EE 200 Signals, Systems & Networks

Assignment 12

1 Sketch these signals, find their Laplace Transforms and sketch the respective ROCs and pole-zero plots. $\delta(t-t_0)$, $u(t-1)$, $u(t+1)-u(t-1)$, $e^{-|t|}$, $\sin\omega_0 t$, $\cos\omega_0 t$, $e^{-at}u(t)\sin\omega_0 t;\ a>0$, $e^{-at}u(t)\cos\omega_0 t;\ a>0$, $e^{-a|t|}\sin\omega_0 t;\ a>0$, $e^{-a|t|}\cos\omega_0 t;\ a>0$, $tu(t)$, $t^k u(t)$, $tu(t)-2(t-2)u(t-2)$, $[tu(t)-2(t-2)u(t-2)+(t-4)u(t-4)]e^{-at}$.

2 Prove your answer. When $h(t)$ is real, its LT $H(s) =:$ $(a)\ -H(s)$ $(b)\ H^*(s)$ $(c)\ H(-s)$ $(d)\ H^*(s^*)$.

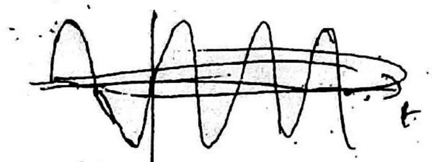
3 Sketch the pole-zero plots of the given transfer functions and *by direct inspection alone,* and no calculations, answer the following questions. First, is the causal time response that of a stable system? Second, find whether the stable time response is causal, anticausal or acausal.
$(a)\ H_1(s) = \frac{(s-1)(s+3)}{(s+1-j2)(s+1+j2)}$ $(b)\ H_2(s) = \frac{(s-2-j1)(s-2+j1)}{(s+3)(s-1)}$ $(c)\ H_3(s) = \frac{(s-j4)(s+j4)}{(s-1-j1)(s-1+j1)}$
$(d)\ H_4(s) = \frac{(s-2)}{(s+2)}$ $(e)\ H_5(s) = H_1(s) + H_4(s)$ $(f)\ H_6(s) = H_1(s)H_2(s)$.

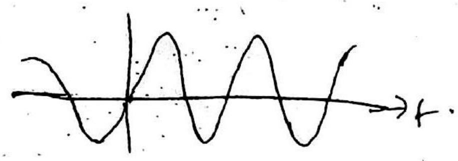
4 The value of $H(s)$ at any point s is best understood in terms of the factorized form of the rational polynomial: $H(s) = \frac{\text{product of distances of } s \text{ from different zeroes}}{\text{product of distances of } s \text{ from different poles}}$. From this visualize what happens when s approaches a pole, and what happens when s approaches a zero. As an example, sketch the magnitude and phase for $H(s) = 1/s$ at points $|s| = 1$ lying on a circle of unit radius around the origin of the s-plane as well as for $H'(s) = (s+5)/(s-5)$ at points $\text{Re}[s] = 0$ which is the $j\omega$ axis.

5 What is the LT of an impulse $\delta(t)$? From this, find the LT of the inverse system $g(t)$ of a system $h(t)$ when $H(s)$ is in rational polynomial form. Relate the pole/zero positions of $H(s)$ and $G(s)$.

6 Construct at least two examples of systems $H_1(s) : R_1$ $H_2(s) : R_2$ where the ROC R of the sum of the two systems is larger than the intersection $R_1 \cap R_2$. Deduce the principle underlying such examples.

$\sin\omega_0 t$





$$\sin\omega_0 t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\sin\omega_0 t\, u(t) \longleftrightarrow \frac{\int_0^\infty e^{j\omega_0 t} e^{-st}\,dt - \int_0^\infty e^{-j\omega_0 t} e^{-st}\,dt}{2j}$$

$$① \rightarrow = \frac{\int_0^\infty e^{-t(s-j\omega)}\,dt - \int_0^\infty e^{-t(s+j\omega_0)}\,dt}{2j}$$

$$= \frac{+\frac{1}{s-j\omega} - \frac{1}{s+j\omega}}{2j}$$

$$= \frac{\omega_0}{s^2+\omega_0^2}$$

Roc is

For ① to be convergent.

