EE200 2018 Assignment 8

1 Given two complex exponentials of radian frequency ω_1, ω_2, find the product of the two signals. Is it also a complex exponential? What is its period? How does its period relate to the periods of the constituents? Now consider two periodic signals $x_1(t), x_2(t)$ with respective periods T_1, T_2. that posess an FS expansion. Generalize your earlier result to comment on the period of $x_1(t)x_2(t)$ in terms of T_1 and T_2. If $T_1 = T_2$, find an expression for the FS coefficients of the product in terms of the respective FS coefficients x_{1k}, x_{2k}.

2 We call $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ an *impulse train*. Given a continuous signal $x(t)$, consider the system H that produces output $y(t) = x(t)p(t)$ which is a so-called *modulated* impulse train. Answer the following questions.

(a) If $x(t)$ is periodic with period T_0, will $y(t)$ be periodic in general? If not, are there special conditions under which $y(t)$ can be periodic?

(b) Is the system H a time-invariant system, *ie*, do shifts in the input appear as equal shifts in the output? If this is not generally true, are there special conditions under which certain shifts int the input will indeed appear as equal shifts in the output?

(c) Also examine whether H is linear, causal, stable and whether it posesses memory.

3 Consider some given time period T. How many different complex exponentials $e^{j\omega t}$ exist that are harmonic with T? Consider next the corresponding discrete situation. How many different discrete complex exponentials $e^{j2\pi nk/N}$ can be found that have a given period of N?

4 Let $x(t)$ be of period T with a Fourier Series x_k; $-\infty < k < \infty$. Find the Fourier Series of the following signals.

(a) $y(t) = x(\alpha t); 0 < \alpha < 1$. Repeat for $\alpha > 1$.

(b) $\sum_{k=-\infty}^{\infty} w(t - k\alpha T)$ where $w(t) = x(t)(u(t + T/2) - u(t - T/2))$ and $\alpha > 1$. Repeat for $0 < \alpha < 1$

5 Suppose f is some function. Let $f(t) \leftrightarrow F(\omega)$. Find the time function $f'(t)$ that is the inverse FT of $f(\omega)$. Similarly, find the FT of $F'(t)$ of $F(\omega)$.

6 (a) Find the Fourier Transform of the unit step $u(t)$. (**Hint:** Carry out the even-odd decomposition $u(t) = u_e(t) + u_o(t)$ and find the FTs of each term separately. Finally use the linearity of the FT.)

(b) Often, we wish to study the *step response* $s(t)$ of a system, which is the output of the system when $u(t)$ is the input. Using (a) above, devise a means of finding the step response of a system whose impulse response is $h(t)$. How does the step response throw light on the stability of the system?

(c) Using (a) above, find the FT of $\int_{-\infty}^{t} x(t')dt'$ if $x(t) \leftrightarrow X(\omega)$

7 (a) Use the previous problem to obtain the FT of the signum function, $\text{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$.

(b) From the above, find the FT of $x(t) \star (\pi t)^{-1}$ and of $x(t) \star (\pi t)^{-1} \star (\pi t)^{-1}$.

8 A certain linear system has the property that if $x(t) \to y(t)$, then $x(-t) \to y(-t)$. Show that the following is valid for any input-output pair:

$$x(t) = x_e(t) + x_o(t) \to y(t) = y_e(t) + y_o(t) \quad iff \quad x(t) = \alpha x_e(t) + \beta x_o(t) \to y(t) = \alpha y_e(t) + \beta y_o(t).$$

Does the expression above remind you of the definition of linearity? Figure out why. Then figure out how the present statement is actually different.

1. Given two exponentials of radian frequency ω_1 and ω_2, find product of the two.

$$x_1(t) = e^{j\omega_1 t} \qquad x_2(t) = e^{j\omega_2 t}$$

Product - $x_1(t) \cdot x_2(t) = e^{j\omega_1 t} \cdot e^{j\omega_2 t}$

$$= e^{j(\omega_1+\omega_2)t} \quad \text{— (1)}$$

Is it also a complex exponential?
product

Yes. (refer (1)).

What is its Period? & how does it relate to T's of constituents?

