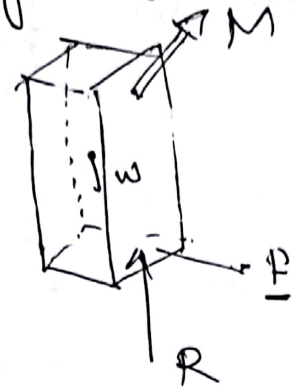
ESO204A Dynamics (Newton-Euler mechanics)

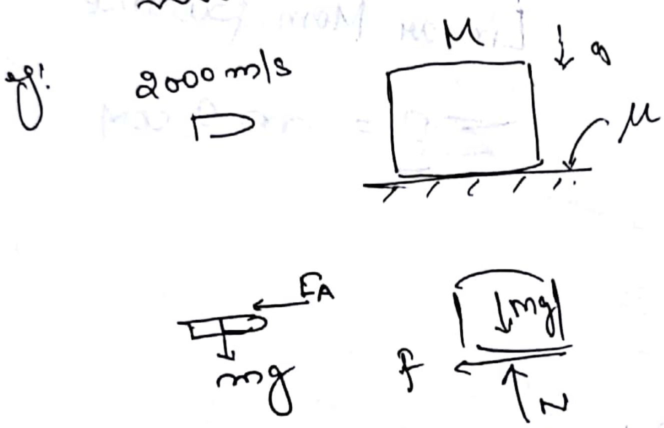
Free body Diagrams



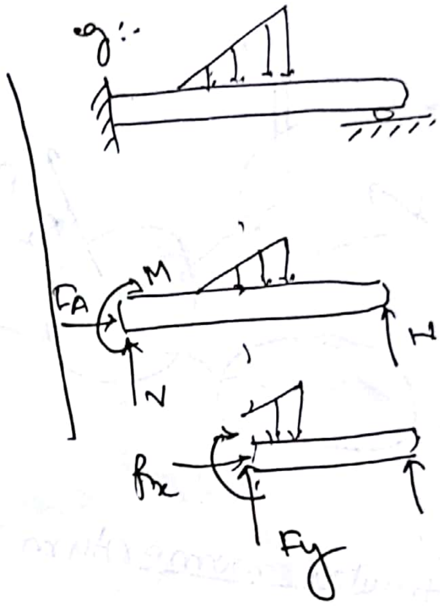
Internal forces & moments do not appear in FBD.

Linear momentum balance

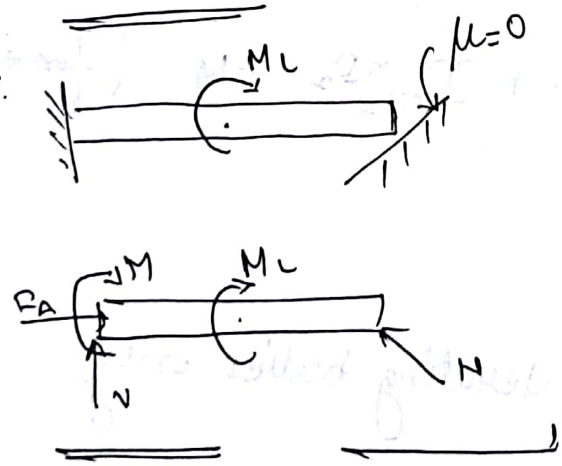
e.g.

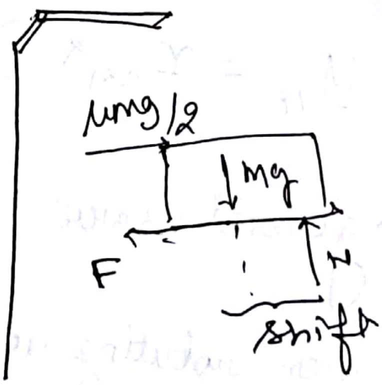


e.g.:

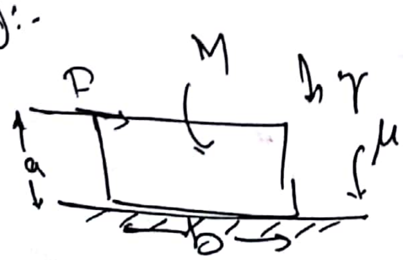


e.g.

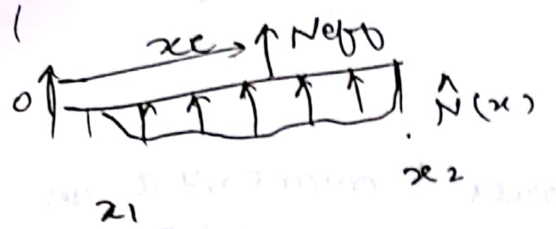




e.g.:



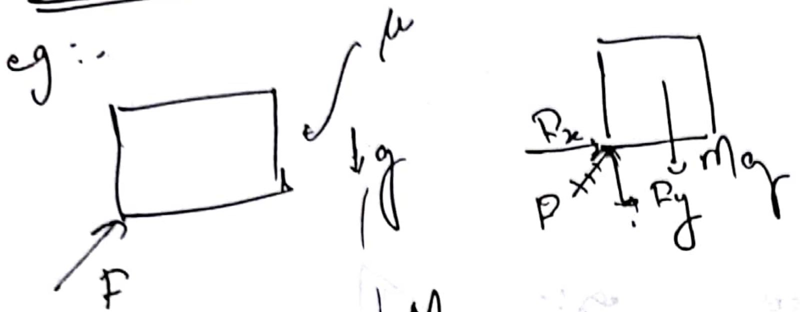
$$F = \frac{\mu mg}{2}$$

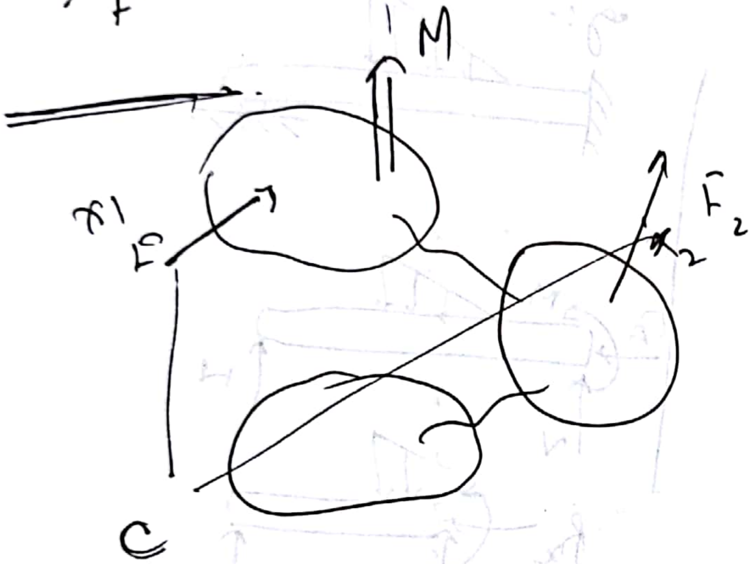


$$N_{eff} = \int_{x_1}^{x_2} \hat{N}(x)\,dx$$

$$x_c N_{eff} = \int_{x_1}^{x_2} x \hat{N}(x)\,dx$$

eg :-





Linear Mom. Balance

$$\Sigma \underline{F} = m \underline{a}_{com}$$

Angular momentum Balance

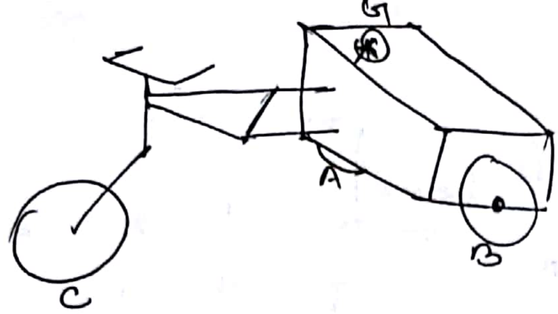
$$\underline{\dot{H}}_{/c} = \underline{M}_{/c} = \underline{r}_{cx_1} \times \underline{F}_1 + \underline{r}_{cx_2} \times f_2 + \underline{M} \quad (\text{for eg})$$

Finding $H_{/c}$ in general : later

for non-rotating accelerating bodies only

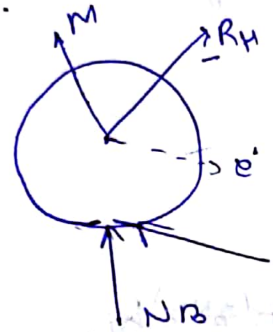
$$\underline{H}_{/c} = \sum_i \underline{r}_{c/cm_i} \times m_i a_{cm_i}$$

eg:

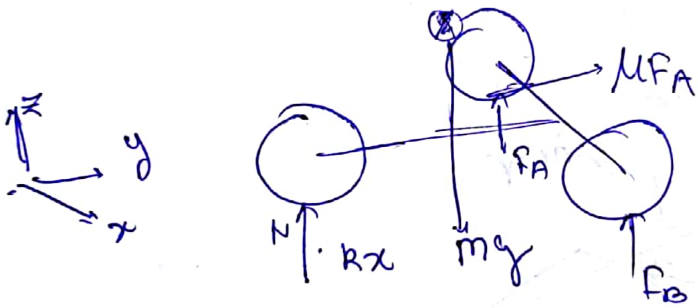


moving forward
handle straight
wheels massless
μ, g

moving forward 'A' suddenly locked 'B' & 'C' rotating free & do not slip. what is the friction force ground on wheel B?

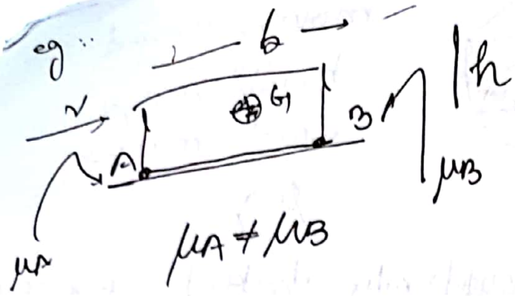


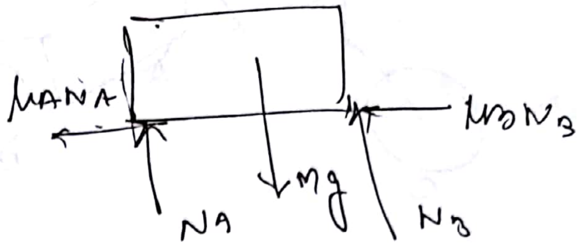
$M \cdot \hat{e}' = 0$ (frictionless bearing)



$A \times B \times C = B(A \cdot C) - C(A \cdot B)$

eg:





$$ma_x = -(\mu_A N_A + \mu_B N_B)$$

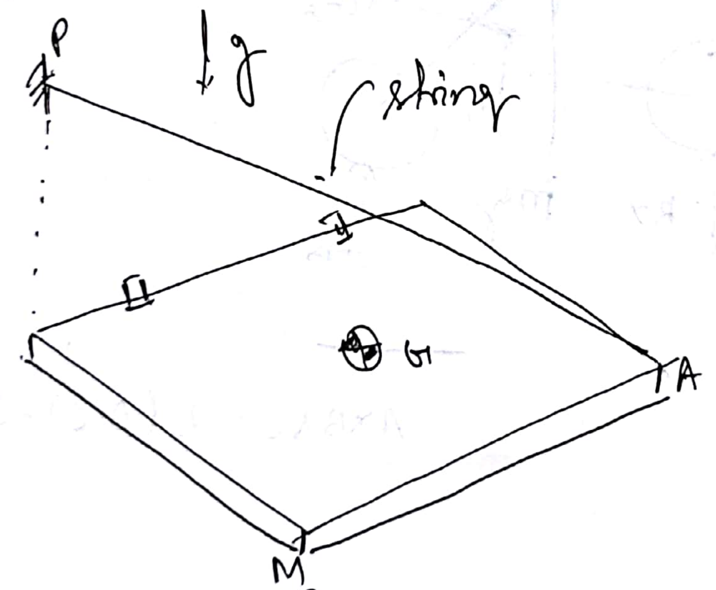
$$ma_y = 0 = -mg + N_A + N_B$$

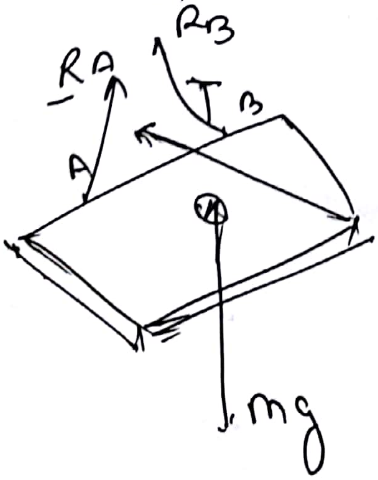
AMB

$$\sum M_{/G} = 0$$

$$N_A \, b/2 - N_B \, b/2 + (\mu_A N_A + \mu_B N_B) \, b/2$$

eg:





$$= \underline{T} = T\hat{e}_{AP}$$

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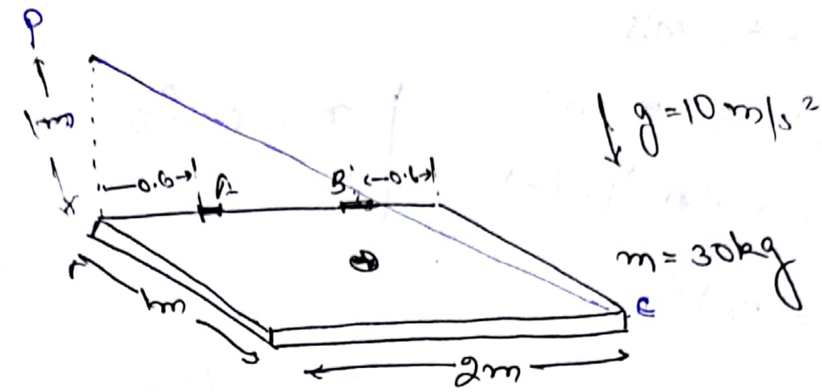
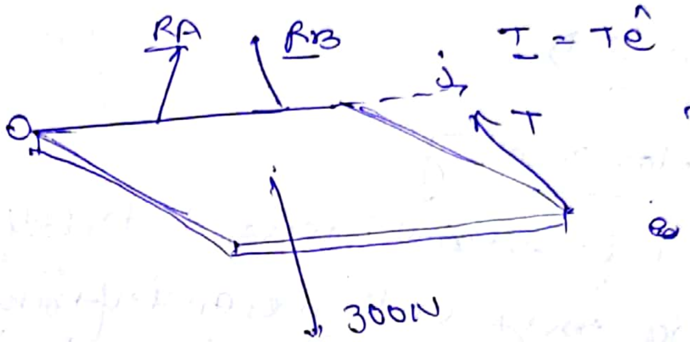
eg:-



$\hat{a}_{CP} = \dfrac{-\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{1+4+1}}$

$T = T\hat{e}$

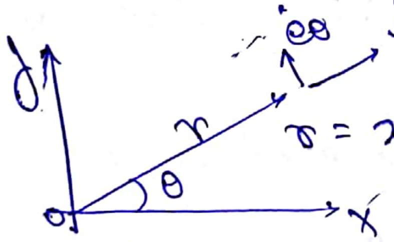
$r_{CP} = -\hat{i} - 2\hat{j} + \hat{k}$



$$(\Sigma \underline{M}/_O)\cdot\hat{j} = 0 \Rightarrow \left[\left(\tfrac{1}{2}\hat{i} + \hat{j}\right) \times (-300\hat{k}) + (\hat{i} + 2\hat{j}) \times \frac{T}{\sqrt{6}}(-\hat{i} - 2\hat{j} + \hat{k})\right]\cdot\hat{j}$$

- Planar Motion :-

$v = \dot{r}, \quad a = \dot{v}$



$r = x\hat{i} + y\hat{j}$

$\boxed{r = r\hat{e}_r}$

$v = \dot{r} = \dot{r}\hat{e} + r\dot{\hat{e}}_r$

$\boxed{v = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}$

$\dot{v} = a = \ddot{r}\hat{e}(r) = \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$

$$\boxed{a = (\ddot{r} - \dot{\theta}^2 r)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta}$$

$\ddot{r} = -4\hat{i} + 7\hat{j}$ (m/s²)

eg:.. $r = 4\hat{i} + 3\hat{j}$ (m)

$\dot{r} = 3\hat{i} + 4\hat{j}$ (m/s)

$|r| = r = 5$ (m)

$\theta = \tan^{-1}(3/4)$

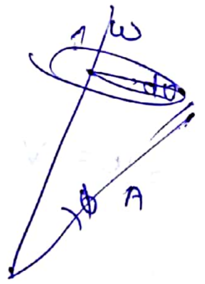
$|\dot{r}| = r|\dot{\theta}|$

$\dot{r} = r\dot{\theta}\hat{e}_\theta$

Motion in 3-D

→ Angular velocity

In an infinitesimal interval dt tentatively accept that the rigid body undergoes an infinitesimal rotation $d\theta$ about some axis $\hat{n}$ and we define ang. vel of the body to be $\omega = \frac{d\theta}{dt}\hat{n}$

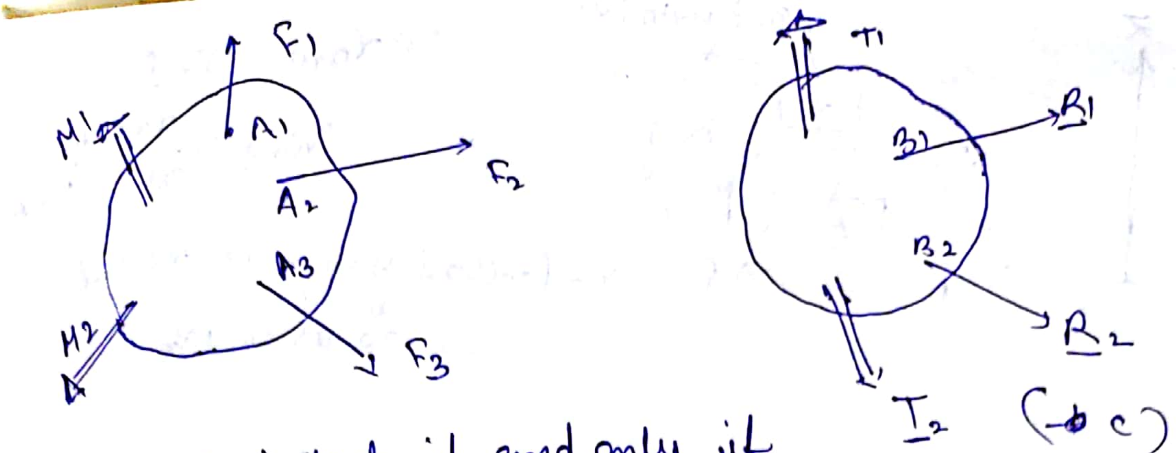


$|A|\sin\phi$

$$\frac{dA}{dt} = \frac{d\theta}{dt}\hat{n} \times A$$

$$\frac{dA}{dt} = \omega \times A$$

$$\left(\frac{dA}{dt}\right)_{XYZ} = \omega_{xyz/XYZ} \times A, \text{ for A fixed in xyz}$$

