Target : JEE (Main + Advanced)

Indefinite Integration

CONTENT

INDEFINITE INTEGRATION

JEE (ADVANCED) SYLLABUS

Integration as the inverse process of differentiation, indefinite integrals of standard functions. Integration by parts, integration by the methods of substitution and partial fractions.

JEE (MAIN) SYLLABUS

Integral as an anti - derivative. Fundamental integrals involving algebraic, trigonometric, exponential and logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities

Indefinite Integration

But just as much as it is easy to find the differential of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not. Bernoulli, Johann

If f & g are functions of x such that $g'(x) = f(x)$, then indefinite integration of f(x) with respect to x is defined and denoted as $\int f(x)\,dx = g(x) + C$, where C is called the **constant of integration**.

Standard Formula:

(i) $\int (ax + b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1$

(ii) $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\,\ell n\,|ax + b| + C$

(iii) $\int e^{ax+b}\,dx = \dfrac{1}{a}\,e^{ax+b} + C$

(iv) $\int a^{px+q}\,dx = \dfrac{1}{p}\,\dfrac{a^{px+q}}{\ell n\,a} + C;\ a > 0$

(v) $\int \sin(ax + b)\,dx = -\dfrac{1}{a}\cos(ax + b) + C$

(vi) $\int \cos(ax + b)\,dx = \dfrac{1}{a}\sin(ax + b) + C$

(vii) $\int \tan(ax + b)\,dx = \dfrac{1}{a}\,\ell n\,|\sec(ax + b)| + C$

(viii) $\int \cot(ax + b)\,dx = \dfrac{1}{a}\,\ell n\,|\sin(ax + b)| + C$

(ix) $\int \sec^2(ax + b)\,dx = \dfrac{1}{a}\tan(ax + b) + C$

(x) $\int \mathrm{cosec}^2(ax + b)\,dx = -\dfrac{1}{a}\cot(ax + b) + C$

(xi) $\int \sec(ax + b).\tan(ax + b)\,dx = \dfrac{1}{a}\sec(ax + b) + C$

(xii) $\int \mathrm{cosec}(ax + b).\cot(ax + b)\,dx = -\dfrac{1}{a}\,\mathrm{cosec}(ax + b) + C$

(xiii) $\int \sec x\,dx = \ell n\,|\sec x + \tan x| + C$ **OR** $\ell n\left|\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right| + C$

(xiv) $\int \mathrm{cosec}\,x\,dx = \ell n\,|\mathrm{cosec}\,x - \cot x| + C$ **OR** $\ell n\left|\tan\dfrac{x}{2}\right| + C$ **OR** $-\ell n\,|\mathrm{cosec}\,x + \cot x| + C$

(xv) $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\dfrac{x}{a} + C$

(xvi) $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

(xvii) $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right| + C$ **OR** $\sinh^{-1}\frac{x}{a} + C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right| + C$ **OR** $\cosh^{-1}\frac{x}{a} + C$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ \ln\left|\frac{a+x}{a-x}\right| + C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ \ln\left|\frac{x-a}{x+a}\right| + C$

(xxii) $\int \sqrt{a^2-x^2}\ dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(xxiii) $\int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ \ell n\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C$

(xxiv) $\int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ \ell n\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

(xxv) $\int e^{ax}.\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}\ (a\sin bx - b\cos bx) + C$

(xxvi) $\int e^{ax}.\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}\ (a\cos bx + b\sin bx) + C$

Theorems on integration

(i) $\int C\ f(x).dx = C\int f(x).dx$

(ii) $\int (f(x) \pm g(x))\ dx = \int f(x)dx \pm \int g(x)\ dx$

(iii) $\int f(x)dx = g(x) + C_1 \Rightarrow \int f(ax+b)dx = \frac{g(ax+b)}{a} + C_2$

Example # 1 Evaluate : $\int 3x^6 dx$

Solution : $\int 3x^6 dx = \frac{3}{7}x^7 + C$

Example # 2 Evaluate : $\int \left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right) dx$

Solution : $\int \left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right) dx$

$= \int x^3\ dx + \int 5x^2\ dx - \int 4dx + \int \frac{7}{x}dx + \int \frac{2}{\sqrt{x}}dx$

$= \int x^3\ dx + 5\int x^2\ dx . -4 . \int 1 . dx + 7 . \int \frac{1}{x}dx + 2. \int x^{-1/2}dx$

$= \frac{x^4}{4} + 5.\ \frac{x^3}{3} - 4x + 7\ \ell n\ |x| + 2\left(\frac{x^{1/2}}{1/2}\right) + C = \frac{x^4}{4} + \frac{5}{3}x^3 - 4x + 7\ \ell n\ |x| + 4\sqrt{x} + C$

Example # 3 Evaluate : $\int 2^{x\log_2 3}\ dx$

Solution : We have, $\int 2^{x\log_2 3}\ dx = \int 3^x dx = \frac{3^x}{\ell n 3} + C$

Example # 4 Evaluate : $\int \frac{4^x+5^x}{7^x}\, dx$

Solution : $\int \frac{4^x+5^x}{7^x}\, dx = \int \left(\frac{4^x}{7^x}+\frac{5^x}{7^x}\right) dx = \int \left[\left(\frac{4}{7}\right)^x + \left(\frac{5}{7}\right)^x\right] dx = \frac{(4/7)^x}{\ell n\left(\frac{4}{7}\right)} + \frac{(5/7)^x}{\ell n\left(\frac{5}{7}\right)} + C$

Example # 5 Evaluate : $\int \frac{\cos 7x - \cos 8x}{1+2\cos 5x} dx$

Solution : We have, $\int \frac{\cos 7x - \cos 8x}{1+2\cos 5x} dx = \frac{1}{2}\int \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{\sin\frac{5x}{2} + 2\cos 5x \sin\frac{5x}{2}} dx$

$$= \frac{1}{2} \int \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}} dx$$

$$= \frac{1}{2} \int \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{21x}{2} + \sin\frac{9x}{2}\right)}{\sin\frac{15x}{2}} dx$$

$$= \frac{1}{2} \int \frac{2\sin\frac{15x}{2}\cos 2x - 2\sin\frac{15x}{2}\cos 3x}{\sin\frac{15x}{2}} dx = \int \cos 2x - \cos 3x) dx = \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + C$$

Example # 6 Evaluate : $\int \frac{x^3}{(x+1)^2} dx$

Solution : $\int \frac{x^3}{(x+1)^2} dx = \int \frac{x^3+1-1}{(x+1)^2} dx = \int \frac{x^3+1}{(x+1)^2} dx - \int \frac{1}{(x+1)^2} dx$

$$= \int \frac{(x+1)(x^2-x+1)}{(x+1)^2} dx - \int \frac{1}{(x+1)^2} dx = \int \frac{x^2-x+1)}{(x+1)} dx - \int \frac{1}{(x+1)^2} dx$$

$$= \int \left(x - 2 + \frac{3}{x+1}\right) dx - \int \frac{1}{(x+1)^2} dx = \frac{x^2}{2} - 2x + 3\,\ell n(x+1) + \frac{1}{x+1} + C$$

Example # 7 : Evaluate : $\int \frac{1}{4+9x^2}\, dx$

Solution : We have

$$\int \frac{1}{4+9x^2}\, dx = dx\ \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2} = \frac{1}{9}\int \frac{1}{(2/3)^2+x^2} dx$$

$$= \frac{1}{9} \cdot \frac{1}{(2/3)} \tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$$

Example # 8 : Evaluate : $\int \cos x \cos 2x\ \ dx$

Solution : $\int \cos x \cos 2x\ \ dx = \frac{1}{2}\int 2\cos x\ \cos 2x\ \ dx = \frac{1}{2}\int (\cos 3x + \cos x)\ dx = \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C$

Self Practice Problems :

(1) Evaluate : $\int \tan^2 x\ dx$ (2) Evaluate : $\int \frac{1}{1+\sin x}\, dx$

Ans. (1) $\tan x - x + C$ (2) $\tan x - \sec x + C$

Integration by Substitution

If we substitute $\phi(x) = t$ in an integral then

(i) everywhere x will be replaced in terms of new variable t.

(ii) dx also gets converted in terms of dt.

Example # 9 : Evaluate : $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$

Solution : We have,

$\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$

$= \int \frac{(x+\cos x) - x + x\sin x}{x(x+\cos x)}\,dx = \int\left(\frac{1}{x} - \frac{1-\sin x}{x+\cos x}\right)dx = \int \frac{1}{x}\,dx - \int \frac{1-\sin x}{x+\cos x}dx$

$= \int \frac{1}{x}\,dx - \int \frac{1}{x+\cos x} d\,(x+\cos x) = \ell n|x| - \ell n\,|x + \cos x| + C.$

Example # 10 : Evaluate : $\int \frac{(\ell nx)^n}{x}dx$

Solution : We have $\int \frac{(\ell nx)^n}{x}dx = \int (\ell nx)^n \frac{1}{x}dx = \int (\ell nx)^n d(\ell nx) = \frac{(\ell nx)^{n+1}}{n+1} + C$

Example # 11 : Evaluate : $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}dx$

Solution : We have , $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}dx = \int (\sin^{-1}x)^3 d(\sin^{-1}x) = \frac{(\sin^{-1}x)^4}{4} + C$

Example # 12 : Evaluate : $\int \frac{x}{x^4 + 2x^2 + 2}\,dx$

Solution : We have,

$I = \int \frac{x}{x^4+2x^2+2}\,dx = \int \frac{x}{(x^2)^2 + 2x^2 + 2}\,dx$ {Put $x^2 = t \Rightarrow x.dx = \frac{dt}{2}$ }

$\Rightarrow I = \frac{1}{2}\int \frac{1}{t^2+2t+2}\,dt = \frac{1}{2}\int \frac{1}{(t+1)^2+1}\,dt = \frac{1}{2}\tan^{-1}(t+1) + C$

$= \frac{1}{2}\tan^{-1}(x^2+1) + C$

Note:

(i) $\int [f(x)]^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C$

(ii) $\int \frac{f'(x)}{[f(x)]^n}\,dx = \frac{(f(x))^{1-n}}{1-n} + C,\ n \neq 1$

(iii) $\int \frac{dx}{x(x^n+1)}$; $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(iv) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$; $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(v) $\int \int \frac{dx}{x^n\left(1+x^n\right)^{1/n}}$; take x^n common as x and put $1 + x^{-n} = t$.

Self Practice Problems :

(3) Evaluate : $\int \frac{\sec^2 x}{1+\tan x} dx$ (4) Evaluate : $\int \frac{\sin(\ell n x)}{x} dx$

Ans. (3) $\ell n\ |1 + \tan x| + C$ (4) $-\cos(\ell n\ x) + C$

Integration by Parts : Product of two functions f(x) and g(x) can be integrated using formula :

$$\int (f(x)\ g(x))\ dx = f(x)\int (g(x))\,dx - \int\left(\frac{d}{dx}(f(x))\ \int (g(x))dx\right)dx$$

(i) when you find integral $\int g(x)\ dx$ then it will **not** contain arbitarary constant.

(ii) $\int g(x)\ dx$ should be taken as same at both places.

(iii) The choice of f(x) and g(x) can be decided by ILATE guideline.
the function will come later is taken an integral function (g(x)).

I → Inverse function
L → Logarithmic function
A → Algebraic function
T → Trigonometric function
E → Exponential function

Example # 13 : Evaluate : $\int \sec^{-1} x\ dx$

Solution : Put $\sec^{-1} x = t$ so that $x = \sec t$ and $dx = \sec t \tan t\ dt$

$\therefore \int \sec^{-1} x\ dx = \int \underset{\text{I}}{t}(\underset{\text{II}}{\sec t}\ \tan t)dt = t\,(\sec t) - \int 1.\sec t\ dt$

$= t \sec t - \ell n\ |\sec t + \tan t)| + C$

$= t \sec t - \ell n\ |\sec t + \sqrt{\sec^2 t - 1}\ | + C = x\,(\sec^{-1}x) - \ell n\ |x + \sqrt{x^2-1}\ | + C$

Example # 14 : Evaluate : $\int x\ \ell n(1+x)\ dx$

Solution : Let $I = \int x\ \ell n(1+x)\ dx = \frac{x^2}{2}.\ell n\,(x+1) - \int \frac{1}{x+1}.\frac{x^2}{2}\ dx$

$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\ dx = \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\ dx$

$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\int\left(\frac{x^2-1}{x+1} + \frac{1}{x+1}\right) dx = \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\int\left((x-1) + \frac{1}{x+1}\right)\ dx$

$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \ell n\,|\,x+1|\right] + C$

Example # 15 : Evaluate : $\int e^{2x}\sin 3x\ dx$

Solution : Let $I = \int e^{2x}\sin 3x\ dx$

$= e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\left(-\frac{\cos 3x}{3}\right)dx = -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$

$= -\frac{1}{3}\ e^{2x}\ \cos 3x + \frac{2}{3}\left[e^{2x}\frac{\sin 3x}{3} - \int 2e^{2x}\frac{\sin 3x}{3}\ dx\right]$

$= -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{9}\ e^{2x}\ \sin 3x - \frac{4}{9}\int e^{2x}\sin 3x\ dx$

$\Rightarrow I = -\frac{1}{3}\ e^{2x}\ \cos 3x + \frac{2}{9}e^{2x}\ \sin 3x - \frac{4}{9}\ I \quad \Rightarrow \quad I + \frac{4}{9}\ I = \frac{e^{2x}}{9}\ (2\sin 3x - 3\cos 3x)$

$\Rightarrow \frac{13}{9}\ I = \frac{e^{2x}}{9}\ (2\sin 3x - 3\cos 3x) \Rightarrow I = \frac{e^{2x}}{13}\ (2\sin 3x - 3\cos 3x) + C$

Note :

(i) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$ (ii) $\int [f(x) + xf'(x)]\, dx = x f(x) + C$

Example # 16 : Evaluate : $\int e^x \dfrac{(x^2 - 2x + 2)}{(x^2+2)^2}\, dx$

Solution : Given integral = $\int e^x \dfrac{(x^2 - 2x + 2)}{(x^2+2)^2}\, dx = \int e^x \left\{\dfrac{1}{x^2+2} + \dfrac{(-2x)}{(x^2+2)^2}\right\} = \dfrac{e^x}{x^2+2} + C$

