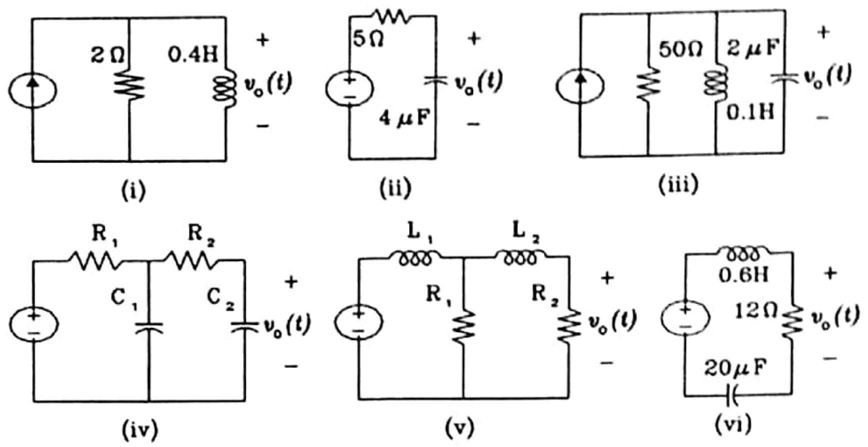
2018

EE 200 Signals, Systems & Networks 2014

Assignment 2

1 Consider the following circuits (i) to (vi). In all cases, assume a switch in series with every voltage source, that is initially open, but closed at $t = 0$ and a switch across every current source that is initially closed, and opened at $t = 0$. For all 6 circuits, set up the KCL/KVL integrodifferential equation in terms of $v_o(t)$. Find the order of each circuit.



2 For (i), assume an initial inductor current of I flowing upwards through the inductor through a closed parallel switch across the inductor (not shown). The switch is also opened at $t = 0$ along with the switch across the 5A current source. For (ii), assume an initial capacitor voltage of V, positive above. Obtain the solution $v_o(t) = y(t)$, separating the respective transient and steady state components $y_T(t)$ and $y_S(t)$. Also separate $y(t)$ into the 'pure state' response $y_C(t)$ and the 'pure input' response $y_I(t)$. For this exercise, consider the inputs to be current source for (i) as $i(t) = I_s;\ t \geq 0$ and the voltage source for (ii) as $v(t) = V_s;\ t \geq 0$.

3 Now consider the previous question, Q.2 with the following changes.

(a) Current source for (i) is $i(t) = I_m \cos\omega t;\ t \geq 0$ and voltage source for (ii) is $v(t) = V_m \cos\omega t;\ t \geq 0$.

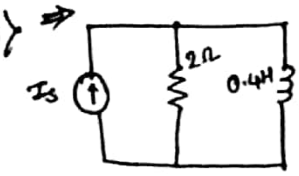
(b) Current source fror (i) is $i(t) = I_s + I_m \cos\omega t;\ t \geq 0$ and voltage source for (ii) is $v(t) = V_s + V_m \cos\omega t;\ t \geq 0$. Can you get this answer directly using the solutions of (a) above and of Q.2? Show how.

4 For circuits (iii) to (vi), for which the integrodifferential equations were formed in Q.1, differentiate throughout against time, if required, to obtain the pure differential equation, in homogeneous form (ignore the source) for $t \geq 0$. Next, write the characteristic equation, and solve it algebraically, to find the roots. Irrespective of the element values, what is the pattern observed with regard to the roots?

5 Whenever the total number of inductors + the total number of capacitors in the circuit is N, you would observe that we get an equation of corresponding order. For circuits (iii) to (vi), we thus get second order equations. Second order differential equations lead naturally to second degree characteristic algebraic equations whose roots we need to find. Consider the case when the roots are a conjugate pair of purely imaginary numbers. Find the form of the solution, and discuss what is unusual about it.

6 Among the circuits (iii) to (vi), find which exhibit the phenomenon of resonance, and which do not. Try to establish a connection between this and the answer to Q.6. Use your own definition of resonance based upon your past knowledge.

