Calculus Warm-Up Quiz — Day 3

Estimation of the Linear Projection Model

Complete before lecture. 10 minutes.

Name: ____

Today we derive the OLS estimator using matrix calculus: differentiating linear and quadratic forms. We also verify the second-order condition. This quiz reviews the scalar versions of these skills.

1. Expanding a Quadratic. Let $f(\beta) = (y - x\beta)^2$ where y and x are known constants.

(a) Expand $f(\beta)$ fully.

(b) Identify the constant, linear, and quadratic terms in β.

2. Derivative of a Sum of Squares. Consider $S(\beta) = \sum_{i=1}^{3}(y_i - x_i\beta)^2$ with data:

i	y_i	x_i
1	4	2
2	5	3
3	7	4

(a) Write $S(\beta) = \sum y_i^2 - 2\beta \sum x_i y_i + \beta^2 \sum x_i^2$. Compute each sum.

(b) Find $\dfrac{dS}{d\beta}$ and set it to zero. Solve for $\hat{\beta}$.

(c) Compute $\dfrac{d^2S}{d\beta^2}$. Confirm it is positive.

3. Derivative of Linear and Quadratic Forms (Scalar Versions).
Let a be a scalar and z, q be constants. Compute:

(a) $\dfrac{d}{da}(za) =$

(b) $\dfrac{d}{da}(qa^2) =$

(c) $\dfrac{d}{da}(qa^2 - 2za) =$ Set this to zero and solve for a.

4. Matrix Calculus Preview. In today's lecture, we will differentiate

$$SSE(\boldsymbol{\beta}) = \boldsymbol{y'y} - 2\boldsymbol{y'X\beta} + \boldsymbol{\beta' X'X\beta}$$

The matrix calculus rules we will use are:

- $\nabla_{\boldsymbol{\beta}}(\boldsymbol{y'X\beta}) = \boldsymbol{X'y}$ (linear form $\rightarrow$ coefficient vector)
- $\nabla_{\boldsymbol{\beta}}(\boldsymbol{\beta'X'X\beta}) = 2\boldsymbol{X'X\beta}$ (quadratic form, symmetric matrix)

Using these rules, compute $\nabla_{\boldsymbol{\beta}} SSE(\boldsymbol{\beta})$ and set it to $\mathbf{0}$.

Answer Key — Day 3

1. (a) $f(\beta) = y^2 - 2xy\beta + x^2\beta^2$

(b) Constant: y^2. Linear in β: $-2xy\beta$. Quadratic in β: $x^2\beta^2$.

2. (a) $\sum y_i^2 = 16 + 25 + 49 = 90$. $\sum x_i y_i = 8 + 15 + 28 = 51$. $\sum x_i^2 = 4 + 9 + 16 = 29$.
So $S(\beta) = 90 - 102\beta + 29\beta^2$.

(b) $S'(\beta) = -102 + 58\beta = 0 \implies \hat{\beta} = 102/58 = 51/29 \approx 1.759$.

(c) $S''(\beta) = 58 > 0$. Confirmed: the critical point is a minimum.

3. (a) z

(b) $2qa$

(c) $2qa - 2z = 0 \implies a = z/q$. (This is the scalar version of $\hat{\beta} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y}$.)

4. $\nabla_{\boldsymbol{\beta}} SSE = \mathbf{0} - 2\boldsymbol{X'y} + 2\boldsymbol{X'X\beta} = \mathbf{0}$
$\implies 2\boldsymbol{X'X\beta} = 2\boldsymbol{X'y}$
$\implies \hat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y}$