EE 200 Signals, Systems & Networks — Assignment 7

1 Prove that convolution has the following properties.

(a) associativity

(b) commutativity

(c) distribution over +

(d) $y(t) = x(t) * h(t) \Rightarrow \dot{x}(t) * h(t) = x(t) * \dot{h}(t) = \dot{y}(t)$ ($\dot{x}(t), \dot{h}(t)$ cont., $\dot{f}(t)) = df(t)/dt$)

(e) $y(t) = x(t) * h(t) \Rightarrow \dot{x}(t) * \dot{h}(t) = \ddot{x}(t) * h(t) = x(t) * \ddot{h}(t) = \ddot{y}(t)$ ($\ddot{x}(t), \ddot{h}(t)$ cont., $\ddot{f}(t)) = d^2 f(t)/dt^2$)

(f) Let $A(f) = \int_{-\infty}^{+\infty} f(t)dt$. Show that $y(t) = x(t) * h(t) \Rightarrow A(y) = A(x)A(h)$. State and prove the discrete version of this result.

2 A different interpretation of the convolution operation from the one discussed in the class is that $y(t) = x(t) * h(t)$ is the inner product of the *convolver* $x(\tau)$ with the *convolvend* $h(\tau)$ after the latter is time reversed and centered about t, *viz*, $h(t-\tau)$, where the inner product of any 2 functions $f(t), g(t)$ is given by $\int_{t=-\infty}^{t=+\infty} f(t)g(t)dt$. Write out the corresponding interpretation for the case of discrete convolution. Verify for the example solved in the lecture that this yields the same rësult.

3 We define the *support* of any signal $x(t)$ or $x[n]$ as the smallest interval outside which the signal is zero. Thus, the support of $\sin t$ is $(-\infty, \infty)$ and the support of $\delta[n+1] - 2\delta[n-3]$ is $[-1, 3]$. Let us denote the support of $x(t)$ by (t_{xL}, t_{xR}) or $(t_{xL}, t_{xR}]$ or $[t_{xL}, t_{xR})$ or $[t_{xL}, t_{xR}]$ as applicable, and that of $x[n]$ by $[n_{xL}, n_{xR}]$. The *support time* of $x(t)$ is denoted as $t_{xR} - t_{xL}$ and of $x[n]$ by $n_{xR} - n_{xL}$. Show that the convolution of two signals $x(t), x'(t)$ with finite support intervals T, T' has a finite support interval of $T + T'$. Similarly, prove the corresponding result for discrete signals: the convolution of two discrete signals having finite supports N, N' is $N + N' - 1$.

4 Develop relations between the boundary points $t_{xL}, t_{xR}, t_{hL}, t_{hR}$ and t_{yL}, t_{yR}. Similarly, develop the corresponding relations between $n_{xL}, n_{xR}, n_{hL}, n_{hR}$ and n_{yL}, n_{yR} for discrete convolution.

5 Under the new interpretation of convolution, the value of $y(t)$ is equal to the area under $x(\tau)h(t-\tau)$. Use this to construct two examples of $y(t) = x(t) * h(t)$ which is finite but not bounded, though $x(t), h(t)$ are both bounded. Find constraints on $x(t), h(t)$ that will ensure that $y(t)$ remains finite for all t. Find a sufficient constraint to be applied upon $x(t), h(t)$ to ensure that $y(t)$ remains bounded, and not just finite. Compare these constraints with those obtained above to keep $y(t)$ finite.

6 After the above problems, can you comment on $y(t)$ when $x(t)$ is periodic and $h(t)$ is finitely supported and both are bounded? What will happen when both $x(t), h(t)$ are non negative, periodic and bounded?

7 Following the consequences of the above, we seek a way out for the specific case of convolving periodic signals. The *periodic convolution* of two signals $x(t), y(t)$ of period T is defined as $x(t) \circledast y(t) = \int_{t=-T/2}^{t=T/2} x(\tau)y(t-\tau)d\tau$. The convolution is now bounded because the integration limits have been restricted to exactly one period. What if the indicated integration interval $(-T/2, T/2]$ is replaced by any other contiguous T-length time interval of the form $(\Delta, T+\Delta]$? Use this definition to convolve a T-periodic signal $x(t)$ with a constant $y(t) = y_0$, using T as the convolution interval. Next, use any finite convolution interval T to convolve two constant signals $x(t) = x_0$ and $y(t) = y_0$. Express your result in terms of T in both cases.

8 Let $x_i(t); i = 1, 2$ be periodic signals of the same period T, and let each cycle of $x_i(t); i = 1, 2$ be nonzero only over $t \leq T/2$ and zero over the remaining part of width $T/2 < t \leq T$ of the cycle. Define $x_i'(t) = \begin{cases} x_i(t); & t \leq T \\ 0; & t < 0, t > T \end{cases}$; $i = 1, 2$. Show that $x_1(t) \circledast x_2(t) = x_1'(t) * x_2'(t); t \leq T$.

