Linear Models Lecture 4: Algebra of Bias

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Why Partition $\boldsymbol{X}$?

- In applied work, we rarely care equally about every regressor. We typically have:
  - A **treatment** or variable of interest ($\boldsymbol{X}_1$), and
  - **Controls** we include to avoid omitted variable bias ($\boldsymbol{X}_2$).
- Partitioning $\boldsymbol{X} = [\boldsymbol{X}_1 \quad \boldsymbol{X}_2]$ lets us answer three questions:
  1. What is the formula for $\hat{\beta}_1$ *holding* $\boldsymbol{X}_2$ *constant*? $\rightarrow$ Frisch-Waugh-Lovell.
  2. What happens to $\hat{\beta}_1$ if we *omit* $\boldsymbol{X}_2$? $\rightarrow$ Omitted variable bias formula.
  3. How sensitive is $\hat{\beta}_1$ to *unobserved* confounders? $\rightarrow$ Cinelli-Hazlett sensitivity analysis.
- The algebra of partitioned matrices (Schur complement) is the common tool behind all three results.

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Partitioned Regression

- We have seen that it is possible to partition the matrix $\boldsymbol{X}$ and β into subcomponents:

$$\boldsymbol{y} = \boldsymbol{X}_1\beta_1 + \boldsymbol{X}_2\beta_2 + \boldsymbol{e}$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \end{bmatrix}$$

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix}$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

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\boldsymbol{X}'\boldsymbol{X} &= \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \\
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\beta &= \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}
\end{aligned}
$$

- We used this partition to study how including (or failing to include) control variables affects a multivariate regression.

Formula for Inverse of $\boldsymbol{X}'\boldsymbol{X}$

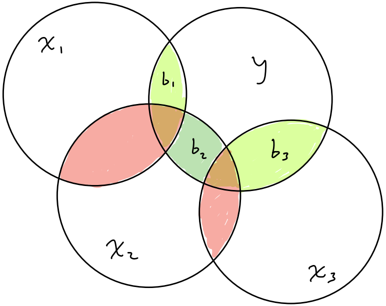
In the next few slides, we will derive the following formula:

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{bmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1} & -(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1} \\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} & (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1} \end{bmatrix}$$

Where

$$\boldsymbol{P}_1 = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'$$
$$\boldsymbol{P}_2 = \boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'$$

Intuition



$$
\begin{aligned}
(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1} &= (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{P}_2\boldsymbol{X}_1)^{-1} \\
&= (\underbrace{\boldsymbol{X}_1'\boldsymbol{X}_1}_{\text{variance of } \boldsymbol{X}_1} - \underbrace{\boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{P}_2\boldsymbol{X}_1}_{\text{Projection of } \boldsymbol{X}_1 \text{ onto the column space of } \boldsymbol{X}_2})^{-1}
\end{aligned}
$$

Finding Inverse of Partitioned Matrix

Call the $n \times n$ partitioned matrix $\boldsymbol{A}$ and its inverse $\boldsymbol{A}^{-1}$, $\boldsymbol{B}$:

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_1 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} = \boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$$

We know that if $\boldsymbol{B} = \boldsymbol{A}^{-1}$, then $\boldsymbol{AB} = \boldsymbol{I}$

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12} + \boldsymbol{A}_{12}\boldsymbol{B}_{22} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} & \boldsymbol{A}_{21}\boldsymbol{B}_{12} + \boldsymbol{A}_{22}\boldsymbol{B}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0}_{k,n-k} \\ \boldsymbol{0}_{n-k,k} & \boldsymbol{I}_{n-k} \end{pmatrix}$$

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$$\begin{aligned} \boldsymbol{AB} &= \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12} + \boldsymbol{A}_{12}\boldsymbol{B}_{22} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} & \boldsymbol{A}_{21}\boldsymbol{B}_{12} + \boldsymbol{A}_{22}\boldsymbol{B}_{22} \end{pmatrix} \quad = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0}_{k,n-k} \\ \boldsymbol{0}_{n-k,k} & \boldsymbol{I}_{n-k} \end{pmatrix} \end{aligned}$$

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Inverse of Partitioned Matrix (finding $\boldsymbol{B}_{11}$)

We can use the following two equations to find $\boldsymbol{B}_{11}$ in terms of $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{22}$, $\boldsymbol{A}_{21}$:

$$\begin{aligned}
\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} &= \boldsymbol{I}_k \\
\boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} &= \boldsymbol{0}_{n-k,k} \\
\boldsymbol{B}_{21} &= -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{B}_{11} \\
\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}[-\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{B}_{11}] &= \boldsymbol{I}_k \\
(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})\boldsymbol{B}_{11} &= \boldsymbol{I}_k
\end{aligned}$$

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Inverse of Partitioned Matrix (finding $\boldsymbol{B}_{11}$)

$$(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})\boldsymbol{B}_{11} = \boldsymbol{I}_k$$

$$(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1} = \boldsymbol{B}_{11} \qquad \text{(inverse of Schur complement)}$$

$$(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1)^{-1} = \boldsymbol{B}_{11} \qquad \text{(plugging in for } \boldsymbol{A}\text{s)}$$

$$(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1} = \boldsymbol{B}_{11} \qquad \text{(Sub in notation for } \boldsymbol{P}_2.\text{)}$$

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\end{aligned}
$$

Result: Formula for Inverse of Partitioned Matrix

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$$

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{bmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1} & -(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1} \\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} & (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1} \end{bmatrix}$$

