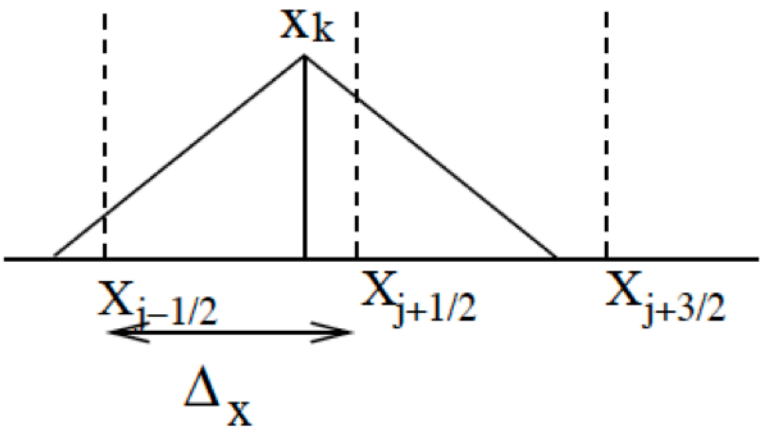
Project 3 Field Solver 2 (1D FDTD) (to be modified)

The bi-linear particle-grid interpolation can be described as follows:
Particle k is centered at x_k. Its width is $2\Delta_x$. It is closest to the center of the cell with index j. Its charge is mainly assigned to the cell $X_{j+\frac{1}{2}}$. A smaller fraction is here assigned to the cell $X_{j-\frac{1}{2}}$



The charge density of particle at x_k is interpolated to the nodes $X_{j-\frac{1}{2}}$ and $X_{j+\frac{1}{2}}$ as

$$\rho_{j-1/2} + \rho_{j+1/2} = \rho_k \text{ and } \rho_{j+1/2}/\rho_k = (x_k - x_{j-1/2})/\Delta x \text{ and } \rho_{j-1/2}/\rho_k = (x_{j+1/2} - x_k)/\Delta x. \tag{1}$$

This bi-linear interpolation is called the cloud-in-cell (CIC) method.

Electromagnetic PIC codes use J rather than ρ to update E and B. The charge density is sometimes used to test if ρ and E computed from Ampere's law fulfills also Gauss' law. We solve the normalized Maxwell's equations in 1D:

$$\frac{\partial}{\partial t}E_y(x,t) = -\frac{\partial}{\partial x}B_z(x,t) - J_y, \qquad \frac{\partial}{\partial t}B_z(x,t) = -\frac{\partial}{\partial x}E_y(x,t).$$
$$\frac{\partial}{\partial t}E_z(x,t) = \frac{\partial}{\partial x}B_y(x,t) - J_z, \qquad \frac{\partial}{\partial t}B_y(x,t) = +\frac{\partial}{\partial x}E_z(x,t).$$
$$\frac{\partial}{\partial t}E_x(x,t) = -J_x. \tag{2}$$

In 1D along x:

$$\nabla \cdot \mathbf{B} = \frac{d}{dx}B_x = 0 \text{ and } \frac{d}{dt}B_x = \frac{d}{dz}E_y - \frac{d}{dy}E_z = 0. \tag{3}$$

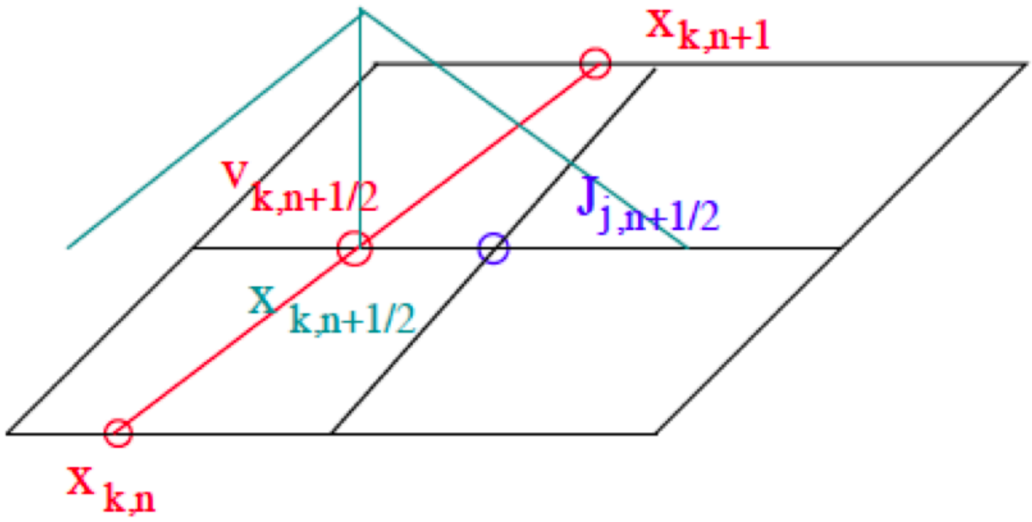
Thus, the Bx component remains constant in space and time. The three equations that use the current are the electromagnetic eq.:

$$\frac{\partial}{\partial t}E_y(x,t) = -\frac{\partial}{\partial x}B_z(x,t) - J_y, \quad \frac{\partial}{\partial t}E_z(x,t) = \frac{\partial}{\partial x}B_y(x,t) - J_z, \tag{4}$$

and the electrostatic eq. is

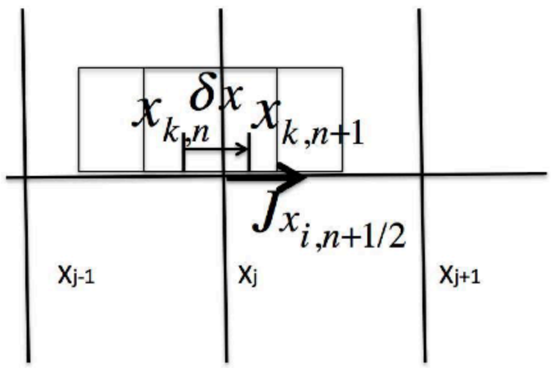
$$\frac{\partial}{\partial t}E_x(x,t) = -J_x. \tag{5}$$

The current has to be defined on the grid. The current is related to the time-derivative of the electric field $\Rightarrow$ J is defined at the same time as B. It is thus defined at the same time as the particle velocities. The current in the electromagnetic equations is added to the spatial derivative of the magnetic field $\Rightarrow$ it is defined at the same position as E.



We obtain from $x_k(n\Delta t)$ and $x_k((n+1)\Delta t)$ the position $x_k(\left(n+\frac{1}{2}\right)\Delta t)$. We use this position to interpolate the micro-current of the k'th CP to $J_{J,n+\frac{1}{2}}$. A loop sums the micro-current over all CP's and we have the macroscopic current $\Rightarrow$ update the system with Ampere's law.

The micro-current Jx,y,z of x_k from t=$n\Delta t$ to t=$(n+1)\Delta t$ can be calculated as show in the following figure.

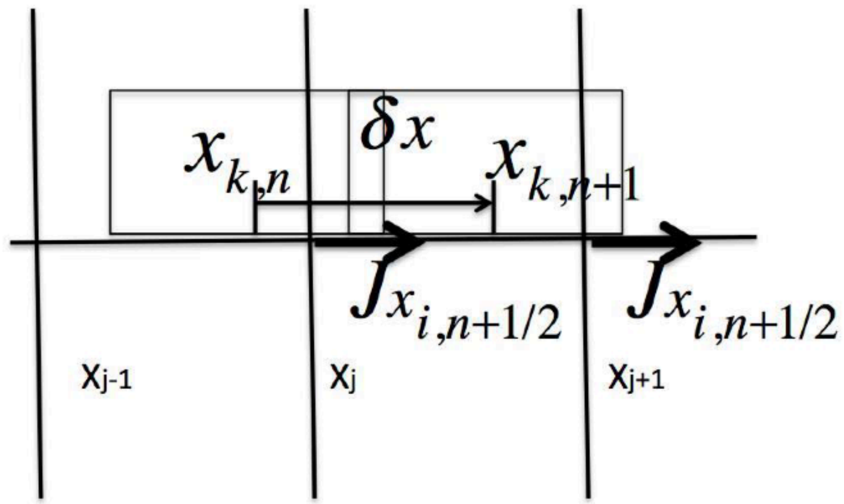


The computational particle x_k is a finite sized particle and we assume the total amount of particle charge is uniformly distributed from $x_k - \frac{1}{2}$ to

$x_k + \frac{1}{2}$. The particle has the width of Δx. The micro-current Jx,y,z by the CP x_k only crossing Xj can be calculated as follows:

$$J_{Xi,x,y,z} = \rho v_{x,y,z} \Delta_t = \rho \delta x, y, z \quad (6)$$

If the particle x_k with the width Δx cross the other boundary, for example, the particle crosses from the grid Xj to Xj+1. In this case the micro-current have to be assigned to both Xj and Xj+1 proportional to the length the CP cross the grids.