Q1 (a) Associativity in convolution:

i.e. $[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)]$

LHS: $[x(t) * h(t)] * g(t) = \left[\int_{-\infty}^{\infty} x(\tau_1)\, h(t-\tau_1)\, d\tau_1\right] * g(t)$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau_1)\, h(\tau_2-\tau_1)\, d\tau_1\right] g(t-\tau_2)\, d\tau_2$$

Assuming $\tau_2 - \tau_1 = \tau$ and using change of order of integration:

LHS: $$= \int_{-\infty}^{\infty} x(\tau_1) \left[\int_{-\infty}^{\infty} h(\tau)\, g(t-\tau-\tau_1)\, d\tau\right] d\tau_1$$

$$\int_{-\infty}^{\infty} h(\tau)\, g(t-\tau-\tau_1)\, d\tau = h(t-\tau_1) * g(t-\tau_1)$$

$$= y(t-\tau_1) \quad [\text{let}] \text{ ------ } (1)$$

Then LHS $= \int_{-\infty}^{\infty} x(\tau_1)\, y(t-\tau_1)\, d\tau_1$

$= x(t) * y(t)$

$= x(t) * [h(t) * g(t)]$ using equation (1)

$=$ RHS.

(b) Commutativity in convolution:

i.e. $x(t) * h(t) = h(t) * x(t)$

LHS: $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

let $t - \tau = m \Rightarrow -d\tau = dm$

$\Rightarrow x(t) * h(t) = -\int_{\infty}^{-\infty} x(t-m)\, h(m)\, dm$

$= \int_{-\infty}^{\infty} x(t-m)\, h(m)\, dm$

$= \int_{-\infty}^{\infty} h(m)\, x(t-m)\, dm$

$= h(t) * x(t)$

$=$ R.H.S.

(c) Distribution over addition:

i.e. $x(t) * [h(t) + g(t)] = [x(t) * h(t)] + [x(t) * g(t)]$

L.H.S: $x(t) * [h(t) + g(t)] = \int_{-\infty}^{\infty} x(\tau)[h(t-\tau) + g(t-\tau)] d\tau$

$= \int_{-\infty}^{\infty} [x(\tau) h(t-\tau) + x(\tau) g(t-\tau)] d\tau$

$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau$

$= [x(t) * h(t)] + [x(t) * g(t)]$

$=$ RHS

(d) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

differentiating with respect to time 't'

$$\frac{dy(t)}{dt} = \int_{-\infty}^{\infty} x(\tau) \frac{d}{dt}[h(t-\tau)] d\tau \quad \text{——— (2)}$$

$$\Rightarrow \dot{y}(t) = \int_{-\infty}^{\infty} x(\tau) \dot{h}(t-\tau) d\tau = x(t) * \dot{h}(t) \quad \text{——— (3)}$$

Now, using the commutativity over convolution

$x(t) * \dot{h}(t) = \dot{x}(t) * h(t) = \dot{y}(t)$

proved

(e) Using equation (2) and differentiating again wrt 't'

$$\frac{d^2 y(t)}{dt^2} = \int_{-\infty}^{\infty} x(\tau) \frac{d^2}{dt^2} h(t-\tau)] d\tau$$

$$\Rightarrow \ddot{y}(t) = \int_{-\infty}^{\infty} x(\tau) \ddot{h}(t-\tau) d\tau = x(t) * \ddot{h}(t)$$

Similarly $\ddot{y}(t) = \ddot{x}(t) * h(t) = x(t) * \ddot{h}(t)$ proved

Using the result of part (d)

$\dot{y}(t) = \dot{x}(t) * h(t) = \int_{-\infty}^{\infty} \dot{x}(\tau) h(t-\tau) d\tau$

differenting w.r.t 't'

$$\frac{d^2 y(t)}{dt^2} = \int_{-\infty}^{\infty} \dot{x}(\tau) \frac{d}{dt} h(t-\tau) d\tau = \int_{-\infty} \dot{x}(\tau) \dot{h}(x-\tau) d\tau$$

$\Rightarrow \ddot{y}(t) = \dot{x}(t) * \dot{h}(t)$

(f) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$A(y) = \int_{-\infty}^{\infty} y(t)\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\, dt$$

changing the order of Integration

$$A(y) = \int_{-\infty}^{\infty} x(\tau)\, d\tau \cdot \int_{-\infty}^{\infty} h(t-\tau)\, dt$$

In the second integration let $t-\tau = m \Rightarrow dt = dm$

$$A(y) = \int_{-\infty}^{\infty} x(\tau)\, d\tau \cdot \int_{-\infty}^{\infty} h(m)\, dm$$

$$A(y) = A(x) \cdot A(h)$$ Proved.

