EE 200
Signals, Systems & Networks

Assignment 8

1 If a periodic signal $x(t) = x(t-T)$ satisfies $x(t) = -x(t-T/2)$, it is said to possess *half wave symmetry*. Show that for such a signal, all the even-numbered FS coefficients are equal to zero: $x_k = 0; k$ even.

2 (a) Find the Fourier series expansion of a square wave $x(t) = x(t-T) = \begin{cases} 1; & 0 \le t < T' \\ -1; & T' \le t < T \end{cases}$. The ratio $d = T'/T$ is called the *duty cycle* of the square wave. Find how the FS coefficients change as $d \to 1$.

(b) Instead, let $x(t) = x(t-T) = \begin{cases} V_a; & 0 \le t < T' \\ V_b; & T' \le t < T \end{cases}$ where $T'V_a = -(T-T')V_b$. What is the average value of $x(t)$? Find how the FS coefficients change as $d \to 1$.

3 Let $h(t), h'(t)$ be the impulse responses of two LTI systems which satisfy $h(t) * h'(t) = \delta(t)$: the two systems are said to be *inverses* of one another.

(a) Does *any* LTI system have an inverse?

(b) Try to find the conditions under which a system can have an inverse.

(c) Find the relationship between $H(\omega), H'(\omega)$.

4 Find the Continuous Time Fourier Transform (CTFT) of a rectangular time pulse: $x(t) = \begin{cases} 1; & |t| \le T_1 \\ 0; & |t| > T_1 \end{cases}$. How will it differ from the CTFT of $x'(t) = \begin{cases} 1; & |t| < T_1 \\ 0; & |t| \ge T_1 \end{cases}$? Next find the inverse CTFT of a rectangular pulse: $X(\omega) = \begin{cases} 1; & |\omega| \le W \\ 0; & |\omega| > W \end{cases}$. Use these results to show that if $x(t)$ has finite support, $X(\omega)$ will necessarily have infinite support on the ω axis.

5 So far, we have been using the term 'causal' only to describe a system: by analogy with the property of the impulse response of an LTI causal system, a signal $x(t)$ is said to be *causal* if $x(t) = 0;\ t < 0$ and said to be *anticausal* if $x(t) = 0;\ t \ge 0$.

(a) Find the FT of $u(t) + u(-t)$.

(b) Any signal $x(t)$ can be subjected to a *causal-anticausal* decomposition: $x(t) = x_c(t) + x_a(t) = x(t)(u(t) + u(-t))$. Find the inner product $\langle x_c(t), x_a(t)\rangle$. Find $X_c(\omega)$ and $X_a(\omega)$. Find the inner product $\langle X_c(\omega), X_a(\omega)\rangle$

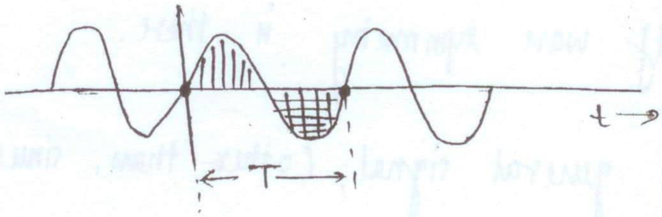
6 If $x(t) = x_e(t) + x_o(t)$ is its even-odd decomposition, find the respective CTFTs $X_e(\omega), X_o(\omega)$ of the components. If we have a causal signal $x(t) \leftrightarrow X(\omega)$, express $X(\omega)$ in terms of $X_e(\omega)$ alone. Then express $X(\omega)$ in terms of $X_o(\omega)$ alone.

Solution

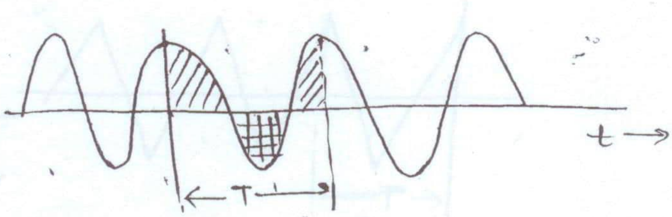
Assignment #7

Sol^n 1 :- Consider a sinusoidal signal

sin wot



or cos wot



Clearly these signals are half wave symmetric signals, as we can see clearly that for these,