Example # 17 : Evaluate : $\int e^x \left(\dfrac{1-\sin x}{1-\cos x}\right) dx$

Solution : Given integral $= \int e^x \left(\dfrac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$

$= \int e^x \left(\dfrac{1}{2}\operatorname{cosec}^2\dfrac{x}{2} - \cot\dfrac{x}{2}\right) dx = -\, e^x \cot\dfrac{x}{2} + C$

Example # 18 : Evaluate : $\int \left[\ell n\ (\ell nx) + \dfrac{1}{(\ell nx)^2}\right] dx$

Solution : Let $I = \int \left[\ell n\ (\ell nx) + \dfrac{1}{(\ell nx)^2}\right] dx$ {put $x = e^t \Rightarrow dx = e^t\, dt$}

$\therefore$ $I = \int e^t \left(\ell nt + \dfrac{1}{t^2}\right) dt \int e^t \left(\ell nt - \dfrac{1}{t} + \dfrac{1}{t} + \dfrac{1}{t^2}\right) = dt$

$= e^t \left(\ell nt - \dfrac{1}{t}\right) + C = x\left[\ell n\ (\ell nx) - \dfrac{1}{\ell nx}\right] + C$

Self Practice Problems :

(5) Evaluate : $\int x \sin x\ dx$ (6) Evaluate : $\int x^2 e^x\ dx$

Ans. (5) $-x\cos x + \sin x + C$ (6) $x^2 e^x - 2xe^x + 2e^x + C$

Integration of type $\int \dfrac{dx}{ax^2+bx+c}$, $\int \dfrac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

Example # 19 : Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Solution : We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\ dx = \int \sqrt{(x+1)^2+2^2}$

$= \dfrac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \dfrac{1}{2}\cdot(2)^2\ \ell n\,|(x+1) + \sqrt{(x+1)^2+2^2}\,| + C$

$= \dfrac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\ \ell n\,|(x+1) + \sqrt{x^2+2x+5}\,| + C$

Example # 20 : Evaluate : $\int \dfrac{dx}{\sqrt{2-6x-9x^2}}\ dx$

Solution : $\int \frac{dx}{\sqrt{2-6x-9x^2}}\ dx = \int \frac{1}{\sqrt{3-(3x+1)^2}}\ dx = \frac{1}{3}\sin^{-1}\left(\frac{3x+1}{\sqrt{3}}\right) + C$

Self Practice Problems :

(7) Evaluate : $\int \frac{1}{2x^2+x-1}\ dx$ (8) Evaluate : $\int \frac{8x-11}{\sqrt{5+2x-x^2}}\ dx$

Ans. (7) $\frac{1}{3}\ \ell n\left|\frac{2x-1}{2x+2}\right| + C$ (8) $-8\sqrt{5+2x-x^2} - 3\sin^{-1}\frac{x-1}{\sqrt{6}} + C$

Integration of type

$$\int \frac{px+q}{ax^2+bx+c}dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,\ \int (px+q)\sqrt{ax^2+bx+c}\ dx$$

Express px + q = A (differential co–efficient of denominator) + B.

Example # 21 : Evaluate : $\int \frac{2x-3}{x^2+3x-18}dx$

Solution : Let $2x - 3 = \lambda\frac{d}{dx}(x^2 + 3x - 18) + \mu$

Then $2x - 3 = \lambda(2x + 3) + \mu$

Comparing the coefficients of like power of x, we get.

$2\lambda = 2$, and $3\lambda + \mu = -3$ $\Rightarrow \lambda = 1$ and $\mu = -6$

So, $\int \frac{2x-3}{x^2+3x-18}dx = \int \frac{2x+3-6}{x^2+3x-18}dx = \int \frac{2x+3}{x^2+3x-18}dx - 6\int \frac{1}{x^2+3x-18}dx$

$= \ell n|x^2 + 3x - 18| - 6\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}-18}dx = \ell n|x^2 + 3x - 18| - 6\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx$

$= \ell n|x^2 + 3x - 18| - 6.\ \frac{1}{2\left(\frac{9}{2}\right)}\ \ell n\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C = \ell n|x^2 + 3x - 18| - \frac{2}{3}\ \ell n\left|\frac{x-3}{x+6}\right| + C$

Example # 22 : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\ dx$

Solution : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx = \int \frac{2x+4}{\sqrt{x^2+4x+1}}dx - \int \frac{1}{\sqrt{x^2+4x+1}}dx$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}}\ dx,$ where $t = (x^2 + 4x + 1)$ for I[st] integral

$= 2\sqrt{t} - \ell n\,|(x + 2) + \sqrt{x^2+4x+1}| + C = 2\sqrt{x^2+4x+1} - \ell n\,|x + 2 + \sqrt{x^2+4x+1}| + C$

Example # 23 : Evaluate : $\int x\sqrt{1+x-x^2}dx$

Solution : Let $x = \lambda\ .\ \frac{d}{dx}(1+x-x^2) + \mu.$

$\Rightarrow x = \lambda(1-2x) + \mu$

Comparing the coefficients of like powers of x, we get

$1 = -2\lambda$ and $\lambda + \mu = 0 \Rightarrow \lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}$ $\therefore$ $x = -\frac{1}{2}(1-2x) + \frac{1}{2}$

so, $\int x\sqrt{1+x-x^2}dx$

$$= \int\left\{-\frac{1}{2}(1-2x)+\frac{1}{2}\right\}\sqrt{1+x-x^2dx} = -\frac{1}{2}\int(1-2x)\sqrt{1+x-x^2}\ dx + \frac{1}{2}\int\sqrt{1+x-x^2}\ dx$$

$$= -\frac{1}{2}\int\sqrt{1+x-x^2}\ d(1+x-x^2) + \frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\ dx,$$

$$= -\frac{1}{3}\ (1+x-x^2)^{3/2} + \frac{1}{2}\left[\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2\sin^{-1}\frac{x-1/2}{\sqrt{5}/2}\right] + C$$

$$= -\frac{1}{3}\ (1+x-x^2)^{3/2} + \frac{1}{2}\left[\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right] + C$$

Self Practice Problems :

(9) Evaluate : $\int\frac{3-4x}{2x^2-3x+1}\ dx$ (10) Evaluate : $\int\frac{6x-5}{\sqrt{3x^2-5x+1}}\ dx$

(11) Evaluate : $\int(x-1)\sqrt{1+x+x^2}\ \ dx$

Ans. (9) $-\ell n|2x^2-3x+1| + C$ (10) $2\ \sqrt{3x^2-5x+1} + C$

(11) $\frac{1}{3}\ (x^2+x+1)^{3/2} - \frac{3}{8}\ (2x+1)\ \sqrt{1+x+x^2}\ - \frac{9}{16}\log(2x+1+2\sqrt{x^2+x+1}) + C$

Integration of Rational Algebraic Functions by using Partial Fractions:

PARTIAL FRACTIONS :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) $\geq$ degree of g(x) then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide f(x) by g(x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\Psi(x)}{g(x)}$, where $\phi(x)$ and $\Psi(x)$ are polynomials such that the degree of $\Psi(x)$ is less than that of g(x). Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

CASE-I $\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$

CASE-II $\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2}$

CASE-III $\frac{ax^2+bx+c}{(x-\alpha)(x^2+\beta^2)} = \frac{A}{x-\alpha} + \frac{Bx+C}{x^2+\beta^2}$

where A, B, C can be evaluated by subsitution or by comparing coefficients.

Example # 24 : Resolve $\frac{1}{2x^3+3x^2-3x-2}$ into partial fractions.

Solution : We have, $\frac{1}{2x^3+3x^2-3x-2} = \frac{1}{(x-1)(x+2)(2x+1)}$

Let $\dfrac{1}{2x^3+3x^2-3x-2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{2x+1}$. Then,

$\Rightarrow$ $1 = A(x + 2)(2x + 1) + B(x - 1)(2x + 1) + C(x - 1)(x + 2)$ ..(i)

Putting $x - 1 = 0$ or $x = 1$ in (i), we get $\Rightarrow$ $A = \dfrac{1}{9}$,

Putting $x = -2$ in (i), we obtain $B = \dfrac{1}{9}$

Putting $x = -\dfrac{1}{2}$ in (i), we obtain $C = -\dfrac{4}{9}$

$\therefore$ $\dfrac{1}{2x^3+3x^2-3x-2} = \dfrac{1}{(x-1)(x+2)(2x+1)} = \dfrac{1}{9(x-1)} + \dfrac{1}{9(x+2)} - \dfrac{4}{9(2x+1)}$

Example # 25 : Resolve $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Solution : Here the given function is an improper rational function. On dividing we get

$\dfrac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 + \dfrac{(-x+4)}{(x^2-5x+6)}$ ....(i)

we have, $\dfrac{-x+4}{x^2-5x+6} = \dfrac{-x+4}{(x-2)(x-3)}$

So, let $\dfrac{-x+4}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$, then

$-x + 4 = A(x - 3) + B(x - 2)$ ....(ii)

Putting $x - 3 = 0$ or $x = 3$ in (ii), we get

$1 = B(1)$ $\Rightarrow$ $B = 1$.

Putting $x - 2 = 0$ or $x = 2$ in (ii), we get

$2 = A(2 - 3) \Rightarrow A = -2$

$\therefore \dfrac{-x+4}{(x-2)(x-3)} = \dfrac{-2}{x-2} + \dfrac{1}{x-3}$

Hence $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 - \dfrac{2}{x-2} + \dfrac{1}{x-3}$

Example # 26 : Evaluate : $\displaystyle\int \frac{3x+1}{(x-1)^3(x+1)}dx$

Solution : Let $\dfrac{3x+1}{(x-1)^3(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2} + \dfrac{D}{(x-1)^3}$ .... (i)

Multiplying both sides by $(x + 1)$ and then putting $x = -1$, we get

$A = \dfrac{-2}{(-2)^3} = \dfrac{1}{4}$

Multiplying both sides by $(x - 1)^3$ and then putting $x = 1$, we get

$D = \dfrac{4}{2} = 2$

From (i) , we get

$3x + 1 = A(x - 1)^3 + B(x - 1)^2(x + 1) + C(x - 1)(x + 1) + D(x + 1)$

putting $x = 0$, we get

$1 = -A + B - C + D$

$\Rightarrow$ $1 = -\dfrac{1}{4} + B - C + 2$ $\Rightarrow B - C = \dfrac{-3}{4}$

Putting $x = 2$, we get

$7 = A + 3B + 3C + 3D$

$\Rightarrow \quad 7 = \frac{1}{4} + 3B + 3C + 6 \quad \Rightarrow 3B + 3C = \frac{3}{4} \Rightarrow B + C = \frac{1}{4}$

Solving $B + C = \frac{1}{4}$ and $B - C = \frac{-3}{4}$, we get $B = -\frac{1}{4}$, $C = \frac{1}{2}$

Substituting the values of A, B, C and D in (i), we get

$$\Rightarrow \quad \frac{3x+1}{(x-1)^3(x+1)} = \frac{1}{4}.\frac{1}{x+1} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{2}{(x-1)^3}$$

$$\Rightarrow \quad \int \frac{3x+1}{(x-1)^3(x+1)}dx = \frac{1}{4}\int \frac{1}{x+1}dx - \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{(x-1)^2}dx + 2\int \frac{1}{(x-1)^3}dx$$

$$= \frac{1}{4}\ell n\,|x+1| - \frac{1}{4}\ell n|x-1| - \frac{1}{2(x-1)} - \frac{1}{(x-1)^2} + C$$

Example # 27 : Evaluate : $\int \frac{1}{\sin x(2\cos^2 x - 1)}dx$

Solution : Putting cosx = t, we get

$$I = \int \frac{1}{\sin x(2\cos^2 x-1)}dx = \int \frac{1}{\sin x(2t^2-1)} \times -\frac{dt}{\sin x} = -\int \frac{1}{(1-t^2)(2t^2-1)}dt$$

$$\therefore \quad I = -\int \left(\frac{1}{1-t^2} + \frac{2}{2t^2-1}\right)dt = -\int \frac{1}{1-t^2}dt - 2\int \frac{1}{2t^2-1}dt$$

$$= -\frac{1}{2}\,\ell n\left|\frac{1+t}{1-t}\right| - \frac{\sqrt{2}}{2}\,\ell n\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C = -\frac{1}{2}\,\ell n\left|\frac{1+\cos x}{1-\cos x}\right| - \frac{1}{\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$$

Example # 28 : Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution : Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting x = 1 in (i), we get $-1 = A(1 + 1)^2 \Rightarrow A = -$

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0$, $C - B = 0$, $2A + B - C + D = 0$, $C + E - B - D = 2$ and $A - C - E = -3$.

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$B = \frac{1}{4} = C$, $D = \frac{1}{4}$ and $E = \frac{5}{2}$ $\quad \therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$

Example # 29 : Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Solution : We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

Then, $2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ...(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting $x = 0$ in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting $x = -1$ in (i), we get $-2 = A + 2B - 2C$. $\quad \Rightarrow \quad -2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{(-2/3)\,x + 2/3}{x^2+x+1} \text{ or } \frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$$

Self Practice Problems :

(12) (i) Evaluate : $\int \frac{1}{(x+2)(x+3)} dx$ (ii) Evaluate : $\int \frac{dx}{(x+1)(x^2+1)}$

Ans. (12) (i) $\ell n \left|\frac{x+2}{x+3}\right| + C$ (ii) $\frac{1}{2}$ $\ell n\ |x + 1| - \ell n\ (x^2 + 1) + \frac{1}{2} \tan^{-1} (x) + C$

Integration of type

$$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1} dx \text{ where K is any constant.}$$

Divide Nr & Dr by x^2 & put $x \mp \frac{1}{x} = t$.