Period represented by T

Since period T_1 for first exponential is $= \frac{2\pi}{\omega_1}$

T_2 for Second Comp. exp $= \frac{2\pi}{\omega_2}$

Period T for the product will be the Lowest common multiple (LCM) of the two periods.

Now if $x_1(t)$ $x_2(t)$ are given which are each periodic. With periods T_1 & T_2 respectively, & possess FS expansion
Comment on period of $x_1(t)\, x_2(t)$ in terms of T_1 and T_2.

Given $x_1(t) = x_1(t-T_1)$ & $x_2(t) = x_2(t-t_2)$

The period of $x(t) = x_1 \cdot x_2(t)$ will be LCM of T_1, T_2 if LCM exists.
If not; then $x(t)$ is non-periodic.

If $T_1 = T_2$, find an expression for the FS coefficients of product in terms of respective FS coefficients.

$$y(t) = x_1(t)\, x_2(t) = \left[\sum_k x_{1k}\, e^{jkw_0 t}\right]\left[\sum_{l} x_{2l}\, e^{jlw_0 t}\right]$$

$$Y_m = \sum_{m} \frac{1}{T}\int_0^T y(t) \cdot e^{-jw_0 t}\, dt.$$

$$= \frac{1}{T}\int_0^T \left[\sum_k x_{1k} \cdot e^{jw_0 kt}\right]\left[\sum_l x_{2l} \cdot e^{jlw_0 t}\right] e^{-jw_0 mt}$$

$$= \frac{1}{T}\sum_{k}\sum_{l}\int_0^T x_{1k} \cdot x_{2l} \cdot e^{j(k+l-m)w_0 t}\, dt$$

Q2 : Given $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ an impulse train.

Given a continuous signal $x(t)$. consider H that produces $y(t) = x(t)\,p(t)$. — The modulated impulse Train.

(a) If $x(t)$ is periodic ; with period $\underline{T_0}$ will $y(t)$ be periodic in general?

Answer — $y(t)$ will be periodic iff there exists a LCM of T_0 & T.

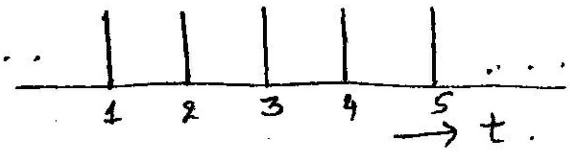
$T_0 \rightarrow$ Period of $x(t)$

$T =$ Period of impulse train.

(b) If H is time invariant system ; will shifts in inputs produce shifts in o/p ?. Generally or under specific conditions..

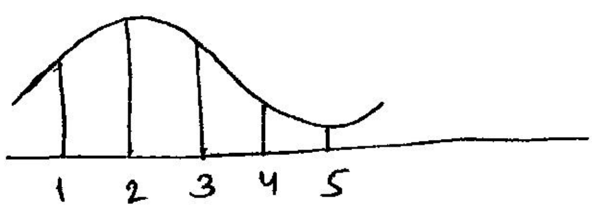
Answer —

We have a pulse train infinite in nature



let $x(t)$ be $\rightarrow$

The modulated wave will be



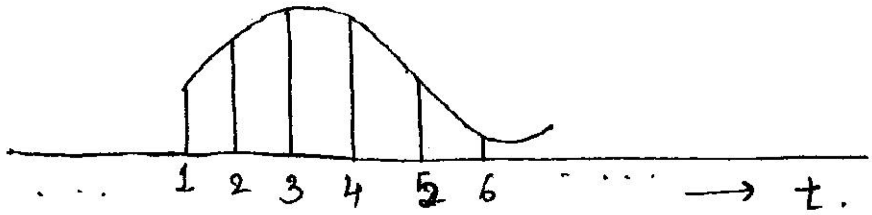
$$= \frac{1}{T} \sum_{k} \sum_{l} x_{1k} \cdot x_{2l} \int_{0}^{T} e^{j(k+l-m)\omega_0 t} \, dt .$$