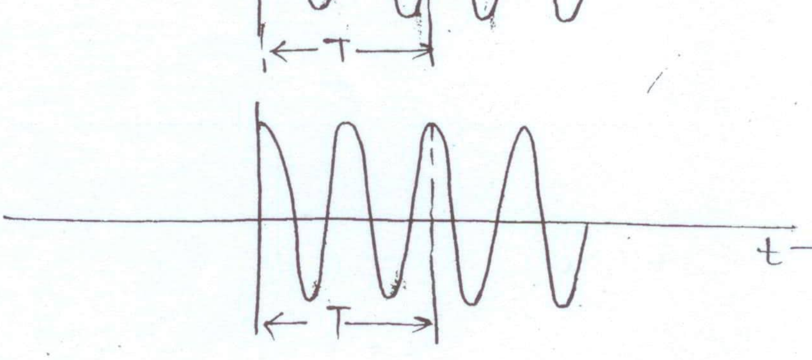
$$x(t) = -x(t - T/2)$$

* Now, take even harmonics of these signals,

for example: sin 2wot, sin 4wot ...

or cos 2wot, cos 4wot ...

sin 2wot



Here in one fundamental period, if we see, we won't find ~~it~~ any half wave symmetry.

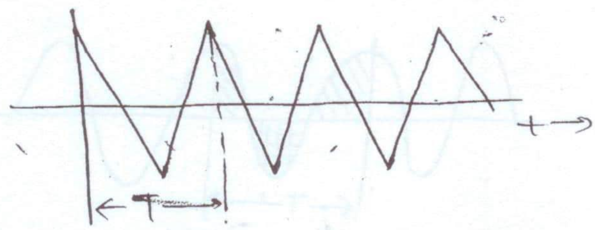
If we go for odd harmonics, like

$\sin 3\omega_0 t$, $\sin 5\omega_0 t$ . .

$\cos 3\omega_0 t$, $\cos 5\omega_0 t$ - -

for these signals, half wave symmetry is there.

Now if we take any general signal (other than sinusoidal)



Clearly for this half wave symmetric signal, all the fourier coefficients should be half wave symmetric.

So for that even harmonics of sin & cosine should not be there.

So, for any half wave symmetric signal,

All the even numbered F.S. coefficients are equal to zero; $x_K = 0$; K even.

Q.2) a)

$$a_0 = \frac{1}{T}\left[\int_0^{T'} dt + \int_{T'}^{T}(-dt)\right]$$

$$= \frac{T' - (T - T')}{T}$$

$$= 2d - 1 \quad \because d = T'/T$$

$$a_n = \frac{2}{T}\left[\int_0^{T'} \cos(n\omega_0 t)\,dt - \int_{T'}^{T} \cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\Big|_0^{T'} - \frac{\sin(n\omega_0 t)}{n\omega_0}\Big|_{T'}^{T}\right]$$

$$= \frac{2}{n\omega_0 T}\left[2\sin(n\omega_0 T') - \sin(n\omega_0 T)\right]$$

$$= \frac{2}{n\omega_0 T}\left[2\sin(n\omega_0 dT) - \sin(n\omega_0 T)\right]$$

$$b_n = \frac{2}{T}\left[\int_0^{T'} \sin(n\omega_0 t)\,dt - \int_{T'}^{T} \sin(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{n\omega_0 T}\left[-\cos(n\omega_0 t)\Big|_0^{T'} + \cos(n\omega_0 t)\Big|_{T'}^{T}\right]$$

$$= \frac{2}{n\omega_0 T}\left[1 + \cos(n\omega_0 T) - 2\cos(n\omega_0 dT)\right]$$

b) Given $x(t) = x(t-T) = V_a \;;\; 0 \leq t < T'$

$$= V_b \;;\; T' \leq t < T.$$

Average value of $x(t) = \frac{1}{T}\left\{\int_0^{T'} V_a\,dt + \int_{T'}^{T} V_b\,dt\right\}$

$$= \frac{1}{T}\left\{V_a T' + V_b(T - T')\right\}$$

$$= 0 \quad \because V_a T' = -V_b(T - T')$$

a_0 = avg value of $x(t) = 0$.

$$a_n = \frac{2}{T}\left[\int_0^{T'} V_a \cos(n\omega_o t)\,dt + \int_{T'}^{T} V_b \cos(n\omega_o t)\,dt\right]$$

$$= \frac{2}{n\omega_o T}\left[(V_a - V_b)\sin(n\omega_o T') + V_b \sin(n\omega_o T)\right]$$

$$= \frac{2}{n\omega_o T}\left[(V_a - V_b)\sin(n\omega_o dT) + V_b \sin(n\omega_o T)\right]$$

$$b_n = \frac{2}{T}\left[\int_0^{T'} V_a \sin(n\omega_o t)\,dt + \int_{T'}^{T} V_b \sin(n\omega_o t)\,dt\right]$$

$$= \frac{2}{n\omega_o T}\left[-V_a\cos(n\omega_o T') + V_a - V_b\cos(n\omega_o T) + V_b\cos(n\omega_o T')\right]$$

$$= \frac{2}{n\omega_o T}\left[(V_b - V_a)\cos(n\omega_o T') + V_a - V_b\cos(n\omega_o T)\right]$$

Note: We can also obtain exponential coefficients.

