Target : JEE (Main + Advanced)

Differential Equation

CONTENT

DIFFERENTIAL EQUATION

JEE (Advanced) Syllabus

Formation of ordinary differential equations, solution of homogeneous differential equations, variables separable method, linear first order differential equations

JEE (Main) Syllabus

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type : $\frac{dy}{dx} + p(x)y = q(x)$

Differential Equation

It is not certain that everything is uncertain ....Pascal, Blaise

Introduction :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

1.1 Ordinary Differential Equation : If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for example $\frac{dy}{dx} + \frac{dz}{dx} = y + z$,

$$\frac{dy}{dx} + xy = \sin x\ ,\quad \frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x\ ,$$

$$k\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2},\quad y = x\frac{dy}{dx} + k\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

1.2 Partial differential equation : If the dependent variables depend on two or more indepen dent variables, then it is known as partial differential equation

for example $y^2\frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial y^2} = ax,\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Order and Degree of a Differential Equation:

2.1 Order : Order is the highest differential appearing in a differential equation.

2.2 Degree :

It is determined by the highest degree of the highest order derivative present in it after the differential equation is cleared of radicals and fractions so far as the derivatives are concerned.

Note : In the differential equation, all the derivatives should be expressed in the polynomial form

$$f_1(x, y)\left[\frac{d^my}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \text{........} f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

Example # 1: Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = \log_e\left(\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2\right)$

(iii) $\tan^{-1}\left(x\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = y$ (iv) $e^{y'''} - xy'' + y = 0$

Solution : (i) $\left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^6$ $\therefore$ order = 2, degree = 4

(ii) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = e^y$ $\therefore$ order = 3, degree = 1

(iii) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \tan y$ $\therefore$ order = 2, degree = 1

(iv) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

$\because$ equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

Self Practice Problems :

(1) Find order and degree of the following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ (ii) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

(iii) $\left[\left(\frac{dy}{dx}\right)^{1/2} + y\right]^2 = \frac{d^2y}{dx^2}$

Ans. (1) (i) order = 1, degree = 2 (ii) order = 1, degree = 2
(iii) order = 2, degree = 2

Formation of Differential Equation:

Differential equation corresponding to a family of curve will have :

(a) Order exactly same as number of essential arbitrary constants in the equation of curve.
(b) No arbitrary constant present in it.

The differential equation corresponding to a family of curve can be obtained by using the following steps:

(i) Identify the number of essential arbitrary constants in equation of curve.

NOTE : If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant.

(ii) Differentiate the equation of curve till the required order.
(iii) Eliminate the arbitrary constant from the equation of curve and additional equations obtained in step (ii) above.

Example # 2 : Form a differential equation of family of straight lines passing through (0,2)

Solution : Family of straight lines passing through (0,2) is y = mx + 2 where'm' is a parameter.

Differentiating w.r.t. x

$\frac{dy}{dx} = m$

Eliminating 'm' from both equations, we obtain

$y = x\frac{dy}{dx} + 2$ which is the required differential equation.

Example # 3 : Form a differential equation of family of parabolas having x axis as line of symmetry and tangent at vertex is y-axes

Solution : Let equation of parabola

$y^2 = 4ax$ ....(i)

$2y\frac{dy}{dx} = 4a$ ....(ii)

by (i) and (ii)

$\Rightarrow y^2 = 2yx\frac{dy}{dx}$ $\Rightarrow$ $y = 2x\frac{dy}{dx}$

Self Practice Problems :

(2) Obtain a differential equation of the family of curves y = a sin (bx + c) where a and c being arbitrary constant.

(3) Show that the differential equation of the system of parabolas $y^2 = 4a(x - b)$ is given by

$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

(4) Form a differential equation of family of parabolas with focus as origin and axis of symmetry along the x-axis.

Ans. (2) $\frac{d^2y}{dx^2} + b^2y = 0$ (4) $y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$

Solution of a Differential Equation:

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation

NOTE : The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be three types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive) :** A relation in x and y satisfying a given differential equation and involving exactly same number of arbitrary constants as order of differential equation.

(ii) **Particular Solution :** A solution obtained by assigning values to one or more than one arbitrary constant of general solution.

(iii) **Singular Solution :** It is not obtainable from general solution. Geometrically, **Singular solution** acts as an envelope to **General solution**.

4.1. Differential Equation of First Order and First Degree :

A differential equation of first order and first degree is of the type $\frac{dy}{dx}$ + f(x, y) = 0, which can also be written as : Mdx + Ndy = 0, where M and N are functions of x and y.

Solution methods of First Order and First Degree Differential Equations :

5.1 Variables separable : If the differential equation can be put in the form, $f(x)\,dx = \phi(y)\,dy$we say that variables are separable and solution can be obtained by integrating each side separately. A general solution of this will be $\int f(x)\,dx = \int \phi(y)\,dy + c$, where c is an arbitrary constant.

Example # 4 : Solve the differential equation $(1 + x)\,y\,dx = (y - 1)\,x\,dy$

Solution : The equation can be written as -

$$\left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy \quad\Rightarrow\quad \int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\ell n\,x + x = y - \ell ny + c \quad\Rightarrow\quad \ell ny + \ell nx = y - x + c \quad\Rightarrow\quad xy = ce^{y-x}$$

Example # 5 : Solve : $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

Solution : The given differential equation is $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

$$\frac{ydy}{(y+1)} = \frac{e^x}{(e^x+1)}$$

Integrating both sides

$\Rightarrow$ $y - \log|y + 1| = \log(e^x + 1) + \log k \Rightarrow y = \log|(y + 1)(e^x + 1)| + \log k \Rightarrow (y + 1)(e^x + 1) = e^y c$

Example # 6 : $\frac{dy}{dx} = \frac{x(2\ell nx+1)}{\sin y + y\cos y}$ Solve :

Solution : $\frac{dy}{dx} = \frac{x(2\ell nx+1)}{\sin y + y\cos y}$

$(\sin y + y\cos y)dy = x(2\ell nx + 1)dx$

Integrating both sides

$$\Rightarrow -\cos y + \{(y\sin y) + \cos y\} = 2 \times \left\{\frac{x^2}{2}\ln x - \frac{1}{2}\int\frac{x}{1}dx\right\} + \frac{x^2}{2} \Rightarrow y\sin y = x^2\,\ell n\,x$$

5.1.1 Polar coordinates transformations :

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

(a) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$ (ii) $dx^2 + dy^2 = dr^2$ (iii) $x\,dy - y\,dx = r^2 d\theta$

(b) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$ (ii) $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Example # 7 : Solve the differential equation xdx + ydy = x (xdy – ydx)

Solution : Taking $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$2x\,dx + 2y\,dy = 2r\,dr$

$x\,dx + y\,dy = r\,dr$ ....(i)

$\frac{y}{x} = \tan\theta \Rightarrow \frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$

$x\,dy - y\,dx = x^2\sec^2\theta \cdot d\theta$

$x\,dy - y\,dx = r^2\,d\theta$ ....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\,dr = r\cos\theta \cdot r^2\,d\theta$

$\frac{dr}{r^2} = \cos\theta\,d\theta \Rightarrow -\frac{1}{r} = \sin\theta + \lambda \Rightarrow -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda \Rightarrow \frac{y+1}{\sqrt{x^2+y^2}} = c$

where $-\lambda' = c \Rightarrow (y+1)^2 = c(x^2+y^2)$

5.1.2 Equations Reducible to the Variables Separable form : If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is $\frac{dy}{dx} = f(ax + by + c)$ $a, b \neq 0$. To solve this, put $ax + by + c = t$.

Example # 8 : Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution : Putting $4x + y + 1 = t \Rightarrow 4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$

Given equation becomes

$\frac{dt}{dx} - 4 = t^2 \Rightarrow \frac{dt}{t^2+4} = dx$ (Variables are separated)

Integrating both sides,

$\int\frac{dt}{4+t^2} = \int dx \Rightarrow \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

Example # 9 : Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution : $\frac{dy}{dx} = \sin(x + y)$

putting $x + y = t$

$\frac{dy}{dx} = \frac{dt}{dx} - 1 \therefore \frac{dt}{dx} - 1 = \sin t \Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \frac{dt}{1+\sin t} = dx$

Integrating both sides,

$\int\frac{dt}{1+\sin t} = \int dx \Rightarrow \int\frac{1-\sin t}{\cos^2 t}dt = x + c \Rightarrow \int(\sec^2 t - \sec t\,\tan t)\,dt = x + c$

$\tan t - \sec t = x + c \Rightarrow -\frac{1-\sin t}{\cos t} = x + c \Rightarrow \sin t - 1 = x\cos t + c\cos t$

substituting the value of t

$\sin(x + y) = x\cos(x + y) + c\cos(x + y) + 1$

Self Practice Problems :

(5) Solve the differential equation $x^2 y \frac{dy}{dx} = (x + 1)(y + 1)$

(6) Solve the differential equation $\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{ydx - xdy}{x^2}$

(7) Solve : $\frac{dy}{dx} = e^{x+y} + x^2e^y$

(8) Solve : $xy\frac{dy}{dx} = 1 + x + y + xy$

(9) Solve $\frac{dy}{dx} = 1 + e^{x-y}$

(10) $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(11) Find the solution of the differential equation $(x + y)^2 \frac{dy}{dx} = 1$, satisfying the condition $y(1) = 0$

Ans. (5) $y - \ell n(y + 1) = \ell nx - \frac{1}{x} + c$ (6) $\sqrt{x^2 + y^2} + \frac{y}{x} = c$

(7) $-\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$ (8) $y = x + \ell n|x(1 + y)| + c$

(9) $e^{y-x} = x + c$ (10) $\log\left|\tan\frac{x+y}{2} + 1\right| = x + c$

(11) $y + \frac{\pi}{4} = \tan^{-1}(x + y)$

5.2 Homogeneous Differential Equations :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equaiton and can be solved easily by putting y = vx.

Example # 10 : Solve $x\frac{dy}{dx} = y + x\tan\left(\frac{y}{x}\right)$

Solution : $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ (Homogeneous differential equation)

put $y = vx$ $\Rightarrow$ $v + x\frac{dv}{dx} = v + \tan v$ $\Rightarrow \cot v.dv = \frac{dx}{x}$

Integrating both sides we have

$\ell n \sin v = \ell nx + \ell nc$ $\Rightarrow$ $\sin v = cx$ $\Rightarrow$ $\sin\left(\frac{y}{x}\right) = cx$

Example # 11 : Solve : $x^2dy + y(x + y)dx = 0$ given that y = 1 when x = 1

Solution : $\frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$ put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ $\Rightarrow$ $v + x\frac{dv}{dx} = -v - v^2 \Rightarrow \frac{dv}{(2+v)v} = -\frac{dx}{x}$

$\Rightarrow$ $\frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = \int -\frac{dx}{x}$ $\Rightarrow$ $\frac{1}{2}\ell n\left(\frac{v}{v+2}\right) + \ell nx = \ell nC$

$\Rightarrow$ $x\sqrt{\frac{v}{v+2}} = C$ $\Rightarrow$ $x\sqrt{\frac{y}{y+2x}} = C$

When x = 1 then y = 1 $\Rightarrow C = \frac{1}{\sqrt{3}}$ $\Rightarrow 3x^2y = (y + 2x)$

5.2.1 Equations Reducible to the Homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ ....(1)

can be made homogeneous (in new variables X and Y) by substituting x = X + h and y = Y + k, where h and k are constants to obtain, $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$ ....(2)

These constants are chosen such that ah + bk + c = 0, and Ah + Bk + C = 0. Thus we obtain the following differential equation $\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$

The differential equation can now be solved by substituting Y = vX.