1. This product is non-zero only when $\underline{k+l-m=0}$ for each value of 'm' when its evaluated.

2. In particular $\int_{0}^{T} e^{j(k+l-m)\omega_0 t} dt = T\delta_{k+l,m}$.

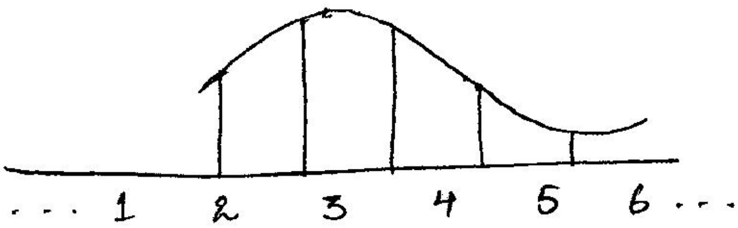
Thus, we get $\boxed{y_m = \sum_{k=-\infty}^{\infty} x_{1k} \, x_{2,m-k}.}$

Now lets shift x(t) by $\frac{1}{2}t$. & see its effect on o/p. ie; the modulated signal.



Result - Shift in i/p did not produce "equal" shift in o/p.

Again lets shift x(t) by $\underline{t}$ & redraw a modified o/p (modulated signal)



Result - Here shift in x(t) has a similar change in modulated signal.
If we keep on doing this, it will be more clear.

Conclusion ; When shift in x(t) is a integer number (1, 2, 3 ...) then a similar shift in o/p will be observed. For fractional shift in x(t) ($\frac{1}{2}$, $\frac{1}{3}$, ...) such similar change in o/p will not be observed.

(c) Examine if H is

Linear - Yes

Causal - Yes

Stable - No

Possesses memory - No.

(3) Given time period T, Now ~~if~~ ~~$e^{j\omega t}$~~ ~~is~~ for Continous, ~~so~~, there are infinitely different complex exponentials $e^{j\omega t}$ exist that are harmonic with T.

eg: $e^{j\omega t}$ of frequencies

$$\frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \frac{4}{T}, \dots \frac{n}{T}, \dots$$

Now, for the corresponding discrete situation, we have only N different discrete complex exponentials $e^{j\frac{2\pi kn}{N}}$ of a given period ~~as~~ N as.

Discrete-time sinusoids separated by an integral multiples of 2π frequencies are identical as.

$$\exp\left[j\{(w_0+2\pi)n+\theta\}\right] = \exp\left[j(w_0 n+2\pi n+\theta)\right]$$
$$= \exp\{j(w_0 n+\theta)\}$$

due to the periodic nature of $\exp(jx)$

eg. $e^{j\frac{2\pi}{N}k\cdot 1}, e^{j\frac{2\pi k 2}{N}}, e^{j\frac{2\pi k 3}{N}}, \dots e^{j\frac{2\pi k(N-1)}{N}}, e^{j\frac{2\pi k N}{N}}$

$$e^{j\frac{2\pi kN}{N}} = e^{j\frac{2\pi k}{N}}$$

(4) $x(t)$ is a periodic signal with period T

with fourier series coeff. x_k $-\infty < k < \infty$

$$x_k = \frac{1}{T}\int x(t)\, e^{-\frac{jk2\pi t}{T}}\, dt \longrightarrow (A)$$

(a) $y(t) = x(\alpha t)$

since $x(t+T) = x(t)$

Hence Period of $y(t)$ is $\frac{T}{\alpha}$

$$y_k = \frac{1}{(T/\alpha)}\int_{(T/\alpha)} x(\alpha t)\, e^{-\frac{jk2\pi t}{(T/\alpha)}}\, dt$$

Put $\alpha t = t'$, $\alpha\, dt = dt'$

$$\Rightarrow y_k = \frac{\alpha}{T}\int_T x(t')\, e^{-\frac{jk2\pi t'}{T}}\, \frac{dt'}{\alpha}$$

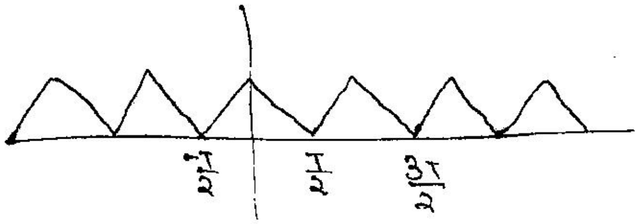
$$\Rightarrow y_k = \frac{1}{T}\int_T x(t')\, e^{-\frac{jk2\pi t'}{T}}\, dt'$$