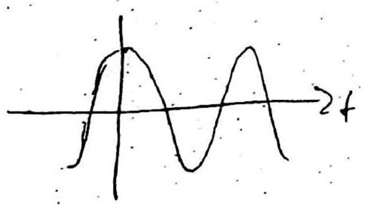
$(s-j\omega_0) > 0$ & $(s+j\omega_0) > 0$

$\Rightarrow Re(s) > 0$.

THIS ANSWER IS FOR $[\sin\omega_0 t\ u(t)]$.
Because for $\sin\omega_0 t$ we canot find LT as it is double sided.

$\cos\omega_0 t$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$



$$\cos\omega_0 t\, u(t) \longleftrightarrow \frac{\int_0^\infty e^{j\omega_0 t} e^{-st}\,dt + \int_0^\infty e^{-j\omega_0 t} e^{-st}\,dt}{2}$$

$$\frac{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}}{2}$$

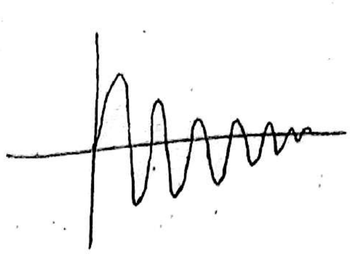
$$= \frac{s}{s^2+\omega_0^2}.$$

Roc is

$Re(s) > 0$.

similarly for $\cos\omega_0 t$ we cant find Laplace tranform as it is double sided. so above is the solution for $[\cos(\omega_0 t)\ u(t)]$.

$\Rightarrow e^{-at} u(t) \sin\omega_0 t \; ; \; a > 0$



$\sin\omega_0 t \, u(t) \longleftrightarrow \dfrac{\omega_0}{s^2 + \omega_0^2}$

$e^{-at} \sin\omega_0 t \, u(t) \longleftrightarrow \dfrac{\omega_0}{(s+a)^2 + \omega_0^2}$

ROC is $Re(s) > -a$.

$$e^{-at} \sin\omega_0 t \, u(t) \longleftrightarrow \frac{\int_0^\infty e^{-at} e^{j\omega_0 t} e^{-st} \, dt - \int_0^\infty e^{-at} e^{-j\omega_0 t} e^{-st} \, dt}{2j}$$

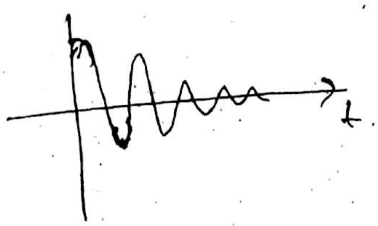
$$\frac{\int_0^\infty e^{-t(s+a-j\omega)} \, dt - \int_0^\infty e^{-t(s+a+j\omega)} \, dt}{2j}$$

For convergence.

$(s+a) > 0$

$\Rightarrow Re(s) > -a$.

$\Rightarrow e^{-at} u(t) \cos\omega_0 t$



$$e^{-at} u(t) \cos\omega_0 t \longleftrightarrow \frac{\int_0^\infty e^{-at} e^{j\omega_0 t} e^{-st} \, dt + \int_0^\infty e^{-at} e^{-j\omega_0 t} e^{-st} \, dt}{2}$$

$$① \leftarrow \frac{\int_0^\infty e^{-t(a+s-j\omega)} \, dt + \int_0^\infty e^{-t(s+a+j\omega)} \, dt}{2}$$

For convergence.

from ①

$Re(s+a) > 0$.

