Target : JEE (Main + Advanced)

Definite Integration & Its Application

CONTENT

DEFINITE INTEGRATION & ITS APPLICATION

JEE (Advanced) Syllabus

Definite integrals and their properties, application of the Fundamental Theorem of Integral Calculus.

JEE (Main) Syllabus

Integral as limit of a sum. Fundamental Theorem of Calculus. Properties of definite integrals. Evaluation of definite integrals, determining areas of the regions bounded by simple curves in standard form.

Definite Integration & its Application

Newton-Leibnitz formula.

Let $\frac{d}{dx}$ (F(x)) = f(x) $\forall$ x $\in$ (a, b). Then $\int_a^b f(x)\ dx = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$.

Note : 1. If a > b, then $\int_a^b f(x)\ dx = \lim_{x\to b^+} F(x) - \lim_{x\to a^-} F(x)$.

2. If F(x) is continuous at a and b, then = F(b) – F(a)

Example # 1 : Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Solution : $\because \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = [\ell n(x+1) - \ell n(x+2)]_1^2 = \ell n3 - \ell n4 - \ell n2 + \ell n3 = \ell n\left(\frac{9}{8}\right)$$

Self practice problems :

Evaluate the following

(1) $\int_1^2 \frac{5x^2}{x^2+4x+3}\ dx$ (2) $\int_0^{\pi/4} \left(2\sec^2 x + x^3 + 2\right)\ dx$ (3) $\int_0^{\pi/3} \frac{x}{1+\sec x}\ dx$

Ans. (1) $5 - \frac{5}{2}\left(9\ell n\ \frac{5}{4} - \ell n\ \frac{3}{2}\right)$ (2) $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$ (3) $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\ \ell n\left(\frac{2}{\sqrt{3}}\right)$

Property (1) $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt$

i.e. definite integral is independent of variable of integration.

Property (2) $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Property (3) $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, where c may lie inside or outside the interval [a, b].

Example # 2 : If f(x) = $\begin{cases} x+3 & : \ x<3 \\ 3x^2+1 & : \ x \ge 3 \end{cases}$, then find $\int_2^5 f(x)\ dx$.

Solution $\int_2^5 f(x)\ dx = \int_2^3 f(x)\ dx + \int_3^5 f(x)\ dx = \int_2^3 (x+3)\ dx + \int_3^5 (3x^2+1)\ dx = \left[\frac{x^2}{2}+3x\right]_2^3 + \left[x^3+x\right]_3^5$

$= \frac{9-4}{2} + 3\ (3-2) + 5^3 - 3^3 + 5 - 3 = \frac{211}{2}$

Example # 3 : Evaluate $\int_2^8 |x-5|\ dx$.

Solution : $\int_2^8 |x-5|\,dx = \int_2^5 (-x+5)\ dx + \int_5^8 (x-5)\,dx = 9$

Example # 4 : Show that $\int_0^2 (2x+1)\ dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Solution : L.H.S. $= x^2 + x\,]_0^2 = 4 + 2 = 6$; R.H.S. $= 25 + 5 - 0 + (4 + 2) - (25 + 5) = 6$

$\therefore$ L.H.S. = R.H.S

Self practice problems :

Evaluate the following

(4) $\int_0^4 (|x-1| + |x-3|)dx$ (5) $\int_{-2}^4 [x]dx$, where [x] is integral part of x.

(6) $\int_0^9 \left[\sqrt{t}\right]$ dt.

Ans. (4) 10 (5) 3 (6) 13

Property (4) $\int_{-a}^{a} f(x)\ dx = \int_0^a (f(x) + f(-x))\ dx = \begin{cases} 2\int_0^a f(x)\ dx\ , & \text{if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0\ , & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

Example # 5 : Evaluate $\int_{-1}^{1} \frac{3^x + 3^{-x}}{1+3^x}\ dx$

Solution : $\int_{-1}^{1} \frac{3^x + 3^{-x}}{1+3^x}\ dx = \int_0^1 \left(\frac{3^x + 3^{-x}}{1+3^x} + \frac{3^{-x} + 3^x}{1+3^{-x}}\right) dx = \int_0^1 \left(\frac{3^x + 3^{-x}}{1+3^x} + \frac{3^x(3^{-x} + 3^x)}{1+3^x}\right)$

$= \int_0^1 (3^x + 3^{-x})\ dx = \left(\frac{3^x}{\ell n3} - \frac{3^{-x}}{\ell n3}\right)_0^1 = \left(\frac{3}{\ell n3} - \frac{3^{-1}}{\ell n3}\right) - \left(\frac{1}{\ell n3} - \frac{1}{\ell n3}\right) = \frac{1}{\ell n3}\left[3 - \frac{1}{3}\right] = \frac{8}{3\ell n3}$

Example # 6 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$.

Solution : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx = 2\int_0^{\frac{\pi}{2}} \cos x\ \ dx = 2$ ($\because$ cos x is even function)

Example # 7 : Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx$.

Solution : Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right)$ $\Rightarrow$ $f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function $\therefore$ $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

Self practice problems :

Evaluate the following

(7) $\int_{-1}^{1} |x|\ dx$. (8) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\ dx$. (9) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x dx}{1+2^x}$ dx.

Ans. (7) 1 (8) 0 (9) $\ell n\left(\sqrt{2}+1\right)$

Property (5) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Further $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Example # 8 : Prove that $\int_0^{\frac{\pi}{2}} \frac{g\,(\sin x)}{g\,(\sin x)+g\,(\cos x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{g\,(\cos x)}{g\,(\sin x)+g\,(\cos x)}\,dx = \frac{\pi}{4}$.

Solution : Let I = $\int_0^{\frac{\pi}{2}} \frac{g\,(\sin x)}{g\,(\sin x)+g\,(\cos x)}\,dx \Rightarrow$ I = $\int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)}dx$

$$= \int_0^{\frac{\pi}{2}} \frac{g\,(\cos x)}{g\,(\cos x)+g\,(\sin x)}dx$$

on adding, we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g\,(\sin x)}{g\,(\sin x)+g\,(\cos x)}+\frac{g\,(\cos x)}{g\,(\cos x)+g\,(\sin x)}\right)dx = \int_0^{\frac{\pi}{2}} dx \quad \Rightarrow I = \frac{\pi}{4}$$

Self practice problems:

Evaluate the following

(10) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx.$ (11) $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}\,dx.$

(12) $\int_0^{\frac{\pi}{2}} \frac{x\sin x\ \cos x}{\sin^4 x+\cos^4 x}\,dx.$ (13) $\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{dx}{1+\sqrt{\cot x}}$

Ans. (10) π (11) $\frac{\pi}{2\sqrt{2}}\log_e\left(1+\sqrt{2}\right)$ (12) $\frac{\pi^2}{16}$ (13) $\frac{\pi}{6}$

Property (6) $\int_0^{2a} f(x)\,dx = \int_0^{a} (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx\ , & \text{if } f(2a-x)=f(x) \\ 0\ , & \text{if } f(2a-x)=-f(x) \end{cases}$

Example # 9 : Evaluate $\int_0^{\pi} \cot x.\cos 2x\,dx$

Solution : Let f(x) = cotx cos2x

$\Rightarrow f(\pi - x) = \cot(\pi-x)\cos 2(\pi-x) = -\cot x \cos 2x = -f(x)$

$\therefore \int_0^{\pi} \cot x \cos 2x\,dx = 0$

Example # 10 : Evaluate $\int_0^{\pi} \frac{dx}{1+3\cos^2 x}\,dx.$

Solution : Let $f(x) = \frac{1}{1+3\cos^2 x} \quad \Rightarrow \quad f(\pi - x) = f(x) \quad \Rightarrow \int_0^{\pi} \frac{dx}{1+3\cos^2 x}$

$$= 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+3\cos^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\ dx}{1+\tan^2 x+3} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\ dx}{4+\tan^2 x} = \left[\tan^{-1}\left(\frac{\tan x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$\because \tan\frac{\pi}{2}$ is undefined, we take limit = $\lim_{x\to\pi/2-} \tan^{-1}\left(\frac{\tan x}{2}\right) - \tan^{-1}\left(\frac{\tan 0}{2}\right) = \pi/2 - 0 = \pi/2$

Example # 11 : Evaluate : $\int_0^{\infty} (\cot^{-1} x)^2 dx$

Solution : Let I = $\int_0^{\infty} (\cot^{-1} x)^2 dx$ $\Rightarrow$ Let x = cotθ $\Rightarrow$ dx = – cosec²θ dθ

$$\therefore I = \int_{\frac{\pi}{2}}^{0} \theta^2(-\text{cosec}^2\theta)d\theta \Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \theta^2(\text{cosec}^2\theta)d\theta$$

$$= \left(\theta^2(-\cot\theta)\right)_0^{\pi/2} + 2\int_0^{\frac{\pi}{2}} \theta \cot\theta \, d\theta \quad \Rightarrow \quad I = 0 + 2\int_0^{\frac{\pi}{2}} \theta \cot\theta \, d\theta$$

$$= \left(2\theta \ell n \sin\theta\right)_0^{\pi/2} - 2\int_0^{\frac{\pi}{2}} \ell n \sin\theta \, d\theta \qquad \left\{ \begin{array}{l} \text{Standard result} \\ \int_0^{\frac{\pi}{2}} \ell n \sin\theta d\theta = \frac{-\pi}{2} \ell n2 \end{array} \right. \qquad = 0 - 2 \times \left(-\frac{\pi}{2}\right)\ell n2 = \pi \ell n2.$$

Self practice problems :

Evaluate the following

(14) $\int_0^{\infty} \left(\frac{\ell n\ (1+x^2)}{1+x^2}\right) dx.$ (15) $\int_0^{\infty} \frac{\tan^{-1} x}{x(1+x^2)} dx$ (16) $\int_0^1 \ell n\ \sin\left(\frac{\pi}{2}x\right)$ dx.

Ans. (14) $\pi \ell n2$ (15) $\frac{\pi}{2} \ell n2$ (16) $-\ell n2$

Property (7) If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\ dx = n \int_0^{T} f(x)\, dx,\ n \in z$

(ii) $\int_a^{a+nT} f(x)\ dx = n \int_0^{T} f(x)\ dx,\ n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\ dx = (n - m) \int_0^{T} f(x)\ dx,\ m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\ dx = \int_0^{a} f(x)\ dx,\ n \in z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\ dx = \int_a^{b} f(x)\ dx,\ n \in z,\ a, b \in R$

Example # 12 : Evaluate $\int_{-3}^{5} e^{\{x\}}\ dx$, where {.} denotes the fractional part function.

Solution : $\int_{-3}^{5} e^{\{x\}}\ dx = (5-(-3))\int_0^1 e^{\{x\}} dx = 8\int_0^1 e^x dx = 8\left(e^x\right)_0^1 = 8\ (e - 1)$

Example # 13 : Evaluate $\sum_{n=1}^{1000} \int_{n-1}^{n} |\cos 2\pi x| dx$

Solution : $\int_0^1 |\cos 2\pi x|\ dx + \int_1^2 |\cos 2\pi x|\ dx + \ldots\ldots.. + \int_{999}^{1000} |\cos 2\pi x|\ dx = \int_0^{1000} |\cos 2\pi x|\ dx$

Now |cos2πx| is a periodic function of period 1/2

$$I = 2000 \int_0^{\frac{1}{2}} |\cos 2\pi x|\, dx \Rightarrow I = 2000 \times 2 = 4000$$

Self practice problems :

Evaluate the following

(17) $\int_{-1}^{\frac{41}{2}} e^{2x-[2x]}dx$, where [•] denotes the greatest integer function.

(18) $\int_{0}^{\frac{14\pi}{3}} |\sin x|\,dx$ (19) $\int_{\pi}^{\frac{3\pi}{2}} (\sin^4 x + \cos^4 x)dx$

Ans. (17) $\frac{43}{2}$ (e – 1) (18) $\frac{19}{2}$ (19) $\frac{3\pi}{8}$

Leibnitz Theorem : If $F(x) = \int_{g(x)}^{h(x)} f(t)\ dt$, then $\frac{dF(x)}{dx} = h'(x)\, f(h(x)) - g'(x)\, f(g(x))$

Proof : Let $P(t) = \int f(t)\ dt \Rightarrow F(x) = \int_{g(x)}^{h(x)} f(t)\ dt = P(h(x)) - P(g(x))$

$$\Rightarrow \frac{dF(x)}{dx} = P'(h(x))\, h'(x) - P'(g(x))\, g'(x) = f(h(x))\, h'(x) - f(g(x))\, g'(x)$$

Example # 14 : If $F(x) = \int_{x}^{x^2} \sqrt{\tan t}\ dt$, then find F′(x).

Solution : $F'(x) = 2x \cdot \sqrt{\tan x^2} - 1 \cdot \sqrt{\tan x}$

Example # 15 : If $F(x) = \int_{x^2}^{x^3} \frac{1}{\ell nt}dt$ then find F’(e)

Solution : $F'(x) = \frac{3x^2}{\ell nx^3} - \frac{2x}{\ell nx^2} = \frac{x^2}{\ell nx} - \frac{x}{\ell nx} = \frac{x(x-1)}{\ell nx}$ now $F'(e) = \frac{e(e-1)}{\ell ne} = e(e-1)$

Example # 16 : Evaluate : $\lim_{x\to 0^+} \int_{0}^{x^2} \frac{\sin\sqrt{t}\ \tan\sqrt{t}dt}{x^4}$

Solution : Applying L’ hospital rule

$$\lim_{x\to 0^+} \frac{2x\sin x\tan x}{4x^3} \Rightarrow \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin x}{x}\right)\left(\frac{\tan x}{x}\right) = \frac{1}{2}$$

Example # 17 : Let $f(x) = \int_{0}^{x} (t-1)(t-2)^2\ dt$, then find a point of minimum

Solution : $f(x) = \int_{0}^{x} (t-1)(t-2)^2\ dt$

$f'(x) = (x - 1)\ (x - 2)^2$

$$\underset{-\infty}{\longleftarrow}\ \ -\ \ \underset{1}{|}\ \ +\ \ \underset{2}{|}\ \ +\ \ \underset{\infty}{\longrightarrow}$$

$\Rightarrow$ x = 1 is the point of minimum

$f(1) = \int_{0}^{1} (t^3 - 5t^2 + 8t - 4)dt = \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}$. Hence $(1, -\frac{17}{12})$ is a point of minimum

Example # 18 : Evaluate , $\int_0^1 \frac{x^b - 1}{\ell nx}$ 'b' being parameter.

Solution : Let $I(b) = \int_0^1 \frac{x^b - 1}{\ell nx} dx \Rightarrow \frac{dI(b)}{db} = \int_0^1 \frac{x^b \ell nx}{\ell nx} dx + 0 - 0$ (using modified Leibnitz Theorem)

$= \int_0^1 x^b \, dx = \left. \frac{x^{b+1}}{b+1} \right]_0^1 = \Rightarrow I(b) = \ell n\ (b + 1) + c$

$b = 0 \Rightarrow I(0) = 0 \therefore c = 0 \qquad \therefore I(b) = \ell n\ (b+1)$

Example # 19 : Evaluate $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}} dx$, 'a' being parameter.

Solution : Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}} dx \Rightarrow \frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)} \frac{1}{x\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$

Put x = sin t $\Rightarrow$ dx = cos t dt
L.L. : x = 0 $\Rightarrow$ t = 0
U.L. : x = 1 $\Rightarrow$ $t = \frac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a^2\sin^2 t} \frac{1}{\cos t} \cos t \, dt = \int_0^{\frac{\pi}{2}} \frac{dt}{1+a^2\sin^2 t}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 t \, dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}} \tan^{-1} \left.\left(\sqrt{1+a^2}\ \tan t\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{\sqrt{1+a^2}} \cdot \frac{\pi}{2}$$

$\Rightarrow I(a) = \frac{\pi}{2} \ell n \left(a + \sqrt{1+a^2}\right) + c$ But $I(0) = 0 \Rightarrow c = 0 \Rightarrow I(a) = \frac{\pi}{2} \ell n \left(a+\sqrt{1+a^2}\right)$

Self Practice Problems :

(20) If $f(x) = \int_0^{x^3} \sqrt{\cos t}\ dt$, find f′(x).

(21) Find the equation of tangent to the y = F(x) at x = 1 , where $F(x) = \int_x^{x^3} \frac{dt}{\sqrt{1+t^4}}$

(22) If $\int_0^x f(t)dt = x^2 - \int_x^{x^3} \frac{f(t)}{t} dt$ then find f(1)

(23) If $f(x) = \int_x^{x^2} x^2 \ell nt\ dt$ then find f'(e)

(24) If $y = \int_4^{4x^2} t^4 e^{4t} dt$, Find $\frac{d^2y}{dx^2}$

(25) If $y = \int_0^{x^2} \ell n(1+t)dt$, then find $\frac{d^2y}{dx^2}$

(26) If $\int_0^{x^2(1+x)} f(t)dt = x$ then find f(2) (27) Evaluate $\int_0^{\pi} \ell n\ (1+b\cos x)\ dx$, 'b' being parameter.

