Target : JEE (Main + Advanced)

Conic Section

CONTENT

Conic Section

JEE (ADVANCED) SYLLABUS

Parabola : Equations of a parabola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal,Locus Problems.

Ellipse : Ellipse and hyperbola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal.

Hyperbola : Equations of a hyperbola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal. Locus Problems.

JEE (MAIN) SYLLABUS

Parabola : Sections of cones, equation of parabola in standard form, condition for $y = mx + c$ to be a tangent and point (s) of tangency.

Ellipse : Equation of ellipse in standard form, condition for $y = mx + c$ to be a tangent and point (s) of tangency.

Hyperbola : Equation of hyperbola in standard form, condition for $y = mx + c$ to be a tangent and point (s) of tangency.

Conic Section

Everything should be made as simple as possible, but not simpler...... Einstein, Albert

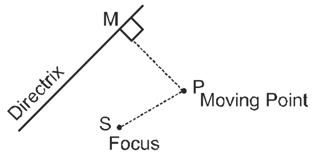
This chapter focusses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone (not necessarily right circular) can be out in various ways by a plane, and thus different types of conic sections are obtained.

Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

1. Definition of Conic Sections:

A conic section or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.



$$\frac{PS}{PM} = e$$

- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

If S is (p, q) & directrix is $\ell x + my + n = 0$

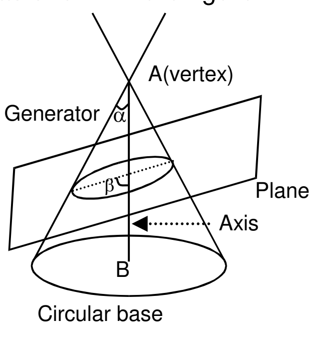
Then PS = $\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ & PM = $\dfrac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$\dfrac{PS}{PM} = e \Rightarrow (\ell^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(\ell x + my + n)^2$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

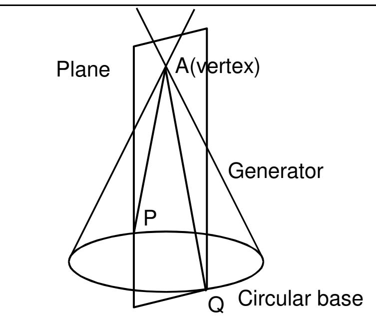
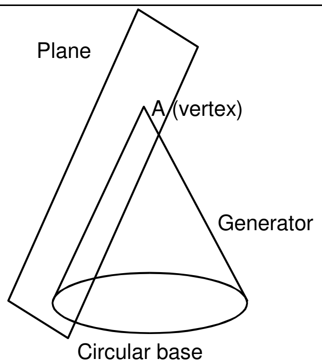
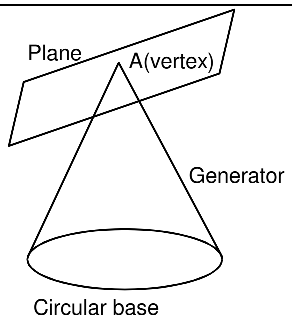
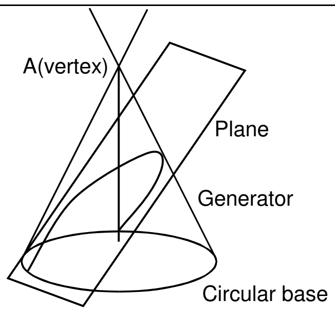
1.1 Section of right circular cone by different planes

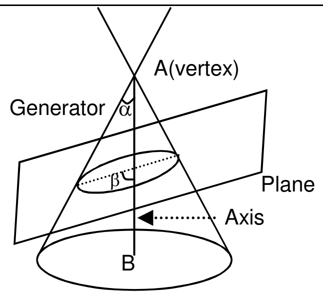
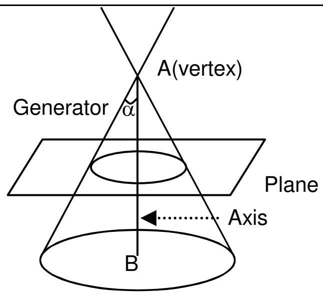
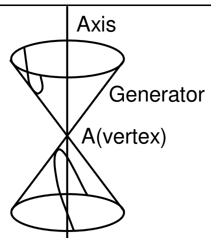
A right circular cone is as shown in the figure – 1



α is angle between generator and axis.
β is angle between plane and axis

Section of a right circular cone by a plane passing through its vertex is a pair of straight lines. Section of a right circular cone by a plane not passing through vertex is either circle or parabola or ellipse or hyperbola which is shown in table below :

Type of conic section	3-D view of section of right circular cone with plane	Condition of conic in definition of conic	condition of conic in $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$
Two distinct real lines	 Plane passes through vertex A and $0 \le \beta < \alpha$	$e > 1$, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 > ab$
Two real same lines	 Plane passes through vertex A and $\beta = \alpha$	$e = 1$, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 = ab,$ (either $g^2 = ac$ or $f^2 = bc$)
Two imaginary lines/point	 Plane passes through vertex A and $\beta > \alpha$	$0 < e < 1$, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 < ab$
Parabola	 Plane does not passes through vertex A and $\beta = \alpha$	$e = 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \ne 0,\ h^2 = ab$

Ellipse	 Circular base Plane does not passes through vertex A and $\alpha < \beta < 90$	$0 < e < 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, h^2 < ab,$ (either $a \neq b$ or $h \neq 0$)
Circle	 Circular base Plane does not passes through vertex A and $\beta = 90$	$e = 0$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0,$ $a = b, h = 0$
Hyperbola	 Plane does not passes through vertex A and $0 \le \beta < \alpha$	$e > 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, h^2 > ab$

Note : (i) Pair of real parallel lines is not the part of conic but it is part of general two degree equation.

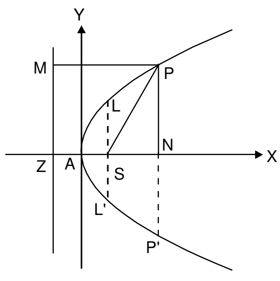
For it $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, h^2 = ab,$ (either $g^2 > ac$ or $f^2 > bc$)

$\Rightarrow$ General two degree equation can represent real curve other than conic section.

(ii) For rectangular hyperbola $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, h^2 > ab, a + b = 0$

2. Elementary Concepts of Parabola

2.1 Definition and terminology of parabola



$y^2 = 4ax$

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix). Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$.

(i) Vertex is (0, 0) **(ii)** focus is (a, 0)

(iii) Axis is y = 0 **(iv)** Directrix is x + a = 0

Focal Distance: The distance of a point on the parabola from the focus.

Focal Chord : A chord of the parabola, which passes through the focus.

Double Ordinate: A chord of the parabola perpendicular to the axis of the symmetry.

Latus Rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.

$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

NOTE :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

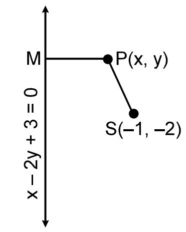
(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

Example # 1: Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix is x – 2y + 3 = 0.

Solution : Let P(x, y) be any point on the parabola whose focus is S(– 1, – 2) and the directrix x – 2y + 3 = 0. Draw PM perpendicular to directrix x – 2y + 3 = 0. Then by definition,

SP = PM



$\Rightarrow$ $SP_2 = PM_2$

$\Rightarrow$ $(x + 1)^2 + (y + 2)^2 = \left(\dfrac{x-2y+3}{\sqrt{1+4}}\right)^2$

$\Rightarrow$ $5\,[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$

$\Rightarrow$ $5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow$ $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

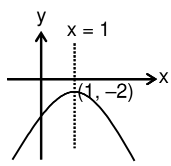
Example#2 : Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches. $x^2 - 2x + 4y + 9 = 0$.

Solution : The given equation is $x^2 - 2x + 4y + 9 = 0$

$\Rightarrow (x - 1)^2 = -4(y + 2)$

which of the form $X_2 = -4bY$

Vertex -



$(X, Y) \equiv (0, 0)$

$(x, y) \equiv (1, -2)$

Axis

$X = 0 \Rightarrow x = 1$

Focus-

$(X, Y) = (0, -b)$

$(x, y) \equiv (1, -1 -2) = (1, -3)$

Directrix -

$Y = b \Rightarrow y + 2 = 1$

$y = -1$

Latusrectum -

The length of the latusrectum of the given parabola is $4b = 4$.

Self Practice Problems :

(1) Find the equation of the parabola whose focus is the point (0, 0)and whose directrix is the straight line $4x - 3y - 2 = 0$.

(2) Find the extremities of latus rectum of the parabola $y = x^2 - 2x + 3$.

(3) Find the latus rectum & equation of parabola whose vertex is origin & directrix is $x + y = 2$.

(4) Find the equation of the parabola whose focus is (–1, 1) and whose vertex is (1, 2). Also find its axis and latusrectum.

Ans. (1) $9x^2 + 16y^2 + 24xy + 16x - 12y - 4 = 0$ (2) $\left(\frac{1}{2}, \frac{9}{4}\right)\left(\frac{3}{2}, \frac{9}{4}\right)$

(3) $4\sqrt{2}$, $x^2 + y^2 - 2xy + 8x + 8y = 0$

(4) $(2y - x - 3)^2 = -20\ (y + 2x - 4)$, Axis $2y - x - 3 = 0$. $LL' = 4\sqrt{5}$.

2.2 Parametric representation of parabola

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Parametric form for : $y^2 = -4ax$ $(-at^2, 2at)$

$x^2 = 4ay$ $(2at , at^2)$

$x^2 = -4ay$ $(2at , -at^2)$

Example # 3: Find the parametric equation of the parabola $(x + 1)^2 = -6\ (y + 2)$

Solution: $\because$ $4a = -6$ $\Rightarrow$ $a = \frac{-3}{2}$, $y + 2 = at^2$

$x + 1 = 2\ at$ $\Rightarrow$ $x = -1 - 3t,\ y = -2 - \frac{3}{2}\ t^2$

Self Practice Problems:

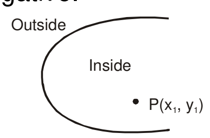
(5) Find the parametric equation of the parabola $x^2 = 4a(y - 1)$

Ans. $x = 2at,\ y = 1 + at^2$

.

2.3 Position of a point relative to a parabola:

The point (x_1 , y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



$S_1 :\ y_1^2 - 4ax_1$

$S_1 < 0 \rightarrow$ Inside

$S_1 > 0 \rightarrow$ Outside

Example # 4 : Check whether the point (4, 5) lies inside or outside the parabola $y^2 = 4x$.

Solution : $y^2 - 4x = 0$

$\because$ $S_1 \equiv y_1^2 - 4x_1 = 25 - 16 = 9 > 0$

$\therefore$ (3, 4) lies outside the parabola.

Self Practice Problems :

(6) Find the set of value's of α for which $(\alpha, -2-\alpha)$ lies inside the parabola $y^2 + 4x = 0$.

Ans. $\alpha \in (-4 - 2\sqrt{3}, -4 + 2\sqrt{3})$

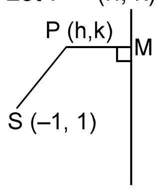
3. Elementary Concepts of Ellipse

3.1 Definition of Ellipse

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point called focus and a fixed line called directrix (not passes through fixed point and all points and line lies in same plane) is constant (e = eccentricity), which is less than one.

Example # 5 : Find the equation to the ellipse whose focus is the point $(-1, 1)$, whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Solution : Let $P \equiv (h, k)$ be moving point,



$$e = \frac{PS}{PM} = \frac{1}{2} \quad \Rightarrow \quad (h+1)^2 + (k-1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ locus of P(h, k) is

$$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$$

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$$

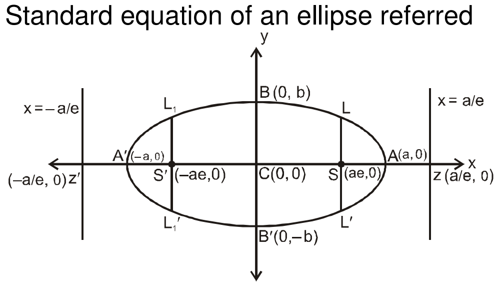
Self Practice Problems :

(7) Find the equation to the ellipse whose focus is (0, 0) directrix is $x + y - 1 = 0$ and $e = \frac{1}{\sqrt{2}}$.

Ans. $3x^2 + 3y^2 - 2xy + 2x + 2y - 1 = 0$.

3.2 Standard Equation of Ellipse

Standard equation of an ellipse referred



to its principal axes along the co–ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2(1 - e^2)$.

Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

Focii : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

Equations of Directrices: $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

Major Axis: The line segment A′A in which the focii S′ & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis: The y–axis intersects the ellipse in the points B′ ≡ (0, – b) & B ≡ (0, b). The line segment B′B is of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices: Point of intersection of ellipse with major axis. A′ ≡ (– a, 0) & A ≡ (a, 0) .

Focal Chord: A chord which passes through a focus is called a focal chord.

Double Ordinate: A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum: The focal chord perpendicular to the major axis is called the latus rectum.

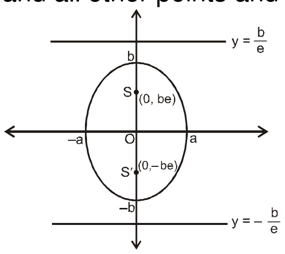
Length of latus rectum **(LL′)** = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a\left(1 - e^2\right)$

= 2 e (distance from focus to the corresponding directrix)

Centre: The point which bisects every chord of the conic drawn through it, is called the centre of the conic. C ≡ (0, 0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Note : (i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.



Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Foci (0, ± be) Directrices : $y = \pm\frac{b}{e}$

$a^2 = b^2 (1 - e^2)$, a < b. ⇒ $e = \sqrt{1 - \frac{a^2}{b^2}}$

Vertices (0, ± b) ; L.R. y = ± be

ℓ (L·R.) = $\frac{2a^2}{b}$, centre : (0, 0)

Example # 6: Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Solution: Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$\therefore \frac{4}{a^2} + \frac{4}{b^2} = 1$ .... (i) and $\frac{9}{a^2} + \frac{1}{b^2} = 1$ ....(ii)

from (i) – 4 (ii), we get

$\frac{4 - 36}{a^2} = 1 - 4 \Rightarrow a^2 = \frac{32}{3}$

from (i), we get

$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32}$ $\Rightarrow$ $b^2 = \frac{32}{5}$

$\therefore$ Ellipse is $3x^2 + 5y^2 = 32$

Example # 7 : Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Solution: Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

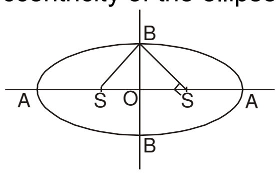
$\because$ $ae = 4$ and $e = \frac{1}{3}$ (Given)

$\therefore$ $a = 12$ and and $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 144\left(1 - \frac{1}{9}\right)$ $\Rightarrow$ $b^2 = 16 \times 8$ $\Rightarrow$ $b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Example # 8 : In the given figure find the eccentricity of the ellipse if SS' subtends right angle at B.



Solution: here $b = ae$ --- (i)

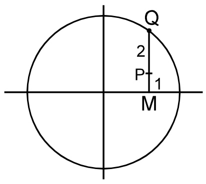
in ellipse $b^2 = a^2 - a^2 e^2$ ----- (ii)

from (i) & (ii) $a^2e^2 = a^2 - a^2 e^2$

$2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

Example # 9 : From a point Q on the circle $x^2 + y^2 = a^2$, perpendicular QM are drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1.

Solution :



Let $Q \equiv (a\cos\theta, a\sin\theta)$

$M \equiv (a\cos\theta, 0)$

Let $P \equiv (h, k)$

$\therefore h = a\cos\theta,\ k = \frac{a\sin\theta}{3}$

$\therefore$ $\left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$ $\Rightarrow$ Locus of P is $\frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1$

Example # 10 : Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $16x^2 + 25y^2 - 96x - 100y + 156 = 0$.

Solution : The given ellipse is $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Let x – 3 = X, y – 2 = Y, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is Y = 0 ⇒ y = 2.
equation of minor axis is X = 0 ⇒ x = 3.
centre (X = 0, Y = 0) ⇒ x = 3, y = 2
C ≡ (3, 2)

Length of semi-major axis a = 5
Length of major axis 2a = 10
Length of semi-minor axis b = 4
Length of minor axis = 2b = 8.
Let 'e' be eccentricity

∴ $b^2 = a^2 (1 - e^2)$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25 - 16}{25}} = \frac{3}{5}.$$

Length of latus rectum = $LL' = \frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

Co-ordinates focii are X = ± ae, Y = 0

⇒ S ≡ (X = 3, Y = 0) & S′ ≡ (X = –3, Y = 0)
⇒ S ≡ (6, 2) & S′ ≡ (0, 2)

Co-ordinate of vertices

Extremities of major axis A ≡ (X = a, Y = 0) & A′ ≡ (X = – a, Y = 0)
⇒ A ≡ (x = 8, y = 2) & A′ = (x = – 2, 2)
A ≡ (8, 2) & A′ ≡ (– 2, 2)
Extremities of minor axis B ≡ (X = 0, Y = b) & B′ ≡ (X = 0, Y = – b)
B ≡ (x = 3, y = 6) & B′ ≡ (x = 3, y = – 2)
B ≡ (3, 6) & B′ ≡ (3, – 2)

Equation of directrix $X = \pm\frac{a}{e}$ $x - 3 = \pm\frac{25}{3}$ ⇒ $x = \frac{34}{3}$ & $x = -\frac{16}{3}$

Self Practice Problems:

(8) Find the equation to the ellipse whose axes are of lengths 6 and 2 and their equations are x – 3y + 3 = 0 and 3x + y – 1 = 0 respectively.

(9) Find the co-ordinates of the focii of the ellipse $4x^2 + 9y^2 = 1$.

(10) A point moves so that the sum of the squares of its distances from two intersecting lines is constant (given that the lines are neither perpendicular nor they make complimentry angle). Prove that its locus is an ellipse.

Hint. : Assume the lines to be y = mx and y = – mx.

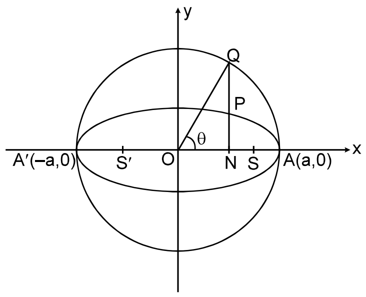
Ans. (8) $3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$, $21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

(9) $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$

3.3 Auxiliary Circle / Eccentric Angle of Ellipse

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x – axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \leq \pi)$. Q ≡ (a cosθ , a sinθ)



$P \equiv (a\cos\theta\ ,\ b\sin\theta)$

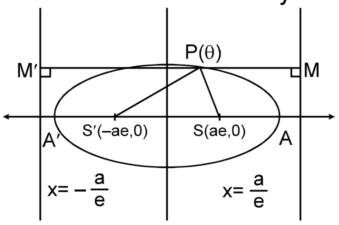
Note that :

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

NOTE : If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Example # 11 : Find the focal distance of a point $P(\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$

Solution : Let 'e' be the eccentricity of ellipse.



$\therefore$ $PS = e\ .\ PM = e\left(\frac{a}{e} - a\cos\theta\right)$

$PS = (a - a\,e\cos\theta)$

and $PS' = e.\ PM' = e\left(a\cos\theta + \frac{a}{e}\right)$

$PS' = a + ae\cos\theta$

$\therefore$ focal distance are $(a \pm ae\cos\theta)$

Note : $PS + PS' = 2a$

$PS + PS' = AA'$

Example # 12 : Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose radius makes angle α with y – axis in clockwise direction.

Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore\ m_{(op)} = \frac{b}{a}\tan\theta = \tan(\pi/2 - \alpha) \Rightarrow \tan\theta = \frac{a}{b}\tan(\pi/2 - \alpha)$

$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{\frac{a^2 + b^2\tan^2\theta}{\sec^2\theta}}$$

$$= \sqrt{\frac{a^2 + b^2\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\frac{a^2 + b^2 \times \frac{a^2}{b^2}\tan^2(\pi/2 - \alpha)}{1 + \frac{a^2}{b^2}\tan^2(\pi/2 - \alpha)}} \Rightarrow OP = \frac{ab}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}}$$

Self Practice Problems :

(11) Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is α

(12) Find the eccentric angle of a point on the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$ whose distance from the centre is 3.

(13) Show that the area of triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

Ans. (11) $r = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ (12) $\pm\frac{\pi}{2}$

3.4 Parametric Representation of Ellipse

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then;
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

Example # 13 : Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Solution : Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} + \frac{y}{4}.\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$\frac{x}{5}.\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}.\sin\left(\frac{3\pi}{4}\right) = 0 \Rightarrow -\frac{x}{5}+\frac{y}{4} = 0 \Rightarrow 4x = 5y$

Example # 14 : If P(α) and P(β) are extremities of a chord of ellipse which passes through the mid-point of the line segment joining focus & centre then prove that its eccentricity

$e = 2.\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|$

Solution : Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

∴ equation of chord is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

above chord passes through (ae/2, 0) or (– ae/2, 0)

∴ $\pm e\cos\left(\frac{\alpha+\beta}{2}\right) = 2\cos\left(\frac{\alpha-\beta}{2}\right)$ ∴ $e = 2\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|$ **Ans.**

Self Practice Problems :

(14) Find the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ on the chord joining two points whose eccentric angles differ by $\frac{\pi}{2}$.

Ans. (14) $2(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

3.5 Position of a Point w.r.t. an Ellipse :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as $S_1 > 0$, $S_1 < 0$ or $S_1 = 0$ where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$.

Example # 15 : Check whether the point P(1, –1) lies inside or outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution : $S_1 \equiv \frac{1}{25} + \frac{1}{16} - 1 < 0$

$\therefore$ Point $P \equiv (1, -1)$ lies inside the ellipse.