Q1) i) ⇒



Apply KCL,

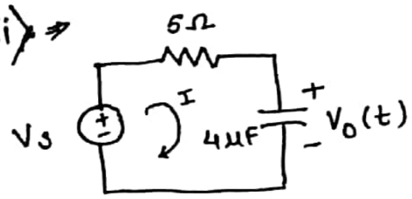
$\Rightarrow I_s = \frac{V_0(t)}{2} + I_L$ and $I_L = \frac{1}{L}\int V_0(t)$,

$\Rightarrow I_s = \frac{V_0(t)}{2} + \frac{1}{L}\int V_0(t)\,dt$.

$\Rightarrow \frac{dI_s}{dt} = \frac{1}{2}\frac{dV_0(t)}{dt} + \frac{V_0(t)}{L}$.

As the above eqn is 1st order differential eqn. Hence the system is 1st order.

ii) ⇒



Apply KVL,

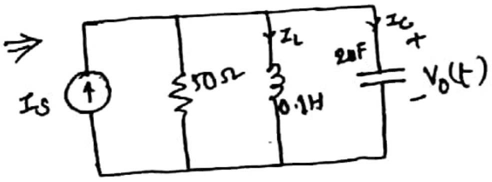
$\therefore V_s = 5I + V_0(t)$ and.

$\therefore I = C\frac{dV_0(t)}{dt}$.

$\Rightarrow V_s = 5\times 4\mu F.\frac{dV_0(t)}{dt} + V_0(t)$.

The above differential eqn is 1st order. Hence system is 1st order.

iii) ⇒



Apply KCL,

$I_s = \frac{V_0(t)}{50} + I_L + I_C$ and

$I_L = \frac{1}{L}\int V_0(t)\,dt$.

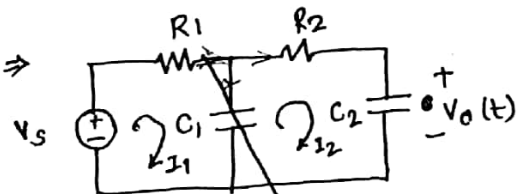
$I_C = C\frac{dV_0(t)}{dt}$.

$\Rightarrow I_s = \frac{V_0(t)}{50} + \frac{1}{L}\int V_0(t)\,dt + C\frac{dV_0(t)}{dt}$. again differentiating

$\Rightarrow \frac{dI_s}{dt} = \frac{1}{50}\frac{dV_0(t)}{dt} + \frac{V_0(t)}{L} + C\frac{d^2V_0(t)}{dt^2}$

differential eqn is 2nd order Hence the system is 2nd order.

iv) ⇒



Apply KVL,

$V_s = R_1 I_1 + \frac{1}{C_1}\int (I_1 - I_2).dt$.

$\frac{1}{C_1}\int (I_2 - I_1).dt + R_2 I_2 + V_0(t) = 0$

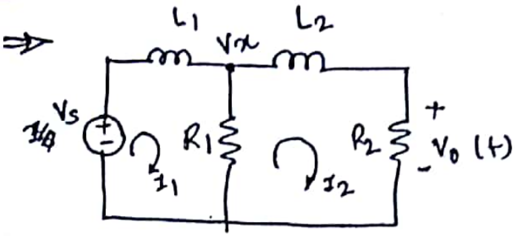
$\Rightarrow \frac{dV_s}{dt} = R_1\frac{dI_1}{dt} + R_2\frac{dI_2}{dt} + \frac{dV_0(t)}{dt}$.

$\Rightarrow$ and $I_2 = C_2 \dfrac{dV_0(t)}{dt}$. Q1 b vi) $\boxed{\text{Done in the Back of Q.1}}$

$$\Rightarrow \frac{dV_s}{dt} = R_1 \frac{dI_1}{dt} + R_2 C_2 \frac{d^2 V_0(t)}{dt^2} + \frac{dV_0(t)}{dt}$$

The above eqn is 2nd order hence the system is 2nd order.

