CONTENTS

RELATIONS, FUNCTIONS & ITF

JEE (ADVANCED) SYLLABUS

Functions : Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions. Even and odd functions, inverse of a function.

Inverse Trigonometric Functions : (principal value only).

Relation : Types of relations, equivalence relations.

JEE (MAIN) SYLLABUS

Functions : Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions. Even and odd functions, inverse of a function.

Inverse Trigonometric Functions : Inverse trigonometrical functions and their properties

Relation : Types of relations, equivalence relations.

RELATIONS, FUNCTIONS & ITF

RELATIONS

ORDERED PAIR :

A pair of objects listed in a specific order is called an ordered pair. It is written by listing the two objects in specific order separating them by a comma and then enclosing the pair in parentheses.

In the ordered pair (a, b), a is called the first element and b is called the second element.

Two ordered pairs are set to be equal if their corresponding elements are equal.

i.e. (a, b) = (c, d) if a = c and b = d.

CARTESIAN PRODUCT :

The set of all possible ordered pairs (a, b), where $a \in A$ and $b \in B$ i.e. $\{(a, b) ; a \in A \text{ and } b \in B\}$ is called the Cartesian product of A to B and is denoted by $A \times B$. Usually $A \times B \neq B \times A$.

Similarly $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$ is called ordered triplet.

RELATION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$. Thus, R is a relation from A to B $\Rightarrow$ $R \subset A \times B$. The subsets is derived by describing a relationship between the first element and the second element of ordered pairs in $A \times B$ e.g. if A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 2, 3, 4, 5} and $R = \{(a, b) : a = b^2, a \in A, b \in B\}$ then R = {(1, 1), (4, 2), (9, 3)}. Here a R b $\Rightarrow$ 1 R 1, 4 R 2, 9 R 3.

NOTE :

(i) Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So total number of subsets of $A \times B$ i.e. number of possible relations from A to B is 2^{mn}.

(ii) A relation R from A to A is called a relation on A.

DOMAIN AND RANGE OF A RELATION :

Let R be a relation from a set A to a set B. Then the set of all first components of coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components of coordinates of the ordered pairs in R is called the range of R.

Thus, $\text{Dom}(R) = \{a : (a, b) \in R\}$ and $\text{Range}(R) = \{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Example # 1 : If A = {1, 2} and B = {3, 4}, then find $A \times B$.

Solution : $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Example # 2 : Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Rightarrow x > y$". Find relation R and its domain and range.

Solution : Under relation R, we have 3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and range (R) = {2, 4, 6}

Example # 3 : Let A = {2, 3, 4, 5, 6, 7, 8, 9}. Let R be the relation on A defined by

$\{(x, y) : x \in A, y \in A \text{ \& } x^2 = y \text{ or } x = y^2\}$. Find domain and range of R.

Solution : The relation R is

R = {(2, 4), (3, 9), (4, 2), (9, 3)}

Domain of R = {2, 3, 4, 9}

Range of R = {2, 3, 4, 9}

Self Practice Problem :

(1) If $(2x + y, 7) = (5, y - 3)$ then find x and y.

(2) If $A \times B = \{(1, 2), (1, 3), (1, 6), (7, 2), (7, 3), (7, 6)\}$ then find sets A and B.

(3) If $A = \{x, y, z\}$ and $B = \{1, 2\}$ then find number of relations from A to B.

(4) Write $R = \{(4x + 3, 1 - x) : x \leq 2, x \in N\}$

Answers

(1)	$x = -\frac{5}{2}$, $y = 10$	(2)	$A = \{1, 7\}$, $B = \{2, 3, 6\}$
(3)	64	(4)	$\{(7, 0), (11, -1)\}$

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

(i) **Void relation :** Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(ii) **Universal relation :** Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(iii) **Identity relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

(iv) **Reflexive relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Note : Every identity relation is reflexive but every reflexive relation in not identity.

(v) **Symmetric relation :** A relation R on a set A is said to be a symmetric relation

iff $(a, b) \in R \Rightarrow (b ,a) \in R$ for all $a, b \in A$. i.e. $a R b \Rightarrow b R a$ for all $a, b \in A$.

(vi) **Transitive relation :** Let A be any set. A relation R on A is said to be a transitive relation

iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a R b$ and $b R c \Rightarrow a R c$ for all a, $b, c \in A$

(vii) **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b \in A$

Example # 4 : Which of the following are identity relations on set A = {1, 2, 3}.
$R_1 = \{(1, 1), (2, 2)\}$, $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$, $R_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Solution: The relation R_3 is identity relation on set A.
R_1 is not identity relation on set A as $(3, 3) \notin R_1$.
R_2 is not identity relation on set A as $(1, 3) \in R_2$

Example # 5 : Which of the following are reflexive relations on set A = {1, 2, 3}.
$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$, $R_2 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$..

Solution : R_1 is a reflexive relation on set A.
R_2 is not a reflexive relation on A because $2 \in A$ but $(2, 2) \notin R_2$.

Example # 6 : Prove that on the set N of natural numbers, the relation R defined by x R y $\Rightarrow$ x is less than y is transitive.

Solution : Because for any $x, y, z \in N$ $x < y$ and $y < z \Rightarrow x < z \Rightarrow$ x R y and y R z $\Rightarrow$ x R z. so R is transitive.

Example # 7 : Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution : Since a relation R in T is said to be an equivalence relation if R is reflexive, symmetric and transitive.
(i) Since every triangle is congruent to itself
$\therefore$ R is reflexive
(ii) $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2 $\Rightarrow$ T_2 is congruent to T_1 $\Rightarrow$ $(T_2, T_1) \in R$
Hence R is symmetric
(iii) Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$ $\Rightarrow$ T_1 is congruent to T_2 and T_2 is congruent to T_3
$\Rightarrow T_1$ is congruent to T_3 $\Rightarrow$ $(T_1, T_3) \in R$
$\therefore$ R is transitive
Hence R is an equivalence relation.

Example # 8 : Show that the relation R in R defined as $R = \{(a, b) : a \le b\}$ is transitive.

Solution : Let $(a, b) \in R$ and $(b, c) \in R$
$\therefore$ $(a \le b)$ and $b \le c$ $\Rightarrow$ $a \le c$ $\therefore$ $(a, c) \in R$ Hence R is transitive.

Example # 9 : Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric.

Solution : Let $(a, b) \in R$ $[\because (1, 2) \in R]$
$\therefore$ $(b, a) \in R$ $[\because (2, 1) \in R]$
Hence R is symmetric.

Self Practice Problem :

(5) Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R$ $\Rightarrow$ x is perpendicular to y. Then prove that R is a symmetric relation on L.

(6) Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Rightarrow$ line ℓ_1 is parallel to line ℓ_2. Prove that R is an equivalence relation.

FUNCTION

Definition :

Function is a rule (or correspondence), from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: A $\to$ B. We read it as "f is a function from A to B".

For example, let A $\equiv$ {–1, 0, 1} and B $\equiv$ {0, 1, 2}.

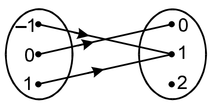
Then A × B $\equiv$ {(–1, 0), (–1, 1), (–1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)}

Now, " f : A $\to$ B defined by f(x) = x^2 " is the function such that

f $\equiv$ {(–1, 1), (0, 0), (1, 1)}

f can also be shown diagramatically by following mapping.

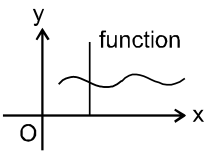
A B



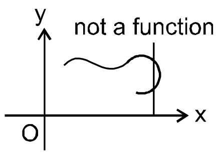
Note : Every function say y = f(x) : A $\to$ B. Here x is independent variable which takes its values from A while 'y' takes its value from B. A relation will be a function if and only if

(i) x must be able to take each and every value of A and

(ii) one value of x must be related to one and only one value of y in set B.



(a)



(b)

Graphically : If any vertical line cuts the graph at more than one point, then the graph does not represent a function.

Example # 10 : (i) Which of the following correspondences can be called a function ?

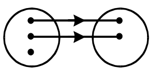
(A) $f(x) = x^3$; {–1, 0, 1} $\to$ {0, 1, 2, 3}

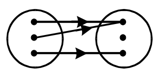
(B) $f(x) = \pm\sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

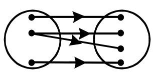
(C) $f(x) = \sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

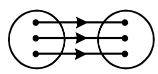
(D) $f(x) = -\sqrt{x}$; {0, 1, 4} $\to$ {–2, –1, 0, 1, 2}

(ii) Which of the following pictorial diagrams represent the function

(A) 

(B) 

(C) 

(D) 

Solution :

(i) f(x) in (C) and (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as f(–1) $\notin$ 2nd set. Hence definition of function is not satisfied. While in case of (B), the given relation is not a function, as f(1) = ± 1 and f(4) = ± 2 i.e. element 1 as well as 4 in 1st set is related with two elements of 2nd set.Hence definition of function is not satisfied.

(ii) B and D. In (A) one element of domain has no image, while in (C) one element of 1st set has two images in 2nd set

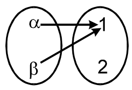
Self practice problem :

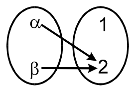
(7) Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and (x,g(x)) is $\sqrt{3}/4$ sq. unit, then the function g(x) may be.

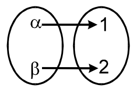
(A) $g(x) = \pm\sqrt{(1-x^2)}$ (B) $g(x) = \sqrt{(1-x^2)}$ (C) $g(x) = -\sqrt{(1-x^2)}$ (D) $g(x) = \sqrt{(1+x^2)}$

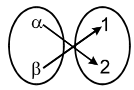
(8) Represent all possible functions defined from {α, β} to {1, 2}.

Answers : (7) B, C

(8) (i) 

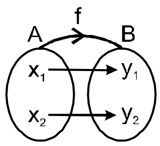
(ii) 

(iii) 

(iv) 

Domain, Co-domain and Range of a Function :

Let y = f(x) : A → B, then the set A is known as the domain of f and the set B is known as co-domain of f.



If x_1 is mapped to y_1, then y_1 is called as image of x_1 under f. Further x_1 is a pre-image of y_1 under f.
If only expression of f (x) is given (domain and co-domain are not mentioned), then domain is **complete** set of those values of x for which f (x) is real, while codomain is considered to be (– ∞, ∞) (except in inverse trigonometric functions).
Range is the complete set of values that y takes. Clearly range is a subset of Co-domain.
A function whose domain and range are both subsets of real numbers is called **a real function**.

Example # 11 : Find the domain of following functions :

(i) $f(x) = \sqrt{x^2 - 5}$ (ii) $\sin(x^3 - x)$

Solution : (i) $f(x) = \sqrt{x^2 - 5}$ is real iff $x^2 - 5 \geq 0$

$\Rightarrow$ $|x| \geq \sqrt{5}$ $\Rightarrow$ $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}\,] \cup [\,\sqrt{5}, \infty)$

(ii) $x^3 - x \in R$ $\therefore$ domain is $x \in R$

Algebraic Operations on Functions :

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in $A \cap B$. Now we define f + g, f – g, (f . g) and (f /g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f.g)(x) = f(x).\, g(x)$
— domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ such that $g(x) \neq 0\}$.

Note : ☞ For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are f(x) > 0 and g(x) must be real.
☞ For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conventional conditions of domain are $f(x) \geq g(x)$ and $f(x) \in N$ and $g(x) \in W$.

Example # 12 : Find the domain of function $f(x) = \dfrac{3}{\sqrt{4-x^2}}\log(x^3 - x)$

Solution : Domain of $\sqrt{4-x^2}$ is [–2, 2] but $\sqrt{4-x^2}$ = 0 for x = ± 2 ⇒ $x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x - 1)(x + 1) > 0$.

∴ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is $\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) \equiv (-1, 0) \cup (1, 2)$.

Self practice problems :

(9) Find the domain of following functions.

(i) $f(x) = \dfrac{1}{\log(2-x)} + \sqrt{x+1}$ (ii) $f(x) = \sqrt{1-x} - \sin\dfrac{2x-1}{3}$

Answers : (i) $[-1, 1) \cup (1, 2)$ (ii) $[-1, 1]$

Methods of determining range :

(i) **Representing x in terms of y**

If y = f(x), try to express as x = g(y), then domain of g(y) represents possible values of y, which is range of f(x).

(ii) **Graphical Method :**

The set of y– coordinates of the graph of a function is the range.

Example # 13 : Find the range of $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$

Solution : $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$y = \dfrac{x^2+x+1}{x^2+x-1}$

⇒ $yx^2 + yx - y = x^2 + x + 1$

⇒ $(y - 1)x^2 + (y - 1)x - y - 1 = 0$

If y = 1, then the above equation reduces to –2 = 0. Which is not true.

Further if $y \neq 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if

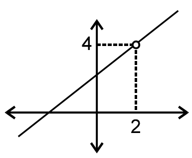
$(y - 1)^2 - 4(y - 1)(-y - 1) \geq 0$

i.e. if $y \leq -3/5$ or $y \geq 1$ but $y \neq 1$

Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

Example # 14 : Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

Solution :

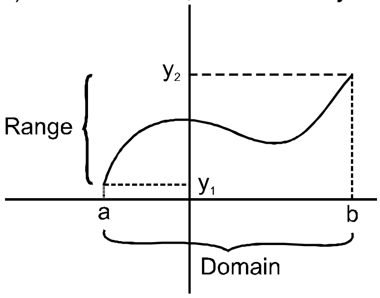


$f(x) = \dfrac{x^2-4}{x-2} = x + 2;\ x \neq 2$

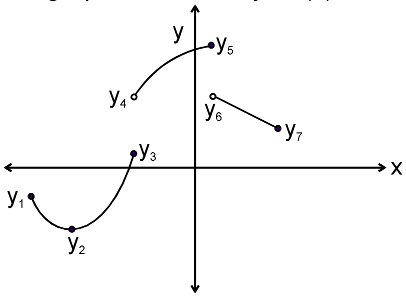
∴ graph of f(x) would be

Thus the range of f(x) is $R - \{4\}$

Further if f(x) happens to be continuous in its domain then range of f(x) is [min f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Example # 15 : Let graph of function y = f(x) is



Then range of above sectionally continuous function is $[y_2, y_3] \cup [y_7, y_6) \cup (y_4, y_5]$

(iii) **Using monotonocity :** Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common function which are increasing or decreasing in the interval where they are **continuous** is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x$, $a > 1$	$\log_a x$, $0 < a < 1$
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\text{cosec}^{-1} x$

For monotonic increasing functions in [a, b]
(i) $f'(x) \geq 0$ (ii) range is [f(a), f(b)]
for monotonic decreasing functions in [a, b]
(i) $f'(x) \leq 0$ (ii) range is [f(b), f(a)]

Example # 16 : Find the range of function $y = \ell n\,(2x - x^2)$

Solution : **Step – 1**
We have $2x - x^2 \in (-\infty, 1]$
Step – 2 Let $t = 2x - x^2$
For ℓnt to be defined accepted values are (0, 1]
Now, using monotonocity of ℓn t,
$\ell n\,(2x - x^2) \in (-\infty, 0]$
$\therefore$ range is $(-\infty, 0]$ **Ans.**

Self practice problems :

(10) Find domain and range of following functions.

(i) $y = x^3$ (ii) $y = \dfrac{x^2 - 2x + 5}{x^2 + 2x + 5}$ (iii) $y = \dfrac{1}{\sqrt{x^2 - x}}$

Answers : (i) domain R; range R (ii) domain R ; range $\left[\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right]$

(iii) domain R – [0, 1] ; range $(0, \infty)$

Classification of Functions :

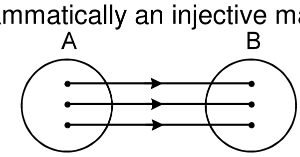
Functions can be classified as "One – One Function (Injective Mapping)" and "Many – One Function" :

One - One Function :

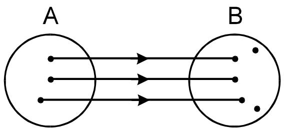
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



OR

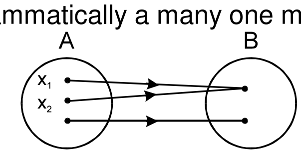


Many - One function :

A function $f : A \to B$ is said to be a many one function if there exist at least two or more elements of A having the same f image in B.

Thus $f : A \to B$ is many one iff there exist atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



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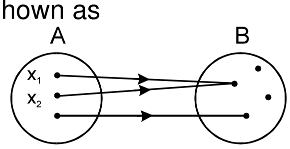


Note : ☞ If a function is one–one, it cannot be many–one and vice versa.

Methods of determining whether a given function is ONE-ONE or MANY-ONE :

(a) If $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE.

(b) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE- ONE.

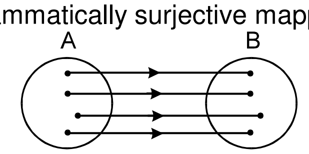
(c) If either $f'(x) \geq 0$, $\forall\ x \in$ domain or $f'(x) \leq 0\ \forall\ x \in$ domain, where equality can hold at discrete point(s) only i.e. strictly monotonic, then function is ONE-ONE, otherwise MANY-ONE.

Note : If f and g both are one-one, then gof and fog would also be one-one (if they exist). Functions can also be classified as "Onto function (Surjective mapping)" and "Into function":

Onto function :

If the function $f : A \to B$ is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall\ b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as



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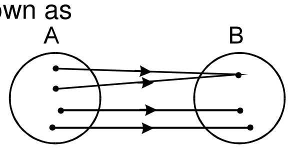
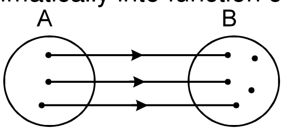


Into function :

If f : A $\rightarrow$ B is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



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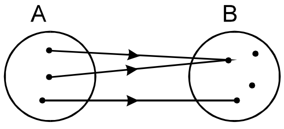
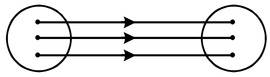


Note : (i) If range $\equiv$ co–domain, then f(x) is onto, otherwise into

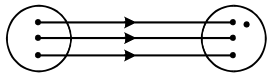
(ii) If a function is onto, it cannot be into and vice versa.

A function can be one of these four types:

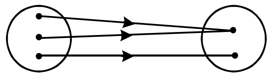
(a) one–one onto (injective and surjective)



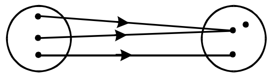
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective and surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements, then the number of different functions defined from
A $\rightarrow$ A is n^n and out of which n! are one one.

(iii) If f and g both are onto, then gof or fog may or may not be onto.

(iv) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g**.

Example # 17 : (i) Find whether f(x) = x + cos x is one-one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for f : R $\rightarrow$ R is ONTO or INTO

(iii) $f(x) = x^2 - 2x$; [0, 3] $\rightarrow$ A. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

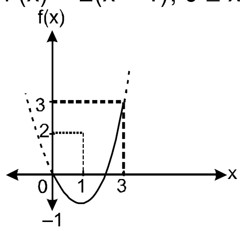
Solution : (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

$\therefore$ $f'(x) \geq 0 \ \forall \ x \in$ complete domain and equality holds at discrete points only

$\therefore$ f(x) is strictly increasing on R. Hence f(x) is one-one.

(ii) As range $\equiv$ codomain, therefore given function is ONTO

(iii) $f'(x) = 2(x - 1)$; $0 \leq x \leq 3$



$\therefore$ $f'(x) = \begin{cases} -ve & ; \ 0 \leq x < 1 \\ +ve & ; \ 1 < x < 3 \end{cases}$

$\therefore$ f(x) is non monotonic. Hence it is not injective.

For f(x) to be surjective, A should be equal to its range. By graph range is [–1, 3]

$\therefore$ A $\equiv$ [–1, 3]

Self practice problems :

(11) For each of the following functions find whether it is one-one or many-one and also into or onto

(i) $f(x) = 2\tan x$; $(\pi/2, 3\pi/2) \to R$ (ii) $f(x) = \dfrac{1}{1+x^2}$; $(-\infty, 0) \to R$

(iii) $f(x) = x^2 + \ell n\, x$

Answers : (i) one-one onto (ii) one-one into (iii) one-one onto

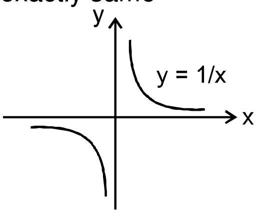
Equal or Identical Functions :

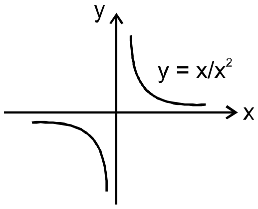
Two functions f and g are said to be identical (or equal) iff :

(i) The domain of f $\equiv$ the domain of g.

