1 Interest Rate Only

This section follows [3].

1.1 Dynamics

The instantaneous risk-free rate is $r_t = X_t^1 + X_t^2$. The state variables are level X_t^1, slope X_t^2, and curvature X_t^3, which follow a system of SDEs ($d\boldsymbol{X}_t = K^{\mathrm{Q}}(-\boldsymbol{X}_t)dt + \Sigma d\boldsymbol{W}_t^{\mathrm{Q}}$):

- Independent factor AFNS:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

- Correlated factor AFNS:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

The zero-coupon bond yield is:

$$y(t,T) = X_t^1 + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} X_t^2 + \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)} \right) X_t^3 - \frac{A(t,T)}{T-t}, \tag{1}$$

where $-\frac{A(t,T)}{T-t}$ is a yield-adjustment term:

$$A(t,T) = \frac{1}{2} \sum_{i=1}^{3} \int_t^T \left(\Sigma^\top \boldsymbol{B}(s,T) \boldsymbol{B}(s,T)^\top \Sigma \right)_{i,i} ds.$$

Assume an affine risk premium, the P-dynamics $d\boldsymbol{X}_t = K^{\mathrm{P}}(\boldsymbol{\theta}^{\mathrm{P}} - \boldsymbol{X}_t)dt + \Sigma d\boldsymbol{W}_t^{\mathrm{P}}$ for

- An independent factor AFNS model are:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & 0 & 0 \\ 0 & \kappa_{22}^{\mathrm{P}} & 0 \\ 0 & 0 & \kappa_{33}^{\mathrm{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right) dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \end{pmatrix}.$$

- A correlated-factor AFNS model are:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & \kappa_{12}^{\mathrm{P}} & \kappa_{13}^{\mathrm{P}} \\ \kappa_{21}^{\mathrm{P}} & \kappa_{22}^{\mathrm{P}} & \kappa_{23}^{\mathrm{P}} \\ \kappa_{31}^{\mathrm{P}} & \kappa_{32}^{\mathrm{P}} & \kappa_{33}^{\mathrm{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right) dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \end{pmatrix}.$$

The measurement equation is:

$$\boldsymbol{y}_t = B\boldsymbol{X}_t + \boldsymbol{A} + \boldsymbol{\epsilon}_t \iff$$
$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} - \begin{pmatrix} \frac{A(\tau_1)}{\tau_1} \\ \frac{A(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A(\tau_N)}{\tau_N} \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_N) \end{pmatrix},$$

where the measurement errors $\epsilon_t(\tau_i)$ are iid noises.

1.2 Estimation

The state transition equation is:

$$\boldsymbol{X}_t = \left[I - \exp(-K^{\mathrm{P}}\Delta t)\right]\boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}}\Delta t)\boldsymbol{X}_{t-1} + \boldsymbol{\eta}_t,$$

with the error structure

$$\begin{pmatrix} \boldsymbol{\eta}_t \\ \boldsymbol{\epsilon}_t \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

where Δt is the time between observations, $H = \mathrm{diag}(\sigma_\epsilon^2(\tau_1), \sigma_\epsilon^2(\tau_2), \ldots, \sigma_\epsilon^2(\tau_N))$, and

$$Q = \int_0^{\Delta t} e^{-K^{\mathrm{P}}s}\Sigma\Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

We apply the Kalman filtering to estimate the parameters:

1. Initialise the filter at the unconditional mean and variance of the state variables under the P−measure:
$$\boldsymbol{X}_0 = \boldsymbol{\theta}^{\mathrm{P}}, \Sigma_0 = \int_0^{\infty} e^{-K^{\mathrm{P}}s}\Sigma\Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

2. Let the information available at time t be $\boldsymbol{Y}_t = (y_1, \ldots, y_t)$. The prediction step is:
$$\boldsymbol{X}_{t|t-1} = [I - \exp(-K^{\mathrm{P}}\Delta t)]\boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}}\Delta t)\boldsymbol{X}_{t-1},$$
$$\Sigma_{t|t-1} = \exp(-K^{\mathrm{P}}\Delta t)\Sigma_{t-1}\exp(-K^{\mathrm{P}}\Delta t)^\top + \int_0^{\Delta t} e^{-K^{\mathrm{P}}s}\Sigma\Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

   We utilise the analytical formulas in [4].