Ans. (20) $3x^2 \sqrt{\cos x^3}$ (21) $\sqrt{2}\,x - y = \sqrt{2}$ (22) 2/3
(23) $e^2(6e–1)$ (24) $2048\, e^{16x^2}$
(25) $\frac{2}{1+x^2}\ [2x^2 + (1 + x^2)\ell n(1 + x^2)]$ (26) $\frac{1}{5}$ (27) $\pi\ \ell n \left(\frac{1+\sqrt{1-b^2}}{2}\right)$

Reduction formulae in definite Integrals:

1. If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Proof : $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$$I_n = \left[-\sin^{n-1} x \ \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \ \sin^{n-2} x \ . \ \cos^2 x \ dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \ . \ (1-\sin^2 x) \ dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \ dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \ dx \Rightarrow I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Note : 1. $\int_0^{\frac{\pi}{2}} \sin^n x \ dx = \int_0^{\frac{\pi}{2}} \cos^n x \ dx$

2. $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)$ ..... I_0 or I_1 according as n is even or odd. $I_0 = \frac{\pi}{2}$, $I_1 = 1$

$$\text{Hence } I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{1}{2}\right) \ . \ \frac{\pi}{2} \ , & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{2}{3}\right) \ . \ 1 \ , & \text{if } n \text{ is odd} \end{cases}$$

2. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Proof : $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \, . \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \ (\sec^2 x - 1) \, dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \ dx = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \qquad \therefore \qquad I_n + I_{n-2} = \frac{1}{n-1}$$

3. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \ . \ \cos^n x \ dx$, then show that $I_{m,n} = \frac{m-1}{m+n} \ I_{m-2,\, n}$

Proof : $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x \ (\sin x \ \cos^n x) \ dx = \left[-\frac{\sin^{m-1} x \ . \ \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1} \ (m-1) \sin^{m-2} x \cos x \, dx$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x . \cos^n x . \cos^2 x \, dx = \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x . \cos^n x - \sin^m x . \cos^n x\right) \ dx$$

$$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n} \ \Rightarrow \ \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$$

Note : 1. $I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right)$ .... $I_{0,n}$ or $I_{1,n}$ according as m is even or odd.

$$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x dx \text{ and } I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x.\cos^n x dx = \frac{1}{n+1}$$

2. <u>Walli's Formula</u>

$$I_{m,n} = \begin{cases} \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........}\ \dfrac{\pi}{2} & \text{when both m, n are even} \\ \\ \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........} & \text{otherwise} \end{cases}$$

Example # 20 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^2 x(\sin x + \cos x)\ dx$.

Solution : Given integral = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \ \cos^2 x \ dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$ ($\because$ $\sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$= 2.\ \frac{1.2}{5.3.1} = \frac{4}{15}$

Example # 21 : Evaluate $\int_0^{\pi} x\sin^7 x\cos^6 x \ dx$

Solution : Let $I = \int_0^{\pi} x\sin^7 x\cos^6 x \ dx$

$I = \int_0^{\pi} (\pi-x)\ \sin^7(\pi-x)\ \cos^6(\pi-x)\ dx = \pi\int_0^{\pi} \sin^7 x\cos^6 x dx - \int_0^{\pi} x\sin^7 x\cos^6 x \ dx$

$\Rightarrow 2I = \pi . 2\int_0^{\frac{\pi}{2}} \sin^7 x.\cos^6 x dx \Rightarrow I = \pi\ \frac{6.4.2.5.3.1}{13.11.9.7.5.3.1} \Rightarrow I = \frac{48\pi}{9009}$

Example # 22 : Evaluate : $\int_0^a x^{5/2}\sqrt{a-x}dx$

Solution : Put $x = a \sin^2\theta \Rightarrow dx = 2a \sin\theta \cos\theta\ d\theta$

Lower limit : $x = 0 \Rightarrow \theta = 0$

Upper limit $x = a \Rightarrow \theta = \frac{\pi}{2}$.

$$\int_0^a x^{5/2}\sqrt{a-x}dx = \int_0^{\frac{\pi}{2}} 2a^4 \sin^6\theta\cos^2\theta d\theta = 2a^4 x\frac{\pi}{2}.\frac{(5.31)(1)}{8.6.4.2} = \frac{5\pi a^4}{128}$$

Self Practice Problems:

Evaluate the following

(28) $\int_0^{\frac{\pi}{2}} \sin^{11} x \ dx$. (29) $\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^4 x \ dx$. (30) $\int_0^1 x^6 \ \sin^{-1} x \ dx$.

(31) $\int_0^a x \ (a^2 - x^2)^{\frac{7}{2}} \ dx$. (32) $\int_0^2 x^{3/2} \ \sqrt{2-x} \ dx$.

Ans. (28) $\frac{128}{693}$ (29) $\frac{8}{315}$ (30) $\frac{\pi}{14} - \frac{16}{245}$ (31) $\frac{a^9}{9}$ (32) $\frac{\pi}{2}$

Property (8) If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \psi(x) \ dx \le \int_a^b f(x) \ dx \le \int_a^b \phi(x) \ dx$$

Property (9) If $m \le f(x) \le M$ for $a \le x \le b$, then $m (b - a) \le \int_a^b f(x)\, dx \le M (b - a)$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b - a) < \int_a^b f(x) \ dx < f(a)(b - a)$ and if f(x) is monotonically increasing in (a, b), then $f(a)(b - a) < \int_a^b f(x)\, dx < f(b)(b - a)$

Property (10) $\left| \int_a^b f(x) \ dx \right| \le \int_a^b | f(x) | \ dx$

Property (11) If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x) \ dx \ge 0$

Example # 23 : For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{9 - x^2}$, $f_2(x) = \frac{1}{9 - 2x^2}$ and $f_3(x) = \frac{1}{9 - x^2 - x^3}$ in ascending order and hence prove that $\frac{1}{6} \ \ell n2 < \int_0^1 \frac{1}{9 - x^2 - x^3} dx < \frac{1}{6\sqrt{2}} \ \ell n5$

Solution : $\because$ $0 < x^3 < x^2$, for all $x \in (0,1)$ $\Rightarrow$ $x^2 < x^2 + x^3 < 2x^2$

$\Rightarrow$ $-2x^2 < - x^2 - x^3 < -x^2$ $\Rightarrow$ $9 - 2x^2 < 9 - x^2 - x^3 < 9 - x^2$

$$\Rightarrow \frac{1}{9 - x^2} < \frac{1}{9 - x^2 - x^3} < \frac{1}{9 - 2x^2}$$

$f_1(x) < f_3(x) < f_2(x)$ for $x \in (0, 1)$

$$\Rightarrow \int_0^1 f_1(x) \ dx < \int_0^1 f_3(x) \ dx < \int_0^1 f_2(x)\, dx$$

$$\Rightarrow \int_0^1 \frac{dx}{9 - x^2} < \int_0^1 \frac{dx}{9 - x^2 - x^3} < \int_0^1 \frac{dx}{9 - 2x^2}$$

$$\Rightarrow \frac{1}{6}\left(\ell n \left| \frac{3 + x}{3 - x} \right| \right)_0^1 < \int_0^1 \frac{dx}{9 - x^2 - x^3} < \frac{1}{6\sqrt{2}} \left(\ell n \left| \frac{3 + 2x}{3 - 2x} \right| \right)_0^1$$

$$\Rightarrow \frac{1}{6} \ell n2 < \int_0^1 \frac{1}{9 - x^2 - x^3} dx < \frac{1}{6\sqrt{2}} \ \ell n5$$

Example # 24 : Prove that $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

Solution : Let $f(x) = \frac{5-x}{9-x^2}$

$\therefore f'(x) = -\frac{(x-9)(x-1)}{(9-x^2)^2} \Rightarrow f'(x) = 0$ or not defined $\Rightarrow x = 1$

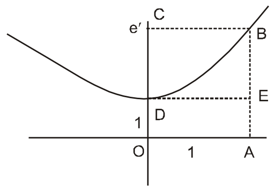
Then $f(0) = 5/9$, $f(1) = \frac{1}{2}$, $f(2) = 3/5$ The greatest and least values of the integrand in the interval [0,2] are respectively , equal to $f(2) = 3/5$ and $f(1) = \frac{1}{2}$

$(2-0)\ \frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < (2-0)\ 3/5$ Hence $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

Example # 25 : Estimate the value of $\int_0^1 e^{x^2}dx$ using (i) rectangle, (ii) triangle.

Solution : (i) By using rectangle

Area OAED $< \int_0^1 e^{x^2}dx <$ Area OABC



$1 < \int_0^1 e^{x^2}dx < 1 \cdot e$

$1 < \int_0^1 e^{x^2}dx < e$

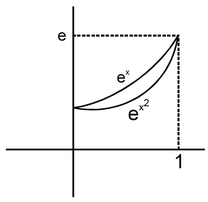
(ii) By using triangle

Area OAED $< \int_0^1 e^{x^2}dx <$ Area OAED + Area of triangle DEB

$1 < \int_0^1 e^{x^2}dx < 1 + \frac{1}{2}.1.(e-1)$ $\qquad 1 < \int_0^1 e^{x^2}dx < \frac{e+1}{2}$

Example # 26 : Estimate the value of $\int_0^1 e^{x^2}dx$ by using $\int_0^1 e^{x}dx$.

Solution : For $x \in (0, 1)$, $e^{x^2} < e^x$



$\Rightarrow \quad 1 \times 1 < \int_0^1 e^{x^2}dx < \int_0^1 e^{x}dx$

$1 < \int_0^1 e^{x^2}dx < e-1$

Self practice problems :

(33) Prove the following : $\int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\ dx$

(34) Prove the following : $0 < \int_0^{\frac{\pi}{2}} \sin^{n+1} x\ dx < \int_0^{\frac{\pi}{2}} \sin^2 x\ \ dx$, $n > 1$

(35) Prove the following : $e^{-\frac{1}{e}} < \int_0^1 x^x dx < 1$

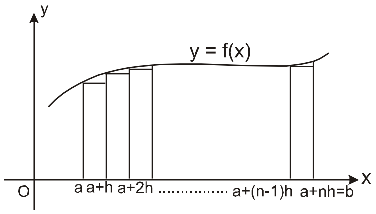
(36) Prove the following : $-\dfrac{1}{2} \le \displaystyle\int_0^1 \frac{x^3 \cos x}{2+x^2}\,dx < \frac{1}{2}$

(37) Prove the following : $1 < \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\ x}\ dx < \sqrt{\frac{\pi}{2}}$

(38) Prove the following : $\dfrac{4}{\pi} < \displaystyle\int_{\pi/4}^{\pi/3} \frac{\tan x}{x} dx < \frac{3\sqrt{3}}{\pi}$

Definite Integral as a limit of sum

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\dfrac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles.

Then, $S_n = hf(a) + hf(a + h) + hf(a + 2h) + ...+hf(a + (n – 1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\displaystyle\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}}\ S_n$

$$\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\ f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)\ r}{n}\right)$$

Note :

1. We can also write

$S_n = hf(a + h) + hf (a + 2h) + ...+ hf(a + nh)$ and $\displaystyle\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right) r\right)$

2. If a = 0, b = 1, $\displaystyle\int_0^1 f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{1}{n}\ f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as definite integral :

Step 1. Replace $\dfrac{r}{n}$ by x, $\dfrac{1}{n}$ by dx and $\underset{n\to\infty}{\text{Lt}}$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{\text{Lt}}\left(\dfrac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\displaystyle\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{pn} \frac{1}{n}\ f\left(\frac{r}{n}\right) = \int_0^p f(x)\ dx \qquad \left(\because\ \underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)\bigg|_{r=1} = 0,\ \underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)\bigg|_{r=np} = p\right)$

Example # 27 : Evaluate $\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+........+\frac{1}{10n}\right]$

Solution : $\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+........+\frac{1}{10n}\right] = \underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{9n}\frac{1}{r+n}$

$= \underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{9n}\frac{1}{n}\frac{1}{\left(\frac{r}{n}\right)+1} = \int_0^9\frac{dx}{x+1} = \left[\ell n\ (x+1)\right]_0^9 = \ell n\ 10$

Example # 28 : Evaluate $\underset{n\to\infty}{\text{Lt}}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+........+\frac{1}{n}\right]$.

Solution : $\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2} = \underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{1}{n}\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$ $\because \underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right) = 0$, when r = 1, lower limit = 0

and $\underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right) = \underset{n\to\infty}{\text{Lt}}\left(\frac{n}{n}\right) = 1$, when r = n, upper limit = 1

$\int_0^1\frac{1+x}{1+x^2}\,dx = \int_0^1\frac{1}{1+x^2}\,dx + \frac{1}{2}\int_0^1\frac{2x}{1+x^2}\,dx = [\tan^{-1}x]_0^1 + \left[\frac{1}{2}\log_e(1+x^2)\right]_0^1$

$= \frac{\pi}{4}+\frac{1}{2}\ell n\ 2$

Example # 29 : Evaluate : $\lim_{n\to\infty}\left(\frac{(2n)\ !}{n\ !\ n^n}\right)^{\frac{1}{n}}$

Solution : Let $y = \lim_{n\to\infty}\left(\frac{(2n)!}{n!\,n^n}\right)^{\frac{1}{n}} \Rightarrow \ell n\ y = \lim_{n\to\infty}\frac{1}{n}\ \ell n\left(\frac{(2n)!}{n!\,n^n}\right)$

$= \lim_{n\to\infty}\frac{1}{n}\ \ell n\left(\frac{2n(2n-1)(2n-2)....(n+1)}{n^n}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ [\ell n(1+r/n)] = \int_0^1\ell n(1+x)dx = (x\ell n(1+x))_0^1 - \int_0^1\frac{x}{1+x}dx$

$= (x\ell n(1+x))_0^1 - (x-\ell n(1+x))_0^1 = \ell n2 - (1-\ell n2) = \ell n4/e \Rightarrow y = 4/e$

Self Practice Problems :

Evaluate the following limits

(39) $\lim_{n\to\infty}\left\{\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{8n}\right\}$

(40) $\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.......+\frac{1}{5n}\right]$

(41) $\lim_{n\to\infty}\frac{1}{n^2}\left[\sin^3\frac{\pi}{4n}+2\sin^3\frac{2\pi}{4n}+3\sin^3\frac{3\pi}{4n}+........+n\sin^3\frac{n\pi}{4n}\right]$

(42) $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

(43) $\lim_{n\to\infty}\left(\tan\frac{\pi}{2n}\ \tan\frac{2\pi}{2n}\ \tan\frac{3\pi}{2n}......\tan\frac{n\pi}{2n}\right)^{\frac{1}{n}}$

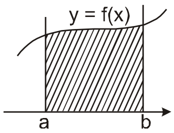
Ans. (39) $\frac{3}{8}$ (40) $\ell n\ 5$ (41) $\frac{\sqrt{2}}{9\pi^2}(52-15\pi)$

(42) $\frac{\pi}{2}$ (43) 1

Area Between The Curve :

Area included between the curve y = f(x), x-axis and the ordinates x = a, x = b

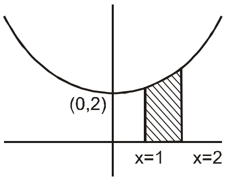
(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve y = f(x), x-axis, x = a and x = b is $\int_a^b f(x)\ dx$



Graph of y = f(x)

Example # 30 : Find the area enclosed between the curve $y = x^2 + 2$, x-axis, x = 1 and x = 2.

Solution :



Graph of $y = x^2 + 2$

Area = $\int_1^2 (x^2+2)dx = \left[\frac{x^3}{3}+2x\right]_1^2 = \frac{13}{3}$

Example # 31 : Find area bounded by the curve $y = \ell n\ x + \tan^{-1} x$ and x-axis between ordinates x = 1 and x = 2.

Solution : $y = \ell n\ x + \tan^{-1}x$

Domain $x > 0$, $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

y is increasing and x = 1, y = $\frac{\pi}{4}$ $\Rightarrow$ y is positive in [1, 2]

$\therefore$ Required area = $\int_1^2 (\ell n\ \ x + \tan^{-1}x)\ dx = \left[x\ell nx - x + x\tan^{-1}x - \frac{1}{2}\ell n(1+x^2)\right]_1^2$

$= 2\ \ell n\ 2 - 2 + 2\tan^{-1}2 - \frac{1}{2}\ \ell n\ 5 - 0 + 1 - \tan^{-1} 1 + \frac{1}{2}\ \ell n\ 2$

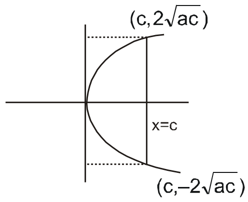
$= \frac{5}{2}\ \ell n\ 2 - \frac{1}{2}\ \ell n\ 5 + 2\tan^{-1}2 - \frac{\pi}{4} - 1$

Note : If a function is known to be positive valued then graph is not necessary.

Example # 32 : The area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by the double ordinate and its distance from the vertex. Find the value of k ?

Solution : Consider $y^2 = 4ax$, $a > 0$ and $x = c$

Area by double ordinate = $2\int_0^c 2\sqrt{a}\sqrt{x}\ dx = \frac{8}{3}\sqrt{a}\ \ c^{3/2}$

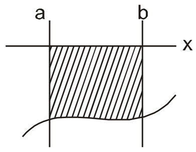


Figure

Area by double ordinate = k (Area of rectangle)

$\frac{8}{3}\sqrt{a}\ \ c^{3/2} = k\ \ 4\sqrt{a}\ \ c^{3/2} \Rightarrow k = \frac{2}{3}$

(b) If $f(x) < 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $-\int_a^b f(x)\ dx$

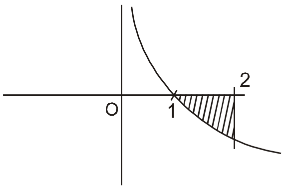


Graph of y = f(x)

Example # 33 : Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between $x = 1$ and $x = 2$

Solution : A rough graph of $y = \log_{\frac{1}{2}} x$ is as follows

Area $= -\int_1^2 \log_{\frac{1}{2}} x\ dx = -\int_1^2 \log_e x \,.\log_{\frac{1}{2}} e\ \ dx$



$= -\log_{\frac{1}{2}} e\ .\ [x\ \ \log_e x - x]_1^2 = -\log_{\frac{1}{2}} e\ .\ (2\log_e 2 - 2 - 0 + 1) = -\log_{\frac{1}{2}} e\ .\ (2\log_e 2 - 1)$

Note : If $y = f(x)$ does not change sign in $[a, b]$, then area bounded by $y = f(x)$, x-axis between ordinates $x = a$, $x = b$ is $\left|\int_a^b f(x)\ dx\right|$

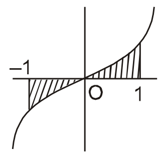
(c) If $f(x) \geq 0$ for $x \in [a,c]$ and $f(x) \leq 0$ for $x \in [c,b]$ $(a < c < b)$ then area bounded by curve $y = f(x)$ and x-axis between $x = a$ and $x = b$ is $\int_a^c f(x)\ dx\ \ -\int_c^b f(x)\ dx$

Example # 34 : Find the area bounded by $y = x^3$ and x- axis between ordinates $x = -1$ and $x = 1$

Solution : Required area $= \int_{-1}^{0} -x^3 dx + \int_0^1 x^3 dx$

$= \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_0^1$

$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$



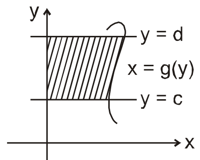
Graph of y = x³

Note : Most general formula for area bounded by curve $y = f(x)$ and x- axis between ordinates $x = a$ and $x = b$ is $\int_a^b |f(x)|\,dx$

Area included between the curve x = g(y), y-axis and the abscissas y = c, y = d

(a) If $g(y) \ge 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and

$y = d$ is $\int_{y=c}^{d} g(y)dy$

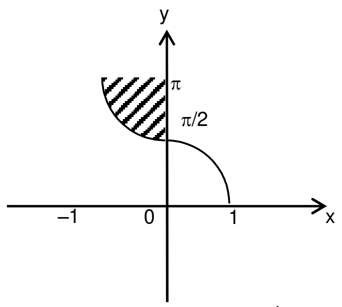


Graph of x = g(y)

Example # 35 : Find area bounded between $y = \cos^{-1}x$ and y–axis between $y = \frac{\pi}{2}$ and $y = \pi$.