(ii) f(x) = g(x), for every x belonging to their common domain.

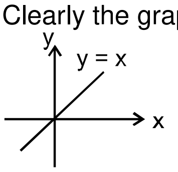
e.g. $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x^2}$ are identical functions. Clearly the graphs of f(x) and g(x) are exactly same

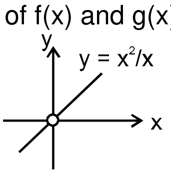




But $f(x) = x$ and $g(x) = \dfrac{x^2}{x}$ are not identical functions.

Clearly the graphs of f(x) and g(x) are different at x = 0.





Example # 18 : Examine whether following pair of functions are identical or not ?

(i) $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = \sec^2 x - \tan^2 x$

Solution : (i) No, as domain of f(x) is R – {1} while domain of g(x) is R

(ii) No, as domain are not same. Domain of f(x) is R while that of g(x) is $R - \left\{(2n+1)\dfrac{\pi}{2};\ n \in I\right\}$

Self practice problems

(12) Examine whether the following pair of functions are identical or not :

(i) $f(x) = \text{sgn}(x)$ and $g(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f(x) = \text{cosec}^2 x - \cot^2 x$ and $g(x) = 1$

Answers : (i) Yes (ii) No

Composite Function :

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and D is the set of values of x such that if $x \in X$, then $f(x) \in Y_2$. If $D \neq \phi$, then the function h defined on D by h(x) = g{f(x)} is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : ☞ Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Pictorially gof(x) can be viewed as under $\longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$

Note that gof(x) exists only for those x when range of f(x) is a subset of domain of g(x).

Properties of Composite Functions :

(a) In general gof $\neq$ fog (i.e. not commutative)

(b) The composition of functions are associative i.e. if three functions f, g, h are such that fo (goh) and (fog) oh are defined, then fo (goh) = (fog) oh.

Example # 19 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$ (ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Solution : (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

<u>For gof(x)</u>

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$ {equal to the domain of f }

gof (x) = g{f(x)} = g ($\sqrt{x+3}$) = 1 + (x+3) = x + 4. Range of gof is $[1, \infty)$.

<u>For fog(x)</u>

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

fog (x) = f{g(x)}= f(1+ x^2) = $\sqrt{x^2+4}$ Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

For gof(x)

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and g{f(x)}= g($\sqrt{x}$) = x – 1. Range of gof is $[-1, \infty)$

For fog(x)

Since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is {$x \in R$, the domain of g : $g(x) \in [0, \infty)$, the domain of f}.

Thus the domain of fog is D = {$x \in R$: $0 \leq g(x) < \infty$}

i.e. D = { $x \in R$: $0 \leq x^2 - 1$}= { $x \in R$: $x \leq -1$ or $x \geq 1$ }= $(-\infty, -1] \cup [1, \infty)$

fog (x) = f{g(x)} = f(x^2–1) = $\sqrt{x^2-1}$ Its range is $[0, \infty)$.

Example # 20 : Let $f(x) = e^x$; $R^+ \to R$ and g(x) = sinx ; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Find domain and range of fog(x)

Solution : Domain of f(x) : $(0, \infty)$ Range of g(x) : $[-1, 1]$

values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$

$\Rightarrow$ $0 < g(x) \leq 1$ $\Rightarrow$ $0 < \sin x \leq 1$ $\Rightarrow$ $0 < x \leq \frac{\pi}{2}$

Hence domain of fog(x) is $x \in (0, \frac{\pi}{2}]$

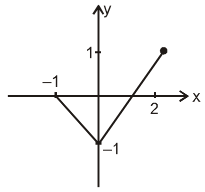
Therefore Domain : $(0, \frac{\pi}{2}]$

Range : (1, e]

Example # 21 : If $f(x) = -1 + |x-2|,\ 0 \le x \le 4$
$g(x) = 2 - |x|,\ -1 \le x \le 3$
Then find fog (x) and gof (x). Also draw their rough sketch.

Solution : $fog(x) = \{-1 + |g(x) - 2|,\ 0 \le g(x) \le 4,\ -1 \le x \le 3$
$= \{-1 + |2 - |x| - 2|,\ 0 \le 2 - |x| \le 4,\ -1 \le x \le 3$
$= \{-1 + |x|,\ -2 \le x \le 2,\ -1 \le x \le 3$

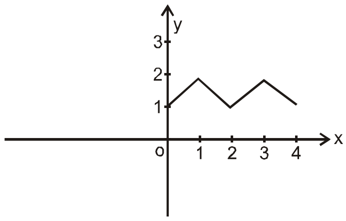
graph of fog(x) is



$$= \begin{cases} -(1+x) & , \ -1 \le x \le 0 \\ x-1 & , \ 0 < x \le 2 \end{cases} \quad ;$$

$gof(x) = \{2 - |f(x)|,\ -1 \le f(x) \le 3,\ 0 \le x \le 4$
$= \{2 - |-1 + |x-2||,\ -1 \le -1 + |x-2| \le 3,\ 0 \le x \le 4$
$= \{2 - |-1 + |x-2||,\ -2 \le x \le 6,\ 0 \le x \le 4$

graph of gof(x) is



$$= \begin{cases} x+1 & , \ 0 \le x < 1 \\ 3-x & , \ 1 \le x \le 2 \\ x-1 & , \ 2 < x \le 3 \\ 5-x & , \ 3 < x \le 4 \end{cases} ;$$

Self practice problems

(13) Define fog(x) and gof(x). Also find their domain and range.
(i) $f(x) = [x]$, $g(x) = \sin x$
(ii) $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$; $g(x) = \sqrt{1-x^2}$

(14) Let $f(x) = e^x : R^+ \to R$ and $g(x) = x^2 - x : R \to R$. Find domain and range of fog (x) and gof (x)

Answers :

(13) (i) gof = sin [x] domain : R range { sin a : a ∈ I}
fog = [sin x] domain : R range : {–1, 0, 1}
(ii) $gof \equiv \sqrt{1-\tan^2 x}$, domain : $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ range : [0, 1]
$fog \equiv \tan \sqrt{1-x^2}$ domain : [–1, 1] range [0, tan 1]

(14) fog (x)
Domain : $(-\infty, 0) \cup (1, \infty)$
Range : $(1, \infty)$

gof (x)
Domain : $(0, \infty)$
Range : $(0, \infty)$

Odd and Even Functions :

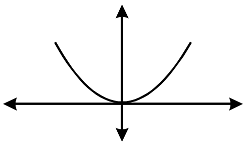
(i) If f (–x) = f (x) for all x in the domain of 'f', then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.
(ii) If f (–x) = –f (x) for all x in the domain of 'f', then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

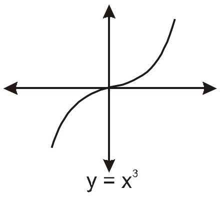
Note : (i) A function may neither be odd nor even. (e.g. $f(x) = e^x$, $\cos^{-1}x$)
(ii) If an odd function is defined at x = 0, then f(0) = 0

Properties of Even/Odd Function

(a) The graph of every even function is symmetric about the y–axis and that of every odd function is symmetric about the origin.

For example graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin


$y = x^2$


$y = x^3$

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even and an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

(c) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(d) If f and g both are even or both are odd, then the function f.g will be even but if any one of them is odd and the other even then f.g will be odd.

(e) If f(x) is even then f′(x) is odd while derivative of odd function is even. Note that same cannot be said for integral of functions.

Example # 22 : Show that $a^x + a^{-x}$ is an even function.

Solution : Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$. Hence f(x) is an even function

Example # 23 : Prove that $f(x) = x\left(\frac{x}{e^x - 1} + \frac{x}{2}\right)$ is odd function

Solution : Let $g(x) = \left(\frac{x}{e^x - 1} + \frac{x}{2}\right)$ then $g(-x) = \left(\frac{-x}{e^{-x} - 1} + \frac{-x}{2}\right) = \left(\frac{x}{e^x - 1} + \frac{x}{2}\right)$

$\Rightarrow$ g(x) is even

hence $f(x) = x.g(x) = x\left(\frac{x}{e^x - 1} + \frac{x}{2}\right)$ is odd function.

Self practice problems

(15) Determine whether the following functions are even / odd / neither even nor odd?

(i) $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(ii) $f : [-2, 3] \to [0, 9]$, $f(x) = x^2$

(iii) $f(x) = x \log\left(x + \sqrt{x^2 + 1}\right)$

Answers (i) Odd (ii) neither even nor odd (iii) Even

Periodic Functions :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

Note : Inverse of a periodic function does not exist.

Properties of Periodic Functions :

(a) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T, then f (ax + b) has a period . $\frac{T}{|a|}$

(c) Every constant function defined for all real x, is always periodic, with no fundamental period.

(d) If f (x) has a period T_1 and g (x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M.}(a,\ p,\ \ell)}{\text{H.C.F.}(b,\ q,\ m)}$

e.g. |sinx| has the period π, | cosx | also has the period π

$\therefore$ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\frac{\pi}{2}$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Example # 24 : Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) f(x) = {x} + sin x, where {.}denotes fractional part function

(iii) $f(x) = 4\cos x \cdot \cos 3x + 2$ (iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Solution : (i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π . Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π {L.C.M. of 4 and 6 is 12}

(ii) Period of sin x $= 2\pi$
Period of {x} = 1
but L.C.M. of 2π and 1 is not possible as their ratio is irrational number
$\therefore$ it is aperiodic

(iii) $f(x) = 4\cos x \cdot \cos 3x + 2$

period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \frac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period

$f(\pi + x) = 4(-\cos x)(-\cos 3x) + 2 = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2} = 12\pi$

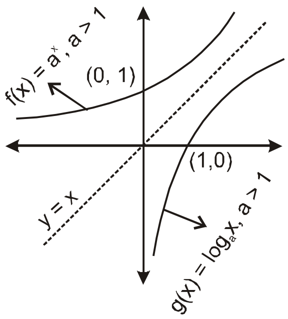
Inverse of a Function :

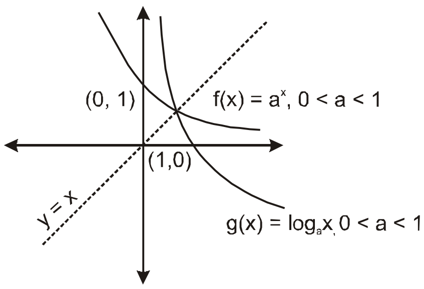
Let y = f(x) : A $\rightarrow$ B be a one-one and onto function. i.e. bijection, then there will always exist bijective function x = g(y) : B $\rightarrow$ A such that if (p, q) is an element of f, (q, p) will be an element of g and the functions f(x) and g(x) are said to be inverse of each other. g(x) is also denoted by $f^{-1}(x)$ and f(x) is denoted by $g^{-1}(x)$

Note : (i) The inverse of a bijection is unique.
(ii) Inverse of an even function is not defined.

Properties of Inverse Function :

(a) The graphs of f and g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.





(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g f(x) = 1/x

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverse of each other. e.g. f(x) = x + 1, g(x) = x + 2. However if fog(x) = gof(x) = x, then $g(x) = f^{-1}(x)$

(d) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(e) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \dfrac{1}{g'(x)}$

Example # 25 : (i) Determine whether $f(x) = \dfrac{2x+3}{4}$ for $f : R \to R$, is invertible or not? If so find it.

(ii) Let $f(x) = x^2 + 2x;\ x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

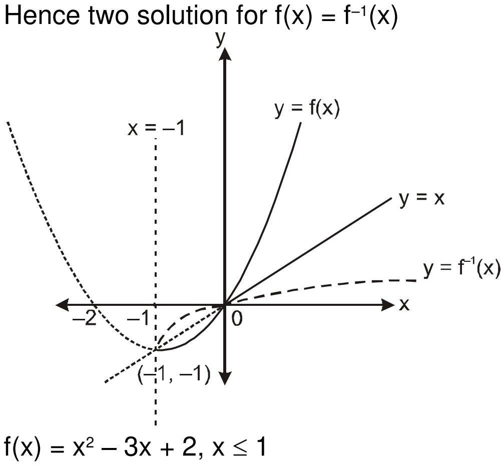
(iii) If $y = f(x) = x^2 - 3x + 2,\ x \leq 1$. Find the value of $g'(2)$where g is inverse of f

Solution : (i) Given function is one-one and onto, therefore it is invertible.

$$y = \frac{2x+3}{4} \quad \Rightarrow \quad x = \frac{4y-3}{2} \qquad \therefore \quad f^{-1}(x) = \frac{4x-3}{2}$$

(ii) $f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$ $\Rightarrow$ $x^2 + 2x = x \Rightarrow x(x+1) = 0 \Rightarrow x = 0, -1$

Hence two solution for $f(x) = f^{-1}(x)$



(iii) $f(x) = x^2 - 3x + 2,\ x \leq 1$

$f(g(x) = g(x)^2 - 3\,g(x) + 2$

$\Rightarrow \quad 2 = g(2)^2 - 3g(2) + 2$

$\Rightarrow \quad g(2) = 0, 3 \leq 1$

so g(2) = 0

$f'(x) = 2x - 3$

$$f(g(x) = x \Rightarrow f'(g(x))\ .\ g'(x) = 1 \Rightarrow g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(0)} = -\frac{1}{3}$$

Self practice problems :

(16) Determine $f^{-1}(x)$, if given function is invertible

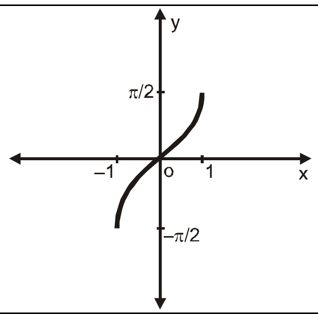
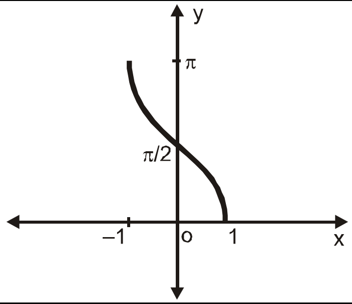
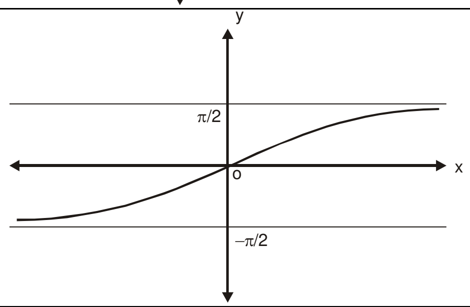
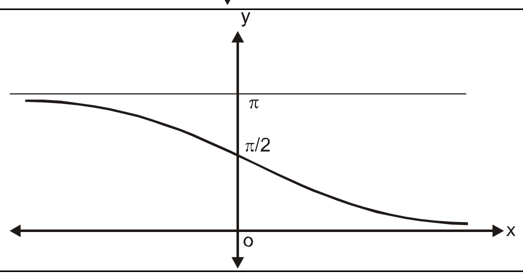
$f : (-\infty, -1) \rightarrow (-\infty, -2)$ defined by $f(x) = -(x + 1)^2 - 2$

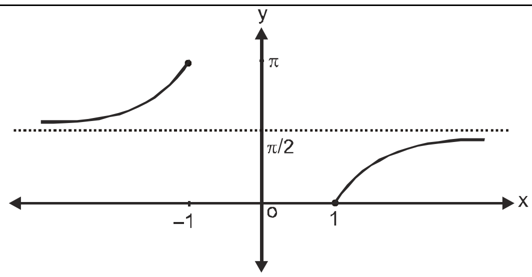
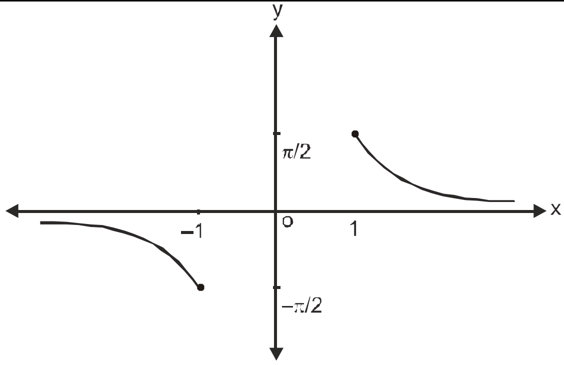
Answers : $-1 - \sqrt{-x-2}$

Inverse Trigonometry Functions

<u>Introduction</u> : The student may be familiar about trigonometric functions viz sin x, cos x, tan x, cosec x, sec x, cot x with respective domains R, R, R – {(2n + 1) π/2}, R – {nπ}, R – {(2n + 1) π/2}, R – {nπ} and respective ranges [–1, 1], [–1, 1], R, R – (–1, 1), R – (–1, 1), R.

Correspondingly, six inverse trigonometric functions (also called inverse circular functions) are defined.

<table>
<tr><td>Inverse Trigonometric Function</td><td>Domain</td><td>Range</td><td>Graph</td></tr>
<tr><td>$f(x) = \sin^{-1}x$ or arcsinx</td><td>[–1, 1]</td><td>[–π/2, π/2]</td><td>
</td></tr>
<tr><td>$f(x) = \cos^{-1}x$ or arccosx</td><td>[–1, 1]</td><td>[0, π]</td><td>
</td></tr>
<tr><td>$f(x) = \tan^{-1}x$ or arctanx</td><td>R</td><td>(–π/2, π/2)</td><td>
</td></tr>
<tr><td>$f(x) = \cot^{-1}x$ or arccotx</td><td>R</td><td>(0, π)</td><td>
</td></tr>
</table>

$f(x) = \sec^{-1}x$ or arcsecx	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$	
$f(x) = \text{cosec}^{-1}x$ or arccosecx	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	

Example # 26 : Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

Solution : $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \mathbf{\frac{1}{\sqrt{3}}}$.

Example # 27 : Find domain of $\sin^{-1}(2x^2 + 1)$

Solution : Let $y = \sin^{-1}(2x^2 + 1)$

For y to be defined $-1 \le (2x^2 + 1) \le 1 \Rightarrow -2 \le 2x^2 \le 0 \Rightarrow x \in \{0\}$

Self practice problems :

(17) Find the value of (i) $\cos\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

(ii) $\text{cosec}\,[\sec^{-1}(\sqrt{2}) + \cot^{-1}(1)]$

(18) Find the domain of

(i) $y = \sec^{-1}(x^2 + 3x + 1)$

(ii) $y = \sin^{-1}\left(\frac{x^2}{1+x^2}\right)$

(iii) $y = \cot^{-1}(\sqrt{x^2 - 1})$

(19) Find the range of (i) $\sin^{-1}|x| + \sec^{-1}|x|$

(ii) $\sin^{-1}\sqrt{x^2 + x + 1}$

Answers :

(17) (i) 0 (ii) 1

(18) (i) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$ (ii) R

(iii) $(-\infty, -1] \cup [1, \infty)$

(19) (i) $\{\pi/2\}$ (ii) $[\pi/3, \pi/2]$

Property 1 : T(T^{-1})

(i) $\sin(\sin^{-1} x) = x, \quad -1 \le x \le 1$

Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.