$Re(s) > -a \Rightarrow \dfrac{\dfrac{1}{s+a-j\omega} + \dfrac{1}{s+a+j\omega}}{2}$

$\Rightarrow \dfrac{s+a}{(s+a)^2 + \omega^2}$

$$e^{-a|t|}\sin\omega_0 t = u(t)e^{-at}\sin\omega_0 t + e^{at}u(-t)\sin\omega_0 t$$

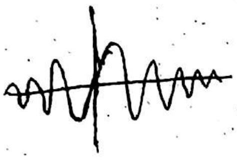
$$e^{-at}u(t)\sin\omega_0 t \longleftrightarrow \frac{\int_0^\infty e^{-at}e^{j\omega t}e^{-st}\,dt - \int_0^\infty e^{-at}e^{-j\omega t}e^{-st}\,dt}{2j}$$

$$\frac{\int_0^\infty e^{-t(s+a-j\omega)}\,dt - \int_0^\infty e^{-t(s+a+j\omega)}\,dt}{2j}$$

For Convergence,

$Re(s+a) > 0$

$Re(s) > -a$.



$$= \frac{\frac{1}{s+a-j\omega_0} - \frac{1}{(s+a+j\omega_0)}}{2j}$$

$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

$$e^{at}u(-t)\sin\omega_0 t \longleftrightarrow \frac{\int_{-\infty}^{0} e^{at}e^{j\omega t}e^{-st}\,dt - \int_{-\infty}^{0} e^{at-j\omega t}e^{-st}\,dt}{2j}$$

$$\frac{\int_{-\infty}^{0} e^{t(a-s+j\omega)}\,dt - \int_{-\infty}^{0} e^{t(a-j\omega-s)}\,dt}{2j}$$

For Convergence

$\cancel{(a+s) < 0}$

$\cancel{(s-a) > 0}$

$\cancel{Re(s) > a}$.

$(a-s) > 0$

$s-a < 0$

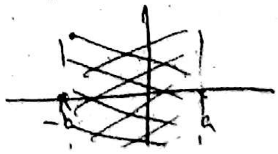
$Re(s) < a$.

$$\frac{\frac{1}{a-s+j\omega} - \frac{1}{(a-s-j\omega)}}{2j}$$

$$\frac{-\omega_0}{(s-a)^2+\omega_0^2}$$

Laplace transform $= \dfrac{\omega_0}{(s+a)^2+\omega_0^2} \quad \dfrac{-\omega_0}{(s-a)^2+\omega_0^2}$

ROC is $-a < Re(s) < a$



$$e^{-a|t|}\cos\omega_0 t = e^{-at}u(t)\cos\omega_0 t + e^{at}u(-t)\cos\omega_0 t$$

$$e^{-at}u(t)\cos\omega_0 t \longleftrightarrow \frac{\int_0^{\infty} e^{-at}\, e^{j\omega_0 t}\, e^{-st}\, dt + \int_0^{\infty} e^{-at}\, e^{-j\omega_0 t}\, e^{-st}\, dt}{2}$$

For Convergence,

$(a+s) > 0$

$Re(s) > -a$,

$$\frac{\int_0^{\infty} e^{-t(a+s-j\omega)}\, dt + \int_0^{\infty} e^{-t(s+a+j\omega)}\, dt}{2}$$

$$\Rightarrow \frac{\frac{1}{s+a-j\omega} + \frac{1}{s+a+j\omega}}{2} \Rightarrow \frac{s+a}{(s+a)^2+\omega_0^2}$$

$$e^{at}u(-t)\cos\omega_0 t \longleftrightarrow \frac{\int_{-\infty}^{0} e^{at}\, e^{j\omega_0 t}\, e^{-st}\, dt + \int_{-\infty}^{0} e^{at}\, e^{-j\omega_0 t}\, e^{-st}\, dt}{2}$$

$$= \frac{\int_{-\infty}^{0} e^{t(a-s+j\omega)}\, dt + \int_{-\infty}^{0} e^{t(a-s-j\omega)}\, dt}{2}$$

To be Convergence

$(a-s) > 0$

$(s-a) < 0$

$Re(s) < a$.

$$\frac{\frac{1}{a-s+j\omega} + \frac{1}{(a-s-j\omega)}}{2}$$

$$= \frac{-(s-a)}{(s-a)^2+\omega_0^2}.$$

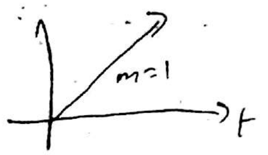
$$\therefore\ x(t) \longleftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2} - \frac{(s-a)}{(s-a)^2+\omega_0^2}$$