Solution : $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$

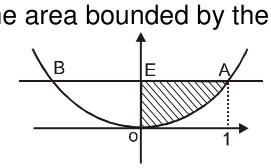
Required area $= -\int_{\frac{\pi}{2}}^{\pi} \cos y\,dy$



Graph of $y = \cos^{-1}x$

$= -\sin y\Big]_{\frac{\pi}{2}}^{\pi} = 1$

Example # 36 : Find the area bounded by the parabola $x^2 = y$, y-axis and the line $y = 1$.

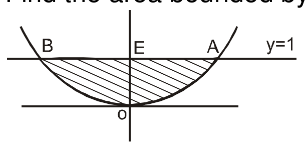


Solution : Graph of $y = x^2$

Area OEBO = Area OAEO = $\int_0^1 |x|\,dy = \int_0^1 \sqrt{y}\,dy = \frac{2}{3}$

Example # 37 : Find the area bounded by the parabola $x^2 = y$ and line $y = 1$.

Solution :



Graph of $y = x^2$

Required area is area OABO

$= 2$ area (OAEO) $= 2\int_0^1 |x|\,dy = 2\int_0^1 \sqrt{y}\,dy = \frac{4}{3}$.

Example # 38 : For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Solution : It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n\left(x+\sqrt{x^2-1}\right)\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}} = \frac{e^{2t_1}-e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2\times\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) = \frac{e^{2t_1}-e^{-2t_1}}{4}$$

$\therefore$ Required area = area ΔOPQ – area (PQRP) = t_1

(b) If $g(y) \le 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)dy$

Note : General formula for area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$

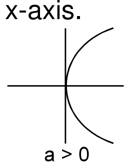
Curve-tracing :

To find approximate shape of a curve, the following phrases are suggested :

(a) Symmetry:

(i) Symmetry about x-axis :

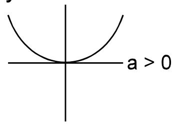
If all the powers of 'y' in the equation are even then the curve (graph) is symmetrical about the x-axis.



E.g. : $y^2 = 4ax$.

(ii) Symmetry about y-axis :

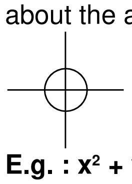
If all the powers of 'x' in the equation are even then the curve (graph) is symmetrical about the y-axis.



E.g. : $x^2 = 4ay$.

(iii) Symmetry about both axis :

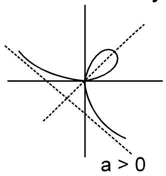
If all the powers of 'x' and 'y' in the equation are even, then the curve (graph) is symmetrical about the axis of 'x' as well as 'y'.



E.g. : $x^2 + y^2 = a^2$.

(iv) Symmetry about the line y = x :

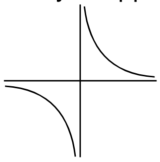
If the equation of the curve remain unchanged on interchanging 'x' and 'y', then the curve (graph) is symmetrical about the line y = x.



a > 0

E.g. : $x^3 + y^3 = 3axy$.

(v) **Symmetry in opposite quadrants :**
If the equation of the curve (graph) remain unaltered when 'x' and 'y' are replaced by '–x' and '–y' respectively, then there is symmetry in opposite quadrants.



E.g. : $xy = c^2$

(b) Find the points where the curve crosses the x-axis and the y-axis.

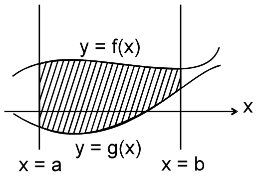
(c) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Examine intervals when f(x) is increasing or decreasing

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$

Area between two curves

If $f(x) \geq g(x)$ for $x \in [a,b]$ then area bounded by curves (graph) y = f(x) and y = g(x) between ordinates x = a and x = b is $\int_a^b \left(f(x) - g(x)\right)\ dx$.



Example # 39 : Find the area enclosed by curve (graph) $y = x^2 + x + 1$ and its tangent at (1,3) between ordinates x = – 2 and x = 1.

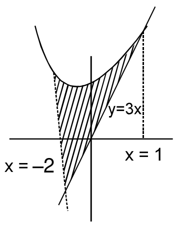
Solution : $\frac{dy}{dx} = 2x + 1$

$\frac{dy}{dx} = 3$ at x = 1

Equation of tangent is

$y - 3 = 3(x - 1)$

$y = 3x$



Required area $= \int_{-2}^{1} (x^2 + x + 1 - 3x)\ dx$

$$= \int_{-2}^{1} (x^2 - 2x + 1)\ dx = \left.\frac{x^3}{3} - x^2 + x\right]_{-2}^{1} = \left(\frac{1}{3} - 1 + 1\right) - \left[-\frac{8}{3} - 4 - 2\right] = 9$$

Note : Area bounded by curves y = f(x) and y = g (x) between ordinates x = a and x = b is $\int_a^b |f(x) - g(x)|\, dx$.

Example # 40 : Find the area of the region bounded by y = sin x, y = cos x and ordinates x = 0, x = $\pi/2$

Solution : $\int_0^{\pi/2} |\sin x - \cos x|\, dx$

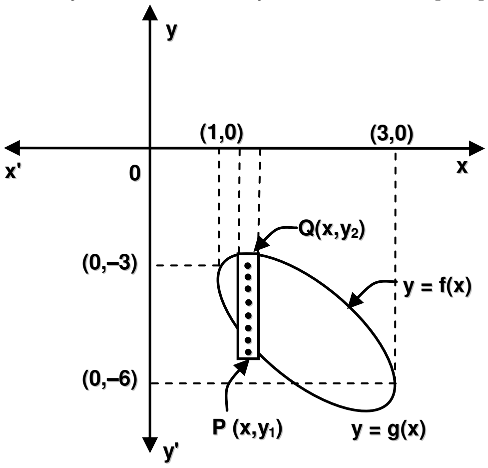
$$\int_0^{\pi/4} (\cos x - \sin x)\ dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\ dx = 2(\sqrt{2} - 1)$$

Example # 41 : Find the area contained by ellipse $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$

Solution : $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$

$2y^2 + 6(1 + x) y + 5x^2 + 7x + 6 = 0 \qquad \Rightarrow y = \dfrac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$

Clearly, the values of y are real for x [1,3]



when x = 1 , we get y = – 3 and, x = 3 $\Rightarrow$ y = – 6

Let $f(x) = \dfrac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}$ and , $g(x) = \dfrac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2}$

required area = $\left|\int_1^3 \{g(x) - f(x)\}dx\right|$

$$= \left|\int_1^3 \sqrt{-x^2 + 4x - 3}\right|dx = \left|\int_1^3 \sqrt{1^2 - (x-2)^2}\right|dx = \left|\left[\frac{1}{2}(x-2)\ \sqrt{-x^2+4x-3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1}\right)\right]_1^3\right|$$

$$= \left|\left[\left\{\frac{1}{2}\sin^{-1} 1\right\} - \left\{\frac{1}{2}\sin^{-1}(-1)\right\}\right]\right| = \frac{\pi}{2} \text{ sq. unit}$$

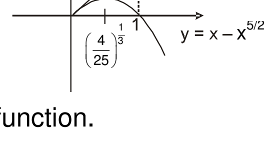
Miscellaneous examples

Example # 42 : Find the area contained between the two arms of curves $(y - x)^2 = x^5$ between x = 0 and x = 1.

Solution : $(y - x)^2 = x^5 \Rightarrow y = x \pm x^{5/2}$

For arm

$y = x + x^{5/2} \Rightarrow \dfrac{dy}{dx} = 1 + \dfrac{5}{2}\ x^{3/2} > 0 \qquad x \geq 0.$



y is increasing function.

For arm

$y = x - x^{5/2} \Rightarrow \frac{dy}{dx} = 1 - \frac{5}{2}\, x^{3/2}$

$\frac{5}{2} = 0 \quad \Rightarrow \quad x = \left(\frac{4}{25}\right)^{1/3}, \; \frac{d^2y}{dx^2} = -\frac{5}{4} \;\; x^{\frac{1}{2}} < 0$ at $x = \left(\frac{4}{25}\right)^{1/3}$

$\therefore$ at $x = \left(\frac{4}{25}\right)^{1/3}$, $y = x - x^{5/2}$ has maxima.

Required area $= \int_0^1 (x + x^{5/2} - x + x^{5/2}) \;\; dx \; = 2 \int_0^1 x^{5/2} dx = \left. \frac{2 \;\; x^{7/2}}{7/2} \right]_0^1 = \frac{4}{7}$

Example # 43 : Let A (m) be area bounded by parabola $y = x^2 + 2x - 3$ and the line $y = mx + 1$. Find the least area A(m).

Solution : Solving we obtain

$x^2 + (2 - m)\, x - 4 = 0$

Let α, β be roots $\Rightarrow \alpha + \beta = m - 2, \; \alpha\beta = -4$

$$A(m) = \left| \int_\alpha^\beta (mx + 1 - x^2 - 2x + 3) dx \right| = \left| \int_\alpha^\beta (-x^2 + (m-2)x + 4) dx \right|$$

$$= \left| \left(-\frac{x^3}{3} + (m-2) \;\; \frac{x^2}{2} + 4x \right)_\alpha^\beta \right| = \left| \frac{\alpha^3 - \beta^3}{3} + \frac{m-2}{2}(\beta^2 - \alpha^2) + 4(\beta - \alpha) \right|$$

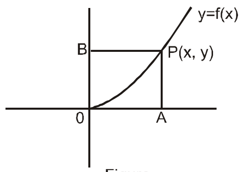
$$= |\beta - \alpha| . \left| -\frac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) + \frac{(m-2)}{2}(\beta + \alpha) + 4 \right|$$

$$= \sqrt{(m-2)^2 + 16} \; \left| -\frac{1}{3}\left((m-2)^2 + 4\right) \; + \; \frac{(m-2)}{2}(m-2) \; + \; 4 \right| = \sqrt{(m-2)^2 + 16} \; \left| \frac{1}{6}(m-2)^2 + \frac{8}{3} \right|$$

$$A(m) \; \left((m-2)^2 + 16\right)^{3/2} = \frac{1}{6} \Rightarrow \text{ Least } A(m) = \frac{1}{6} \; (16)^{3/2} = \frac{32}{3} .$$

Example # 44 : A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point P(x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in m : n, then show that $f(x) = cx^{m/n}$ or $f(x) = cx^{n/m}$ (c-being arbitrary).

Solution : Area (OAPB) = xy $\qquad \Rightarrow$ Area (OAPO) $= \int_0^x f(t) \;\; dt$



Figure

Area (OPBO) $= xy - \int_0^x f(t) \;\; dt \qquad \Rightarrow \frac{\text{Area} \;\; (\text{OAPO})}{\text{Area} \;\; (\text{OPBO})} = \frac{m}{n}$

$n\int_0^x f(t) \;\; dt = m\left(xy - \int_0^x f(t) \;\; dt \right) \qquad \Rightarrow n\int_0^x f(t) \;\; dt = mx \;\; f(x) - m \;\; \int_0^x f(t) \;\; dt$

Differentiating w.r.t. x

$nf(x) = m\, f(x) + mx\, f¢(x) - m\, f(x) \Rightarrow \frac{f'(x)}{f(x)} = \frac{n}{m}\frac{1}{x}$

$f(x) = cx^{n/m}$

similarly $f(x) = cx^{m/n}$

Self practice problems :

(44) Find the area bounded by the curves $y = e^x$, $y = |x - 1|$ and $x = 2$.

(45) Compute the area of the region bounded by the parabolas $y^2 + 8x = 16$ and $y^2 - 24x = 48$.

(46) Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

(47) What is geometrical significance of

(i) $\int_0^{\pi} |\cos x| \, dx$, (ii) $\int_0^{\frac{3\pi}{2}} \cos x \, dx$

(48) Find the area of the region bounded by the x-axis and the curves defined by $y = \tan x$, $\left(\text{where } \frac{-\pi}{3} \le x \le \frac{\pi}{3}\right)$ and $y = \cot x$. $\left(\text{where } \frac{\pi}{6} \le x \le \frac{2\pi}{3}\right)$

(49) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(50) Find the area bounded by the curve $y = x^2 - 2x + 5$, the tangent to it at the point (2, 5) and the axes of coordinates.

(51) Find the area of the region bounded by $y = x - 1$ and $(y - 1)^2 = 4(x + 1)$

(52) Find the area of the region lying in the first quadrant and included between the curves $x^2 + y^2 = 3a^2$. $x^2 = 2ay$ and $y^2 = 2ax$. $a > 0$

(53) Find the area enclosed by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the straight line $y = 3x - 15$.

(54) Find the area bounded by the curves $4y = |4 - x^2|$, $y = 7 - |x|$

(55) Find the area bounded by the curves $x = |y^2 - 1|$ and $y = x - 5$.

(56) Find the area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $2x \le 5$.

Ans. **44.** $(e^2 - 2)$ sq. units **45.** $32\sqrt{\frac{2}{3}}$ sq. units **46.** $2\sqrt{2}$ sq. units

47. (i) Area bounded by $y = \cos x$, x-axis between $x = 0$, $x = \pi$.

(ii) Difference of area bounded by $y = \cos x$, x-axis between $x = 0$, $x = \frac{\pi}{2}$ and area bounded by $y = \cos x$, x-axis between $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$.

48. $\ell n \frac{3}{2}$ **49.** 4 sq. units

50. 8/3 sq. units

51. 64/3 sq. units **52.** $a^2\left[\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right]$ sq. units **53.** $\frac{73}{6}$ sq. units

54. 32 sq. units **55.** $\frac{109}{6}$ sq. units **56.** $\left(\frac{\pi}{6} + \frac{1}{8} - \frac{\sqrt{3}}{8}\right)$ sq. units

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Definite Integration in terms of Indefinite Integration, using substitution and By parts

A-1. Evaluate :

(i) $\int_0^1 \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\, dx$ (ii)✍ $\int_0^1 x\cos(\tan^{-1}x)\,dx$

A-2. Evaluate :

(i) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ (ii) $\int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$ (iii) $\int_0^4 \frac{x^2}{1+x}\, dx$

(iv) $\int_0^{\pi/2} \sqrt{\cos\theta}\sin^3\theta \quad d\theta$

A-3. Evaluate :

(i) $\int_0^1 \sin^{-1}x \quad dx$ (ii) $\int_1^2 \frac{\ell n\ x}{x^2}dx$ (iii) $\int_0^1 x^2\sin^{-1}x \quad dx$.

A-4. Evaluate

(i) $\int_0^{\pi/3} f(x)\ dx$ where f(x) = Minimum {tanx, cot x} $\forall\ x \in \left(0, \frac{\pi}{2}\right)$

(ii) $\int_{-1}^{1}$ f(x) dx where f(x) = min $\{x+1, \sqrt{1-x}\}$

(iii)✍ $\int_{-1}^{1} f(x)\ dx$ where f(x) = minimum (|x| , 1 – |x|, 1/4)

A-5. Evaluate

(i) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ (ii)✍ $\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\, dx$

(iii)✍ $\int_a^b \sqrt{(x-a)(b-x)}\ dx$, a > b (iv) $\int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

A-6. Evaluate :

(i) $\int_0^{\infty} \frac{dx}{e^x+e^{-x}}$ (ii) $\int_0^1 \frac{x}{1+\sqrt{x}}dx$ (iii) $\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$

(iv) $\int_0^{\pi/2} \frac{\sin 2\theta \quad d\theta}{\sin^4\theta+\cos^4\theta}$ (v) $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\quad \sin 2x}dx$

A-7 (i) Find the value of a such that $\int_0^a \frac{1}{e^x+4e^{-x}+5}dx = \ell n\ \sqrt[3]{2}$.

(ii) Find the value of $\int_0^{(\pi/2)^{1/3}} x^5 . \sin x^3\ dx$

Section (B) : Definite Integration using Properties

B-1. Let f(x) = $\ell n\left(\frac{1-\sin x}{1+\sin x}\right)$, then show that $\int_a^b f(x)\ dx = \int_b^a \ell n\left(\frac{1+\sin x}{1-\sin x}\right) dx$.