Example # 12 : Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

Solution : Let x = X + h, y = Y + k

$\frac{dy}{dX} = \frac{d}{dX}(Y + k)$

$\frac{dy}{dX} = \frac{dY}{dX}$ ....(i) $\frac{dx}{dX} = 1 + 0$ ....(ii)

on dividing (i) by (ii) $\frac{dy}{dx} = \frac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4} = \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$$

h & k are such that h + 2k – 5 = 0 & 2h + k – 4 = 0

h = 1, k = 2

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ which is homogeneous differential equation.

Now, substituting Y = vX

$\frac{dY}{dX} = v + X\frac{dv}{dX}$ $\therefore X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$ $\Rightarrow \int \frac{2+v}{1-v^2}\, dv = \int \frac{dX}{X}$

$\int\left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right) dv = \ell n\, X + c$ $\Rightarrow \frac{1}{2}\ell n\,(v+1) - \frac{3}{2}\ell n\,(1-v) = \ell n\, X + c$

$\ell n \left|\frac{v+1}{(1-v)^3}\right| = \ell n X^2 + 2c \Rightarrow \frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2} = e^{2c}$

$X + Y = c'(X - Y)^3$ where $e^{2c} = c'$

$x - 1 + y - 2 = c'(x - 1 - y + 2)^3$

$x + y - 3 = c'(x - y + 1)^3$

Special case :

Case - 1 In equation (1) if $\frac{a}{A} = \frac{b}{B}$, then the substitution ax + by = v will reduce it to the form in which variables are separable.

Example # 13 : Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Solution : Putting u = 2x + 3y

$\frac{du}{dx} = 2 + 3 . \frac{dy}{dx}$ $\Rightarrow$ $\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5} \Rightarrow$ $\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$

$\int \frac{2u-5}{7u-13}dx = \int dx$ $\Rightarrow$ $\frac{2}{7}\int 1.du - \frac{9}{7}\int \frac{1}{7u-13}. du = x + c$

$\Rightarrow \frac{2}{7}u - \frac{9}{7} . \frac{1}{7}\ell n\,(7u - 13) = x + c \Rightarrow 4x + 6y - \frac{9}{7}\ell n\,(14x + 21y - 13) = 7x + 7c$

$\Rightarrow - 3x + 6y - \frac{9}{7}\ell n\,(14x + 21y - 13) = c'$

Case - 2 In equation (1), if b + A = 0, then by a simple cross multiplication equation (1) becomes an **exact differential equation**.

Example # 14 : Solve $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Solution : Cross multiplying,

$2xdy + y\,dy - dy = xdx - 2ydx + 5dx$

$2\,(xdy + y\,dx) + ydy - dy = xdx + 5\,dx$

$2\,d(xy) + y\,dy - dy = xdx + 5dx$

On integrating,

$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c \Rightarrow x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

Case - 3 If the homogeneous equation is of the form :

$yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Self Practice Problems :

Solve the following differential equations

(12) $\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x$ given that $y = 0$ at $x = 1$

(13) $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$

(14) $\frac{dy}{dx} = \frac{2x-y+3}{x+2y+4}$

(15) $(3x - 2y + 1)\,dy + (4y - 6x + 3)dx = 0$

(16) $\frac{dy}{dx} = \frac{3x+2y-5}{3y-2x+5}$

Ans. (12) $\sqrt{x^2+y^2} = e^{\frac{y}{x}\tan^{-1}\frac{y}{x}}$

(13) $x\sin\frac{y}{x} = C$

(14) $y^2 - x^2 + xy + 4y - 3x + C = 0$

(15) $10\,\ell n\,|3x - 2y - 9| = 2y - 4x + C$

(16) $3x^2 + 4xy - 3y^2 - 10x - 10y = C$

5.3 Exact Differential Equation :

The differential equation $M + N\frac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form f(x, y) = c

e.g. $y^2\,dy + x\,dx + \frac{dx}{x} = 0$ is an exact differential equation.

NOTE : (i) The necessary condition for (1) to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) For finding the solution of exact differential equation, following results on exact differentials should be remembered:

(a) $xdy + y\,dx = d(xy)$

(b) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(c) $2(x\,dx + y\,dy) = d\,(x^2 + y^2)$

(d) $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$

(e) $\frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(f) $\frac{xdy + ydx}{xy} = d(\ln xy)$

(g) $\frac{xdy + ydx}{x^2\,y^2} = d\left(-\frac{1}{xy}\right)$

Example # 15 : Solve : $y^2x\,dx + ydx - xdy = 0$

Solution : $\dfrac{y^2xdx + ydx - xdy}{y^2} = 0 \quad \Rightarrow xdx + d\left(\dfrac{x}{y}\right) = 0$

on integrating $\dfrac{x^2}{2} + \dfrac{x}{y} + c = 0$

Example # 16 : Solve : $(x - y)dy + (x + y)dx = dx + dy$

Solution : The given equation can be written as

$(xdy + ydx) - ydy + xdx = dx + dy \quad \Rightarrow \quad d(x.y) - ydy + xdx = dx + dy$

Also integrating each term we get $xy - \dfrac{y^2}{2} + \dfrac{x^2}{2} = x + y + C$

Self Practice Problems :

(17) Solve : $xdy + ydx + xy\,e^y\,dy = 0$

(18) Solve : $ye^{-x/y}\,dx - (xe^{-x/y} + y^3)\,dy = 0$

Ans. (17) $\ell n\,(xy) + e^y = c$ (18) $2e^{-x/y} + y^2 = c$

Linear Differential Equation :

A linear differential equation has the following characteristics :

(i) Dependent variable and its derivative in first degree only and are not multiplied together

(ii) All the derivatives should be in a polynomial form

(iii) The order may be more than one

The m^{th} order linear differential equation is of the form.

$$P_0(x)\,\frac{d^my}{dx^m} + P_1(x)\,\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots\ldots + P_{m-1}(x)\,\frac{dy}{dx} + P_m(x)\,y = \phi(x),$$

where$P_0(x)$, $P_1(x)$...............$P_m(x)$ are called the coefficients of the differential equation.

NOTE : $\dfrac{dy}{dx} + y^2 \sin x = \ln x$ is not a Linear differential equation.

6.1 Linear differential equations of first order :

The differential equation $\dfrac{dy}{dx} + Py = Q$, is linear in y.

(Where P and Q are functions of x only).

Integrating Factor (I.F.) : It is an expression which when multiplied to a differential equation converts it into an exact form.

I.F for linear differential equation = $e^{\int Pdx}$ (constant of integration will not be considered)

$\therefore$ after multiplying above equation by I.F it becomes;

$$\frac{dy}{dx}\,e^{\int Pdx} + Py\,.\,e^{\int Pdx} = Q.\,e^{\int Pdx}$$

$$\Rightarrow \quad \frac{d}{dx}\,(y.e^{\int Pdx}) = Q.e^{\int Pdx} \quad \Rightarrow \quad y.e^{\int Pdx} = \int Q.e^{\int Pdx} + C.$$

NOTE : Some times differential equation becomes linear, if x is taken as the dependent variable, and y as independent variable. The differential equation has then the following form : $\dfrac{dx}{dy} + P_1\,x = Q_1$.

where P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1\,dy}$

Example # 17 : Solve the differential equation $f(x)\,\frac{dy}{dx} = f^2(x) + f(x)\,y + f'(x)\,y$

Solution : $f(x)\,\frac{dy}{dx} = f^2(x) + f(x)\,y + f'(x)\,y$

Given DE can be written as $\frac{dy}{dx} - \left(1 + \frac{f'(x)}{f(x)}\right) y = f(x)$

Which is L.D.E.

I.F. $= e^{-x - \ell n f(x)} = \frac{e^{-x}}{f(x)}$

General solution $y\frac{e^{-x}}{f(x)} = \int f(x)\frac{e^{-x}}{f(x)}\,dx + c = -e^{-x} + c \Rightarrow y = -f(x) + ce^{x} f(x)$

Example # 18 : Solve : $x\,\ell nx\,\frac{dy}{dx} + y = 2\,\ell n\,x$

Solution : $\frac{dy}{dx} + \frac{1}{x\ell nx}\,y = \frac{2}{x} \quad \Rightarrow \quad P = \frac{1}{x\ell nx},\ Q = \frac{2}{x}$

IF $= e^{\int P.dx} = e^{\int \frac{1}{x\ell nx}dx} = e^{\ell n(\ell nx)} = \ell n\,x$

$\therefore$ General solution is $y.(\ell n\,x) = \int \frac{2}{x}.\ell nx.dx + c \Rightarrow y\,(\ell n\,x) = (\ell n\,x)^2 + c$

Example # 19 : Solve the differential equation
$t\,(1 + t^2)\,dx = (x + xt^2 - t^2)\,dt$ and it given that $x = -\pi/4$ at $t = 1$

Solution : $t\,(1 + t^2)\,dx = [x\,(1 + t^2) - t^2]\,dt$

$\frac{dx}{dt} = \frac{x}{t} - \frac{t}{(1+t^2)}$

$\frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$ which is linear in $\frac{dx}{dt}$

Here, $P = -\frac{1}{t}$, $Q = -\frac{t}{1+t^2}$ IF $= e^{-\int \frac{1}{t}dt} = e^{-\ell nt} = \frac{1}{t}$

$\therefore$ General solution is -

$x : \frac{1}{t} = \int \frac{1}{t}.\left(-\frac{t}{1+t^2}\right) dt + c \quad \Rightarrow \quad \frac{x}{t} = -\tan^{-1} t + c$

putting $x = -\pi/4$, $t = 1$

$-\pi/4 = -\pi/4 + c \quad \Rightarrow \quad c = 0$

$\therefore \quad x = -t\tan^{-1} t$

6.2 Equations reducible to linear form

6.2.1 By change of variable.

Often differential equation can be reduced to linear form by appropriate substitution of the non-linear term

Example # 20 : Solve : $y\sin x\frac{dy}{dx} = \cos x\,(\sin x - y^2)$

Solution : The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$$2y\,\frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x.z = \sin x \cos x$$

$\frac{dz}{dx}$ + 2 cot x . z = 2 cos x   which is linear in $\frac{dz}{dx}$

IF = $e^{\int 2\cot x\,dx} = e^{2\ell n \sin x}$ = $\sin^2 x$

∴ General solution is -

z. $\sin^2 x = \int 2\cos x.\sin^2 x.dx + c \Rightarrow y^2 \sin^2 x = \frac{2}{3}\sin^3 x + c$

6.2.2 Bernoulli's equation :

Equations of the form $\frac{dy}{dx}$ + Py = Q.y^n, $n \neq 0$ and $n \neq 1$

where P and Q are functions of x, is called Bernoulli's equation and can be made linear in v by dividing by y^n and putting y^{-n+1} = v. Now its solution can be obtained as in (v).

e.g. : $2 \sin x \frac{dy}{dx} - y \cos x = xy^3 e^x$.