2.1) V) $\Rightarrow$



Apply KCL,

$$V_s = R_1(I_1 - I_2) + V_1$$

$$V_1 = L_1 \frac{dI_1}{dt}$$

and $\quad R_1(I_2 - I_1) + V_0(t) + L_2 \dfrac{dI_2}{dt} = 0.$

$$V_s = V_0(t) + L_2 \frac{dI_2}{dt} + L_1 \frac{dI_1}{dt}$$

$$\therefore V_s = V_0(t) + \frac{L_2}{R_2} \cdot \frac{dV_0(t)}{dt} + L_1 \frac{dI_1}{dt}. \quad \text{——} (1)$$

Now, $\quad V_x = V_0(t) + \dfrac{L_2}{R_2} \dfrac{dV_0(t)}{dt} \quad \Rightarrow \quad I_1 = \dfrac{1}{L_1} \displaystyle\int (V_s - V_x) \cdot dt.$

$$\therefore I_1 = \frac{V_x}{R_1} + I_2 . = \frac{V_0(t)}{R_1} + \frac{L_2}{R_2} \cdot \frac{1}{R_1} \cdot \frac{dV_0(t)}{dt} + \frac{V_0(t)}{R_2}$$

Putting I_1 in (1)

$$\therefore V_s = V_0(t) + \frac{L_2}{R_2} \frac{dV_0(t)}{dt} + \frac{L_1}{R_1} \frac{dV_0(t)}{dt} + \frac{L_1 L_2}{R_1 R_2} \frac{d^2 V_0(t)}{dt^2} + \frac{L_1}{R_2} \frac{dV_0(t)}{dt}$$

Hence the system is 2nd order system.

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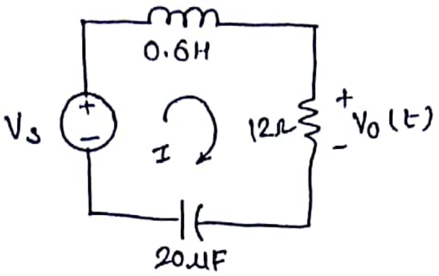
Q1)
Vi) ⇒



Apply KVL,

$$\therefore V_s = \cancel{L\,di(t)} \quad L\frac{dI}{dt} + V_0(t) + \frac{1}{C}\int I \cdot dt.$$

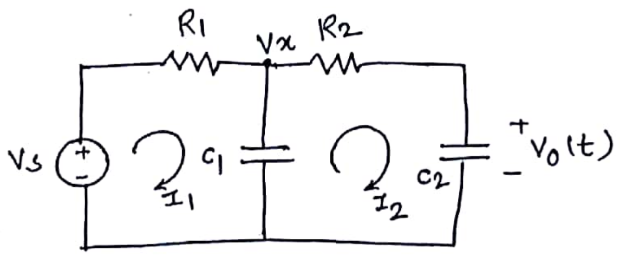
$$\therefore \text{ and } I = \frac{V_0}{12}.$$

$$\therefore V_s = \frac{L}{12}\frac{dV_0}{dt} + V_0(t) + \frac{1}{C}\int \frac{V_0}{12}\cdot dt$$

$$\therefore \frac{dV_s}{dt} = \frac{L}{12}\frac{d^2V_0(t)}{dt^2} + \frac{dV_0(t)}{dt} + \frac{1}{12C}\cdot V_0(t).$$

∴ Hence the system is <u>2nd order</u> system.

a)
iv) ⇒



Apply KVL,

$$\therefore V_s = R_1I_1 + R_2I_2 + V_0(t).$$

$$\therefore V_s = R_1I_1 + R_2C_2\frac{dV_0(t)}{dt} + V_0(t).$$

$$\& \quad I_1 = I_2 + C_1\frac{dV_x}{dt} \qquad \because V_x = R_2I_2 + V_0(t).$$

$$\therefore I_1 = I_2 + C_1R_2\frac{dI_2}{dt} + C_1\frac{dV_0(t)}{dt}$$

Where, $$I_2 = C_2 \frac{dv_0(t)}{dt}.$$

$$\therefore I_1 = C_2 \frac{dv_0(t)}{dt} + C_1 R_2 C_2 \frac{d^2 v_0(t)}{dt^2} + C_1 \frac{dv_0(t)}{dt}$$

$$\therefore V_S = R_1 C_2 \frac{dv_0(t)}{dt} + R_1 C_1 R_2 C_2 \frac{d^2 v_0(t)}{dt^2} + R_1 C_1 \frac{dv_0(t)}{dt} + R_2 C_2 \frac{dv_0(t)}{dt} + V_0(t).$$

$\therefore$ Hence the system is 2nd order system.

