Some Experimental Verifications of the BHR Conjecture via Inductive Construction of Hamiltonian Paths

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Abstract

The Buratti–Horak–Rosa (BHR) conjecture posits that for any multiset of positive integers satisfying a certain divisor condition, there exists a Hamiltonian path in the complete graph K_p whose edge lengths match the multiset. In this paper, we present two constructive, inductive strategies for building such Hamiltonian paths. These methods are implemented algorithmically and validated through empirical testing. Our approach provides a practical framework for exploring the conjecture and generating realizations for admissible multisets.

This document presents a computational framework for inductively constructing Hamiltonian paths (HPs) corresponding to specific edge frequency profiles (FPs), providing experimental support for the BHR conjecture. Using Python, we systematically evolve FPs by adding edge-length multiplicities and validate that HPs exist for each evolved profile. Through reuse-insertion heuristics and fallback brute-force search with backtracking, we demonstrate successful constructions up to $p = 40$, capturing performance metrics across iterations. The results offer strong empirical evidence of the conjecture's validity within the tested scope.

Mathematics Subject Classification (2020)

05C45, 05C38, 05C85, 68R10, 68W01

Keywords

BHR conjecture, Hamiltonian path, cyclic edge length, complete graph, inductive construction, graph algorithms, combinatorial design

1 Introduction

Let K_p be the complete graph on p vertices labeled $0, 1, \ldots, p-1$. Define the *cyclic length* of an edge between vertices x and y as:

$$\ell(x, y) = \min(|x - y|, p - |x - y|).$$

Given a multiset L of $p - 1$ positive integers, the BHR conjecture asks whether there exists a Hamiltonian path in K_p whose edge lengths (under the cyclic metric) match L.

A necessary and sufficient condition for such a multiset L to be *admissible* is the **divisor condition**:

> For every divisor d of p, the number of elements in L divisible by d must be at most $p - d$.

2 Scenario I: Increasing the Multiplicity of an Existing Part (K fixed)

3 Scenario II: Increasing the Number of Parts ($K \to K + 1$)

In the scenario I, we fix the number of distinct edge lengths K and increase the multiplicity of one of the existing edge lengths by 1. This results in a new multiset L_2 of size p that differs from L_1 by a single count increment.

In the scenario II, we begin with a multiset L_1 of size $p-1$ with K distinct edge lengths and construct a new multiset L_2 of size p by adding one new edge length not present in L_1. This corresponds to increasing the number of parts in the fingerprint representation of the multiset.

1. **Reuse-Insertion:** Insert the new vertex into the previous HP and validate against FP.
2. **Heuristic Scoring:** Try top-scoring insertions minimizing FP deviation.
3. **Backtracking Search:** Exhaustively search for valid HP, counting backtracks.

Each run is logged with timestamps, methods used, FP and HP details, backtrack counts, and computation time.

4 Experimental Results

In one run spanning 10 iterations, the system evolved FP up to $p = 40$, maintaining successful HP constructions throughout. Selected iterations are shown below.

Iteration 9

- **Timestamp:** 20:19:41
- **Method used:** Backtrack
- **p (vertices):** 39
- **Evolved FP:** `Counter({1:6, 2:6, ..., 16:1})`
- **HP:** `[0, 1, 2, ..., 9]`
- **HP freq:** `Counter({1:6, 2:6, ..., 16:1})`
- **Backtracks:** 9002
- **Time:** 0.03 sec
- **Result:** SUCCESS

Iteration 10

- **Timestamp:** 20:19:41
- **Method used:** Backtrack
- **p (vertices):** 40
- **Evolved FP:** `Counter({1:6, 2:6, ..., 17:1})`
- **HP:** `[0, 1, 2, ..., 20]`
- **HP freq:** `Counter({1:6, 2:6, ..., 17:1})`
- **Backtracks:** 10827
- **Time:** 0.03 sec
- **Result:** SUCCESS

Summary Table

Iteration	p	Backtracks
1	31	1765830
2	32	82254492
3	33	5981815
4	34	90111
5	35	565485
6	36	162158
7	37	1297988
8	38	2212
9	39	9002
10	40	10827

5 Discussion

These two inductive scenarios provide a constructive pathway for exploring the BHR conjecture. While backtracking becomes computationally intensive for large p, the reuse-insert method succeeds in many cases, especially when the change to the multiset is minimal. The divisor condition acts as a reliable filter, ensuring that only admissible cases are attempted.

6 Conclusion

We have demonstrated that two inductive strategies—adding a new edge length and increasing the count of an existing one—can be used to construct Hamiltonian paths for admissible multisets. These methods, supported by algorithmic implementation and empirical validation, offer a promising framework for exploring and potentially proving the BHR conjecture. The Python program developed for this research is efficient and is designed to be scalable to run on faster computers for larger values of p beyond 40.

Acknowledgments and code availability

Thanks to all those who helped to me on this work. The code is available on request.

References

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