1 Term Structure

This section follows Christensen et al. (2011).

1.1 Dynamics

The instantaneous risk-free rate is $r_t = X_t^1 + X_t^2$. The state variables are level X_t^1, slope X_t^2, and curvature X_t^3, which follow a system of SDEs ($d\boldsymbol{X}_t = K^{\mathrm{Q}}(-\boldsymbol{X}_t)dt + \Sigma d\boldsymbol{W}_t^{\mathrm{Q}}$):

- Independent factor AFNS:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

- Correlated factor AFNS:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

The zero-coupon bond yield is:

$$y(t,T) = X_t^1 + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} X_t^2 + \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)} \right) X_t^3 - \frac{A(t,T)}{T-t}, \tag{1}$$

where $-\frac{A(t,T)}{T-t}$ is a yield-adjustment term:

$$A(t,T) = \frac{1}{2} \sum_{i=1}^{3} \int_t^T \left(\Sigma^\top \boldsymbol{B}(s,T) \boldsymbol{B}(s,T)^\top \Sigma \right)_{i,i} ds.$$

Assume an affine risk premium, the P-dynamics $d\boldsymbol{X}_t = K^{\mathrm{P}}(\boldsymbol{\theta}^{\mathrm{P}} - \boldsymbol{X}_t)dt + \Sigma d\boldsymbol{W}_t^{\mathrm{P}}$ for

- An independent factor AFNS model are:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & 0 & 0 \\ 0 & \kappa_{22}^{\mathrm{P}} & 0 \\ 0 & 0 & \kappa_{33}^{\mathrm{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right) dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \end{pmatrix}.$$

- A correlated-factor AFNS model are:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & \kappa_{12}^{\mathrm{P}} & \kappa_{13}^{\mathrm{P}} \\ \kappa_{21}^{\mathrm{P}} & \kappa_{22}^{\mathrm{P}} & \kappa_{23}^{\mathrm{P}} \\ \kappa_{31}^{\mathrm{P}} & \kappa_{32}^{\mathrm{P}} & \kappa_{33}^{\mathrm{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right) dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \end{pmatrix}.$$

The vector $\boldsymbol{\theta}^{\mathbb{P}}$ is interpreted as the mean vector, while the matrix K is the mean-reversion matrix. The measurement equation is:

$$\boldsymbol{y}_t = B\boldsymbol{X}_t + \boldsymbol{A} + \boldsymbol{\epsilon}_t \iff$$
$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} - \begin{pmatrix} \frac{A(\tau_1)}{\tau_1} \\ \frac{A(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A(\tau_N)}{\tau_N} \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_N) \end{pmatrix},$$

where the measurement errors $\epsilon_t(\tau_i)$ are iid noises.

1.2 Estimation

The state transition equation is:

$$\boldsymbol{X}_t = \left[I - \exp(-K^{\mathrm{P}}\Delta t)\right]\boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}}\Delta t)\boldsymbol{X}_{t-1} + \boldsymbol{\eta}_t,$$

with the error structure

$$\begin{pmatrix} \boldsymbol{\eta}_t \\ \boldsymbol{\epsilon}_t \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

where Δt is the time between observations, $H = \operatorname{diag}(\sigma_\epsilon^2(\tau_1), \sigma_\epsilon^2(\tau_2), \ldots, \sigma_\epsilon^2(\tau_N))$, and

$$Q = \int_0^{\Delta t} e^{-K^{\mathrm{P}}s}\Sigma\Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

We apply the Kalman filtering to estimate the parameters:

1. Initialise the filter at the unconditional mean and variance of the state variables under the P$-$measure:

$$\boldsymbol{X}_0 = \boldsymbol{\theta}^{\mathrm{P}}, \Sigma_0 = \int_0^{\infty} e^{-K^{\mathrm{P}}s}\Sigma\Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

2. Let the information available at time t be $\boldsymbol{Y}_t = (y_1, \ldots, y_t)$. The prediction step is:

$$\begin{aligned} \boldsymbol{X}_{t|t-1} &= [I - \exp(-K^{\mathrm{P}}\Delta t)]\boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}}\Delta t)\boldsymbol{X}_{t-1}, \\ \Sigma_{t|t-1} &= \exp(-K^{\mathrm{P}}\Delta t)\Sigma_{t-1}\exp(-K^{\mathrm{P}}\Delta t)^\top + Q. \end{aligned}$$

   We utilise the analytical formulas in Christensen et al. (2015).

3. Calculate the measurement residuals:

$$\begin{aligned} \boldsymbol{v}_t &= \boldsymbol{y}_t - \boldsymbol{A} - B\boldsymbol{X}_{t|t-1}, \\ \operatorname{Cov}(\boldsymbol{v}_t) &= B\Sigma_{t|t-1}B^\top + H. \end{aligned}$$

4. The optimal Kalman gain is:

$$K_t = \Sigma_{t|t-1} B^\top \mathrm{Cov}(\boldsymbol{v}_t)^{-1}.$$

   Update the estimate:

$$\begin{aligned}\boldsymbol{X}_t &= \boldsymbol{X}_{t|t-1} + K_t \boldsymbol{v}_t,\\ \Sigma_t &= \Sigma_{t|t-1} - K_t B \Sigma_{t|t-1}.\end{aligned}$$

5. The log-likelihood is:

$$\log \ell(y_1, \ldots, y_T) = \sum_{t=1}^{T} \left(-\frac{N}{2} \log(2\pi) - \frac{1}{2} \log\det(\mathrm{Cov}(\boldsymbol{v}_t)) - \frac{1}{2} \boldsymbol{v}_t^\top \mathrm{Cov}(\boldsymbol{v}_t)^{-1} \boldsymbol{v}_t \right),$$

   where N is the number of observed yields.