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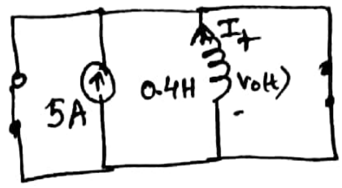
2 (i) For $t < 0$, the ckt. diagram is



Given, $i_L(0^-) = I$

For $t \geq 0$, the ckt. diagram is

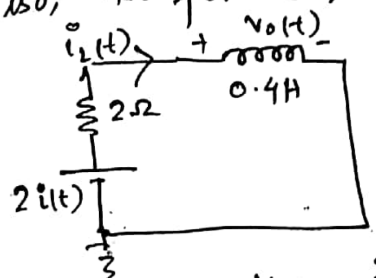
$i(t) = I_s; \quad t \geq 0$

Also, we know that the current across an inductor cannot change instaneously.

as, $V_L = L\left(\frac{di_L}{dt}\right) \rightarrow 0$

so, $i_L(0^-) = i_L(0^+) = i_L(0) = I$

Also, ~~the~~ for $t \geq 0$, the ckt. can be redrawn as



where, $R = 2\Omega$
$L = 0.4H$

so, from the integrodifferential equation of the ckt we have,

$$\frac{2i(t) - V_o(t)}{R} = i_L(t) \quad \text{—(i)}$$

and, $V_o(t) = V_L(t) = L\frac{di_L(t)}{dt}$ —(ii)

also, $i_L(\infty) = i(\infty) = I_s$,

as when the inductor is fully energised it acts like a short circuit.

and, $i(t) = I_s$ for $t \geq 0$

So, we have the boundary conditions as, 2

$i_L(0) = I$

and, $i_L(\infty) = I_s$

and from eqn (i) and (ii) we have.

$$2I_s - L\frac{di_L(t)}{dt} = Ri_L(t)$$

$$\Rightarrow \boxed{\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = \frac{2I_s}{R}}$$

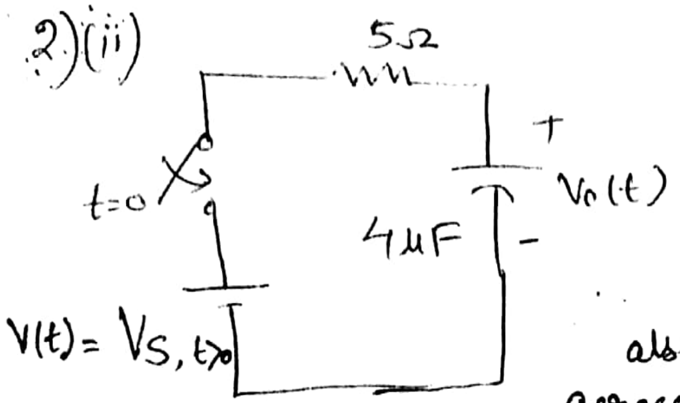
Solving the above with the boundary conditions.
We have,

$$i_L(t) = I_s + (I - I_s)\exp\left(-\frac{t}{L/R}\right) \text{ for } t \geq 0$$

$$\therefore \boxed{\begin{aligned} i_L(t) &= I \quad \text{for } t < 0 \\ &= \underbrace{I_s}_{\text{Steady state}} + \underbrace{(I - I_s)\exp\left(-\frac{t}{L/R}\right)}_{\text{transient state}} \quad \text{for } t \geq 0 \end{aligned}}$$

$$V_o(t) = \underbrace{RIe^{-\frac{t}{L/R}}}_{\text{State Response}} + \underbrace{RI_s(1 - e^{-\frac{t}{L/R}})}_{\text{Input Response}}$$

2)(ii)



Given, $V_{cap}(t) = V_0(t)$

$$V_0(0^-) = V_{cap}(0^-) = V$$

also, we know, the voltage across a capacitor capacitor c cannot changes instaneously as

$$i_{cap} = C \cdot \left(\frac{dv_{cap}}{dt}\right) \to 0$$

so, $V_0(0^+) = V_{cap}(0^+) = V_{cap}(0^-) = V$

Also, $V_{cap}(\infty) = V_0(\infty) = V_S$ state is obtained when the capacitor is fully charged to V_S potential and no more charging takes place.

