CONTENT

Matrices & Determinant

JEE (Advanced) Syllabus

Matrices and Determinant : As a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.Determinant of a square matrix of order up to three.

JEE (Main) Syllabus

Matrices and Determinant : Algebra of matrices, types of matrices, determinants and matrices of order two and three. Properties of determinants, evaluation of determinants, area of triangles using determinants. Adjoin and evaluation of inverse of a square matrix using determinants and elementary transformations, Test of consistency and solution of simultaneous linear equations in two or three variables using determinants and matrices.

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Matrices & Determinant

As for everything else, so for a mathematical theory, beauty can be perceived but not explained..... Cayley Arthur

Introduction :

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is written as m × n and we call it as order m by n

The general m × n matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$.

Notes :

(i) The elements a_{11}, a_{22}, a_{33},........ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as tr(A)

(ii) Capital letters of English alphabets are used to denote matrices.

(iii) Order of a matrix : If a matrix has m rows and n columns, then we say that its order is "m by n", written as "m × n".

Example # 1 : Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2} | i - 3j |$.

Solution : In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$a_{ij} = \frac{1}{2} | i - 3j |$, i = 1, 2, 3 and j = 1, 2

Therefore $a_{11} = \frac{1}{2} | 1 - 3 \times 1 | = 1$ $\quad a_{12} = \frac{1}{2} | 1 - 3 \times 2 | = \frac{5}{2}$

$a_{21} = \frac{1}{2} | 2 - 3 \times 1 | = \frac{1}{2}$ $\quad a_{22} = \frac{1}{2} | 2 - 3 \times 2 | = 2$

$a_{31} = \frac{1}{2} | 3 - 3 \times 1 | = 0$ $\quad a_{32} = \frac{1}{2} | 3 - 3 \times 2 | = \frac{3}{2}$

Hence the required matrix is given hy $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

Types of Matrices :

Row matrix :

A matrix having only one row is called as row matrix (or row vector).General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, ..., a_{1n}]$

This is a matrix of order "$1 \times n$" (or a row matrix of order n)

Column matrix :

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{bmatrix}$

This is a matrix of order "$m \times 1$" (or a column matrix of order m)

Zero matrix :

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

e.g. : (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square matrix :

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$ which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "$n \times n$" (or a square matrix of order n)

Diagonal matrix :

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, ...a_{nn})$.

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

Scalar matrix :

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

e.g. : (i) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

Unit matrix (identity matrix) :

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : (i) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & u & v \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

Lower triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above
the diagonal elements are zero.)

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ x & y & z & 0 \end{bmatrix}$

Comparable matrices :

Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ are comparable

e.g. : (ii) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ are not comparable

Equality of matrices :

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

$A = B$ iff (i) $m = p$, $n = q$

(ii) $a_{ij} = b_{ij}\ \forall\ i$ & j.

Example # 2 : Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find θ so that A = B.

Solution : By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ & $\tan\theta = -1$

$\Rightarrow \quad \theta = (2n + 1)\pi - \frac{\pi}{4}$.

Example # 3 : If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$, then find the values of a, b, c, x, y and z.

Solution : As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$x + 3 = 0 \qquad z + 4 = 6 \qquad 2y - 7 = 3y - 2$

$a - 1 = -3 \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4$

$\Rightarrow \quad a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$

Multiplication of matrix by scalar :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ & $-3A \equiv (-3)\,A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is a diagonal entry of A

Addition of matrices :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.

$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$, $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$

Substraction of matrices :

Let A & B be two matrices of same order. Then A – B is defined as A + (– B) where – B is (– 1) B.

Properties of addition & scalar multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).
Let $M_{m \times n}$ (F) denote the set of all such matrices.

Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F) \Rightarrow A + B \in M_{m \times n}(F)$

(b) $A + B = B + A$

(c) $(A + B) + C = A + (B + C)$

(d) $O = [o]_{m \times n}$ is the additive identity.

(e) For every $A \in M_{m \times n}(F)$, $-A$ is the additive inverse.

(f) $\lambda (A + B) = \lambda A + \lambda B$

(g) $\lambda A = A\lambda$

(h) $(\lambda_1 + \lambda_2)\,A = \lambda_1 A + \lambda_2 A$

Example # 4 : IF $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B

Solution : We have 2A + 3X = 5B.

$\Rightarrow \quad 3X = 5B - 2A$

$\Rightarrow \quad X = \frac{1}{3}(5B - 2A)$

$$\Rightarrow \quad X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Multiplication of matrices :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

Notes :

(1) The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined BA is defined.

(2) In general $AB \neq BA$, even when both the products are defined.

(3) A (BC) = (AB) C, whenever it is defined.

Properties of matrix multiplication :

Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n. (where F is Q, R or C). Then

(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general $AB \neq BA$

(c) (AB) C = A(BC)

(d) I_n, the identity matrix of order n, is the multiplicative identity.
$AI_n = A = I_n A \quad \forall A \in M_n(F)$

(e) For every non singular matrix A (i.e., $|A| \neq 0$) of M_n (F) there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A) B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n(F)$

(h) $(A + B) C = AC + BC \quad \forall A, B, C \in M_n(F)$.

Notes : (1) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(2) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Example # 5 : If A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution : We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $A^3 = AA^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

Now $A^3 - 23A - 40I = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 90-92+0 & 46-46+0 & 63-23-40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Self practice problems :

(1) If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $A(\alpha)\,A(\beta) = A(\alpha + \beta)$.

Hence show that in this case $A(\alpha).\,A(\beta) = A(\beta)\,.\,A(\alpha)$.

(2) Let A = $\begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$ and C = [3 1 2].

Then which of the products ABC, ACB, BAC, BCA, CAB, CBA are defined. Calculate the product whichever is defined.

Answer (2) Only CAB is defined. CAB = [25 100]

Transpose of a matrix :

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A′(or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}$ $\forall$ i & j.

i.e. A′ is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, A′ = $\begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Results : (i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm ..... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_i are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1 A_2 ....A_n)' = A_n' . A_{n-1}' ....A_2' . A_1'$, provided the product is defined.

Symmetric & skew-symmetric matrix : A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j. A square matrix A is said to be skew-symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew-symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Notes :

(1) In a skew-symmetric matrix all the diagonal elements are zero.

$(\because \; a_{ii} = -a_{ii} \quad \Rightarrow \quad a_{ii} = 0)$

(2) For any square matrix A, $A + A'$ is symmetric & $A - A'$ is skew-symmetric.

(3) Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric.

$A = B + C$, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$.

Example # 6 : If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \;\; 3 \;\; -6]$, verify that $(AB)' = B'A'$.

Solution : We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1 \;\; 3 \;\; -6]$$

Then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \;\; 3 \;\; -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2 \;\; 4 \;\; 5], B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \;\; 4 \;\; 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

Example # 7 : Express the matrix B = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution : Here $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let $P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$

Now $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

Now $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

Example # 8 : Show that BAB′ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Solution : Case - IA is symmetric $\Rightarrow$ $A' = A$

$(BAB')' = (B')'A'B' = BAB'$ $\Rightarrow$ BAB' is symmetric.

Case - II A is skew-symmetric $\Rightarrow$ $A' = -A$

$(BAB')' = (B')'A'B'$

$= B(-A)B'$

$= -(BAB')$

$\Rightarrow$ BAB' is skew-symmetric

Self practice problems :

(3) For any square matrix A, show that $A'A$ & AA' are symmetric matrices.

(4) If A & B are symmetric matrices of same order, then show that AB + BA is symmetric and AB – BA is skew-symmetric.

Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$ Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}$, $\begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}$, $\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

Determinant of a square matrix :

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix.

Let $A = [a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|$ is defined as $ad - bc$.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

Minors & Cofactors :

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$

$M_{12} = c,\ C_{12} = -c$

$M_{21} = b,\ C_{21} = -b$

$M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$.

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay$ etc.

Determinant of any order :

Let $A = [a_{ij}]_n$ be a square matrix ($n > 1$). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column).

$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Transpose of a determinant : The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of determinant :

(1) $|A| = |A'|$ for any square matrix A.

i.e. the value of a determinant remains unaltered, if the rows & columns are inter changed,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

(2) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_2 = -D_1$

(3) Let λ be a scalar. Than $\lambda\,|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $E = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $E = KD$

(4) $|AB| = |A|\,|B|$.

(5) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(6) A skew-symmetric matrix of odd order has determinant value zero.