6. Compute MLE by the Nelder-Mead approach.

We use the monthly RBA data from July 1992 to May 2021, maturing every quarter until 10 years. We adapt the R codes from *Arbitrage-free Nelson Siegel model with R code* (n.d.).

1.3 Results

1.3.1 Independent-factor AFNS

Assuming K^{P} is diagonalisable, the estimated parameters for the independent-factor model are:

$$\hat{K}^{\mathrm{P}} = \begin{pmatrix} 0.222361 & 0 & 0 \\ 0 & 0.840951 & 0 \\ 0 & 0 & 0.6043 \end{pmatrix}, \hat{\Sigma} = \begin{pmatrix} 0.00789569 & 0 & 0 \\ 0 & 0.0112751 & 0 \\ 0 & 0 & 0.0137063 \end{pmatrix},$$

$$\hat{\boldsymbol{\theta}}^{\mathrm{P}} = \begin{pmatrix} 0.040404788 \\ -0.024757856 \\ 0.000506779 \end{pmatrix}, \hat{\lambda} = 0.323143.$$

The estimated parameters for the correlated-factor model are:

$$\hat{K}^{\mathrm{P}} = \begin{pmatrix} 0.503160 & 0.114419 & 0.1756810 \\ 0.238044 & 0.283698 & 0.0585722 \\ 0.199730 & 0.094278 & 0.2006165 \end{pmatrix}, \hat{\Sigma} = \begin{pmatrix} 0.0268977 & 0 & 0 \\ -0.0281544 & 0.0014550 & 0 \\ -0.0639422 & 0.0622431 & 0.0013877 \end{pmatrix},$$

$$\hat{\boldsymbol{\theta}}^{\mathrm{P}} = \begin{pmatrix} 0.1357986 \\ -0.0568641 \\ -0.1156273 \end{pmatrix}, \hat{\lambda} = 0.0934927$$

We provide the following justifications to the fitted factor estimates:

- Independent-factor model (figure 1 left):

- There's a downward trend in level X^1, though very negligible. Slope X^2 is consistently below 0. Together they lead to the decrease in interest rate.
- Curvature X^3 gradually becomes negative, implying an inverse U shape followed by an U shape in the yield curve. We have seen interest rate decline after the 2000s, but it will likely bounce back after it hits the bottom line (0).

• Correlated-factor model (figure 1 right):
  - The level X^1 and curvature X^3 symmetrise around the 0-line. At times of interest rate increase, the curvature presumes that it is going to decrease (inverse U shape when $X^3 < 0$).
  - The slope X^2 symmetrises also the level X^1 but with smaller magnitudes, implying that the risk-free rate $X^1 + X^2$ is approximately constant throughout.

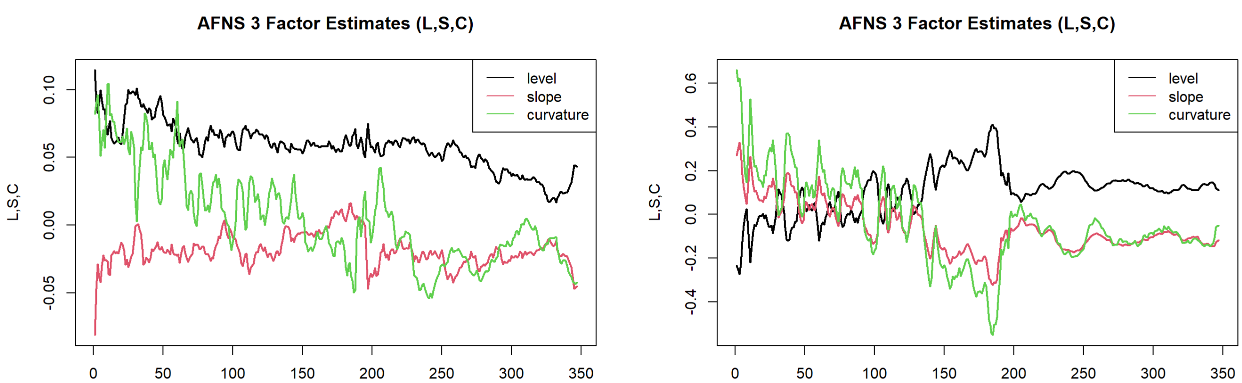

Figure 1: Fitted state variables: independent-factor (left) vs correlated-factor (right).

Fitted long-term rates are more accurate than the short-term rates, due to lower variability (figure 2). Discrepancy appears when the historic yields flatten after sudden rise/fall. The correlated-factor model is more variable, thus it amplifies the fluctuations in the historic yields.