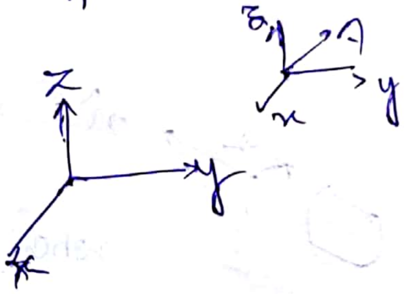


(+C). Equivalent if and only if

$$\sum_i F_i = \sum_j R_j$$

AND

$$\sum_k M_k + \sum_i r_{CA_i} \times F_i = \sum_n T_n + \sum_m r_{CB_m} \times R_m$$



Then
(lim dt → 0)

$$\omega = \frac{d\theta}{dt}\hat{n}$$

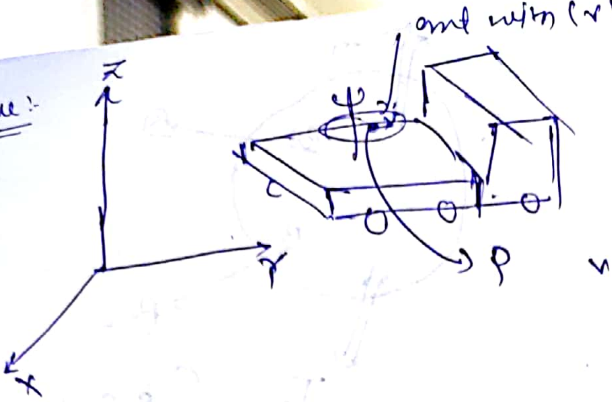
$\omega_{xyz/XYZ}$ (angular velocity)

$$\left(\frac{dA}{dt}\right)_{XYZ} = \omega_{xyz/XYZ} \times A$$

If A is not fixed in xyz

$$\left(\frac{dA}{dt}\right)_{XYZ} = \left(\frac{dA}{dt}\right)_{xyz} + \omega_{xyz/XYZ} \times (A)$$

Que :-



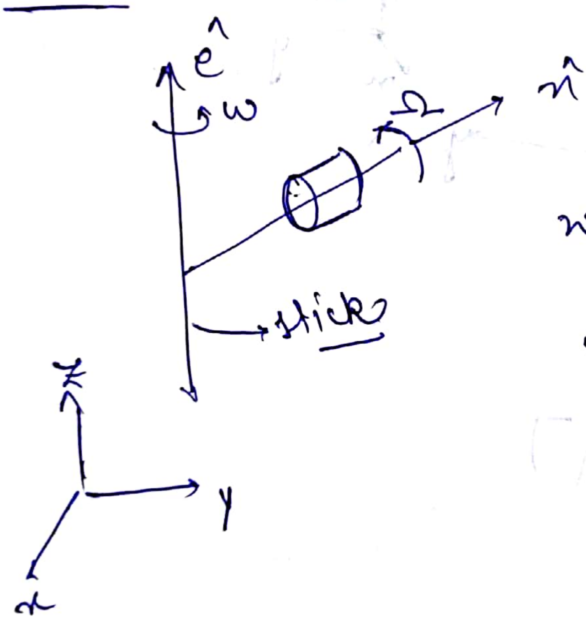
$r_{ant} = r + \rho$

$V_{ant} = \left(\dfrac{d r_{ant}}{dt}\right)_{XYZ}$

v = (velocity of ant ~~out~~ rel. to turntable)

$$= \left(\frac{dr}{dt}\right)_{XYZ} + \left(\frac{d\rho}{dt}\right)_{XYZ} \cancel{\left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho}$$

$$\Rightarrow \left(\frac{dr}{dt}\right)_{XYZ} + \left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho_{/XYZ}$$



what is $\omega_{can}|_{XYZ}$

$\omega_{can}|_{stick} = -\Omega \hat{n}$

$\omega_{stick}|_{XYZ} = \omega \hat{e}$

Let A be any vector fixed in the can

then $\left(\dfrac{dA}{dt}\right)_{XYZ} = \omega_{CAN}|_{XYZ} \times A$

Also

$$\left(\frac{dA}{dt}\right)_{XYZ} = \left(\frac{dA}{dt}\right)_{stick} + \omega_{stick}/_{XYZ} \times A$$

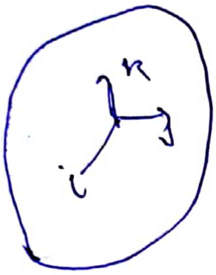
$$\left(\frac{dA}{dt}\right)_{XYZ} = \cancel{\left(\frac{dA}{dt}\right)} \; \omega_{can/stick} \times A + \omega_{stick/XYZ} \times A$$

$$\omega_{can/XYZ} \times A = \left(\omega_{can/stick} + \omega_{stick/XYZ}\right) \times A$$

Because A is arbitrary

$$\omega_{CAN/XYZ} = \omega_{CAN/stick} + \omega_{stick/XYZ}$$

Illustration



Let $\hat{i}, \hat{j}, k$ be RHON unit vectors embedded in the body

$$\omega = \omega_x \hat{i} + \omega_y \hat{k} + \omega_z \hat{k}$$

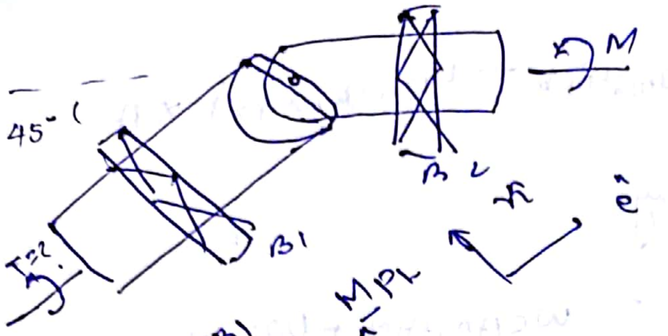
$$\left(\frac{d\hat{i}}{dt}\right)_{XYZ} = \omega \times \hat{i} = -\omega_y \hat{k} + \omega_z \hat{j}$$

$$\left(\frac{dj}{dt}\right) = \omega_x \hat{k} - \omega_z \hat{i}$$

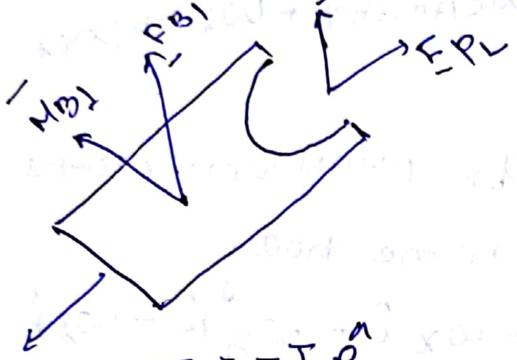
$$\left(\frac{d\hat{k}}{dt}\right) = -\omega_x j + \omega_y i$$

$$\omega = \left(\frac{d\hat{i}}{dt}\right)_{XYZ} \cdot j \cdot \hat{k} + \left(\frac{dj}{dt}\right)_{XYZ} \cdot \hat{k} \cdot i + \left(\frac{d\hat{i}}{dt}\right)_{XYZ} \cdot i, \hat{j}$$

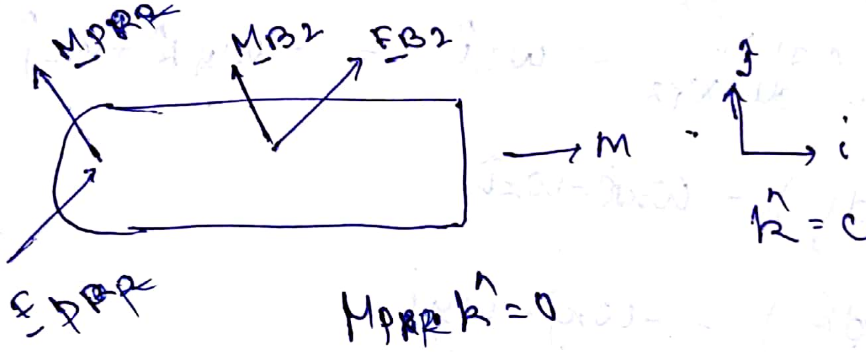
Universal Joint



$$\underline{M}_{B1} \cdot \hat{e} = 0$$
$$\underline{M}_{PL} \cdot \hat{n} = 0$$

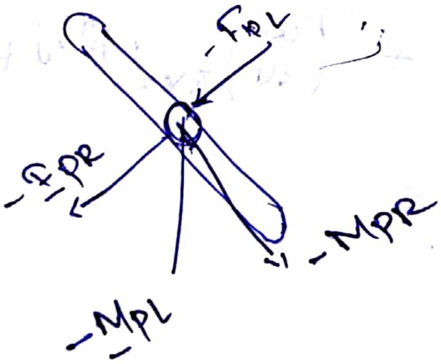


$$\underline{T} = -T\hat{e}$$



$\hat{k}$ = coming out

$$M_{PR} \cdot \hat{k} = 0$$
$$M_{B_2} \cdot \hat{i} = 0$$



$$\underline{M}_{PL} + \underline{M}_{PR} = 0$$
$$(\underline{M}_{PL} + \underline{M}_{PR}) \cdot \hat{k} = 0$$
$$\Rightarrow M_{PL} \cdot \hat{k} = 0$$

$$\Rightarrow \underline{M}_{PR} = M_{PL}\hat{e}$$

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$(\underline{T} + \underline{M}_{B1} + \underline{M}_{PL}) \cdot \hat{e} + 0 = 0$

$-T + M_{PL} = 0$

$\Rightarrow \boxed{M_{PL} = T}$ $\qquad \boxed{\underline{M}_{PR} = -T\hat{e}}$

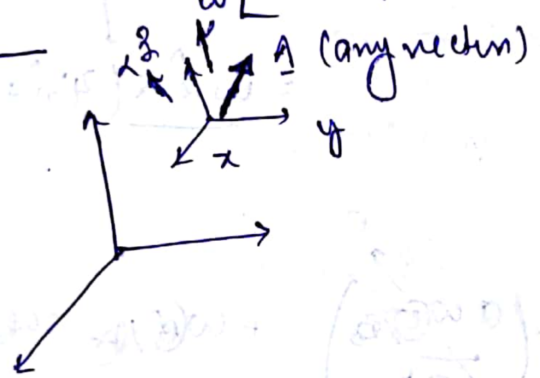
$\underline{M}_{PR} = -\underline{M}_{PL}$

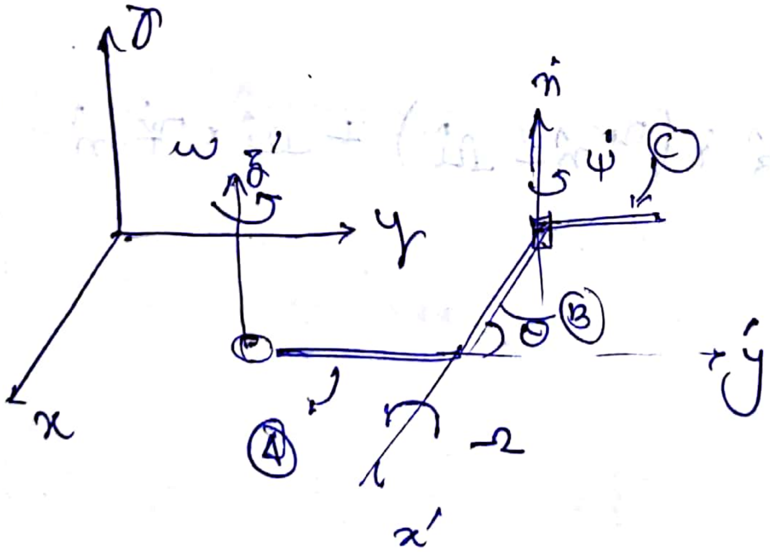
$(\underline{M}_{PR} + \underline{M}_{B2} + \underline{M}) \cdot \hat{i} + 0 = 0$

↑ Force

$-T\hat{e} \cdot \hat{i} + 0 + M = 0$

$\boxed{M = \frac{T}{\sqrt{2}}}$





$\omega, \Omega, \dot{\psi}$ are all constant.

Angular velocity and angular acceleration of ©?

$\omega_{C/B} = \dot{\psi}\hat{n}$; $\omega_{B/A} = \Omega\hat{i}$; $\omega_{A/xyz} = \omega\hat{k}$

$\omega_{C/xyz} = \omega_{A/xyz} + \omega_{C/A}$

$\left(\frac{d}{dt}\omega_{C/xyz}\right)_{xyz} = \alpha(C) = \cancel{\frac{d}{dt}(\omega_{A/xyz})_{xyz}} + \left(\frac{d}{dt}\omega_{C/A}\right)_{xyz}$

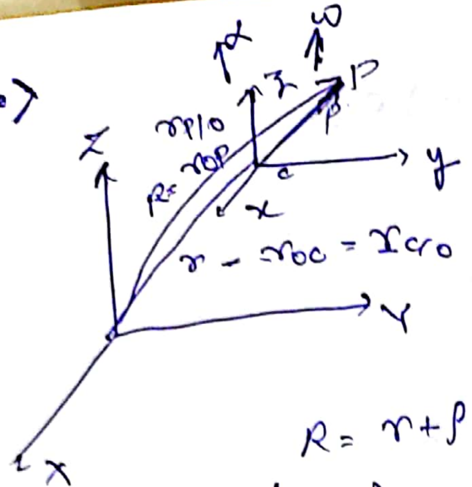
0 As ($\omega\hat{k}$) is const.

$\alpha_C = \left(\frac{d}{dt}\omega_{C/A}\right)_{xyz} = \left(\frac{d}{dt}\omega_{C/A}\right)_A + \underbrace{\omega_{A/xyz} \times \omega_{C/A}}_{= \omega\hat{k} \times (\dot{\psi}\hat{n} + \Omega\hat{i})}$

$\omega_{C/A} = \omega_{B/A} + \omega_{C/B}$

$\frac{d\,\omega_{C/A}}{dt} = \left(\cancel{\frac{d\,\omega_{B/A}}{dt}}\right)_A + \left(\cancel{\frac{d\,\omega_{C/B}}{dt}}\right)_B + \underbrace{\omega_{B/A} \times \omega_{C/B}}_{= \Omega\hat{i} \times \dot{\psi}\hat{n}}$

$\Rightarrow \alpha_{C/xyz} = \omega\hat{k} \times (\dot{\psi}\hat{n} + \Omega\hat{i}) + \Omega\hat{i} \times \dot{\psi}\hat{n}$



$$\left(\frac{dR}{dt}\right)_{XYZ}$$

$$\left(\frac{d^2R}{dt^2}\right)_{XYZ} = \left[\frac{d}{dt}\left(\frac{dR}{dt}\right)_{XYZ}\right]_{XYZ}$$

$$R = r + \rho$$

$$\rightarrow \left(\frac{dR}{dt}\right)_{XYZ} = \left(\frac{dr}{dt}\right)_{XYZ} + \left(\frac{d\rho}{dt}\right)_{XYZ} + \omega \times \rho$$

$$\rightarrow \left(\frac{d^2R}{dt^2}\right)_{XYZ} = \left(\frac{d^2r}{dt^2}\right)_{XYZ} + \left(\frac{d^2\rho}{dt^2}\right)_{xyz} + \omega \times \left(\frac{d\rho}{dt}\right)_{xyz} + \left(\frac{d}{dt}(\omega \times \rho)\right)_{XYZ}$$

$$= \left(\frac{d^2r}{dt^2}\right)_{XYZ} + \left(\frac{d^2\rho}{dt^2}\right)_{xyz} + \omega \times \left(\frac{d\rho}{dt}\right)_{xyz}$$

$$+ \left(\frac{d\omega}{dt}\right)_{XYZ} \times \rho + \underbrace{\omega \times \left(\frac{d\rho}{dt}\right)_{XYZ}}_{\omega \times \left[\left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho\right]}$$

• 5-term Acceleration Formula

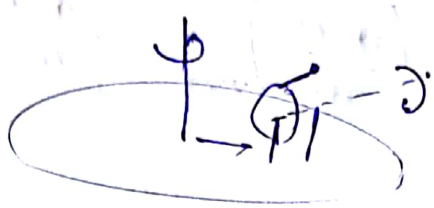
$$\left(\frac{d^2R}{dt^2}\right)_{XYZ} = \left(\frac{d^2r}{dt^2}\right)_{XYZ} + \left(\frac{d^2\rho}{dt^2}\right)_{xyz} + 2\omega \times \left(\frac{d\rho}{dt}\right)_{xyz} + \alpha \times \rho + \omega \times (\omega \times \rho)$$

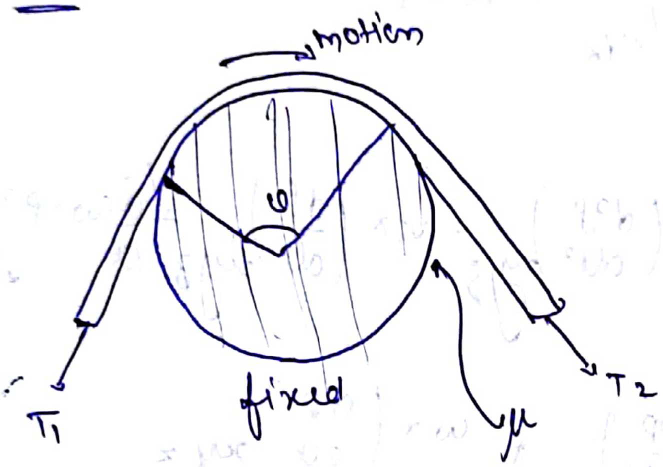
$$\left[\ddot{R} = \ddot{r} + \ddot{\rho} + \underbrace{2\omega \times \rho'}_{\text{coriolis term}} + \alpha \times \rho + \underbrace{\omega \times \omega \times \rho}_{\text{centripetal}}\right]$$

