CONTENTS

COMPLEX NUMBERS

JEE (Advanced) Syllabus

Complex Numbers : Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

JEE (Main) Syllabus

Complex Number : Complex numbers as ordered pairs of reals, Representation of complex numbers in the forma+ib and their representation in a plane, Argand diagram,algebra of complex numbers, modulus and argument (or amplitude)of a complex number, square root of a complex number, triangle inequality.

Complex Numbers

The shortest path between two truths in the real domain passes through the complex domain. ......Hadamard, Jacques

The complex number system

A complex number (z) is a number that can be expressed in the form z = a + ib where a and b are real numbers and $i^2 = -1$. Here 'a' is called as real part of z which is denoted by (Re z) and 'b' is called as imaginary part of z, which is denoted by (Im z).

Any complex number is :

(i) Purely real, if b = 0 ;

(ii) Imaginary, if $b \neq 0$.

(iii) Purely imaginary, if a = 0

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non - negative.

(e) If z = a + ib, then a – ib is called complex conjugate of z and written as $\bar{z} = a - ib$

(f) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e. i > 0, 3 + i < 2 are meaningless.

Self Practice Problems

(1) Write the following as complex number

(i) $\sqrt{-16}$ (ii) $\sqrt{x}$, (x > 0) (iii) $-b + \sqrt{-4ac}$, (a, c> 0)

(2) Write the following as complex number

(i) $\sqrt{x}$ (x < 0) (ii) roots of $x^2 - (2\cos\theta)\,x + 1 = 0$

Answers : (1) (i) 0 + 4i (ii) $\sqrt{x} + 0i$ (iii) $-b + i\sqrt{4ac}$

(2) (i) $0 + i\sqrt{x}$ (ii) $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$

Algebraic Operations:

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by – 1 when it occurs.

1. Addition (a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d) i
2. Subtraction (a + bi) – (c + di) = a + bi – c – di = (a – c) + (b – d) i
3. Multiplication (a + bi) (c + di) = ac + adi + bci + bdi^2 = (ac – bd) + (ad+ bc)i
4. Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Example # 1 : Find the multiplicative inverse of $4 + 3i$.

Solution : Let z be the multiplicative inverse of $4 + 3i$ then

$z(4 + 3i) = 1$

$z = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{16+9} = \frac{4-3i}{25}$. **Ans.** $\frac{4-3i}{25}$

Self Practice Problem :

(3) Simplify $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $n \in I$. **Ans.** 0

Equality In Complex Number :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Example # 2 : Find the value of x and y for which

$(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$, where x, yR

Solution : $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

and $\quad 2x - y = -5 + 2y$

$\quad 2x + 5 = 3y$

when $\quad x = 2 \quad \Rightarrow \quad y = 3$

and $\quad x = -2 \quad \Rightarrow \quad y = 1/3$ **Ans.** (2,3) or (–2,1/3)

Example # 3 : Find the value of expression $x^4 + 4x^3 + 5x^2 + 2x + 3$, when $x = -1 + i$.

Solution : $x = -1 + i$

$(x + 1)^2 = i^2$

$x^2 + 2x + 2 = 0$

now, $x^4 + 4x^3 + 5x^2 + 2x + 3 = (x^2 + 2x + 2)(x^2 + 2x - 1) + 5 = 5$

Example # 4 : Find the square root of $-21 - 20i$

Solution : Let $x + iy = \sqrt{-21-20i}$

$(x + iy)^2 = -21 - 20i$

$x^2 - y^2 = -21$ ----- (i)

$xy = -10$ ----- (ii)

From (i) & (ii)

$x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

when $x = 2$, $y = -5$ and $x = -2$, $y = 5$

$x + iy = (2 - i5)$ or $(-2 + i5)$

Self Practice Problem

(4) Solve for z : $\bar{z} = i z^2$

(5) Given that $x, y \in R$, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

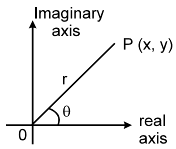
Answers : (4) $\pm \frac{\sqrt{3}}{2} - \frac{1}{2}i, 0, i$ (5) $x = K, y = \frac{3K}{2}$ $K \in R$

Representation of a complex number :

To each complex number there corresponds one and only one point in plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.

(a) Cartesian Form (Geometric Representation) :

Every complex number $z = x + i\,y$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called argument or amplitude.

$|z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \left(\dfrac{y}{x}\right)$ (angle made by OP with positive x–axis)

Note :

(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(iii) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0\,i$ the argument is not defined and this is the only complex number which is only given by its modulus.

(b) Trignometric/Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ

(c) Euler's Formula :

$z = re^{i\theta}$, $|z| = r$, $\arg z = \theta$

$\bar{z} = re^{-i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$.

Note : If θ is real then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(d) **Vectorial Representation :**

Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

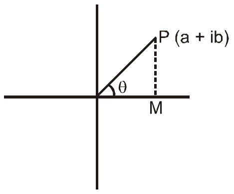
Agrument of a Complex Number :

Argument of a non-zero complex number P(z) is denoted and defined by arg(z) = angle which OP makes with the positive direction of real axis.

If $OP = |z| = r$ and $\arg(z) = \theta$, then obviously $z = r(\cos\theta + i\sin\theta)$, called the polar form of z.

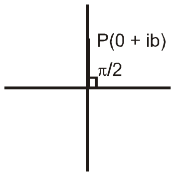
'Argument of z' would mean principal argument of z(i.e. argument lying in $(-\pi, \pi]$ unless the context requires otherwise. Thus argument of a complex number $z = a + ib = r(\cos\theta + i\sin\theta)$ is the value of θ satisfying $r\cos\theta = a$ and $r\sin\theta = b$. Let $\theta = \tan^{-1}\left|\dfrac{b}{a}\right|$

(i) $a > 0, b > 0$



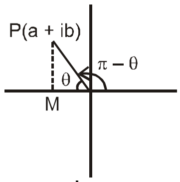
p.v. arg z = θ

(ii) $a = 0, b > 0$



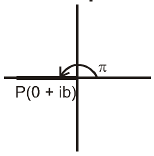
p.v. arg z = $\frac{\pi}{2}$

(iii) $a < 0, b > 0$



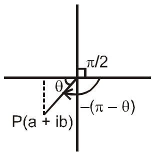
p.v. arg z = $\pi - \theta$

(iv) $a < 0, b = 0$



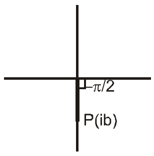
p.v. arg z = π

(v) $a < 0, b < 0$



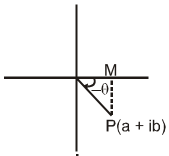
p.v. arg z = $-(\pi - \theta)$

(vi) $a = 0, b < 0$



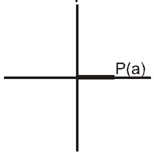
p.v. arg z = $-\frac{\pi}{2}$

(vii) $a > 0, b < 0$



p.v. arg z = $-\theta$

(viii) $a > 0, b = 0$



p.v. arg z = 0

Example # 5 : Solve for z if $(\bar{z})^2 + 2|z| = 0$.

Solution : Let $z = x + iy$

$\Rightarrow \quad x^2 - y^2 - 2ixy + 2\sqrt{x^2 + y^2} = 0 \quad \Rightarrow \quad x^2 - y^2 + 2\sqrt{x^2 + y^2} = 0$ and $2xy = 0$

when $x = 0$, $\Rightarrow \quad -y^2 + 2|y| = 0$

$\Rightarrow \quad y = 0, 2, -2$

$\Rightarrow \quad z = 0, 2i, -2i$

when $y = 0$ $\Rightarrow \quad x^2 + 2|x| = 0$

$\Rightarrow \quad x = 0 \Rightarrow z = 0$

Ans. $z = 0, 2i, -2i$.

Example # 6 : Find the modulus and principal argument of complex number z = 1 + i tan α, $\pi < \alpha < \frac{3\pi}{2}$

Solution : $|z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha| = -\sec\alpha$, where $\pi < \alpha < \frac{3\pi}{2}$

$\text{Arg}(z) = \tan^{-1}\left|\frac{\tan\alpha}{1}\right| = \tan^{-1}(\tan\alpha) = \alpha-\pi$

Ans. $-\sec\alpha$, $\alpha-\pi$

Self Practice Problems

(6) Find the principal argument and |z|. If $z = \frac{(2+i)\ (3-4i)}{3+i}$

(7) Find the |z| and principal argument of the complex number
$z = -8(\cos 310º - i \sin 310°)$

Answers : (6) $-\pi/4$, $\frac{5\sqrt{2}}{2}$ (7) 8, –130°

Demoivre's Theorem :

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where $n \in I$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ .....(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ ....\ \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ ....\ + \theta_n)$ where $n \in N$

(iii) If p, q $\in$ Z and q $\neq$ 0, then $(\cos\theta + i\sin\theta)^{p/q}$ can take 'q' distinct values which are equal to
$\cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$ where k = 0, 1, 2, 3, ..., q – 1

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

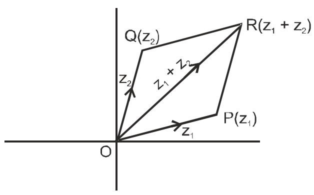
Self practice problems :

(8) Prove the identity: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;

(9) Prove that identity: $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

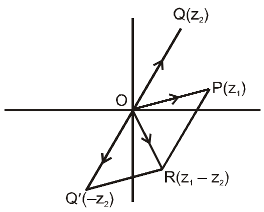
Geometrical Representation of Fundamental Operations :

(i) Geometrical representation of addition.



If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

(ii) Geometric representation of substraction.



(iii) Modulus and argument of multiplication of two complex numbers.

Theorem : For any two complex numbers z_1, z_2 we have $|z_1\ z_2| = |z_1|\ |z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Proof : $z_1 = r_1\ e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1z_2 = r_1r_2\ e^{i(\theta_1+\theta_2)} \quad \Rightarrow \quad |z_1z_2| = |z_1|\ |z_2|$

$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

i.e. to multiply two complex numbers, we multiply their absolute values and add their arguments.

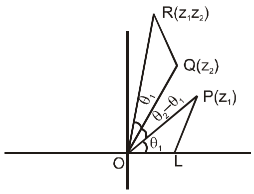
Note : (i) P.V. $\arg(z_1z_2) \neq$ P.V. $\arg(z_1)$ + P.V. $\arg(z_2)$

(ii) $|z_1\ z_2\ ....\ z_n| = |z_1|\ |z_2|\ .....\ |z_n|$

(iii) $\arg(z_1z_2\ ....\ z_n) = \arg z_1 + \arg z_2 + ... + \arg z_n$

(iv) Geometrical representation of multiplication of complex numbers.

Let P, Q be represented by $z_1 = r_1\ e^{i\theta_1}$, $z_2 = r_2\ e^{i\theta_2}$ repectively. To find point R representing complex number z_1z_2 , we take a point L on real axis such that OL = 1 and draw triangle OQR similar to triangle OLP. Therefore



$\dfrac{OR}{OQ} = \dfrac{OP}{OL} \Rightarrow OR = OP.OQ$ i.e. $OR = r_1r_2$ and $Q\hat{O}R = \theta_1$

$L\hat{O}R = L\hat{O}P + P\hat{O}Q + Q\hat{O}R = \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$

Hence, R is represented by $z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)}$

(v) Modulus and argument of division of two complex numbers.

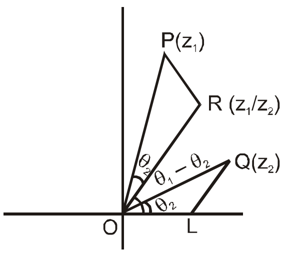
Theorem : If z_1 and z_2 ($\neq 0$) are two complex numbers, then $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note : P.V. $\arg\left(\dfrac{z_1}{z_2}\right) \neq$ P.V. $\arg(z_1)$ – P.V. $\arg(z_2)$

(vi) Geometrical representation of the division of complex numbers.

Let P, Q be represented by $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$ respectively. To find point R representing complex number $\dfrac{z_1}{z_2}$, we take a point L on real axis such that OL = 1 and draw a triangle OPR similar to OQL.

Therefore $\dfrac{OP}{OQ} = \dfrac{OR}{OL} \quad \Rightarrow \quad OR = \dfrac{r_1}{r_2}$ and $L\hat{O}R = L\hat{O}P - R\hat{O}P = \theta_1 - \theta_2$

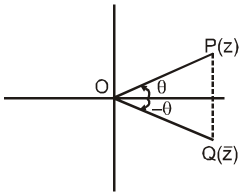


Hence, R is represented by $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\ e^{i(\theta_1-\theta_2)}$

Conjugate of a complex Number :

Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.
In a complex number if we replace i by – i, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.