3. Update
$$\boldsymbol{X}_t = \boldsymbol{X}_{t|t-1} + \Sigma_{t|t-1}B^\top \mathrm{Cov}(\boldsymbol{v}_t)^{-1}\boldsymbol{v}_t,$$
$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1}B^\top \mathrm{Cov}(\boldsymbol{v}_t)^{-1}B\Sigma_{t|t-1},$$

   where

$$\boldsymbol{v}_t = \boldsymbol{y}_t - \boldsymbol{A} - B\boldsymbol{X}_{t|t-1},$$
$$\mathrm{Cov}(\boldsymbol{v}_t) = B\Sigma_{t|t-1}B^\top + H.$$

4. The log-likelihood is:

$$\log \ell(y_1, \ldots, y_T) = \sum_{t=1}^{T} \left(-\frac{N}{2} \log(2\pi) - \frac{1}{2} \log \det(\mathrm{Cov}(\boldsymbol{v}_t)) - \frac{1}{2} \boldsymbol{v}_t^\top \mathrm{Cov}(\boldsymbol{v}_t)^{-1} \boldsymbol{v}_t \right),$$

   where N is the number of observed yields.

5. Compute MLE by the Nelder-Mead approach.

We use the monthly RBA data from July 1992 to May 2021, maturing every quarter until 10 years. We adapt the R codes from [1].

1.3 Results

1.3.1 Independent-factor AFNS

Assuming K^{P} is diagonalisable, the estimated parameters are:

$$\hat{K}^{\mathrm{P}}, \hat{\Sigma}, \hat{\boldsymbol{\theta}}^{\mathrm{P}}, \hat{\lambda}$$

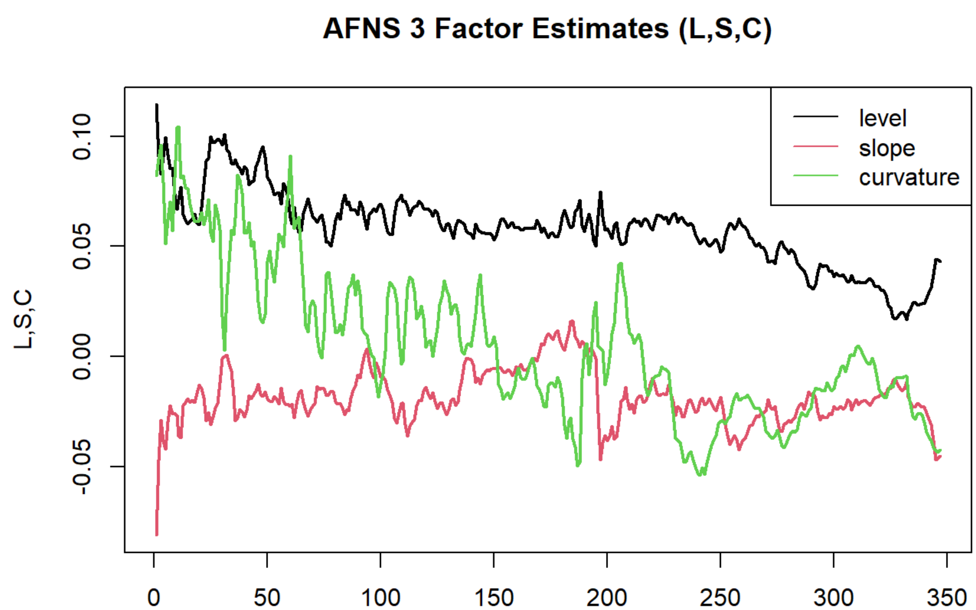

Figure 1: Fitted state variables for the independent factor AFNS.

