Linear Models Lecture 16: IV

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2SLS and IV

- IV formula:

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y$$

- Two stage least squares:
    - Suppose in the first stage we regress

$$X = Z\gamma + v$$

    - In the second stage, we use $\hat{X} = Z\hat{\gamma} = Z(Z'Z)^{-1}Z'X = P_Z X$,

$$\hat{\beta}_{2SLS} = (\hat{X}'\hat{X})^{-1}\hat{X}'y$$

- iv_robust(Y $\sim D + X|Z + X$, data = dat)

Equivalence Between 2SLS and IV

- 2SLS is exactly identical to IV when $l = k$

$$
\begin{aligned}
\hat{\beta}_{2SLS} &= (\hat{X}'\hat{X})^{-1}\hat{X}'y \\
&= (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y \\
&= (Z'X)^{-1}(Z'Z)(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'y \qquad ((ABC)^{-1} = C^{-1}B^{-1}A^{-1}) \\
&= (Z'X)^{-1}(Z'Z)(Z'Z)^{-1}Z'y \\
&= (Z'X)^{-1}Z'y = \hat{\beta}_{IV}
\end{aligned}
$$

Control Function Regression

- Assume that X_2 is endogenous:

$$Y = \boldsymbol{x}_1'\beta_1 + \boldsymbol{x}_2'\beta_2 + e$$

$$\boldsymbol{x}_2 = \Gamma_{12}'\boldsymbol{z}_1 + \Gamma_{22}'\boldsymbol{z}_2 + u_2$$

- The control function approach directly models the error:

$$e = u_2'\alpha + v$$

$$\alpha = (E[u_2 u_2'])^{-1} E[u_2 e]$$

$$E[u_2 v] = 0$$

Control Function Regression

- We then plug this in to the original structural form equation, controlling for the error.

$$
\begin{aligned}
Y &= X_1'\beta_1 + X_2'\beta_2 + e \\
Y &= X_1'\beta_1 + X_2'\beta_2 + u_2'\alpha + v \\
E[X_1 v] &= 0 \\
E[X_2 v] &= 0 \\
E[u_2 v] &= 0
\end{aligned}
$$

- After we control for u_2, the error is uncorrelated with X.
- We estimate this new control with the reduced form residual:

$$\hat{u}_{2i} = \boldsymbol{x}_{2i} - \hat{\Gamma}'_{12}\boldsymbol{z}_1 + \hat{\Gamma}'_{22}\boldsymbol{z}_2$$

- It is like subtracting off the endogenous part.

$$\boldsymbol{Y} = \boldsymbol{X}\hat{\beta} + \hat{\boldsymbol{U}}_e\hat{\alpha} + \hat{v}$$

Decomposing Observed Differences

- We start with the observed difference in average outcomes:

$$\Delta = \mathbb{E}[Y_i \mid D_i = 1] - \mathbb{E}[Y_i \mid D_i = 0]$$

- Using the potential outcomes framework:

$$Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0)$$

implies:

$$\Delta = \mathbb{E}[Y_i(1) \mid D_i = 1] - \mathbb{E}[Y_i(0) \mid D_i = 0]$$

- Add and subtract $\mathbb{E}[Y_i(0) \mid D_i = 1]$:

$$\Delta = (\mathbb{E}[Y_i(1) \mid D_i = 1] - \mathbb{E}[Y_i(0) \mid D_i = 1]) + (\mathbb{E}[Y_i(0) \mid D_i = 1] - \mathbb{E}[Y_i(0) \mid D_i = 0])$$

ATT and Type I Bias

- Now interpret each term:

$$\mathbb{E}[Y_i(1) - Y_i(0) \mid D_i = 1] = \text{ATT}$$
$$\mathbb{E}[Y_i(0) \mid D_i = 1] - \mathbb{E}[Y_i(0) \mid D_i = 0] = \text{Type I Bias (Selection on Levels)}$$

- So the decomposition becomes:

$$\mathbb{E}[Y_i \mid D_i = 1] - \mathbb{E}[Y_i \mid D_i = 0] = \text{ATT} + \text{Type I Bias}$$

- Type I bias arises when treatment is correlated with baseline outcomes $Y(0)$.

