

Faddeev Leverrier Method

- ✓ Recall: eigenvalues are the roots of the *characteristic polynomial* given by, $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$
- ✓ For an $n \times n$ matrix, the polynomial is of the order n
- ✓ This method is an algorithm to obtain the coefficients of the characteristic polynomial:

$$(-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \dots - a_2\lambda^2 - a_1\lambda - a_0) = 0$$

- ✓ Algorithm:
 - ✓ Initialize: $\mathbf{A}_{n-1} = \mathbf{A}$; $a_{n-1} = \text{trace}(\mathbf{A}_{n-1})$
 - ✓ $\mathbf{A}_i = \mathbf{A}(\mathbf{A}_{i+1} - a_{i+1}\mathbf{I})$; $a_i = \frac{\text{trace}(\mathbf{A}_i)}{n-i}$
 - ✓ $i = (n-2), (n-3), \dots, 2, 1, 0$.
- ✓ Compute the roots of the polynomial using *Mueller's* or *Bairstow's* algorithm for the eigenvalues

Faddeev Leverrier Method: Example

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Characteristic polynomial:

$$(-1)^4(\lambda^4 - a_3\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0) = 0$$

Initialize: $A_3 = A$; $a_3 = \text{trace}(A_3) = 12$

$$A_2 = A(A_3 - a_3 I)$$
$$(A_3 - a_3 I) = \begin{bmatrix} -10 & -1 & 0 & 0 \\ -1 & -8 & -1 & 0 \\ 0 & -1 & -8 & -1 \\ 0 & 0 & -1 & -10 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -19 & 6 & 1 & 0 \\ 6 & -30 & 4 & 1 \\ 1 & 4 & -30 & 6 \\ 0 & 1 & 6 & -19 \end{bmatrix}$$
$$a_2 = \frac{\text{trace}(A_2)}{4 - 2} = \frac{-98}{2} = -49$$

Faddeev Leverrier Method: Example

$$A_1 = A(A_2 - a_2 I)$$

$$(A_2 - a_2 I) = \begin{bmatrix} 30 & 6 & 1 & 0 \\ 6 & 19 & 4 & 1 \\ 1 & 4 & 19 & 6 \\ 0 & 1 & 6 & 30 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 54 & -7 & -2 & -1 \\ -7 & 66 & -4 & -2 \\ -2 & -4 & 66 & -7 \\ -1 & -2 & -7 & 54 \end{bmatrix}$$

$$a_1 = \frac{\text{trace}(A_1)}{4 - 1} = \frac{240}{3} = 80$$

$$A_0 = A(A_1 - a_1 I)$$

$$(A_1 - a_1 I) = \begin{bmatrix} -26 & -7 & -2 & -1 \\ -7 & -14 & -4 & -2 \\ -2 & -4 & -14 & -7 \\ -1 & -2 & -7 & -14 \end{bmatrix}; \quad A_0 = \begin{bmatrix} -45 & 0 & 0 & 0 \\ 0 & -45 & 0 & 0 \\ 0 & 0 & -45 & 0 \\ 0 & 0 & 0 & -45 \end{bmatrix}$$

$$a_0 = \frac{\text{trace}(A_0)}{4 - 0} = \frac{-180}{4} = -45$$

$$\lambda^4 - 12\lambda^3 + 49\lambda^2 - 80\lambda + 45 = 0$$

Similarity Transformation

- ✓ Two $n \times n$ matrices A and B are similar if there exists another $n \times n$ invertible matrix S such that $A = SBS^{-1}$ or $B = S^{-1}AS$
- ✓ The process of obtaining the similar matrix B from matrix A using the relation $B = S^{-1}AS$ is called *similarity transformation*!
- ✓ Similar matrices have the same eigenvalues!
- ✓ Some matrices can be diagonalized using similarity transformation
 - ✓ $A = X\Lambda X^{-1}$; where Λ is a diagonal matrix containing eigenvalues and X is a square matrix containing eigenvectors in the columns $AX = X\Lambda$
 $[Ax_1 \quad Ax_2 \quad \dots \quad Ax_n] = [\lambda_1 x_1 \quad \lambda_2 x_2 \quad \dots \quad \lambda_n x_n]$