Example # 30 : Evaluate $\int \frac{x^2+4}{x^4+16} dx$

Solution : $\int \frac{x^2+4}{x^4+16} dx = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}} dx = \int \frac{1}{\left(x-\frac{4}{x}\right)^2+8} d\left(x-\frac{4}{x}\right) = \int \frac{dt}{t^2+(2\sqrt{2})^2}$,

where $t = x - \frac{4}{x}$ $= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C$

Example # 31 : Evaluate : $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$

Solution : $\Rightarrow I = \int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}} dx$ $\left[\begin{array}{l}\text{Multiplying the} \\ N^r \text{ and } D^r \text{ by } (x+1)\end{array}\right]$

$\Rightarrow I = \int \frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}} dx$

$\Rightarrow I = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}} dx$ [Dividing N^r and D^r by x^2)

$\Rightarrow I = \int \frac{2t\ dt}{(t^2+1)\sqrt{t^2}}$ where, $x + \frac{1}{x} + 1 = t^2$ $\Rightarrow I = 2 \int \frac{1}{t^2+1} dt$ $\Rightarrow I = 2\tan^{-1}(t) + C$

$\Rightarrow I = 2 \tan^{-1} \sqrt{x+\frac{1}{x}+1} + C$

Self Practice Problems :

(13) Evaluate : $\int \frac{x^2-1}{x^4-7x^2+1} dx$ (14) Evaluate : $\int \sqrt{\tan x}\ dx$

Ans. (13) $\frac{1}{6} \ell n \left|\frac{x+\frac{1}{x}-3}{x+\frac{1}{x}+3}\right| + C$ (14) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \ell n \left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + C$

where $y = \sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}$

Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \text{ OR } \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}.$$

Put $px + q = t^2$.

Example # 32 : Evaluate : $\int \frac{dx}{(x-4)\sqrt{x+5}}$

Solution : Let I = $\int \frac{dx}{(x-4)\sqrt{x+5}}$ $\qquad$ {Put $x + 5 = t^2 \Rightarrow dx = 2t\,dt$}

$\therefore \quad I = \int \frac{2dt}{(t^2-9)} = \frac{2}{6}\,\ell n\left|\frac{t-3}{t+3}\right| + C = \frac{1}{3}\,\ln\left|\frac{\sqrt{x+5}-3}{\sqrt{x+5}+3}\right| + C$

Example # 33 : Evaluate : $\int \frac{dx}{(x^2+3x+2)\sqrt{x+4}}$

Solution : Let I = $\int \frac{dx}{(x^2+3x+2)\sqrt{x+4}}$

Putting $x + 4 = t^2$, and $dx = 2t\,dt$, we get $I = \int \frac{2t\ dt}{\{(t^2-4)^2+3(t^2-4)+2\}\sqrt{t^2}}$

$\Rightarrow \quad 2\int \frac{dt}{t^4-5t^2+6}\,dt = 2\int \frac{dt}{(t^2-2)(t^2-3)}\,dt = 2\int\left[\frac{1}{t^2-3}-\frac{1}{t^2-2}\right]dt$

$= \frac{1}{\sqrt{3}}\,\ell n\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \frac{1}{\sqrt{2}}\,\ell n\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$ where $t = \sqrt{x+4}$

Integration of type

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ **, put** $ax + b = \frac{1}{t}$ **;** $\quad \int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ **, put** $x = \frac{1}{t}$

Example # 34 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$

Solution : Let I = $\int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$ $\qquad$ {put $x - 1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$ }

$\Rightarrow \quad I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2-\left(\frac{1}{t}+1\right)-1}} = \int -\frac{dt}{\sqrt{-t^2+t+1}} = \int -\frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}$

$= -\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C = -\sin^{-1}\left(\frac{2t-1}{\sqrt{5}}\right) + C$, where $t = \frac{1}{x-1}$

Example # 35 : Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Put $x = \frac{1}{t} \Rightarrow \quad dx = -\frac{1}{t^2}\,dt \quad \Rightarrow \quad I = -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$ $\qquad$ {put $t^2 - 1 = y^2 \Rightarrow t\,dt = y\,dy$ }

$\Rightarrow \quad I = -\int \frac{y\ dy}{(y^2+2)\ y} = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$

$= -\frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

Self Practice Problems :

(15) Evaluate : $\int \frac{dx}{(x+2)\sqrt{x+1}}$

(16) Evaluate : $\int \frac{dx}{(x^2+5x+6)\sqrt{x+1}}$

(17) Evaluate : $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$

(18) Evaluate : $\int \frac{dx}{(2x^2+1)\sqrt{1-x^2}}$

(19) Evaluate : $\int \frac{dx}{(x^2+2x+2)\sqrt{x^2+2x-4}}$

Ans. (15) $2\tan^{-1}\left(\sqrt{x+1}\right) + C$ (16) $2\tan^{-1}\left(\sqrt{x+1}\right) - \sqrt{2}\tan^{-1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right) + C$

(17) $\sin^{-1}\left(\frac{\frac{3}{2} - \frac{1}{x+1}}{\frac{\sqrt{5}}{2}}\right) + C$ (18) $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{3}\ x}\right) + C$

(19) $-\frac{1}{2\sqrt{6}}\ \ell n\left(\frac{\sqrt{x^2+2x-4} - \sqrt{6}\ (x+1)}{\sqrt{x^2+2x-4} + \sqrt{6}\ (x+1)}\right) + C$

Integration of type

$\int \sqrt{\frac{x-\alpha}{\beta - x}}\ dx$ **or** $\int \sqrt{(x-\alpha)(\beta - x)}\ dx$; **put** $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\ dx$ **or** $\int \sqrt{(x-\alpha)(x-\beta)}\ dx$; **put** $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; **put** $x - \alpha = t^2$ **or** $x - \beta = t^2$.

Self Practice Problems

(20) Evaluate : $\int \sqrt{\frac{x-3}{x-4}}\ dx$

(21) Evaluate : $\int \frac{dx}{[(x-1)(2-x)]^{3/2}}$

(22) Evaluate : $\int \frac{dx}{[(x+2)^8(x-1)^6]^{1/7}}$

Ans. (20) $\sqrt{(x-3)(x-4)} + \ell n\left(\sqrt{x-3} + \sqrt{x-4}\right) + C$ (21) $2\left(\sqrt{\frac{x-1}{2-x}} - \sqrt{\frac{2-x}{x-1}}\right) + C$

(22) $\frac{7}{3}\left(\frac{x-1}{x+2}\right)^{1/7} + C$

Integration of trigonometric functions

(i) $\int \frac{dx}{a + b\sin^2 x}$ **OR** $\int \frac{dx}{a + b\cos^2 x}$ **OR** $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put $\tan x = t$.

(ii) $\int \frac{dx}{a + b\sin x}$ **OR** $\int \frac{dx}{a + b\cos x}$ **OR** $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}\ dx.$

Express Nr ≡ A(Dr) + B (Dr) + C & proceed.

Example # 36 : Evaluate: $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

Solution : Let I = $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

Putting sinx = $\frac{2\tan x/2}{1+\tan^2 x/2}$ and , cos x = $\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$,

we get

$$I = \int \frac{1+\frac{2\tan x/2}{1+\tan^2 x/2}}{\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)\left(1+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}dx = \int \frac{(1+\tan^2 x/2+2\tan x/2)(1+\tan^2 x/2)}{2\tan x/2(1+\tan^2 x/2+1-\tan^2 x/2)}dx$$

$$= \int \frac{(1+\tan\ \ x/2)^2 \sec^2 x/2}{4\tan x/2}dx = \int \frac{1+t^2+2t)}{2t}dt, \text{ where } t = \tan\frac{x}{2}$$

$$= \frac{1}{2}\int\left(\frac{1}{t}+t+2\right)dt = \frac{1}{2}\left[\ell n\,|\,t\,|+\frac{t^2}{2}+2t\right]+C = \frac{1}{2}\left[\ell n\,|\tan x/2|+\frac{\tan^2 x/2}{2}+2\tan x/2\right]+C$$

Example # 37 : Evaluate : $\int \frac{dx}{\sin x+\sqrt{3}\cos x}$

Solution : Let 1 = rcosθ and $\sqrt{3}$ = r sinθ ⇒ r = $\sqrt{(1)^2+(\sqrt{3})^2}$ = 2

tanθ = $\sqrt{3}$ ⇒ θ = π/3

$$\therefore \int \frac{dx}{\sin x+\sqrt{3}\cos x} = \frac{1}{r}\int \frac{dx}{\sin x\cos\theta+\cos x\sin\theta} = \frac{1}{r}\int \frac{dx}{\sin(x+\theta)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)dx = \frac{1}{r}\ \ell n\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C = \frac{1}{2}\ \ell n\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$$

Example # 38 : Evaluate : $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\ dx$

Solution : We have,

$$I = \int \frac{3\cos x+2}{\sin x+2\cos x+3}\ dx$$

Let 3 cos x + 2 = λ (sin x + 2 cos x + 3) + μ (cos x – 2 sin x) + ν

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2 \quad \Rightarrow \quad \lambda = \frac{6}{5},\ \mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

$$\therefore \quad I = \int \frac{\lambda(\sin x+2\cos x+3)+\mu(\cos x-2\sin x)+\nu}{\sin x+2\cos x+3}dx$$

$$\Rightarrow \quad I = \lambda\int dx+\mu\int \frac{\cos x-2\sin x}{\sin x+2\cos x+3}dx + \nu\int \frac{1}{\sin x+2\cos x+3}dx$$

$$\Rightarrow \quad I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu I_1$$

where $I_1 = \int \frac{1}{\sin x+2\cos x+3}\ dx$

Putting, sin x = $\frac{2\tan x/2}{1+\tan^2 x/2}$, cos x = $\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3}\, dx \; = \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)}\, dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}\, dx$$

Putting $\tan\frac{x}{2}$ = t and $\frac{1}{2}$ $\sec^2 \frac{x}{2}$ = dt or $\sec^2 \frac{x}{2}$ dx = 2 dt, we get

$$I_1 = \int \frac{2dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)$$

Hence, $I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Example # 39 : Evaluate : $\int \frac{dx}{1+3\cos^2 x}$

Solution : Multiply Nr. & Dr. of given integral by $\sec^2 x$, we get

$$I = \int \frac{\sec^2 x\ dx}{\tan^2 x + 4} = \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$$

Self Practice Problems :

(23) Evaluate : $\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\, dx$

Ans. (23) $\frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + C$

Integration of type $\int \sin^m x.\ \cos^n x\ dx$

Case - I

If m and n are even natural number then converts higher power into higher angles.

Case - II

If at least one of m or n is odd natural number then if m is odd put cosx = t and vice-versa.

Case - III

When m + n is a negative even integer then put tan x = t.

Example # 40 : Evaluate : $\int \cos^5 x \sin^4 x dx$

Solution : Let $I = \int \cos^5 x \sin^4 x dx$ put sinx = t $\Rightarrow$ cosx dx = dt

$\Rightarrow$ $I = \int (1-t^2)^2 . t^4 . dt = \int (t^4 - 2t^2 + 1)\ t^4\, dt = \int (t^8 - 2t^6 + t^4)\ dt$

$= \frac{t^9}{9} - \frac{2t^7}{7} + \frac{t^5}{5} + C$, where t = sinx

Example # 41 : Evaluate : $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx$

Solution : We have ,

$I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx = \int \frac{1}{\cos^{4/3} x \sin^{8/3} x} dx = \int \cos^{-4/3} x \quad \sin^{-8/3} \ xdx$

divide N^r and D^r by $\cos^4 x$

$= \int \frac{\sec^4 x}{\tan^{8/3} x} dx = \int \frac{(1+\tan^2 x)}{\tan^{8/3} x} \sec^2 xdx = \int \frac{1+\tan^2 x}{\tan^{8/3} x} d(\tan x) = \int \frac{1+t^2}{t^{8/3}} dt$ where t = tan x

$= \int (t^{-8/3} + t^{-2/3}) dt = \frac{-3}{5} t^{-5/3} + 3t^{1/3} + C = \frac{-3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C$

Example # 42 : Evaluate : $\int \sin^4 x \cos^2 x dx$

Solution : $\int \sin^4 x \cos^2 x dx = \frac{1}{8} \int 4 \sin^2 x \cos^2 x . 2 \sin^2 xdx = \frac{1}{8} \int \sin^2 2x \ (1 - \cos 2x) dx$

$= \frac{1}{8} \int \sin^2 2xdx - \frac{1}{8} \int \sin^2 2x \ \cos 2x \ dx = \frac{1}{16} \int (1 - \cos 4x) \ dx - \frac{1}{48} \ (\sin 2x)^3$

$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{1}{48} (\sin 2x)^3 + C$

Reduction formula of $\int \tan^n x \ dx$, $\int \cot^n x \ dx$, $\int \sec^n x \ dx$, $\int \operatorname{cosec}^n x \ dx$

1. $I_n = \int \tan^n x \ dx = \int \tan^2 x \ \tan^{n-2} x \ dx = \int (\sec^2 x - 1) \ \tan^{n-2} x \ dx$

$\Rightarrow \quad I_n = \int \sec^2 x \tan^{n-2} x \ dx - I_{n-2} \Rightarrow \quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

2. $I_n = \int \cot^n x \ dx = \int \cot^2 x \ . \ \cot^{n-2} x \ dx = \int (\operatorname{cosec}^2 x - 1) \cot^{n-2} x \ dx$

$\Rightarrow I_n = \int \operatorname{cosec}^2 x \cot^{n-2} x \ dx - I_{n-2} \quad \Rightarrow \quad I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

3. $I_n = \int \sec^n x \ dx = \int \sec^2 x \ \sec^{n-2} x \ dx$

$\Rightarrow \quad I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2) \ \sec^{n-3} x . \sec x \tan x \ dx.$

$\Rightarrow \quad I_n = \tan x \sec^{n-2} x - (n-2) (\sec^2 x - 1) \sec^{n-2} x \ dx$

$\Rightarrow \quad (n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2} \quad \Rightarrow I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

4. $I_n = \int \operatorname{cosec}^n x \ dx = \int \operatorname{cosec}^2 x \ \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad I_n = -\cot x \operatorname{cosec}^{n-2} x + \int (\cot x)(n-2) \ (-\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot x) \ dx$

$\Rightarrow \quad -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \cot^2 x \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int (\operatorname{cosec}^2 x - 1) \ \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad (n-1) I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$

$\Rightarrow \quad I_n = \frac{\cot x \operatorname{cosec}^{n-2} x}{-(n-1)} + \frac{n-2}{n-1} I_{n-2}$

Example # 43 : Obtain the reduction formula for $\int \cos^n x dx$

Solution : Let $I_n = \int \cos^n x dx$

$I_n = \int \cos x \ (\cos x)^{n-1} dx$

$\quad\quad\quad\quad \text{II} \quad\quad \text{I}$

$I_n = (\sin x)(\cos x)^{n-1} - \int (n-1)(\cos x)^{n-2}(-\sin x)\sin x dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1) \int (\cos x)^{n-2}(1-\cos^2 x)dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1) \int (\cos x)^{n-2}dx - (n-1)\int (\cos x)^n dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1)\ I_{n-2} - (n-1)\ I_n$

$I_n + (n-1)I_n = (\sin x)(\cos x)^{n-1} + (n-1)I_{n-2}$

$$I_n = \frac{(\sin x)(\cos x)^{n-1}}{n} + \frac{(n-1)}{n} I_{n-2}\ ,\ n \geq 2$$

Self Practice Problems :

(24) Deduce the reduction formula for $I_n = \int \dfrac{dx}{(1+x^4)^n}$ and Hence evaluate $I_2 = \int \dfrac{dx}{(1+x^4)^2}$.