Using Inverse of Partitioned Matrix formula

Regressing $\boldsymbol{X}_1$ onto $\boldsymbol{X}$ gives us the identity matrix and 0:

$$
\begin{aligned}
(X'X)^{-1}X'X_1 &= (X'X)^{-1}\begin{bmatrix} X_1'X_1 \\ X_2'X_1 \end{bmatrix} \\
&= \begin{bmatrix} B_{11}X_1'X_1 + B_{12}X_2'X_1 \\ B_{21}X_1'X_1 + B_{22}X_2'X_1 \end{bmatrix} \\
&= \begin{bmatrix} (X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_1 - (X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}X_2'X_1 \\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1 \end{bmatrix} \\
&= \begin{bmatrix} (X_1'X_1 - X_1'P_2X_1)^{-1}[X_1'X_1 - X_1'P_2X_1] \\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1 \end{bmatrix} \\
&= \begin{bmatrix} I \\ 0 \end{bmatrix}
\end{aligned}
$$

Using Inverse of Partitioned Matrix formula

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&= \begin{bmatrix}B_{11}X_1'X_1 + B_{12}X_2'X_1\\ B_{21}X_1'X_1 + B_{22}X_2'X_1\end{bmatrix}\\
&= \begin{bmatrix}(X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_1 - (X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}X_2'X_1\\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1\end{bmatrix}\\
&= \begin{bmatrix}(X_1'X_1 - X_1'P_2X_1)^{-1}[X_1'X_1 - X_1'P_2X_1]\\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1\end{bmatrix}\\
&= \begin{bmatrix}I\\ 0\end{bmatrix}
\end{aligned}
$$

Using Inverse of Partitioned Matrix formula

Regressing $\boldsymbol{X}_1$ onto $\boldsymbol{X}$ gives us the identity matrix and 0:

$$
\begin{aligned}
(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}_1 &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\begin{bmatrix}\boldsymbol{X}_1'\boldsymbol{X}_1\\ \boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{B}_{11}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{12}\boldsymbol{X}_2'\boldsymbol{X}_1\\ \boldsymbol{B}_{21}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{22}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_1 - (X_1'X_1 - X_1'P_2X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}X_2'X_1\\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1\end{bmatrix}\\
&= \begin{bmatrix}(X_1'X_1 - X_1'P_2X_1)^{-1}[X_1'X_1 - X_1'P_2X_1]\\ -(X_2'X_2 - X_2'P_1'X_2)^{-1}X_2'X_1 + (X_2'X_2 - X_2'P_1X_2)^{-1}X_2'X_1\end{bmatrix}\\
&= \begin{bmatrix}I\\ 0\end{bmatrix}
\end{aligned}
$$

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Regressing $\boldsymbol{X}_1$ onto $\boldsymbol{X}$ gives us the identity matrix and 0:

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&= \begin{bmatrix}\boldsymbol{B}_{11}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{12}\boldsymbol{X}_2'\boldsymbol{X}_1\\ \boldsymbol{B}_{21}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{22}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 - (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}[\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1]\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}
\end{aligned}
$$

Using Inverse of Partitioned Matrix formula

Regressing $\boldsymbol{X}_1$ onto $\boldsymbol{X}$ gives us the identity matrix and 0:

$$
\begin{aligned}
(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}_1 &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\begin{bmatrix}\boldsymbol{X}_1'\boldsymbol{X}_1\\ \boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{B}_{11}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{12}\boldsymbol{X}_2'\boldsymbol{X}_1\\ \boldsymbol{B}_{21}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{22}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 - (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}[\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1]\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}
\end{aligned}
$$

Using Inverse of Partitioned Matrix formula

Regressing $\boldsymbol{X}_1$ onto $\boldsymbol{X}$ gives us the identity matrix and 0:

$$
\begin{aligned}
(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}_1 &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\begin{bmatrix}\boldsymbol{X}_1'\boldsymbol{X}_1\\ \boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{B}_{11}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{12}\boldsymbol{X}_2'\boldsymbol{X}_1\\ \boldsymbol{B}_{21}\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{B}_{22}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 - (\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}(\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1)^{-1}[\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{P}_2\boldsymbol{X}_1]\\ -(\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\end{bmatrix}\\
&= \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}
\end{aligned}
$$

Schur complement for B_{11}: $\left(10 - \frac{3^2}{8}\right)^{-1} = \left(\frac{71}{8}\right)^{-1} = \frac{8}{71}$ ✓

The denominator $10 - \frac{9}{8} = \frac{71}{8}$ is the variance of education *not explained by* experience.

> The Schur complement isolates the *unique* information in $\boldsymbol{X}_1$ after removing $\boldsymbol{X}_2$. More collinearity → less unique info → larger standard errors.

- Two regressors (education, experience) with cross-product matrix: $\boldsymbol{X}'\boldsymbol{X} = 103$

3 8 *Directinversion* $:(\boldsymbol{X}'\boldsymbol{X})^{-1} = \frac{1}{10 \cdot 8 - 3^2} 8 - 3$

-3 10 $= 1 \frac{}{718 - 3}$

-3 10

Frisch Waugh Lovell in Matrix Terms

- FWL claim is that the regression coefficient $\hat{\beta}_2$ is the same as the result of first regressing $\boldsymbol{y}$ and $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$ and then on one another. That is, we can first project $\boldsymbol{y}$ into the orthogonal complement of the column space of $\boldsymbol{X}_1$, then project it onto $\boldsymbol{X}_2$.