Computation of Eigenvalues

- ✓ Recall: What is an orthogonal matrix?
 - ✓ Each column vector is *orthonormal* to each other
 - ✓ $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$ or $\mathbf{Q}^T = \mathbf{Q}^{-1}$
- ✓ A $n \times n$ matrix is *non-defective* if it has n independent eigenvectors (rank n , non-zero determinant, inverse exists, etc.)
- ✓ For every *non-defective* real matrix \mathbf{A} with real eigenvalues, there is an orthogonal matrix \mathbf{Q} and an upper-triangular matrix \mathbf{U} such that $\mathbf{A} = \mathbf{Q}\mathbf{U}\mathbf{Q}^T$
- ✓ If \mathbf{A} is complex, \mathbf{Q} is unitary such that $\mathbf{A} = \mathbf{Q}\mathbf{U}\mathbf{Q}^H$
- ✓ Therefore, \mathbf{A} and \mathbf{U} are similar matrices, $\mathbf{A} = \mathbf{Q}\mathbf{U}\mathbf{Q}^T$
- ✓ The upper triangular matrix \mathbf{U} contains the eigenvalues of \mathbf{A} on its diagonal.

Computation of Eigenvalues

- ✓ All *non-defective* matrices are similar to upper triangular matrices
- ✓ $A = QUQ^T$ or $U = Q^T A Q$, U contains the eigenvalues of A on its diagonal.
- ✓ Therefore, a non-defective matrix A can be transformed to an upper-triangular matrix U through *similarity transformation* and the diagonal elements of U will be the eigenvalues.
- ✓ This transformation cannot be achieved in one step. It is achieved through a sequence of *similarity transformation* using *QR-decomposition*!

Computation of Eigenvalues

- ✓ **QR-decomposition:** A *non-defective* matrix A can be decomposed into an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$
- ✓ A sequence of matrix is generated through *similarity transformation* as follows:
 - ✓ Initialize: $A_0 = A$
 - ✓ QR-Decomposition: $A_k = Q_k R_k$
 - ✓ Similarity Transformation:
$$A_{k+1} = Q_k^T A Q_k = Q_k^T Q_k R_k Q_k = R_k Q_k$$
- ✓ Stopping criteria: $\max_j \left| \frac{\lambda_j^{k+1} - \lambda_j^k}{\lambda_j^k} \right| \leq \varepsilon$
- ✓ How to perform QR-decomposition?
 - ✓ Detailed process: proof by induction, inefficient computation
 - ✓ Gram-Schmidt Orthogonalization: efficient computation

QR-Decomposition

✓ We seek a decomposition of the form: $A = QR$

$$\begin{array}{cccccc}
 \mathbf{a}^{(1)} & \mathbf{a}^{(2)} & & \mathbf{a}^{(j)} & & \mathbf{a}^{(n)} \\
 \left[\begin{array}{cccccc}
 a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\
 a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn}
 \end{array} \right]
 \end{array}$$

$$= \begin{array}{cccccc}
 & \mathbf{q}^{(1)} & \mathbf{q}^{(2)} & & \mathbf{q}^{(j)} & & \mathbf{q}^{(n)} \\
 \left[\begin{array}{cccccc}
 q_{11} & q_{12} & \dots & q_{1j} & \dots & q_{1n} \\
 q_{21} & q_{22} & \dots & q_{2j} & \dots & q_{2n} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 q_{i1} & q_{i2} & \dots & q_{ij} & \dots & q_{in} \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 q_{n1} & q_{n2} & \dots & q_{nj} & \dots & q_{nn}
 \end{array} \right]
 \end{array}$$

$$\begin{array}{cccccc}
 & \mathbf{r}^{(1)} & \mathbf{r}^{(2)} & & \mathbf{r}^{(j)} & & \mathbf{r}^{(n)} \\
 \left[\begin{array}{cccccc}
 r_{11} & r_{12} & \dots & r_{1j} & \dots & r_{1n} \\
 0 & r_{22} & \dots & r_{2j} & \dots & r_{2n} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \dots & r_{jj} & \dots & r_{in} \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \dots & 0 & \dots & r_{nn}
 \end{array} \right]
 \end{array}$$