(25) If $I_{m,n} = \int (\sin x)^m (\cos x)^n dx$ then prove that

$$I_{m,n} = \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n} \cdot I_{m,n-2}$$

Ans. (24) $I_n = \dfrac{x}{4(n-1)(1+x^4)^{n-1}} + \dfrac{4n-5}{4(n-1)} I_{n-1}$

$$I_2 = \frac{x}{4\ (1+x^4)} + \left(\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}} \right) - \frac{1}{4\sqrt{2}}\ \ell n \left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right) \right) + C$$

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Integration using Standard Integral :

A-1. Integrate with respect to x :

(i) $(2x + 3)^5$ (ii) sin 2x (iii) $\sec^2 (4x + 5)$

(iv) sec (3x + 2) (v) tan (2x + 1) (vi) 2^{3x+4}

(vii) $\dfrac{1}{2x+1}$ (viii) $e^{4x + 5}$

A-2. Integrate with respect to x :

(i) $\sin^2x$ (ii) $\cos^3x$ (iii) sin 2x cos 3x

(iv) $4\sin x \cos \dfrac{x}{2} \cos \dfrac{3x}{2}$ (v) $\dfrac{1}{\sqrt{x+3} - \sqrt{x+2}}$

Section (B) : Integration using Substitution :

B-1. Integrate with respect to x :

(i) $x \sin x^2$ (ii) $\dfrac{x}{x^2+1}$ (iii) $\sec^2 x \tan x$ (iv) $\dfrac{e^x+1}{e^x+x}$

(v) $\dfrac{1-\sin x}{x+\cos x}$ (vi) $\dfrac{e^{2x}}{e^{2x}-2}$ (vii) $\dfrac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x}$ (viii) $\dfrac{\sec x}{\ell n \ (\sec x + \tan x)}$

(ix) $\dfrac{x}{\sqrt{x+2}}$ (x) $\left(e^x + \dfrac{1}{e^x}\right)^2$ (xi) $(e^x + 1)^2 e^x$ (xii)✍ $\dfrac{1}{x\left(x^5+1\right)}$

(xiii)✍ $\dfrac{1}{x^5(1+x^5)^{\frac{1}{5}}}$ (xiv)✍ $\dfrac{\sqrt{x^2-8}}{x^4}$

B-2. Find the value of $\displaystyle\int \frac{d(x^2+1)}{\sqrt{(x^2+2)}}$.

B-3.✍ Evaluate the following :

(i) $\displaystyle\int \left(\frac{x\cos x - \sin x}{x \sin x}\right) dx$ (ii) $\displaystyle\int \left(\frac{\frac{x}{x+1} - \ln(x+1)}{x(\ln(x+1))}\right) dx$

Section (C) : Integration by parts :

C-1. Integrate with respect to x :

(i) $x\, \ell n\, x$ (ii) $x \sin^2 x$ (iii) $x \tan^{-1} x$ (iv) $\ell n\, x$

(v)✍ $\sec^3 x$ (vi) $2x^3\ e^{x^2}$ (vii) $\sin^{-1}\sqrt{x}$ (viii) $\dfrac{x^2 \tan^{-1} x}{1+x^2}$

(ix) $e^x \sin x$ (x) $e^x (\sec^2 x + \tan x)$

C-2. Find the antiderivative of $f(x) = \ln (\ln x) + (\ln x)^{-2}$ whose graph passes through (e, e).

Section (D) : Algebraic integral :

D-1. Integrate with respect to x :

(i) $\dfrac{1}{x^2+4}$ (ii) $\dfrac{1}{x^2+5}$ (iii) $\dfrac{1}{x^2+2x+5}$

(iv) $\dfrac{2x+1}{x^2+3x+4}$ (v) $\dfrac{x^3-1}{x^3+x}$ (vi) $\dfrac{1}{\sqrt{x^2-4}}$

(vii) $\sqrt{x^2+4}$ (viii) $\sqrt{x^2+2x+5}$ (ix) $(x-1)\sqrt{1-x-x^2}$

(x) $x^5\sqrt{a^3+x^3}$

D-2. Integrate with respect to x :

(i) $\dfrac{1}{(x+1)(x+2)}$ (ii) $\dfrac{1}{(x^2+1)(x+3)}$

(iii) $\dfrac{3x+2}{(x+1)^2(x+2)}$ (iv) $\dfrac{1}{(x+1)(x+2)(x+3)}$

D-3. Integrate with respect to x :

(i) $\dfrac{1}{x^4+x^2+1}$ (ii) $\dfrac{1+x^2}{1+x^4}$ (iii) $\dfrac{1-x^2}{1-x^2+x^4}$

D-4. Integrate with respect to x :

(i) $\dfrac{1}{(x+1)\sqrt{x+2}}$ (ii) $\dfrac{1}{(x^2-4)\sqrt{x+1}}$

(iii) $\dfrac{1}{(x+1)\sqrt{x^2+2}}$ (iv) $\dfrac{1}{(x^2+1)\sqrt{x^2+2}}$

D-5. **Evaluate the following :**

(i) $\displaystyle\int\sqrt{\left(\frac{1+x}{x}\right)}\,dx$ (ii) $\displaystyle\int\sqrt{\left(\frac{x-1}{x+1}\right)}\,dx$ (iii) $\displaystyle\int\left(\frac{x\sqrt{1+x}}{\sqrt{1-x}}\right)dx$

Section (E) : Integration of trigonometric functions :

E-1. Integrate with respect to x :

(i) $\dfrac{1}{2+\cos x}$ (ii) $\dfrac{1}{2-\cos x}$ (iii) $\dfrac{2\sin x+2\cos x}{3\cos x+2\sin x}$

(iv) $\dfrac{1}{1+\sin x+\cos x}$ (v) $\dfrac{1}{2+\sin^2 x}$ (vi) $\dfrac{\operatorname{cosec}^2 x.\sin x}{(\sin x - \cos x)}$

(vii) $\dfrac{\sin^4 x}{\cos^2 x}$

E-2. Evaluate the following

(i) $\displaystyle\int\left(\frac{\sin x+\cos x}{9+16\sin 2x}\right)dx$ (ii) $\displaystyle\int\left(\frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}\right)dx$

E-3. If $\displaystyle\int\sqrt{\frac{\cos^3 x}{\sin^{11} x}}\,dx = -2\left(A\tan^{\frac{-9}{2}}x + B\tan^{\frac{-5}{2}}x\right) + C$, then find A and B.

Section (F) : Reduction formulae

F-1. If $I_n = \int \frac{1}{(x^2+a^2)^n}dx$ then prove that $I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}$

F-2. If $I_n = \int x^n (a-x)^{1/2}dx$ then prove that $I_n = \frac{2an}{2n+3} I_{n-1} - \frac{2x^n(a-x)^{3/2}}{2n+3}$

PART - II : ONLY ONE OPTION CORRECT TYPE

*** In each question C is arbitrary constant**

Section (A) : Integration using Standard Integral :

A-1. Integrate with respect to x : $\sqrt{x+1}$

(A) $\frac{(x+1)^{3/2}}{2}$ + C (B) $\frac{3(x+1)^{3/2}}{2}$ + C (C) $\frac{(x+1)^{3/2}}{3}$ + C (D) $\frac{2(x+1)^{3/2}}{3}$ + C

A-2 Integrate with respect to x : $\frac{1}{\sqrt{2x+1}}$

(A) $\sqrt{2x+1}$ + C (B) $(2x+1)^{3/2}$ + C (C) $-\sqrt{2x+1}$ + C (D) $\frac{1}{(2x+1)^{3/2}}$ + C

A-3. If $\int \frac{1}{1+\sin x} dx = \tan\left(\frac{x}{2}+a\right)$ + C, then

(A) $a = -\frac{\pi}{4}$, $C \in R$ (B) $a = \frac{\pi}{4}$, $C \in R$ (C) $a = \frac{5\pi}{4}$, $C \in R$ (D) $a = \frac{\pi}{3}$, $C \in R$

A-4. If $\int (\sin 2x - \cos 2x)\, dx = \frac{1}{\sqrt{2}} \sin(2x-a) + C$, then

(A) $a = \frac{5\pi}{4}$, $C \in R$ (B) $a = -\frac{5\pi}{4}$, $C \in R$ (C) $a = \frac{\pi}{4}$, $C \in R$ (D) $a = \frac{\pi}{2}$, $C \in R$

A-5. The value of $\int \frac{\cos 2x}{\cos x} dx$ is equal to

(A) 2 sin x – ℓn |sec x + tan x| + C (B) 2 sin x – ℓn |sec x – tan x| + C
(C) 2 sin x + ℓn |sec x + tan x| + C (D) sin x – ℓn |sec x – tan x| + C

A-6. If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx$ = A cos 4x + B; where A & B are constants, then

(A) A = – 1/4 & B may have any value (B) A = – 1/8 & B may have any value
(C) A = – 1/2 & B = – 1/4 (D) A = B = 1/2

Section (B) : Integration using Substitution :

B-1. The value of $\int \frac{a^{\sqrt{x}}}{\sqrt{x}} dx$ is equal to

(A) $\frac{a^{\sqrt{x}}}{\sqrt{x}}$ + C (B) $\frac{2a^{\sqrt{x}}}{\ell n\ a}$ + C (C) $2a^{\sqrt{x}} . \ell n\ a$ + C (D) $2a^{\sqrt{x}}$ + C

B-2. The value of $\int 5^{5^{5^x}} . 5^{5^x} . 5^x \, dx$ is equal to

(A) $\frac{5^{5^{5^x}}}{(\ell n \ 5)^3} + C$ (B) $5^{5^{5^x}} (\ell n\ 5)^3 + C$ (C) $\frac{5^{5^{5^x}}}{(\ell n \ 5)^3} + C$ (D) $\frac{5^{5^{5^x}}}{(\ell n\ 5)^2} + C$

B-3. The value of $\int \frac{\sqrt{\tan x}}{\sin x \ \cos x} dx$ is equal to

(A) $2\sqrt{\tan x} + C$ (B) $2\sqrt{\cot x} + C$ (C) $\frac{\sqrt{\tan x}}{2} + C$ (D) $\sqrt{\tan x} + C$

B-4. If $\int \frac{2^x}{\sqrt{1-4^x}} dx = K \sin^{-1}(2^x) + C$, then the value of K is equal to

(A) $\ell n\ 2$ (B) $\frac{1}{2} \ell n\ 2$ (C) $\frac{1}{2}$ (D) $\frac{1}{\ell n \ 2}$

B-5. If $y = \int \frac{dx}{(1+x^2)^{3/2}}$ and y = 0 when x = 0, then value of y when x = 1, is:

(A) $\sqrt{\frac{2}{3}}$ (B) $\sqrt{2}$ (C) $3\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

B-6. The value of $\int \tan^3 2x \sec 2x \, dx$ is equal to :

(A) $\frac{1}{3}\sec^3 2x - \frac{1}{2}\sec 2x + C$ (B) $-\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$

(C) $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$ (D) $\frac{1}{3}\sec^3 2x + \frac{1}{2}\sec 2x + C$

B-7. If $\int x^{13/2} . (1+x^{5/2})^{1/2} dx = P(1 + x^{5/2})^{7/2} + Q(1 + x^{5/2})^{5/2} + R(1 + x^{5/2})^{3/2} + C$, then P,Q and R are

(A) $P = \frac{4}{35}$, $Q = -\frac{8}{25}$, $R = \frac{4}{15}$ (B) $P = \frac{4}{35}$, $Q = \frac{8}{25}$, $R = \frac{4}{15}$

(C) $P = -\frac{4}{35}$, $Q = -\frac{8}{25}$, $R = \frac{4}{15}$ (D) $P = \frac{4}{35}$, $Q = -\frac{8}{25}$, $R = -\frac{4}{15}$

B-8. The value of $\int \frac{1-x^7}{x(1+x^7)} dx$ is equal to

(A) $\ell n|x| + \frac{2}{7}\ell n|1+x^7| + C$ (B) $\ell n|x| - \frac{2}{7}\ell n|1-x^7| + C$

(C) $\ell n|x| - \frac{2}{7}\ell n|1+x^7| + C$ (D) $\ell n|x| + \frac{2}{7}\ell n|1-x^7| + C$

Section (C) : Integration by parts :

C-1. The value of $\int (x-1)\ e^{-x} dx$ is equal to

(A) $-xe^x + C$ (B) $xe^x + C$ (C) $-xe^{-x} + C$ (D) $xe^{-x} + C$

C-2. The value of $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to

(A) $x\,e^{\tan^{-1}x}+C$ (B) $x^2\,e^{\tan^{-1}x}+C$ (C) $\frac{1}{x}\,e^{\tan^{-1}x}+C$ (D) $x\,e^{\cot^{-1}x}+C$

C-3. The value of $\int [f(x)g''(x)-f''(x)g(x)]\,dx$ is equal to

(A) $\frac{f(x)}{g'(x)}+C$ (B) $f'(x)\,g(x)-f(x)\,g'(x)+C$

(C) $f(x)\,g'(x)-f'(x)\,g(x)+C$ (D) $f(x)\,g'(x)+f'(x)\,g'(x)+C$

C-4. $\int \frac{x\,\ell n\,x}{(x^2-1)^{3/2}}\,dx$ equals

(A) $\text{arc}\sec x-\frac{\ell n\,x}{\sqrt{x^2-1}}+C$ (B) $\sec^{-1}x+\frac{\ell n\,x}{\sqrt{x^2-1}}+C$