ROC is $-a < Re(s) < a$

$$\Rightarrow t u(t) \longleftrightarrow \int_0^\infty t e^{-st} dt.$$

$$\left. \frac{t e^{-st}}{-s} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt.$$

$$\left. \frac{t e^{-st}}{-s} \right]_0^\infty + \left. (-1/s^2) e^{-st} \right]_0^\infty$$



$$\Rightarrow \quad 1/s^2$$

For Convergence.

in e^{-st} $\quad -s < 0$

$s > 0$

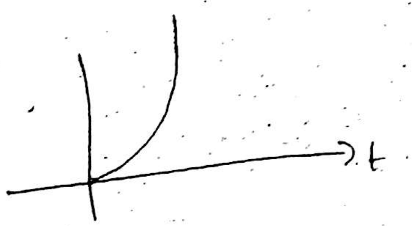
$\therefore Re(s) > 0.$

$$\Rightarrow t^k u(t) \longleftrightarrow \int_0^\infty t^k e^{-st} dt.$$

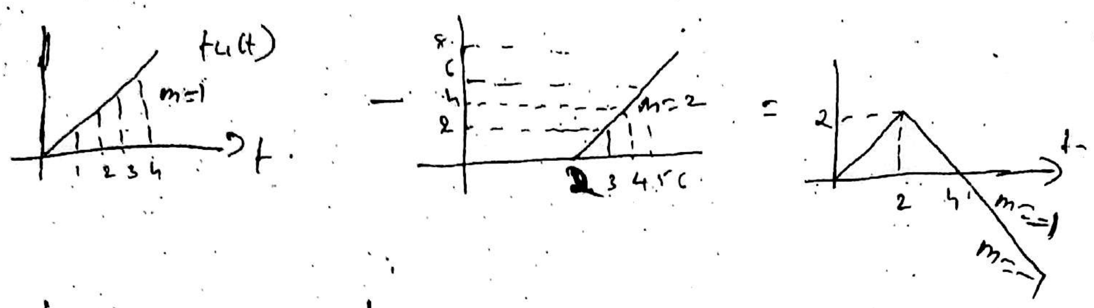
$$\left. \frac{t^k e^{-st}}{-s} \right]_0^\infty - - - - \quad \left. -\frac{k!}{s^{k+1}} e^{-st} \right]_0^\infty$$

$$\Rightarrow \quad \frac{k!}{s^{k+1}}$$

ROC is $\quad Re(s) > 0.$



→ $tu(t) - 2(t-2)u(t-2)$



$tu(t) \longleftrightarrow \frac{1}{s^2}$

$(t-2)u(t-2) \longleftrightarrow \frac{e^{-2s}}{s^2}$

ROC is $Re(s) > 0$

$x(t) \longleftrightarrow \frac{1-2e^{-2s}}{s^2}$

$$\Rightarrow \left[t u(t) - 2(t-2)u(t-2) + (t-4)u(t-4)\right] e^{-at}$$

$$\cancel{t} e^{-at} u(t) \longleftrightarrow \cancel{\frac{t}{(s)}} \qquad \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(a+s)t} dt$$

$$\cancel{(t u(t))} \; e^{-at} u(t) \longleftrightarrow \frac{1}{(s+a)} \qquad = \frac{1}{(s+a)}$$

Roc is $Re(s) > -a$.

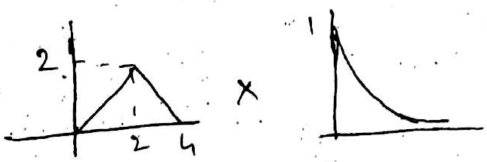
$$t e^{-at} u(t) \longleftrightarrow \frac{-d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$$

$$(t-2)u(t-2) e^{-a(t-2)} \times e^{-2a} \longleftrightarrow \frac{e^{-2a} \times e^{-2s}}{(s+a)^2}$$

$$(t-4)u(t-4) e^{-a(t-4)} e^{-4a} \longleftrightarrow \frac{e^{-4a} \times e^{-4s}}{(s+a)^2}$$

$$\therefore LT[\text{of given}] = \frac{1}{(s+a)^2}\left[1 - 2e^{-2a} e^{-2s} + e^{-4a} e^{-4s}\right]$$

Roc is $Re(s) > -a$



Product of two signals.

