Linear Models Lecture 7: Classic Normal Regression (small sample)

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Using Distributional assumptions

- Statistical inference aims to make statements about unobserved parameters β, σ etc.
- In this lecture, we do so by imposing distributional assumptions on our population with a *parametric model*
- In particular, we will be using Maximum Likelihood Estimation (MLE)

Parametric Model

- A parametric model for X is a complete probability function that depends on an unknown parameter vector θ.
- In the continuous case, we can write it as a probability density function $f(x|\theta)$.
- E.g. If $X \sim N(\mu, \sigma^2)$, $f(x|\mu, \sigma^2) = \sigma^{-1}\phi((x-\mu)/\sigma)$, the parameters are $\mu \in \mathbb{R}$ and $\sigma^2 > 0$.
- Recall, $\phi(z)$ is the density for the standard normal, $N(0,1) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})$

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Specified Parametric Models

- A model is called **correctly specified** when there is a unique parameter value θ_0 such that $f(x|\theta_0) = f(x)$, the true data distribution.
- For example, if the true density is

$$f(x) = 2\exp(-2x)$$

  - the exponential model $f(x|\lambda) = \lambda^{-1}\exp(-x/\lambda)$ is a correctly specific model with $\lambda_0 = 1/2$
  - the lognormal model $f(x|\mu,\ \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}}\exp(-\frac{(\ln x-\mu)^2}{2\sigma^2})$ is mispecified, cannot equal $2\exp(-2x)$ under any parameter value.

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Likelihoods

- Call $f(\theta|\boldsymbol{X})$ the **probability density function** of some model and parameters θ, given the data $\boldsymbol{X}$.
- If we reverse the order $f(\boldsymbol{X}|\theta)$ we have a *likelihood*, how probable is the data given the θ.
- Example: Binomial model of term lengths of candidates, $P(x,p) = \binom{n}{x} p^x (1-p)^{n-x}$.
- Suppose we have data $\boldsymbol{x} = \{1,\ 0,\ 1,\ 2,\ 0\}$.
- The Joint Likelihood of the data is:

$$\begin{aligned} P(\boldsymbol{x}|\ p) &= \binom{2}{1} p^1 (1-p)^1 \binom{2}{0} p^0 (1-p)^2 \binom{2}{1} p^1 (1-p)^1 \binom{2}{2} p^2 (1-p)^0 \binom{2}{0} p^0 (1-p)^2 \\ &= 4p^4(1-p)^6 \end{aligned}$$

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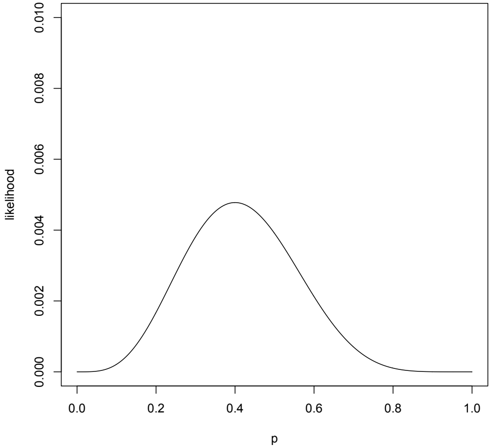
Likelihood for Binomial x



Maximum Likelihood Estimator (from Hansen Probability)

Definition

The **maximum likelihood estimator** $\hat{\theta}$ of θ is the value that maximizes the likelihood:

$$\mathcal{L}_n(\theta) \equiv f(X_1, X_2, \ldots X_n|\theta)$$

Call $\ell_n = \sum_{i=1}^{n} \log f(X_i|\theta)$ the log-likelihood.