Example # 16 : Find the set of value(s) of 'α' for which the point $P(2\alpha, -3\alpha)$ lies inside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

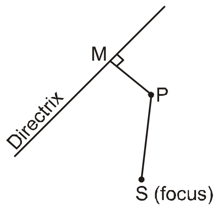
Solution : If $P(2\alpha, -3\alpha)$ lies inside the ellipse

$\therefore$ $S_1 < 0$

$\Rightarrow \frac{\alpha^2}{4} + \frac{\alpha^2}{1} - 1 < 0 \quad \Rightarrow \quad -\frac{2}{\sqrt{5}} < \alpha < \frac{2}{\sqrt{5}} \quad \therefore \quad \alpha \in \left(-\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

4. Elementary Concepts of Hyperbola

Hyperbolic curves are of special importance in the field of science and technology especially astronomy and space studies. In this chapter we are going to study the characteristics of such curves.

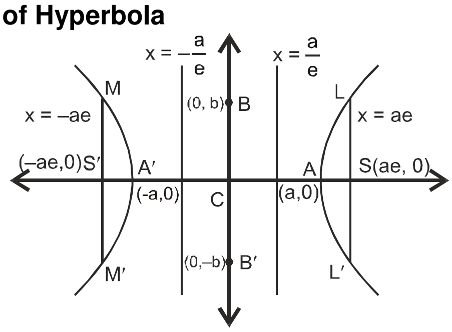


4.1 Definition of Hyperbola

A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is a fixed constant greater than 1.

$\frac{PS}{PM} = e > 1$, e – eccentricity

4.2 Standard equation of Hyperbola



Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

- **Eccentricity (e) :** $e^2 = 1 + \frac{b^2}{a^2}$
- **Foci :** $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.
- **Equations of directrices :** $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

- **Transverse axis :**
  The line segment A′A of length 2a in which the foci S′ & S both lie is called the transverse axis of the hyperbola.
- **Conjugate axis :**
  The line segment B′B of length 2b between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.
- **Principal axes :**
  The transverse & conjugate axis together are called principal axes of the hyperbola.
- **Vertices :**
  $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$
- **Focal chord :**
  A chord which passes through a focus is called a focal chord.
- **Double ordinate :**
  A chord perpendicular to the transverse axis is called a double ordinate.
- **Latus rectum :**
  Focal chord perpendicular to the transverse axis is called latus rectum. Its length (ℓ) is

given by $\ell = \dfrac{2b^2}{a} = \dfrac{(C.A.)^2}{T.A.} = 2a\,(e^2 - 1)$.

Note : (i) Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

(ii) End points of latus rectum are $L \equiv \left(ae, \dfrac{b^2}{a}\right)$, $L' \equiv \left(ae, -\dfrac{b^2}{a}\right)$, $M \equiv \left(-ae, \dfrac{b^2}{a}\right)$, $M' \equiv \left(-ae, -\dfrac{b^2}{a}\right)$

- **Centre:**
  The point which bisects every chord of the conic, drawn through it, is called the centre of the conic. $C \equiv (0,0)$ the origin is the centre of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

General note :
Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

Example #17: Find the equation of the hyperbola whose directrix is x + 2y = 1, focus (2,1) and eccentricity $\sqrt{3}$.

Solution: Let P(x,y) be any point on the hyperbola.
Draw PM perpendicular from P on the directrix.
Then by definition SP = e PM

$\Rightarrow$ $(SP)^2 = e^2\,(PM)^2$

$\Rightarrow$ $(x-2)^2 + (y-1)^2 = 3\left\{\dfrac{x+2y-1}{\sqrt{4+1}}\right\}^2 \Rightarrow 2x^2 - 7y^2 - 12xy - 14x + 2y + 22 = 0$

Which is the required hyperbola.

Example # 18: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution: Let the equation of hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus–rectum = $\dfrac{2b^2}{a}$. According to question $\dfrac{2b^2}{a} = \dfrac{1}{2}\,(2a)$

$\Rightarrow$ $2b^2 = a^2$ $(\because\ b^2 = a^2(e^2 - 1))$

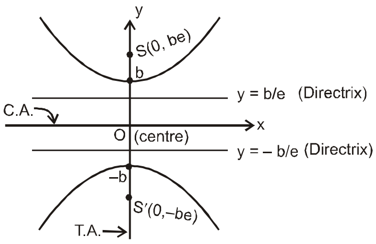
$\Rightarrow$ $2a^2(e^2 - 1) = a^2$ $\Rightarrow$ $2e^2 - 2 = 1$ $\Rightarrow$ $e^2 = \dfrac{3}{2}$

$\therefore$ $e = \sqrt{\dfrac{3}{2}}$ Hence the required eccentricity is $\sqrt{\dfrac{3}{2}}$.

4.3 Conjugate hyperbola :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **conjugate hyperbolas** of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.



Equation : $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$a^2 = b^2 (e^2 - 1) \Rightarrow e = \sqrt{1 + \frac{a^2}{b^2}}$

Vertices$(0, \pm b)$; ℓ (L.R.) $= \frac{2a^2}{b}$

Note : (a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similar if they have the same eccentricity.

(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Example # 19 : Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Solution : The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore\ a^2 = 9, b^2 = 16 \Rightarrow a = 3, b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum $= \frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{4}{(5/4)} \Rightarrow y = \pm \frac{16}{5}$$

Self Practice Problems :

(15) Find the equation of the hyperbola whose foci are (6, 4) and (– 4, 4) and eccentricity is 2.

(16) Obtain the equation of a hyperbola with coordinates axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.

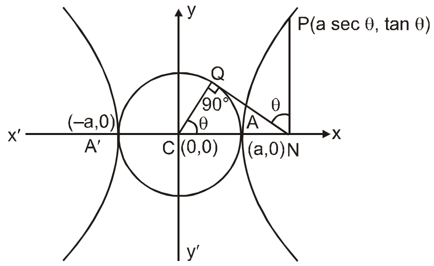
(17) The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. Find the equation of the hyperbola if its eccentricity is 2.

Ans. (15) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ (16) $\frac{x^2}{16}-\frac{y^2}{9}=1$, $\frac{y^2}{16}-\frac{x^2}{9}=1$

(17) $3x^2 - y^2 - 12 = 0$.

4.4 Auxiliary Circle of Hyperbola

A circle drawn with centre C and transverse axis as a diameter is called the **auxiliary circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.
Note from the following figure that P & Q are called the **"corresponding points"** of the hyperbola & the auxiliary circle.



4.5 Parametric representation of Hyperbola

The equations x = a sec θ & y = b tan θ together represent the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where θ is a parameter.

Note that if P(θ) ≡ (a sec θ, b tan θ) is on the hyperbola then,
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the hyperbola joining the two points P(α) & Q(β) is given by

$$.\ \frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

4.6 Position of a point 'P' w.r.t. a hyperbola :

The quantity $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is positive, zero or negative according as the point (x_1, y_1) lies inside, on or outside the curve.

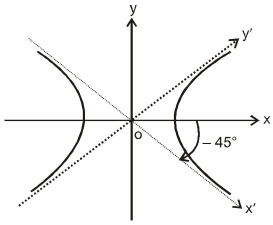
Example # 20 : Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution : Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,
So the point (5,–4) lies inside the hyperbola $9x^2 - y^2 = 1$..

5. Rectangular hyperbola (equilateral hyperbola) :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is.
Since a = b
equation becomes $x^2 - y^2 = a^2$
whose asymptotes are y = ± x.

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

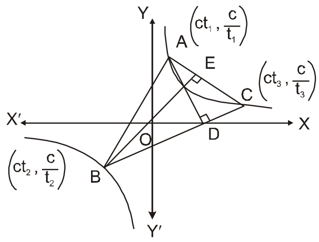
which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

It is referred to its asymptotes as axes of co–ordinates.
Vertices : (c, c) & $(-c, -c)$;
Foci : $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$ & $\left(-\sqrt{2}\,c, -\sqrt{2}\,c\right)$,
Directrices : $x + y = \pm \sqrt{2}\,c$
Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.
Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Example # 21 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now slope of BC is $\frac{c(t_3 - t_2)}{c(t_2 - t_3)t_2 t_3} = -\frac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of Altitude AD is $y - \frac{c}{t_1} = t_2t_3(x - ct_1)$

or $t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3$ .....(1)

Similarly equation of altitude BE is

$t_2y - c = x\, t_1t_2t_3 - ct_1t_2^2t_3$ ......(2)

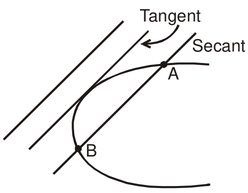
Solving (1) and (2),

we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ Which lies on $xy = c^2$.

6. Line & a parabola :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > cm$, $a = cm$, $a < cm$ respectively.

$\Rightarrow$ condition of tangency is, $c = a/m$.



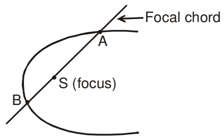
Length of the chord intercepted by the parabola on the line $y = m\,x + c$ is :

$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a - mc)}$

NOTE :

1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\, y + 2\, at_1 t_2 = 0$.

2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$.Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



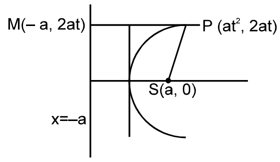
3. Length of the focal chord making an angle α with the x– axis is $4a\text{cosec}^2\,\alpha$.

Example # 22 : Discuss the position of line $y = x + 3$ with respect to parabola $y^2 = 4(x + 2)$.

Solution : Solving we get $(x + 3)^2 = 4(x + 2) \Rightarrow (x - 1)^2 = 0$
so $y = x + 3$ is tangent to the parabola.

Example # 23 : Prove that focal distance of a point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ $(a > 0)$ is $a(1 + t^2)$.

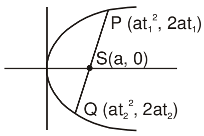
Solution :



$\because \quad PS = PM = a + at^2 \qquad PS = a\,(1 + t^2)$.

Example # 24 : If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Solution : Let parabola is $y^2 = 4ax$



since P, S & Q are collinear $\quad \therefore \quad m_{PQ} = m_{PS}$

$\Rightarrow \quad \frac{2}{t_1 + t_2} = \frac{2t_1}{t_1^2 - 1} \quad \Rightarrow \quad t_1^2 - 1 = t_1^2 + t_1t_2 \;\Rightarrow\; t_1t_2 = -1$

Example # 25 : If the endpoint t_1, t_2 of a chord satisfy the relation $t_1\, t_2 = -3$, then prove that the chord of $y^2 = 4x$ always passes through a fixed point. Find the point ?

Solution : Equation of chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$y - 2at_1 = \frac{2}{t_1 + t_2}\,(x - at_1^2) \quad \Rightarrow \quad (t_1 + t_2)\, y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$y = \frac{2}{t_1 + t_2}\,(x - 3) \;\; (\because \;\; t_1t_2 = -3) \qquad \therefore$ This line passes through a fixed point (3, 0).

Self Practice Problems :

(18) If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct point's then set of value's of 'λ' is

(19) Find the midpoint of the chord $x + y = 2$ of the parabola $y^2 = 4x$.

(20) If one end of focal chord of parabola $y^2 = 16x$ is (16, 16) then coordinate of other end is.

(21) If PSQ is focal chord of parabola $y^2 = 4ax$ $(a > 0)$, where S is focus then prove that $\frac{1}{PS} + \frac{1}{SQ} = \frac{1}{a}$

(22) Find the length of focal chord whose one end point is $(ap^2, 2ap)$

Ans. (18) $(-\infty, 1/3)$ (19) $(4, -2)$ (20) $(1, -4)$ (22) $a\left(p + \frac{1}{p}\right)^2$

6.1 Tangents to the parabola $y^2 = 4ax$:

Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.

In replacement method, following changes are made to the second degree equation to obtain T.

$x^2 \to x\,x_1$, $y^2 \to y\,y_1$, $2xy \to xy_1 + x_1y$, $2x \to x + x_1$, $2y \to y + y_1$

So, it follows that the targents are :

(i) $y\,y_1 = 2a\,(x + x_1)$ at the point (x_1, y_1) ;

(ii) $y = mx + \dfrac{a}{m}$ $(m \neq 0)$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(iii) $t\,y = x + a\,t^2$ at $(at^2, 2at)$.

(iv) Point of intersection of the tangents at the point t_1 & t_2 is $\{ at_1t_2 , a(t_1 + t_2) \}$.

Example # 26 : Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \dfrac{a}{m}$

Solution: Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \dfrac{a}{m} \quad \Rightarrow \quad y = mx + a\left(m + \dfrac{1}{m}\right)$

But the given tangent is $y = mx + c \qquad \therefore c = am + \dfrac{a}{m}$

Example # 27 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution : Slope of required tangent's are $m = \dfrac{3 \pm 1}{1 \mp 3} \Rightarrow m_1 = -2,\ m_2 = \dfrac{1}{2}$

$\because$ Equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$.

$\therefore$ tangent's $y = -2x - 1$ at $\left(\dfrac{1}{2}, -2\right) \Rightarrow y = \dfrac{1}{2}x + 4$ at $(8, 8)$

Example # 28 : Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is $y = mx + \dfrac{9}{4m}$

Since it passes through (4, 10)

$\therefore 10 = 4m + \dfrac{9}{4m} \quad \Rightarrow \quad 16m^2 - 40m + 9 = 0 \qquad m = \dfrac{1}{4}, \dfrac{9}{4}$

$\therefore$ equation of tangent's are $y = \dfrac{x}{4} + 9$ & $y = \dfrac{9}{4}x + 1$.

Example # 29 : Find the equations to the common tangents of the parabolas $(y - 1)^2 = 4ax$ and $x^2 = 4b(y - 1)$.

Solution : Equation of tangent to $(y - 1)^2 = 4ax$ is

$(y - 1) = mx + \dfrac{a}{m}$ ....(i)

Equation of tangent to $x^2 = 4b(y - 1)$ is

$x = m_1(y - 1) + \dfrac{b}{m_1} \quad \Rightarrow \quad (y - 1) = \dfrac{1}{m_1}x - \dfrac{b}{(m_1)^2}$ ....(ii)

for common tangent, (i) & (ii) must represent same line.

$\therefore \quad \dfrac{1}{m_1} = m$ & $\dfrac{a}{m} = -\dfrac{b}{m_1^2} \Rightarrow \dfrac{a}{m} = -bm^2 \Rightarrow m = \left(-\dfrac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is $y = \left(-\dfrac{a}{b}\right)^{1/3} x + a\left(-\dfrac{a}{b}\right)^{1/3} + 1$.

Self Practice Problems:

(23) Find equation tangent to parabola $y^2 = 4x$ whose intercept on y–axis is 2.

(24) Prove that perpendicular drawn from focus upon any tangent of a parabola lies on the tangent at the vertex.

(25) Prove that image of focus in any tangent to parabola lies on its directrix.

(26) Prove that the area of triangle formed by three tangents to the parabola $y^2 = 4ax$ is half the area of triangle formed by their points of contacts..

Ans. (23) $y = \frac{x}{2} + 2$

7. Line and an Ellipse :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as $c^2 < a^2m^2 + b^2$, $c^2 = a^2m^2 + b^2$ or $c^2 > a^2m^2 + b^2$

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

NOTE: The equation to the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

Example # 30 : Find the set of value(s) of 'λ' for which the line $x + y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at two distinct points.

Solution : Solving given line with ellipse, we get $\frac{x^2}{16} + \frac{(x+\lambda)^2}{9} = 1$

$25x^2 + 39\lambda x + 16\lambda^2 - 144 = 0$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ $D > 0$

$\therefore$ $(32\lambda)^2 - 4.25(16\lambda^2 - 144) > 0 \quad \Rightarrow \quad \lambda \in (-5, 5)$

Self Practice Problems :

(27) Find the value of 'λ' for which $2x - y + \sqrt{109}\,\lambda = 0$ touches the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Ans. (27) $\lambda = \pm 1$

7.1 Tangents to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.

(b) Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(c) Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

Note : (i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Example # 31 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are parallel to the line $x - 2y + \sqrt{7} = 0$

Solution: Slope of tangent = m = $\frac{1}{2}$

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2+3}$

$\because \quad m = \frac{1}{2} \qquad \therefore\ y = \frac{1}{2}x \pm \sqrt{1+3} \quad \Rightarrow \quad 2y = x \pm 4$

Example # 32 : A tangent to the ellipse $9x^2 + 16y^2 - 144 = 0$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Solution: The given ellipse is $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1 \Rightarrow \quad a = 4, b = 3$

Let $P \equiv (a\cos\theta, b\sin\theta) \quad \therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$A \equiv (a\sec\theta, 0)$

$B \equiv (0, b\,\text{cosec}\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore \quad a\cos\theta = \frac{a\sec\theta}{4} \quad \Rightarrow \quad \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$

and $\quad b\sin\theta = \frac{3b\,\text{cosec}\theta}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1 \quad \Rightarrow \quad bx + \sqrt{3}\,ay = 2ab \quad \Rightarrow \quad 3x + 4\sqrt{3}\,y = 24$

Example # 33 : Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by $\frac{\pi}{3}$ is an ellipse having the same eccentricity.

Solution : Let P (h, k) be the point of intersection of tangents at $A(\theta)$ and $B(\beta)$ to the ellipse.

$\therefore \quad h = \dfrac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)}$ & $k = \dfrac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \Rightarrow \left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = \sec^2\left(\frac{\theta-\beta}{2}\right)$

but given that $\theta - \beta = \frac{\pi}{3}$

$\therefore \quad$ locus is $\dfrac{x^2}{a^2\sec^2\left(\frac{\pi}{6}\right)} + \dfrac{y^2}{b^2\sec^2\left(\frac{\pi}{6}\right)} = 1$ which is ellipse having same eccentricity.

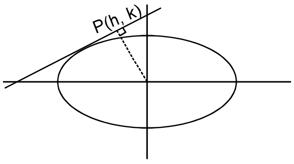
Example # 34 : If the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $3x^2 + 4y^2 = 12$ is $(x^2 + y^2)^2 = ax^2 + by^2$, then find a + b.

Solution : Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{4m^2 + 3}$ ....(i)
from centre

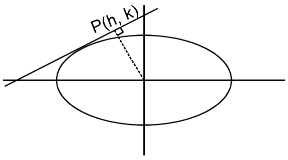
$\therefore \quad \frac{k}{h} . m = -1 \quad \Rightarrow \quad m = -\frac{h}{k}$ ....(ii)

$\because$ P(h, k) lies on tangent

$\therefore \quad k = mh + \sqrt{4m^2 + 3}$ ....(iii)



from equation (ii) & (iii), we get



$\left(k + \frac{h^2}{k}\right)^2 = \frac{4h^2}{k^2} + 3 \quad \Rightarrow$ locus is $(x^2 + y^2)^2 = 4x^2 + 3y^2$

Self Practice Problems :

(28) Show that the locus of the point of intersection of the tangents at the extremities of any focal chord of an ellipse is the directrix corresponding to the focus.

(29) Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

(30) Prove that the portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

(31) Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(32) If y_1 is ordinate of a point P on the ellipse then show that the angle between its focal radius and tangent at it, is $\tan^{-1}\left(\frac{b^2}{aey_1}\right)$.

(33) Find the eccentric angle of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangent at which, is equally inclined to the axes.

Ans. (31) $\frac{2a^3}{\sqrt{a^2 - b^2}}$ (33) $\theta = \pm \tan^{-1}\left(\frac{b}{a}\right)$, $\pi - \tan^{-1}\left(\frac{b}{a}\right)$, $-\pi + \tan^{-1}\left(\frac{b}{a}\right)$

8. Line and a hyperbola :

The straight line y = mx + c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > a^2m^2 - b^2$ or $c^2 = a^2m^2 - b^2$ or $c^2 < a^2m^2 - b^2$, respectively.

NOTE: The equation to the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining the two points P(α) & Q(β) is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

8.1 Tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(i) Slope form : $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm'.

(ii) Point form: Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(iii) Parametric form: Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point.

(a sec θ, b tan θ) is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Note : (i) Point of intersection of the tangents at $P(\theta_1)$ & $Q(\theta_2)$ is $\left(a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}},\ b\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangent touches the hyperbola at the extremities of a diameter.

Example # 35 : Find c, if x + y = c touch the hyperbola $\frac{x^2}{4} - y^2 = 1$.

Solution: Solving line and hyperbola we get

$x^2 - 4(c-x)^2 = 4$

$3x^2 + 8cx + 4c^2 + 4 = 0$

D = 0

$64c^2 - 4.3.4(c^2-1) = 0$

$c^2 - 3 = 0$

$c = \pm\sqrt{3}$

Example # 36 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $\sqrt{3}x + y + \sqrt{5} = 0$

Solution : $y = mx \pm \sqrt{36m^2 - 9}$, where $m = \frac{1}{\sqrt{3}}$

∴ equation of tangents are $y = \frac{x}{\sqrt{3}} \pm \sqrt{3}$ ⇒ $\sqrt{3}y = x \pm 3$

Example # 37 : Find the point of contact if $3x - \sqrt{7}y - 9 = 0$ is tangent to $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution : Let the point of contact is (x_1, y_1). The equation of tangent is

$\frac{xx_1}{16} - \frac{yy_1}{9} - 1 = 0$...........(i)

The given equation of tangent is $3x - \sqrt{7}y - 9 = 0$...........(ii)

From Equ (i) & (ii)

$\frac{x_1}{16\times3} = \frac{y_1}{9\sqrt{7}} = \frac{1}{9}$ ⇒ $(x_1, y_1) = \left(\frac{16}{3}, \sqrt{7}\right)$

9. Line and Rectangular hyperbola :

Equation of a chord joining the points P (t_1) & Q(t_2) is $x + t_1 t_2 y = c(t_1 + t_2)$.

Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + ty = 2c$.

Example # 38 : A, B, C are three points on the rectangular hyperbola $xy = c^2$, find
(i) The area of the triangle ABC
(ii) The area of the triangle formed by the tangents at A, B and C.

Solution : Let co–ordinates of A,B and C on the hyperbola $xy = c^2$ are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$, and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle ABC $= \frac{1}{2}\left[\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right]$

$$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right| = \frac{c^2}{2t_1t_2t_3}\left|t_1^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$$

$$= \frac{c^2}{2t_1t_2t_3} \mid (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \mid$$

(ii) Equations of tangents at A,B,C are

$x + yt_1^2 - 2ct_1 = 0$

$x + yt_2^2 - 2ct_2 = 0$

and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area $= \frac{1}{2\,|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2$ ....(1)

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (1) $\frac{1}{2\left|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)\right|} = 4c^2.(t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$

$$= 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$$

$\therefore$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

Example # 39 : Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

Solution : Let rectangular hyperbola is $x^2 - y^2 = a^2$

Let equations of PQ and DE are

$y = mx + c$ ......(1)

and $y = m_1x + c_1$ ......(2)

respectively.

Be any two focal chords of any rectangular hyperbola $x^2 - y^2 = a^2$ through its focus. We have to prove PQ = DE. Since $PQ \perp DE$.

$\therefore \quad mm_1 = -1$ ......(3)

Also PQ passes through S $(a\sqrt{2}, 0)$ then from (1),

$0 = ma\sqrt{2} + c$

or $\quad c^2 = 2a^2m^2$ ......(4)

Let (x_1,y_1) and (x_2,y_2) be the co–ordinates of P and Q then

$(PQ)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ ......(5)

Since (x_1,y_1) and (x_2,y_2) lie on (1)

$\therefore \quad y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

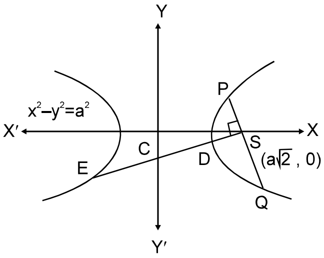
$\therefore \quad (y_1 - y_2) = m(x_1 - x_2)$ ....(6)

From (5) and (6)

$(PQ)^2 = (x_1 - x_2)^2 (1 + m^2)$ ....(7)

Now solving $y = mx + c$ and $x^2 - y^2 = a^2$ then $x^2 - (mx + c)^2 = a^2$

or $\quad (m^2 - 1)x^2 + 2mcx + (a^2 + c^2) = 0$



$\therefore \quad x_1 + x_2 = -\dfrac{2mc}{m^2-1}$ and $\quad x_1x_2 = = \dfrac{a^2+c^2}{m^2-1}$

$\Rightarrow \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = \dfrac{4m^2c^2}{(m^2-1)^2} - \dfrac{4(a^2+c^2)}{(m^2-1)}$

$= \dfrac{4\{a^2+c^2-a^2m^2\}}{(m^2-1)^2} = \dfrac{4a^2(m^2+1)}{(m^2-1)^2} \quad \{\because c^2 = 2a^2m^2\}$

From (7), $(PQ)^2 = 4a^2\left(\dfrac{m^2+1}{m^2-1}\right)^2$

Similarly, $(DE)^2 = 4a^2\left(\dfrac{m_1^2+1}{m_1^2-1}\right)^2 = 4a^2\left(\dfrac{\left(-\dfrac{1}{m}\right)^2+1}{\left(-\dfrac{1}{m}\right)^2-1}\right)^2 = 4a^2\left(\dfrac{m^2+1}{m^2-1}\right)^2 = (PQ)^2$

$(\because mm_1 = -1)$ Thus $(PQ)^2 = (DE)^2 \Rightarrow PQ = DE$.

Hence perpendicular focal chords of a rectangular hyperbola are equal.

Self Practice Problems :

(34) Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

(35) For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

(36) Find the equation of the tangent to the hyperbola $x^2 - y^2 = 1$which is parallel to the line $4y = 5x + 7$.

Ans. (35) $\lambda = \pm 2\sqrt{5}$ (36) $4y = 5x \pm 3$

10 **Pair of tangents :** The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the curve S = 0 is $SS_1 = T^2$

Curve(S=0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Combined equation of tangents from external point (x_1, y_1) to S=0
Parabola $(y^2 - 4ax = 0)$	$T \equiv y y_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$SS_1 = T^2$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$	$S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$	$SS_1 = T^2$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$	$S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$	$SS_1 = T^2$

Example # 40 : Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point P(–1, 2)

Solution : We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore$ $(y^2 - 4x)(4 + 4) = (2y - 2(x - 1))^2$

$\Rightarrow$ $8y^2 - 32x = 4y^2 + 4x^2 + 4 - 8xy + 8y - 8x \Rightarrow y^2 - x^2 + 2xy - 6x - 2y = 1$

Example # 41 : Find the locus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(i) $|m_1 - m_2| = 2$ (ii) $\theta_1 + \theta_2 = \pi/3$

Solution : Equation of tangent to $y^2 = 4ax$, is $y = mx + \frac{a}{m}$

Let it passes through P(h, k)

$\therefore$ $m^2h - mk + a = 0$

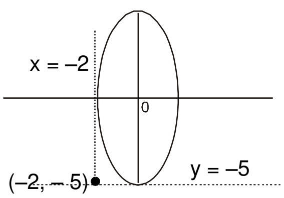
(i) $m_1 + m_2 = \frac{k}{h}$ and $m_1 \cdot m_2 = \frac{a}{h}$ $\Rightarrow$ $|m_1 - m_2| = 2 \Rightarrow (m_1 + m_2)^2 - 4m_1m_2 = 4$

$\frac{k^2}{h^2} - 4\frac{a}{h} = 4 \Rightarrow 4ax = 4x^2$

(ii) $\tan \pi/3 = \frac{m_1+m_2}{1-m_1 m_2} = \frac{k/h}{1-a/h}$ $\Rightarrow$ $y = (x - a)\sqrt{3}$

Example # 42 : How many real tangents can be drawn from the point (–2, –5) to the ellipse $\frac{x^2}{4}+\frac{y^2}{25} = 1$. Find the equation of these tangents & angle between them.

Solution :



By direct observation

x = –2, y = –5 are tangents.

Example # 43 : Find the locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents
equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right)+........=0 \qquad ....(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0$$

$\Rightarrow$ $k^2 - b^2 + h^2 - a^2 = 0$ $\Rightarrow$ locus is $x^2 + y^2 = a^2 + b^2$

Example # 44 : How many real tangents can be drawn from the point (2, 1) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Find the equation of these tangents.

Solution : Given point $P \equiv (2, 1)$ Hyperbola $S \equiv \frac{x^2}{16}-\frac{y^2}{9}-1=0$

$\because$ $S_1 \equiv \frac{4}{16}-\frac{1}{9}-1=-\frac{31}{36}<0$ $\Rightarrow$ Point $P \equiv (2, 1)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(2, 1).
Equation of pair of tangents is $SS_1 = T_2$

$$\Rightarrow \left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)\left(\frac{1}{4}-\frac{1}{9}-1\right)=\left(\frac{2x}{16}-\frac{y}{9}-1\right)^2 \Rightarrow 144\,(9x^2-16y^2-144)+(9x-9y-72)^2=0$$

Example # 45 : Find the locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents. Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}-\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(-\frac{k^2}{b^2}-1\right)-\frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right)+........=0 \qquad ....(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(-\frac{k^2}{b^2}-1\right)-\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0 \Rightarrow -k^2-b^2-h^2+a^2=0 \Rightarrow \text{locus is } x^2+y^2=a^2-b^2$$

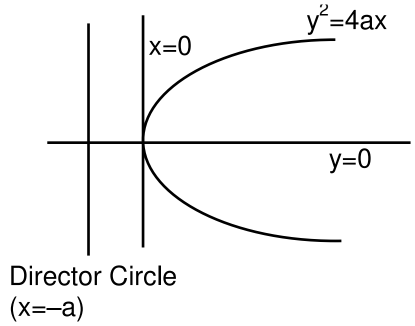
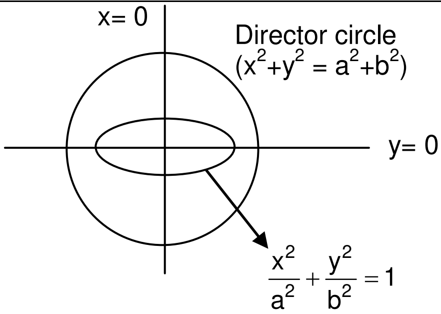
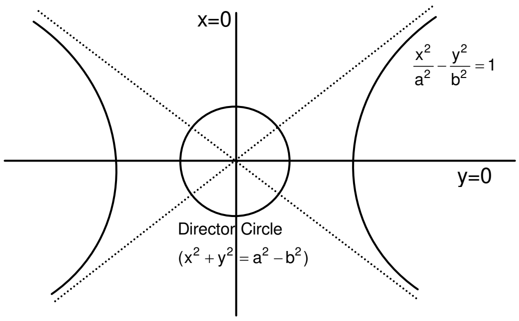
Self Practice Problems :

(37) If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.
(i) $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$ (a constant) (ii) $\cos\theta_1 \cos\theta_2 = \lambda$ (a constant)

(38) Find the locus of point of intersection of the tangents drawn at the extremities of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ans. (37) (i) $y^2 - 2ax = \lambda x^2$, (ii) $x^2 = \lambda^2 \{(x-a)^2 + y^2\}$ (38) $x = \pm\frac{a}{e}$

11. **Director circle:** Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle**.**

Curve(S=0)	Locus of Director Circle of (S=0)	Figure
Parabola ($y^2 - 4ax = 0$)	$x + a = 0$	
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 + b^2$	
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 - b^2$	

Note: For hyperbola, if $b^2 < a^2$, then the director circle is real.

If $b^2 = a^2$ (i.e. rectangular hyperbola), then the radius of the director circle is zero and it reduces to a point circle at the origin. In this case centre is the only point from which two perpendicular tangents can be drawn on the curve.

If $b^2 > a^2$, then the radius of the director circle is imaginary, so that there is no such circle and so no pair of tangents at right angle can be drawn to the curve.

Example # 46 : Find the point of the line $x - y = 0$ for from where perpendicular tangent can be drawn to $\frac{x^2}{9} + y^2 = 1$

Solution : Solving director circle $x^2 + y^2 = 10$ & $x - y = 0$ $\Rightarrow$ $(\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5})$

Self Practice Problems :

(39) Find the angle between the tangent drawn from (–2, 1) to $x^2 + 4y^2 = 4$

Ans. (39) $\frac{\pi}{2}$

12. **Chord of contact:** Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the curve $S = 0$ is $T = 0$

Curve(S=0)	T for point (x_1, y_1) & S = 0	equation of chord of contact from external point (x_1, y_1) to S=0 is T = 0
Parabola $(y^2 - 4ax = 0)$	$T \equiv y y_1 - 2a(x + x_1)$	$yy_1 - 2a(x + x_1) = 0$
Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right)$	$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$
Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\right)$	$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$
Rectangular Hyperbola $(xy - c^2 = 0)$	$T = \frac{xy_1 + yx_1}{2} - c^2$	$\frac{xy_1 + yx_1}{2} - c^2 = 0$

NOTE : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$

Example # 47 : Find the length of chord of contact of the tangents drawn from point (–2, 3) to the parabola $y^2 = 8x$.

Solution : Let tangent at $P(t_1)$ & $Q(t_2)$ meet at (–2, 3)

$\therefore \quad 2t_1t_2 = -2 \qquad \& \qquad 2(t_1 + t_2) = 3$

$\because PQ = \sqrt{(2t_1^2 - 2t_2^2)^2 + (4(t_1 - t_2))^2}$

$= 2\sqrt{((t_1 + t_2)^2 - 4t_1t_2)((t_1 + t_2)^2 + 4)} = \sqrt{\frac{(3^2 - 4.2(-2))(3^2 + 4.2^2)}{2^2}} = 25/2$

Example # 48 : If the line x – y – 1 = 0 intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0$ .....(i)

But given is x – y – 1 = 0

$\therefore \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \Rightarrow h = -1, k = 4 \quad \therefore \quad$ point $\equiv (-1, 4)$

Example # 49 : Find the locus of point whose chord of contact w.r.t to the parabola $y^2 = 4bx$ is the tangents of the circle $x^2 + y^2 = a^2$.

Solution : Let it is chord of contact for parabola $y_2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$y = \frac{2b}{k}x + \frac{2bh}{k}$ .....(i)

(i) is tangents to $x^2 + y^2 = a^2 \Rightarrow \left|\frac{\frac{2bh}{k}}{\sqrt{1 + \frac{4b^2}{k^2}}}\right| = a \Rightarrow 4b^2x^2 = a^2(y^2 + 4b^2)$

Example # 50 : If tangents to the circle $x^2 + y^2 = b^2$ intersect the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution : Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ ....(i)

which touches the circle $x^2 + y^2 = b^2$

$$\therefore \quad \frac{1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} = b \Rightarrow \text{ required locus is } \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{b^2}$$

Example # 51: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution: Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ Equation of chord of contact AB is $\frac{xh}{a^2} - \frac{yk}{b^2} = 1$ ......(i)

Which touches the parabola

Equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m} \Rightarrow mx - y = -\frac{a}{m}$ ......(ii)

equation (i) & (ii) as must be same

$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(-\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \Rightarrow m = \frac{h}{k}\frac{b^2}{a^2} \quad \& \ m = -\frac{ak}{b^2}$$

$$\therefore \quad \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \Rightarrow \text{ locus of P is } y^2 = -\frac{b^4}{a^3} . x$$

Self Practice Problems :

(40) Prove that locus of a point whose chord of contact w.r.t. parabola passes through focus is directrix

(41) If from a variable point 'P' on the line $2x - y - 1 = 0$ pair of tangent's are drawn to the parabola $x^2 = 8y$ then prove that chord of contact passes through a fixed point, also find that point.

(42) Find the locus of point of intersection of tangents at the extremities of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(43) Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Ans. (41) (8,1) (42) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$ (43) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$

13. Chord with a given middle point:

Equation of the chord of the curve S = 0 whose middle point is (x_1, y_1) is $T = S_1$.

Curve(S=0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Chord with middle point (x_1, y_1) for S=0 is $T = S_1$
Parabola ($y^2 - 4ax = 0$)	$T = y y_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$y y_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_1}{a^2} + \frac{yy_1}{b^2}$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$
Rectangular Hyperbola ($xy - c^2 = 0$)	$T = \frac{xy_1 + yx_1}{2} - c^2$	$S_1 = x_1y_1 - c^2$	$xy_1 + yx_1 = 2x_1y_1$

Example # 52 : Find the locus of middle point of the chord of the parabola $y^2 = 16x$ which pass through a given point (7, –2).

Solution : Let P(h, k) be the mid point of chord of parabola $y^2 = 16x$
so equation of chord is $yk - 8(x + h) = k^2 - 16h$.
Since it passes through (7, –2)
$-2k - 8(7 + h) = k^2 - 16h$
$\therefore$ Required locus is
$y^2 + 2y - 8x + 56 = 0$

Example # 53 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which is parallel to line $y = mx + c$

Solution : Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,
so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.
but slope $= \frac{2a}{k} = m$ $\quad\therefore\quad$ locus is $y = \frac{2a}{m}$

Example # 54 : Find the locus of the mid - point of chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Which are focal chords of $y^2 = 4ax$

Solution : Let P ≡ (h, k) be the mid-point
$\therefore$ equation of chord whose mid-point is given $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$
since it is a focal chord,
$\therefore$ it passes through focus (a, 0)
$\Rightarrow \frac{h}{a} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ $\Rightarrow$ required locus is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a}$

Example # 55 : Find the mid point of chord x + 2y = 4 of ellipse $9x^2 + 36y^2 = 324$

Solution : Let (h,k) be mid point of chord . So $T = S_1$
$9xh + 36yk = 9h^2 + 36k^2$ ------ (i) $\quad$ x + 2y = 4 ------ (ii)
From (i) and (ii)
$\frac{9h}{1} = \frac{36k}{2} = \frac{9h^2 + 36k^2}{4}$ $\Rightarrow$ (h, k) = (2,1)

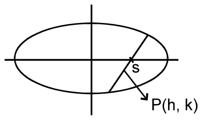
Example # 56 : Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution : Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given is $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)



If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Example # 57 : Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line x + y = b.

Solution : Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b)

$$\therefore \quad \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0 \quad \alpha = 0, \quad \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}} \qquad \therefore \qquad a \neq \pm b$$

Example # 58 : Find the locus of the mid point of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Solution : let (h,k) be the mid–point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{b^2} - \frac{k^2}{b^2} - 1 \qquad \text{or} \qquad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \qquad ....(1)$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (1) is obtained by making homogeneous hyperbola with the help of (1)

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2 = \frac{h^2}{a^4} x^2 + \frac{k^2}{b^4} y^2 - \frac{2hk}{a^2b^2} xy \qquad ....(2)$$

The lines represented by (2) will be at right angle if coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0 \Rightarrow \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = \frac{h^2}{a^4}+\frac{h^2}{a^4}$$

hence, the locus of (h,k) is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$

Self Practice Problems :

(44) Find the mid point of chord $x - y - 2 = 0$ of parabola $y^2 = 4x$.

(45) Find the locus of mid - point of chord of parabola $y^2 = 4ax$ which touches the parabola $x^2 = 4by$.

(46) Find the equation of the chord $\frac{x^2}{36} + \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(47) Find the locus of the mid-points of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(48) Find the equation of the chord $\frac{x^2}{36} - \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(49) Find the point 'P' from which pair of tangents PA & PB are drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ in such a way that (5, 2) bisect AB

(50) From the points on the circle $x^2 + y^2 = a^2$, tangent are drawn to the hyperbola $x^2 - y^2 = a^2$, prove that the locus of the middle points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$.

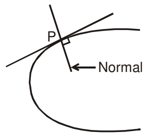
Ans. (44) (4, 2) (45) $y (2ax - y^2) = 4a^2b$ (46) $x + 2y = 4$

(47) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right) = (a^2 - b^2)^2$ (48) $x = 2y$ (49) $\left(\frac{20}{3}, \frac{8}{3}\right)$

14 NORMAL

14.1 Normal to the parabola :

Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}$ $\Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

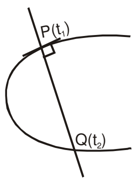
(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; **(ii)** $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

NOTE :

(i) Point of intersection of normals at t_1 & t_2 is $(a (t + t + t_1t_2 + 2), - a t_1 t_2 (t_1 + t_2))$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point



t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$

Example # 59 : If the normal at point 't_1' intersects the parabola again at 't_2' then find value of $|t_1.t_2 + t_1^2|$.

Solution : Slope of normal at P = $-t_1$ and slope of chord PQ = $\dfrac{2}{t_1+t_2}$

$\therefore\ -t_1 = \dfrac{2}{t_1+t_2} \Rightarrow t_2 = -t_1 - \dfrac{2}{t_1} \Rightarrow \quad t_2 = -t_1 - \dfrac{2}{t_1} \Rightarrow t_2.t_1 = -t_1^2 - 2 \quad \Rightarrow |t_1.t_2 + t_1^2| = 2$

Example # 60 : If the normals at points t_1, t_2 meet at the point t_3 on the parabola then find value of $(t_1 + t_2 + t_3)^2 + (t_1 . t_2)^2$

Solution : Since normal at t_1 & t_2 meet the curve at t_3

$\therefore \quad t_3 = -t_1 - - \dfrac{2}{t_1}$ .....(i)

$t_3 = -t_2 - \dfrac{2}{t_2}$ .....(ii)

$\Rightarrow \quad (t_1^2 + 2)\, t_2 = t_1\, (t_2^2 + 2) \quad \Rightarrow \quad t_1t_2\,(t_1 - t_2) + 2\,(t_2 - t_1) = 0$

$\because \quad t_1 \neq t_2\,,\ \ t_1t_2 = 2$ ......(iii)

Hence (i) $t_1\, t_2 = 2$

from equation (i) & (iii), we get $t_3 = -t_1 - t_2$

Hence (ii) $t_1 + t_2 + t_3 = 0$ ....(iv)

from (iii) & (iv) $(t_1 + t_2 + t_3)^2 + (t_1 . t_2)^2 = 4$

Example # 61 : Find the locus of the point N from which 3 normals are drawn to the parabola $y^2 = 4ax$ are such that

(i) Two of them are equally inclined to y-axis

(ii) Two of them have product of their slops is equal to 2.

Solution : Equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Let the normal passes through N(h, k)

$\therefore \quad k = mh - 2am - am^3 \quad \Rightarrow \quad am^3 + (2a - h)\, m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's of above equation

$\therefore \quad m_1 + m_2 + m_3 = 0$ ......(i)

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a}$ ......(ii)

$m_1m_2m_3 = -\dfrac{k}{a}$ ......(iii)

(i) If two normal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore \quad m_3 = 0 \quad \Rightarrow \quad y = 0$

(ii) If two normal's have product of their slops = 2

$\therefore \quad m_1\, m_2 = 2$

from (3) $\quad m_3 = -\dfrac{k}{2a}$ .....(iv)

from (2) $\quad 2 - \dfrac{k}{2a}\,(m_1 + m_2) = \dfrac{2a-h}{a}$ .....(v)

from (1) $\quad m_1 + m_2 = \dfrac{k}{2a}$ .....(vi)

from (5) & (6), we get

$k_2 = 4ax$

Self Practice Problems:

(51) Find the points of the parabola $y^2 = 4ax$ at which the normal is inclined at 45° to the axis.
(52) If chord drawn from point P(9, –6) on the parabola $y^2 = 4x$ is normal at point Q then Q = ?
(53) Find the length of normal chord at point 't' to the parabola $y^2 = 4ax$.
(54) If the normals at 3 points P, Q & R on the parabola $(x - 3)^2 = y + 2$ are concurrent, then show that
(i) The sum of slopes of normals is zero,
(ii) The locus of centroid of ΔPQR is $x - 3 = 0$.