(C) $\cos^{-1}x-\frac{\ell n\,x}{\sqrt{x^2-1}}+C$ (D) $\sec x-\frac{\ell n\,x}{\sqrt{x^2-1}}+C$

C-5. The value of $\int (x\,e^{\ell n \sin x}-\cos x)\,dx$ is equal to:

(A) $x\cos x+C$ (B) $\sin x-x\cos x+C$ (C) $-e^{\ell n x}\cos x+C$ (D) $\sin x+x\cos x+C$

Section (D) : Algebraic integral :

D-1. The value of $\int \frac{dx}{x^2+x+1}$ is equal to

(A) $\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$ (B) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

(C) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$ (D) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

D-2. The value of $\int \frac{1}{x^2(x^4+1)^{3/4}}\,dx$ is equal to

(A) $\left(1+\frac{1}{x^4}\right)^{1/4}+C$ (B) $(x^4+1)^{1/4}+C$ (C) $\left(1-\frac{1}{x^4}\right)^{1/4}+C$ (D) $-\left(1+\frac{1}{x^4}\right)^{1/4}+C$

D-3. The value of $\int \frac{dx}{x\sqrt{1-x^3}}$ is equal to

(A) $\frac{1}{3}\,\ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C$ (B) $\frac{1}{3}\,\ell n\left|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right|+C$

(C) $\frac{1}{3}\,\ell n\left|\frac{1}{\sqrt{1-x^3}}\right|+C$ (D) $\frac{1}{3}\,\ell n\,|1-x^3|+C$

D-4. The value of $\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx$ is equal to

(A) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right)-\sec^{-1}(e^x)+C$ (B) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right)+\sec^{-1}(e^x)+C$

(C) $\ell n\left(e^x-\sqrt{e^{2x}-1}\right)-\sec^{-1}(e^x)+C$ (D) $\ell n\left(e^x+\sqrt{e^{2x}-1}\right)-\sin^{-1}(e^x)+C$

D-5. If $\int \frac{dx}{x^4 + x^3} = \frac{A}{x^2} + \frac{B}{x} + \ell n \left|\frac{x}{x+1}\right| + C$, then

(A) $A = \frac{1}{2}$, $B = 1$ (B) $A = 1$, $B = -\frac{1}{2}$ (C) $A = -\frac{1}{2}$, $B = 1$ (D) $A = -\frac{1}{2}$, $B = \frac{1}{2}$

Section (E) : Integration of trigonometric functions :

E-1. The value of $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\, dx$ is equal to

(A) $\frac{-1}{\sin x + \cos x} + C$ (B) $\ell n\,(\sin x + \cos x) + C$

(C) $\ell n\,(\sin x - \cos x) + C$ (D) $\ell n\,(\sin x + \cos x)^2 + C$

E-2 The value of $\int [1 + \tan x\ .\ \tan(x + \alpha)]\, dx$ is equal to

(A) $\cos \alpha\ .\ \ell n \left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$ (B) $\tan \alpha\ .\ \ell n \left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$

(C) $\cot \alpha\ .\ \ell n \left|\frac{\sec(x+\alpha)}{\sec x}\right| + C$ (D) $\cot \alpha\ .\ \ell n \left|\frac{\cos(x+\alpha)}{\cos x}\right| + C$

E-3✍ The value of $\int \sqrt{\sec x - 1}\, dx$ is equal to

(A) $2\, \ell n \left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$ (B) $\ell n \left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

(C) $-\,2\, \ell n \left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$ (D) $-\,2\, \ell n \left(\sin\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

E-4. The value of $\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$ is equal to

(A) $\sqrt{2}\left(\sqrt{\cos x} + \frac{1}{5}\tan^{5/2} x\right) + C$ (B) $\sqrt{2}\left(\sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x\right) + C$

(C) $\sqrt{2}\left(\sqrt{\tan x} - \frac{1}{5}\tan^{5/2} x\right) + C$ (D) $\sqrt{2}\left(\sqrt{\cos x} - \frac{1}{5}\tan^{5/2} x\right) + C$

E-5. Antiderivative of $\frac{\sin^2 x}{1 + \sin^2 x}$ w.r.t. x is :

(A) $x - \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}\tan x\right) + C$ (B) $x - \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$

(C) $x - \sqrt{2} \arctan\left(\sqrt{2}\tan x\right) + C$ (D) $x - \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$

E-6. Integrate $\frac{1}{1 - \cot x}$

(A) $\frac{1}{2}\log|\sin x - \cos x| + \frac{1}{2}x + C$ (B) $\frac{1}{2}\log|\sin x + \cos x| + \frac{1}{2}x + C$

(C) $\frac{1}{2}\log|\sin x + \cos x| - \frac{1}{2}x + C$ (D) $\frac{1}{2}\log|\sin x - \cos x| - \frac{1}{2}x + C$

E-7. $I = \int \frac{dx}{\sin x + \sec x}$ is equal to

(A) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\text{sinx - cosx}}{\sqrt{3}-(\text{sinx - cosx})}\right| + \tan^{-1}(\text{sinx + cosx}) + C$

(B) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\text{sinx - cosx}}{\sqrt{3}-(\text{sinx - cosx})}\right| + \tan^{-1}(\text{sinx - cosx}) + C$

(C) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\text{sinx + cosx}}{\sqrt{3}-(\text{sinx - cosx})}\right| + \tan^{-1}(\text{sinx + cosx}) + C$

(D) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\text{sinx - cosx}}{\sqrt{3}-(\text{sinx + cosx})}\right| + \tan^{-1}(\text{sinx + cosx}) + C$

Section (F) : Reduction formulae

F-1. If $I_n = \int \frac{e^x}{x^n}dx$ and $I_n = \frac{-e^x}{k_1x^{n-1}} + \frac{1}{k_2-1}I_{n-1}$, then $(k_2 - k_1)$ is equal to :

(A) 0 (B) 1 (C) 2 (D) 3

F-2. If $I_n = \int \cot^n x \ dx$ and $I_0 + I_1 + 2(I_2 + ...+ I_8) + I_9 + I_{10} = A\left(u + \frac{u^2}{2} + ... + \frac{u^9}{9}\right) + C$, where u = cotx and C is an arbitrary constant, then

(A) A = 2 (B) A = – 1 (C) A = 1 (D) A is dependent on x

PART - III : MATCH THE COLUMN

1. **Column – I** **Column – II**

(A) If $F(x) = \int \frac{x+\sin x}{1+\cos x}dx$ and F(0) = 0, then the value of $F(\pi/2)$ is (p) $\frac{\pi}{2}$

(B) Let $F(x) = \int e^{\sin^{-1}x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right)dx$ and F(0) = 1, (q) $\frac{\pi}{3}$

If $F(1/2) = \frac{k\sqrt{3}\ e^{\pi/6}}{\pi}$, then the value of k is

(C) Let $F(x) = \int \frac{dx}{(x^2+1)\ (x^2+9)}$ and F(0) = 0, (r) $\frac{\pi}{4}$

if $F(\sqrt{3}) = \frac{5}{36}k$, then the value of k is

(D) Let $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$ and F(0) = 0 (s) π

if $F(\pi/4) = \frac{2k}{\pi}$, then the value of k is

2. If $I = \int \frac{dx}{a+b \quad \cos x}$, where a, b > 0 and a + b = u, a – b = v, then match the following column

Column – I		**Column – II**	
(A)	v = 0	(p)	$I = \frac{1}{\sqrt{uv}} \quad \ell n \left\| \frac{\sqrt{u}+\sqrt{v} \quad \tan\frac{x}{2}}{\sqrt{u}-\sqrt{v} \quad \tan\frac{x}{2}} \right\| + C$
(B)	v > 0	(q)	$I = \frac{2}{\sqrt{uv}} \tan^{-1}\left(\sqrt{\frac{v}{u}} \quad \tan \ \frac{x}{2}\right) + C$
(C)	v < 0	(r)	$I = \frac{1}{\sqrt{-u \quad v}} \quad \ell n \left\| \frac{\sqrt{u}+\sqrt{-v} \quad \tan\frac{x}{2}}{\sqrt{u}-\sqrt{-v} \quad \tan\frac{x}{2}} \right\| + C$
		(s)	$\frac{2}{u} \tan \frac{x}{2} + C$

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

*** In each question C is arbitrary constant**

1.✍ Value of $\int \frac{1}{\sin(x-a)\cos(x-b)} dx$ is equal to

(A) $\frac{1}{\cos(a-b)} \ \ell n \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$ (B) $\frac{1}{\cos(a-b)} \ \ell n \left| \frac{\cos(x-b)}{\sin(x-a)} \right| + C$

(C) $\frac{1}{\sin(a-b)} \ \ell n \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$ (D) $\frac{1}{\sin(a+b)} \ \ell n \left| \frac{\cos(x-a)}{\sin(x-b)} \right| + C$

2. $\int$ tan x. tan 2x. tan 3x dx =

(A) $-\ell n|\cos x| - \frac{1}{2}\ell n|\sec 2x| + \frac{1}{3}\ell n|\sec 3x| + C$

(B) $-\ell n|\sec x| - \frac{1}{2}\ell n|\sec 2x| + \frac{1}{3}\ell n|\sec \ 3x| + C$

(C) $\ell n|\cos x| + \ell n|\cos 2x| + \ell n|\cos 3x| + C$

(D) $\ell n|\sec x| + \frac{1}{2}\ell n|\sec 2x| + \frac{1}{3}|\sec 3x| + C$

3. The value of $\int$ (sin x. cos x. cos 2x. cos 4x. cos 8x. cos 16 x) dx is equal to

(A) $\frac{\sin 16x}{1024} + C$ (B) $-\frac{\cos 32x}{1024} + C$ (C) $\frac{\cos 32x}{1096} + C$ (D) $-\frac{\cos 32x}{1096} + C$

4. $\int x \sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}\, dx =$

(A) $\frac{1}{2} a^2 \cos^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 + x^4} + C$

(B) $\frac{1}{2} \sin^{-1}\left(\frac{x^2}{a^2}\right) + \sqrt{a^4 + x^4} + C$

(C) $\frac{1}{2} a^2 \sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + C$

(D) $\frac{1}{2} \cos^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + C$

5. The value of $\int \sqrt{\dfrac{x-1}{x+1}} \cdot \dfrac{1}{x^2}\, dx$ is equal to

(A) $\sin^{-1}\frac{1}{x} + \frac{\sqrt{x^2-1}}{x} + C$

(B) $\frac{\sqrt{x^2-1}}{x} + \cos^{-1}\frac{1}{x} + C$

(C) $\sec^{-1} x - \frac{\sqrt{x^2-1}}{x} + C$

(D) $\tan^{-1}\sqrt{x^2+1} - \frac{\sqrt{x^2-1}}{x} + C$

6. The value of $\int \dfrac{\ell n|x|}{x\sqrt{1+\ell n|x|}}\, dx$ equals :

(A) $\frac{2}{3}\sqrt{1+\ell n|x|}\,(\ell n|x| - 2) + C$

(B) $\frac{2}{3}\sqrt{1+\ell n|x|}\,(\ell n|x| + 2) + C$

(C) $\frac{1}{3}\sqrt{1+\ell n|x|}\,(\ell n|x| - 2) + C$

(D) $2\sqrt{1+\ell n|x|}\,(3\,\ell n|x| - 2) + C$

7. The value of $\int \dfrac{1}{[(x-1)^3(x+2)^5]^{1/4}}\, dx$ is equal to

(A) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

(B) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + C$

(C) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

(D) $\frac{1}{3}\left(\frac{x+1}{x-1}\right)^{1/4} + C$

8. The value of $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ is equal to

(A) $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} + \cos^{-1}\left(\sqrt{x}\right) + C$

(B) $\sqrt{x}\sqrt{1-x} + 2\sqrt{1-x} + \cos^{-1}\left(\sqrt{x}\right) + C$

(C) $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} - \cos^{-1}\left(\sqrt{x}\right) + C$

(D) $\sqrt{x}\sqrt{1-x} + 2\sqrt{1-x} - \cos^{-1}\left(\sqrt{x}\right) + C$

9. $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\, dx$ is equal to

(A) $(a + x) \text{ arc tan}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

(B) $(a + x) \text{ arc tan}\sqrt{\frac{x}{a}} + \sqrt{ax} + C$

(C) $(a - x) \text{ arc tan}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

(D) $(a + x) \text{ arc cot}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

10. The value of $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\left(x + \sqrt{x}\right) dx$ is equal to :

(A) $2e^{\sqrt{x}}\,[\sqrt{x} - x + 1\,] + C$

(B) $2e^{\sqrt{x}}\,[x - 2\sqrt{x} + 1\,] + C$

(C) $2e^{\sqrt{x}}\,[x - \sqrt{x} + 1\,] + C$

(D) $2e^{\sqrt{x}}\,(x + \sqrt{x} + 1\,) + C$

11. If I = $\int \frac{2}{x}(x^{\ell nx})(\ell nx)^3 dx = Ax^{\ell nx}(\ell nx)^2 - B\,x^{\ell nx} + C$, then $\frac{A}{B}$ is equal to :
(A) 1 (B) –1 (C) 2 (D) –2

12. The value of $\int e^{\tan\theta}(\sec\theta - \sin\theta)\,d\theta$ is equal to
(A) $-e^{\tan\theta}\sin\theta + C$ (B) $e^{\tan\theta}\sin\theta + C$ (C) $e^{\tan\theta}\sec\theta + C$ (D) $e^{\tan\theta}\cos\theta + C$

13. The value of $\int\left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}dx$ is equal to:
(A) $x\,\ell n(1 + \sin x) + C$ (B) $\ell n(1 + \sin x) + C$
(C) $-x\,\ell n(1 + \sin x) + C$ (D) $\ell n(1 - \sin x) + C$

14. The value of $\int x.\frac{\ell n\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals:
(A) $\sqrt{1+x^2}\;\ell n\left(x+\sqrt{1+x^2}\right) - x + C$ (B) $\frac{x}{2}.\ell n^2\left(x+\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}} + C$
(C) $\frac{x}{2}.\ell n^2\left(x+\sqrt{1+x^2}\right) + \frac{x}{\sqrt{1+x^2}} + C$ (D) $\sqrt{1+x^2}\;\ell n\left(x+\sqrt{1+x^2}\right) + x + C$

15. If $\int\frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}\;f(x) + A\;\ell n\left|x+\sqrt{x^2+1}\right| + C$, then
(A) $f(x) = \tan^{-1}x$, A = –1 (B) $f(x) = \tan^{-1}x$, A = 1
(C) $f(x) = 2\tan^{-1}x$, A = –1 (D) $f(x) = 2\tan^{-1}x$, A = 1

16. $\int\frac{x+\sqrt{x+1}}{x+2}dx$ is equal to
(A) $(x+1) - 2\sqrt{x+1} + 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+1} + C$
(B) $(x+1) + 2\sqrt{x+2} - 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+2} + C$
(C) $(x+1) + 2\sqrt{x+1} - 2\,\ell n|x+2| - 2\tan^{-1}\sqrt{x+1} + C$
(D) $(x+1) + 2\sqrt{x+2} - 2\,\ell n|x+1| + 2\tan^{-1}\sqrt{x+2} + C$

17. The value of $\int\sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}\,dx$, where $0 < \alpha < x < \pi$, is equal to
(A) $2\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$ (B) $\sqrt{2}\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$
(C) $2\sqrt{2}\,\ell n\left(\cos\frac{\alpha}{2} - \cos\frac{x}{2}\right) + C$ (D) $-2\sin^{-1}\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right) + C$

18. If I = $\int\frac{\sin x + \sin^3 x}{\cos 2x}dx = A\cos x + B\,\ell n|f(x)| + C$, then
(A) $A = \frac{1}{4}$, $B = \frac{-1}{\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$ (B) $A = -\frac{1}{2}$, $B = \frac{-3}{4\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$
(C) $A = -\frac{1}{2}$, $B = \frac{3}{\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$ (D) $A = \frac{1}{2}$, $B = \frac{-3}{4\sqrt{2}}$, $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

19. The value of $\int \frac{1}{\cos^6 x + \sin^6 x} dx$ is equal to

(A) $\tan^{-1}(\tan x + \cot x) + C$
(B) $-\tan^{-1}(\tan x + \cot x) + C$
(C) $\tan^{-1}(\tan x - \cot x) + C$
(D) $-\tan^{-1}(\tan x - \cot x) + C$

20. Consider the following statements :

S_1 : The antiderivative of every even function is an odd function.