Maximizing the likelihood

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\begin{aligned}
P(\mathbf{x}|\ p) &= 4p^4(1-p)^6 \\
\frac{\partial}{\partial p} P(\mathbf{x}|\ p) &= 16p^3(1-p)^6 - 24p^4(1-p)^5 \\
&= 0 \\
3p^4(1-p)^5 &= 2p^3(1-p)^6 \\
3p &= 2(1-p) \\
5p &= 2 \\
p^* &= \frac{2}{5}
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Invariance (Useful result)

- Recall that usually, $E[g(x)] \neq g(E[x])$, that is, functions of unbiased estimators will not be unbiased.
- If $\hat{\theta}$ is the MLE of θ, then for any transformation $\beta = h(\theta)$, the MLE of β is $\hat{\beta} = h(\hat{\theta})$

Invariance Example $f(X|\lambda) = \lambda^{-1}\exp(-X/\lambda)$.

$$\mathcal{L}_n(\lambda) = \prod_{i=1}^{n} \lambda^{-1}\exp(-X_i/\lambda) = \lambda^{-n}\exp(\frac{1}{\lambda}\sum_{i=1}^{n} X_i)$$

$$\log \mathcal{L}_n(\lambda) = -n\log\lambda - \frac{1}{\lambda}\sum_{i=1}^{n} X_i$$

$$\frac{d\ell_n(\lambda)}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} X_i$$

$$\sum_{i=1}^{n} X_i = n\lambda$$

The MLE is $\hat{\lambda} = \bar{X}_n$

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Invariance Example $\beta = 1/\lambda$

- Set $\beta = 1/\lambda$, so $h(\lambda) = 1/\lambda$.
- The log density of this model is $\log f(x|\lambda) = \log[\beta \exp(-x\beta)] = \log \beta - x\beta$.
- The log likelihood is $n \log \beta - \beta n\bar{X}_n$
- Take the derivative with respect to β:

$$n/\hat{\beta} - n\bar{X}_n = 0$$

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Score: the slope of the log likelihood

- The **likelihood score** is the derivative of the log-likelihood function:

$$S_n(\theta) = \frac{\partial}{\partial\theta}\ell_n(\theta)$$

- The score is a function of θ and tells us how sensitive the log-likelihood is to the parameter, and equals zero at the optimum.
- The **efficient score** is the derivative of the log likelihood for a single observation, evaluated at $\boldsymbol{x} = X_1, X_2, \ldots X_n$ and the true parameter vector

$$S = \frac{\partial}{\partial\theta}\log f(\boldsymbol{x}|\theta_0)$$

- The efficient score fixes θ at the true value θ_0.

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Hessian: the curvature of the log likelihood

- The **likelihood Hessian** is the negative second derivative:

$$\mathscr{H}_n(\theta) = -\frac{\partial^2}{\partial\theta\partial\theta'}\ell_n(\theta)$$

- The Hessian matrix is used to calculate the variance.

Fisher Information

- The **Fisher information** is the variance of the efficient score (score evaluated at true θ_0).

$$\mathcal{I}_\theta = \mathbb{E}[SS']$$

- The **expected Hessian** is the expectation of the Hessian for a single observation:

$$\mathcal{H}_\theta = -\mathbb{E}[\frac{\partial^2}{\partial\theta\partial\theta'} \log f(X|\theta)]$$

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Cramér-Rao Lower Bound

- The term efficient refers to an estimator which has minimum variance.
- If $\tilde{\theta}$ is an unbiased estimator of θ, then $var[\tilde{\theta}] \geq (n\mathcal{I}_\theta)^{-1}$

Variance Estimators

- The sample Hessian Estimator depends on calculating the second derivatives of the log-likelihood:

$$\hat{\mathcal{H}}_\theta = \frac{1}{n}\sum_{i=1}^{n} -\frac{\partial}{\partial\theta\partial\theta'} \log f(X_i|\hat{\theta})$$

$$\hat{\mathbf{V}}_1 = \hat{\mathcal{H}}_\theta^{-1}$$

- The Outer Product Estimator is based on the Fisher Information:

$$\hat{\mathcal{I}}_\theta = \frac{1}{n}\sum_{i=1}^{n}(\frac{\partial}{\partial\theta} \log f(X_i|\hat{\theta}))(\frac{\partial}{\partial\theta} \log f(X_i|\hat{\theta}))'$$

$$\hat{\mathbf{V}}_2 = \hat{\mathcal{I}}_\theta^{-1}$$

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Example: Normal with known mean, unknown variance

$X \sim N(0, \theta)$, where $\theta \equiv \sigma^2$. $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = \theta_0$, $\mathbb{E}[X^4] = 3\theta_0^2$.