$$\boldsymbol{y} = \boldsymbol{X}_1\hat{\beta}_1 + \boldsymbol{X}_2\hat{\beta}_2 + e$$

$$\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 + \boldsymbol{M}_1 e$$

$$\hat{\beta}_2 = ((\boldsymbol{M}_1\boldsymbol{X}_2)'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}(\boldsymbol{X}_2\boldsymbol{M}_1)'\boldsymbol{M}_1\boldsymbol{y}$$

$$= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}$$

- Where $\boldsymbol{M}_1\boldsymbol{q}$ produces the residuals of any variable $\boldsymbol{q}$ regressed on $\boldsymbol{X}_1$

Frisch Waugh Lovell in Matrix Terms

- FWL claim is that the regression coefficient $\hat{\beta}_2$ is the same as the result of first regressing $\boldsymbol{y}$ and $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$ and then on one another. That is, we can first project $\boldsymbol{y}$ into the orthogonal complement of the column space of $\boldsymbol{X}_1$, then project it onto $\boldsymbol{X}_2$.

$$\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}_1\hat{\beta}_1 + \boldsymbol{X}_2\hat{\beta}_2 + e \\
\boldsymbol{M}_1\boldsymbol{y} &= \boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 + \boldsymbol{M}_1 e \\
\hat{\beta}_2 &= ((\boldsymbol{M}_1\boldsymbol{X}_2)'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}(\boldsymbol{X}_2\boldsymbol{M}_1)'\boldsymbol{M}_1\boldsymbol{y} \\
&= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}$$

- Where $\boldsymbol{M}_1\boldsymbol{q}$ produces the residuals of any variable $\boldsymbol{q}$ regressed on $\boldsymbol{X}_1$

Frisch Waugh Lovell in Matrix Terms

- FWL claim is that the regression coefficient $\hat{\beta}_2$ is the same as the result of first regressing $\boldsymbol{y}$ and $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$ and then on one another. That is, we can first project $\boldsymbol{y}$ into the orthogonal complement of the column space of $\boldsymbol{X}_1$, then project it onto $\boldsymbol{X}_2$.

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}_1\hat{\beta}_1 + \boldsymbol{X}_2\hat{\beta}_2 + e \\
\boldsymbol{M}_1\boldsymbol{y} &= \boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 + \boldsymbol{M}_1 e \\
\hat{\beta}_2 &= ((\boldsymbol{M}_1\boldsymbol{X}_2)'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}(\boldsymbol{X}_2\boldsymbol{M}_1)'\boldsymbol{M}_1\boldsymbol{y} \\
&= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}
$$

- Where $\boldsymbol{M}_1\boldsymbol{q}$ produces the residuals of any variable $\boldsymbol{q}$ regressed on $\boldsymbol{X}_1$

Frisch Waugh Lovell in Matrix Terms

- FWL claim is that the regression coefficient $\hat{\beta}_2$ is the same as the result of first regressing $\boldsymbol{y}$ and $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$ and then on one another. That is, we can first project $\boldsymbol{y}$ into the orthogonal complement of the column space of $\boldsymbol{X}_1$, then project it onto $\boldsymbol{X}_2$.

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}_1\hat{\beta}_1 + \boldsymbol{X}_2\hat{\beta}_2 + e \\
\boldsymbol{M}_1\boldsymbol{y} &= \boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 + \boldsymbol{M}_1 e \\
\hat{\beta}_2 &= ((\boldsymbol{M}_1\boldsymbol{X}_2)'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}(\boldsymbol{X}_2\boldsymbol{M}_1)'\boldsymbol{M}_1\boldsymbol{y} \\
&= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}
$$

- Where $\boldsymbol{M}_1\boldsymbol{q}$ produces the residuals of any variable $\boldsymbol{q}$ regressed on $\boldsymbol{X}_1$

Frisch Waugh Lovell in Matrix Terms

- FWL claim is that the regression coefficient $\hat{\beta}_2$ is the same as the result of first regressing $\boldsymbol{y}$ and $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$ and then on one another. That is, we can first project $\boldsymbol{y}$ into the orthogonal complement of the column space of $\boldsymbol{X}_1$, then project it onto $\boldsymbol{X}_2$.

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}_1\hat{\beta}_1 + \boldsymbol{X}_2\hat{\beta}_2 + e \\
\boldsymbol{M}_1\boldsymbol{y} &= \boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 + \boldsymbol{M}_1 e \\
\hat{\beta}_2 &= ((\boldsymbol{M}_1\boldsymbol{X}_2)'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}(\boldsymbol{X}_2\boldsymbol{M}_1)'\boldsymbol{M}_1\boldsymbol{y} \\
&= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}
$$

- Where $\boldsymbol{M}_1\boldsymbol{q}$ produces the residuals of any variable $\boldsymbol{q}$ regressed on $\boldsymbol{X}_1$

Normal Form Equations (FWL proof part I)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$\begin{aligned} (\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_1'\boldsymbol{y} \\ (\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 &= \boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \\ \hat{\beta}_1 &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \end{aligned}$$

(This is the omitted variable bias formula)

$$\hat{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)$$

Normal Form Equations (FWL proof part I)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 = \boldsymbol{X}_1'\boldsymbol{y}$$

$$(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 = \boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2$$

$$\hat{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2$$

(This is the omitted variable bias formula)

$$\hat{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)$$

Normal Form Equations (FWL proof part I)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 = \boldsymbol{X}_1'\boldsymbol{y}$$

$$(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 = \boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2$$

$$\hat{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2$$

(This is the omitted variable bias formula)

$$\hat{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)$$

Normal Form Equations (FWL proof part I)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$\begin{aligned} (\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_1'\boldsymbol{y} \\ (\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 &= \boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \\ \hat{\beta}_1 &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \\ &\qquad \text{(This is the omitted variable bias formula)} \\ \hat{\beta}_1 &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2) \end{aligned}$$