Ans. (51) (a, –2a), (a, 2a) (52) (9, –6) (53) $\ell = \dfrac{4a(t^2+1)^{\frac{3}{2}}}{t^2}$

14.2 Normal to Ellipse

(i) Equation of the normal at (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$.

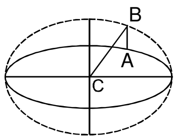
(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is;

$ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \dfrac{(a^2 - b^2)\,m}{\sqrt{a^2 + b^2m^2}}$.

Example # 62 : A and B are corresponding points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at A to the ellipse meets CB in R, where C is the centre of the ellipse. Prove that locus of R is a circle of radius a + b.

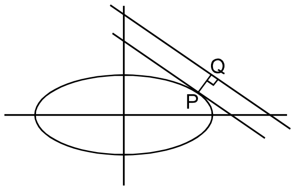
Solution : Let $A \equiv (a\cos\theta, b\sin\theta)$
$\therefore$ $B \equiv (a\cos\theta, a\sin\theta)$



Equation of normal at A is
$(a\sec\theta)\,x - (b\operatorname{cosec}\theta)\,y = a^2 - b^2$ ....(i)
equation of CB is $y = \tan\theta\,.\,x$ ....(ii)
Solving equation (i) & (ii), we get $(a - b)\,x = (a^2 - b^2)\cos\theta$
$x = (a + b)\cos\theta$, & $y = (a + b)\sin\theta$
$\therefore$ $R \equiv ((a + b)\cos\theta, (a + b)\sin\theta) = (h, k)$
$h^2 + k^2 = (a + b)^2 \Rightarrow x^2 + y^2 = (a + b)^2$

Example # 63 : Find the shortest distance between the line $3x + 4y = 12$ and the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Solution : Shortest distance occurs between two non-intersecting curve always along common normal.
Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.
$\therefore$ Tangent at 'P' is parallel to given line
$\therefore$ Equation of tangent parallel to given line is $(y = mx \pm \sqrt{a^2m^2 + b^2})$
$y = \dfrac{3x}{4} \pm 3\sqrt{2}$



$\therefore$ minimum distance = distance between

$3x + 4y = 12$ & $3x + 4y = 3\sqrt{2}$

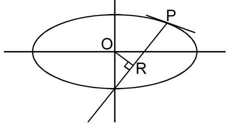
$\Rightarrow$ shortest distance $= \left|\dfrac{12-3\sqrt{2}}{5}\right|$

Example # 64 : Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

Solution : Let the equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Equation of normal at P (θ) is $(a\sec\theta)x - (b\,\text{cosec}\,\theta)y - a^2 + b^2 = 0$

distance of normal from centre



$$= OR = \frac{|a^2-b^2|}{\sqrt{a^2+b^2+(a\tan\theta)^2+(b\cot\theta)^2}} = \frac{|a^2-b^2|}{\sqrt{(a+b)^2+(a\tan\theta-b\cot\theta)^2}}$$

$\because (a + b)^2 + (a\tan\theta - b\cot\theta)^2 \geq (a + b)^2$ or $\leq \dfrac{|a^2-b^2|}{\sqrt{(a+b)^2}} \Rightarrow |OR| \leq (a - b)$

Self Practice Problems :

(55) Find the value(s) of 'a' for which the line $x + y = a$ is a normal to the ellipse $3x^2 + 4y^2 = 12$

(56) If the normal at the point P(θ) to the ellipse $\dfrac{x^2}{14}+\dfrac{y^2}{5} = 1$ intersects it again at the point Q(2θ) then find the value of $\cos\theta$

Ans. (55) $a = \pm\dfrac{1}{\sqrt{7}}$ (56) $-\dfrac{2}{3}$

14.3 Normal to Hyperbola

(a) The equation of the normal to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point P (x_1, y_1) on it is

$\dfrac{a^2x}{x_1}+\dfrac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(b) The equation of the normal at the point P ($a\sec\theta$, $b\tan\theta$) on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is

$\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

(c) Equation of normals in terms of its slope 'm' are $y = mx \pm \dfrac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$.

14.4 Normal to Rectangular hyperbola

Equation of the normal at P (t) is $xt^3 - yt = c(t^4 - 1)$

Example # 65 : A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N. find a locus of point R on segment MN such that NR : RM = 2 :1.

Solution : The equation of normal at the point Q (a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

ax cos ϕ + by cot ϕ = $a^2 + b^2$ ....(1)

The normal (1) meets the x–axis in

$M\left(\frac{a^2+b^2}{a}\sec\phi,\ 0\right)$ and y–axis in $N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$

Let R (h, k) is point whose locus we have to find. as NR : RM = 2 :1.

$\Rightarrow$ $h = \frac{2}{3}\frac{(a^2+b^2)}{a}\sec\phi,\ k = \frac{(a^2+b^2)}{3b}\tan\phi$

we know that

$\sec^2\phi - \tan^2\phi = 1 \ \Rightarrow \ \frac{9a^2}{4(a^2+b^2)^2}x^2 - \frac{9b^2}{(a^2+b^2)^2}y^2 = 1 \Rightarrow \frac{a^2x^2}{4} - b^2y^2 = \frac{(a^2+b^2)^2}{9}$

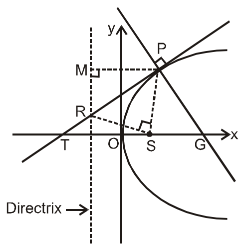
Self Practice Problems :

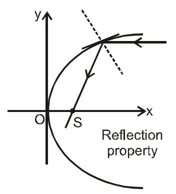
(57) Prove that the line lx + my – n = 0 will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

15. Important Highlights of Parabola :

(i) If the tangent & normal at any point ‘P’ of the parabola intersect the axis at T & G then ST = SG = SP where ‘S’ is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.

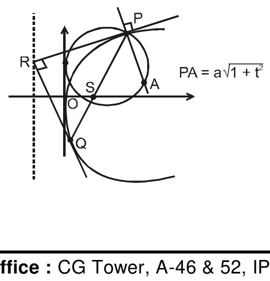




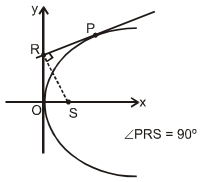
(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus.

See figure above.

(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P (at^2, 2at) as diameter touches the tangent at the vertex and intercepts a chord of length a $\sqrt{1+t^2}$ on a normal at the point P.



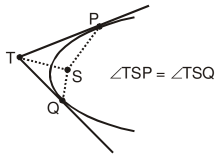
(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.



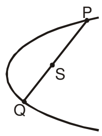
(v) If the tangents at P and Q meet in T, then:

$\Rightarrow$ TP and TQ subtend equal angles at the focus S.

$\Rightarrow$ $ST^2 = SP.\ SQ$ & $\Rightarrow$ The triangles SPT and STQ are similar.



(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



$$\frac{2(PS)(SQ)}{PS+SQ} = 2a$$

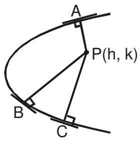
(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h, k) to the parabola $y^2 = 4ax$ then

$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0.$

$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = -\frac{k}{a}$.

Where m_1, m_2, & m_3 are the slopes of the three concurrent normals. Note that



A, B, C $\rightarrow$ Conormal points

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

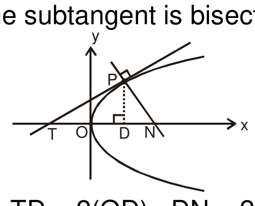
$\Rightarrow$ algebraic sum of the ordinates of the three conormal points on the parabola is zero

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn froma point P (h, k) is

$h > 2a$ & $k^2 < \frac{4}{27a}(h - 2a)^3$.

(ix) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex..



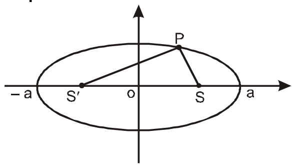
TD = 2(OD) , DN = 2a

(x) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum. See figure above.

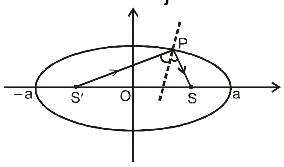
16. Important Highlights of Ellipse :

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

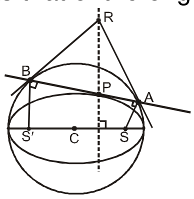
(i) If P be any point on the ellipse with S & S′ as its foci then ℓ (SP) + ℓ (S′P) = 2a.



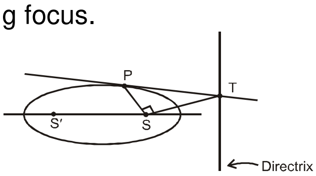
(ii) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice−versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



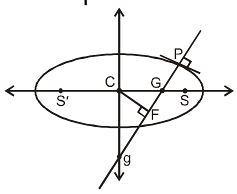
(iii) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b² and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of Intersection is a similiar ellipse as that of the original one.



(iv) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.



(v) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be Perpendicular upon this normal, then



(i) PF. PG = b² (ii) PF. Pg = a²

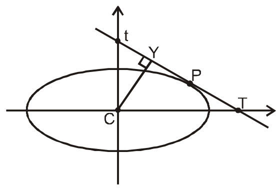
(iii) PG. Pg = SP. S′ P (iv) CG. CT = CS²

(v) Locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[Where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(vi) The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(vii) If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

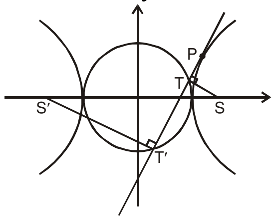


(i) T t. PY = $a^2 - b^2$ and
(ii) least value of T t is a + b.

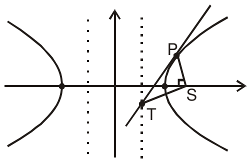
17. Important Highlights of Hyperbola:

(i) Difference of focal distances is a constant, i.e. |PS – PS′| = 2a

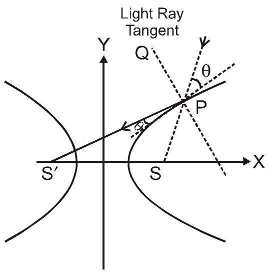
(ii) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



(iii) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

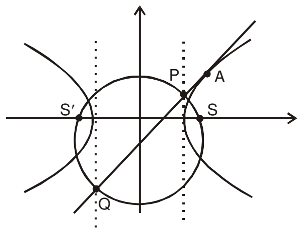


(iv) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$& the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2}$ = 1 (a > k > b > 0) are confocal and therefore orthogonal.

(v) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



(vi) A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\, t_2\, t_3}, -c\,t_1\, t_2\, t_3\right)$.

(vii) If a circle and the rectangular hyperbola xy = c^2 meet in the four points t_1, t_2, t_3 & t_4, then

(a) $t_1\, t_2\, t_3\, t_4 = 1$

(b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

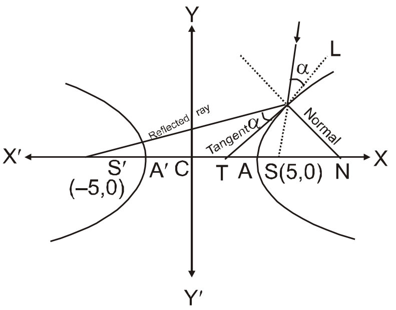
(c) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1\, t_2\, t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1\, t_2\, t_3\right)\right\}$$

Example # 66 : A ray originating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lying in first quadrant.

Solution : Given hyperbola is $9x^2 - 16y^2 = 144$. This equation can be rewritten as $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ....(1)

Since x co–ordinate of P is 8. Let y co–ordinate of P is α. $\because$ $(8,\alpha)$ lies on (1)



$\therefore$ $\frac{64}{16} - \frac{\alpha^2}{9} = 1$ $\Rightarrow$ $\alpha^2 = 27 \Rightarrow$ $\alpha = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence co-ordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passes through P $(8, 3\sqrt{3})$ and S′(–5,0)

$\therefore$ Its equation is $y - 3\sqrt{3} = \frac{0 - 3\sqrt{3}}{-5 - 8}(x - 8)$ or $13y - 39\sqrt{3} = 3\sqrt{3}\,x - 24\sqrt{3}$

or $3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$.

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Elementary concepts of Parabola

A-1. Find the value of λ for which the equation $\lambda x^2 + 4xy + y^2 + \lambda x + 3y + 2 = 0$ represents a parabola

A-2. Find
(i) The vertex, axis, focus, directrix, length of latusrectum of the parabola $x^2 + 2y - 3x + 5 = 0$.
(ii) The equation of the parabola whose focus is (1, 1) and the directrix is $x + y + 1 = 0$.
(iii) The equation to the parabola whose focus is (1, –1) and vertex is (2, 1).
(iv) The equation of the directrix of the parabola $x^2 - 4x - 3y + 10 = 0$.

A-3. Find the equation of the parabola the extremities of whose latus rectum are (1, 2) and (1, –4).

A-4. Find the axis, vertex, focus, directrix and equation of latus rectum of the parabola
$9y^2 - 16x - 12y - 57 = 0$

A-5. Find the locus of a point whose sum of the distances from the origin and the line x = 2 is 4 units.

A-6. Find the value of α for which point $(\alpha, 2\alpha + 1)$ doesn't lie outside the parabola $y = x^2 + x + 1$.

A-7. Find the set of values of α in the interval $[\pi/2, 3\pi/2]$, for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$.

A-8. If a circle be drawn so as always to touch a given straight line and also a given circle externally then prove that the locus of its centre is a parabola.(given line and given circle are non intersecting)

Section (B) : Elementary concepts of Ellipse & Hyperbola

B-1. Find the eccentricity of an ellipse of which distance between the focii is 10 and that of focus and corresponding directrix is 15.

B-2 If focus and corresponding directrix of an ellipse are (3, 4) and $x + y - 1 = 0$ respectively and eccentricity is $\frac{1}{2}$ then find the co-ordinates of extremities of major axis.

B-3 Find the set of those value(s) of 'α' for which the point $\left(7 - \frac{5}{4}\alpha, \alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

B-4. Write the parametric equation of ellipse $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$.

B-5. Find the set of possible value of α for which point $P(\alpha, 3\alpha)$ lies on the smaller region of the ellipse $9x^2 + 16y^2 = 144$ divided by the line $3x + 4y = 12$.

B-6. Find the equation of the ellipse having its centre at the point (2, –3), one focus at (3, –3) and one vertex at (4, –3).

B-7.✍ Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi-minor axis is $\sqrt{5}$.

B-8. Find
(i) The centre, eccentricity, foci and directrices of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.
(ii) The equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

B-9. For the hyperbola $x^2/100 - y^2/25 = 1$, prove that
(i) eccentricity = $\sqrt{5}/2$
(ii) SA . S′A = 25, where S & S′ are the foci & A is the vertex .

B-10. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.

B-11. Find
(i) The foci of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$
(ii) Equation of the hyperbola if vertex and focus of hyperbola are (2, 3) and (6, 3) respectively and eccentricity e of the hyperbola is 2

B-12. Find the position of the point (2, 5) relative to the hyperbola $9x^2 - y^2 = 1$.

B-13. Find the equation of auxiliary circle, of conic which passes through (1, 1) & is having foci (4, 5) & (2, 3).

B-14. Find the eccentricity of the hyperbola with its principal axes along the co-ordinate axes and which passes through (3, 0) & $\left(3\sqrt{2}, 2\right)$.

B-15. If m is a variable, then prove that the locus of the point of intersection of the lines $\frac{x}{3} - \frac{y}{2} = m$ and $\frac{x}{3} + \frac{y}{2} = \frac{1}{m}$ is a hyperbola.

B-16. Given the base of a triangle and the ratio of the tangent of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

B-17. Show that for rectangular hyperbola $xy = c^2$, length of transverse axis, length of conjugate axis and length of latus rectum are equal to $2\sqrt{2}\,c$

B-18. Prove that the distance of the point $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on the ellipse $x^2/6 + y^2/2 = 1$ from the centre of the ellipse is 2, if $\theta = 5\pi/4$

B-19. Find the eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the x-axis and the straight line $4x + 5y = 20$ on the y-axis and whose axes lie along the coordinates axes.

B-20. If the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ & the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide then find the value of b^2.

Section (C) : Position of line, Equation of chord and various forms of tangents of parabola

C-1. A line $y = x + 5$ intersect the parabola $(y - 3)^2 = 8(x + 2)$ at A & B. Find the length of chord AB.

C-2. Chord joining two distinct points $P(\alpha^2, k_1)$ and $Q\left(k_2, -\frac{16}{\alpha}\right)$ on the parabola $y^2 = 16x$ always passes through a fixed point. Find the co-ordinate of fixed point.

C-3. Find the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

C-4. Two perpendicular chords are drawn from the origin 'O' to the parabola $y = x^2$, which meet the parabola at P and Q Rectangle POQR is completed. Find the locus of vertex R.

C-5. Prove that the straight line $\ell x + my + n = 0$ touches the parabola $y^2 = 4ax$ if $\ell n = am^2$.

C-6. Find the range of c for which the line $y = mx + c$ touches the parabola $y^2 = 8 (x + 2)$.

C-7. Find the equation of that tangent to the parabola $y^2 = 7x$ which is parallel to the straight line $4y - x + 3 = 0$. Find also its point of contact.

C-8. A parabola $y = ax^2 + bx + c$ crosses the x – axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. Find the length of a tangent from the origin to the circle.

C-9. If tangent at P and Q to the parabola $y^2 = 4ax$ intersect at R then prove that mid point of R and M lies on the parabola, where M is the mid point of P and Q.

Section (D) : Position of line, Equation of chord and various forms of tangents of Ellipse & Hyperbola

D-1. Find the length of chord $x - 2y - 2 = 0$ of the ellipse $4x^2 + 16y^2 = 64$.

D-2. Find the locus of the middle points of chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are drawn through the positive end of the minor axis.

D-3. Check whether the line $4x + 5y = 40$ touches the ellipse $\frac{x^2}{50} + \frac{y^2}{32} = 1$ or not. If yes, then also find its point of contact.

D-4. An ellipse passes through the point (4, – 1) and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with co–ordinate axes.

D-5. Find the equation of the tangents at the ends of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and also show that they pass through the points of intersection of the major axis and directrices.

D-6. Any tangent to an ellipse is cut by the tangents at the ends of major axis in the points T and T′. Prove that the circle, whose diameter is T T′ will pass through the focii of the ellipse.

D-7. If 'P' be a moving point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ in such a way that tangent at 'P' intersect $x = \frac{25}{3}$ at Q then circle on PQ as diameter passes through a fixed point. Find that fixed point.

D-8. AB is a chord to the curve $S \equiv \frac{x^2}{9} + \frac{y^2}{16} - 1 = 0$ with A (3, 0) and C is a point on line AB such that AC : AB = 2 : 1 then find the locus of C.

D-9. Find the length of chord $x - 3y - 3 = 0$ of hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

D-10. For what value of λ, does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$?

D-11. If the straight line $2x + \sqrt{2}y + n = 0$ touches the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then find the value of n.

D-12. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

D-13. AB is a chord to the curve $S \equiv x^2 - y^2 - 16 = 0$ with A (4, 0) and C is a point on line segment AB such that AC : AB = 1 : 2 then find the locus of C.

D-14. The curve $xy = c(c > 0)$ and the circle $x^2 + y^2 = 25$ touch at two points, then find the distance between the points of contact.

D-15. If the tangent on the point $(3 \sec \phi, 4 \tan \phi)$ (which is in first quadrant) of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is perpendicular to $3x + 8y - 12 = 0$, then find the value of ϕ is (in degree).

Section (E) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Parabola

E-1. Find the equation of tangents to the parabola $y^2 = 9x$, which pass through the point (4, 10).

E-2. If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.

(i) $\theta_1 + \theta_2 = \alpha$ (a constant)

(ii) $\theta_1 + \theta_2 = \frac{\pi}{2}$

(iii) $\tan \theta_1 + \tan\theta_2 = \lambda$ (is constant)

E-3. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. Find the point on this line from which the other tangents to the parabola is perpendicular to the given tangent.

E-4. From the point (α, β) two perpendicular tangents are drawn to the parabola $(x - 7)^2 = 8y$. Then find the value of β.

E-5. Find the locus of the middle point of the focal chord of the parabola $y^2 = 4x$.

Section (F): Pair of tangents, Director circle, chord of contact and chord with given middle point of Ellipse & Hyperbola

F-1. Find the equation of tangents to the ellipse $\frac{x^2}{50} + \frac{y^2}{32} = 1$ which passes through a point (15, – 4).

F-2. If $3x + 4y = 12$ intersect the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at P and Q, then find the point of intersection of tangents at P and Q.

F-3. Find the equation of chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose mid point is (3, 1).

F-4. If m_1 & m_2 are the slopes of the tangents to the hyperbola $x^2/25 - y^2/16 = 1$ which passes through the point (4, 2), find the value of (i) $m_1 + m_2$ & (ii) $m_1 m_2$.

F-5. Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

F-6. Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$.

F-7. Chords of the hyperbola, $x^2 - y^2 = a^2$ touch the parabola, $y^2 = 4ax$. Prove that the locus of their middle points is the curve, $y^2(x - a) = x^3$.

F-8. Find the condition so that the line $px + qy = r$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\frac{\pi}{4}$.

Section (G) : Equation of normal, co-normal points of parabola

G-1. Find equation of all possible normals to the parabola $x^2 = 4y$ drawn from point (1, 2).

G-2. If $ax + by = 1$ is a normal to the parabola $y^2 = 4Px$, then prove that $Pa^3 + 2aPb^2 = b^2$.