Q. 2 given $h(t) = h^*(t)$

$$\mathcal{L}(h(t)) = \int_{-\infty}^{\infty} e^{-st} h(t)\, dt = H(s) \longrightarrow (1)$$

$$\mathcal{L}(h^*(t)) = \int_{-\infty}^{\infty} e^{-st} h^*(t)\, dt$$

$$= \left[\int_{-\infty}^{\infty} e^{-s^*t} h(t)\, dt\right]^*$$

$$= \{H(s^*)\}^* \longrightarrow (2)$$

But $h(t) = h^*(t)$

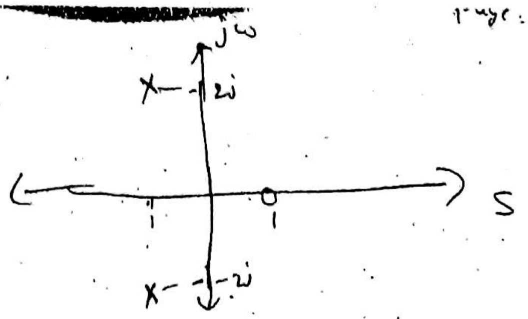
hence

$$\mathcal{L}(h(t)) = \mathcal{L}(h^*(t))$$

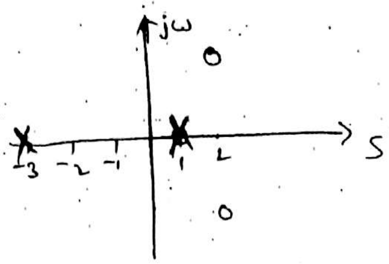
Hence From (1) & (2)

Ans $\boxed{H(s) = H^*(s^*)}$

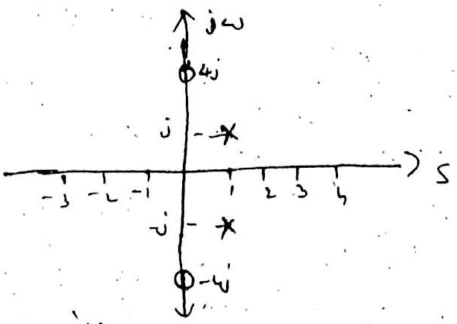
3 (a) $H_1(s) = \dfrac{s-1}{(s+1-j2)(s+1+j2)}$



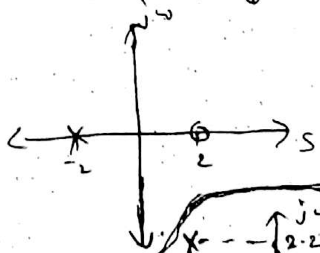
(b) $\dfrac{(s-2-j1)(s-2+j1)}{(s+3)(s-1)}$



(c) $H_3(s) = \dfrac{(s-j4)(s+j4)}{(s-1-j)(s-1+j)}$

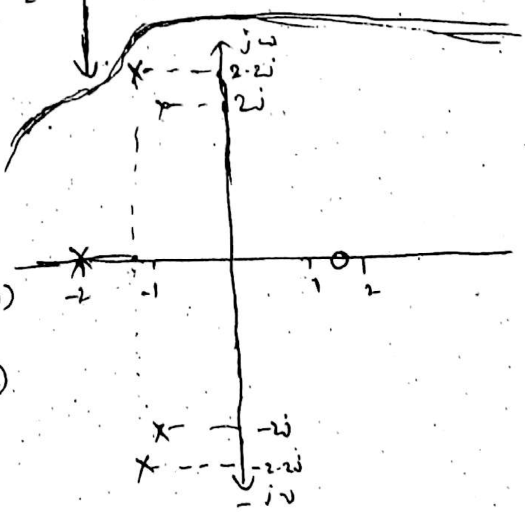


(d) $H_4(s) = \dfrac{(s-2)}{(s+2)}$



(e) $H_5(s) = H_1(s) + H_4(s)$

$$= \frac{(s-1.71)(s+1.37-2.2j)(s+1.37+2.23j)}{(s+1-2j)(s+1+2j)(s+2)}$$



(f) $H_6(s) = H_2(s)H_4(s)$. → It contains all the poles & zeros of $H_2(s)$ & $H_4(s)$.