Normal Form Equations (FWL proof part I)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$\begin{aligned}
(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_1'\boldsymbol{y} \\
(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 &= \boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \\
\hat{\beta}_1 &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} - (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 \\
&\qquad \text{(This is the omitted variable bias formula)} \\
\hat{\beta}_1 &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)
\end{aligned}$$

Normal Form Equations (FWL part II)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_2$:

$$\begin{aligned}
(\boldsymbol{X}_2'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
(\boldsymbol{X}_2'\boldsymbol{X}_1)[(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)] + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
\boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
(\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y} \\
\boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{y} \\
\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y} \\
\hat{\beta}_2 &= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}$$

Normal Form Equations (FWL part II)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_2$:

$$(\boldsymbol{X}_2'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 = \boldsymbol{X}_2'\boldsymbol{y}$$

$$(\boldsymbol{X}_2'\boldsymbol{X}_1)[(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)] + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 = \boldsymbol{X}_2'\boldsymbol{y}$$

$$\boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 = \boldsymbol{X}_2'\boldsymbol{y}$$

$$(\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 = \boldsymbol{X}_2'\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y}$$

$$\boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{X}_2\hat{\beta}_2 = \boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{y}$$

$$\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 = \boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}$$

$$\hat{\beta}_2 = (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}$$

Normal Form Equations (FWL part II)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_2$:

$$\begin{aligned}
(\boldsymbol{X}_2'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
(\boldsymbol{X}_2'\boldsymbol{X}_1)[(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)] + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
\boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
(\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} - \boldsymbol{X}_2'\boldsymbol{P}_1\boldsymbol{y} \\
\boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'(1 - \boldsymbol{P}_1)\boldsymbol{y} \\
\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y} \\
\hat{\beta}_2 &= (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y}
\end{aligned}$$

Normal Form Equations (FWL part II)

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_2$:

$$\begin{aligned}
(\boldsymbol{X}_2'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
(\boldsymbol{X}_2'\boldsymbol{X}_1)[(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{y} - \boldsymbol{X}_2\hat{\beta}_2)] + (\boldsymbol{X}_2'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_2'\boldsymbol{y} \\
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\end{aligned}$$

Frisch-Waugh-Lovell (FWL)

- Under $\boldsymbol{y} = \boldsymbol{X}\beta + \boldsymbol{e}$,

$$
\begin{aligned}
\hat{\boldsymbol{y}} &= b_0 + b_1\boldsymbol{x}_1 + \ldots + b_{k-1}\boldsymbol{x}_{k-1} + b_k\boldsymbol{x}_k + b_{k+1}\boldsymbol{x}_{k+1} + \ldots + b_n\boldsymbol{x}_n \\
\tilde{\boldsymbol{y}} &= d_0 + d_1\boldsymbol{x}_1 + \ldots + d_{k-1}\boldsymbol{x}_{k-1} + 0 + d_{k+1}\boldsymbol{x}_{k+1} + \ldots + d_n\boldsymbol{x}_n \\
\hat{\boldsymbol{x}}_k &= c_0 + c_1\boldsymbol{x}_1 + \ldots + c_{k-1}\boldsymbol{x}_{k-1} + 0 + c_{k+1}\boldsymbol{x}_{k+1} + \ldots + c_n\boldsymbol{x}_n
\end{aligned}
$$

- FWL: The regression coefficient b_k is equivalent to a regression coefficient b_1^* produced by regressing the residualised outcome $\boldsymbol{e}_y = \boldsymbol{y} - \tilde{\boldsymbol{y}}$ on the residualised $\boldsymbol{x}_k$: $\boldsymbol{e}_{x_k} = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k$.

$$
\boldsymbol{e}_y = b_0^* + b_1^*\boldsymbol{e}_{x_k} + \boldsymbol{e}
$$

Frisch-Waugh-Lovell (FWL)

- Under $\boldsymbol{y} = \boldsymbol{X}\beta + \boldsymbol{e}$,

$$\begin{aligned}
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\tilde{\boldsymbol{y}} &= d_0 + d_1\boldsymbol{x}_1 + \ldots + d_{k-1}\boldsymbol{x}_{k-1} + \quad 0 \quad + d_{k+1}\boldsymbol{x}_{k+1} + \ldots + d_n\boldsymbol{x}_n \\
\hat{\boldsymbol{x}}_{\boldsymbol{k}} &= c_0 + c_1\boldsymbol{x}_1 + \ldots + c_{k-1}\boldsymbol{x}_{k-1} + \quad 0 \quad + c_{k+1}\boldsymbol{x}_{k+1} + \ldots + c_n\boldsymbol{x}_n
\end{aligned}$$

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$$\boldsymbol{e}_y = b_0^* + b_1^*\boldsymbol{e}_{x_k} + \boldsymbol{e}$$

Frisch-Waugh-Lovell (FWL)

- Under $\boldsymbol{y} = \boldsymbol{X}\beta + \boldsymbol{e}$,

$$\begin{aligned}
\hat{\boldsymbol{y}} &= b_0 + b_1\boldsymbol{x}_1 + \ldots + b_{k-1}\boldsymbol{x}_{k-1} + b_k\boldsymbol{x}_k \quad + b_{k+1}\boldsymbol{x}_{k+1} + \ldots + b_n\boldsymbol{x}_n \\
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\end{aligned}$$