But from (A) we can say

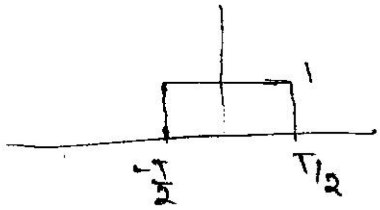
Ans. $\boxed{y_k = x_k \quad \forall \alpha}$ with Fundamental Frequency $\omega_0 = \frac{2\pi\alpha}{T}$

(b) $\sum_{k=-\infty}^{\infty} w(t - k\alpha T)$

$$w(t) = x(t)\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right]$$



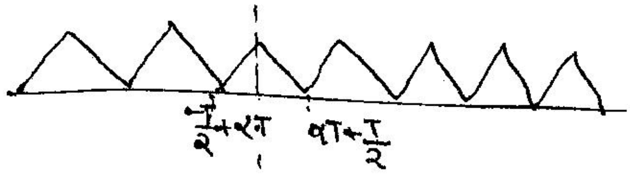
$x(t)$



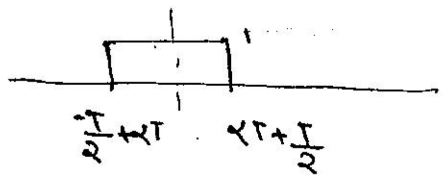
$u(t+\frac{T}{2}) - u(t-\frac{T}{2})$

For $\alpha > 1$:-

$w(t - k\alpha T)$

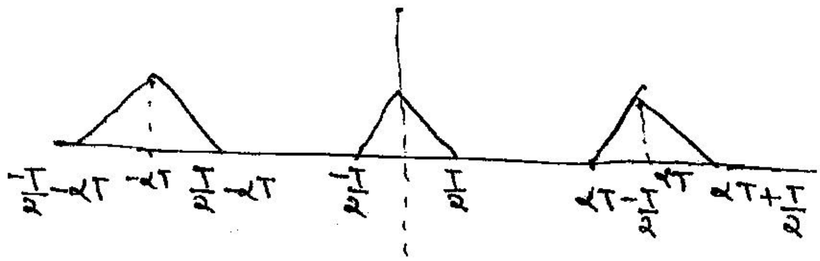


$x(t-\alpha T)$



$u(t-\alpha T+\frac{T}{2}) - u(t-\alpha T-\frac{T}{2})$

$\sum_{k=-\infty}^{\infty} w(t-k\alpha T)$



Period of such signal αT

$$Y_k = \frac{1}{\alpha T}\int_{\alpha T} w(t)\, e^{-\frac{jk2\pi t}{T\alpha}}\, dt$$

$$= \frac{1}{\alpha T}\int_{-\frac{T}{2}}^{T/2} x(t)\, e^{-\frac{jk2\pi t}{T\alpha}}\, dt$$

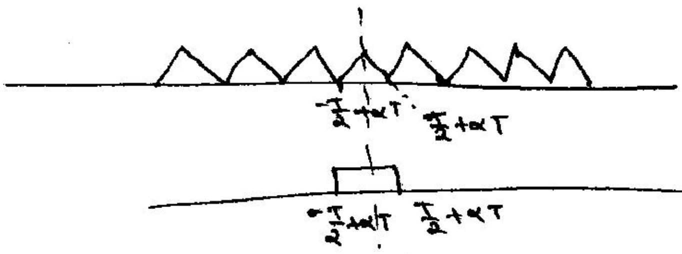
$$\boxed{Y_k = \frac{1}{\alpha T}\int_{-T/2}^{T/2} x(t)\, e^{-\left(\frac{k}{\alpha}\right) j\frac{2\pi t}{T}}\, dt}$$

Ans.

