1 VAR Dynamics

The vector of state variables is $\boldsymbol{z}_t = (r_t^{(1)}, r_t^{(40)} - r_t^{(1)}, h_t, y_t, g_t, c_t, a_t, d_t)^\top$. Its stationary counterpart is $\tilde{\boldsymbol{z}}_t = (r_t^{(1)} - r_{t-1}^{(1)}, r_t^{(40)} - r_t^{(1)}, h_t, y_t - y_{t-1}, g_t, c_t, a_t, d_t)^\top$, which follows a VAR(1) process (cf. VAR_Framework.pdf):

$$\tilde{\boldsymbol{z}}_{t+1} = \boldsymbol{\alpha} + \tilde{\Phi}\tilde{\boldsymbol{z}}_t + \Sigma^{1/2}\boldsymbol{\epsilon}_{t+1},$$

where $\tilde{\boldsymbol{\alpha}}$ is an 8×1 vector, $\tilde{\Phi}$ is an 8×8 coefficient matrix, and $\Sigma^{1/2}$ is the Cholesky decomposition of the residual covaraince matrix Σ.

However, $\boldsymbol{z}_t$ essentially follows a VAR(2) process

$$\boldsymbol{z}_{t+1} = \boldsymbol{\alpha} + \Phi_1\boldsymbol{z}_t + \Phi_2\boldsymbol{z}_{t-1} + \Sigma^{1/2}\boldsymbol{\epsilon}_{t+1},$$

with $\Phi_1 = \tilde{\Phi} + \Psi$, where Ψ is zero everywhere except $\Psi_{1,1} = \Psi_{4,4} = 1$, and Φ_2 is zero except the first and fourth columns being the corresponding columns of $-\tilde{\Phi}$.

2 Stochastic Discount Factors

2.1 The Model

Following [1], [2], [3], we develop stochastic discount factors, that is, pricing kernels, to price all nominal asset in the market.

Let ξ_{t+1} be the Radon-Nikodym derivative which converts the risk-neutral measure Q to the data-generating measure P; let X_{t+1} be the payoff of an asset. The following holds

$$\mathrm{E}_t^{\mathrm{Q}}(X_{t+1}) = \mathrm{E}_t^{\mathrm{P}}(\xi_{t+1}X_{t+1})/\xi_t.$$

Assume that ξ_{t+1} follows the log-normal process:

$$\xi_{t+1} = \xi_t \exp\left(-\frac{1}{2}\boldsymbol{\lambda}_t^\top\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t^\top\boldsymbol{\epsilon}_{t+1}\right),$$

where $\boldsymbol{\lambda}_t$ are the market prices of risk process associated with the sources of uncertainty $\boldsymbol{\epsilon}_t$. We parametrise $\boldsymbol{\lambda}_t$ as an affine process:

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \lambda_1\boldsymbol{z}_t,$$

where $\boldsymbol{\lambda}_0 \in \mathrm{R}^8$ affects the long-run mean yields and $\lambda_1 \in \mathrm{R}^{8\times 8}$ affects the time-varying risk premiums.

The pricing kernel is defined as:

$$s_{t+1} = \frac{\xi_{t+1}}{\xi_t}\exp(-r_t) = \exp\left(-\boldsymbol{e}_1^\top\boldsymbol{z}_t - \frac{1}{2}\boldsymbol{\lambda}_t^\top\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t^\top\boldsymbol{\epsilon}_{t+1}\right), \tag{1}$$

where r_t is the one-period short rate and $\boldsymbol{e}_1 = (1, 0, 0, 0, 0, 0, 0, 0)^\top$.

The price of an asset P_t paying X_{t+1} at time $t+1$ is:

$$P_t = \mathrm{E}_t^{\mathrm{P}}(s_{t+1}X_{t+1}).$$

In particular, the price of an n-period nominal bond $p_t^{(n)}$ satisfies the recursive equation:

$$p_t^{(n)} = \mathrm{E}_t^{\mathrm{P}}\left(s_{t+1}p_{t+1}^{(n-1)}\right),$$

with the terminal condition $p_t^{(0)} = 1$.

The bond price is then an exponential linear function of the state vector:

$$p_t^{(n)} = \exp\left(A_n + \boldsymbol{B}_n^\top \boldsymbol{z}_t + \boldsymbol{C}_n^\top \boldsymbol{z}_{t-1}\right).$$

The absence of arbitrage can be achieved by setting

$$\begin{aligned}
A_{n+1} &= A_n + \boldsymbol{B}_n^\top(\boldsymbol{\alpha} - \Sigma^{1/2}\boldsymbol{\lambda}_0) + \frac{1}{2}\boldsymbol{B}_n^\top \Sigma \boldsymbol{B}_n, \\
\boldsymbol{B}_{n+1} &= -\boldsymbol{e}_1 + (\Phi_1 - \Sigma^{1/2}\lambda_1)^\top \boldsymbol{B}_n + \boldsymbol{C}_n, \\
\boldsymbol{C}_{n+1} &= \Phi_2^\top \boldsymbol{B}_n.
\end{aligned}$$

To ensure the consistency between this equation and the observables $r^{(1)}, r^{(40)}$ in the VAR model [3], we require initial values to be $A_1 = 0, \boldsymbol{B}_1 = -\boldsymbol{e}_1, \boldsymbol{C}_1 = \boldsymbol{0}$, and terminal values to be $A_{40} = 0, \boldsymbol{B}_{40} = -40\boldsymbol{e}_1 - 40\boldsymbol{e}_2, \boldsymbol{C}_{40} = \boldsymbol{0}$.