G-3. Find the equation of normal to the parabola $x^2 = 4y$ at (6, 9).

G-4. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then prove that $p^2 = 2$.

G-5. If a line $x + y = 1$ cut the parabola $y^2 = 4ax$ in points A and B and normals drawn at A and B meet at C (C does not lies on parabola). The normal to the parabola from C other, than above two meet the parabola in D, then find D

G-6. If normal of circle $x^2 + y^2 + 6x + 8y + 9 = 0$ intersect the parabola $y^2 = 4x$ at P and Q then find the locus of point of intersection of tangent's at P and Q.

Section (H) : Equation of normal, co-normal points of Ellipse & Hyperbola

H-1. If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e^4 + e^2 - 1 = 0$

H-2. A ray emanating from the point $(-4, 0)$ is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection.

H-3. The tangent & normal at a point on $x^2/a^2 - y^2/b^2 = 1$ cut the y – axis respectively at A & B. Prove that the circle on AB as diameter passes through the focii of the hyperbola .

H-4. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. Show that the locus of the middle point of LM is a hyperbola of eccentricity $\frac{e}{\sqrt{(e^2 - 1)}}$.

Section (I) : Miscelleneous problems

I-1. Find the equation of a circle touching the parabola $y^2 = 8x$ at (2, 4) and passes through (0, 4).

I-2. An ellipse and a hyperbola have the same centre origin, the same foci and the minor-axis of the one is the same as the conjugate axis of the other. If e_1, e_2 be their eccentricities respectively, then find value of $\frac{1}{e_1^2} + \frac{1}{e_2^2}$.

I-3. $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then find the value of 'b' and the other common tangent.

I-4. The line $y = x$ intersects the hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$ at the points P and Q. Then find eccentricity of ellipse with PQ as major axis and miror axis of length $\frac{5}{\sqrt{2}}$.

I-5. Find the equation of common tangent to circle $x^2 + y^2 = 5$ and ellipse $x^2 + 9y^2 = 9$.

I-6. If latus rectum of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is double ordinate of parabola $y^2 = 4ax$, then find the value of a.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Elementary concepts of Parabola

A-1. The equation of the parabola whose focus is $(-3, 0)$ and the directrix is $x + 5 = 0$ is:
(A) $y^2 = 4(x-4)$ (B) $y^2 = 2(x+4)$ (C) $y^2 = 4(x-3)$ (D) $y^2 = 4(x+4)$

A-2. If (2, 0) is the vertex & y – axis is the directrix of a parabola, then its focus is:
(A) (2, 0) (B) (–2, 0) (C) (4, 0) (D) (–4, 0)

A-3. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is:
(A) 4 (B) 2 (C) 1/5 (D) 2/5

A-4. A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is:
(A) $x^2 - 6x - 8y + 25 = 0$ (B) $y^2 - 8x - 6y + 25 = 0$
(C) $x^2 - 6x + 8y - 25 = 0$ (D) $x^2 + 6x - 8y - 25 = 0$

A-5. Which one of the following equations parametrically represents equation to a parabolic profile?
(A) $x = 3\cos t;\ y = 4\sin t$ (B) $x^2 - 2 = -2\cos t;\ y = 4\cos^2\frac{t}{2}$
(C) $\sqrt{x} = \tan t;\ \sqrt{y} = \sec t$ (D) $x = \sqrt{1 - \sin t}\ ;\ y = \sin\frac{t}{2} + \cos\frac{t}{2}$

A-6. The points on the parabola $y^2 = 12x$ whose focal distance is 4, are
(A) $(2, \sqrt{3}), (2, -\sqrt{3})$ (B) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$ (C) (1, 2), (2, 1) (D) $(2, 2\sqrt{3}), (3, -2\sqrt{3})$

A-7. Find the all possible values of α such that point $P(\alpha, \alpha)$ is outside the parabola $y = x^2 + x + 1$ and inside the circle $x^2 + y^2 = 50$.
(A) $(-5, \infty)$ (B) $(-\infty, \infty)$ (C) $(-1, 5)$ (D) $(-5, 5)$

A-8. If on a given base, a triangle be described such that the sum of the tangents of the base angles is a constant, then the locus of the vertex is :
(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola

A-9. **Statement-1 :** For triangle whose two vertices are ends of a double ordinate for a parabola and third vertex lies on axis of same parabola incentre, circumcentre, centroid are collinear.
Statement-2 : In isosceles triangle incentre, circumcentre; orthocentre, centroid all lie on same line.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

A-10. The length of the latus rectum of the parabola whose focus is $\left(\frac{u^2}{2g}\sin 2\alpha, -\frac{u^2}{2g}\cos 2\alpha\right)$ and directrix is $y = \frac{u^2}{2g}$, is
(A) $\frac{u^2}{g}\cos^2\alpha$ (B) $\frac{u^2}{g}\cos 2\alpha$ (C) $\frac{2u^2}{g}\cos 2\alpha$ (D) $\frac{2u^2}{g}\cos^2\alpha$

A-11. The distance between the focus and directrix of the conic $\left(\sqrt{3}x - y\right)^2 = 48\left(x + \sqrt{3}y\right)$ is :
(A) 24 (B) 48 (C) 6 (D) 12

A-12. If one end of a focal chord of the parabola $y^2 = 4x$ is (1, 2), the other end doesn't lie on
(A) $x^2 y + 2 = 0$ (B) $xy + 2 = 0$ (C) $xy – 2 = 0$ (D) $x^2 + xy – y – 1 = 0$

A-13. The angle made by a double ordinate of length 8a at the vertex of the parabola $y^2 = 4ax$ is :
(A) $\pi/3$ (B) $\pi/2$ (C) $\pi/4$ (D) $\pi/6$

Section (B) : Elementary concepts of Ellipse & Hyperbola

B-1. The equation of the ellipse whose focus is (1, –1), directrix is the line x – y – 3 = 0 and the eccentricity is $\frac{1}{2}$, is
(A) $7x^2 + 2xy + 7y^2 – 10x + 10y + 7 = 0$ (B) $7x^2 + 2xy + 7y^2 + 7 = 0$
(C) $7x^2 + 2xy + 7y^2 + 10x – 10y – 7 = 0$ (D) $7x^2 + 4xy + 7y^2 – 10x + 10y + 7 = 0$

B-2. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is
(A) $\frac{5}{6}$ (B) $\frac{3}{5}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{5}}{3}$

B-3. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if
(A) $r > 2$ (B) $2 < r < 5$ (C) $r > 5$ (D) $r \in (2, 5) – \{3.5\}$

B-4. The length of the latus rectum of the ellipse $9x^2 + 4y^2 = 1$, is
(A) $\frac{3}{2}$ (B) $\frac{8}{3}$ (C) $\frac{4}{9}$ (D) $\frac{8}{9}$

B-5. The equation of the ellipse with its centre at (1, 2), focus at (6, 2) and passing through the point (4, 6) is
(A) $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$ (B) $\frac{(x-1)^2}{20} + \frac{(y-2)^2}{45} = 1$
(C) $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$ (D) $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{25} = 1$

B-6. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$
(A) outside the ellispe
(B) on the ellipse
(C) on the major axis
(D) on the minor axis

B-7. With respect to the hyperbola $(3x - 3y)^2 - (2x + 2y)^2 = 36$
(A) (3,2) lies on conjugate axis
(B) (3,2) lies on tranverse axis
(C) (3,2) lies inside hyperbola
(D) (3,2) lies outside hyperbola

B-8. Equation of auxilliary circle of the ellipse $2x^2 + 6xy + 5y^2 = 1$ is
(A) $(x - 1)^2 + y^2 = 7-3\sqrt{5}$
(B) $x^2 + y^2 = \frac{7+3\sqrt{5}}{2}$
(C) $x^2 + y^2 = \frac{2}{7+3\sqrt{5}}$
(D) $(x - 1)^2 + y^2 = \frac{4}{7+3\sqrt{5}}$

B-9. **Statement-1 :** Eccentricity of ellipse whose length of latus rectum is same as distance between foci is 2sin18°.
Statement-2 : For $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, eccentricity $e = \sqrt{1-\frac{b^2}{a^2}}$ `
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

B-10. The curve represented by x = 3 (cos t + sin t), y = 4 (cos t – sin t), is
(A) ellipse
(B) parabola
(C) hyperbola
(D) circle

B-11. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is
(A) 1
(B) $\sqrt{2}$
(C) 2
(D) 1/2

B-12. Which of the following pair, may represent the eccentricities of two conjugate hyperbolas, for all $\alpha \in (0, \pi/2)$?
(A) $\sin\alpha, \cos\alpha$
(B) $\tan\alpha, \cot\alpha$
(C) $\sec\alpha, \text{cosec}\,\alpha$
(D) $1 + \sin\alpha, 1 + \cos\alpha$

B-13.✍ For hyperbola represented by $16x^2 - 3y^2 - 32x + 12y - 44 = 0$, which of the following statement is INCORRECT
(A) the length of whose transverse axis is $4\sqrt{3}$
(B) the length of whose conjugate axis is 8
(C) whose centre is (1, 2)
(D) whose eccentricity is $\sqrt{\frac{19}{3}}$

B-14. **Statement-1 :** If $\sec\theta$, $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ represent eccentricity of a hyperbola then eccentricity of its conjugate hyperbola is given by $\text{cosec}\theta$.
Statement-2 : If e_1, e_2 are eccentricities of two hyperbolas which are conjugate to each other then $e_1^{-2} + e_2^{-2} = 1$
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

B-15. The eccentricity of the hyperbola whose conjugate axis is equal to half the distance between the foci, is:

(A) $\frac{4}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{5}{\sqrt{3}}$

B-16. Identify the following statements for true/false (T/F) in order

S1 : A latus rectum of an ellipse is a line passing through a focus

S2 : A latus rectum of an ellipse is a line through the centre

S3 : A latus rectum of an ellipse is a line perpendicular to the major axis

S4 : A latus rectum of an ellipse is a line parallel to the minor axis

(A) TFTF (B) TTFF (C) TFTT (D) FFFF

B-17. If P ($\sqrt{2}$ sec θ, $\sqrt{2}$ tan θ) is a point on the hyperbola whose distance from the origin is $\sqrt{6}$ where P is in the first quadrant then θ =

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{15}$

B-18. A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its

(A) orthocenter (B) circum centre (C) centroid (D) incentre

B-19. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of orthocentre of the ΔPQR are

(A) (x_4, y_4) (B) $(x_4, - y_4)$ (C) $(-x_4, - x_4)$ (D) $(- x_4, - y_4)$

B-20. The co-ordinates of a focus of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ is

(A) (–1, 1) (B) (6, 1) (C) (4, 1) (D) (– 6, –1)

B-21. The set of values of 'a' for which $(13x - 1)^2 + (13y - 2)^2 = a(5x + 12y - 1)^2$ represents an ellipse is

(A) $1 < a < 2$ (B) $0 < a < 1$ (C) $2 < a < 3$ (D) $3 < a < 4$

B-22. Find the equation of latus rectum of rectangular hyperbola $xy = c^2$

(A) $x - y \pm 2\sqrt{2}\,c = 0$ (B) $x - y \pm \sqrt{2}\,c = 0$ (C) $x + y \pm 2\sqrt{2}\,c = 0$ (D) $x + y \pm \sqrt{2}\,c = 0$

Section (C) : Position of line, Equation of chord and various forms of tangents of parabola

C-1. The locus of point of trisections of the focal chords of the parabola, $y^2 = 4x$ is:

(A) $y^2 = x - 1$ (B) $9y^2 = 4.(3x - 4)$ (C) $y^2 = 2(1 - x)$ (D) None of these

C-2. The latus rectum of a parabola whose focal chord is PSQ such that SP = 3 and SQ = 2 is given by:

(A) 24/5 (B) 12/5 (C) 6/5 (D) 23/5

C-3. Identify following statements for true/false (T/F) in order

S1 : The circles on focal radii of a parabola as diameter touch the tangent at the vertex

S2 : The circles on focal radii of a parabola as diameter touch the axis

S3 : A circle described on any focal chord of the parabola as its diameter will touch the directrix of the parabola

S4 : A circle described on any focal chord of the parabola as its diameter will touch the axis of the parabola

(A) TTFF (B) TFTF (C) FFTT (D) FTFT

C-4. The length of the chord $y = \sqrt{3}\,x - 2\sqrt{3}$ intercepted by the parabola $y^2 = 4(x - 1)$ is

(A) $4\sqrt{3}$ (B) $\frac{16}{3}$ (C) $\frac{8}{3}$ (D) $\frac{4}{\sqrt{3}}$

C-5. If y = 2x – 3 is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then 'a' is equal to, where a ≠ 0 :

(A) 1 (B) –1 (C) $\frac{14}{3}$ (D) $\frac{-14}{3}$

C-6. An equation of a tangent common to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is
(A) x – y + 1 = 0 (B) x + y – 1 = 0 (C) x + y + 1 = 0 (D) y = 0

C-7. Equation of a tangent to the parabola $y^2 = 12x$ which make an angle of 45° with line y = 3x + 77 is
(A) 2x – 4y + 3 = 0 (B) x – 2y + 12 = 0 (C) 4x + 2y + 5 = 0 (D) 2x + y – 12 = 0

C-8. Identify the following statements for true/false (T/F) in order
S1 : The tangents at the extremities of a focal chord of a parabola are perpendicular
S2 : The tangents at the extremities of a focal chord of a parabola are parallel
S3 : The tangents at the extremities of a focal chord of a parabola intersect on the directrix
S4 : The tangents at the extremities of a focal chord of a parabola intersect at the vertex
(A) TFTF (B) TTFF (C) TTTT (D) FFFF

Section (D) : Position of line, Equation of chord and various forms of tangents of Ellipse & Hyperbola

D-1. If the line y = 2x + c be a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, then c is equal to

(A) ± 4 (B) ± 6 (C) ± 1 (D) ± 8

D-2. The distance of the point of contact from the origin of the line $y = x - \sqrt{7}$ with the ellipse $3x^2 + 4y^2 = 12$, is

(A) $\sqrt{3}$ (B) 2 (C) $\frac{5}{\sqrt{7}}$ (D) $\frac{5}{7}$

D-3. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point P, then eccentric angle of P is
(A) 0 (B) 45° (C) 60° (D) 90°

D-4. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its corresponding point Q on the auxiliary circle, lies on the line :

(A) x = a/e (B) x = 0 (C) y = 0 (D) $y = \frac{b}{e}$

D-5. A chord is drawn to the hyperbola xy = 4 from a point A(2, 2) which cuts it again at point B. The locus of point P such that AP : PB = 2 :1
(A) (3x – 2) (3y – 2) = 16 (B) (2x – 3)(2y – 3) = 16
(C) xy = 2 (D) (3x – 2)(2y – 3) = 16

D-6. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line 5x + 2y –10 = 0 is :
(A) zero (B) 1 (C) 2 (D) 4

D-7. The tangent at any point $P(x_1, y_1)$ on the hyperbola $xy = c^2$ meets the co-ordinate axes at points Q & R. The circumcentre of ΔOQR has co-ordinates.

(A) (0, 0) (B) (x_1, y_1) (C) $\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$ (D) $\left(\frac{2x_1}{3}, \frac{2y_1}{3}\right)$

D-8. The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular to the line $y = x$ are :
(A) $y = -x \pm 7$ (B) $y = -x \pm 3$ (C) $y = -x \pm 4$ (D) none of these

Section (E) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Parabola

E-1. The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

E-2. The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A & B. The co-ordinates of the point of intersection of the tangents drawn at the points A & B are:
(A) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (B) $\left(-\frac{5}{2}, -\frac{7}{2}\right)$ (C) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (D) $\left(-\frac{7}{2}, -\frac{5}{2}\right)$

E-3. The locus of the middle points of the focal chords of the parabola, $y^2 = 4x$ is:
(A) $y^2 = x - 1$ (B) $y^2 = 2(x - 1)$ (C) $y^2 = 2(1 - x)$ (D) $y^2 = 2(x + 1)$

Section (F) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Ellipse & Hyperbola

F-1. The equation of the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ included between the co-ordinate axes is the curve:
(A) $9x^2 + 16y^2 = 4x^2y^2$ (B) $16x^2 + 9y^2 = 4x^2y^2$
(C) $3x^2 + 4y^2 = 4x^2y^2$ (D) $9x^2 + 16y^2 = x^2y^2$

F-2. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1) is
(A) $4x + 5y + 13 = 0$ (B) $4x + 5y = 13$ (C) $5x + 4y + 13 = 0$ (D) $5x + 4y = 13$

F-3 Point, from which tangents to the ellipse $5x^2 + 4y^2 = 20$ are not perpendicular, is:
(A) $(1, 2\sqrt{2})$ (B) $(2\sqrt{2}, 1)$ (C) $(2, \sqrt{5})$ (D) $(\sqrt{5}, 3)$

F-4. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is
(A) $3x - 4y = 4$ (B) $3y - 4x + 4 = 0$ (C) $4x - 4y = 3$ (D) $3x - 4y = 2$

F-5. The chords passing through L(2, 1) intersects the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at P and Q. If the tangents at P and Q intersects at R then Locus of R is
(A) $x - y = 1$ (B) $9x - 8y = 72$ (C) $x + y = 3$ (D) $9x + 8y = 7$

F-6. The number of points from where a pair of perpendicular tangents can be drawn to the hyperbola, $x^2 \sec^2\alpha - y^2 \operatorname{cosec}^2\alpha = 1$, $\alpha \in (0, \pi/4)$, is :
(A) 0 (B) 1 (C) 2 (D) infinite

F-7. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is:
(A) $y + mx = 0$ (B) $y - mx = 0$ (C) $my - x = 0$ (D) $my + x = 0$

F-8. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to:
(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

F-9. The locus of the mid points of the chords passing through a fixed point (α, β) of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is :

(A) a circle with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ (B) an ellipse with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(C) a hyperbola with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ (D) straight line passing through

Section (G) : Equation of normal, co-normal points of parabola

G-1. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in

(A) AP (B) GP (C) HP (D) none of these

G-2. Equation of the normal to the parabola, $y^2 = 4ax$ at its point $(am^2, 2am)$ is:

(A) $y = -mx + 2am + am^3$ (B) $y = mx - 2am - am^3$
(C) $y = mx + 2am + am^3$ (D) none

G-3. At what point on the parabola $y^2 = 4x$ the normal makes equal angles with the axes?

(A) (4, 4) (B) (9, 6) (C) (4, – 1) (D) (1, 2)

G-4. The line $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is

(A) 12 (B) – 12 (C) 24 (D) – 24

G-5. The equation of the other normal to the parabola $y^2 = 4ax$ which passes through the intersection of those at $(4a, -4a)$ & $(9a, -6a)$ is:

(A) $5x - y + 115a = 0$ (B) $5x + y - 135a = 0$ (C) $5x - y - 115a = 0$ (D) $5x + y + 115 = 0$

G-6. The normal chord of a parabola $y^2 = 4ax$ at the point $P(x_1, x_1)$ does not subtends a right angle at the

(A) focus (B) point (12a, 0)
(C) one of the end of the latus rectum (D) (a, 2a)

G-7. If three normals can be drawn to the curve $y^2 = x$ from point (c, 0) then 'c' can be equal to

(A) 0 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 2

G-8. The locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is

(A) $y^4 - 2a(x - 2a).\ y^2 + 8a^4 = 0$ (B) $y^4 + 2a(x - 2a).\ y^2 + 8a^4 = 0$
(C) $y^4 - 2a(x + 2a).\ y^2 + 8a^4 = 0$ (D) $y^4 - 2a(x - 2a).\ y^2 - 8a^4 = 0$

Section (H) : Equation of normal, co-normal points of Ellipse & Hyperbola

H-1. If the line $x\cos\alpha + y\sin\alpha = p$ be normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

(A) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = a^2 - b^2$ (B) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = (a^2 - b^2)^2$
(C) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = a^2 - b^2$ (D) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = (a^2 - b^2)^2$

H-2. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t', then

(A) $t' = -\frac{1}{t^3}$ (B) $t' = -\frac{1}{t}$ (C) $t' = \frac{1}{t^2}$ (D) $t'^2 = -\frac{1}{t^2}$

H-3. If the focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) is normal at $(a\cos\theta, b\sin\theta)$ then eccentricity of the ellipse is (it is given that $\sin\theta \neq 0$)
(A) $|\sec\theta|$ (B) $|\cos\theta|$ (C) $|\sin\theta|$ (D) None of these

H-4. The locus of the foot of perpendicular drawn from the centre of the hyperbola $x^2 - y^2 = 25$ to its normal.
(A) $100x^2y^2 = (x^2 + y^2)^2 (y^2 - x^2)$ (B) $10x^2y^2 = (x^2 + y^2)^2 (y^2 - x^2)$
(C) $200x^2y^2 = (x^2 - y^2)^2 (y^2 + x^2)$ (D) $100x^2y^2 = (x^2 - y^2)^2 (y^2 + x^2)$

H-5. The value of $|\lambda|$, for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$ is
(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{8}$

Section (I) : Miscelleneous problems

I-1. The feet of the perpendicular drawn from focus upon any tangent to the parabola, $y = x^2 - 2x - 3$ lies on
(A) $y + 4 = 0$ (B) $y = 0$ (C) $y = -2$ (D) $y + 1 = 0$

I-2. If F_1 & F_2 are the feet of the perpendiculars from the focii S_1 & S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to :
(A) 2 (B) 3 (C) 4 (D) 5

I-3. P & Q are corresponding points on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R where C is centre of the ellipse. Then ℓ(CR) is
(A) 5 units (B) 6 units (C) 7 units (D) 8 units

I-4. The equation of common tangent of $x^2 + y^2 = 2$ and $y^2 = 8x$ is
(A) $x - y + 2 = 0$ (B) $x + y + 1 = 0$ (C) $x - y + 1 = 0$ (D) $x + y - 2 = 0$

I-5. The equation of common normal to the circle $x^2 + y^2 - 12x + 16 = 0$ and parabola $y^2 = 4x$ is
(A) $y = 0$ (B) $2x - y = 12$ (C) $2x + y = 12$ (D) All of the above

I-6. Equation of common tangent to ellipse $5x^2 + 2y^2 = 10$, and hyperbola $11x^2 - 3y^2 = 33$ is
(A) $y = \pm 3x \pm \sqrt{21}$ (B) $y = \pm x \pm 3$
(C) $y = \pm 4x \pm \sqrt{37}$ (D) $3x \pm y = 12$

I-7. The equation of common tangent to the parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$ is
(A) $2x \pm y + 1 = 0$ (B) $2x \pm y - 1 = 0$
(C) $x \pm 2y + 1 = 0$ (D) $x \pm 2y - 1 = 0$

I-8. $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then the value of 'b' and the other common tangent are given by :
(A) $b = \sqrt{3}$; $x + 2y + 4 = 0$ (B) $b = 3$; $x + 2y + 4 = 0$
(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$ (D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

PART - III : MATCH THE COLUMN

1. **Match the column**

Column – I		Column – II	
(A)	If the mid point of a chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is (0, 3), then length of the chord is $\frac{4k}{5}$, then k is	(p)	6
(B)	Eccentric angle of one of the points on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is $\frac{k\pi}{4}$, then k is	(q)	8
(C)	If 'e' is eccentricity and ℓ is the length of latus rectum of the ellipse $9x^2 + 5y^2 - 30y = 0$, then $4(e + \ell)$ is	(r)	3
(D)	Sum of distances of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ from the focii	(s)	16

2. AB is a chord of the parabola $y^2 = 4ax$ joining $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$. Match the following

Column – I		Column – II	
(A)	AB is a normal chord	(p)	$t_2 = -t_1 + 2$
(B)	AB is a focal chord	(q)	$t_2 = -\frac{4}{t_1}$
(C)	AB subtends 90º at (0, 0)	(r)	$t_2 = -\frac{1}{t_1}$
(D)	AB is inclined at 45º to the axis of parabola	(s)	$t_2 = -t_1 - \frac{2}{t_1}$

3. A tangent having slope $-\frac{4}{3}$ touches the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ at point P and intersects the major and minor axes at A & B respectively, O is the centre of the ellipse

Column - I		Column - II	
(A)	Distance between the parallel tangents having slopes $-\frac{4}{3}$, is	(p)	24
(B)	Area of ΔAOB is	(q)	7/24
(C)	If the tangent in first quadrant touches the ellipse at (h, k) then value of hk is	(r)	48/5
(D)	If equation of the tangent intersecting positive axes is $\ell x + my = 1$, then $\ell + m$ is equal to	(s)	12

4.