For discreate case:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

Sum : $S[y] = \sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$

changing the order of summation

$$S[y] = \underbrace{\sum_{m=-\infty}^{\infty} x[m]}_{S[x]} \underbrace{\sum_{n=-\infty}^{\infty} h[n-m]}_{S[h]}$$

proved.

$$\Rightarrow S[y] = S[x] \cdot S[h]$$

Q 2 In discrete domain convolution is defined as:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = x[n] * h[n]$$

And inner product ~~of~~ of $x[m]$ and $g[m]$ is given by

$$\langle x, g\rangle = \sum_{m=-\infty}^{\infty} x[m]\, g[m] = y[n] \quad = \text{inner product}$$

where $g[m] = h[n-m]$

which is time reversed and centred about n the function h after

* Assuming the functions $x[n]$ and $h[n]$ are real valued.

Q3

Suppose $x(t)$ is finite in the interval (t_{xL}, t_{xR})

$$x(t) = \begin{cases} \text{finite} & ; \forall \; t \in (t_{xL}, t_{xR}) \\ 0 & ; \text{otherwise} \end{cases}$$

and $$x'(t) = \begin{cases} \text{finite} & ; \forall \; t \in (t_{x'L}, t_{x'R}) \\ 0 & ; \text{otherwise} \end{cases}$$

Then $x'(t-\tau) \neq 0$ for $t_{x'L} \leq t-\tau < t_{x'R}$

$$\Rightarrow t_{x'L} + \tau < t < t_{x'R} + \tau$$

While convolving $x(t)$ and $x'(t)$, we get the first point for $\tau = t_{xL}$ i.e. $t_{x'L} + t_{xL} < t$

and last point for $\tau = t_{xR}$ i.e. $t < t_{x'R} + t_{xR}$

overall the output is non zero for

$$t_{x'L} + t_{xL} < t < t_{x'R} + t_{xR}$$

$$\Rightarrow t_{x'L} + t_{xL} < t < t_{x'L} + T' + t_{xL} + T$$

$$\Rightarrow t_{x'L} + t_{xL} < t < t_{x'L} + t_{xL} + \underbrace{T + T'} \quad \text{— (1)}$$

Thus the range of interval for finite support is $T+T'$.

Discrete case:

suppose $$x[m] \begin{cases} \neq 0 & \text{for } (N_0 \leq m \leq N_1) \\ = 0 & \text{otherwise} \end{cases}$$

and $$x'[m] = \begin{cases} \neq 0 & \text{for } (N_2 \leq m \leq N_3) \\ = 0 & \text{otherwise} \end{cases}$$

then $x'[n-m] \neq 0$ for $N_2 \leq n-m \leq N_3$

or for $N_2 + m \leq n \leq m + N_3$

After convolving the minimum value of n will be $N_2 + N_0$ and similarly maximum value of $n = N_1 + N_3$

$$\Rightarrow x[m] * x'[m] \neq 0 \quad , \quad N_2 + N_0 \leq n \leq N_1 + N_3$$

But $N_1 - N_0 + 1 = N$ and $N_3 - N_2 + 1 = N'$

$$\Rightarrow x[m] * x'[m] \neq 0 \quad ; \quad N_0 + N_2 \leq n \leq N_0 + N + N_2 + N' - 2$$

Thus the support interval is $(N + N' - 1)$ — (2)

Q4. Using equatinon (I) from question 3

$$\underbrace{t_{x'L} + t_{xL}}_{t_{yL}} < t < \underbrace{t_{xR} + t_{xL} + T + T'}_{t_{yR}} \quad \text{(Range of } y\text{)}$$

$t_{yL} = t_{xL} + t_{x'L} \longrightarrow$ Sum of lower limits

$y_{YR} = t_{xR} + t_{x'R} \longrightarrow$ sum of upper limits

Using equation (I) from question 3

$$\underbrace{N_0 + N_2}_{n_{yL}} \leq n \leq \underbrace{(N_0 + N - 1) + (N_2 + N' - 1)}_{n_{yR}}$$

$n_{YL} = N_0 + N_2 = n_{xL} + n_{x'L} \longrightarrow$ Sum of lower limits

$n_{YR} = (N_0 + N - 1) + (N_2 + N' - 1) = n_{xR} + n_{x'R}$

Sum of upper limits

Q5 Example 1 : Lets take $x(t) = u(t)$ and $h(t) = u(t)$ } bounded

$$y(t) = x(t) * h(t) = u(t) * u(t)$$
$$= r(t) \quad [\text{unit ramp func}^n]$$

finite but not bounded

Example 2 : Let $x(t) = u(t)$

$$h(t) = (1 + \sin(t))\, u(t)$$
$$y(t) = x(t) * h(t) = r(t) + (1 - \cos t)\, u(t)$$

finite but unbounded

for y(t) to remain finite :
x(t), h(t) should be single sided funct^n (semiinfinite support)
ie they shouldn't extend to infinity on both the sides of the axis but only on one side. Should ~~not~~ be ~~oscillatory about~~ non-negative.

for y(t) to remain finite and bounded :
At least One out of x(t) or h(t) should have a finite support. Both of them should be bounded.