$\Rightarrow$ $\sin\theta = x$, by meaning of the symbol $\Rightarrow$ $\sin(\sin^{-1} x) = x$

Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1} x) = x, \quad -1 \le x \le 1$

(iii) $\tan(\tan^{-1} x) = x,\ x \in R$

(iv) $\cot(\cot^{-1} x) = x,\ x \in R$

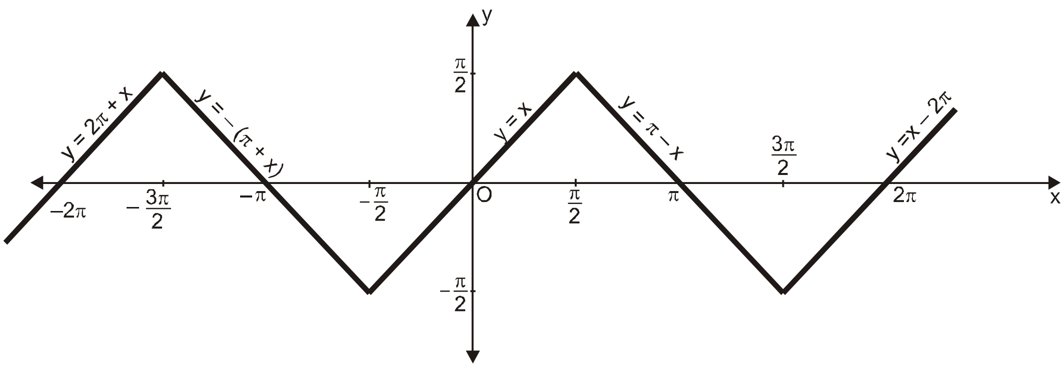
(v) $\sec(\sec^{-1} x) = x, \quad x \le -1, x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1} x) = x, \quad |x| \ge 1$

Property 2 : T^{-1}(T)

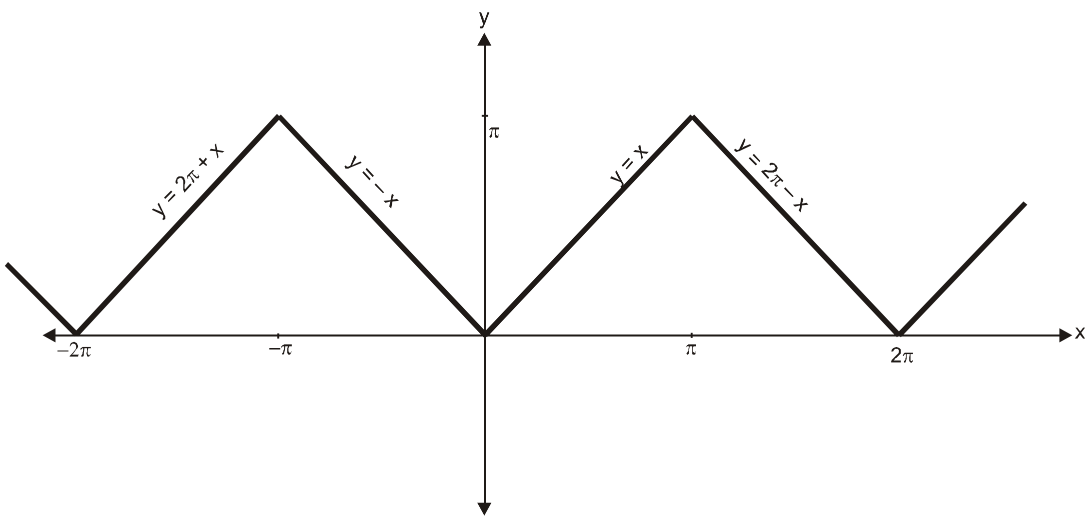
(i) $$\sin^{-1}(\sin x) = \begin{cases} -2n\pi + x, & x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2] \\ (2n+1)\ \pi - x, & x \in [(2n+1)\ \pi - \pi/2,\ (2n+1)\pi + \pi/2],\ n \in \mathbb{Z} \end{cases}$$

Graph of $y = \sin^{-1}(\sin x)$



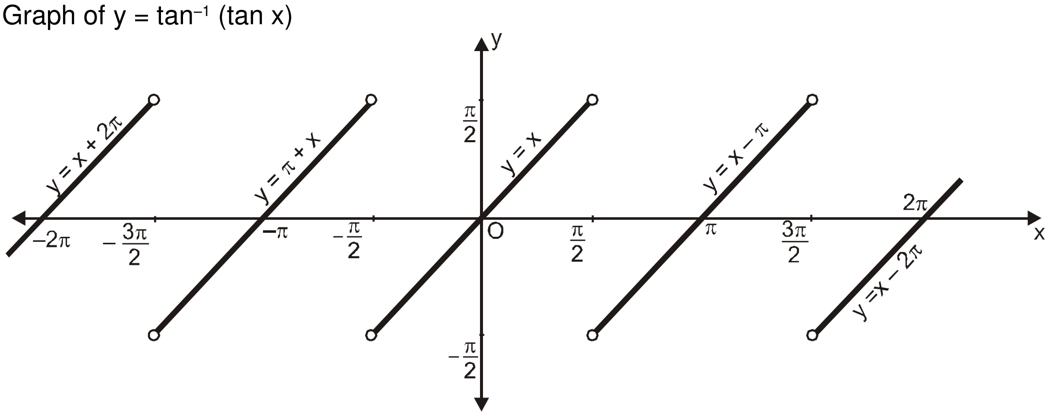
(ii) $$\cos^{-1}(\cos x) = \begin{cases} -2n\pi + x, & x \in [2n\pi,\ (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\ \pi,\ 2n\pi],\ n \in I \end{cases}$$

Graph of $y = \cos^{-1}(\cos x)$



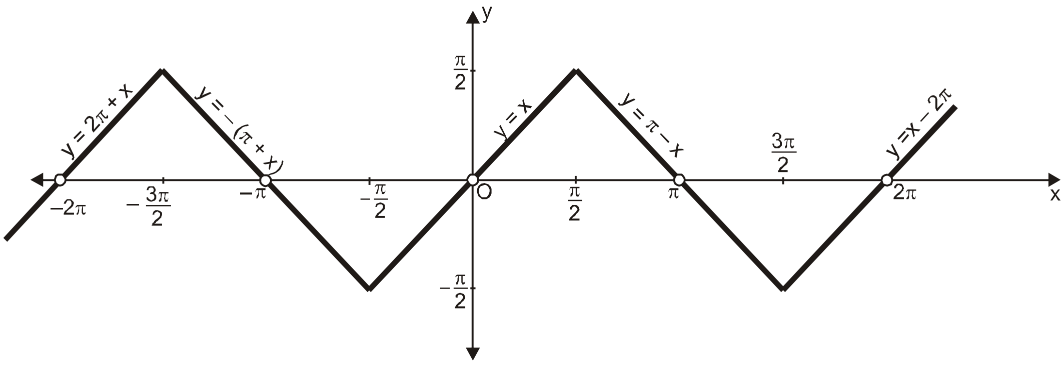
(iii) $\tan^{-1}(\tan x) = -n\pi + x,\ n\pi - \pi/2 < x < n\pi + \pi/2,\ n \in z$

Graph of $y = \tan^{-1}(\tan x)$



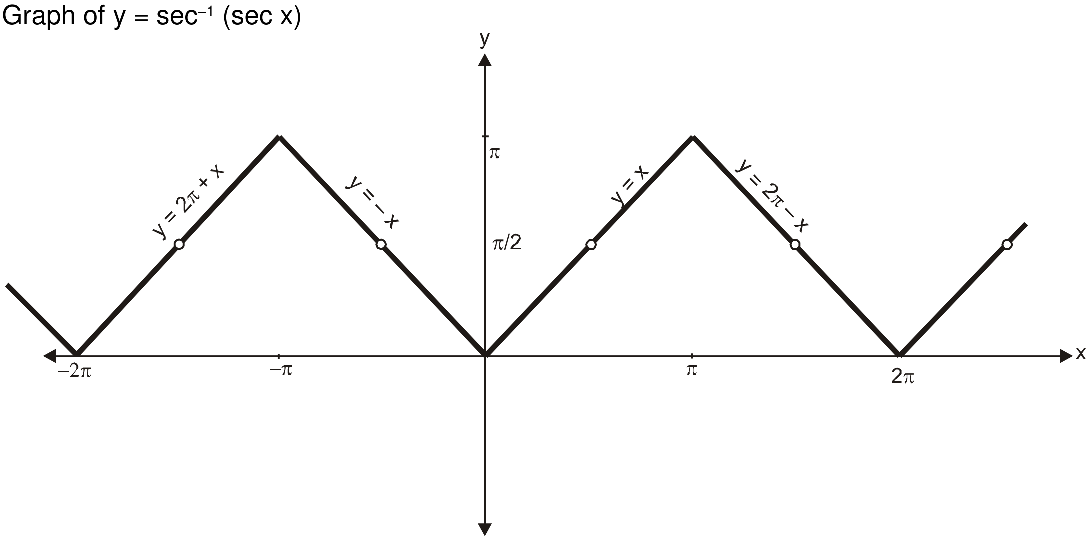
(iv) $\text{cosec}^{-1}(\text{cosec } x)$ is similar to $\sin^{-1}(\sin x)$

Graph of $y = \text{cosec}^{-1}(\text{cosec } x)$



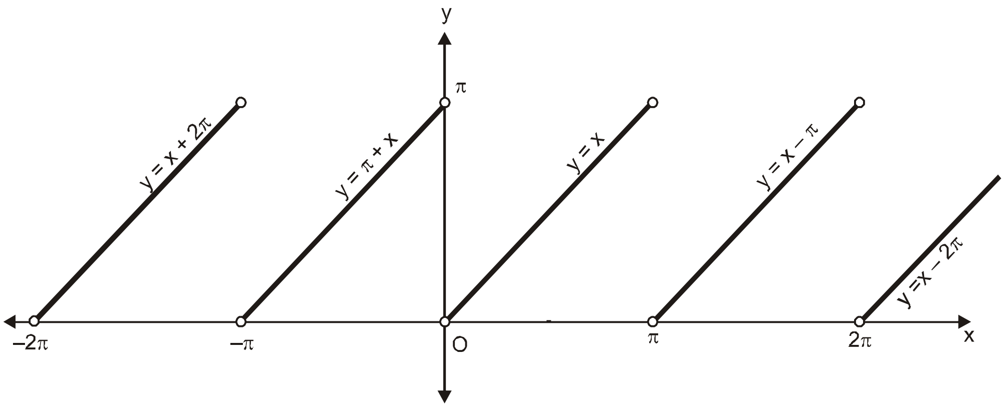
(v) $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$

Graph of $y = \sec^{-1}(\sec x)$



(vii) $\cot^{-1}(\cot x) = -n\pi + x,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$

Graph of $y = \cot^{-1}(\cot x)$



Remark : $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Property 3 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\operatorname{cosec}^{-1}x$ are symmetric about origin.

Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x.$

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x.$

Property 4 : "π/2"

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$, $-1 \le x \le 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$ $\Rightarrow$ $\sin A = x$ and $\cos B = x$

$\Rightarrow$ $\sin A = \cos B$ $\Rightarrow$ $\sin A = \sin(\pi/2 - B)$

$\Rightarrow$ $A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $A + B = \pi/2$.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$, $x \in R$ (iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}$, $|x| \ge 1$

Example # 28 : Find the value of $\operatorname{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\}$.

Solution :

$\because$ $\cot(\cot^{-1}x) = x$, $\forall\, x \in R$

$\therefore$ $\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4}$

$\operatorname{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\} = \operatorname{cosec}\left(\dfrac{3\pi}{4}\right) = .\ \sqrt{2}$

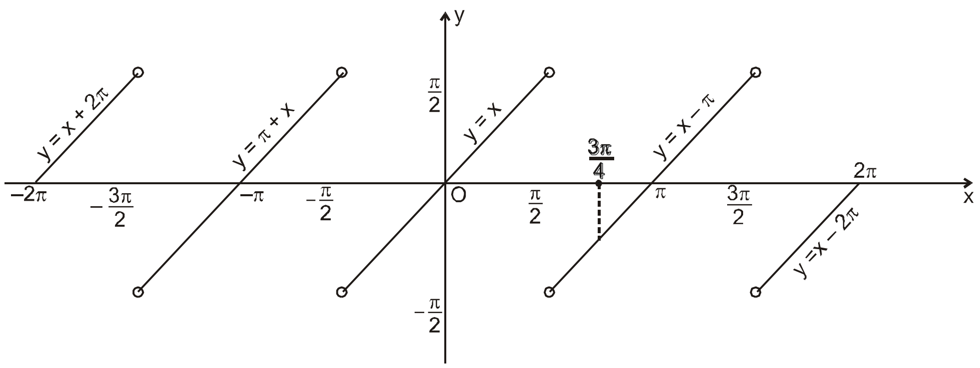
Example # 29 Find the value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

Solution : $\because$ $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

As $\dfrac{3\pi}{4} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ $\therefore$ $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) \neq \dfrac{3\pi}{4}$

$\therefore$ $\dfrac{3\pi}{4} \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$,

then $\tan^{-1}(\tan x) = x - \pi$

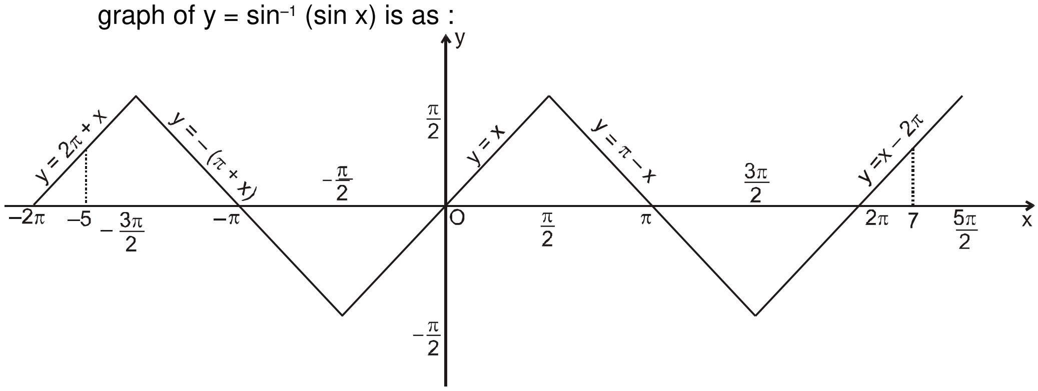
$\therefore$ $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4} - \pi = -\dfrac{\pi}{4}$

Example # 30 : Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Solution. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ $\because$ $2\pi < 7 < \dfrac{5\pi}{2}$

graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \dfrac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore$ $\sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because$ $-2\pi < -5 < -\dfrac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that $\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example # 31 : Solve $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Solution : $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2} \quad \Leftrightarrow \quad f(x) = g(x)$ and $-1 \le f(x), g(x) \le 1$

$x^2 - 2x + 1 = x^2 - x \quad \Leftrightarrow \quad x = 1$, accepted as a solution

Self practice problems :

(20) Find the value of (i) $\cos\left\{\sin\left(\sin^{-1}\frac{\pi}{6}\right)\right\}$

(ii) $\sin\left\{\cos\left(\cos^{-1}\frac{3\pi}{4}\right)\right\}$ (iii) $\cos^{-1}(\cos 13)$

(iv) $\cos^{-1}(-\cos 4)$ (v) $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$ (vi) $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$

(21) Find $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$, $\cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$

(22) Solve the following equations (i) $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ (ii) $4\sin^{-1}x = \pi - \cos^{-1}x$

(iii) Solve $\sin^{-1}(x^2 - 2x + 3) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Answer : (20) (i) $\frac{\sqrt{3}}{2}$ (ii) not defined (iii) $13 - 4\pi$

(iv) $4 - \pi$ (v) $\frac{\pi}{8}$ (vi) $\left(\frac{1}{4} - \frac{\pi}{2}\right)$

(21) $3\pi - \theta$, $\theta - 2\pi$, $\theta - 3\pi$, $\theta - 2\pi$

(22). (i) $x = 1$ (ii) $x = \frac{1}{2}$ (iii) No solution

Interconversion & Simplification

Interconversion of any trigonometric ratio inverse means its conversion in remaining five trigonometric ratio inverse. Example

$$\sin^{-1}x = \begin{cases} \text{for } x \in (0,1) & \text{for } x \in (-1,0) \\ \cos^{-1}\sqrt{1-x^2} & -\cos^{-1}\sqrt{1-x^2} \\ \tan^{-1}\frac{x}{\sqrt{1-x^2}} & \tan^{-1}\frac{x}{\sqrt{1-x^2}} \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & -\pi + \cot^{-1}\frac{\sqrt{1-x^2}}{x} \\ \sec^{-1}\frac{1}{\sqrt{1-x^2}} & -\sec^{-1}\frac{1}{\sqrt{1-x^2}} \\ \operatorname{cosec}^{-1}\frac{1}{x} & \operatorname{cosec}^{-1}\frac{1}{x} \end{cases}$$

Example # 32 : Convert (i) $\tan^{-1}3$, (ii) $\sin^{-1}(-1/3)$ in terms of cosine inverse.

Sol. (i) Let $\theta = \tan^{-1}3 \Rightarrow \tan\theta = 3 \Rightarrow \cos\theta = \frac{1}{\sqrt{10}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{10}}$

(ii) $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$

Let $\theta = \sin^{-1}(1/3) \Rightarrow \sin\theta = \frac{1}{3} \Rightarrow \cos\theta = \frac{2\sqrt{2}}{3} \Rightarrow \theta = \cos^{-1}\frac{2\sqrt{2}}{3}$

$\Rightarrow \quad \sin^{-1}(-1/3) = -\cos^{-1}\frac{2\sqrt{2}}{3}$

Example # 33 : Show that $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Sol. Let $\cot^{-1}x = \theta$ $(x = \cot\theta)$ $\Rightarrow$ $\theta \in (0, \pi)$

Now $\tan^{-1}(1/x) = \tan^{-1}\tan(\theta)$ $= \begin{cases} \theta & \theta \in \left(0, \dfrac{\pi}{2}\right) \\ \theta - \pi & \theta \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$

$= \begin{cases} \cot^{-1}x & x > 0 \\ \cot^{-1}x - \pi & x < 0 \end{cases}$

$\Rightarrow$ $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Example # 34 : Show that $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$

Sol : Let $\tan^{-1}x = \theta(x = \tan\theta) \Rightarrow \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ $\Rightarrow$ $2\theta \in (-\pi, \pi)$

Now $\sin^{-1}\dfrac{2x}{1+x^2} = \sin^{-1}\sin 2\theta = \begin{cases} -\pi - 2\theta & 2\theta \in \left(-\pi, -\dfrac{\pi}{2}\right] \text{ or } \theta \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right] \\ 2\theta & 2\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \text{ or } \theta \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right] \\ \pi - 2\theta & 2\theta \in \left[\dfrac{\pi}{2}, \pi\right) \text{ or } \theta \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right) \end{cases}$

$= \begin{cases} 2\tan^{-1}x & \text{if } x \in [-1, 1] \\ \pi - 2\tan^{-1}x & \text{if } x \ge 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x \le -1 \end{cases}$

Example # 35 : Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1}x$

Solution : Let $\cos^{-1}x = \theta(x = \cos\theta) \Rightarrow$ $\theta \in [0, \pi]$ $\Rightarrow$ $3\theta = [0, 3\pi]$

Now $y = \cos^{-1}(4x^3 - 3x) = \cos^{-1}\cos 3\theta$

$= \begin{cases} 3\theta & 3\theta \in [0, \pi] \text{ or } \theta \in \left[0, \dfrac{\pi}{3}\right] \\ 2\pi - 3\theta & 3\theta \in [\pi, 2\pi] \text{ or } \theta \in \left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right] \\ 3\theta - \pi & 3\theta \in [2\pi, 3\pi] \text{ or } \theta \in \left[\dfrac{2\pi}{3}, \pi\right] \end{cases}$

$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x & ; \dfrac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & ; -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ -2\pi + 3\cos^{-1}x & ; -1 \le x < -\dfrac{1}{2} \end{cases}$

Example # 36 : Simplify **(i)** $\sin\left(\pi\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right)$

(ii) $\sin\left(2\tan^{-1}\ \frac{1}{2}\right)$

(iii) $\cos\ (2\cos^{-1}(1/5) + \sin^{-1}(1/5))$

Solution : **(i)** Let $y = \tan\ \left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(A)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

(A) can be written as

$y = \tan\ \left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\ \left(\cot^{-1}\frac{2}{3}\right)$

$\because$ $\cot^{-1} x = \tan^{-1}\ \frac{1}{x}$ if $x > 0$

$\therefore$ $y = -\tan\ \left(\tan^{-1}\frac{3}{2}\right)$ $\Rightarrow$ $y = -\frac{3}{2}$ so $\sin\left(\pi\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right) = \sin\ \left(-\frac{3\pi}{2}\right) = 1$

(ii) $\sin\left(2\tan^{-1}\ \frac{1}{2}\right) = 2\sin\ \left(\tan^{-1}\frac{1}{2}\right)\cos\ \left(\tan^{-1}\frac{1}{2}\right) = 2\sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) \times \cos\ \left(\cos^{-1}\frac{2}{\sqrt{5}}\right)$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$

(iii) $\cos\ \left(2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right) = \cos\ \left(\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$

$= \cos\ \left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\ \left(\cos^{-1}\left(\frac{1}{5}\right)\right)$ ....(i)

$$= -\sqrt{1 - \left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}.$$

Self practices problem :

(23) Define (i) $\cos^{-1}\left(\frac{1\ -\ x^2}{1\ +x^2}\right)$ in terms of $\tan^{-1}x$

(ii) $\tan^{-1}\ \left(\frac{3x - x^3}{1 - 3x^2}\right)$ in terms of $\tan^{-1} x$

(24) Find the value of (i) $\sec\ \left(\cos^{-1}\left(\frac{2}{3}\right)\right)$, (ii) $\mathrm{cosec}\ \left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$,

(iii) $\tan\ \left(\mathrm{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$, (iv) $\sec\ \left(\cot^{-1}\frac{16}{63}\right)$, (v) $\sin\ \left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$

(vi) $\tan\ \left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$,

(25) If $x \in (-1, 1)$ and $2\tan^{-1} x = \tan^{-1} y$ then find y in term of x.