Column - I		Column - II	
(A)	Number of positive integral values of b for which tangent parallel to line y = x + 1 can be drawn to hyperbola $\frac{x^2}{5}-\frac{y^2}{b^2}=1$ is	(p)	16
(B)	The equation of the hyperbola with vertices (3, 0) and (–3, 0) and semi-latusrectum 4, is given by is $4x^2 - 3y^2 = 4k$, then k =	(q)	2
(C)	The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola $\frac{x^2}{25}-\frac{y^2}{3}=1$ is $\sqrt{k}$, then k is	(r)	4
(D)	An equation of a tangent to the hyperbola, $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\frac{\pi}{4}$ with the transverse axis is $y = x + \lambda$, $(\lambda > 0)$, then 2λ is	(s)	9

Exercise-2

✍ Marked questions are recommended for Revision.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. The vertex of a parabola is the point (a, b) and latus rectum is of length 1. If the axis of the parabola is along the positive direction of y-axis, then its equation is :

(A) $(x + a)^2 = \frac{1}{2}(2y - 2b)$ (B) $(x - a)^2 = \frac{1}{2}(2y - 2b)$

(C) $(x + a)^2 = \frac{1}{4}(2y - 2b)$ (D) $(x - a)^2 = \frac{1}{8}(2y - 2b)$

2.✍ Length of the focal chord of the parabola $y^2 = 4ax$ at a distance p from the vertex is:

(A) $\frac{2a^2}{p}$ (B) $\frac{a^3}{p^2}$ (C) $\frac{4a^3}{p^2}$ (D) $\frac{p^2}{a}$

3. The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the points Q and R is:

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$ (C) $\frac{2A}{a}$ (D) $\frac{4A}{a}$

4.✍ AB is a chord of the parabola $y^2 = 4ax$ with vertex at A. BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is

(A) a (B) 2a (C) 4a (D) 8a

5.✍ If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$. then the chords P_1P_2 and Q_1Q_2 intersect on

(A) tangent at the vertex of the parabola (B) the directrix of the parabola

(C) at x = –2a (D) y = 2a and x = –2a

6. The vertex of the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is

(A) $\left(\frac{1}{9},\frac{2}{9}\right)$ (B) $\left(\frac{8}{9},\frac{1}{9}\right)$ (C) $\left(\frac{2}{9},\frac{8}{9}\right)$ (D) $\left(\frac{1}{9},\frac{1}{9}\right)$

7.✍ AB, AC are tangents to a parabola $y^2 = 4ax$. p_1, p_2 & p_3 are the lengths of the perpendiculars from A, B & C respectively on any tangent to the curve, then p_2, p_1, p_3 are in:

(A) A.P. (B) G.P. (C) H.P. (D) none of these

8. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point (1, 2) is

(A) $(x - 1)^2 = 4(y - 2)$ (B) $(x + 3)^2 = 4(y + 2)$

(C) $(x + 1)^2 = 4(y - 1)$ (D) $(x - 1)^2 = 4(y - 1)$

9. A normal chord of the parabola subtending a right angle at the vertex makes an acute angle θ with the x – axis, then θ =

(A) arc tan 2 (B) arc tan $\sqrt{2}$ (C) arc cot $\sqrt{2}$ (D) arc cot2

10.✍ If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet passes through a fixed point whose co-ordinates are:

(A) (– 2a, 0) (B) (a, 0) (C) (2a, 0) (D) (–a, 0)

11. If a parabola whose length of latus rectum is 4a touches both the coordinate axes then the locus of its focus is
(A) $xy = a^2 (x^2 + y^2)$ (B) $x^2y^2 = a^2 (x^2 + y^2)$
(C) $x^2 - y^2 = a^2 (x^2 + y^2)$ (D) $x^2y^2 = a^2 (x^2 - y^2)$

12. T is a point on the tangent to a parabola $y^2 = 4ax$ at its point P. TL and TN are the perpendiculars on the focal radius SP and the directrix of the parabola respectively. Then:
(A) SL = 2 (TN) (B) 3 (SL) = 2 (TN) (C) SL = TN (D) 2 (SL) = 3 (TN)

13. In the parabola $y^2 = 4ax$, the locus of middle points of all chords of constant length c is
(A) $(4ax - y^2)(y^2 - 4a^2) = a^2c^2$ (B) $(4ax + y^2)(y^2 + 4a^2) = a^2c^2$
(C) $(4ax + y^2)(y^2 - 4a^2) = a^2c^2$ (D) $(4ax - y^2)(y^2 + 4a^2) = a^2c^2$

14. Through the vertex 'O' of the parabola $y^2 = 4ax$, variable chords OP and OQ are drawn at right angles. If the variable chord PQ intersects the axis of x at R, then the distance OR is equal to
(A) 2a (B) 3a (C) 4a (D) 8a

15. From the focus of the parabola, $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex & focus of the parabola. The equation of the circle is
(A) $(x - 2)^2 + y^2 = 9$ (B) $(x - 2)^2 + y^2 = 3$ (C) $(x - 2)^2 + y^2 = 2$ (D) $(x - 2)^2 + y^2 = 1$

16. If from the vertex of a parabola $y^2 = 4ax$, a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, locus of the further angle of the rectangle is
(A) $y^2 = 8a (x - 8a)$. (B) $y^2 = 4a (x + 8a)$. (C) $y^2 = - 4a (x - 8a)$. (D) $y^2 = 4a (x - 8a)$.

17. A line of fixed length (a + b) moves so that its ends are always on two fixed perpendicular straight lines. The locus of the point which divided this line into portions of lengths a & b, is :
(A) an ellipse (B) an hyperbola (C) a circle (D) a straight line

18. Coordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0) respectively. The vertex A is varying in such a way that $4 \tan \frac{B}{2} . \tan \frac{C}{2} = 1$. Then locus of A is
(A) $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$ (B) $\frac{(x-5)^2}{16} + \frac{y^2}{25} = 1$
(C) $\frac{(x-5)^2}{25} + \frac{y^2}{9} = 1$ (D) $\frac{(x-5)^2}{9} + \frac{y^2}{25} = 1$

19. The locus of point of intersection of tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points, the sum of whose eccentric angles is constant, is :
(A) a hyperbola (B) an ellipse (C) a circle (D) a straight line

20. An ellipse with major axis 4 and minor axis 2 touches both the coordinate axis, then Locus of its centre is
(A) $x^2 - y^2 = 5$ (B) $x^2.y^2 = 5$ (C) $\frac{x^2}{4} + y^2 = 5$ (D) $x^2 + y^2 = 5$

21. An ellipse with major axis 4 and minor axis 2 touches both the coordinate axes, then locus of its focus is
(A) $(x^2 - y^2) (1 + x^2y^2) = 16 x^2 y^2$ (B) $(x^2 - y^2) (1 - x^2y^2) = 16 x^2 y^2$.
(C) $(x^2 + y^2) (1 + x^2y^2) = 16 x^2 y^2$ (D) $(x^2 + y^2) (1 - x^2y^2) = 16 x^2 y^2$.

22. A series of concentric ellipses E_1, E_2, ..., E_n are drawn such that E_n touches the extremities of the major axis of E_{n-1} and the foci of E_n coincide with the extremities of minor axis of E_{n-1}. If the eccentricity of the ellipses is independent of n, then the value of the eccentricity is

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}-1}{16}$ (C) $\frac{\sqrt{5}-1}{2}$ (D) $\frac{\sqrt{5}-1}{8}$

23. If e and e′ are the eccentricities of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, then the point $\left(\frac{1}{e},\frac{1}{e'}\right)$ lies on the circle :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $x^2 + y^2 = 3$ (D) $x^2 + y^2 = 4$

24. P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then OT. ON is equal to :

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2

25. Tangent at any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the axes at A and B respectively. If the rectangle OAPB (where O is origin) is completed then locus of point P is given by

(A) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ (B) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (C) $\frac{a^2}{y^2}-\frac{b^2}{x^2}=1$ (D) none of these

26. If the chord of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles, then $\frac{x_1\,x_2}{y_1\,y_2}$ is equal to

(A) $-\frac{a^2}{b^2}$ (B) $-\frac{b^2}{a^2}$ (C) $-\frac{b^4}{a^4}$ (D) $-\frac{a^4}{b^4}$

27^. The sides AC and AB of a triangle ABC touch the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at C and B respectively. If the vertex A lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the side BC

(A) must touch the ellipse
(B) must cut the ellipse at two distinct points
(C) may not touch the ellipse
(D) may cut the ellipse at two distinct points

28. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ to the circle $x^2 + y^2 = 9$, then the locus of mid-point of the chord of contact is

(A) $\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$ (B) $\frac{x^2}{9}+\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$

(C) $\frac{x^2}{9}+\frac{y^2}{4}=\left(\frac{x^2-y^2}{9}\right)^2$ (D) $\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2-y^2}{9}\right)^2$

29. If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 such that ΔOAB (O is the origin) is an equilateral triangle, then the eccentricity 'e' of the hyperbola

(A) is greater than $\frac{2}{\sqrt{3}}$ (B) is less than $\frac{2}{\sqrt{3}}$

(C) is equal to $\frac{2}{\sqrt{3}}$ (D) is less than $\frac{1}{\sqrt{3}}$

30. Let two variable ellipse E_1 and E_2 touches each other externally at (0, 0). Their common tangent at (0, 0) is y = x. If one of the focus at E_1 & one of the focus of E_2 always lies or line y = 2x then find locus of other focus of E_1 & E_2.

(A) y = 4x (B) y = –2x (C) y = x/2 (D) y = –x/2

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. If $(a^2, a - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, – 4), then the number of all possible integral values of a is :

2. If ℓ is the distance between focus and directrix of the parabola $9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$ then 6ℓ is :

3. The number of integral values of a for which the point (–2a, a + 1) will be an interior point of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$, is :

4. A variable chord PQ of the parabola, $y^2 = 4x$ is drawn parallel to the line y = x. If the parameters of the points P & Q on the parabola be p & q respectively, then (p + q) equal to.

5. The parabola whose axis is parallel to the y-axis and which passes through the points (0, 4), (1, 9) and (–2,6), also passes through $(2, \alpha)$ then the value of α is :

6. Through the vertex O of the parabola $y^2 = 8x$, a perpendicular is drawn to any tangent meeting it at P & the parabola at Q, then the value of OP. OQ is

7. The centre of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point (6, 9) is (α, β) then $|\alpha - \beta|$ is

8. Points A, B & C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B & C, taken in pairs, intersect at points P, Q & R. the ratio of the areas of the triangles ABC & PQR is $\frac{\lambda}{\mu}$ where λ and μ are co-prime number then $\lambda + \mu$ is

9. A normal is drawn to a parabola $y^2 = 4ax$ at any point other than the vertex and it cuts the parabola again at a point whose distance from the vertex is not less than $\lambda\sqrt{6}\,a$, then the value of λ is

10. If three normal are drawn through (c, 0) to $y^2 = 4x$ and two of which of perpendicular then the value of c is

11. P & Q are the points with eccentric angles θ & $\theta + \pi/6$ on the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$, then the area of the triangle OPQ is :

12. If P is a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$=1 whose focii are S and S′ and e_1 is the eccentricity and the locus of the incentre of ΔPSS′ is an ellipse whose eccentricity is e_2 , then the value of $\left(1+\frac{1}{e_1}\right)e_2^2$ is:

13. If $\left(0, 3+\sqrt{5}\right)$ is a point on the ellipse whose foci are (2, 3), (–2, 3) then the length of semimajor axis is :

14. A circle has the same centre as an ellipse & passes through the focii F_1 & F_2 of the ellipse, such that the two curves intersect at 4 points. Let 'P' be any one of their points of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the focii is :

15. Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) is

16. If 'r' be the radius of largest circle with centre (3, 0) that can be inscribed in the ellipse $9x^2 + 25y^2 = 225$, then $4\sqrt{7}$ r is equal to

17. Minimum length of the intercept made by the axes on the tangent to the ellipse $\frac{x^2}{81} + \frac{y^2}{36} = 1$ is equal to

18. If the distance of the centre of the ellipse $4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$ from the origin is λ times its eccentricity, then $5\lambda^2$ is :

19. The radius of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$ is $\sqrt{\frac{\alpha}{\beta}}$ where α and β are prime number, then $\alpha + \beta$ is

20. Common tangents are drawn to the parabola $y^2 = 4x$ & the ellipse $3x^2 + 8y^2 = 48$ touching the parabola at A & B and the ellipse at C & D, then the area of the quadrilateral ABCD is $\lambda\sqrt{2}$ the λ is equal to

21. A circle of radius r is concentric with the ellipse $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ and the common tangent is inclined to the major axis at an angle of $\tan^{-1}\sqrt{\left(\frac{r^2 - \beta^2}{\alpha^2 - r^2}\right)}$; $r \in (b, a)$ then the value of $|\alpha| + |\beta|$ is

22. If CF is perpendicular from the centre of the ellipse $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ to the tangent at P, and G is the point where the normal at P meets the major axis, then the product $CF \cdot PG$ is

23. The eccentricity of an ellipse whose focii are (2, 4) & (14, 9) and touches x-axis is $\frac{\lambda}{\sqrt{313}}$ then the value of λ is

24. If two points P & Q on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then $\frac{1}{CP^2} + \frac{1}{CQ^2} = \lambda\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ where λ is :

25. If $7x^2 + pxy + qy^2 + rx - sy + t = 0$ is the eqaution of the hyperbola whose one focus is (–1, 1), eccentricity = 3 and the equation of the corresponding directrix is $x - y + 3 = 0$, then the value of 't' is :

26. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ & $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. The value of $2(a^2 + b^2)$ is :

27. If m_1 and m_2 are slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}$ = 1 which passes through the point of contact of $3x – 4y = 5$ and $x^2 – 4y^2 = 5$ then $32(m_1 + m_2 – m_1m_2)$ = ...

28. Tangents are drawn from the point $(\alpha, 2)$ to the hyperbola $3x^2 – 2y^2 = 6$ and are inclined at angles θ & ϕ to the x –axis . If $\tan\theta . \tan\phi = 2$, then the value of $2\alpha^2 – 7$ is

29. C the centre of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. The tangents at any point P on this hyperbola meets the striaght lines $4x – 3y = 0$ and $4x + 3y = 0$ in the points Q and R respectively. Then CQ . CR =

30. If radii of director circles of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ = 1 are 2r and r respectively and e_e and e_h be the eccentricities of the ellipse and the hyperbola respectively then $4e_h^2 – e_e^2$ is equal to

31. The length of that focal chord of the hyperbola $xy = 8$ which touches the circle $x^2 + y^2 = 8$ is.

32. The sum of lengths of perpendiculars drawn from focii to any real tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ is always greater than a, then find maximum value of a.

33. Let tangent at point A, B and vertex (V) of parabola is $x – 2y + 1 = 0$, $3x + y + 4 = 0$ and $y = x$ respectively. If focus of parabola is $\left(\frac{a}{7}, \frac{b}{7}\right)$ then find the value of (a + 5b).

34. If common tangent of $x^2 + y^2 = r^2$ and $\frac{x^2}{16}+\frac{y^2}{9}=1$ forms square then find its area.

35. Let $x^2 + y^2 = r^2$ and $xy = 1$ intersect at A & B in first quadrant, If AB = $\sqrt{14}$ then find the value of r.

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Let A be the vertex and L the length of the latus rectum of the parabola, $y^2 – 2y – 4x – 7 = 0$. The equation of the parabola with A as vertex, 2L the length of the latus rectum and the axis at right angles to that of the given curve is:

(A) $x^2 + 4x + 8y – 4 = 0$
(B) $x^2 + 4x – 8y + 12 = 0$
(C) $x^2 + 4x + 8y + 12 = 0$
(D) $x^2 + 8x – 4y + 8 = 0$

2. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(A) Latus rectum is half the latus rectum of the original parabola
(B) Vertex is (a/2, 0)
(C) Directrix is y-axis
(D) Focus has the co-ordinates (a,0)

3. P is a point on the parabola $y^2 = 4ax$ $(a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x–axis at a point whose co–ordinates are:

(A) (– 3a, 0) (B) (– a, 0) (C) (– 2a, 0) (D) (a, 0)

4. Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then:

(A) a > 0, b > 0 (B) a > 0, b < 0 (C) a < 0, b > 0 (D) a < 0, b < 0

5. P is a point on the parabola $y^2 = 4x$ where abscissa and ordinate are equal. Equation of a circle passing through the focus and touching the parabola at P is:
(A) $x^2 + y^2 - 13x + 2y + 12 = 0$ (B) $x^2 + y^2 - 3x - 18y + 2 = 0$
(C) $x^2 + y^2 + 13x - 2y - 14 = 0$ (D) $x^2 + y^2 - x = 0$

6^. Subset of complete set of values of m for which a chord of slope m of the circle $x^2 + y^2 = 4$ touches parabola $y^2 = 4x$, can be
(A) $\left(-\infty, -\sqrt{\frac{\sqrt{2}-1}{2}}\right)$ (B) (0 , 1/2) (C) $\left(\sqrt{\frac{\sqrt{2}-1}{2}}, \infty\right)$ (D) (–1/2, 0)

7. Locus of the centre of the circle passing through the vertex and the mid-points of perpendicular chords from the vertex of the parabola $y^2 = 4ax$ is.
(A) is a parabola with vertex (–a, a) (B) is a parabola with latus rectum a
(C) is a parabola with vertex (2a,0) (D) is a parabola with latus rectum $\frac{a}{2}$

8. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = C$.
(A) cannot represent a real pair of straight lines for any value of C
(B) represents an ellipse, if C > 0
(C) no locus, if C < 0
(D) a point, if C = 0

9. If P is a point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose focii are S and S′. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then
(A) PS + PS′ = 2a, if a > b
(B) PS + PS′ = 2b, if a < b
(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$
(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{a^2-b^2}}{b^2}\,[a - \sqrt{a^2-b^2}\,]$ when a > b

10. Let A(α) and B(β) be the extremeties of a chord of an ellipse . If the slope of AB is equal to the slope of the tangent at a point C(θ) on the ellipse, then the value of θ, is
(A) $\frac{\alpha+\beta}{2}$ (B) $\frac{\alpha-\beta}{2}$ (C) $\frac{\alpha+\beta}{2} + \pi$ (D) $\frac{\alpha-\beta}{2} - \pi$

11. Let F_1, F_2 be two focii of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P then
(A) PN bisects $\angle F_1PF_2$ (B) PT bisects $\angle F_1PF_2$
(C) PT bisects angle $(180° - \angle F_1PF_2)$ (D) None of these

12. If ℓ_1 be the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ and λ_1 be the length of the intercept of the common tangent between the coordinate axes then
(A) $\lambda_1 = \frac{14}{\sqrt{3}}$ (B) Equation of ℓ_1 is $2x + \sqrt{3}y = 4\sqrt{7}$
(C) $\lambda_1 = \frac{4}{\sqrt{3}}$ (D) Equation of ℓ_1 is $x + \sqrt{3}y = 4\sqrt{7}$