so, from the integrodifferential eqn. of the ck.t. we have,

$$\frac{v(t) - V_{cap}(t)}{R} = i_{cap} = C\,\frac{d\,V_{cap}(t)}{dt}$$

also, $V_{cap}(t) = V_0(t)$

$$\Rightarrow \boxed{RC\frac{dV_0(t)}{dt} + V_0(t) = V(t)}$$

with boundary conditions, $V_0(0) = V$

and, $V_0(\infty) = V_S$

and, $V(t) = V_S$ for $t>0$

Solving the eqn. we get,

$$\boxed{\begin{aligned} V_0(t) &= V \quad \text{for } t<0 \\ &= \underbrace{V_S}_{\text{Steady State}} + \underbrace{(V - V_S)\exp\left(-\frac{t}{RC}\right)}_{\text{Transient State}} \quad \text{for } t \geq 0 \end{aligned}}$$

Ans

where $\boxed{\begin{aligned} R &= 5\Omega \\ C &= 4\mu F \end{aligned}}$

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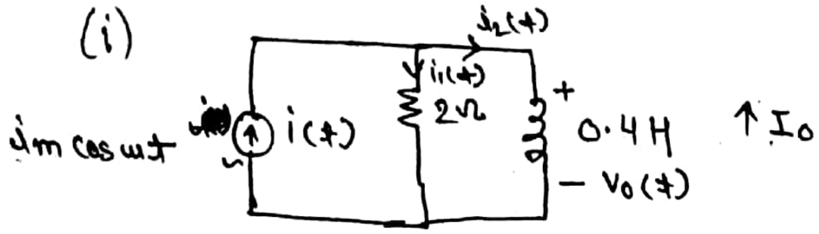
$$V_o(t) = \underbrace{V e^{-\frac{t}{RC}}}_{\text{State Response}} + \underbrace{V_s\left(1 - e^{-\frac{t}{RC}}\right)}_{\text{input Response}}$$

3(a)

(i)



we can perform AC Analysis as AC source is present

$$i(t) = i_1(t) + i_2(t)$$

2Ω, $Jw\, 0.4\, \Omega$ are in parallel

hence

$$i_2(t) = \frac{2\, i(t)}{2 + 0.4 Jw}$$

$$i_2(t) = \frac{2\, i_m \cos wt}{2 + 0.4 Jw}$$

$$v_o(t) = i_2(t)\, 0.4 Jw$$

$$\boxed{v_o(t) = \frac{0.4Jw \times 2\, i_m \cos wt}{2 + 0.4Jw}} \longrightarrow (I)$$

Since there is a initial current I_o upward at $t=0$, so from previous question state Response part -

$$I_s = I_o e^{-\frac{t}{4R}} = I_o e^{-5t}$$

$$\boxed{V_o(t) = + I_o R e^{-5t}} \longrightarrow (II)$$

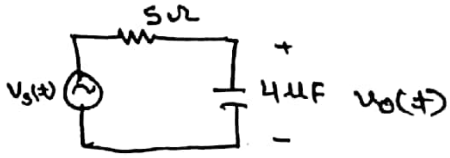
Combining (I) & (II)

$$\boxed{v_o(t) = \frac{0.4j\omega \times 2\,i_m \cos\omega t}{2+0.4j\omega} + 2I_0 e^{-5t} \quad t \geq 0}$$

From (I) & (II)

Ans $\boxed{v_o(t) = \underbrace{I_m \cos \omega t \,(2+0.4J\omega)}_{\text{steady state}} + \underbrace{2I_o\,(e^{-5t})}_{\text{Transient}}}$ $t \geq 0$

(II) $v(t) = V_m \cos \omega t \quad t > 0$



$$v_o(t) = \frac{v(t) \times \frac{1}{J\omega C}}{\frac{1}{J\omega C} + R}$$

5Ω & 4μF are in Series, so voltage will divide

$$= \frac{V_m \cos \omega t \times \frac{1}{J \times \omega \times 4 \times 10^{-6}}}{(1 + 5 \times J\omega \times 4 \times 10^{-6})\, \frac{1}{J\omega \times 4 \times 10^{-6}}}$$

$$\boxed{v_o(t) = \frac{V_m \cos \omega t}{(1 + 20 J\omega\, 10^{-6})}} \longrightarrow (I)$$