Example # 21 : Solve : $\frac{dy}{dx} = x^3y^3 - xy$ (Bernoulli's equation)

Solution : Dividing both sides by y^3

$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$

Putting t = $1/y^2$ $\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$ $\Rightarrow \frac{dt}{dx} - 2x\,t = -2x^3$

I.F. = $e^{-\int 2x dx} = e^{-x^2}$

General solution is

$t e^{-x^2} = \int -2x^3 e^{-x^2} dx + C \Rightarrow \frac{e^{-x^2}}{y^2} = -e^{-x^2}(-x^2 - 1) + C \Rightarrow \frac{1}{y^2} = (x^2 + 1) + C e^{x^2}$

Self Practice Problems :

(19) Solve : $x(x^2 + 1)\frac{dy}{dx} = y(1 - x^2) + x^2 \ell n x$

(20) Solve : $(x + 2y^3)\frac{dy}{dx} = y$

(21) Solve : $x\frac{dy}{dx} + y = y^2 \log x$

(22) Solve the differential equation $xy^2\left(\frac{dy}{dx}\right) - 2y^3 = 2x^3$ given y = 1 at x = 1

Ans. (19) $\left(\frac{x^2+1}{x}\right) y = x\,\ell n\, x - x + c$ (20) $x = y(c + y^2)$

(21) $y(1 + cx + \log x) = 1$ (22) $y^3 + 2x^3 = 3x^6$

Higher Degree Equation :

The differential equation y = mx + f(m) (where m=$\frac{dy}{dx}$) ....(1),

is known as Clairaut's Equation.

To solve (1), differentiate it w.r.t. x, which gives $\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx}$

$x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx} = 0$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ....(2) or $x + f'(m) = 0$ ....(3)

NOTE : (i)If m is eliminated between (1)and(2),thesolutionobtained is a general solution of(1)
(ii) If m is eliminated between (1) and (3), then solution obtained does not contain any arbitrary constant and is not particular solution of (1). This solution is called singular solution of (1).

Example # 22 : Solve : $y = mx + m - m^3$ where, $m = \frac{dy}{dx}$

Solution : $y = mx + m - m^3$ ..... (i)

The given equation is in clairaut's form.

Now, differentiating wrt. x -

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx} \Rightarrow \qquad m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x + 1 - 3m^2) = 0 \Rightarrow \frac{dm}{dx} = 0 \Rightarrow \qquad m = c \qquad ..... (ii)$$

$$\text{or} \qquad x + 1 - 3m^2 = 0 \qquad \Rightarrow \qquad m^2 = \frac{x+1}{3} \qquad ..... (iii)$$

Eliminating 'm' between (i) & (ii) is called the general solution of the given equation.

$y = cx + c - c^3$ where, 'c' is an arbitrary constant.

Again, eliminating 'm' between (i) & (iii) is called singular solution of the given equation.

$y = m(x + 1 - m^2)$

$$y = \pm\left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right) \qquad \Rightarrow \qquad y = \pm\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1)$$

$$y = \pm\, 2\left(\frac{x+1}{3}\right)^{3/2} \qquad \Rightarrow \qquad y^2 = \frac{4}{27}(x+1)^3 \Rightarrow 27y^2 = 4(x+1)^3$$

Self Practice Problems :

(23) Solve the differential equation $y = mx + 2/m$ where, $m = \frac{dy}{dx}$

(24) Solve : $\sin px \cos y = \cos px \sin y + p$ where $p = \frac{dy}{dx}$

Ans. (23) General solution : $y = cx + 2/c$ where c is an arbitrary constant
Singular solution : $y^2 = 8x$

(24) General solution : $y = cx - \sin^{-1}(c)$ where c is an arbitrary constant.

Singular solution : $y = \sqrt{x^2-1} - \sin^{-1}\sqrt{\frac{x^2-1}{x^2}}$

Example # 23 : The normal to a given curve at each point (x,y) on the curve passes through the point (3,0). If the curve contains, the point (3,4) find its equation.

Solution : Equation of normal at any point (x,y) is

$$\frac{dy}{dx}(Y - y) + (X - x) = 0$$

Passes through (3,0)

$$\Rightarrow \qquad (3 - x)\frac{dy}{dx} - y = 0 \Rightarrow ydy = (3 - x)dx \Rightarrow \frac{y^2}{2} = 3x - \frac{x^2}{2} + C \qquad .... (i)$$

The curve contains the point (3,4)

$$\Rightarrow \qquad 8 = 9 - \frac{9}{2} + C \Rightarrow C = 7/2$$

By equation (i)

$$\frac{y^2}{2} = 3x - \frac{x^2}{2} + \frac{7}{2} \Rightarrow x^2 + y^2 - 6x - 7 = 0$$

Example # 24 : The slope of the tangent to a curve at any point (x,y) on it given by $\frac{y}{x} - \cot\frac{y}{x} \cdot \cos\frac{y}{x}$, $(x > 0, y > 0)$ and curve passes through the point$(1,\pi/4)$. Find the equation of the curve.

Solution : Let y = f(x) be the curve

given that $\frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x} \cdot \cos\frac{y}{x}$ (homogeneous differential equation)

put y = vx

$\Rightarrow \quad v + x\frac{dv}{dx} = v - \cot v.\cos v \Rightarrow \tan v.\sec v\, dv = \frac{-dx}{x}$

Integrating both sides, we have

$\Rightarrow \quad \sec v = -\ell n|x| + C \Rightarrow \sec\frac{y}{x} + \ell n|x| = C$

Passes through $\left(1, \frac{\pi}{4}\right) \Rightarrow C = \sqrt{2}$

The curve is $\Rightarrow \sec\frac{y}{x} + \ell n|x| = \sqrt{2}$

Example # 25 : Assume that a spherical rain drop evaporates at a rate proportional to its surface area. if its radius originally is 3mm and 1 hr later has been reduced to 2mm, find an expression for radius of the rain drop at any time.

Solution : Let r be radius , V be volume and S be surface area of rain drop at any time t.

Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

given $\frac{dV}{dt} \propto S \Rightarrow \frac{dV}{dt} = kS$, k is constant of proportionality

$\Rightarrow \frac{4}{3}.3\pi r^2 \frac{dr}{dt} = k4\pi r^2 \quad \Rightarrow \frac{dr}{dt} = k \quad \Rightarrow dr = kdt$

Integrating both sides we have r = kt + C

when t = 0, r = 3 $\Rightarrow$ C = 3

when t = 1 hr, r = 2 $\Rightarrow$ k = – 1

Hence r = 3 – t **Ans.**

Self Practice Problems :

(25) The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

Ans. (25) k log 2 , where k is constant of proportionality

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Degree & Order, Differential equation formation

A-1. Find the order and degree of the following differential equations -

(i) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + y^4 = 0$

(ii) $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = y$

(iii) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

(iv) $\left(\frac{dy}{dx}\right) + y = \frac{1}{\frac{dy}{dx}}$

(v) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

(vi)✍ $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = x\frac{d^3y}{dx^3}$

(vii)✍ $\frac{d^2y}{dx^2} = \sin\left(x+\frac{dy}{dx}\right)$

A-2. Identify the order of the following equations, (where a, b, c, d are parameters)

(i) $(\sin a)\, x + (\cos a)\, y = \pi$

(ii) $y^2 = 4a\, e^{x+b}$

(iii) $\ell n\, (ay) = be^x + c$

(iv)✍ $y = \tan\left(\frac{\pi}{4}+ax\right)\ \tan\left(\frac{\pi}{4}-ax\right) + ce^{bx+d}$

A-3. Form differential equations to the curves

(i)✍ $y^2 = m\,(n^2 - x^2)$, where m, n are arbitrary constants.

(ii) $ax^2 + by^2 = 1$, where a & b are arbitrary constants.

(iii)✍ $xy = ae^{-x} + be^x$

A-4. (i) Form diffrential equation of all circles touching both positive co-ordinate axes.

(ii) Form diffrential equation of all straight lines at a distance unity from (2, 0)

(iii) Form D.E of locus of a point whose distance from origin is equal to distance from line $x + y + \lambda = 0$ where λ is a variable parameter.

Section (B) : Variable separable, Homogeneous equation, polar substitution

B-1. Solve the following differential equations

(i) $(1 + \cos x)\, dy = (1 - \cos x)\, dx$

(ii) $\frac{dy}{dx} - x \sin^2 x = \frac{1}{x \log x}$

(iii) $\frac{dy}{dx} = \frac{x(2\ell nx+1)}{\sin y + y\cos y}$

B-2. Solve :

(i) $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(ii) $\frac{dy}{dx} + e^{x-y} + e^{y-x} = 1$

B-3.✍ Solve :

(i) $\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1+x^2-y^2}{x^2-y^2}}$

(ii) $\frac{xdx + ydy}{\sqrt{x^2+y^2}} = \frac{xdy - ydx}{x^2}$

B-4. Solve :

(i) $x^2\,dy + y(x + y)\,dx = 0$, given that $y = 1$, when $x = 1$

(ii) $y\cos\frac{y}{x}(xdy - ydx) + x\sin\frac{y}{x}(xdy + ydx) = 0$, when $y(1) = \frac{\pi}{2}$.

B-5. Find the equation of the curve satisfying $\frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{x^2 + 2xy - y^2}$ and passing through (1, –1).

B-6. Identify the conic whose differential equation is $(1 + y^2)\,dx - xydy = 0$ and passing through (1, 0). Also find its focii and eccentricity

B–7. If a curve passes through the point $(1, \pi/4)$ and its slope at any point (x, y) on it is given by $y/x - \cos^2(y/x)$, then find the equation of the curve.

B-8. (i) The temperature T of a cooling object drops at a rate which is proportional to the difference T – S, where S is constant temperature of the surrounding medium.

Thus, $\frac{dT}{dt} = -k\,(T - S)$, where $k > 0$ is a constant and t is the time. Solve the differential equation if it is given that $T(0) = 150$.

(ii) The surface area of a spherical balloon, being inflated changes at a rate proportional to time t. If initially its radius is 3 units and after 2 seconds it is 5 units, find the radius after t seconds.

(iii) The slope of the tangent at any point of a curve is λ times the slope of the straight line joining the point of contact to the origin. Formulate the differential equation representing the problem and hence find the equation of the curve.

B-9. Find the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point.

B-10. Find the curve such that the ordinate of any of its points is the geometric mean between the abscissa and the sum of the abscissa and subnormal at the point.

Section (C) : Linear upon linear, Linear diff. eq. & bernaullis diff. eq.

