CA167: Practical X09

<u>Note</u>: This practical forms the third continuous assessment test, following the lab exams of semester 1. It is important that the work submitted be the student's own and a clear statement to this effect should be included; s***pecifically, the front page of the submission should be a completed copy of page 5 of DCU's "Academic Integrity and Plagiarism Policy"***
(https://www4.dcu.ie/sites/default/files/registry/docs/IntegrityPlagiarism.pdf)

1. Mathematical model

Consider the problem of a sphere of radius a, which is at an initial temperature of T_0. If the material is characterized by a diffusivity α^2 and it is placed in $0\ °\mathrm{C}$ air, the sphere will slowly cool by convection and radiation. If the rate of temperature decrease at the surface is characterized by an experimental constant h, the solution for the temperature at the centre of the sphere is

$$T_c = T_0 \left(\frac{ah}{\sigma}\right) \sum_{n=1}^{\infty} C_n^{-\beta_n^2 t} \qquad (*)$$

where

$$\beta_n^2 = \theta_n \frac{\alpha^2}{a^2} \qquad (**)$$

$$C_n = \frac{4Sin(\theta_n)}{2\theta_n - Sin(2\theta_n)} \qquad (***)$$

and θ_n is the n^th root of the equation

$$Tan(\theta_n) = \frac{\theta_n}{1 - \frac{ah}{\sigma}} \qquad (****)$$

2.Statement of the programming problem

Write a Matlab program that takes a set of parameter values as input (see below on sets of values). These values should be read from a text file. Then, the program should (A) Determine and report the first 10 roots of Equation (****).

(B) Then use Equation (*) to compute and report the temperature at the centre of the sphere for values of time t = 0 to t = 3600 seconds (that is, one hour) in steps of 60 seconds. The program should generate a plot of the centre temperature.

2. 1 Specific guidance on the root-finding step (A)

(**1**) Express Equation (****) in the form $f(\theta) = 0$ so that the first 10 roots of this are what is required. You will also need to work out a formula for the derivative $f'(\theta)$

(**2**) Finding the roots θ_n

(a) **First Root**: Write Matlab code that will start at $\theta = 0.1$ and step in values of 0.1 to find the first time that $f(\theta)$ changes sign. (*HINT: This should be near 0.4 for the parameter values of Set 1 – see below*). Then use the Newton-Raphson method to refine this rough estimate of the first root so that it is known to an accuracy of at least 10^{-5}.

(b) **Subsequent Roots**: You must be careful about how to find the remaining 9 roots that are required. One cannot proceed to look for sign changes in $f(\theta)$ because there are discontinuities at multiples of $\pi/2$. Instead, for all roots after the first, you should begin the Newton-Raphson method with the initial guess

$$\theta_n = (n - \frac{1}{2})\pi(1 - \frac{1}{(n-\frac{1}{2})^2 \pi^2 k^2}), (n = 2, 3, ..., 9)$$

where $$k = 1 - (\frac{ah}{\sigma})$$

3. Values of the parameters (2 sets)

Set 1: First, use the following values of the parameters

T_0	α^2	h	σ	a
250°C	1.2×10^{-5} m^2/sec	23.0 W/m^2-°C	46.0 W/m^2-°C	0.1m

Set 2: Next, modify the values in Set 1 and re-run the program to see what the impact of the modifications is. The choice of modification is up to you (*so it isanticipated that students will make different choices*). Note that this may impact on the value of the rough estimate of the first root of Equation (****).

Correction to Specification

Following questions from some students, I have re-checked the formulae in the specification. It turns out, unfortunately, that the textbook I used for this problem had an error in equation (*). Having calculated the solution myself, I see that this equation should read

$$T_c = T_0\left(\frac{ah}{\sigma}\right)\sum_{n=1}^{\infty} C_n e^{-\beta_n^2 t} \qquad (*)$$

This error does not impact at all on the root finding part of the question.