The density is:

$$f(x|\theta) = \frac{1}{(2\pi\theta)^{1/2}} \exp(-\frac{x^2}{2\theta})$$

The log density is:

$$\log f(x|\theta) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\theta) - \frac{x^2}{2\theta}$$

the first and second derivatives are:

$$\frac{d}{d\theta}\log f(x|\theta) = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} = \frac{x^2 - \theta}{2\theta^2}$$

$$\frac{d^2}{d\theta^2}\log f(x|\theta) = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

Example: Normal log likelihood

$$\ell_n(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n} X_i^2$$

$$\frac{d}{d\theta}\ell_n(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$$

The MLE is $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Example: Normal with known mean, unknown variance

$$\frac{d}{d\theta} \log f(x|\theta) = \frac{x^2 - \theta}{2\theta^2}$$

The efficient score $S = \frac{x^2 - \theta_0}{2\theta_0^2}$

$$var[S] = \frac{\mathbb{E}[(X^2 - \theta_0)^2]}{4\theta_0^4} = \frac{\mathbb{E}[X^4 - 2X^2\theta_0 + \theta_0^2]}{4\theta_0^4} = \frac{3\theta_0^2 - 2\theta_0^2 + \theta_0^2}{4\theta_0^4} = \frac{1}{2\theta_0^2}$$

expected Hessian

$$\mathcal{H}_\theta = \mathbb{E}[-\frac{d}{d\theta^2}\log f(X|\theta_0)] = \mathbb{E}[-\frac{d}{d\theta}\frac{x^2-\theta_0}{2\theta_0^2}] = \mathbb{E}[\frac{2x^2-\theta_0}{2\theta_0^3}] = \frac{1}{2\theta_0^2}$$

$$\mathcal{I}_\theta = var[S] = \frac{1}{2\theta_0^2} = \mathcal{H}_\theta$$

The total Fisher information for n observations is $I_n(\theta_0) = n \times \mathcal{I}_\theta = \frac{n}{2\theta_0^2}$

The variance of $\hat{\theta}$ is $\frac{1}{I_n(\theta_0)} = 2\theta_0^2/n$

So we can get the plug in estimator for the standard error as $\sqrt{2\hat{\theta}_0^2/n}$.

expected Hessian

```
for(i in 1:100000){
 x <- rnorm(100, sd=sqrt(5))
thetahat[i] <- sum(x^2)/100
diff[i] <- thetahat[i]-5
se[i] <- sqrt(2*thetahat[i]^2/100) }
mean(thetahat)
5.002307
sqrt(var(diff))
0.7102136
mean(se)
0.707433
```

Robust Variance Estimator

- Under misspecification, $\mathscr{I}_\theta \neq \mathscr{H}(\theta)$
- A consistent estimator for the variance is:

$$\hat{\boldsymbol{V}} = \hat{\mathscr{H}}^{-1}\hat{\mathscr{I}}\hat{\mathscr{H}}^{-1}$$

- This is calculated by the sandwich package.

Summary: The MLE Toolkit

We now have four key ingredients:

1. **Likelihood** $\rightarrow$ a model-based measure of how well parameters fit data.
2. **Score** $\rightarrow$ first derivative of ℓ; equals zero at the MLE.
3. **Fisher Information** $\rightarrow$ curvature of ℓ; determines how precisely we can estimate θ.
4. **Cramér-Rao bound** $\rightarrow$ no unbiased estimator can beat $(n\mathcal{I}_\theta)^{-1}$.

Next: apply this machinery to the normal linear regression model and see that OLS *is* the MLE.

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Likelihood methods for linear model

- The Normal Regression model assumes that $y \sim N(\mu_Y, \sigma^2)$, or that $e \sim N(0, \sigma^2)$.

$$
\begin{aligned}
f(y|\boldsymbol{x}) &= \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{1}{2\sigma^2}(y - \boldsymbol{x}'\boldsymbol{\beta})^2\right) \\
f(y_1, \dots y_n|\boldsymbol{x}_1, \dots \boldsymbol{x}_n) &= \prod_{i=1}^{n} f(y_i|x_i) \\
&= \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{1}{2\sigma^2}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right) \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right) \\
&\equiv \mathcal{L}_n(\beta, \sigma^2)
\end{aligned}
$$

