CONTENT

Solution of Triangle

JEE (Advanced) Syllabus

Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle.

Solution of Triangle

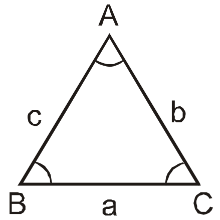
According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect.

..................Proclus

Sine Rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides

i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



Example # 1 : How many triangles can be constructed with the data : a = 5, b = 7, sin A = 3/4

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{5}{3/4} = \frac{7}{\sin B}$

$\Rightarrow$ $\sin B = \frac{21}{20} > 1$ not possible

$\therefore$ no triangle can be constructed.

Example # 2 : If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution : We have $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

$\Rightarrow \sin(B + C)\sin(B-C) = \sin(A + B)\sin(A-B)$ $\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\Rightarrow b^2 - c^2 = a^2 - b^2$ $\Rightarrow a^2, b^2, c^2$ are in A.P.

Self Practice Problems :

(1) In a ΔABC, the sides a, b and c are in A.P. , then prove that $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) : \cot\frac{B}{2} = 2 : 3$

(2) If the angles of ΔABC are in the ratio 1 : 2 : 3, then find the ratio of their corresponding sides

(3) In a ΔABC prove that $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

Ans. (2) $1 : \sqrt{3} : 2$

Cosine Formula :

In any ΔABC

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 + 2bc\cos(B + C)$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ **(iii)** $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Example # 3 : In a triangle ABC, A, B, C are in A.P. Show that $2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2-ac+c^2}}$.

Solution : $A + C = 2B \Rightarrow A + B + C = 3B \Rightarrow B = 60º$

$\therefore \cos 60º = \frac{a^2+c^2-b^2}{2ac} \Rightarrow a^2 - ac + c^2 = b^2$

$$\Rightarrow \frac{a+c}{\sqrt{a^2-ac+c^2}} = \frac{a+c}{b} = \left[\frac{\sin A+\sin C}{\sin B}\right] = \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B}$$

$= 2\cos\frac{A-C}{2}$ $\quad (\because A + C = 2B)$

Example # 4 : In a ΔABC , prove that $a (b\cos C - c \cos B) = b^2 - c^2$

Solution : Since $\cos C = \frac{a^2+b^2-c^2}{2ab}$ & $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$\therefore$ L.H.S. $= a\left\{b\left(\frac{a^2+b^2-c^2}{2ab}\right) - c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right\}$

$= \frac{a^2+b^2-c^2}{2} - \frac{(a^2+c^2-b^2)}{2} = (b^2 - c^2)$ = R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Example # 5 : The sides of ΔABC are AB = $\sqrt{13}$ cm, BC = $4\sqrt{3}$ cm and CA = 7 cm. Then find the value of $\sin\theta$ where θ is the smallest angle of the triangle.

Solution : Angle opposite to AB is smallest . Therefore,

$\cos\theta = \frac{49+48-13}{2.7.4\sqrt{3}} = \frac{\sqrt{3}}{2} \quad \Rightarrow \sin\theta = \frac{1}{2}$

Self Practice Problems :

(4) If in a triangle ABC, $3 \sin A = 6 \sin B = 2\sqrt{3} \sin C$, Then find the angle A.

(5) If two sides a, b and angle A be such that two triangles are formed, then find the sum of two values of the third side.

Ans. (4) 90º (5) 2b cosA

Projection Formula :

In any ΔABC

(i) $a = b \cos C + c \cos B$ **(ii)** $b = c \cos A + a \cos C$ **(iii)** $c = a \cos B + b \cos A$

Example # 6 : If in a ΔABC, $c \cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show that a, b, c are in A.P.

Solution : $c (1 + \cos A) + a (1 + \cos C) = 3b$

$\Rightarrow a + c + (c \cos A + a\cos C) = 3b$

$\Rightarrow a + c + b = 3b$

$\Rightarrow a + c = 2b$

Example # 7 : In a ΔABC, prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

Solution : $\because$ L.H.S. $= (b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$

$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (c \cos B + b \cos C)$

$= a + b + c$

= R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Self Practice Problems :

(6) The roots of $x^2 - 2\sqrt{3}x + 2 = 0$ represent two sides of a triangle. If the angle between them is $\frac{\pi}{3}$, then find the perimeter of triangle.

(7) In a triangle ABC, if cos A + cosB + cos C = 3/2, then show that the triangle is an equilateral triangle.

(8) In a ΔABC, prove that $\frac{\cos A}{c\cos B + b\cos C} + \frac{\cos B}{a\cos C + c\cos A} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2+b^2+c^2}{2abc}$.

Ans. (6) $2\sqrt{3} + \sqrt{6}$

Napier's Analogy - tangent rule :

In any ΔABC

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ **(ii)** $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Example # 8 : Find the unknown elements of the ΔABC in which a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 90°.

Solution : $\because$ a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 90°

$\because$ A + B + C = 180°

$\therefore$ A + B = 90° ....(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$= \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 45° = \frac{2}{2\sqrt{3}}\cot 45° \Rightarrow \qquad \tan\left(\frac{A-B}{2}\right) = \frac{1}{\sqrt{3}}$$

$\therefore$ $\frac{A-B}{2} = \frac{\pi}{6}$

$\Rightarrow$ $A - B = \frac{\pi}{3}$ ....(ii)

From equation (i) and (ii), we get $A = \frac{5\pi}{12}$ and $B = \frac{\pi}{12}$

Now, $c = \sqrt{a^2+b^2} = 2\sqrt{2}$

$\therefore$ $c = 2\sqrt{2}$, $A = \frac{5\pi}{12}$, $B = \frac{\pi}{12}$ Ans.

Self Practice Problems :

(9) In a ΔABC if b = 3, c = 5 and cos (B – C) = $\frac{7}{25}$, then find the value of $\sin\frac{A}{2}$.

(10) If in a ΔABC, we define $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$, then show that x + y + z = – xyz.