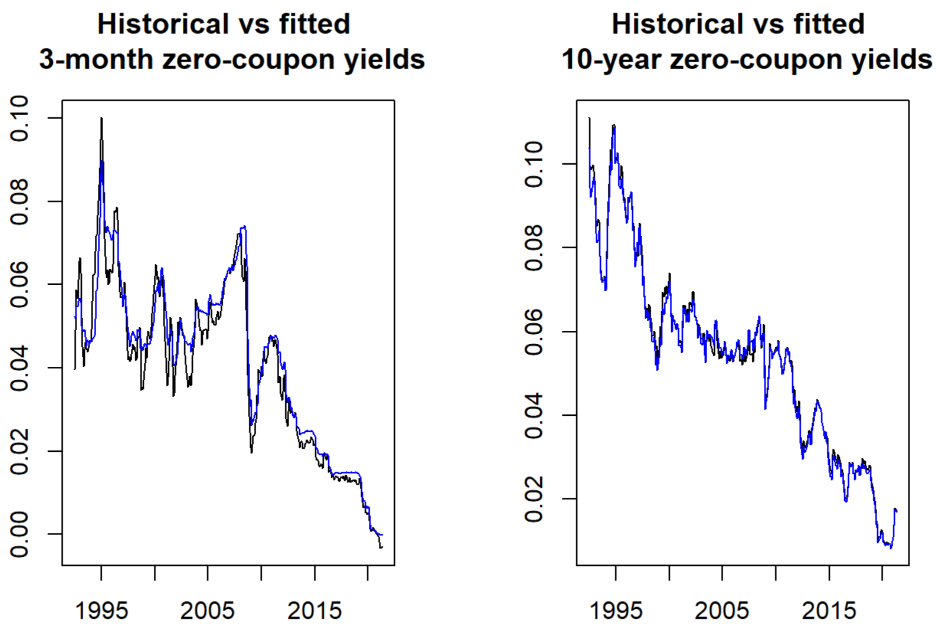

Figure 2: Fitted zero-coupon bond rates for the independent factor AFNS.

1.3.2 Correlated-factor AFNS

Assuming K^{P} is diagonalisable, the estimated parameters are:

$$\hat{K}^{\mathrm{P}}, \hat{\Sigma}, \hat{\boldsymbol{\theta}}^{\mathrm{P}}, \hat{\lambda}$$

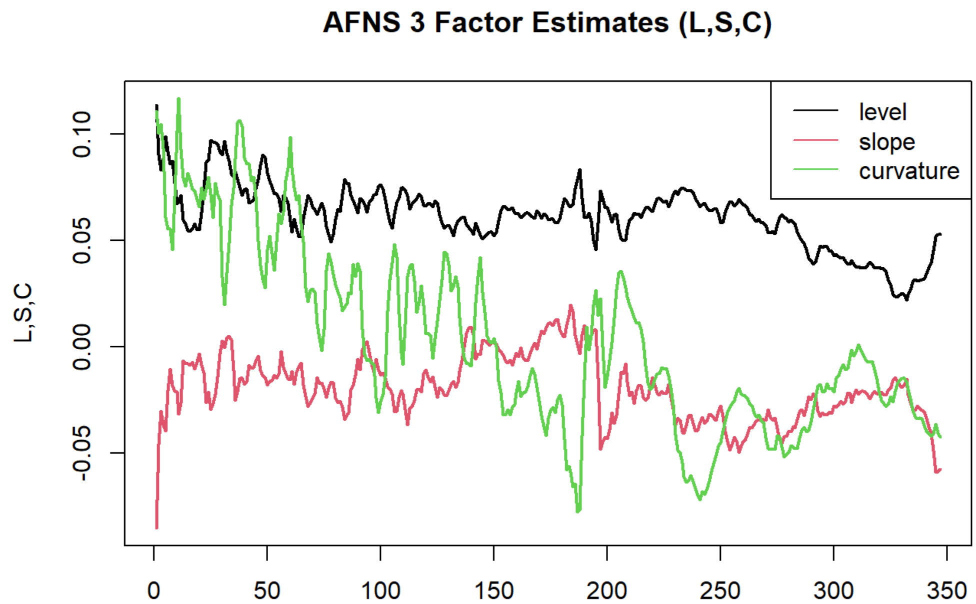

Figure 3: Fitted state variables for the correlated factor AFNS.