Since capacitor has initial voltage that will decay through Resistor R. hence

$$i_c(t) = C\frac{dV_o}{dt}$$

$$v_o = -R \times i_c = -RC\frac{dv_o}{dt} \longrightarrow ①$$

$$\Rightarrow \boxed{v_o = V_o\,(e^{-\frac{t}{RC}}) = V_o\,(e^{-\frac{t}{5 \times 4 \times 10^{-6}}})} \rightarrow (II)$$

using (I) & (II)

$$\boxed{v_o(t) = \frac{V_m \cos \omega t}{1 + 20 j\omega \times 10^{-6}} + \underbrace{V_o\, e^{-\frac{t}{20\times 10^{-6}}}}_{\text{Transient}} \quad t \geq 0}$$

Using (I) & (II)

Ans $$\boxed{v_0(t) = \frac{V_m \cos \omega t}{(1+20 J\omega \times 10^{-6})} + \underbrace{V_0 \left(e^{-\frac{t}{20\times 10^{-6}}}\right)}_{\text{Transient}}}$$

$t > 0$

3 (b) (i) $i(t) = I_s + I_m \cos \omega t$

we can decompose $i(t)$ into $i_1(t)$ & $i_2(t)$ since system is linear

$$i(t) = i_1(t) + i_2(t)$$

$$i_1(t) = I_s$$

$$i_2(t) = I_m \cos \omega t$$

Now from previous question & part (a) we can write

$$\boxed{v_0(t) = i_m \cos \omega t \,(2+0.4 J\omega) + 2 I_s - 2 I_s e^{-5t}} \longrightarrow \text{I}$$

Now solving for Initial condition as it was in part (a)

$$\boxed{v_0^t = 2 I_0 (e^{-5t})} \longrightarrow \text{II}$$

combining (I) & (II)

$$\boxed{v_0(t) = i_m \cos \omega t \,(2+0.4 J\omega) + 2 I_s - 2 I_s e^{-5t} + 2 I_0 e^{-5t}}$$

Ans $$\Rightarrow \boxed{V_0(t) = i_m \cos \omega t \,(2+0.4 J\omega) + 2 I_s (1 - e^{-5t}) + 2 I_0 e^{-5t}}$$

3(b)(ii)

$v(t) = V_S + V_m \cos \omega t$

Since circuit is linear so we can decompose $v(t)$ into $v_1(t)$, $v_2(t)$ as follows

$v_1(t) = V_S$

$v_2(t) = V_m \cos \omega t$

Now from previous question (2) and part (a)

$v_0(t) = v_{0_1}(t) + v_{0_2}(t)$

$$\boxed{v_0(t) = V_S(1 - e^{-\frac{t}{20\times10^{-6}}}) + \frac{V_m \cos \omega t}{1 + j\omega\, 20\times10^{-6}}} \rightarrow \mathrm{I}$$

Since capacitor has initial voltage V_0

hence from part (b) solving for transient

$$\boxed{v_0(t) = V_0\, e^{-\frac{t}{20\times10^{-6}}}} \rightarrow \mathrm{II}$$

Combining (I) & (II) we get

Ans. $$\boxed{v_0(t) = V_S\left(1 - e^{-\frac{t}{20\times10^{-6}}}\right) + \frac{V_m \cos \omega t}{1 + j\omega\, 20\times10^{-6}} + V_0\, e^{-\frac{t}{20\times10^{-6}}}}$$

Q 4 Homogeneous differential eq^ns for circuits (iii) to (vi) (ignoring source)

(iii) $$50LC\frac{d^2 v_0(t)}{dt^2} + L\frac{dv_0(t)}{dt} + 50 v_0(t) = 0$$

(iv) $$R_1C_1R_2C_2\frac{d^2 v_0(t)}{dt^2} + (R_1C_1 + R_2C_2 + R_1C_2)\frac{dv_0(t)}{dt} + v_0(t) = 0$$

(v) $$L_1L_2\frac{d^2 v_0(t)}{dt^2} + (R_1L_2 + R_2L_1 + R_1L_1)\frac{dv_0(t)}{dt} + R_1R_2v_0(t) = 0$$