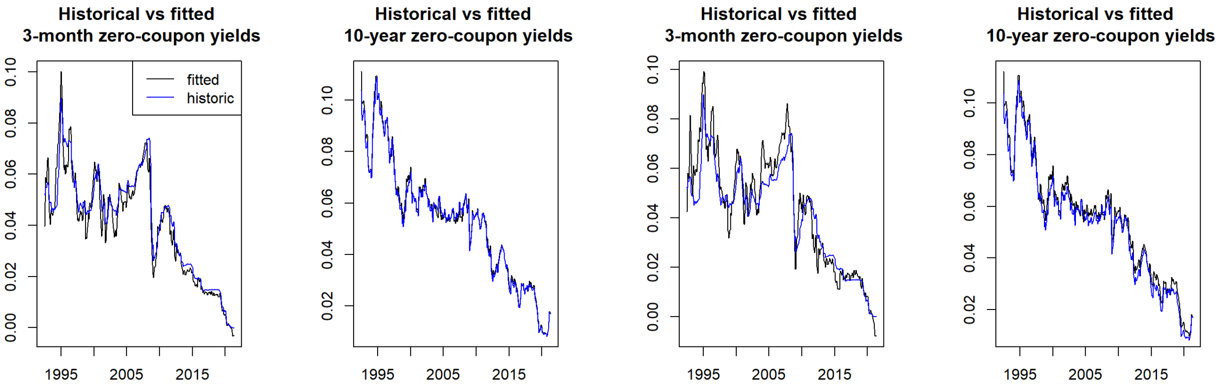

Figure 2: Fitted zero-coupon rates: independent-factor (left) vs correlated-factor (right).

1.4 Simulation

Fix the time step $\Delta t > 0$. Our simulations follow the state transition equation:

$$\boldsymbol{X}_{t+\Delta t} = [I - \exp(-\hat{K}^{\mathrm{P}}\Delta t)]\boldsymbol{\theta}^{\mathrm{P}} + \exp(-\hat{K}^{\mathrm{P}}\Delta t)\boldsymbol{X}_t + \boldsymbol{\eta}_t, \boldsymbol{\eta}_t \sim \mathcal{N}(\boldsymbol{0}, \widehat{Q}).$$

We set $\Delta t = \frac{1}{12}$ because the minimum data frequency is monthly. This gives a trajectory of the state variables: $\hat{\boldsymbol{X}}_0, \hat{\boldsymbol{X}}_{\Delta t}, \hat{\boldsymbol{X}}_{2\Delta t}, \ldots$. We use it with formula (1) to simulate the yields of a maturity-τ month zero-coupon bond.

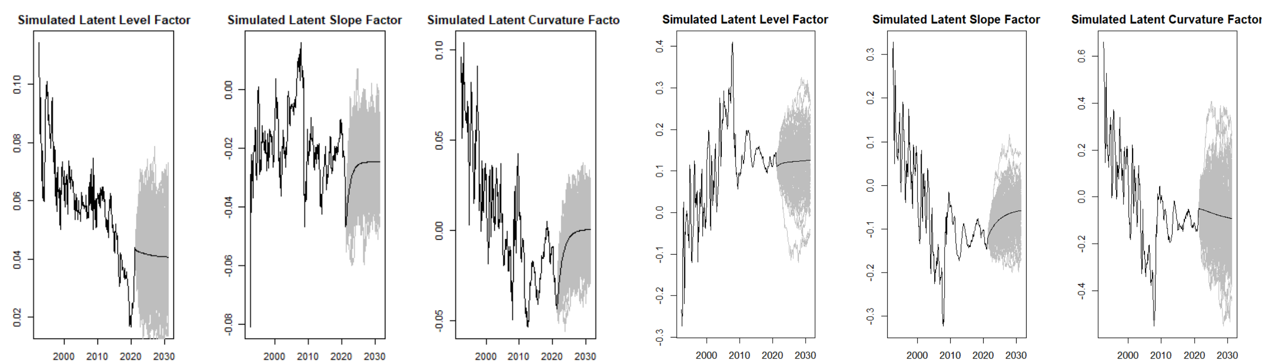

Figure 3: Simulated state variables: independent-factor (left) vs correlated-factor (right).

Simulated trajectories for the independent-factor model gradually level off, covering the range of the latest values. The correlated-factor is overly optimistic, but it exhibits some cyclicity and avoids the negative yields that the independent-factor model doesn't (figures 3, 4). I suppose this is highly influenced by the recent COVID trends: the correlated-factor model reflects the recovery in interest rate, and it assumes this will continue into the future (which is unlikely).

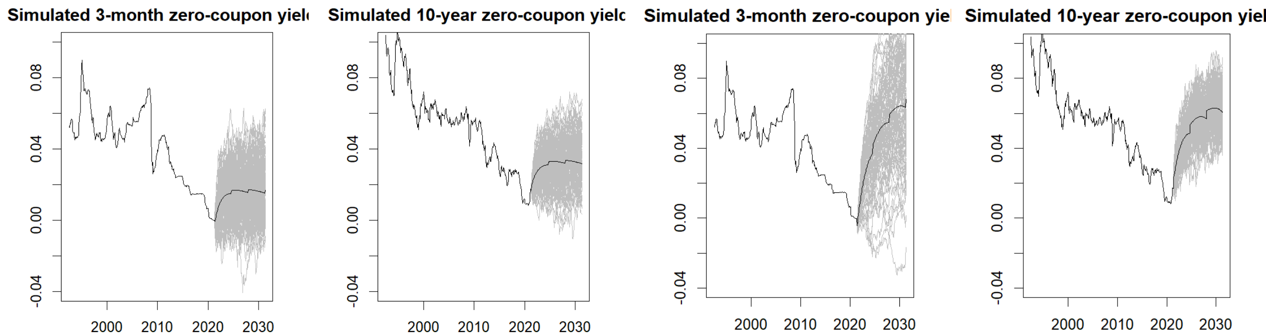

Figure 4: Simulated zero-coupon bond rates: independent-factor (left) vs correlated-factor (right).