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$$\boldsymbol{e}_y = b_0^* + b_1^*\boldsymbol{e}_{x_k} + \boldsymbol{e}$$

```r
mod0 <- lm(prestige ~ education + income +  women, data=Prestige)
mod1a <- lm(prestige ~  income  + women, data = Prestige)
mod1b <- lm(education ~ income  + women, data = Prestige)
eprest <- lm(resid(mod1a)~resid(mod1b))
coef(eprest)
 (Intercept) resid(mod1b)
2.445994e-15 4.362425
coef(mod0)
 (Intercept)    education        income         women
-7.524222154  4.362424649  0.001172269 -0.012946077
```

Applications Frisch-Waugh-Lovell (FWL)

- Practical: Plotting data/ coefficients from multivariate regression in 2d.
- Theoretical: Basis for sensitivity tests to evaluate the effects of omitted variables.
- Pedagogical: Improving understanding of the linear model.

Trace of Projection and Annihilator Matrix

- Recall, the **trace** of a matrix is the sum of the diagonal elements and the sum of the eigenvalues.
- The **rank** of a matrix is the maximum number of linearly independent column vectors.
- The trace of $\boldsymbol{P}$ is its rank k
  - The projection matrix is **idempotent**: $\boldsymbol{P} = \boldsymbol{P}^2$.
  - If λ is an eigenvalue of $\boldsymbol{P}$, $\boldsymbol{P}v = \lambda v$
  - Applying P again: $\boldsymbol{P}^2 v = \lambda \boldsymbol{P} v$, which means that $\boldsymbol{P}^2 v = \lambda\lambda v$.
  - $\boldsymbol{P}^2 v = \boldsymbol{P} v = \lambda^2 v$, or $\lambda = \lambda^2$. Only true for $\lambda = 0$ or $\lambda = 1$.
  - The eigenvalue 1 corresponds to a direction preserved by the projection, the dimension of the column space of $\boldsymbol{X}$.
- $tr(\boldsymbol{M}) = tr(\boldsymbol{I}) - tr(\boldsymbol{P}) = n - k$

Alternative proof that Trace of Projection is k

- Theorem: $\text{tr}\boldsymbol{P} = k$.

$$\begin{aligned}
\text{tr}\boldsymbol{P} &= \text{tr}\left(\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\right) \\
&= \text{tr}\left((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\right) \\
&= \text{tr}(\boldsymbol{I}_k) \\
&= k
\end{aligned}$$

Leverage

- The i'th diagonal element of $\boldsymbol{P}$ is $h_{ii} = X_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}X_i$ is called the *leverage* of the i'th observation.
- h_{ii} ranges between 0 and 1, measures how unusual the i^{th} observation X_i is relative to other observations.
- A regression design is called **balanced** when the leverage values are roughly equal.
- Recall, the sum of $h_{ii} = k$
- A regression is perfectly balanced if $\max(h_{ii}) = k/n$

Leave-One-Out Regression

- The Leave-One-Out Regression estimates the projection model excluding an observation i, repeating for each observation.

$$
\begin{aligned}
\hat{\beta}_{(-i)} &= \left(\sum_{j \neq i} X_j X_j'\right)^{-1} \left(\sum_{j \neq i} X_j Y_j\right) \\
&= \left(\boldsymbol{X'X} - X_j X_i'\right)^{-1} \left(\boldsymbol{X'X} - X_i Y_i\right) \\
&= \left(\boldsymbol{X}_{(-i)}' \boldsymbol{X}_{(-i)}\right)^{-1} \boldsymbol{X}_{(-i)} \boldsymbol{y}_{(-i)}
\end{aligned}
$$

- Where $\boldsymbol{X}_{(-i)}$ excludes row i.
- $\hat{\beta}_{(-i)}$ is not a function of i, so it can be used for prediction.

$$\tilde{Y}_i = X_i' \hat{\beta}_{(i)}$$

Leave-One-Out Regression

- Calculating $\hat{\beta}_{(-i)}$ comes for free from our projection model:

$$\hat{\beta}_{(-i)} = \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i \tilde{e}_i$$

$$\tilde{e}_i = (1 - h_{ii})^{-1} \hat{e}_i$$

- We can define a modified annihilator matrix:

$$\boldsymbol{M}^* \equiv (\boldsymbol{I}_n - \text{diag}\{h_{11}, \ldots, h_{nn}\})^{-1}$$

Which allows us to rewrite:

$$\tilde{\boldsymbol{e}} = \boldsymbol{M}^* \hat{\boldsymbol{e}}$$

- We observe the residuals $\hat{\boldsymbol{e}}$ and we know how much each observation affects the regression, so we can just subtract that out.

Proof

- We will now derive the formula for $\hat{\beta}_{(-i)}$ in terms of $\tilde{e}_i$

$$
\begin{aligned}
\hat{\beta}_{(-i)} &= (\boldsymbol{X}'\boldsymbol{X} - X_i X_i')^{-1}(\boldsymbol{X}'\boldsymbol{y} - X_i Y_i) \\
(\boldsymbol{X}'\boldsymbol{X} - X_i X_i')\hat{\beta}_{(-i)} &= (\boldsymbol{X}'\boldsymbol{X} - X_i X_i')(\boldsymbol{X}'\boldsymbol{X} - X_i X_i')^{-1}(\boldsymbol{X}'\boldsymbol{y} - X_i Y_i) \\
\boldsymbol{X}'\boldsymbol{X}\hat{\beta}_{(-i)} - X_i X_i'\hat{\beta}_{(-i)} &= (\boldsymbol{X}'\boldsymbol{y} - X_i Y_i) \\
(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{X}\hat{\beta}_{(-i)} - X_i X_i'\hat{\beta}_{(-i)}) &= (\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{y} - X_i Y_i) \\
\hat{\beta}_{(-i)} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i X_i'\hat{\beta}_{(-i)} &= \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i Y_i \\
\hat{\beta}_{(-i)} &= \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i Y_i + (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i X_i'\hat{\beta}_{(-i)} \\
\hat{\beta}_{(-i)} &= \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i (Y_i - X_i'\hat{\beta}_{(-i)}) \\
\hat{\beta}_{(-i)} &= \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i \tilde{e}_i
\end{aligned}
$$