Type II Bias: Selection on Gains

- If treatment effects $\tau_i = Y_i(1) - Y_i(0)$ vary across individuals, and:

$$\mathbb{E}[\tau_i \mid D_i = 1] \neq \mathbb{E}[\tau_i]$$

then:

$$\text{ATT} \neq \text{ATE} \quad \Rightarrow \quad \text{Type II Bias (Selection on Gains)}$$

- This occurs when treatment status is correlated with unobserved factors that affect the **magnitude of the treatment effect**.
- **Summary:**
  - Type I Bias: D_i correlated with $Y_i(0)$
  - Type II Bias: D_i correlated with τ_i
  - Both biases can exist simultaneously, and confound causal interpretation

Causal Heterogeneity and Bias in Returns to Education

- Consider the model of log wages:

$$lwage_i = \alpha + \beta_i \cdot educ_i + u_{i1}$$

where β_i is the individual-specific return to education.

- Assume:

$$\beta_i = \gamma + v_i \quad \text{(mean return + heterogeneity)}$$

- Substituting in:

$$lwage_i = \alpha + \gamma \cdot educ_i + v_i \cdot educ_i + u_i$$

- If $educ_i$ is endogenous and correlated with v_i, then **OLS is biased**:

$$\text{OLS identifies } \mathbb{E}[\beta_i \mid educ_i], \text{ not } \gamma$$

- This reflects **causal heterogeneity**: individuals with higher returns may select more schooling.

LATE in the Education Context

- Suppose we instrument education with background characteristics $\boldsymbol{z}_i$ that shift schooling but not wages directly:

$$educ_i = \boldsymbol{z}_i \boldsymbol{\pi} + v_i$$

- The IV estimand recovers:

$$\text{LATE} = \mathbb{E}[\beta_i \mid \text{Compliers}]$$

  i.e., the **average return to education for individuals whose schooling decisions are influenced by $\boldsymbol{z}_i$**.

- In general:

$$\frac{\mathbb{E}[lwage_i \mid z_i = 1] - \mathbb{E}[lwage_i \mid z_i = 0]}{\mathbb{E}[educ_i \mid z_i = 1] - \mathbb{E}[educ_i \mid z_i = 0]}$$

  identifies the "local" return — not the ATE or the return for everyone.

Angrist-Imbens IV Framework for Education

- Identification of LATE requires four assumptions:
  1. **Independence:** $(Y_i(0), Y_i(1), D_i(0), D_i(1)) \perp Z_i$
  2. **Exclusion restriction:** Z_i affects $lwage_i$ only through $educ_i$
  3. **Monotonicity:** $educ_i(1) \geq educ_i(0)$ for all i (no one reduces schooling when instrument increases it)
  4. **First stage:** $\mathbb{E}[educ_i \mid Z_i = 1] \neq \mathbb{E}[educ_i \mid Z_i = 0]$
- Under these assumptions:

$$\text{2SLS estimates } \mathbb{E}[g_{i1} \mid \text{Compliers}]$$

Compliance Types under Binary Instrument

Principal Strata

Always-Takers
$D(0) = 1, \ D(1) = 1$ — Not affected by Z

Compliers
$D(0) = 0, \ D(1) = 1$ — **LATE identified here**

Never-Takers
$D(0) = 0, \ D(1) = 0$ — Not affected by Z

Defiers (ruled out)
$D(0) = 1, \ D(1) = 0$ — Excluded by monotonicity

Who Are the Compliers? What Are Their Gains?

- In the model of heterogeneous returns to education:

$$\beta_i = \gamma + v_i, \quad \text{where } v_i \text{ captures individual-specific deviations}$$

- The IV estimator identifies:

$$\mathbb{E}[\beta_i \mid \text{Compliers}] = \gamma + \mathbb{E}[v_i \mid \text{Compliers}]$$

- **Who are the compliers?**
  - Individuals whose educational choices are influenced by the instrument z_i
  - They are at the *margin of attending college*: those who attend if and only if encouraged
- **What are their gains?**
  - Depends on the relationship between the instrument and v_i
  - If those at the margin have lower motivation, preparation, or ability, then:

$$\mathbb{E}[v_{i1} \mid \text{Compliers}] < 0 \quad \Rightarrow \quad \text{LATE} < \text{ATE}$$

- **Conclusion:** IV estimates are local — their policy relevance depends on *who the compliers are*, and what their unobserved gains v_i look like.