Log Likelihood of the normal linear model

$$log\mathcal{L}_n(\beta,\sigma^2) = -\frac{n}{2}log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \boldsymbol{x}_i'\beta)^2 \equiv \ell_n(\beta,\sigma^2)$$

Classic Normal Regression Model

- The Classic Normal Regression Model consists of the following assumptions:
  1. $\boldsymbol{y} = \boldsymbol{X'}\beta + \boldsymbol{e}$
  2. $\boldsymbol{e}|\boldsymbol{X} \sim N(\boldsymbol{0},\ \sigma^2\boldsymbol{I})$
     - or equivalently: $\boldsymbol{y} \sim N(\boldsymbol{X}\beta,\ \sigma^2\boldsymbol{I})$
  3. rank$(\boldsymbol{X}) = K$.

  In this model, $N(\boldsymbol{\mu},\ \boldsymbol{\Sigma})$ is the multivariate normal density function whose pdf is:

$$f(\boldsymbol{e}) = (2\pi)^{-n/2}(\sigma^2)^{-n/2}\exp(-\frac{1}{2\sigma^2}\boldsymbol{e}'\boldsymbol{e})$$

Background on Multivariate Normal

- A multivariate normal's parameters are a vector of means and a variance covariance matrix.
- Linear functions of a multinormal vector $\boldsymbol{y}$ are also normal.
- If $\boldsymbol{z} = \boldsymbol{g} + \boldsymbol{H}\boldsymbol{y}$, where $\boldsymbol{g}$ and $\boldsymbol{H}$ are non-random, and $\boldsymbol{H}$ has full row rank, then

$$\boldsymbol{z} \sim N(\boldsymbol{g} + \boldsymbol{H}\boldsymbol{\mu},\ \boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{H}')$$

- In the case of MVN, independence is the same as uncorrelated.

Bivariate Normal CEF

- Given two random variables, they are distributed bivariate normal if $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}\right)$
- Call $X^* = \frac{X-\mu_X}{\sigma_X}$, $Y^* = \frac{Y-\mu_Y}{\sigma_Y}$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} exp\left(-\frac{1}{2}(X^*)^2\right)$$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{2\pi}\sigma_Y}\sqrt{1-\rho^2} exp\left(-\frac{1}{2[1-\rho^2]}\left[(X^*)^2 - 2\rho(X^*)(Y^*) + (Y^*)^2\right]\right)$$

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Bivariate Normal CEF: Derivation

Dividing the joint by the marginal and completing the square:

$$f_{Y|X}(y|x) \propto \exp\left(-\frac{1}{2(1-\rho^2)}(Y^* - \rho X^*)^2\right)$$
$$= \exp\left(-\frac{1}{2}\left(\frac{Y - \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X)}{\sigma_Y\sqrt{1-\rho^2}}\right)^2\right)$$

This is the kernel of a normal density with:

- Conditional mean: $\mathbb{E}[Y|X = x] = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = \mu_Y + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(x - \mu_X)$
- Conditional variance: $\sigma_Y^2(1 - \rho^2)$

Key insight: Under joint normality, the CEF is linear and the regression coefficient is $\beta = \text{Cov}(Y, X)/\text{Var}(X)$. This is a classical motivation for linear regression.

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Roadmap: From Likelihood to Inference

- We have the normal linear model: $\boldsymbol{y}|\boldsymbol{X} \sim N(\boldsymbol{X}\beta,\ \sigma^2\boldsymbol{I})$.
- Now we solve for the MLE and show it equals OLS.
- Then we connect the score of the normal model to moment conditions — the bridge to GMM.
- Finally, the distributional results (t, χ^2, F) that make exact small-sample inference possible.

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Solving for the MLE

- Given the likelihood, we can solve for the MLE $(\hat{\beta}_{MLE},\ \hat{\sigma}^2_{MLE})$

$$\frac{\partial}{\partial \beta}[-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \mathbf{x}_i'\beta)^2]$$

$$\frac{2}{2\hat{\sigma}^2_{MLE}}\sum_{i=1}^{n} X_i(Y_i - \mathbf{x}_i'\hat{\beta}_{MLE}) = 0$$

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Overall Likelihood

- $\hat{\beta}_{MLE} = (\sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i')^{-1} (\sum_{i=1}^{n} \boldsymbol{x}_i Y_i) = \hat{\beta}_{ols}$.
- $\hat{\sigma}^2_{MLE} = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \boldsymbol{x}_i' \hat{\beta}_{mle})^2 = \hat{\sigma}_{ols}$.
- $\log \mathcal{L}(\hat{\beta}_{MLE}, \ \hat{\sigma}^2_{MLE}) = -\frac{n}{2} \log(2\pi\hat{\sigma}^2_{mle}) - n/2$
- You will see the "log likelihood" reported as a measure of fit, logLik()
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- AIC gives an asymptotically unbiased estimator of the expected relative Kullback-Leibler divergence when approximating an unknown distribution and a given model.