B-2. Evaluate :

(i) $\int_0^2 [x^2]\ dx$ (where [.] denotes greatest integer function)

(ii) $\int_0^{\pi} \sqrt{1+\sin 2x}\,dx$ (iii) $\int_0^2 f(x)dx$ where $f(x) = \begin{cases} 2x+1 & 0 \le x < 1 \\ 3x^2 & 1 \le x \le 2 \end{cases}$

(iv) $\int_0^4 |x^2 - 4x + 3|dx$ (v) $\int_0^{\infty} [\cot^{-1} x]dx$ (where [.] denotes greatest integer function)

(vi) $\int_{-5}^5 |x+2|\,dx$ (vii) $\int_{-1}^1 [\cos^{-1} x]dx$ (where [.] denotes greatest integer function)

B-3. Evaluate :

(i) $\int_{-1}^1 e^{|x|}\ dx$ (ii) $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx$ (iii) $\int_{-\pi/4}^{\pi/4} \frac{x+\pi/4}{2-\cos 2x}\ dx$

(iv) $\int_{-1}^1 \sin^5 x \cos^4 x\ \ dx$ (v) $\int_{-\pi/2}^{\pi/2} \frac{g(x)-g(-x)}{f(-x)+f(x)}dx$

B-4. Evaluate

(i) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\ dx$ (ii) $\int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\ dx$ (iii) $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\ dx$

(iv) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}\ dx$ (v) $\int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2}\ dx$

B-5. Evaluate :

(i) $\int_0^{2\pi} \{\sin(\sin x) + \sin(\cos x)\}dx$ (ii) $\int_0^{\pi} \frac{dx}{5+4\cos 2x}$

(iii) $\int_0^{\pi/2} (2\ \ \ell n \sin x - \ell n\ \ \sin 2x)\,dx$ (iv) $\int_0^{\infty} \ell n\left(x+\frac{1}{x}\right).\frac{dx}{1+x^2}$

B-6. Evaluate :

(i) $\int_{-1}^2 \{2x\}\ \ dx$ (where function {.} denotes fractional part function)

(ii) $\int_0^{10\pi} (|\sin x| + |\cos x|)\ \ dx$

(iii) $\dfrac{\int_0^n [x]\ dx}{\int_0^n \{x\}\ dx}$, where [x] and {x} are integral and fractional parts of x and n $\in$ N

(iv) $\int_0^{2n\pi} \left(|\sin x| - \left[\left|\frac{\sin x}{2}\right|\right]\right)\ dx$ (where [ ] denotes the greatest integer function and n $\in$ I)

B-7. If f(x) is a function defined $\forall$ x$\in$R and f(x) + f(–x) = 0 $\forall$ x$\in$ $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has period T, then prove that $\phi(x) = \int_a^x f(t)\ dt$ is also periodic with period T.

Section (C) : Leibnitz formula and Wallis' formula

C-1. (i) If $f(x) = 5^{g(x)}$ and $g(x) = \int_2^{x^2} \frac{t}{\ell n\ (1+t^2)}\ dt$, then find the value of $f'(\sqrt{2})$.

(ii) The value of $\lim\limits_{x\to 0} \frac{\frac{d}{dx}\int_0^{x^3}\sqrt{\cos t}\ dt}{1-\sqrt{\cos x}}$

(iii) Find the slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1} t^2\, dt$ at $x = \frac{1}{\sqrt[4]{2}}$

C-2. (i) If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$, then prove that $f'(x) = 0\ \forall\ x \in R$.

(ii) Find the value of x for which function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\ dt$ has a local minimum

C-3. If $y = \int_1^x x\sqrt{\ell n\, t}\, dt$

then find the value of $\frac{d^2y}{dx^2}$ at x = e

C-4. $\lim\limits_{n\to\infty} \frac{\int_{1/(n+1)}^{1/n} \tan^{-1}(nx)dx}{\int_{1/(n+1)}^{1/n} \sin^{-1}(nx)dx}$ is equal to

C-5. Let f be a differentiable function on R and satisfying the integral equation

$x\int_0^x f(t)dt - \int_0^x tf(x-t)dt = e^x - 1\quad \forall\ x \in R$, then f(1) equals to

C-6. Evaluate :

(i) $\int_{-\pi/2}^{\pi/2} \sin^2 x\cos^2 x(\sin x + \cos x)\ dx$ (ii) $\int_0^{\pi} x\sin^5 x\ dx$

(iii) $\int_0^2 x^{3/2}\sqrt{2-x}\ dx$ (iv) $\int_0^{2\pi} x\ (\sin^2 x\cos^2 x)\ dx$

SECTION (D) : ESTIMATION & MEAN VALUE THEOREM

D-1. Prove the following inequalities :

(i) $\dfrac{\sqrt{3}}{8} < \displaystyle\int_{\pi/4}^{\pi/3} \dfrac{\sin x}{x} dx < \dfrac{\sqrt{2}}{6}$ (ii) $4 \le \displaystyle\int_{1}^{3} \sqrt{(3+x^3)}\, dx \le 2\sqrt{30}$

D-2. Show that

(i) $\dfrac{1}{10\sqrt{2}} < \displaystyle\int_{0}^{1} \dfrac{x^9}{\sqrt{1+x}} dx < \dfrac{1}{10}$ (ii) $\dfrac{1}{2}\ell n2 < \displaystyle\int_{0}^{1} \dfrac{\tan x}{1+x^2} dx < \dfrac{\pi}{2}$

D-3. (i) Show that $\displaystyle\int_{0}^{2} \sin x.\cos\sqrt{x}\, dx = 2\sin c.\cos\sqrt{c}$ for some $c \in (0, 2)$

(ii) f(x) is a continuous function $x \in R$, then show that $\displaystyle\int_{1}^{4} f(x)dx = 2\alpha f(\alpha^2)$ some $\alpha \in (1, 2)$

Section (E) : Integration as a limit of sum and reduction formula

E-1. Evaluate :

(i) $\displaystyle\lim_{n\to\infty} \sum_{r=0}^{n-1} \dfrac{1}{\sqrt{n^2-r^2}}$ (ii) $\displaystyle\lim_{n\to\infty} \dfrac{3}{n}\left[1+\sqrt{\dfrac{n}{n+3}}+\sqrt{\dfrac{n}{n+6}}+\sqrt{\dfrac{n}{n+9}}+..........+\sqrt{\dfrac{n}{n+3(n-1)}}\right]$

(iii) $\displaystyle\lim_{n\to\infty} \dfrac{1}{n^4}\left(\sum_{r=1}^{2n}(3nr^2+2n^2r)\right)$

E-2. (i) If $I_n = \displaystyle\int_{0}^{\pi/4} \tan^n x \; dx$, then show that $I_n + I_{n-2} = \dfrac{1}{n-1}$

(ii) $I_n = \displaystyle\int_{0}^{\pi/2} (\sin x)^n dx,\ n \in N$. Show that $I_n = \dfrac{n-1}{n} I_{n-2} \quad \forall n \ge 2$

Section (F) : Area Under Curve

F-1. Find the area enclosed between the curve $y = x^3 + 3$, $y = 0$, $x = -1$, $x = 2$.

F-2. (i) Find the area bounded by $x^2 + y^2 - 2x = 0$ and $y = \sin\dfrac{\pi x}{2}$ in the upper half of the circle.

(ii) Find the area bounded by the curve $y = 2x^4 - x^2$, x-axis and the two ordinates corresponding to the minima of the function.

(iii) Find area of the curve $y^2 = (7 - x)(5 + x)$ above x–axis and between the ordinates $x = -5$ and $x = 1$.

F-3. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

F-4. Find the area bounded by the y-axis and the curve $x = e^y \sin \pi y$, $y = 0$, $y = 1$.

F-5. (i) Find the area bounded in the first quadrant between the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ and the line $3x + 4y = 12$

(ii) Find the area of the region bounded by $y = \{x\}$ and $2x - 1 = 0$, $y = 0$, ({ } stands for fraction part)

F-6. Compute the area of the figure bounded by straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$ and $y = 2x - x^2$

F-7. Let $f(x) = \sqrt{\tan x}$. Show that area bounded by y = f(x), y = f(c), x = 0 and x = a, $0 < c < a < \frac{\pi}{2}$ is minimum when $c = \frac{a}{2}$

F-8. Find the area included between the parabolas $y^2 = x$ and $x = 3 - 2y^2$.

F-9. A tangent is drawn to the curve $x^2 + 2x - 4ky + 3 = 0$ at a point whose abscissa is 3. This tangent is perpendicular to x + 3 = 2y. Find the area bounded by the curve, this tangent and ordinate x = – 1

F-10. (i) Draw graph of $y = (\tan x)^n$, $n \in \left[0, \frac{\pi}{4}\right]$ N, $x \in \left[0, \frac{\pi}{4}\right]$. Hence show

$0 < (\tan x)^{n+1} < (\tan x)^n$, $x \in \left(0, \frac{\pi}{4}\right)$

(ii) Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines x = 0, y = 0 and x = π/4. Prove that for n > 2, $A_n + A_{n-2} = 1/(n-1)$ and deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

PART - II : ONLY ONE OPTION CORRECT TYPE

SECTION (A) : D.I. IN TERMS OF INDEFINITE INTIGRATION, USING SUBSTITUTION AND BY PARTS

A-1. If $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$, then x can be equal to :

(A) $\frac{2}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) 2 (D) $\frac{4}{\sqrt{3}}$

A-2. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, where $0 < \alpha < \frac{\pi}{2}$, is equal to:

(A) $\sin\alpha$ (B) $\alpha \sin \alpha$ (C) $\frac{\alpha}{2\sin\alpha}$ (D) $\frac{\alpha}{2}\sin\alpha$

A-3. If $f(x) = \begin{cases} x & x<1 \\ x-1 & x\ge 1 \end{cases}$, then $\int_0^2 x^2 f(x)\, dx$ is equal to :

(A) 1 (B) $\frac{4}{3}$ (C) $\frac{5}{3}$ (D) $\frac{5}{2}$

A-4. If f(0) = 1, f(2) = 3, f′(2) = 5 and f′(0) is finite, then $\int_0^1 x\ .\ f''(2x)\, dx$ is equal to

(A) zero (B) 1 (C) 2 (D) 3

A-5. $\int_0^{\pi} |1+2\cos x|\, dx$ is equal to :

(A) $\frac{2\pi}{3}$ (B) π (C) 2 (D) $\frac{\pi}{3}+2\sqrt{3}$

A-6. The value of $\int_{-1}^{3} (|x-2|+[x])dx$ is ([x] stands for greatest integer less than or equal to x)

(A) 7 (B) 5 (C) 4 (D) 3

A-7. $\int_0^{\infty} [2e^{-x}]\ dx$, where [.] denotes the greatest integer function, is equal to :

(A) 0 (B) $\ell n\ 2$ (C) e^2 (D) $2e^{-1}$

A-8. $\int_{\ell n\pi - \ell n 2}^{\ell n \pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}$ dx is equal to

(A) $\sqrt{3}$ (B) $-\sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $-\frac{1}{\sqrt{3}}$

A-9. If $I_1 = \int_e^{e^2} \frac{dx}{\ell n\ x}$ and $I_2 = \int_1^2 \frac{e^x}{x}$ dx, then

(A) $I_1 = I_2$ (B) $2\ I_1 = I_2$ (C) $I_1 = 2\ I_2$ (D) $I_1 + I_2 = 0$

A-10. $\int_0^{\pi/4} \frac{x\ .\ \sin x}{\cos^3 x}$ dx equals to :

(A) $\frac{\pi}{4} + \frac{1}{2}$ (B) $\frac{\pi}{4} - \frac{1}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{4} + 1$

A-11. The value of the definite integral $\int_{\frac{3}{2}}^{\frac{9}{4}} \left[\sqrt{2x - \sqrt{5(4x-5)}} + \sqrt{2x + \sqrt{5(4x-5)}}\right]$ dx is equal to

(A) $4\sqrt{5} - \frac{2\sqrt{2}}{5}$ (B) $4\sqrt{5}$ (C) $4\sqrt{3} - \frac{4}{3}$ (D) $\frac{3\sqrt{5}}{\sqrt{8}}$

A-12. If $\int_{\ell n 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$, then x is equal to

(A) 4 (B) $\ell n\ 8$ (C) $\ell n\ 4$ (D) $\ell n\ 2$

A-13. $\int_0^{\infty} \frac{x^2+1}{x^4+7x^2+1}$ dx =

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Section (B) : Definite Integration using Properties

B-1. Suppose for every integer n, . $\int_n^{n+1} f(x)dx = n^2$ The value of $\int_{-2}^{4} f(x)dx$ is :

(A) 16 (B) 14 (C) 19 (D) 21

B-2. Let f : R → R, g : R → R be continuous functions. Then the value of integeral

$\int_{\ell n \lambda}^{\ell n 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x) - f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x) + g(-x)}$ dx is :

(A) depend on λ (B) a non-zero constant (C) zero (D) 2

B-3. $\int_{-1}^{1} \cot^{-1}\left(\frac{x+x^3}{1+x^4}\right) dx$ is equal to

(A) 2π (B) $\frac{\pi}{2}$ (C) 0 (D) π

B-4. $\int_{-2}^{0} \{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\ dx$ is equal to

(A) – 4 (B) 0 (C) 4 (D) 6

B-5. $\int_{-1}^{1} x\ell n(1+e^x)dx$ =

(A) 0 (B) ℓn(1 + e) (C) ℓn(1 + e) – 1 (D) 1/3

B-6. If $\int_{-1}^{3/2} |x\sin\pi x|\ dx = \frac{k}{\pi^2}$, then the value of k is :

(A) $3\pi + 1$ (B) $2\pi + 1$ (C) 1 (D) 4

B-7. The value of definite integral is $\int_{0}^{\frac{\pi^2}{4}} \frac{dx}{1+\sin\sqrt{x}+\cos\sqrt{x}}$

(A) $\pi \ln 2$ (B) $\frac{\pi \ln 2}{2}$ (C) $\frac{\pi \ln 2}{4}$ (D) $2\pi \ln 2$

B-8. $\int_{2-\ell n 3}^{3+\ell n 3} \frac{\ell n\ (4+x)}{\ell n\ (4+x)+\ell n\ (9-x)} dx$ is equal to :

(A) cannot be evaluated (B) is equal to $\frac{5}{2}$

(C) is equal to 1 + 2 ℓn 3 (D) is equal to $\frac{1}{2}$ + ℓn 3

B-9. The value of the definite integral I = $\int_{0}^{\pi} x\sqrt{1+|\cos x|}\ dx$ is equal to

(A) $2\sqrt{2}\ \pi$ (B) $\sqrt{2}\ \pi$ (C) $2\ \pi$ (D) 4π

B-10. The value of $\int_{0}^{\pi/2} \ell n|\tan x+\cot x|\ dx$ is equal to :

(A) $\pi\ \ell n\ 2$ (B) $-\pi\ \ell n\ 2$ (C) $\frac{\pi}{2}\ \ell n\ 2$ (D) $-\frac{\pi}{2}\ \ell n\ 2$

B-11. Let $I_1 = \int_{0}^{1} \frac{e^x dx}{1+x}$ and $I_2 = \int_{0}^{1} \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then $\frac{I_1}{I_2}$ is

(A) 3/e (B) e/3 (C) 3e (D) 1/3e

B-12. The value of $\int_{0}^{[x]} \{x\}\ dx$ (where [.] and { . } denotes greatest integer and fraction part function respectively) is

(A) $\frac{1}{2}[x]$ (B) 2[x] (C) $\frac{1}{2[x]}$ (D) [x]

B-13. If $\int_0^{11} \frac{11^x}{11^{[x]}}\, dx = \frac{k}{\log 11}$, (where [ ] denotes greatest integer function) then value of k is

(A) 11 (B) 101 (C) 110 (D)121

Section (C) : Leibnitz formula and Wallis' formula

C-1. $f(x) = \int_x^{x^2} \frac{e^t}{t} dt$, then f' (1) is equal to:

(A) e (B) 2e (C) $2e^2 - 2$ (D) $e^2 - e$

C-2. $f(x) = \int_0^x (t-1)(t-2)^2(t-3)^3(t-4)^5\ dt$ (x > 0) then number of points of extremum of f(x) is

(A) 4 (B) 3 (C) 2 (D) 1

C-3. $\underset{h \to 0}{\text{Limit}} \frac{\int_a^{x+h} \ell n^2 t\ dt - \int_a^x \ell n^2 t\ dt}{h}$ equals to :

(A) 0 (B) $\ell n^2 x$ (C) $\frac{2\ell n x}{x}$ (D) does not exist

C-4. The value of the function $f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\,\ell n t)dt$, where $f'(x)$ vanishes is:

(A) e^{-1} (B) 0 (C) $2e^{-1}$ (D) $1 + 2e^{-1}$

C-5. If $\int_a^y \cos t^2 dt = \int_a^{x^2} \frac{\sin t}{t} dt$, then the value of $\frac{dy}{dx}$ is

(A) $\frac{2\sin^2 x}{x\cos^2 y}$ (B) $\frac{2\sin x^2}{x\cos y^2}$ (C) $\frac{2\sin x^2}{x\left(1-2\sin \frac{y^2}{2}\right)}$ (D) $\frac{\sin x^2}{2y}$

C-6. If $\int_{\sin x}^{1} t^2\ (f(t))\ dt = (1 - \sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) 1/3 (B) $1/\sqrt{3}$ (C) 3 (D) $\sqrt{3}$

C-7. The value of $\underset{a\to\infty}{\text{Lim}} \frac{1}{a^2} \int_0^a \ln(1+e^x)\ dx$ equals

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) non-existent

C-8. $f(x) = \int_1^x \frac{\sin x \cos y}{y^2 + y + 1} dy$, then

(A) $f'(x) = 0\ \forall\ x = \frac{n\pi}{2}$, $n \in Z$ (B) $f'(x) = 0\ \forall\ x = (2n+1)\frac{\pi}{2}$, $n \in Z$

(C) $f'(x) = 0\ \forall\ x = n\pi$, $n \in Z$ (D) $f'(x) \neq 0\ \forall\ x \in R$

C-9. $\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx$ is equal to :

(A) $\frac{6}{35}$ (B) $\frac{2}{21}$ (C) $\frac{2}{15}$ (D) $\frac{2}{35}$

C-10. $\int_0^1 x^2(1-x)^3\,dx$ is equal to

(A) $\frac{1}{60}$ (B) $\frac{1}{30}$ (C) $\frac{2}{15}$ (D) $\frac{\pi}{120}$

SECTION (D) : ESTIMATION & MEAN VALUE THEOREM

D-1. Let $I = \int_1^3 \sqrt{x^4 + x^2}\,dx$, then

(A) $I > 6\sqrt{10}$ (B) $I < 2\sqrt{2}$ (C) $2\sqrt{2} < I < 6\sqrt{10}$ (D) $I < 1$