Ans. (9) $\frac{1}{\sqrt{10}}$

Trigonometric Functions of Half Angles :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s\,(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$, where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Area of Triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

Example # 9 : If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then show that $p_1^{-1} + p_2^{-1} - p_3^{-1} = \frac{s-c}{\Delta}$

Solution : We have

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$$

$$= \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$$

Example#10 : In a ΔABC if b sinC(b cosC + c cosB) = 64, then find the area of the ΔABC.

Solution : $\because$ b sinC (b cosC + c cosB) = 64 ....(i) given

$\because$ From **projection rule**, we know that
a = b cosC + c cosB put in (i), we get
ab sinC = 64 ....(ii)

$\because$ $\Delta = \frac{1}{2}ab\sin C$ $\quad\therefore\quad$ from equation (ii), we get

$\therefore$ Δ = 32 sq. unit

Example#11 : If A,B,C are the angle of a triangle, then prove that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$

Solution : $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

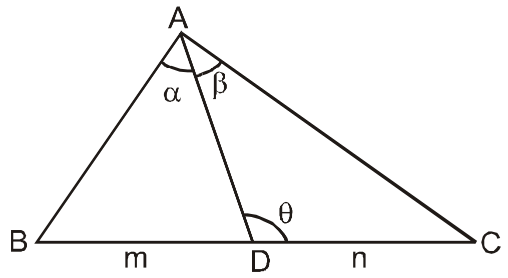
$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\sqrt{s}(s-a+s-b+s-c)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s}{\Delta}\ (3s-2s) = \frac{s^2}{\Delta}$$

m - n Rule : In any triangle ABC if D be any point on the base BC, such that BD : DC :: m : n and if $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$, then

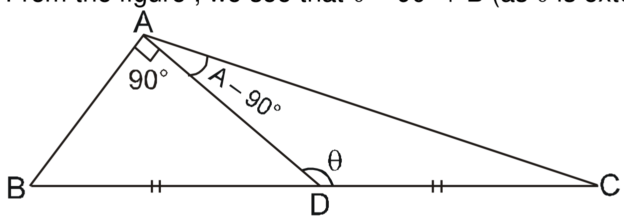
$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

$$n\cot B - m\cot C$$



Example#12 : In a ΔABC . AD divides BC in the ratio 2 : 1 such that at $\angle BAD = 90^\circ$ then prove that $\tan A + 3\tan B = 0$

Solution : From the figure , we see that $\theta = 90^\circ + B$ (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

$(2 + 1) \cot (90^\circ + B) = 2. \cot 90^\circ - 1.\cot (A - 90^\circ)$

$\Rightarrow \quad -3 \tan B = \cot (90^\circ - A)$

$\Rightarrow \quad -3 \tan B = \tan A$

$\Rightarrow \quad \tan A + 3 \tan B = 0$ **Hence proved.**

Example#13 : The base of a Δ is divided into three equal parts . If α, β, γ be the angles subtended by these parts at the vertex, prove that :
$(\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) = 4\text{cosec}^2\beta$

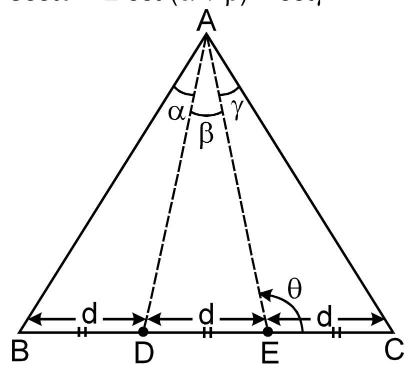
Solution : Let point D and E divides the base BC into three equal parts i.e. BD = DE = EC = d (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex. Now in ΔABC

$\because \quad BE : EC = 2d : d = 2 : 1$

$\therefore$ from **m-n rule**, we get

$(2 + 1) \cot\theta = 2 \cot (\alpha + \beta) - \cot\gamma$

$\Rightarrow \quad 3\cot\theta = 2 \cot (\alpha + \beta) - \cot\gamma$ ....(i)



again

$\because$ in ΔADC

$\because \quad DE : EC = d : d = 1 : 1$

$\therefore$ if we apply **m-n rule** in ΔADC, we get

$(1 + 1) \cot\theta = 1. \cot\beta - 1 \cot\gamma$

$2\cot\theta = \cot\beta - \cot\gamma$ ....(ii)

from (i) and (ii), we get $\dfrac{3\cot\theta}{2\cot\theta} = \dfrac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma}$

$\Rightarrow \quad 3\cot\beta - 3\cot\gamma = 4\cot (\alpha + \beta) - 2 \cot\gamma$

$\Rightarrow \quad 3\cot\beta - \cot\gamma = 4 \cot (\alpha + \beta)$

$\Rightarrow$ $3\cot\beta - \cot\gamma = 4\left\{\dfrac{\cot\alpha.\cot\beta - 1}{\cot\beta + \cot\alpha}\right\}$

$\Rightarrow$ $3\cot^2\beta + 3\cot\alpha \cot\beta - \cot\beta \cot\gamma - \cot\alpha \cot\gamma = 4 \cot\alpha \cot\beta - 4$

$\Rightarrow$ $4 + 3\cot^2\beta = \cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\alpha \cot\gamma$

$\Rightarrow$ $4 + 4\cot^2\beta = \cot\alpha \cot\beta + \cot\alpha \cot\gamma + \cot\beta \cot\gamma + \cot^2\beta$

$\Rightarrow$ $4(1 + \cot^2\beta) = (\cot\alpha + \cot\beta) (\cot\beta + \cot\gamma)$

$\Rightarrow$ $(\cot\alpha + \cot\beta) (\cot\beta + \cot\gamma) = 4\text{cosec}^2\beta$

Self Practice Problems :

(11) In a ΔABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11-6\sqrt{3}}}$ unit and it divides angle A into the angles of 30° and 45°. Prove that the side BC is of length 2 unit.

