筑波大学
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Numerical Simulation

Hometask 3

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1 Prove Instability using Von Neumann Stability Analysis

The task is to analyze the stability of the forward-time central-space (FTCS) scheme for the hyperbolic equation given by:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)$$

using Von Neumann Stability Analysis. We are to demonstrate that this scheme is unstable.

1.1 Analysis

Using the Von Neumann stability analysis, we introduce the ansatz:

$$u_j^n = \xi^n e^{ikj\Delta x}$$

Substituting into the discretized equation and simplifying, we get:

$$\xi^{n+1} = \xi^n \left[1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right]$$

This yields the amplification factor ξ as:

$$\xi(k) = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

To assess stability, we need $|\xi| \leq 1$. Calculating $|\xi|$, we find:

$$|\xi| = \sqrt{1 + \left(\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right)^2}$$

Since $\sin^2(k\Delta x) \leq 1$, it follows that $|\xi| \geq 1$ for all k, except possibly at $k = 0$. This proves that the solution is unstable as $|\xi| > 1$ for all $-\pi/\Delta x < k < \pi/\Delta x$ except for $k = 0$.

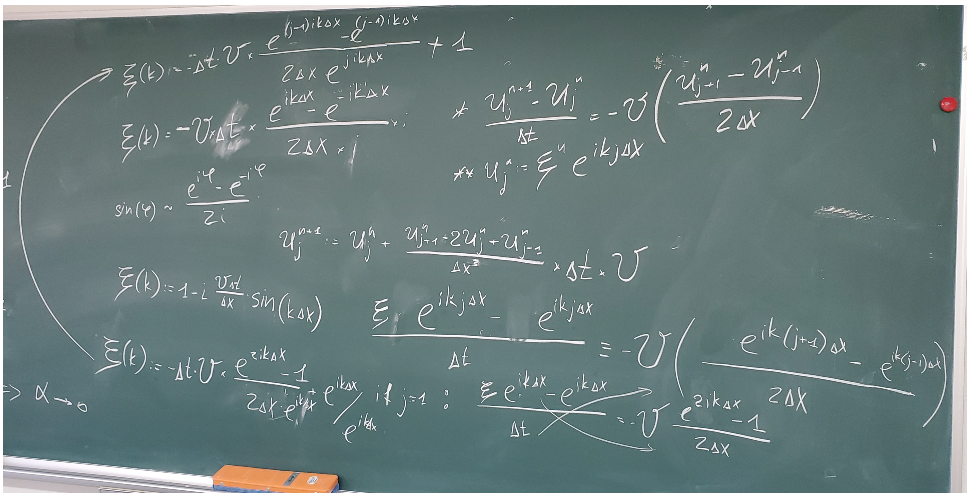

Figure 1: Initial part of the solution derivation.

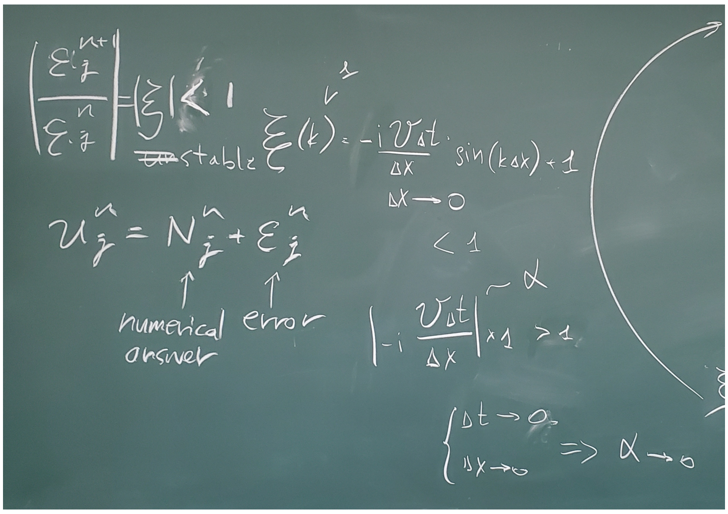

Figure 2: Completion of the solution derivation showing the amplification factor calculation.

