Target : JEE (Main + Advanced)

Application of Derivatives

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APPLICATION OF DERIVATIVES

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JEE (ADVANCED) SYLLABUS

Tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, Rolle's theorem and Lagrange's mean value theorem.

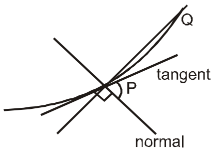
JEE (MAIN) SYLLABUS

Rolle's and Lagrange's Mean Value Theorems. Applications of derivatives: Rate of change of quantities, monotonic - increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

APPLICATION OF DERIVATIVES

Tangent and Normal

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...I.e. $Q \to P$, secant PQ will become tangent at P. A line through P



Figure

perpendicular to tangent is called normal at P.

Geometrical Meaning of $\frac{dy}{dx}$

As $Q \to P$, $h \to 0$ and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{Q\to P}$ slope of chord PQ = $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

Equation of tangent and normal

$\left.\frac{dy}{dx}\right]_{(x_1,\, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve y = f(x). Hence the equation of tangent at (x_1, y_1) is given by

$(y - y_1) = f'(x_1)\,(x - x_1)$; when, $f'(x_1)$ is real.

Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

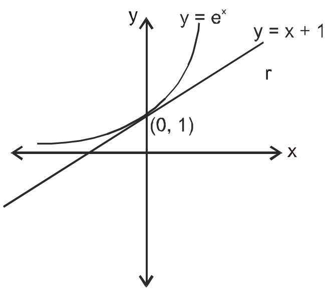
If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

If $\lim\limits_{h\to 0} \frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Example # 1 Find equation of tangent to $y = e^x$ at x = 0. Hence draw graph

Solution : At x = 0 $\Rightarrow$ $y = e^0 = 1$

$\frac{dy}{dx} = e^x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$



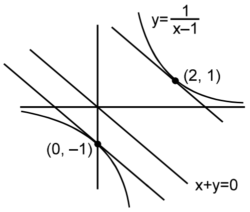
Hence equation of tangent is

1 (x – 0) = (y – 1)

$\Rightarrow$ y = x + 1

Example # 2 Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line x + y = 0.

Solution : Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$

$\Rightarrow \quad -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow \quad x_1 = 0 \quad$ or $\quad 2$

$\Rightarrow \quad y_1 = -1 \quad$ or $\quad 1$

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0) = (y+1)$ and $-1(x-2) = (y-1)$

$\Rightarrow \quad x + y + 1 = 0$ and $y + x = 3$

Example # 3 Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Solution : In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is (– 2, 2)

$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at (– 2, 2) is .

Hence equation of normal is

$\frac{1}{3}(x+2) = y - 2 \quad \Rightarrow \quad 3y = x + 8$

Example # 4 Prove that sum of intercepts of the tangent at any point to the curve represented by $x = 3\cos^4\theta$ & $y = 3\sin^4\theta$ on the coordinate axis is constant.

Solution : Let $P(3\cos^4\theta, 3\sin^4\theta)$ be a variable point on the given curve.

$\Rightarrow \quad m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3.4\sin^3\theta.\cos\theta}{-3.4\cos^3\theta\sin\theta} = -\tan^2\theta$

$\Rightarrow$ equation of tangent at point P is

$y - 3\sin^4\theta = -\tan^2\theta\,(x - 3\cos^4\theta)$

$\Rightarrow \quad \frac{x}{3\cos^2\theta} + \frac{y}{3\sin^2\theta} = 1$

$\Rightarrow$ sum of x-axis intercept and y-axis intercept $= 3\cos^2\theta + 3\sin^2\theta = 3$ (which is constant)

Self Practice Problems :

(1) Find the slope of the normal to the curve x = 1 – a sin θ, y = b cos² θ at $\theta = \frac{\pi}{2}$.

(2) Find the equation of the tangent and normal to the given curves at the given points.
(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)
(ii) $y^2 = \frac{x^3}{4-x}$ at (2, – 2).

(3) Prove that area of the triangle formed by any tangent to the curve $xy = c^2$ and coordinate axes is constant.

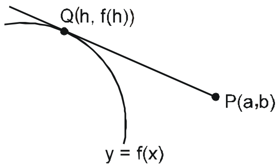
(4) A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q . Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

Ans. (1) $-\frac{a}{2b}$ (2) (i) Tangent : y = 2x + 1, Normal :x + 2y = 7
(ii) Tangent : 2x + y = 2, Normal :x – 2y = 6

Tangent and Normal from an external point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h)-b}{h-a}$$

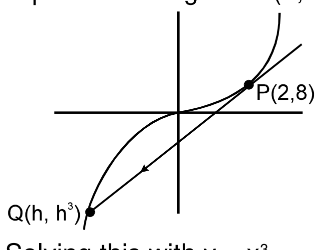


Figure

And equation of tangent is $y - b = \frac{f(h)-b}{h-a}(x-a)$

Example # 5 Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Solution : Equation of tangent at (2, 8) is y = 12x – 16



Solving this with $y = x^3$
$x^3 - 12x + 16 = 0$
This cubic will give all points of intersection of line and curve $y = x^3$ i.e., point P and Q. (see figure)
But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations :
sum of roots ⇒ 2 + 2 + h = 0 ⇒ h = – 4
Hence coordinates of Q are (– 4, – 64)

Self Practice Problems :

(5) How many tangents are possible from (1, 1) to the curve $y - 1 = x^3$. Also find the equation of these tangents.