(7) If a determinant has all the elements zero in any row or column, then its value is zero,

i.e. $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(8) If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(9) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(10) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$. Then $D_2 = D_1$

(11) Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

Example # 9 Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution : Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$= (a + b + c)\ ((b - c)\ (a - b) - (c - a)^2)$

$= (a + b + c)\ (ab + bc - ca - b^2 - c^2 + 2ca - a^2)$

$= (a + b + c)\ (ab + bc + ca - a^2 - b^2 - c^2) \equiv 3abc - a^3 - b^3 - c^3$

Example # 10 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution : Given determinant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a - b)\ (b - c)\ [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$

$= (a - b)\ (b - c)\ [c(ab + bc + ca) - a(ab + bc + ca)]$

$= (a - b)\ (b - c)\ (c - a)\ (ab + bc + ca)$

Self practice problems

(5) Find the value of $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$.

(6) Simplify . $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

(7) Prove that $\begin{vmatrix} a-b-c & 2\,a & 2\,a \\ 2\,b & b-c-a & 2\,b \\ 2\,c & 2\,c & c-a-b \end{vmatrix} = (a+b+c)^3$.

(8) Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$ by using factor theorem .

Answers : (5) 0 (6) 0

Application of determinants : Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is:

$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If D = 0 then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines :
$a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Singular & non singular matrix : A square matrix A is said to be singular or non-singular according as |A| is zero or non-zero respectively.

Cofactor matrix & adjoint matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$

then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.

Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

Properties of cofactor A and adj A :

(a) $A \,.\, \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A. In particular, for 3×3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Example # 11 : For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Solution : $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $\qquad$ $\text{cof } A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

$\text{adj } A = (\text{cof } A)' = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$ which is symmetric.

Inverse of a matrix (reciprocal matrix) :

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}\text{adj } A$ is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}. We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right) A$, for A is non-singular

$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{ adj } A.$

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non-singular.

2. A^{-1} is always non-singular.

3. If $A = \text{dia }(a_{11}, a_{22}, \ldots\ldots, a_{nn})$ where $a_{ii} \neq 0\ \forall$ i, then $A^{-1} = \text{diag }(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.

4. $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also $\text{adj }(A') = (\text{adj } A)'$.

5. $(A^{-1})^{-1} = A$ if A is non-singular.

6. Let k be a non-zero scalar & A be a non-singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.

7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.

8. Let A be a non-singular matrix. Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.

9. A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.

10. $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.

11. In general AB = **0** does not imply A = **0** or B = **0**. But if A is non-singular and AB = **0**, then B = **0**. Similarly B is non-singular and AB = **0** $\Rightarrow$ A = **0**. Therefore, AB = **0** $\Rightarrow$ either both are singular or one of them is **0**.

Example # 12 : If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A \text{ adj } A = |A|\, I$. Also find A^{-1}

Solution : We have $|A| = 1\,(16-9) - 3\,(4-3) + 3\,(3-4) = 1 \neq 0$

Now $C_{11} = 7$, $C_{12} = -1$, $C_{13} = -1$, $C_{21} = -3$, $C_{22} = 1$, $C_{23} = 0$, $C_{31} = -3$, $C_{32} = 0$, $C_{33} = 1$

Therefore $\text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|. I$

Also $A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Example # 13 : Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using the equation, find A^{-1} .

Solution : We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $A A - 4A = - I$

or $AA(A^{-1}) - 4 A A^{-1} = - I A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A (A A^{-1}) - 4I = - A^{-1}$ or $AI - 4I = - A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Example # 14 : For two non-singular matrices A & B, show that adj (AB) = (adj B) (adj A)

Solution : We have $(AB)(\text{adj }(AB)) = |AB|\, I_n$

$= |A|\, |B|\, I_n$

$A^{-1} (AB)(\text{adj }(AB)) = |A|\, |B|\, A^{-1}$

$\Rightarrow$ B adj (AB) = |B| adj A $\quad (\because \quad A^{-1} = \frac{1}{|A|} \text{adj } A)$

$\Rightarrow$ B^{-1} B adj (AB) = |B| B^{-1} adj A

$\Rightarrow$ adj (AB) = (adjB) (adj A)

Self practice problems :

(9) If A is non-singular, show that adj (adj A) = $|A|^{n-2}$ A.

(10) Prove that adj (A^{-1}) = (adj A)$^{-1}$.

(11) For any square matrix A, show that |adj (adj A) | = $|A|^{(n-1)^2}$.

(12) If A and B are non-singular matrices, show that $(AB)^{-1} = B^{-1} A^{-1}$.

Elementary row transformation of matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remarks : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

Finding inverse using Elementry operations

(i) Using row transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

(ii) Using column transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the second matrix B of the product AB on RHS.

Example # 15 : Obtain the inverse of the matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementary operations.

Solution : Write A = IA, i.e. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \rightarrow R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \rightarrow R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \rightarrow R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (applying $R_3 \rightarrow \frac{1}{2} R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$ (Applying $R_1 \rightarrow R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$ A (Applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

System of linear equations & matrices : Consider the system

$a_{11} x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22} x_2 + ...+ a_{2n} x_n = b_2$

....

$a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ ..... & ..... & .... & ..... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ ... \\ b_n \end{bmatrix}$.

Then the above system can be expressed in the matrix form as AX = B.
The system is said to be consistent if it has atleast one solution.

System of linear equations and matrix inverse:

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix.

Results :

(1) If A is non-singular, solution is given by $X = A^{-1}B$.

(2) If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.

(3) If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

Homogeneous system and matrix inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O.
($\because$ in this case $b_1 = b_2 = ... b_n = 0$), where A is a square matrix.

Results :

(1) If A is non-singular, the system has only the trivial solution (zero solution) X = 0

(2) If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non-trivial solutions.

Example # 16 : Solve the system $\begin{matrix} x+y+z=6 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ using matrix inverse.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

Then the system is AX = B.

|A| = 6. Hence A is non singular.

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1} B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad x = 1,\ y = 2,\ z = 3.$$

Self practice problems:

(13) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the inverse of A using |A| and adj A.

(14) Find real values of λ and μ so that the following systems has
(i) unique solution (ii) infinitely many solutions (iii) No solution.
$x + y + z = 6$
$x + 2y + 3z = 1$
$x + 2y + \lambda z = \mu$

(15) Find λ so that the following homogeneous system have a non zero solution
$x + 2y + 3z = \lambda x$
$3x + y + 2z = \lambda y$
$2x + 3y + z = \lambda z$

Answers : (13) $\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{vmatrix}$ (14) (i) $\lambda \neq 3$, $\mu \in R$ (ii) $\lambda = 3$, $\mu = 1$ (iii) $\lambda = 3$, $\mu \neq 1$ (15) $\lambda = 6$

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix

A-1. Construct a 3 × 2 matrix whose elements are given by $a_{ij} = 2i - j$.

A-2. If $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, find x, y, z, w.

A-3. Let $A + B + C = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}$, $4A + 2B + C = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$ and $9A + 3B + C = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}$ then find A

A-4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$, will AB be equal to BA. Also find AB & BA.

A-5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^3 = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$

A-6. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ show that $(I + A) = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

A-7. Given $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If $x \in R$ Then for what values of y, $F(x + y) = F(x)\,F(y)$.

A-8. Let $A = [a_{ij}]_{n \times n}$ where $a_{ij} = i^2 - j^2$. Show that A is skew-symmetric matrix.

A-9. If $C = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 & 9 \\ 4 & 2 & 8 \\ 6 & 5 & 3 \end{bmatrix}$, then trace of $C + C^3 + C^5 + ... + C^{99}$ is

Section (B) : Determinant of Matrix

B-1. If the minor of three-one element (i.e. M_{31}) in the determinant $\begin{vmatrix} 0 & 1 & \sec\alpha \\ \tan\alpha & -\sec\alpha & \tan\alpha \\ 1 & 0 & 1 \end{vmatrix}$ is 1 then find the value of α. $(0 \le \alpha \le \pi)$.

B-2. Using the properties of determinants, evalulate:

(i) $\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$ (ii) $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$. (iv) $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

B-3. Prove that :

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(ii) $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a+b+c)(a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(iv) If $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = (a+b)(b+c)(c+a)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

B-4. If a, b, c are positive and are the p^{th}, q^{th}, r^{th} terms respectively of a G.P., show without expanding that,

$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$.

B-5. Find the non – zero roots of the equation,

(i) $\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & c \end{vmatrix} = 0$. (ii) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

B-6. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$.

B-7. Show that $\begin{vmatrix} a_1l_1+b_1m_1 & a_1l_2+b_1m_2 & a_1l_3+b_1m_3 \\ a_2l_1+b_2m_1 & a_2l_2+b_2m_2 & a_2l_3+b_2m_3 \\ a_3l_1+b_3m_1 & a_3l_2+b_3m_2 & a_3l_3+b_3m_3 \end{vmatrix} = 0$.

B-8. If $\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + \ldots$, then find the value of A and B.