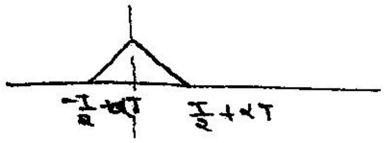
$$y_k = \frac{1}{\alpha} x\left(\frac{k}{\alpha}\right) \quad \text{if } \frac{k}{\alpha} \in Z$$

$$y_k = \frac{1}{\alpha T} \int_T x(t)\, e^{-\frac{j2\pi t k}{\alpha T}}\, dt \quad \text{else}$$

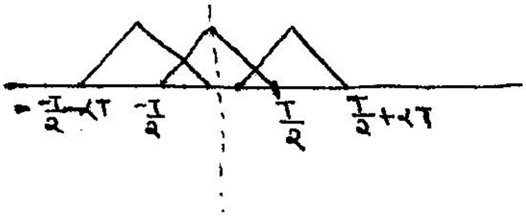
For $0<\alpha<1$



$\sum_{k=-\infty}^{\infty} w(t-k\alpha T)$



all shift of this



signal will have period αT

$$y_k = \frac{1}{\alpha T} \int_{\alpha T/2}^{\alpha T/2} \left\{ \sum_{k_1} w(t-k_1\alpha T) \right\} e^{-\frac{j2\pi t k}{\alpha T}}\, dt$$

$$m = \left\lfloor \frac{1}{\alpha} \right\rfloor$$

$$\boxed{y_k = \frac{1}{\alpha T} \int_{\alpha T/2}^{\alpha T/2} \sum_{l=-m}^{m} w(t-l\alpha T)\, e^{-\frac{j2\pi t k}{\alpha T}}\, dt}$$

using shift property for Fourier series

$$x(t-t_0) \longrightarrow a_k e^{-j\frac{k 2\pi}{T} t_0}$$

If for $\alpha > 1$ coeff. are Y_k', then $\{ Y_k'$ has fundamental in Period $(m+1)\alpha T \}$

$$Y_k = \sum_{l=0}^{m} a_k e^{-\frac{j 2\pi k}{T} l \alpha T} Y_k'$$

Ans. $\boxed{Y_k = \sum_{l=0}^{m} Y_k' e^{-j 2\pi k l \alpha} \qquad m = \left\lfloor \frac{1}{\alpha} \right\rfloor}$

Assignment - 7

5: Duality proof:

given $f(t) \longleftrightarrow F(\omega)$

we have to find $f'(t)$ that is the inverse FT of $f(\omega)$ & also FT $F(t)$ of $F(\omega)$

(A) Fourier transform of $f(t) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \rightarrow (1)$

$$IFT\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \rightarrow (2)$$

$$f(t) \longleftrightarrow F(\omega)$$

$$? \longleftrightarrow f(\omega)$$

$$IFT\{f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{j\omega t}\, d\omega = f'(t)$$

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega$$

$\cancel{= F(t)}$.

By interchanging the role of t and ω in above equation and comparing the result with the (1) eq, we obtain

$$FT\{F(t)\} = 2\pi f(-\omega)$$

$$\frac{1}{2\pi} FT\{F(t)\} = f(-\omega).$$

$$IFT\{f(-\omega)\} = \frac{F(t)}{2\pi}$$

$$IFT\{f(\omega)\} = \frac{F(-t)}{2\pi} = f'(t).$$

(11) To find the fourier Transform of $F(\omega)$.

$$F'(t) = FT\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging t & ω

$$= \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} F(t)\, e^{j(-\omega)t}\, dt$$

$\rightarrow$ by (2)

$$= 2\pi f(-\omega)$$

$$\begin{tikzcd}
\frac{F(-\omega)}{2\pi} \arrow[d, "IFT"'] & f(t) \arrow[l, "IFT"'] \arrow[r, "FT"] & F(\omega) \arrow[r, "FT"] & 2\pi f(-t) \arrow[d, "FT"] \\
f(-t)/2\pi \arrow[d, "IFT"'] & & & 2\pi F(-\omega) \arrow[d, "FT"] \\
F(w)/4\pi^2 & & & \pi^2 f(t)
\end{tikzcd}$$

6. (a) Fourier transform of unit step $u(t)$:

Even decomposition of $x(t) = \frac{x(t) - x(-t)}{2}$.

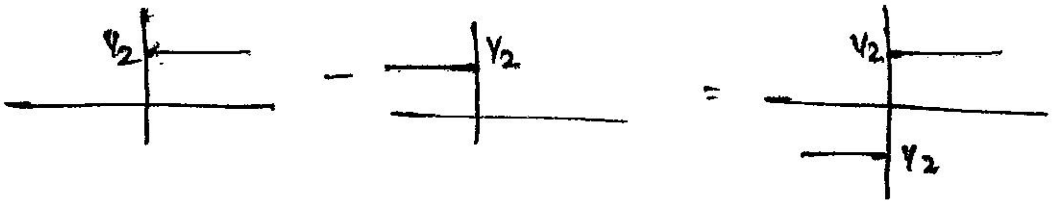
odd decomposition of $x(t) = \frac{x(t) + x(-t)}{2}$.

