EE 200 Assignment 11

Signals, Systems & Networks

1 $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_1(\omega)$ are bandlimited signals satisfying $X_1(\omega) = 0;\ |\omega| \geq W_1$ and $X_2(\omega) = 0;\ |\omega| \geq W_2$. Find the sampling period for faithful representation of the following signals. $y_1(t) = x_1(t) + x_2(t)$ $\quad y_2(t) = x_1(t)x_2(t)$ $\quad y_3(t) = x_1(t) * x_2(t)$.

2 We develop the principle of a *fractional delay*. Given a sample train $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$ of a W-bandlimited $x(t) \leftrightarrow X(\omega)$ how do we obtain the signal $x_{s\epsilon}(t) = x_s(t - \epsilon)$ where $\epsilon < T_s$? Find an expression for, and also sketch the frequency response $H_\epsilon(\omega)$ of the system that will produce $x_{s\epsilon}(t)$.

3 Now study the following technique called *time division multiplexing*. By the use of the sampling theorem, we can transmit more than one signal *simultaneously* on a cable (or store it in a storage medium such as tape) if the cable (or tape) possesses sufficient bandwidth. Suppose $x(t) \leftrightarrow X(\omega)$ and $x'(t) \leftrightarrow X'(\omega)$ are both W-bandlimited signals. We first generate the signals $x_s(t) = x(t)p(t)$ and $x'_s(t) = x'(t)p(t)$. Next, we delay $x'(t)$ by $\epsilon = T_s/2$, using a so called *half sample delay* of the kind discussed in the previous problem. Finally, we construct the combined signal $y_s(t) = x_s(t) + x'(t - T_s/2)$. Sketch $x_s(t) + x'_s(t)$ and its CTFT. What is the bandwidth of central alias of the combined signal? Obtain an expression for $Y(\omega)$, the FT of the continuous-time signal $y(t)$ obtained from $y_s(t)$ by reconstruction using an ideal lowpass filter of suitable bandwidth. If we wish to extract $x(t), x'(t)$ separately, what do we need to do?

4 The above technique may be extended to multiplex more than just two signals: the N different W-bandlimited signals $x_i(t);\ i = 1, 2, \ldots, N$ are each first sampled to obtain the sample trains $x_{si}(t);\ i = 1, 2, \ldots, N$ which are then respectively delayed by $\epsilon_i = (i-1)T_s/N; =;\ i = 1, 2, \ldots, N$. The combined sample train is then $y_s(t) = \sum_{i=1}^{N} x_{si}(t - \epsilon_i)$. Obtain an expression for $Y(\omega)$, the FT of the continuous-time signal $y(t)$ obtained from $y_s(t)$ by reconstruction using an ideal lowpass filter of suitable bandwidth.

5 Suppose $x(t)$ is a bandpass signal with a magnitude spectrum satisfying $|X(\omega)| = 0; |\omega| \leq W_L$ as well as $|X(\omega)| = 0; |\omega| \geq W_H$. We can of course safely sample this at $T > \pi/W_H$. Depending upon the values of W_L, W_H can there exist a greater sampling period than T at which the signal may be sampled so that perfect reconstruction with an appropriate ideal filter is still possible? What would be this appropriate filter? Construct an example, choosing suitable values of W_L, W_H and demonstrate by sketching all the relevant spectra.

EE200: Assignment Solution.