Recall

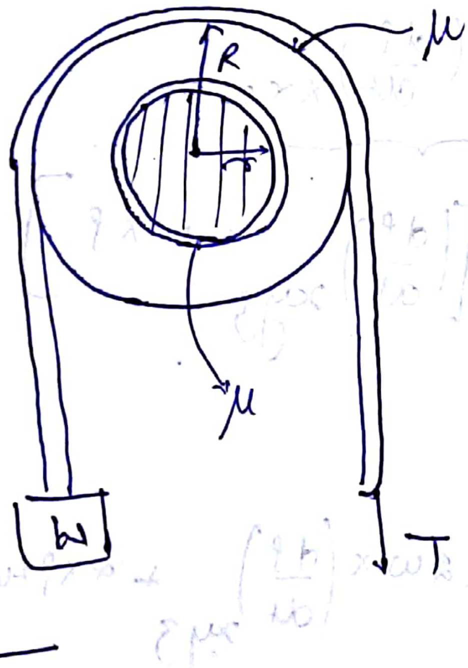
$$(\ddot{r} - \dot{\theta}^2 r)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

Que



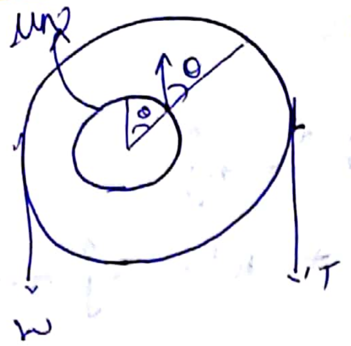


$$\frac{T_2}{T} = e^{\mu\theta}$$



If rope slips and pulley sticks

$$\boxed{T = We^{\mu x}}$$

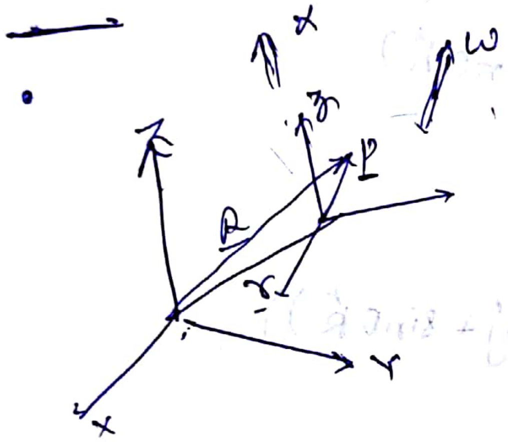


$\theta = \tan^{-1}\mu$

$$T(R - r\sin(\tan^{-1}\mu)) = W(R + r\sin\tan^{-1}\mu)$$

$$T = \frac{W(R + r\sin\tan^{-1}\mu)}{(R - r\sin\tan^{-1}\mu)}$$

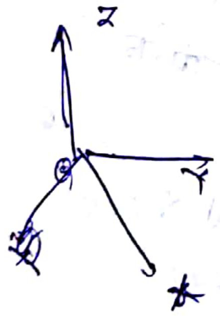
$$\boxed{\xrightarrow{\mu\to\infty} T = \frac{W(R+r)}{(R-r)}}$$

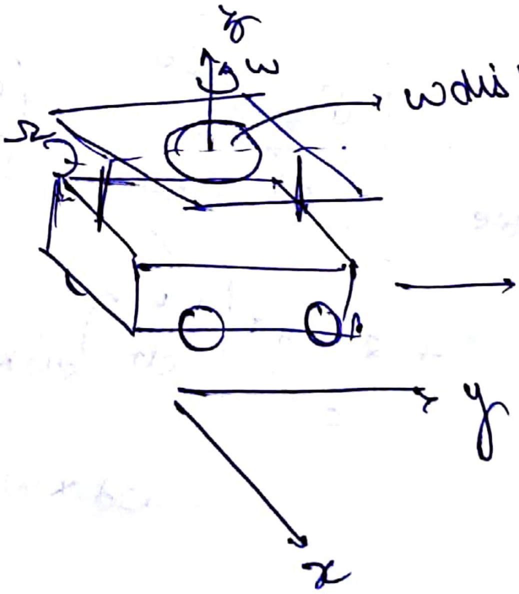


$$\ddot{R} = \ddot{r} + \ddot{\rho}' + \overbrace{2\omega\times\dot{\rho}'}^{\text{coriolis}} + \alpha\times\rho$$

$$\dot{R} = \dot{r} + \dot{\rho} + \omega\times\rho$$

$$+ \underbrace{\omega\times\omega\times\rho}_{\text{centripetal}}$$





$\omega_{disk} = \omega\hat{k} - \Omega\hat{j}$

straight line motion

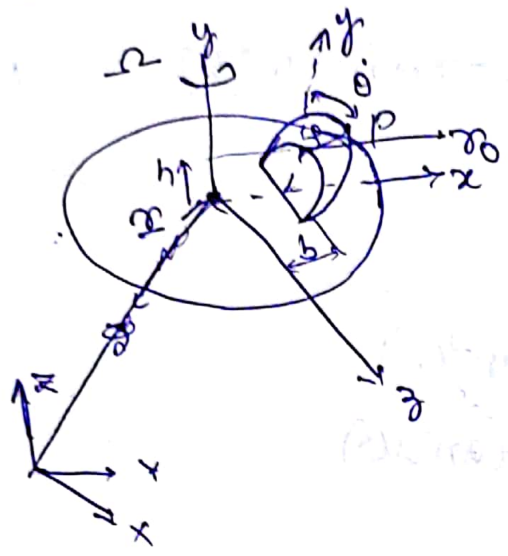
$\alpha_{disk} = -\Omega\hat{j}\times\omega\hat{k}$

$= -\omega\Omega\hat{i}$

Qu:

$\Omega, \dot{\theta}$ const

6



$$\rho = b\hat{i} + h\hat{j} + r_0(\cos\theta\,\hat{j} + \sin\theta\,\hat{k})$$

$$\dot{\rho} = \dot{\theta}\hat{i} \times r_0(\cos\theta\,\hat{j} + \sin\theta\,\hat{k})$$

$$R = r + b\hat{i} + h\hat{j} + r_0(\cos\theta\,\hat{j} + \sin\theta\,\hat{k})$$

(1) Turntable

$$\rho = b\hat{i} + h\hat{j} + r_0(\cos\theta\,\hat{j} + \sin\theta\,\hat{k})$$

$$\omega = \Omega\hat{j} \qquad \alpha = 0$$

$$\ddot{\rho} = \dot{\theta}\hat{i} \times (\dot{\theta}\hat{i} \times r_0(\cos\theta\,\hat{j} + \sin\theta\,\hat{k}))$$

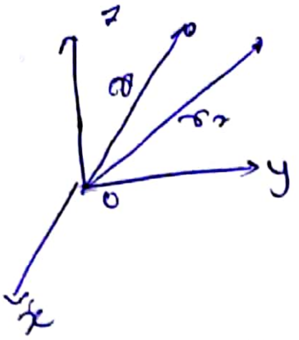
$$\Rightarrow \left(\frac{d^2 r_{op}}{dt^2}\right)_{table} = \cancel{\left(\frac{d^2 r_{oc}}{dt^2}\right)_{table}}^{0} + \cancel{\left(\frac{d^2 r_{cp}}{dt^2}\right)_{disk}}^{0}$$

$\dot{\theta}$ is const

$$+ 2\,\cancel{\omega_d \times \left(\frac{d\,r_{cp}}{dt}\right)_{disk}}_{0} + \cancel{\alpha_d \times r_{cp}}_{0}$$

$$+ \omega_d \times \omega_d \times \omega_{cp}$$

~: Kinetics :~



- Define the total mass (of system of particles)

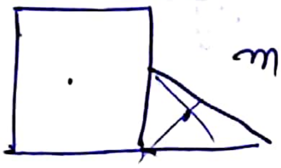
as $m_{total} = \sum_{i=1}^{N} m_i$

Define centre of mass

via, $m_{tot} \cdot r_{cM/o} = \sum_{i=1}^{N} m_i \underline{r_i}$

- $$\underline{r}_{cM} = r_{cM/o} = \frac{\sum_{i=1}^{N} m_i r_i}{\sum_{i=1}^{N} m_i}$$

eg:.



$m_1 r_{cM_1} + m_2 r_{cM_2} = (m_1+m_2) r_{cM}$

- $$\underline{V}_{cM} = \left(\frac{d}{dt}\underline{r}_{cM}\right)_{xyz \text{ (inertial normally)}} = \frac{\sum_i m_i \underline{v}_i}{\sum_i m_i}$$

- $m_{tot} \cdot V_{cm}$ is called the linear momentum of the sys

$$\frac{d}{dt} m_{tot} V_{cm} = \sum \underline{F}_{ext}$$

(Based on a Linear mom. bala

$m_{total} \cdot a_{com}$; $a_{cm} = \left(\frac{d}{dt} V_{cm}\right)_{xyz} = \frac{\sum m_i \ddot{\underline{r}}_i}{\sum m_i}$

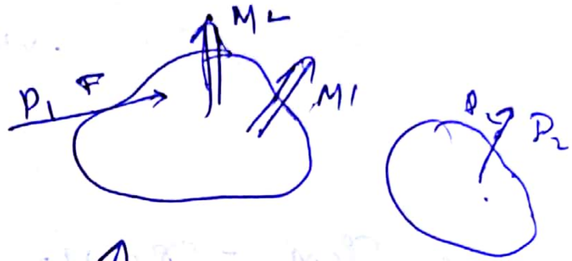
- Angular Momentum

Absolute Quantities only

$$\underline{H}_{/C} = \sum_{i=1}^{N} \underline{r}_{i/C} \times m_i \underline{v}_i$$

$$\mathring{H}_{/C} = \sum_i \cancel{v_i \times m_i v_i} + \sum r_i \times m_i \underline{a_i}$$

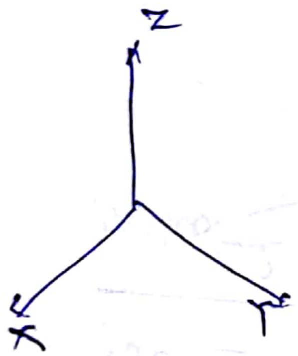
$\overset{\circ}{0}$



Angular Momentum Balance

$$\mathring{H}_{/C} = \sum_i r_{i/C} \times m_i \underline{a_i} = \sum_k r_{P_k/C} \times F_i + \sum_j M_j$$

- Let us chose and fix a coordinate system



$$\underline{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$\underline{a} \equiv \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \equiv a = [\underline{a}]$$

Linear transformation of vector a

$$\underline{a} \longrightarrow \underline{\underline{L}} \cdot \underline{a}$$

vector

tensor

3x3 matrix $\longrightarrow La \longrightarrow$ 3x1 matrix

Consider

$$\underline{a} \times \underline{b} \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_x \end{bmatrix}$$

skew symmetric matrix

$\underline{S}(\underline{a}) \equiv S(a)$

NOTE:

$$\sum m_i r_{cm} = \sum m_i r_i = \sum m_i (r_{i/cm}) + \sum m_i (r_{cm})$$

$$\dot{\underline{H}}_{/C} = \sum_i r_{i/C} \times m_i a_i$$

write $\underline{r}_{i/C} = \underline{r}_{i/CM} + \underline{r}_{CM/C}$

$$\underline{a}_{i/\text{ground}} = a_{i/C} = a_{i/CM} + a_{CM/C}$$

$$\rightarrow \dot{\underline{H}}_{/C} = \sum_i \underline{r}_{i/CM} \times m_i (a_{i/CM} + a_{CM/C}) + \sum_i r_{CM/C} \times m_i (a_{i/CM} + \underline{a}_{CM/C})$$

$$\rightarrow \dot{\underline{H}}_{/C} = \sum r_{i/CM} \times m_i \underline{a}_{i/CM} + \cancelto{0}{\sum_i \underline{r}_{i/CM} \times m_i a_{cm}} + \cancelto{0}{\sum_i r_{CM/C} \times m_i a_{i/CM}} + \underbrace{\sum_i r_{CM/C} \times m_i a_{CM}}_{r_{CM/C} \times M_{total}\, a_{com}}$$

→ For a rigid body

$$\underline{a_{i/CM}} = \alpha \times r_{i/CM} + \omega \times \omega \times r_{i/CM}$$

Take a step back to $\underline{H}_{i/C} = \sum r_{i/C} \times m_i \underline{v}_i$

• Special case of a single rigid body

$a_{i/CM} = \underline{\alpha} \times \underline{r}_{i/CM} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{i/CM}$

back to $H_{/C}$ for a system of particles

$$\sum_i r_{i/C} \times m_i \underline{v}_i = \sum_i r_{i/CM} \times m_i v_{i/CM} + \sum_i r_{CM/C} \times m_i v_{i/CM}$$
$$+ \sum_i r_{i/CM} m_i v_{CM} + \sum_i r_{CM/C} \times m_i v_{CM}$$

$$\boxed{\underline{H}_{/C} = \sum_i \underline{r}_{i/CM} \times m_i \underline{v}_{i/CM} + r_{CM/C} \times M_{tot} V_{CM}}$$

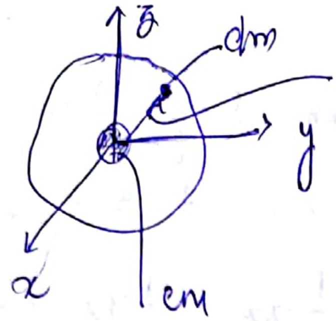
For a single rigid body, $v_{i/CM} = \omega \times r_{i/CM}$

Consider $\sum_i r_{i/CM} \times m_i (\omega \times \underline{r}_{i/CM})$

$$= \sum_i (-r_{i/CM} \times m_i (r_{i/CM} \times \underline{\omega}))$$

$$\equiv \sum_i -m_i S(r_{i/CM}) S(r_{i/CM}) \omega$$

$$= \underbrace{\sum_i m_i S(r_{i/CM})^T S(r_{i/CM})}_{[\underline{\underline{I}}_{CM}]} \omega$$



$$r_{i/cm} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad S(r) = S\left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}\right)$$

$$= \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$S^T S = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

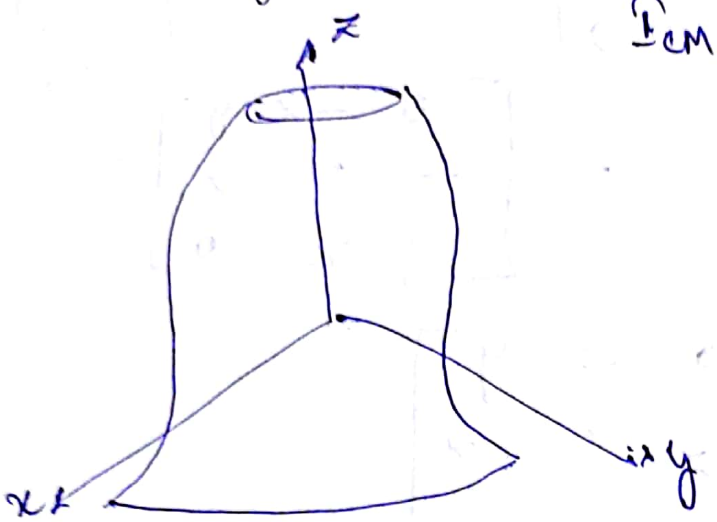
$$= \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

$$\underline{\underline{I}}_{cm} = \int_{\text{all mass}} \left[\quad \right] dm$$

$$H_{/c} = r_{cm/c} \times m_{tot} \underline{V}_{cm} + \underline{\underline{I}}_{cm} \cdot \underline{\omega}$$

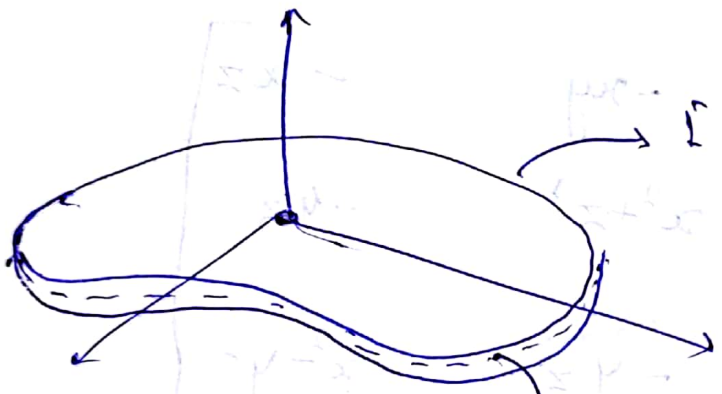
$$\dot{\underline{H}}_{/c} = r_{cm/c} \times m_{tot}\, \underline{a}_{cm} + \underline{\underline{I}}_{cm} \cdot \alpha + \omega \times \underline{\underline{I}}_{cm} \times \underline{\omega}$$

Axisymmetric Body



$$\underline{I}_{CM} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

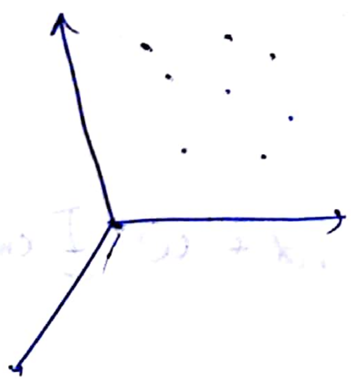
$I_{xy} = I_{xz} = I_{yz} = 0$



$I_{xz} = I_{yz} = 0$

symmetric plane

System of particles



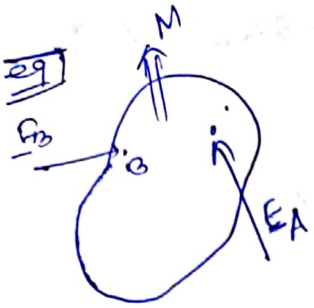
$\underline{H}_{/C} = \sum_i r_{i/C} \times m_i v_i$

$= r_{CM/C} \times m_i v_{cm} + \underbrace{\sum_i r_{i/cm} \times v_{i/cm}}_{\text{For a rigid body}}$

$= \underline{\underline{I}}_{cm} \omega$

$$[\underline{\underline{I}}_{cM}] = I_{cM} = \int_{Mass} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} dm$$

eg



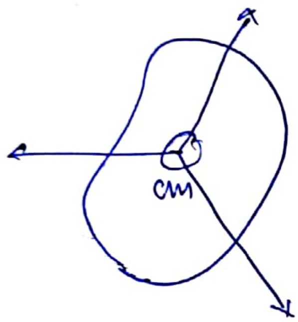
AMB for this rigid body is

$$\sum M_{/C} = r_{CB} \times F_B + r_{CA} \times F_A + M$$

$$= r_{cM/C} \times m a_{cM} + \underline{\underline{I}}_{cM} \cdot \alpha + \omega \times \underline{\underline{I}}_{cM} \cdot \omega$$

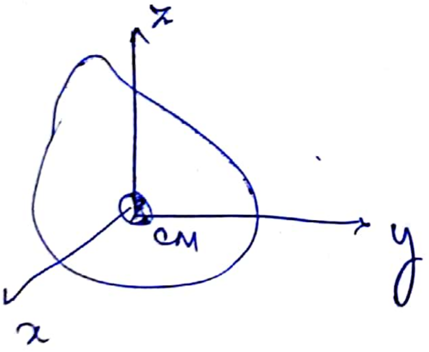
$\dot{c}$

→ The three eigen values of the moment of inertia matrix are called the principle.