✓ Denote: $\mathbf{a}^{(k)}$, $\mathbf{q}^{(k)}$ and $\mathbf{r}^{(k)}$ as the k^{th} column vectors of matrices A , Q and R

QR-Decomposition: proof by induction

✓ $k = 1: \mathbf{a}^{(1)} = \mathbf{Q}\mathbf{r}^{(1)} = r_{11}\mathbf{q}^{(1)}$

✓ \mathbf{Q} is orthogonal: $\|\mathbf{q}^{(1)}\|_2 = 1$

✓ $\|\mathbf{a}^{(1)}\|_2 = |r_{11}|\|\mathbf{q}^{(1)}\|_2 = r_{11}$

✓ $\mathbf{q}^{(1)} = (1/r_{11})\mathbf{a}^{(1)}$

✓ $k = 2: \mathbf{a}^{(2)} = \mathbf{Q}\mathbf{r}^{(2)} = r_{12}\mathbf{q}^{(1)} + r_{22}\mathbf{q}^{(2)}$

✓ \mathbf{Q} is orthogonal: $\mathbf{q}^{(1)T}\mathbf{a}^{(2)} = r_{12}\mathbf{q}^{(1)T}\mathbf{q}^{(1)} + r_{22}\mathbf{q}^{(1)T}\mathbf{q}^{(2)} = r_{12}$

✓ Also: $\|\mathbf{a}^{(2)} - r_{12}\mathbf{q}^{(1)}\|_2 = |r_{22}|\|\mathbf{q}^{(2)}\|_2 = r_{22}$

✓ $\mathbf{q}^{(2)} = (1/r_{22})(\mathbf{a}^{(2)} - r_{12}\mathbf{q}^{(1)})$

QR-Decomposition: proof by induction

- ✓ $k = 1: \mathbf{a}^{(1)} = \mathbf{Q}\mathbf{r}^{(1)} = r_{11}\mathbf{q}^{(1)}$
 - ✓ \mathbf{Q} is orthogonal: $\|\mathbf{q}^{(1)}\|_2 = 1$
 - ✓ $\|\mathbf{a}^{(1)}\|_2 = |r_{11}|\|\mathbf{q}^{(1)}\|_2 = r_{11}$
 - ✓ $\mathbf{q}^{(1)} = (1/r_{11})\mathbf{a}^{(1)}$

- ✓ $k = 2: \mathbf{a}^{(2)} = \mathbf{Q}\mathbf{r}^{(2)} = r_{12}\mathbf{q}^{(1)} + r_{22}\mathbf{q}^{(2)}$
 - ✓ \mathbf{Q} is orthogonal: $\mathbf{q}^{(1)T}\mathbf{a}^{(2)} = r_{12}\mathbf{q}^{(1)T}\mathbf{q}^{(1)} + r_{22}\mathbf{q}^{(1)T}\mathbf{q}^{(2)} = r_{12}$
 - ✓ Also: $\|\mathbf{a}^{(2)} - r_{12}\mathbf{q}^{(1)}\|_2 = |r_{22}|\|\mathbf{q}^{(2)}\|_2 = r_{22}$
 - ✓ $\mathbf{q}^{(2)} = (1/r_{22})(\mathbf{a}^{(2)} - r_{12}\mathbf{q}^{(1)})$