$\therefore$ $\text{Even}\{u(t)\} = \frac{u(t) + u(-t)}{2}$

$\text{odd}\{u(t)\} = \frac{u(t) - u(-t)}{2}$.

$u(t) = \text{Even}\{u(t)\} + \text{odd}\{u(t)\}$

(i) FT { odd part } in t/i $= FT\left\{ \frac{u(t) - u(-t)}{2} \right\}$



$= \frac{1}{2} \text{sgn}(t)$.

$FT\{\text{odd}\{u(t)\}\} = FT\{\frac{1}{2} \text{sgn}(t)\}$

$= \frac{1}{2} \times \frac{2}{j\omega} = \frac{1}{j\omega}$

(ii) $FT\{\text{even}\{u(t)\}\} = FT\left\{ \frac{u(t) + u(-t)}{2} \right\}$

$= FT\{\frac{1}{2}\}$

$\delta(t) \longleftrightarrow 1$

$1 \longleftrightarrow 2\pi \delta(\omega)$.

$\therefore$ $FT\{\text{even}\{u(t)\}\} = FT\{1/2\}$

$= \frac{1}{2} \times 2\pi \delta(\omega)$

$= \pi \delta(\omega)$.

$\therefore$ $FT\{u(t)\} = FT\{\text{even part}\} + FT\{\text{odd part}\}$

$= \frac{1}{j\omega} + \pi \delta(\omega)$

(b)

if i/p is $\delta(t)$ - Impulse then o/p is called impulse response $h(t)$.
if i/p to the system is $\delta(t)$ - unit step then o/p is called step response $s(t)$.

$$y(t) = x(t) * h(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

Similarly $$s(t) = \int_{-\infty}^{t} h(t)\, dt$$

(c) FT of $\int_{-\infty}^{t} x(t')\, dt'$

$$FT\{x(t) * u(t)\}$$

$$= X(\omega) \cdot U(\omega)$$

$$= X(\omega) \cdot \left[\frac{1}{j\omega} + \pi \delta(\omega) \right]$$

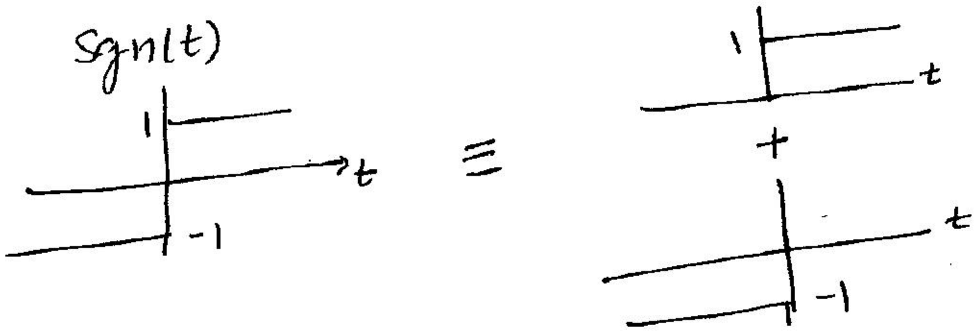
$$= \frac{X(\omega)}{j\omega} + \pi \delta(\omega) X(\omega)$$

$$= \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$$

Q 7:

(a) $u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$

sgn(t)



$$\mathrm{sgn}(t) = u(t) - u(-t)$$

$$FT\{\mathrm{sgn}(t)\} = U(\omega) - U(-\omega)$$

$$= \pi\delta(\omega) + \frac{1}{j\omega} - \left(\pi\delta(-\omega) - \frac{1}{j\omega}\right)$$

$$= \frac{2}{j\omega}$$

hence $\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$.