Geometrical representation of conjugate of complex number.



$|z| = |\bar{z}|$

$\arg(\bar{z}) = -\arg(z)$

General value of $\arg(\bar{z}) = 2n\pi$ – P.V. $\arg(z)$

Properties

(i) If $z = x + iy$, then $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

(ii) $z = \bar{z} \Leftrightarrow$ z is purely real

(iii) $z + \bar{z} = 0 \Leftrightarrow$ z is purely imaginary

(iv) Relation between modulus and conjugate. $|z|^2 = z\bar{z}$

(v) $\bar{\bar{z}} = z$

(vi) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{(z_1 z_2)} = \bar{z}_1\,\bar{z}_2$, In general $\overline{(z^n)} = (\bar{z})^n$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(\bar{z}_1)}{(\bar{z}_2)}$ $(z_2 \neq 0)$

Theorem : Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs

Note : If $w = f(z)$, then $\bar{w} = f(\bar{z})$

Theorem : $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2)$
$= |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

Example # 7 : If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$

Solution : $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right) = 0 \quad \Rightarrow \quad \dfrac{z-1}{z+1} + \overline{\left(\dfrac{z-1}{z+1}\right)} = 0$

$\Rightarrow \quad \dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1} = 0 \quad \Rightarrow \quad z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0$

$\Rightarrow \quad z\bar{z} = 1 \quad \Rightarrow \quad |z|^2 = 1 \quad \Rightarrow \quad |z| = 1$ **Hence proved**

Example # 8 : If z_1 and z_2 are two complex numbers and $c > 0$, then prove that
$|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

Solution : We have to prove : $|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \leq c|z_1|^2 + c^{-1}|z_2|^2$ or $c|z_1|^2 + \dfrac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \geq 0$

(using $\mathrm{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$) or $\left(\sqrt{c}\,|z_1| - \dfrac{1}{\sqrt{c}}|z_2|\right)^2 \geq 0$ which is always true.

Example # 9 : Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

Solution : $z_1 = r(\cos\theta + i\sin\theta)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$z_2 = r(\cos\phi + i\sin\phi)$, $-\pi < \phi < 0$

$$\Rightarrow \quad \frac{z_1+z_2}{z_1-z_2} = -i\cot\left(\frac{\theta-\phi}{2}\right), \qquad -\frac{\pi}{4} < \frac{\theta-\phi}{2} < \frac{3\pi}{4}$$

Hence purely imaginary.

Self Practice Problem

(10) If $|z + \alpha| > |\bar{\alpha}z + 1|$ and $|\alpha| > 1$, then show that $|z| < 1$.

(11) If $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, then show that $f(z) = \bar{z}^2 + 2iz$

Distance, Triangular Inequality

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

In triangle OAC

OC ≤ OA + AC

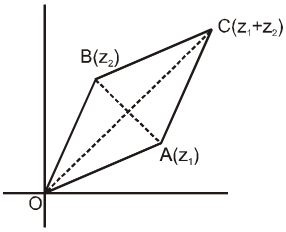
OA ≤ AC + OC

AC ≤ OA + OC

using these in equalities we have $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Similarly from triangle OAB

we have $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$



Note :

(a) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(b) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

Example # 10 : If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

Solution : We have $9 = |z - (5 + 7i)|$ = distance between z and 5 + 7i.

Thus locus of z is the circle of radius 9 and centre at 5 + 7i. For such a z (on the circle), we have to find its greatest and least distance as from 2 + 3i, which obviously 14 and 4.

Example # 11 : Find the minimum value of $|z| + |z - 2|$

Solution : $|z| + |z - 2| \geq |z + 2 - z|$

$|z| + |z - 2| \geq 2$

Example # 12 : If $\theta_i \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5, and $z^4 \cos\theta_1 + z^3 \cos\theta_2 + z^2 \cos\theta_3. + z \cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$,

then show that $|z| > \frac{3}{4}$

Solution : Given that $\cos\theta_1 . z^4 + \cos\theta_2 . z^3 + \cos\theta_3 . z^2 + \cos\theta_4 . z + \cos\theta_5 = 2\sqrt{3}$

or $|\cos\theta_1 . z^4 + \cos\theta_2 . z^3 + \cos\theta_3 . z^2 + \cos\theta_4 . z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1 . z^4| + |\cos\theta_2 . z^3| + |\cos\theta_3 . z^2| + \cos\theta_4 . z| + |\cos\theta_5|$

$\because$ $\theta_i \in [\pi/6, \pi/3]$

$\therefore$ $\frac{1}{2} \le \cos\theta_i \le \frac{\sqrt{3}}{2}$

$$2\sqrt{3} \le \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$$

$3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

Case I : If $|z| \ge 1$, then above result is automatically true

Case II : If $|z| < 1$, then

$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + ...\infty$

$3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \frac{3}{4}$ Hence by both cases, $|z| > \frac{3}{4}$

Example # 13 : $\left|z - \frac{3}{z}\right| = 2$, then find maximum and minimum value of $|z|$.

Solution : $\left||z| - \left|\frac{3}{z}\right|\right| \le \left|z - \frac{3}{z}\right|$

Let $|z| = r$

$\left|r - \frac{3}{r}\right| \le 2 \quad \Rightarrow -2 \le r - \frac{3}{r} \le 2$

$r^2 + 2r - 3 \ge 0$ ....(i) and $r^2 - 2r - 3 \le 0$ ....(ii)

$\Rightarrow$ $r \in [1, 3]$

from (i) and (ii)

$|z|_{max} = 3$ and $|z|_{min} = 1$.

Self Practice Problem

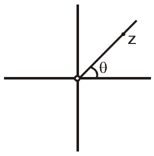
(12) $|z - 3| < 1$ and $|z - 4i| > M$ then find the positive real value of M for which there exist at least one complex number z satisfying both the equation.

(13) If z lies on circle $|z| = 2$, then show that $\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}$

Answers : (12) $M \in (0, 6)$

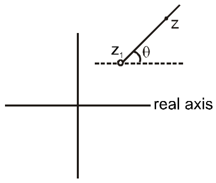
Important results :

(i) arg z = θ represnets points (non-zero) on ray



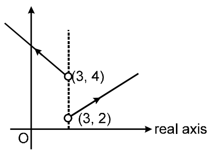
eminating from origin making an angle θ with positive direction of real axis

(ii) arg $(z - z_1) = \theta$ represents points $(\neq z_1)$ on ray eminating from z_1 making an angle θ with positive direction of real axis



Example # 14 : Solve for z, which satisfy Arg $(z - 3 - 2i) = \frac{\pi}{6}$ and Arg $(z - 3 - 4i) = \frac{2\pi}{3}$.

Solution : From the figure, it is clear that there is no z, which satisfy both ray

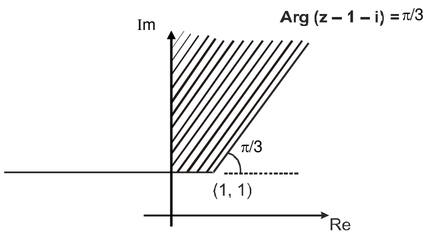


Example # 15 : Sketch the region given by

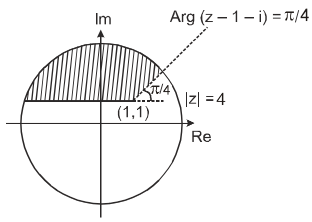
(i) $\pi/2 \geq$ Arg $(z - 1 - i) \geq \pi/3$ (ii) $|z| \leq 4$ & Arg $(z - i - 1) > \pi/4$

Solution :

(i)



(ii)



Self Practice Problems

(14) Sketch the region given by

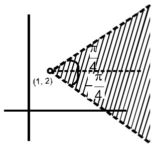
(i) $|\text{Arg}(z - i - 2)| < \pi/4$ (ii) Arg $(z + 1 - i) \leq \pi/6$

(15) Consider the region $|z - 4 - 3i| \leq 3$. Find the point in the region which has

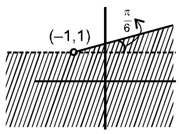
(i) max $|z|$ (ii) min $|z|$

(iii) max arg (z) (iv) min arg (z)

Answers : (14) (i)



(ii)

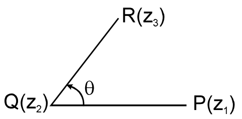


(15) (i) $\frac{32}{5} + i\frac{24}{5}$ (ii) $\frac{8}{5} + i\frac{6}{5}$ (iii) $\frac{28}{25} + i\frac{96}{25}$ (iv) 4 + 0i

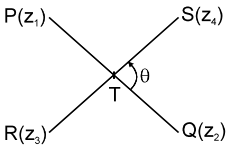
Rotation theorem :

(i) If $P(z_1)$ and $Q(z_z)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1\, e^{i\theta}$ where $\theta = \angle POQ$

(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$



(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers and $\angle STQ = \theta$, then $\dfrac{z_3 - z_4}{z_1 - z_2} = \left|\dfrac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$



Example # 16 : If arg $\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ then interpret the locus.

Solution : $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4} \Rightarrow \arg\left(\dfrac{1-z}{-1-z}\right) = \dfrac{\pi}{4}$

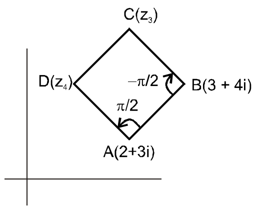
Here arg $\left(\dfrac{1-z}{-1-z}\right)$ represents the angle between lines joining –1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

Example # 17 : If A(2 + 3i) and B(3 + 4i) are two vertices of a square ABCD (take in anticlock wise order) then find C and D.

Solution : Let affix of C and D are z_3 and z_4 respectively.

Considering $\angle DAB = 90^{\circ}$ and AD = AB

we get $\dfrac{z_4 - (2+3i)}{(3+4i) - (2+3i)} = \dfrac{AD}{AB}\, e^{\frac{i\pi}{2}}$



$\Rightarrow \quad z_4 - (2 + 3i) = (1 + i)\, i \quad \Rightarrow \quad z_4 = 2 + 3i + i - 1 = 1 + 4i$

and $\quad \dfrac{z_3 - (3+4i)}{(2+3i) - (3+4i)} = \dfrac{CB}{AB}\, e^{-\frac{i\pi}{2}} \quad \Rightarrow \quad z_3 = 3 + 4i - (1 + i)(-i)$

$\Rightarrow \quad z_3 = 3 + 4i + i - 1 = 2 + 5i$

Self Practice Problems

(16) Let ABC be an isosceles triangle inscribed in the circle $|z| = r$ with AB = AC. If z_1, z_2, z_3 represent the points A, B, C respectively, show that $z_2 z_3 = z_1^2$

(17) Check that z_1z_2 and z_3z_4 are parallel or, not
where, $z_1 = 1 + i$ $z_3 = 4 + 2i$
$z_2 = 2 - i$ $z_4 = 1 - i$

(18) P is a point on the argand diagram on the circle with OP as diameter, two point Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that $z_2^2 \cos 2\theta = z_1z_3\cos^2\theta$

(19) If a, b, c ; u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)\, a + rb$, $w = (1 - r)\, u + rv$ where r is a complex number show that the two triangles are similiar.

Answers : (17) z_1z_2 and z_3z_4 are not parallel.

Cube Root of Unity :

(i) The cube roots of unity are 1, $\dfrac{-1 + i\sqrt{3}}{2}$, $\dfrac{-1 - i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :
$\cos 0 + i \sin 0$; $\cos \dfrac{2\pi}{3} + i \sin, \dfrac{2\pi}{3}$ $\cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

(v) The following factorisation should be remembered :
($a, b, c \in R$ & ω is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Example # 18 : Find the value of $\omega^{200} + \omega^{198} + \omega^{193}$.
Solution : $\omega^{200} + \omega^{198} + \omega^{193}$
$\omega^2 + 1 + \omega = 0$.