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R Example: Log-Likelihood and AIC

```
data(swiss)
mod1 <- lm(Fertility ~ Education, data = swiss)
mod2 <- lm(Fertility ~ Education + Agriculture,
           data = swiss)
mod3 <- lm(Fertility ~ Education + Agriculture
           + Catholic + Infant.Mortality, data=swiss)
# Log-likelihoods (higher = better fit)
sapply(list(mod1, mod2, mod3), logLik)
#  -168.3   -166.0   -155.3
# AIC = -2*logLik + 2*k (lower = better)
sapply(list(mod1, mod2, mod3), AIC)
#   342.5    340.0    322.6
```

AIC penalizes complexity: mod3 wins because the likelihood improvement outweighs the $2k$ penalty.

Score of the Normal Regression Model (Hansen 5.14)

The likelihood scores are the derivatives of the log-likelihood:

$$\frac{\partial}{\partial \beta}\ell_n(\beta, \sigma^2) = \frac{1}{\sigma^2}\sum_{i=1}^{n} X_i(Y_i - X_i'\beta) = \frac{1}{\sigma^2}\boldsymbol{X'e}$$

$$\frac{\partial}{\partial \sigma^2}\ell_n(\beta, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i - X_i'\beta)^2$$

Setting the score for β to zero:

$$\frac{1}{\sigma^2}\boldsymbol{X'e} = 0 \iff \boldsymbol{X'e} = 0 \iff \boldsymbol{X'}(\boldsymbol{Y} - \boldsymbol{X}\beta) = 0$$

These are exactly the OLS normal equations! The MLE for β equals the OLS estimator because the score FOC is proportional to the least squares FOC.

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Scores as Moment Conditions

The score for β evaluated at the true parameter is:

$$S_i(\beta_0) = \frac{\partial}{\partial \beta} \log f(Y_i | X_i, \beta_0, \sigma^2) = \frac{1}{\sigma^2} X_i e_i$$

The population moment condition from the CEF is:

$$\mathbb{E}[X_i e_i] = 0$$

Key observation: Setting the score to zero in the sample,

$$\frac{1}{n}\sum_{i=1}^{n} S_i(\hat{\beta}) = 0 \quad \Longleftrightarrow \quad \frac{1}{n}\sum_{i=1}^{n} X_i \hat{e}_i = 0$$

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Information Matrix of Normal Regression (Hansen 5.14)

The Fisher information matrix for the normal regression model is:

$$\mathcal{I} = \text{var}\begin{bmatrix} \frac{\partial}{\partial\beta}\ell(\beta,\sigma^2) \\ \frac{\partial}{\partial\sigma^2}\ell(\beta,\sigma^2) \end{bmatrix} = \begin{pmatrix} \frac{1}{\sigma^2}\boldsymbol{X'X} & \mathbf{0} \\ \mathbf{0} & \frac{n}{2\sigma^4} \end{pmatrix}$$

- The matrix is block diagonal: estimation of β and σ^2 are independent.
- The Cramér-Rao lower bound for β is:

$$\mathcal{I}_\beta^{-1} = \sigma^2(\boldsymbol{X'X})^{-1}$$

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From Scores to the Sandwich (connecting to Lecture 6)

Under **correct specification** ($e \sim N(0, \sigma^2)$):

$$\mathcal{I}_\theta = \mathcal{H}_\theta \quad \Longrightarrow \quad V = \mathcal{H}^{-1} = \mathcal{I}^{-1} = \sigma^2(\boldsymbol{X'X})^{-1}$$

One formula suffices. This is the classical OLS variance.