C-1. Solve :

(i) $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$

(ii) $\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$

(iii) $(2x + 3y - 5)\,dy + (3x + 2y - 5)\,dx = 0$

(iv) $4\frac{dy}{dx} = \frac{\sqrt{3}x - 4y + 7}{x - y}$

C-2. Solve :

(i) $\frac{dy}{dx} = y\tan x - 2\sin x$

(ii) $(1 + y + x^2y)\,dx + (x + x^3)dy = 0$

(iii) $(x + 3y^2)\frac{dy}{dx} = y,\ y > 0$

(iv) $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$

C-3 Solve :

(i) $x\frac{dy}{dx} + y = x^2y^4$

(ii) $2\frac{dy}{dx} = \frac{y^2 - x}{xy + y}$

(iii) $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

(iv) $y\,y'\sin x = \cos x\,(\sin x - y^2)$

C-4. (a) Find the integrating factor of the following equations

(i) $(x \log x)\dfrac{dy}{dx} + y = 2 \log x$

(ii) $\dfrac{dy}{dx} = y \tan x - y^2 \sec x$, is

(b) If the integrating factor of $x(1 - x^2)\, dy + (2x^2 y - y - ax^3)\, dx = 0$ is $e^{\int p \cdot dx}$, then P is equal to

Section (D) : Exact differential equation, Higher degree & Higher Order differential equation

D-1 Solve the following differential equations

(i) $xdy - ydx = x^3dy + x^2ydx$

(ii) $x^2y(2xdy + 3ydx) = dy$

(iii) $x\, dy - ydx = x^{10}y^4\,(3ydx + 4x\, dy)$

D-2. Solve

(i) $y(x^2y + e^x)\, dx = e^x\, dy$

(ii) $2y \sin x \dfrac{dy}{dx} + y^2 \cos x + 2x = 0$

(iii) $(1 + x\sqrt{x^2 + y^2})\, dx + y\,(-1 + \sqrt{x^2 + y^2})\, dy = 0$

D-3. Solve

(i) $\left(y - x\dfrac{dy}{dx}\right)\left(\dfrac{dy}{dx} - 1\right) = \dfrac{dy}{dx}$ (ii) $y + x.\dfrac{dy}{dx} = x^4\left(\dfrac{dy}{dx}\right)^2$

D-4. Solve (here $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$)

(i) $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$

(ii) $\dfrac{d^3y}{dx^3} = 8\dfrac{d^2y}{dx^2}$ satisfying $y(0) = \dfrac{1}{8}$, $y_1(0) = 0$ and $y_2(0) = 1$.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Degree & Order, Differential equation formation

A-1. The order and degree of the differential equation

$$r = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$ are respectively

(A) 2, 2 (B) 2, 3 (C) 2, 1 (D) none of these

A-2. The order of the differential equation whose general solution is given by

$y = (C_1 + C_2) \sin (x + C_3) - C_4 e^{x + C_5}$ is

(A) 5 (B) 4 (C) 2 (D) 3

A-3. The order and degree of differential equation of all tangent lines to parabola $x^2 = 4y$ is

(A) 1, 2 (B) 2, 2 (C) 3, 1 (D) 4, 1

A-4. If p and q are order and degree of differential equation $y^2\left(\frac{d^2y}{dx^2}\right)^2 + 3x\left(\frac{dy}{dx}\right)^{1/3} + x^2y^2 = \sin x$, then :

(A) $p > q$ (B) $\frac{p}{q} = \frac{1}{2}$ (C) $p = q$ (D) $p < q$

A-5. Family $y = Ax + A^3$ of curve represented by the differential equation of degree
(A) three (B) two (C) one (D) four

A-6. The differential equation whose solution is $(x - h)^2 + (y - k)^2 = a^2$ is (a is a constant)

(A) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\frac{d^2y}{dx^2}$ (B) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(C) $\left[1+\left(\frac{dy}{dx}\right)\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$ (D) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = a^2\left(\frac{d^2y}{dx^2}\right)^3$

A-7. The differential equations of all conics whose centre lie at the origin is of order :
(A) 2 (B) 3 (C) 4 (D) none of these

A-8. The differential equation for all the straight lines which are at a unit distance from the origin is

(A) $\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$ (B) $\left(y + x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(C) $\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$ (D) $\left(y + x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

Section (B) : Variable separable, Homogeneous equation, polar substitution

B-1. If $\frac{dy}{dx} = e^{-2y}$ and y = 0 when x = 5, the value of x for y = 3 is

(A) e^5 (B) $e^6 + 1$ (C) $\frac{e^6+9}{2}$ (D) $\log_e 6$

B-2. If $\phi(x) = \phi'(x)$ and $\phi(1) = 2$, then $\phi(3)$ equals
(A) e^2 (B) $2e^2$ (C) $3e^2$ (D) $2e^3$

B-3. If $\frac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then function y is

(A) $e^{(1-x)^2/2}$ (B) $e^{(1+x)^2/2} - 1$ (C) $\log_e(1 + x) - 1$ (D) $1 + x$

B-4. The value of $\lim\limits_{x\to\infty} y(x)$ obtained from the differential equation $\frac{dy}{dx} = y - y^2$, where $y(0) = 2$ is

(A) 1 (B) –1 (C) 0 (D) $\frac{2}{2-e}$

B-5. The solution of $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ {where $x, y \in (-1, 1)$} is
(A) $\sin^{-1}x \sin^{-1}y = C$ (B) $\sin^{-1}x = C\sin^{-1}y$ (C) $\sin^{-1}x - \sin^{-1}y = C$ (D) $\sin^{-1}x + \sin^{-1}y = C$

B-6. Integral curve satisfying $y' = \frac{x^2+y^2}{x^2-y^2}$, $y(1) = 2$, has the slope at the point (1, 2) of the curve, equal to

(A) $-\frac{5}{3}$ (B) – 1 (C) 1 (D) $\frac{5}{3}$

B-7. Solution of differential equation xdy – y dx = 0 represents :
(A) rectangular hyperbola (B) straight line passing through origin
(C) parabola whose vertex is at origin (D) circle whose centre is at origin

B-8. The slope of a curve at any point is the reciprocal of twice the ordinate at that point and it passes through the point (4, 3). The equation of the curve is
(A) $x^2 = y + 5$ (B) $y^2 = x - 5$ (C) $y^2 = x + 5$ (D) $x^2 = y + 5$

B-9. Solution of differential equation $x(xdx - ydy) = 4\sqrt{x^2 - y^2}\ (xdy - ydx)$ is

(A) $\sqrt{x^2 - y^2} = Ae^{4\sin^{-1}\left(\frac{x}{y}\right)}$ (B) $\sqrt{x^2 + y^2} = Ae^{4\cos^{-1}x}$

(C) $\sqrt{x^2 - y^2} = Ae^{4\tan^{-1}\left(\frac{y}{x}\right)}$ (D) $\sqrt{x^2 - y^2} = Ae^{4\sin^{-1}\left(\frac{y}{x}\right)}$

B-10. Let normal at point P on curve intersect on x-axis at N and foot of P on x-axis is P'. If P'N is always constant for any point P on curve, then equation of curve is
(A) $y = ax + b$ (B) $y^2 = 2ax + b$ (C) $ay^2 - x^2 = a$ (D) $ay^2 + x^2 = a$

Section (C) : Linear upon linear, Linear diff. eq. & bernaullis diff. eq.

C-1. Solution of D.E. $\frac{dy}{dx} = \frac{2x+5y}{2y-5x+3}$ is, (if (y(0) = 0)
(A) $x^2 - y^2 + 5xy - 3y = 0$ (B) $x^2 + y^2 + 5xy - 3y = 0$
(C) $x^2 - y^2 + 5xy + 3y = 0$ (D) $x^2 - y^2 - 5xy - 3y = 0$

C-2. Solution of D.E. $\frac{dy}{dx} = \frac{3x+4y+3}{12x+16y-4}$ is
(A) $y=4x + \ell n|3x + 4y| + C$ (B) $4y=x + \ell n\ |3x + 4y| + C$
(C) $y=\ell n|3x + 4y| + C$ (D) $x + y = \ell n|3x + 4y| + C$

C-3. Solution of D.E. $\frac{dv}{dt} + \frac{k}{m}\ v = -g$ is

(A) $v = ce^{-\frac{k}{m}t} - \frac{mg}{k}$ (B) $v = c - \frac{mg}{k}\ e^{-\frac{k}{m}t}$ (C) $v\,e^{-\frac{k}{m}t} = c - \frac{mg}{k}$ (D) $v\,e^{\frac{k}{m}t} = c - \frac{mg}{k}$

C-4. Solution of differential equation $4y^3\frac{dy}{dx} + \frac{y^4}{x} = x^3$ is

(A) $y^4.x^5 = \frac{x}{5} + C$ (B) $y^4 = \frac{x^5}{5} + C$

(C) $y^4.x = x^5 + C$ (D) $y^4.x = \frac{x^5}{5} + C$

C-5. Solution of differential equation $\sin y.\frac{dy}{dx} + \frac{1}{x}\cos y = x^4 \cos^2 y$ is
(A) $x \sec y = x^6 + C$ (B) $6x \sec y = x + C$ (C) $6x \sec y = x^6 + C$ (D) $6x \sec y = 6x^6 + C$

Section (D) : Exact differential equation, Higher degree & Higher Order differential equation

D-1. Solution of differential equation $\frac{dy}{dx} = \frac{2x^3y + 3x^4 + y}{x - x^4}$ is

(A) $x^2y + x^3 = \frac{y}{x} + C$ (B) $x^2y + 2x^3 = \frac{y}{x} + C$

(C) $x^2y + x^3 = \frac{2y}{x} + C$ (D) $y + x^3 = \frac{y}{x} + C$

D-2. Solution of differential equation $xdy = \frac{xy}{\sqrt{1-x^2}}dx - ydx$ is

(A) $\ell n(x+y) = \sin^{-1}x + C$ (B) $\ell n(xy) = \sin^{-1}x + C$ (C) $2\ell n(xy) = \sin^{-1}x + C$ (D) $\ell n(xy) = 2\sin^{-1}x + C$

D-3. Solution of differential equation $x^6dy + 3x^5ydx = xdy - 2y\,dx$ is

(A) $x^3y = \frac{y}{x^2} + C$ (B) $x^3y = \frac{2y}{x^2} + C$

(C) $x^3y^2 = \frac{y}{x^2} + C$ (D) $x^3 = \frac{y}{x^2} + C$

D-4. Solution of $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$ is

(A) $y = 3x^2 + 9$ (B) $y = 3x + 9$ (C) $y = \frac{4}{3}x^2$ (D) $y = 9x + 3$

D-5. The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point (0, 1) and having slope of tangnet at x = 0 as 3, is (Here $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$)

(A) $y = x^2 + 3x + 2$ (B) $y^2 = x^2 + 3x + 1$ (C) $y = x^3 + 3x + 1$ (D) none of these

PART - III : MATCH THE COLUMN

1. Match the following

	Column - I		Column - II
(A)	Solution of $y - \frac{xdy}{dx} = y^2 + \frac{dy}{dx}$ is	(p)	$xy^2 = 2y^5 + c$
(B)	Solution of $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is	(q)	$\sec y = x + 1 + ce^x$
(C)	Solution of $\sec^2 y\,dy + \tan y\,dx = dx$ is	(r)	$(x + 1)(1 - y) = cy$
(D)	Solution of $\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$ is	(s)	$\tan y = 1 + ce^{-x}$