Section (C) : Cofactor matrix, adj matrix and inverse of matrix

C-1. If A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, C = $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and AB – CD = 0 find D.

C-2. (i) Prove that (adj adj A) = $|A|^{n-2}$ A
(ii) Find the value of |adj adj adj A| in terms of |A|

C-3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

C-4. If A is a symmetric and B skew symmetric matrix and (A + B) is non-singular and C = $(A + B)^{-1}$ (A – B), then prove that
(i) $C^T (A + B) C = A + B$ (ii) $C^T (A - B) C = A - B$

C-5. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then find values of a & c.

Section (D) : Charactristic equation and system of equations

D-1. For the matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ find a & b so that $A^2 + aA + bI = 0$. Hence find A^{-1}.

D-2. Find the total number of possible square matrix A of order 3 with all real entries, whose adjoint matrix B has characterstics polynomial equation as $\lambda^3 - \lambda^2 + \lambda + 1 = 0$.

D-3. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, show that $A^3 = (5A - I)(A - I)$

D-4. Apply Cramer's rule to solve the following simultaneous equations.
(i) $2x + y + 6z = 46$
$5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$
(ii) $x + 2y + 3z = 2$
$x - y + z = 3$
$5x - 11y + z = 17$

D-5. Solve using Cramer's rule: $\frac{4}{x+5} + \frac{3}{y+7} = -1$ & $\frac{6}{x+5} - \frac{6}{y+7} = -5$.

D-6. Find those values of c for which the equations:
$2x + 3y = 3$
$(c+2)x + (c+4)y = c+6$
$(c+2)^2 x + (c+4)^2 y = (c+6)^2$ are consistent.
Also solve above equations for these values of c.

D-7. Solve the following systems of linear equations by matrix method.
(i) $2x - y + 3z = 8$
$-x + 2y + z = 4$
$3x + y - 4z = 0$

(ii) $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

D-8. Investigate for what values of λ, μ the simultaneous equations
$x + y + z = 6$; $x + 2y + 3z = 10$ & $x + 2y + \lambda z = \mu$ have;
(a) A unique solution
(b) An infinite number of solutions.
(c) No solution.

D-9. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.

D-10. Compute A^{-1}, if $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$. Hence solve the matrix equations

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}.$$

D-11. Which of the following statement(s) is/are true:

S1 : The system of equations $\begin{matrix} 4x - 5y - 2z = 2 \\ 5x - 4y + 2z = 3 \\ 2x + 2y + 8z = 1 \end{matrix}$ is Inconsistent.

S2 : A matrix 'A' has 6 elements. The number of possible orders of A is 6.

S3 : For any 2×2 matrix A, if A (adjA) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then |A| = 10.

S4 : If A is skew symmetric, then $B'AB$ is also skew symmetric.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix,

A-1. $\begin{bmatrix} x^2 + x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$ then x is equal to -

(A) – 1 (B) 2 (C) 1 (D) No value of x

A-2. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

(A) $AB = \begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ (B) $AB = [-2 \ \ -1 \ \ 4]$ (C) $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (D) AB does not exist

A-3. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then B =

(A) $I\cos\theta + J\sin\theta$ (B) $I\cos\theta - J\sin\theta$ (C) $I\sin\theta + J\cos\theta$ (D) $-I\cos\theta + J\sin\theta$

A-4. In an upper triangular matrix $A = [a_{ij}]_{n \times n}$ the elements $a_{ij} = 0$ for
(A) $i < j$ (B) $i = j$ (C) $i > j$ (D) $i \leq j$

A-5. If A = diag (2, –1, 3), B = diag (–1, 3, 2), then A^2B =
(A) diag (5, 4, 11) (B) diag (– 4, 3, 18) (C) diag (3, 1, 8) (D) B

A-6. If A is a skew- symmetric matrix, then trace of A is
(A) 1 (B) – 1 (C) 0 (D) none of these

A-7. Let $A = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$ such that det(A) = r where p, q, r all prime numbers, then trace of A is equal to
(A) 6 (B) 5 (C) 2 (D) 3

A-8. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 31 \\ 62 \end{bmatrix}$.
(Where I is the (2 × 2) identity matrix), then the product of all elements of matrix V is
(A) 2 (B) 1 (C) 3 (D) –2

A-9. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, B = [a b c] and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+1)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$
Where a, b, c and x ∈ R, Given that tr (AB) = tr(C), then the value of (a + b + c).
(A) 7 (B) 2 (C) 1 (D) 4

Section (B) : Determinant of Matrix

B-1. If A and B are square matrices of order 3 such that |A| = – 1, |B| = 3, then |3AB| is equal to
(A) – 9 (B) – 81 (C) – 27 (D) 81

B-2. Let $A = \begin{bmatrix} \cos^{-1}x & \cos^{-1}y & \cos^{-1}z \\ \cos^{-1}y & \cos^{-1}z & \cos^{-1}x \\ \cos^{-1}z & \cos^{-1}x & \cos^{-1}y \end{bmatrix}$ such that |A| = 0, then maximum value of x + y + z is
(A) 3 (B) 0 (C) 1 (D) 2

B-3. The absolute value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is:
(A) $16\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) none

B-4. If α, β & γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ =
(A) p (B) q (C) $p^2 - 2q$ (D) none

B-5. If a, b, c > 0 & x, y, z ∈ R, then the determinant $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$ =
(A) $a^xb^yc^z$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D) zero

B-6. If a, b & c are non-zero real numbers, then D = $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ =

(A) abc (B) $a^2b^2c^2$ (C) $bc+ca+ab$ (D) zero

B-7. The determinant $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$ =

(A) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (B) 2 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (C) 3 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (D) none of these

B-8. If x, y, z $\in$ R & $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$ then value of x is

(A) – 2 (B) – 3 (C) 2 (D) 3

B-9. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is:

(A) 0 (B) independent of θ (C) independent of ϕ (D) independent of θ & ϕ both

B-10. Let A be set of all determinants of order 3 with entries 0 or 1, B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value –1. Then
STATEMENT -1 : The number of elements in set B is equal to number of elements in set C.
and
STATEMENT-2 : $(B \cap C) \subset A$

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Section (C) : Cofactor matrix, adj matrix and inverse of matrix.

C-1. If A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj A =

(A) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

C-2. Identify statements S_1, S_2, S_3 in order for true(T)/false(F)

S_1 : If A = $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then adj A = A'

S_2 : If A = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

S_3 : If B is a non-singular matrix and A is a square matrix, then det $(B^{-1}AB)$ = det (A)

(A) TTF (B) FTT (C) TFT (D) TTT

C-3. If A, B are two n × n non-singular matrices, then
(A) AB is non-singular (B) AB is singular
(C) $(AB)^{-1} = A^{-1} B^{-1}$ (D) $(AB)^{-1}$ does not exist

C-4. Let A = $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that A = BX, then X is equal to
(A) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (D) none of these

C-5. Let **A** = $\begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ be a matrix, then (det **A**) x (adj **A**$^{-1}$) is equal to
(A) $O_{3 \times 3}$ (B) I_3 (C) $\begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}$

C-6. **STATEMENT-1 :** If A = $\begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+xc & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$ and B = $\begin{bmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{bmatrix}$, then $|A| = |B|^2$.
STATEMENT-2 : If A^c is cofactor matrix of a square matrix A of order n then $|A^c| = |A|^{n-1}$.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Section (D) : Characteristic equation and system of equations

D-1. If A = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ is a root of polynomial $x^3 - 6x^2 + 7x + k = 0$, then the value of k is
(A) 2 (B) 4 (C) –2 (D) 1

D-2 If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where bc ≠ 0) satisfies the equations $x^2 + k = 0$, then
(A) a + d = 0 & k = |A| (B) a – d = 0 & k = |A|
(C) a + d = 0 & k = –|A| (D) a + d ≠ 0 & k = |A|

D-3. If the system of equations x + 2y + 3z = 4, x + py + 2z = 3, x + 4y + μz = 3 has an infinite number of solutions and solution triplet is
(A) p = 2, μ = 3 and $(5 - 4\lambda, \lambda - 1, \lambda)$ (B) p = 2, μ = 4 and $(5 - 4\lambda, \frac{\lambda-1}{2}, 2\lambda)$
(C) 3p = 2μ and $(5 - 4\lambda, \lambda - 1, 2\lambda)$ (D) p = 4, μ = 2 and $(5 - 4\lambda, \frac{\lambda-1}{2}, \lambda)$

D-4. Let λ and α be real. Find the set of all values of λ for which the system of linear equations have infinite solution ∀ real values of α.
$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$
$x + (\cos\alpha)y + (\sin\alpha)z = 0$
$-x + (\sin\alpha)y + (\cos\alpha)z = 0$
(A) $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ (B) – 1 (C) $(-5, -\sqrt{2})$ (D) None of these

D-5. Let A = $\begin{bmatrix} a & o & b \\ 1 & e & 1 \\ c & o & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ where a,b, c, d, e $\in$ {0, 1}

then number of such matrix A for which system of equation AX = O have unique solution.