D-2. $I = \int_0^{2\pi} e^{\sin^2 x + \sin x + 1}\,dx$, then

(A) $\pi e^3 < I < 2\pi e^5$ (B) $2\pi e^{3/4} < I < 2\pi e^3$ (C) $2\pi e^3 < I < 2\pi e^4$ (D) $0 < I < 2\pi$

D-3. Let $f''(x) \ge 0$, $f'(x) > 0$, $f(0) = 3$ & $f(x)$ is defined in $[-2, 2]$. If $f(x)$ is non-negative, then

(A) $\int_{-1}^{0} f(x)dx > 6$ (B) $\int_{-2}^{2} f(x)dx > 12$ (C) $\int_{-2}^{2} f(x)dx \ge 12$ (D) $\int_{-1}^{1} f(x)dx > 12$

D-4. Let mean value of $f(x) = \frac{1}{x+c}$ over interval (0, 2) is $\frac{1}{2}\ell n3$ then positive value of c is

(A) 1 (B) $\frac{1}{2}$ (C) 2 (D) $\frac{3}{2}$

SECTION (E) : INTEGRATION AS A LIMIT OF SUM AND REDUCTION FORMULA

E-1. $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^3}{r^4 + n^4}\right)$ equals to :

(A) $\ell n\, 2$ (B) $\frac{1}{2}\ell n\, 2$ (C) $\frac{1}{3}\ell n\, 2$ (D) $\frac{1}{4}\ell n\, 2$

E-2. $\text{Lt}_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{n}{r^2 - n^2}$ is equal to :

(A) $\ell n \sqrt{\frac{2}{3}}$ (B) $\ell n \sqrt{\frac{3}{2}}$ (C) $\ell n \frac{2}{3}$ (D) $\ell n \frac{3}{2}$

E-3. $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right) \ldots \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ is equal to :

(A) $\frac{e^{\pi/2}}{2e^2}$ (B) $2\,e^2\,e^{\pi/2}$ (C) $\frac{2}{e^2}\,e^{\pi/2}$ (D) $2\,e^{\pi}$

E-4. $\lim\limits_{n\to\infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{(n-1)\pi}{n}\right]$ is equals to :
(A) 0 (B) π (C) 2 (D) 3

E-5. Let $I_n = \int_0^1 (1-x^3)^n dx,\ (n \in N)$ then
(A) $3n\ I_n = (3n - 1)\ I_{n-1}\ \forall\ n \ge 2$ (B) $(3n - 1)I_n = 3n\ I_{n-1} \forall\ n \ge 2$
(C) $(3n - 1)I_n = (3n + 1)\ I_{n-1}\ \forall\ n \ge 2$ (D) $(3n + 1)I_n = 3n\ I_{n-1}\ \forall\ n \ge 2$

Section (F) : Area Under Curve

F-1. The area bounded by the x-axis and the curve $y = 4x - x^2 - 3$ is
(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{8}{3}$

F-2. The area of the figure bounded by right of the line $y = x + 1$, $y = \cos x$ and x–axis is:
(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{5}{6}$ (D) $\frac{3}{2}$

F-3. Area bounded by curve $y^3 - 9y + x = 0$ and y-axis is
(A) $\frac{9}{2}$ (B) 9 (C) $\frac{81}{2}$ (D) 81

F-4. Let $f:[0, \infty) \to R$ be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t\ f^2(t)\ dt$, $x \ge 0$.
The area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$ is ————
(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{1}{2}$

F-5. The area bounded by the curve $y = e^x$ and the lines $y = |x - 1|$, $x = 2$ is given by:
(A) $e^2 + 1$ (B) $e^2 - 1$ (C) $e^2 - 2$ (D) $e - 2$

F-6. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is:
(A) $\frac{4 + 3\ \ell n\ 3}{2}$ (B) $\frac{4 - 3\ \ell n\ 3}{2}$ (C) $\frac{3}{2} + \ell n3$ (D) $\frac{1}{2} + \ell n3$

F-7. The area bounded by the curve $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ is
(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

F-8. The area of the region bounded by $x = 0$, $y = 0$, $x = 2$, $y = 2$, $y \le e^x$ and $y \ge \ell n\ x$, is
(A) $6 - 4\ \ell n\ 2$ (B) $4\ \ell n\ 2 - 2$ (C) $2\ \ell n\ 2 - 4$ (D) $6 - 2\ \ell n\ 2$

F-9. The area between two arms of the curve $|y| = x^3$ from $x = 0$ to $x = 2$ is
(A) 2 (B) 4 (C) 8 (D) 16

F-10. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = \frac{1}{4}$ is
(A) 4 sq. units (B) $\frac{1}{6}$ sq. units (C) $\frac{4}{3}$ sq. units (D) $\frac{1}{3}$ sq. units

PART - III : MATCH THE COLUMN

1. Let $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^T(\sin x+\sin ax)^2dx = L$ then

Column - I	**Column- II**
(A) for a = 0, the value of L is	(p) 0
(B) for a = 1 the value of L is	(q) 1/2
(C) for a = – 1 the value of L is	(r) 3/2
(D) $\forall$ a $\in$ R – {–1, 0, 1} the value of L is	(s) 2
	(t) 1

2.

Column – I	**Column – II**
(A) Area bounded by region $0 \le y \le 4x - x^2 - 3$ is	(p) 32/3
(B) The area of figure formed by all the points satisfying the inequality $y^2 \le 4(1 - \lvert x\rvert)$ is	(q) 1/2
(C) The area bounded by $\lvert x\rvert + \lvert y\rvert \le 1$ and $\lvert x\rvert \ge 1/2$ is	(r) 4/3
(D) Area bounded by $x \le 4 - y^2$ and $x \ge 0$ is	(s) 16/3

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. The value of $\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$, (where { } denotes fractional part of x) is equal to :

(A) $\frac{19}{36}$ (B) $\frac{19}{144}$ (C) $\frac{19}{72}$ (D) $\frac{19}{18}$

2.✍ If $\int_0^{100} f(x)\,dx = a$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right) =$

(A) 100 a (B) a (C) 0 (D) 10 a

3. $\lim\limits_{t\to\left(\frac{\pi}{2}\right)^-}\int_0^t \tan\theta\sqrt{\cos\theta}\,\ell n(\cos\theta)d\theta$ is equal to:

(A) –4 (B) 4 (C) –2 (D) Does not exists

4. If f(x) = $\begin{cases} 0 \ , & \text{where } x = \frac{n}{n+1},\ n = 1, 2, 3..... \\ 1 \ , & \text{else where} \end{cases}$, then the value of $\int_0^2 f(x)\,dx$.

(A) 1 (B) 0 (C) 2 (D) ∞

5. If $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, then $\int_0^\infty e^{-ax^2}dx$ where a > 0 is :

(A) $\frac{\sqrt{\pi}}{2}$ (B) $\frac{\sqrt{\pi}}{2a}$ (C) $2\frac{\sqrt{\pi}}{a}$ (D) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

6. If $\sum_{i=1}^{4}\left(\sin^{-1}x_i+\cos^{-1}y_i\right)$ = 6π, then $\int_{\sum_{i=1}^{4}x_i}^{\sum_{i=1}^{4}y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)dx$ is equal to

(A) 0 (B) $e^4 + e^{-4}$ (C) $\ln\left(\frac{17}{12}\right)$ (D) $e^4 - e^{-4}$

7. The tangent to the graph of the function y = f(x) at the point with abscissa x =1 form an angle of $\pi/6$ and at the point x = 2, an angle of $\pi/3$ and at the point x = 3, an angle of $\pi/4$ with positive x-axis. The value of $\int_1^3 f'(x)\ f''(x)\ dx + \int_2^3 f''\ (x)\ dx$ (f''(x) is supposed to be continuous) is :

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D) $\frac{4}{3}-\sqrt{3}$

8. Let A = $\int_0^1 \frac{e^t}{1+t}$ dt, then $\int_{a-1}^{a}\frac{e^{-t}}{t-a-1}$ dt has the value :

(A) Ae^{-a} (B) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

9. $\int_1^2 x^{2x^2+1}(1+2\ell n x)dx$ is equal to

(A) 256 (B) 255 (C) $\frac{255}{2}$ (D) 128

10. If f(x) is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\ dx$ equals to :

(A) $\sin\theta + \operatorname{cosec}\theta$ (B) $\sin^2\theta$ (C) $\operatorname{cosec}^2\theta$ (D) none of these

11. If $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}=0$, where C_0, C_1, C_2 are all real, the equation $C_2x^2 + C_1x + C_0 = 0$ has:

(A) atleast one root in (0, 1)
(B) one root in (1, 2) & other in (3, 4)
(C) one root in (–1, 1) & the other in (– 5, – 2)
(D) both roots imaginary

12. If f(x) = $\int_0^x (2\cos^2 3t + 3\sin^2 3t)$ dt, f(x + π) is equal to :

(A) f(x) + 2f(π) (B) f(x) + 2f $\left(\frac{\pi}{2}\right)$ (C) f(x) + 4f$\left(\frac{\pi}{4}\right)$ (D) 2f(x)

13. Let f (x) = $\int_0^x \frac{dt}{\sqrt{1+t^3}}$ and g (x) be the inverse of f (x), then which one of the following holds good?

(A) $2g'' = g^2$ (B) $2g'' = 3g^2$ (C) $3g'' = 2g^2$ (D) $3g'' = g^2$

14. Let f(x) is differentiable function satisfying $2\int_1^2 f(tx)dt$ = x + 2 , $\forall$ x $\in$ R Then $\int_0^1 (8f(8x) - f(x) - 21x)$ dx equals to

(A) 3 (B) 5 (C) 7 (D) 9

15. Let $I_n = \int_0^1 x^n (\tan^{-1} x)\,dx$, $n \in N$, then

(A) $(n + 1)I_n + (n - 1)I_{n-2} = \frac{\pi}{4} + \frac{1}{n} \quad \forall \quad n \geq 3$ (B) $(n + 1)I_n + (n - 1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n} \quad \forall \quad n \geq 3$

(C) $(n + 1)I_n - (n - 1)I_{n-2} = \frac{\pi}{4} + \frac{1}{n} \quad \forall \quad n \geq 3$ (D) $(n + 1)I_n - (n - 1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n} \quad \forall \quad n \geq 3$

16. If, $u_n = \int_0^{\pi/2} x^n \sin x\,dx$, then the value of $u_{10} + 90\,u_8$ is :

(A) $9\left(\frac{\pi}{2}\right)^8$ (B) $\left(\frac{\pi}{2}\right)^9$ (C) $10\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^9$

17. The value of $\int_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\,dt$, where $x \in (\pi/6, \pi/3)$, is equal to :

(A) 0 (B) 2 (C) 1 (D) cannot be determined

18. Let $A_1 = \int_0^x \left(\int_0^u f(t)\,dt\right) du$ and $A_2 = \int_0^x f(u).(x - u)\,du$ then $\frac{A_1}{A_2}$ is equal to :

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) –1

19. $\lim_{n\to\infty} \left(\sin\frac{\pi}{2n} \,.\, \sin\frac{2\pi}{2n} \,.\, \sin\frac{3\pi}{2n} ....\sin\frac{(n-1)\pi}{n}\right)^{1/n}$ is equal to :

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

20. Area bounded by the region consisting of points (x, y) satisfying $y \leq \sqrt{2-x^2}$, $y^2 \geq x$, $\sqrt{y} \geq -x$ is

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) $\pi/4$

21. The area enclosed between the curves

$y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis is

(A) 2 (B) 1 (C) 4 (D) 3

22. The area bounded by the curve $x = a\cos^3 t$, $y = a \sin^3 t$ is

(A) $\frac{3\pi a^2}{8}$ (B) $\frac{3\pi a^2}{16}$ (C) $\frac{3\pi a^2}{32}$ (D) $3\pi a^2$

23. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ and $x = 2\pi$ is

(A) 4π (B) 8π (C) 4 (D) 8

24. P(2, 2), Q(–2, 2), R(–2, –2) & S(2, –2) are vertices of a square. A parabola passes through P, S & its vertex lies on x-axis. If this parabola bisects the area of the square PQRS, then vertex of the parabola is

(A) (–2, 0) (B) (0, 0) (C) $\left(-\frac{3}{2}, 0\right)$ (D) (–1, 0)

25. The ratio in which the curve $y = x^2$ divides the region bounded by the curve; $y = \sin\left(\frac{\pi x}{2}\right)$ and the x–axis as x varies from 0 to 1, is :

(A) $2: \pi$ (B) $1: 3$ (C) $3: \pi$ (D) $(6 - \pi): \pi$

26. If $f(x) = \sin x, \ \forall x \in \left[0, \frac{\pi}{2}\right]$, $f(x) + f(\pi - x) = 2. \ \forall x \in \left(\frac{\pi}{2}, \pi\right]$ and $f(x) = f(2\pi - x), \ x \in (\pi, 2\pi]$, then the area enclosed by $y = f(x)$ and x-axis is

(A) π (B) 2π (C) 2 (D) 4

27. The area bounded by the curves $y = x e^x$, $y = x e^{-x}$ and the line $x = 1$

(A) $\frac{2}{e}$ (B) $1 - \frac{2}{e}$ (C) $\frac{1}{e}$ (D) $1 - \frac{1}{e}$

28. Obtain the area enclosed by region bounded by the curves $y = x \ln x$ and $y = 2x - 2x^2$.

(A) 7/6 (B) 7/24 (C) 12/7 (D) 7/12

29. The area of the region on plane bounded by $\max(|x|, |y|) \le 1$ and $xy \le \frac{1}{2}$ is

(A) $1/2 + \ln 2$ (B) $3 + \ln 2$ (C) 31/4 (D) $1 + 2 \ln 2$

30. Consider the following statements :

S_1 : The value of $\int_0^{2\pi} \cos^{-1}(\cos x) \ dx$ is π^2

S_2 : Area enclosed by the curve $|x - 2| + |y + 1| = 1$ is equal to 3 sq. unit

S_3 : If $\frac{d}{dx} f(x) = g(x)$ for $a \le x \le b$, then $\int_a^b f(x)g(x)dx$ equals to $\frac{[f(b)]^2 - [f(a)]^2}{2}$.

S_4 : Area of the region $R \equiv \{(x, y) \ ; x^2 \le y \le x\}$ is $\frac{1}{6}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TFTT (B) TTTT (C) FFFF (D) TFTF

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. $\int_2^4 \frac{3x^2+1}{(x^2-1)^3} dx = \frac{\lambda}{n^2}$ where $\lambda, n \in N$ and $\gcd(\lambda, n) = 1$, then find the value of $\lambda + n$

2. Let $U = \int_{\pi/6}^{\pi/2} \min . (\sqrt{3}\sin x, \cos x) \ dx$ and $V = \int_{-3}^{5} x^2 \ \text{sgn}\,(x - 1)dx$. If $V = \lambda U$, then find the value of λ.

[**Note :** sgn k denotes the signum function of k.]

3. Let f(x) be a function satisfying $f(x) = f\left(\frac{100}{x}\right) \forall x > 0$. If $\int_1^{10} \frac{f(x)}{x} dx = 5$ then find the value of $\int_1^{100} \frac{f(x)}{x} dx$

4. Evaluate

$$\frac{2005\displaystyle\int_0^{1002}\frac{dx}{\sqrt{1002^2-x^2}+\sqrt{1003^2-x^2}}+\int_{1002}^{1003}\sqrt{1003^2-x^2}\,dx}{\displaystyle\int_0^1\sqrt{1-x^2}\,dx}$$ = k, then find the sum of squares of digits of natural number k.

5. If $\displaystyle\int_0^{\pi/2}\sqrt{\sin 2\theta}.\sin\theta\, d\theta=\frac{\pi}{n}$ then find n

6. Let $I_1=\displaystyle\int_0^{\pi/4}(1+\tan x)^2dx$, $I_2=\displaystyle\int_0^1\frac{dx}{(1+x)^2(1+x^2)}$

then find the value of $\frac{I_1}{I_2}$

7. Find the value of $\ell n\left(\displaystyle\int_0^1 e^{t^2+t}(2t^2+t+1)dt\right)$

8. If $\displaystyle\int_{-1}^0\frac{x}{x+1+e^x}dx$ is equal to $-\ell nk$, then find the value of k.

9. If f, g, h be continuous functions on [0, a] such that f (a – x) = f (x), g (a – x) = – g (x) and 3 h (x) – 4 h (a – x) = 5, then find the value of $\displaystyle\int_0^a f(x)\,g(x)\,h(x)\,dx$

10. If f(x) = $\frac{\sin x}{x}$ $\forall$ x $\in$ (0, π], If $\frac{\pi}{k}\displaystyle\int_0^{\pi/2}f(x)\ f\left(\frac{\pi}{2}-x\right)dx=\int_0^{\pi}f(x)\ dx$ then find the value of k.

11. Evaluate: $3\displaystyle\int_0^{\pi}\frac{a^2\sin^2x+b^2\cos^2x}{a^4\sin^2x+b^4\cos^2x}dx$, where $a^2+b^2=\frac{3\pi}{4}$, $a^2\neq b^2$ and $ab\neq 0$.

12. $\displaystyle\int_0^{2\pi}|\sqrt{15}\sin x+\cos x|\,dx$

13. Let a be a real number in the interval [0, 314] such that $\displaystyle\int_{-\pi+a}^{3\pi+a}|x-a-\pi|\sin\left(\frac{x}{2}\right)dx=-16$, then determine number of such values of a.

14. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)=\frac{\pi}{n}$, find 'n' $\left(\text{Note that } \tan^{-1}x+c=\displaystyle\int\frac{1}{1+x^2}dx\right)$

15. If $f(x) = x + \int_0^1 t(x+t)\ f(t)\ dt$, then the value of the definite integral $\int_0^1 f(x)\ dx$ can be expressed in the form of rational as $\frac{p}{q}$ (where p and q are coprime). Find (p + q).