3

1) Given that system has causal time response.

$\Rightarrow$ ROC is $Re(s) > \sigma_{max}$.

For the causal system, the condition for stability is that ROC of $H(s)$ contains $j\omega$ axis.

(a) $\frac{s-1}{(s+1+j2)(s+1+j2)}$ $\Rightarrow$ ROC is $Re(s) > -1$

rightsided & contains $j\omega$ axis

$\therefore$ It is stable.

(b) $H_2(s) = \frac{(s-2-j1)(s-2+j1)}{(s+3)(s-1)}$ $\Rightarrow$ ROC is $Re(s) > 1$

right sided & doesnot contain $j\omega$ axis

$\therefore$ It is unstable.

(c) $H_3(s) = \frac{(s-j4)(s+j4)}{(s-1-j1)(s-1+j1)}$ $\Rightarrow$ ROC is $Re(s) > 1$

right sided & doesnot contain $j\omega$ axis.

$\therefore$ It is unstable.

(d) $H_4(s) = \frac{s-2}{s+2}$ $\Rightarrow$ ROC is $Re(s) > -2$

right sided & contains $j\omega$ axis.

$\therefore$ It is stable.

(e) $H_5(s) = H_1(s) + H_4(s)$ $\Rightarrow$ ROC is $(Re(s) > -1) \cap (Re(s) > -2)$

$\Rightarrow Re(s) > -1$

right sided & contains $j\omega$ axis

(f) $H_6(s) = H_2(s) H_4(s)$ $\Rightarrow$ ROC is $Re(s) > 1$

rightsided & doesnot contain $j\omega$ axis.

$\therefore$ It is unstable

3 (ii). Given that system is stable.
$\Rightarrow$ ROC contains $j\omega$ axis.

A system is causal if the ROC is $Re(s) > \sigma$
A system is anticausal if the ROC is $Re(s) < \sigma$
A system is acausal if the ROC is $\sigma_1 < Re(s) < \sigma_2$.

(a) ROC is $Re(s) > -1$ ($\because$ it is stable, it contains $j\omega$ axis)
$\Rightarrow$ ~~ROC~~ It is rightsided.
$\therefore$ It is causal.

~~(b) ROC is $Re(s) < 1$. ($\because$ It is stable, it contains $j\omega$ axis)~~
~~$\Rightarrow$ It is left sided~~
~~$\therefore$ It is anticausal.~~

(b) ~~(a)~~ ROC is $-3 < Re(s) < 1$ ($\because$ It is stable, it contains $j\omega$ axis)

$\Rightarrow$ It is acausal.

(c). ROC is $Re(s) < 1$
It is left sided.
$\therefore$ It is anticausal.

(d). ROC is $Re(s) > -2$
$\Rightarrow$ It is causal.

(e) ROC is $(Re(s) > -1) \cap (Re(s) > -2)$
$\Rightarrow$ ROC is $Re(s) > -1$
$\Rightarrow$ It is causal.

(f) ROC is $-3 < Re(s) < 1$
$\Rightarrow$ It is acausal

3 (iii)

(a) $H_1(s)\Big|_{s=0} \neq 0$; $H_1(s)\Big|_{s=\infty} = 0$.

∴ It is lowpass filter.

(b) $H_2(s)\Big|_{s=0} \neq 0 \; \& \neq 1$; $H_2(s)\Big|_{s=\infty} \neq 0 \; \& \; 1$

∴ It is ~~bandpass filter~~ bandstop filter

(c) $H_3(s)\Big|_{s=0} \neq 0 \neq 1$; $H_3(s)\Big|_{s=\infty} \neq 0$

∴ It is ~~bandpass filter~~ bandstop filter.

(d) $H_4(s)\Big|_{s=0} = 1$; $H_4(s)\Big|_{s=\infty} = 1$

∴ It is allpass filter.

(e) $H_5(s)\Big|_{s=0} \neq 0, \neq 1$; $H_5(s)\Big|_{s=\infty} \neq 0; \neq 1$

It is ~~bandpass filter~~ bandstop filter.