2 Term Structure, Home Value Indexes, Stock Prices

This section is based on *Reverse Mortgage : Long Term Care Project* (n.d.). [1] Granger causality between these variables are proven in "VAR_Framework.pdf".

Assume that the interest rate follows AFNS as above, both the home value indexes and stock prices follow the GBM. Denote the additional state variables as: logged home value index NSW $X_t^4 = \log(H_t)$ and logged stock price $X_t^5 = \log(S_t)$. The new state variables are correlated with X_t^1, X_t^2, X_t^3.

2.1 Dynamics

The Q-dynamics are:

$$dX_t^4 = \left(X_t^1 + X_t^2 - \frac{\sigma_4^2}{2} \right) dt + \sigma_4 \sum_{i=1}^{3} \rho_{4i} dW_{it}^{\mathrm{Q}} + \sigma_4 \sqrt{1-\rho_4^2} dW_{4t}^{\mathrm{Q}},$$
$$dX_t^5 = \left(X_t^1 + X_t^2 - \frac{\sigma_5^2}{2} \right) dt + \sigma_5 \sum_{i=1}^{4} \rho_{5i} dW_{it}^{\mathrm{Q}} + \sigma_5 \sqrt{1-\rho_5^2} dW_{5t}^{\mathrm{Q}}.$$

The standard Brownian motions W_{4t}^{Q} and W_{5t}^{Q} are correlated with other BMs by:

$$\mathrm{Cov}\left(dW_{4t}^{\mathrm{Q}}, dW_{it}^{\mathrm{Q}} \right) = \rho_{4i} dt, i = 1, 2, 3,$$
$$\mathrm{Cov}\left(dW_{5t}^{\mathrm{Q}}, dW_{it}^{\mathrm{Q}} \right) = \rho_{5i} dt, i = 1, 2, 3, 4.$$

The ρ's are correlation coefficients which take value from -1 to 1. Let $\rho_4^2 = \sum_{i=1}^{3} \rho_{4i}^2 \leq 1$, and $\rho_5^2 = \sum_{i=1}^{4} \rho_{5i}^2 \leq 1$. The state variables $\boldsymbol{X}_t = (X_t^1, X_t^2, X_t^3, X_t^4, X_t^5)$ follow the system of SDEs $d\boldsymbol{X}_t = (\boldsymbol{\theta}^{\mathrm{Q}} - K^{\mathrm{Q}} \boldsymbol{X}_t)dt + \Sigma d\boldsymbol{W}_t^{\mathrm{Q}}$:

[1] *Reverse Mortgage : Long Term Care Project* (n.d.) assumes the same variance for ALL zero-coupon bond maturities, but here assumes different variances.

- Independent factor:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\sigma_4^2}{2} \\ -\frac{\sigma_5^2}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +$$

$$\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ \sigma_4\rho_{41} & \sigma\rho_{42} & \sigma_4\rho_{43} & \sigma_4\sqrt{1-\rho_4^2} & 0 \\ \sigma_5\rho_{51} & \sigma_5\rho_{52} & \sigma_5\rho_{53} & \sigma_5\rho_{54} & \sigma_5\sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \\ dW_{4t}^{\mathrm{Q}} \\ dW_{5t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

- Correlated factor:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\sigma_4^2}{2} \\ -\frac{\sigma_5^2}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +$$

$$\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 \\ \sigma_4\rho_{41} & \sigma\rho_{42} & \sigma_4\rho_{43} & \sigma_4\sqrt{1-\rho_4^2} & 0 \\ \sigma_5\rho_{51} & \sigma_5\rho_{52} & \sigma_5\rho_{53} & \sigma_5\rho_{54} & \sigma_5\sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{Q}} \\ dW_{2t}^{\mathrm{Q}} \\ dW_{3t}^{\mathrm{Q}} \\ dW_{4t}^{\mathrm{Q}} \\ dW_{5t}^{\mathrm{Q}} \end{pmatrix}, \lambda > 0.$$

The measure change is facilitated by the Girsanov's Theorem: $d\boldsymbol{W}_t^{\mathrm{Q}} = d\boldsymbol{W}_t^{\mathrm{P}} + \Gamma_t dt$. Assume an essentially affine risk premium:

$$\begin{aligned} \Gamma_t &= \boldsymbol{\gamma}^0 + \gamma^1 \boldsymbol{X}_t \\ &= \begin{pmatrix} \gamma_1^0 \\ \gamma_2^0 \\ \gamma_3^0 \\ \gamma_4^0 \\ \gamma_5^0 \end{pmatrix} + \begin{pmatrix} \gamma_{11}^1 & \gamma_{12}^1 & \gamma_{13}^1 & \gamma_{14}^1 & \gamma_{15}^1 \\ \gamma_{21}^1 & \gamma_{22}^1 & \gamma_{23}^1 & \gamma_{24}^1 & \gamma_{25}^1 \\ \gamma_{31}^1 & \gamma_{32}^1 & \gamma_{33}^1 & \gamma_{34}^1 & \gamma_{35}^1 \\ \gamma_{41}^1 & \gamma_{42}^1 & \gamma_{43}^1 & \gamma_{44}^1 & \gamma_{45}^1 \\ \gamma_{51}^1 & \gamma_{52}^1 & \gamma_{53}^1 & \gamma_{54}^1 & \gamma_{55}^1 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix}. \end{aligned}$$