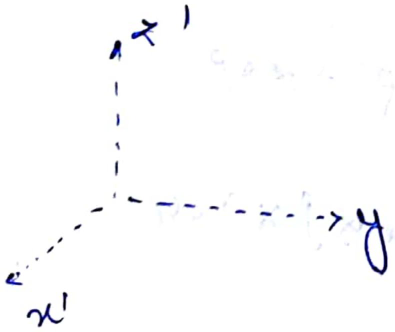
$$I_{cM} = \begin{bmatrix} \int (y^2+z^2) dm & 0 & 0 \\ 0 & \int (x^2+z^2) dm & 0 \\ 0 & 0 & \int (x^2+y^2) dm \end{bmatrix}$$

I_{11} $\quad$ I_{22} $\quad$ I_{33}



eg. $\int x'y' \, dm = \int (x_{cM}+x)(y_{cM}+y) dm$

$$= \int x_{cM} y_{cM} dm + \int x_{cM} \overset{0}{y} dm + \int \cancel{x} y_{cM} dm + \int xy \, dm$$



$$\left[I_{0,xy} = I_{cM,xy} - m_{tot} x_{cM} y_{cM} \right]$$

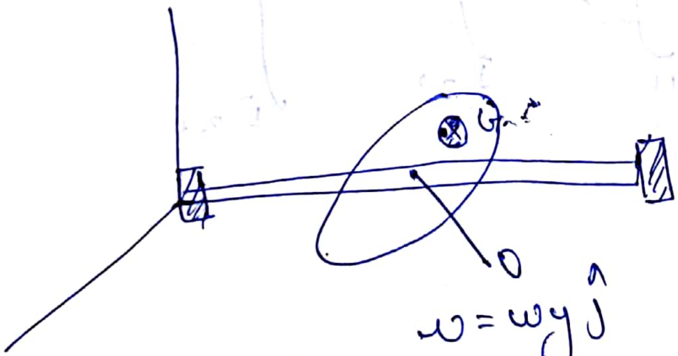
$$\int (y'^2 + z'^2)\, dm = \int \left[(y_{cm}+y)^2 + (z_{cm}+z)^2 \right] dm$$

$$= \int \left(y_{cm}^2 + 2y_{cm}y + y^2 + z_{cm}^2 + 2z_{cm}z + z^2 \right) dm$$

$$= m_{tot}\left(y_{cm}^2 + z_{cm}^2\right) + \underline{\underline{\int (y^2+z^2)\, dm}} \quad I_{xx}$$

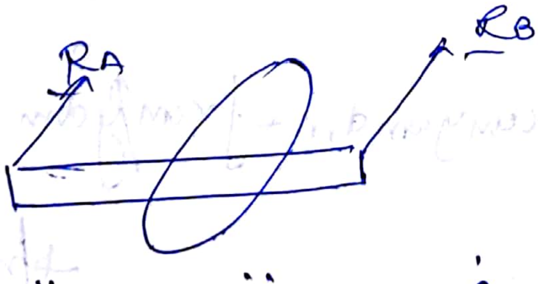
$$I_{O,xx} = I_{cm,xx} + m_{tot}\left(y_{cm}^2 + z_{cm}^2\right)$$

Principal Axis + Principle moment of inertia



$\omega = \omega_y \hat{j}$

$\alpha = \alpha \hat{j}$

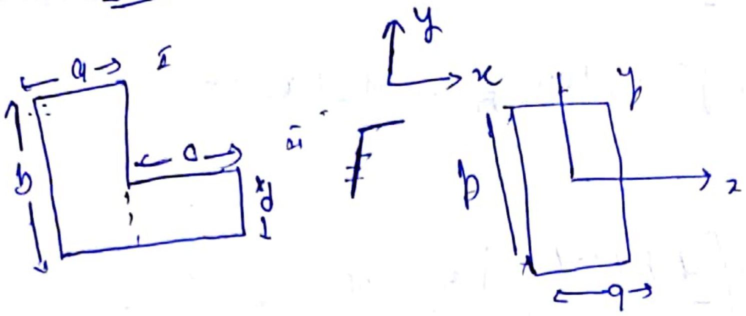


$$\ddot{R} = \ddot{r}' + \ddot{\rho} + 2\underline{\omega} \times \dot{\underline{\rho}} + \underline{\omega} \times \underline{\omega} \times \underline{\rho} + \alpha \times \rho$$

$$\underline{a}_G = \omega_y \hat{j} \times (\omega_y \hat{j}) \times \underline{r}_{OG} + \alpha \hat{j} \times r_{OG}$$

Tutorial

I cm for a thin sheet



$$I_{xx} = \int_{mass} (y^2+z^2)\, dm = \rho \int y^2\, dA$$

$$= \rho \int_{-p/2}^{+p/2} \int_{-q/2}^{+q/2} y^2\, dx\, dy$$

$$= \rho \int_{-p/2}^{p/2} \int \left[y^2 x \right]_{-q/2}^{q/2}$$

$$= \rho \left[\frac{y^3}{3} \right]_{-p/2}^{p/2} \left[x \right]_{-q/2}^{q/2} = \frac{\rho q p^3}{12}$$

$$I_{yy} = \frac{\rho p q^3}{12}$$

centre of mass ⇒ $ab\left(\frac{a}{2}\right) + cd\left(a + c/2\right) = (ab+cd)\, X_{cm}$

$ab\left(b/2\right) + cd\left(d/2\right) = (ab+cd)\, Y_{cm}$

$$(\vec{I}_{xx})_{cm} \quad [I_{xx}]_{I} + [I_{xx}]_{II}$$

$$= (I_{xx})_{cmI} + M_1 [\quad] + (I_{xx})_{cmI}$$

$$+ M_2 [\quad]$$

$$(I_{yy})_{cm} = [I_{yy}]_{I} + [I_{yy}]_{II}$$

$$= [I_{yy}]_{cm_I} + [x_1]^2 + (I_{yy})_{\circledast} + M_2[x_2]^2$$

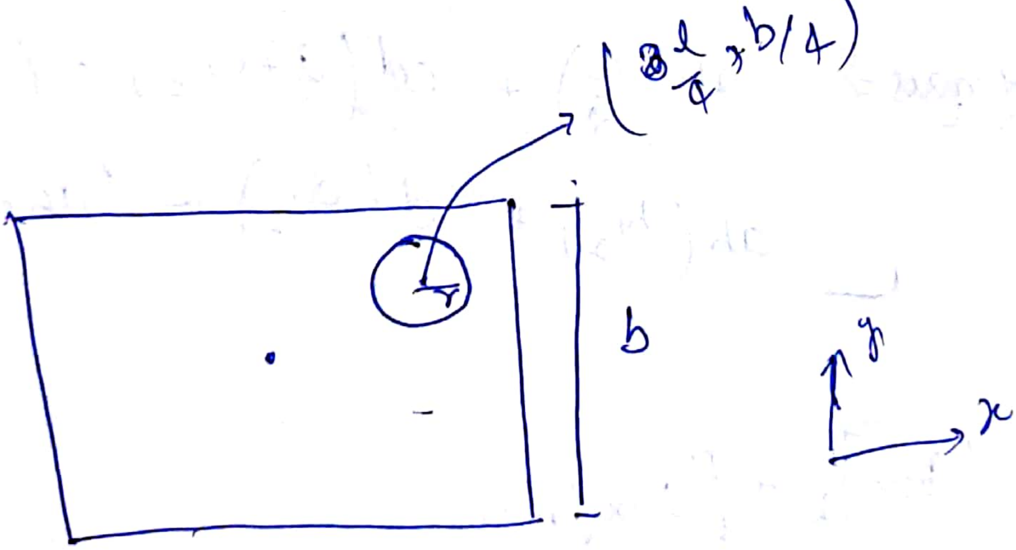
$$= (I_{yy})_{cm_I} + M_1\left[\frac{a^2b+c^2d}{2(ab+cd)}\right]^2 + M_2\left[\frac{ab(a+c)}{2(ab+cd)}\right]^2$$

$$\begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

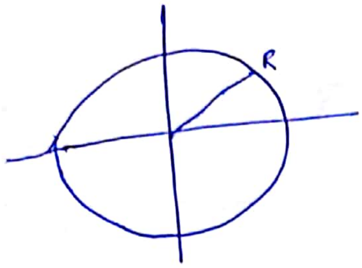
$$(I_{xy})_{cm} = (I_{xy})_{I} + (I_{xy})_{II}$$

$$= (I_{xy})_{cm_I} - M_1[x_1][y_1] + (I_{yy})_{cm_2} - M_2(x_2)(y_2)$$

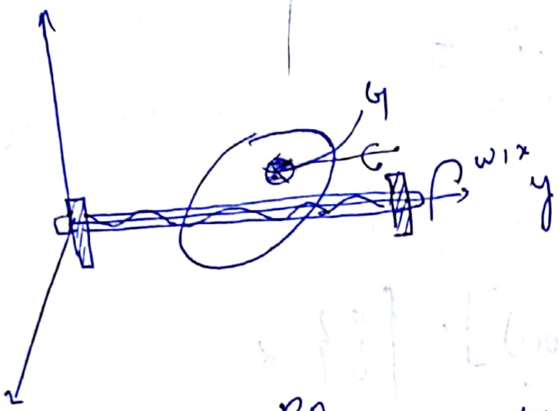
Que!



For a circle:



$I_{xx} =$



$\underline{\omega} = \omega \hat{j}$

$\underline{\alpha} = \alpha \hat{j}$

$a_G = \underline{\omega} \times \underline{\omega} \times \underline{r}_{AG} + \underline{\alpha} \times \underline{r}_{AG}$

$a_G = \underline{\omega} \times \underline{\omega} \times \underline{r}_{OG} + \underline{\alpha} \times \underline{r}_{OG}$

$\underbrace{\underline{R_A} + \underline{R_B}}_{} = m a_G$

non zero if $r_{OG} \neq 0$

$\Big[\underline{\dot{H}}_{/A} = \underline{r}_{AG} \times m a_G + \underline{\underline{I}}_{cm} \cdot \alpha + \underline{\omega} \times \underline{\underline{I}}_{cm} \underline{\omega}$

$= \underbrace{\underline{M_A} + \underline{M_B} + \underline{r}_{AB} \times \underline{R_B}}_{} \Big] \cdot j$

Angular momentum about rotor axis

$$-\Big[\underline{r}_{AO} \times m \big[\underbrace{\alpha \times \underline{r}_{OG}}_{\text{along } \hat{i}} + \underbrace{\omega \times \underline{\omega} \times \underline{r}_{OG}}_{r_O} \big]$$

$$+ \underline{r}_{OG} \times m \big[\alpha \times \underline{r}_{OG} + \underbrace{\underline{\omega} \times \underline{\omega}}_{r_O} \times \underline{r}_{OG} \big]$$

$$+ \underline{\underline{I}}_{CM} \cdot \alpha + \underbrace{\underline{\omega} \times \underline{\underline{I}}_{CM} \cdot \underline{\omega}}_{0} \Big] \cdot \hat{j}$$

For convenience assume r_{OG} is ~~always~~ along $\hat{k}$.

$$\left[\underline{\underline{I}}_{CM} \cdot \underline{\alpha} + \underline{r}_{OG} \times \underline{\alpha} \times \underline{r}_{OG} \right] \cdot \hat{j} = 0$$

In matrix notation

$$(0,1,0) \Big[\underbrace{I_{CM} + m S(r_{OG})^T S(r_{OG})}_{B} \Big] \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \cdot \alpha$$

$$B = B^T > 0$$

Using

$$u^T B u \geq 0$$

$$u^T B u = 0 \text{ only if } u = 0$$

$$\Rightarrow \boxed{\alpha = 0}$$

To find $\underline{R}_A$ & $\underline{R}_B$ (A good approx $M_A = M_B = 0$)

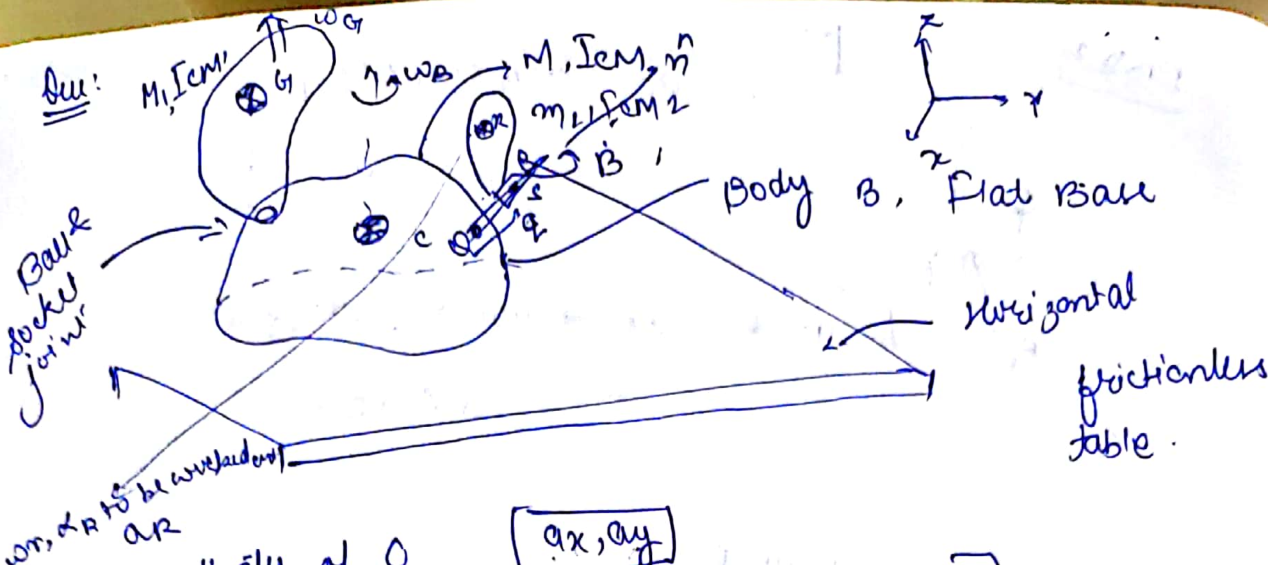
$$a_G = \underline{\omega} \times \underline{\omega} \times \underline{r}_{OG}$$

AMB $\quad (R_B \cdot \hat{j} = 0)$

$$\dot{H}_{/A} = \underline{r}_{AB} \times \underline{R}_B$$

$$= \underline{r}_{AG} \times m a_G + \underline{\omega} \times \underline{I}_{CM} \underline{\omega}$$

Qu:

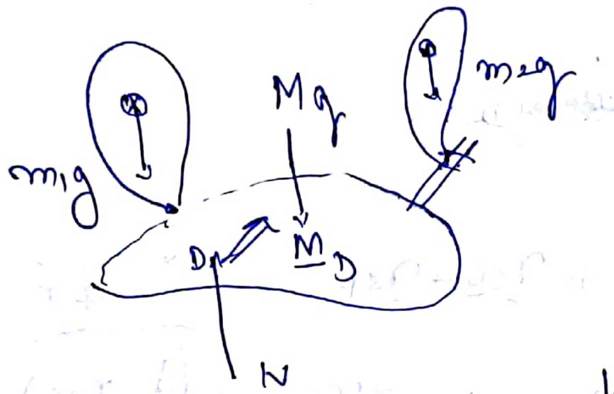


x, y coordinates of O $\quad \boxed{a_x, a_y}$

v_x, v_y, and $\omega_B = \omega_B \hat{k}$ $\quad \boxed{\alpha_B = \alpha_B \hat{k}}$ $\quad [\ddot{q}, \alpha_G]$

q and $\dot{q}$ given, ω_G given $\qquad \dot{\beta}$ given $\boxed{\ddot{\beta} = ?}$

FBDs



$M_D \cdot \hat{k} = 0$

N & M_D are the net force & moment from the ground, acting at D

$$\left[\sum F = M(\underbrace{a_x \hat{i} + a_y \hat{j}}_{a_C}) + m_1 \underline{a}_G + m_2 \underline{a}_R \right] \begin{matrix} \to \cdot \hat{i} = 0 \\ \to \cdot \hat{j} = 0 \end{matrix}$$

$$\left[\dot{H}_{/D} \right] \cdot \hat{k} = 0$$

$$\Downarrow$$

$$r_{DC} \times M \underline{a}_C + \underline{\underline{I}}_{CM} \cdot \underline{\alpha}_B + \underline{\omega}_B \times (\underline{\underline{I}}_{CM} \cdot \omega_B)$$

$$+ r_{DG} \times m_1 a_G + \underline{\underline{I}}_{CM_1} \cdot \underline{\alpha}_G + \omega_G \times I_{CM_1} \cdot \omega_G$$

$$+ \underline{r}_{DR} \times m_2 \underline{a}_R + \underline{\underline{I}}_{CM_2} \cdot \underline{\alpha}_R + \omega_R \times \underline{\underline{I}}_{CM} \cdot \omega_R$$

