Calculus Warm-Up Quiz — Day 7

Maximum Likelihood Estimation and Normal Regression

Complete before lecture. 10 minutes.

Name: ____________________

Today we maximize likelihood functions. This requires the product rule, the derivative of $\log x$, and the chain rule applied to exponentials. We also compute second derivatives (the Hessian) to characterize curvature.

1. Product Rule. Compute the derivative.

(a) $\dfrac{d}{dp}\left[p^3(1-p)^2\right] =$

(b) Set your answer equal to zero and solve for p. (Hint: factor out $p^2(1-p)$.)

2. Derivatives of $\log$ **and** $1/x$**.** Compute:

(a) $\dfrac{d}{d\lambda}\left[-3\log\lambda\right] =$

(b) $\dfrac{d}{d\lambda}\left[-\dfrac{5}{\lambda}\right] =$

(c) $\dfrac{d}{d\lambda}\left[-3\log\lambda - \dfrac{5}{\lambda}\right] =$ Set this to zero and solve for λ.

3. Log-Likelihood of the Normal.
The log-density of a $N(\mu, \sigma^2)$ random variable x is:

$$\log f(x|\mu, \sigma^2) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

(a) Treating σ^2 as known, compute $\dfrac{\partial}{\partial\mu} \log f(x|\mu, \sigma^2)$.

(b) For n independent observations, the log-likelihood is $\ell(\mu) = \sum_{i=1}^{n} \log f(x_i|\mu, \sigma^2)$. Set $\dfrac{d\ell}{d\mu} = 0$ and solve for $\hat{\mu}$.

4. Second Derivative (Hessian Preview).
From Question 2(c), let $\ell(\lambda) = -3 \log \lambda - 5/\lambda$.

(a) Compute $\ell'(\lambda)$ (you already did this).

(b) Compute $\ell''(\lambda)$. (Differentiate $\ell'(\lambda)$ again.)

(c) Evaluate $\ell''(\lambda)$ at $\hat{\lambda}$ from 2(c). Is it negative? (This confirms a maximum.)

5. Chain Rule with Exponentials. Compute:

(a) $\dfrac{d}{dx} e^{-2x} =$

(b) $\dfrac{d}{dx} e^{-x^2/2} =$

Answer Key — Day 7

1. (a) $3p^2(1-p)^2 - 2p^3(1-p) = p^2(1-p)[3(1-p)-2p] = p^2(1-p)(3-5p)$

(b) $p^2(1-p)(3-5p) = 0$. Solutions: $p = 0$, $p = 1$, $p = 3/5$. The interior maximum is $p = 3/5$.

2. (a) $-3/\lambda$

(b) $5/\lambda^2$

(c) $-3/\lambda + 5/\lambda^2 = 0 \implies 5/\lambda^2 = 3/\lambda \implies \lambda = 5/3$.

3. (a) $\dfrac{\partial}{\partial \mu} \log f = \dfrac{x-\mu}{\sigma^2}$

(b) $\dfrac{d\ell}{d\mu} = \dfrac{1}{\sigma^2} \sum_{i=1}^{n}(x_i - \mu) = 0 \implies n\mu = \sum x_i \implies \hat{\mu} = \bar{x} = \dfrac{1}{n} \sum x_i$.

4. (a) $\ell'(\lambda) = -3/\lambda + 5/\lambda^2$

(b) $\ell''(\lambda) = 3/\lambda^2 - 10/\lambda^3$

(c) At $\hat{\lambda} = 5/3$: $\ell''(5/3) = 3/(25/9) - 10/(125/27) = 27/25 - 270/125 = 27/25 - 54/25 = -27/25 < 0$. Confirmed: maximum.

5. (a) $-2e^{-2x}$

(b) $-x\, e^{-x^2/2}$