough. More information one can get from a smooth (or analytic) normal form. A number of smooth normal form (for various λ) can be found in [RT16, Appendix]. Here we present only one result of this kind:

Key words: Singularities of smooth mappings, projection, normal form

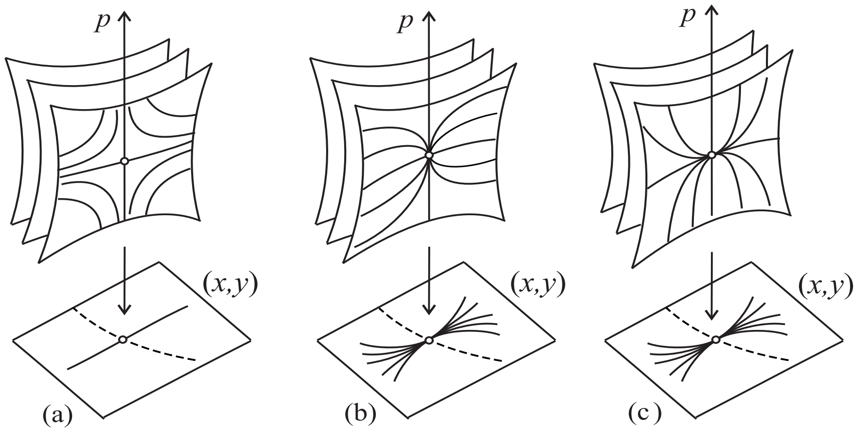

Figure 2.1: Trajectories of the field (2.2) on the invariant leaf and their projections on the (x, y)-plane. From left to right: $\lambda < 0$ (a), $0 < \lambda < 1$ (b), $\lambda > 1$ (c). The curve Σ on the (x, y)-plane is depicted as dashed line.

Theorem 2.3. *If $\lambda(q_0, p_i)$ is positive and not integer or it is negative and not rational, than the germ of the direction field of Equation (2.2) at (q_0, p_i) is C^k-smooth equivalent to the germ*

$$\dot{\xi} = \xi, \quad \dot{\eta} = a(\zeta)\eta, \quad \dot{\zeta} = 0 \tag{2.5}$$

at the origin, with any integer $k \geq 1$.

From what is said above, it follows that solutions of Equation (2.1) entering $q_0 \in \Sigma$ with the admissible direction p_i are the projections of trajectories of the field of Equation (2.2) that enter (q_0, p_i) and lie on the invariant leaf corresponding to the invariant plane $\zeta = 0$ in Equation (2.5). See Figure 2.1.

Example 2.4. Consider the differential equation $2xy'' = y'$, which describes geodesics in the pseudo-Riemannian metric $ds^2 = xdx^2 + dy^2$. This metric changes its signature on the line $x = 0$, which in the singular curve Σ of the equation $2xy'' = y'$, whose solutions are given by the formula $y = k|x|^{3/2} + c$, where c, k are arbitrary constants. Therefore, geodesics entering a singular point $q_0 = (0, y_0)$ are semicubic parabolas

$$y = y_0 + k|x|^{3/2}, \quad k \in \mathbb{R}, \tag{2.6}$$

with the common tangential direction $p = 0$.

Let us see how the general theory works in this case. The vector field of Equation (2.2) reads

$$\dot{x} = x, \quad \dot{y} = xp, \quad \dot{p} = p/2. \tag{2.7}$$

It is not hard to see that the field of Equation (2.7) has the first integral $U = y - \frac{2}{3}xp$, whose level sets constitute the invariant foliation. The restriction of the field of Equation (2.7) to every invariant leaf $U = c$ is a node; see (b) of Figure 2.1. Projecting trajectories lying on the leaf $U = y_0$ down to the (x, y)-plane, we get the family of semicubic parabolas of Equation (2.6).

Conclusion. Further we considered the projections of curves lying on invariant leaves of the field of Equation (2.2), but it is interesting to study the projections of the leaves themselves. Remark that such projections do not belong to known classes of mappings (fold, cusp, etc.). An important feature of such projections is that every leaf contains a line (parallel to the p-axis) whose projection is a point. This justifies the name "flag projections": this line plays a role of the flagpole, while the leaf looks like the flag cloth.

Choosing the variables (x, p) as coordinates on the leaf, one can formulate the problem as follows.