1.2 Solution Figures

During the class I have solved this task as it is shown on the figures 1 and 2.

1.3 Detailed Solution

We consider the forward-time central-space (FTCS) approximation for the hyperbolic equation:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right)$$

This equation can be rearranged to express the future time step u_j^{n+1} as:

$$u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x} \left(u_{j+1}^n - u_{j-1}^n \right)$$

To apply the Von Neumann stability analysis, we introduce the ansatz:

$$u_j^n = \xi^n e^{ikj\Delta x}$$

Substituting this ansatz into the rearranged equation gives:

$$\xi^{n+1} e^{ikj\Delta x} = \xi^n e^{ikj\Delta x} - \frac{v\Delta t}{2\Delta x} \left(\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x} \right)$$

This can be simplified by dividing through by $\xi^n e^{ikj\Delta x}$, resulting in:

$$\xi = 1 - \frac{v\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

Using the identity for the difference of complex exponentials, $e^{ix} - e^{-ix} = 2i\sin(x)$, we rewrite:

$$\xi = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

The amplification factor ξ is thus given by:

$$\xi = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

To check for stability, we calculate $|\xi|$, the magnitude of the amplification factor:

$$|\xi| = \sqrt{1 + \left(\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right)^2}$$

Since $\sin^2(k\Delta x) \leq 1$, it follows that:

$$|\xi| \geq 1$$

This result implies $|\xi| > 1$ for any non-zero $k\Delta x$, indicating that the scheme is unstable as the error grows exponentially with each time step. The only exception occurs at $k = 0$, where $\xi = 1$ and the scheme does not amplify errors, but this is a trivial case that does not affect the overall instability conclusion.

2 Stability Analysis of the Diffusion Equation

This section provides a comprehensive analysis of the stability of the numerical scheme used to approximate the diffusion equation. We employ the forward-time centered-space (FTCS) discretization and the Von Neumann stability criterion to evaluate the stability of the scheme.

2.1 Theoretical Background

The diffusion equation, when discretized using the FTCS approach, is represented as:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right)$$

This formulation leads to the update equation:

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

2.2 Von Neumann Stability Analysis

Introducing the Fourier series ansatz for u_j^n:

$$u_j^n = \xi^n e^{ikj\Delta x}$$

where ξ is the growth factor per time step, and substituting it into the update equation, we derive:

$$\xi^{n+1} e^{ikj\Delta x} = \xi^n e^{ikj\Delta x} + \frac{D\Delta t}{\Delta x^2}\left(\xi^n e^{ik(j+1)\Delta x} - 2\xi^n e^{ikj\Delta x} + \xi^n e^{ik(j-1)\Delta x}\right)$$

which simplifies to:

$$\xi = 1 + \frac{D\Delta t}{\Delta x^2}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

Using the identity for the sum of exponentials, $e^{ix} + e^{-ix} = 2\cos(x)$:

$$\xi = 1 + \frac{2D\Delta t}{\Delta x^2}(\cos(k\Delta x) - 1)$$

and by substituting $\cos(k\Delta x) = 1 - 2\sin^2\left(\frac{k\Delta x}{2}\right)$:

$$\xi = 1 - \frac{4D\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

2.3 Stability Condition

For the scheme to be stable, the magnitude of ξ, $|\xi|$, must be less than or equal to 1:

$$|\xi| = \left|1 - \frac{4D\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right| \leq 1$$

The critical condition for stability occurs at the maximum of $\sin^2\left(\frac{k\Delta x}{2}\right) = 1$:

$$\frac{4D\Delta t}{\Delta x^2} \leq 2$$

$$\Delta t \leq \frac{\Delta x^2}{2D}$$

2.4 Conclusion

The maximum permissible time step for the stability of the FTCS scheme in simulating the diffusion equation is therefore:

$$\Delta t \leq \frac{\Delta x^2}{2D}$$

This condition ensures the numerical scheme remains stable without amplifying errors exponentially over time.

References

1. **Mamanchuk N., University of Tsukuba**, Github, June 14, 2024. Available online: `https://github.com/RIFLE`