Challenges with IV

- The IV estimator is among the most common tools of econometrics.
- However, it has several weaknesses.
  - Imprecision
  - Small sample Bias
  - Sensitivity to Weak Instruments

Problems with IV estimator: Imprecision

- Suppose Z and X are mean 0, $y = X\beta + e$,

$$Z'X = X'Z = \sum z_i x_i = n * cov(z,x)$$

$$Z'Z = \sum z_i^2 = n * var(z)$$

$$X'X = \sum x_i^2 = n * var(x)$$

$$\hat{\beta}_{IV} = (Z'X)^{-1} Z'y$$

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'y$$

$$Avar(\hat{\beta}_{OLS}) = \sigma_e^2 (X'X)^{-1}$$

$$Avar(\hat{\beta}_{IV}) = \sigma_e^2 (Z'X)^{-1} Z'Z (X'Z)^{-1}$$

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Problems with IV estimator: Imprecision

$$Avar(\hat{\beta}_{OLS}) = \sigma_e^2 (X'X)^{-1} = \frac{\sigma_e^2}{n} \frac{1}{var(x)}$$

$$Avar(\hat{\beta}_{IV}) = \sigma_e^2 (Z'X)^{-1} Z'Z (X'Z)^{-1} = \frac{\sigma_e^2}{n^2} \frac{n * var(z)}{cov(x,z)^2}$$

$$= \frac{\sigma_e^2}{n} \frac{1}{var(x)} \frac{var(x) var(z)}{cov(x,z)^2}$$

$$= \frac{\sigma_e^2}{n} \frac{1}{var(x)} \frac{1}{\rho_{xz}^2}$$

$$= Avar(\hat{\beta}_{OLS}) \frac{1}{\rho_{xz}^2}$$

- As $\rho_{xz}^2 \to 0$, $Avar(\hat{\beta}_{IV}) \to \infty$

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- As $\rho_{xz}^2 \to 0$, $Avar(\hat{\beta}_{IV}) \to \infty$

Problems with IV estimator: Bias

- IV is often is neither biased nor unbiased because it does not even have an expectation.
- Kiviet has shown that the IV estimator has M moments, the number of overidentifying restrictions. If $q = 0$, IV has no expectation.

$$
\begin{aligned}
y &= X\beta + e \\
X &= Z\pi + v \\
\hat{\beta}_{IV} &= (X'P_Z X)^{-1} X'P_z y \\
&= \beta + (X'P_Z X)^{-1} X'P_z e \\
&= \beta + (X'P_Z X)^{-1} (\pi'Z' + v')P_z e \\
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Form of small sample bias

$$
\begin{aligned}
E(\hat{\beta}_{IV}) - \beta &\approx E(X'P_Z X)^{-1} E(\pi' Z' e) + E(X'P_Z X)^{-1} E(v' P_z e) \\
&= E(X'P_Z X)^{-1} \pi' E(Z'e) + E(X'P_Z X)^{-1} E(v' P_z e) \\
&= (E(X'P_Z X))^{-1} E(v' P_z e) \\
&= (E(\pi' Z' + v') P_z (Z\pi + v)))^{-1} E(v' P_z e) \\
&= (E(\pi' Z' Z\pi + \pi' Z' v + v' Z\pi + v' P_z v))^{-1} E(v' P_z e) \\
&= (E(\pi' Z' Z\pi) + E(v' P_z v))^{-1} E(v' P_z e) \qquad \text{(b/c } E(Z'e) = E(Z'v) = 0) \\
&= (E(\pi' Z' Z\pi) + \sigma_v^2 p)^{-1} \sigma_{ev}^2 p \\
&= \frac{1}{\left(\frac{E(\pi' Z' Z\pi)/p}{\sigma_v^2} + 1\right)} \frac{\sigma_{ev}^2}{\sigma_v^2}
\end{aligned}
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&= (E(X'P_ZX))^{-1}E(v'P_ze) \\
&= (E(\pi'Z' + v')P_z(Z\pi + v)))^{-1}E(v'P_ze) \\
&= (E(\pi'Z'Z\pi + \pi'Z'v + v'Z\pi + v'P_zv))^{-1}E(v'P_ze) \\
&= (E(\pi'Z'Z\pi) + E(v'P_zv))^{-1}E(v'P_ze) \qquad \text{(b/c } E(Z'e) = E(Z'v) = 0) \\
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Form of small sample bias