2. Match the following

	Column - I		Column - II
(A)	$xdy = y(dx + ydy)$, $y(1) = 1$ and $y(x_0) = -3$, then $x_0 =$	(p)	$\frac{1}{4}$
(B)	If y(t) is solution of $(t + 1)\frac{dy}{dt} - ty = 1$, $y(0) = -1$, then $y(1) =$	(q)	-15
(C)	$(x^2 + y^2)dy = xydx$ and $y(1) = 1$ and $y(x_0) = e$, then $x_0 =$	(r)	$-\frac{1}{2}$
(D)	$\frac{dy}{dx} + \frac{2y}{x} = 0$, $y(1) = 1$, then $y(2) =$	(s)	$\sqrt{3}\,e$

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1.✍ The differential equation of all parabola having their axis of symmetry coinciding with the x-axis is

(A) $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$ (B) $y\frac{d^2x}{dy^2}+\left(\frac{dx}{dy}\right)^2=0$ (C) $y\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$ (D) none of these

2.✍ If $y_1(x)$ and $y_2(x)$ are two solutions of $\frac{dy}{dx}$ + f(x) y = r(x) then $y_1(x) + y_2(x)$ is solution of :

(A) $\frac{dy}{dx}$ + f(x) y = 0 (B) $\frac{dy}{dx}$ + 2f(x) y = r(x)

(C) $\frac{dy}{dx}$ + f(x) y = 2 r(x) (D) $\frac{dy}{dx}$ + 2f (x) y = 2r(x)

3. If $y_1(x)$ is a solution of the differential equation $\frac{dy}{dx}$ + f(x) y = 0, then a solution of differential equation $\frac{dy}{dx}$ + f(x) y = r (x) is

(A) $\frac{1}{y(x)}\int y_1(x)\,dx$ (B) $y_1(x)\int \frac{r(x)}{y_1(x)}\,dx$ (C) $\int r(x)y_1(x)\,dx$ (D) none of these

4. The solution of $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$ is

(A) $\frac{x}{y}+e^{x^3}=C$ (B) $\frac{x}{y}-e^{x^3}=0$ (C) $-\frac{x}{y}+e^{x^3}=C$ (D) $\frac{y}{x}+e^{x^3}=c$

5. The solution of the differential equation $(x^2\sin^3 y - y^2\cos x)\,dx + (x^3\cos y\sin^2 y - 2y\sin x)\,dy = 0$ is

(A) $x^3\sin^3 y = 3y^2\sin x + C$ (B) $x^3\sin^3 y + 3y^2\sin x = C$

(C) $x^2\sin^3 y + y^3\sin x = C$ (D) $2x^2\sin y + y^2\sin x = C$

6. Solve : $\frac{xdy}{dx}-\frac{x^3e^{\frac{-y}{x}}}{2}(\pi+2)=y+2x^2$

(A) $e^{\frac{y}{x}}=-\frac{\pi+1}{8}(1+2x)+ce^{2x}$ (B) $e^{\frac{y}{x}}=-\frac{\pi+1}{8}(1+2x)+ce^{3x}$

(C) $e^{\frac{x}{y}}=-\frac{\pi+1}{8}(1+2x)+ce^{3x}$ (D) . $e^{\frac{y}{x}}=-\frac{\pi+2}{8}(1+2x)+ce^{2x}$

7. Solution of differential equation xy(my dx + nx dy) = $\frac{xdy+ydx}{x^my^n}$, given m + n = 1 , is

(A) $x^{m+1}.y^{n+1} + 1 = c(x/y)$ (B) $x^{m+1}.y^{n+1} + 1 = cxy$

(C) $x^{m+1}.y^{n+1} - 1 = cxy$ (D) $x^m.y^n + 1 = cxy$.

8. The equation of the curve which is such that the portion of the axis of x cut off between the origin and tangent at any point is proportional to the ordinate of that point is

(A) x = y (b – a log y) (B) log x = by² + a

(C) x² = y (a – b log y) (D) y = x (b – a log x)

(a is constant of proportionality)

9. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Then equation of the curve is.
(A) $x^2 + y^2 = 2x$ (B) $2x^2 + y^2 = 3x$ (C) $x^2 + 2y^2 = 3x$ (D) $x^2 - y^2 = x - 1$

10. f(x) is a continuous and differentiable function defined in $\in [0, \infty)$. If f(0) = 1 and $f'(x) > 3f(x)\ \forall\ x \geq 0$ then
(A) $f(x) \leq e^{3x}\ \forall x \geq 0$ (B) $f(x) \leq e^{-3x}\ \forall x \geq 0$ (C) $f(x) > e^{3x}\ \forall x \geq 0$ (D) $f(x) \geq e^{3x}\ \forall x \geq 0$

11. Solution of the equation $x\int_0^x y(t)dt = (x+1)\int_0^x t\,y(t)dt$, x > 0 is

(A) $y = \frac{c}{x^3}e^{\frac{1}{x}}$ (B) $y = \frac{c}{x^3}e^{-1/x}$ (C) $x = \frac{c}{y^3}e^{\frac{1}{y}}$ (D) $x = \frac{c}{y^3}e^{-\frac{1}{y}}$

12. The solution of diffferntial equation $(1 - x^2)\frac{dy}{dx} + xy = ax$ is

(A) $\frac{(y-a)^2 + c^2x^2}{c^2} = 1$ (B) $\frac{(y+a)^2 + c^2x^2}{c^2} = 1$
(C) $\frac{(y+a)^2 - c^2x^2}{c^2} = 1$ (D) $\frac{(y+a)^2 + c^2x^2}{c^2} = -1$

13. Find the curve which passes through the point (2, 0) such that the segment of the tangent between the point of tangency & the y–axis has a constant length equal to 2.

(A) $y = \pm\left[\sqrt{4-x^2} - 2\ell n\frac{2-\sqrt{4-x^2}}{x}\right]$ (B) $y = \pm\left[\sqrt{4-x^2} + 2\ell n\frac{2-\sqrt{4-x^2}}{x}\right]$

(C) $y = \pm\left[\sqrt{4-x^2} + 2\ell n\frac{2+\sqrt{4-x^2}}{x}\right]$ (D) $y = \pm\left[\sqrt{4-x^2} - 2\ell n\frac{2+\sqrt{4-x^2}}{x}\right]$

14. A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlet are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservior at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservior B. After how many hours do both the reservoirs have the same quantity of water?
(A) $2\log_{4/3} 2$ (B) $-\log_{2/3} 2$ (C) $\log_3 2$ (D) $\log_{4/3} 2$

15. A tank contiains 20 kg of salt dissolved in 5000 L of water. Brine that contains .03 kg of salt per litre of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour ?
(A) $150 - 130\ e^{-50/200}$ (B) $130 - 150\ e^{-30/200}$ (C) $130 - 150\ e^{-50/200}$ (D) $150 - 130\ e^{-30/200}$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. If differential equations of the curves $c(y + c)^2 = x^3$, where ‘c’ is any arbitrary constant is $12y(y')^2 + ax = bx(y')^3$ then (a + b) is equal to

2. The order of the differential equation of the family of ellipse having fixed centre and given eccentricity , is :

3. If y(x) satisfies the equation $y'(x) = y(x) + \int_0^1 y\,dx$ & $y(0) = 1$ then value of $y\left(\ell n\frac{11-3e}{2}\right)$

4. Let c_1 and c_2 be two integral curves of the differential equation $\frac{dy}{dx} = \frac{x^2-y^2}{x^2+y^2}$. A line passing through origin meets c_1 at $P(x_1, y_1)$ and c_2 at $Q(x_2, y_2)$. If $c_1 : y = f(x)$ and $c_2 : y = g(x)$ then find the value of $\frac{f'(x_1)}{g'(x_2)}$

5. If solution of the differential equation $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$ is $x = ce^{\sin y} - k(1 + \sin y)$, then k =

6. If y(x) satisfies the differential equation ; $\cos^2x\,(dy/dx) - (\tan 2x)\,y = \cos^4x$, $|x| < \frac{\pi}{4}$, and $y(0) = 0$ then $\frac{64y\left(\frac{\pi}{6}\right)}{3\sqrt{3}}$ is equal to

7. Let y_1 and y_2 are two different solutions of the equation $y' + P(x)\,.\,y = Q(x)$.
Such that the linear combination $\alpha y_1 + \beta y_2$ is also solution of given differential equation . Then value of $\alpha + \beta$ is

8. Let the curve y = f(x) passes through (4, –2) satisfy the differential equation,
$y(x + y^3)\,dx = x(y^3 - x)\,dy$ & let $y = g(x) = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$, $0 \le x \le \frac{\pi}{2}$, If the area of the region bounded by curves y = f(x), y = g(x) and x = 0 is $\frac{1}{8}\left(\frac{3\pi}{a}\right)^4$ where $a \in N$ then a is equal to

9. If the equation of curve passing through (3, 4) and satisfying the differential equation
$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ is $Ax + By + 2 = 0$ then value of A – B is

10. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. If equation of tangent to the curve at (1, 3) is ax + by + 5 = 0 then value of $a^2 + b^2$ is equal to

11. A curve passing through point (1, 2) possessing the following property; the segment of the tangent between the point of tangency & the x–axis is bisected at the point of intersection with the y–axis. If A is area bounded by the curve & line x = 1 then $9A^2$ is equal to

12. Two cylindrical tanks in which initially one is filled with water to the height of 1 m and other is empty, are connected by a pipe at the bottom. Water is allowed to flow from filled tank to the empty tank through the pipe. The rate of flow of water through the pipe at any time is $a\sqrt{2g(h_1 - h_2)}$, where 'h_1' and 'h_2' are the heights of water level (above pipe) in the tanks at that time and 'g' is acceleration due to gravity. If the cross sectional area of the filled and empty tanks be A and A/2 and that of the pipe be 'a', and if $\frac{1}{\lambda}\frac{A}{a}\sqrt{\frac{2}{g}}$ is the time when the level of water in both tanks will be same (neglect the volume of the water in pipe), then λ is :

13. If $f(x) = e^{-1/x}$, x > 0. Let for each positive integer n, P_n be the polynomial such that $\frac{d^n f(x)}{dx^n} = P_n\left(\frac{1}{x}\right)e^{-1/x}$ for all x > 0 and if $P_{n+1}(x) = x^2\left[\alpha.P_n(x) - \beta.\frac{d}{dx}P_n(x)\right]$, then $\alpha + \beta$ is :

14. If y = f (x) be a curve passing through (e, e^e) and which satisfy the differential equation $(2ny + xy \log x)dx - x \log x\, dy = 0$, value of $\int_{1/e}^{e} g(x)\,dx$ where $g(x) = \lim_{n\to\infty} f(x)$, is :

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. The differential equation of all circles in a plane must be $\left(y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}, ...\text{etc.}\right)$

(A) $y_3\left(1+y_1^2\right) - 3y_1y_2^2 = 0$
(B) of order 3 and degree 1
(C) of order 3 and degree 2
(D) $y_3^2\left(1-y_1^2\right) - 3y_1y_2^2 = 0$

2. Correct statement is/are
(A) The differential equation of all conics whose axes coincide with the axes of co-ordinates is of order 2.
(B) The differential equation of all staright lines which are at a fixed distance p from origin is of degree 2.
(C) The differential equation of all parabola each of which has a latus rectum 4a & whose axes are parallel to y-axis is of order 2.
(D) The differential equation of all parabolas of given vertex, is of order 3.