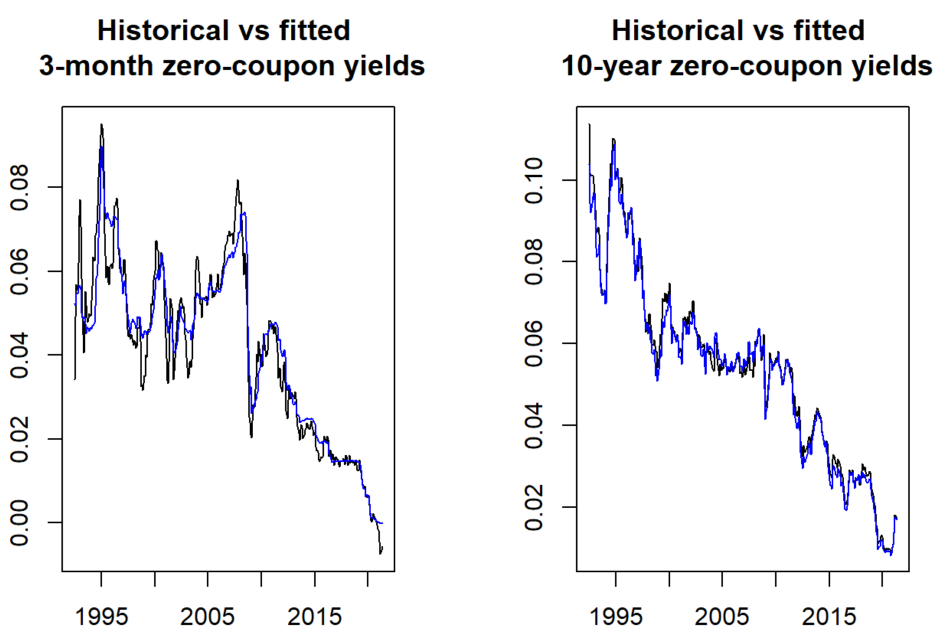

Figure 4: Fitted state variables for the correlated factor AFNS.

1.4 Testing

1.5 Simulation

Fix the time step $h > 0$. Discretise the SDEs under the P measure:

$$\boldsymbol{X}_{t+h} - \boldsymbol{X}_t = K^{\mathrm{P}}(\boldsymbol{\theta}^{\mathrm{P}} - \boldsymbol{X}_t)h + \Sigma\sqrt{h}\boldsymbol{Z}, \boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, I).$$

This gives a trajectory of the state variables: $\boldsymbol{X}_0, \boldsymbol{X}_h, \boldsymbol{X}_{2h}, \ldots$. We use it with formula (1) to simulate the yields of a maturity-τ month zero-coupon bond.

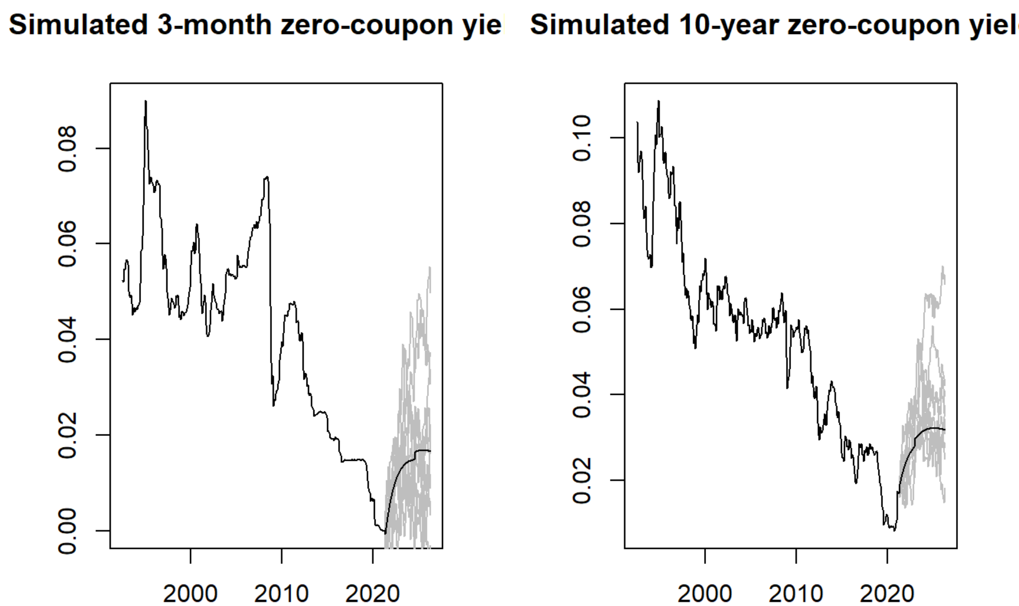

Figure 5: Simulated zero-coupon bond rates for the independent factor AFNS.