(A) 16 (B) 6 (C) 5 (D) none

D-6. If the system of equations ax + y + z = 0,. x + by + z = 0 and x + y + cz = 0, where a, b, c $\neq$ 1, has a non–trivial solution, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is

(A) 1 (B) 2 (C) 3 (D) 4

PART - III : MATCH THE COLUMN

1. **Column I** **Column II**

(A) $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ = 0 then x = (p) 2

(B) If A is a square matrix of order 3 × 3 and k is a scalar, then adj (kA) = k^m adj A, then m is (q) – 2

(C) If A = $\begin{bmatrix} 2 & \mu \\ \mu^2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} \gamma & 7 \\ 49 & \delta \end{bmatrix}$ here (A – B) is upper triangular matrix then number of possible values of μ are (r) 1

(D) If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ = k abc (a + b + c)^3 then the value of k is (s) $-\frac{9}{8}$

2. **Column – I** **Column – II**

(A) If A and B are square matrices of order 3 × 3, where |A| = 2 and |B| = 1, then $|(A^{-1}) \cdot \text{adj}(B^{-1}) \cdot \text{adj}(2A^{-1})|$ = (p) 7

(B) If A is a square matrix such that A^2 = A and $(I + A)^3 = I + kA$, then k is equal to (q) 8

(C) Matrix $\begin{bmatrix} a & b & (a\alpha-b) \\ b & c & (b\alpha-c) \\ 2 & 1 & 0 \end{bmatrix}$ is non invertible ($b^2 \neq ac$) if -2α is (r) 0

(D) If A = $[a_{ij}]_{3\times3}$ is a scalar matrix with $a_{11} = a_{22} = a_{33} = 2$ and A(adjA) = kI_3 then k is (s) – 1

Exercise-2

Marked questions are recommended for Revision.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. Two matrices A and B have in total 6 different elements (none repeated) . How many different matrices A and B are possible such that product AB is defined.

(A) 5(6!) (B) 3(6!) (C) 12(6!) (D) 8 (6!)

2. If AB = O for the matrices

A = $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and B = $\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ then $\theta - \phi$ is

(A) an odd multiple of $\frac{\pi}{2}$ (B) an odd multiple of π

(C) an even multiple of $\frac{\pi}{2}$ (D) 0

3. If X = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then value of X^n is, (where n is natural number)

(A) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (B) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$ (C) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (D) $\begin{bmatrix} 2n+1 & -4n \\ n & -(2n-1) \end{bmatrix}$

4. If A and B are two matrices such that AB = B and BA = A, then $A^2 + B^2$ =

(A) 2AB (B) 2BA (C) A + B (D) AB

5. Find number of all possible ordered sets of two (n × n) matrices **A** and **B** for which **AB** – **BA** = I

(A) infinite (B) n^2 (C) n! (D) zero

6. If B, C are square matrices of order n and if A = B + C, BC = CB, C^2 = O, then which of following is true for any positive integer N.

(A) $A^{N+1} = B^N (B + (N + 1) C)$ (B) $A^N = B^N (B + (N + 1) C)$

(C) $A^{N+1} = B (B + (N + 1) C)$ (D) $A^{N+1} = B^N (B + (N + 2) C)$

7. How many 3 × 3 skew symmetric matrices can be formed using numbers –2, –1, 1, 2, 3, 4, 0 (any number can be used any number of times but 0 can be used at most 3 times)

(A) 8 (B) 27 (C) 64 (D) 54

8. If A is a skew - symmetric matrix and n is an even positive integer, then A^n is

(A) a symmetric matrix (B) a skew-symmetric matrix

(C) a diagonal matrix (D) none of these

9. Number of 3 × 3 non symmetric matrix A such that $A^T = A^2 - I$ and $|A| \neq 0$, equals to

(A) 0 (B) 2 (C) 4 (D) Infinite

10. Matrix **A** is such that $\mathbf{A}^2 = 2\mathbf{A} - I$, where I is the identity matrix. Then for n ≥ 2, $\mathbf{A}^n$ =

(A) $n\mathbf{A} - (n-1)I$ (B) $n\mathbf{A} - I$ (C) $2^{n-1}\mathbf{A} - (n-1)I$ (D) $2^{n-1}\mathbf{A} - I$

11. If P = $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and Q = PAP^T and x = $P^TQ^{2005}P$, then x is equal to

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

12. Let Δ = $\begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then

(A) Δ is independent of θ (B) Δ is indepedent of ϕ
(C) Δ is a constant (D) none of these

13. Δ = $\begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ is equal to

(A) $(a-b)^2 (b-c)^2 (c-a)^2$ (B) $2(a-b)(b-c)(c-a)$
(C) $4(a-b)(b-c)(c-a)$ (D) $(a+b+c)^3$

14. If D = $\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ then D =

(A) $1+a^2+b^2+c^2$ (B) $a^2+b^2+c^2$ (C) $(a+b+c)^2$ (D) none

15. Value of the Δ = $\begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$ is

(A) 0 (B) $(a^3-1)(a^6-1)(a^9-1)$
(C) $(a^3+1)(a^6+1)(a^9+1)$ (D) $a^{15}-1$

16. If $\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & f \\ 4x & 2y & 2z \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix}$, then the value of $\Delta_1 - \Delta_2$ is

(A) $x + \frac{y}{2} + z$ (B) 2 (C) 0 (D) 3

17. From the matrix equation AB = AC, we conclude B = C provided:
(A) A is singular (B) A is non–singular (C) A is symmetric (D) A is a square

18. Let A = $\begin{bmatrix} -2 & 7 & \sqrt{3} \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $A^4 = \lambda.I$, then λ is

(A) – 16 (B) 16 (C) 8 (D) –8

19. If A is 3 × 3 square matrix whose characterstic polynomial equations is $\lambda^3 - 3\lambda^2 + 4 = 0$ then trace of adjA is
(A) 0 (B) 3 (C) 4 (D) – 3

20. If a, b, c are non zeros, then the system of equations
$$(\alpha + a)x + \alpha y + \alpha z = 0$$
$$\alpha x + (\alpha + b)y + \alpha z = 0$$
$$\alpha x + \alpha y + (\alpha + c)z = 0$$
has a non-trivial solution if
(A) $\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$ (B) $\alpha^{-1} = a + b + c$
(C) $\alpha + a + b + c = 1$ (D) none of these

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. Let X be the solution set of the equation
$A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the unit matrix and $X \subset N$ then the minimum value of $\sum_x (\cos^x \theta + \sin^x \theta)$, $\theta \in R$ is :

2. If A is a diagonal matrix of order 3 × 3 is commutative with every square matrix of order 3 × 3 under multiplication and tr(A) = 12, then the value of |A| is :

3. **A,** is a (3×3) diagonal matrix having integral entries such that det(A) = 120, number of such matrices is 10n. Then n is :

4. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \geq 0$, where a, b, c $\in R^+$, then $\frac{a+b}{c}$ is

5. If a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9 are in H.P. and **D** = $\begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, then the value of **21D** is

6. If $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = k(\alpha a + \beta b + \gamma c)^3$, then $(2\alpha + \beta - \gamma)^k$ is ($\alpha, \beta, \gamma, k \in z^+$)

7. If A is a square matrix of order 3 and A′ denotes transpose of matrix A, A′ A = I and det A = 1, then det (A – I) must be equal to

8. Suppose A is a matrix such that $A^2 = A$ and $(I + A)^6 = I + kA$, then k is

9. If $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = 64$, then (ab + bc + ac) is :

10. Let f(x) = $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ then the maximum value of f(x) is

11. If $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$ and $\sum_{n=1}^{N} U_n = \lambda \sum_{n=1}^{N} n^2$, then λ is

12. The absolute value of a for which system of equations, $a^3x + (a + 1)^3y + (a + 2)^3z = 0$, $ax + (a + 1) y + (a + 2) z = 0$, $x + y + z = 0$, has a non–zero solution is:

13. Consider the system of linear equations in x, y, z:

$$(\sin 3\theta)\, x - y + z = 0$$
$$(\cos 2\theta)\, x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

Number of values of $\theta \in (0, \pi)$ for which this system has non – trivial solution, is

14. The value of ' 2k ' for which the set of equations $3x + ky - 2z = 0$, $x + ky + 3z = 0$, $2x + 3y - 4z = 0$ has a non – trivial solution over the set of rational is:

15. $A_1 = [\, a_1]$

$$A_2 = \begin{bmatrix} a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots A_n = [\ldots\ldots]$$