(26) Find the value of $\sin(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Answers : (23) (i) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

(ii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1}x \ ; & -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x \ ; & -\infty < x < -\frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x \ ; & \frac{1}{\sqrt{3}} < x < \infty \end{cases}$

(24) (i) $\frac{3}{2}$ (ii) $-\sqrt{3}$ (iii) $\frac{4}{5}$ (iv) $\frac{65}{16}$ (v) $\frac{2\sqrt{5}}{5}$ (vi) $\frac{-7}{17}$

(25) $y = \frac{2x}{1-x^2}$

(26) $\frac{1}{5}$

Identities on addition and subtraction :

S.No.	Identities	Condition
(1)	$\tan^{-1}x + \tan^{-1}y = \pi/2$	$x, y > 0$ & $xy = 1$
(2)	$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$	$x, y \ge 0$ & $xy < 1$
(3)	$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$	$x, y \ge 0$ & $xy > 1$
(4)	$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$	$x \ge 0, y \ge 0$
(5)	$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$	$x \ge 0, y \ge 0$ and $(x^2 + y^2) \le 1$
(6)	$\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$	$x \ge 0, y \ge 0$ and $(x^2 + y^2) \ge 1$
(7)	$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$	$x, y \in [0, 1]$
(8)	$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$x, y \in [0, 1]$
(9)	$\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$0 \le x < y \le 1$
(10)	$\cos^{-1}x - \cos^{-1}y = -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$0 \le y < x \le 1$

Some useful Results :

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(iv) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

Example # 37 : Show that $\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{15}{17} = \frac{\pi}{2} + \cos^{-1}\frac{84}{85}$

Solution : $\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5}$

$\because$ $\frac{3}{5} > 0,\ \frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore$ $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}} + \frac{15}{17}\sqrt{1-\frac{9}{25}}\right)$

$= \pi - \sin^{-1}\left(\frac{3}{5}\ .\ \frac{8}{17} + \frac{15}{17}\ .\ \frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right) = \pi - \frac{\pi}{2} + \cos^{-1}\frac{84}{85} = \frac{\pi}{2} + \cos^{-1}\frac{84}{85}$

Example # 38 : Evaluate $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1$

Solution : $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1 = \tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1$

$\because$ $9 > 0,\ \frac{5}{4} > 0$ and $\left(9 \times \frac{5}{4}\right) > 1$

$\therefore$ $\tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1 = \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9.\frac{5}{4}}\right) + \cot^{-1}1 = \pi + \tan^{-1}(-1) + \cot^{-1}1$

$= \pi - \frac{\pi}{4} + \cot^{-1}1 = \pi$.

Self practice problems:

(27) Evaluate $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

(28) If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}\lambda$, then find 'λ'

(29) Prove that $2\cos^{-1}\frac{3}{\sqrt{13}} + \cot^{-1}\frac{16}{63} + \frac{1}{2}\cos^{-1}\frac{7}{25} = \pi$

Answers. (27) $\frac{\pi}{2}$ (28) $\lambda = -\frac{19}{9}$ (29) $x = \frac{1}{2}$

Exercise-1

✍ Marked questions are recommended for Revision.

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Ordered pair , Cartesian product, Relation, Domain and Range of Relation

A-1. If A = {2, 4, 5}, B = {7, 8, 9}, then find n(A × B).

A-2. If A = {x : $x^2 - 5x + 6 = 0$}, B = {2, 4}, C = {4, 5} then find A × (B ∩ C).

A-3.✍ A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. Find the number of possible relations which can be defined from A to B.

A-4. If A = {2, 3, 4, 5}, B = {1, 3, 5, 7} and a relation R : A →B such that y = 2x –3, x∈A, y∈B, then find R.

A-5. Let R be a relation defined as R = { (x, y) : y = $\sqrt{(x-1)^2}$, x ∈ Z and $-3 \le x \le 3$} then find
(i) Domain of R (ii) Range of R (iii) Relation R

A-6. The Certesian product A × A has 16 elements S = {(a,b) ∈ A × A| a < b}. (–1,2) and (0,1) are two elements belonging to S. Find the set containing the remaining elements of S.

Section (B) : Types of Relation

B-1. Identify the type of relation among reflexive, symmetric and transitive.
(i) R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on set A = {1, 2, 3}.
(ii) P = {(x, y) | $x^2 + y^2 = 1$, x, y ∈ R}

B-2. Prove that the relation "less than" in the set of natural number is transitive but not reflexive and symmetric.

B-3. Let A = {p, q , r}. Which of the following is an equivalence relation on A ?
(i) R = {(p, q), (q, r), (p, r) (p, p)}
(ii) R = {(p, p), (q, q), (r, r) (q, p)}
(iii) R = {(p, p), (q, q), (r, r)}
(iv) R = {(p, p), (q, q), (r, r), (p, q), (q, r), (p, r)}
(v) R = {(p, p), (q, q), (r, r), (p, q), (q, p)}

B-4. Let R be a relation on the set N be defined by {(x, y)| x, y ∈ N, 2x + y = 41}. Then prove that R is neither reflexive nor symmetric and nor transitive.

B-5. Let n be a fixed positive integer. Define a relation R on the set of integers Z, aRb ⇔ n|(a – b). Then prove that R is equivalence.

B-6.✍ Let S be a set of all square matrices of order 2. If a relation R defined on set S such that AR B ⇒ AB = BA, then identify the type of relation of R (A, B ∈ S) among reflexive, symmetric and transitive.

Section (C) : Definition of function, Domain and Range, Classification of Functions

C-1. Check whether the followings represent function or not
(i) $x^2 + y^2 = 36$, y ∈ [0, 6] (ii) $x^2 + y^2 = 36$, x ∈ [0, 1]
(iii) $x^2 + y^2 = 36$, x ∈ [–6, 6] (iv) $x^2 + y^2 = 36$

C-2. Find the domain of each of the following functions :

(i) $f(x) = \dfrac{x^3 - 5x + 3}{x^2 - 1}$

(ii) $f(x) = \sqrt{\sin(\cos x)}$

(iii) $f(x) = \dfrac{1}{\sqrt{x + |x|}}$

(iv) $f(x) = e^{x + \sin x}$

(v) $f(x) = \dfrac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$

(vi) $f(x) = \sqrt{\dfrac{\log_2(x - 2)}{\log_{1/2}(3x - 1)}}$

(vii) $f(x) = \ell n\,[x^2 + x + 1]$, where [.] GIF.

(viii) $f(x) = \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6 + 35x - 6x^2}}$

C-3. Find the domain of definitions of the following functions :

(i) $f(x) = \sqrt{3 - 2^x - 2^{1-x}}$

(ii) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$

(iii) $f(x) = (x^2 + x + 1)^{-3/2}$

(iv) $f(x) = \sqrt{\dfrac{x - 2}{x + 2}} + \sqrt{\dfrac{1 - x}{1 + x}}$

(v) $f(x) = \sqrt{\tan x - \tan^2 x}$

(vi) $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$

(vii) $f(x) = \sqrt{\ell og_{1/4}\left(\dfrac{5x - x^2}{4}\right)}$

(viii) $f(x) = \ell og_{10}(1 - \ell og_{10}(x^2 - 5x + 16))$

C-4. Find the range of each of the following functions :

(i) $f(x) = |x - 3|$ (ii) $f(x) = \dfrac{x}{1 + x^2}$ (iii) $f(x) = \sqrt{16 - x^2}$ (iv) $f(x) = \dfrac{|x - 4|}{x - 4}$

C-5. Find the domain and the range of each of the following functions :

(i) $f(x) = \dfrac{1}{\sqrt{4 + 3\sin x}}$ (ii) $f(x) = x\,!$ (iii) $f(x) = \dfrac{x^2 - 9}{x - 3}$ (iv) $f(x) = \sin^2(x^3) + \cos^2(x^3)$

C-6. Find the range of each of the following functions : (where {.} and [.] represent fractional part and greatest integer part functions respectively)

(i) $f(x) = 5 + 3\sin x + 4\cos x$

(ii) $f(x) = \dfrac{1}{1 + \sqrt{x}}$

(iii) $f(x) = 2 - 3x - 5x^2$

(iv) $f(x) = 3|\sin x| - 4|\cos x|$

(v) $f(x) = \dfrac{\sin x}{\sqrt{1 + \tan^2 x}} + \dfrac{\cos x}{\sqrt{1 + \cot^2 x}}$

(vi) $f(x) = \ell n\left(\dfrac{\sqrt{8 - x^2}}{x - 2}\right)$

(vii) $f(x) = \left[\dfrac{1}{\sin\{x\}}\right]$

C-7. Find the range of the following functions : (where {.} and [.] represent fractional part and greatest integer part functions respectively)

(i) $f(x) = 1 - |x - 2|$

(ii). $f(x) = \dfrac{1}{\sqrt{16 - 4^{x^2 - x}}}$

(iii) $f(x) = \dfrac{1}{2 - \cos 3x}$

(iv) $f(x) = \dfrac{x + 2}{x^2 - 8x - 4}$

(v) $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

(vi) $f(x) = 3\sin\sqrt{\dfrac{\pi^2}{16} - x^2}$

(vii) $f(x) = x^4 - 2x^2 + 5$

(viii) $f(x) = x^3 - 12x$, where $x \in [-3, 1]$

(ix) $f(x) = \sin^2 x + \cos^4 x$

(x) $f(x) = [\sin x + [\cos x + [\tan x + [\sec x]]]]$ Here $x \in (0, \pi/4)$

(xi) $f(x) = \sec^2 x - \tan^2 x + \sin(\sin x + \cos x)$

C-8. Find whether the following functions are one-one or many-one & into or onto if f : D $\rightarrow$ R where D is its domain.

(i) $f(x) = |x^2 + 5x + 6|$ (ii) $f(x) = |\ell nx|$

(iii) $f(x) = \sin 4x : \left(-\frac{\pi}{8}, \frac{\pi}{8}\right) \rightarrow (-1, 1)$ (iv) $f(x) = x + \frac{1}{x}$, $x \in (0, \infty)$

(v) $f(x) = \sqrt{1 - e^{\left(\frac{1}{x}-1\right)}}$ (vi) $f(x) = \frac{3x^2}{4\pi} - \cos \pi x$

(vii) $f(x) = \frac{1+x^6}{x^3}$ (viii) $f(x) = x \cos x$ (ix) $f(x) = \frac{1}{\sin\sqrt{|x|}}$

C-9. Classify the following functions f(x) defined in R $\rightarrow$ R as injective, surjective, both or none.

(i) $f(x) = x|x|$ (ii) $f(x) = \frac{x^2}{1+x^2}$ (iii) $f(x) = x^3 - 6x^2 + 11x - 6$

C-10. Check whether the following functions is/are many-one or one-one & into or onto

(i) $f(x) = \tan(2 \sin x)$ (ii) $f(x) = \tan(\sin x)$

C-11. Let f : A $\rightarrow$ A where A = {x : $-1 \le x \le 1$}. Find whether the following functions are bijective.

(i) $x - \sin x$ (ii) $x|x|$ (iii) $\tan \frac{\pi x}{4}$ (iv) x^4

C-12. Let A be a set of n distinct elements. Then find the total number of distinct functions from A to A ? How many of them are onto functions ?

Section (D) : Identical functions, Composite functions

D-1. Check whether following pairs of functions are identical or not ?

(i) $f(x) = \sqrt{x^2}$ and $g(x) = \left(\sqrt{x}\right)^2$ (ii)_ $f(x) = \tan x$ and $g(x) = \frac{1}{\cot x}$

(iii) $f(x) = \sqrt{\frac{1 + \cos 2x}{2}}$ and $g(x) = \cos x$ (iv) $f(x) = x$ and $g(x) = e^{\ell nx}$

D-2. Find for what values of x, the following functions would be identical.

$f(x) = \log(x - 1) - \log(x - 2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$

D-3. Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that fog $\neq$ gof

D-4. Let $f(x) = x^2$, $g(x) = \sin x$, $h(x) = \sqrt{x}$, then verify that [fo (goh)] (x) and [(fog) oh] (x) are equal.

D-5. Find fog and gof, if

(i) $f(x) = e^x$; $g(x) = \ell n\, x$ (ii) $f(x) = |x|$; $g(x) = \sin x$

(iii) $f(x) = \sin x$; $g(x) = x^2$ (iv) $f(x) = x^2 + 2$; $g(x) = 1 - \frac{1}{1-x}$, $x \neq 1$

D-6. If $f(x) = \ell n(x^2 - x + 2)$; $R^+ \rightarrow R$ and

$g(x) = \{x\} + 1$; $[1, 2] \rightarrow [1, 2]$, where {x} denotes fractional part of x.

Find the domain and range of f(g(x)) when defined.

D-7. If $f(x) = \begin{cases} 1+x^2 ; & x \le 1 \\ x+1 ; & 1 < x \le 2 \end{cases}$ and $g(x) = 1 - x$; $-2 \le x \le 1$, then define the function fog(x).

D-8. If $f(x) = \dfrac{x+2}{x+1}$ and $g(x) = \dfrac{x-2}{x}$, then find the domain of
(i) fog(x) (ii) gof(x) (iii) fof(x) (iv) fogof(x)

D-9. If $f(x) = \begin{cases} \sqrt{2}x & x \in Q - \{0\} \\ 3x & x \in Q^c \end{cases}$, then define fof(x) and hence define fofof.....f(x) where f is 'n' times.

D-10. Let $f(x)\begin{cases} x+1 & x \le 4 \\ 2x+1 & 4 < x \le 9 \\ -x+7 & x > 9 \end{cases}$ and $g(x) = \begin{cases} x^2 & -1 \le x < 3 \\ x+2 & 3 \le x \le 5 \end{cases}$ then, find f(g(x)).

D-11. If $f(x) = \dfrac{4^x}{4^x + 2}$, then show that f(x) + f(1 – x) = 1

Section (E) : Even/Odd Functions & Periodic Functions

E-1. Determine whether the following functions are even or odd or neither even nor odd :
(i) $\sin(x^2 + 1)$ (ii) $x + x^2$ (iii) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$
(iv) $f(x) = \sin x + \cos x$ (v) $f(x) = (x^2 - 1)\,|x|$
(vi) $f(x) = \begin{cases} |\ell n e^x| & ; \quad x \le -1 \\ [2+x]+[2-x] & ; \quad -1 < x < 1 \\ e^{\ell nx} & ; \quad x \ge 1 \end{cases}$, where [.] is GIF.

E-2. Examine whether the following functions are even or odd or neither even nor odd, where [ ] denotes greatest integer function.
(i) $f(x) = \dfrac{(1+2^x)^7}{2^x}$ (ii) $f(x) = \dfrac{\sec x + x^2 - 9}{x \sin x}$
(iii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ (iv) $f(x) = \begin{cases} x\,|x|, & x \le -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x\,|x|, & x \ge 1 \end{cases}$

E-3. Which of the following functions are not periodic (where [.] denotes greatest integer function) :
(i) $f(x) = \sin\sqrt{x}$ (ii) $f(x) = x + \sin x$
(iii) $f(x) = [\sin 3x] + |\cos 6x|$

E-4. Find the fundamental period of the following functions :
(i) $f(x) = 2 + 3\cos(x - 2)$ (ii) $f(x) = \sin 3x + \cos^2 x + |\tan x|$
(iii) $f(x) = \sin\dfrac{\pi x}{4} + \sin\dfrac{\pi x}{3}$ (iv) $f(x) = \cos\dfrac{3}{5}x - \sin\dfrac{2}{7}x.$
(v) $f(x) = \dfrac{1}{1-\cos x}$ (vi) $f(x) = \dfrac{\sin 12x}{1+\cos^2 6x}$
(vii) $f(x) = \sec^3 x + \mathrm{cosec}^3 x$

Section (F) : Inverse of a function

F-1. Let f : D → R, where D is the domain of f. Find the inverse of f, if it exists

(i) $f(x) = 1 - 2^{-x}$ (ii) $f(x) = \left(4 - (x-7)^3\right)^{1/5}$

(iii) $f(x) = \ell n\left(x + \sqrt{1+x^2}\right)$

(iv) Let f : [0, 3] → [1, 13] is defined by $f(x) = x^2 + x + 1$, then find $f^{-1}(x)$.

F-2. Let f : R → R be defined by $f(x) = \dfrac{e^{2x} - e^{-2x}}{2}$. Is f(x) invertible ? If yes, then find its inverse.

F-3. (a) If $f(x) = -x|x|$, then find $f^{-1}(x)$ and hence find the number of solutions of $f(x) = f^{-1}(x)$.

(b) Solve $2x^2 - 5x + 2 = \dfrac{5 - \sqrt{9+8x}}{4}$, where $x < \dfrac{5}{4}$

F-4. If g is inverse of $f(x) = x^3 + x + \cos x$, then find the value of $g'(1)$.

F-5. If $f(x) = \begin{cases} (\alpha - 1)x & x \in Q^c \\ -\alpha^2 x + \alpha + 3x - 1 & x \in Q \end{cases}$ and $g(x) = \begin{cases} x & x \in Q^c \\ 1 - x & x \in Q \end{cases}$ are inverse to each other then find all possible values of α.

Section (G) : Definition, graphs and fundamentals & Inverse Trigonometry

G-1. Find the domain of each of the following functions :

(i) $f(x) = \dfrac{\sin^{-1} x}{x}$ (ii) $f(x) = \sqrt{1-2x} + 3\sin^{-1}\left(\dfrac{3x-1}{2}\right)$ (iii) $f(x) = 2^{\sin^{-1} x} + \dfrac{1}{\sqrt{x-2}}$

G-2. Find the range of each of the following functions :

(i) $f(x) = \ell n(\sin^{-1} x)$ (ii) $f(x) = \sin^{-1}\left(\dfrac{\sqrt{3x^2+1}}{5x^2+1}\right)$

(iii) $f(x) = \cos^{-1}\left(\dfrac{(x-1)(x+5)}{x(x-2)(x-3)}\right)$

G-3. Find the simplified value of the following expressions :

(i) $\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right]$ (ii) $\tan\left[\cos^{-1}\dfrac{1}{2} + \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)\right]$

(iii) $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right\}\right]$

G-4. (i) If $\sum_{i=1}^{n} \cos^{-1}\alpha_i = 0$, then find the value of $\sum_{i=1}^{n} i \cdot \alpha_i$

(ii) If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then show that $\sum_{i=1}^{2n} x_i = 2n$

G-5. Solve the following inequalities:

(i) $\cos^{-1} x > \cos^{-1} x^2$ (ii) $\text{arccot}^2 x - 5\,\text{arccot}\, x + 6 > 0$

(iii) $\sin^{-1} x > -1$ (iv) $\cos^{-1} x < 2$ (v) $\cot^{-1} x < -\sqrt{3}$

G–6. Let f : $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$ defined by f (x) = 2 cos²x + $\sqrt{3}$ sin2x + 1. Find B such that f^{-1} exists. Also find $f^{-1}(x)$.

Section (H) : Trig (trig⁻¹x), trig⁻¹ (trig x), trig⁻¹ (–x) and Property (π/2)

H-1. Evaluate the following inverse trigonometric expressions :

(i) $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ (ii) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

(iii) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ (iv) $\sec^{-1}\left(\sec\frac{7\pi}{4}\right)$

H-2. Find the value of the following inverse trigonometric expressions :

(i) $\sin^{-1}(\sin 4)$ (ii) $\cos^{-1}(\cos 10)$

(iii) $\tan^{-1}(\tan(-6))$ (iv) $\cot^{-1}(\cot(-10))$

(v) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\left(\cos\frac{9\pi}{10} - \sin\frac{9\pi}{10}\right)\right)$

H-3. Find the value of following expressions :

(i) $\cot(\tan^{-1} a + \cot^{-1} a)$ (ii) $\sin(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$

H-4. Solve the inequality $\tan^{-1} x > \cot^{-1} x$.