The continuously compounding yield $r_t^{(n)}$ on an n-quarter zero-coupon bond is an affine function of the state vector:

$$r_t^{(n)} = -\frac{\log p_t^{(n)}}{n} = -\frac{A_n}{n} - \frac{\boldsymbol{B}_n^\top}{n}\boldsymbol{z}_t - \frac{\boldsymbol{C}_n^\top}{n}\boldsymbol{z}_{t-1}.$$

2.2 Estimation

Given the VAR dynamics, we can estimate the risk parameters $\boldsymbol{\lambda}_0, \lambda_1$ by minimising the sum squared error:

$$\min_{\boldsymbol{\lambda}_0, \lambda_1} \sum_{t=1}^{T}\sum_{n=1}^{N}\left(\hat{r}_t^{(n)} - r_t^{(n)}\right)^2.$$

Different to [1], we calibrate the model to zero-coupon bonds which mature every quarter from 3-month to 10-year over the same time window as VAR (112 quarters):

$$\min_{\boldsymbol{\lambda}_0, \lambda_1} \sum_{t=1}^{112}\sum_{n=1}^{40}\left(\hat{r}_t^{(n)} - r_t^{(n)}\right)^2.$$

2.3 Results

The estimated parameters are presented in table 1.

λ_0	λ_1							
-0.03009	-0.00677	0.00190	0.00376	0.00615	0.00116	0.00082	0.00363	0.00511
-0.01263	0.00874	0.00330	0.00514	0.00769	0.00494	0.01166	0.00778	0.01118
-0.01793	-0.00400	0.00189	0.00345	-0.00518	0.00723	-0.00883	0.00291	0.01206
0.02454	0.01599	0.01590	0.01790	0.01134	0.00546	0.01049	0.00909	0.01011
-0.01413	-0.00320	0.00489	0.00568	-0.00123	0.00773	0.00787	0.00216	-0.01536
-0.01316	-0.00164	0.01035	0.00173	0.00064	-0.00192	0.00954	0.00140	0.01201
-0.03393	-0.02448	-0.01474	-0.00440	0.04978	-0.00573	-0.00387	-0.00522	0.01668
-0.02980	-0.03279	-0.00689	0.00840	0.00221	-0.00160	-0.00581	0.00300	0.01806

Table 1: Estimated parameters in the market price of risk.

We simulated 10,000 trajectories of the pricing kernels over the next 100 years. Figure 1 shows the fitted historical and simulated stochastic discount factors. Observe that the range for both the historical and simulated SDFs are larger than the range in [1], probably because a larger time window was considered. The SDF around 2020 was greater than 1, which is consistent with the interest rate being negative. Similar to [1], the range of simulated trajectories almost doubled that of historical SDFs.

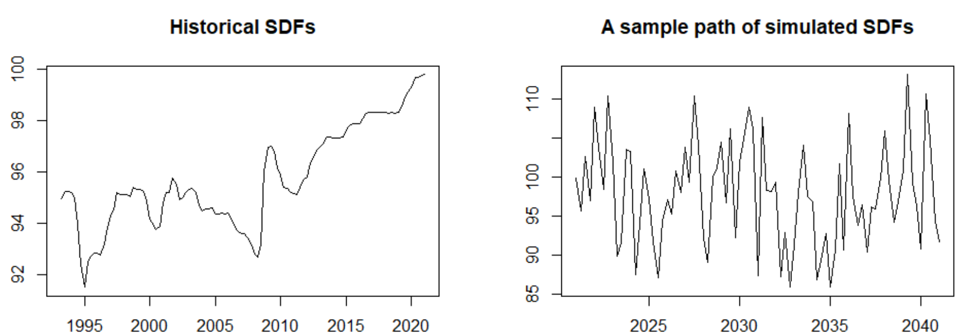

Figure 1: Stochastic Discount Factors.

Table 2 reports the correlations between the historical stochastic discount factors and the state variables.

Correlation	$r^{(1)}$	$r^{(40)} - r^{(1)}$	h_t	y_t	g_t	c_t	a_t	d_t
SDF	-0.99993	0.04220	0.20235	0.40931	-0.13179	-0.29297	0.00139	-0.03617

Table 2: Correlations between the stochastic discount factors and state variables.

References

[1] Daniel H Alai, Hua Chen, Daniel Cho, Katja Hanewald, and Michael Sherris. Developing equity release markets: Risk analysis for reverse mortgages and home reversions. *North American Actuarial Journal*, 18(1):217–241, 2014.

[2] Andrew Ang and Monika Piazzesi. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics*, 50(4):745–787, 2003.

[3] Andrew Ang, Monika Piazzesi, and Min Wei. What does the yield curve tell us about gdp growth? *Journal of econometrics*, 131(1-2):359–403, 2006.