Substituting it into the Q-dynamics yields:

$$\begin{aligned} d\boldsymbol{X}_t &= (\boldsymbol{\theta}^{\mathrm{Q}} - K^{\mathrm{Q}} \boldsymbol{X}_t)dt + \Sigma \left[d\boldsymbol{W}_t^{\mathrm{P}} + (\boldsymbol{\gamma}^0 + \gamma^1 \boldsymbol{X}_t)dt \right] \\ &= \left[(\boldsymbol{\theta}^{\mathrm{Q}} + \Sigma\boldsymbol{\gamma}^0) - (K^{\mathrm{Q}} - \Sigma\gamma^1)\boldsymbol{X}_t \right] dt + \Sigma d\boldsymbol{W}_t^{\mathrm{P}}. \end{aligned}$$

We can make further assumptions to $\boldsymbol{\gamma}^0, \gamma^1$ such that the P-dynamics preserve the structure of the corresponding Q-dynamics:

$$d\boldsymbol{X}_t = [\boldsymbol{\theta}^{\mathrm{P}} - K^{\mathrm{P}} \boldsymbol{X}_t]dt + \Sigma d\boldsymbol{W}^{\mathrm{P}}.$$

- Independent factor:

$$
\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \\ \theta_4^{\mathrm{P}} \\ \theta_5^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22}^{\mathrm{P}} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{33}^{\mathrm{P}} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +
$$

$$
\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ \sigma_4\rho_{41} & \sigma\rho_{42} & \sigma_4\rho_{43} & \sigma_4\sqrt{1-\rho_4^2} & 0 \\ \sigma_5\rho_{51} & \sigma_5\rho_{52} & \sigma_5\rho_{53} & \sigma_5\rho_{54} & \sigma_5\sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \\ dW_{4t}^{\mathrm{P}} \\ dW_{5t}^{\mathrm{P}} \end{pmatrix}
$$

- Correlated factor:

$$
\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \left[\begin{pmatrix} \theta_1^{\mathrm{P}} \\ \theta_2^{\mathrm{P}} \\ \theta_3^{\mathrm{P}} \\ \theta_4^{\mathrm{P}} \\ \theta_5^{\mathrm{P}} \end{pmatrix} - \begin{pmatrix} \kappa_{11}^{\mathrm{P}} & \kappa_{12}^{\mathrm{P}} & \kappa_{13}^{\mathrm{P}} & 0 & 0 \\ \kappa_{21}^{\mathrm{P}} & \kappa_{22}^{\mathrm{P}} & \kappa_{23}^{\mathrm{P}} & 0 & 0 \\ \kappa_{31}^{\mathrm{P}} & \kappa_{32}^{\mathrm{P}} & \kappa_{33}^{\mathrm{P}} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right] dt +
$$

$$
\begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 \\ \sigma_4\rho_{41} & \sigma\rho_{42} & \sigma_4\rho_{43} & \sigma_4\sqrt{1-\rho_4^2} & 0 \\ \sigma_5\rho_{51} & \sigma_5\rho_{52} & \sigma_5\rho_{53} & \sigma_5\rho_{54} & \sigma_5\sqrt{1-\rho_5^2} \end{pmatrix} \begin{pmatrix} dW_{1t}^{\mathrm{P}} \\ dW_{2t}^{\mathrm{P}} \\ dW_{3t}^{\mathrm{P}} \\ dW_{4t}^{\mathrm{P}} \\ dW_{5t}^{\mathrm{P}} \end{pmatrix}
$$

The measurement equation is:

$$
\boldsymbol{y}_t = B\boldsymbol{X}_t + A + \boldsymbol{\epsilon}_t \iff
$$

$$
\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \\ \log h_t \\ \log s_t \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} & 0 & 0 \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} - \begin{pmatrix} \frac{A(\tau_1)}{\tau_1} \\ \frac{A(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A(\tau_N)}{\tau_N} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_N) \\ \epsilon_H \\ \epsilon_S \end{pmatrix},
$$

where the first N equations follow from the derived bond yield in Christensen et al. (2011), while the last two equations are the observable logged home value index and stock prices (recall $X_t^4 = \log H_t, X_t^5 = \log S_t$).

2.2 Estimation

The state transition equation is:

$$
\boldsymbol{X}_t = \int_0^{\Delta t} \exp(-K^{\mathrm{P}}s)ds\boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}}\Delta t)\boldsymbol{X}_{t-1} + \boldsymbol{\eta}_t,
$$

with the error structure

$$\begin{pmatrix} \boldsymbol{\eta}_t \\ \epsilon_t \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

where $H = \text{diag}(\sigma_\epsilon^2, \sigma_\epsilon^2, \ldots, \sigma_\epsilon^2, \sigma_H^2, \sigma_S^2)$, and

$$Q = \int_0^{\Delta t} e^{-K^{\mathrm{P}} s} \Sigma \Sigma^\top e^{-(K^{\mathrm{P}})^\top s} ds.$$

Apply the Kalman filtering method for estimation.