13. Let E_1 and E_2 be two ellipses $\frac{x^2}{a^2}+y^2=1$ and $x^2+\frac{y^2}{a^2}=1$ (where a is parameter) the locus of points of intersection of the ellipses E_1 and E_2 is a set of curves
(A) $y = x$, $y = -x$, $x^2 + y^2 = 1$ (B) $y = 2x$, $y = -2x$, $x^2 + y^2 = 4$
(C) $(4x^2 - y^2)(x^2 + y^2 - 4) = 0$ (D) $(x^2 - y^2)(x^2 + y^2 - 1) = 0$

14. If (5, 12) and (24, 7) are the foci of a conic, passing through the origin then the eccentricity of conic is
(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

15. The equation of a hyperbola with co-ordinate axes as principal axes, if the distances of one of its vertices from the foci are 3 & 1 can be :
(A) $3x^2 - y^2 = 3$ (B) $x^2 - 3y^2 + 3 = 0$ (C) $x^2 - 3y^2 - 3 = 0$ (D) $x^2 - 3y^2 - 6 = 0$

16. A point moves such that the sum of the squares of its distances from the two sides of length 'a' of a rectangle is twice the sum of the squares of its distances from the other two sides of length 'b'. The locus of the point can be :
(A) a circle (B) an ellipse (C) a hyperbola (D) a pair of lines

17. If $(3\sin\alpha, 2\cos\alpha)$ lies on the same side as that of origin w.r.t conic $2x^2 - 3y^2 = 6$, then $\sin\alpha$ may be
(A) $-\sqrt{\frac{4}{5}}$ (B) $\sqrt{\frac{2}{5}}$ (C) $\frac{1}{\sqrt{5}}$ (D) $\frac{2}{15}$

18. Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.
(A) $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$
(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

19. If two distinct tangents can be drawn from the point $(\alpha, 2)$ on different branches of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$, then the range of α is subset of
(A) $\left[-\frac{3}{2},\frac{3}{2}\right]$ (B) $[-2,2]$ (C) $[-1,1]$ (D) $\left(-\frac{1}{2},\frac{1}{2}\right)$

20. A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S. Then $CP^2 + CQ^2 + CR^2 + CS^2 =$
(A) 16 if $r = \sqrt{2}$ (B) 16 if r = 2 (C) 2 if r = 1 (D) 4 if r = 1

21. $x^2 + y^2 = 16$ is the auxilliary circle of
(A) $9x^2 - 16y^2 - 144 = 0$ (B) $16x^2 - 9y^2 + 144 = 0$
(C) $9(x - y)^2 - 16(x + y)^2 - 288 = 0$ (D) $16(x - y)^2 - 9(x + y)^2 + 288 = 0$

22. If the chord joining the points whose eccentric angles are 'α' and 'β' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a focal chord then
(A) $\pm e\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$
(B) $\pm e\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$
(C) $\tan(\alpha/2)\tan(\beta/2) + \left(\frac{ke-1}{ke+1}\right) = 0$ where $k = \pm 1$
(D) $\tan(\alpha/2)\tan(\beta/2) + \left(\frac{ke+1}{ke-1}\right) = 0$ where $k = \pm 1$

23. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then
(A) PG = PC (B) Pg = PC (C) PG = Pg (D) Gg = PC

24. Circles are drawn on chords of rectangular hyperbola $xy = c^2$ parallel to the line y = x as diameters. All such circles pass through two fixed points whose co-ordinates are :
(A) (c, c) (B) (c, –c) (C) (–c, c) (D) (–c, –c)

25. If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then
(A) $t_1 t_2 t_3 t_4 = 1$
(B) The arthmetic mean of the four points bisects the distance between the centres of the two curves.
(C) The geometrical mean of the four points bisects the distance between the centres of the two curves.
(D) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3\right)\right\}$$

26. Two confocal parabola intersect at A and B. If their axis are parallel to x-axis and y-axis respectively, then slope of chord AB can be :
(A) 1 (B) –1 (C) 2 (D) –2

PART - IV : COMPREHENSION

Comprehension # 1 (For Q.No. 1 to 3)

Consider three lines y axis, y = 2 and $\ell x + my = 1$ where (ℓ, m) lies on $y^2 = 4x$. answer the following :

1. Locus of circum centre of triangle formed by given three lines is a parabola whose vertex is
(A) (–2, 3/2) (B) (2, –3/2) (C) (–2, –3/2) (D) (2, –5/2)

2. Area of triangle formed by vertex and end points of latus rectum of parabola obtained in questions (1) is
(A) $\frac{1}{2^8}$ (unit)^2 (B) $\frac{1}{2^9}$ (unit)^2 (C) $\frac{1}{2^{10}}$ (unit)^2 (D) $\frac{1}{2^7}$ (unit)^2

3. Any point on the parabola obtained in question (1) can be represented as
(A) $\left(2+\frac{1}{32}t^2, \frac{3}{2}+\frac{t}{16}\right)$ (B) $\left(2+\frac{t^2}{32}, \frac{-3}{2}+\frac{1}{16}t^2\right)$ (C) $\left(-2+\frac{1}{32}t^2, \frac{3}{2}+\frac{t}{16}\right)$ (D) $\left(-2+\frac{1}{16}t^2, \frac{3}{2}+\frac{t}{5}\right)$

Comprehension # 2

Let PQ be a variable focal chord of the parabola $y^2 = 4ax$ where vertex is A. Locus of, centroid of triangle APQ is a parabola 'P_1'

4. Latus rectum of parabola P_1 is
(A) $\frac{2a}{3}$ (B) $\frac{4a}{3}$ (C) $\frac{8a}{3}$ (D) $\frac{16a}{3}$

5. Vertex of parabola P_1 is
(A) $\left(\frac{2a}{3}, 0\right)$ (B) $\left(\frac{4a}{3}, 0\right)$ (C) $\left(\frac{8a}{3}, 0\right)$ (D) $\left(\frac{a}{3}, 0\right)$

6. Let Δ_1 is the area of triangle formed by joining points T_1, T_2 and T_3 on parabola P_1 and Δ_2 be the area of triangle T formed by tangents at T_1, T_2 and T_3, then
(A) $\Delta_2 = 2\Delta_1$
(B) $\Delta_1 = 4\Delta_2$
(C) orthocentre of triangle T lies on x = a/3.
(D) Both (A) and (C) are correct.

Comprehension # 3

Two tangents PA and PB are drawn from a point P(h, k) to the ellipse E : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b). Angle of the tangents with the positive x - axis are θ_1 and θ_2. Normals at A and B are intersecting at Q point.
On the basis of above information answer the following questions.

7. Locus of P, if $\tan\theta_1 . \tan\theta_2 = 4$, is

(A) $\frac{y-b}{x-a} = 2(x+a)$ (B) $y^2 - b^2 = 2(x^2 + a^2)$ (C) $\frac{y+b}{x+a} = \frac{4(x-a)}{y-b}$ (D) $\frac{y+b}{x-b}=\frac{x+a}{y-b}$

8. Circumcentre of ΔQAB is

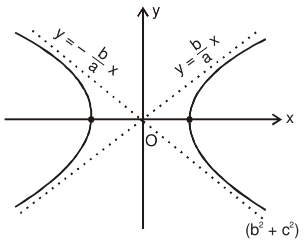
(A) mid point of AB (B) mid point of PQ (C) orthocentre of ΔPAB (D) can't say

9. Locus of P, if $\cot\theta_1 + \cot\theta_2 = \lambda$, is

(A) $2xy = \lambda(y^2 - b^2)$ (B) $2xy - \lambda(b^2 - y^2) = 0$ (C) $xy = \lambda$ (D) $x^2 + xy = \lambda$

Comprehension # 4

Asymptotes are lines whose distance from the curve at infinity tends to zero Let y = mx + c is asymptote of H $\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1$. Solving the two equations, we have $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$. Both roots of this equation must be infinite so $m = \pm\frac{b}{a}$ and c = 0 which implies that $y = \frac{b}{a}\pm x$ are asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. Note that no real tangent can be drawn to the hyperbola from its centre and only one real tangent can be drawn from a point lying on its asymptote other than centre. Further combined equation of asymptotes is A = $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0$ and conjugate hyperbola C = $\frac{x^2}{a^2}-\frac{y^2}{b^2} + 1 = 0$. Hence 2A = H + C, as we can see, equation of A, H and C vary only by a constant, for asymptotes which can be evaluated by applying condition of pair of lines.



10. The points of contact of tangents drawn to the hyperbola $\frac{x^2}{3}-\frac{y^2}{2} = 1$ from point (2, 1) are

(A) (3, 2), (1, 5) (B) (3, 2), $\left(\frac{9}{5},\frac{2}{5}\right)$ (C) (1, 2), (3, 4) (D) (3, 2), (3, 4)

11. The number of real distinct tangents drawn to hyperbola $4x^2 - y^2 = 4$ from point (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

12. The number of real distinct tangents drawn from point (1, 2) to hyperbola $x^2 - y^2 - 2x + 4y - 4 = 0$ is

(A) 1 (B) 2 (C) 3 (D) None of these

13. The asymptotes of $xy - 3y - 2x = 0$ is

(A) x + 2 = 0 and y + 3 = 0 (B) x – 2 = 0 and y – 3 = 0
(C) x – 3 = 0 and y – 2 = 0 (D) x + 3 = 0 and y + 2 = 0

Comprehension # 5

Equation of the transverse and conjugate axis of a hyperbola are respectively x + 2y – 3 = 0, 2x – y + 4 = 0 and their respectively lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$ then answer following :

14. If $x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is equation of given hyperbola where h, b, g, f, c all are integers then the sum h + b + g + f + c =
(A) 3 (B) 4 (C) 5 (D) 6

15. Equation of one of the directrix is
(A) $2x - y + 4 + \sqrt{\frac{3}{2}} = 0$ (B) $x + 2y + 4 - \sqrt{\frac{2}{3}} = 0$
(C) $2x - y = \sqrt{\frac{3}{2}}$ (D) $2x - y + 4 + \sqrt{\frac{3}{2}} = \sqrt{3}$

16. Coordinates of one of possible focus of hyperbola is
(A) $\left(-1+\frac{2}{\sqrt{6}}, 2-\frac{1}{\sqrt{6}}\right)$ (B) $\left(\left(-1+\frac{2}{\sqrt{5}}\right), \left(2-\frac{1}{\sqrt{5}}\right)\right)$
(C) $\left(\left(-1-\frac{2}{\sqrt{5}}\right), \left(2+\frac{1}{\sqrt{5}}\right)\right)$ (D) $\left(\left(-1-\frac{2}{\sqrt{5}}\right), \left(2-\frac{1}{\sqrt{5}}\right)\right)$

Exercise-3

Marked questions are recommended for Revision.
*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1*. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose **[IIT-JEE 2009, Paper-2, (4, –1), 80]**
(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is x = 0 (C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0)

2. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse
$x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is **[IIT-JEE 2009, Paper-1, (3, –1)/ 80]**

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

3*. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then **[IIT-JEE 2009, Paper-1, (4, –1)/ 80]**
(A) b + c = 4a (B) b + c = 2a
(C) locus of points A is an ellipse (D) locus of point A is a pair of straight lines

4. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectum of the given ellipse at the points **[IIT-JEE 2009, Paper-2, (3, –1)/ 80]**
(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$ (C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

5. Match the conics in **Column - I** with the statements/expressions in **Column - II.**
[IIT-JEE-2009, Paper-1, (8, 0), 80]

Column - I	**Column - II**
(A) Circle	(p) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\|z + 2\| - \|z - 2\| = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \|z\|^2 + 1$

6*. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then
[IIT-JEE 2009, Paper-2, (4, –1), 80]

(A) Equation of ellipse is $x^2 + 2y^2 = 2$
(B) The foci of ellipse are $(\pm 1, 0)$
(C) Equation of ellipse is $x^2 + 2y^2 = 4$
(D) The foci of ellipse are $(\pm, \sqrt{2}\ 0)$

7*. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
[IIT-JEE-2010, Paper-1(3, 0)/84]

(A) $-\frac{1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $-\frac{2}{r}$

Comprehension # 1 (Q.8 - 10)

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at point A and B.

8. The coordinates of A and B are **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) (3, 0) and (0, 2)
(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)
(D) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

9. The orthocentre of the triangle PAB is **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(5, \frac{8}{7}\right)$ (B) $\left(\frac{7}{5}, \frac{25}{8}\right)$ (C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

10. The equation of the locus of the point whose distances from the point P and the line AB are equal, is
(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$
[IIT-JEE 2010, Paper-2, (3, –1), 79]

Comprehension

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

11. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is
(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$
(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$ **[IIT-JEE-2010, Paper-1(3, –1)/84]**

12. Equation of the circle with AB as its diameter is **[IIT-JEE-2010, Paper-1(3, –1)/84]**
(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$
(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

13. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then find the eccentricity of the hyperbola.
[IIT-JEE-2010, Paper-1(3, 0)/84]

14. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

15. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is **[IIT-JEE 2011, Paper-2, (3, –1), 80]**
(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

16*. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by
(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$
[IIT-JEE 2011, Paper-2, (4, 0), 80]

17*. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then **[IIT-JEE 2011, Paper-1, (4, 0), 80]**
(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
(B) a focus of the hyperbola is (2, 0)
(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

18. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is **[IIT-JEE 2011, Paper-2, (3, –1), 80]**
(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

19. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**
(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

20. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contacts of the tangents on the hyperbola are **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

(A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ (C) $\left(3\sqrt{3},\ -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3},\ 2\sqrt{2}\right)$

21. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is.
[IIT-JEE 2012, Paper-1, (4, 0), 70]

Paragraph for Question Nos. 22 to 23

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line y = 2x + a, a > 0. **[IIT-JEE - 2013, Paper-2, (3,–1), 60]**

22. Length of chord PQ is
(A) 7a (B) 5a (C) 2a (D) 3a

23. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then tan θ =
(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

24. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral PQRS is
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]
(A) 3 (B) 6 (C) 9 (D) 15

Paragraph For Questions 25 and 26

Let a, r, s, t be nonzero real numbers. Let P(at^2, 2at), Q, R (ar^2, 2ar) and S(as^2, 2as) be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0) **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

25. The value of r is
(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

26. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is
(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$

27. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line x + y + 4 = 0. If A and B are the points of intersection of C with the line y= – 5, then the distance between A and B is
[JEE (Advanced) 2015, P-1 (4, 0) /88]

28. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

29*. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola, if P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P ?
[JEE (Advanced) 2015, P-1 (4, –2)/ 88]

(A) $(4,\ 2\sqrt{2})$ (B) $(9,\ 3\sqrt{2})$ (C) $\left(\frac{1}{4},\ \frac{1}{\sqrt{2}}\right)$ (D) $(1,\ \sqrt{2})$

30. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is.

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

31*. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that PQ = PR = $\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) $e_1^2 + e_2^2 = \frac{43}{40}$
(B) $e_1e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(C) $|e_1^2 - e_2^2| = \frac{5}{8}$
(D) $e_1e_2 = \frac{\sqrt{3}}{4}$

32*. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$
(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$
(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$
(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

33. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then **[JEE (Advanced) 2016, Paper-2, (3, –1)/60]**

(A) SP = $2\sqrt{5}$
(B) SQ : QP = $\sqrt{5}+1 : 2$
(C) the x-intercept of the normal to the parabola at P is 6
(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

34. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced) 2017, Paper-1,(4, –2)/61]**

(A) a, 4, 1 (B) 2a, 4, 1 (C) a, 4, 2 (D) 2a, 8, 1

35. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k?

[JEE(Advanced) 2017, Paper-1,(4, –2)/61]

(A) p = –1, h = 1, k = –3
(B) p = 2, h = 3, k = –4
(C) p = –2, h = 2, k = –4
(D) p = 5, h = 4, k = –3

Answer Q.36, Q.37 and Q.38 by appropriately matching the information given in the three columns of the following table. **[JEE(Advanced) 2017, Paper-1,(3, –1)/61]**

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, repectively.		
Column-1	**Column-2**	**Column-3**
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

36. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1, 1), then which of the following options is the only CORRECT combination for obtaining its equation ?
[JEE(Advanced) 2017, Paper-1,(3, –1)/61]
(A) (I) (ii) (Q) (B) (I) (i) (P) (C) (III) (i) (P) (D) (II) (ii) (Q)

37. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination? **[JEE(Advanced) 2017, Paper-1,(3, –1)/61]**
(A) (IV) (iv) (S) (B) (II) (iv) (R) (C) (IV) (iii) (S) (D) (II) (iii) (R)

38. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?
[JEE(Advanced) 2017, Paper-1,(3, –1)/61]
(A) (III) (i) (P) (B) (I) (ii) (Q) (C) (II) (iv) (R) (D) (III) (ii) (Q)

39. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018, Paper-2,(3, –1)/60]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{16}(\pi - 2)$

40. Let H : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

[JEE(Advanced) 2018, Paper-2,(3, –1)/60]

LIST-I	**LIST-II**
(P) The length of the conjugate axis of H is	(1) 8
(Q) The eccentricity of H is	(2) $\frac{4}{\sqrt{3}}$
(R) The distance between the foci of H is	(3) $\frac{2}{\sqrt{3}}$
(S) The length of the latus rectum of H is	(4) 4

The correct option is:
(A) P → 4; Q → 2; R → 1; S → 3
(B) P → 4; Q → 3; R → 1; S → 2
(C) P → 4; Q → 1; R → 3; S → 2
(D) P → 3; Q → 4; R → 2; S → 1

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle alingent with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is : **[AIEEE 2009 (4, –1), 144]**
(1) $x^2 + 12y^2 = 16$ (2) $4x^2 + 48y^2 = 48$ (3) $4x^2 + 64y^2 = 48$ (4) $x^2 + 16y^2 = 16$

2*. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is : **[AIEEE 2011, I, (4, –1), 120]**
(1) $3x^2 + 5y^2 - 32 = 0$ (2) $5x^2 + 3y^2 - 48 = 0$ (3) $3x^2 + 5y^2 - 15 = 0$ (4) $5x^2 + 3y^2 - 32 = 0$

3. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by : **[AIEEE 2011, II, (4, –1), 120]**
(1) $x^2 - 3y^2 = 3$ (2) $3x^2 - y^2 = 3$ (3) $-x^2 + 3y^2 = 3$ (4) $-3x^2 + y^2 = 3$

4.✍ **Statement-1 :** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$. **[AIEEE - 2013, (4, – 1) 120]**

Statement-2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.
(1) Statement-1 is false, Statement-2 is true.
(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(4) Statement-1 is true, statement-2 is false.

5. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ is semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : **[AIEEE-2012, (4, –1)/120]**
(1) $4x^2 + y^2 = 4$ (2) $x^2 + 4y^2 = 8$ (3) $4x^2 + y^2 = 8$ (4) $x^2 + 4y^2 = 16$

6. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having centre at (0, 3) is **[AIEEE - 2013, (4, – 1)]**
(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$
(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

7. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is: **[JEE(Main) 2014, (4, – 1), 120]**
(1) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 + y^2)^2 = 6x^2 - 2y^2$ (3) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

8. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[JEE(Main) 2014, (4, – 1), 120]**
(1) $\frac{1}{8}$ (2) $\frac{2}{3}$ (3) $\frac{1}{2}$ (4) $\frac{3}{2}$

9. The area (in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse , $\frac{x^2}{9}+\frac{y^2}{5}=1$ is **[JEE(Main) 2015, (4, – 1), 120]**
(1) $\frac{27}{4}$ (2) 18 (3) $\frac{27}{2}$ (4) 27

10. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is **[JEE(Main) 2015, (4, – 1), 120]**
(1) $x^2 = y$ (2) $y^2 = x$ (3) $y^2 = 2x$ (4) $x^2 = 2y$

11. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is : **[JEE(Main) 2016, (4, – 1), 120]**
(1) $x^2 + y^2 - x + 4y - 12 = 0$ (2) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(3) $x^2 + y^2 - 4x + 9y + 18 = 0$ (4) $x^2 + y^2 - 4x + 8y + 12 = 0$

12. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE(Main) 2016, (4, – 1), 120]**
(1) $\frac{4}{\sqrt{3}}$ (2) $\frac{2}{\sqrt{3}}$ (3) $\sqrt{3}$ (4) $\frac{4}{3}$

13. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is **[JEE(Main) 2017, (4, – 1), 120]**
(1) $2y - x = 2$ (2) $4x - 2y = 1$ (3) $4x + 2y = 7$ (4) $x + 2y = 4$

14. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point : **[JEE(Main) 2017, (4, – 1), 120]**
(1) $(3\sqrt{2}, 2\sqrt{3})$ (2) $(2\sqrt{2}, 3\sqrt{3})$ (3) $(\sqrt{3}, \sqrt{2})$ (4) $(-\sqrt{2}, -\sqrt{3})$

15. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is : **[JEE(Main) 2018, (4, – 1), 120]**
(1) 85 (2) 95 (3) 195 (4) 185

16. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is : **[JEE(Main) 2018, (4, – 1), 120]**
(1) $60\sqrt{3}$ (2) $36\sqrt{5}$ (3) $45\sqrt{5}$ (4) $54\sqrt{3}$

17. Tangent and normal are drawn at P(16,16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is : **[JEE(Main) 2018, (4, – 1), 120]**
(1) 3 (2) $\frac{4}{3}$ (3) $\frac{1}{2}$ (4) 2

18. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbola $xy = 2$ is :

[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]

(1) $x + 2y + 4 = 0$ (2) $x - 2y + 4 = 0$ (3) $4x + 2y + 1 = 0$ (4) $x + y + 1 = 0$

19. If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c) ?

[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]

(1) (1, 1, 3) (2) $\left(\frac{1}{2},2,3\right)$ (3) $\left(\frac{1}{2},2,0\right)$ (4) (1, 1, 0)

20. The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}\,y + 4\sqrt{2} = 0$ is :

[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

(1) $6\sqrt{3}$ (2) $8\sqrt{2}$ (3) $3\sqrt{2}$ (4) $2\sqrt{11}$

21. Let $S = \left\{(x,y) \in R^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1\right\}$, where $r \neq \pm 1$. Then S represents :

(1) a hyperbola whose eccentricity $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$.