16. If f(x) = (ax + b) e^x satisfies the equation : $f(x) = \int_0^x e^{x-y} f'(y)dy - (x^2 - x + 1)e^x$, find $(a^2 + b^2)$

17. If the minimum of the following function f(x) defined at $0 < x < \frac{\pi}{2}$.

$f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ is equal to $\ell n(a + \sqrt{b})$ where a, b $\in$ N and b is not a perfect square then find the value of (a + b)

18. If $f(\pi) = 2$ and $\int_0^{\pi} (f(x) + f''(x))\ \sin x\ dx = 5$, then find the value of f(0)
(it is given that f(x) is continuous in $[0, \pi]$)

19. If $f(x) = 2x^3 - 15x^2 + 24x$ and $g(x) = \int_0^x f(t)\ dt + \int_0^{5-x} f(t)\ dt$ $(0 < x < 5)$. Find the number of integers for which g(x) is increasing.

20. Let $f(x) = \begin{cases} 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \\ (2-x)^2 & \text{if } 2 < x \le 3 \end{cases}$ and function $F(x) = \int_0^x f(t)\ dt$. If number of points of discontinuity in [0, 3] and non-differentiablity in (0, 3) of F(x) are α and β respectively, then $(\alpha - \beta)$ is equal to.

21. Find the value of m (m > 0) for which the area bounded by the line y = mx + 2 and $x = 2y - y^2$ is 9/2 square units.

22. Find area bounded by $y = f^{-1}(x)$, x = 10, x = 4 and x-axis
given that area bounded by y = f(x), x = 2, x = 6 and x-axis is 30 sq. units, where f(2) = 4 and f(6) = 10.
(given f(x) is an invertible function)

23. Consider a line $\ell : 2x - \sqrt{3}\,y = 0$ and a parameterized C : x = tan t, $y = \frac{1}{\cos t}\left(0 \le t < \frac{\pi}{2}\right)$

If the area of the part bounded by ℓ, C and the y-axis is equal to $\frac{1}{4}\ell n(a + \sqrt{b})$, where a, b, $\in$ N, b, is not perfect square then find the value of (a + b)

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$ equals to :

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) is same as $\int_0^{\infty} \frac{dx}{(1+x)(1+x^2)}$

(D) cannot be evaluated

2. The value of integral $\int_a^b \frac{|x|}{x}\, dx$, a < b is :

(A) b – a if a > 0 (B) a – b if b < 0 (C) b + a if a < 0 < b (D) | b | – | a |

3. If $I_n = \int_0^1 \frac{dx}{\left(1+x^2\right)^n}$; $n \in N$, then which of the following statements hold good?

(A) $2n\, I_{n+1} = 2^{-n} + (2n - 1)\, I_n$ (B) $I_2 = \frac{\pi}{8} + \frac{1}{4}$

(C) $I_2 = \frac{\pi}{8} - \frac{1}{4}$ (D) $I_3 = \frac{\pi}{16} - \frac{5}{48}$

4. The value of integral $\int_0^{\pi} xf\ (\sin x)\, dx$ is :

(A) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\ dx$ (B) $\pi \int_0^{\pi/2} f(\sin x) dx$ (C) 0 (D) $2\pi \int_0^{\pi/4} f(\sin x)\, dx$

5. If $I = \int_0^{2\pi} \sin^2 x dx$, then

(A) $I = 2\int_0^{\pi} \sin^2 x dx$ (B) $I = 4\int_0^{\pi/2} \sin^2 x dx$ (C) $I = \int_0^{2\pi} \cos^2 x dx$ (D) $I = 8\int_0^{\pi/4} \sin^2 x dx$

6. Given f is an odd function defined everywhere, periodic with period 2 and integrable on every interval. Let $g(x) = \int_0^x f(t)\, dt$. Then :

(A) g(2n) = 0 for every integer n (B) g(x) is an even function
(C) g(x) and f(x) have the same period (D) g(x) is an odd function

7. Let f : R → R be defined as $f(x) = \int_{-1}^{e^x} \frac{dt}{1+t^2} + \int_1^{e^{-x}} \frac{dt}{1+t^2}$, then

(A) f(x) is periodic (B) $f(f(x)) = f(x)\ \forall\ x \in R$

(C) $f(1) = f'(1) = \frac{\pi}{2}$ (D) f(x) is unbounded

8. If $a, b \in R^+$ then $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(k+an)(k+bn)}$ is equal to

(A) $\frac{1}{a-b}\ \ln\frac{b(b+1)}{a(a+1)}$ if $a \neq b$ (B) $\frac{1}{a-b}\ \ln\frac{a(b+1)}{b(a+1)}$ if $a \neq b$

(C) non existent if a = b (D) $\frac{1}{a(1+a)}$ if a = b

9. Let $f(x) = \int_x^{x+\frac{\pi}{3}} |\sin\theta|\, d\theta \quad (x \in [0, \pi])$

(A) f(x) is strictly increasing in this interval (B) f(x) is differentiable in this interval

(C) Range of f(x) is $\left[2-\sqrt{3}, 1\right]$ (D) f(x) has a maxima at $x = \frac{\pi}{3}$

10. If f(x) is integrable over [1, 2], then $\int_1^2 f(x)\,dx$ is equal to :

(A) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ (B) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$

(C) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)$ (D) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right)$

11. Let $I_n = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}dx$ where n > 2 , then

(A) $I_n < \frac{\pi}{6}$ (B) $I_n > \frac{\pi}{6}$ (C) $I_n < \frac{1}{2}$ (D) $I_n > \frac{1}{2}$

12. If f(x) = $2^{\{x\}}$, where {x} denotes the fractional part of x. Then which of the following is true ?

(A) f is periodic (B) $\int_0^1 2^{\{x\}}dx = \frac{1}{\ell n2}$ (C) $\int_0^1 2^{\{x\}}dx = \log_2 e$ (D) $\int_0^{100} 2^{\{x\}}dx = 100\log_2 e$

13. Let f(x) = $\int_0^x |2t-3|\,dt$, then f is

(A) continuous at x = 3/2 (B) continuous at x = 3

(C) differentiable at x = 3/2 (D) differentiable at x = 0

14. Let $I_n = \int_0^{\pi} (\sin x)^n dx$, $n \in N$, then

(A) I_n is rational if n is odd (B) I_n is irrational if n is even

(C) I_n is an increasing sequence (D) I_n is a decreasing sequence

15. Let f(x) be a function satisfying f(x) + f(x + 2) = 10 $\forall x \in R$, then

(A) f(x) is a periodic function (B) f(x) is aperiodic function

(C) $\int_{-1}^{7} f(x)dx = 20$ (D) $\int_{-1}^{7} f(x)dx = 40$

16. Let $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x}dx$, $n \in N$, then

(A) $I_{n+2} + I_n = 2I_{n+1}$ (B) $I_n = I_{n+1}$

(C) $I_n = n\pi$ (D) $I_1, I_2, I_3, ...I_n$ are in Harmonic progression

17. Let f(x) be a continuous function and $I = \int_1^9 \sqrt{x}\, f(x)\, dx$, then

(A) There exists some $c \in (1, 9)$ such that $I = 8\sqrt{c}\, f(c)$

(B) There exists some $p, q \in (1, 3)$ such that $I = 2[p^2 f(p^2) + q^2 f(q^2)]$

(C) There exists some $\alpha \in (1, 9)$ such that $I = 9\sqrt{\alpha}\, f(\alpha)$

(D) If $f(x) \geq 0\ \forall x \in [1, 9] \Rightarrow I > 0$

18. Let $A = \int_1^{e^2} \frac{\ell nx}{\sqrt{x}}dx$, then

(A) $A > 2\left(e - \frac{1}{e}\right)$ (B) $A < (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$ (C) $A > (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$ (D) $A < \left(e^2 - 1\right)\frac{2}{e}$

19. Let $f(a, b) = \int_a^b (x^2 - 4x + 3)dx$, (b > a) then

(A) f(a, 3) is least when a = 1
(B) f(4, b) is an increasing function $\forall b \geq 4$
(C) f(0, b) is least for b = 2
(D) $\min\{f(a, b)\} = -\frac{4}{3}$

20. Let $I = \int_2^{\infty}\left(\frac{\lambda x}{x^2+1} - \frac{1}{2x+1}\right)dx$ & I is a finite real number, then

(A) $\lambda = \frac{1}{2}$ (B) $\lambda = 1$ (C) $I = \frac{1}{2}\ell n\left(\frac{5}{2}\right)$ (D) $I = \frac{1}{4}\ell n\left(\frac{5}{4}\right)$

21. Let f(x) be a strictly increasing, non-negative function such that f''(x) < 0 $\forall x \in (\alpha, \beta)$ & $I = \int_{\alpha}^{\beta} f(x)dx$ $(\beta > \alpha)$, then

(A) $I < f\left(\frac{\alpha+\beta}{2}\right)(\beta - \alpha)$
(B) $I > f\left(\frac{\alpha+\beta}{2}\right)(\beta - \alpha)$
(C) $I > \frac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$
(D) $I < \frac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$

22. $I_1 = \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$, $I_2 = \int_0^{\pi}\frac{x^3 \sin x}{(\pi^2 - 3\pi x + 3x^2)(1+\cos^2 x)}dx$, then

(A) $I_1 = \frac{\pi^2}{8}$ (B) $I_1 = \frac{\pi^2}{4}$ (C) $I_1 = I_2$ (D) $I_1 > I_2$

23. Let $L_1 = \lim\limits_{x\to 0^+}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x - \sin x}$, $L_2 = \lim\limits_{x\to 0^-}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x - \sin x}$, then identify the correct option(s).

(A) $L_1 = 4$ (B) $L_1 + L_2 = 8$ (C) $L_1 + L_2 = 0$ (D) $|L_2| = |L_1|$

24. $\lim\limits_{n\to\infty}\frac{(1^k + 2^k + 3^k + ... + n^k)}{(1^2 + 2^2 + ... + n^2)(1^3 + 2^3 + ... + n^3)}$ = F(k), then $(k \in N)$

(A) F(k) is finite for $k \leq 6$ (B) F(5) = 0 (C) $F(6) = \frac{12}{7}$ (D) $F(6) = \frac{5}{7}$

25. Let $T_n = \sum_{r=1}^{n}\frac{n}{r^2 - 2r.n + 2n^2}$, $S_n = \sum_{r=0}^{n-1}\frac{n}{r^2 - 2r.n + 2n^2}$, then

(A) $T_n > S_n \ \forall n \in N$ (B) $T_n > \frac{\pi}{4}$ (C) $S_n < \frac{\pi}{4}$ (D) $\lim\limits_{n\to\infty} S_n = \frac{\pi}{4}$

26. $f(x) = \int_0^1 f(tx)dt$, where f '(x) is a continuous function such that f(1) = 2, then
(A) f(x) is a periodic function (B) f '(x) = 0
(C) f(x) is an even function (D) f(x) is an odd function

27. Area bounded by y = $\sin^{-1}x$, y = $\cos^{-1}x$, y = 0 in first quadrant is equal to :
(A) $\int_0^{1/\sqrt{2}} (\sin^{-1} x)dx + \int_{1/\sqrt{2}}^{1} (\cos^{-1} x)dx$ (B) $\int_{\pi/4}^{\pi/2} (\sin y - \cos y)dy$
(C) $\int_0^{\pi/4} (\cos y - \sin y)dy$ (D) $(\sqrt{2} - 1)$ sq.unit

28. Let f(x) be a non-negative, continuous and even function such that area bounded by x-axis, y-axis & y = f(x) is equal to $(x^2 + x^3)$ sq. units $\forall x \geq 0$, then
(A) $\sum_{r=1}^{n} f'(r) = 3n^2 + 5n \; \forall \; n \in N$ (B) $\sum_{r=1}^{n} f'(r) = 6n^2 + 5n \; \forall \; n \in N$
(C) $f(x) = 3x^2 + 2x \; \forall \; x \leq 0$ (D) $f(x) = 3x^2 - 2x \; \forall \; x \leq 0$

29. Let 'c' be a positive real number such that area bounded by y = 0 y = $[\tan^{-1}x]$ from x = 0 to x = c is equal to area bounded by y = 0, y = $[\cot^{-1} x]$, from x = 0 to x = c (where [*] represents greatest integer function), then
(A) c = tan1 + cot1 (B) c = 2cosec2 (C) c = tan1 – cot1 (D) c = –2 cot2

30. Area bounded by y = $x^2 - 2|x|$ and y = –1 is equal to
(A) $2\int_0^1 (2x - x^2)dx$
(B) $\frac{2}{3}$ sq. units
(C) $\frac{2}{3}$ (Area of rectangle ABCD) where points A, B, C, D are (–1, –1), (–1, 0), (1, 0) & (1, –1)
(D) $\frac{2}{3}$ (Area of rectangle ABCD) where points A, B, C, D are (–1, –1), (–1, 0), (1, 0) & (1, –1)

PART - IV : COMPREHENSION

Comprehension # 1

If $y = \int_{u(x)}^{v(x)} f(t)\,dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x)\, f^2(v(x)) - u'(x)\, f^2(u(x))$ and the equation of the tangent at (a, b) as $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$

1. If $y = \int_x^{x^2} t^2\,dt$, then equation of tangent at x = 1 is
(A) y = x + 1 (B) x + y = 1 (C) y = x – 1 (D) y = x

2. If $F(x) = \int_1^x e^{t^2/2}(1 - t^2)\,dt$, then $\frac{d}{dx}$ F(x) at x = 1 is
(A) 0 (B) 1 (C) 2 (D) – 1

3. If $y = \int_{x^3}^{x^4} \ell nt \; dt$, then $\lim_{x\to 0^+} \frac{dy}{dx}$ is

(A) 0 (B) 1 (C) 2 (D) – 1

Comprehension # 2

Let $g(t) = \int_{x_1}^{x_2} f(t, \; x) \; dx$. Then $g'(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial t}(f(t, x))\, dx$. Consider $f(x) = \int_0^{\pi} \frac{\ell n \; (1+x\cos\theta)}{\cos\theta}\, d\theta$.

4. Range of f(x) is

(A) $(0, \pi)$ (B) $(0, \pi^2)$ (C) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (D) $\left(\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right)$

5. The number of critical points of f(x), in the interior of its domain, is

(A) 0 (B) 1 (C) 2 (D) infinitely many

6. f(x) is

(A) discontinuous at x = 0 (B) differentiable at x = 1

(C) continuous at x = 0 (D) None of these

Comprehension # 3

If length of perpendicular drawn from points of a curve to a straight line approaches zero along an infinite branch of the curve, the line is said to be an asymptote to the curve. For example, y-axis is an asymptote to y = ℓnx & x-axis is an asymptote to y = e^{-x}.

Asymptotes parallel to x-axis :

If $\lim_{x\to\infty} f(x) = e$ (a finite number) then y = e is an asymptote to y = f(x). Similarly if $\lim_{x\to-\infty} f(x) = \alpha$, then y = α is also an asymptote.

Asymptotes parallel to y-axis :

If $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$, then x = a is an asymptote to y = f(x).

7. Number of asymptotes parallel to co-ordinate axes for the function $f(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$ is equal to :

(A) 1 (B) 2 (C) 3 (D) 4

8. Area bounded by $y = \frac{2x}{x^2+1}$, it's asymptote and ordinates at points of extremum is equal to (in square unit)

(A) ℓn2 (B) 2ℓn2 (C) ℓn3 (D) 2ℓn3

9. Area bounded by $y = x^2e^{-x}$ and it's asymptote in first quadrant is equal to (in square unit)

(A) 2e (B) e (C) 1 (D) 2

Exercise-3

✍ **Marked questions are recommended for Revision.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

* **Marked Questions may have more than one correct option.**

1*. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$, n = 0, 1, 2, ..., then **[IIT-JEE - 2009,Paper-2, (4, –1), 80]**

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$ (C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

2. Let f : R → R be a continuous function which satisfies $f(x) = \int_0^x f(t)\,dt$. Then the value of f(ℓn 5) is

[IIT-JEE - 2009,Paper-2, (4, –1), 80]

3*. Area of the region bounded by the curve y = e^x and lines x = 0 and y = e is

[IIT-JEE 2009, P-1, (4, –1), 80]

(A) e – 1 (B) $\int_1^e \ell n\,(e+1-y)\,dy$ (C) $e - \int_0^1 e^x\,dx$ (D) $\int_1^e \ell n\,y\,dy$

4. The value of $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\,\ell n\,(1+t)}{t^4+4}\,dt$ is **[IIT-JEE-2010, Paper-1 (3, –1)/84]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

5.✍ The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$ is (are) **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) $\frac{22}{7} - \pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

6*.✍ Let f be a real-valued function defined on the interval (0, ∞) by $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true? **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) $f''(x)$ exists for all x ∈ (0, ∞)

(B) $f'(x)$ exists for all x ∈ (0, ∞) and f' is continuous on (0, ∞), but not differentiable on (0, ∞)

(C) there exists α > 1 such that $|f'(x)| < |f(x)|$ for all x ∈ (α, ∞)

(D) there exists β > 0 such that $|f(x)| + |f'(x)| \le \beta$ for all x ∈ (0, ∞)

7.✍ For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd,} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is

8. Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\ dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

[IIT-JEE-2010, Paper-2 (5, –2)/84]

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

Comprehension (9 to 11)

Consider the polynomial
$f(x) = 1 + 2x + 3x^2 + 4x^3$
Let s be the sum of all distinct real roots of f(x) and let t = |s|

9. The real number s lies in the interval. **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

10. The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval

[IIT-JEE 2010, Paper-2, (3, –1), 79]

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) (9, 10) (D) $\left(0, \frac{21}{64}\right)$

11. The function f′(x) is **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) increasing in $\left(-t, \frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in (–t, t)

(D) decreasing in (–t, t)

12. The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ell n6 - x^2)} dx$ is **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\frac{1}{4} \ell n \frac{3}{2}$ (B) $\frac{1}{2} \ell n \frac{3}{2}$ (C) $\ell n \frac{3}{2}$ (D) $\frac{1}{6} \ell n \frac{3}{2}$

13. Let the straight line x = b divide the area enclosed by $y = (1 - x)^2$, y = 0, and x = 0 into two parts R_1 ($0 \le x \le b$) and R_2($b \le x \le 1$) such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

14. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that f(x) = f(1 – x) for all $x \in [-1, 2]$.
Let $R_1 = \int_{-1}^{2} x\ f(x)\ dx$, and R_2 be the area of the region bounded by y = f(x), x = –1, x = 2, and the x-axis. Then **[IIT-JEE 2011, Paper-2, (3, –1), 80]**

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

15*. If S be the area of the region enclosed by $y = e^{-x^2}$, y = 0, x = 0, and x = 1. Then

[IIT-JEE 2012, Paper-1, (4, 0), 70]

(A) $S \ge \frac{1}{e}$ (B) $S \ge 1 - \frac{1}{e}$ (C) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (D) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

16. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\ \frac{\pi+x}{\pi-x}\right)\cos x\ \ dx$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

Paragraph for Question Nos. 17 to 18

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1}-\ell nt\right) f(t)\ dt$ for all $x \in (1, \infty)$.