(6) Find the equation of tangent to the hyperbola $y = \dfrac{x+9}{x+5}$ which passes through (0, 0) origin

Ans. (5) $y = 1$, $4y = 27x - 23$ (6) $x + y = 0$; $25y + x = 0$

Derivative as rate of change

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of charge of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition : The average rate of change of a function f(x) with respect to x over an interval [a, a + h] is defined as $\dfrac{f(a+h)-f(a)}{h}$

Definition : The instantaneous rate of change of f(x) with respect to x is defined as $f'(x) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

Note : To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

Example # 6 How fast the area of a circle increases when its radius is 5cm;
(i) with respect to radius (ii) with respect to diameter

Solution : (i) $A = \pi r^2$, $\dfrac{dA}{dr} = 2\pi r$

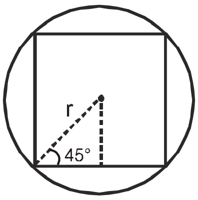
$\therefore \left.\dfrac{dA}{dr}\right]_{r=5} = 10\pi\ cm^2/cm.$

(ii) $A = \dfrac{\pi}{4} D^2$, $\dfrac{dA}{dD} = \dfrac{\pi}{2} D$

$\therefore \left.\dfrac{dA}{dD}\right]_{D=10} = \dfrac{\pi}{2} . 10 = 5\pi\ cm^2/cm.$

Example # 7 If area of circle increases at a rate of $2cm^2/sec$, then find the rate at which area of the inscribed square increases.

Solution : Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$\frac{dA_1}{dt} = 2\pi r\,\frac{dr}{dt}$, $\quad\frac{dA_2}{dt} = 4r\,.\,\frac{dr}{dt}$

$\because \quad 2 = 2\pi r\,.\,\frac{dr}{dt} \quad \Rightarrow \quad r\,\frac{dr}{dt} = \frac{1}{\pi}$

$\therefore \quad \frac{dA_2}{dt} = 4\,.\,\frac{1}{\pi} = \frac{4}{\pi}$ cm^2/sec

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm^2/sec.

Example # 8 The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution. Let at some time t, the length of edge is x cm.

$v = x^3 \quad \Rightarrow \quad \frac{dv}{dt} = 3x^2\,\frac{dx}{dt} \quad$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow \quad \frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

Now $\quad S = 6x^2$

$\frac{dS}{dt} = 12x\,\frac{dx}{dt} \quad \Rightarrow \quad \frac{dS}{dt} = 12x.\,\frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\frac{dS}{dt} = 7$ cm^2/sec.

Example # 9 Sand is pouring from pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution. $V = \frac{1}{3}\,\pi r^2 h$

but $\quad h = \frac{r}{6}$

$\Rightarrow \quad V = \frac{1}{3}\,\pi\,(6h)^2\,h$

$\Rightarrow \quad V = 12\pi\,h^3$

$\frac{dV}{dt} = 36\pi\,h^2.\,\frac{dh}{dt}$

when, $\frac{dV}{dt} = 12$ cm^3/s $\quad$ and $\quad h = 4$ cm

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}$ cm/sec.

Self Practice Problems :

(7) Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm.

(8) A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall.

(9) Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm^3/s. Find the rate at which slant height of water is decreasing when the height of water is $\sqrt{2}$ cm.

(10) A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment.

Ans. (7) 60π cm²/sec (8) $\frac{8}{3}$ cm/sec (9) $\frac{1}{\sqrt{2\pi}}$ cm/sec. (10) 140 ft/min.

Error and Approximation :

Let y = f(x) be a function. If these is an error δx in x then corresponding error in y is $\delta y = f(x + \delta x) - f(x)$.

We have $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{dy}{dx} = f'(x)$

We define the differential of y, at point x, corresponding to the increment δx as $f'(x)\,\delta x$ and denote it by dy.

i.e. $dy = f'(x)\,\delta x$.

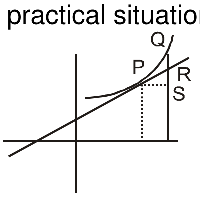
Let $P(x, f(x))$, $Q((x + \delta x), f(x + \delta x))$ (as shown in figure)

$\delta y = QS$,

$\delta x = PS$,

$dy = RS$

In many practical situations, it is easier to evaluate dy but not δy.