Example # 19 If W is an imaginary cube root of unity then find the value of $\dfrac{1}{1+2w} + \dfrac{1}{2+w} - \dfrac{1}{1+w}$

Solution :
$$\frac{1}{1+w+w} + \frac{1}{1+(1+w)} - \frac{1}{1+w} = \frac{1}{-w^2+w} + \frac{1}{1-w^2} - \frac{1}{-w^2}$$
$$= \frac{1}{w(1-w)} + \frac{1}{(1-w^2)} + \frac{1}{w^2} = \frac{w(1+w)+w^2+1-w^2}{w^2(1-w^2)} = \frac{1+w+w^2}{w^2(1-w^2)} = 0$$
Ans. 0

Self Practice Problem

(20) Find $\sum_{r=0}^{100}(1+\omega^r+\omega^{2r})$

(21) It is given that n is an odd integer greater than three, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$

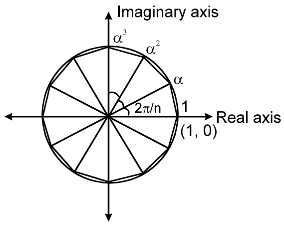
(22) If $x = a + b$, $y = a\alpha + b\beta$, $z = a\beta + b\alpha$ where α, β are imaginary cube roots of unity show that $xyz = a^3 + b^3$

(23) If $x^2 - x + 1 = 0$, then find the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$

Answers : (20) 102 (23) 8

n^{th} Roots of Unity :

If 1, α_1, α_2, α_3.... α_{n-1} are the n, n^{th} root of unity then :



(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2)......(1 - \alpha_{n-1}) = n$ &

$(1 + \alpha_1)(1 + \alpha_2).......(1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1.\ \alpha_1.\ \alpha_2.\ \alpha_3........\ \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Example # 20 : Find the roots of the equation $z^5 = -32i$, whose real part is negative.

Solution : $z^5 = -32i$

$z^5 = 2^5 e^{i(4n-1)\frac{\pi}{2}}$, n = 0, 1, 2, 3, 4.

$z = 2e^{i(4n-1)\frac{\pi}{10}}$

$z = 2e^{-i\frac{\pi}{10}},\ 2e^{i\frac{3\pi}{10}},\ 2e^{i\frac{7\pi}{10}},\ 2e^{i\frac{11\pi}{10}},\ 2e^{i\frac{15\pi}{10}}$ roots with negative real part are $2e^{i\frac{7\pi}{10}}, 2e^{i\frac{11\pi}{10}}$.

Example # 21 : Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution : $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=0}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$- \sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) + 1 = 0 – 0 + 1 = 1

Self Practice Problems

(24) If 1, α_1, α_2 , α_3 , α_4 are the fifth roots of unity then find $\sum_{i=1}^{4}\frac{1}{2-\alpha_i}$

(25) If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ and ω is a complex cube root of unity then prove that $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}=3\omega^2$

(26) Find all values of $(-256)^{1/4}$. Interpret the result geometrically.

Answers : (24) $\frac{49}{31}$

(26) $4\left[\cos\left(\frac{2r+1}{4}\right)\pi+i\sin\left(\frac{2r+1}{4}\right)\pi\right]$, r = 0, 1, 2, 3; vertices of a square in a circle of radius 4 & centre (0, 0)

The Sum Of The Following Series Should Be Remembered :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots.. + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots.. + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

Geometrical Properties :

Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$z = \frac{mz_2+nz_1}{m+n}$ where m, n > 0

If C divides PQ in the ratio m : n externally then $z = \frac{mz_2-nz_1}{m-n}$

Note : If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where a + b + c = 0 and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(1) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the Δ ABC = $\frac{z_1+z_2+z_3}{3}$:

(ii) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1+(b\sec B)z_2+(c\sec C)z_3}{a\sec A+b\sec B+c\sec C} \text{ or } \frac{z_1\tan A+z_2\tan B+z_3\tan C}{\tan A+\tan B+\tan C}$$

(iii) Incentre of the Δ ABC = $(az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of the Δ ABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(2) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the positive x− axis.

(3) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t\,(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1 (1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear. The above equation on manipulating, takes the form $\bar{\alpha} z + \alpha \bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(7) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg \dfrac{z - z_2}{z - z_1} = \pm \dfrac{\pi}{2}$

or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(8) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real $\Rightarrow$ $\dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \dfrac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$.

(9) $\text{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.

(10) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(11) If $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ where $k \in (0, 1) \cup (1, \infty)$, then locus of z is circle.

(12) If $\big||z - z_1| - |z - z_2|\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

Match the following columns :

	Column - I		**Column - II**
(i)	If $\mid z - 3+2i \mid - \mid z + i \mid = 0$, then locus of z represents	(i)	circle
(ii)	If arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then locus of z represents...	(ii)	Straight line
(iii)	if $\mid z - 8 - 2i \mid + \mid z - 5 - 6i \mid = 5$ then locus of z represents	(iii)	Ellipse
(iv)	If arg $\left(\frac{z-3+4i}{z+2-5i}\right) = \frac{5\pi}{6}$, then locus of z represents	(iv)	Hyperbola
(v)	If $\mid z - 1 \mid + \mid z + i \mid = 10$ then locus of z represents	(v)	Major Arc
(vi)	$\mid z - 3 + i \mid - \mid z + 2 - i \mid = 1$ then locus of z represents	(vi)	Minor arc
(vii)	$\mid z - 3i \mid = 25$	(vii)	Perpendicular bisector of a line segment
(viii)	arg $\left(\frac{z-3+5i}{z+i}\right) = \pi$	(viii)	Line segment

Ans.

I	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
II	(ii),(vii)	(v)	(viii)	(vi)	(iii)	(iv)	(i)	(viii)

Example # 22 : If z_1, z_2 & z_3 are the affixes of three points A, B & C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2 - i)\, z_1 + iz_3| = |z_1| + |(1 - i)\, z_1 + iz_3|$ then prove that Δ ABC in a right angled.

Solution : $|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow$ z_1, z_2 and origin will be collinear and z_1, z_2 will be opposite side of origin

Similarly $|(2 - i)\, z_1 + iz_3| = |z_1| + |(1 - i)\, z_1 + iz_3|$

$\Rightarrow$ z_1 and $(1 - i)\, z_1 + iz_3 = z_4$ say, are collinear with origin and lies on same side of origin. Let $z_4 = \lambda z_1$, λ real

then $(1 - i)\, z_1 + iz_3 = \lambda z_1$

$\Rightarrow$ $i\,(z_3 - z_1) = (\lambda - 1)\, z_1 \Rightarrow \frac{(z_3 - z_1)}{-z_1} = (\lambda - 1)\, I \Rightarrow \frac{z_3 - z_1}{0 - z_1} = me^{i\pi/2}$, $m = \lambda - 1$

$\Rightarrow$ $z_3 - z_1$ is perpendicular to the vector $0 - z_1$.

i.e. also z_2 is on line joining origin and z_1

so we can say the triangle formed by z_1 , z_2 and z_3 is right angled.

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Algebra of Complex Numbers and Its Representation and Demoivre's Theorem

A-1. Find the real values of x and y for which the following equation is satisfied :

(i) $\dfrac{(1+i)\,x-2i}{3+i}+\dfrac{(2-3i)\,y+i}{3-i}=i$

(ii) $\dfrac{x}{1+2i}+\dfrac{y}{3+2i}=\dfrac{5+6i}{8i-1}$

(iii) $(2 + 3i)\, x^2 - (3 - 2i)\, y = 2x - 3y + 5i$

(iv) $4x^2 + 3xy + (2xy - 3x^2)\, i = 4y^2 - (x^2/2) + (3xy - 2y^2)\, i$

A-2_. Let $z = \dfrac{1+2(\sin\theta)i}{1-(\sin\theta)i}$

(i) Find the number of values of $\theta\in[0, 4\pi]$ such that z is purely imaginary.
(ii) Find the sum of all values of $\theta\in[0, 4\pi]$ such that z is purely real.

A-3. (i) Find the real values of x and y for which $z_1 = 9y^2 - 4 - 10ix$ and $z_2 = 8y^2 - 20i$ are conjugate complex of each other.
(ii) Find the value of $x^4 - x^3 + x^2 + 3x - 5$ if $x = 2 + 3i$

A-4. Find
(i) the square root of 7 + 24 i (ii) $\sqrt{i}+\sqrt{-i}$

A-5. Solve the following for z :
$z^2 - (3 - 2i)\, z = (5i - 5)$

A-6. Simplify and express the result in the form of a + bi :

(i) $-i\,(9 + 6i)\,(2 - i)^{-1}$ (ii) $\left(\dfrac{4i^3-i}{2i+1}\right)^2$

(iii) $\dfrac{1}{(1-\cos\theta)+2\,i\sin\theta}$ (iv) $\left(\sqrt{3}+i\right)e^{-i\frac{\pi}{6}}$

A-7. Convert the following complex numbers in Eulers form

(i) $z = -\pi$ (ii) $z = 5i$ (iii) $z = -\sqrt{3} - i$ (iv) $z = -2\left(\cos\dfrac{\pi}{5}+i\sin\dfrac{\pi}{5}\right)$

A-8. Find the modulus, argument and the principal argument of the complex numbers.

(i) $z = 1 + \cos\dfrac{18\pi}{25} + i\sin\dfrac{18\pi}{25}$ (ii) $z = -2\,(\cos 30° + i \sin 30°)$

(iii) $(\tan 1 - i)^2$ (iv) $\dfrac{i-1}{i\left(1-\cos\dfrac{2\pi}{5}\right)+\sin\dfrac{2\pi}{5}}$

A-9. Dividing polynomial f(z) by z – i, we get the remainder i and dividing it by z + i, we get the remainder 1 + i. Find the remainder upon the division of f(z) by $z^2 + 1$.

A-10. If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$, then find
(i) $a^2 + b^2$ (ii) b

A-11. If n is a positive integer, prove the following
(i) $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$.
(ii) $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} \cdot \cos\frac{n\pi}{4}$

A-12. Show that $e^{i2m\theta}\left(\frac{i\cot\theta + 1}{i\cot\theta - 1}\right)^m = 1$.

A-13. If $x_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$, prove that $x_1 x_2 x_3$....... upto infinity = i.

Section (B) : Argument / Modulus / Conjugate Properties and Triangle Inequality

B-1. If z = x + iy is a complex number such that $z = (a + ib)^2$ then
(i) find $\bar{z}$
(ii) show that $x^2 + y^2 = (a^2 + b^2)^2$

B-2. If z_1 and z_2 are conjugate to each other, then find arg $(-z_1z_2)$.

B-3. If z ($\neq -1$) is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then find |z|

B-4. If |z – 2| = 2 |z – 1|, where z is a complex number, prove $|z|^2 = \frac{4}{3}$ Re (z) using
(i) polar form of z, (ii) z = x + iy, (iii) modulus, conjugate properties

B-5. For any two complex numbers z_1, z_2 and any two real numbers a, b show that
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

B-6. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

B-7. If k > 0, |z| = |w| = k and $\alpha = \frac{z - \bar{w}}{k^2 + z\bar{w}}$, then find Re($\alpha$).

B-8. (i) If $w = \frac{z+i}{z-i}$ is purely real then find arg z.
(ii) If $w = \frac{z+4i}{z+2i}$ is purely imaginary then find |z + 3i|.

B-9. If $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$ and $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove the following
(i) a + b + c = 0 (ii) ab + bc + ca = 0
(iii) $a^2 + b^2 + c^2 = 0$ (iv) $\Sigma\cos 2\alpha = 0 = \Sigma\sin 2\alpha$

B-10. If $|z - 1 + i| + |z + i| = 1$ then find range of principle argument of z.

Section (C) : Geometry of Complex Number and Rotation Theorem

C-1. If $|z - 2 + i| = 2$, then find the greatest and least value of $|z|$.

C-2. If $|z + 3| \le 3$ then find minimum and maximum values of
(i) $|z|$ (ii) $|z - 1|$ (iii) $|z + 1|$

C-3. Interpret the following locus in $z \in C$.
(i) $1 < |z - 2i| < 3$ (ii) $\text{Im}(z) \ge 1$
(iii) $\text{Arg}(z - 3 - 4i) = \pi/3$ (iv) $\text{Re}\left(\frac{z+2i}{iz+2}\right) \le 4$ $(z \ne 2i)$

C-4. If O is origin and affixes of P, Q, R are respectively z, iz, z + iz. Locate the points on complex plane. If $\Delta PQR = 200$ then find
(i) $|z|$ (ii) sides of quadrilateral OPRQ

C-5. The three vertices of a triangle are represented by the complex numbers, 0, z_1 and z_2. If the triangle is equilateral, then show that $z_1^2 + z_2^2 = z_1z_2$. Further if z_0 is circumcentre then prove that $z_1^2 + z_2^2 = 3z_0^2$.

C-6. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then show that $a^2 = 3b$.

C-7. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1) / (z - z_2)$ is $\pi/4$, then find the length of arc of the locus.