2 Interest Rate, Home Value Indexes, Stock Prices

This section is based on [2]. Granger causality between these variables are proven in "VAR_Framework.pdf".

Assume that the interest rate follows AFNS as above, both the home value indexes and stock prices follow the GBM. Denote the additional state variables as: logged home value index NSW $X_t^4 = \log(H_t)$ and logged stock price $X_t^5 = \log(S_t)$. The new state variables are correlated with X_t^1, X_t^2, X_t^3.

2.1 Dynamics

The Q-dynamics are:

$$dX_t^4 = \left(X_t^1 + X_t^2 - \frac{\sigma_4^2}{2} \right) dt + \sigma_4 \sum_{i=1}^{3} \rho_{4i} dW_{it}^{\mathrm{Q}} + \sigma_4 \sqrt{1-\rho_4^2} dW_{4t}^{\mathrm{Q}},$$

$$dX_t^5 = \left(X_t^1 + X_t^2 - \frac{\sigma_5^2}{2} \right) dt + \sigma_5 \sum_{i=1}^{4} \rho_{5i} dW_{it}^{\mathrm{Q}} + \sigma_5 \sqrt{1-\rho_5^2} dW_{5t}^{\mathrm{Q}}.$$

The standard Brownian motions W_{4t}^{Q} and W_{5t}^{Q} are correlated with other BMs by:

$$\mathrm{Cov}\left(dW_{4t}^{\mathrm{Q}}, dW_{it}^{\mathrm{Q}} \right) = \rho_{4i} dt, i = 1, 2, 3,$$

$$\mathrm{Cov}\left(dW_{5t}^{\mathrm{Q}}, dW_{it}^{\mathrm{Q}} \right) = \rho_{5i} dt, i = 1, 2, 3, 4.$$

The ρ's are correlation coefficients which take value from -1 to 1. Let $\rho_4^2 = \sum_{i=1}^{3} \rho_{4i}^2 \leq 1$, and $\rho_5^2 = \sum_{i=1}^{4} \rho_{5i}^2 \leq 1$.

The state variables $\boldsymbol{X}_t = (X_t^1, X_t^2, X_t^3, X_t^4, X_t^5)$ follow the system of SDEs $d\boldsymbol{X}_t = (\boldsymbol{\theta}^{\mathrm{Q}} - K^{\mathrm{Q}} \boldsymbol{X}_t) dt + \Sigma d\boldsymbol{W}_t^{\mathrm{Q}}$:

- Independent factor:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\sigma_4^2}{2} \\ -\frac{\sigma_5^2}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +$$

$$\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ \sigma_4 \rho_{41} & \sigma \rho_{42} & \sigma_4 \rho_{43} & \sigma_4 \sqrt{1-\rho_4^2} & 0 \\ \sigma_5 \rho_{51} & \sigma_5 \rho_{52} & \sigma_5 \rho_{53} & \sigma_5 \rho_{54} & \sigma_5 \sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \\ dW_{4t}^{\mathrm{Q}} \\ dW_{5t}^{\mathrm{Q}} \end{pmatrix}$$

- Correlated factor:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\sigma_4^2}{2} \\ -\frac{\sigma_5^2}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +$$

$$\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 \\ \sigma_4 \rho_{41} & \sigma \rho_{42} & \sigma_4 \rho_{43} & \sigma_4 \sqrt{1-\rho_4^2} & 0 \\ \sigma_5 \rho_{51} & \sigma_5 \rho_{52} & \sigma_5 \rho_{53} & \sigma_5 \rho_{54} & \sigma_5 \sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \\ dW_{4t}^{\mathrm{Q}} \\ dW_{5t}^{\mathrm{Q}} \end{pmatrix}$$