Radius of Circumcirlce :

If R be the circumradius of ΔABC, then $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4\Delta}$

Example#14 : In a ΔABC , prove that $\sin 2A + \sin 2B + \sin 2C = 2\Delta/R^2$

Solution : In a ΔABC, we know that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

and $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

$= \dfrac{4abc}{8R^3} = \dfrac{16\Delta R}{8R^3} = \dfrac{2\Delta}{R^2}$

Example#15 : In a ΔABC if a = 22 cm, b = 28 cm and c = 36 cm, then find its circumradius.

Solution : $\because$ $R = \dfrac{abc}{4\Delta}$ ....(i)

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because$ $s = \dfrac{a+b+c}{2} = 43$ cm

$\therefore$ $\Delta = \sqrt{43\times 21\times 15\times 7} = 21\sqrt{215}$

$\therefore$ $R = \dfrac{22\times 28\times 36}{4\times 21\sqrt{215}} = \dfrac{264}{\sqrt{215}}$ cm

Example#16 : In a ΔABC, if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

Solution : We have : $8R^2 = a^2 + b^2 + c^2$

$\Rightarrow 8R^2 = [4R^2 \sin^2 A + 4R^2 \sin^2 B + 4R^2 \sin C]$ [$\because$ a = 2R sin A etc.]

$\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C \Rightarrow (1 - \sin^2 A) - \sin^2 B + (1 - \sin^2 C) = 0$

$\Rightarrow (\cos^2 A - \sin^2 B) + \cos^2 C = 0 \Rightarrow \cos (A + B) \cos (A - B) + \cos^2 C = 0$

$\Rightarrow -\cos C \cos (A - B) + \cos^2 C = 0 \Rightarrow - \cos C \{\cos (A - B) - \cos C\} = 0$

$\Rightarrow -\cos C[\cos (A - B) + \cos(A + B)] = 0 \Rightarrow - 2\cos A \cos B \cos C = 0$

$\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow A = \dfrac{\pi}{2}$ or $B = \dfrac{\pi}{2}$ or $C = \dfrac{\pi}{2}$

$\Rightarrow \Delta$ABC is a right angled triangle.

Example#17 : $\dfrac{b^2 - c^2}{2a} = R \sin (B - C)$

Solution : $\dfrac{b^2 - c^2}{2a} = \dfrac{4R^2(\sin^2 B - \sin^2 C)}{4R\sin A} = \dfrac{R\sin(B+C)\sin(B-C)}{\sin A} = R \sin (B - C)$

Self Practice Problems :

(12) In a ΔABC, prove that (a + b) = 4R $\cos\left(\frac{A-B}{2}\right)\cos\frac{C}{2}$

(13) In a ΔABC , if b = 15 cm and cos B = $\frac{4}{5}$, find R.

(14) In a triangle ABC if α, β, γ are the distances of the vertices of triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = r^2$

Ans. (13) 12.5

Radius of The Incircle :

If 'r' be the inradius of ΔABC, then

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Radius of The Ex-Circles :

If r_1, r_2 , r_3 are the radii of the ex-circles of ΔABC opposite to the vertex A, B, C respectively, then

(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$;

(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

Example # 18 : $\cos A + \cos B + \cos C = \left(1+\frac{r}{R}\right)$

Solution : LHS = cosA + cosB + cosC

$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$

$= 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\sin\frac{C}{2}\right\}+1 = 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}+1$

$= 2\sin\frac{C}{2}\left\{2\sin\frac{A}{2}\sin\frac{B}{2}\right\} + 1 = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= 1 + \frac{1}{R}\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 1 + \frac{r}{R}$ = RHS

Example # 19 : In a triangle ABC, find the value of $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$.

Solution : $\frac{b-c}{\left(\frac{\Delta}{s-a}\right)}+\frac{c-a}{\left(\frac{\Delta}{s-b}\right)}+\frac{a-b}{\left(\frac{\Delta}{s-c}\right)}$

$= \frac{1}{\Delta}$ [(b – c) (s – a) + (c – a) (s – b) + (a – b) (s – c)]

$= \frac{1}{\Delta}$ [s(b – c + c – a + a – b) –a (b – c) – b(c – a) – c (a – b)] = 0

Self Practice Problems :

(15) In a triangle ABC, r_1, r_2, r_3 are in HP. If its area is 24 cm^2 and its perimeter is 24 cm. then find lengths of its sides.

(16) In a triangle ABC, a : b : c = 4 : 5 : 6 . Find the ratio of the radius of the circumcircle to that of the incircle.

(17) In a ΔABC, prove that $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

(18) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC, then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

Ans. (15) 6, 8 , 10 (16) 16 : 7

Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle A = $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$;

(ii) Length of median from the angle A = $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

& (iii) Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

Example#20 : In ΔABC, AD & BE are its two median . If AD = 4 , $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then find the length of BE and area of ΔABC.

Solution : AP = $\frac{2}{3}$; AD = $\frac{8}{3}$; PD = $\frac{4}{3}$; Let PB = x

$\tan 60^\circ = \frac{8/3}{x}$ or $x = \frac{8}{3\sqrt{3}}$

Area of $\Delta ABP = \frac{1}{2} \times \frac{8}{3} \times \frac{8}{3\sqrt{3}} = \frac{32}{9\sqrt{3}}$

$\therefore$ Area of $\Delta ABC = 3 \times \frac{32}{9\sqrt{3}} = \frac{32}{3\sqrt{3}}$

Also, BE = $\frac{3}{2}x = \frac{4}{\sqrt{3}}$

Self Practice Problem :

(19) In a ΔABC if $\angle A = 90^\circ$, b = 5 cm, c = 12 cm. If 'G' is the centroid of triangle, then find circumradius of ΔGAB.