(vi) $$LC\frac{d^2 v_0(t)}{dt^2} + 12C\frac{dv_0(t)}{dt} + v_0(t) = 0.$$

Characteristic equations and their roots

(iii) $$50LCs^2 + Ls + 50 = 0$$

$$s = \frac{-L \pm \sqrt{L^2 - 10^4 LC}}{100LC}$$

(iv) char. equation

$$R_1C_1R_2C_2\, s^2 + (R_1C_1 + R_2C_2 + R_1C_2)\, s + 1 = 0$$

roots

$$s = \frac{-(R_1C_1 + R_2C_2 + R_1C_2) \pm \sqrt{(R_1C_1 + R_2C_2 + R_1C_2)^2 - 4R_1C_1R_2C_2}}{2R_1C_1R_2C_2}$$

(v) char. equation.

$$L_1L_2\, s^2 + (R_1L_2 + R_2L_1 + R_1L_1)\, s + R_1R_2 = 0$$

roots

$$s = \frac{-(R_1L_2 + R_2L_1 + R_1L_1) \pm \sqrt{(R_1L_2 + R_2L_1 + R_1L_1)^2 - 4R_1R_2L_1L_2}}{2L_1L_2}$$

(vi) Char. equation

$$LCs^2 + 12Cs + 1 = 0$$

Roots $$s = \frac{-12C \pm \sqrt{144C^2 - 4LC}}{2LC}$$

~~Roots of all the four cases will be complex.~~

Roots are fn. of inverse time constants, $\left(RC, \frac{R}{L}\right)$

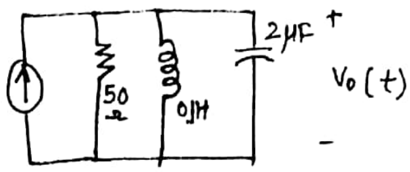
Complex roots only can exists in case both L & C are present.

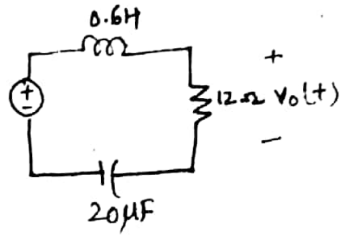
5.

Given : the fact that

- No. of inductors + No. of capacitors in a circuit = Order of $=^n$.

For ckt iii and For ckt (vi)





- we thus get 2^{nd} order differential equations.
- Second order differential $=^n$ lead naturally to 2^{nd} degree characteristic eq^n whose roots we need to find.
- Consider the case when roots are a conjugate pair of purely imaginary numbers.
- find the form of solution, and discuss what is unusual about it.

W.r.t to given conditions;

The general solution to differential $=^n$ will have 2 parts.

$$x(t) = x_{homo} + x_{Particular}.$$

$x_{homogenous}$ → due to initial $cond^n$ in circuit.

$x_{particular}$ → Independent voltage & current sources for $t > 0$.

Now 2^{nd} order differential $=^n$.

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x(t) = f(t)$$

$x(t)$ can be either $v(t)$ or $i(t)$.

To find natural response; set $f(t) = 0$

$$\therefore \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x(t) = 0 .$$

corresponding characteristic $\Rightarrow$

$$s^2 + a_1 s + a_0 = 0 .$$

characteristic roots or natural frequencies corresponding to $\Rightarrow$ are

$$s_1, s_2 = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_0}}{2}$$

Cases : If $[(a_1)^2 > 4a_0]$

Roots will be real & distinct.

Soln will be of form

$$x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Response of ckt will be <u>Overdamped</u>

for given case :

Roots are complex. $[\, a_1^2 < 4a_0 \,]$

$$\text{So} \quad s_1, s_2 = \alpha \pm i\beta \quad \text{--- (1)}$$

Response of system will be <u>underdamped</u>.

if roots are purely imaginary then $\alpha = 0$ in (1)

$$\Rightarrow \quad s_1, s_2 = \pm i\beta .$$

and solution is <u>UNDAMPED</u>. This means transient solution persists forever (not really Transient).

6. Following from the conclusion of Q5 above, a circuit can sometimes oscillate endlessly even when no input is applied. This is tantamount to resonance. Next we note, following the solution to Q4 that only when <u>both</u> L & C are present can roots have an imaginary component, that allows oscillation.

Therefore only (iii) & (vi) can exhibit resonance.