Setting of the problem. Consider the germ (at the origin) of a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$\begin{pmatrix} x \\ p \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} xF(x,p) \\ x \end{pmatrix} \tag{2.8}$$

with smooth or analytic function F. The Jacobi matrix of this mapping is

$$J_f = \begin{pmatrix} F + xF_x & xF_p \\ 1 & 0 \end{pmatrix},$$

whence the set of critical points $\Sigma = \{(x,p) : |J_f| = 0\}$ is given by the equation $xF_p = 0$. The set Σ contains the p-axis ($x = 0$). We remark that the kernel of J_f is tangent to the axis $x = 0$ at all its points. This property makes the mapping of Equation (2.8) completely non-generic in the space of smooth mappings, and RL-normal forms for such germs are not studied.

First of all, one can make the following obvious simplification. Let us present the function F as $F(x,p) = f_0(x) + pf(x,y)$, then the left change of variables $y \mapsto y - zf_0(z)$ brings the germ of Equation (2.8) to the form

$$\begin{pmatrix} x \\ p \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} xpf(x,p) \\ x \end{pmatrix}. \tag{2.9}$$

Assume that $f(0) \neq 0$, then the critical set Σ coincides with the p-axis. Then the right change of variables $p \mapsto pf(x,p)$ brings the germ of Equation (2.9) to the normal form:

$$\begin{pmatrix} x \\ p \end{pmatrix} \mapsto \begin{pmatrix} xp \\ x \end{pmatrix}.$$

Problem 2.5. The case $f(0) = 0$, i.e., the germs of Equation (2.9) with functions

$$f(x,p) = xf_1(x,p) + pf_2(x,p)$$

are not studied yet. It is not even known whether RL-classification of such mappings has functional invariants or not.

The author is grateful to the reviewer, who recommended a number of sources related to the problem: [AFKTW23], [CMTS23, Chapter 2], and [Zah91, Zah94].

Bibliography

[ABCAMM19] E. Artal Bartolo, J. I. Cogolludo-Agustín, and J. Martín-Morales. Triangular curves and cyclotomic Zariski tuples. *Collectanea Mathematica*, 71(3):427–441, November 2019. 4

[ABCT08] E. Artal Bartolo, J. I. Cogolludo, and H.-o. Tokunuga. A survey on Zariski pairs. In *Proceedings of the 2nd international conference on Algebraic Geometry in East Asia, Hanoi, Vietnam, 2005*. Tokyo: Mathematical Society of Japan, 2008. 4

[ABS$^+$23] M. Amram, S. Bannai, T. Shirane, U. Sinichkin, and H. o Tokunaga. The realization space of a certain conic line arrangement of degree 7 and a π_1-equivalent Zariski pair. *arxiv.org/abs/2307.01736, to appear in Israel Journal of Mathematics*, 2023. 4

[ABST20] E. Artal Bartolo, S. Bannai, T. Shirane, and H. Tokunaga. Torsion divisors of plane curves with maximal flexes and Zariski pairs. *arxiv.org/abs/2005.12673*, 2020. 4

[AFKTW23] D. Abramovich, A. Frühbis-Krüger, M. Temkin, and J. Włodarczyk. *New Techniques in Resolution of Singularities*. Birkhäuser Cham, 2023. 8

[AGM23] M. Amram, C. Gong, and J.-L. Mo. On the galois covers of degenerations of surfaces of minimal degree. *Mathematische Nachrichten*, 296(4):1351–1365, 2023. 3

[AGM24a] M. Amram, C. Gong, and J.-L. Mo. On galois covers of a union of Zappatic surfaces of type R_k. *Collectanea Mathematica*, November 2024. 3

[AGM24b] M. Amram, C. Gong, and J. Mo. Deformations of Zappatic stable surfaces and their Galois covers. With appendix by János Kollár. *arxiv.org/abs/2402.06017*, 2024. 3

[AGR$^+$24] M. Amram, C. Gong, P. K. Roy, U. Sinichkin, and U. Vishne. The fundamental group of Galois covers of surfaces with octahedral envelope. *arxiv.org/abs/2303.05241, to appear in Homology, Homotopyand Applications*, 2024. 3

[ALV12] M. Amram, R. Lawrence, and U. Vishne. Artin covers of the braid groups. *Journal of Knot Theory and Its Ramifications*, 21(07), 2012. 3

[Amr25] M. Amram. Non-planar degenerations and related fundamental groups. *European Journal of Mathematics*, 11(1), December 2025. 3