(2) an ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$.

(3) a hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0 < r < 1$.

(4) an ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, when $r > 1$.

[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

22. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices then the mid points of the tangents intercepted between the coordinate axes lie on the curve :

[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]

(1) $\frac{x^2}{4} + \frac{y^2}{2} = 1$ (2) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (3) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$ (4) $\frac{x^2}{2} + \frac{y^2}{4} = 1$

23. Let S and S' be the foci of an ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and are $(\Delta S'BS) = 8$ sq. units, then the length of a latus

[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]

(1) 2 (2) $4\sqrt{2}$ (3) 4 (4) $2\sqrt{2}$

Answers

EXERCISE # 1

PART-I

Section (A) :

A-1. 4

A-2. (i) vertex $\equiv \left(\frac{3}{2}, -\frac{11}{8}\right)$, focus $\left(\frac{3}{2}, -\frac{15}{8}\right)$

axis $x = \frac{3}{2}$, directrix $y = -\frac{7}{8}$, length of latus rectum = 2.

(ii) $x^2 - 2xy + y^2 - 6x - 6y + 3 = 0$

(iii) $4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$

(iv) $y = 5/4$

A-3. $(y + 1)^2 = 3(2x + 1)$ & $(y + 1)^2 = -3(2x - 5)$

A-4. $y = \frac{2}{3}, \left(-\frac{61}{16}, \frac{2}{3}\right), \left(-\frac{485}{144}, \frac{2}{3}\right), \quad x = \frac{-613}{144}, x = -\frac{485}{144}$

A-5. $\sqrt{x^2 + y^2} + |x - 2| = 4$

A-6. $\alpha \in [0, 1]$

A-7. $\alpha \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$

Section (B) :

B-1. $\left(e = \frac{1}{2}\right)$

B-2 ((2, 3) & (6, 7))

B-3 $\left(\frac{12}{5}, \frac{16}{5}\right)$

B-4. $(x = 3 + 5\cos\theta,\ y = -2 + 4\sin\theta)$

B-5. $\frac{4}{5} < \alpha < \frac{4}{\sqrt{17}}$

B-6. $3x^2 + 4y^2 - 12x + 24y + 36 = 0$

B-7. $5x^2 + 9y^2 - 54y + 36 = 0$

B-8. (i) Centre (–1, 2), $e = \frac{5}{3}$, foci = (4, 2), (–6, 2), $x = -\frac{14}{5}$ and $x = \frac{4}{5}$

(ii) $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

B-10. $3x^2 - y^2 - 12 = 0$

B-11. (i) (4, 1), (–6, 1) (ii) $\frac{(x+2)^2}{16} - \frac{(y-3)^2}{48} = 1$

B-12. Inside

B-13. $(x - 3)^2 + (y - 4)^2 = \left(\frac{5 \pm \sqrt{5}}{2}\right)^2$

B-14. $\frac{\sqrt{13}}{3}$

B-19. $\frac{\sqrt{7}}{4}$

B-20. 16

Section (C) :

C-1. $8\sqrt{2}$

C-2. (4, 0), (–4, 0)

C-3. $y^2 - 2ax + 8a^2 = 0$

C-4. $y = x^2 + 2$

C-6. $(-\infty, -4] \cup [4, \infty)$

C-7. $x - 4y + 28 = 0$ at (28, 14)

C-8. $\sqrt{\frac{c}{a}}$

Section (D) :

D-1. $\sqrt{35}$ **D-2.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{y}{b}$ **D-3.** (yes, (5, 4))

D-4. $x^2 + 64y^2 = 80$ & $x^2 + 4y^2 = 20$ **D-5.** $ex \pm y = a, -ex \pm y = a$

D-7. (3, 0) **D-8.** $16x^2 + 9y^2 + 96x = 432$ or $16x^2 + 9y^2 - 288x + 720 = 0$

D-9. $\frac{8}{3}\sqrt{10}$ **D-10.** $\lambda = \pm 6$ **D-11.** $n = \pm 2$ **D-12.** $x + y \pm 3\sqrt{3} = 0$

D-13. $x^2 - y^2 - 4x = 0$ **D-14.** 10 **D-15.** 30

Section (E) :

E-1. $4y = 9x + 4$, $4y = x + 36$ **E-2.** (i) $y = (x - a)\tan\alpha$, (ii) $x = a$, (iii) $y = \lambda x$

E-3. (–2, 0) **E-4.** –2 **E-5.** $y^2 = 2x - 2$

Section (F):

F-1. $4x + 5y = 40$, $4x - 35y = 200$. **F-2.** $\left(\frac{25}{4}, \frac{16}{3}\right)$ **F-3.** $48x + 25y - 169 = 0$

F-4. **(i)** –16/9 **(ii)** –20/9 **F-5.** $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

F-6. $(x^2 + y^2)^2 = 16x^2 - 9y^2$ **F-8.** $a^2p^2 + b^2q^2 = r^2\sec^2\frac{\pi}{8} = (4 - 2\sqrt{2})\, r^2$

Section (G) :

G-1. $x + y = 3$ **G-3.** $x + 3y = 33$ **G-5.** (4a, 4a) **G-6.** $x + 2y - 3 = 0$

Section (H) :

H-2. $12x + 5y = 48$; $12x - 5y = 48$

Section (I) :

I-1. $(x - 2)^2 + (y - 4)^2 + 2(x - y + 2) = 0$ **I-2.** 2 **I-3.** $b = \sqrt{3}$; $x + 2y + 4 = 0$

I-4. $\frac{2\sqrt{2}}{3}$ **I-5.** $y = x \pm \sqrt{10}$, $y = -x \pm \sqrt{10}$ **I-6.** $\frac{64}{75}$

PART - II

Section (A) :

A-1. (D) **A-2.** (C) **A-3.** (D) **A-4.** (A) **A-5.** (B) **A-6.** (B) **A-7.** (D)

A-8. (B) **A-9.** (A) **A-10.** (D) **A-11.** (D) **A-12.** (C) **A-13.** (B)

Section (B) :

B-1.	(A)	**B-2.**	(B)	**B-3.**	(D)	**B-4.**	(C)	**B-5.**	(A)	**B-6.**	(C)	**B-7.**	(D)
B-8.	(B)	**B-9.**	(C)	**B-10.**	(A)	**B-11.**	(B)	**B-12.**	(C)	**B-13.**	(A)	**B-14.**	(D)
B-15.	(C)	**B-16.**	(C)	**B-17.**	(A)	**B-18.**	(A)	**B-19.**	(D)	**B-20.**	(C)	**B-21.**	(B)
B-22.	(C)												

Section (C) :

C-1.	(D)	**C-2.**	(A)	**C-3.**	(B)	**C-4.**	(B)	**C-5.**	(D)	**C-6.**	(C)	**C-7.**	(B)
C-8.	(A)												

Section (D) :

D-1.	(B)	**D-2.**	(C)	**D-3.**	(B)	**D-4.**	(C)	**D-5.**	(A)	**D-6.**	(A)	**D-7.**	(B)
D-8.	(B)												

Section (E) :

E-1.	(B)	**E-2.**	(C)	**E-3.**	(B)

Section (F) :

F-1.	(A)	**F-2.**	(B)	**F-3**	(D)	**F-4.**	(A)	**F-5.**	(B)	**F-6.**	(D)	**F-7.**	(A)
F-8.	(D)	**F-9.**	(C)										

Section (G) :

G-1.	(B)	**G-2.**	(A)	**G-3.**	(D)	**G-4.**	(C)	**G-5.**	(B)	**G-6.**	(D)	**G-7.**	(D)
G-8.	(A)												

Section (H) :

H-1.	(D)	**H-2.**	(A)	**H-3.**	(D)	**H-4.**	(A)	**H-5.**	(A)

Section (I) :

I-1.	(A)	**I-2.**	(B)	**I-3.**	(C)	**I-4.**	(A)	**I-5.**	(D)	**I-6.**	(C)	**I-7.**	(A)
I-8.	(A)												

PART - III

1.	(A) $\rightarrow$ (q),	(B) $\rightarrow$ (r),	(C) $\rightarrow$ (s),	(D) $\rightarrow$ (q)
2.	(A) $\rightarrow$ (s),	(B) $\rightarrow$ (r),	(C) $\rightarrow$ (q),	(D) $\rightarrow$ (p)
3.	(A) $\rightarrow$ (r),	(B) $\rightarrow$ (p),	(C) $\rightarrow$ (s),	(D) $\rightarrow$ (q)
4.	(A) $\rightarrow$ (q),	(B) $\rightarrow$ (s),	(C) $\rightarrow$ (s),	(D) $\rightarrow$ (r)

EXERCISE # 2

PART-I

1.	(B)	**2.**	(C)	**3.**	(C)	**4.**	(C)	**5.**	(B)	**6.**	(C)	**7.**	(B)
8.	(C)	**9.**	(B)	**10.**	(B)	**11.**	(B)	**12.**	(C)	**13.**	(D)	**14.**	(C)

15.	(A)	**16.**	(D)	**17.**	(A)	**18.**	(A)	**19.**	(D)	**20.**	(D)	**21.**	(C)
22.	(C)	**23.**	(A)	**24.**	(B)	**25.**	(A)	**26.**	(D)	**27.**	(A)	**28.**	(A)
29.	(A)	**30.**	(C)										

PART-II

1.	1	**2.**	3	**3.**	0	**4.**	2	**5.**	18	**6.**	16	**7.**	23
8.	3	**9.**	4	**10.**	3	**11.**	2	**12.**	2	**13.**	3	**14.**	13
15.	65	**16.**	21	**17.**	15	**18.**	9	**19.**	14	**20.**	55	**21.**	7
22.	9	**23.**	13	**24.**	1	**25.**	77	**26.**	57	**27.**	22	**28.**	4
29.	0025	**30.**	6	**31.**	16	**32.**	6	**33.**	6	**34.**	50	**35.**	3

PART-III

1.	(AB)	**2.**	(ABCD)	**3.**	(AD)	**4.**	(AD)	**5.**	(AD)	**6.**	(AC)	**7.**	(BC)
8.	(ABCD)	**9.**	(ABC)	**10.**	(AC)	**11.**	(AC)	**12.**	(AB)	**13.**	(AD)	**14.**	(AD)
15.	(AB)	**16.**	(CD)	**17.**	(BCD)	**18.**	(ACD)	**19.**	(AB)	**20.**	(BD)		
21.	(ABCD)	**22.**	(ACD)	**23.**	(ABC)	**24.**	(AD)	**25.**	(ABD)	**26.**	(AB)		

PART - IV

1.	(A)	**2.**	(B)	**3.**	(C)	**4.**	(B)	**5.**	(A)	**6.**	(C)	**7.**	(C)
8.	(B)	**9.**	(A)	**10.**	(B)	**11.**	(A)	**12.**	(D)	**13.**	(C)	**14.**	(A)
15.	(A)	**16.**	(A)										

EXERCISE # 3

PART-I

1*.	(AD)	**2.**	(D)	**3*.**	(BC)	**4.**	(C)						
5.	(A) $\rightarrow$ (p),		(B) $\rightarrow$ (s, t),		(C) $\rightarrow$ (r),		(D) $\rightarrow$ (q, s)			**6*.**	(AB)	**7*.**	(CD)
8.	(D)	**9.**	(C)	**10.**	(A)	**11.**	(B)	**12.**	(A)	**13.**	2	**14.**	(2)
15.	(C)	**16*.**	(ABD)	**17*.**	(BD)	**18.**	(B)	**19.**	(C)	**20.**	(AB)	**21.**	(4)
22.	(B)	**23.**	(D)	**24.**	(D)	**25.**	(D)	**26.**	(B)	**27.**	4	**28.**	2
29*.	(AD)	**30.**	4	**31*.**	(AB)	**32*.**	(ABD)	**33.**	(ACD)	**34.**	(ACD)	**35.**	(B)
36.	(A)	**37.**	(B)	**38.**	(A)	**39.**	(AC)	**40.**	(B)				

PART - II

1.	(1)	**2*.**	(1,2)	**3.**	(2)	**4.**	(2)	**5.**	(4)	**6.**	(1)	**7.**	(1)
8.	(3)	**9.**	(4)	**10.**	(4)	**11.**	(4)	**12.**	(2)	**13.**	(2)	**14.**	(2)
15.	(2)	**16.**	(3)	**17.**	(4)	**18.**	(1)	**19.**	(1)	**20.**	(1)	**21.**	(2)
22.	(2)	**23.**	(3)										

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

1. Prove that in a parabola the angle θ that the latus rectum subtends at the vertex of the parabola isindependent of the latus rectum and lies between $\frac{2\pi}{3}$ & $\frac{3\pi}{4}$

2. A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; then axes being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$

3. Find the points of intersection of the curves whose parametric equations are $x = t^2 + 1$, $y = 2t$ and $x = 2s$, $y = 2/s$.

4. If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1r_2)^{4/3} = 16a^2(r_1^{2/3} + r_2^{2/3})$.

5. Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

6. A chord is a normal to a parabola and is inclined at an angle θ to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3\theta \operatorname{cosec}^3\theta$

7. From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then find the locus of P .

8. TP and TQ are tangents to the parabola and the normals at P and Q meet at a point R on the curve ; prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x - a)$.

9. From an external point P, tangents are drawn to the parabola; find the equation of the locus of P when these tangents make angles θ_1 and θ_2 with the axis, such that $\cos\theta_1$
$\cos\theta_2 = \mu$, which is constant.

10. A pair of tangents are drawn to the parabola which are equally inclined to a straight line whose inclination to the axis is α ; prove that the locus of their point of intersection is the straight line $y = (x - a) \tan 2\alpha$.

11. Prove that the normals at the points, where the straight line $\ell x + my = 1$ meets the parabola $y^2 = 4ax$, meet on the normal at the point $\left(\frac{4am^2}{\ell^2}, \frac{4am}{\ell}\right)$ on the parabola.

12. Prove that the equation to the circle, which passes through the focus and touches the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$, is $x^2 + y^2 - ax(3t^2 + 1) - ay(3t - t^3) + 3a^2t^2 = 0$.
Prove also that the locus of its centre is the curve $27ay^2 = (2x - a)(x - 5a)^2$.

13. Two tangents to the parabola $y^2 = 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2 = 8(x + 2)$.

14. Find locus of a point P if the three normals drawn from it to the parabola $y^2 = 4ax$ are such that two of them make complementry angles with the axis of the parabola

15. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

16. If tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\phi, 2\sin\phi)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of t & ϕ.

17. Find the locus of centre of a family of circles passing through the vertex of the parabola $y^2 = 4ax$, and cutting the parabola orthogonally at the other point of intersection.

18. Let A, B, C be three points on the parabola $y^2 = 4ax$. If the orthocentre of the triangle ABC is at the focus then show that the circumcircle of ΔABC touches the y-axis.

19. If α, β are eccentric angles of the extremities of a focal chord of an ellipse, then eccentricity of the ellipse is

20. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with vertices having eccentric angles α, β, γ respectively is (x_1, y_1), then find $\Sigma \cos\alpha\cos\beta + \Sigma \sin\alpha\sin\beta$.

21. Find the locus of extremities of latus rectum of the family of ellipse $b^2x^2 + y^2 = a^2b^2$ where b is a parameter $(b^2 < 1)$.

22. A point moves such that the sum of the square of the distances from two fixed straight lines intersecting at angle 2α is a constant. Prove that the locus is an ellipse of eccentricity
$\frac{\sqrt{\cos 2\alpha}}{\cos\alpha}$ if $\alpha < \frac{\pi}{4}$ and $\frac{\sqrt{-\cos 2\alpha}}{\sin\alpha}$ if $\alpha > \frac{\pi}{4}$

23. A straight line PQ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = r^2 (b < r < a)$. RS is a focal chord of the ellipse. If RS is parallel to PQ and meets the circle at points R and S. Find the length of RS.

24. Prove that the sum of the eccentric angles of the extremities of a chord of an ellipse, which is drawn in a given direction is constant and is equal to twice the eccentric angle of the point at which the tangent is parallel to the given direction.

25. If the normals at $\alpha, \beta, \gamma, \delta$ on an ellipse are concurrent, prove that $(\Sigma\cos\alpha)(\Sigma\sec\alpha) = 4$
$(\Sigma\cos\alpha)(\Sigma\sec\alpha) = 4$

26. Show that the equation of the pair of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points of intersection with the line, $px + qy + 1 = 0$ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \cdot (p^2a^2 + q^2b^2 - 1) = (px + qy + 1)^2$.

27. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = a + b$ at the points P and Q; prove that the tangents at P and Q are at right angles.

28. Find the locus of the point, the chord of contact of the tangents drawn from which to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the circle $x^2 + y^2 = c^2$, where $c < b < a$.

29. A chord of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles of extremities are α and β, intersects its director circle at point A and B. Tangents at A and B intersect at point P. Find the equation of circumcircle of triangle ABP.

30. A tangent is drawn at any fixed point P on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and if chord of contact of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ with respect to any point on this tangent passes through a fixed point, then prove that the line joining this fixed point to the point P never subtends right angle at the origin.

31. If the parabola $y^2 = 4ax$ cuts the ellipse $\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1$ in three distinct points then show that the eccentricity of the ellipse e belongs to $\left(\frac{1}{\sqrt{2}},\ 1\right)$.

32. Find the number of integral points lying on or inside the ellipse $2x^2 + 6xy + 6y^2 - 1 = 0$.

33. The equations of the transverse and conjugate axes of a hyperbola are respectively $x + 2y - 3 = 0$, $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $2/\sqrt{3}$. Then find the equation of the hyperbola .

34. If P is any point common to the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$ and the circle having line segment joining its focii as diameter then find the sum of focal distances of point P.

35. The transverse axis of a hyperbola is of length 2a and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2 : 1. Find the equation of the hyperbola.

36. If $x \cos \alpha + y \sin \alpha = p$, a variable chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$ subtends a right angle at the centre of the hyperbola, then the chords touch a fixed circle, find the radius of the circle.

37. If the distance between the centres of the hyperbolas :
$x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ ..... (i)
$9x^2 - 16y^2 - 18x - 32y - 151 = 0$ ..... (ii) is d then $125\ d^2$ = ...

38. Find an equation of the hyperbola whose directrix is the normal to circle $x^2 + y^2 - 4x - 6y + 9 = 0$ having slope is 2 and eccentricity is equal to radius of given circle where focus of hyperbola is point of contact of given circle with y-axis.

39. PQ is the chord joining the points whose eccentric angles are ϕ_1 and ϕ_2 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, If $\phi_1 - \phi_2 = 2\alpha$, where α is constant, prove that PQ touches the hyperbola $\cos^2\alpha\ \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$.

40. Find the locus of the mid-points of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ which subtend a right angle at the origin is

41. If a chord joining the points P $(a \sec \theta, a \tan \theta)$ & Q $(a \sec \phi, a \tan \phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan \phi = \tan \theta\ (4 \sec^2 \theta - 1)$.

42. Chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are tangents to the circle drawn on the line joining the foci as diameter . Find the locus of the point of intersection of tangents at the extremities of the chords .

43. From any point on the hyperbola $H_1 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the hyperbola $H_2 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. Then find the area cut-off by the chord of contact on the asymptotes of H_2 .

44. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the x-axis at A; C is the mid point of PQ & 'O' is the origin. Then prove that the Δ ACO is isosceles.

45. If the normals at (x_i, y_i), i = 1, 2, 3, 4 on the rectangular hyperbola, $xy = c^2$, meet at the point (α, β) show that

(i) $\Sigma x_i = \alpha$ (ii) $\Sigma y_i = \beta$ (iii) $\Pi x_i = \Pi y_i = -c^4$,

(iv) $\Sigma x_i^2 = \alpha^2$ (v) $\Sigma y_i^2 = \beta^2$

46. If α, β, γ & δ be the eccentric angles of feet of four co–normal points of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from any point in its plane then prove that $\alpha + \beta + \gamma + \delta$ is odd integral multiple of π.

47. Prove that a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cannot be normal to its conjugate hyperbola.

48. Let P be a point from where perpendicular tangents are drawn to the circle $2x^2 + 2y^2 - a^2 = 0$. Let a line from P perpendicular to OP is drawn which intersect hyperbola $x^2 - y^2 = a^2$ at Q and R. Find number of all possible positions of P such that product of ordinates of points Q and R is.

(i) $\frac{3}{2}a^2$ (ii) a^2 (iii) $\frac{a^2}{2}$

Answers

3. (2, 2) **7.** $x - y - 1 = 0$ **9.** $x^2 = \mu^2 \{(x - a)^2 + y^2\}$ **14.** $y^2 = a(x - a)$

16. $\phi = \pi - \tan^{-1} 2,\ t = -\frac{1}{\sqrt{5}}$; $\phi = \pi + \tan^{-1}2,\ t = \frac{1}{\sqrt{5}}$; $\phi = \pm\frac{\pi}{2},\ t = 0$

17. $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$ **19.** $\frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$ **20.** $\frac{9{x_1}^2}{2a^2} + \frac{9{y_1}^2}{2b^2}$ $-\frac{3}{2}$

21. $x^2 \pm ay = a^2$ **23.** RS = 2b **28.** $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$

33. $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$ **34.** $2\sqrt{66}$ **35.** $5x^2 - 4y^2 = 5a^2$

36. $\sqrt{2}\,a$ **37.** 0025 **38.** $11x^2 - y^2 - 16xy - 16x + 38y - 41 = 0$

42. $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$ **43.** 4 ab

48. (i) 4 (ii) 2 (iii) 0