S_2 : Primitive of $\frac{3x^4 - 1}{(x^4 + x + 1)^2}$ w.r.t. x is $\frac{x}{x^4 + x + 1}$ + C.

S_3 : $\int \frac{1}{\sqrt{\sin^3 x \cos x}} dx = \frac{-2}{\sqrt{\tan x}}$ + C.

S_4 : The value of $\int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\right)$ dx is equal to $-2\sqrt{a^2 - x^2}$ + C

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) FFTT (B) TTTT (C) FFFF (D) TFTF

21. If $I_n = \int (\sin x + \cos x)^n dx$, snd $I_n = \frac{1}{n}(\sin x + \cos x)^{n-1}(\sin x - \cos x) + \frac{2k}{n} I_{n-2}$ then k =

(A) (n+ 1) (B) (n – 1) (C) (2n+ 1) (A) (2n– 1)

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

*** In each question C is arbitrary constant**

1. If f(x) = $\int \frac{2\sin x - \sin 2x}{x^3}$ dx, where $x \neq 0$, then $\underset{x \to 0}{\text{Limit}}\ f'(x)$ has the value

2. If $\int \sin^4 x \cos^4 x\, dx = \frac{1}{128}\left[ax - \sin 4x + \frac{1}{8}.\sin 8x\right] + C$ then value of 'a' equal to :

3. Let f(x) be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r. to x. If f(10) = 60 then twice of sum of digits of the value of f(13) is.

4. If $\int \frac{\sqrt{4+x^2}}{x^6} dx = \frac{(a+x^2)^{3/2}.(x^2-b)}{120x^5}$ + C then a + b equals to :

5. If $\int \sqrt{\frac{x}{a^3 - x^3}} dx = \frac{d}{b}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)$ + C, (where b & d are coprime integer) then b + d equals to.

6. If $\int \frac{x\ dx}{\sqrt{1 + x^2 + \sqrt{(1+x^2)^3}}} = k\sqrt{1 + \sqrt{1 + x^2}}$ + C then k equals to :

7. If $\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x} dx = e^{\sin x} f(x) + C$ such that f(0) = –1 then $\frac{\pi}{3} - f\left(\frac{\pi}{3}\right)$ is equal to :

8. Let g(x) = $\int \frac{1+2\cos x}{(\cos x + 2)^2}$ dx and g(0) = 0 then value of 32 $g\left(\frac{\pi}{2}\right)$ is.

9. If $f(x) = \sqrt{x-1}$; $g(x) = e^x$ and $\int fog(x)dx = Afog(x) + B\tan^{-1}(fog(x)) + C$ then $A^3 + B^2$ equals

10. If $\int \dfrac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi - 4\sin\phi}\, d\phi = p\,\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + q\tan^{-1}(\sin\phi - r) + C$ then p + q + r equal to :

11. If $\int \dfrac{(x-1)^2}{x^4+x^2+1}dx = \dfrac{1}{\sqrt{a}}\tan^{-1}\left(\dfrac{x^2-1}{x\sqrt{3}}\right) - \dfrac{b}{\sqrt{a}}\tan^{-1}\left(\dfrac{2x^2+1}{\sqrt{3}}\right) + C$ then $a^2 + b^2$ equals to :

12. If $\int \dfrac{1+x\cos x}{x\left(1-x^2e^{2\sin x}\right)}dx = k\,\ell n\sqrt{\dfrac{x^2e^{2\sin x}}{1-x^2e^{2\sin x}}} + C$ then k is equal to :

13. If $\int \dfrac{x^4+1}{x\left(x^2+1\right)^2}dx = A\,\ell n\,|x| + \dfrac{B}{1+x^2} + C$, then A + B equals to :

14. If $\int \dfrac{1}{1-\sin^4 x}dx = \dfrac{1}{a\sqrt{b}}\tan^{-1}\left(\sqrt{a}\tan x\right) + \dfrac{1}{b}\tan x + C$ then $\dfrac{a}{b}$ is equal to :

15. If $\int \dfrac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx = p\sin x - \dfrac{q}{\sin x} - r\tan^{-1}(\sin x) + C$ then p + 2q + r is equal to :

16. If $\int \dfrac{dx}{\sqrt{\sin^3 x\cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + C$, where C is an arbitrary constant of integration, then the values of $a^2 + 9b$ equals to :

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

*** In each question C is arbitrary constant**

1. The value of $\int 2^{mx}.3^{nx}dx$ (when m, n $\in$ N) is equal to :

(A) $\dfrac{2^{mx}+3^{nx}}{m\ell n2 + n\ell n3} + C$ (B) $\dfrac{e^{(m\ell n2 + n\ell n3)x}}{m\ell n2 + n\ell n3} + C$ (C) $\dfrac{2^{mx}.3^{nx}}{\ell n\left(2^m.3^n\right)} + C$ (D) $\dfrac{(mn).2^x.3^x}{m\ell n2 + n\ell n3} + C$

2. If $f\left(\dfrac{1-x}{1+x}\right) = x$ and $g(x) = \int f(x)dx$ then

(A) g(x) is continuous in domain

(B) g(x) is discontinuous at two points in its domain

(C) $\lim\limits_{x\to\infty} g'(x) = -1$

(D) $\int g(x)dx = -\dfrac{x^2}{2} + (2x+1)\,\lambda n\left(\dfrac{1+x}{e}\right) + C$

3. If $= \int \tan^5 x dx = A\tan^4 x - \dfrac{1}{2}\tan^2 x + B\ell n|\sec x| + C$ then

(A) $A = \dfrac{1}{4}$ (B) $A = \dfrac{1}{2}$ (C) B = 1 (D) B = – 1

4. The value of $\int\{1+2\tan x(\tan x+\sec x)\}^{1/2}\,dx$ is equal to

(A) $\ell n\,|\sec x\,(\sec x - \tan x)| + C$
(B) $\ell n\,|\text{cosec}\,x\,(\sec x + \tan x)| + C$
(C) $\ell n\,|\sec x\,(\sec x + \tan x)| + C$
(D) $-\ell n\,|\cos x(\sec x - \tan x)| + C$

5. The value of $\int \dfrac{\ell n\left(\dfrac{x-1}{x+1}\right)}{x^2-1}\,dx$ is equal to

(A) $\dfrac{1}{2}\,\ell n^2\,\dfrac{x-1}{x+1} + C$ (B) $\dfrac{1}{4}\,\ell n^2\,\dfrac{x-1}{x+1} + C$ (C) $\dfrac{1}{2}\,\ell n^2\,\dfrac{x+1}{x-1} + C$ (D) $\dfrac{1}{4}\,\ell n^2\,\dfrac{x+1}{x-1} + C$

6. The value of $\int \dfrac{\ell n\,(\tan x)}{\sin x\,\cos x}\,dx$ is equal to

(A) $\dfrac{1}{2}\,\ell n^2\,(\cot x) + C$
(B) $\dfrac{1}{2}\,\ell n^2\,(\sec x) + C$
(C) $\dfrac{1}{2}\,\ell n^2\,(\sin x\,\sec x) + C$
(D) $\dfrac{1}{2}\,\ell n^2\,(\cos x\,\text{cosec}\,x) + C$

7. The value of $\int \dfrac{\cos^3 x}{\sin^2 x + \sin x}\,dx$ is equal to :

(A) $\ell n\,|\sin x| + \sin x + C$
(B) $\ell n\,|\sin x| - \sin x + C$
(C) $-\,\ell n\,|\text{cosec}\,x| - \sin x + C$
(D) $-\,\ell n\,|\sin x| + \sin x + C$

8. If $\int \dfrac{(x-1)\,dx}{x^2\sqrt{2x^2-2x+1}} = \dfrac{\sqrt{f(x)}}{g(x)} + C$, where f(x) is a quadratic expression and g(x) is a monic linear expression.

(A) $f(x) = 2x^2 - 2x + 1$
(B) $g(x) = x + 1$
(C) $g(x) = x$
(D) $f(x) = 2x^2 - 2x$

9. If $\int e^{3x}\cos 4x\,dx = e^{3x}(A\sin 4x + B\cos 4x) + C$ then :

(A) 4A = 3B (B) 2A = 3B (C) 3A = 4B (D) 4A + 3B = 1

10. $I = \int \dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$ equals to

(A) $-x + \dfrac{2}{\pi}\,(2x-1)\sin^{-1}\sqrt{x} + \dfrac{2}{\pi}\sqrt{x-x^2} + C$

(B) $x - \dfrac{4x}{\pi}\cos^{-1}\sqrt{x} - \dfrac{2}{\pi}\sin^{-1}\sqrt{x} + \dfrac{2}{\pi}\sqrt{x}\,\sqrt{1-x} + C$

(C) $-x + \dfrac{2}{\pi}\,(2x+1)\cos^{-1}\sqrt{x} + \dfrac{2}{\pi}\sqrt{x}\,\sqrt{1-x} + C$

(D) $x - \dfrac{4x}{\pi}\sin^{-1}\sqrt{x} + C$

11. If $\int \dfrac{x^2-x+1}{(1+x^2)^{3/2}}\,e^x dx = e^x f(x) + C$ then

(A) f(x) is a an even function
(B) f(x) is a bounded function
(C) Range of f(x) is (0, 1]
(D) f(x) has two points of exterma.

12. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\,dx = Ax + B\,\ell n\,|9e^{2x} - 4| + C$, then

(A) A + 18B = 16 (B) 18B – A = 19
(C) A –18B = 17 (D) A + 18B = 32

13. The value of $\int \frac{x^2 + \cos^2 x}{1 + x^2} \operatorname{cosec}^2 x\,dx$ is equal to:

(A) $\cot x - \cot^{-1} x + C$ (B) $C - \cot x + \cot^{-1} x$
(C) $-\tan^{-1} x - \frac{\operatorname{cosec} x}{\sec x} + C$ (D) $\frac{1}{\tan^{-1} x} - \cot x + C$

14. The value of $\int \frac{dx}{\sqrt{x - x^2}}$; $\left(x > \frac{1}{2}\right)$ is equal to

(A) $2\sin^{-1}\sqrt{x} + C$ (B) $\sin^{-1}(2x - 1) + C$
(C) $C - 2\cos^{-1}(2x - 1)$ (D) $\cos^{-1} 2\sqrt{x - x^2} + C$

15. $\int \frac{x^3 - 1}{x^3 + x}\,dx$ is equal to

(A) $x - \ell n\,|x| + \ell n\,(x^2 + 1) - \tan^{-1}x + C$
(B) $x - \ell n\,|x| + \frac{1}{2}\,\ell n\,(x^2 + 1) - \tan^{-1}x + C$
(C) $x + \ell n\,|x| + \frac{1}{2}\,\ell n\,(x^2 + 1) + \tan^{-1}x + C$
(D) $x + \ell n\,\sqrt{\frac{x^2 + 1}{x^2}} + \cot^{-1}x + C$

16. The value of $2\int \sin x \,.\, \operatorname{cosec} 4x\,dx$ is equal to

(A) $\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{4}\,\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$ (B) $\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{2}\,\ell n\left|\frac{1+\sin x}{\cos x}\right| + C$
(C) $\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| - \frac{1}{4}\,\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$ (D) $-\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| + \frac{1}{4}\,\ell n\left|\frac{1-\sin x}{1+\sin x}\right| + C$

17. If $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\,dx = p\,f(x) + q\,g(x) + C$, then which of the following may be correct?