Section (I) : Interconversion/Simplification

I-1. Evaluate the following expressions :

(i) $\sin\left(\cos^{-1}\frac{3}{5}\right)$ (ii) $\tan\left(\cos^{-1}\frac{1}{3}\right)$

(iii) $\operatorname{cosec}\left(\sec^{-1}\frac{\sqrt{41}}{5}\right)$ (iv) $\tan\left(\operatorname{cosec}^{-1}\frac{65}{63}\right)$

(v) $\sin\left(\frac{\pi}{6} + \cos^{-1}\frac{1}{4}\right)$ (vi) $\cos\left(\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{2}{3}\right)$

(vii) $\sec\left(\tan\left\{\tan^{-1}\left(-\frac{\pi}{3}\right)\right\}\right)$ (viii) $\cos\tan^{-1}\sin\cot^{-1}\left(\frac{1}{2}\right)$

(ix) $\tan\left[\cos^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(\frac{3}{4}\right) - \sec^{-1}3\right]$

I-2. Find the value of $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$

I-3. If $\tan^{-1}x + \cot^{-1}\frac{1}{y} + 2\tan^{-1}z = \pi$, then prove that $x + y + 2z = xz^2 + yz^2 + 2xyz$

I-4. If $\cos^{-1}x + 2\sin^{-1}x + 3\cot^{-1}y + 4\tan^{-1}y = 4\sec^{-1}z + 5\operatorname{cosec}^{-1}z$, then prove that $\sqrt{z^2-1} = \frac{\sqrt{1-x^2} - xy}{x + y\sqrt{1-x^2}}$

I-5. Consider, $f(x) = \tan^{-1}\left(\frac{2}{x}\right)$, $g(x) = \sin^{-1}\left(\frac{2}{\sqrt{4+x^2}}\right)$ and $h(x) = \tan(\cos^{-1}(\sin x))$, then show that

$(h(f(x)) + h(g(x))) = \begin{cases} 0 , & x < 0 \\ x , & x > 0 \end{cases}$

I-6. Prove each of the following relations :

(i) $\tan^{-1} x = -\pi + \cot^{-1}\frac{1}{x} = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = -\cos^{-1}\frac{1}{\sqrt{1+x^2}}$ when $x < 0$.

(ii) $\cos^{-1}x = \sec^{-1}\frac{1}{x} = \pi - \sin^{-1}\sqrt{1-x^2} = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$ when $-1<x<0$

I-7. Express in terms of

(i) $\tan^{-1}\frac{2x}{1-x^2}$ to $\tan^{-1} x$ for $x > 1$

(ii) $\sin^{-1}(2x\sqrt{1-x^2})$ to $\sin^{-1} x$ for $1 \geq x > \frac{1}{\sqrt{2}}$

(iii) $\cos^{-1}(2x^2 - 1)$ to $\cos^{-1}x$ for $-1 \leq x < 0$

I-8. Simplify $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$, if $x > y > 1$.

I-9. Solve for x

(i) $\cos(2\sin^{-1}x) = \frac{1}{3}$

(ii) $\cot^{-1} x + \tan^{-1} 3 = \frac{\pi}{2}$

(iii) $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

(iv) $\sin^{-1}x + \sin^{-1}2x = \frac{2\pi}{3}$

Section (J) : Addition and Subtraction Rule

J-1. Prove that

(i) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\frac{77}{85}$

(ii) $\tan^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13} = \cos^{-1}\frac{33}{65}$

(iii) $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cot^{-1} 3 = \frac{\pi}{4}$

(iv) $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

J-2. Find the sum of each of the following series :

(i) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13}$ .... upto n terms.

(ii) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + ... + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + ...$upto infinite terms

(iii) $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}} + ... + \sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + ...$ upto infinite terms

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Ordered pair , Cartesian product, Relation, Domain and Range of Relation

A-1. If A = {a, b}, B = {c, d}, C = {d, e}, then {(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)} is equal to

(A) $A \cap (B \cup C)$ (B) $A \cup (B \cap C)$ (C) $A \times (B \cup C)$ (D) $A \times (B \cap C)$

A-2. If A={1, 2, 3} and B = {1, 2} and C = {4, 5, 6}, then what is the number of elements in the set $A \times B \times C$?

(A) 8 (B) 9 (C) 15 (D) 18

A-3. Let A = {a, b, c} and B = {1, 2}. Consider a relation R defined from set A to set B. Then R can equal to set

(A) A (B) B (C) $A \times B$ (D) $B \times A$

A-4. Let R be relation from a set A to a set B, then
(A) $R = A \cup B$ (B) $R = A \cap B$ (C) $R \subseteq A \times B$ (D) $R \subseteq B \times A$

A-5. Let X = {1, 2, 3, 4, 5} and Y = {1, 3, 5, 7, 9}. Which of the following is not a relation from X to Y
(A) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$ (B) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(C) $R_3 = \{(1, 1), (1, 3) (3, 5), (3, 7), (5, 7)\}$ (D) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

A-6. The relation R defined in A = {1, 2, 3} by a R b if $-5 \le a^2 - b^2 \le 5$. Which of the following is false?
(A) R = {(1, 2), (2, 2), (3, 3), (2, 1), (2, 3), (3, 2)} (B) Co-domain of R = {1, 2, 3}
(C) Domain of R = {1, 2, 3} (D) Range of R = {1, 2, 3}

Section (B) : Types of Relation

B-1. The relation R defined in N as aRb $\Leftrightarrow$ b is divisible by a is
(A) Reflexive but not symmetric (B) Symmetric but not transitive
(C) Symmetric and transitive (D) Equivalence relation

B-2. In the set A = {1, 2, 3, 4, 5} a relation R is defined by $R = \{(x, y) \mid x, y \in A \text{ and } x < y\}$. Then R is
(A) Reflexive (B) Symmetric (C) Transitive (D) Equivalence relation

B-3. Which one of the following relations on R is equivalence relation-
(A) $x R_1 y \Leftrightarrow x^2 = y^2$ (B) $x R_2 y \Leftrightarrow x \ge y$ (C) $x R_3 y \Leftrightarrow x \mid y$ (x divides y) (D) $x R_4 y \Leftrightarrow x < y$

B-4. Let R_1 be a relation defined by $R_1 = \{(a, b) \mid a \ge b ; a, b \in R\}$. Then R_1 is
(A) An equivalence relation on R (B) Reflexive, transitive but not symmetric
(C) Symmetric, Transitive but not reflexive (D) Neither transitive nor reflexive but symmetric

B-5. Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. The R is
(A) Reflexive (B) Symmetric (C) Transitive (D) equivalence relation

B-6. Let S be the set of all real numbers. Then the relation R = {(a, b) : 1 + ab > 0} on S is
(A) Reflexive and symmetric but not transitive (B) Reflexive, transitive but not symmetric
(C) Symmetric, transitive but not reflexive (D) Reflexive, transitive and symmetric

B-7. Consider the following :
1. If $R = \{(a, b) \in N \times N$: a divides b in N} then the relation R is reflexive and symmetric but not transitive.
2. If A = {1, 2, 3, 4, 5, 6} and $R = \{(S_1, S_2) : S_1, S_2$ are subsets of A, $S_1 \not\subset S_2\}$, then the relation R is not reflexive, not symmetric and not transitive.

Which of the statements is/are correct ?
(A) 1 only (B) 2 only (C) Both 1 and 2 (D) Neither 1 nor 2

B-8. Let R be a relation over the set N × N and it is defined by (a, b) R (c, d) $\Rightarrow$ a + d = b + c. Then R is
(A) Symmetric only (B) Transitive only (C) Reflexive only (D) Equivalence only

B-9. Let L be the set of all straight lines in the Euclidean plane. Two lines ℓ_1 and ℓ_2 are said to be related by the relation R if ℓ_1 is parallel to ℓ_2. Then R is
(A) Symmetric only (B) Transitive only (C) Reflexive only (D) Equivalence only

B-10. Let $R = \{(x, y) : x, y \in A, x + y = 5\}$ where A = {1, 2, 3, 4, 5} then R is
(A) Reflexive (B) symmetric (C) Transitive (D) Equivalence

B-11. Let S be a set of all square matrices of order 2. If a relation R defined on set S such that AR B $\Rightarrow$ AB = O, where O is zero square matirx of order 2, then relation R is (A, B $\in$ S)
(A) Reflexive (B) Transitive
(C) Symmetric (D) Not equivalence

Section (C) : Definition of function, Domain and Range, Classification of Functions

C-1. The domain of the function $f(x) = \dfrac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{x^2+2x+8}}$ is

(A) (1, 4) (B) (– 2, 4) (C) (2, 4) (D) [2, ∞)

C-2. Range of $f(x) = \ell n\,(3x^2 - 4x + 5)$ is

(A) $\left[\ell n\dfrac{11}{3},\ \infty\right)$ (B) $[\ell n\,10\ ,\ \infty)$ (C) $\left[\ell n\dfrac{11}{6},\ \infty\right)$ (D) $\left[\ell n\dfrac{11}{12},\ \infty\right)$

C-3. Range of $f(x) = 4^x + 2^x + 1$ is

(A) (0, ∞) (B) (1, ∞) (C) (2, ∞) (D) (3, ∞)

C-4. Range of $f(x) = \log_{\sqrt{5}}\,(\sqrt{2}\,(\sin x - \cos x) + 3)$ is

(A) [0, 1] (B) [0, 2] (C) $\left[0,\ \dfrac{3}{2}\right]$ (D) [1, 2]

C-5. Let $f : R \to R$ be a function defined by $f(x) = \dfrac{2x^2 - x + 5}{7x^2 + 2x + 10}$, then f is :

(A) one – one but not onto
(B) onto but not one – one
(C) onto as well as one – one
(D) neither onto nor one – one

C-6. Let $f : R \to R$ be a function defined by $f(x) = x^3 + x^2 + 3x + \sin x$. Then f is:

(A) one – one and onto
(B) one – one and into
(C) many one and onto
(D) many one and into

C-7. Domain of definition of the function $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$, is :

(A) (1,2)
(B) (–1,0) ∪ (1,2)
(C) (1,2) ∪ (2, ∞)
(D) (–1,0) ∪ (1,2) ∪ (2, ∞)

C-8. If $f : [0, \infty) \to [0, \infty)$, and $f(x) = \dfrac{x}{1+x}$, then f is:

(A) one-one and onto
(B) one-one but not onto
(C) onto but not one-one
(D) neither one-one nor onto

C-9. Range of the function $f(x) = \dfrac{(x-2)^2}{(x-1)(x-3)}$ is

(A) (1, ∞) (B) (–∞, 1) (C) R – (0, 1] (D) (0, 1]

C-10. Range of the function $f(x) = \dfrac{x-2}{x^2-4x+3}$ is

(A) (–∞, 0) (B) R (C) (0, ∞) (D) R – {0}

C-11. **Statement - 1** If f (x) and g (x) both are one one and f(g (x)) exists, then f(g (x)) is also one one.
Statement - 2 If $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$, then f(x) is one-one.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

C-12. **Statement - 1** If y = f (x) is increasing in [α, β], then its range is [f (α), f (β)]
Statement - 2 Every increasing function need not to be continuous.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

C-13. If the functions f(x) and g(x) are defined on R→R such that $f(x) = \begin{cases} 0 & , \quad x \in \text{rational} \\ x & , \quad x \in \text{irrational} \end{cases}$,

$g(x) = \begin{cases} 0 & , \quad x \in \text{irrational} \\ x & , \quad x \in \text{rational} \end{cases}$, then (f – g) (x) is
(A) one-one and onto
(B) neither one-one nor onto
(C) one-one but not onto
(D) onto but not one-one

Section (D) : Identical functions, Composite functions

D-1. Which of the following pair of functions are identical –
(A) $f(x) = \sin^2x + \cos^2x$ and g(x) = 1
(B) $f(x) = \sec^2x - \tan^2x$ and g(x) = 1
(C) $f(x) = \text{cosec}^2x - \cot^2 x$ and g(x) = 1
(D) $f(x) = \ell nx^2$ and $g(x) = 2\ell nx$

D-2. Let f(x) be a function whose domain is [– 5, 7]. Let g(x) = |2x + 5|, then domain of (fog) (x) is
(A) [– 4, 1]
(B) [– 5, 1]
(C) [– 6, 1]
(D) [– 5, 7]

D-3. Let g (x) = 1 + x – [x] and $f(x) = \begin{cases} -1 & , \quad x < 0 \\ 0 & , \quad x = 0 \\ 1 & , \quad x > 0 \end{cases}$. Then for all x, f (g (x)) is equal to (where [.] denotes greatest integer function)
(A) x
(B) 1
(C) f (x)
(D) g (x)

Section (E) : Even/Odd Functions & Periodic Functions

E-1. The function $f(x) = \log\left(\dfrac{1+\sin x}{1-\sin x}\right)$ is
(A) even
(B) odd
(C) neither even nor odd
(D) both even and odd

E-2. The function $f(x) = [x] + \dfrac{1}{2}$, $x \notin I$ is a/an (where [.] denotes greatest integer function)
(A) Even
(B) odd
(C) neither even nor odd
(D) Even as well as odd

E-3. The graph of the function y = f(x) is symmetrical about the line x = 2, then :
(A) f(x + 2) = f(x – 2)
(B) f(2 + x) = f(2 – x)
(C) f(x) = f(–x)
(D) f(x) = –f(–x)

E-4. Fundamental period of f(x) = sec (sin x) is
(A) $\dfrac{\pi}{2}$
(B) 2π
(C) π
(D) aperiodic

E-5. If $f(x) = \sin\left(\sqrt{[a]}\, x\right)$ (where [.] denotes the greatest integer function) has π as its fundamental period, then
(A) a = 1
(B) a = 9
(C) $a \in [1, 2)$
(D) $a \in [4, 5)$

E-6. Find the area below the curve $y = \left[\sqrt{2+2\cos 2x}\right]$ but above the x-axis in $[-3\pi, 6\pi]$ is (where [.] denotes the greatest integer function) :
(A) 2π square units (B) π square units (C) 6π square units (D) 8π square units

Section (F) : Inverse of a function

F-1. The inverse of the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is
(A) $\dfrac{1}{2}\ \ell n\ \dfrac{1+x}{1-x}$ (B) $\dfrac{1}{2}\ \ell n\dfrac{2+x}{2-x}$ (C) $\dfrac{1}{2}\ \ell n\ \dfrac{1-x}{1+x}$ (D) $2\ \ell n\ (1 + x)$

F-2. If $f: [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \dfrac{1}{x}$, then $f^{-1}(x)$ equals:
(A) $\dfrac{x + \sqrt{x^2 - 4}}{2}$ (B) $\dfrac{x}{1 + x^2}$ (C) $\dfrac{x - \sqrt{x^2 - 4}}{2}$ (D) $1 - \sqrt{x^2 - 4}$

F-3. If $f : R \rightarrow R$ is an invertible function such that f(x) and $f^{-1}(x)$ are also mirror image to each other about the line $y = -x$, then
(A) f(x) is odd
(B) f(x) and $f^{-1}(x)$ may not be mirror image about the line y = x
(C) f(x) may not be odd
(D) f(x) is even

F-4. If $f(x) = \dfrac{ax+b}{cx+d}$, then (fof) (x) = x, provided that
(A) d + a = 0 (B) d – a = 0 (C) a = b = c = d = 1 (D) a = b = 1

F-5. Let $f(x) = \begin{cases} x & -1 \le x \le 1 \\ x^2 & 1 < x \le 2 \end{cases}$ the range of $h^{-1}(x)$, where h(x) = fof(x) is
(A) $\left[-1, \sqrt{2}\right]$ (B) [–1, 2] (C) [–1, 4] (D) [–2, 2]

F-6. **Statement – 1** All points of intersection of y = f (x) and $y = f^{-1}(x)$ lies on y = x only.
Statement – 2 If point P (α, β) lies on y = f (x), then Q (β, α) lies on $y = f^{-1}(x)$.
Statement – 3 Inverse of invertible function is unique and its range is equal to the function domain.
Which of the following option is correct for above statements in order
(A) T T F (B) F T T (C) T T T (D) T F T

Section (G) : Definition, graphs and fundamentals of Inverse Trigonometric functions

G-1. The domain of definition of $f(x) = \sin^{-1}(|x - 1| - 2)$ is:
(A) $[-2, 0] \cup [2, 4]$ (B) $(-2, 0) \cup (2, 4)$ (C) $[-2, 0] \cup [1, 3]$ (D) $(-2, 0) \cup (1, 3)$

G-2. The function $f(x) = \cot^{-1}\sqrt{(x + 3)\,x} + \cos^{-1}\sqrt{x^2 + 3x + 1}$ is defined on the set S, where S is equal to:
(A) {0, 3} (B) (0, 3) (C) {0, – 3} (D) [– 3, 0]

G-3. Domain of $f(x) = \cos^{-1}x + \cot^{-1}x + \mathrm{cosec}^{-1}x$ is
(A) [– 1, 1] (B) R (C) $(-\infty, -1] \cup [1, \infty)$ (D) {– 1, 1}

G-4. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is
(A) $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$ (B) $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ (C) $\left\{\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right\}$ (D) $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$

G-5. $\text{cosec}^{-1}(\cos x)$ is real if

(A) $x \in [-1, 1]$ (B) $x \in R$

(C) x is an odd multiple of $\frac{\pi}{2}$ (D) x is a multiple of π

G-6. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued 'x' is:

(A) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (C) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (D) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

G-7. The solution of the equation $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) - \frac{\pi}{6} = 0$ is

(A) x = 2 (B) x = – 4 (C) x = 4 (D) x = 3

G-8. Number of solutions of the equation $\cot^{-1}\sqrt{4-x^2} + \cos^{-1}(x^2-5) = \frac{3\pi}{2}$ is :

(A) 2 (B) 4 (C) 6 (D) 8

Section (H) : Trig (trig^{-1}x), trig^{-1} (trig x), trig^{-1} (–x) and Property (π/2)

H-1. If $\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is equal to

(A) x (B) $\pi - x$ (C) $2\pi + x$ (D) $2\pi - x$

H-2. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) π

H-3. If $x \ge 0$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then

(A) $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$ (B) $0 \le \theta \le \frac{\pi}{4}$ (C) $0 \le \theta < \frac{\pi}{2}$ (D) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

H-4. Number of solutions of equation $\tan^{-1}(e^{-x}) + \cot^{-1}(|\ell nx|) = \pi/2$ is :

(A) 0 (B) 1 (C) 3 (D) 2

Section (I) : Interconversion/Simplification

I-1. The numerical value of $\cot\left(2\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5}\right)$ is

(A) $\frac{-4}{3}$ (B) $\frac{-3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

I-2. **STATEMENT-1 :** $\tan^2(\sec^{-1}2) + \cot^2(\text{cosec}^{-1}3) = 11$.

STATEMENT-2 : $\tan^2\theta + \sec^2\theta = 1 = \cot^2\theta + \text{cosec}^2\theta$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

I-3. If α is a real root of the equation $x^3 + 3x - \tan 2 = 0$, then $\cot^{-1}\alpha + \cot^{-1}\frac{1}{\alpha} - \frac{\pi}{2}$ can be equal to

(A) 0 (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$

I-4. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right) + \tan^{-1}y = \frac{2\pi}{3}$, then :

(A) maximum value of $x^2 + y^2$ is $\frac{49}{3}$ (B) maximum value of $x^2 + y^2$ is 4

(C) minimum value of $x^2 + y^2$ is $\frac{1}{2}$ (D) minimum value of $x^2 + y^2$ is 3

I-5. If $x < 0$, then value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is equal to

(A) $\frac{\pi}{2}$ (B) $-\frac{\pi}{2}$ (C) 0 (D) $-\pi$

I-6. If $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(A) 0 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{3}}{2}$

I-7. The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$ is

(A) $\frac{-7}{17}$ (B) $\frac{7}{17}$ (C) $\frac{17}{7}$ (D) $-\frac{2}{3}$

Section (J) : Addition and Subtraction Rule

J-1. If $f(x) = \tan^{-1}\left(\frac{\sqrt{3}x - 3x}{3\sqrt{3} + x^2}\right) + \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$, $0 \le x \le 3$, then range of f(x) is

(A) $\left[0, \frac{\pi}{2}\right)$ (B) $\left[0, \frac{\pi}{4}\right]$ (C) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ (D) $\left[0, \frac{\pi}{3}\right]$

J-2. **STATEMENT-1 :** If $a > 0$, $b > 0$, $\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) = \frac{\pi}{2}$. $\Rightarrow$ $x = \sqrt{ab}$.

STATEMENT-2 : If $m, n \in N$, $n \ge m$, then $\tan^{-1}\left(\frac{m}{n}\right) + \tan^{-1}\left(\frac{n-m}{n+m}\right) = \frac{\pi}{4}$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

J-3. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to-

(A) $2\sin 2\alpha$ (B) 4 (C) $4\sin^2\alpha$ (D) $-4\sin^2\alpha$

PART - III : MATCH THE COLUMN

1. Match the relation defined on set A = {a,b,c} in column I with the corresponding type in column II

Column I		**Column II**	
(A)	{a,b), (b,a)	(p)	symmetric but not reflexive and transitive
(B)	{(a,b), (b,a), (a,a), (b,b)}	(q)	equivalence
(C)	{(a,b), (b,c), (a,c)}	(r)	symmetric and transitive but not reflexive
(D)	{(a,a), (b,b), (c,c)}	(s)	transitive but not reflexive and symmetric

2.