$C_0 = a_0$

$C_n = \frac{a_n - jb_n}{2}$

$C_{-n} = \frac{a_n + jb_n}{2}$

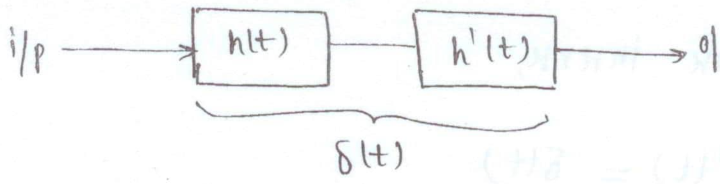
We can also obtain polar coefficient.

$d_0 = a_0$

$d_n = \sqrt{a_n^2 + b_n^2}$.

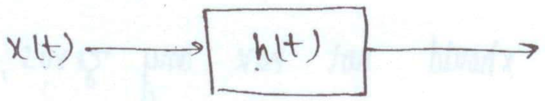
$\theta_n = \tan^{-1}(-b_n/a_n)$.

Sol^n 3 :-



(a) No, ~~All LTI systems~~ does not ~~h~~ Here is an example of a many to one system, which does not have an inverse.

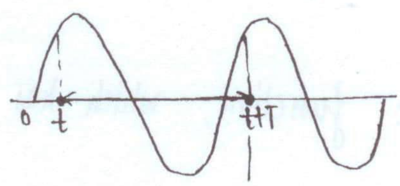
Take a system, which averages the i/p signal in one fundamental period.



$$o/p = \frac{1}{T}\int_{t}^{t+T} x(t')\,dt'$$

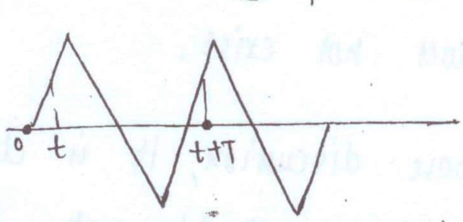
suppose if

$x(t) = \sin w_0 t$



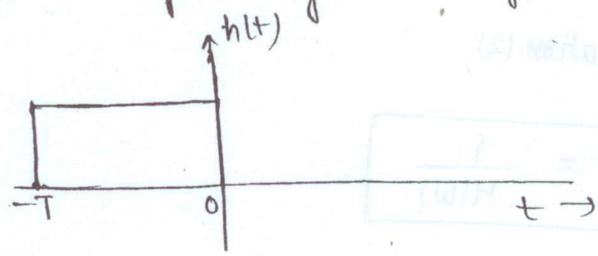
or

$x(t) =$



for these on averaging for one period the o/p will become zero.

for this to possess, impulse response of the system is a rectangular



function, ~~with~~ shown in figure.

Clearly this is a shifted rectangular function. So fourier transform of this will be a sinc function in multiplication with an exponential term (to compensate shift).

Refer Prob. 4 Solution.

So, $H(\omega) =$ (exponential term) $\times$ sinc function ————

Now, for system, to have inverse,

$$h(t) * h'(t) = \delta(t)$$

Take fourier transform,

$$H(\omega) \cdot H'(\omega) = 1$$

or $$H'(\omega) = \frac{1}{H(\omega)} \quad \text{———— (2)}$$

Now, from eq^n (2),

$H'(\omega)$ to exist, $H(\omega)$ should not have any zeros, because at those points, $H'(\omega)$ will not be defined.

but from eq^n (1),

$H(\omega)$ contains a sinc function, which has zeros at regular intervals.

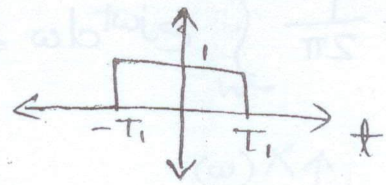
So for this system, inverse does not exist.

Sol^n 3(b):- Clearly from the above discussion, it is clear that for a system to have an inverse, it should not have any zeros.

