Calculus Warm-Up Quiz — Day 2

CEF and Best Linear Predictor

Complete before lecture. 10 minutes.

Name: _____

Today we derive the Best Linear Predictor by minimizing $\mathbb{E}[(Y - a - bX)^2]$ with respect to two parameters a and b. We also interpret regression coefficients as partial derivatives. This quiz reviews those skills.

1. **Expanding a Squared Expression.** Expand $(Y - a - bX)^2$ into individual terms.

2. **Multi-Parameter Optimization.** Let $g(a, b) = a^2 + b^2 + 2ab - 4a - 6b$.

   (a) Find $\dfrac{\partial g}{\partial a}$ and set it to zero.

   (b) Find $\dfrac{\partial g}{\partial b}$ and set it to zero.

   (c) Solve the two equations simultaneously for a and b.

3. **Differentiating Through a Sum/Expectation.**
   Suppose $\mathbb{E}[Y] = 3$, $\mathbb{E}[X] = 2$, $\mathbb{E}[X^2] = 5$, $\mathbb{E}[XY] = 8$.
   Consider $M(a, b) = \mathbb{E}[(Y - a - bX)^2] = \mathbb{E}[Y^2] - 2a\mathbb{E}[Y] - 2b\mathbb{E}[XY] + a^2 + 2ab\mathbb{E}[X] + b^2\mathbb{E}[X^2]$.

   (a) Compute $\dfrac{\partial M}{\partial a}$ and set it to zero. Express a in terms of b.

(b) Compute $\frac{\partial M}{\partial b}$ and set it to zero. Substitute part (a) and solve for b.

4. Marginal Effects as Partial Derivatives.
Consider the regression model $\mathbb{E}[Y|X, Z] = 5 + 3X - 2Z + 0.5\,XZ$.

(a) What is $\frac{\partial\,\mathbb{E}[Y|X,Z]}{\partial X}$? (This is the marginal effect of X.)

(b) What is the marginal effect of X when $Z = 4$?

(c) What is the marginal effect of X when $Z = 0$?

Answer Key — Day 2

1. $Y^2 - 2aY - 2bXY + a^2 + 2abX + b^2X^2$

2. (a) $\frac{\partial g}{\partial a} = 2a + 2b - 4 = 0$

(b) $\frac{\partial g}{\partial b} = 2b + 2a - 6 = 0$

(c) From (a): $a + b = 2$. From (b): $a + b = 3$. These are inconsistent, so the system has no finite critical point. (This signals the function has no interior minimum—it is unbounded below along certain directions. In lecture, the positive definiteness of $\mathbb{E}[XX']$ rules this out for the BLP problem.)

3. (a) $\frac{\partial M}{\partial a} = -2\mathbb{E}[Y] + 2a + 2b\mathbb{E}[X] = -6 + 2a + 4b = 0$, so $a = 3 - 2b$.

(b) $\frac{\partial M}{\partial b} = -2\mathbb{E}[XY] + 2a\mathbb{E}[X] + 2b\mathbb{E}[X^2] = -16 + 4a + 10b = 0$.
Substitute $a = 3 - 2b$: $-16 + 4(3 - 2b) + 10b = -16 + 12 - 8b + 10b = -4 + 2b = 0$, so $b = 2$.
Then $a = 3 - 2(2) = -1$.

4. (a) $\frac{\partial \mathbb{E}[Y|X,Z]}{\partial X} = 3 + 0.5Z$

(b) $3 + 0.5(4) = 5$

(c) $3 + 0.5(0) = 3$