(b) $x(t) \longleftrightarrow X(\omega)$

$\frac{1}{\pi t} \longleftrightarrow ?$

using Duality. If $\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$

then $\frac{2}{jt} \longleftrightarrow 2\pi\,\mathrm{sgn}(-\omega)$

$\frac{1}{\pi t} \longleftrightarrow -j\,\mathrm{sgn}(\omega)$

By using property of convolution in Time domain

$$x(t) * \frac{1}{\pi t} \longleftrightarrow X(\omega)\left(-j\,\mathrm{sgn}(\omega)\right)$$

$$x(t) * \frac{1}{\pi t} * \frac{1}{\pi t} \longleftrightarrow X(\omega)\left(-j\,\mathrm{sgn}(\omega)\right)\left(-j\,\mathrm{sgn}(\omega)\right)$$

$$= -X(\omega)$$

8) $x(t) \rightarrow y(t)$, $x(-t) \longrightarrow y(-t)$

$x(t) = x_e(t) + x_o(t)$

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

Where, $x_e(t) = \frac{x(t) + x(-t)}{2}$ & $x_o(t) = \frac{x(t) - x(-t)}{2}$

Since, the system $x(t) \longrightarrow y(t)$ is linear. with the property $x(-t) \rightarrow y(-t)$.

also, $$y(t) = \underbrace{\frac{y(t) + y(-t)}{2}}_{y_e(t)} + \underbrace{\frac{y(t) - y(-t)}{2}}_{y_o(t)}$$

Since, $y_e(t) = y_e(-t)$ & $y_o(t) = -y_o(-t)$

now, $x(t) = x_e(t) + x_o(t) \longrightarrow y(t) = y_e(t) + y_o(t)$ iff it satisfies the ~~princip~~ properties of linearity, i.e, Superposition and homogeneousness.

let $$x'(t) = \alpha x_e(t) + \beta x_o(t)$$

$$= \alpha\left(\frac{x(t) + x(-t)}{2}\right) + \beta\left(\frac{x(t) - x(-t)}{2}\right)$$

$$= \left(\frac{\alpha+\beta}{2}\right)x(t) + \left(\frac{\alpha-\beta}{2}\right)x(-t)$$

$$x'(t) \rightarrow y'(t) = \left(\frac{\alpha+\beta}{2}\right)x(t) + \left(\frac{\alpha-\beta}{2}\right)x(-t) \rightarrow$$

$$\left(\frac{\alpha+\beta}{2}\right)y(t) + \left(\frac{\alpha-\beta}{2}\right)y(-t)$$

or, $$\alpha\left(\frac{y(t) + y(-t)}{2}\right) + \beta\left(\frac{y(t) - y(-t)}{2}\right)$$

$$= \alpha y_e(t) + \beta y_o(t)$$

$x(t) \rightarrow y(t)$, $x(-t) \rightarrow y(-t)$

$$x(t) = x_e(t) + x_o(t)$$

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

Where, $x_e(t) = \frac{x(t) + x(-t)}{2}$ & $x_o(t) = \frac{x(t) - x(-t)}{2}$

Since, the system $x(t) \rightarrow y(t)$ is linear. with the property $x(-t) \rightarrow y(-t)$.

also, $$y(t) = \underbrace{\frac{y(t) + y(-t)}{2}}_{y_e(t)} + \underbrace{\frac{y(t) - y(-t)}{2}}_{y_o(t)}$$

Since, $y_e(t) = y_e(-t)$ & $y_o(t) = -y_o(-t)$

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let $$x'(t) = \alpha x_e(t) + \beta x_o(t)$$

$$= \alpha\left(\frac{x(t) + x(-t)}{2}\right) + \beta\left(\frac{x(t) - x(-t)}{2}\right)$$

$$= \left(\frac{\alpha+\beta}{2}\right)x(t) + \left(\frac{\alpha-\beta}{2}\right)x(-t)$$

$$x'(t) \rightarrow y'(t) = \left(\frac{\alpha+\beta}{2}\right)x(t) + \left(\frac{\alpha-\beta}{2}\right)x(-t) \rightarrow$$

$$\left(\frac{\alpha+\beta}{2}\right)y(t) + \left(\frac{\alpha-\beta}{2}\right)y(-t)$$

or, $$\alpha\left(\frac{y(t) + y(-t)}{2}\right) + \beta\left(\frac{y(t) - y(-t)}{2}\right)$$

$$= \alpha y_e(t) + \beta y_o(t)$$