Proof

- We will now prove the relationship between $\hat{e}_i$ and $\tilde{e}_i$.

$$
\begin{aligned}
\hat{\beta}_{(-i)} &= \hat{\beta} - (\boldsymbol{X}'\boldsymbol{X})^{-1} X_i \tilde{e}_i \\
X_i'\hat{\beta}_{(-i)} &= X_i'\hat{\beta} - X_i'(\boldsymbol{X}'\boldsymbol{X})^{-1} X_i \tilde{e}_i \\
X_i'\hat{\beta}_{(-i)} &= X_i'\hat{\beta} - h_{ii}\tilde{e}_i \\
Y_i - X_i'\hat{\beta}_{(-i)} &= Y_i - X_i'\hat{\beta} + h_{ii}\tilde{e}_i \\
\tilde{e}_i &= \hat{e}_i + h_{ii}\tilde{e}_i \\
\tilde{e}_i - h_{ii}\tilde{e}_i &= \hat{e}_i \\
\tilde{e}_i &= (1 - h_{ii})^{-1}\hat{e}_i
\end{aligned}
$$

Check your understanding

- For what observation would $\hat{\beta}_{(-i)} = \hat{\beta}$?

Influential observations

- An observation i is influential if its omission changes the parameter of interest.
- $\hat{\beta} - \hat{\beta}_{(-i)} = (\boldsymbol{X}'\boldsymbol{X})^{-1}X_i\tilde{e}_i$.
- Premultiply by X_i', and we get that

$$\hat{Y}_i - \tilde{Y}_i = X_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}X_i\tilde{e}_i = h_{ii}\tilde{e}_i$$

- So i is influential if h_{ii} is big and $|\tilde{e}_i|$ is big.
- This warrants investigation: it could indicate a data entry error, an outlier from a different population, or a genuine extreme case that deserves scrutiny.

Leave-One-Out in Practice

```r
library(carData)
mod <- lm(prestige ~ education +
            income + women, data=Prestige)
# Leverage values
h <- hatvalues(mod)
which.max(h)  # highest-leverage obs
# Studentized residuals
rst <- rstudent(mod)
# Flag: |rstudent| > 2
Prestige$flag <- abs(rst) > 2
```

- `hatvalues()` returns the diagonal of $\boldsymbol{P}$.
- `rstudent()` gives $\tilde{e}_i/\hat{\sigma}_{(-i)}$: the leave-one-out residual scaled by its SE.
- Both high leverage *and* $|\text{rstudent}| > 2$ warrants investigation.

High leverage + large residual = investigate. It may be a data error, an outlier, or the most interesting observation in your dataset.

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High leverage $+$ large residual $=$ investigate. It may be a data error, an outlier, or the most interesting observation in your dataset.

What is Bias?

- Formally, an estimator is unbiased if the expectation of the conditional distribution of our estimator is the true parameter.
- We will discuss an "unbiased" estimator next week, which will assume we have correctly specified our model.
- What to do if we might not have correctly specified our model?

How do we choose X?

- What to include? Include control variables that are 1) predictive of the outcome and 2) determined prior to the treatment.
- Common examples: racial composition of a county, gender of a respondent, distance to the ocean.
- What should the functional form be? Logs, levels or differences?

Theoretical Framework: Factory Closures and Voting Behavior

- **Observation:** A local factory closure is visible evidence that trade may harm the local economy.
- **Mechanism:** Voters interpret a closure as an indication that trade liberalization negatively affects their community.
- **Prediction:** Exposure to a factory closure increases the likelihood of voting for a protectionist candidate.
- **Moderating Factor:** The effect of a factory closure depends on the *industrial composition* of the region—areas with a higher share of trade-dependent industries are more affected.

Econometric Model Specification

- **Outcome Variable:** Y_i = Indicator for voting protectionist (e.g., 1 if yes, 0 otherwise)
- **Key Predictor:** Closure_i = Indicator for a factory closure in region i
- **Control Variable:** IndComp_i = Measure of industrial composition in region i
- **Interaction:** $\text{Closure}_i \times \text{IndComp}_i$ captures how the impact of closures varies with industrial structure

$$Y_i = \alpha + \beta\,\text{Closure}_i + \gamma\,\text{IndComp}_i + \delta\,(\text{Closure}_i \times \text{IndComp}_i) + \epsilon_i$$

- β: Baseline effect of a factory closure on voting behavior.
- γ: Direct effect of industrial composition on voting.
- δ: Differential effect of a closure when industrial composition favors exposure

Tools to address bias

- Preregistration: eg OSF.io, to defining the entire data-collection and data-analysis protocol ahead of time
- Pre-publication replication, preregistered protocol for a followup.
- Training and test split: fixing the specification on a subset of the data, testing on the remaining data.