FBD 2

$$\left[\, a_p = a_c + \underline{\omega}_B \times \underline{\omega}_B \times \underline{r}_{cp} + \alpha_B \times r_{cp} \,\right]$$

$$\left[\, \underline{a}_G = \underline{a}_p + \underline{\omega}_G \times \underline{\omega}_G \times \underline{r}_{pG} + \underline{\alpha}_G \times \underline{r}_{pG} \,\right]$$

Angular mom. about P.

$$\left[\, \underline{r}_{pG} \times m_1 \underline{a}_G + \underline{\underline{I}}_{CM_1} \cdot \underline{\alpha}_G + \underline{\omega}_G \times I_{CM_1} \cdot \underline{\omega}_G = \underline{r}_{pG} \times (-m_1 g)\hat{k} \,\right]$$

~~$a_R \,|\, \text{body } B + \omega \cdot \text{body } B$~~

$$\underline{r}_{OR} = \underline{r}_{OC} + \underline{r}_{CS} + \underline{r}_{SR}$$

$$\left(\frac{d}{dt}\underline{r}_{OR}\right)_{XYZ} = \underline{V}_c + \underline{\omega}_B \times \underline{r}_{CS} + \overbrace{\left(\frac{d}{dt}r_{CS}\right)_B}^{\dot{q}\hat{n}}$$

$$+ \underbrace{\left(\frac{d}{dt}\underline{r}_{SR}\right)_B}_{\dot{\beta}\hat{n} \times \underline{r}_{SR}} + \underline{\omega}_B \times \underline{r}_{SR}$$

5-term acc. formula

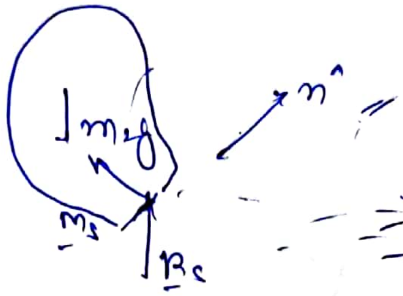
$$\underline{a}_R \;\; \ddot{R} = \underbrace{\ddot{\underline{r}}}_{a_c} + \ddot{\underline{\rho}} + \underbrace{\underline{\alpha}}_{\alpha_B} \times \underline{\rho} + 2\underline{\omega} \times \dot{\underline{\rho}} + \underbrace{\underline{\omega} \times \underline{\omega}}_{\omega_B} \times \underline{\rho}$$

$a_c +$; ρ: r_{CR}

$$\dot{\underline{\rho}} = \left(\frac{d}{dt}\, r_{cR}\right)_{\text{Body } B} = \dot{q}\,\hat{n} + 0 + \dot{\beta}\,\hat{n} \times r_{sR}$$

$$\ddot{\underline{\rho}} = \ddot{q}\,\hat{n} + 0 + \ddot{\beta}\,\hat{n} \times \underline{r}_{sR} + 0 + \dot{\beta}\hat{n} \times \dot{\beta}\hat{n} \times \underline{r}_{sR}$$

→ FBD 3



$$\underline{R}_s \cdot \hat{n} = 0 \,;\; \underline{M}_s \cdot \hat{n} = 0$$

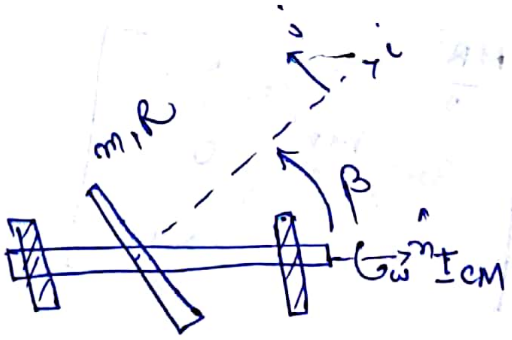
$$\left[-m_2 g\,\hat{k} + \underline{R}_s = m_2 a_R\right] \cdot \hat{n}$$

Ang. Mom. balance about s

$$\left[\dot{H}_{/s} = \underline{r}_{sR} \times m_2 \underline{a}_R + \underline{\underline{I}}_{cM_2} \cdot \alpha_R + \underline{\omega}_R \times \underline{\underline{I}}_{cM_2} \cdot \omega_R\right.$$

$$\left.= r_{sR} \times (-m_2 g)\hat{k} + \underbrace{\underline{M}_s}_{\text{drop out}}\right] \cdot \hat{n} =$$

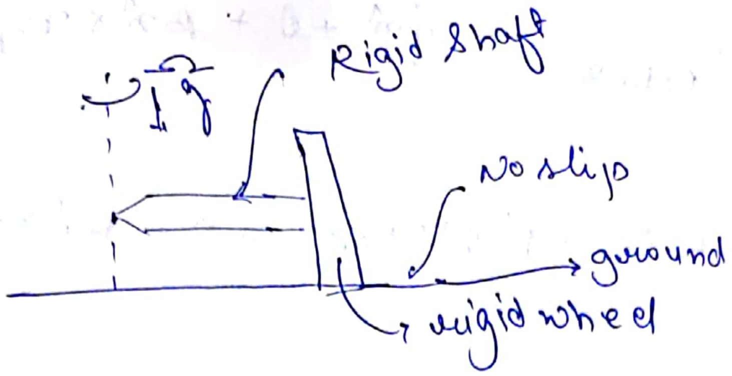
example:-

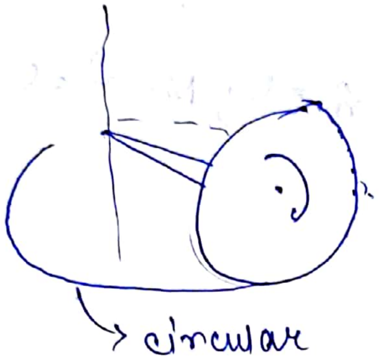


$$I_{cM} = \begin{bmatrix} \frac{mR^2}{2} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{4} \end{bmatrix}$$

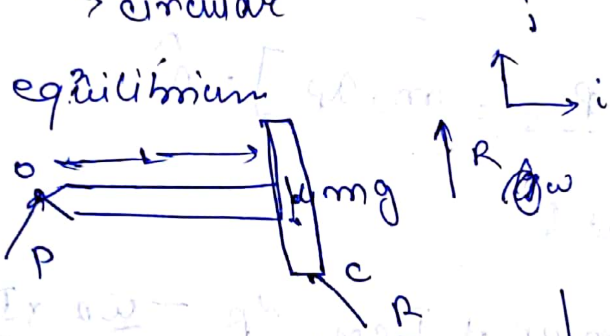
$$\omega = \omega\hat{n}, \quad \hat{n} = \cos\beta\,\hat{i} - \sin\beta\,\hat{j} \,)$$

eg:





Static eqilibrium



$$\underline{P} + \underline{R} - mg\hat{j} = 0$$

$$r_{oc} \times (\underline{R} - mg\hat{j}) = 0$$

$$I_{CM}\cdot\omega = \frac{mR^2}{2}\omega i + \frac{mR^2}{4}\Omega j$$
$$= -m\frac{\Omega RL}{2} i + mR^2\Omega j$$

In motion

$$I_{CM} = \begin{bmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{4} \end{bmatrix}$$

$$\underline{\omega}_{disk} = \Omega j + \omega i$$

no slip requires $v_c = 0$

$\therefore \underline{\omega}_{disk} \times r_{oc} = 0$

$(\Omega \hat{j} + \omega \hat{i}) \times (L\hat{i} - R\hat{j}) = 0$

$\Rightarrow \frac{L}{\omega} = -\frac{R}{\Omega}$ $\Rightarrow \boxed{\omega = -\frac{\Omega L}{R}}$

$\alpha_{disk} = \Omega \hat{j} \times \omega \hat{i} = \Omega^2 \frac{L}{R} \hat{k}$

$a_G = -\Omega^2 L \hat{i}$

AMB about O.

$r_{OG} \times m \cancel{a_G} + \underline{\underline{I}}_{cm} \cdot \alpha + \underline{\omega} \times \underline{\underline{I}}_{cm} \cdot \omega = r_{oc} \times (\underline{R} - mg\hat{j})$

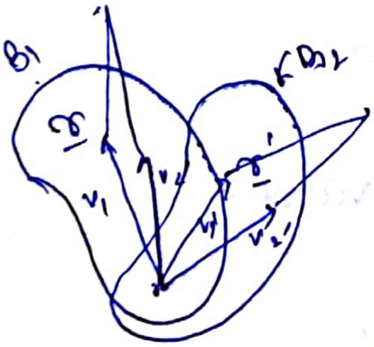
$\frac{mR^2}{4} \cdot \frac{\Omega^2 L}{R} \hat{k} + (\quad\quad) \hat{k} =$

$\Rightarrow \underline{R} \cdot \hat{k} = 0$; $\underline{R} \cdot \hat{i} = 0$ (Reasonable Engineering assumpti

:—·→

• Euler's Theorem :

The most general displacement of a rigid body with one point fixed is a pure rotation about some axis ~~thro~~. passing through that point.



$(B_1 \to B_2)$

$\underline{r}' = f(\underline{r})$

claim mapping f is linear.

$(f(\alpha_1 v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$ for linearity)

We can choose and fix a coordinate system (RHON)

$\underline{v} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$ or $\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$ on V $\qquad v \longrightarrow v'$ is linear

$v' = Rv$ for some R, all v

$v'^T v' = v^T v$ for all v (R fixed)

$v^T R^T R v - v^T v = 0 \quad \forall v$

$v^T (R^T R - I) v = 0 \quad \forall v$

$\Rightarrow R^T R = I$

$[\det(R)^2] = 1 \Rightarrow \det(R) = \pm 1$

→ Let the eigen values of R be $\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 \lambda_2 \lambda_3 = 1$

R has atleast one real eigen value call it λ

Let $Ru = \lambda u$

$u^T R^T R u = \lambda^2 \cdot u^T u$

$\lambda^2 = 1$

So $\lambda = \pm 1$

i) 2 eigen values complex

$\lambda \sigma \bar{\sigma} = 1$

$\exists$ real

1	1	1	✓
1	-1	1	×
1	-1	-1	✓
-1	-1	-1	×

$|D| = 1$

ii) 3 real eigen values.

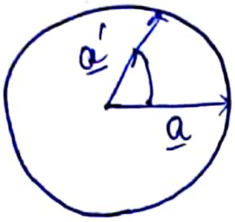
$\lambda = 1$ is ~~real~~ always an eigen value of R

This means $\exists u$ st $Ru = u$ etc.

$r \to r'$ $\quad b + a \to b + a'$

$$b = (\hat{n}\cdot r)\hat{n} \equiv nn^T r$$

$$a = r - (\hat{n}\cdot r)\hat{n} \equiv (I - nn^T) r$$



$$a' = \cos\phi\, a + \sin\phi\, \hat{n} \wedge a$$

$$r' = a + b$$

$$r' = nn^T r + \cos\phi\,(I - nn^T) r + \sin\phi \cdot S(n)\,(I - \underbrace{\cancel{nn^T}}_{\text{goes away}}) r$$

$$r' = nn^T r + \cos\phi\,(I - nn^T) r + \sin\phi\, S(n) r$$

$$= \underbrace{[\cos\phi\, I + (1-\cos\phi)\, nn^T + \sin\phi\, S(n)]}_{R(n,\phi)} r$$

$$R(n,\phi) = nn^T + \cos\phi\,(I - nn^T) + \sin\phi\, S(n)$$

or $\cos\phi\, I + (1-\cos\phi)\, nn^T + \sin\phi\, S(n)$

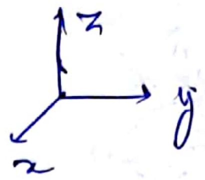
(check = $RR^T = I$) $\quad R^T = nn^T + \cos\phi\,(I - nn^T) - \sin\phi\, S(n)$

$\hat{n} = \hat{j}$, $n = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ $\qquad nn^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$S(n) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ $\qquad S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$

$R = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$

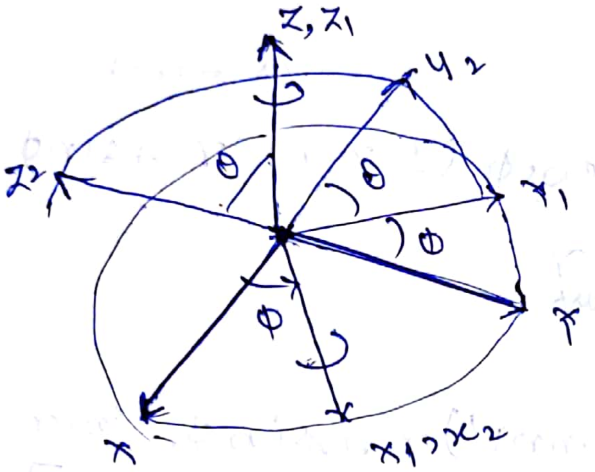
Rotation in 3D using Euler Angles



→ 3-1-3 sequence (Pin stating)

① ϕ- rotation about body fixed z- axis.

② θ - rotation about one-~~stated~~ rotated x-axis (body fixed)



3 → z
1 → x
2 → y

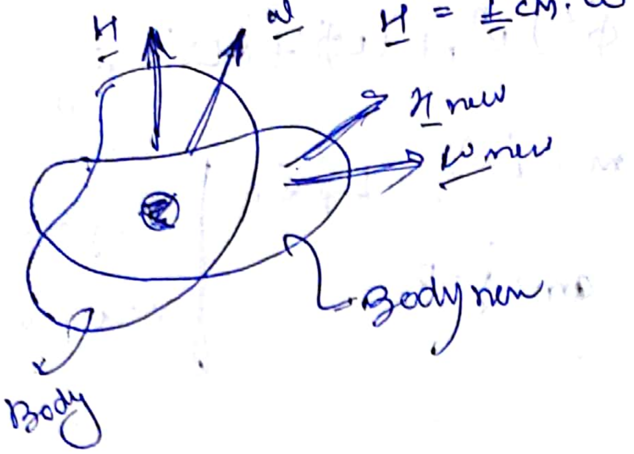
③ ψ- rotation about the twice - rotated z-axis

$$e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad e_2 \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad e_3 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

For 3-1-3

$$R_1 = R(e_3, \phi) \; ; \; R_2 = R(R_1, e_1 \theta), \quad R_3 = R(R_2 R_1 e_3, \psi)$$

$$\underline{R_{net} = R_3 R_2 R_1}$$



$\underline{H}_{new} = \underline{I}_{cm,new} \cdot \underline{\omega}_{new}$

$\omega_{new} = \underline{\underline{R}} \cdot \underline{\omega}$

$\underline{H}_{new} = \underline{\underline{R}} \cdot H$

$\underline{\underline{R}} \cdot \underline{H} = \underline{I}_{cm,new} \cdot R \cdot \omega$

$= \underline{I}_{cm,new} \cdot R \cdot \omega$

$$R^T I_{cm,new} R\omega = I_{cm} \cdot \omega \quad \forall \omega$$

$I_{cm} = R^T I_{cm,new} R$

$$\boxed{I_{cm,new} = R I_{cm} R^T}$$

Consider an arbitrarily rotating body

$RR^T = I$

diff^n with time

$\dot{R}R^T + R\dot{R}^T = 0$

$\underbrace{\dot{R}R^T} = -R\dot{R}^T = -(\dot{R}R^T)^T$ $\quad\quad \dot{R}R^T = W$

skew symmetric matrix

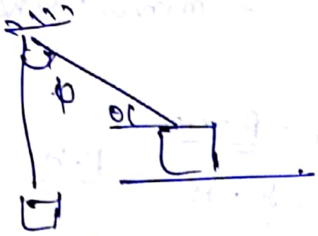
Consider a fixed vector r (body fixed)

r_{ref}

$r = R\, r_{ref}$

$\dot{r} = \dot{R}\, r_{ref} = \dot{R}R^T r$

Qu:



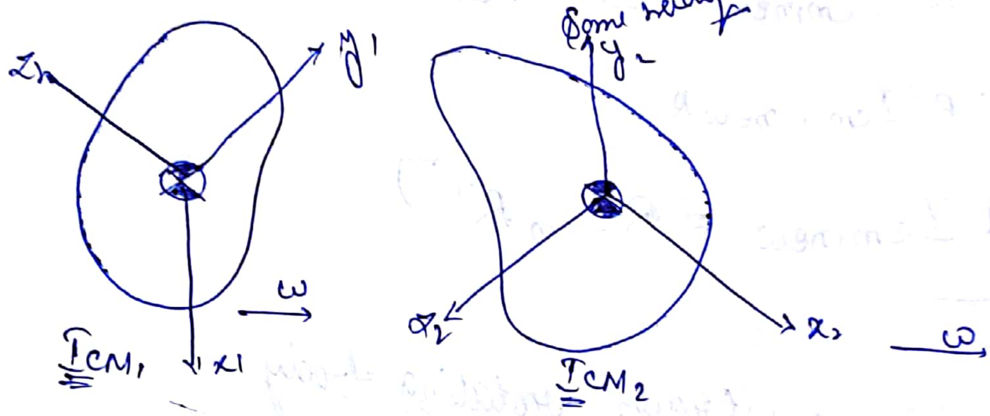
$$\underline{a}_M = (\ddot{L} - L\dot{\phi}^2)\,\hat{e}_r + (L\ddot{\phi} + 2\dot{L}\dot{\phi})\,\hat{e}_\phi$$

$$\underline{V}_M = \dot{L}\hat{e}_r + L\dot{\phi}\hat{e}_\phi$$

$a_m \cdot j = 0$

$v \cdot j = 0$

$v \cdot i = 0$



$\underline{\underline{I}}_{CM_1}$ $\qquad$ $\underline{\underline{I}}_{CM_2}$

G is the CM

$$H_{/C} = r_{CM} \times m\underline{V}_{CM}$$

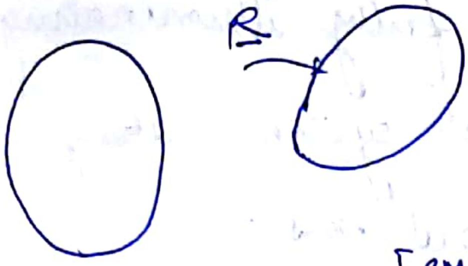
$+C$

$$\Sigma \underline{M}_{/C} = \dot{H}_{/C} = \left(\frac{d\underline{H}_C}{dt}\right)_{\text{ground}} = r_{cm/C} \times m\,\underline{a}_{cm}$$

$$+ \underbrace{\left(\frac{d}{dt}\,\underline{\underline{I}}_{cm}\cdot\omega\right)_{\text{Grnd}}}$$

$\underline{\underline{I}}_{cm} \cdot \dot{\omega} + \omega \times \underline{\underline{I}}_{cm} \cdot \omega$

Scanned by CamScanner



$\underline{\underline{I}}_{cm}$

$$\underline{\underline{I}}_{cm,new} = \underline{R}\,\underline{\underline{I}}_{cm,old}\,R^T$$

K.E. of a rigid body $\frac{1}{2}\sum_i m_i \underline{v}_i\cdot\underline{v}_i$; $v_i = \underline{v}_{cm} + v_{i/cm}$

$$\frac{1}{2}\sum_i m_i(\underline{v}_{cm}+\underline{v}_{i/cm})\cdot(\underline{v}_{cm}+\underline{v}_{i/cm})$$

$$= \frac{1}{2}\Big(\sum_i v_{cm}\cdot v_{cm}, + 2\sum m_i(\underline{v}_i\cdot v_{cm} + \sum_i m_i v_{i/cm}\cdot v_{i/cm}\Big)$$

$$v_{i/cm} = \omega\times r_{i/cm} = -r_{i/cm}\times\underline{\omega} = -S(r_{i/cm})\omega$$

$$\underline{v}_{i/cm}\cdot v_{i/cm} = \omega^T S^T(r_{i/cm})\,S(r_{i/cm})\,\omega$$

$$m_i\underline{v}_{i/cm}\cdot\underline{v}_{i/cm} = m_i\,\omega^T S^T(r_{i/cm})S(r_{i/cm})\,\omega$$

$$KE_{rotation} = \frac{1}{2}\omega^T I_{cm}\cdot\omega \text{ or } \frac{1}{2}\underline{\omega}\cdot\underline{\underline{I}}_{cm}\cdot\omega$$

Force-free, torque free rigid body

$$\underline{\underline{I}}_{cm}\cdot\alpha + \omega\times\underline{\underline{I}}_{cm}\cdot\omega = 0$$

Let $I_{11} > I_{22} > I_{33}$ be the three principle moment of inertia.

