Assignment (3)

Problem 16a

Newton Raphson Method $x_{k+1} = \phi(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$.

α is a zero of $f(x)$ with multiplicity $m > 1$.

Express $f(x) = (x-\alpha)^m g(x)$. ——— (1)

$f'(\alpha) = f''(\alpha) = \cdots = f^{m-1}(\alpha) = 0, \quad g(\alpha) \neq 0$.

$$\phi(\alpha) = \phi(x_k) + (\alpha - x_k)\phi'(x_k) + \frac{(\alpha - x_k)^2}{2!}\phi''(\xi); \ \xi \in (\alpha, x_k) \quad \text{——— (2)}$$

$$\Rightarrow (\alpha - x_{k+1}) = (\alpha - x_k)\phi'(x_k) + \frac{(\alpha - x_k)^2}{2!}\phi''(\xi)$$

$$\epsilon_{k+1} = \epsilon_k \phi'(x_k) + \frac{\epsilon_k^2}{2!}\phi''(\xi). \quad \text{——— (3)}$$

Now $\phi = x - \frac{f}{f'} \Rightarrow \phi' = 1 - \frac{f'^2 - ff''}{f'^2} = \frac{ff''}{f'^2}$ ——— (4)

$$f' = (x-\alpha)^m g' + m(x-\alpha)^{m-1} g = (x-\alpha)^m \cdot \left[g' + \frac{mg}{(x-\alpha)}\right]. \quad \text{——— (5)}$$

$$f'' = (x-\alpha)^m g'' + 2m(x-\alpha)^{m-1} g' + m(m-1)(x-\alpha)^{m-2} \cdot g$$

$$= (x-\alpha)^m \left[g'' + \frac{2mg'}{(x-\alpha)} + \frac{m(m-1)g}{(x-\alpha)^2}\right] \quad \text{——— (6)}$$

From (4) $$\phi' = \frac{g\left[g'' + \frac{2mg'}{(x-\alpha)} + \frac{m(m-1)g}{(x-\alpha)^2}\right]}{\left[g' + \frac{mg}{(x-\alpha)}\right]^2}.$$

$$= \frac{gg''(x-\alpha)^2 + 2mgg'(x-\alpha) + m(m-1)g^2}{g'^2(x-\alpha)^2 + 2mgg'(x-\alpha) + m^2 g^2}.$$

$\lim_{x \to \alpha} \phi' = \frac{\cancel{m}(m-1)\cancel{g^2}}{m^{\cancel{2}}\cancel{g^2}} = \frac{m-1}{m} \neq 0$ So, from (3), it is 1st order.

Prob. 16b -

$$\phi = x - \frac{mf}{f'}$$

$$\phi' = 1 - \frac{mf'^2 - mff''}{f'^2}$$

$$= \frac{(1-m)f'^2 + mff''}{f'^2}$$

$$= (1-m) + m\frac{ff''}{f'^2}$$

$$= (1-m) + \cancel{m}\cdot\frac{m-1}{\cancel{m}} = 0$$

so, at least 2nd order from 3

Tutorial Problem-4

17. Writing in the form, $u(x) = \frac{f(x)}{f'(x)}$

$$x_{n+1} = \phi(x_n) = \frac{x'_{n+1} + x''_{n+1}}{2} = \tfrac{1}{2}\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) + \tfrac{1}{2}\left(x_n - \frac{u(x_n)}{u'(x_n)}\right)$$

$$\phi(x) = x - \tfrac{1}{2}u(x) - \tfrac{1}{2}\frac{u(x)}{u'(x)}$$

$$u = f/f', \quad u' = 1 - \frac{ff''}{(f')^2}$$

$$\phi' = 1 - \tfrac{1}{2}u' - \tfrac{1}{2}\left(1 - \frac{uu''}{(u')^2}\right)$$

$$\phi'' = -\tfrac{1}{2}u'' + \tfrac{1}{2}\frac{(u')^2(u'u'' + uu''') - uu''\cdot 2u'u''}{(u')^4}$$

Now, since, $f(\xi) = 0$, $u(\xi) = 0$.

$$u'(\xi) = 1, \quad \phi(\xi) = \xi$$

$$\phi'(\xi) = 1 - \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$\Rightarrow \phi''(\xi) = -\tfrac{1}{2}u'' + \tfrac{1}{2}u'' = 0$$

Now,

$$\phi(\xi) = \phi(x_n) + (\xi - x_n)\phi'(x_n) + \frac{(\xi - x_n)^2}{2}\phi''(x_n) + \frac{(\xi - x_n)^3}{3!}\phi'''(x_n) + \text{HOT}$$

$$\phi(x_n) = x_{n+1}, \quad \phi(\xi) = \xi$$

$$\Rightarrow \quad (\xi - x_{n+1}) = \epsilon_{n+1} = \epsilon_n\cdot\phi'(x_n) + \epsilon_n^2\cdot\frac{\phi''(x_n)}{2} + \frac{\epsilon_n^3}{6}\phi'''(x_n) + \text{HOT}$$

$\lim_{x_n \to \xi}$ ϕ'' & $\phi' = 0$.

$$\Rightarrow \epsilon_{n+1} \propto \epsilon_n^3$$