C-8. Let I : $\text{Arg}\left(\frac{z-8i}{z+6}\right) = \pm\frac{\pi}{2}$

II : $\text{Re}\left(\frac{z-8i}{z+6}\right) = 0$

Show that locus of z in I or II lies on $x^2 + y^2 + 6x - 8y = 0$. Hence show that locus of z can also be represented by $\frac{z-8i}{z+6} + \frac{\bar{z}+8i}{\bar{z}+6} = 0$. Further if locus of z is expressed as $|z + 3 - 4i| = R$, then find R.

C-9. Show that $z\bar{z} + (4 - 3i)z + (4 + 3i)\bar{z} + 5 = 0$ represents circle. Hence find centre and radius.

C-10. If z_1 & z_2 are two complex numbers & if $\arg\frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$ but $|z_1 + z_2| \ne |z_1 - z_2|$ then identify the figure formed by the points represented by 0, z_1, z_2 & $z_1 + z_2$.

Section (D) : Cube root and nth Root of Unity.

D-1. If ω ($\ne 1$) be a cube root of unity and $(1 + \omega^4)^n = (1 + \omega^2)^n$ then find the least positive integral value of n

D-2. When the polynomial $5x^3 + Mx + N$ is divided by $x^2 + x + 1$, the remainder is 0. Then find M + N.

D-3. Show that $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... to 2n factors $= 2^{2n}$

D-4. Let ω is non-real root of $x^3 = 1$

(i) If $P = \omega^n$, ($n \in N$) and

$Q = ({}^{2n}C_0 + {}^{2n}C_3 + ...) + ({}^{2n}C_1 + {}^{2n}C_4 + ...)\omega + ({}^{2n}C_2 + {}^{2n}C_5 + ...)\omega^2$ then find $\frac{P}{Q}$.

(ii) If $P = 1 - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^3}{8}......$ upto ∞ terms and $Q = \frac{1-\omega^2}{2}$ then find value of PQ.

D-5. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ and $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

D-6. Solve $(z - 1)^4 - 16 = 0$. Find sum of roots. Locate roots, sum of roots and centroid of polygon formed by roots in complex plane.

D-7. Find the value(s) of the following

(i) $\left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)^3$ (ii) $\left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)^{3/4}$

Hence find continued product if two or more distinct values exists.

D-8. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, then find the value

of $\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4}$ (where ω is imaginary cube root of unity.)

D-9. $a = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Algebra of Complex Numbers and Its Representation and Demoivre's Theorem

A-1. If z is a complex number such that $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then z is equal to

(A) $-2\sqrt{3} + 2i$ (B) $2\sqrt{3} + i$ (C) $2\sqrt{3} - 2i$ (D) $-\sqrt{3} + i$

A-2. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for

(A) $x = n\pi$ (B) $x = 0$ (C) $x = \frac{n\pi}{2}$ (D) no value of x

A.3. The least value of n ($n \in N$), for which $\left(\frac{1+i}{1-i}\right)^n$ is real, is

(A) 1 (B) 2 (C) 3 (D) 4

A-4. In G.P. the first term & common ratio are both $\frac{1}{2}\left(\sqrt{3}+i\right)$, then the modulus of n^{th} term is :

(A) 1 (B) 2^n (C) 4^n (D) 3^n

A-5. If z = (3 + 7i) (p + iq), where p, q $\in$ I – {0}, is purely imaginary, then minimum value of $|z|^2$ is

(A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

A-6. If z = x + iy and $z^{1/3}$ = a – ib then $\frac{x}{a} - \frac{y}{b} = k\left(a^2 - b^2\right)$ where k =

(A) 1 (B) 2 (C) 3 (D) 4

A-7. If z = $\frac{\pi}{4}$ $(1 + i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{amp(z)}\right)$ equals

(A) 1 (B) π (C) 3π (D) 4

A-8. The set of values of a $\in$ R for which $x^2 + i(a - 1)x + 5 = 0$ will have a pair of conjugate imaginary roots is

(A) R (B) {1}
(C) {a : $a^2 - 2a + 21 > 0$} (D) {0}

A-9. Let z is a complex number satisfying the equation, $z^3 - (3 + i)z + m + 2i = 0$, where m $\in$ R. Suppose the equation has a real root α, then find the value of $\alpha^4 + m^4$

(A) 32 (B) 16 (C) 8 (D) 64

A-10. The expression $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n - \frac{1+i\tan n\alpha}{1-i\tan n\alpha}$ when simplified reduces to :

(A) zero (B) 2 sin n α (C) 2 cos n α (D) none

A-11. If (cosθ + i sinθ) (cos 2θ + i sin 2θ) ... (cos nθ + i sin nθ) = 1, then the value of θ is

(A) $\frac{3m\pi}{n(n+1)}$, m $\in$ Z (B) $\frac{2m\pi}{n(n+1)}$, m $\in$ Z (C) $\frac{4m\pi}{n(n+1)}$, m $\in$ Z (D) $\frac{m\pi}{n(n+1)}$, m $\in$ Z

A-12. Let principle argument of complex number be re-defined between (π, 3π], then sum of principle arguments of roots of equation $z^6 + z^3 + 1 = 0$ is

(A) 0 (B) 3π (C) 6π (D) 12π

Section (B) : Argument / Modulus / Conjugate Properties and Triangle Inequality

B-1. If |z| = 1 and $\omega = \frac{z-1}{z+1}$ (where z $\neq$ –1), the Re(ω) is

(A) 0 (B) $-\frac{1}{|z+1|^2}$ (C) $\left|\frac{z}{z+1}\right| . \frac{1}{|z+1|^2}$ (D) $\frac{\sqrt{2}}{|z+1|^2}$

B-2. If $a^2 + b^2 = 1$, then $\frac{(1 + b + ia)}{(1 + b - ia)}$ =

(A) 1 (B) 2 (C) b + ia (D) a + ib

B-3. If (2 + i)(2 + 2i) (2 + 3i) ...... (2 + ni) = x + iy, then the value of 5.8.13. ....(4 + n^2)

(A) ($x^2 + y^2$) (B) $\sqrt{(x^2 + y^2)}$ (C) 2($x^2 + y^2$) (D) (x + y)

B-4. If z = x + iy satisfies amp (z – 1) = amp (z + 3) then the value of (x – 1) : y is equal to

(A) 2 : 1 (B) 1 : 3 (C) – 1 : 3 (D) does not exist

B-5. If $z_1 = -3 + 5i$; $z_2 = -5 - 3i$ and z is a complex number lying on the line segment joining z_1 & z_2, then arg(z) can be :

(A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

B-6. If $(1 + i)z = (1 - i)\bar{z}$ then z is

(A) $t(1 - i), t \in R$ (B) $t(1 + i), t \in R$ (C) $\frac{t}{1+i}, t \in R^+$ (D) $\frac{t}{1-i}, t \in R^+$

B-7. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals

(A) ω (B) $-\omega$ (C) $\bar{\omega}$ (D) $-\bar{\omega}$

B-8. If z_1 and z_2 are two non-zero complex numbers such that $\left|\frac{z_1}{z_2}\right| = 2$ and $\arg(z_1z_2) = \frac{3\pi}{2}$, then $\frac{\bar{z}_1}{z_2}$ is equal to

(A) 2 (B) –2 (C) –2i (D) 2i

B-9. Number of complex numbers z such that $|z| = 1$ and $|z/\bar{z} + \bar{z}/z| = 1$ is ($\arg(z) \in [0, 2\pi]$)

(A) 4 (B) 6 (C) 8 (D) more than 8

B-10. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to

(A) 1 (B) 3 (C) 0 (D) 2

B-11. The number of solutions of the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$ is

(A) 4 (B) 3 (C) 2 (D) 1

B-12. If $|z^2 - 1| = |z^2| + 1$, then z lies on :

(A) the real axis (B) the imaginary axis (C) a circle (D) an ellipse

B-13. If $|z - 2i| + |z - 2| \geq ||z| - |z - 2 - 2i||$, then locus of z is

(A) circle (B) line segment (C) point (D) complete x-y plane

Section (C) : Geometry of Complex Number and Rotation Theorem

C-1. The complex number $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on :

(A) the x-axis (B) the straight line $y = 5$
(C) a circle passing through the origin (D) the y-axis

C-2. The inequality $|z - 4| < |z - 2|$ represents :

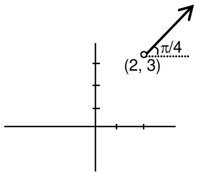
(A) $\text{Re}(z) > 0$ (B) $\text{Re}(z) < 0$ (C) $\text{Re}(z) > 2$ (D) $\text{Re}(z) > 3$

C-3. Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre is represented by the complex number :

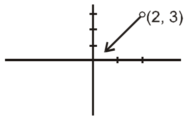
(A) $z_1 + z_2 - z_3$ (B) $z_2 + z_3 - z_1$ (C) $z_3 + z_1 - z_2$ (D) $z_1 + z_2 + z_3$

C-4. If Arg $(z - 2 - 3i) = \frac{\pi}{4}$, then the locus of z is

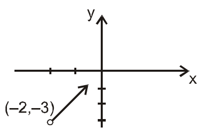
(A)



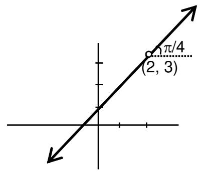
(B)



(C)



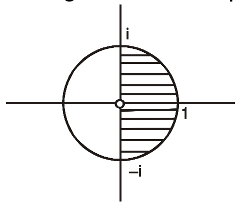
(D)



C-5. The system of equations $\left.\begin{array}{ll} |z+1-i| & = 2 \\ \text{Re } z & \geq 1 \end{array}\right\}$, where z is a complex number has :

(A) no solution
(B) exactly one solution
(C) two distinct solutions
(D) infinite solution

C-6. The locus of z which lies in shaded region is best represented by



(A) $|z| \leq 1, \ \frac{-\pi}{2} \leq \arg z \leq \frac{\pi}{2}$
(B) $|z| = 1, \ \frac{-\pi}{2} \leq \arg z \leq 0$
(C) $|z| \geq 0, \ 0 \leq \arg z \leq \frac{\pi}{2}$
(D) $|z| \leq 1, \ \frac{-\pi}{2} \leq \arg z \leq \pi$

C-7. The equation $|z - 1|^2 + |z + 1|^2 = 2$ represents
(A) a circle of radius '1'
(B) a straight line
(C) the ordered pair (0, 0)
(D) None of these

C-8. If $\arg\left(\frac{z-2}{z-4}\right) = \frac{\pi}{3}$ then locus of z is :
(A) equileteral triangle
(B) arc of circle
(C) arc of ellipse
(D) two rays making angle $\frac{\pi}{3}$ between them

C-9. The region of Argand diagram defined by $|z - 1| + |z + 1| \leq 4$ is :
(A) interior of an ellipse
(B) exterior of a circle
(C) interior and boundary of an ellipse
(D) exterior of ellipse

C-10. The vector $z = -4 + 5i$ is turned counter clockwise through an angle of 180º & stretched 1.5 times. The complex number corresponding to the newly obtained vector is :
(A) $6 - \frac{15}{2}i$ (B) $-6 + \frac{15}{2}i$ (C) $6 + \frac{15}{2}i$ (D) $-6 - \frac{15}{2}i$

C-11. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if :
(A) $z_1 + z_4 = z_2 + z_3$ (B) $z_1 + z_3 = z_2 + z_4$ (C) $z_1 + z_2 = z_3 + z_4$ (D) $z_1 z_3 = z_2 z_4$

C-12. Complex numbers z_1 , z_2 , z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C and $(z_1 - z_2)^2 = k(z_1 - z_3)(z_3 - z_2)$, then find k
(A) 1 (B) 2 (C) 3 (D) –2

C-13. If z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$ and if $z_1 = 1 + i\sqrt{3}$, then
(A) $z_2 = -2, z_3 = 1 + i\sqrt{3}$ (B) $z_2 = 2, z_3 = 1 - i\sqrt{3}$
(C) $z_2 = -2, z_3 = 1 - i\sqrt{3}$ (D) $z_2 = 1 - i\sqrt{3}, z_3 = -1 - i\sqrt{3}$

Section (D) : Cube root of unity and n^th Root of Unity.