Column – I		**Column – II**	
(A)	If S be set of all triangles and $f : S \to R^+$, $f(\Delta)$ = Area of Δ, then f is	(p)	one-one
(B)	$f : R \to \left[\frac{3\pi}{4}, \pi\right)$ and $f(x) = \cot^{-1}(2x - x^2 - 2)$, then f(x) is	(q)	many one
(C)	If $f : R \to R$ such that $f(x) = \frac{2x^2 - x + 1}{7x^2 - 4x + 4}$, then f(x) is	(r)	onto function
(D)	$f : R \to R$ and $f(x) = e^{px} \sin qx$ where $p, q \in R^+$, then f(x) is	(s)	into function

3. **Match The column**

(A)	If f(x) is even & g(x) is odd	(p)	then fog must be odd
(B)	If g(x) is periodic	(q)	then fog must be manyone
(C)	If f(x) & g(x) are bijective	(r)	then fog is periodic
(D)	If f(x) is into	(s)	then fog is injective
		(t)	then fog is into

4. Let $f(x) = \sin^{-1} x$, $g(x) = \cos^{-1} x$ and $h(x) = \tan^{-1} x$. For what complete interval of variation of x the following are true.

Column – I		**Column – II**	
(A)	$f\left(\sqrt{x}\right) + g\left(\sqrt{x}\right) = \pi/2$	(p)	$[0, \infty)$
(B)	$f(x) + g\left(\sqrt{1-x^2}\right) = 0$	(q)	[0, 1]
(C)	$g\left(\frac{1-x^2}{1+x^2}\right) = 2h(x)$	(r)	$(-\infty, 1)$
(D)	$h(x) + h(1) = h\left(\frac{1+x}{1-x}\right)$	(s)	[– 1, 0]

5. **Match the column**

	Column - I		**Column - II**
(A)	Let a, b, c be three positive real numbers $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$, then θ is equal to	(p)	π
(B)	The value of the expression $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$ is equal to	(q)	$-\frac{\pi}{2}$
(C)	If x < 0, then $\frac{1}{2}\{\cos^{-1}(2x^2 - 1) + 2\cos^{-1} x\}$ is equal to	(r)	$-\pi$
(D)	The value of $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$ is equal to	(s)	$\frac{\pi}{2}$

Exercise-2

Marked questions are recommended for Revision.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. For real numbers x and y, we write x R y $\Rightarrow$ x – y + $\sqrt{2}$ is an irrational number. Then the relation R is-
(A) Reflexive (B) Symmetric (C) Transitive (D) Equivalence relation

2. Let A = N × N be the Certesian product of N and N. Let
S = {((m, n), (p, q)) $\in$ A × A : m + q = n + p}
Consider the following statements:
I.If ((m,n), (p , q)) $\in$ S, and ((p,q), (r, s)) $\in$ S then ((r,s), (m,n)) $\in$ S
II.There exists at least one element ((m,n), (p, q)) $\in$ S such that ((p , q), (m, n)) $\notin$ S
Which of the statements given above is / are correct ?
(A) I only (B) II only (C) Both I and II (D) Neither I nor II.

3. Let A = Z, the set of integers. Let R_1 = {(m, n) $\in$ Z × Z : (m + 4n) is divisible by 5 in Z}.
Let R_2 = {(m, n) $\in$ Z × Z : (m + 9n) is divisible by 5 in Z}.
Which one of the following is correct ?
(A) R_1 is a proper subset of R_2 (B) R_2 is a proper subset of R_1
(C) $R_1 = R_2$ (D) R_1 is not a symmetric relation on Z

4. Let X be the set of all persons living in a state. Elements x, y in X are said to be related if 'x < y', whenever y is 5 years older than x. Which one of the following is correct?
(A) The relation is an equivalence relation
(B) The relation is transitive only
(C) The relation is transitive and symmetric, but not reflexive
(D) The relation is neither reflexive, nor symmetric, nor transitive

5. The domain of the function $f(x) = \log_{1/2}\left(-\log_2\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$ is:
(A) $0 < x < 1$ (B) $0 < x \le 1$ (C) $x \ge 1$ (D) null set

6. If $q^2 - 4pr = 0$, $p > 0$, then the domain of the function $f(x) = \log(px^3 + (p+q)x^2 + (q+r)x + r)$ is:
(A) $R - \left\{-\frac{q}{2p}\right\}$ (B) $R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$
(C) $R - \left[(-\infty, -1) \cap \left\{-\frac{q}{2p}\right\}\right]$ (D) R

7. Let $f(x) = \frac{x-[x]}{1+x-[x]}$, R $\to$ A is onto then find set A. (where {.} and [.] represent fractional part and greatest integer part functions respectively)
(A) $\left(0, \frac{1}{2}\right]$ (B) $\left[0, \frac{1}{2}\right]$ (C) $\left[0, \frac{1}{2}\right)$ (D) $\left(0, \frac{1}{2}\right)$

8. Let f be a real valued function defined by $f(x) = \frac{e^x - e^{-|x|}}{e^x + e^{|x|}}$, then the range of f(x) is :
(A) R (B) [0, 1] (C) [0, 1) (D) $\left[0, \frac{1}{2}\right)$

9. The range of the function $f(x) = \log_{\sqrt{2}}\left(2 - \log_2\left(16\sin^2 x + 1\right)\right)$ is
(A) $(-\infty, 1)$ (B) $(-\infty, 2)$ (C) $(-\infty, 1]$ (D) $(-\infty, 2]$

10. Which of the following pair of functions are identical ?

(A) $\sqrt{1+\sin x}$, $\sin\frac{x}{2}+\cos\frac{x}{2}$ (B) x, $\frac{x^2}{x}$

(C) $\sqrt{x^2}$, $\left(\sqrt{x}\right)^2$ (D) $\ell n\, x^3 + \ell n\, x^2$, $5\,\ell n\, x$

11. If domain of f(x) is $(-\infty, 0]$, then domain of $f(6\{x\}^2 - 5\{x\} + 1)$ is (where $\{\cdot\}$ represents fractional part function).

(A) $\bigcup\limits_{n\in I}\left[n+\frac{1}{3}\ ,\ n+\frac{1}{2}\right]$ (B) $(-\infty, 0)$ (C) $\bigcup\limits_{n\in I}\left[n+\frac{1}{6}\ ,\ n+1\right]$ (D) $\bigcup\limits_{n\in I}\left[n-\frac{1}{2}, n-\frac{1}{3}\right]$

12. Let f: $(e, \infty) \to R$ be defined by $f(x) = \ell n(\ell n(\ell n\, x))$, then

(A) f is one one but not onto (B) f is onto but not one - one

(C) f is one-one and onto (D) f is neither one-one nor onto

13. If f (x) = 2 [x] + cos x, then f: R → R is: (where [ ] denotes greatest integer function)

(A) one–one and onto (B) one–one and into

(C) many–one and into (D) many–one and onto

14. If f : R → R be a function such that $f(x) = \begin{cases} x|x|-4 & ; x \in Q \\ x|x|-\sqrt{3} & ; x \notin Q \end{cases}$, then f(x) is

(A) one-one, onto (B) many one, onto (C) one-one, into (D) many one, into

15. $f(x) = |x-1|$, f: $R^+ \to R$, $g(x) = e^x$, g: $[-1, \infty) \to R$. If the function fog (x) is defined, then its domain and range respectively are:

(A) $(0, \infty)$ and $[0, \infty)$ (B) $[-1, \infty)$ and $[0, \infty)$

(C) $[-1, \infty)$ and $\left[1-\frac{1}{e}, \infty\right)$ (D) $[-1, \infty)$ and $\left[\frac{1}{e}-1, \infty\right)$

16. Let f : (2, 4) → (1, 3) be a function defined by $f(x) = x - \left[\frac{x}{2}\right]$ (where [.] denotes the greatest integer function), then $f^{-1}(x)$ is equal to :

(A) 2x (B) $x + \left[\frac{x}{2}\right]$ (C) x + 1 (D) x – 1

17. The mapping f : R → R given by $f(x) = x^3 + ax^2 + bx + c$ is a bijection if

(A) $b^2 \le 3a$ (B) $a^2 \le 3b$ (C) $a^2 \ge 3b$ (D) $b^2 \ge 3a$

18. If the function f: $[1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then f^{-1} is

(A) $(1/2)^{x(x-1)}$ (B) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

(C) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (D) Not defined

19. Let f : N → N, where $f(x) = x + (-1)^{x-1}$, then the inverse of f is.

(A) $f^{-1}(x) = x + (-1)^{x-1}$, $x \in N$ (B) $f^{-1}(x) = 3x + (-1)^{x-1}$, $x \in N$

(C) $f^{-1}(x) = x$, $x \in N$ (D) $f^{-1}(x) = (-1)^{x-1}$, $x \in N$

20. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$, $x \neq 0$ is equal to

(A) x (B) 2x (C) $\frac{2}{x}$ (D) $\frac{x}{2}$

21. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, where $\frac{\pi}{2} < x < \pi$, is:

(A) $\pi - \frac{x}{2}$ (B) $\frac{\pi}{2} + \frac{x}{2}$ (C) $\frac{x}{2}$ (D) $2\pi - \frac{x}{2}$

22. The domain of the function $f(x) = \sin^{-1}\left(\frac{1+x^3}{2x^{3/2}}\right) + \sqrt{\sin(\sin x)} + \log_{(3\{x\}+1)}(x^2+1)$,

where {.} represents fractional part function, is:

(A) $x \in \{1\}$ (B) $x \in R - \{1, -1\}$ (C) $x > 3, x \neq I$ (D) $x \in \phi$

23. A function g(x) satisfies the following conditions

(i) Domain of g is $(-\infty, \infty)$ (ii) Range of g is $[-1, 7]$

(iii) g has a period π and (iv) $g(2) = 3$

Then which of the following may be possible.

(A) $g(x) = 3 + 4\sin(n\pi + 2x - 4), n \in I$ (B) $g(x) = \begin{cases} 3 & ; \ x = n\pi \\ 3+4\sin x & ; \ x \neq n\pi \end{cases}$

(C) $g(x) = 3 + 4\cos(n\pi + 2x - 4), n \in I$ (D) $g(x) = 3 - 8\sin(n\pi + 2x - 4), n \in I$

24. The complete solution set of the inequality $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \leq 0$, where [.] denotes greatest integer function, is

(A) $(-\infty, \cot 3]$ (B) $[\cot 3, \cot 2]$ (C) $[\cot 3, \infty)$ (D) $(-\infty, \cot 2]$

25. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds for

(A) $x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$ (B) $x > 2 + \sqrt{9-2\pi}$

(C) $x < 2 - \sqrt{9-2\pi}$ (D) $x \in \phi$

26. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals

(A) 1/2 (B) 1 (C) – 1/2 (D) –1

27. $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$. then sin x is equal to -

(A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$ (C) $\tan\alpha$ (D) $\cot\left(\frac{\alpha}{2}\right)$

28. The Inverse trigonometric equation $\sin^{-1}x = 2\sin^{-1}\alpha$, has a solution for

(A) $-\frac{\sqrt{3}}{2} < \alpha < \frac{\sqrt{3}}{2}$ (B) all real values of α (C) $|\alpha| \leq \frac{1}{\sqrt{2}}$ (D) $|\alpha| \geq \frac{1}{\sqrt{2}}$

29. If $f(x) = \cot^{-1}x$: $R^+ \to \left(0, \frac{\pi}{2}\right)$

and $g(x) = 2x - x^2$: $R \to R$. Then the range of the function f(g(x)) wherever define is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{4}\right]$ (C) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left\{\frac{\pi}{4}\right\}$

30. Given the functions $f(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)}$, $g(x) = \mathrm{cosec}^{-1}\left(\frac{4-2\cos x}{3}\right)$ and the function

h(x) = f(x) defined only for those values of x, which are common to the domains of the functions f(x) and g(x). The range of the function h(x) is :

(A) $[e^{\frac{\pi}{6}}, e^{\pi}]$ (B) $[e^{-\frac{\pi}{6}}, e^{\pi}]$ (C) $(e^{\frac{\pi}{6}}, e^{\pi})$ (D) $[e^{-\frac{\pi}{6}}, e^{\frac{\pi}{6}}]$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. The domain of the function y = $\sqrt{\sin x + \cos x}$ + $\sqrt{7x - x^2 - 6}$ is $\left[p, \frac{q\pi}{4}\right] \cup \left[\frac{r\pi}{4}, s\right]$ then value of p + q + r + s is

2. The domain of f(x) such that the f(x) = $\dfrac{\left[x + \frac{1}{2}\right]}{\left[x - \frac{1}{2}\right]}$ is prime is $[x_1, x_2)$, then the value of $2(x_1^{\ 2} + x_2^2)$. [Where [.] denotes greatest integer function less than or equal to x]

3. Number of integers in the range of the function f(x) = $\dfrac{x^3 + 2x^2 + 3x + 2}{x^3 + 2x^2 + 2x + 1}$; $x \in R - \{0\}$ is :

4. Range of the function f(x) = |sin x |cos x| + cos x |sin x|| is [a, b] then (a + b) is equal to

5. If f and g are two distinct linear functions defined on R such that they map[–1, 1] onto [0, 2] and h : R – {–1, 0, 1} → R defined by h(x) = $\dfrac{f(x)}{g(x)}$, then |h(h(x)) + h(h(1/x))| > n. Then maximum integral value of n is :

6. If f(x) = $\dfrac{1}{1-x}$, g(x) = f (f(x)), h(x) = f(f(f(x))), then the absolute value of f(x) . g(x) . h(x), where x ≠ 0, 1, is

7. If f(x) = $ax^7 + bx^3 + cx - 5$; a, b, c are real constants and f(–7) = 7 then maximum value of |f(7)+17cosx| is

8. If f(x) = $\dfrac{4a-7}{3}$ $x^3 + (a - 3)x^2 + x + 5$ is a one-one function, then number of possible integral values of a is

9. Number of solutions of the equation $e^{-\sin^2 x}$ = tan2x in [0, 10π] is

10. Let f(x)= $([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x$ – (tan x) sgn (x) be an even function ∀ x ∈ R, then the sum of all possible values of '3a' is
(where [·] denotes G.I. F and {·} fractional part functional part function)

11. Let f be a one–one function with domain {21, 22, 23} and range {x,y,z}. It is given that exactly one of the following statements is true and the remaining two are false. f(21) = x; f(22) ≠ x ; f(23) ≠ y. Then $f^{-1}(x)$ is :

12. Let f : $[-\sqrt{2} + 1, \sqrt{2} + 1] \to \left[\frac{-\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right]$ be a function defined by f(x) = $\dfrac{1-x}{1+x^2}$.

If $f^{-1}(x) = \begin{cases} \dfrac{-1+\lambda\left(\sqrt{4x - 4x^2 + 1}\right)}{2x}, & x \neq 0 \\ \mu \quad , & x = 0 \end{cases}$, then $\lambda + \mu$ is.

13. The number of real solutions of the equation $x^3 + 1 = 2\sqrt[3]{2x-1}$, is :

14. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y, z \le 1$, then find the value of $x^2 + y^2 + z^2 + 2xyz$

15. The sum of absolute value of all possible values of x for which $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\dfrac{226}{227}}$.

16. If $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is:

17. If $x \in (0, 1)$ and $f(x) = \sec\left\{\tan^{-1}\left(\dfrac{\sin(\cos^{-1}x)+\cos(\sin^{-1}x)}{\cos(\cos^{-1}x)+\sin(\sin^{-1}x)}\right)\right\}$, then $\sum\limits_{r=2}^{10} f\left(\dfrac{1}{r}\right)$ is

18. If $\dfrac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \dfrac{\pi}{4}$, then $\tan\theta$ is equal to

19. The number of real solutions of equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$, $-10\pi \le x \le 10\pi$, is/are

20. The number of solution(s) of the equation, $\sin^{-1}x + \cos^{-1}(1-x) = \sin^{-1}(-x)$, is/are

21. Find the value of $3\sum\limits_{n=1}^{\infty}\left\{\dfrac{1}{\pi}\sum\limits_{k=1}^{\infty}\cot^{-1}\left(1+2\sqrt{\sum\limits_{r=1}^{k} r^3}\right)\right\}^n$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Let A = {1, 2, 3, 4} and R be a relation in A given by R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 1), (1, 3)}, then relation R is
(A) Reflexive (B) Symmetric (C) Equivalence (D) Reflexive and Symmetric

2. For $n, m \in N$, $n \mid m$ means that n is a factor of m, then relation | is
(A) Reflexive (B) symmetric (C) Transitive (D) Equivalence

3. If $f(x) = \sin \ell n\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$, then
(A) domain of f(x) is $(-2, 1)$
(B) domain of f(x) is $[-1, 1]$
(C) range of f(x) is $[-1, 1]$
(D) range of f(x) is $[-1, 1)$

4. D is domain and R is range of $f(x) = \sqrt{x-1} + 2\sqrt{3-x}$, then
(A) D : [1 , 3] ;
(B) D : $(-\infty, 1] \cup [3, \infty)$,
(C) R : $\left[1, \sqrt{3}\right]$
(D) R : $\left[\sqrt{2}, \sqrt{10}\right]$

5. If $[2\cos x] + [\sin x] = -3$, then the range of the function, $f(x) = \sin x + \sqrt{3}\cos x$ in $[0, 2\pi]$ lies in (where [ ] denotes greatest integer function)
(A) $[-\sqrt{3}, \sqrt{3})$ (B) $[-2, -\sqrt{3}]$ (C) $[-3, -1]$ (D) $[-2, -\sqrt{3})$

6. Let $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective.
(A) $f(x) = \begin{cases} \tan^{-1}\dfrac{1}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$
(B) $g(x) = x^3$
(C) $h(x) = \sin 2x$
(D) $k(x) = \sin(\pi x/2)$

7. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If f(x) divided by $x^3 - x$, then the remainder is some function of x say g(x). Then g(x) is an :
(A) one-one function (B) many one function (C) into function (D) onto function

8. The function f : X $\to$ Y, defined by $f(x) = x^2 - 4x + 5$ is both one–one and onto if
(A) X = [2, ∞) & Y = [1, ∞)
(B) X = ($-\infty$, 2] & Y = [1, ∞)
(C) X = [3, ∞) & Y = [2, ∞)
(D) X = ($-\infty$, 2] & Y = (1, ∞)

9. f : N $\to$ N where $f(x) = x - (-1)^x$ then f is :
(A) one-one (B) many-one (C) onto (D) into

10. Which one of the following pair of functions are **NOT** identical ?
(A) $e^{(\ell nx)/2}$ and $\sqrt{x}$
(B) tan (tanx) and cot (cotx)
(C) $\cos^2x + \sin^4x$ and $\sin^2x + \cos^4x$
(D) $\frac{|x|}{x}$ and sgn (x), where sgn(x) stands for signum function.