17. Which of the following is true ? **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) g is increasing on $(1, \infty)$
(B) g is decreasing on $(1, \infty)$
(C) g is increasing on (1, 2) and decreasing on $(2, \infty)$
(D) g is decreasing on (1, 2) and increasing on $(2, \infty)$

18. Consider the statements :
P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$
Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$
Then
(A) both P and Q are true (B) P is true and Q is false
(C) P is false and Q is true (D) both P and Q are false

19.* If $f(x) = \int_0^x e^{t^2}(t-2)\ \ (t-3)\ \ dt$ for all $x \in (0, \infty)$, then

(A) f has a local maximum at x = 2 **[IIT-JEE 2012, Paper-2, (4, 0), 66]**
(B) f is decreasing on (2, 3)
(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$
(D) f has a local minimum at x = 3

20. The area enclosed by the curves y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0,\ \frac{\pi}{2}\right]$ is

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) $4\ \left(\sqrt{2}-1\right)$ (B) $2\sqrt{2}\ \left(\sqrt{2}-1\right)$ (C) $2\ \left(\sqrt{2}+1\right)$ (D) $2\sqrt{2}\ \left(\sqrt{2}+1\right)$

21. Let $f : \left[\frac{1}{2},\ 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2 f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)\ \ dx$ lies in the interval

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1) (C) $\left(\frac{e-1}{2},\ e-1\right)$ (D) $\left(0\ ,\ \frac{e-1}{2}\right)$

22. For $a \in R$ (the set of all real numbers), $a \neq -1$, $\lim_{n\to\infty}\frac{(1^a+2^a+....+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]}=\frac{1}{60}$. Then a =

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$;

23*. Let $f:[a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a, \\ \int_a^x f(t)dt & \text{if } a \le x \le b, \\ \int_a^b f(t)dt & \text{if } x > b. \end{cases}$$, Then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A) $g(x)$ is continuous but not differentiable at a
(B) $g(x)$ is differentiable on R
(C) $g(x)$ is continuous but not differentiable at b
(D) $g(x)$ is continuous and differentiable at either a or b but not both

24*. Let $f: (0, \infty) \to R$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$ **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(B) $f(x)$ is monotonically decreasing on $(0, 1)$

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on R

25. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

26. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosec}x)^{17}dx$ is equal to **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u+e^{-u})^{16}du$ (B) $\int_0^{\log(1+\sqrt{2})} (e^u+e^{-u})^{17}du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u-e^{-u})^{17}du$ (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u-e^{-u})^{16}du$

Paragraph For Questions 27 and 28

Given that for each $a \in (0, 1)$

$$\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}\ dt$$

exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0, 1). **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

27. The value of $g\left(\frac{1}{2}\right)$ is

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

28. The value of $g'\left(\frac{1}{2}\right)$ is

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

29. **List I** **List II**

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

	List I		List II
P.	The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying f(0) = 0 and $\int_0^1 f(x)dx = 1$, is	**1.**	8
Q.	The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.**	2
R.	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	**3.**	4
S.	$\frac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals	**4.**	0

30. Let f: R $\to$ R be a function defined by f(x) = $\begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$ where [x] is the greatest integer less than or equal to x. If I = $\int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}$ dx , then the value of (4I–1) is

[JEE (Advanced) 2015, P-1 (4, 0) /88]

31. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}$ x takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

32. Let f: R $\to$ R be a continuous odd function, which vanishes exactly at one point and f(1) $= \frac{1}{2}$. Suppose that F(x) = $\int_{-1}^{x} f(t)\ dt$ for all x $\in$ [–1, 2] and G(x) = $\int_{-1}^{x} t\,|f(f(t))|\ dt$ for all x $\in$ [–1, 2]. If $\lim_{x\to1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is.

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

33*. Let f(x) = 7tan^8 x + 7tan^{6}x – 3tan^{4}x – 3tan^{2}x for all x $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are) **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) $\int_0^{\pi/4} xf(x)\ dx = \frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)\ dx = 0$

(C) $\int_0^{\pi/4} xf(x)\ dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)\ dx = 1$

34. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all x $\in$ R with $f\left(\frac{1}{2}\right) = 0$. If m $\le \int_{1/2}^{1} f(x)\ dx \le$ M, then the possible values of m and M are **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) m = 13, M = 24 (B) m = $\frac{1}{4}$, M = $\frac{1}{2}$

(C) m = – 11, M = 0 (D) m = 1, M = 12

35*. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\ ?$$ **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) a = 2, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$ (B) a = 2, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) a = 4, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$ (D) a = 4, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

Paragraph For Questions 36 and 37

Let F : R → R be a thrice differentiable function. Supose that F(1) = 0, F(3) = –4 and F'(x) < 0 for all x ∈ (1/2, 3). Let f(x) = xF(x) for all x ∈ R.

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

36*. The correct statement(s) is(are)

(A) f '(1) < 0 (B) f (2) < 0

(C) f '(x) ≠ 0 for any x ∈ (1, 3) (D) f '(x) = 0 for some x ∈ (1, 3)

37*. If $\int_1^3 x^2\ F'(x)dx = -12$ and $\int_1^3 x^3\ F''(x)dx = 40$, then the correct expression(s) is(are)

(A) 9f '(3) + f '(1) – 32 = 0 (B) $\int_1^3 f(x)dx = 12$

(C) 9f '(3) – f '(1) + 32 = 0 (D) $\int_1^3 f(x)dx = -12$

38. Let F(x) = $\int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\ \ dt$ for all x ∈ R and f: $\left[0,\frac{1}{2}\right]$ → [0, ∞) be a continuous function. For a ∈, $\left[0,\frac{1}{2}\right]$ if F′(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

39. The total number of distinct x ∈ (0, 1] for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is

[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

40. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ is equal to **[JEE (Advanced) 2016, Paper-2, (3, –1)/62]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\pi/2}$ (D) $\pi^2 + e^{\pi/2}$

41. Area of the region $\left\{(x,y) \in R^2 : y \ge \sqrt{|x+3|}\ , 5y \le x+9 \le 15\right\}$ is equal to

[JEE (Advanced) 2016, Paper-2, (3, 1)/62]

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

42. Let $f(x) = \lim_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all x > 0. Then

[JEE (Advanced) 2016, Paper-2, (4, –2)/62]

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$ (C) $f'(2) \le 0$ (D) $\dfrac{f'(3)}{f(3)} \ge \dfrac{f'(2)}{f(2)}$

43*. Let f : R → (0, 1) be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval (0, 1) ? **[JEE(Advanced) 2017, Paper-1,(4, –2)/61]**

(A) $e^x - \int_0^x f(t)\sin t\, dt$ (B) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\, dt$

(C) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$ (D) $x^9 - f(x)$

44. Let f : R → R be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right) = 3$ and f′(0) = 1. If

$g(x) = \int_x^{\frac{\pi}{2}}[f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]\,dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$

[JEE(Advanced) 2017, Paper-1,(3, 0)/61]

45*. If $I = \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

(A) $I > \log_e 99$ (B) $I < \log_e 99$ (C) $I < \frac{49}{50}$ (D) $I > \frac{49}{50}$

46*. If the line x = α divides the area of region R = {(x, y) ∈ R^2 : $x^3 \le y \le x$, $0 \le x \le 1$} into two equal parts, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

(A) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $\frac{1}{2} < \alpha < 1$ (D) $0 < \alpha \le \frac{1}{2}$

47. If $g(x) = \int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

(A) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = -2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(\frac{\pi}{2}\right) = -2\pi$

48. For each positive integer n, let $y_n = \frac{1}{n}((n + 1)(n + 2) \ldots (n + n))^{1/n}$.

For x ∈ R, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is

________. **[JEE(Advanced) 2018, Paper-1,(3, 0)/60]**

49. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ (n > 1). If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is **[JEE(Advanced) 2018, Paper-1,(3, 0)/60]**

50. The value of the integral $\int_0^{\frac{1}{2}}\dfrac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}}dx$ is ____. **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. $\int_0^{\pi}[\cot x]dx$, where [·] denotes the greatest integer function, is equal to : **[AIEEE 2009 (4, –1), 144]**

(1) 1 (2) – 1 (3) $-\frac{\pi}{2}$ (4) $\frac{\pi}{2}$

2. The area of the region bounded by the parabola $(y-2)^2 = x-1$, the tangent to the parabola at the point (2, 3) and the x-axis is **[AIEEE 2009 (8, –2), 144]**

(1) 6 sq unit (2) 9 sq unit (3) 12 sq unit (4) 3 sq unit

3. Let p(x) be a function defined on **R** such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, p(0) = 1 and p(1) = 41. Then $\int_0^1 p(x)dx$ equals **[AIEEE 2010 (8, –2), 144]**

(1) 21 (2) 41 (3) 42 (4) $\sqrt{41}$

4. The area bounded by the curves y = cos x and y = sinx between the ordinates x = 0 and $x = \frac{3\pi}{2}$ is **[AIEEE 2010 (4, –1), 144]**

(1) $4\sqrt{2}+2$ (2) $4\sqrt{2}-1$ (3) $4\sqrt{2}+1$ (4) $4\sqrt{2}-2$

5. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\ \sin t\ dt$. Then f has : **[AIEEE 2011, I, (4, –1), 120]**

(1) local maximum at π and 2π.
(2) local minimum at π and 2π
(3) local minimum at π and local maximum at 2π.
(4) local maximum at π and local minimum at 2π.

6. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x\ [x^2]\,dx$ is : **[AIEEE 2011, II, (4, –1), 120]**

(1) 0 (2) $\frac{3}{2}$ (3) $\frac{3}{4}$ (4) $\frac{5}{4}$

7. The area of the region enclosed by the curves y = x, x = e, $y = \frac{1}{x}$ and the positive x-axis is **[AIEEE 2011, I, (4, –1), 120]**

(1) $\frac{1}{2}$ square units (2) 1 square units (3) $\frac{3}{2}$ square units (4) $\frac{5}{2}$ square units

8. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is : **[AIEEE 2011, II, (4, –1), 120]**

(1) $\frac{32}{3}$ (2) $\frac{16}{3}$ (3) $\frac{8}{3}$ (4) 0

9. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line y = 2 is : **[AIEEE-2012, (4, –1)/120]**

(1) $20\sqrt{2}$ (2) $\frac{10\sqrt{2}}{3}$ (3) $\frac{20\sqrt{2}}{3}$ (4) $10\sqrt{2}$

10.* If $g(x) = \int_0^x \cos 4t \, dt$, then $g(x + \pi)$ equals **[AIEEE-2012, (4, –1)/120]**

(1) $\frac{g(x)}{g(\pi)}$ (2) $g(x) + g(\pi)$ (3) $g(x) - g(\pi)$ (4) $g(x) . g(\pi)$

11. Statement-I : The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$. **[AIEEE - 2013, (4, –1),360]**

Statement-II : . $\int_a^b f(x)dx = \int_a^b f(a+b-x) \; dx$

(1) Statement-I is true; Statement-II is true; Statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true; Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true; Statement-II is false.
(4) Statement-I is false; Statement-II is true.

12. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is : **[AIEEE - 2013, (4, –1),360]**

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

13. The integral $\int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}} \, dx$ equals : **[JEE(Main)2014,(4, – 1), 120]**

(1) $4\sqrt{3}-4$ (2) $4\sqrt{3}-4-\frac{\pi}{3}$ (3) $\pi - 4$ (4) $\frac{2\pi}{3}-4-4\sqrt{3}$

14. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :
[JEE(Main)2014,(4, – 1), 120]

(1) $\frac{\pi}{2}-\frac{2}{3}$ (2) $\frac{\pi}{2}+\frac{2}{3}$ (3) $\frac{\pi}{2}+\frac{4}{3}$ (4) $\frac{\pi}{2}-\frac{4}{3}$

15. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)} dx$ is equal to **[JEE(Main)2015,(4, – 1), 120]**

(1) 2 (2) 4 (3) 1 (4) 6

16. The area (in sq. units) of the region described by $\{(x, y); y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is
[JEE(Main)2015,(4, – 1), 120]

(1) $\frac{7}{32}$ (2) $\frac{5}{64}$ (3) $\frac{15}{64}$ (4) $\frac{9}{32}$

17. The area (in sq.units) of the region $\{(x,y) : y^2 \ge 2x \text{ and } x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is
[JEE(Main)2016,(4, – 1), 120]

(1) $\pi - \frac{8}{3}$ (2) $\pi - \frac{4\sqrt{2}}{3}$ (3) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$ (4) $\pi - \frac{4}{3}$

18. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2).......3n}{n^{2n}}\right)^{1/n}$ is equal to : **[JEE(Main)2016,(4, – 1), 120]**

(1) $\frac{27}{e^2}$ (2) $\frac{9}{e^2}$ (3) 3 log3 – 2 (4) $\frac{18}{e^4}$

19. The area (in sq. units) of the region $\{(x, y) : x \geq 0 , x + y \leq 3 , x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is :

[JEE(Main) 2017, (4, – 1), 120]

(1) $\frac{59}{12}$ (2) $\frac{3}{2}$ (3) $\frac{7}{3}$ (4) $\frac{5}{2}$

20. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to **[JEE(Main) 2017, (4, – 1), 120]**

(1) –2 (2) 2 (3) 4 (4) –1

21. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x} dx$ is : **[JEE(Main) 2018, (4, – 1), 120]**

(1) 4π (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{8}$ (4) $\frac{\pi}{2}$

22. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$, and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x=\alpha$, $x = \beta$ and $y = 0$, is

[JEE(Main) 2018, (4, – 1), 120]

(1) $\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$ (2) $\frac{1}{2}\left(\sqrt{2}-1\right)$ (3) $\frac{1}{2}\left(\sqrt{3}-1\right)$ (4) $\frac{1}{2}\left(\sqrt{3}+1\right)$

23. Let $I = \int_a^b (x^4 - 2x^2)dx$. If I is minimum then the ordered pair (a, b) is :

[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]

(1) $(0, \sqrt{2})$ (2) $(\sqrt{2}, -\sqrt{2})$ (3) $(-\sqrt{2}, \sqrt{2})$ (4) $(-\sqrt{2}, 0)$

24. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest less than or equal to t, is :

[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

(1) $\frac{1}{12}(7\pi - 5)$ (2) $\frac{3}{10}(4\pi - 3)$ (3) $\frac{3}{20}(4\pi - 3)$ (4) $\frac{1}{12}(7\pi + 5)$

25. The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^{x}\right\} \log_e x \, dx$ is equal to

[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]

(1) $\frac{3}{2}-\frac{1}{e}-\frac{1}{2e^2}$ (2) $\frac{3}{2}-e-\frac{1}{2e^2}$ (3) $\frac{1}{2}-e-\frac{1}{e^2}$ (4) $-\frac{1}{2}+\frac{1}{e}-\frac{1}{2e^2}$

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) $-\frac{10}{21}$ (ii) $\sqrt{2}-1$ **A-2.** (i) π (ii) $\frac{\pi}{4}$ (iii) $4 + \ell n\ 5$ (iv) $\frac{8}{21}$

A-3. (i) $\frac{\pi-2}{2}$ (ii) $\frac{1}{2}\ell n\left(\frac{e}{2}\right)$ (iii) $\frac{\pi}{6}-\frac{2}{9}$ **A-4.** (i) $\ell n\ (\sqrt{3})$ (ii) 7/6 (iii) $\frac{3}{8}$

A-5. (i) $\frac{\pi}{2} - \ell n\ 2$ (ii) $\frac{4-\pi}{4\sqrt{2}}$ (iii) $-\frac{\pi}{8}\ (b-a)^2$ (iv) $\pi\left(1-\frac{1}{\sqrt{3}}\right) - \ell n\ 4$

A-6. (i) $\frac{\pi}{4}$ (ii) $\frac{5}{3} - 2\ \ell n2$ (iii) $\ell n\left(\frac{9}{8}\right)$ (iv) $\frac{\pi}{2}$ (v) $\frac{1}{20}\ \ell n\ 3$

A-7 (i) $\ell n\ 11$ (ii) $\frac{1}{3}$

Section (B) :

B-2. (i) $5 - \sqrt{2} - \sqrt{3}$ (ii) $2\sqrt{2}$ (iii) 9 (iv) 4 (v) cot 1 (vi) 29
(vii) cos 1 + cos 2 + cos 3 + 3

B-3. (i) $2e - 2$ (ii) $2-\sqrt{2}$ (iii) $\frac{\pi^2}{6\sqrt{3}}$ (iv) 0 (v) 0

B-4. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{a}{2}$ (iv) $(a+b)\ \frac{\pi}{4}$ (v) 0

B-5. (i) 0 (ii) $\frac{\pi}{3}$ (iii) $-\frac{\pi}{2}\ \ell n\ 2$ (iv) $\pi\ell n2$

B-6. (i) $\frac{3}{2}$ (ii) 40 (iii) $n - 1$ (iv) $4n$

Section (C) :

C-1. (i) $4\sqrt{2}$ (ii) 12 (iii) $\left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$ **C-2.** (ii) 1, 3 **C-3.** 5/2 **C-4.** $\frac{1}{2}$

C-5. e **C-6.** (i) $\frac{4}{15}$ (ii) $\frac{8\pi}{15}$ (iii) $\frac{\pi}{2}$ (iv) $\frac{\pi^2}{4}$

Section (E) :

E-1. (i) $\frac{\pi}{2}$ (ii) 2 (iii) 12

Section (F) :

F-1. $\frac{51}{4}$ sq. unit **F-2.** (i) $\frac{\pi}{2}-\frac{4}{\pi}$ (ii) $\frac{7}{120}$ (iii) 9π

F-3. 4/3 sq. units **F-4.** $\frac{(e+1)\ \pi}{1+\pi^2}$ **F-5.** (i) $3(\pi-2)$ (ii) $\frac{1}{8}$ **F-6.** $\left(\frac{3}{\log_e 2}-\frac{4}{3}\right)$ sq. units

F-8. 4 sq. units. **F-9.** $\frac{16}{3}$ sq. units.