D-1. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter then the value of λ is
(A) 4 (B) 3 (C) 2 (D) $\sqrt{2}$

D-2. If $x = a + b + c$, $y = a\alpha + b\beta + c$ and $z = a\beta + b\alpha + c$, where α and β are imaginary cube roots of unity, then xyz =
(A) $2(a^3 + b^3 + c^3)$ (B) $2(a^3 - b^3 - c^3)$ (C) $a^3 + b^3 + c^3 - 3abc$ (D) $a^3 - b^3 - c^3$

D-3. If 1, ω, ω^2 are the cube roots of unity, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$ is equal to-
(A) 0 (B) 1 (C) ω (D) ω^2

D-4. If $x^2 + x + 1 = 0$, then the numerical value of
$\left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \left(x^4+\frac{1}{x^4}\right)^2 + \ldots\ldots + \left(x^{27}+\frac{1}{x^{27}}\right)^2$ is equal to
(A) 54 (B) 36 (C) 27 (D) 18

D-5. If $a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots\ldots,$ $b = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots\ldots,$ $c = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots\ldots$
then find $a^3 + b^3 + c^3 - 3abc$.
(A) 1 (B) 2 (C) 3 (D) 4

D-6. If equation $(z - 1)^n = z^n = 1 (n \in N)$ has solutions, then n can be :
(A) 2 (B) 3 (C) 6 (D) 9

D-7. If α is non real and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to
(A) 4 (B) 2 (C) 1 (D) 8

D-8. If $\alpha = e^{i8\pi/11}$ then Real $(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ equals to :
(A) (B) 1 (C) $-\frac{1}{2}$ (D) –1

PART - III : MATCH THE COLUMN

1. **Match the column**

Column – I (Complex number Z)	**Column – II (Principal argument of Z)**
(A) $Z = \dfrac{(1+i)^5\left(1+\sqrt{3}i\right)^2}{-2i\left(-\sqrt{3}+i\right)}$.	(p) π
(B) $Z = \sin\dfrac{6\pi}{5} + i\left(1+\cos\dfrac{6\pi}{5}\right)$ is	(q) $-\dfrac{7\pi}{18}$
(C) $Z = 1 + \cos\left(\dfrac{11\pi}{9}\right) + i\sin\left(\dfrac{11\pi}{9}\right)$	(r) $\dfrac{9\pi}{10}$
(D) $Z = \sin x \sin(x-60)\sin(x+60)$ where $x \in \left(0, \dfrac{\pi}{3}\right)$ and $x \in R$	(s) $-\dfrac{5\pi}{12}$
	(t) 0

2. **Column I** — **Column II (one of the values of z)**

Column I	**Column II (one of the values of z)**
(A) $z^4 - 1 = 0$	p. $z = \cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}$
(B) $z^4 + 1 = 0$	q. $z = \cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}$
(C) $iz^4 + 1 = 0$	r. $z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$
(D) $iz^4 - 1 = 0$	s. $z = \cos 0 + i\sin 0$

3. Which of the condition/ conditions in column II are satisfied by the quadrilateral formed by z_1, z_2, z_3, z_4 in order given in column I ?

Column - I	**Column-II**
(A) Parallelogram	(p) $z_1 - z_4 = z_2 - z_3$
(B) Rectangle	(q) $\lvert z_1 - z_3\rvert = \lvert z_2 - z_4\rvert$
(C) Rhombus	(r) $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real
(D) Square	(s) $\dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
	(t) $\dfrac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

4. Let z_1 lies on | z | = 1 and z_2 lies on | z | = 2.

Column – I	**Column – II**
(A) Maximum value of $\lvert z_1 + z_2\rvert$	(p) 3
(B) Minimum value of $\lvert z_1 - z_2\rvert$	(q) 1
(C) Minimum value of $\lvert 2z_1 + 3z_2\rvert$	(r) 4
(D) Maximum value of $\lvert z_1 - 2z_2\rvert$	(s) 5

Exercise-2

Marked questions are recommended for Revision.

PART - I : ONLY ONE OPTION CORRECT TYPE

1. $\sin^{-1}\left\{\frac{1}{i}(z-1)\right\}$, where z is nonreal, can be the angle of a triangle if

(A) Re(z) = 1, Im(z) = 2 (B) Re(z) = 1, 0 < Im (z) ≤ 1

(C) Re(z) + Im(z) = 0 (D) Re(z) = 2, 0 < Im (z) ≤ 1

2. If $|z|^2 - 2iz + 2c(1 + i) = 0$, then the value of z is, where c is real.

(A) $z = c + 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c\in[-1-\sqrt{2}, -1+\sqrt{2}]$

(B) $z = c - 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c\in[-1-\sqrt{2}, -1+\sqrt{2}]$

(C) $z = 2c + 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c\in[-1-\sqrt{2}, -1+\sqrt{2}]$

(D) $z = c + 1\ i(-1 \pm)\sqrt{1-2c-c^2}$, where $c\in[-1-\sqrt{2}, 1+\sqrt{2}]$

3. If $(a + ib)^5 = \alpha + i\beta$, then $(b + ia)^5$ is equal to

(A) $\beta + i\alpha$ (B) $\alpha - i\beta$ (C) $\beta - i\alpha$ (D) $-\alpha - i\beta$

4. Let z be non real number such that $\frac{1+z+z^2}{1-z+z^2} \in R$, then value of 7|z| is

(A) 1 (B) 3 (C) 5 (D) 7

5. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|z_1 + z_2 + z_3| = 2$, then the value of $|4z_2z_3 + 9z_3z_1 + 16z_1z_2|$

(A) 24 (B) 48 (C) 96 (D) 120

6. The minimum value of |3z–3| + |2z–4| equal to

(A) 1 (B) 2 (C) 3 (D) 4

7. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

(A) is less than 6 (B) is more than 3

(C) is less than 12 (D) lies between 6 and 12

8. Let O = (0, 0) ; A=(3, 0) ; B = (0, –1) and C=(3, 2),then minimum value of |z|+|z–3|+|z+i|+|z–3–2i| occur at

(A) intersection point of AB and CO (B) intersection point of AC and BO

(C) intersection point of CB and AO (D) mean of O, A, B, C

9. Given z is a complex number with modulus 1. Then the equation $[(1 + ia)/(1 - ia)]^4 = z$ in 'a' has

(A) all roots real and distinct (B) two real and two imaginary

(C) three roots real and one imaginary (D) one root real and three imaginary

10. The real values of the parameter 'a' for which at least one complex number z = x + iy satisfies both the equality $|z - ai| = a + 4$ and the inequality $|z - 2| < 1$.

(A) $\left(-\frac{21}{10}, -\frac{5}{6}\right)$ (B) $\left(-\frac{7}{2}, -\frac{5}{6}\right)$ (C) $\left(\frac{5}{6}, \frac{7}{2}\right)$ (D) $\left(-\frac{21}{10}, \frac{7}{2}\right)$

11. The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are:
(A) $\frac{1}{2}\left(7 \pm i\sqrt{3}\right)$ (B) $\frac{1}{2}\left(3 \pm i\sqrt{7}\right)$ (C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

12. The equation of the radical axis of the two circles represented by the equations,
$|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is :
(A) $3iz - 3i\bar{z} - 2 = 0$ (B) $3iz - 3i\bar{z} + 2 = 0$ (C) $iz - i\bar{z} + 1 = 0$ (D) $2iz - 2i\bar{z} + 3 = 0$

13. If $\log_{1/2}\left(\frac{|z-1| + 4}{3|z - 1| - 2}\right) > 1$, then the locus of z is
(A) Exterior to circle with center 1 + i0 and radius 10
(B) Interior to circle with center 1 + i0 and radius 10
(C) Circle with center 1 + i0 and radius 10
(D) Circle with center 2 + i0 and radius 10

14. Points z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 ($z_3 \neq z_1$) is represented by :
(A) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$ (B) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 - z_2)$
(C) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_2 - z_1)$ (D) none of these

15. If $p = a + b\omega + c\omega^2$; $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$ where $a, b, c \neq 0$ and ω is the non-real complex cube root of unity, then :
(A) $p + q + r = a + b + c$ (B) $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$
(C) $p^2 + q^2 + r^2 = 2(pq + qr + rp)$ (D) None of these

16. The points $z_1 = 3 + \sqrt{3}\,i$ and $z_2 = 2\sqrt{3} + 6\,i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is :
(A) $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$ (B) $z = 5 + 5\,i$
(C) $z = -1 - i$ (D) none

17. Let ω be the non real cube root of unity which satisfy the equation $h(x) = 0$ where $h(x) = x\,f(x^3) + x^2g(x^3)$. If h(x) is polynomial with real coefficient then which statement is incorrect.
(A) $f(1) = 0$ (B) $g(1) = 0$ (C) $h(1) = 0$ (D) $g(1) \neq f(1)$

18. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ be the n^{th} roots of unity, then the value of
$\sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdot \sin\frac{3\pi}{n} \ldots\ldots \sin\frac{(n-1)\pi}{n}$ equals
(A) $\frac{n}{2^n}$ (B) $\frac{n}{2^{n-1}}$ (C) $\frac{n+1}{2^{n-1}}$ (D) $\frac{n}{2^{n+1}}$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer then find the value of $\frac{N}{2}$

2. Let z,w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $zw = \pi$. If Re(z) < 0 and principal arg $z = \frac{a\pi}{b}$ then find the value of a + b. (where a & b are co-prime natural numbers)

3. If $x = 9^{1/3}\, 9^{1/9}\, 9^{1/27} \ldots\infty$, $y = 4^{1/3}\, 4^{-1/9}\, 4^{1/27} \ldots \infty$, and $z = \sum_{r=1}^{\infty}(1+i)^{-r}$ and principal argument of $P = (x + yz)$ is $-\tan^{-1}\left(\frac{\sqrt{a}}{b}\right)$ then determine $a^2 + b^2$. (where a & b are co-prime natural numbers)

4. $z_1, z_2 \in c$ and $z_1^2 + z_2^2 \in R$,
$z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$
If $z_1^2 + z_2^2 = \lambda$ then determine λ^2

5. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in C$ (where C is the set of complex numbers), then find product of maximum and minimum value of $|w|$.

6. A function 'f' is defined by $f(z) = (4 + i)z^2 + \alpha z + \gamma$ for all complex number z, where α and γ are complex numbers if f(1) and f(i) are both real and the smallest possible values of $|\alpha| + |\gamma|$ is p then determine p^2.

7. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\arg(z) - \arg(\omega) = \frac{\pi}{2}$, then find the value of $5\, i\, \bar{z}\, \omega$.

8. Number of complex number satisfying $|z| = \max\{|z - 1|, |z + 1|\}$.

9. If z_1 & z_2 both satisfy the relation, $z + \bar{z} = 2\,|z - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find the imaginary part of $(z_1 + z_2)$.

10. If $a_1, a_2, a_3 \ldots a_n, A_1, A_2, A_3 \ldots A_n$, k are all real numbers and number of imaginary roots of the equation $\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + \ldots\ldots + \frac{A_n^2}{x - a_n} = k$ is α. Then find the value of $\alpha + 15$.

11. How many complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$.

12. If a variable circle S touches $S_1 : |z - z_1| = 7$ internally and $S_2 : |z - z_2| = 4$ externally while the curves S_1 & S_2 touch internally to each other, $(z_1 \neq z_2)$. If the eccentricity of the locus of the centre of the curve S is 'e' find the value of 11e.

13. Given that, $|z - 1| = 1$, where ' z ' is a point on the argand plane. $\frac{z-2}{2z} = \alpha\, i \tan(\arg z)$. Then determine $\frac{1}{\alpha^4}$.

14. Area of the region formed by $|z| \le 4$ & $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{3}$ on the Argand diagram is expressed in the form of $\frac{a\pi}{b}$. Then find the value of ab (where a & b are co-prime natural number)

15. The points A, B, C represent the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. If

$(z_2 - z_3)^2 = \lambda\,(z_3 - z_1)\,(z_1 - z_2)\,\sin^2\frac{\alpha}{2}$ then determine λ.

16. If ω and ω^2 are the non-real cube roots of unity and a, b, c $\in$ R such that $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega} = 2\omega^2$ and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2} = 2\omega$. If $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} = \lambda$ then determine λ^4

17. If $L = \lim\limits_{n\to\infty}\left[\frac{n}{(1-n\omega)(1-n\omega^2)}+\frac{n}{(2-n\omega)(2-n\omega^2)}+......+\frac{n}{(n-n\omega)(n-n\omega^2)}\right]$

then find the value of $\frac{\pi}{\sqrt{3}L}$ {where ω is non real cube root of unity}.

18. The value of $\sum\limits_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right) = \alpha$ then find α^4

19. Let $Z_r = \left(e^{i\frac{2\pi}{15}}\right)^r$. If $\arg\left(\frac{1+Z_1+Z_2+Z_3+.........+Z_7}{1+Z_8+Z_9+Z_{10}+.......+Z_{14}}\right) = \frac{a\pi}{b}$, then b – a equals. (where a & b are co-prime natural number)

20. If $A_1, A_2, ..., A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then find the value of n

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i.
(A) b = 51 (B) a = –6 (C) c = –70 (D) d = 170

2. The quadratic equation $z^2 + (p + ip')\,z + q + iq' = 0$; where p, p′, q, q′ are all real.
(A) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.
(B) if the equation has two equal roots then $pp' = 2q'$.
(C) if the equation has two equal roots then $p^2 - p'^2 = 4q$
(D) if the equation has one real root then $p'^2 - pp'q' + q'^2 = 0$.

3. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is :
(A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$ (C) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

4. If amp $(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then
(A) $z_1 + z_2 = 0$ (B) $z_1z_2 = 1$ (C) $z_1 = \bar{z}_2$ (D) $z_1 = z_2$

5. Let z_1 and z_2 are two complex numbers such that $(1 - i)z_1 = 2z_2$ and $\arg(z_1z_2) = \frac{\pi}{2}$, then $\arg(z_2)$ is equal to

(A) $3\pi/8$ (B) $\pi/8$ (C) $5\pi/8$ (D) $- 7\pi/8$

6. If $|z_1 + z_2|^2 = |z_1|^2 + |z_z|^2$ (where z_1 and z_2 are non-zero complex numbers), then

(A) $\frac{z_1}{z_2}$ is purely real
(B) $\frac{z_1}{z_2}$ is purely imaginary
(C) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$
(D) amp $\frac{z_1}{z_2}$ may be equal to $\frac{\pi}{2}$

7. a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$. If two roots of the equation $P(z) = 0$ are 2 and i. Then which of the following are true.

(A) $a = -\frac{11}{2}$ (B) $b = 5$ (C) $c = -\frac{11}{2}$ (D) $a = -11$

8. If $Z = \frac{(1+i)(1+2i)(1+3i)......(1+ni)}{(1-i)(2-i)(3-i)......(n-i)}$, $n \in N$ then principal argument of Z can be

(A) 0 (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{2}$ (D) π

9. For complex numbers z and w, if $|z|^2w - |w|^2 z = z - w$. Which of the following can be true :

(A) $z = w$ (B) $z\bar{w} = 1$ (C) $z = w + 2$ (D) $\bar{z}w = 1$

10. If z satisfies the inequality $|z - 1 - 2i| \leq 1$, then which of the following are true.

(A) maximum value of $|z| = \sqrt{5}+1$
(B) minimum value of $|z| = \sqrt{5}-1$
(C) maximum value of $\arg(z) = \pi/2$
(D) minimum value of $\arg(z) = \tan^{-1}\left(\frac{3}{4}\right)$

11. The curve represented by $z = \frac{3}{2+\cos\theta+i\sin\theta}$, $\theta \in [0, 2\pi)$

(A) never meets the imaginary axis
(B) meets the real axis in exactly two points
(C) has maximum value of $|z|$ as 3
(D) has minimum value of $|z|$ as 1

12. POQ is a straight line through the origin O . P and Q represent the complex number a + i b and c + i d respectively and OP = OQ. Then which of the follwoing are true :

(A) $|a + i b| = | c + i d|$
(B) $a + c = b + d$
(C) $\arg (a + i b) = \arg (c + i d)$
(D) none of these

13. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then which of the following are true.

(A) $|z_{2050}| = \sqrt{3}$ (B) $|z_{2017}| = \sqrt{2}$ (C) $|z_{2016}| = 1$ (D) $|z_{2111}| = \sqrt{2}$

14. If $|z_1| = |z_2| = ... = |z_n| = 1$ then which of the following are true.

(A) $\bar{z}_1 = \dfrac{1}{z_1}$

(B) $|z_1 + z_2 + ... + z_n| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + ... + \dfrac{1}{z_n}\right|$.

(C) Centroid of polygon with 2n vertices $z_1, z_2, ... z_n\ \dfrac{1}{z_1}, \dfrac{1}{z_2}, ... \dfrac{1}{z_n}$ (need not be in order) lies on real axis.

(D) Centroid of polygon with 2n vertices $z_1, z_2, ... z_n\ \dfrac{1}{z_1}, \dfrac{1}{z_2}, ... \dfrac{1}{z_n}$ (need not be in order) lies on imaginary axis.

15. If $2\cos\theta = x + \dfrac{1}{x}$ and $2\cos\phi = y + \dfrac{1}{y}$, then

(A) $x^n + \dfrac{1}{x^n} = 2\cos(n\theta),\ n \in z$

(B) $\dfrac{x}{y} + \dfrac{y}{x} = 2\cos(\theta - \phi)$

(C) $xy + \dfrac{1}{xy} = 2\cos(\theta + \phi)$

(D) $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\theta + n\phi),\ m, n \in z$

16. If $\left|\dfrac{z-\alpha}{z-\beta}\right| = k,\ k > 0$ where, $z = x + iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex numbers. Then which of the following are true

(A) if $k \neq 1$ then locus is a circle whose centre is $\left(\dfrac{k^2\beta - \alpha}{k^2 - 1}\right)$

(B) if $k \neq 1$ then locus is a circle whose radius is $\left|\dfrac{k(\alpha - \beta)}{1 - k^2}\right|$

(C) if $k = 1$ then locus is perpendicular bisector of line joining $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$

(D) if $k \neq 1$ then locus is a circle whose centre is $\left(\dfrac{k^2\alpha - \beta}{k^2 - 1}\right)$

17. The locus of equation $\text{Arg}\left(\dfrac{z-1-2i}{z+3+i}\right) = \dfrac{\pi}{3}$ represents part of circle in which

(A) centre is $\left[-1 - \dfrac{\sqrt{3}}{2} + i\left(\dfrac{1}{2} + \dfrac{2}{\sqrt{3}}\right)\right]$

(B) radius is $\dfrac{5}{\sqrt{3}}$

(C) centre is $\left[-1 - \dfrac{\sqrt{3}}{2} - i\left(\dfrac{1}{2} + \dfrac{2}{\sqrt{3}}\right)\right]$

(D) radius is $\dfrac{\sqrt{5}}{3}$

18. The equation $||z + i| - |z - i|| = k$ represents

(A) a hyperbola if $0 < k < 2$

(B) a pair of ray if $k > 2$

(C) a straight line if $k = 0$

(D) a pair of ray if $k = 2$

19. The equation $|z - i| + |z + i| = k,\ k > 0$, can represent

(A) an ellipse if $k > 2$

(B) line segment if $k = 2$

(C) an ellipse if $k = 5$

(D) no locus if $k = 1$

20. If $|z_1| = |z_2| = |z_3| = 1$ and z_1, z_2, z_3 are represented by the vertices of an equilateral triangle then

(A) $z_1 + z_2 + z_3 = 0$ (B) $z_1z_2z_3 = 1$

(C) $z_1z_2 + z_2z_3 + z_3z_1 = 0$ (D) $z_2^3 + z_3^3 = 2z_1^3$

21. Let z_1, z_2, z_3 be three distinct complex numbers satisfying, $|z_1-1| = |z_2-1| = |z_3-1| = 1$. Let A, B & C be the points representing vertices of equilateral triangle in the Argand plane corresponding to z_1, z_2 and z_3 respectively. Which of the following are true

(A) $z_1 + z_2 + z_3 = 3$ (B) $z_1^2 + z_2^2 + z_3^2 = 3$

(C) area of triangle = $\frac{3\sqrt{3}}{4}$ (D) $z_1z_2 + z_2z_3 + z_3z_1 = 1$

22. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ be the n^{th} roots of unity, then which of the following are true

(A) $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + \ldots\ldots + \frac{1}{1-\alpha_{n-1}} = \frac{n-1}{2}$

(B) $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots\ldots(1-\alpha_{n-1}) = n.$

(C) $(2-\alpha_1)(2-\alpha_2)(2-\alpha_3)\ldots\ldots(2-\alpha_{n-1}) = 2^n - 1$

(D) $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + \ldots\ldots + \frac{1}{1-\alpha_{n-1}} = \frac{n}{2}$

23. Which of the following are true.

(A) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots. + {}^nC_n \cos(n+1)x = 2^n . \cos^n \frac{x}{2} . \cos\left(\frac{n+2}{2}\right)x$

(B) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots. + {}^nC_n \sin(n+1)x = 2^n . \cos^n \frac{x}{2} . \sin\left(\frac{n+2}{2}\right)x$

(C) $1 + {}^nC_1 \cos x + {}^nC_2 \cos 2x + \ldots. + {}^nC_n \cos nx = 2^n . \cos^n \frac{x}{2} . \cos\left(\frac{nx}{2}\right)$

(D) ${}^nC_1 \sin x + {}^nC_2 \sin 2x + \ldots. + {}^nC_n \sin nx = 2^n . \cos^n \frac{x}{2} . \sin\left(\frac{nx}{2}\right)$

24. If α, β, γ are distinct roots of $x^3 - 3x^2 + 3x + 7 = 0$ and ω is non-real cube root of unity, then the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$ can be equal to :

(A) ω^2 (B) $2\omega^2$ (C) $3\omega^2$ (D) 3ω

25. If z is a complex number then the equation $z^2 + z|z| + |z^2| = 0$ is satisfied by
(ω and ω^2 are imaginary cube roots of unity)

(A) $z = k\omega$ where $k \in R$ (B) $z = k\omega^2$ where k is non negative real

(C) $z = k\omega$ where k is positive real (D) $z = k\omega^2$ where $k \in R$.

26. If α is imaginary n^{th} $(n \geq 3)$ root of unity. Which of the following are true.

(A) $\sum_{r=1}^{n-1}(n-r)\,\alpha^r = \frac{n\alpha}{1-\alpha}$ (B) $\sum_{r=1}^{n-1}(n-r)\sin\frac{2r\pi}{n} = \frac{n}{2}\cot\frac{\pi}{n}$.

(C) $\sum_{r=1}^{n-1}(n-r)\,\cos\frac{2r\pi}{n} = -\frac{n}{2}$ (D) $\sum_{r=1}^{n-1}(n-r)\,\alpha^r = \frac{n}{1-\alpha}$

27. Which of the following is true
(A) roots of the equation $z^{10} - z^5 - 992 = 0$ with real part positive = 5
(B) roots of the equation $z^{10} - z^5 - 992 = 0$ with real part negative = 5
(C) roots of the equation $z^{10} - z^5 - 992 = 0$ with imaginary part non-negative = 6
(D) roots of the equation $z^{10} - z^5 - 992 = 0$ with imaginary part negative = 4

PART - IV : COMPREHENSION

Comprehension # 1 (Q. No. 1 - 2)

Let $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$. For sum of series $C_0 + C_1 + C_2 + ...$, put x = 1. For sum of series $C_0 + C_2 + C_4 + C_6 + ...$, or $C_1 + C_3 + C_5 + ...$ add or substract equations obtained by putting x = 1 and x = – 1.
For sum of series $C_0 + C_3 + C_6 + ...$ or $C_1 + C_4 + C_7 + ...$ or $C_2 + C_5 + C_8 + ...$ we subsitute x = 1, $x = \omega$, $x = \omega^2$ and add or manipulate results.
Similarly, if suffixes differe by 'p' then we substitute p^{th} roots of unity and add.

1. $C_0 + C_3 + C_6 + C_9 +........ =$
(A) $\frac{1}{3}\left[2^n - 2\cos\frac{n\pi}{3}\right]$ (B) $\frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$ (C) $\frac{1}{3}\left[2^n - 2\sin\frac{n\pi}{3}\right]$ (D) $\frac{1}{3}\left[2^n + 2\sin\frac{n\pi}{3}\right]$

2. $C_1 + C_5 + C_9 +.... =$
(A) $\frac{1}{4}\left[2^n - 2^{n/2}2\cos\frac{n\pi}{4}\right]$ (B) $\frac{1}{4}\left[2^n + 2^{n/2}2\cos\frac{n\pi}{4}\right]$
(C) $\frac{1}{4}\left[2^n - 2^{n/2}2\sin\frac{n\pi}{4}\right]$ (D) $\frac{1}{4}\left[2^n + 2^{n/2}2\sin\frac{n\pi}{4}\right]$

Comprehension # 2 (Q. No. 3 to 6)

As we know $e^{i\theta} = \cos\theta + i\sin\theta$ and $(\cos\theta_1 + i\sin\theta_1).(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$
Let $\alpha, \beta, \gamma \in R$ such that $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

3. $\Sigma \sin(\alpha + \beta) = \Sigma \cos(\alpha + \beta) =$
(A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3\cos(\alpha + \beta + \gamma)$ (D) 3

4. $\Sigma\cos(2\alpha - \beta - \gamma)$
(A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3\cos(\alpha + \beta + \gamma)$ (D) 3

5. $\Sigma \cos 3\alpha =$
(A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3\cos(\alpha + \beta + \gamma)$ (D) 3

6. If $\theta \in R$ then $\frac{\Sigma\cos^3(\theta+\alpha)}{\Pi\cos(\theta+\alpha)} =$
(A) 0 (B) $3\cos\alpha \cos\beta \cos\gamma$ (C) $3\cos(\alpha + \beta + \gamma)$ (D) 3

Comprehension # 3(Q. No. 7 to 8)
ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. Let the points D and M represent complex numbers 1 + i and 2 – i respecetively.
If θ is arbitary real, then $z = re^{i\theta}$, $R_1 \le r \le R_2$ lies in annular region formed by concentric circles $|z| = R_1$, $|z| = R_2$.

