

Cubic

①

$$h_i = x_i - x_{i-1} \quad g_i = \frac{y_i - y_{i-1}}{h_i}$$

$m+1$ data points
 $4m \rightarrow$ unknown.
 $3m-3 \rightarrow$ conditions.

$$x_i - y_i = k_i - m_i$$

$$g_i(x) \quad g_i'(x) \quad g_i''(x)$$

$$\text{map } [x_{i-1}, x_i] \text{ to } [0, 1]$$

$$t = \frac{x - x_{i-1}}{x_i - x_{i-1}} = \frac{x - x_{i-1}}{h_i} \Rightarrow \frac{dt}{dx} = \frac{1}{h_i}$$

fit a cubic polynomial $p_i(t) = at^3 + bt^2 + ct + d_i$

$$\text{So, } g_i'(x) = p_i'(t) \cdot \frac{dt}{dx} \Rightarrow p_i'(t) = h_i g_i'(x)$$

$$g_i''(x) = \frac{1}{h_i^2} p_i''(t) \Rightarrow p_i''(t) = h_i^2 g_i''(x)$$

$$\Rightarrow p_i(0) = g_i(x_{i-1}) = y_{i-1} = g_{i+1}(x_i) = p_{i+1}(0) \rightarrow 2m$$

$$\Rightarrow \frac{1}{h_i} p_i'(0) = g_i'(x_{i-1}) = k_i = g_{i+1}'(x_i) = \frac{1}{h_{i+1}} p_{i+1}'(0) \rightarrow (m-1)$$

$$\Rightarrow \frac{1}{h_i^2} p_i''(0) = g_i''(x_{i-1}) = m_i = g_{i+1}''(x_i) = \frac{1}{h_{i+1}^2} p_{i+1}''(0) \rightarrow (m-1)$$

2 extra conditions. $g_i''(x_0) = g_{n+1}''(x_n) = 0 \rightarrow 2$

~~First derivative unknown~~

$$p_i(0) = g_i(x_{i-1}) = y_{i-1} \Rightarrow d_i = y_{i-1} \quad \text{--- ①}$$

$$p_i(1) = g_i(x_i) = y_i \Rightarrow a_i + b_i + c_i = y_i - y_{i-1} = h_i g_i \quad \text{--- ②}$$

First Derivative Unknown:-

$$p_i'(t) = 3at^2 + 2bt + c_i$$

$$p_i'(0) = h_i g_i'(x_{i-1}) = h_i k_{i-1}; \quad p_i'(1) = h_i g_i'(x_i) = h_i k_i$$

$$\Rightarrow c_i = h_i k_{i-1} \quad \text{--- ③}$$

$$3a_i + 2b_i + c_i = h_i k_i \quad \text{--- ④}$$

$$\text{or } 3a_i + 2b_i = h_i (k_i - k_{i-1})$$

$$2a_i + 2b_i = 2(y_i - y_{i-1}) - 2h_i k_{i-1}$$

$$a_i = h_i (k_i + k_{i-1}) - 2(y_i - y_{i-1}) = h_i (k_i + k_{i-1} - 2g_i)$$

$$a_i^0 + b_i^0 = h_i (y_i - y_{i-1}) - h_i k_{i-1}$$

$$b_i^0 = -h_i (k_i + 2k_{i-1}) + 3(y_i - y_{i-1}) = -h_i (k_i + 2k_{i-1} - 3g_i)$$

$$q_i''(x_i) = q_{i+1}''(x_i)$$

$$\frac{p_i''(1)}{h_i^2} = \frac{p_{i+1}''(0)}{h_{i+1}^2}$$

$$\Rightarrow p_i''(t) = 6a_i t + 2b_i$$

$$\Rightarrow \frac{6a_i + 2b_i}{h_i^2} = \frac{2b_{i+1}}{h_{i+1}^2}$$

$$\Rightarrow \frac{6(k_i + k_{i-1} - 2g_i)}{h_i} + \frac{2(-k_i - 2k_{i-1} + 3g_i)}{h_i} = \frac{2(-k_{i+1} - 2k_i + 3g_{i+1})}{h_{i+1}}$$

$$\Rightarrow h_{i+1} k_{i-1} + 2(h_{i+1} + h_i)k_i + h_i k_{i+1} = 3(h_i g_{i+1} + h_{i+1} g_i)$$

$$i = 1, \dots, n-1$$

$$2b_1 = 0 \quad k_1 + 2k_0 = 3g_1$$

$$6a_n + 2b_n = 0$$

$$6h_n(k_n + k_{n-1} - 2g_n) + 2h_n(k_n + 2k_{n-1} - 3g_n) = 0$$

$$4k_n + 2k_{n-1} = 6g_n$$

$$\Rightarrow k_{n-1} + 2k_n = 3g_n$$

$$2k_0 + k_1 = 3g_1$$

2nd Derivative Unknown:

(3)

$$b_i''(t) = 6a_i t + 2b_i$$

$$b_i''(0) = h_i^2 q_i''(x_{i-1}) = h_i^2 m_{i-1} = 2b_i$$

$$\Rightarrow b_i = \frac{h_i^2}{2} m_{i-1}$$

$$b_i''(1) = h_i^2 q_i''(x_i) = h_i^2 m_i = 6a_i + h_i^2 m_{i-1}$$

$$\Rightarrow a_i = \frac{h_i^2}{6} (m_i - m_{i-1})$$

From ②

$$C_i = h_i g_i - \frac{h_i^2}{2} m_{i-1} - \frac{h_i^2}{6} (m_i - m_{i-1})$$
$$= h_i g_i - \frac{h_i^2}{6} (m_i + 2m_{i-1})$$

$$q_i'(x_i) = q_{i+1}'(x_i)$$

$$\Rightarrow \frac{b_i'(1)}{h_i} = \frac{b_{i+1}'(0)}{h_{i+1}}$$

$$\Rightarrow \frac{3a_i + 2b_i + C_i}{h_i} = \frac{C_{i+1}}{h_{i+1}}$$

$$\Rightarrow \frac{h_i}{2} (m_i - m_{i-1}) + h_i m_{i-1} + g_i - \frac{h_i}{6} (m_i + 2m_{i-1}) = g_{i+1} - \frac{h_{i+1}}{6} (m_{i+1} + 2m_i)$$

$$\Rightarrow \left(-\frac{h_i}{2} + h_i - \frac{h_i}{3}\right) m_{i-1} + \left(\frac{h_i}{2} - \frac{h_i}{6} + \frac{h_{i+1}}{3}\right) m_i + \left(\frac{h_{i+1}}{6}\right) m_{i+1} = g_{i+1} - g_i$$

$$\Rightarrow h_i m_{i-1} + 2(h_i + h_{i+1}) m_i + h_{i+1} m_{i+1} = 6(g_{i+1} - g_i)$$

$$i = 1, 2, \dots, n-1$$

$$m_0 = 0$$

$$m_n = 0$$

(n+1) equations, n unknowns.