Sol^n 3(c):- from equation (2),

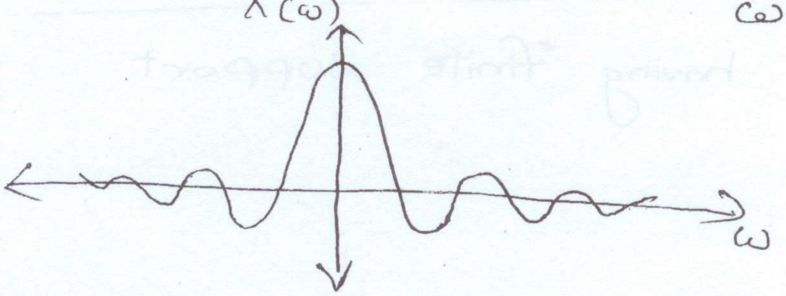
$$\boxed{H'(\omega) = \frac{1}{H(\omega)}}$$

Q.4) $x(t) = \begin{cases} 1 & ; \ |t| \le T_1 \\ 0 & ; \ |t| > T_1 \end{cases}$



$$X(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega}$$

$$= \frac{2\sin(\omega T_1)}{\omega}$$



Consider $x'(t) = \begin{cases} 1 & ; \ |t| < T_1 \\ 0 & ; \ |t| > T_1 . \end{cases}$

$x(t)$, $x'(t)$ differ only at 2pts which are discontinuties. Thus in any case the CTFT in both cases ~~ref~~ will reconstruct signals which average to the midpoints of the discontinuties.

$x(t) \xrightarrow{CTFT} X(\omega) \xrightarrow{CTFT^{-1}} \hat{x}(t).$

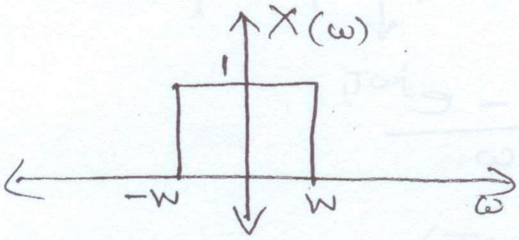
$x'(t) \xrightarrow{CTFT} X'(\omega) \xrightarrow{CTFT^{-1}} \hat{x}'(t).$

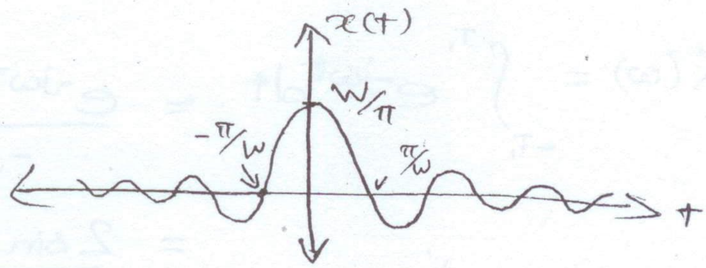
Though $x(t) \neq x'(t)$, $\hat{x}(t) = \hat{x}'(t)$.

Also $= \begin{cases} x(t) = x'(t) & ; \ |t| \neq T_1 . \\ \tfrac{1}{2} & ; \ |t| = T_1 . \end{cases}$

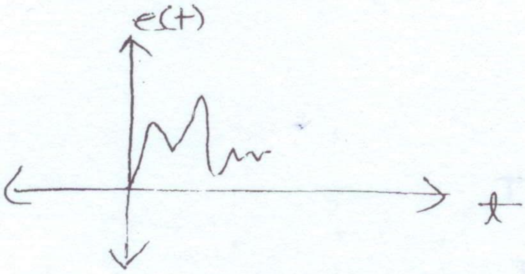
$X(\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t}$$

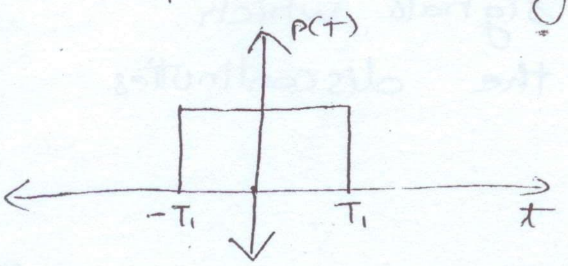




Consider any signal e(t) having finite support as shown below.



Any signal of the form e(t), having finite support, can be written as $e(t) = e(t) * p(t)$, ① where p(t) is rectangular pulse as shown below



Taking fourier transform of above eq^n we get on Right side as $\frac{e(\omega) * p(\omega)}{2\pi}$. As we all know that $p(\omega)$ = sinc fun^n which has got infinete support. And convolution of signal with signal having infinete support makes convolution infinete.

∴ Left side e(ω) becomes infinete. Hence proved.