7. A possible representation of point A is

(A) $3 - \frac{i}{2}$ (B) $3 + \frac{i}{2}$ (C) $1 + \frac{3}{2}i$ (D) $3 - \frac{3}{2}i$

8. If z is any point on segment DM then $w = e^{iz}$ lies in annular region formed by concentric circles.

(A) $|w|_{min} = 1$, $|w|_{max} = 2$ (B) $|w|_{min} = \frac{1}{e}$, $|w|_{max} = e$

(C) $|w|_{min} = \frac{1}{e^2}$, $|w|_{max} = e^2$ (D) $|w|_{min} = \frac{1}{2}$, $|w|_{max} = 1$

Comprehension # 4 (Q. No. 9 to 10)

Logarithm of a complex number is given by

$$\log_e(x + iy) = \log_e(|z|e^{i\theta})$$
$$= \log_e|z| + \log_e e^{i\theta}$$
$$= \log_e |z| + i\theta$$
$$= \log_e \sqrt{(x^2 + y^2)} + i \arg(z)$$

$\therefore$ $\log_e(z) = \log_e|z| + i \arg(z)$

In general $\log_e (x + iy) = \frac{1}{2} \log_e (x^2 + y^2) + i \left(2n\pi + \tan^{-1}\frac{y}{x}\right)$ where $n \in I$.

9. Write $\log_e (1 + \sqrt{3}\, i)$ in (a + ib) form

(A) $\log_e 2 + i(2n\pi + \frac{\pi}{3})$ (B) $\log_e 3 + i(n\pi + \frac{\pi}{3})$

(C) $\log_e 2 + i(2n\pi + \frac{\pi}{6})$ (D) $\log_e 2 + i(2n\pi - \frac{\pi}{3})$

10. Find the real part of $(1 - i)^{-i}$.

(A) $e^{\pi/4 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$ (B) $e^{-\pi/4 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$

(C) $e^{-\pi/4 + 2n\pi} \cos(\log_e 2)$ (D) $e^{-\pi/2 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$

Exercise-3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

Marked questions are recommended for Revision.

*** Marked Questions may have more than one correct option.**

1. Let $z = \cos\theta + i \sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}\ (z^{2m-1})$ at $\theta = 2^\circ$ is

[IIT-JEE-2009, Paper-I, (3, – 1), 80]

(A) $\frac{1}{\sin 2^\circ}$ (B) $\frac{1}{3\sin 2^\circ}$ (C) $\frac{1}{2\sin 2^\circ}$ (D) $\frac{1}{4\sin 2^\circ}$

2. Let z = x + iy be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is **[IIT-JEE-2009, Paper-I, (3, – 1), 80]**

(A) 48 (B) 32 (C) 40 (D) 80

3*. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)\, z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (B) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (D) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z_2 - z_1)$

[IIT-JEE-2010, Paper-1, (3, 0)/84]

4. Let ω be the complex number $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to **[IIT-JEE-2010, Paper-1, (3, 0)/84]**

5. Match the statements in **Column-I** with those in **Column-II.** **[IIT-JEE-2010, Paper-2, (8, 0)/79]**
[Note : Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column-I	**Column-II**
(A) The set of points z satisfying $|z - i|z|| = |z + i|z||$ is contained in or equal to	(p) an ellipse with eccentricity $\dfrac{4}{5}$
(B) The set of points z satisfying $|z + 4| + |z - 4| = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $|w| = 2$, then the set of points $z = w - \dfrac{1}{w}$ is contained in or equal to	(r) the set of point z satisfying $|\text{Im } z| \le 1$
(D) If $|w| = 1$, then the set of points $z = w + \dfrac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $|\text{Re } z| \le 2$
	(t) the set of points z satisfying $|z| \le 3$

6. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is
[IIT-JEE 2011, Paper-1, (4, 0), 80]

7. Let $\omega = e^{\frac{\pi i}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$
$$a + b\omega + c\omega^2 = y$$
$$a + b\omega^2 + c\omega = z.$$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

8. Let z be a complex number such that the imaginary part of z is non zero and $a = z^2 + z + 1$ is real. Then a cannot take the value **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) –1 (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{4}$

9. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lies on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

10.* Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1,2,3,....\}$. Further $H_1 = \left\{z \in C : \text{Re } z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re } z < -\frac{1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1O\, z_2 =$ **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

11.* Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) 57 (B) 55 (C) 58 (D) 56

Paragraph for Question Nos. 12 to 13

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}\, i}{1 - \sqrt{3}\, i}\right] > 0\right\}$ and

$S_3 : \{z \in C : \text{Re } z > 0\}$.

12. Area of S = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

13. $\min\limits_{z \in S} |1 - 3i - z| =$ **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

14. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, ...9$. **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

List I		**List II**	
P.	For each z_k there exists a z_j such that $z_k . z_j = 1$	**1.**	True
Q.	There exists a $k \in \{1,2,....,9\}$ such that $z_1 . z = z_k$ has no solution z in the set of complex numbers.	**2.**	False
R.	$\frac{\lvert 1-z_1 \rvert \lvert 1-z_2 \rvert ..... \lvert 1-z_9 \rvert}{10}$ equals	**3.**	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	**4.**	2

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

15. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12} \lvert \alpha_{k+1} - \alpha_k \rvert}{\sum_{k=1}^{3} \lvert \alpha_{4k-1} - \alpha_{4k-2} \rvert}$ is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

16. Let $z = \frac{-1+\sqrt{3}\,i}{2}$, where $i = \sqrt{-1}$ and $r, s \in \{1,2,3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is
[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

17. Let $a, b \in R$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{ z \in R : z = \frac{1}{a + ibt}, t \in R, t \neq 0 \right\}$, where $i = \sqrt{-1}$.
If $z = x + iy$ and $z \in S$, then (x, y) lies on **[JEE (Advanced) 2016, Paper-2, (4, –2)/62]**
(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0$, $b \neq 0$
(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a < 0$, $b \neq 0$
(C) the x-axis for $a \neq 0$, $b = 0$
(D) the y-axis for $a = 0$, $b \neq 0$

18. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?
[JEE(Advanced) 2017, Paper-1,(4, –2)/61]
(A) $1 - \sqrt{1+y^2}$ (B) $-1 - \sqrt{1-y^2}$ (C) $1 + \sqrt{1+y^2}$ (D) $-1 + \sqrt{1-y^2}$

19. For a non-zero complex number z, let arg(z) denote the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) **FALSE** ?

(A) $\text{Arg}(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$ **[JEE(Advanced) 2018, Paper-1,(4, –2)/60]**

(B) The function $f : R \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line.

20. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in R, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then,which of the following statement(s) is (are) TRUE ?

(A) If L has exactly one element, then $|s| \neq |t|$ **[JEE(Advanced) 2018, Paper-2,(4, –2)/60]**

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of |z| is equal to : **[AIEEE 2009, (4, –1), 144]**

(1) $\sqrt{5}+1$ (2) 2 (3) $2+\sqrt{2}$ (4) $\sqrt{3}+1$

2. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$ **[AIEEE 2010, (4, –1), 144]**

(1) – 1 (2) 1 (3) 2 (4) –2

3. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals **[AIEEE 2010, (4, –1), 120]**

(1) 1 (2) 2 (3) ∞ (4) 0

4. If $\omega(\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals **[AIEEE 2011, I, (4, –1), 120]**

(1) (0, 1) (2) (1, 1) (3) (1, 0) (4) (–1, 1)

5. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that : **[AIEEE- 2011, I, (4, –1), 120]**

(1) $\beta \in (0, 1)$ (2) $\beta \in (-1, 0)$ (3) $|\beta| = 1$ (4) $\beta \in (1, \infty)$

6. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals :

(1) $-\theta$ (2) $\frac{\pi}{2}-\theta$ (3) θ (4) $\pi-\theta$

[AIEEE - 2013, (4, –1), 120]

7. If z a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$:

(1) is strictly greater than 5/2 **[JEE(Main) 2014, (4, – 1), 120]**
(2) is strictly greater than 3/2 but less than 5/2
(3) is equal to 5/2
(4) lie in the interval (1, 2)

8. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a :

[JEE(Main) 2015, (4, – 1), 120]

(1) straight line parallel to x-axis (2) straight line parallel to y-axis
(3) circle of radius 2 (4) circle of radius $\sqrt{2}$

9. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main) 2016, (4, – 1), 120]**

(1) $\frac{\pi}{6}$ (2) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (3) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\frac{\pi}{3}$

10. Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to : **[JEE(Main) 2017, (4, – 1), 120]**

(1) $-z$ (2) z (3) –1 (4) 1

11. If $\alpha, \beta \in C$ are the distinct roots, of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to :
[JEE(Main) 2018, (4, – 1), 120]

(1) 1 (2) 2 (3) –1 (4) 0

12. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to :
[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

(1) 512 (2) –256 (3) 256 (4) –512

13. Let z be a complex number such that $|z| + z = 3 + i$, (where $i = \sqrt{-1}$) then $|z|$ is equal to :
[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]

(1) $\frac{\sqrt{34}}{3}$ (2) 5/4 (3) 5/3 (4) $\frac{\sqrt{41}}{4}$

14. Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then the minimum value of $|z_1 - z_2|$ is : **[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

Answers

EXERCISE - 1

PART – I

A-1. (i) 3, –1 (ii) x = 1 and y = 2 (iii) (1, 1) $\left(0, \frac{5}{2}\right)$ (iv) $x = K,\ y = \frac{3K}{2}$, $K \in R$

A-2_. (i) 8 (ii) 10π **A-3.** (i) [(–2, 2) ; (–2, –2)] (ii) – (77 + 108 i)

A-4. (i) ± (4 + 3i) (ii) $\pm\sqrt{2} + 0i$ or $0 \pm \sqrt{2}$ I **A-5.** z = (2 + i) or (1 – 3i)

A-6. (i) $\frac{21}{5} - \frac{12}{5}i$ (ii) 3 + 4 i (iii) $\frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)} + i\frac{-\cot\frac{\theta}{2}}{1+3\cos^2\frac{\theta}{2}}$ (iv) 2

A-7. (i) $\pi e^{i\pi}$ (ii) $5e^{i\frac{\pi}{2}}$ (iii) $2e^{-i\frac{5\pi}{6}}$ c (iv) $2e^{-i\frac{4\pi}{5}}$

A-8. (i) $|z| = 2\cos\frac{9\pi}{25}$ Principal Arg $z = \frac{9\pi}{25}$, arg $z = \frac{9\pi}{25} + 2k\pi$, $k \in I$

(ii) Modulus = 2, Arg = $2k\pi - \frac{5\pi}{6}$, $k \in I$, Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\sec^2 1$, arg = $2k\pi + (2 - \pi)$, Principal Arg = $(2 - \pi)$

(iv) Modulus = $\frac{1}{\sqrt{2}}$ cosec $\frac{\pi}{5}$, arg z = $2k\pi + \frac{11\pi}{20}$, Principal Arg = $\frac{11\pi}{20}$

A-9. $\frac{iz}{2} + \frac{1}{2} + i$ **A-10.** (i) 4 (ii) $\sqrt{3}$

Section (B) :

B-1. (i) $(a - ib)^2$ **B-2.** π **B-3.** 1 **B-7.** 0 **B-8.** (i) $\pm\frac{\pi}{2}$ (ii) 1

B-10. arg $z \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$

Section (C) :

C-1. $\sqrt{5} + 2$ & $\sqrt{5} - 2$ **C-2.** (i) 0, 6 (ii) 1, 7 (iii) 0, 5

C-3. (i) The region between the concentric circles with centre at (0, 2) & radii 1 and 3 units

(ii) The part of the complex plane on or above the line y = 1

(iii) a ray emanating from the point (3 + 4i) directed away from the origin & having equation, $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$, $x > 3$