(A) $p = 1;\ q = \frac{1}{\sqrt{3}}$; $f(x) = x;\ g(x) = \ell n\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(B) $p = 1;\ q = -\frac{1}{\sqrt{3}}$; $f(x) = x;\ g(x) = \ell n\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(C) $p = 1;\ q = -\frac{2}{\sqrt{3}}$; $f(x) = x;\ g(x) = \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right|$

(D) $p = 1;\ q = -\frac{1}{\sqrt{3}}$; $f(x) = x;\ g(x) = \ell n\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right|$

18. If $\int \frac{dx}{5+4\cos x} = P\tan^{-1}\left(m\tan\frac{x}{2}\right) + C$ then :

(A) P = 2/3 (B) m = 1/3 (C) P = 1/3 (D) m = 2/3

19. The value of $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$ is equal to:

(A) $\cot^{-1}(\cot^2 x) + C$ (B) $-\cot^{-1}(\tan^2 x) + C$
(C) $\tan^{-1}(\tan^2 x) + C$ (D) $-\tan^{-1}(\cos 2x) + C$

PART - IV : COMPREHENSION

Comprehension # 1 (Q.No. 1 to 3)

Let $I_{n,m} = \int \sin^n x \cos^m x.dx$. Then we can relate $I_{n,m}$ with each of the following

(i) $I_{n-2,m}$ (ii) $I_{n+2,m}$ (iii) $I_{n,m-2}$
(iv) $I_{n,m+2}$ (v) $I_{n-2,m+2}$ (vi) $I_{n+2,m-2}$

Suppose we want to establish a relation between $I_{n,m}$ and $I_{n,m-2}$, then we set

$P(x) = \sin^{n+1}x \cos^{m-1}x$ ....(1)

In $I_{n,m}$ and $I_{n,m-2}$ the exponent of cosx is m and m – 2 respectively, the minimum of the two is m – 2, adding 1 to the minimum we get m – 2 + 1 = m – 1. Now choose the exponent m – 1 of cosx in P(x). Similarly choose the exponent of sin x for P (x)

Now differentiating both sides of (1), we get

$P'(x) = (n+1)\sin^n x\cos^m x - (m-1)\sin^{n+2}x\cos^{m-2}x$
$= (n+1)\sin^n x\cos^m x - (m-1)\sin^n x(1-\cos^2 x)\cos^{m-2}x$
$= (n+1)\sin^n x\cos^m x - (m-1)\sin^n x\cos^{m-2}x + (m-1)\sin^n x\cos^m x$
$= (n+m)\sin^n x\cos^m x - (m-1)\sin^n x\cos^{m-2}x$

Now integrating both sides, we get

$\sin^{n+1}x\cos^{m-1}x = (n+m)I_{n,m} - (m-1)I_{n,m-2}$.

Similarly we can establish the other relations.

1. The relation between $I_{4,2}$ and $I_{2,2}$ is

(A) $I_{4,2} = \frac{1}{6}(-\sin^3x\cos^3x + 3I_{2,2})$ (B) $I_{4,2} = \frac{1}{6}(\sin^3x\cos^3x + 3I_{2,2})$
(C) $I_{4,2} = \frac{1}{6}(\sin^3x\cos^3x - 3I_{2,2})$ (D) $I_{4,2} = \frac{1}{6}(-\sin^3x\cos^3x + 2I_{2,2})$

2. The relation between $I_{4,2}$ and $I_{6,2}$ is

(A) $I_{4,2} = \frac{1}{5}(\sin^5x\cos^3x + 8I_{6,2})$ (B) $I_{4,2} = \frac{1}{5}(-\sin^5x\cos^3x + 8I_{6,2})$
(C) $I_{4,2} = \frac{1}{5}(\sin^5x\cos^3x - 8I_{6,2})$ (D) $I_{4,2} = \frac{1}{5}(\sin^5x\cos^3x + 8I_{6,2})$

3. The relation between $I_{4,2}$ and $I_{4,4}$ is

(A) $I_{4,2} = \frac{1}{3}(\sin^5x\cos^3x + 8I_{4,4})$ (B) $I_{4,2} = \frac{1}{3}(-\sin^5x\cos^3x + 8I_{4,4})$
(C) $I_{4,2} = \frac{1}{3}(\sin^5x\cos^3x - 8I_{4,4})$ (D) $I_{4,2} = \frac{1}{3}(\sin^5x\cos^3x + 6I_{4,4})$

Comprehension # 2 (Q. No. 4 to 6)

It is known that

$$\sqrt{\tan x}+\sqrt{\cot x}=\begin{cases}\dfrac{\sqrt{\sin x}}{\sqrt{\cos x}}+\dfrac{\sqrt{\cos x}}{\sqrt{\sin x}} & \text{if } 0<x<\dfrac{\pi}{2}\\[2ex] \dfrac{\sqrt{-\sin x}}{\sqrt{-\cos x}}+\dfrac{\sqrt{-\cos x}}{\sqrt{-\sin x}} & \text{if } \pi<x<\dfrac{3\pi}{2}\end{cases},$$

$$\frac{d}{dx}\left(\sqrt{\tan x}-\sqrt{\cot x}\right)=\frac{1}{2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)(\tan x+\cot x),\ \forall\ x\in\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right)$$

and $\dfrac{d}{dx}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)=\dfrac{1}{2}\left(\sqrt{\tan x}-\sqrt{\cot x}\right)(\tan x+\cot x),\ \forall\ x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\pi,\dfrac{3\pi}{2}\right)$.

4. Value of integral $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\ dx$, where $x\in\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right)$ is

(A) $\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+C$
(B) $\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{2}}\right)+C$
(C) $-\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+C$
(D) $-\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{2}}\right)+C$

5. Value of the integral $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\ dx$, where $x\in\left(0,\frac{\pi}{2}\right)$, is

(A) $\sqrt{2}\sin^{-1}(\cos x-\sin x)+C$
(B) $\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$
(C) $\sqrt{2}\sin^{-1}(\sin x+\cos x)+C$
(D) $-\sqrt{2}\sin^{-1}(\sin x+\cos x)+C$

6. Value of the integral $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\ dx$, where $x\in\left(\pi,\frac{3\pi}{2}\right)$, is

(A) $\sqrt{2}\sin^{-1}(\cos x-\sin x)+C$
(B) $\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$
(C) $\sqrt{2}\sin^{-1}(\sin x+\cos x)+C$
(D) $-\sqrt{2}\sin^{-1}(\sin x+\cos x)+C$

Exercise-3

✍ **Marked questions are recommended for Revision.**

*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1.✍ Integrate, $\displaystyle\int\frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$. **[IIT-JEE 1999, Part-2, (7, 0), 120]**

2. Let $f(x)=\int e^x(x-1)(x-2)\,dx$ then f decreases in the interval : **[IIT-JEE 2000, Scr, (1, 0), 35]**

(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, +\infty)$

3. Evaluate, $\displaystyle\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. **[IIT-JEE 2001, Main, (5, 0), 100]**

4. For any natural number m, evaluate,
$\int \left(x^{3m} + x^{2m} + x^{m}\right)\left(2x^{2m} + 3x^{m} + 6\right)^{1/m} dx,\ x > 0.$ **[IIT-JEE 2002, Main, (5, 0), 60]**

5. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}$ dx is equal to **[IIT-JEE 2006, (3, –1), 184]**

(A) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$ (B) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$

(C) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$ (D) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

6. Let f(x) = $\frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and g(x) = $\underbrace{(f \circ f \circ \dots \circ f)}_{\text{f occurs n times}}$ (x). Then $\int x^{n-2} g(x)\, dx$ equals

[IIT-JEE 2007, Paper-2, (3, – 1), 81]

(A) $\frac{1}{n(n-1)}\left(1+nx^n\right)^{1-\frac{1}{n}} + K$ (B) $\frac{1}{(n-1)}\left(1+nx^n\right)^{1-\frac{1}{n}} + K$

(C) $\frac{1}{n(n+1)}\left(1+nx^n\right)^{1+\frac{1}{n}} + K$ (D) $\frac{1}{(n+1)}\left(1+nx^n\right)^{1+\frac{1}{n}} + K$

7. Let F(x) be an indefinite integral of $\sin^2 x$. **[IIT-JEE 2007, Paper-1, (3, – 1), 81]**
STATEMENT-1 : The function F(x) satisfies $F(x + \pi) = F(x)$ for all real x.
because
STATEMENT-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.
(A) Statement-1 is True, Statement-2 is True ; Statement-2 **is** a correct explanation for Statement-1
(B)Statement-1 is True, Statement-2 is True; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

8. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}$ dx, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}$ dx . Then, for an arbitrary constant C, the value of J – I is equal to : **[IIT-JEE 2008, Paper-2, (3, – 1), 81]**

(A) $\frac{1}{2}\ell n\left|\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right| + C$ (B) $\frac{1}{2}\ell n\left|\frac{e^{2x} + e^{x} + 1}{e^{2x} - e^{x} + 1}\right| + C$

(C) $\frac{1}{2}\ell n\left|\frac{e^{2x} - e^{x} + 1}{e^{2x} + e^{x} + 1}\right| + C$ (D) $\frac{1}{2}\ell n\left|\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right| + C$

9. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}$dx equals (for some arbitrary constant K)

(A) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$ **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(B) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. If the integral $\int \frac{5\tan x}{\tan x - 2}$ dx = x + a ℓn |sin x – 2 cos x| + k, then a is equal to :

[AIEEE-2012, (4, –1)/120]

(1) – 1 (2) – 2 (3) 1 (4) 2

2. If $\int f(x)\ dx = \psi(x)$, then $\int x^5 f(x^3)\ dx$ is equal to **[AIEEE - 2013, (4, –1),360]**

(1) $\frac{1}{3}\left[x^3\psi(x^3) - \int x^2\psi(x^3)\ dx\right] + C$ (2) $\frac{1}{3}x^3\psi(x^3) - 3\int x^3\psi(x^3)\ dx + C$

(3) $\frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\ dx + C$ (4) $\frac{1}{3}\left[x^3\psi(x^3) - \int x^3\psi(x^3)\ dx\right] + C$

3. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$ is equal to : **[JEE(Main) 2014, (4, – 1), 120]**

(1) $(x + 1)\ e^{x+\frac{1}{x}} + c$ (2) $-x\,e^{x+\frac{1}{x}} + c$ (3) $(x - 1)\ e^{x+\frac{1}{x}} + c$ (4) $x\ e^{x+\frac{1}{x}} + c$

4. The integral $\int \frac{dx}{x^2\left(x^4+1\right)^{3/4}}$ equals **[JEE(Main) 2015, (4, – 1), 120]**

(1) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$ (2) $(x^4 + 1)^{1/4} + c$ (3) $-(x^4 + 1)^{1/4} + c$ (4) $-\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$

5. The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3} dx$ is equal to **[JEE(Main) 2016, (4, –1), 120]**

(1) $\frac{x^{10}}{2(x^5+x^3+1)^2} + C$ (2) $\frac{x^5}{2(x^5+x^3+1)^2} + C$ (3) $\frac{-x^{10}}{2(x^5+x^3+1)^2} + C$ (4) $\frac{-x^5}{(x^5+x^3+1)^2} + C$

where C is an arbitrary constant

6. Let $I_n = \int \tan^n x\, dx$, (n > 1). If $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to **[JEE(Main) 2017, (4, –1), 120]**

(1) $\left(-\frac{1}{5}, 1\right)$ (2) $\left(\frac{1}{5}, 0\right)$ (3) $\left(\frac{1}{5}, -1\right)$ (4) $\left(-\frac{1}{5}, 0\right)$

7. The integral $\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$ is equal to :

[JEE(Main) 2018, (4, –1), 120]

(1) $\frac{1}{1+\cot^3 x} + C$ (2) $\frac{-1}{1+\cot^3 x} + C$ (3) $\frac{1}{3(1+\tan^3 x)} + C$ (4) $\frac{-1}{3(1+\tan^3 x)} + C$

(where C is a constant of integration)

8. Let n ≥ 2 be a natural number and 0 < θ < π / 2. Then $\int \frac{(\sin^n\theta - \sin\theta)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}$ dθ is equal to :

(where C is a constant of integration) **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

(1) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}} + C$ (2) $\frac{n}{n^2-1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

(3) $\frac{n}{n^2+1}\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$ (4) $\frac{n}{n^2-1}\left(1 + \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C$

9. The integral $\int \cos(\log_e x)dx$ is equal to : (where C is a constant of integration

[JEE(Main) 2019, Online (12-01-19),P-1 (4, – 1), 120]

(1) $x\left[\cos(\log_e x) - \sin(\log_e x)\right] + C$
(2) $\frac{x}{2}\left[\sin(\log_e x) - \cos(\log_e x)\right] + C$
(3) $x[\cos(\log_e x) + \sin(\log_e x)] + C$
(4) $\frac{x}{2}\left[\cos(\log_e x) + \sin(\log_e x)\right] + C$

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. (i) $\frac{(2x+3)^6}{12} + C$ (ii) $-\frac{\cos 2x}{2} + C$ (iii) $\frac{\tan(4x+5)}{4} + C$

(iv) $\frac{1}{3}\ \ell n\ |\sec(3x+2) + \tan(3x+2)| + C$ (v) $\frac{1}{2}\ \ell n\ |\sec(2x+1)| + C$

(vi) $\frac{2^{3x+4}}{3\ \ell n\ 2} + C$ (vii) $\frac{1}{2}\ \ell n\ |2x+1| + C$ (viii) $\frac{e^{4x+5}}{4} + C$

A-2. (i) $\frac{x}{2} - \frac{1}{4}\sin 2x + C$ (ii) $\frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$

(iii) $-\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$ (iv) $\cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + C$

(v) $\frac{2}{3}\left((x+3)^{3/2} + (x+2)^{3/2}\right) + C$

Section (B) :

B-1. (i) $-\frac{1}{2}\cos x^2 + C$ (ii) $\frac{1}{2}\ \ell n\ |x^2+1| + C$

(iii) $\frac{1}{2}(\tan x)^2 + C$ or $\frac{\sec^2 x}{2} + C$ (iv) $\ell n\ |e^x + x| + C$

(v) $\ell n\ |x + \cos x| + C$ (vi) $\frac{1}{2}\ \ell n\ |e^{2x} - 2| + C$

(vii) $\frac{1}{2}\ \ell n\ |x^2 + \sin 2x + 2x| + C$ (viii) $\ell n\ |\ \ell n(\sec x + \tan x)\ | + C$

(ix) $\frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + C$ (x) $\frac{1}{2}(e^{2x} - e^{-2x}) + 2x + C$

(xi) $\frac{1}{3}e^{3x} + e^{2x} + e^x + C$ (xii) $-\frac{1}{5}\ \ell n\left|1 + \frac{1}{x^5}\right| + C$

(xiii) $-\frac{1}{4}\left(1 + \frac{1}{x^5}\right)^{4/5} + C$ (xiv) $\frac{(x^2-8)^{3/2}}{24\ x^3} + C$

B-2. $2\sqrt{(x^2+2)} + C$ **B-3.** (i) $\ln\left(\frac{\sin x}{x}\right) + C$ (ii) $\ln\left(\frac{\ln(x+1)}{x}\right) + C$