Ans. (19) $\frac{13\sqrt{601}}{30}$ cm

The Distances of The Special Points from Vertices and Sides of Triangle :

(i) Circumcentre (O) : OA = R and O_a = R cos A

(ii) Incentre (I) : IA = r cosec $\frac{A}{2}$ and I_a = r

(iii) Excentre (I_1) : I_1 A = r_1 cosec $\frac{A}{2}$ and $I_{1a} = r_1$

(iv) Orthocentre (H) : HA = 2R cos A and H_a = 2R cos B cos C

(v) Centroid (G) : GA = $\frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and $G_a = \frac{2\Delta}{3a}$

Example#21 : If p_1,p_2, p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, prove that :

(i) $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=\frac{1}{R}$ (ii) $\frac{bp_1}{c}+\frac{cp_2}{a}+\frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R}$

Solution : (i) use $\frac{1}{p_1}=\frac{a}{2\Delta},\frac{1}{p_2}=\frac{b}{2\Delta},\frac{1}{p_3}=\frac{c}{2\Delta}$

$\therefore$ LHS = $\frac{1}{2\Delta}$ (a cosA + b cosB + c cosC)

$= \frac{R}{2\Delta}$ (sin 2A + sin 2B + sin 2C) = $\frac{4R\sin A\ \sin B\ \sin C}{2\Delta}$

$= \frac{4R}{2\Delta}.\frac{a}{2R}.\frac{b}{2R}.\frac{c}{2R}=\frac{1}{4\Delta R^2}abc=\frac{1}{4\Delta R^2}.(4R\Delta)=\frac{1}{R}$ = RHS

(ii) LHS = $\frac{bp_1}{c}+\frac{cp_2}{a}+\frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R} = \frac{2b\Delta}{ac}+\frac{2c\Delta}{ab}+\frac{2a\Delta}{bc} = \frac{2\Delta(a^2+b^2+c^2)}{abc}$

$= \frac{2\Delta(a^2+b^2+c^2)}{4\Delta R} = \frac{a^2+b^2+c^2}{2R}$

Self Practice Problems :

(20) If I be the incentre of ΔABC, then prove that IA . IB . IC = abc tan $\frac{A}{2}$ tan $\frac{B}{2}$ tan $\frac{C}{2}$.

(21) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

Exercise-1

✍ **Marked questions are recommended for Revision.**

SUBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, prove that :

(i) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$

(ii) $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(iv)✍ $(a - b)^2 \cos^2\frac{C}{2} + (a + b)^2 \sin^2\frac{C}{2} = c^2$

(v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$

(vi) $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.

A-3. The angles of a ΔABC are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle$A.

A-4.✍ If $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$, prove that sides a, c, b of the triangle ABC are in A.P.

A-5. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then prove that a^2, b^2, c^2 are in A.P.

A-6. In a triangle ABC, prove that for any angle θ, $b \cos (A - \theta) + a \cos (B + \theta) = c \cos \theta$.

A-7.✍ With usual notations, if in a Δ ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

A-8. Let a, b and c be the sides of a ΔABC. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R.

A-9.✍ If in a triangle ABC, the altitude AM be the bisector of $\angle$BAD, where D is the mid point of side BC, then prove that $(b^2 - c^2) = a^2/2$.

A-10. If in a triangle ABC, $\angle C = 60^\circ$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

A-11.✍ In a triangle ABC, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the ΔABD is $\sqrt{3}$ times the area of the ΔBCD, find the $\angle$ABD.

A-12.✍ In a scalene triangle ABC, D is a point on the side AB such that $CD^2 = AD. DB$, if $\sin A. \sin B = \sin^2\frac{C}{2}$ then prove that CD is internal bisector of $\angle$C.

A-13. In triangle ABC, D is on AC such that AD = BC , BD = DC, $\angle$DBC = 2x, and $\angle$BAD = 3x, all angles are in degrees, then find the value of x.

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. In a ΔABC, prove that

(i) $2\left[a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right]$ = c + a – b.

(ii) $\dfrac{\cos^2\frac{A}{2}}{a}+\dfrac{\cos^2\frac{B}{2}}{b}+\dfrac{\cos^2\frac{C}{2}}{c}=\dfrac{s^2}{abc}$

(iii) $4\left(bc.\cos^2\frac{A}{2}+ca.\cos^2\frac{B}{2}+ab.\cos^2\frac{C}{2}\right)=(a+b+c)^2$

(iv) $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$

(v) $4\Delta(\cot A+\cot B+\cot C)=a^2+b^2+c^2$

(vi) $\left(\dfrac{2abc}{a+b+c}\right).\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}=\Delta$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\frac{A}{2}+\tan\frac{C}{2}$ in terms of cot(B/2).

B-3. If in a Δ ABC, a = 6, b = 3 and cos(A – B) = 4/5, then find its area.

B-4. If in a triangle ABC, $\angle$A = 30º and the area of triangle is $\dfrac{\sqrt{3}\,a^2}{4}$, then prove that either B = 4C or C = 4B.

Section (C) Inradius and Exradius

C-1. In any ΔABC, prove that

(i) $Rr(\sin A+\sin B+\sin C)=\Delta$

(ii) $a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\dfrac{\Delta}{R}$

(iii) $\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca}=\dfrac{1}{2Rr}$.

(iv) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\dfrac{r}{2R}$

(v) $a\cot A+b\cot B+c\cot C=2(R+r)$

C-2. In any ΔABC, prove that

(i) $r.r_1.r_2.r_3=\Delta^2$

(ii) $r_1+r_2-r_3+r=4R\cos C$.

(iii) $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}=\dfrac{a^2+b^2+c^2}{\Delta^2}$

(iv) $\left(\dfrac{1}{r}+\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)^2=\dfrac{4}{r}\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$

(v) $\dfrac{bc-r_2r_3}{r_1}=\dfrac{ca-r_3r_1}{r_2}=\dfrac{ab-r_1r_2}{r_3}=r$

C-3. Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3-x^2(4R+r)+xs^2-rs^2=0$.

C-4. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

C-5. If the area of a triangle is 100 sq.cm, r_1 = 10 cm and r_2 = 50 cm, then find the value of (b – a).