For y(t) to be finite, we need both x(t) & h(t) to be semifinite and nonnegative so that the area under $x(\tau)\, h(t-\tau)$ keeps on increasing.

We want one out of $x(t)$ or $h(t)$ to have a finite support and both of them to be bounded because this ensures area under $x(\tau)\, h(t-\tau)$ to be bounded.

Q6 $x(t)$ is periodic & $h(t)$ finitely supported

$y(t)$ is bounded, finite and also periodic with period T.

When $x(t)$, $h(t)$ are both periodic, non-negative and bounded, the o/p $y(t)$ is non-negative, periodic and unbounded.

Q7 $x(t) \circledast y(t) = \int_{t=-T/2}^{T/2} x(\tau)\, y(t-\tau)\, d\tau$

Lets put $\tau = -\Delta - T/2$

$$x(t) \circledast y(t) = \int_{t=-T/2}^{t=T/2} x(-\Delta - T/2)\, y(t+\Delta+T/2)\, d(-\Delta - T/2)$$

Assuming $t + \Delta + T/2 = t'$ ∴

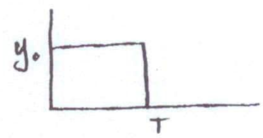
$$x(t) \circledast y(t) = \int_{t'=\Delta}^{t'=\Delta+T} y(t)\, x(t-t')\, d(t-t')$$

$$= \int_{\Delta}^{\Delta+T} y(t')\, x(t-t')\, dt'$$

$= y(t) * x(t)$ for any period T

Case 1 : $x(t)$ is periodic with period T

$y(t) = y_0$



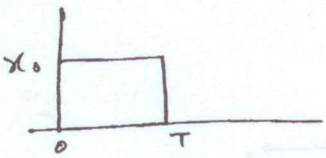
$$x(t) \circledast y(t) = \int_{0}^{T} x(\tau)\, y(t-\tau)\, d\tau$$

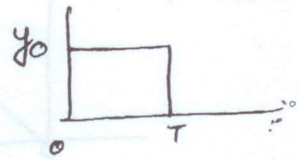
$\cancel{= y_0 \int_{t=0}^{t=T} x(\tau)\, d\tau}$ ~~for period T~~

we will keep one signal the same and flip & shift the other from 0 – T. Area under multiplicatⁿ gives convolution

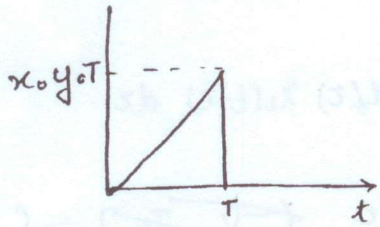
With 2 constant signals

$x(t) = x_0$ & $y(t) = y_0$



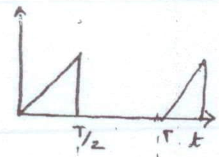


$$x(t) \circledast y(t) = \int_0^T x(\tau)\, y(t-\tau_0)\, d\tau$$

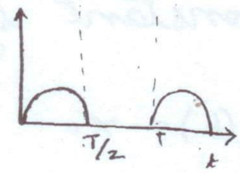


$$\text{Max value} = \int_0^T x_0 y_0 \, d\tau = x_0 y_0 T$$

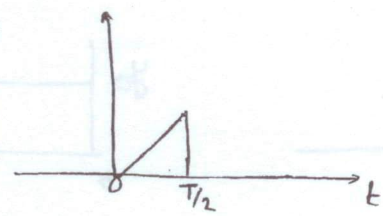
Q8 Let $x_1(t)$:



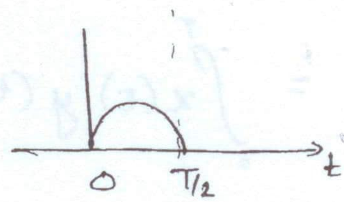
And $x_2(t)$:



Now $x_1'(t)$:



and $x_2'(t)$:



$$y(t) = x_1(t) \circledast x_2(t) = \int_{t=0}^{t=T} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Thus $y(t)$ lies in the range $\cancel{[-T/2, T/2]}$ $[0, T]$

And $$y'(t) = x_1'(t) * x_2'(t) = \int_{-\infty}^{\infty} x_1'(\tau)\, x_2'(t-\tau)\, d\tau$$

Also lies in the range $[0, T]$

Thus $y(t) = y'(t)$; $t \leq T$