Proportional to the length the CP cross the grid Xj, Xj+1, the micro-current can be calculated as follows:

$$J_{Xj,x} = \rho \left(\frac{\Delta x}{2} - \left(x_{k,n} - X_j \right) \right) (7)$$

$$J_{Xj+1,x} = \rho \left(x_{k,n+1} - \frac{\Delta x}{2} - X_j \right) \quad (8)$$

The same proportion of micro-current are also assigned to $J_{Xi,y,z}$ and $J_{Xi+1,y,z}$, respectively.

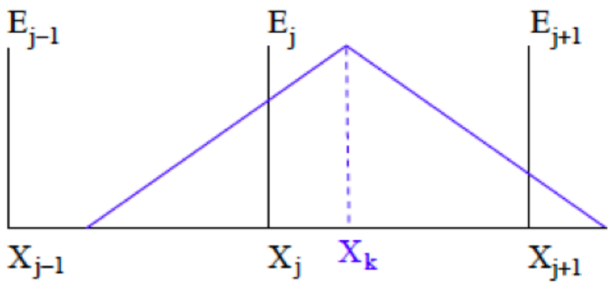
$$J_{Xj,y,z} = \frac{\rho \delta_{y,z} \left(\frac{\Delta x}{2} - \left(x_{k,n} - X_j \right) \right)}{\delta_x} \quad (9)$$

$$J_{Xj+1,y.z} = \frac{\rho \delta_{y,z} \left(x_{k,n+1} - \frac{\Delta x}{2} - X_j \right)}{\delta_x} \quad (10)$$

The CP can move also in negative direction. After we collect all micro-current of all CPs to obtain all currents at all grids, we can now solve Eq. (4,5) to obtain new B and E.

The shape function $S(x_k - x)$ of a CP, which is centered at x_k, is used to interpolate the electric field to the particle position.

The electric field $E_{j,n}$ on the grid nodes j is defined at times nt
We can use the position x_k, which is defined at nt, to interpolate $E_{j,n}$ on the grid to $E(x_k [nt])$.

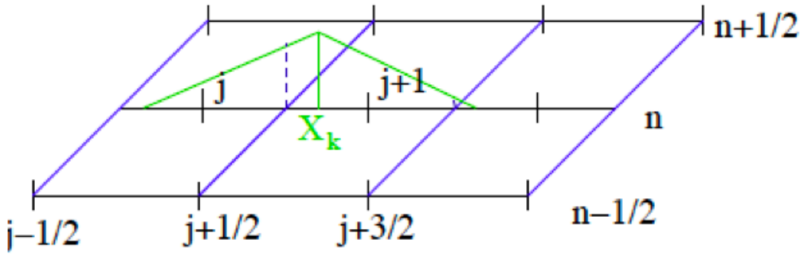


The interpolated electric field (cloud-in-cell method) is

$$\mathbf{E}(x_k) = \mathbf{E}_j \cdot \frac{(x_{j+1} - x_k)}{\Delta_x} + \mathbf{E}_{j+1} \cdot \frac{(x_k - x_j)}{\Delta_x}$$

The electric field at the time nt and at the position x_k is the one, which we use to advance the particle velocity (Boris scheme).

We use the same CIC shape function $S(xk - x)$ for the k′ th CP.
The Boris pusher needs $B(xk [nt])$. The particle position xk is defined at nt, while $B(j + 1/2, n + 1/2)$ is defined at half-integer times and positions.



Faraday's law allows us to compute Bj+1/2,n+1/2 from Bj+1/2,n−1/2 with no knowledge of ρ , j $\Rightarrow$ We have the magnetic field at both times.

Let $W_1 = S(x_k - x_{j+1/2})$ and $W_2 = S(x_k - x_{j+3/2})$ $\quad\Rightarrow\quad$ $\mathbf{B}(x_k[n\Delta_t]) = (W_1/2)\left(\mathbf{B}_{j+1/2,n-1/2} + \mathbf{B}_{j+1/2,n+1/2}\right) + (W_2/2)\left(\mathbf{B}_{j+3/2,n-1/2} + \mathbf{B}_{j+3/2,n+1/2}\right)$

Project: Modify Project 2 to add current J in the program. You can add one ion and electron at the center with velocities +1 and -1 to calculate the current and solve the field equation with current J.