Section (D) Miscellaneous

D-1. If α, β, γ are the respective altitudes of a triangle ABC, prove that

(i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = \frac{2ab}{(a + b + c)\ \Delta}\cos^2\frac{C}{2}$

D-2. If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of tan A.tan C.

D-3. A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $\left(\sqrt{3}-1\right)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.

D-4. If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that tan ($\angle$ADB) = $\frac{4\Delta}{a^2 - c^2}$.

Exercise-2

Marked questions are recommended for Revision.

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, A : B : C = 3 : 5 : 4. Then a + b + c $\sqrt{2}$ is equal to
(A) 2b (B) 2c (C) 3b (D) 3a

A-2*. In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is
(A) right angled (B) isosceles (C) obtuse angled (D) equilateral

A-3. If in a Δ ABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :
(A) right angled (B) isosceles (C) equilateral (D) obtuse angled

A-4. In a ΔABC $\frac{bc\sin^2 A}{\cos A + \cos B\cos C}$ is equal to
(A) $b^2 + c^2$ (B) bc (C) a^2 (D) a^2 + bc

A-5 Given a triangle ΔABC such that $\sin^2A + \sin^2C = 1001.\sin^2B$. Then the value of $\frac{2(\tan A + \tan C).\tan^2 B}{\tan A + \tan B + \tan C}$ is
(A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D) $\frac{1}{250}$

A-6. If in a triangle ABC, (a + b + c) (b + c – a) = k. bc, then :
(A) k < 0 (B) k > 6 (C) 0 < k < 4 (D) k > 4

A-7. In a triangle ABC, a: b: c = 4: 5: 6. Then 3A + B equals to :
(A) 4C (B) 2π (C) π – C (D) π

A-8. The distance between the middle point of BC and the foot of the perpendicular from A is :

(A) $\frac{-a^2+b^2+c^2}{2a}$ (B) $\frac{b^2-c^2}{2\,a}$ (C) $\frac{b^2+c^2}{\sqrt{bc}}$ (D) $\frac{b^2+c^2}{2\,a}$

A-9*. If in a triangle ABC, cos A cos B + sin A sin B sin C = 1, then the triangle is

(A) isosceles (B) right angled (C) equilateral (D) None of these

A-10. Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that BP = PQ = QC. If AP = 3 and AQ = 4, then length BC is equal to

(A) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7

A-11. In ΔABC, $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then ΔABC is

(A) isosceles but not necessarily equilaterial
(B) equilateral
(C) right angled but not neccessarily isosceles
(D) right angled isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. If in a triangle ABC, right angle at B, s – a = 3 and s – c = 2, then

(A) a = 2, c = 3 (B) a = 3, c = 4 (C) a = 4, c = 3 (D) a = 6, c = 8

B-2. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are :

(A) in A.P. (B) in G.P. (C) in H.P. (D) None

B-3. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :

(A) R, R, R (B) $\sqrt{2}R,\ \sqrt{2}R,\ \sqrt{2}R$ (C) 2R, 2R, 2R (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$

B-4. In a Δ ABC if b + c = 3a, then $\cot\frac{B}{2} \cdot \cot\frac{C}{2}$ has the value equal to:

(A) 4 (B) 3 (C) 2 (D) 1

B-5. In a ΔABC, A = $\frac{2\pi}{3}$, b – c = $3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm^2. Then 'a' is

(A) $6\sqrt{3}$ cm (B) 9 cm (C) 18 cm (D) 7 cm

B-6.* The diagonals of a parallelogram are inclined to each other at an angle of 45º, while its sides a and b(a > b) are inclined to each other at an angle of 30º, then the value of $\frac{a}{b}$ is

(A) 2cos36º (B) $\sqrt{\frac{3+\sqrt{5}}{4}}$ (C) $\frac{3+\sqrt{5}}{4}$ (D) $\frac{\sqrt{5}+1}{2}$

B-7. If in a ΔABC, $\Delta = a^2 - (b-c)^2$, then tan A is equal to

(A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2

B-8*. If in a ΔABC, a = 5, b = 4 and cos (A – B) = $\frac{31}{32}$, then

(A) c = 6 (B) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$

(C) area of ΔABC= $\frac{15\sqrt{7}}{4}$ (D) c = 8

B-9. If R denotes circumradius, then in ΔABC, $\dfrac{b^2 - c^2}{2a\ R}$ is equal to

(A) cos (B – C) (B) sin (B – C) (C) cos B – cos C (D) sin(B + C)

B-10*. Which of the following holds good for any triangle ABC?

(A) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$ (B) $\dfrac{\sin A}{a} + \dfrac{\sin B}{b} + \dfrac{\sin C}{c} = \dfrac{3}{2R}$

(C) $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$ (D) $\dfrac{\sin 2A}{a^2} = \dfrac{\sin 2B}{b^2} = \dfrac{\sin 2C}{c^2}$

B-11. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:

(A) $\dfrac{9\sqrt{3}\ (1+\sqrt{3})}{\pi^2}$ (B) $\dfrac{9\sqrt{3}\ (\sqrt{3}-1)}{\pi^2}$ (C) $\dfrac{9\ \sqrt{3}\ (1+\sqrt{3})}{2\ \pi^2}$ (D) $\dfrac{9\ \sqrt{3}\ (\sqrt{3}-1)}{2\ \pi^2}$

B-12. In a ΔABC, a = 1 and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle$ A is

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{2}$ (C) $\dfrac{\pi}{6}$ (D) $\dfrac{\pi}{4}$

B-13*. Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :

(A) $\dfrac{2-\sqrt{3}}{\sqrt{3}}$ (B) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ (C) $\dfrac{2+\sqrt{3}}{\sqrt{3}}$ (D) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

B-14. Triangle ABC is isosceles with AB = AC and BC = 65 cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm is)

(A) 1254 (B) 1950 (C) 2535 (D) 5070

Section (C) Inradius and Exradius

C-1. In a Δ ABC, the value of $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:

(A) $\dfrac{r}{R}$ (B) $\dfrac{R}{2r}$ (C) $\dfrac{R}{r}$ (D) $\dfrac{2r}{R}$

C-2. In a triangle ABC, if a : b : c = 3 : 7 : 8, then R : r is equal to

(A) 2 : 7 (B) 7 : 2 (C) 3 : 7 (D) 7 : 3

C-3*. If $r_1 = 2r_2 = 3r_3$, then

(A) $\dfrac{a}{b} = \dfrac{4}{5}$ (B) $\dfrac{a}{b} = \dfrac{5}{4}$ (C) $\dfrac{a}{c} = \dfrac{3}{5}$ (D) $\dfrac{a}{c} = \dfrac{5}{3}$

C-4*. In a ΔABC, following relations hold good. In which case(s) the triangle is a right angled triangle?