1. Initialise the filter at:

$$\boldsymbol{X}_t = \begin{pmatrix} (K^{\mathrm{P}}_{1:3,1:3})^{-1} \boldsymbol{\theta}^{\mathrm{P}}_{1:3} \\ \log H_0 \\ \log S_0 \end{pmatrix}, \Sigma_0 = \int_0^\infty \exp(-K^{\mathrm{P}} s) \Sigma \Sigma^\top \exp(-(K^{\mathrm{P}})^\top s) ds.$$

2. Let the information available at time t be $\boldsymbol{Y}_t = (y_1, \ldots, y_t)$. The prediction step is:

$$\boldsymbol{X}_{t|t-1} = \int_0^{\Delta t} \exp(-K^{\mathrm{P}} s) ds \boldsymbol{\theta}^{\mathrm{P}} + \exp(-K^{\mathrm{P}} \Delta t) \boldsymbol{X}_{t-1},$$
$$\Sigma_{t|t-1} = \exp(-K^{\mathrm{P}} \Delta t) \Sigma_{t-1} \exp(-K^{\mathrm{P}} \Delta t)^\top + Q.$$

3. Calculate the measurement residuals:

$$\boldsymbol{v}_t = \boldsymbol{y}_t - \boldsymbol{A} - B \boldsymbol{X}_{t|t-1},$$
$$\text{Cov}(\boldsymbol{v}_t) = B \Sigma_{t|t-1} B^\top + H.$$

4. The optimal Kalman gain is:

$$K_t = \Sigma_{t|t-1} B^\top \text{Cov}(\boldsymbol{v}_t)^{-1}.$$

   Update the estimate:

$$\boldsymbol{X}_t = \boldsymbol{X}_{t|t-1} + K_t \boldsymbol{v}_t,$$
$$\Sigma_t = \Sigma_{t|t-1} - K_t B \Sigma_{t|t-1}.$$

5. The log-likelihood is:

$$\log \ell(y_1, \ldots, y_T) = \sum_{t=1}^T \left(-\frac{N}{2} \log(2\pi) - \frac{1}{2} \log \det(\text{Cov}(\boldsymbol{v}_t)) - \frac{1}{2} \boldsymbol{v}_t^\top \text{Cov}(\boldsymbol{v}_t)^{-1} \boldsymbol{v}_t \right),$$

   where N is the number of observed yields.

6. Compute MLE by the Nelder-Mead approach.

2.3 Results

The estimated parameters for the independent-factor model are:

$$\hat{K}^{\mathrm{P}} = \begin{pmatrix} 0.0168809 & 0 & 0 & 0 & 0 \\ 0 & 0.132465 & 0 & 0 & 0 \\ 0 & 0 & 0.528005 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}, \hat{\boldsymbol{\theta}}^{\mathrm{P}} = \begin{pmatrix} 0.06288723 \\ -0.48560096 \\ 0.20092702 \\ 0.00526758 \\ 0.04260104 \end{pmatrix},$$

$$\hat{\Sigma} = \begin{pmatrix} 0.00893424 & 0 & 0 & 0 & 0 \\ 0 & 0.02365144 & 0 & 0 & 0 \\ 0 & 0 & 1.51340765 & 0 & 0 \\ 0.00705073 & 0.01047179 & 0.00170855 & 0.02602095 & 0 \\ 0.12350992 & 0.00583818 & -0.01134868 & -0.00650384 & 0.0589449 \end{pmatrix}, \quad \hat{\lambda} = 0.00120714$$

The estimated parameters for the correlated-factor model are:

$$\hat{K}^{\mathrm{P}} = \begin{pmatrix} -0.689053 & 0.520699 & -0.955178 & 0 & 0 \\ 1.401675 & 1.153088 & 0.240643 & 0 & 0 \\ -1.057659 & 4.539859 & 2.446048 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}, \hat{\boldsymbol{\theta}}^{\mathrm{P}} = \begin{pmatrix} 0.0476599 \\ 0.4011195 \\ -0.6895536 \\ 0.8074529 \\ 0.2549644 \end{pmatrix},$$

$$\hat{\Sigma} = \begin{pmatrix} 0.0117026 & 0 & 0 & 0 & 0 \\ 0.2144641 & 0.101125 & 0 & 0 & 0 \\ -0.2258828 & -0.154428 & 0.0169528 & 0 & 0 \\ -0.6775545 & -0.108429 & -0.0293253 & 1.187086 & 0 \\ -3.7251824 & -0.883950 & -0.4093372 & 0.531852 & 2.73094 \end{pmatrix}, \quad \hat{\lambda} = 1.20156.$$

Observations from figure 5:

- Independent-factor model:
  - Level X^1 and slope X^2 are constant, with approximately the same magnitudes, giving a constant risk-free rate $r = X^1 + X^2$.
  - Curvature X^3 fluctuates around a mean of approximately 0 (except during GFC, curvature was negative).
- Correlated-factor model:
  - Level X^1 decreases.
  - Both slope X^2 and curvature X^3 were negative.