Convenient (although not fully illuminating) to use body-fixed coordinate system ~~along~~ coord axis along principal axis.

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + S(\omega) \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Euler-Equation for a torque-free rigid body

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_{33}\omega_3 \end{vmatrix} = \hat{i}(\quad) + \hat{j}(\quad) + \hat{k}(\quad)$$

$$I_{11}\dot{\omega}_1 = (I_{22} - I_{33})\,\omega_2\omega_3$$

$$I_{22}\dot{\omega}_2 = (I_{33} - I_{11})\,\omega_3\omega_1$$

$$I_{33}\dot{\omega}_3 = (I_{11} - I_{22})\,\omega_1\omega_2$$

Stability of pure spin $\omega_1 \neq 0 = \omega_1^*$ $\quad \omega_2 = \omega_3 = 0$

Let $\omega_1 = \omega_1^* + \varepsilon\tilde{\omega}_1$

$$\omega_2 = \varepsilon\tilde{\omega}_2$$

$$\omega_3 = \varepsilon\tilde{\omega}_3$$

$I \cdot \varepsilon \dot{\omega}_1^* = (\quad) \varepsilon^2 \tilde{\omega}_2 \tilde{\omega}_3$

$I \cdot \varepsilon \dot{\tilde{\omega}}_2 = (\quad) \varepsilon \tilde{\omega}_3 \tilde{\omega}_1^*$

$I \cdot \varepsilon \dot{\tilde{\omega}}_3 = (\quad) \cdot \varepsilon \omega_1^* \tilde{\omega}_2$

$I_{11} \dot{\omega}_1 = 0$

$I_{22} \dot{\tilde{\omega}}_2 = \omega_1^* \tilde{\omega}_3 (I_{33} - I_{11})$

$I_{33} \dot{\tilde{\omega}}_3 = \omega_1^* \tilde{\omega}_2 (I_{11} - I_{22})$

$\dot{\tilde{\omega}}_2 = -k_1 \tilde{\omega}_3$

$\dot{\tilde{\omega}}_3 = k_2 \tilde{\omega}_2$

$$\left[\ddot{\tilde{\omega}}_3 = -k_1 k_2 \tilde{\omega}_3 \right]$$

Similarly for ω_3 and ω_2

$\omega_2 = \omega_2^* + \varepsilon \tilde{\omega}_2$

$\omega_1 = \varepsilon \tilde{\omega}_1$

$\omega_3 = \varepsilon \tilde{\omega}_3 \implies \dot{\tilde{\omega}}_1 = M \tilde{\omega}_1 \quad M > 0$

- $RR^T = I$

$\dot{R}R^T + R\dot{R}^T = 0$

$\dot{R}R^T + (\dot{R}R^T)^T = 0$

$\dot{R}R^T$ is $S(\omega)$

$r \to r'(t) = R(t) r$

$\dot{r}'(t) = \dot{R} r = \underbrace{\dot{R}R^T}\, r'$

$\frac{dr'}{dt} = \omega \times r'$

→ Rotation don't commute $R_1 R_2 \neq R_2 R_1$ in general

→ But infinitesimal rotation do commute

$R(n, \theta) = nn^T + \cos\theta (I - nn^T) + \sin\theta\, S(n)$

For infinitesimal θ, $R(n, \theta) = I + \theta S(n)$ to first order in θ

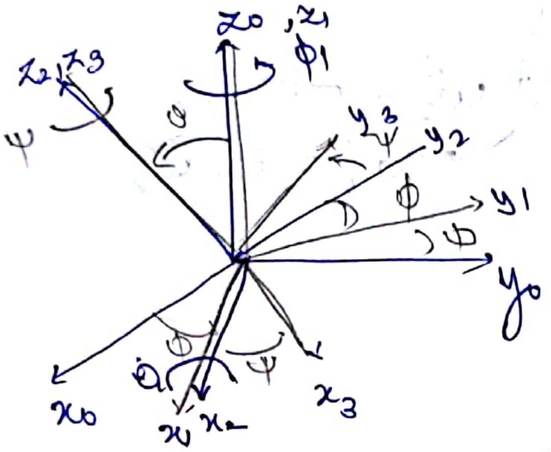
$R(n_1, \theta_1) R(n_2 \theta_2) = [I + \theta_1 S(n_1)][I + \theta_2 S(n_2)]$

$= I + \theta_1 S(n_1) + \theta_2 S(n_2)$ to first order

$= R(n_2, \theta_2) R(n_1, \theta_1)$

- The integral of $\underline{\omega}(t)$ is not a physically meaningful quantity.

→ 3-1-3 Euler Angles (again)



$x_0 y_0 z_0$ is Ground

$x_1 y_1 z_1$ = is intermediate frame F_1

$x_2 y_2 z_2$ = " " F_2

$x_3 y_3 z_3$ is the body.

$$\underline{\omega}_{Body/Ground} = \omega_{Body/F_2} + \omega_{F_2/F_1} + \omega_{F_1/Ground}$$

$R_1 = R(e_3, \phi)$ $\quad R_2 = R(R_1 e_1, \theta)$ $\quad$; $R_3 = R(R_2 R_1 e_3, \psi)$

$\underline{\omega}_{Body/F_2} = \dot{\psi}$

$$\omega_{Body/F_2} = \dot{\psi} R_2 R_1 e_3$$

$$\omega_{F_2/F_1} = \dot{\theta} R_1 e_1$$

$$\omega_{F_1/Ground} = \dot{\phi} e_3$$

$$\frac{d}{dt}\begin{Bmatrix}\phi\\ \theta\\ \psi\end{Bmatrix} = A^{-1}\omega$$

$$\frac{d}{dt}\begin{Bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{Bmatrix} = \dot{\omega}$$

$$\begin{bmatrix} e_3 & R_1 e_1 & R_2 R_1 e_3 \end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{Bmatrix} = \omega$$

$$A\dot{p} = \omega$$

$$\dot{p} = A^{-1}\omega$$

Singularity of A

perpendicular

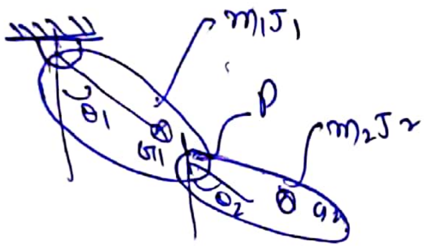
$[e_3 \;\; R_1 e_1 \;\; R_2 e_3]$

guaranteed perpendicular

Singular when $\theta = n\pi$

- Planar problems :

I_1, I_2 refer to $I_{cm,zz}$

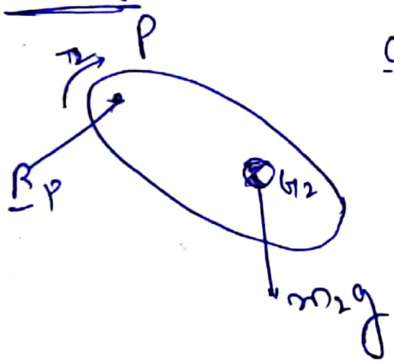


g

Joints have frictional torques

Fixed eqn of motions

FBD 1)



$\underline{a}_p = \underline{\omega}_1 \times \underline{\omega}_1 \times \underline{r}_{op} + \alpha_1 \times \underline{r}_{op}$

where $\underline{\omega}_1 = \dot{\theta}_1 \hat{k}$, $\alpha_1 = \ddot{\theta}_1 \hat{k}$

$\underline{a}_{G_2} = \underline{a}_p + \underline{\omega}_2 \times \underline{\omega}_2 \times \underline{r}_{PG_2} + \underline{\alpha}_2 \times \underline{r}_{PG_2}$

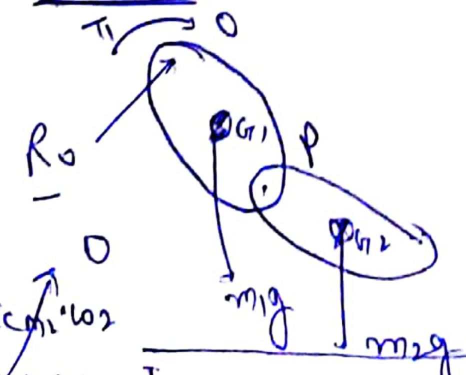
where $\omega_2 = \dot{\theta}_2 \hat{k}$; $\alpha_2 = \ddot{\theta}_2 \hat{k}$

$\dot{\underline{H}}_{/P} = \underline{r}_{PG_2} \times (-m_2 g \hat{j}) - T_2 \hat{k}$

$= \underline{r}_{PG_2} \times m_2 \underline{a}_{G_2} + \underbrace{\underline{\underline{I}}_{cm_2} \alpha_2}_{J_2 \ddot{\theta}_2 \hat{k}} + \omega_2 \underline{\underline{I}}_{cm}' \omega_2$

O

FBD 2)



$\underline{a}_{G_1} = \underline{\omega}_1 \times \underline{\omega}_1 \times \underline{r}_{OG_1} + \underline{\alpha}_1 \times \underline{r}_{OG_1}$

$\underline{r}_{OG_2} = \underline{r}_{OP} + \underline{r}_{PG_2}$

$\dot{\underline{H}}_{/O} = -T_1 \hat{k} + \underline{r}_{OG_1} \times (-m_1 g \hat{j})$

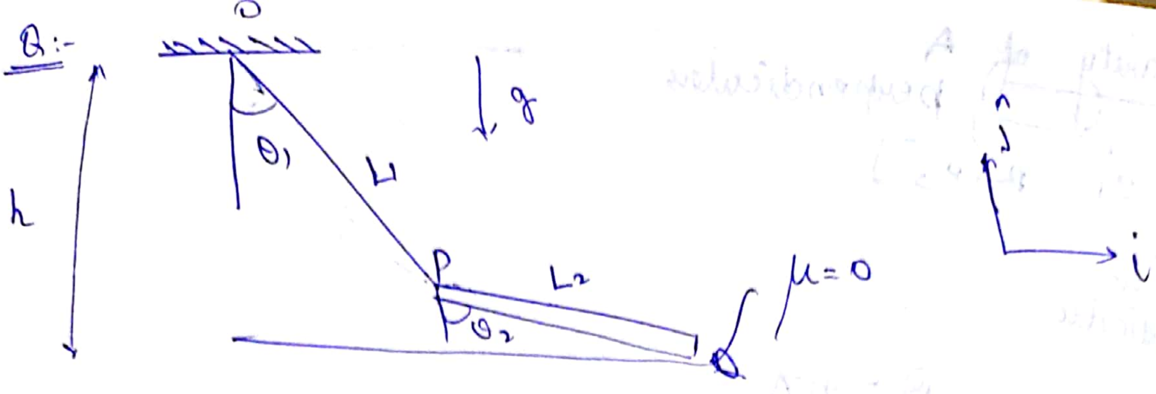
$+ \underline{r}_{OG_2} \times (-m_2 g \hat{j})$

$= \underline{r}_{OG_1} \times m_1 a_{G_1} + \underline{r}_{OG_2} \times m_2 a_{G_2}$

$+ \omega_2 \times I_{cm_2}' \omega_2$

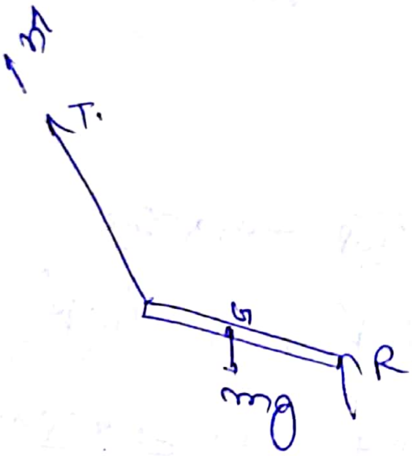
$+ \omega_1 \times I_{cm_1}' \omega_1 + I_{G_1} \alpha_1 + I_{cm} \alpha_2$

Q:-



2 dof problem
1-dimensional

FBD



$\hat{n} = -\sin\theta_1 \hat{i} + \cos\theta_1 \hat{j}$

Given $\theta_1, \dot{\theta}_1$ Find $\ddot{\theta}_1$

$L_1\cos\theta_1 + L_2\cos\theta_2 = h$ $\Rightarrow$ solve for θ_2

$\underline{v}_P = \underline{\omega}_1 \times \underline{r}_{OP}$

$v_Q = v_P + \underline{\omega}_2 \times \underline{r}_{PQ}$

$v_Q \cdot \hat{j} = 0$

given $\underline{\omega}_2$ and θ_2

$\boxed{a_Q \cdot \hat{j} = 0}$

r_{PG} is known

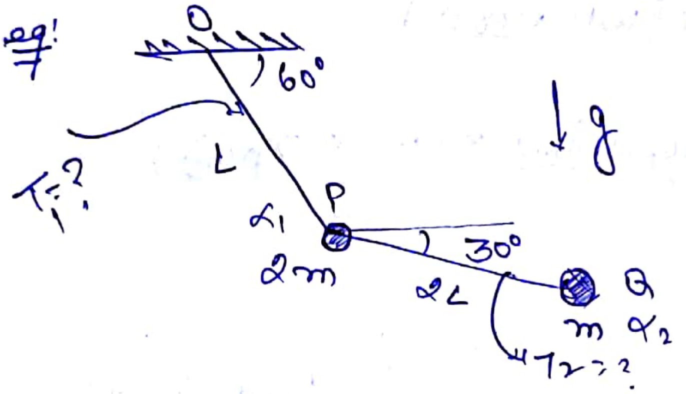
$a_G = a_P + \underline{\omega_2} \times \underline{\omega_2} \times r_{PG} + \alpha_2 \times r_{PG}$

$[m\underline{a_G} = T\hat{n} + N\hat{j}]$

~~A~~

$\dot{\underline{H}}_{/P}$

eg: 7



$\underline{a_P} = \underline{\alpha_1} \times \underline{r_{OP}}$; $a_Q = \underline{a_P} + \underline{\alpha_2} \times r_{PQ}$

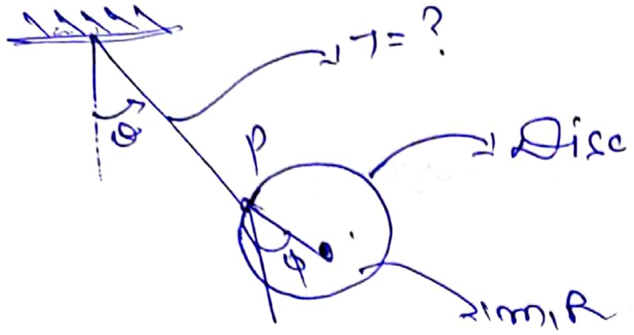
$$\dot{\underline{H}}_{/P} = \underline{r_{PQ}} \times (-mg\hat{j}) = \underline{r_{PQ}} \times m\underline{a_Q} + \cancel{I_{cm}} \cdot \underline{\alpha_2}$$

$I_{cm} = 0$ Because its a point mass

$$\dot{H}_{/O} = \underline{r_{OP}} \times (-2mg\hat{j}) + r_{OQ} \times (-mg\hat{j})$$

$$= I_{cm_1} \cdot \alpha_1 + I_{cm_2} \cdot \alpha_2 + r_{OP} \times m a_P$$

$$+ r_{OQ} \times m a_Q$$



$\theta = 30°$; $\phi = 45°$

$\dot{\theta} = 1$ rad/s $\quad \dot{\phi} = 1$ rad/sec

$$\underline{a}_P = \underline{\alpha}_1 \times \underline{r}_{OP} + \underline{\omega}_1 \times (\underline{\omega}_1 \times \underline{r}_{OP})$$