3. Solution of the differential equation $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$ is

(A) $\tan^{-1} y + \sin^{-1} x = c$
(B) $\tan^{-1} x + \sin^{-1} y = c$
(C) $\tan^{-1} y \,.\, \sin^{-1} x = c$
(D) $\cot^{-1}\frac{1}{y} + \cos^{-1}\sqrt{1-x^2} = c$

4. The solution of (x + y + 1) dy = dx are
(A) $x + y + 2 = Ce^y$
(B) x + y + 4= = C log y
(C) log (x + y + 2) = Cy
(D) log (x + y + 2) = C + y

5. The solution of $\frac{dx}{dy} + y = ye^{(n-1)x}$, $(n \neq 1)$

(A) $\frac{1}{n-1}\,\ell n\left(\frac{e^{(n-1)x}-1}{e^{(n-1)x}}\right) = \frac{y^2}{2} + C$
(B) $e^{(1-n)x} = 1 + ce^{(n-1)\frac{y^2}{2}}$
(C) $\ell n\left(1 + ce^{(n-1)\frac{y^2}{2}}\right) + nx + 1 = 0$
(D) $e^{(n-1)x} = ce^{(n-1)+\frac{(n-1)y^2}{2}} + 1$

6. Correct statement is/are

(A) f(x, y) = $x^2e^{\frac{x}{y}} + \frac{y^3}{x} + y^2\ell n\left(\frac{y}{x}\right)$ is a homogenous function of degree two.

(B) f(x, y) = $\frac{\sin y + x}{\sin 2y + x\cos y}$ is homogenous function of degree one.

(C) $x\sin\left(\frac{y}{x}\right)dy + \left(y\sin\frac{y}{x} - x\right)dx = 0$ is a homogenous differential equation.

(D) f(x, y) = $e^{\frac{y}{x}} + \tan\frac{y}{x}$ is homogenous function of degree zero.

7. Solution of differential equation $f(x)\dfrac{dy}{dx} = f^2(x) + f(x)\,y + f'(x)\,y$ is

(A) $y = f(x) + ce^x$ (B) $y = -f(x) + ce^x$ (C) $y = -f(x) + ce^x f(x)$ (D) $y = cf(x) + e^x$

8. The solution of $x^2 y_1^2 + xy\,y_1 - 6y^2 = 0$ are (here $y_1 = dy/dx$)

(A) $y = Cx^2$ (B) $x^2 y = C$ (C) $\frac{1}{2}\ell ny = C + \log x$ (D) $x^3 y = C$

9. The solution of differential equation $\left(\dfrac{dy}{dx}\right)^2 - \dfrac{dy}{dx}(e^x + e^{-x}) + 1 = 0$ is

(A) $y = e^x + c$
(B) $y = -e^{-x} + c$
(C) $y = 2e^x + 3e^{-x} + c$
(D) $ye^x + 1 = ce^x$

10. If $y = e^{-x}\cos x$ and $y_n + k_n y = 0$, where $y_n = \dfrac{d^n y}{dx^n}$ and k_n, $n \in N$ are constants.

(A) $k_4 = 4$ (B) $k_8 = -16$ (C) $k_{12} = 20$ (D) $k_{16} = -24$

11. A solution of the differential equation $y_1 y_3 = 3y_2^2$ can be (where $y_n = \dfrac{d^n y}{dx^n}$)

(A) $x = A_1y^2 + A_2 y + A_3$
(B) $x = A_1 y + A_2$
(C) $x = A_1 y^2 + A_2 y$
(D) $y = A_1 x^2 + A_2x + A_3$

12. A differentiable function satisfies equation $f(x) = \int_0^x \big(f(t)\cos t - \cos(t - x)\big)dt$ then

(A) $f''\left(\dfrac{\pi}{2}\right) = e$
(B) $\lim\limits_{x\to-\infty} f(x) = 1$
(C) $f(x)$ has minimum value $1 - e^{-1}$
(D) $f'(0) = -1$

13. Let f(x) is a continuous function which takes positive values for $x \geq 0$ and satisfy $\int_0^x f(t)dt = x\sqrt{f(x)}$ with $f(1) = \dfrac{1}{2}$ then

(A) $f(x) = \dfrac{1}{\left[1+\left(1-\sqrt{2}\right)x\right]^2}$

(B) $f\left(\cot\dfrac{\pi}{8}\right) = \dfrac{1}{4}$

(C) Area bounded by f(x) and x-axis between x = 0 to $x = \sqrt{2} + 1$ is $\dfrac{1}{2\left(\sqrt{2}-1\right)}$ square units.

(D) $f\left(\sin\dfrac{\pi}{4}\right) = 2$

14. Let f(x), $x \geq 0$ be a non negative continuous function & let $F(x) = \int_0^x f(t)\,dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq c\,F(x)$ for all $x \geq 0$ then

(A) $f(x) = 0\ \forall\ x \geq 0$
(B) $f(0) = 0$
(C) $e^{-cx}F(x)$ is a non-increasing function on $[0, \infty)$
(D) $F(x) \leq 0\ \forall\ x \leq 0$

15. A curve passing through (1 , 0) such that the ratio of the square of the intercept cut by any tangent off the y–axis to the subnormal is equal to the ratio of the product of the co–ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

(A) $x = e^{2\sqrt{y/x}}$ (B) $y = e^{2\sqrt{x/y}}$ (C) $x = e^{-2\sqrt{y/x}}$ (D) $x = e^{-2\sqrt{x/y}}$

16. The differential equation $\frac{d^2y}{dx^2} + y + \cot^2 x = 0$ must be satisfied by

(A) $y = 2 + c_1 \cos x + \sqrt{c_2}\ \sin x$

(B) $y = \cos x \,.\, \ell n\left(\tan\frac{x}{2}\right) + 2$

(C) $y = 2 + c_1 \cos x + c_2 \sin x + \cos x \log\left(\tan\frac{x}{2}\right)$

(D) all the above

PART - IV : COMPREHENSION

Comprehension # 1

Differential equations are solved by reducing them to the exact differential of an expression in x & y i.e., they are reduced to the form d(f(x, y)) = 0

e.g. :

$$\frac{xdx + ydy}{\sqrt{x^2+y^2}} = \frac{ydx - xdy}{x^2}$$

$$\Rightarrow \quad \frac{1}{2}\frac{2xdx + 2ydy}{\sqrt{x^2+y^2}} = -\frac{xdy - ydx}{x^2} \quad \Rightarrow \quad d\left(\sqrt{x^2+y^2}\right) = -d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad d\left(\sqrt{x^2+y^2} + \frac{y}{x}\right) = 0$$

$\therefore$ solution is $\sqrt{x^2+y^2} + \frac{y}{x} = c.$

Use the above method to answer the following question (3 to 5)

1. The general solution of $(2x^3 - xy^2)\, dx + (2y^3 - x^2y)\, dy = 0$ is

(A) $x^4 + x^2y^2 - y^4 = c$ (B) $x^4 - x^2y^2 + y^4 = c$
(C) $x^4 - x^2y^2 - y^4 = c$ (D) $x^4 + x^2y^2 + y^4 = c$

2. General solution of the differential equation $\frac{xdy}{x^2+y^2} + \left(1 - \frac{y}{x^2+y^2}\right) dx = 0$ is

(A) $x + \tan^{-1}\left(\frac{y}{x}\right) = c$ (B) $x + \tan^{-1}\frac{x}{y} = c$ (C) $x - \tan^{-1}\left(\frac{y}{x}\right) = c$ (D) none of these

3. General solution of the differential equation $e^y\, dx + (xe^y - 2y)\, dy = 0$ is

(A) $xe^y - y^2 = c$ (B) $ye^x - x^2 = c$ (C) $ye^y + x = c$ (D) $xe^y - 1 = cy^2$

Comprehension # 2

In order to solve the differential equation of the form $a_0 \dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1}y}{dx^{n-1}} + ... + a_n y = 0$, where a_0, a_1, a_2 are constants.

We take the auxiliary equation $a_0 D^n + a_1 D^{n-1} + ... + a_n = 0$

Find the roots of this equation and then solution of the given differential equation will be as given in the following table.

	Roots of the auxiliary equation	Corresponding complementary function
1.	One real root α_1	$c_1 e^{\alpha_1 x}$
2.	Two real and different roots α_1 and α_2	$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$
3.	Two real and equal roots α_1 and α_1	$(c_1 + c_2 x)\, e^{\alpha_1 x}$
4.	Three real and equal roots $\alpha_1, \alpha_1, \alpha_1$	$(c_1 + c_2 x + c_3 x^2)\, e^{\alpha_1 x}$
5.	One pair of imaginary roots $\alpha \pm i\beta$	$(c_1 \cos \beta x + c_2 \sin \beta x)\, e^{\alpha x}$
6.	Two pair of equal imaginary roots $\alpha \pm i\beta$ and $\alpha \pm i\beta$	$[(c_1 + c_2 x) \cos \beta x + (c_1 + c_2 x) \sin \beta x]\, e^{\alpha x}$

Solution of the given differential equation will be y = sum of all the corresponding parts of the complementary functions.

4. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$.