The measure change is facilitated by the Girsanov's Theorem: $d\boldsymbol{W}_t^{\mathrm{Q}} = d\boldsymbol{W}_t^{\mathrm{P}} + \Gamma_t dt$. Assume an essentially affine risk premium:

$$
\begin{aligned}
\Gamma_t &= \boldsymbol{\gamma}^0 + \gamma^1 \boldsymbol{X}_t \\
&= \begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \\ \gamma_3^0 \\ \gamma_4^0 \\ \gamma_5^0 \end{pmatrix} + \begin{pmatrix} \gamma_{11}^1 & \gamma_{12}^1 & \gamma_{13}^1 & \gamma_{14}^1 & \gamma_{15}^1 \\ \gamma_{21}^1 & \gamma_{22}^1 & \gamma_{23}^1 & \gamma_{24}^1 & \gamma_{25}^1 \\ \gamma_{31}^1 & \gamma_{32}^1 & \gamma_{33}^1 & \gamma_{34}^1 & \gamma_{35}^1 \\ \gamma_{41}^1 & \gamma_{42}^1 & \gamma_{43}^1 & \gamma_{44}^1 & \gamma_{45}^1 \\ \gamma_{51}^1 & \gamma_{52}^1 & \gamma_{53}^1 & \gamma_{54}^1 & \gamma_{55}^1 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix}.
\end{aligned}
$$

Substituting it into the Q-dynamics yields:

$$
\begin{aligned}
d\boldsymbol{X}_t &= (\boldsymbol{\theta}^{\mathrm{Q}} - K^{\mathrm{Q}} \boldsymbol{X}_t)dt + \Sigma \left[d\boldsymbol{W}_t^{\mathrm{P}} + (\boldsymbol{\gamma}^0 + \gamma^1 \boldsymbol{X}_t)dt \right] \\
&= \left[(\boldsymbol{\theta}^{\mathrm{Q}} + \Sigma \boldsymbol{\gamma}^0) - (K^{\mathrm{Q}} - \Sigma \gamma^1) \boldsymbol{X}_t \right] dt + \Sigma d\boldsymbol{W}_t^{\mathrm{P}}.
\end{aligned}
$$

We can make further assumptions to $\boldsymbol{\gamma}^0, \gamma^1$ such that the P-dynamics preserve the structure of the corresponding Q-dynamics:

- Independent factor:

$$
\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \\ \theta_4^{\mathrm{P}} \\ \theta_5^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22}^{\mathrm{P}} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{33}^{\mathrm{P}} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +
$$
$$
\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ \sigma_4 \rho_{41} & \sigma \rho_{42} & \sigma_4 \rho_{43} & \sigma_4 \sqrt{1-\rho_4^2} & 0 \\ \sigma_5 \rho_{51} & \sigma_5 \rho_{52} & \sigma_5 \rho_{53} & \sigma_5 \rho_{54} & \sigma_5 \sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \\ dW_{4t}^{\mathrm{P}} \\ dW_{5t}^{\mathrm{P}} \end{pmatrix}
$$

- Correlated factor:

$$
\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \\ \theta_4^{\mathrm{P}} \\ \theta_5^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & \kappa_{12}^{\mathrm{P}} & \kappa_{13}^{\mathrm{P}} & 0 & 0 \\ \kappa_{21}^{\mathrm{P}} & \kappa_{22}^{\mathrm{P}} & \kappa_{23}^{\mathrm{P}} & 0 & 0 \\ \kappa_{31}^{\mathrm{P}} & \kappa_{32}^{\mathrm{P}} & \kappa_{33}^{\mathrm{P}} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +
$$
$$
\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 \\ \sigma_4 \rho_{41} & \sigma \rho_{42} & \sigma_4 \rho_{43} & \sigma_4 \sqrt{1-\rho_4^2} & 0 \\ \sigma_5 \rho_{51} & \sigma_5 \rho_{52} & \sigma_5 \rho_{53} & \sigma_5 \rho_{54} & \sigma_5 \sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \\ dW_{4t}^{\mathrm{P}} \\ dW_{5t}^{\mathrm{P}} \end{pmatrix}
$$

The measurement equation is:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \\ H_t \\ S_t \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} & 0 & 0 \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} - \begin{pmatrix} \frac{A(\tau_1)}{\tau_1} \\ \frac{A(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A(\tau_N)}{\tau_N} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_N) \\ \epsilon_H \\ \epsilon_S \end{pmatrix},$$

2.2 Estimation

2.3 Results

2.4 Testing

2.5 Simulation

References

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