Under **misspecification** (heteroskedasticity, non-normality):

$$\mathcal{I}_\theta \neq \mathcal{H}_\theta \quad \Longrightarrow \quad V = \mathcal{H}^{-1}\mathcal{I}\mathcal{H}^{-1}$$

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This is the heteroskedasticity-robust variance from Lecture 6 (HC0–HC3)!

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Preview: From MLE Scores to GMM

Maximum likelihood solves k score equations in k unknowns: $\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(Y_i|X_i,\theta)=0$

Method of moments solves m moment conditions in k unknowns: $\frac{1}{n}\sum_{i=1}^{n}g(Y_i,X_i,\theta)=0$

- When $m=k$ and $g=$ score: GMM = MLE.
- When $m>k$: GMM uses an *optimal weighting matrix* to combine them.
- IV example: $\mathbb{E}[Z_ie_i]=0$ gives more moment conditions than parameters.

Takeaway: MLE $\subset$ GMM. Score equations are one set of moment conditions. Robust SEs arise when the likelihood is misspecified.

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Concept	Formula	Role
Score	$S_i = \frac{1}{\sigma^2} X_i e_i$	FOC $\rightarrow$ OLS normal eqns
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Proof of normality of $\hat{\boldsymbol{\beta}}$ in CNRM

We can show that $\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta},\ \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$

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$$\text{Rank}((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}') = \text{Rank}(\boldsymbol{X}) = K$$

So $\hat{\beta} - \beta$ is a full-row-rank linear transformation of the multinormal vector $\boldsymbol{e}$

$$\hat{\beta} - \beta \sim N(\mathbf{0},\ \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1})$$

Distribution of Residuals (Hansen 5.7)

Recall $\hat{\boldsymbol{e}} = \boldsymbol{M}\boldsymbol{e}$ where $\boldsymbol{M} = \boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$.

Since $\boldsymbol{e}|\boldsymbol{X} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$ and $\hat{\boldsymbol{e}}$ is a linear function of $\boldsymbol{e}$:

$$\hat{\boldsymbol{e}}|\boldsymbol{X} \sim N(\boldsymbol{0},\ \sigma^2 \boldsymbol{M})$$

Key fact: $\hat{\beta}$ and $\hat{\boldsymbol{e}}$ are independent (conditional on $\boldsymbol{X}$).

Proof: Their joint covariance is

$$\mathrm{Cov}(\hat{\beta} - \beta,\ \hat{\boldsymbol{e}}) = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' \cdot \sigma^2 \boldsymbol{I} \cdot \boldsymbol{M} = \sigma^2 (\boldsymbol{X}'\boldsymbol{X})^{-1} \underbrace{\boldsymbol{X}'\boldsymbol{M}}_{=\boldsymbol{0}} = \boldsymbol{0}$$

Since they are jointly normal and uncorrelated, they are independent.

Why this matters: It means $s^2 = \hat{\boldsymbol{e}}'\hat{\boldsymbol{e}}/(n-k)$ is independent of $\hat{\beta}$. This is what allows us to form the t-statistic as a ratio of independent normal and χ^2 components.

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Distribution of residual sum of squares $s^2_{\hat{e}} = \frac{\hat{\boldsymbol{e}}'\hat{\boldsymbol{e}}}{(N-K)}$ in CNRM

Thm: Let $\boldsymbol{z} \sim N(\boldsymbol{0},\ \boldsymbol{I})$, let $\boldsymbol{A}$ be idempotent. Then $\boldsymbol{z}'\boldsymbol{A}\boldsymbol{z} \sim \chi^2(\nu)$ where $\nu = \text{Rank}(\boldsymbol{A})$.

$$\hat{\boldsymbol{e}}|\boldsymbol{X} \sim N(\boldsymbol{0},\ \sigma^2\boldsymbol{I})$$
$$\hat{\boldsymbol{e}}/\sigma|\boldsymbol{X} \sim N(\boldsymbol{0},\ \boldsymbol{I})$$

Recall

$$\begin{aligned}(N-K)s^2_{\hat{e}} &= \hat{\boldsymbol{e}}'\hat{\boldsymbol{e}} \\ &= \hat{\boldsymbol{e}}\boldsymbol{M}\hat{\boldsymbol{e}} \\ \frac{\hat{\boldsymbol{e}}'}{\sigma}\boldsymbol{M}\frac{\hat{\boldsymbol{e}}}{\sigma} &\sim \chi^2(N-K) \qquad (\boldsymbol{M} \text{ is idempotent})\end{aligned}$$