11. If the graph of the function f (x) = $\frac{a^x - 1}{x^n(a^x+1)}$ is symmetric about y-axis, then n is equal to:
(A) 1/5 (B) 1/3 (C) 1 / 4 (D) – 1 / 3

12. If f(x) = $\begin{cases} x^2 & x \le 1 \\ 1-x & x > 1 \end{cases}$ & composite function h(x) = |f(x)| + f(x + 2), then
(A) $h(x) = 2x^2 + 4x + 4 \quad \forall\ x \le -1$
(B) $h(x) = x^2 + x + 1 \quad \forall\ -1 < x \le 1$
(C) $h(x) = x^2 - x - 1 \quad \forall\ -1 < x \le 1$
(D) $h(x) = -2 \quad \forall\ x > 1$

13. Let f(x) = $\begin{cases} 0 & \text{for } x = 0 \\ x^2 \sin\left(\frac{\pi}{x}\right) & \text{for } -1 < x < 1\ (x \ne 0) \\ x\ |x| & \text{for } x > 1 \text{ or } x < -1 \end{cases}$, then:
(A) f(x) is an odd function
(B) f(x) is an even function
(C) f(x) is neither odd nor even
(D) $f'(x)$ is an even function

14. If f : [–2, 2] $\to$ R where f(x) = $x^3 + \tan x + \left[\frac{x^2+1}{P}\right]$ is a odd function, then the value of parametric P, where [.] denotes the greatest integer function, can be
(A) 5 < P < 10 (B) P < 5 (C) P > 5 (D) P = 15

15. If f: R $\to$ [– 1, 1], where f (x) = $\sin\left(\frac{\pi}{2}[x]\right)$, (where [.] denotes the greatest integer function), then
(A) f (x) is onto (B) f (x) is into (C) f (x) is periodic (D) f (x) is many one

16. If f (x) = $\frac{2x\ (\sin x + \tan x)}{2\left[\frac{x+2\pi}{\pi}\right] - 3}$ then it is, (where [.] denotes the greatest integer function)
(A) odd (B) Even (C) many one (D) one-one

17. Identify the statement(s) which is/are incorrect ?
(A) the function f(x) = sinx + cosx is neither odd nor even
(B) the fundamental period of f(x) = cos (sin x) + cos (cos x) is π
(C) the range of the function f(x) = cos (3 sin x) is [– 1, 1]
(D) f(x) = 0 is a periodic function with period 2

18. If $F(x) = \dfrac{\sin \pi [x]}{\{x\}}$, then F (x) is: (where { . } denotes fractional part function and [ ] denotes greatest integer function and sgn (x) is a signum function)
(A) periodic with fundamental period 1
(B) even
(C) range is singleton
(D) identical to $\text{sgn}\left(\text{sgn}\dfrac{\{x\}}{\sqrt{\{x\}}}\right) - 1$

19. Let $f : R \to R$ and $g : R \to R$ be two one-one and onto functions such that they are mirror images of each other about the line y = a. If h(x) = f(x) + g(x), then h(x) is
(A) one-one
(B) into
(C) onto
(D) many-one

20. Which of following pairs of functions are identical.
(A) $f(x) = e^{\ell n \sec^{-1} x}$ and $g(x) = \sec^{-1} x$
(B) $f(x) = \tan(\tan^{-1} x)$ and $g(x) = \cot(\cot^{-1} x)$
(C) f(x) = sgn (x) and g(x) = sgn (sgn (x))
(D) $f(x) = \cot^2 x. \cos^2 x$ and $g(x) = \cot^2 x - \cos^2 x$

21. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$, then
(A) $x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 0$
(B) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$
(C) $x^{50} + y^{25} + z^{5} = 0$
(D $\dfrac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$)

22. If X = cosec $\tan^{-1}$ cos $\cot^{-1}$ sec $\sin^{-1}$ a and Y = sec $\cot^{-1}$ sin $\tan^{-1}$ cosec $\cos^{-1}$ a; where $0 \le a < 1$. Find the relation between X and Y. Then
(A) X = Y
(B) $Y = \sqrt{3 - a^2}$
(C) $X \neq Y$
(D) X = 2Y

23. If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for
(A) $\sin^{-1} \alpha$
(B) $\cos^{-1} \alpha$
(C) $\sec^{-1} \alpha$
(D) $\text{cosec}^{-1} \alpha$

24. For the function $f(x) = \ell n (\sin^{-1} \ell og_2 x)$,
(A) Domain is $\left[\dfrac{1}{2}, 2\right]$
(B) Range is $\left(-\infty, \ell n\dfrac{\pi}{2}\right]$
(C) Domain is (1, 2]
(D) Range is R

25. In the following functions defined from [–1, 1] to [–1, 1] , then functions which are not bijective are
(A) $\sin(\sin^{-1}x)$
(B) $\dfrac{2}{\pi}\sin^{-1}(\sin x)$
(C) $(\text{sgn } x)\ \ell n\ e^x$
(D) $x^3 \text{ sgn } x$

26. The expression $\dfrac{1}{\sqrt{2}}\left\{\dfrac{\sin\cot^{-1}\cos\tan^{-1} t}{\cos\tan^{-1}\sin\cot^{-1}\sqrt{2}\,t}\right\}.\left\{\sqrt{\dfrac{1+2t^2}{2+t^2}}\right\}$ can take the value
(A) 1/2
(B) –5
(C) 1
(D) 3/4

27. If 0 < x < 1, then $\tan^{-1}\dfrac{\sqrt{1-x^2}}{1+x}$ is equal to:
(A) $\dfrac{1}{2}\cos^{-1} x$
(B) $\cos^{-1}\sqrt{\dfrac{1+x}{2}}$
(C) $\cos^{-1}\sqrt{\dfrac{1-x}{2}}$
(D) $\dfrac{1}{2}\sin^{-1}x$

28. If $f(x) = \cos^{-1}x + \cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\}$, then

(A) $f\left(\frac{2}{3}\right) = \frac{\pi}{3}$ (B) $f\left(\frac{2}{3}\right) = \frac{\pi}{2}$

(C) $f\left(\frac{1}{3}\right) = \frac{\pi}{3}$ (D) $f\left(\frac{1}{3}\right) = 2\cos^{-1}\frac{1}{3} - \frac{\pi}{3}$

29. $\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4 - 2n^2 + 2}$ is equal to:

(A) $\tan^{-1} 2 + \tan^{-1} 3$ (B) $4\tan^{-1} 1$ (C) $\pi/2$ (D) $\sec^{-1}\left(-\sqrt{2}\right)$

30. If $\sin^2 (2\cos^{-1} (\tan x)) = 1$ then x may be

(A) $x = \pi + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (B) $x = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(C) $x = -\pi + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (D) $x = -\pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

31. If $\sin^{-1} x + 2\cot^{-1} (y^2 - 2y) = 2\pi$, then

(A) $x + y = y^2$ (B) $x^2 = x + y$ (C) $y = y^2$ (D) $x^2 - x + y = y^2$

PART - IV : COMPREHENSION

Comprehension # 1

Given a function $f : A \to B$; where A = {1, 2, 3, 4, 5} and B = {6, 7, 8}

1. Find number of all such functions y = f(x) which are one-one ?

(A) 0 (B) 3^5 (C) ${}^5P_3$ (D) 5^3

2. Find number of all such functions y = f(x) which are onto

(A) 243 (B) 93 (C) 150 (D) none of these

3. The number of mappings of $g(x) : B \to A$ such that $g(i) \le g(j)$ whenever $i < j$ is

(A) 60 (B) 140 (C) 10 (D) 35

Comprehension # 2

Let the domain and range of inverse circular functions are defined as follows

	Domain	**Range**
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

4. ✍ $\sin^{-1}x < \dfrac{3\pi}{4}$ then solution set of x is

(A) $\left(\dfrac{1}{\sqrt{2}}, 1\right]$ (B) $\left(-\dfrac{1}{\sqrt{2}}, -1\right]$ (C) $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$ (D) none of these

5. ✍ If $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, cosec^{-1} cosec x is

(A) $2\pi - x$ (B) $\pi + x$ (C) $\pi - x$ (D) $-\pi - x$

6. ✍ If $x \in [-1, 1]$, then range of $\tan^{-1}(-x)$ is

(A) $\left[\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right]$ (B) $\left[\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right]$ (C) $[-\pi, 0]$ (D) $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

Exercise-3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**
✍ Marked questions are recommended for Revision.

1. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is **[IIT-JEE 2009,P-2,(4, –1), 80]**

2. ✍ If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[IIT-JEE 2009,P-2,(4, –1), 80]**

3. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying (f o g o g o f) (x) = (g o g o f) (x), where (f o g) (x) = f(g(x)), is **[IIT-JEE 2011, Paper-2, (3, –1), 80]**

(A) $\pm\sqrt{n\pi}$, $n \in \{0, 1, 2, ...\}$ (B) $\pm\sqrt{n\pi}$, $n \in \{1, 2, ...\}$

(C) $\dfrac{\pi}{2} + 2n\pi$, $n \in \{.....-2, -1, 0, 1, 2, ...\}$ (D) $2n\pi$, $n \in \{...., -2, -1, 0, 1, 2, ...\}$

4. Let $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$. Then the value of $\dfrac{d}{d(\tan\theta)}(f(\theta))$ is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

5. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

(A) one-one and onto (B) onto but not one-one
(C) one-one but not onto (D) neither one-one nor onto

[IIT-JEE 2012, Paper-1, (3, –1), 70]

6*. Let $f : (-1, 1) \to \text{IR}$ be such that $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value(s) of $f\left(\dfrac{1}{3}\right)$ is (are) **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) $1 - \sqrt{\dfrac{3}{2}}$ (B) $1 + \sqrt{\dfrac{3}{2}}$ (C) $1 - \sqrt{\dfrac{2}{3}}$ (D) $1 + \sqrt{\dfrac{2}{3}}$

7. The value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$ is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) $\dfrac{23}{25}$ (B) $\dfrac{25}{23}$ (C) $\dfrac{23}{24}$ (D) $\dfrac{24}{23}$

8. Match List I with List II and select the correct answer using the code given below the lists :

List - I		List - II
P	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$ takes value	1. $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q.	If cos x + cos y + cox z = 0 = sin x + sin y + sin z then possible value of cos $\frac{x-y}{2}$ is	2. $\sqrt{2}$
R.	If cos $\left(\frac{\pi}{4}-x\right)$cos 2x + sin x sin 2x sec x = cos xsin 2x sec x + cos$\left(\frac{\pi}{4}+x\right)$ cos 2x then possible value of sec x is	3. $\frac{1}{2}$
S.	If cot $\left(\sin^{-1}\sqrt{1-x^2}\right)$ = sin $\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then possible value of x is	4. 1

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

9*. Let f: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to R$ be given by f(x) = (log(sec x + tanx))^3. Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) is an odd function
(B) f(x) is a one-one function
(C) f(x) is an onto function
(D) f(x) is an even function

10. Let f : [0, 4π] → [0, π] be defined by f(x) = cos^{-1} (cos x). The number of points x ∈ [0, 4π] satisfying the equation f(x) = $\frac{10-x}{10}$ is **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

11*. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) cos β > 0
(B) sin β < 0
(C) cos(α + β) > 0
(D) cos α < 0

12. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is __________. **[JEE(Advanced) 2018, Paper-1,(3, 0)/60]**

(Here, the inverse trigonometric functions sin^{-1}x and cos^{-1}x assume values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and [0, π], respectively).

13. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto function form Y to X, then the value of $\frac{1}{5!}(\beta-\alpha)$ is ______ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

14. Let $E_1 = \left\{x \in R : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \to R$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to R$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$

[JEE(Advanced) 2018, Paper-2,(3, –1)/60]

LIST-I	**LIST-II**
(P) The range of f is	(1) $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
(Q) The range of g contains	(2) (0, 1)
(R) The domain of f contains	(3) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(S) The domain of g is	(4) $(-\infty, 0) \cup (0, \infty)$
	(5) $\left(-\infty, \frac{e}{e-1}\right]$
	(6) $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is
(A) $P \to 4; Q \to 2; R \to 1; S \to 1$
(B) $P \to 3; Q \to 3; R \to 6; S \to 5$
(C) $P \to 4; Q \to 2; R \to 1; S \to 6$
(D) $P \to 4; Q \to 3; R \to 6; S \to 5$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Consider the following relations : **[AIEEE-2010, (4, – 1), 144]**
R : {(x, y)| x ,y are real numbers and x = wy for some rational number w}
S = { $\left(\frac{m}{n}, \frac{p}{q}\right)$ | m, n, p and q are integers such that n, q ≠ 0 and qm = pn}
Then
(1) neither R nor S is an equivalence relation
(2) S is an equivalence relation but R is not an equivalence relation
(3) R and S both are equivalence relations
(4) R is an equivalence relation but S is not an equivalence relation

2.✍ Let R be the set of real numbers. **[AIEEE-2011(Part-I), (4, – 1), 120]**
Statement-1 : A = {(x, y) ∈ R × R : y – x is an integer} is an equivalence relation on R.
Statement-2 : B = {(x, y) ∈ R × R : x = αy for some rational number α} is an equivalence relation on R.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

3.✍ Consider the following relation R on the set of real square matrices of order 3.
R = {(A, B)|A = P^{-1} BP for some invertible matrix P}. **[AIEEE-2011(Part-II), (3, – 1), 120]**
Statement -1 : R is equivalence relation.
Statement - 2 : For any two invertible 3 × 3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$.
(1) Statement-1 is true, statement-2 is a correct explanation for statement-1.
(2) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is false.
(4) Statement-1 is false, statement-2 is true.

4. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is : **[AIEEE 2011, I, (4, –1), 120]**

(1) $(-\infty, \infty)$ (2) $(0, \infty)$ (3) $(-\infty, 0)$ (4) $(-\infty, \infty) - \{0\}$

5. Let f be a function defined by $f(x) = (x-1)^2 + 1$, $(x \geq 1)$. **[AIEEE 2011, II, (4, –1), 120]**
Statement - 1 : The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$.
Statement - 2 : f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \geq 1$.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true .

6.✍ If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[AIEEE - 2013, (4, –1),120]**
(1) $x = y = z$ (2) $2x = 3y = 6z$ (3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$

7. If g is the inverse of a function f and $f'(x) = \dfrac{1}{1+x^5}$, then g'(x) equal to :

[JEE(Main)2014,(4, – 1), 120]

(1) $\dfrac{1}{1+\{g(x)\}^5}$ (2) $1 + \{g(x)\}^5$ (3) $1 + x^5$ (4) $5x^4$

8. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, where $|x| < \dfrac{1}{\sqrt{3}}$. Then a value of y is

[JEE(Main)2015,(4, – 1), 120]

(1) $\dfrac{3x - x^3}{1 - 3x^2}$ (2) $\dfrac{3x + x^3}{1 - 3x^2}$ (3) $\dfrac{3x - x^3}{1 + 3x^2}$ (4) $\dfrac{3x + x^3}{1 + 3x^2}$

9. If $f(x) + 2f\left(\dfrac{1}{x}\right) = 3x$, $x \neq 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S : **[JEE(Main)2016,(4, – 1), 120]**

(1) contains exactly one element (2) contains exactly two elements.
(3) contains more than two elements. (4) is an empty set.

10. Two sets A and B are as under : $A = \{(a, b) \in R \times R : |a - 5| < 1 \text{ and } |b - 5| < 1\}$;
$B = \{(a, b) \in R \times R : 4(a-6)^2 + 9(b-5)^2 \leq 36\}$. Then; **[JEE(Main)2018,(4, – 1), 120]**
(1) $A \cap B = \phi$ (an empty set) (2) Neither $A \subset B$ nor $B \subset A$
(3) $B \subset A$ (4) $A \subset B$

11. If $\cos^{-1}\left(\dfrac{2}{3x}\right) + \cos^{-1}\left(\dfrac{3}{4x}\right) = \dfrac{\pi}{2}\left(x > \dfrac{3}{4}\right)$ then x is equal to :

[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

(1) $\dfrac{\sqrt{145}}{12}$ (2) $\dfrac{\sqrt{145}}{10}$ (3) $\dfrac{\sqrt{146}}{12}$ (4) $\dfrac{\sqrt{145}}{11}$

12.✍ For $x \in R - \{0, 1\}$, let $f_1(x) = \dfrac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \dfrac{1}{1-x}$ be three given functions. If a function, J(x) satisfies $(f_2 \circ J \circ f_1)(x) = f_3(x)$ is equal to : **[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]**

(1) $\dfrac{1}{x}f_3(x)$ (2) $f_1(x)$ (3) $f_3(x)$ (4) $f_2(x)$

13. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is : **[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**

(1) $\frac{19}{21}$ (2) $\frac{21}{19}$ (3) $\frac{22}{23}$ (4) $\frac{23}{22}$

14. The number of functions f from {1, 2, 3, …., 20}, onto {1,2,3,………..,20} such that f(k) is a multiple of 3, wherever k is a multiple of 4, is : **[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]**

(1) $5! \times 6!$ (2) $(15)! \times 6!$ (3) $6^5 \times (15)!$ (4) $5^6 \times 15$

15. Let Z be the set of integers. If $A = \{x \in Z : 2^{(x+2)(x^2-5x+6)} = 1]$ and B = $\{x \in Z : -3 < 2x - 1 < 9\}$ then the number of subsets of the set A × B, is – **[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]**

(1) 2^{18} (2) 2^{12} (3) 2^{15} (4) 2^{10}

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. 9 **A-2.** {(2, 4), (3, 4)} **A-3.** 2^{12} **A-4.** R = { (2, 1), (3, 3), (4, 5), (5, 7)}

A-5. (i) {– 3, – 2, – 1, 0, 1, 2, 3} (ii) {0, 1, 2, 3, 4}
(iii) {(–3, 4), (–2, 3), (–1, 2), (0, 1), (1, 0), (2, 1), (3, 2)}

A-6. {(–1,0), (–1,1), (0,2), (1,2)}

Section (B) :

B-1. (i) Reflexive and transitive but not symmetric.
(ii) neither reflexive nor transitive but it is symmetric

B-3. (iii) & (v) **B-6.** Reflexive and symmetric but not transitive

Section (C) :

C-1. (i) yes (ii) no (iii) no (iv) no

C-2. (i) R – {–1, 1} (ii) $2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}$, $n \in I$ (iii) $(0, \infty)$ (iv) R (v) $[-2, 0) \cup (0, 1)$
(vi) (2, 3] (vii) $(-\infty, -1] \cup [0, \infty)$ (viii) $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$

C-3. (i) [0, 1] (ii) [– 1, 1] (iii) R (iv) ϕ
(v) $\bigcup\limits_{n\in I} \left[n\pi, n\pi + \frac{\pi}{4}\right]$ (vi) R – {2nπ}, $n \in I$ (vii) $(0, 1] \cup [4,5)$ (viii) (2, 3)

C-4. (i) $[0, \infty)$ (ii) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (iii) [0, 4] (iv) {–1, 1}

C-5. (i) Domain : R, Range : $\frac{1}{\sqrt{7}} \le y \le 1$ (ii) Domain : $N \cup \{0\}$, Range : (n! : n = 0, 1, 2,...}
(iii) Domain R – {3}, Range : R – {6} (iv) Domain : R, Range : {1}

C-6. (i) [0, 10] (ii) (0, 1] (iii) $(-\infty\ \frac{49}{20},]$ (iv) [–4, 3] (v) [–1, 1] (vi) R
(vii) $n \in N$

C-7. (i) $(-\infty, 1]$ (ii) $\left[\frac{1}{\sqrt{16 - 1/\sqrt{2}}}, \infty\right)$ (iii) $\left[\frac{1}{3}, 1\right]$
(iv) $\left(-\infty, -\frac{1}{4}\right] \cup \left[-\frac{1}{20}, \infty\right)$ (v) $\left[\frac{1}{3}, 3\right]$ (vi) $\left[0, \frac{3}{\sqrt{2}}\right]$
(vii) $[4, \infty)$ (viii) [–11, 16] (ix) $\left[\frac{3}{4}, 1\right]$
(x) 1 (xi) $\left[1 - \sin\sqrt{2}, 1 + \sin\sqrt{2}\right]$

C-8. (i) many-one & into (ii) many-one & into (iii) one-one & onto (iv) many-one & into
(v) one – one & into (vi) many-one & into (vii) many-one & into (viii) many-one & onto
(ix) many-one & into

C-9. (i) bijective (injective as well as surjective) (ii) neither surjective nor injective
(iii) surjective but not injective

C-10. (i) many-one & onto (ii) many-one & into

C-11. (i) No (ii) Yes (iii) Yes (iv) No **C-12.** n^n , n!