- ✓ k^{th} step: $\mathbf{a}^{(k)} = \mathbf{Q}\mathbf{r}^{(k)} = r_{1k}\mathbf{q}^{(1)} + r_{2k}\mathbf{q}^{(2)} \dots + r_{kk}\mathbf{q}^{(k)}$
 - ✓ \mathbf{Q} is orthogonal: $\mathbf{q}^{(i)T}\mathbf{a}^{(k)} = r_{1k}\mathbf{q}^{(i)T}\mathbf{q}^{(1)} + r_{2k}\mathbf{q}^{(i)T}\mathbf{q}^{(2)} \dots + r_{kk}\mathbf{q}^{(i)T}\mathbf{q}^{(k)} = r_{ik}$
for $i = 1, 2, \dots (k-1)$
 - ✓ Also: $\|\mathbf{a}^{(k)} - r_{1k}\mathbf{q}^{(1)} - r_{2k}\mathbf{q}^{(2)} \dots - r_{k-1,k}\mathbf{q}^{(k-1)}\|_2 =$
 $|r_{kk}|\|\mathbf{q}^{(k)}\|_2 = r_{kk}$
 - ✓ $\mathbf{q}^{(k)} = (1/r_{kk})(\mathbf{a}^{(k)} - r_{1k}\mathbf{q}^{(1)} - r_{2k}\mathbf{q}^{(2)} \dots + r_{k-1,k}\mathbf{q}^{(k-1)})$

QR-Decomposition: proof by induction

- ✓ Proceeding this way up to step n , all n columns of \mathbf{Q} and all the elements of \mathbf{R} can be computed. This concludes the proof that, $\mathbf{A} = \mathbf{Q}\mathbf{R}$ can be constructed!
- ✓ However, the algorithm in proof is tedious and inefficient!
- ✓ It is easier to construct the \mathbf{Q} and \mathbf{R} independently, directly from \mathbf{A} using Gram-Schmidt orthogonalization!
- ✓ For eigenvalues:
 - ✓ Initialize: $\mathbf{A}_0 = \mathbf{A}$
 - ✓ QR-Decomposition at each step: $\mathbf{A}_k = \mathbf{Q}_k\mathbf{R}_k$
 - ✓ Similarity Transformation:

$$\mathbf{A}_{k+1} = \mathbf{Q}_k^T \mathbf{A} \mathbf{Q}_k = \mathbf{Q}_k^T \mathbf{Q}_k \mathbf{R}_k \mathbf{Q}_k = \mathbf{R}_k \mathbf{Q}_k$$

- ✓ Stopping criteria: $\max_j \left| \frac{\lambda_j^{k+1} - \lambda_j^k}{\lambda_j^k} \right| \leq \varepsilon$

QR-Decomposition: Algorithm

- ✓ For a given matrix A , formulate Q using Gram-Schmidt procedure (MTH 102) as follows:

- ✓ Initialize: $\mathbf{q}^{(1)} = \frac{\mathbf{a}^{(1)}}{\|\mathbf{a}^{(1)}\|_2}$

- ✓ $\mathbf{z}^{(k+1)} = \mathbf{a}^{(k+1)} - \sum_{i=1}^k \left(\mathbf{a}^{(k+1)T} \mathbf{q}^{(i)} \right) \mathbf{q}^{(i)}$

- ✓ $\mathbf{q}^{(k+1)} = \frac{\mathbf{z}^{(k+1)}}{\|\mathbf{z}^{(k+1)}\|_2}$

- ✓ Compute the elements of matrix R as follows:

- ✓ $r_{ij} = \mathbf{q}^{(i)T} \mathbf{a}^{(j)}$

Example: QR-Decomposition

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 3 & 5 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \quad \text{and} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{3^2 + 3^2 + 2^2} = 4.6904 \quad y_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 0.6396 \\ 0.6396 \\ 0.4264 \end{bmatrix}$$

$$z_2 = x_2 - (x_2^T y_1) y_1 = \begin{bmatrix} -0.2273 \\ 0.7727 \\ -0.8182 \end{bmatrix} \quad \|z_2\| = 1.1481, \quad y_2 = \frac{z_2}{\|z_2\|} = \begin{bmatrix} -0.1980 \\ 0.6730 \\ -0.7126 \end{bmatrix}$$