Where $a_r = [\, \log_2 r \,]$ ([.] denotes greatest integer). Then trace of A_{10}

16. If $\left\{ \frac{1}{2}\left(A - A' + I\right) \right\}^{-1} = \frac{2}{\lambda} \begin{bmatrix} \lambda-13 & -\frac{\lambda}{3} & \frac{\lambda}{3} \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$ for $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$, then λ is :

17. Given $A = \begin{bmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{bmatrix}$ For $\alpha \in R - \{a, b\}$, A^{-1} exists and $A^{-1} = A^2 - 5bA + cI$, when $\alpha = 1$. The value of a + 5b + c is :

18. Let a, b, c positive numbers. Find the number of solution of system of equations in x, y and z

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;$- \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has finitely many solutions

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Which one of the following is wrong ?
(A) The elements on the main diagonal of a symmetric matrix are all zero
(B) The elements on the main diagonal of a skew - symmetric matrix are all zero
(C) For any square matrix A, A A′ is symmetric
(D) For any square matrix A, $(A + A')^2 = A^2 + (A')^2 + 2AA'$

2. Which of the following is true for matrix A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(A) A + 4I is a symmetric matrix
(B) $A^2 - 4A + 5I_2 = 0$
(C) A – B is a diagonal matrix for any value of α if B = $\begin{bmatrix} \alpha & -1 \\ 2 & 5 \end{bmatrix}$
(D) A – 4I is a skew symmetric matrix

3. Suppose a_1, a_2, a_3 are in A.P. and b_1, b_2, b_3 are in H.P. and let

$$\Delta = \begin{vmatrix} a_1 - b_1 & a_1 - b_2 & a_1 - b_3 \\ a_2 - b_1 & a_2 - b_2 & a_2 - b_3 \\ a_3 - b_1 & a_3 - b_2 & a_3 - b_3 \end{vmatrix}, \text{ then}$$

(A) Δ is independent of a_1, a_2, a_3,
(B) $a_1 - \Delta, a_2 - 2\Delta, a_3 - 3\Delta$ are in A.P.
(C) $b_1 + \Delta, b_2 + \Delta^2, b_3 + \Delta$ are in H.P.
(D) Δ is independent of b_1, b_2, b_3

4. Let $\theta = \frac{\pi}{5}$, X = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, O is null maxtrix and I is an identity matrix of order 2 × 2, and if $I + X + X^2 + ... + X^n = O$, then n can be

(A) 9
(B) 19
(C) 4
(D) 29

5. If $\Delta = \begin{vmatrix} x & 2y - z & -z \\ y & 2x - z & -z \\ y & 2y - z & 2x - 2y - z \end{vmatrix}$, then

(A) x – y is a factor of Δ
(B) $(x - y)^2$ is a factor of Δ
(C) $(x - y)^3$ is a factor of Δ
(D) Δ is independent of z

6. Let a, b > 0 and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then

(A) a + b – x is a factor of Δ
(B) $x^2 + (a + b)x + a^2 + b^2 - ab$ is a factor of Δ
(C) $\Delta = 0$ has three real roots if a = b
(D) a + b + x is a factor of Δ

7. The determinent $\Delta = \begin{vmatrix} b & c & b\alpha + c \\ c & d & c\alpha + d \\ b\alpha + c & c\alpha + d & a\alpha^3 - c\alpha \end{vmatrix}$ is equal to zero if

(A) b, c, d are in A.P.
(B) b, c, d are in G.P.
(C) b, c, d are in H.P.
(D) α is a root of $ax^3 - bx^2 - 3cx - d = 0$

8. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by

(A) x + 3
(B) $(1 + x)^2$
(C) x^2
(D) $x^2 + 1$

9. If A is a non–singular matrix and A^T denotes the transpose of A, then:

(A) $|A| \neq |A^T|$
(B) $|A. A^T| = |A|^2$
(C) $|A^T. A| = |A^T|^2$
(D) $|A|+|A^T| \neq 0$

10. Let $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$, then

(A) f(x) is independent of x

(B) $f'(\pi/2) = 0$

(C) $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$

(D) tangent to the curve y = f(x) at x = 0 is y = 0

11. Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$, then

(A) $1 - x^3$ is a factor of Δ

(B) $(1 - x^3)^2$ is factor of Δ

(C) $\Delta(x) = 0$ has 4 real roots

(D) $\Delta'(1) = 0$

12. Let $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then (where $f^n(x)$ denotes n^{th} derivative of f(x))

(A) f^n (1) is indepedent of a

(B) f^n (1) is indepedent of n

(C) f^n (1) depends on a and n

(D) $y = a(x - f^n(1))$ represents a straight line through the origin

13. If D is a determinant of order three and Δ is a determinant formed by the cofactors of determinant D ; then

(A) $\Delta = D^2$

(B) D = 0 implies $\Delta = 0$

(C) if D = 27, then Δ is perfect cube

(D) if D = 27, then Δ is perfect square

14. Let A, B, C, D be real matrices such that $A^T = BCD$; $B^T = CDA$; $C^T = DAB$ and $D^T = ABC$ for the matrix M = ABCD, then find M^{2016} ?

(A) M (B) M^2 (C) M^3 (D) M^4

15. Let **A** and **B** be two 2 × 2 matrix with real entries. If **AB** = **O** and tr(**A**) = tr(**B**) = 0 then

(A) **A** and **B** are comutative w.r.t. operation of multiplication.

(B) **A** and **B** are not commutative w.r.t. operation of multiplication.

(C) **A** and **B** are both null matrices.

(D) **BA** = 0

16. If $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, then

(A) | A | = 2

(B) A is non-singular

(C) Adj. A = $\begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$

(D) A is skew symmetric matrix

17. If A and B are square matrices of order 3, then the true statement is/are (where I is unit matrix).

(A) det (– A) = – det A

(B) If AB is singular then atleast one of A or B is singular

(C) det (A + I) = 1 + det A

(D) det (2A) = 2^3 det A

18. Let **M** be a 3 × 3 non-singular matrix with det(**M**) = 4. If $\mathbf{M}^{-1}$ adj(adj **M**) = $\mathbf{k}^2\mathbf{I}$, then the value of **'k'** may be

(A) +2 (B) 4 (C) –2 (D) –4

19. If AX = B where A is 3 × 3 and X and B are 3×1 matrices then which of the following is correct?
(A) If |A| = 0 then AX = B has infinite solutions
(B) If AX = B has infinite solutions then |A| = 0
(C) If (adj(A)) B = 0 and |A| ≠ 0 then AX = B has unique solution
(D) If (adj(A)) B ≠ 0 & |A| = 0 then AX = B has no solution

PART - IV : COMPREHENSION

Comprehension # 1

Let $\mathscr{A}$ be the set of all 3×3 symmetric matrices whose entries are 1,1,1,0,0,0,–1, –1, –1. B is one of the matrix in set $\mathscr{A}$ and

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad U = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Number of such matrices B in set $\mathscr{A}$ is λ, then λ lies in the interval
(A) (30, 40) (B) (38, 40) (C) (34, 38) (D) (25, 35)

2. Number of matrices B such that equation BX = U has infinite solutions
(A) is at least 6 (B) is not more than 10 (C) lie between 8 to 16 (D) is zero.

3. The equation BX = V
(A) is inconsistent for atleast 3 matrices B.
(B) is inconsistent for all matrices B.
(C) is inconsistent for at most 12 matrices B.
(D) has infinite number of solutions for at least 3 matrices B.

Comprehension # 2

Some special square matrices are defined as follows :

Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$. A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$.

Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A'A = I = AA'$.