Normal Form Equations (FWL proof part I)

OLS with two sets of variables $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

$$\begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{bmatrix}$$

Derivation of $\hat{\beta}_1$:

$$\begin{aligned}
(\boldsymbol{X}_1'\boldsymbol{X}_1)\hat{\beta}_1 + (\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\beta}_2 &= \boldsymbol{X}_1'\boldsymbol{y} \\
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\end{aligned}$$

Normal Form Equations (FWL proof part I)

OLS with two sets of variables $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

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Unpacking the Bias

$$\underbrace{(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)}_{\text{Imbalance of } X_2 \text{ w.r.t. } X_1.} \quad \underbrace{\hat{\beta}_2}_{\text{Partial impact of } X_2 \text{ on } Y}$$

- $\hat{\beta}_2$ reflects the causal effects of X_2 as well as any spurious associations between $\boldsymbol{X}_2$ and Y.
- $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)$ are the coefficients of a regression of $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$, which is to say, how well does treatment predict confounders?
- Cinelli and Hazlett (2020) call this "impact" $\hat{\gamma}$ times "imbalance" $\hat{\delta}$
- The problem is that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are multivariate, so this is difficult to sign.

Reparameterizing in terms of partial R^2

- Cinelli and Hazlett (2020) propose studying omitted variables using R^2 as a metric.
- Suppose the full linear regression is as follows:

$$Y = \boldsymbol{X}_1\beta_1 + \boldsymbol{X}_2\beta_2 + \boldsymbol{e}$$

- $\boldsymbol{P}_1 = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'$ is the projection matrix onto $\boldsymbol{X}_1$.
- $\boldsymbol{M}_1 = \boldsymbol{I} - \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'$ is the residual maker for $\boldsymbol{X}_1$.
- In general the partial R^2 measures the proportion of the variance in Y that is uniquely explained by a set of predictors $\boldsymbol{X}_2$ after accounting for $\boldsymbol{X}_1$.

Formula for R^2 with one variable

$$
\begin{aligned}
R^2_{Y\sim \boldsymbol{Z}} &= \frac{SSR}{TSS} && \text{(definition (1) of } R^2\text{)} \\
&= \frac{var(\hat{Y})}{var(Y)} && \text{(} SSR \text{ is variance explained)} \\
&= 1 - \frac{var(e)}{var(Y)} && \text{(since } var(Y) = var(\hat{Y}) + var(e)\text{)} \\
&= corr(Y, \hat{Y})^2 && \text{(definition (2) of } R^2\text{)} \\
&= corr(Y, \boldsymbol{Z})^2 && \text{(} \hat{Y} \text{ is linear prediction from } \boldsymbol{Z}\text{)}
\end{aligned}
$$

Example in R

```r
library(carData)
attach(Prestige)
mod <- lm(prestige ~ education)
summary(mod)$r.square
# [1] 0.7228007
var(predict(mod)) / var(prestige)
1 - var(resid(mod)) / var(prestige)
cor(prestige, predict(mod))^2
cor(prestige, education)^2
```

Formula for partial R^2

- Using Residual Sum of Squares:
  - $R^2_{\boldsymbol{Y}\sim\boldsymbol{X}_2|\boldsymbol{X}_1} = 1 - \frac{\text{RSS}(\boldsymbol{X}_1,\boldsymbol{X}_2)}{\text{RSS}(\boldsymbol{X}_1)}$
  - $\text{RSS}(\boldsymbol{X}_1) = ||\boldsymbol{M}_1 Y||^2 = Y\boldsymbol{M}_1'\boldsymbol{M}_1 Y = Y'\boldsymbol{M}_1 Y$
  - $\text{RSS}(\boldsymbol{X}_1, \boldsymbol{X}_2) = ||\boldsymbol{M}_{1,2} Y||^2 = Y'\boldsymbol{M}_{1,2} Y$
- Using Sum of Squared Residuals:
  - $R^2_{\boldsymbol{Y}\sim\boldsymbol{X}_2|\boldsymbol{X}_1} = \frac{\text{SSR}(\boldsymbol{X}_2|\boldsymbol{X}_1)}{\text{RSS}(\boldsymbol{X}_1)}$
  - SSR, the explained sum of squares, the amount explained by $\boldsymbol{X}_2$ once accounting for $\boldsymbol{X}_1$

$$\begin{aligned}\text{SSR}(\boldsymbol{X}_2|\boldsymbol{X}_1) &= Y'\boldsymbol{M}_1\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1 Y \\ &= Y'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\beta}_2\end{aligned}$$

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Cinelli Hazlett (2020) formulation

We model $Y = \hat{\beta}_1 D + e$, so $\hat{\beta}_1 = \frac{cov(D,\ Y)}{var(D)}$.

What if there is regression of Y on D and there is a a single omitted variable Z?

$$\begin{aligned}\widehat{\text{bias}} &= (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}(\boldsymbol{X}_1'\boldsymbol{X}_2)\hat{\boldsymbol{\beta}}_2 \\ &= \left(\frac{cov(D,Z)}{var(D)}\right)\left(\frac{cov(Z^{\perp D},\ Y^{\perp D})}{var(Z^{\perp D})}\right)\end{aligned}$$

Lets write this in terms of D and two new formalisms for the disturbance: $Z^{\perp D}$ the part of Z not predicted by D, and $Y^{\perp D}$ the part of Y not predicted by D.