1. $x_1(t) \longleftrightarrow X_1(W)$ and, $x_2(t) \longleftrightarrow X_2(W)$

with, $X_1(\omega) = 0; \ |\omega| \geq W_1$, $X_2(\omega) = 0 \quad |\omega| \geq W_2$

(i) $y_1(t) = x_1(t) + x_2(t)$

$Y_1(\omega) = X_1(\omega) + X_2(\omega)$ [Linearity of Transform]

Bandwidth of $Y_1(\omega) = \max\{W_1, W_2\}$

$\therefore$ the sampling period $\boxed{T_{S1} = \dfrac{2\pi}{\max\{W_1, W_2\}}}$

(ii) $y_2(t) = x_1(t)\, x_2(t)$ [Product in time domain $\leftrightarrow$ Convolution in freq. domain]

$Y_2(W) = X_1(\omega) * X_2(\omega)$

Bandwidth of $Y_2(\omega) = W_1 + W_2$

$\therefore$ the sampling period $\boxed{T_{S2} = \dfrac{2\pi}{W_1 + W_2}}$

(iii) $y_3(t) = x_1(t) * x_2(t)$

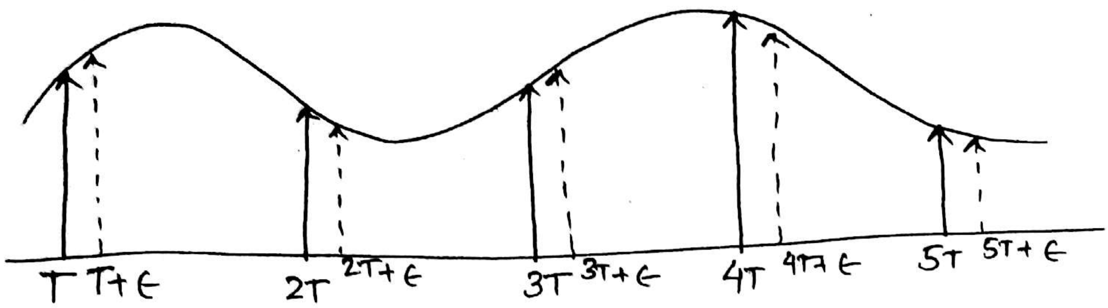
$Y_3(\omega) = X_1(W) \cdot X_2(W)$

Bandwidth of $Y_3(\omega) = \min\{W_1, W_2\}$

$\therefore$ the sampling period $\boxed{T_{S3} = \dfrac{2\pi}{\min\{W_1, W_2\}}}$

(2.) $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$ of a W-band limited

$x(t) \leftrightarrow X(\omega)$ $\qquad x_{s\epsilon}(t) = x_s(t-\epsilon)$ where $\epsilon < T_s$



$X_s(\omega) \leftrightarrow x_s(t)$

and also, $x_s(t-\epsilon) \leftrightarrow X_s(\omega)\exp(-j\omega\epsilon)$

as, $X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\omega t}\, dt$.

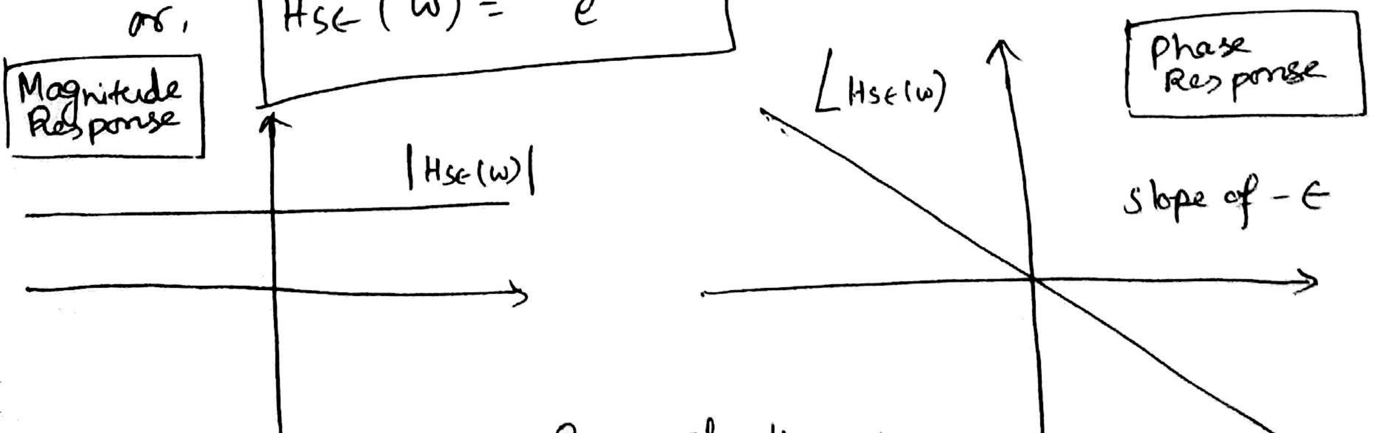
$X_{s\epsilon}(\omega) = \int_{-\infty}^{\infty} x_s(t-\epsilon)\, e^{-j\omega t}\, dt$ $\qquad \left[\begin{array}{l} t-\epsilon = y \\ dt = dy \end{array}\right.$