(A) $y = (c_1 + c_2 x)e^x$ (B) $y = (c_1 e^x + c_2 e^x)$ (C) $y = (c_1 x)e^x$ (D) none of these

5. Solve $\dfrac{d^2y}{dx^2} + a^2 y = 0$.

(A) $y = (c_1 \cos ax + c_2 \sin ax)e^{ax}$ (B) $y = c_1 \cos ax + c_2 \sin ax$

(C) $y = c_1 e^{ax} + c_2 e^{-ax}$ (D) none of these

6. Solve $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

(A) $y = (c_1 + c_2 x + c_3 x^2)\, e^x$ (B) $y = x\, (c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

(C) $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ (D) none of these

Comprehension # 3 (Q.No. 7 to 9)

Let f(x) be a differentiable function, satisfying $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$

7. Graph of y = f(x) cuts x -axis at

(A) $x = -\dfrac{1}{2}\ell n5$ (B) $x = \dfrac{1}{2}\ell n5$ (C) $x = -\ell n5$ (D) $x = \ell n5$

8. Area enclosed by y = f(x) in the second quadrant is

(A) $3 + \dfrac{1}{2}\ell n\sqrt{5}$ (B) $2 + \dfrac{1}{2}\ell n5$ (C) $3 - \sqrt{5}$ (D) 3

9. Area enclosed by $y = f(x)$, $y = f^{-1}(x)$, $x + y = 2$ and $x + y = -\dfrac{1}{2}\ell n5$ is

(A) $8 + \dfrac{1}{8}(\ell n5)^2$ (B) $8 - 2\sqrt{5} + \dfrac{1}{8}(\ell n5)^2$ (C) $2\sqrt{5} - \dfrac{1}{8}(\ell n5)^2$ (D) $8 + 2\sqrt{5} - \dfrac{1}{8}(\ell n5)^2$

Exercise-3

✍ **Marked questions are recommended for Revision.**

* **Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1.✍ Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt,\ 0 \le x \le 1$ and f(0) = 0, then **[IIT-JEE-2009, Paper-1, (3, –1), 80]**

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

2. Match the statements/expressions in **Column - I** with the open intervals in **Column - II** **[IIT-JEE-2009, Paper-1, (8, 0), 80]**

Column - I	**Column - II**
(A) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2\ \ y' + y = 0$	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

3. Match the statements/expressions given in **Column - I** with the values given in **Column - II** **[IIT-JEE-2009, Paper-2, (8, 0), 80]**

Column - I	**Column – II**
(A)✍ The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p) 1
(B) Value(s) of k for which the planes kx + 4y + z = 0, 4x + ky + 2z = 0 and 2x + 2y + z=0 intersect in a straight line	(q) 2
(C) Value(s) of k for which |x – 1| + |x – 2| + |x + 1| + |x + 2| = 4k has integer solution(s)	(r) 3
(D) If y′ = y + 1 and y(0) = 1, then value(s) of y (ℓn 2)	(s) 4
	(t) 5

4. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to **[IIT-JEE 2010, Paper-1, (3, 0), 84]**

5.✍ Let f : [1, ∞) → [2, ∞) be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \ge 1$, then the value of f(2) is **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

6.✍ Let $y'(x) + y(x)\,g'(x) = g(x)\,g'(x)$, y(0) = 0, $x \in$ **R** , where f′(x) denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is
[IIT-JEE 2011, Paper-2, (4, 0), 80]

7.* If y(x) satisfies the differential equation $y' - y\tan x = 2x\sec x$ and y(0) = 0, then
[IIT-JEE 2012, Paper-1, (4, 0), 70]

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

8. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, x > 0. Then the equation of the curve is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\mathrm{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

9. The function y = f(x) is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in (–1, 1) satisfying f(0) = 0. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

10.✍ Let f : [0, 2] → R be a function which is continuous on [0, 2] and is differentiable on (0, 2) with f(0) = 1. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0, 2]$. If F'(x) = f'(x) for all $x \in (0, 2)$, then F(2) equals
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) e – 1 (D) e^4

11*. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If y(0) = 2, then which of the following statements is (are) true ? **[JEE (Advanced) 2015, P-1 (4, –2)/ 88]**
(A) y(–4) = 0
(B) y(–2) = 0
(C) y(x) has a critical point in the interval (–1, 0)
(D) y(x) has no critical point in the interval (–1, 0)

12*. Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' (here y' = , y" =), then which of the following statements is (are) true?
[JEE (Advanced) 2015, P-1 (4, –2)/ 88]
(A) $P = y + x$ (B) $P = y - x$
(C) $P + Q = 1 - x + y + y' + (y')^2$ (D) $P - Q = x + y - y' - (y')^2$

13. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then **[JEE (Advanced) 2016, Paper-1, (3, –1)/62]**
(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$ (B) $\lim\limits_{x\to 0^+} x f\left(\frac{1}{x}\right) = 2$ (C) $\lim\limits_{x\to 0^+} x^2 f'(x) = 0$ (D) $|f(x)| \leq 2$ for all $x \in (0, 2)$

14. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point
(1, 3). Then the solution curve **[JEE (Advanced) 2016, Paper-1, (4, –2)/62]**
(A) intersects y = x + 2 exactly at one point
(B) intersects y = x + 2 exactly at two points
(C) intersects $y = (x + 2)^2$
(D) does NOT intersect $y = (x + 3)^2$

15. If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx$, $x > 0$ and $y(0) = \sqrt{7}$, then y(256) = **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**
(A) 16 (B) 3 (C) 9 (D) 80

16*. Let $f : R \to R$ and $g : R \to R$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})\, g'(x)$ for all $x \in R$, and f(1) = g(2) = 1, then which of the following statement(s) is (are) TRUE?
[JEE(Advanced) 2018, Paper-1,(4, –2)/60]
(A) $f(2) < 1 - \log_e 2$ (B) $f(2) > 1 - \log_e 2$
(C) $g(1) > 1 - \log_e 2$ (D) $g(1) < 1 - \log_e 2$

17*. Let $f : [0, \infty) \to R$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$
for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are)) TRUE?
(A) The curve y = f(x) passes through the point (1, 2)
(B) The curve y = f(x) passes through the point (2, –1) **[JEE(Advanced) 2018, Paper-1,(4, –2)/60]**
(C) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\}$ is $\frac{\pi - 2}{4}$
(D) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\}$ is $\frac{\pi - 1}{4}$

18*. Let $f : (0, \pi) \to R$ be a twice differentiable function such that $\lim\limits_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$ for all $x \in (0, \pi)$. If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?
[JEE(Advanced) 2018, Paper-2,(4, –2)/60]
(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$ (B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$
(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$ (D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

19. Let f : R → R be a differentiable function with f(0) = 0. If y = f(x) satisfies the differential equation

$$\frac{dy}{dx} = (2 + 5y)(5y - 2),$$

then the value of $\lim_{x \to -\infty} f(x)$ is _______ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

20. Let f : R → R be a differentiable function with f(0) = 1 and satisfying the equation

$$f(x + y) = f(x)\, f'(y) + f'(x) f(y) \text{ for all } x, y \in R.$$

Then, the value of $\log_e(f(4))$ is ______ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$ where c_1 and c_2 are arbitary constants is **[AIEEE 2009 (4, –1), 144]**

(1) $y' = y^2$ (2) $y'' = y'\, y$ (3) $y.y'' = y'$ (4) $y.y'' = (y')^2$

2. Solution of the differential equation cosx dy = y(sinx – y) dx, $0 < x < \frac{\pi}{2}$ is **[AIEEE 2010 (4, –1), 144]**

(1) y sec x = tan x + c (2) y tan x = sec x + c (3) tanx = (sec x + c)y (4) secx = (tanx + c) y

3. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is **[AIEEE 2011, I, (4, –1), 120]**

(1) $T^2 - \frac{1}{k}$ (2) $I - \frac{kT^2}{2}$ (3) $I - \frac{k(T-t)^2}{2}$ (4) e^{-kT}

4. If $\frac{dy}{dx} = y + 3 > 0$ and y(0) = 2, then y(ℓn2) is equal to : **[AIEEE 2011, I, (4, –1), 120]**

(1) 7 (2) 5 (3) 13 (4) –2

5. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact is given by : **[AIEEE 2011, II, (4, –1), 120]**

(1) 2y – 3x = 0 (2) $y = \frac{6}{x}$ (3) $x^2 + y^2 = 13$ (4) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$

6. Consider the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If y (1) = 1, then x is given by : **[AIEEE 2011, II, (4, –1), 120]**

(1) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (2) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (3) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$ (4) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$

7. The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\, p(t) - 450$. If p(0) = 850, then the time at which the population becomes zero is : **[AIEEE-2012, (4, –1)/120]**

(1) 2 ℓn 18 (2) ℓn 9 (3) $\frac{1}{2}\ell$n 18 (4) ℓn 18

8. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is **[AIEEE - 2013, (4, –1),360]**
(1) 2500 (2) 3000 (3) 3500 (4) 4500

9. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}$ p(t) – 200 . If p(0) = 100, then p(t) equals : **[JEE(Main)2014,(4, – 1), 120]**
(1) $600 - 500\,e^{t/2}$ (2) $400 - 300\,e^{-t/2}$ (3) $400 - 300\,e^{t/2}$ (4) $300 - 200\,e^{-t/2}$

10. Let y(x) be the solution of the differential equal $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$. Then y(e) is equal to **[JEE(Main)2015,(4, – 1), 120]**
(1) e (2) 0 (3) 2 (4) 2e

11. If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to **[JEE(Main)2016,(4, – 1), 120]**
(1) $-\frac{4}{5}$ (2) $\frac{2}{5}$ (3) $\frac{4}{5}$ (4) $-\frac{2}{5}$

12. If $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to : **[JEE(Main)2017,(4, – 1), 120]**
(1) $\frac{1}{3}$ (2) $-\frac{2}{3}$ (3) $-\frac{1}{3}$ (4) $\frac{4}{3}$

13. Let y = y(x) be the solution of the differential equation
$\sin x\frac{dy}{dx} + y\cos x = 4x$, $x \in (0,\pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to **[JEE(Main)2018,(4, – 1), 120]**
(1) $-\frac{8}{9}\pi^2$ (2) $-\frac{4}{9}\pi^2$ (3) $\frac{4}{9\sqrt{3}}\pi^2$ (4) $\frac{-8}{9\sqrt{3}}\pi^2$

14. Let $f : [0, 1] \to R$ be such that f(xy) = f(x).f(y), for all $x, y \in [0, 1]$, and $f(0) \neq 0$. if y = y(x) satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with y(0) = 1, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to :
[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]
(1) 5 (2) 2 (3) 3 (4) 4

15. The curve amongst the family of curves represented by the differential equation, $(x^2 - y^2)dx + 2xy\,dy = 0$ which passes through (1, 1), is : **[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**
(1) a circle with centre on the x-axis.
(2) a hyperbola with transverse axis along the x-axis
(3) an ellipse with major axis along the y-axis.
(4) a circle with centre on the y-axis

16. If y(x) is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$, x > 0, where $y(1) = \frac{1}{2}e^{-2}$, then
[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]
(1) $y(\log_e 2) = \log_e 4$
(2) $y(\log_e 2) = \frac{\log_e 2}{4}$
(3) y(x) is decreasing in $\left(\frac{1}{2}, 1\right)$
(4) y(x) is decreasing in (0, 1)

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) (2, 2) (ii) (3, 2) (iii) 1, 1
(iv) 1, 2 (v) 3, degree is not applicable
(vi) 3, 2 (vii) 2, degree is not applicable

A-2. (i) 1 (ii) 1 (iii) 2 (iv) 2

A-3. (i) $xyy_2 + (xy_1 - y)\, y_1 = 0$ (ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ (iii) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$

A-4. (i) $x^2 + y^2 - 2\,\frac{x+yy'}{1+y'}\,(x+y) + \frac{(x+yy')^2}{(1+y')^2} = 0$ (ii) $(1 + y'^2) = (y - (x-2)\, y')^2$

(iii) $(1 + y')\sqrt{x^2+y^2} = \pm\sqrt{2}(x+yy')$

Section (B) :

B-1. (i) $y = 2\tan x/2 - x + c$ (ii) $y = \frac{x^2}{4} - \frac{1}{4}\,x\sin 2x - \frac{1}{8}\cos 2x + \log|\log x| + c$

(iii) $y\sin y = x^2\ell n x + c$

B-2. (i) $\log\left|\tan\left(\frac{x+y}{2}\right)+1\right| = x + c$ (ii) $\tan^{-1}(e^{y-x}) + x = c$

B-3. (i) $\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = \frac{c(x+y)}{\sqrt{x^2-y^2}}$ (ii) $\sqrt{x^2+y^2} = \frac{y}{x} + c$