$$
\begin{aligned}
E(\hat{\beta}_{IV}) - \beta &\approx E(X'P_Z X)^{-1}E(\pi' Z' e) + E(X'P_Z X)^{-1}E(v'P_z e) \\
&= E(X'P_Z X)^{-1}\pi' E(Z'e) + E(X'P_Z X)^{-1}E(v'P_z e) \\
&= (E(X'P_Z X))^{-1}E(v'P_z e) \\
&= (E(\pi' Z' + v')P_z(Z\pi + v)))^{-1}E(v'P_z e) \\
&= (E(\pi' Z' Z\pi + \pi' Z' v + v' Z\pi + v' P_z v))^{-1}E(v'P_z e) \\
&= (E(\pi' Z' Z\pi) + E(v'P_z v))^{-1}E(v'P_z e) \qquad (\text{b/c } E(Z'e) = E(Z'v) = 0) \\
&= (E(\pi' Z' Z\pi) + \sigma_v^2 p)^{-1}\sigma_{ev}^2 p \\
&= \frac{1}{\left(\frac{E(\pi' Z' Z\pi)/p}{\sigma_v^2} + 1\right)} \frac{\sigma_{ev}^2}{\sigma_v^2}
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F-test

$$E(\hat{\beta}_{IV}) - \beta \approx \frac{1}{\left(\frac{E(\pi' Z' Z \pi)/p}{\sigma_v^2} + 1\right)} \frac{\sigma_{ev}^2}{\sigma_v^2} \approx \frac{1}{(1 + F_{p,n-p})} \frac{\sigma_{ev}^2}{\sigma_v^2}$$

- F is the test where the null is that all instrument coefficients are 0.
- The bias of IV only goes away if $F \to \infty$
- The bias of IV is the OLS bias as $F \to 0$.
- Adding useless instruments increases p, which decreases F and increases the bias.

Weak instruments

Suppose we have a single x and a single instrument z. An instrument is weak if ρ_{zx} is small.

$$
\begin{aligned}
plim \hat{\beta}_{OLS} &= plim \frac{cov(x, y)}{var(x)} = plim \frac{cov(x, \alpha + \beta + e)}{var(x)} \\
&= \beta + plim \frac{cov(x, e)}{var(x)} = \beta + plim \frac{cov(x, e)}{\sqrt{var(x)}\sqrt{var(e)}} \frac{\sqrt{var(e)}}{\sqrt{var(x)}} \\
&= \beta + \rho_{xe} \frac{\sigma_e}{\sigma_x}
\end{aligned}
$$

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\begin{aligned}
plim\hat{\beta}_{IV} &= plim\frac{cov(x, \alpha + \beta + e)}{cov(z, x)} \\
&= \beta + plim\frac{cov(z, e)}{cov(z, x)} = \beta + plim\frac{\frac{cov(z,e)}{\sqrt{var(x)}\sqrt{var(e)}}}{\frac{cov(z,x)}{\sqrt{var(x)}\sqrt{var(z)}}}\frac{\sqrt{var(e)}}{\sqrt{var(x)}} \\
&= \beta + \frac{\rho_{ze}}{\rho_{zx}}\frac{\sigma_e}{\sigma_x} \\
&= \beta + \frac{\rho_{ze}}{\rho_{zx}\rho_{xe}}\rho_{xe}\frac{\sigma_e}{\sigma_x} = \beta + \frac{\rho_{ze}}{\rho_{zx}\rho_{xe}}ABias(\hat{\beta}_{OLS})
\end{aligned}
$$