(f) $H_6(s)\Big|_{s=0} \neq 0; \neq 1$; $H_6(s)\Big|_{s=\infty} \neq 0; \neq 1$

It is bandstop filter.

Sol. 4

The value of H(S) when s approaches a pole = ∞

The value of H(S) when s approaches a zero = 0

Since;

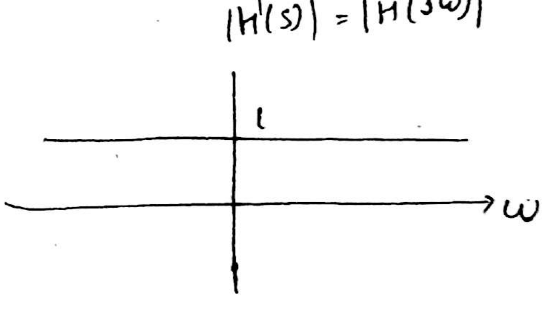
$$\text{Value of } H(s)\Big|_{s=s_0} = \frac{\text{Product of dist. of } s_0 \text{ from diff. zeros}}{\text{Product of dist. of } s_0 \text{ from diff. poles}}$$

So when s_0 approaches a pole, distance from that pole becomes zero. and hence the denominator becomes zero.

when s_0 approaches a zero, distance from that zero to s_0 becomes zero. and hence the Numerator of the above rational polynomial becomes zero.

(i)
$$H'(s) = \frac{s+5}{s-5}$$

$$|H'(s)| = |H'(j\omega)|$$



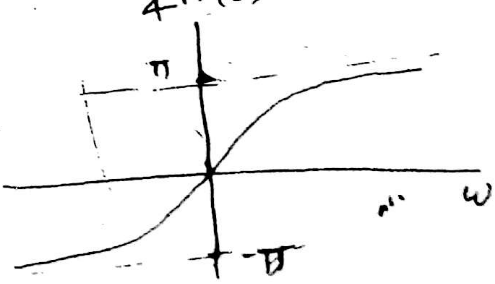
$$H'(s)\Big|_{Re(s)=0} = \frac{j\omega+5}{j\omega-5}$$

$$|H(s)| = \frac{\sqrt{5^2+\omega^2}}{\sqrt{5^2+\omega^2}} = 1$$

$$\measuredangle H(s) = \tan^{-1}\frac{\omega}{5} - \tan^{-1}\frac{\omega}{-5}$$

$$= 2\tan^{-1}\frac{\omega}{5}$$

$$\measuredangle H'(s) = \measuredangle H'(j\omega)$$



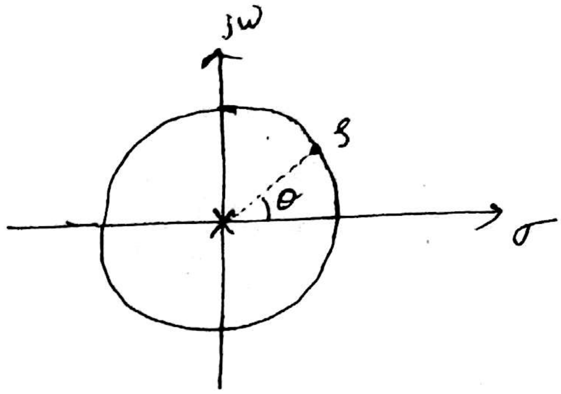
— 4(ii) corrected.

(ii) $H(s) = \frac{1}{s}$, $|s| = 1$

$\omega = s \sin\theta$

$\sigma = s \cos\theta$

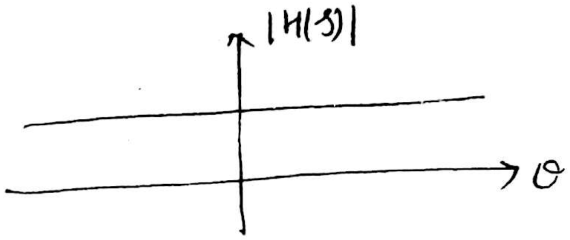
$\frac{\omega}{\sigma} = \tan\theta$.