Solution (5)(a): fourier transform of $u(t) + u(-t)$

Method 1: Clearly $u(t) + u(-t) = 1 \xleftrightarrow{F.T.} 2\pi\delta(\omega)$

or

Method 2:

As discussed in the class,

$$u(t) \xrightarrow{F.T.} \frac{1}{j\omega} + \pi\delta(\omega)$$

so $$u(-t) \xrightarrow{F.T.} \frac{-1}{j\omega} + \pi\delta(-\omega)$$

but since $\delta(\omega)$ is an even function,

so $$u(t) + u(-t) \xrightarrow{F.T.} 2\pi\delta(\omega).$$

Ans.

Sol^n 5(b): $x(t) = x_c(t) + x_a(t)$

$x(t) = x(t).u(t) + x(t).u(-t)$

Inner product: $\langle x_c(t), x_a(t)\rangle$

$= \langle x(t).u(t), x(t).u(-t)\rangle$

$= (x(t))^2 \langle u(t), u(-t)\rangle$ $\quad \because$ Inner product of orthogonal signals is zero!

$= (x(t))^2 \times 0$

$= 0$ Ans

$X_c(\omega) =$ F.T. of $x_c(t) =$ F.T. of $x(t).u(t)$

$$= \frac{1}{2\pi}\left[X(\omega) * \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)\right]$$ Ans.

$$X_a(\omega) = \text{F.T. of } x_a(t)$$

$$= \text{F.T. of } x(t).u(-t)$$

$$= \frac{1}{2\pi}\left[X(\omega) * \left(\frac{-1}{j\omega} + \pi\,\delta(\omega)\right)\right] \quad \underline{\text{Ans}}$$

<u>Inner product</u>: $\langle X_c(\omega), X_a(\omega) \rangle$

$$= \left\langle \frac{1}{2\pi}\left(X(\omega) * U(\omega)\right), \frac{1}{2\pi}\left(X(\omega) * U(-\omega)\right) \right\rangle$$

$$= \left(\frac{1}{2\pi}\right)^2 \langle X(\omega) * U(\omega), X(\omega) * U(-\omega) \rangle$$

<u>Ans</u>.

Q.6) $x(t) = x_e(t) + x_o(t)$

$x(-t) = x_e(t) - x_o(t)$

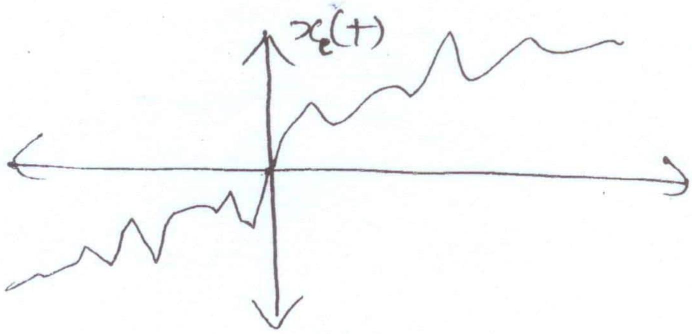
$x_e(t) = \frac{x(t) + x(-t)}{2} \Rightarrow X_e(\omega) = \frac{X(\omega) + X(-\omega)}{2}$.

$x_o(t) = \frac{x(t) - x(-t)}{2} \Rightarrow X_o(\omega) = \frac{X(\omega) - X(-\omega)}{2}$.

for causal s

for $t<0$, $x(t) = 0$; hence $x_e(t) = -x_o(t)$.

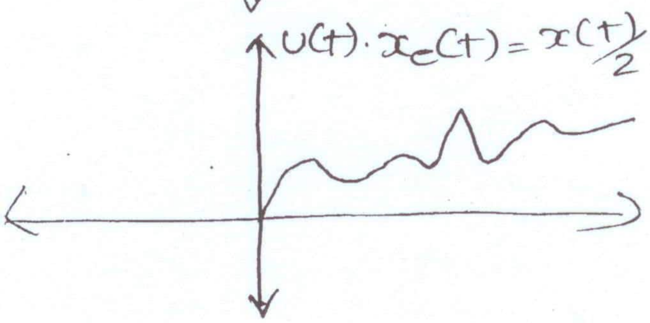
$\therefore$ for $t>0$ $x_e(t) = x_o(t) = \frac{1}{2} x(t)$.



$\therefore x_e(t) \cdot u(t) = \frac{x(t)}{2}$.

Taking F.T. on both sides.

$$\frac{X_e(\omega) * U(\omega)}{2\pi} = \frac{1}{2} X(\omega)$$



Similarly for odd signals, we have.

$$\frac{X_o(\omega) * U(\omega)}{2\pi} = \frac{1}{2} X(\omega).$$