4. If A and B are two square matrices such that AB = A & BA = B, then A & B are
(A) Idempotent matrices (B) Involutory matrices
(C) Orthogonal matrices (D) Nilpotent matrices

5. If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then

(A) $\alpha = \pm \dfrac{1}{\sqrt{2}}$ (B) $\beta = \pm \dfrac{1}{\sqrt{6}}$ (C) $\gamma = \pm \dfrac{1}{\sqrt{3}}$ (D) all of these

6. The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutory matrix
(C) nilpotent matrix (D) none of these

Exercise-3

✍ **Marked questions are recommended for Revision.**
* **Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

Comprehension # 1 (Q. No. 1 to Q. No. 3)

Let $\mathcal{A}$ be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

1.✍ The number of matrices in $\mathcal{A}$ is **[IIT-JEE 2009, Paper-1, (4, –1), 80]**
(A) 12 (B) 6 (C) 9 (D) 3

2.✍ The number of matrices A in $\mathcal{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, is **[IIT-JEE 2009, Paper-1, (4, –1), 80]**
(A) less than 4 (B) at least 4 but less than 7
(C) at least 7 but less than 10 (D) at least 10

3.✍ The number of matrices A in $\mathcal{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent, is
(A) 0 (B) more than 2 (C) 2 (D) 1
[IIT-JEE 2009, Paper-1, (4, –1), 80]

Comprehension # 2 (Q. No. 4 to 6)

Let p be an odd prime number and T_p be the following set of 2 × 2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, ..., p-1\} \right\}$$

4. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p - 1$

5. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[Note : The trace of matrix is the sum of its diagonal entries.] **[IIT-JEE 2010, Paper-1, (3, –1), 84]**
(A) $(p-1)(p^2-p+1)$ (B) $p^3-(p-1)^2$ (C) $(p-1)^2$ (D) $(p-1)(p^2-2)$

6. The number of A in T_p such that det (A) is not divisible by p is **[IIT-JEE 2010, Paper-1, (3, –1), 84]**
(A) $2p^2$ (B) p^3-5p (C) p^3-3p (D) p^3-p^2

7.✍ The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z)\cos 3\theta = (xyz)\sin 3\theta$$

$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is **[IIT-JEE 2010, Paper-1, (3, 0), 84]**

8. Let k be a positive real number and let **[IIT-JEE 2010, Paper-2, (3, 0), 79]**

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$$. If det (adj A) + det (adj B) = 10^6, then

[k] is equal to
(**Note** : adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

9. Let M and N be two 2n × 2n non-singular skew-symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ is equal to
(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

Comprehension # 3 (10 to 12) **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

Let a, b and c be three real numbers satisfying

$$[a\ b\ c] \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0] \qquad \text{...........(E)}$$

10. If the point P(a, b, c), with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a + b + c is
(A) 0 (B) 12 (C) 7 (D) 6

11. Let ω be a solution of $x^3 - 1 = 0$ with Im (ω) > 0. if a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to
(A) – 2 (B) 2 (C) 3 (D) – 3

12. Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is
(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

13. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form, $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is **[IIT-JEE 2011, Paper-2, (3, –1), 80]**
(A) 2 (B) 6 (C) 4 (D) 8

14. Let M be a 3 × 3 matrix satisfying

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$. Then the sum of the diagonal entries of M is

[IIT-JEE 2011, Paper-2, (4, 0), 80]

15. Let P = $[a_{ij}]$ be a 3 × 3 matrix and let Q = $[b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**
(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

16. If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

[IIT-JEE 2012, Paper-2, (3, –1), 66]

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D) PX = – X

17*. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

[IIT-JEE 2012, Paper-2, (4, 0), 66]

(A) –2 (B) –1 (C) 1 (D) 2

18.* For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

(A) $N^T M N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric
(B) M N – N M is skew symmetric for all symmetric matrices M and N
(C) M N is symetric for all symmetric matrices M and N
(D) (adj M) (adj N) = adj(MN) for all invertible matrices M and N

19*. Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) the first column of M is the transpose of the second row of M
(B) the second row of M is the transpose of first column of M
(C) M is a diagonal matrix with nonzero entries in the main diagonal
(D) the product of entries in the main diagonal of M is not the square of an integer

20*. Let M and N be two 3 × 3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) determinant of $(M^2 + MN^2)$ is 0
(B) there is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(C) determinant of $(M^2 + MN^2) \geq 1$
(D) for a 3 × 3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

21*. Let X and Y be two arbitrary, 3 × 3, non-zero, skew-symmetric matrices and Z be an arbitrary 3 × 3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ?

[JEE (Advanced) 2015, Paper-1 (4, –2)/88]

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

22*. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$?

[JEE (Advanced) 2015, Paper-1 (4, –2)/88]

(A) – 4 (B) 9 (C) – 9 (D) 4

23*. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that PQ = k I, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

[JEE (Advanced) 2016, Paper-1 (4, –2)/62]

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$ (C) $\det(P \text{ adj}(Q)) = 2^9$ (D) $\det(Q \text{ adj}(P)) = 2^{13}$

24. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

[JEE (Advanced) 2016, Paper-1 (3, 0)/62]

25. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\dfrac{q_{31}+q_{32}}{q_{21}}$ equals **[JEE (Advanced) 2016, Paper-2 (3, –1)/62]**

(A) 52 (B) 103 (C) 201 (D) 205

26*. Let $a, \lambda, \mu \in R$. Consider the system of linear equations

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

[JEE (Advanced) 2016, Paper-2 (4, –2)/62]

Which of the following statement(s) is(are) correct ?

(A) if $a = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $a \neq -3$, then the system has a a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, the the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

27*. Which of the following is(are) NOT the square of a 3×3 matrix with real entries?

[JEE(Advanced) 2017, Paper-1,(4, –2)/61]

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. For a real number α, if the system **[JEE(Advanced) 2017, Paper-1,(3, 0)/61]**

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

29. How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of $M^T M$ is 5 ? **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

(A) 198 (B) 162 (C) 126 (D) 135

30*. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_2 \in R$ and the system of equation (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced) 2018, Paper-2,(4, –2)/60]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

31. Let P be a matrix of order 3 × 3 such that all the entries in P are from the set {–1, 0, 1}. Then, the maximum possible value of the determinant of P is _____ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let A be a 2 × 2 matrix.
Statement-1 : adj(adj (A)) = A.
Statement-2 : |adj A| = |A| **[AIEEE 2009 (4, –1), 144]**
(1) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement-1 is false, Statement-2 is true.
(4) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

2. Let a, b, c be such that $b(a + c) \neq 0$. If $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0$. Then the value of 'n' is - **[AIEEE 2009 (4, –1), 144]**
(1) zero (2) any even integer (3) any odd integer (4) any integer

3. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is
(1) 5 (2) 6 (3) at least 7 (4) less than 4
[AIEEE 2010 (8, –2), 144]

4. Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix.
Tr(A) = sum of diagonal elements of A and |A| = determinant of matrix A. **[AIEEE 2010 (4, –1), 144]**
Statement -1 : Tr(A) = 0
Statement -2 : |A| = 1
(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

5. Consider the system of linear equations : **[AIEEE 2010 (4, –1), 144]**
$x_1 + 2x_2 + x_3 = 3$
$2x_1 + 3x_2 + x_3 = 3$
$3x_1 + 5x_2 + 2x_3 = 1$
The system has
(1) exactly 3 solutions (2) a unique solution (3) no solution (4) infinite number of solutions

6. Let A and B be two symmetric matrices of order 3. **[AIEEE 2011, I, (4, –1), 120]**
Statement-1 : A(BA) and (AB)A are symmetric matrices.
Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

7. The number of values of k for which the linear equations **[AIEEE 2011, I, (4, –1), 120]**
$4x + ky + 2z = 0$
$kx + 4y + z = 0$
$2x + 2y + z = 0$
posses a non-zero solution is :
(1) 3 (2) 2 (3) 1 (4) zero

8. If $\omega \neq 1$ is the complex cube root of unity and matrix H = $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then H^{70} is equal to - **[AIEEE 2011, I, (4, –1), 120]**

(1) 0 (2) – H (3) H^2 (4) H

9. If the trivial solution is the only solution of the system of equations **[AIEEE 2011, II, (4, –1), 120]**

$x - ky + z = 0$
$kx + 3y - kz = 0$
$3x + y - z = 0$

then the set of all values of k is :

(1) R – {2, –3} (2) R – { 2 } (3) R – { –3 } (4) {2, –3}

10. **Statement - 1 :** Determinant of a skew-symmetric matrix of order 3 is zero. **[AIEEE 2011, II, (4, –1), 120]**

Statement - 2 : For any matrix A, det $(A)^T$ = det(A) and det (–A) = – det(A).
Where det (B) denotes the determinant of matrix B. Then :

(1) Both statements are true (2) Both statements are false
(3) Statement-1 is false and statement-2 is true. (4) Statement-1 is true and statement-2 is false

11. Let A = $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012, (4, –1)/120]**

(1) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (4) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

12.✍ Let P and Q be 3 × 3 matrices P ≠ Q. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to : **[AIEEE-2012, (4, –1)/120]**

(1) – 2 (2) 1 (3) 0 (4) – 1

13. The number of values of k, for which the system of equations : **[AIEEE - 2013, (4, – 1) 120]**

$(k + 1)x + 8y = 4k$
$kx + (k + 3)y = 3k - 1$

has no solution, is

(1) infinite (2) 1 (3) 2 (4) 3

14. If P = $\begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3 × 3 matrix A and |A| = 4, then α is equal to : **[AIEEE - 2013, (4, – 1) 120]**

(1) 4 (2) 11 (3) 5 (4) 0

15.✍ If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to **[JEE(Main) 2014, (4, – 1), 120]**

(1) 1 (2) –1 (3) $\alpha\beta$ (4) $\dfrac{1}{\alpha\beta}$

16. If A is an 3 × 3 non-singular matrix such that AA′ = A′A and $B = A^{-1}A'$, then BB′ equals : **[JEE(Main) 2014, (4, – 1), 120]**

(1) B^{-1} (2) $(B^{-1})'$ (3) I + B (4) I

17. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to : **[JEE(Main) 2015, (4, – 1), 120]**
(1) (2, – 1) (2) (–2, 1) (3) (2, 1) (4) (–2, – 1)

18. The set of all value of λ for which the system of linear equations :
$2x_1 - 2x_2 + x_3 = \lambda x_1$
$2x_1 - 3x_2 + 2x_3 = \lambda x_2$
$-x_1 + 2x_2 = \lambda x_3$
has a non-trivial solution, **[JEE(Main) 2015, (4, –1), 120]**
(1) is an empty set (2) is a singleton
(3) contains two elements (4) contains more than two elements

19. The system of linear equations **[JEE(Main) 2016, (4, –1), 120]**
$x + \lambda y - z = 0$
$\lambda x - y - z = 0$
$x + y - \lambda z = 0$
has a non-trivial solution for :
(1) Exactly one value of λ. (2) Exactly two values of λ.
(3) Exactly three values of λ. (4) Infinitely many values of λ.

20. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj A = A A^T, then 5a + b is equal to **[JEE(Main) 2016, (4, –1), 120]**
(1) 5 (2) 4 (3) 13 (4) – 1

21. It S is the set of distinct values of 'b' for which the following system of linear equations
$x + y + z = 1$ **[JEE(Main) 2017, (4, –1), 120]**
$x + ay + z = 1$
$ax + by + z = 0$
has no solution, then S is :
(1) an empty set (2) an infinite set
(3) a finite set containing two or more elements (4) a singleton

22. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to **[JEE(Main) 2017, (4, –1), 120]**
(1) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (2) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (3) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (4) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

23. If the system of linear equations **[JEE(Main) 2018, (4, –1), 120]**
$x + ky + 3z = 0$
$3x + ky - 2z = 0$
$2x + 4y - 3z = 0$
has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to :
(1) – 30 (2) 30 (3) –10 (4) 10

24. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x-A)^2$ then the ordered pair (A, B) is equal to :
[JEE(Main) 2018, (4, –1), 120]
(1) (–4, 5) (2) (4, 5) (3) (–4, –5) (4) (–4, 3)

25. The system of linear equations **[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]**

$x + y + z = 2$

$2x + 3y + 2z = 5$

$2x + 3y + (a^2 + 1)z = a + 1$

(1) is inconsistent when a = 4
(2) has infinitely many solutions for a = 4
(3) is inconsistent when $|a| = \sqrt{3}$
(4) has a unique solution for $|a| = \sqrt{3}$

26. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to :

[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

(1) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
(2) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
(3) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
(4) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

27. Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin\theta - 2) \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is : **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

(1) –5
(2) $2(\sqrt{2} + 2)$
(3) $2(\sqrt{2} + 1)$
(4) –7

28. Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I_3$, then |p| is **[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]**

(1) $\frac{1}{\sqrt{5}}$
(2) $\frac{1}{\sqrt{3}}$
(3) $\frac{1}{\sqrt{6}}$
(4) $\frac{1}{\sqrt{2}}$

Answers

EXERCISE # 1

PART-I

Section (A) :

A-1. $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ **A-2.** (x, y, z, w) = (1, 2, 4, 5) **A-3.** $\begin{bmatrix} 2 & -1/2 \\ 4 & -1 \end{bmatrix}$

A-4. $AB = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$, $BA = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$ **A-7.** $y \in R$ **A-9.** Zero

Section (B) :

B-1. $0, \frac{3\pi}{4}, \pi$ **B-2.** (i) 0 (ii) 0 (iii) 0 (iv) $5(3\sqrt{2} - 5\sqrt{3})$

B-5. (i) $x = -2\,b/a$ (ii) 4 **B-8.** A = 0, B = 0

Section (C) :

C-1. $\begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$ **C-2.** (ii) $|A|^{(n-1)^3}$ **C-3.** $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ **C-5.** a = 1, c = – 1

Section (D) :

D-1. $a = -4, b = 1, A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ **D-2.** 0

D-4. (i) x = 3, y = 4, z = 6 (ii) $x = -\frac{5k}{3} + \frac{8}{3}$, $y = -\frac{2k}{3} - \frac{1}{3}$, $z = k$, where $k \in R$

D-5. x = – 7, y = – 4 **D-6.** for c = 0, x = – 3, y = 3 ; for c = – 10, $x = -\frac{1}{2}$, $y = \frac{4}{3}$

D-7. **(i)** x = 2, y = 2, z = 2 **(ii)** x = 1, y = 3, z = 5 **D-8.** **(a)** $\lambda \neq 3$ **(b)** $\lambda = 3, \mu = 10$ **(c)** $\lambda = 3, \mu \neq 10$

D-9. x = 3, y = – 2, z = – 1 **D-10.** x = 1, y = 2, z = 3, $A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix}$

D-11. S1, S3, S4

PART-II

Section (A) :

A-1. (A) **A-2.** (D) **A-3.** (A) **A-4.** (C) **A-5.** (B) **A-6.** (C) **A-7.** (A)
A-8. (A) **A-9.** (A)

Section (B) :

B-1. (B) **B-2.** (A) **B-3.** (A) **B-4.** (D) **B-5.** (D) **B-6.** (D) **B-7.** (B)
B-8. (C) **B-9.** (B) **B-10.** (B)

Section (C) :

C-1. (A) **C-2.** (C) **C-3.** (A) **C-4.** (A) **C-5.** (C) **C-6.** (A)

Section (D) :

D-1. (A) **D-2** (A) **D-3.** (D) **D-4.** (B) **D-5.** (B) **D-6.** (A)

PART-III

1.	(A) → (s),	(B) → (p),	(C) → (p),	(D) → (p)
2.	(A) → (q),	(B) → (p),	(C) → (s),	(D) → (q)

EXERCISE # 2

PART-I

1.	(D)	**2.**	(A)	**3.**	(D)	**4.**	(C)	**5.**	(D)	**6.**	(A)	**7.**	(C)
8.	(A)	**9.**	(A)	**10.**	(A)	**11.**	(A)	**12.**	(B)	**13.**	(A)	**14.**	(A)
15.	(A)	**6.**	(C)	**17.**	(B)	**18.**	(B)	**19.**	(A)	**20.**	(A)		

PART - II

1.	2	**2.**	64	**3.**	36	**4.**	2	**5.**	50	**6.**	4	**7.**	0
8.	63	**9.**	4	**10.**	6	**11.**	2	**12.**	1	**13.**	**2**	**14.**	33
15.	80	**16.**	$\lambda = 39$	**17.**	17	**18.**	8						

PART - III

1.	(AD)	**2.**	(BC)	**3.**	(ABCD)	**4.**	(ABD)	**5.**	(AB)	**6.**	(ABC)	**7.**	(BD)
8.	(AC)	**9.**	(BCD)	**10.**	(BCD)	**11.**	(ABD)	**12.**	(ABD)	**13.**	(ABCD)		
14.	(BD)	**15.**	(AD)	**16.**	(BC)	**17.**	(ABD)	**18.**	(AC)	**19.**	(BCD)		

PART - IV

1*.	(AC)	**2*.**	(AC)	**3*.**	(AC)	**4.**	(A)	**5.**	(D)	**6.**	(C)

EXERCISE # 3

PART-I

1.	(A)	**2.**	(B)	**3.**	(B)	**4.**	(D)	**5.**	(C)	**6.**	(D)	**7.**	3
8.	4	**9.**	(C)	**10.**	(D)	**11.**	(A)	**12 .**	(B)	**13.**	(A)	**14.**	**9**
15.	(D)	**16.**	(D)	**17*.**	(AD)	**18.***	(CD)	**19*.**	(CD)	**20*.**	(AB)	**21*.**	(CD)
22*.	(BC)	**23*.**	(BC)	**24.**	2	**25.**	(B)	**26*.**	(BCD)	**27*.**	(AC)	**28.**	1
29.	(A)	**30*.**	(AD)	**31.**	4								

PART - II

1.	(1)	**2.**	(3)	**3.**	(3)	**4.**	(2)	**5.**	(3)	**6.**	(2)	**7.**	(2)
8.	(4)	**9.**	(1)	**10.**	(4)	**11.**	(4)	**12.**	(3)	**13.**	(2)	**14.**	(2)
15.	(1)	**16.**	(4)	**17.**	(4)	**18.**	(3)	**19.**	(3)	**20.**	(1)	**21.**	(4)
22.	(2)	**23.**	(4)	**24.**	(1)	**25.**	(3)	**26.**	**(2)**	**27.**	(1)	**28.**	(4)