$= \int_{-\infty}^{\infty} x_s(y)\, e^{-j\omega(y+\epsilon)}\, dy = e^{-j\omega\epsilon} \int_{-\infty}^{\infty} x_s(y)\, e^{-j\omega y}\, dy.$

$= e^{-j\omega\epsilon} X_s(\omega)$

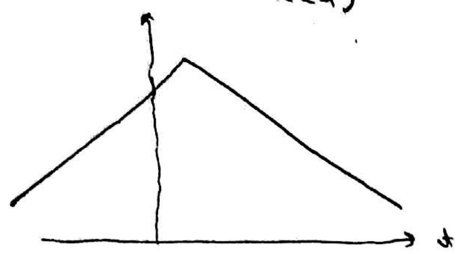
$\therefore\ X_{s\epsilon}(\omega) = X_s(\omega)\cdot e^{-j\omega\epsilon} = X_s(\omega)\cdot H_{s\epsilon}(\omega)$

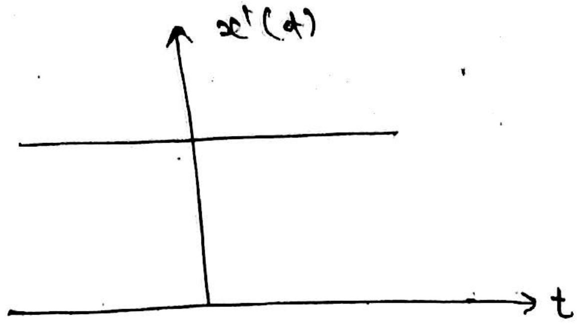
or, $\boxed{H_{s\epsilon}(\omega) = e^{-j\omega\epsilon}}$

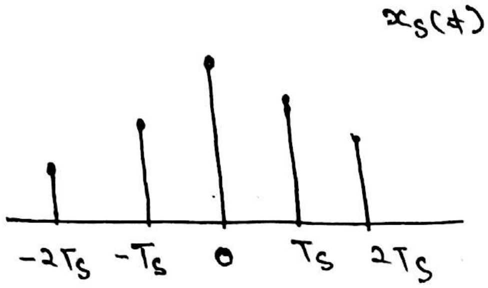


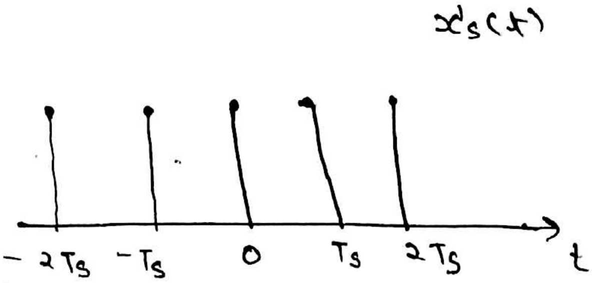
Freq. Response of $H_{s\epsilon}(\omega)$