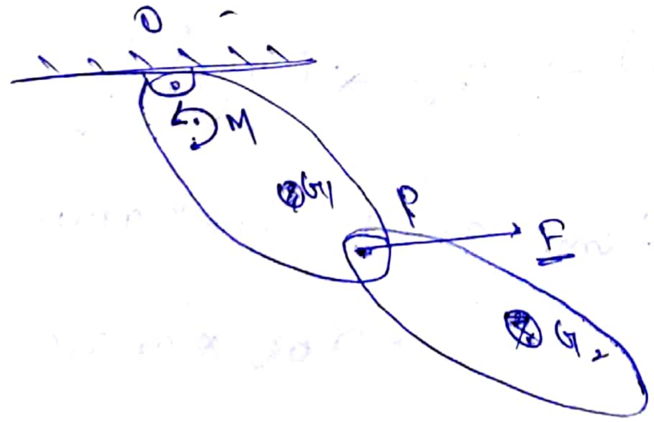
$$\underline{a}_G = \underline{a}_P + \underline{\alpha}_2 \times \underline{r}_{PG} + (\underline{\omega}_2 \times (\omega_2 \times r_{PG}))$$

$\dot{H}_{/P} =$

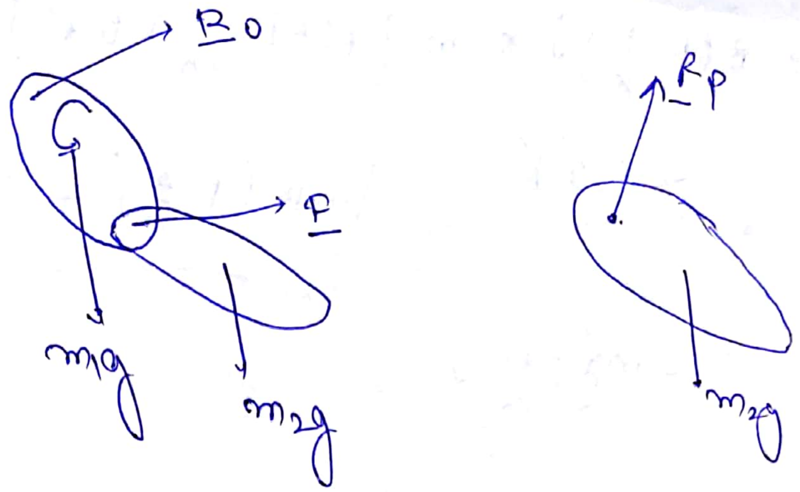
Find α_1 and α_2

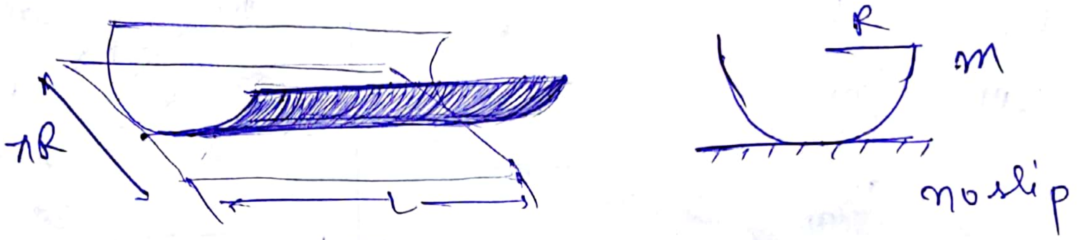
$\dot{H}_{/O} =$

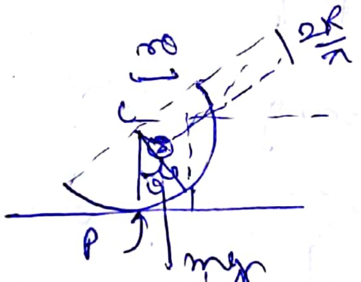
eg:



FBD







FBD

$\underline{V}_c = -R\dot{\theta}\,\hat{i}$ $\quad \underline{a}_c = -R\ddot{\theta}\,\hat{i}$ $\quad \omega = \dot{\theta}\,\hat{k}$ $\quad \alpha = \ddot{\theta}\,\hat{k}$

$$J = I_{zz} = MR^2 - m\cdot\frac{4}{\pi^2}R^2 = MR^2\left(1 - \frac{4}{\pi^2}\right)$$

$$\underline{a}_G = \underline{a}_c + \underline{\omega}\times\underline{\omega}\times\underline{r}_{cG} + \underline{\alpha}\times\underline{r}_{cG}$$

θ

$$H'_{/P} = r_{PG}\times(-mg\hat{j}) = r_{PG}\times m a_G + J\ddot{\theta}\hat{k}$$

$$\Rightarrow -mg\left(\frac{2R}{\pi}\right)\sin\theta\,\hat{k} = \left[R\hat{j} - \frac{2R}{\pi}(\sin\theta\,\hat{i} - \cos\theta\,\hat{j})\right] \times m\left[-R\ddot{\theta}\,\hat{i} + \frac{2R}{\pi}\ddot{\theta}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\right] + J\ddot{\theta}\,\hat{k}$$

neglect $\omega\times\omega\times r_{cG}$ for small osc. because $\dot{\theta}^2$

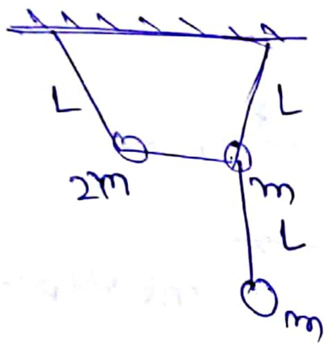
$$\underbrace{\qquad} = R\left(1-\frac{2}{\pi}\right)\hat{j} \times m\left[R\left(-1+\frac{2}{\pi}\right)\ddot{\theta}\,\hat{i}\right] + J\ddot{\theta}\,\hat{k} = \underbrace{\left[mR^2\left(1-\frac{2}{\pi}\right)^2 + J\right]\ddot{\theta}}$$

$$\underbrace{\qquad} = -mg\,\frac{2R}{\pi}\,\theta$$

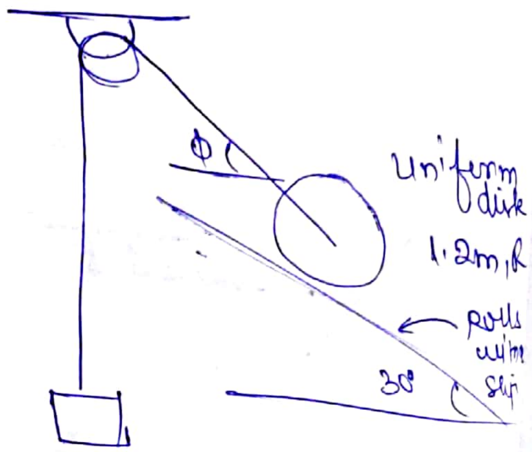
$\Leftrightarrow$

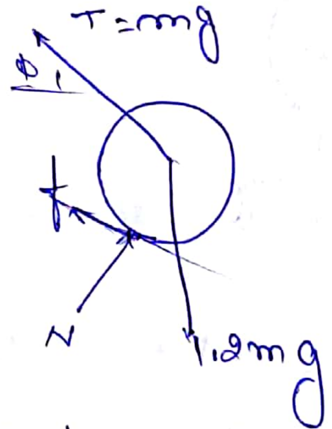
---

eq:

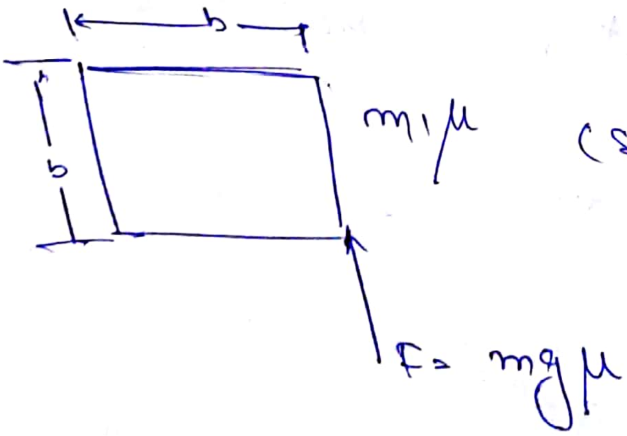


eq:



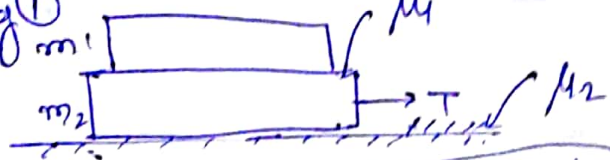


$f = 0$

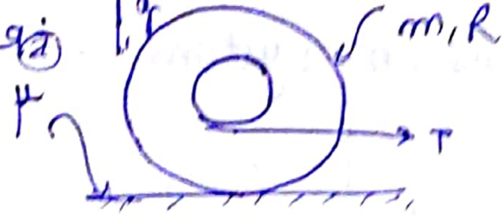


(Simplification)
Uniform contact pressure)

eg ①

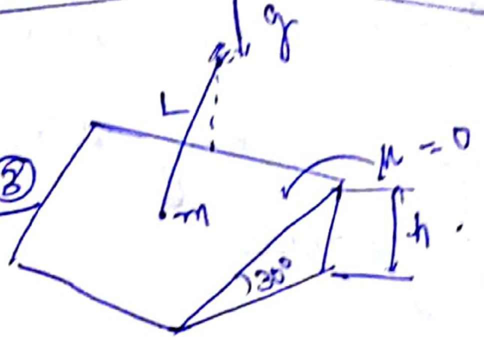


eg ②



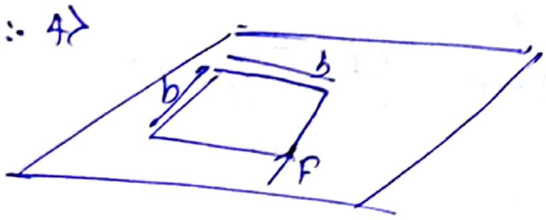
eqn - min. T needed (from rest) to cause slip at μ-surface

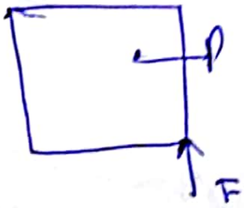
eg: ③



Q small oscillations

eg:- 4)





$F = \mu m g$

$$\underline{a}_P = \underline{a}_G + \alpha \times r_{GP}$$

Let $\underline{f}$ = (friction force per unit area)

$$\underline{f} = -\mu \frac{mg}{b^2} \cdot \frac{a_P}{|a_P|}$$

$$\underline{F} + \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \underline{f} \, dx\, dy = m a_G$$

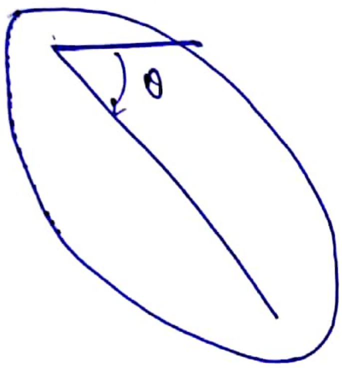
$$\underline{r}_{GC} \times \underline{F} + \iint_{Area} \underline{r}_{GP} \times \underline{f} \, dx\, dy = J \alpha \hat{k}$$

Problems on systems where there are consumable quantities

m, J

Released at rest

Find : 1) the angular velocity at the lowest position
2) Time taken to get there.



$$KE = \frac{1}{2} J\dot{\theta}^2 + \frac{1}{2} m (L\dot{\theta})^2$$

$$= \frac{1}{2} [J + mL^2] \dot{\theta}^2 = \frac{1}{2} J_p \dot{\theta}^2$$

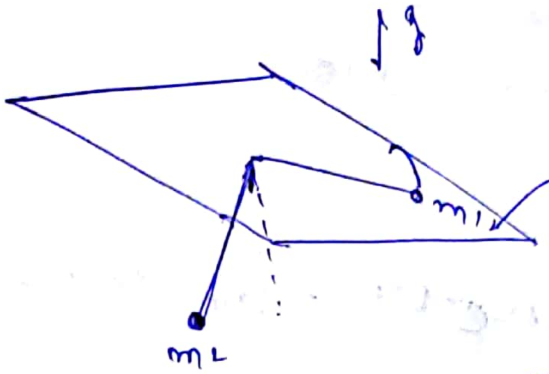
$$PE = -mgL \sin\theta$$

$$\frac{1}{2} J_p \dot{\theta}^2 - mgL \sin\theta = \text{const} = 0$$

$$\dot{\theta} = \sqrt{\frac{2}{J_p} mgL} \sqrt{\sin\theta}$$

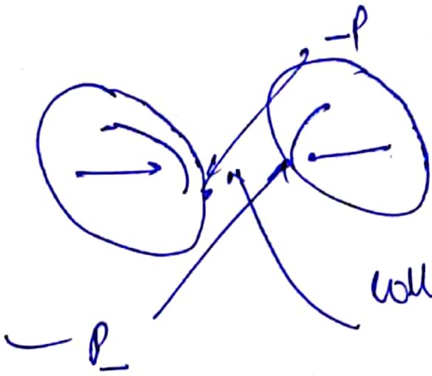
$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \sqrt{\frac{2mgL}{J_p}} \cdot T$$

Que:



$P \Rightarrow$ Impulse

Impact (collisions)

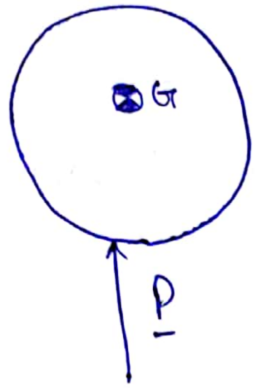


$$\underline{r}_{GC} \wedge \underline{F} = \underline{\underline{I}}_{cm} \cdot \underline{\alpha} + \underbrace{\underline{\omega} \times \underline{\underline{I}}_{cm} \cdot \underline{\omega}}_{\text{small in comp. }(\alpha)\text{ neglect}}$$

$$\underline{r}_{GC} \times \underline{F} = \underline{\underline{I}}_{cm} \cdot \underline{\alpha}$$

Integrate over brief impact duration, neglect changes in

$\underline{r}_{GC}$ & $\underline{\underline{I}}_{cm}$, $\Rightarrow \underline{r}_{GC} \times \underline{P} = \underline{\underline{I}}_{cm} \cdot (\Delta \underline{\omega})$



LMB

$m\underline{a}_G = \underline{P}(t)$

$m \Delta \underline{v}_G = \underline{P}$

(very small) neglect

AMB

$\underline{r}_{GC} \times P(t) = \underline{\underline{I}}_{cm} \cdot \alpha + \cancel{\omega \times I_{cm} \cdot \omega}$

$\underline{r}_{GC} \times P = \underline{I}_{cm} \cdot \Delta\omega$

neglect

$\underline{a}_C = \underline{a}_G + \underline{\alpha} \times \underline{r}_{GC} + \cancel{\underline{\omega} \times \underline{\omega} \times \underline{r}_{GC}}$

$\Delta \underline{v}_C = \Delta \underline{v}_G - \underline{r}_{GC} \times \Delta \underline{\omega}$

In matrix notation

$S(r_{GC})\, P = \underline{I}_{cm} \Delta\omega$

$\Delta v_C = \Delta v_G - S(r_{GC})\, \Delta\omega$

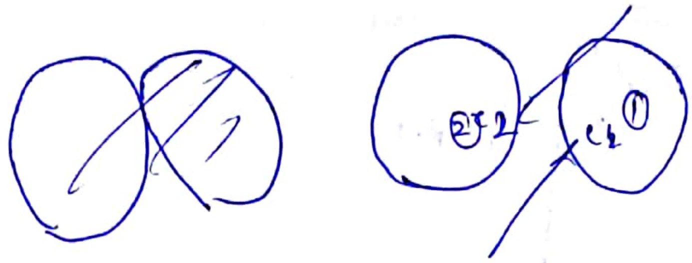
$\Delta v_G = \frac{P}{m} = \frac{1}{m} I_d P$ $\qquad$ I_d = Identity

$\Delta\omega = I_{cm}^{-1} \cdot S(r_{GC})\, P$

$\Delta v_C = \frac{1}{m} I_d P + S(r_{GC})^T I_{cm}^{-1} S(r_{GC})\, P$

$= WP$ $\qquad$ $W = \frac{1}{m} \cdot I_d + S(r_{GC})^T I_{cm}^{-1} S(r_{GC})$

$W = W^T > 0$ (Symmetric position definite matrix)



$\Delta V_{C1} = \omega_1 \cdot P$

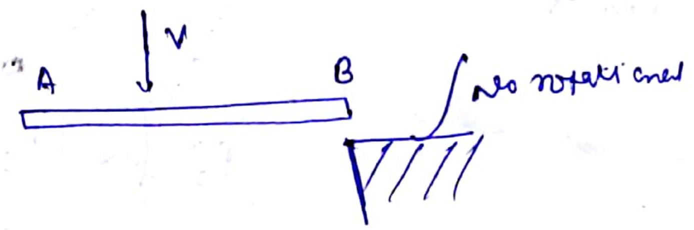
$\Delta V_{C2} = -\omega_2 P$

let $V_{c,rel} = V_{c1} - V_{c2}$

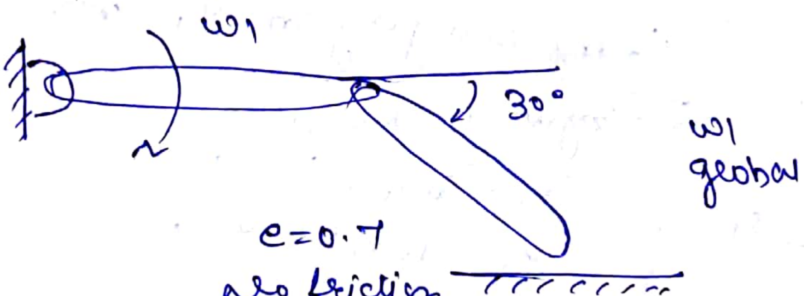
$$\boxed{\Delta V_{c,rel} = (\omega_1+\omega_2) P}$$