High Level Problems (HLP)

1. If $a^2 + b^2 + c^2 = 1$, then prove that $\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$

is independent of a, b, c

2. If a, b, c, x, y, z $\in$ R, then prove that,

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = . \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

3. If a_1, a_2, a_3 are distinct real roots of the equation $px^3 + px^2 + qx + r = 0$ such that $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = 0$,

then Prove that $\frac{r}{p} < 0$

4. Prove that $\Delta = \begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix} = (\alpha\beta' - \alpha'\beta)\,(\beta\gamma' - \beta'\gamma)\,(\gamma\alpha' - \gamma'\alpha)$

5. If

$$\begin{aligned} ax_1^2 + by_1^2 + cz_1^2 &= d \\ ax_2^2 + by_2^2 + cz_2^2 &= d \\ ax_3^2 + by_3^2 + cz_3^2 &= d \end{aligned} \quad \text{and} \quad \begin{aligned} ax_2x_3 + by_2y_3 + cz_2z_3 &= f \\ ax_3x_1 + by_3y_1 + cz_3z_1 &= f \\ ax_1x_2 + by_1y_2 + cz_1z_2 &= f \end{aligned}$$

then prove that $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = (d-f)^2 \dfrac{d\ + 2f}{abc}$, where a, b, c $\neq$ 0.

6. If

$$\begin{aligned} (x_1-x_2)^2 + (y_1-y_2)^2 &= a^2 \\ (x_2-x_3)^2 + (y_2-y_3)^2 &= b^2, \\ (x_3-x_1)^2 + (y_3-y_1)^2 &= c^2 \end{aligned}$$

prove that $4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a + b + c)\,(b + c - a)\,(c + a - b)(a + b - c)$.

7. Let $A = \begin{bmatrix} \cos^{-1}x & \cos^{-1}y & \cos^{-1}z \\ \cos^{-1}y & \cos^{-1}z & \cos^{-1}x \\ \cos^{-1}z & \cos^{-1}x & \cos^{-1}y \end{bmatrix}$ such that $|A| = 0$, then find the maximum value of x + y + z

8. If $y = \frac{u}{v}$, where u & v are functions of ' x ', show that, $v^3 \dfrac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$.

9. If α, β be the real roots of $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n$, then prove that $as_n + bs_{n-1} + cs_{n-2} = 0$ for all $n \geq 2$, $n \in N$. Hence or otherwise prove that $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} \geq 0$ for all real a, b, c.

10. Let a > 0, d > 0. Find the value of determinant $\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$

11. Let $\vec{a}_r = x_r\hat{i} + y_r\hat{j} + z_r\hat{k}$, r = 1, 2, 3 be three mutually perpendicular unit vectors, then find the value of $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

12. If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then find the value of k.

13. If the determinant $\begin{vmatrix} a+p & \ell+x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then find the value of K =

14. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then find the value of $abc(ab+bc+ca) - (a+b+c)$.

15. If a, b, c are complex numbers and $z = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix}$ then show that z is purely imaginary.

16. If $f(x) = \log_{10}x$ and $g(x) = e^{i\pi x}$ and $h(x) = \begin{vmatrix} f(x)g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$, then find the value of h(10).

17. If a, b, c, are real numbers, and $D = \begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$ then show that D is purely real.

18. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ then find $\lim\limits_{n\to\infty} \frac{1}{n}A^n$.

19. Let **P** = $\begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix}$ and α, β, γ be non-zero real numbers such that $\alpha p^6 + \beta p^3 + \gamma I$ is the zero matrix. Then find value of $(\alpha^2 + \beta^2 + \gamma^2)^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$

20. Consider an odd order square symmetric matrix A = $[a_{i\,j}]_{nxn}$. It's element in any row are 1, 2, ..., n in some order, then prove that a_{11}, a_{22}, ..., a_{nn} are numbers 1, 2, 3, ..., n in some order.

21. Let **A** = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; **B** = $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and **C** = 3**A** + 7**B**

Prove that

(i) $(\mathbf{A} + \mathbf{B})^{2013} = \mathbf{A}^{2013} + \mathbf{B}^{2013}$

(ii) Prove that $\mathbf{A}^n = 3^{n-1}\,\mathbf{A}$; $\mathbf{B}^n = 3^{n-1}\,\mathbf{B}$; $\mathbf{C}^n = 3^{2n-1}\,\mathbf{A} + \mathbf{7.21}^{n-1}\,\mathbf{B}$.

22. Let **'A'** is (4×4) matrix such that sum of elements in each row is 1. Find out sum of all the elements in $\mathbf{A}^{10}$.

23. Let A = $\begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$, then prove that A^{-1} exists if $3x + \lambda \neq 0, \lambda \neq 0$

24. Prove that if A and B are n × n matrices, then det $(I_n - AB)$ = det $(I_n - BA)$.

25. Let A be an n × n matrix such that $A^n = \alpha A$ where α is a real number different from 1 and – 1. Prove that the matrix $A + I_n$ is invertible.

26. Let p and q be real numbers such that $x^2 + px + q \neq 0$ for every real number x. Prove that if n is an odd positive integer, then $X^2 + pX + qI_n \neq 0_n$ for all real matrices X of order n × n.

27. Let **A, B, C** be three 3 × 3 matrices with real entries. If **BA** + **BC** + **AC** = I and det**(A + B)** = 0 then find the value of det**(A + B + C – BAC)**.

28. If $|z_1| = |z_2| = 1$, then prove that $\begin{bmatrix} z_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{z}_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

29. If A and B are two square matrices such that $B = -A^{-1}BA$, then show that $(A + B)^2 = A^2 + B^2$

30. If $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and AX = U has infinitely many solution. Prove that BX = V has no unique solution, also prove that if afd ≠ 0, then BX = V has no solution.

31. If the system of equations x = cy + bz, y = az + cx and z = bx + ay has a non-zero solution and at least one of a, b, c is a proper fraction, prove that $a^3 + b^3 + c^3 < 3$ and abc > – 1.

32. If D = diag $\{d_1, d_2, ..., d_n\}$, then prove that f(D) = diag $\{f(d_1), f(d_2), ..., f(d_n)\}$, where f(x) is a polynomial with scalar coefficient.

33. Given the matrix **A** = $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and **X** be the solution set of the equation A^x = **A**, where $x \in \mathbf{N} - \{1\}$. Evaluate $\Pi \dfrac{x^3+1}{x^3-1}$; where the continued product extends $\forall\ x \in \mathbf{X}$.

Comprehension (Q. NO. 34 to 36)

Any non-zero vector, X, is said to be characteristic vector of a matrix A, if there exist a number λ such that AX = λX. And then λ is said to be a charactristic root of the matrix A corresponding to the characteristic vector X and vice versa.
Also $AX = \lambda X \quad \Rightarrow \quad (A - \lambda I)X = 0$
Since $X \neq 0 \quad \Rightarrow \quad |A - \lambda I| = 0$
Thus every characteristic root λ of a matrix A is a root of its characteristic equation.

34. Prove that the two matrices A and P^{-1} AP have the same characteristic roots and hence show that square matrices AB & BA have same characteristic roots if at least one of them is invertible.

35. If q is a characteristic root of a non singular matrix A, then prove that $\dfrac{|A|}{q}$ is a characteristic root of a adj A.

36. Show that if $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are n characteristic roots of a square matrix A of order n, then the roots of the matrix A^2 be $\alpha_1^2, \alpha_2^2, \ldots\ldots\ldots \alpha_n^2$.

37. IF threre are three square matrices A, B, C of same order satisying the equation $A^3 = A^{-1}$ and let $B = -A^{3^n}$ and $C = A^{3^{(n+4)}}$ then prove that det(B + C) = 0, $n \in N$

38. If A is a non-singular matrix satisfying AB – BA = A, then prove that det.(B + I) = det.(B – I)

39. If rank is a number associated with a matrix which is the highest order of non-singular sub matrix then prove that

(i) Rank of the matrix A = $\begin{bmatrix} 1 & 3 & -2 \\ 4 & -1 & 0 \\ 2 & -7 & 4 \end{bmatrix}$ is 2

(ii) If the matrix A = $\begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix}$ has rank 3, then $y \neq -(a + b + c)$ and $y \neq 0$

(iii) If A & B are two square matrices of order 3 such that rank of matrix AB is two, then atleast one of A & B is singular.

HLP Answers

7. 3 **10.** $\dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$ **11.** ± 1 **12.** -1 **13.** 8

14. 0 **16.** 0 **18.** $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ **19.** 1 **22.** 4 **27.** 0

33. $\dfrac{3}{2}$