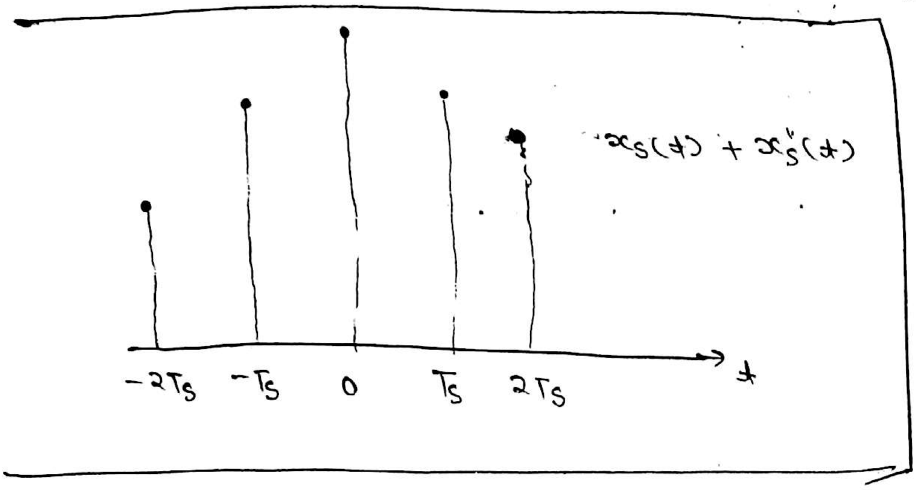
3.