(iv) Region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$

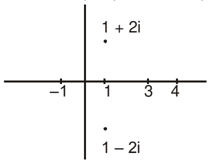
C-4. (i) $|z| = 20$ (ii) $OP = OQ = PR = QR = 20$ **C-7.** $9\dfrac{\pi}{\sqrt{2}}$ **C-8.** 5 **C-9.** $-4 - 3i,\ 2\sqrt{5}$

C-10. a rhombous but not a square

Section (D) :

D-1. 3 **D-2.** -5 **D-3.** 4^n **D-4.** (i) 1 (ii) 1

D-6. $z = -1, 3, 1 - 2i, 1 + 2i$



Sum = 4
centroid = 1

D-7. (i) -1 (ii) $e^{(6n+1)\frac{\pi}{4}i}$, n = 0, 1, 2, 3. Continued product = 1

D-8. ω **D-9.** $x^2 + x + 2 = 0$

EXERCISE - 2

PART - II

Section (A) :

A-1. (A) **A-2.** (D) **A.3.** (B) **A-4.** (A) **A-5.** (D) **A-6.** (D) **A-7.** (D)
A-8. (B) **A-9.** (A) **A-10.** (A) **A-11.** (C) **A-12.** (D)

Section (B) :

B-1. (A) **B-2.** (C) **B-3.** (A) **B-4.** (D) **B-5.** (D) **B-6.** (A) **B-7.** (D)
B-8. (D) **B-9.** (C) **B-10.** (C) **B-11.** (A) **B-12.** (B) **B-13.** (D)

Section (C) :

C-1. (A) **C-2.** (D) **C-3.** (D) **C-4.** (A) **C-5.** (B) **C-6.** (A) **C-7.** (C)
C-8. (B) **C-9.** (C) **C-10.** (A) **C-11.** (B) **C-12.** (B) **C-13.** (C)

Section (D) :

D-1. (B) **D-2.** (C) **D-3.** (A) **D-4.** (A) **D-5.** (A) **D-6.** (C) **D-7.** (A)
D-8. (C)

PART - III

1. (A) $\rightarrow$ (s), (B) $\rightarrow$ (r), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p) **2.** A $\rightarrow$ s; B $\rightarrow$ r; C $\rightarrow$ p; D $\rightarrow$ q.

3. a $\rightarrow$ p, r; b $\rightarrow$ p,q,r, t; c $\rightarrow$ p,r,s; d $\rightarrow$ p,q,r,s, t. **4.** (A) $\rightarrow$ (p), (B) $\rightarrow$ (q), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)

EXERCISE - 2

PART - I

1.	(B)	**2.**	(A)	**3.**	(A)	**4.**	(D)	**5.**	(B)	**6.**	(B)	**7.**	(C)	**8.**	(C)
9.	(A)	**10.**	(A)	**11.**	(B)	**12.**	(B)	**13.**	(A)	**14.**	(C)	**15.**	(C)	**16.**	(B)
17.	(D)	**18.**	(B)												

PART - II

1.	99	**2.**	7	**3.**	13	**4.**	25	**5.**	1	**6.**	2	**7.**	5
8.	0	**9.**	2	**10.**	15	**11.**	0	**12.**	3	**13.**	16	**14.**	60
15.	4	**16.**	16	**17.**	3	**18.**	11	**9.**	1	**20.**	7		

PART - III

1.	(ABCD)	**2.**	(ABC)	**3.**	(BD)	**4.**	(BC)	**5.**	(BD)	**6.**	(BCD)
7.	(ABC)	**8.**	(ABCD)	**9.**	(ABD)	**10.**	(ABCD)	**11.**	(ABCD)	**12.**	(AB)
13.	(BCD)	**14.**	(ABC)	**15.**	(ABCD)	**16.**	(ABC)	**17.**	(AB)	**18.**	(ACD)
19.	(ABCD)	**20.**	(ACD)	**21.**	(ABC)	**22.**	(ABC)	**23.**	(ABCD)	**24.**	(CD)
25.	(BC)	**26.**	(ABC)	**27.**	(ABCD)						

PART - IV

1.	(B)	**2.**	(D)	**3.**	(A)	**4.**	(D)	**5.**	(C)	**6.**	(D)	**7.**	(A)
8.	(B)	**9.**	(A)	**10.**	(B)	**9.**	(A)	**10.**	(B)				

EXERCISE - 3

PART - I

1. (D) **2.** (A) **3*.** (ACD) **4.** 1

5. (A) - (q,r), (B)-(p), (C) - (p,s,t), (D) - (q,r,s,t) **6.** 5

7. Bonus (w = $e^{i\pi/3}$ is a typographical error, because of this the answer cannot be an integer.)

(if w = $e^{i\frac{2\pi}{3}}$ then answer comes out to be 3)

8.	(D)	**9.**	(C)	**10.***	(CD)	**11.***	(BCD)	**11.**	(B)	**13.**	(C)	**14.**	(C)		
15.	4	**16.**	1	**17.**	(ACD)	**18.**	(BD)	**19.**	(ABD)	**20.**	(ACD)				

PART - II

1.	(1)	**2.**	(2)	**3.**	(1)	**4.**	(2)	**5.**	(4)	**6.**	(3)		
7.	(4)	**8.**	(3)	**9.**	(3)	**10.**	(1)	**11.**	(1)	**12.**	(2)	**13.**	(3)
14.	(1)												

High Level Problems (HLP)

1. If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$ where a_1, a_2, a_3, a_4 are real coefficient different from zero, has a purely imaginary root, then find the value of $\frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3}$.

2. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then find the value of $|8z_2z_3 + 27z_3z_1 + 64z_1z_2|$

3. If $|z|^2 + \bar{A}\;z^2 + A\bar{z}^2 + B\bar{z} + \bar{B}z + c = 0$ represents a pair of intersecting lines with angle of intersection 'θ' then find the value of $|A|$.

4. If $z^2 + \alpha z + \beta = 0$ (α, β are complex numbers) has a real root then prove that $(\bar{\alpha} - \alpha)(\alpha\bar{\beta} - \bar{\alpha}\beta) = (\beta - \bar{\beta})^2$.

5. If z_1, z_2, z_3 be three complex number such that
$|z_1| = |z_2| = |z_3| = 1$ and $\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0$
then sum of all the possible values of $|z_1 + z_2 + z_3|$.

6. Number of complex number (z) satisfying $|z|^2 = |z|^{n-2}z^2 + |z|^{n-2}z + 1$ such that $\text{Re}(z) \neq -\frac{1}{2}$ and $n = 2\lambda + 1$, $\lambda \in N$.

7. Let z_1 & z_2 be any two arbitrary complex numbers then prove that

(i) $|z_1 + z_2| = \left|\frac{z_1}{|z_1|}\,|z_2| + \frac{z_2}{|z_2|}\,|z_1|\right|$ (ii) $|z_1 + z_2| \geq \frac{1}{2}\left(|z_1| + |z_2|\right)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$.

8. Prove that

(i) $\left|\frac{z}{|z|} - 1\right| \leq |\arg z|$. (ii) $|z - 1| \leq ||z| - 1| + |z|\,|\arg z|$.

9. Prove that
$|\text{Img}(z^n)| \leq n\,|\text{Img}(z)||z|^{n-1}$, $n \in I^{\oplus}$

10. If $|z-1| + |z+3| \leq 8$ then find the range of values of $|z-4|$.

11. Show that all the roots of the equation $a_1z^3 + a_2z^2 + a_3z + a_4 = 3$, where $|a_i| \leq 1$, i = 1, 2, 3, 4 lie outside the circle with centre origin and radius 2/3.

12. Consider the locus of the complex number z in the Argand plane is given by $\text{Re}(z) - 2 = |z - 7 + 2i|$. Let $P(z_1)$ and $Q(z_2)$ be two complex number satisfying the given locus and also satisfying
$\arg\left(\frac{z_1 - (2+\alpha i)}{z_2 - (2+\alpha i)}\right) = \frac{\pi}{2}\ (\alpha \in R)$ then find the minimum value of PQ

13. Find the mirror image of the curve $\left|\frac{z-z_1}{z-z_2}\right| = a$, $a \in R^+$ $a \neq 1$ about the line $|z - z_1| = |z - z_2|$.

14. Let z_1 and z_2 are the two compelx numbers satisfying $|z - 3 - 4i| = 3$. Such that $\text{Arg}\left(\frac{z_1}{z_2}\right)$ is maximum then find the value of $|z_1 - z_2|$.

15. If z_1 and z_2 are the two complex numbers satisfying $|z - 3 - 4i| = 8$ and $\text{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ then find the range of the values of $|z_1 - z_2|$.

16. If $|z - z_1| = |z_1|$ and $|z - z_2| = |z_2|$ be the two circles and the two circles touch each other then prove that $\text{Img}\left(\frac{z_1}{z_2}\right) = 0$

17. If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p, q, r are the modulus of non-zero complex numbers u, v, w respectively, prove that, $\arg \frac{w}{v} = \arg \left(\frac{w-u}{v-u}\right)^2$.

18. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ & $|z_2| = 4$, if affix of A, B, C are z_1, z_2, $\left(\frac{z_2 - iz_1}{1-i}\right)$ respectively. Then find the area of ΔABC

19. Find the locus of mid-point of line segment intercepted between real and imaginary axes, by the line $a\bar{z} + \bar{a}z + b = 0$, where 'b' is real parameter and 'a' is a fixed complex number such that $\text{Re}(a) \neq 0$, $\text{Im}(a) \neq 0$.

20. Given $z_1 + z_2 + z_3 = A$, $z_1 + z_2\,\omega + z_3\,\omega^2 = B$, $z_1 + z_2\,\omega^2 + z_3\,\omega = C$, where ω is cube root of unity,

(a) express z_1, z_2, z_3 in terms of A, B, C.

(b) prove that, $|A|^2 + |B|^2 + |C|^2 = 3\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right)$.

(c) prove that $A^3 + B^3 + C^3 - 3ABC = 27z_1 z_2 z_3$

21. If $w \neq 1$ is n^{th} root of unity, then find the value of $\sum_{k=0}^{n-1} |z_1 + w^k \; z_2|^2$

22. Let a, b, c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$, $(a, b, c \neq 1)$. Find the value of k.

23. If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then find the value of, $f(x) + f(\alpha x) + ... + f(\alpha^6 x)$ independent of α.

24. Given, $z = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are, $\alpha = z + z^3 + ... + z^{2n-1}$ and $\beta = z^2 + z^4 + ... + z^{2n}$.

25. Prove that $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + ... + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When जब $n \in N$.

26. Proof that
(i) $\sin\frac{\pi}{2k+1}\sin\frac{2\pi}{2k+1}........\sin\frac{k\pi}{2k+1} = \frac{\sqrt{2k+1}}{2^k}$
(ii) $\cos\frac{\pi}{2k+1}\cos\frac{2\pi}{2k+1}........\cos\frac{k\pi}{2k+1} = \frac{1}{2^k}$

27. If Z_r, r = 1, 2, 3, ..., 2m, $m \in N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + ... + Z + 1 = 0$, then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

28. The points represented by the complex numbers a, b, c lie on a circle with centre O and radius r. The tangent at c cuts the chord joining the points a, b at z. Show that $z = \frac{a^{-1} + b^{-1} - 2c^{-1}}{a^{-1}b^{-1} - c^{-2}}$

29. Show that for the given complex numbers z_1 and z_2 and for a real constant c the equation

$$(z_1 + \lambda z_2)\bar{z} + (\bar{z}_1 + \lambda\bar{z}_2)z + c = 0$$

represents a family of concurrent lines and and also find the fixed point of the family.
(where λ is a real parameter)

30. Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundary.

Answers

1. 1 **2.** 96 **3.** $\frac{\sec\theta}{2}$ **5.** 3 **6.** 2 **10.** [1, 9] **12.** 10

13. $\left|\frac{z - z_2}{z - z_1}\right| = a$ **14.** $\frac{24}{5}$ **15.** $\left[\sqrt{103} - 5, \sqrt{103} + 5\right]$ **18.** $\frac{25}{4}$ **19.** $\bar{a}\bar{z} + az = 0$

20. (a) $z_1 = \frac{A + B + C}{3}$, $z_2 = \frac{A + B\omega^2 + C\omega}{3}$, $z_3 = \frac{A + B\omega + C\omega^2}{3}$

21. $n(|z_1|^2 + |z_2|^2)$ **22.** $-\omega$ or $-\omega^2$ **23.** $7A_0 + 7A_7x^7 + 7A_{14}x^{14}$

24. $z^2 + z + \frac{\sin^2 n\theta}{\sin^2\theta} = 0$, where $\theta = \frac{2\pi}{2n+1}$ **29.** $z = \frac{cz_2}{z_1\bar{z}_2 - z_2\bar{z}_1}$