(A) $r_2 + r_3 = r_1 - r$ (B) $a^2 + b^2 + c^2 = 8\ R^2$ (C) $r_1 = s$ (D) $2\ R = r_1 - r$

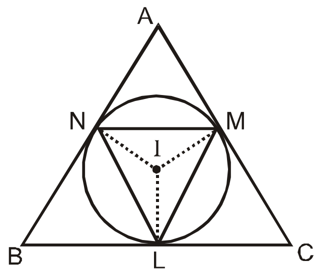
C-5. The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then |a – b| equals

(A) 1 (B) 2 (C) 8 (D) 9

C-6. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:

(A) A.P. (B) G.P. (C) H.P. (D) none of these

C-7. If the incircle of the Δ ABC touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



(A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$

C-8. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ is equal to :

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3

C-9. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to

(A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D) $\frac{a-b}{c}$

C-10. In any ΔABC, $\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{Rs^2}$ is always equal to

(A) 8 (B) 27 (C) 16 (D) 4

C-11*. In a triangle ABC, right angled at B, then

(A) $r = \frac{AB+BC-AC}{2}$ (B) $r = \frac{AB+AC-BC}{2}$ (C) $r = \frac{AB+BC+AC}{2}$ (D) $R = \frac{s-r}{2}$

C-.12*. With usual notations, in a Δ ABC the value of $\Pi\,(r_1 - r)$ can be simplified as:

(A) $abc\,\Pi\tan\frac{A}{2}$ (B) $4rR^2$ (C) $\frac{(abc)^2}{R(a+b+c)^2}$ (D) $4Rr^2$

C-13. **STATEMENT-1 :** In a triangle ABC, the harmonic mean of the three exradii is three times the inradius.
STATEMENT-2 : In any triangle ABC, $r_1 + r_2 + r_3 = 4R$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Section (D) Miscellaneous

D-1. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to :
(A) 0 (B) 1 (C) 2 (D) 1/2

D-2. In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex 'A' from the side BC is (in cm)
(A) $\frac{25}{13}$ (B) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$

D-3. If AD, BE and CF are the medians of a ΔABC, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to
(A) 4 : 3 (B) 3 : 2 (C) 3 : 4 (D) 2 : 3

D-4*. In a triangle ABC, with usual notations the length of the bisector of internal angle A is :
(A) $\frac{2bc\cos\frac{A}{2}}{b+c}$ (B) $\frac{2bc\sin\frac{A}{2}}{b+c}$ (C) $\frac{abc\ \ \operatorname{cosec}\frac{A}{2}}{2R\ \ (b+c)}$ (D) $\frac{2\Delta}{b+c}\cdot\operatorname{cosec}\frac{A}{2}$

D-5. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides BC, CA and AB respectively. If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$, then the value of '$\lambda$' is:
(A) 1/4 (B) 1/2 (C) 1 (D) 2

D-6. In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to
(A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D) $\frac{\Delta}{R}$

D-7. AA_1, BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1, G and B_1 are concyclic, then
(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

D-8. If 'ℓ' is the length of median from the vertex A to the side BC of a ΔABC, then
(A) $4\ell^2 = b^2 + 4ac\cos B$ (B) $4\ell^2 = a^2 + 4bc\cos A$ (C) $4\ell^2 = c^2 + 4ab\cos C$ (D) $4\ell^2 = b^2 + 2c^2 - 2a^2$

D-9*. The product of the distances of the incentre from the angular points of a Δ ABC is:
(A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(a\,b\,c)R}{s}$ (D) $\frac{(a\,b\,c)r}{s}$

D-10. In a triangle ABC, B = 60° and C = 45°. Let D divides BC internally in the ratio 1 : 3,
then value of $\frac{\sin\angle BAD}{\sin\angle CAD}$ is
(A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$

D-11*. In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:
(A) $\tan\theta = 3\tan B$ (B) $3\tan\theta = \tan C$ (C) $\frac{6\tan\theta}{\tan^2\theta-9} = \tan A$ (D) angle B = angle C

D-12. **STATEMENT-1 :** If R be the circumradius of a ΔABC, then circumradius of its excentral $\Delta I_1I_2I_3$ is 2R.

STATEMENT-2 : If circumradius of a triangle be R, then circumradius of its pedal triangle is $\frac{R}{2}$.

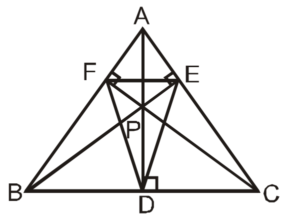
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

PART-II (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 4)

The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.