Example: QR-Decomposition

$$z_3 = x_3 - (x_3^T y_1) y_1 - (x_3^T y_2) y_2 = \begin{bmatrix} -0.1379 \\ 0.0690 \\ 0.1034 \end{bmatrix}$$

$$\|z_3\| = 0.1857, \quad y_3 = \frac{z_3}{\|z_3\|} = \begin{bmatrix} -0.7428 \\ 0.3714 \\ 0.5571 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.6396 & -0.1980 & -0.7428 \\ 0.6396 & 0.6730 & 0.3714 \\ 0.4264 & -0.7126 & 0.5571 \end{bmatrix}$$

Example: QR-Decomposition

$$r_{11} = y_1^T x_1 = 4.6904, \quad r_{12} = y_1^T x_2 = 6.6092, \quad r_{13} = y_1^T x_3 = 1.7056, \quad r_{22} = y_2^T x_2 = 1.1481, \\ r_{23} = y_2^T x_3 = -0.2375 \quad \text{and} \quad r_{33} = y_3^T x_3 = 0.1857$$

$$R = \begin{bmatrix} 4.6904 & 6.6092 & 1.7056 \\ 0 & 1.1481 & -0.2375 \\ 0 & 0 & 0.1857 \end{bmatrix}$$

Example: Eigenvalues by Similarity Transformation

k	$A_k = R_k Q_k$			Q_k			R_k			e (%)
0	3.0000	4.0000	1.0000	0.6396	-0.1980	-0.7428	4.6904	6.6092	1.7056	
	3.0000	5.0000	1.0000	0.6396	0.6730	0.3714	0.0000	1.1481	-0.2375	
	2.0000	2.0000	1.0000	0.4264	-0.7126	0.5571	0.0000	0.0000	0.1857	
1	7.9545	2.3043	-0.0792	0.9968	-0.0757	-0.0257	7.9801	2.3703	-0.0546	866
	0.6331	0.9420	0.2941	0.0793	0.9765	0.2004	0.0000	0.7721	0.2723	
	0.0792	-0.1323	0.1034	0.0099	-0.2018	0.9794	0.0000	0.0000	0.1623	
2	8.1420	1.7215	0.2166	1.0000	-0.0078	-0.0006	8.1423	1.7269	0.2199	34.92
	0.0640	0.6990	0.4214	0.0079	0.9988	0.0482	0.0000	0.6863	0.4116	
	0.0016	-0.0328	0.1589	0.0002	-0.0482	0.9988	0.0000	0.0000	0.1790	
3	8.1556	1.6505	0.2983	1.0000	-0.0007	0.0000	8.1557	1.6509	0.2986	11.08
	0.0055	0.6656	0.4442	0.0007	0.9999	0.0130	0.0000	0.6645	0.4416	
	0.0000	-0.0086	0.1788	0.0000	-0.0130	0.9999	0.0000	0.0000	0.1845	
4	8.1568	1.6414	0.3199	1.0000	-0.0001	0.0000	8.1568	1.6415	0.3199	3.1
	0.0004	0.6587	0.4502	0.0001	1.0000	0.0036	0.0000	0.6587	0.4495	
	0.0000	-0.0024	0.1845	0.0000	-0.0036	1.0000	0.0000	0.0000	0.1861	
5	8.1568	1.6398	0.3259	1.0000	0.0000	0.0000	8.1568	1.6398	0.3259	0.88
	0.0000	0.6570	0.4519	0.0000	1.0000	0.0010	0.0000	0.6570	0.4517	
	0.0000	-0.0007	0.1861	0.0000	-0.0010	1.0000	0.0000	0.0000	0.1866	
6	8.1569	1.6395	0.3276	1.0000	0.0000	0.0000	8.1569	1.6395	0.3276	0.25
	0.0000	0.6565	0.4524	0.0000	1.0000	0.0003	0.0000	0.6565	0.4523	
	0.0000	-0.0002	0.1866	0.0000	-0.0003	1.0000	0.0000	0.0000	0.1867	
7	8.1569	1.6394	0.3281							0.07
	0.0000	0.6564	0.4525							
	0.0000	-0.0001	0.1867							