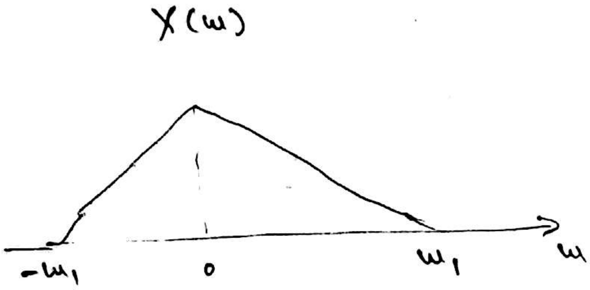


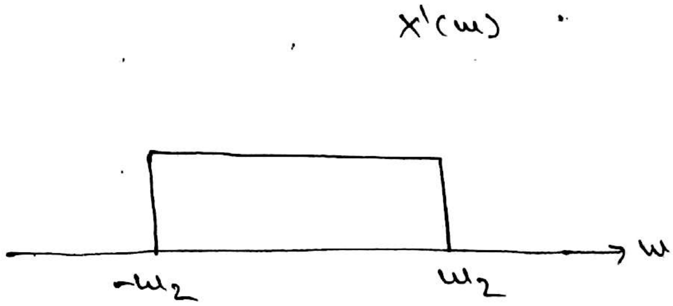




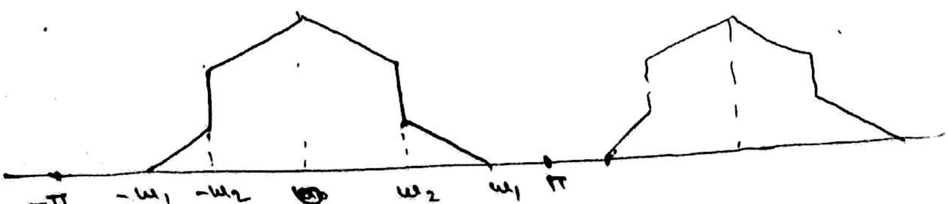








$X_s(\omega) + X'_s(\omega)$. If $\omega_1 > \omega_2$

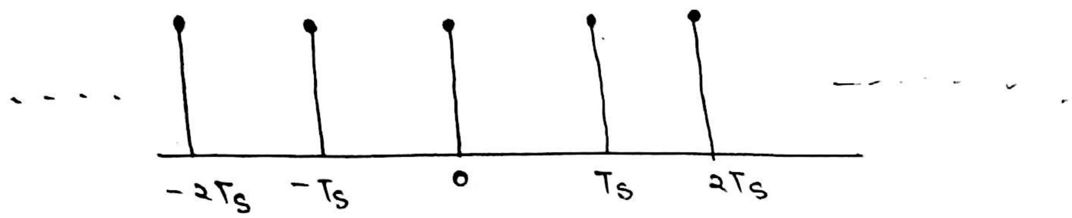


$\boxed{\text{Bandwidth of center alias} = \max(\omega_1, \omega_2) =}$

$$\boxed{Y(\omega) = X(\omega) + e^{-j\omega \frac{T_s}{2}} X'(\omega)}$$

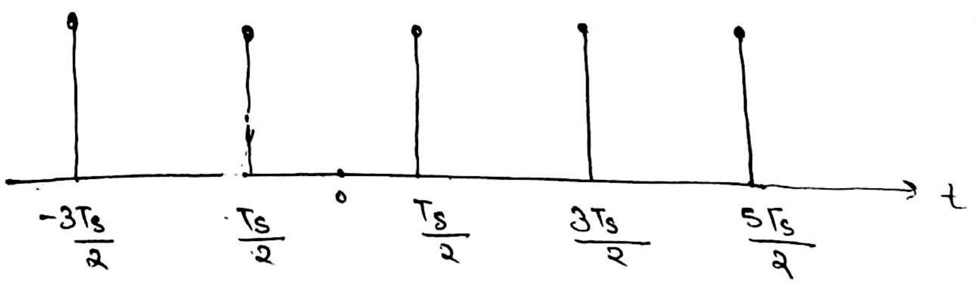
To Recover $x(t)$ from $Y(t)$ we will multiply following Impulse train with $Y(t)$

$P(t)$



we can use this sampled $Y_s(t)$ to Reconstruct $x(t)$ using filtering with Ideal low pass filter of suitable bandwidth.

To Recover $x'(t)$ from $y(t)$, we use following sampling Train



Now using this sampled $Y_s'(t)$ we can recover $x_s'(t)$ hence $x'(t)$ using ~~Fourier Transform and~~ lowpass filtering of suitable bandwidth filters.

④

$$Y_s(t) = \sum_{i=1}^{N} x_{s_i}(t-\varepsilon_i)$$

$$Y_s(j\omega) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x_{s_i}(t-\varepsilon_i)\, e^{-j\omega t}\, dt$$

$$Y_s(j\omega) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x_{s_i}(t')\, e^{-j\omega(t'+\varepsilon_i)}\, dt'$$

$$= \sum_{i=1}^{N} e^{-j\omega\varepsilon_i} \int_{-\infty}^{\infty} x_{s_i}(t')\, e^{-j\omega t'}\, dt'$$

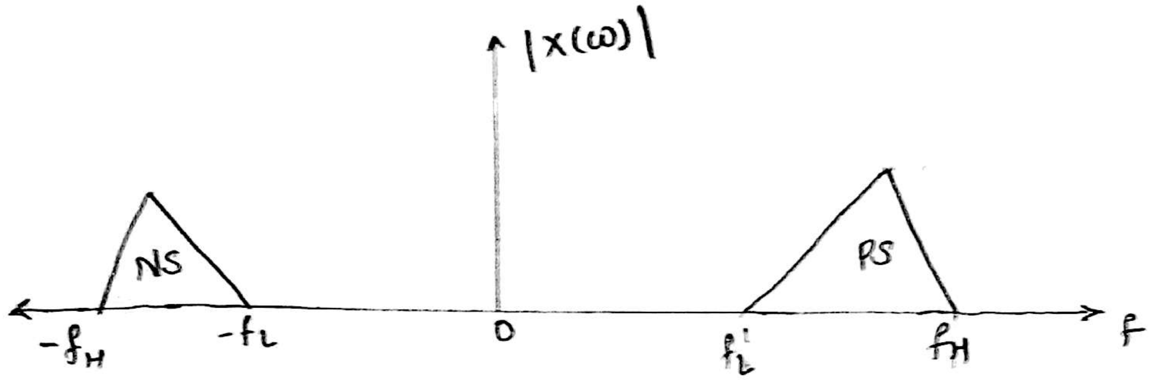
$$Y_s(j\omega) = \sum_{i=1}^{N} e^{-j\omega\varepsilon_i} \sum_{k=-\infty}^{\infty} X_i(\omega - k\omega_s)$$