Cinelli Hazlett (2020) – Part 1: Model Setup

We start with the regression model:

$$Y = \hat{\beta}_1 D + e, \qquad \hat{\beta}_1 = \frac{\text{cov}(D, Y)}{\text{var}(D)}.$$

Now consider the case where a variable Z is omitted from the regression of Y on D. The omitted variable bias is given by:

$$\widehat{\text{bias}} = \frac{\text{cov}(D, Z)}{\text{var}(D)} \cdot \frac{\text{cov}(Z^{\perp D}, Y^{\perp D})}{\text{var}(Z^{\perp D})}.$$

Note: The first fraction reflects how D and Z co-move, while the second captures the effect of Z on Y once D's influence is removed.

Cinelli Hazlett (2020) – Part 2: Covariance to Correlation Conversion

For the First Term:

$$\frac{\text{cov}(D, Z)}{\text{var}(D)} = \frac{\text{corr}(D, Z)\, sd(D)\, sd(Z)}{sd(D)^2} = \frac{\text{corr}(D, Z)\, sd(Z)}{sd(D)}.$$

Note: We used the identity

$$\text{cov}(X, Y) = \text{corr}(X, Y)\, sd(X)\, sd(Y),$$

and noted that $\text{var}(D) = sd(D)^2$.

For the Second Term:

$$\frac{\text{cov}(Z^{\perp D}, Y^{\perp D})}{\text{var}(Z^{\perp D})} = \frac{\text{corr}(Z^{\perp D}, Y^{\perp D})\, sd(Z^{\perp D})\, sd(Y^{\perp D})}{sd(Z^{\perp D})^2} = \frac{\text{corr}(Z^{\perp D}, Y^{\perp D})\, sd(Y^{\perp D})}{sd(Z^{\perp D})}.$$

Note: Similarly, we express the covariance in terms of correlation and standard deviations, with $\text{var}(Z^{\perp D}) = sd(Z^{\perp D})^2$.

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Cinelli Hazlett (2020) – Part 3: Final Bias Expression

Combining the two unpacked terms, we obtain:

$$\widehat{\text{bias}} = \left(\frac{\text{corr}(D, Z)\, sd(Z)}{sd(D)} \right) \left(\frac{\text{corr}(Z^{\perp D}, Y^{\perp D})\, sd(Y^{\perp D})}{sd(Z^{\perp D})} \right).$$

Rearranging, this becomes:

$$\widehat{\text{bias}} = \frac{\text{corr}(D, Z)\ \text{corr}(Z^{\perp D}, Y^{\perp D})\, sd(Z)\, sd(Y^{\perp D})}{sd(D)\, sd(Z^{\perp D})}.$$

Replacing partial R^2:

$$|\widehat{\text{bias}}| = \sqrt{\left(\frac{R^2_{D\sim Z} R^2_{Y\sim Z|D}}{1 - R^2_{D\sim Z}} \right) \left(\frac{sd(Y^{\perp D})}{sd(D)} \right)}.$$

What Does the Bias Formula Tell Applied Researchers?

- The bias depends on two quantities, both expressed as partial R^2:
  1. $R^2_{D\sim Z}$: How strongly does the omitted variable predict the **treatment**?
  2. $R^2_{Y\sim Z|D}$: How strongly does the omitted variable predict the **outcome**, after accounting for treatment?
- If *either* partial R^2 is small, the bias is small—a confounder must predict both D and Y to be dangerous.
- In practice, you ask: "Is it plausible that an unobserved variable explains $x\%$ of the residual variation in D and $y\%$ of the residual variation in Y?"
- Observed covariates serve as benchmarks: if the strongest observed predictor has $R^2_{Y\sim Z|D} = 0.05$, an omitted confounder would need to be far stronger to overturn your result.

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Example: Simulation Setup for Sensitivity Analysis

Simulation Design:

- Generate N observations with treatment D, observed confounder Z, and outcome:

$$Y = \beta_D D + \beta_Z Z + e, \quad e \sim N(0, 1).$$

- We first show that omitting Z biases the estimate of β_D.
- Then we include Z and ask: how robust is our estimate to *additional* unobserved confounders?

Step 1: Simulate Data and Compare Models

```r
set.seed(123)
N <- 1000; beta_D <- 2; beta_Z <- 3
D <- rbinom(N, 1, 0.5); Z <- rnorm(N)
Y <- beta_D * D + beta_Z * Z + rnorm(N)

# Naive model (omitting Z) -- biased
coef(lm(Y ~ D))              # beta_D != 2

# Full model (including Z) -- unbiased
model_full <- lm(Y ~ D + Z)
coef(model_full)              # beta_D ~ 2
```

The naive model is biased because Z is correlated with both D and Y. The full model recovers the true effect—but what if there were *another* confounder we could not observe?

Step 2: Running the Sensitivity Analysis

```
library(sensemakr)

# Sensitivity analysis on the FULL model
sensitivity <- sensemakr(model_full, treatment = "D",
                         benchmark_covariates = "Z")
summary(sensitivity)
plot(sensitivity)
```

Key question: How strong would an *unobserved* confounder need to be—relative to the observed confounder Z—to explain away the estimated effect of D?

Interpreting the Sensitivity Analysis Output

Key Diagnostics:

- **Robustness Value:** The minimum strength (in terms of partial R^2) that an unobserved confounder must have with both the treatment and outcome to fully explain away the estimated effect.
- **Benchmark Comparison:** Using the observed Z as a benchmark helps gauge whether such a confounder is plausible. If an omitted variable would need to be $3\times$ stronger than Z, the result is likely robust.
- **Contour Plot:** Visualizes all combinations of $R^2_{D\sim U|X}$ and $R^2_{Y\sim U|D,X}$ that would reduce the estimated effect to zero.