That is, in a linear regression model, $\frac{(N-K)s^2_{\hat{e}}}{\sigma^2} \sim \chi^2_{n-k}$

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R Example: χ^2 Distribution of s^2

```
set.seed(1); n <- 50; k <- 3; sigma2 <- 4
X <- cbind(1, matrix(rnorm(n*(k-1)), n, k-1))
beta <- c(2, -1, 0.5)
scaled_s2 <- replicate(10000, {
  y <- X %*% beta + rnorm(n, sd = sqrt(sigma2))
  ehat <- resid(lm(y ~ X - 1))
  sum(ehat^2) / sigma2   # (n-k)*s^2 / sigma^2
})
```

```
# Compare to chi-squared(n-k)
mean(scaled_s2)      # ~47 (= n-k)
var(scaled_s2)       # ~94 (= 2*(n-k))
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Theory: $\mathbb{E}[\chi^2_{47}] = 47$, $\text{Var}[\chi^2_{47}] = 94$.

The simulation matches.

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Distribution of OLS Estimates

Under the classical linear model assumptions, the OLS estimator is normally distributed:

$$\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta},\ \sigma^2(\boldsymbol{X'X})^{-1})$$

Focus on the j-th coefficient:

$$\hat{\beta}_j \sim N\left(\beta_j,\ \sigma^2\left[(\boldsymbol{X'X})^{-1}\right]_{jj}\right)$$

Therefore, the standardized version is:

$$\frac{\hat{\beta}_j - \beta_j}{\sqrt{\sigma^2\left[(\boldsymbol{X'X})^{-1}\right]_{jj}}} \sim N(0,1)$$

From Normal to t: Estimated Variance

In practice, we do not know the true error variance σ^2. The unbiased estimator is:

$$s_{\hat{e}}^2 = \frac{1}{n-k}\sum_{i=1}^{n} \hat{e}_i^2 = \frac{\hat{\boldsymbol{e}}'\hat{\boldsymbol{e}}}{n-k}$$

We plug in this estimate to standardize:

$$T = \frac{\hat{\beta}_j - \beta_j}{\sqrt{s_{\hat{e}}^2\,[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}}$$

This is no longer standard normal, but it has a **t-distribution**.

The t Statistic and its Distribution

Recall from probability theory:

$$\frac{Z}{\sqrt{W/(n-k)}} \sim t(n-k) \quad \text{where} \quad Z \sim N(0,1),\ W \sim \chi^2(n-k),\ Z \perp W$$

In our case:

$$\frac{\hat{\beta}_j - \beta_j}{\sqrt{\sigma^2 \left[(\boldsymbol{X'X})^{-1}\right]_{jj}}} \sim N(0,1)$$

$$\frac{(n-k)s_{\hat{e}}^2}{\sigma^2} \sim \chi^2(n-k)$$

So the statistic:

$$T = \frac{\hat{\beta}_j - \beta_j}{\sqrt{s_{\hat{e}}^2 \left[(\boldsymbol{X'X})^{-1}\right]_{jj}}} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\frac{(n-k)s_{\hat{e}}^2}{\sigma^2}\sigma^2 \left[(\boldsymbol{X'X})^{-1}\right]_{jj}/(n-k)}} \sim t(n-k)$$

Limits of t-statistic

- The T statistic follows the t distribution under homoskedasticity (by using s^2) and i.i.d. normality of e.
- Without normality, we can still say the OLS estimators are unbiased, but our exact distributions will not apply.
- The generic T statistic is $\frac{\hat{\beta}-\beta_0}{SE(\hat{\beta})}$.
- We will not have an exact finite-sample distribution of T when we use HC0-HC3 errors.
- Instead, we will use large sample approximations.

The F-test as a Likelihood Ratio Test (Hansen 5.13)

Consider testing $\mathbb{H}_0 : \beta_2 = 0$ in $Y = X_1'\beta_1 + X_2'\beta_2 + e$.

The likelihood ratio statistic compares the maximized log-likelihoods:

$$\text{LR} = 2\left(\ell_n(\hat{\beta}, \hat{\sigma}^2) - \ell_n(\tilde{\beta}_1, \tilde{\sigma}^2)\right) = n \log\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)$$

where $\tilde{\sigma}^2$ is from the restricted (null) model. This is equivalent to the F-statistic:

$$F = \frac{(\tilde{\sigma}^2 - \hat{\sigma}^2)/q}{\hat{\sigma}^2/(n-k)} \sim F_{q,\,n-k} \quad \text{under } \mathbb{H}_0$$

- $q = \dim(\beta_2)$ is the number of restrictions.
- Under the null, F has an *exact* F-distribution in the normal model.
- This justifies the F-test reported by `anova()` and regression output.
- The t-test is a special case: when $q = 1$, $F = T^2$ and $F_{1,n-k} = t^2_{n-k}$.

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Summary: Small-Sample Distributions

Under the classical normal regression model:

- $\hat{\beta}|\boldsymbol{X} \sim N(\beta,\ \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$ *(linear fn of normal)*
- $\hat{\mathbf{e}}|\boldsymbol{X} \sim N(\mathbf{0},\ \sigma^2\boldsymbol{M}), \quad \hat{\beta} \perp \hat{\mathbf{e}}$ *(projection)*
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Confidence Intervals

- $\hat{\beta}$ is a **point estimate** for a coefficient β.
- We can instead estimate an interval, $\hat{C} = [\hat{L},\ \hat{U}]$ which contains the true value with high probability.
- An interval estimate $\hat{C}$ is called a $1-\alpha$ confidence interval when $Pr(\beta \in \hat{C}) = 1-\alpha$. The value $1-\alpha$ is called the **coverage probability**.
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- $Pr(\beta \in \hat{C})$ is the probability that the random set $\hat{C}$ covers or contains β.

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Confidence Intervals in Practice

$$Pr(\beta \in \hat{C}) = Pr(-c \leq T(\beta) \leq c)$$
$$Pr(\beta \in \hat{C}) = 2F(c) - 1$$

Our goal is to set this coverage probablity equal to $1 - \alpha$, or $F(c) = 1 - \alpha/2$. If $\alpha = .05$, we solve $c = F^{-1}(1 - .05/2)$. In case of a normal, c=1.96 ≈ 2

$$\hat{C} = [\hat{\beta} - 2 * s(\hat{\beta}),\ \hat{\beta} + 2 * s(\hat{\beta})]$$

t test and p-values

- A theory is said to have *testable implications* if it can be falsified.
- For example, a theory may be false if $\beta = \beta_0$. This is called a "null hypothesis" $\mathbb{H}_0$.
- We further specify the complement of $\mathbb{H}_0$ as $\mathbb{H}_1$.
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$$\text{Reject } \mathbb{H}_0 \text{ if } |T| > c$$

- A p-value indexes a test's strength of rejection of the null.
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Example

	Dependent variable: Fertility
Education (%)	-0.98^{***}
	(0.15)
Agriculture (%)	-0.15^{**}
	(0.07)
Catholic (%)	0.12^{***}
	(0.03)
Infant.Mortality (%)	1.08^{***}
	(0.38)
Constant	62.10^{***}
	(9.60)
Observations	47
R^2	0.70
Adjusted R^2	0.67
Residual Std. Error	7.17 (df = 42)
F Statistic	24.42^{***} (df = 4; 42)
Note:	*p<0.1; **p<0.05; ***p<0.01

Did education share predict the average fertility across Swiss districts in 1888?

$$\hat{\beta} = -.98, \quad \beta_0 = 0, \quad se(\hat{\beta}) = 0.15, \quad Df = 47 - 5$$

$$t = \frac{\hat{\beta} - \beta_0}{se(\hat{\beta})} = \frac{-.98 - 0}{0.15} = -6.61$$

$$c = F^{-1}(1 - .05/2) = \text{qt}(1 - .025, 41) = 2.02$$

$$|t| > 2.02$$

$$p = 2(1 - F_{n-k}(|T|)) = 2 * (1 - \text{pt}(6.61, 42)) = 5.2 \times 10^{-8}$$

For each 1 percentage point of post-primary attendance, fertility rate is (.7, 1.3) percentage points lower

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