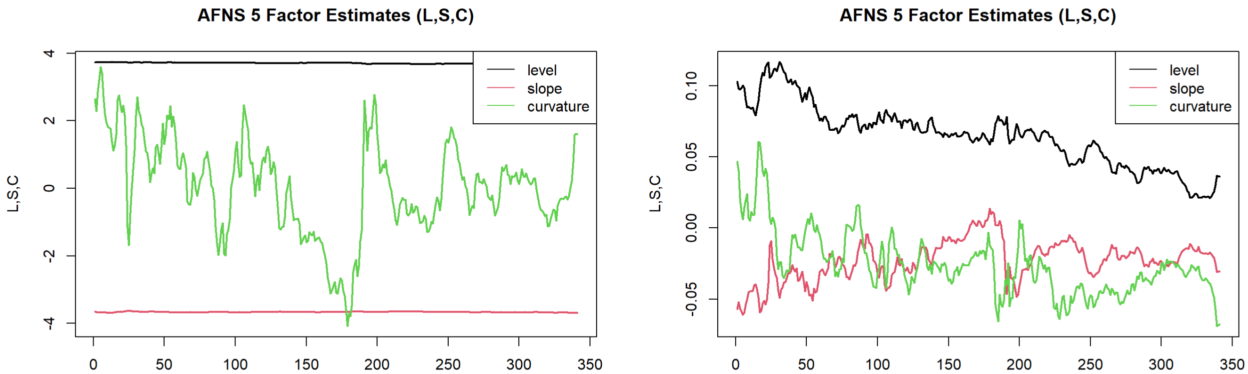

Figure 5: Fitted state variables: independent-factor (left) vs correlated-factor (right).

Long-term bond yields are more accurate (figure 6).

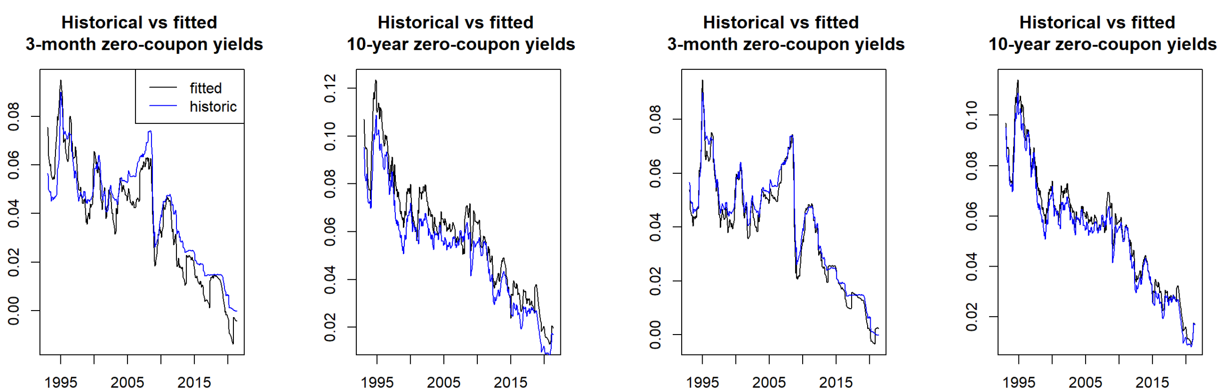

Figure 6: Fitted zero-coupon rates: independent-factor (left) vs correlated-factor (right).

Both models provide exact fits to home indexes X^4 and stock prices X^5 (figure 7).

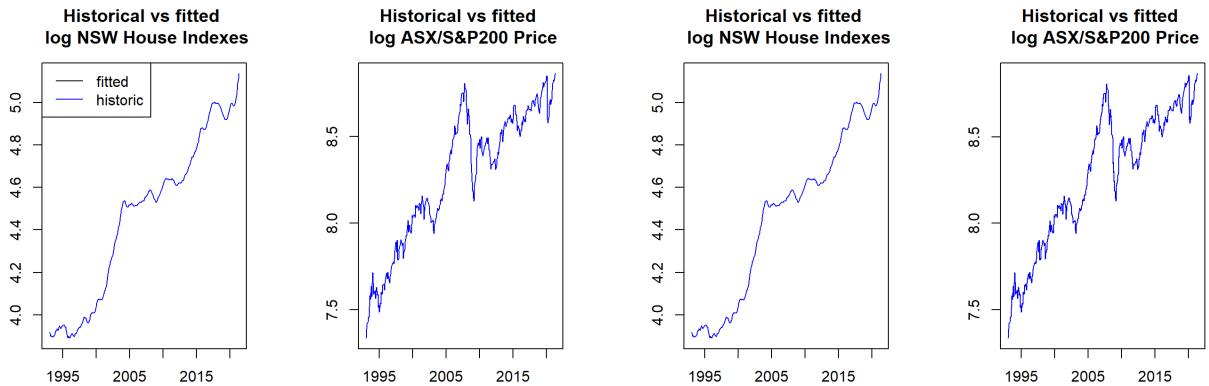

Figure 7: Fitted logged home index and S&P/ASX200 prices: independent-factor (left) vs correlated-factor (right).

2.4 Simulation

Fix the time step $\Delta t > 0$. Our simulation is based on the state transition equation:

$$\boldsymbol{X}_{t+\Delta t} = \int_0^{\Delta t} \exp(-\widehat{K}^{\mathrm{P}} s) ds \widehat{\boldsymbol{\theta}}^{\mathrm{P}} + \exp(-\widehat{K}^{\mathrm{P}} \Delta t) \boldsymbol{X}_t + \boldsymbol{\eta}_t, \ \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \widehat{Q}).$$

This gives a trajectory of the state variables: $\hat{\boldsymbol{X}}_0, \hat{\boldsymbol{X}}_{\Delta t}, \hat{\boldsymbol{X}}_{2\Delta t}, \ldots$. The NSW house value indexes and S&P/ASX200 prices can be obtained from exponentiating the simulated $\boldsymbol{X}_4, \boldsymbol{X}_5$. The maturity-$\tau$ month zero-coupon bond yields are calculated from formula (1).