Answer The Following Questions :



1. Angle of triangle DEF are
(A) $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ (B) $\pi + 2A$, $\pi + 2B$ and $\pi + 2C$
(C) $\pi - A$, $\pi - B$ and $\pi - C$ (D) $2\pi - A$, $2\pi - B$ and $2\pi - C$

2*. Sides of triangle DEF are
(A) b cosA, a cosB, c cosC (B) a cosA, b cosB, c cosC
(C) R sin 2A, R sin 2B, R sin 2C (D) a cotA, b cotB, c cotC

3. Circumraii of the triangle PBC, PCA and PAB are respectively
(A) R, R, R (B) 2R, 2R, 2R (C) R/2, R/2, R/2 (D) 3R, 3R, 3R

4*. Which of the following is/are correct
(A) $\frac{\text{Perimeter of } \Delta DEF}{\text{Perimeter of } \Delta ABC} = \frac{r}{R}$ (B) Area of $\Delta DEF = 2\,\Delta \cos A \cos B \cos C$
(C) Area of $\Delta AEF = \Delta \cos^2 A$ (D) Circum-radius of $\Delta DEF = \frac{R}{2}$

Comprehension # 2 (Q. 5 to 8)
The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1I_2I_3$

5. Incentre I of Δ ABC is the .... of the excentral $\Delta I_1 I_2 I_3$.
(A) Circumcentre (B) Orthocentre (C) Centroid (D) None of these

6. Angles of the $\Delta I_1 I_2 I_3$ are
(A) $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$ (B) $\frac{\pi}{2} + \frac{A}{2}$, $\frac{\pi}{2} + \frac{B}{2}$ and $\frac{\pi}{2} + \frac{C}{2}$
(C) $\frac{\pi}{2} - A$, $\frac{\pi}{2} - B$ and $\frac{\pi}{2} - C$ (D) None of these

7. Sides of the $\Delta I_1 I_2 I_3$ are

(A) $R\cos\frac{A}{2}$, $R\cos\frac{B}{2}$ and $R\cos\frac{C}{2}$ (B) $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$

(C) $2R\cos\frac{A}{2}$, $2R\cos\frac{B}{2}$ and $2R\cos\frac{C}{2}$ (D) None of these

8. Value of $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 =$

(A) $4R^2$ (B) $16R^2$ (C) $32R^2$ (D) $64R^2$

PART-III (MATCH THE COLUMN)

1. Match the column

Column– I		Column–II	
(A)	In a ΔABC, $2B = A + C$ and $b^2 = ac$. Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to	(p)	8
(B)	In any right angled triangle ABC, the value of $\frac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)	(q)	1
(C)	In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to	(r)	5
(D)	In a ΔABC, $a = 5$, $b = 3$ and $c = 7$, then the value of $3\cos C + 7\cos B$ is equal to	(s)	9

2. **Match the column**

Column – I		**Column – II**	
(A)	In a ΔABC, $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	(p)	27
(B)	In any ΔABC, minimum value of $\frac{r_1\ r_2\ r_3}{r^3}$ is equal to	(q)	7
(C)	In a ΔABC, $a = 5$, $b = 4$ and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to	(r)	6
(D)	In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8\cos B)$ is equal to	(s)	11

Exercise-3

✍ Marked Questions may have for Revision Questions.

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

* Marked Questions may have more than one correct option.

1. Given an isosceles triangle, whose one angle is 120° and radius of its incircle is $\sqrt{3}$ unit. Then the area of the triangle in sq. units is **[IIT-JEE-2006, Main.,(3, –1)/184]**

(A) $7 + 12\sqrt{3}$ (B) $12 - 7\sqrt{3}$ (C) $12 + 7\sqrt{3}$ (D) 4π

2.* Internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC, then **[IIT-JEE-2006, Main.,(5, –1)/184]**

(A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (D) the triangle AEF is isosceles

3.✍ Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30º. Find the absolute value of the difference between the areas of these triangles. **[IIT-JEE 2009, Paper-2, (4, –1), 80]**

4*. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then **[IIT-JEE 2009, Paper-1, (4, –1)/ 80]**

(A) b + c = 4a (B) b + c = 2a

(C) locus of points A is an ellipse (D) locus of point A is a pair of straight lines

5. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

6.✍ Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

(A) $-\left(2+\sqrt{3}\right)$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

[IIT-JEE 2010, Paper-1, (3, 0), 84]

7.✍ Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to **[IIT-JEE 2010, Paper-2, (3, 0), 79]**

8. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and c = $\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

[IIT-JEE 2012, Paper-2, (3, –1), 66]

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

9.* In a triangle PQR, P is the largest angle and cosP = $\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 16 (B) 18 (C) 24 (D) 22

10. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

11*. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and 2s = x + y + z. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

[JEE (Advanced) 2016, Paper-1, (4, –2)/62]

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

12*. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

[JEE(Advanced) 2018, Paper-1,(4, –2)/60]

(A) $\angle QPR = 45^\circ$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side 'a', is : **[AIEEE – 2003 (3, 0), 225]**

(1) $a \cot\left(\frac{\pi}{n}\right)$ (2) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ (3) $a \cot\left(\frac{\pi}{2n}\right)$ (4) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$

2. If in a triangle ABC, $a \cos^2\left(\frac{C}{2}\right) + c \cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides a, b and c : **[AIEEE – 2003 (3, 0), 225]**

(1) are in A.P. (2) are in G.P. (3) are in H.P. (4) satisfy a + b = c.

3. In a triangle ABC, medians AD and BE are drawn. If AD = 4, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of the ΔABC is : **[AIEEE – 2003 (3, 0), 225]**

(1) $\frac{8}{3}$ (2) $\frac{16}{3}$ (3) $\frac{32}{3\sqrt{3}}$ (4) $\frac{64}{3}$.

4. The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0 < \alpha < \frac{\pi}{2}$. Then the greatest angle of the triangle is : **[AIEEE – 2004 (3, 0), 225]**

(1) 60º (2) 90º (3) 120º (4) 150º

5. In a triangle ABC, let $\angle C = \pi/2$, if r is the inradius and R is the circumradius of the triangle ABC, then 2(r+R) equals : **[AIEEE - 2005 (3, 0), 225]**

(1) c + a (2) a + b + c (3) a + b (4) b + c

6. If in a ΔABC, the altitudes from the vertices A,B,C on opposite sides are in H.P., then sinA, sinB, sinC are in : **[AIEEE - 2005 (3, 0), 225]**

(1) HP (2) Arithemetico-Geometric Progression
(3) AP (4) GP

7. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is **[AIEEE - 2010 (4, –1), 144]**

(1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$. (2) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.