[ASS$^+$23] M. Amram, R. Shwartz, U. Sinichkin, S.-L. Tan, and H.-o. Tokunaga. Zariski pairs of conic-line arrangements of degrees 7 and 8 via fundamental groups. *arxiv.org/abs/2106.03507*, 2023. 4

[BT20] S. Bannai and H.-o. Tokunaga. Zariski tuples for a smooth cubic and its tangent lines. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 96(2), February 2020. 4

[CCFM07] A. Calabri, C. Ciliberto, F. Flamini, and R. Miranda. On the K^2 of degenerations of surfaces and the multiple point formula. *Ann. Math. (2)*, 165(2):335–395, 2007. 3

[CCFM08] A. Calabri, C. Ciliberto, F. Flamini, and R. Miranda. On degenerations of surfaces. *https://arxiv.org/abs/math/0310009*, 2008. 3

[CLM93] C. Ciliberto, A. Lopez, and R. Miranda. Projective degenerations of K_3 surfaces, Gaussian maps, and Fano threefolds. *Inventiones Mathematicae*, 114(1):641–667, December 1993. 3

[CMM22] J. L. Cisneros-Molina and D. Mond. Multiple points of a simplicial map and image-computing spectral sequences. *Journal of Singularities*, 24:190–212, 2022. 5

[CMTS23] J. L. Cisneros Molina, L. D. Tráng, and J. Seade, editors. *Handbook of geometry and topology of singularities IV*. Springer Cham, 2023. 8

[GC21] R. Giménez Conejero. *Singularities of germs and vanishing homology*. PhD thesis, Universitat de València, 2021. 5

[GC22] R. Giménez Conejero. Isotypes of ICIS and images of deformations of map germs. *arXiv:2207.05196*, July 2022. 5

[GM93] V. Goryunov and D. Mond. Vanishing cohomology of singularities of mappings. *Compositio Mathematica*, 89(1):45–80, 1993. 5

[Hou07] K. Houston. A general image computing spectral sequence. In *Singularity theory*, pages 651–675. World Sci. Publ., Hackensack, NJ, 2007. 5

[HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant Manifolds*. Springer Berlin, Heidelberg, 1977. 6

[Lib85] A. Libgober. Fundamental groups of the complements to plane singular curves. In *Algebraic Geometry–Bowdoin, Part 2*. 1985. 3

[Lib07] A. Libgober. *Problems in topology of the complements to plane singular curves*. Singularities in Geometry and Topology. Proceedings of the Trieste Singularity Summer School and Workshop, ICTP, 2007. 3

[Lib14] A. Libgober. Topology of algebraic curves. an approach via dessins d'enfants, by Alex Degtyarev. Technical report, American Mathematical Society 51, 2014. 3

[MM89] W. L. Marar and D. Mond. Multiple point schemes for corank 1 maps. *Journal of the London Mathematical Society. Second Series*, 39(3):553–567, 1989. 5

[MNB20] D. Mond and J. J. Nuño-Ballesteros. *Singularities of mappings*, volume 357 of *Grundlehren der mathematischen Wissenschaften*. Springer, Cham, 2020. 5

[Rem09] A. O. Remizov. Geodesics on 2-surfaces with pseudo-Riemannian metric: singularities of changes of signature. *Mat. Sb.*, 200(3):385, apr 2009. 6

[Rem15] A. O. Remizov. On the local and global properties of geodesics in pseudo-Riemannian metrics. *Differential Geometry and its Applications*, 39:36–58, 2015. 6

[Rem20] A. O. Remizov. On solutions of singular differential equations of the second order. *ArXiv:2006.15949*, June 2020. 6

[RT16] A. O. Remizov and F. Tari. Singularities of the geodesic flow on surfaces with pseudo-riemannian metrics. *Geometriae Dedicata*, 185(1):131–153, June 2016. 6

[SS21] W. M. Seiler and M. Seiß. Singular initial value problems for scalar quasi-linear ordinary differential equations. *Journal of Differential Equations*, 281:258–288, 2021. 6

[Zah91] A. Zaharia. On simple germs with non-isolated singularities. *Mathematica Scandinavica*, 68(2):187–192, 1991. 8

[Zah94] A. Zaharia. Topological properties of certain singularities with critical locus a 2-dimensional complete intersection. *Topology and its Applications*, 60(2):153–171, 1994. 8