Pole-zero plot

<u>magnitud</u>

$|H(s)| = \frac{1}{|s|} = 1$



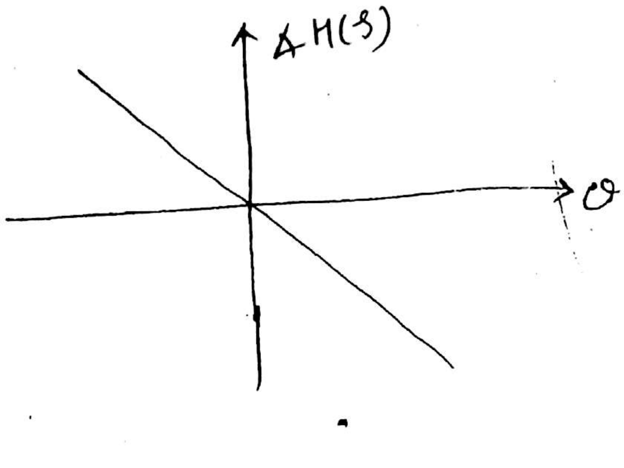
<u>phase</u>

$H(s) = \frac{1}{\sigma + j\omega}$

$\angle H(s) = -\tan^{-1}\left(\frac{\omega}{\sigma}\right)$

$= -\tan^{-1}(\tan\theta)$ $\quad \because \tan\theta = \frac{\omega}{\sigma}$

$= -\theta$



Assignment-10

(5) Laplace transform of $\delta(t)$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) dt$$

$$= 1$$

Area under impulse is unity

LT of $\delta(t) = 1$

→ $g(t)$ is the inverse system of $h(t)$

$LT(h(t)) = H(s)$

$LT(g(t)) = G(s)$

$G(s) = \frac{1}{H(s)}$ as it is inverse.

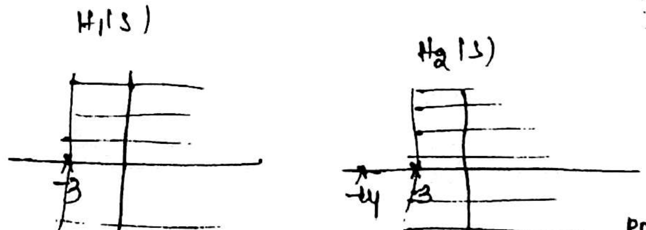
Poles of $H(s)$ will become zero's of $G(s)$

and zero's of $H(s)$ will become poles of $G(s)$

(6) Example 1:

consider $H_1(s) = \frac{1}{s+3}$ ROC $Re(s) > -3$

$H_2(s) = \frac{1}{(s+3)(s+4)}$ $Re(s) > -3$.



ROC is right of the right most pole.

consider $H(s) = H_1(s) - H_2(s)$

$$= \frac{1}{s+3} - \frac{1}{(s+3)(s+4)}$$

$$= \frac{s+4-1}{(s+3)(s+4)}$$

$$= \frac{s+3}{(s+3)(s+4)}$$

$$= \frac{1}{s+4}$$

ROC of $H(s) = Re(s) > -4$

$H_1(s)$ ROC $Re(s) > -3$

$H_2(s)$ ROC $Re(s) > -3$

But $H(s)$ ~~should~~ ROC $Re(s) > -4$.

ROC of $H(s)$ is greater ... the

ROC $(H_1(s)) \cap$ ROC $(H_2(s))$

... underlying in such examples is

... cancellation occurs ...

... ROC will become larger than the

Q.6

Example ①

$$H_1(s) = \frac{1}{s}, \quad Re(s) > 0 \; : R_1$$

$$H_2(s) = \frac{1}{s+1}, \quad Re(s) > -1 \; : R_2$$

then

$$H_1(s) - H_2(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$= \frac{s+1-1}{s(s+1)}$$

$$= \frac{\cancel{s}}{\cancel{s}(s+1)}$$

$$= \frac{1}{s+1} \qquad ROC: Re(s) > -1$$

thus ROC of $H_1(s) - H_2(s)$ is larger than $R_1 \cap R$