Weak/Bad instruments are worse than OLS

$$\frac{ABias(\hat{\beta}_{OLS})}{ABias(\hat{\beta}_{IV})} > 1 \rightarrow \frac{\rho_{ze}}{\rho_{zx}\rho_{xe}} > 1$$

If $\frac{\rho_{ze}}{\rho_{zx}\rho_{xe}} \geq 1$, then IV is more biased than OLS.
Suppose $\rho_{xu} = .5$, so X is super endogenous, Z is barely endogenous: $\rho_{zu} = 0.01$.
Small $\rho_{zx} = 0.019$ gives $\frac{ABias(\hat{\beta}_{OLS})}{ABias(\hat{\beta}_{IV})} = 1.052$.

Testing power of instruments

$$\frac{ABias(\hat{\beta}_{OLS})}{ABias(\hat{\beta}_{IV})} \approx \frac{1}{F}$$

F statistic of 100 means IV is 1% as biased as OLS.

Testing endogeneity via Durbin-Hausman-Wu test

- If X is exogenous, then both OLS and IV are consistent, but OLS is BLUE.
- Asymptotically, the difference between OLS and IV should converge to zero.

$$H = (\hat{\beta}_{IV} - \hat{\beta}_{OLS})'[Avar(\hat{\beta}_{IV}) - Avar(\hat{\beta}_{OLS})]^{-1}(\hat{\beta}_{IV} - \hat{\beta}_{OLS}) \sim \chi^2_{dim(\beta)}$$

- Rejecting null says that OLS and IV are not close to one another, so either X is endogneous or Z is an invalid instrument.

Heckman Selection

- IV estimates the LATE under fairly reasonable assumptions.
- Under stronger distributional assumptions we can get the ATE and the ATT.

$$\begin{pmatrix} u_i^1 \\ u_i^0 \\ e_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{01} & \sigma_{1e} \\ & \sigma_0^2 & \sigma_{0e} \\ & & 1 \end{bmatrix}\right)$$

Conditional Expectations of Joint Normals

Consider two jointly normal random variables (X, Y).

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right)$$

Then, the conditional expectation of X given $Y > c$ is:

$$E[X \mid Y > c] = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(Y)}} \cdot \frac{\phi\left(\frac{c}{\sqrt{\text{Var}(Y)}}\right)}{1 - \Phi\left(\frac{c}{\sqrt{\text{Var}(Y)}}\right)}$$

Here, $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and CDF of the standard normal distribution, respectively.

Application to Selection Model

We focus on deriving:

$$E[u_i^1 \mid e_i > -Z_i'\gamma]$$

Using the previous result with $X = u_i^1$, $Y = e_i$, and threshold $c = -Z_i'\gamma$:

$$E[u_i^1 \mid e_i > -Z_i'\gamma] = \sigma_{1e}\frac{\phi(Z_i'\gamma)}{\Phi(Z_i'\gamma)}$$

$$E[u_i^0 \mid e_i < -Z_i'\gamma]$$

$$E[u_i^0 \mid e_i < -Z_i'\gamma] = -\sigma_{1e}\frac{\phi(Z_i'\gamma)}{1-\Phi(Z_i'\gamma)}$$

Inverse Mills Ratio

Define the **Inverse Mills Ratio** $\lambda(\cdot)$ as:

$$\lambda(Z_i'\gamma) = \frac{\phi(Z_i'\gamma)}{\Phi(Z_i'\gamma)}$$

Thus, the conditional expectation is:

$$E[u_i^1 \mid e_i > -Z_i'\gamma] = \sigma_{1e}\lambda(Z_i'\gamma)$$

- σ_{1e} captures correlation between selection and outcome errors.
- $\lambda(Z_i'\gamma)$ measures the intensity of selection at given values of the selection index $Z_i'\gamma$.
- We estimate γ with a probit.