B-4. (i) $3x^2y = 2x + y$ (ii) $xy\sin(y/x) = \frac{\pi}{2}$

B-5. $x + y = 0$ **B-6.** Conic: $x^2 - y^2 = 1$ (hyperbola) focii : $(\pm\sqrt{2}, 0)$, $e = \sqrt{2}$

B–7. $\tan y/x = 1 - \log x.$

B-8. (i) $\frac{T-S}{150-S} = e^{-kt}$ (ii) $r = \sqrt{4t^2+9}$ units (iii) $y = kx^{\lambda}$ where, k is some constant

B-9. $\sqrt{x^2+y^2} = ce^{\pm\tan^{-1}\frac{y}{x}}$ **B-10.** $y^2 = \frac{x^4+c}{2x^2}$ or $y^2 + 2x^2\ell n x = cx^2$

Section (C) :

C-1. (i) $x^2 + y^2 - xy + x - y = c$ (ii) $y - 2x + \frac{3}{8}\ell n(24y + 16x + 23) = c$

(iii) $4xy + 3(x^2 + y^2) - 10(x + y) = c$ (iv) $8xy - 4y^2 = \sqrt{3}x^2 + 14x + C$

C-2. (i) $y = \cos x + k\sec x$ (ii) $yx = -\tan^{-1}x + c$

(iii) $\frac{x}{y} = 3y + C$ (iv) $(1 + x^2)y = \sin x + C$

C-3 (i) $\frac{1}{y^3} = 3x^2 + kx^3$ (ii) $y^2 + (1 + x)\,\ell n(1 + x) + 1 + c(1 + x)$

(iii) $e^y = ce^{-e^x} + e^x - 1$ (iv) $y^2 \sin^2 x = \frac{2}{3}\sin^3 x + c$

C-4. (a) (i) $\pm\ell n\ x$ (ii) $\pm\sec x$ (b) $\frac{(2x^2-1)}{x(1-x^2)}$

Section (D) :

D-1 (i) $(y/x) = xy + C$ (ii) $x^3y^2 = y + c$ (iii) $\frac{1}{4}\left(\frac{y}{x}\right)^4 = \frac{x^6.y^8}{2} + C$

D-2. (i) $\frac{1}{y}e^x = -\frac{x^3}{3} + c$ (ii) $y^2 \sin x = -x^2 + c$ (iii) $x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$

D-3. (i) General solution : $y = cx + \frac{c}{c-1}$

Singular solution : $y = (\sqrt{x} \pm 1)^2$

(ii) General solution : $xy + c = c^2x$

Singular solution : $4x^2y + 1 = 0$

D-4. (i) $c_1e^x + c_2$ (ii) $64y = (e^{8x} - 8x) + 7$

PART - II

Section (A) :

A-1. (A) **A-2.** (D) **A-3.** (A) **A-4.** (D) **A-5.** (A) **A-6.** (B)
A-7. (B) **A-8.** (C)

Section (B) :

B-1. (C) **B-2.** (B) **B-3.** (B) **B-4.** (A) **B-5.** (D)

B-6. (A) **B-7.** (B) **B-8.** (C) **B-9.** (D) **B-10.** (B)

Section (C) :

C-1. (A) **C-2.** (B) **C-3.** (A) **C-4.** (D) **C-5.** (C)

Section (D) :

D-1. (A) **D-2.** (B) **D-3.** (A) **D-4.** (B) **D-5.** (C)

PART - III

1. (A) $\to$ (r), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (q) **2.** (A) $\to$ (q), (B) $\to$ (r), (C) $\to$ (s), (D) $\to$ (p)

EXERCISE - 2

PART - I

1. (A) **2.** (C) **3.** (B) **4.** (A) **5.** (A) **6.** (D)

7. (B) **8.** (A) **9.** (A) **10.** (D) **11.** (B) **12.** (A)

13. (B) **14.** (D) **15.** (D)

PART – II

1. 35 **2.** 2 **3.** 4 **4.** 1 **5.** 2 **6.** 8

7. $\alpha + \beta = 1$ **8.** 16 **9.** 4 **10.** 25 **11.** 64 **12.** 3

13. 2 **14.** 0

PART - III

1. (AB) **2.** (ABC) **3.** (AD) **4.** (AD) **5.** (AB) **6.** (ACD) **7.** (C)

8. (ACD) **9.** (ABD) **10.** (AB) **11.** (ABC) **12.** (AD) **13.** (BC)

14. (ABC) **15.** (AC) **16.** (BC)

PART - IV

1. (B) **2.** (A) **3.** (A) **4.** (A) **5.** (B) **6.** (C) **7.** (A)

8. (C) **9.** (B)

EXERCISE - 3

PART - I

1. (C) **2.** (A) $\rightarrow$ (p, q, s), (B) $\rightarrow$ (p, t), (C) $\rightarrow$ (p, q, r, t), (D) $\rightarrow$ (s)

3. (A) $\rightarrow$ (p), (B) $\rightarrow$ (q, s), (C) $\rightarrow$ (q, r, s, t), (D) $\rightarrow$ (r) **4.** 9

5. Bonus (Taking x = 1, the integral becomes zero, whereas the right side of the equation gives 5. Therefore, the function f does not exist.)

6. 0 **7.*** (AD) **8.** (A) **9.** (B) **10.** (B) **11*.** (AC)

12*. (BC) **13.** (A) **14.** (A,D) **15.** (B) **16.** (BC) **17.** (BC)

18. (BCD) **19.** (0.4) **20.** 2

PART - II

1. (4) **2.** (4) **3.** (2) **4.** (1) **5.** (2) **6.** (3) **7.** (1)

8. (3) **9.** (3) **10.** (3) **11.** (3) **12.** (1) **13.** (1) **14.** (3)

15. (1) **16.** (3)

High Level Problems (HLP)

1. Solve the differential equation $\frac{dy}{dx}$ = (sinx – siny) $\frac{\cos x}{\cos y}$

2. Solve : (1 + xy) y + (1 – xy) x $\frac{dy}{dx} = 0$

3. Use the substitution y^2 = a – x to reduce the equation $y^3\frac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it. (where 'a' is variable)

5. Solve : $\frac{dy}{dx} - y\ell n2 = 2^{\sin x}(\cos x - 1)\ell n2$, y being bounded when $x \to \infty$.

6. Solve : $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{1}{\left(1+x^2\right)^2}$ given that y = 0, when x = 1.

7. Solve the differential equation, $(x^2 + 4y^2 + 4xy)\, dy = (2x + 4y + 1)\, dx$.

8. Solve the following differential equations.

(i) $3\frac{dy}{dx} + \frac{2y}{x+1} = \frac{x^3}{y^2}$ (ii) $x^2 y - x^3\frac{dy}{dx} = y^4 \cos x$

9. Find the integral curve of the differential equation $x(1 - x\ell ny)\frac{dy}{dx} + y = 0$ which passes through (1, 1/e).

10. Solve the following differential equations.

(i) $(x^2 + y^2 + a^2)\, y\frac{dy}{dx} + x\,(x^2 + y^2 - a^2) = 0$ (ii) (1 + tany) (dx – dy) + 2x dy = 0

11. If y_1 & y_2 be solutions of the differential equation $\frac{dy}{dx}$ + Py = Q, where P & Q are functions of x alone, and $y_2 = y_1 z$, then prove that $z = 1 + ae^{-\int \frac{Q}{y_1}dx}$,'a' being an arbitrary constant.

12. Let y_1 and y_2 are two different solutions of the equation $y' + P(x)\,.\,y = Q(x)$.
Prove that $y = y_1 + C(y_2 - y_1)$ is the general solution of the same equation (C is a constant)

13. Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\frac{x^4 + 2xy - 1}{1+x^2}$.

14. A curve y = f(x) passes through the point p (1, 1). The normal to the curve at P is; a (y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve & the normal to the curve at P.

15. Consider a curved mirror y = f(x) passing through (8, 6) having the property that all rays emerging from origin after getting reflected from the mirror becomes parallel to x - axis. Find the equation of curve (s)

16. Find the curve for which sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co-ordinates of the point of tangency, the proportionality factor is equal to k.

17. Find the curve $y = f(x)$ where $f(x) \geq 0$, $f(0) = 0$, bounding a curvilinear trapezoid with the base $[0, x]$ whose area is proportional to $(n + 1)^{th}$ power of $f(x)$. It is known that $f(1) = 1$

18. Find the nature of the curve for which the length of the normal at the point P is equal to the radius vector of the point P.

19. A country has a food deficit of 10 %. Its population grows continuously at a rate of 3 % per year. Its annual food production every year is 4 % more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self–sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to, $\frac{\ell n\ 10 - \ell n\ 9}{\ell n\,(1.04) - 0.03}$.

20. Solution of Differential equation $\left\{\frac{y^2}{(x-y)^2} - \frac{1}{x}\right\} dx + \left\{\frac{1}{y} - \frac{x^2}{(x-y)^2}\right\} dy = 0$ is

21. Solution of the differential equation $(xdy - ydx)(x+y)^2 = 4xy(x^2+y^2)(xdx - ydy)$ is

22. Solve . $x^2\frac{dy}{dx} + y^2 e^{\frac{x(y-x)}{y}} = 2y(x-y)$

23.✍ Solution of differential equation $xe^{-\frac{y}{x}}dy - \left(ye^{-\frac{y}{x}} + x^3\right)dx = 0$ is

HLP Answers

1. $\sin y = \sin x - 1 + c\, e^{-\sin x}$

2. $\ell n\frac{x}{y} - \frac{1}{xy} = c$

3. $\frac{1}{2}\ell n\,|x^2 + a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c$ where, $a = x + y^2$

5. $y = 2^{\sin x}$

6. $y(1 + x^2) = \tan^{-1}x - \frac{\pi}{4}$

7. $y = \ell n\left((x+2y)^2 + 4(x+2y) + 2\right) - \frac{3}{2\sqrt{2}}\ell n\left(\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right) + c$

8. (i) $y^3(x+1)^2 = \frac{x^6}{6} + \frac{2}{5}x^5 + \frac{1}{4}x^4 + c$ (ii) $x^3 y^{-3} = 3\sin x + c$

9. $x(ey + \ell ny + 1) = 1$

10. (i) $(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = c$ (ii) $x\, e^y(\cos y + \sin y) = e^y \sin y + C$

13. $y = (x - 2\tan^{-1}x)(1 + x^2)$

14. $e^{a(x-1)}$, $\frac{1}{a}\left[a - \frac{1}{2} + e^{-a}\right]$ sq. unit

15. $y^2 = 4(1 + x)$ or $y^2 = 36(9 - x)$

16. $y = \pm\frac{1}{k}\ \ell n\,|c(k^2x^2 - 1)|$

17. $y = x^{1/n}$

18. Rectangular hyperbola or circle.

19. 19

20. $\ell n\left(\frac{y}{x}\right) + \frac{xy}{y-x} = C$

21. $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ell n\left(\frac{x}{y}\right) + x^2 - y^2 + c$

22. $x(x-y) = y\ln(ce^x - 1)$

23.✍ $2e^{-\frac{y}{x}} + x^2 = C$