Section (C) :

C-1. (i) $\frac{x^2}{2}\ \ell n\, x - \frac{x^2}{4} + C$ (ii) $\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$

(iii) $\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$ (iv) $x(\ell nx - 1) + C$

(v) $\frac{\sec x \tan x}{2} + \frac{1}{2}\ \ell n\,|\sec x + \tan x| + C$ (vi) $(x^2 - 1)\ e^{x^2} + C$

(vii) $x \sin^{-1}\sqrt{x} + \frac{\sqrt{x}\sqrt{1-x}}{2} - \frac{1}{2}\sin^{-1}\sqrt{x} + C$ (viii) $x\tan^{-1}x - \frac{1}{2}\ell n(1+x^2) - \frac{(\tan^{-1}x)^2}{2} + C$

(ix) $\frac{e^x}{2}(\sin x - \cos x) + C$ (x) $e^x \tan x + C$

C-2. $y = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + 2e$

Section (D) :

D-1. (i) $\frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (ii) $\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C$ (iii) $\frac{1}{2}\tan^{-1}\left(\frac{(x+1)}{2}\right) + C$

(iv) $\ell n\,|x^2 + 3x + 4| - \frac{4}{\sqrt{7}}\tan^{-1}\frac{2x+3}{\sqrt{7}} + C$ (v) $x - \arctan x + \ell n\frac{\sqrt{1+x^2}}{x} + C$

(vi) $\ell n\,|x + \sqrt{x^2-4}\,| + C$ (vii) $\frac{x}{2}\sqrt{x^2+4} + 2\,\ell n\,|x + \sqrt{x^2+4}\,| + C$

(viii) $\frac{x+1}{2}\sqrt{x^2+2x+5} + 2\,\ell n\left|x+1 + \sqrt{x^2+2x+5}\right| + C$

(ix) $-\frac{(1-x-x^2)^{3/2}}{3} - \frac{3}{8}(2x+1)\sqrt{1-x-x^2} - \frac{15}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$

(x) $\frac{2}{15}(a^3+x^3)^{5/2} - \frac{2a^3}{9}(a^3+x^3)^{3/2} + C$

D-2. (i) $\ell n\left|\frac{x+1}{x+2}\right| + C$ (ii) $\frac{1}{10}\ell n\,|x+3| - \frac{1}{20}\ell n\,|x^2+1| + \frac{3}{10}\tan^{-1}x + C$

(iii) $4\ell n|x+1| + \frac{1}{(x+1)} - 4\ell n|x+2| + C$ (iv) $\frac{1}{2}\ell n\,|x+1| - \ell n\,|x+2| + \frac{1}{2}\ell n\,|x+3| + C$

D-3. (i) $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) - \frac{1}{4}\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$ (ii) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\ x}\right) + C$

(iii) $-\frac{1}{2\sqrt{3}}\ell n\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right| + C$

D-4. (i) $\ell n\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$ (ii) $\frac{1}{4\sqrt{3}}\ell n\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}(t) + C$, where $t = \sqrt{x+1}$

(iii) $-\frac{1}{\sqrt{3}}\ell n\left|\left(t-\frac{1}{3}\right) + \sqrt{\left(t-\frac{1}{3}\right)^2 + \frac{2}{9}}\right| + C$, where $t = \frac{1}{x+1}$

(iv) $-\tan^{-1}\sqrt{\frac{x^2+2}{x^2}} + C$

D-5. (i) $\frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + \sqrt{x^2+x}+C$

(ii) $\sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right| + C$ (iii) $\frac{1}{2}\sin^{-1}x-\frac{x}{2}\sqrt{1-x^2}-\sqrt{1-x^2}+C$

Section (E) :

E-1. (i) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right)+C$ (ii) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\ \tan\ \frac{x}{2}\right)+C$

(iii) $\frac{10}{13}x-\frac{2}{13}\ell n\ |3\cos x + 2\sin x| + C$ (iv) $\ell n\left|1+\tan\frac{x}{2}\right| + C$

(v) $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+C$ (vi) $\ell n\,|1-\cot x| + C$

(vii) $\tan x + \frac{1}{4}\sin 2x - \frac{3x}{2} + C$

E-2. (i) $\frac{1}{40}\ell n\left(\frac{4(\sin x-\cos x)+5}{4(\sin x+\cos x)-5}\right)+C$ (ii) $\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+C$

E-3. $A=\frac{1}{9}$, $B=\frac{1}{5}$

PART - II

Section (A) :

A-1. (D) **A-2** (A) **A-3.** (A) **A-4.** (B) **A-5.** (A) **A-6.** (B)

Section (B) :

B-1. (B) **B-2.** (C) **B-3.** (A) **B-4.** (D) **B-5.** (D) **B-6.** (C)
B-7. (A) **B-8.** (C)

Section (C) :

C-1. (C) **C-2.** (A) **C-3.** (C) **C-4.** (A) **C-5.** (C)

Section (D) :

D-1. (B) **D-2.** (D) **D-3.** (A) **D-4.** (A) **D-5.** (C)

Section (E) :

E-1. (B) **E-2** (C) **E-3.** (C) **E-4.** (B) **E-5.** (A) **E-6.** (A) **E-7.** (A)

Section (F) :

F-1. (B) **F-2.** (B)

PART - III

1. (A) $\rightarrow$ (p), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s) **2.** (A) $\rightarrow$ (s) ; (B) $\rightarrow$ (q) ; (C) $\rightarrow$ (r)

EXERCISE - 2

PART - I

1.	(A)	**2.**	(B)	**3.**	(B)	**4.**	(C)	**5.**	(C)	**6.**	(A)	**7.**	(A)
8.	(A)	**9.**	(A)	**10.**	(C)	**11.**	(A)	**12.**	(D)	**13.**	(A)	**14.**	(A)
15.	(A)	**16.**	(C)	**17.**	(D)	**18.**	(D)	**19.**	(C)	**20.**	(A)	**21.**	(B)

PART - II

1.	1	**2.**	a = 3	**3.**	12	**4.**	10	**5.**	5	**6.**	2	**7.**	2
8.	16	**9.**	12	**10.**	11	**11.**	13	**12.**	1	**13.**	2	**14.**	1
15.	11	**16.**	10										

PART - III

1.	(BC)	**2.**	(AC)	**3.**	(AC)	**4.**	(CD)	**5.**	(BD)	**6.**	(ACD)	**7.**	(BC)
8.	(AC)	**9.**	(CD)	**10.**	(AB)	**11.**	(ABC)	**12.**	(AB)	**13.**	(BC)	**14.**	(ABD)
15.	(BD)	**16.**	(ABD)	**17.**	(AD)	**18.**	(AB)	**19.**	(ABCD)				

PART - IV

1.	(A)	**2.**	(A)	**3.**	(B)	**4.**	(A)	**5.**	(B)	**6.**	(A)

EXERCISE - 3

PART - I

1. $\frac{3}{2}\tan^{-1}x - \frac{1}{2}\ell n(1+x) + \frac{1}{4}\ell n(1+x^2) + \frac{x}{1+x^2} + C$ **2.** (C)

3. $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\ell n(4x^2+8x+13) + C$ **4.** $\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{6(m+1)} + C$

5.	(D)	**6.**	(A)	**7.**	(D)	**8.**	(C)	**9.**	(C)

PART - II

1.	(4)	**2.**	(3)	**3.**	(4)	**4.**	(4)	**5.**	(1)	**6.**	(2)	**7.**	(4)
8.	(2)	**9.**	(4)										

High Level Problems (HLP)

1. Evaluate : $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\, dx$

2. Evaluate : $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$

3. Evaluate : $\int \sqrt{x + \sqrt{x^2 + 2}}\ dx$

4. Evaluate : $\int \frac{dx}{\left(x^3 + 3x^2 + 3x + 1\right)\sqrt{x^2 + 2x - 3}}$

5. Evaluate : $\int \frac{(\cos\ \ 2x - 3)}{\cos^4 x\sqrt{4 - \cot^2 x}}\ dx$

6. Evaluate : $\left[\frac{\sqrt{x^2+1}\ \{\ell n(x^2+1) - 2\ell nx\}}{x^4}\right] dx$

7. Evaluate : $\int \frac{x}{(7x - 10 - x^2)^{3/2}}\ dx$

8. If $\int \frac{x\cos\alpha + 1}{\left(x^2 + 2x\cos\alpha + 1\right)^{3/2}}\ dx = \frac{f(x)}{\sqrt{g(x)}}$ + C then find f(x) and g(x).

9. Evaluate : cos x. e^x. x^2 dx

10. Evaluate : $\int e^x \left(\frac{x^3 - x + 2}{(x^2+1)^2}\right) dx$

11. Evaluate : $\int \frac{x^2}{(x\sin x + \cos x)^2}\ dx$

12. Evaluate : $\int \sin 4x . e^{\tan^2 x} dx$

13. Evaluate : $\int$ $\tan^{-1}$ x. ℓn (1 + x^2) dx.

14. Evaluate : $\int$ $e^x \frac{1 + n x^{n-1} - x^{2n}}{\left(1 - x^n\right)\sqrt{1 - x^{2n}}} dx$

15. Evaluate : $\int$ cos 2x ℓn (1 + tan x) dx

16. Evaluate : $\int \frac{dx}{\left(a + b\cos x\right)^2}$, (a > b)

17. Evaluate : $\int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$

18. Integrate: $\int \frac{(5x^2-12)\,dx}{(x^2-6x+13)^2}$

19. If $\int \frac{3x^2+2x}{x^6+2x^5+x^4+2x^3+2x^2+5}\,dx = F(x)$, then find the value of [F(1) – F(0)], where [.] represents greatest integer function.

20. Evaluate : $\int \frac{\ell n\ (1+\sin^2 x)}{\cos^2 x}\,dx$

21. Evaluate : $\int \frac{1+\cos\alpha\ \cos x}{\cos\alpha+\cos x}\,dx$

22. Evaluate : $\int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$

23. Evaluate : $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

24. Evaluate $\int \frac{(\cos 2x)^{1/2}}{\sin x}\,dx$

25. Evaluate $\int \frac{\sin^3 \frac{x}{2}}{\cos\frac{x}{2}\sqrt{\cos^3 x+\cos^2 x+\cos x}}\,dx$

26. If $\int \frac{x^2}{x^4+3x^2+9}\,dx = A\tan^{-1}\left(\frac{x^2-3}{3x}\right)+\frac{B}{\sqrt{3}}\,\ell n\left|\frac{x^2-\sqrt{3}\,x+3}{x^2+\sqrt{3}\,x+3}\right|$ + c, then find the value of 12(A + B).

27. Evaluate $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$

28. Evaluate $\int \sqrt[3]{\tan x}\,dx$

29. Evaluate : $\int \sqrt{\frac{\operatorname{cosec} x-\cot x}{\operatorname{cosec} x+\cot x}}\cdot\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$

Answers

1. $-\frac{1}{2}\sin 2x + C$

2. $-\left(\sin x + \frac{\sin 2x}{2}\right) + C$

3. $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - \frac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}} + C$

4. $\frac{\sqrt{x^2+2x-3}}{8(x+1)^2} + \frac{1}{16}\cdot\cos^{-1}\left(\frac{2}{x+1}\right) + C$

5. $C - \frac{1}{3}\tan x.(2+\tan^2 x).\ \sqrt{4-\cot^2 x}$

6. $\frac{2\ (x^2+1)\sqrt{x^2+1}}{9x^3}.\left[1-\frac{3}{2}\ell n\left(1+\frac{1}{x^2}\right)\right] + C$

7. $\frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + C$

8. $x;\ x^2 + 2x\cos\alpha + 1$

9. $\frac{1}{2}e^x[(x^2-1)\cos x + (x-1)^2 . \sin x] + C$

10. $e^x\left(\frac{x+1}{x^2+1}\right) + C$

11. $\frac{\sin x - x\cos x}{x\sin x + \cos x} + C$

12. $-2\cos^4 x.\ e^{\tan^2 x} + C$

13. $x\tan^{-1}x.\ \ell n(1+x^2) + (\tan^{-1}x)^2 - 2x\tan^{-1}x + \ell n(1+x^2) - \left(\ell n\sqrt{1+x^2}\right)^2 + C$

14. $e^x\sqrt{\frac{1+x^n}{1-x^n}} + C$

15. $\frac{1}{2}[\sin 2x.\ell n(1+\tan x) - x + \ell n|\sin x + \cos x|] + C$

16. $-\frac{b\sin x}{(a^2-b^2)(a+b\cos x)} + \frac{2a}{(a^2-b^2)^{3/2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} + C$

17. $-\frac{\sqrt{2-x-x^2}}{x} + \frac{\sqrt{2}}{4}\ln\left|\frac{4-x+2\sqrt{2}\sqrt{2-x-x^2}}{x}\right| - \sin^{-1}\left(\frac{2x+1}{3}\right) + K$

18. $\frac{13x-159}{8(x^2-6x+13)} + \frac{53}{16}\tan^{-1}\frac{x-3}{2} + C$

19. 0

20. $\tan x\ \ell n(1+\sin^2 x) - 2x + \sqrt{2}\tan^{-1}(\sqrt{2}.\tan x) + C.$

21. $x\cos\alpha + \sin\alpha\ \ell n\left|\frac{\cos\frac{1}{2}(\alpha - x)}{\cos\frac{1}{2}(\alpha + x)}\right| + C$

22. $-\frac{\cos x}{b + a\sin x} + C$

23. $\frac{-2}{\alpha-\beta}.\sqrt{\frac{x-\beta}{x-\alpha}} + C$

24. $\sqrt{2}\log\left[\frac{\sqrt{\cot^2 x - 1} + \sqrt{2\cot^2 x}}{\sqrt{\cot^2 x + 1}}\right] - \log\left[\cot x + \sqrt{\cot^2 x - 1}\right] + c$

25. $\sec^{-1}\left(\sqrt{\cos x} + \frac{1}{\sqrt{\cos x}}\right) + c.$

26. 5

27. $\frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$

28. $-\frac{1}{2}\log(1+\tan^{2/3}x) + \frac{1}{4}\log(\tan^{4/3}x - \tan^{2/3}x + 1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2\tan^{2/3}x - 1}{\sqrt{3}} + c$

29. $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right) + C$