(3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$. (4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

8. ABCD is a trapezium such that AB and CD are parallel and BC $\perp$ CD. If $\angle ADB = \theta$, BC = p and CD = q, then AB is equal to : **[AIEEE - 2013, (4, –1),120]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ (3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

9. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is : **[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**

(1) 9 : 7 (2) 7 : 1 (3) 3 : 1 (4) 5 : 3

10. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is **[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]**

(1) $\frac{c}{\sqrt{3}}$ (2) $\frac{3}{2}y$ (3) $\frac{c}{3}$ (4) $\frac{y}{\sqrt{3}}$

Answers

EXERCISE - 1

Section (A) :

A-2. $x > 5$ **A-3.** 75° **A-8.** $\frac{P}{2\sqrt{R}}$ **A-11.** 30° **A-13.** 10°

Section (B)

B-2. $\frac{2}{3}\cot\frac{B}{2}$ **B-3.** 9 sq. unit

Section (C) :

C-4. 6, 8, 10 cm **C-5.** 8

Section (D) :

D-2. 3 **D-3.** $\sqrt{2}$

EXERCISE - 2

Section (A) :

A-1. (C) **A-2*.** (AB) **A-3.** (C) **A-4.** (C) **A-5.** (D) **A-6.** (C) **A-7.** (D)

A-8. (B) **A-9.** (AB) **A-10.** (A) **A-11.** (A)

Section (B) :

B-1. (B) **B-2.** (A) **B-3.** (A) **B-4.** (C) **B-5.** (B) **B-6.** (AD) **B-7.** (B)

B-8*. (ABC) **B-9.** (B) **B-10*.** (AB) **B-11.** (A) **B-12.** (C) **B-13*.** (AC) **B-14.** (C)

Section (C) :

C-1. (A) **C-2.** (B) **C-3*.** (BD) **C-4*.** (ABCD) **C-5.** (A) **C-6.** (A) **C-7.** (C)

C-8. (B) **C-9.** (D) **C-10.** (D) **C-11*.** (AD) **C-.12*.** (ACD) **C-13.** (C)

Section (D)

D-1.	(B)	**D-2.**	(B)	**D-3.**	(C)	**D-4*.**	(ACD)	**D-5.**	(A)	**D-6.**	(D)
D-7.	(C)	**D-8.**	(B)	**D-9*.**	(BD)	**D-10.**	(C)	**D-11*.**	(ACD)	**D-12.**	(A)

PART-II

1.	(A)	**2*.**	(BC)	**3.**	(A)	**4*.**	(ABCD)	**5.**	(B)	**6.**	(A)	**7.**	(B)
8.	(B)												

PART-III

1. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)

2. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q)

EXERCISE - 3

PART - I

1.	(C)	**2.***	(ABCD)	**3.**	4	**4*.**	(BC)	**5.**	(D)	**6.**	(B)	**7.**	3
8.	(C)	**9.***	(BD)	**10.**	(B)	**11.**	(ACD)	**12.**	(BCD)				

PART - II

1.	(2)	**2.**	(1)	**3.**	(3)	**4.**	(3)	**5.**	(3)	**6.**	(3)	**7.**	(2)
8.	(1)	**9.**	(2)	**10.**	(1)								

High Level Problems (HLP)

Marked questions are recommended for Revision.

1. In ΔABC , P is an interior point such that $\angle PAB = 10º$ $\angle PBA = 20º$, $\angle PCA = 30º$, $\angle PAC = 40º$ then prove that ΔABC is isosceles

2. In a triangle ABC, if a tan A + b tan B = (a + b) tan $\left(\frac{A+B}{2}\right)$, prove that triangle is isosceles.

3. In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is is the area of triangle), then prove that $\angle A$ is obtuse

4. If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.

5. In a Δ ABC, $\angle$ C = 60° and $\angle$ A = 75°. If D is a point on AC such that the area of the Δ BAD is $\sqrt{3}$ times the area of the Δ BCD, find the $\angle$ ABD.

6. In a ΔABC, if a, b and c are in A.P., prove that cos A.cot $\frac{A}{2}$, cos B.cot $\frac{B}{2}$, and cos C.cot $\frac{C}{2}$ are in A.P.

7. In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,
$\pi : \cot\left(\frac{A}{2}\right) . \cot\left(\frac{B}{2}\right) . \cot\left(\frac{C}{2}\right)$.

8. In ΔABC, prove that $a^2 (s - a) + b^2 (s - b) + c^2 (s - c) = 4R\Delta \left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

9. In any ΔABC, prove that

(i) $(r_3 + r_1)(r_3 + r_2)\sin C = 2 r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$

(ii) $\frac{\tan\frac{A}{2}}{(a-b)(a-c)} + \frac{\tan\frac{B}{2}}{(b-a)(b-c)} + \frac{\tan\frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}$

(iii) $(r + r_1)\tan\frac{B-C}{2} + (r + r_2)\tan\frac{C-A}{2} + (r + r_3)\tan\frac{A-B}{2} = 0$

(iv) $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

10. In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$

11. If the inradius in a right angled triangle with integer sides is r. Prove that
(i) If r = 4, the greatest perimeter (in units) is 90
(ii) If r = 5, the greatest area (in sq. units) is 330

12. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then prove that the triangle is right angled.

13. DEF is the triangle formed by joning the points of contact of the incircle with the sides of the triangle ABC; prove that

(i) its sides are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$ and $2r\cos\frac{C}{2}$,

(ii) its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$

and

(iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2}\frac{r}{R}\Delta$.

14. Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

15. OA and OB are the equal sides of an isoscles triangle lying in the first quadrant making angles θ and ϕ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is cosec β – cot β where $\beta = \phi + \theta$. (Where O is origin)

16. The hypotenuse BC = a of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A. Also h is the altitude of the triangle through A. Prove that $\tan\alpha = \frac{4nh}{a(n^2-1)}$.

HLP Answers

5. $\angle$ ABD = 30°