PART - II

SECTION (A) :

A-1. (A) **A-2.** (C) **A-3.** (C) **A-4.** (C) **A-5.** (D) **A-6.** (A) **A-7.** (B)
A-8. (A) **A-9.** (A) **A-10.** (B) **A-11.** (D) **A-12.** (C) **A-13.** (C)

Section (B) :

B-1. (C) **B-2.** (C) **B-3.** (D) **B-4.** (C) **B-5.** (D) **B-6.** (A) **B-7.** (B)
B-8. (D) **B-9.** (C) **B-10.** (A) **B-11.** (C) **B-12.** (A) **B-13.** (C)

Section (C) :

C-1. (A) **C-2.** (B) **C-3.** (B) **C-4.** (D) **C-5.** (B) **C-6.** (C) **C-7.** (C)
C-8. (B) **C-9.** (D) **C-10.** (A)

SECTION (D) :

D-1. (C) **D-2.** (B) **D-3.** (C) **D-4.** (A)

SECTION (E) :

E-1. (D) **E-2.** (B) **E-3.** (C) **E-4.** (C) **E-5.** (D)

Section (F) :

F-1. (C) **F-2.** (D) **F-3.** (C) **F-4.** (A) **F-5.** (C) **F-6.** (B) **F-7.** (A)
F-8. (A) **F-9.** (C) **F-10.** (D)

PART - III

1. A-q, B-s, C-p, D-t **2.** (A)$\rightarrow$(r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

EXERCISE - 2

PART - I

1. (C) **2.** (B) **3.** (A) **4.** (C) **5.** (D) **6.** (A) **7.** (D)
8. (B) **9.** (C) **10.** (D) **11.** (A) **12.** (B) **13.** (B) **14.** (C)
15. (B) **16.** (C) **17.** (C) **18.** **(B)** **19.** (C) **20.** (A) **21.** (A)
22. (A) **23.** (D) **24.** (D) **25.** (D) **26.** (B) **27.** (A) **28.** (D)
29. (B) **30.** (A)

PART - II

1. 61 **2.** 64 **3.** 10 **4.** **29** **5.** 4 **6.** 4 **7.** 2
8. 2 **9.** 0 **10.** 2 **11.** 8 **12.** 16 **13.** 25 **14.** 4
15. 65 **16.** 5 **17.** 11 **18.** 3 **19.** 2 **20.** 0 **21.** 1
22. 22 sq.unit **23.** 55

PART - III

1.	(AC)	**2.**	(ABCD)	**3.**	(AB)	**4.**	(AB)	**5.**	(ABC)	**6.**	(ABC)	**7.**	(AB)
8.	(BD)	**9.**	(BCD)	**10.**	(BC)	**11.**	(AD)	**12.**	(ABCD)	**13.**	(ABCD)		
14.	(ABD)	**15.**	(AD)	**16.**	(AC)	**17.**	(AB)	**18.**	(BD)	**19.**	(ABD)	**20.**	(AD)
21.	(AC)	**22.**	(BC)	**23.**	(ACD)	**24.**	(ABC)	**25.**	(ABCD)	**26.**	(ABC)		
27.	(ABCD)	**28.**	(AD)	**29.**	(AB)	**30.**	(BD)						

PART - IV

1.	(C)	**2.**	(A)	**3.**	(A)	**4.**	(D)	**5.**	(A)	**6.**	(BC)	**7.**	(C)
8.	(B)	**9.**	(D)										

EXERCISE - 3

PART - I

1*.	(ABC)	**2.**	0	**3*.**	(BCD)	**4.**	(B)	**5.**	(A)	**6*.**	(BC)		
7.	4	**8.**	(B)	**9.**	(C)	**10.**	(A)	**11.**	(B)	**12.**	(A)	**13.**	(B)
14.	(C)	**15.***	(ABD)	**16.**	(B)	**17.**	(B)	**18.**	(C)	**19.**	(ABCD)	**20.**	(B)
21.	(D)	**22.**	(B)	**23.***	(AC)	**24.***	(ACD)	**25.**	2	**26.**	(A)	**27.**	(A)
28.	**(D)**	**29.**	(D)	**30.**	0	**31.**	9	**32.**	7	**33*.**	(AB)	**34*.**	(D)
35*.	(AC)	**36*.**	(ABC)	**37*.**	(CD)	**38.**	3	**39.**	1	**40.**	(A)	**41.**	(C)
42*.	(BC)	**43.**	(CD)	**44.**	(2)	**45.**	(BD)	**46.**	(AC)	**47.**	(BONUS)		
48.	(1)	**49.**	(4)	**50.**	(2)								

PART - II

1.	(3)	**2.**	(2)	**3.**	(1)	**4.**	(4)	**5.**	(4)	**6.**	(3)	**7.**	(3)
8.	(2)	**9.**	(3)	**10.***	(2) or (3)	**11.**	(4)	**12.**	(1)	**13.**	(2)	**14.**	(3)
15.	(3)	**16.**	(4)	**17.**	(1)	**18.**	(1)	**19.**	(4)	**20.**	(2)	**21.**	(2)
22.	(3)	**23.**	(3)	**24.**	(3)	**25.**	(2)						

High Level Problems (HLP)

1. Find the integral value of a for which $\int_0^{\frac{\pi}{2}} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^{\frac{\pi}{2}} x\cos x dx = 2$

2. Evaluate :

$$\int_0^{\pi} \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}\, dx$$

3. Let α & β be distinct positive roots of the equation tanx = 2x, then evaluate $\int_0^1 \sin(\alpha x).\sin(\beta x)dx$

4. Evaluate :

$$\lim_{a\to\left(\frac{\pi}{2}\right)^-} \int_0^a (\cos x)\ln(\cos x)dx$$

5. Find the value of a(0 < a < 1) for which the following definite integral is minimized.

$$\int_0^{\pi} |\sin x - ax|\, dx$$

6. Find the $\lim\limits_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{\frac{1}{n}}$

where ${}^iC_j$ is a binomial coefficient which means $\frac{i.(i-1)....(i-j+1)}{j.(j-1)....2.1}$

7. Show that $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell n x}{x}dx = \ell n a .\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

8. Evaluate $\lim\limits_{n\to\infty} n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} (2014\sin x + 2015\cos x)|x|\, dx$

9. Let sequence $\{a_n\}$ be defined as

$a_1 = \frac{\pi}{4}$, $a_n = \int_0^{\frac{1}{2}} (\cos(\pi x) + a_{n-1})\cos\pi x\, dx$, (n = 2, 3, 4, ...)

then evaluate $\lim\limits_{n\to\infty} a_n$

10. Find f(x) if it satisfies the relation $f(x) = e^x + \int_0^1 (x + ye^x)\, f(y)\, dy$.

11. Evaluate : $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$.

12. Evaluate $\int_0^1 \frac{1}{(5+2x-2x^2)\ (1+e^{(2-4x)})}dx$

13. Prove that for any positive integer k ;
$\frac{\sin 2kx}{\sin x}$ = 2 [cos x + cos 3x +..... + cos (2k – 1)x]. Hence prove that;

$$\int_0^{\pi/2} \sin(2kx).\cot x\, dx = \frac{\pi}{2}.$$

14. Prove that $\underset{n\to\infty}{\text{Limit}} \frac{1}{n}\left[\cos^{2p}\frac{\pi}{2n} + \cos^{2p}\frac{2\pi}{2n} + \cos^{2p}\frac{3\pi}{2n} + ... + \cos^{2p}\frac{\pi}{2}\right] = \prod_{r=1}^{p}\frac{2r-1}{2r}$,
where Π denotes the continued product and $p \in N$.

15. If n > 1, evaluate $\int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$

16. Let f(x) be a continuous function $\forall$ x $\in$ R, except at x = 0 such that $\int_0^a f(x)dx$, a $\in$ R^+ exists. If $g(x) = \int_x^a \frac{f(t)}{t}dt$, prove that $\int_0^a g(x)\ dx = \int_0^a f(x)\ dx$

17. Given that $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\log_e(n^2+r^2) - 2\log_e n}{n} = \log_e 2 + \frac{\pi}{2} - 2$, then
evaluate : $\lim_{n\to\infty}\frac{1}{n^{2m}}\ [(n^2+1^2)^m (n^2+2^2)^m .... (2n^2)^m]^{1/n}$.

18. For a natural number n, let $a_n = \int_0^{\pi/4}(\tan x)^{2n}dx$
Now answer the following questions :
(1) Express a_{n+1} in terms of a_n
(2) Find $\lim_{n\to\infty} a_n$
(3) Find $\lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}\ (a_k + a_{k-1})$

19. Given that $\lim_{x\to 0}\frac{\int_0^x \frac{t^2}{\sqrt{a+t}}dt}{bx-\sin x} = 1$, then find the values of a and b

20. Prove that $m\sin x + \int_0^x \sec^m t\ \ dt > (m+1)x\ \ \forall\ x \in \left(0, \frac{\pi}{2}\right)\ m \in N$

21. f(x) is differentiable function: g(x) is double differentiable function such that |f(x)| ≤ 1 and g(x) = $f'(x)$. If $f^2(0) + g^2(0) = 9$ then show that there exists some C $\in$ (–3, 3) such that g(c) g'' (c) < 0

22. Draw a graph of the function f (x) = $\cos^{-1}(4x^3 - 3x)$, x $\in$ [–1, 1] and find the area enclosed between the graph of the function and the x–axis as x varies from 0 to 1.

23. Consider a square with vertices at (1, 1), (– 1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.

24. If [x] denotes the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4\left[\sqrt{x}\right]$ y and $y^2 = 4\left[\sqrt{y}\right]$ x that lie within the square {(x, y) | $1 \le x \le 4$, $1 \le y < 4$} Find the area of the part of the square that is enclosessd by the two curves and the line x + y = 3

25. Find the area of the region bounded by y = f(x) , y = | g(x) | and the lines x = 0, x = 2, where 'f' , 'g' are continuous functions satisfying f(x + y) = f(x) + f(y) – 8xy $\forall$ x, y $\in$ R and g(x + y) = g(x) + g(y) + 3xy(x+y) x, y $\in$ R also f'(0) = 8 and g'(0) = – 4.

26. Let f(x) = $\begin{cases} -2 & , \ -3 \le x \le 0 \\ x-2 & , \ 0 < x \le 3 \end{cases}$, where g(x) = min {f(|x|) + |f(x)|, f(|x|) – |f(x)|}

Find the area bounded by the curve g(x) and the x-axis between the ordinates x = 3 and x = –3.

27. Find the area of region $\{(x, \ y) : 0 \le y \le x^2 + 1, \ \ 0 \le y \le x + 1, \ \ 0 \le x \ \le 2\}$.

28. A curve y = f(x) passes through the point P(1, 1), the normal to the curve at P is a(y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

29. Find the area bounded by y = [– 0. 01 x^4 – 0.02 x^2], (where [.] G.I.F.) and curve $3x^2 + 4y^2 = 12$, which lies below y = – 1.

30. Let ABC be a triangle with vertices A(6, 2($\sqrt{3}$ + 1)), B(4, 2) and C(8, 2). If R be the region consisting of all these points and point P inside ΔABC which satisfy d(P, BC) $\ge$ max. {d(P, AB), d(P, AC)} where d(P, L) denotes the distance of the point P from the line L. Sketch the region R and find its area.

31. Find the area of the region which contains all the points satisfying the condition |x – 2y| + |x + 2y| $\le$ 8 and xy $\ge$ 2.

32. Find the area of the region which is inside the parabola y = – x^2 + 6 x – 5, out side the parabola y = – x^2 + 4 x – 3 and left of the straight line y = 3 x – 15.

33. Consider the curve C: y = sin 2x – $\sqrt{3}$ |sin x|, C cuts the x – axis at (a, 0), x $\in$ (– π, π).

A_1 : The area bounded by the curve C and the positive x – axis between the origin and the line x = a.

A_2 : The area bounded by the curve C and the negative x – axis between the line x = a and the origin.

Prove that $A_1 + A_2 + 8 A_1 A_2 = 4$.

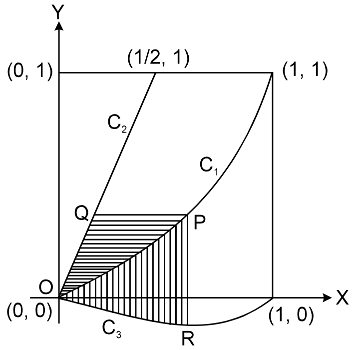
34. Area bounded by the line y = x, curve y = f(x), (f(x) > x $\forall$ x > 1) and the lines x = 1, x = t is (t + $\sqrt{1+t^2}$ – (1 + $\sqrt{2}$) $\forall$ t > 1. Find f(x) for x > 1.

35. Consider the two curves y = 1/x^2 and y = 1/[4 (x – 1)].

(i) At what value of 'a' (a > 2) is the reciprocal of the area of the figure bounded by the curves, the lines x = 2 and x = a equal to 'a' itself ?

(ii) At what value of 'b' (1 < b < 2) the area of the figure bounded by these curves, the lines x = b and x = 2 equal to 1 – 1/b.

36. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x).



37. Given the parabola C : $y = x^2$. If the circle centred at y axis with radius 1 touches parabola C at two distinct points, then find the coordinate of the center of the circle K and the area of the figure surrounded by C and K.

38. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB.

39_. f(x) and g(x) are polynomials of degree 2 such that $\left|\int_{a_1}^{a_2}(f(x)-1)dx\right| = \left|\int_{b_1}^{b_2}(g(x)-1)dx\right|$

where a_1, a_2 ($a_2 > a_1$) are roots of equation f(x) = 1 and b_1, b_2 ($b_2 > b_1$) are roots of equation g(x) = 1. If $f''(x)$ and $g''(x)$ are positive constant and

$\left|\int_{a_1}^{a_2} f(x))dx\right| = (a_2 - a_1) - \left|\int_{b_1}^{b_2}(f(x)-1)dx\right|$ but $\left|\int_{b_1}^{b_2} g(x))dx\right| \ne (b_2 - b_1) - \left|\int_{b_1}^{b_2}(g(x)-1)dx\right|$ then

(A) $|f''(x)| < |g''(x)|$ (B) $|f''(x)| > |g''(x)|$ (C) $a_2 - a_1 > b_2 - b_1$ (D) $a_2 - a_1 > b_2 - b_1$

40_. Let L = 4x – 5y, $L_i = \frac{x}{10} + \frac{y}{8} - \frac{i}{n}$, $L_i = \frac{x}{10} + \frac{y}{8} + \frac{i}{n}$, and $E = \frac{x^2}{50} + \frac{y^2}{32} - 1$.

Let A_i represents the area of region common between $L_{i-1} > 0$, $L_i < 0$, $E < 0$ and $L < 0$;
A'_i represents the area of region common between $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ and $L < 0$;
B_i represents the area of region common between $L_{i-1} > 0$, $L_i < 0$, $E < 0$ and $L > 0$;
B'_i represents the area of region common between $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ and $L > 0$, then value of
$(A_1 + A'_2 + A_3 + A'_4 + \ldots..) + (B_1 + B'_2 + B_3 + B'_4 + \ldots..)$ is equal to.

Answers

1. -1 **2.** $\frac{\pi}{3}+2\sqrt{3}$ **3.** 0 **4.** $\ell n2 - 1$ **5.** $a = \frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{\sqrt{2}}\right)$

6. $\frac{27}{16}$ **8.** 2015 **9.** $\frac{\pi}{4(\pi-1)}$ **10.** $\frac{-3e^x}{2(e-1)} - 3x$

11. $\frac{\pi}{4}\ \ell n\ \left(2+\sqrt{3}\right)+\frac{\pi^2}{12}-\frac{\pi}{\sqrt{3}}$ **12.** $\frac{1}{2}\frac{1}{\sqrt{11}}\ \ell n\ \frac{\sqrt{11}+1}{\sqrt{11}-1}$ **15.** $\frac{n}{n^2-1}$

17. $\left(\frac{2\sqrt{e^{\pi}}}{e^2}\right)^m$ **18.** (1) $\frac{1}{2n+1}-a_n$ (2) 0 (3) $\frac{\pi}{4}$

19. $a = 4, b = 1$ **22.** $3(\sqrt{3}-1)$ sq. units **23.** $\frac{1}{3}\left(16\sqrt{2}-20\right)$

24. $\frac{19}{6}$ **25.** $\frac{4}{3}$ **26.** $\frac{23}{2}$ **27.** $\frac{23}{6}$ **28.** $y = e^{a(x-1)}$, $\left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

29. $2\sqrt{3}\sin^{-1}\sqrt{\frac{2}{3}}-\frac{2\sqrt{2}}{\sqrt{3}}$ **30.** $\frac{4\sqrt{3}}{3}$ **31.** $2(6 - 2\log 4)$ **32.** $\frac{73}{6}$ **34.** $1 + x + \frac{x}{\sqrt{1+x^2}}$.

35. (i) $a = 1 + e^2$ (ii) $b = 1 + e^{-2}$ **36.** $f(x) = x^3 - x^2$

37. centre $\left(0, \frac{5}{4}\right)$ and area $= \frac{3\sqrt{3}}{4}-\frac{\pi}{3}$ **38.** $\frac{125}{3}$ square units. **39.** (AC) **40.** 20π