Section (D) :

D-1. (i) No (ii) No (iii) No (iv) No **D-2.** $(2, \infty)$

D-4. $[fo(goh)](x) = [(fog)oh](x) = \sin^2 \sqrt{x}$

D-5. (i) $fog = x, x > 0$; $gof = x, x \in R$ (ii) $|\sin x|, \sin|x|$

(iii) $\sin(x^2), (\sin x)^2$ (iv) $\dfrac{3x^2-4x+2}{(x-1)^2}, \dfrac{x^2+2}{x^2+1}$

D-6. Domain : [1 , 2] ; Range : $[\ell n2, \ell n4)$ **D-7.** $f(g(x)) = \begin{bmatrix} 2-2x+x^2, & 0 \le x \le 1 \\ 2-x, & -1 \le x < 0 \end{bmatrix}$

D-8. **(i)** $x \in R - \{0, 1\}$ **(ii)** $x \in R - \{-2, -1\}$

(iii) $x \in R - \left\{-\dfrac{3}{2}, -1\right\}$ **(iv)** $x \in R - \{-2, -1\}$

D-9. $fof(x) = \begin{cases} 3\sqrt{2}\ x & x \in Q - \{0\} \\ 3^2\ x & x \in Q^c \end{cases}$, $fofof.....f(x) = \begin{cases} 3^{n-1}\sqrt{2}x & x \in Q - \{0\} \\ 3^n x & x \in Q^c \end{cases}$

D-10. $f(g(x)) = \begin{cases} x^2+1 & x \in [-1, 2] \\ 2x^2+1 & x \in (2, 3) \\ 2x+5 & x \in [3, 5] \end{cases}$

Section (E) :

E-1. (i) even, (ii) neither even nor odd (iii) even, (iv) neither even nor odd
(v) even (vi) even

E-2. (i) neither even nor odd (ii) even (iii) odd (iv) even

E-4. (i) 2π (ii) 2π (iii) 24 (iv) 70π (v) 2π (vi) $\pi/6$ (vii) 2π

Section (F) :

F-1. **(i)** f^{-1} Does not exists **(ii)** $f^{-1} : R \to R$; $f^{-1} = 7 + (4 - x^5)^{1/3}$

(iii) $f^{-1} : R \to R$; $f^{-1} = \dfrac{e^x - e^{-x}}{2}$ **(iv)** $f^{-1}(x) = \dfrac{-1+\sqrt{4x-3}}{2}$

F-2. $f^{-1} : R \to R, f^{-1}(x) = \dfrac{1}{2}\ \ell n\,(x + \sqrt{x^2+1})$ **F-3.** **(a)** $f^{-1}(x) = \begin{cases} \sqrt{-x} & x \le 0 \\ -\sqrt{x} & x > 0 \end{cases}$, 3 **(b)** $x = \dfrac{3-\sqrt{5}}{2}$

F-4. 1 **F-5.** $\alpha = 2$

Section (G) :

G-1. (i) $[-1, 1] - \{0\}$ (ii) $\left[-\dfrac{1}{3}, \dfrac{1}{2}\right]$ (iii) ϕ **G-2.** (i) $(-\infty, \ell n\,\pi/2]$(ii) $(0, \pi/2]$ (iii) $[0, \pi]$

G-3. (i) 1 (ii) $\dfrac{1}{\sqrt{3}}$ (iii) $\dfrac{\pi}{6}$ **G-4.** (i) $n\left(\dfrac{n+1}{2}\right)$

G-5. (i) $[-1, 0)$ (ii) $(-\infty, \cot 3)\ U\ (\cot 2, \infty)$
(iii) $-\sin 1 < x \le 1$ (iv) $\cos 2 < x \le 1$ (v) no solution

G–6. $B = [0, 4]$; $f^{-1}(x) = \dfrac{1}{2}\left(\sin^{-1}\left(\dfrac{x-2}{2}\right) - \dfrac{\pi}{6}\right)$

Section (H) :

H-1. (i) $-\dfrac{\pi}{6}$ (ii) $-\dfrac{\pi}{3}$ (iii) $\dfrac{3\pi}{4}$ (iv) $\dfrac{\pi}{4}$

H-2. (i) $\pi - 4$ (ii) $4\pi - 10$ (iii) $2\pi - 6$ (iv) $4\pi - 10$ (v) $\dfrac{17\pi}{20}$

H-3. (i) 0 (ii) 1 **H-4.** $x > 1$

Section (I) :

I-1. (i) $\frac{4}{5}$ (ii) $2\sqrt{2}$ (iii) $\frac{\sqrt{41}}{4}$ (iv) $\frac{63}{16}$ (v) $\frac{1+3\sqrt{5}}{8}$ (vi) $\frac{6-4\sqrt{5}}{15}$ (vii) 2 (viii) $\frac{\sqrt{5}}{3}$ (ix) $\frac{1}{2\sqrt{2}}$

I-2. $\frac{\pi}{2}$ **I-7.** (i) $2\tan^{-1}x - \pi$ (ii) $\pi - 2\sin^{-1}x$ (iii) $2\pi - 2\cos^{-1}x$

I-8. $\frac{1+xy}{x-y}$

I-9 (i) $\pm\frac{1}{\sqrt{3}}$ (ii) $x = 3$ (iii) $\pm\frac{1}{\sqrt{2}}$ (iv) $x = \frac{1}{2}$

Section (J) :

J-2. (i) $\tan^{-1}(x + n) - \tan^{-1} x$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{2}$

PART - II

Section (A) :

A-1. (C) **A-2.** (D) **A-3.** (C) **A-4.** (C) **A-5.** (D) **A-6.** (A)

Section (B) :

B-1. (A) **B-2.** (C) **B-3.** (A) **B-4.** (B) **B-5.** (B) **B-6.** (A)
B-7. (B) **B-8.** (D) **B-9.** (D) **B-10.** (B) **B-11.** (D)

Section (C) :

C-1. (D) **C-2.** (A) **C-3.** (B) **C-4.** (B) **C-5.** (D) **C-6.** (A) **C-7.** (D)
C-8. (B) **C-9.** (C) **C-10.** (B) **C-11.** (A) **C-12.** (D) **C-13.** (A)

Section (D) :

D-1. (A) **D-2.** (C) **D-3.** (B)

Section (E) :

E-1. (B) **E-2.** (B) **E-3.** (B) **E-4.** (C) **E-5.** (D) **E-6.** (C)

Section (F) :

F-1. (A) **F-2.** (A) **F-3.** (A) **F-4.** (A) **F-5.** (A) **F-6.** (B)

Section (G) :

G-1. (A) **G-2.** (C) **G-3.** (D) **G-4.** (C) **G-5.** (D) **G-6.** (A) **G-7.** (C)
G-8. (A)

Section (H) :

H-1. (D) **H-2.** (B) **H-3.** (D) **H-4.** (D)

Section (I) :

I-1. (B) **I-2.** (C) **I-3.** (C) **I-4.** (A) **I-5.** (B) **I-6.** (B) **I-7.** (A)

Section (J) :

J-1. (B) **J-2.** (B) **J-3.** (C)

PART – III

1. (1) $\rightarrow$ (p), (2) $\rightarrow$ (r), (3) $\rightarrow$ (s), (4) $\rightarrow$ (q)

2. (A) $\rightarrow$ (q,r), (B) $\rightarrow$ (q,r), (C) $\rightarrow$ (q,s), (D) $\rightarrow$ (q,r),

3. (A$\rightarrow$ q ; B$\rightarrow$ r,q ; C$\rightarrow$ s ; D$\rightarrow$ t)

4. (A) $\rightarrow$ (q), (B) $\rightarrow$ (s), (C) $\rightarrow$ (p), (D) $\rightarrow$ (r),

5. (A) $\rightarrow$ (p), (B) $\rightarrow$ (p), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)

EXERCISE - 2

PART - I

1.	(A)	**2.**	(A)	**3.**	(C)	**4.**	(D)	**5.**	(D)	**6.**	(B)	**7.**	(C)
8.	(D)	**9.**	(D)	**10.**	(D)	**11.**	(A)	**12.**	(C)	**13.**	(C)	**14.**	(D)
15.	(B)	**16.**	(C)	**17.**	(B)	**18.**	(B)	**19.**	(A)	**20.**	(C)	**21.**	(B)
22.	(D)	**23.**	(A)	**24.**	(A)	**25.**	(A)	**26.**	(B)	**27.**	(A)	**28.**	(C)
29.	(C)	**30.**	(A)										

PART - II

1.	17	**2.**	17	**3.**	0	**4.**	1	**5.**	2	**6.**	1	**7.**	34
8.	7	**9.**	20	**10.**	35	**11.**	22	**12.**	2	**13.**	3	**14.**	1
15.	30	**16.**	5	**17.**	54	**18.**	3	**19.**	20	**20.**	1	**21.**	1

PART – III

1.	(ABD)	**2.**	(AC)	**3.**	(AC)	**4.**	(AD)	**5.**	(BCD)	**6.**	(BD)	**7.**	(AD)
8.	(ABC)	**9.**	(AC)	**10.**	(ABD)	**11.**	(ABD)	**12.**	(ACD)	**13.**	(AD)	**14.**	(ACD)
15.	(BCD)	**16.**	(AC)	**17.**	(BC)	**18.**	(ABCD)	**19.**	(BD)	**20.**	(BCD)	**21.**	(AB)
22.	(AB)	**23.**	(CD)	**24.**	(BC)	**25.**	(BCD)	**26.**	(AD)	**27.**	(AB)	**28.**	(AD)
29.	(AD)	**30.**	(ABCD)	**31.**	(CD)								

PART - IV

1.	(A)	**2.**	(C)	**3.**	(D)	**4.**	(A)	**5.**	(C)	**6.**	(B)

EXERCISE - 3

PART - I

1.	7	**2.**	2	**3.**	(A)	**4.**	**1**	**5.**	(B)	**6*.**	(AB)	**7.**	(B)
8.	(B)	**9.**	(ABC)	**10.**	**3**	**11.**	(BCD)	**12.**	2	**13.**	119	**14.**	(A)

PART – II

1.	(2)	**2.**	(3)	**3.**	(2)	**4.**	(3)	**5.**	(1)	**6.**	(1)	**7.**	(2)
8.	(1)	**9.**	(2)	**10.**	(4)	**11.**	(1)	**12.**	(3)	**13.**	(2)	**14.**	(2)
15.	(3)												

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

✍ Marked Questions may have for Revision Questions.

1. Find the domain of the function f(x) = $\sqrt{-\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right)}$

2. Let f (x) = $(x^{12} - x^9 + x^4 - x + 1)^{-1/2}$. The domain of the function is :

3. Find the values of 'a' in the domain of the definition of the function , f (a) = $\sqrt{2a^2 - a}$ for which the roots of the equation , $x^2 + (a + 1) x + (a - 1) = 0$ lie between -2 & 1 .

4. The domain of the function f (x) = $\sqrt{\dfrac{1}{(|x|-1)\cos^{-1}(2x+1)\ .\ \tan 3x}}$ is:

5. Find domain of the following functions

(i) f(x) = $\sqrt{\log_{1/3}\log_4([x]^2-5)}$, where [.] denotes greatest integer function.

(ii) f (x) = $\dfrac{1}{[|x-1|] + [|12-x|] - 11}$, where [x] denotes the greatest integer not greater than x.

(iii) f(x) = $(x+0.5)^{\log_{(0.5+x)}\frac{x^2+2x-3}{4x^2-4x-3}}$

(iv) f (x) = $\dfrac{5}{\left[\dfrac{x-1}{2}\right]} - 3^{\sin^{-1}x^2} + \dfrac{(7x+1)!}{\sqrt{x+1}}$, where [.] denotes greatest integer function.

(v) $3^y + 2^{x^4} = 2^{4x^2-1}$

6. The range of the function f (x) = $\sin^{-1}\left[x^2+\frac{1}{2}\right] + \cos^{-1}\left[x^2-\frac{1}{2}\right]$, where [ ] is the greatest integer function, is:

7. Find the range of f(x) = $\dfrac{1}{2\{-x\}} - \{x\}$, (where {.} represents fractional part of x)

8. If f : R $\to$ R ; $f(x) = \dfrac{\sqrt{x^2+1}-3x}{\sqrt{x^2+1}+x}$ then find the range of f(x).

9. If a function is defined as $f(x) = \sqrt{\log_{h(x)} g(x)}$, where g(x) = |sinx| + sinx , h(x) = sinx + cosx , $0 \le x \le \pi$ .Then find the doman of f(x).

10. Find the domain and range of the following functions.

(i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\dfrac{|x|}{x}}$, where [.] denotes the greatest integer function .

(ii) $f(x) = \sqrt{\log_{1/2}\log_2\left[x^2+4x+5\right]}$ where [.] denotes the greatest integer function

(iii) $f(x) = \sin^{-1}\left[\log_2\left(\dfrac{x^2}{2}\right)\right]$, where [.] denotes greatest integer function .

(iv) $f(x) = \log_{[x-1]} \sin x$, where [ ] denotes greatest integer function .

(v) $f(x) = \tan^{-1}\left(\sqrt{[x]+[-x]}\right) + \sqrt{2-|x|} + \dfrac{1}{x^2}$, (where [ ] denotes greatest integer function)

11. If $f(x) = \dfrac{\sin^2 x + 4\sin x + 5}{2\sin^2 x + 8\sin x + 8}$, then range of f(x) is

12. Find range of the function $f(x) = \log_2\left[3x - \left[x + \left[x + \left[x\right]\right]\right]\right]$

(where [·] is greatest integer function)

13. If $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = \sin \pi x + 8\left\{\dfrac{x}{2}\right\}$ where {·} denotes fractional part function then the find range of f(g(x))

14. If the range of the function $f(x) = \left\{\dfrac{x}{4}\right\} + \cos\pi\left(\dfrac{(1-2[x])}{2}\right) + \sin\left(\dfrac{\pi[x]}{2}\right)$ is $\left[\dfrac{\alpha}{4},\dfrac{\beta}{4}\right) \cup \left[\dfrac{\gamma}{4},\dfrac{\delta}{4}\right) \cup \left[\dfrac{2\gamma+1}{4},\dfrac{\delta}{2}\right)$, (where {.} and [.] represent fractional part and greatest integer part functions respectively), then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is

15. The fundamental period of $\sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cos\frac{\pi}{3}[x]$, where [.] denotes the integral part of x, is.

16. Consider the function g(x) defined as g(x), $\left(x^{\left(2^{2011}-1\right)}-1\right) = (x+1)(x^2+1)(x^4+1)...........(x^{2^{2010}}+1)-1$

($|x| \neq 1$). Then the value of g(2) is equal to

17. It is given that f(x) is a function defined on N, satisfying f(1) = 1 and for any $x \in N$

$f(x + 5) \geq f(x) + 5$ and $f(x + 1) \leq f(x) + 1$

If g(x) = f(x) + 1 – x, then g(2016) equals

18. Find the integral solutions to the equation [x] [y] = x + y. Show that all the non-integral solutions lie on exactly two lines. Determine these lines. Here [.] denotes greatest integer function.

19. Let $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if f(x) is an integer whenever x is integer, then the numbers 2A, A + B and C are all integers. Conversely, prove that if the numbers 2A, A + B and C are all integer then f(x) is an integer whenever x is an integer.

20. Suppose X and Y are two sets and $f : X \to Y$ is a function. For a subset A of X, define f(A) to be the subset $\{f(a) : a \in A\}$ of Y. For a subset B of Y, define $f^{-1}(B)$ to be the subset $\{x \in X : f(x) \in B\}$ of X. Then prove the followings

(i) Statement "$f^{-1}(f(A)) = A$ for every $A \subset X$" is false

(ii) Statement "$f^{-1}(f(A)) = A$ for every $A \subset X$ if only if f(X) = Y" is false

(iii) Statement "$f(f^{-1}(B)) = B$ for every $B \subset Y$" is false

(iv) Statement "$f(f^{-1}(B)) = B$ for every $B \subset Y$ if only if f(X) = Y" is true

21. Let $g : R \to (0, \pi/3]$ is defined by $g(x) = \cos^{-1}\left(\frac{x^2-k}{1+x^2}\right)$. Then find the possible values of 'k' for which g is surjective.

22. Let $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ are the solutions of the equations cosx = x, cos(sinx) = x and sin(cosx) = x respectively, then show that $\gamma < \alpha < \beta$

23. Let $f(x) = \log_2\log_3\log_4\log_5(\sin x + a^2)$. Find the set of values of a for which domain of f(x) is R.

24. $\tan^{-1}(\tan\theta) = \begin{cases} \pi+\theta & , \ -\frac{3\pi}{2} < \theta < -\frac{\pi}{2} \\ \theta & , \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\pi+\theta & , \ \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$, $\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta & , \ -\frac{3\pi}{2} \le \theta < -\frac{\pi}{2} \\ \theta & , \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \pi-\theta & , \ \frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases}$,

$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \ -\pi \le \theta < 0 \\ \theta & , \ 0 \le \theta \le \pi \\ 2\pi-\theta & , \ \pi < \theta \le 2\pi \end{cases}$

Based on the above results, prove each of the following :

(i) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

(ii) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$

(iii) $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

25. Express cot ($\text{cosec}^{-1}x$) as an algebraic function of x.

26. Express $\sin^{-1}x$ in terms of (i) $\cos^{-1}\sqrt{1-x^2}$ (ii) $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ (iii) $\cot^{-1}\frac{\sqrt{1-x^2}}{x}$

27. If $f(x) = \begin{bmatrix} x & , & x < 1 \\ x^2 & , & 1 \le x \le 4 \\ 8\sqrt{x} & , & x > 4 \end{bmatrix}$, then find $f^{-1}(x)$.

28. $\sin^{-1}\left(\frac{x^2}{4}+\frac{y^2}{9}\right) + \cos^{-1}\left(\frac{x}{2\sqrt{2}}+\frac{y}{3\sqrt{2}}-2\right)$ equals to :

29. If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha+\beta = \pi$. What the value of $\alpha+\beta$ will be if $x > 1$?

30. Solve $\{\cos^{-1} x\} + [\tan^{-1} x] = 0$ for real values of x. Where { . } and [.] are fractional part and greatest integer functions respectively.

31. Find the set of all real values of x satisfying the inequality $\sec^{-1}x > \tan^{-1}x$.

32. Find the solution of $\sin^{-1}\sqrt{\frac{x}{1+x}} - \sin^{-1}\frac{x-1}{x+1} = \sin^{-1}\frac{1}{\sqrt{1+x}}$.

33. (i) Find all positive integral solutions of the equation, $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$.
(ii) If 'k' be a positive integer, then show that the equation:
$\tan^{-1} x + \tan^{-1} y = \tan^{-1} k$ has no non–zero integral solution.

34. Determine the integral values of 'k' for which the system, $(\tan^{-1} x)^2 + (\cos^{-1} y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \frac{\pi}{2}$ possess solution and find all the solutions.

HLP Answers

1. $(-4, -3) \cup (4, \infty)$ **2.** $(-\infty, \infty)$ **3.** $a \in \left(-\frac{1}{2}, 0\right] \cup \left[\frac{1}{2}, 1\right)$ **4.** $\left(-\frac{\pi}{6}, 0\right)$

5. **(i)** $[-3, -2) \cup [3, 4)$ **(ii)** $R - \{(0, 1) \cup \{1, 2, ..., 12\} \cup (12, 13)\}$

(iii) $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$ **(iv)** $\left(\frac{n}{7}, n \in I, -1 \le n \le 6\right)$

(v). $\left(\frac{-\sqrt{3}-1}{\sqrt{2}}, \frac{-\sqrt{3}+1}{\sqrt{2}}\right) \cup \left(\frac{\sqrt{3}-1}{\sqrt{2}}, \frac{\sqrt{3}+1}{\sqrt{2}}\right)$

6. π **7.** $[\sqrt{2} - 1, \infty)$ **8.** $(-1, \infty)$ **9.** $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

10. **(i)** **D : $[2, \infty)$; R : $\{\pi/2\}$** **(ii)** **D :** $\left(-2-\sqrt{2}, -3\right]$ **U ,** $\left[-1, -2+\sqrt{2}\right)$ **; R {0}]**

(iii) **D :** $\left(-\sqrt{8}, -1\right] \cup \left[1, \sqrt{8}\right)$ **; R :** $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$ **]**

(iv) **D : $[3, \pi) \cup \bigcup_{n=1}^{\infty} (2n\pi, 2n\pi + \pi)$; R : $(-\infty, 0]$ }**

(v) **D :** $\{-2, -1, 1, 2\}$ **; R :** $\left\{\frac{1}{4}, 2\right\}$

11. $\left[\frac{5}{9}, 1\right]$ **12.** **{0, 1}** **13.** $\left(\frac{1}{65}, 1\right]$ **14.** 15 **15.** 24

16. 2 **17.** 1 **18.** Integral solution (0, 0); (2, 2). x + y = 6, x + y = 0

21. $k = -\frac{1}{2}$ **23.** $a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$ **25.** $\cot(\operatorname{cosec}^{-1}x) = \begin{cases} -\sqrt{x^2-1} & \text{if } x \le -1 \\ \sqrt{x^2-1} & \text{if } x \ge 1 \end{cases}$

26. (i) $\sin^{-1}x = \begin{cases} -\cos^{-1}\sqrt{1-x^2}, & \text{if } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1 \end{cases}$ (ii) $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, for all $x \in (-1, 1)$

(iii) $\sin^{-1}x = \begin{cases} \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0 \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1 \end{cases}$ **27.** $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \frac{x^2}{64}, & x > 16 \end{cases}$

28. $\frac{3\pi}{2}$ **29.** $-\pi$ **30.** $\{1, \cos 1\}$ **31.** $\{x : x \in (-\infty, -1]\}$ **32.** $x \ge 0$

33. (i) Two solutions (1, 2) (2, 7) **34.** $k = 1, x = \tan(1 - \sqrt{7})\frac{\pi}{4}, y = \cos(\sqrt{7} + 1)\frac{\pi}{4}$