Observations from the simulated paths (figures 8, 9, 10):

- Bond yields bounce back after COVID.
- The correlated-factor model is more optimistic in future stock/house prices (figure 10), while more pessimistic in future coupon bond yields (figure 9).
- Confidence bounds for simulation are wider than the 3-factor models.

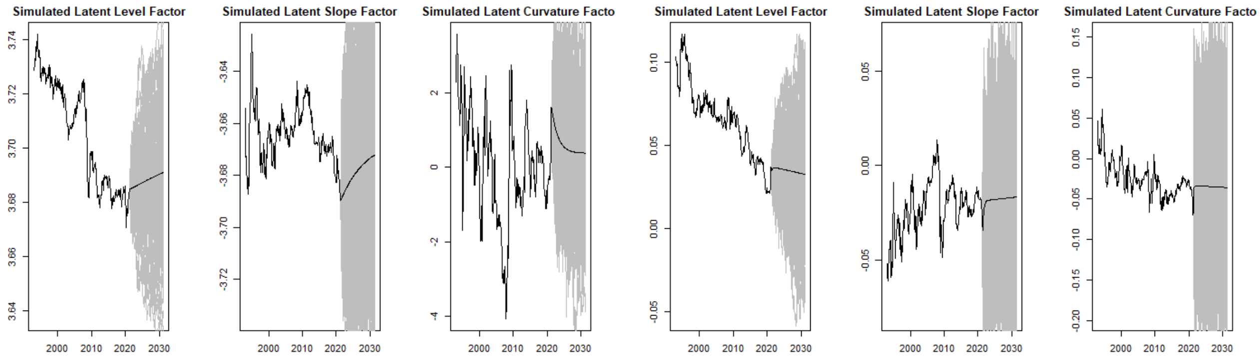

Figure 8: Simulated state variables: independent-factor (left) vs correlated-factor (right).

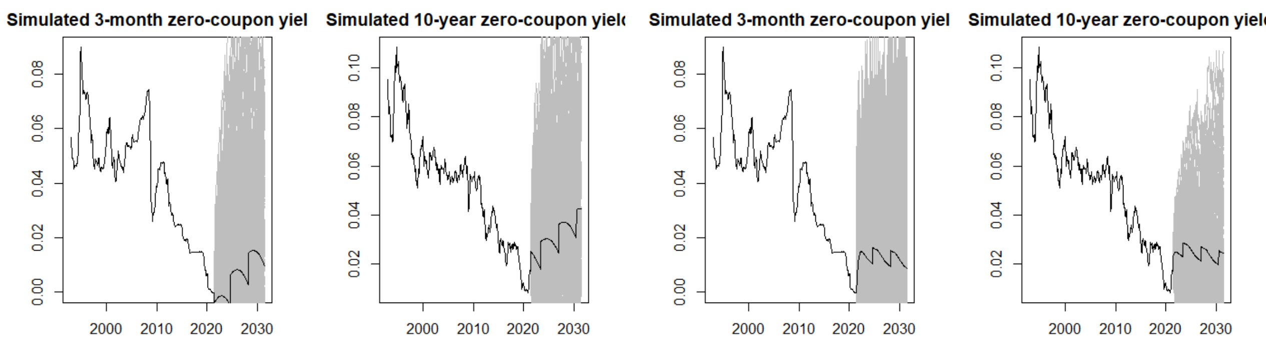

Figure 9: Simulated zero-coupon bond rates: independent-factor (left) vs correlated-factor (right).

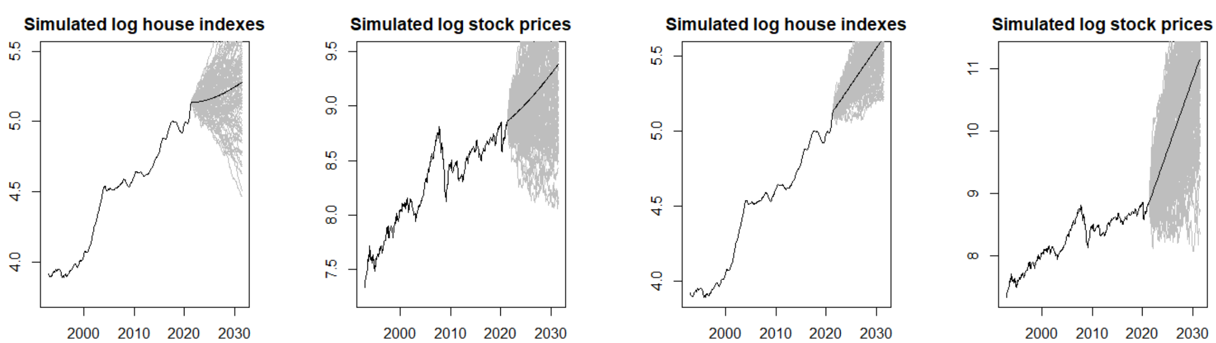

Figure 10: Simulated logged home index and S&P/ASX200 prices: independent-factor (left) vs correlated-factor (right).

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