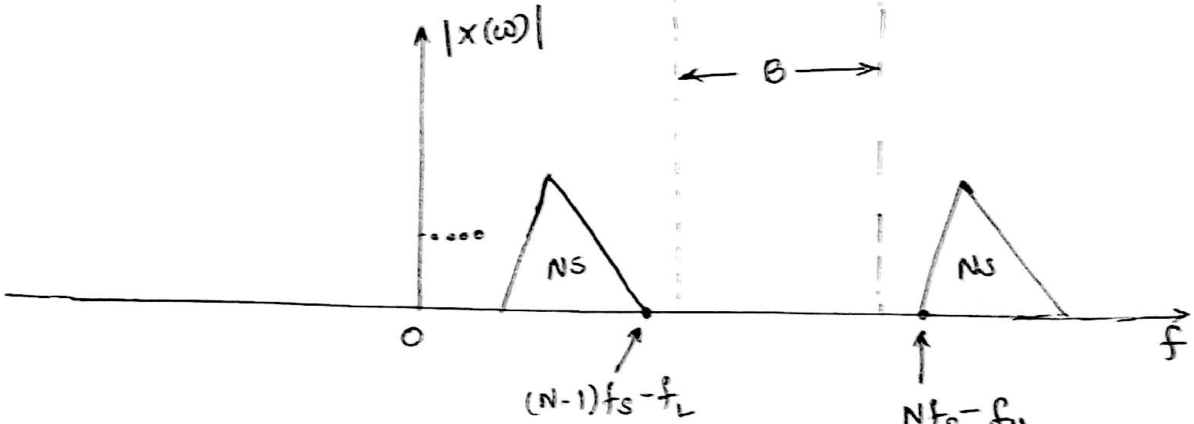
Using filtering with Ideal low pass filter of suitable bandwidth we pick up ~~of~~ zeroth component (assuming No aliasing).

$$Y_s(\omega) = \sum_{i=1}^{N} e^{-j\omega\varepsilon_i}\, X_i(\omega)$$

$$\boxed{Y(\omega) = \sum_{i=1}^{N} e^{-j\omega\frac{(i-1)T_s}{N}}\, X_i(\omega)}$$

25) ⇒ $x(t)$ is a bandpass signal, The positive-freq. part and negative-freq. part of spectrum are called PS & NS resp. PS and NS are shifted in such a way that we cause no overlap.





→ The product of $x(t)$ and the dc component of sampling waveform leaves PS unmoved.

Fig. shows the right shifted patterns of NS due to (N-1) and Nth harmonic of sampling waveform. To avoide overlap it is necessary that,

$$(N-1)\,f_s - f_L \le f_L .$$

and $\quad N f_s - f_H \ge f_H . \qquad$ and $B = f_H - f_L$

$$\therefore\ (N-1)\,f_s \le 2(f_H - B),$$

$$\&\ \ N f_s \ge 2 f_H .$$

Let $k = f_H/B$ then,

$$\boxed{f_s \le 2B\left(\frac{K-1}{N-1}\right)}$$

&

$$\boxed{f_s \ge 2B\left(\frac{K}{N}\right)}.$$

The original $x(t)$ can be reconstructed from sampled $x(nT_s)$ by using band pass filter with non-zero value between f_L and f_H frequency

⇒ If we select the minimum value of f_s to be $f_s = 2(f_H - f_L) = 2B$ then the shifted PS pattern will not overlap PS. The left shifted NS cannot cause any overlap with PS.