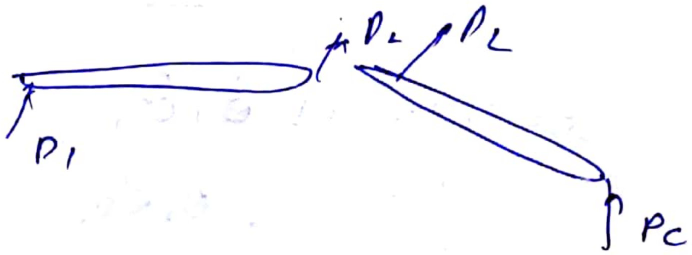
$$\boxed{P = M \Delta V_{c,rel}} \qquad M = (W_1 + W_2)^{-1}$$

eg:-

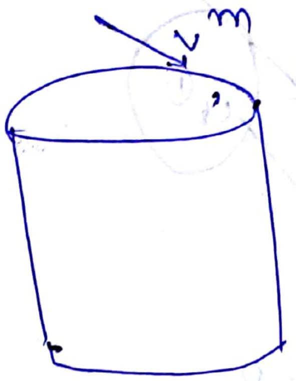


eg:-

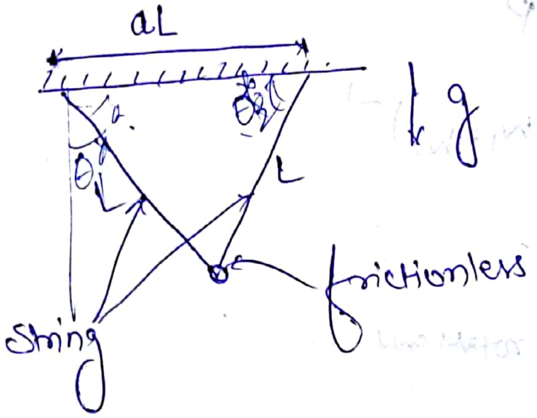




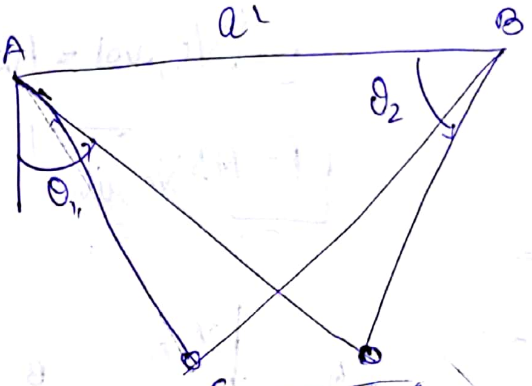
eg:



Other planar problems :-



$\rightarrow$



$\theta_2^* = \boxed{\cos^{-1}\left(a/2\right)}$ — $\hat{e}_{r_1}$

$\theta_1^* = \pi/2 - \theta_2^*$ — $\hat{e}_{\theta_1}$

$\theta_1^* + \psi_1 = \theta_1$

$\theta_2^* + \psi_2 = \theta_2$

ψ_2 and s_1 depend on ψ_1 (non-linear fun^n)

try to get away with

$\psi_2 = a_1\psi_1 + (\ )\psi_1^2 \dots$

$s_1 = a_2\psi_1 + (\ )\psi_1^2 + \dots$

$L_1 = L + s_1$

$L_2 = 2L - L_1 = L - s_1$

$\dot{L}_2 = -\dot{L}_1 = -\dot{s}_1$

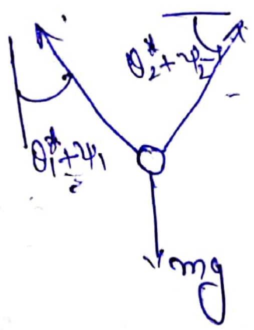
$\underline{v}_C = \dot{L}_1 \hat{e}_{r_1} + L_1 \dot{\theta}_1 \hat{e}_{\theta_1}$

$\underline{v}_C = \dot{L}_2 \hat{e}_{r_2} + L_2 \dot{\theta}_2 \hat{e}_{\theta_2}$

$\left.\begin{matrix} \hat{e}_{r_1}, \hat{e}_{\theta_1} \\ \hat{e}_{r_2}, \hat{e}_{\theta_2} \end{matrix}\right\}$ Equil position values can be used & θ_2 interms of θ_1 ψ_1 and ψ_2

$\underline{a}_c = (\ddot{L}_1 - \cancel{L_1}\dot{\theta}_1^2)\hat{e}_{r_1} + (L_1\ddot{\theta}_1 + 2\cancel{\dot{L}_1}\dot{\theta}_1)\hat{e}_{\theta_1}$

$\underline{a}_c = (\ddot{L}_2 - L_2\cancel{\dot{\theta}_2^2})\hat{e}_{r_2} + (L_2\ddot{\theta}_2 + 2\cancel{\dot{L}_2}\dot{\theta}_2)\hat{e}_{\theta_2}$



Stability

eg: $\dot{x} = x + \sin t$

$x(0) = -1/2$

$x = -\frac{\sin t}{2} - \frac{\cos t}{2}$

let $x = -\frac{\sin t}{2} - \frac{\cos t}{2} + \xi$

$\dot{x} = -\frac{\cos t}{2} + \frac{\sin t}{2} + \dot{\xi}$.

$\dot{\xi} = \xi$

$\ddot{\theta} + c\dot{\theta} + \sin\theta = A\sin\omega t$ (periodically, damped oscillation) #

let us say $p(t)$ is a periodic soln of #

(period $= 2\pi/\omega$)

let $\theta = p(t) + \xi$; $\ddot{\theta} = \ddot{p} + \ddot{\xi}$; $\dot{\theta} = \dot{p} + \dot{\xi}$

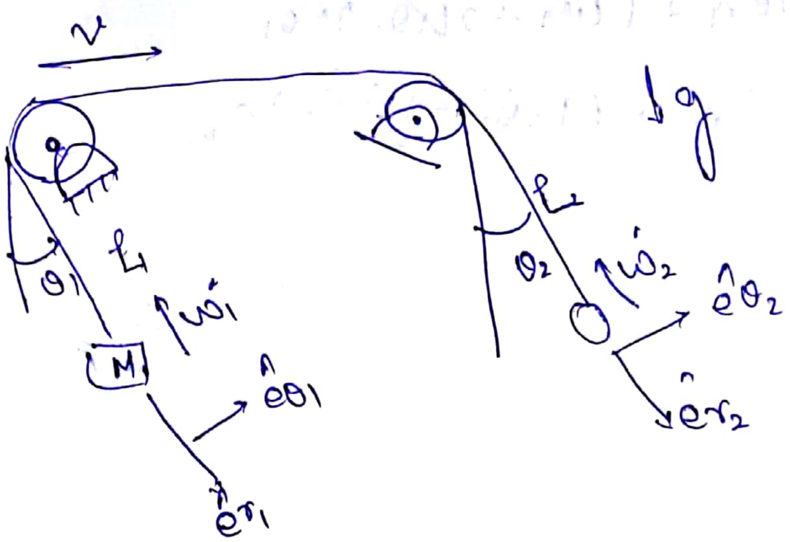
$\ddot{p} + \ddot{\xi} + c\dot{p} + c\dot{\xi} + \sin(p + \xi) = A\sin\omega$

$\sin p + \cos p \cdot \xi + (\ )\xi^2 + \cdots$

$\ddot{p} + \ddot{\xi} + c\dot{p} + c\dot{\xi} + \sin p + \cos(p)\,\xi = A\sin\omega t$

$\ddot{\xi} + c\dot{\xi} + \cos(p(t))\,\xi = 0$

Sol:-



For first mass

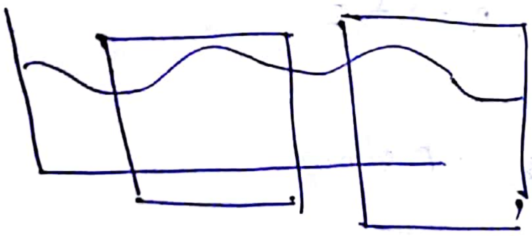
$$\underline{a_1} = (\ddot{L}_1 - L_1\dot{\theta}_1^2)\hat{e}_{r_1} + (L_1\ddot{\theta}_1 + 2\dot{L}_1\dot{\theta}_1)\hat{e}_\theta$$

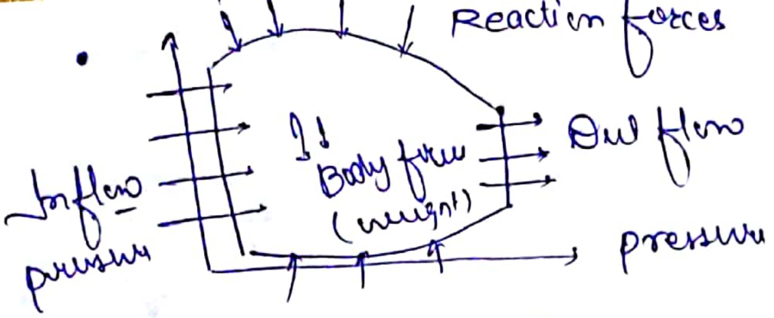
$$M\underline{a_1} = -T\hat{e}_{r_1} - Mg\hat{j}$$

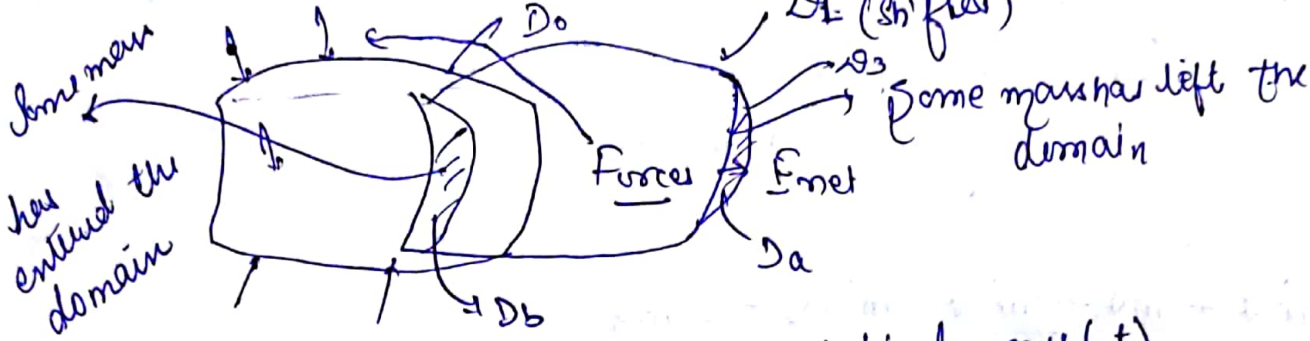
Same for mass 2

$$\dot{L}_1 = v - \omega_1 \quad ; \quad \dot{L}_2 = v - \omega_2$$

$$\ddot{L}_1 = \dot{v} - \dot{\omega}_1 \quad ; \quad \ddot{L}_2 = \dot{v} - \dot{\omega}_2$$







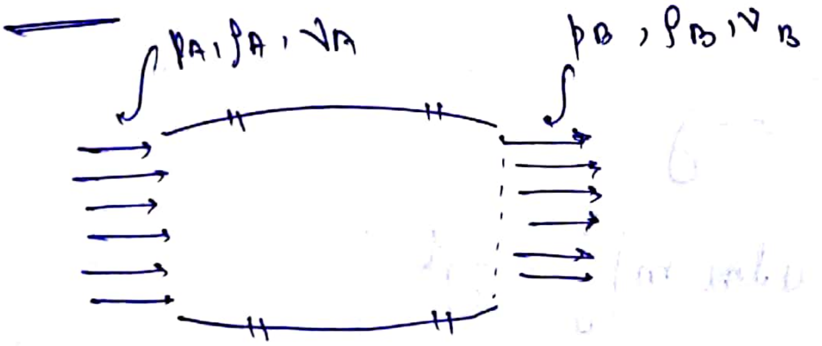
$F_{net}\,\Delta t$ = Momentum of fixed mass $(t+\Delta t)$ − Mom. of fixed mass (t)

$$F. \quad = \underbrace{\int_{D_3} \rho(t+\Delta t)\,\underline{v}(t+\Delta t)\, d\,vol_3}\ -\ \int_{D_0} \rho(t)\,\underline{v}(t)\, d\,vol_0$$

$$\int_{D_1} \rho(t+\Delta t)\,\underline{v}(t+\Delta t)\, d\,vol_1 + \int_{D_a} \rho(t+\Delta t)\,\underline{v}(t+\Delta t)\, d\,vol_a - \int_{D_b} \rho(t+\Delta t)\,\underline{v}(t+\Delta t)\, d\,vol_b$$

correct up to 1st order

F_{net} = Rate of change of momentum inside vol. of interest + Net rate of outward flow of mom.



Que:

$$F_{net} = -mg = \frac{d}{dt}(mv) + m'(v-u)$$

$m' = -\dot{m}$

← ejects mass at rate. $m' > 0$

ejected vel. $v-u$

$$\cancel{\dot{m}v} + m\dot{v} + \cancel{m'v} - m'u = -mg$$

$$m\dot{v} + \dot{m}u = -mg$$

for $g=0$ $\quad m\dot{v} + \dot{m}u = 0$

$$\dot{v} = -\frac{\dot{m}}{m}u$$

$$v_f - v_{in} = -u \log m \Big|_{m_{in}}^{m_f}$$

$$\Delta v = u \ln \frac{m_i}{m_f}$$

for $g = \text{const}$

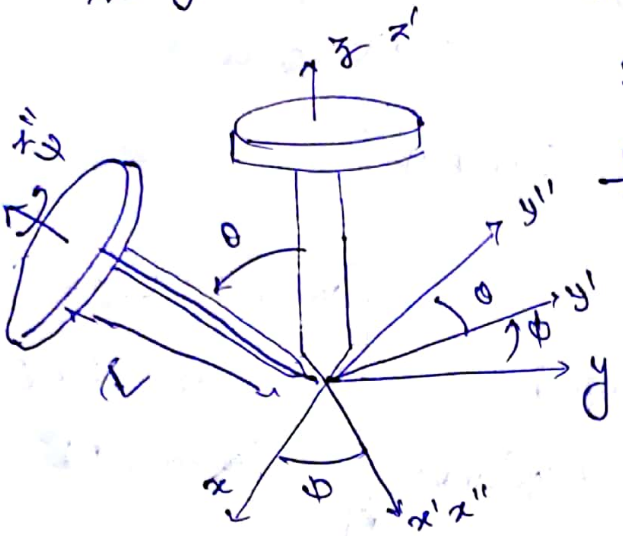
$$\dot{v} + \frac{\dot{m}u}{m} = -g$$

$$v\Big|_0^t + u \ln m\Big|_0^t = -gt$$

$$v = u \ln \frac{m_i}{m_f} - gt$$

$$\frac{v+gt}{u} = \ln \frac{m_i}{m_f}$$

Steady precession of a gyroscope



3.1.3 Euler Angles $(\phi - \theta - \psi)$

$$\underline{\Omega} = \dot\phi \hat k + \dot\theta \hat i'' + \dot\psi \hat k''$$

Consider a moving frame F that rotate about $\hat k$ at rate $\dot\phi$

Steady precession means $\dot\theta = 0$, $\dot\phi = \text{const}$, $\dot\psi = \text{const}$

$M_{/p} = M \hat i''$ (Required for steady precession.

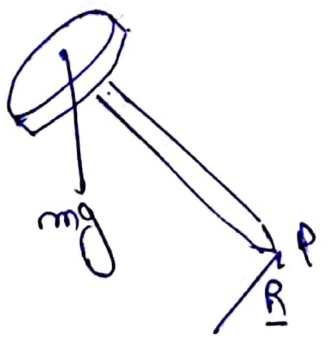
$$[I_{cm}]_{x''y''z''} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\underline{\alpha} = \dot\phi \hat k \times \dot\psi \hat k'' = \dot\phi \dot\psi \sin\theta \hat i''$$

$$\underline{a}_{cm} = \dot\phi \hat k \times \dot\phi \hat k \times L \hat k''$$
$$= \dot\phi^2 L \sin\theta\, \hat k \times \hat i'' = \dot\phi^2 L \sin\theta \hat j''$$

$$\hat j' = \cos\theta \hat j'' - \sin\theta \hat k'', \quad \hat k = \cos\theta \hat k'' + \sin\theta \hat j''$$

AMB about P



$\underline{M}_{/p} = M \hat i''$ (if only weight, $N = mgL\sin\theta$)

$$= \underbrace{\underline{r}_{pG} \times m \underline{a}_{cm}}_{-L^2 m \dot\phi^2 \sin\theta \cos\theta \hat i''} + \underbrace{\underline{I}_{cm} \cdot \underline{\alpha}}_{I \dot\phi \dot\psi \sin\theta \hat i''} + \omega \times \underline{\underline{I}}_{cm} \cdot \omega$$

$$\Rightarrow \underbrace{-L^2 m \dot\phi^2 \sin\theta \cos\theta \hat i'' + I \dot\phi \dot\psi \sin\theta \hat i''}_{\Phi}$$

$\omega = \dot\psi \hat{k}'' + \dot\phi(\cos\theta \hat{k}'' + \sin\theta \hat{j}'') = \dot\phi \sin\theta \hat{j}'' + (\dot\psi + \dot\phi\cos\theta)\hat{k}''$ Ajemmu

$\underline{\underline{I}}_{cm} \cdot \underline{\omega} = I\dot\phi \sin\theta \hat{j}'' + I_0 (\dot\psi + \dot\phi \cos\theta)\hat{k}''$

$\underline{\omega} = p_1 \hat{j}'' + p_2 \hat{k}''$

$I \cdot \omega = I p_1 \hat{j}'' + I_0 p_2 \hat{k}''$

$= I\omega + (I_0 - I)p_2 \hat{k}''$

$\Rightarrow \quad \not{C} + (I_0 - I_1)\dot\phi \sin\theta(\dot\psi + \dot\phi\cos\theta)\hat{j}''$

$$M = I_0 \dot\phi \dot\psi \sin\theta + (\underbrace{I_0 - I - mL^2}_{\Delta I})\dot\phi^2 \sin\theta\cos\theta$$

For weight only :-

$M = mgL\sin\theta$

Assuming $\sin\theta \neq 0$

$I_0 \dot\phi \dot\psi + (\Delta I)\dot\phi^2 \cos\theta = mgL$