Example # 10. Find the approximate value of $25^{1/3}$.

Sol. Let $y = x^{1/3}$

Let x = 27 and $\Delta x = -2$

Now $\Delta y = (x + \Delta x)^{1/3} - x^{1/3} = (25)^{1/3} - 3$

$\frac{dy}{dx}\,\Delta x = 25^{1/3} - 3$

At x = 27, $25^{1/3} = 3 - 0.074 = 2.926$

Monotonicity of a function :

Let f be a real valued function having domain D(DR) and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes : (i) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non decreasing). But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non increasing). Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)

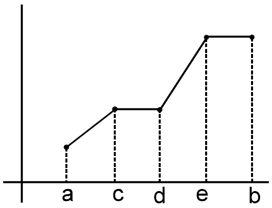
(i) If $f'(x) > 0 \ \forall \ x \in I$, then f is strictly increasing in I

(ii) If $f'(x) < 0 \ \forall \ x \in I$, then f is strictly decreasing in I

Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0, \ \forall \ x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.

$\{f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I $\}$

Let us consider another function whose graph is shown below for $x \in (a, b)$.

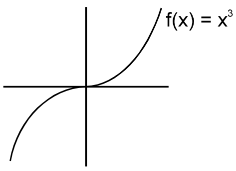


Here also $f'(x) \geq 0$ for all $x \in (a, b)$. But, note that in this case, $f'(x) = 0$ holds for all $x \in (c, d)$ and (e,b). Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

Example # 11 : Let $f(x) = x^3$. Find the intervals of monotonicity.

Solution : $f'(x) = 3x^2$

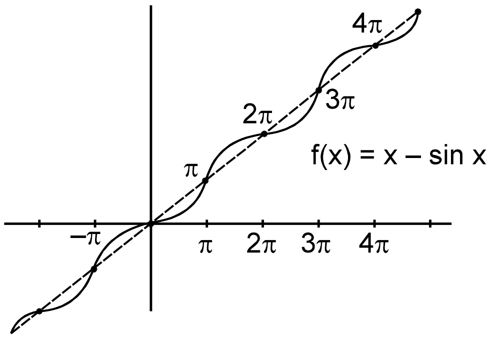
$f'(x) > 0$ everywhere except at $x = 0$. Hence f(x) will be strictly increasing function for $x \in R$ {see figure}



Example # 12 : Let $f(x) = x - \sin x$. Find the intervals of monotonicity.

Solution : $f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



Example # 13 : Find the intervals in which $f(x) = x^3 - 2x^2 - 4x + 7$ is increasing.

Solution : $f(x) = x^3 - 2x^2 - 4x + 7$

$f'(x) = 3x^2 - 4x - 4$

$f'(x) = (x - 2)(3x + 2)$

for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad x \in \left(-\infty, -\frac{2}{3}\right] \cup [2, \infty)$

Example # 14 : Find the intervals of monotonicity of the following functions.

(i) $f(x) = x^2 (x - 2)^2$ (ii) $f(x) = x \ln x$

Solution : (i) $f(x) = x^2 (x - 2)^2 \Rightarrow f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$

$-$ | $+$ | $-$ | $+$

0 1 2

Hence increasing in $[0, 1]$ and in $[2, \infty)$

and decreasing for $x \in (-\infty, 0]$ and $[1, 2]$

(ii) $f(x) = x \ln x$

$f'(x) = 1 + \ln x$

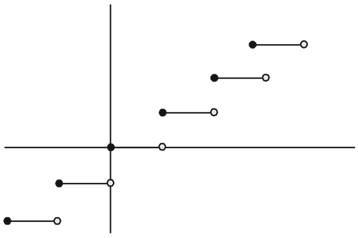
$f'(x) \geq 0 \quad \Rightarrow \quad \ln x \geq -1 \quad \Rightarrow \quad x \geq \frac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\frac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \frac{1}{e}\right]$.

Note : If a function $f(x)$ is increasing in (a, b) and $f(x)$ is continuous in $[a, b]$, then $f(x)$ is increasing on $[a, b]$

Example # 15 : $f(x) = [x]$ is a step up function. Is it a strictly increasing function for $x \in R$.

Solution : No, $f(x) = [x]$ is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



Example # 16 : If $f(x) = \sin^4 x + \cos^4 x + bx + c$, then find possible values of b and c such that $f(x)$ is monotonic for all $x \in R$

Solution : $f(x) = \sin^4 x + \cos^4 x + bx + c$

$f'(x) = 4 \sin^3 x \cos x - 4\cos^3 x \sin x + b = -\sin 4x + b.$

<u>**Case - (i)**</u> **:** for M.I. $f'(x) \geq 0$ for all $x \in R$

$\Leftrightarrow \quad b \geq \sin 4x$ for all $x \in R \quad \Leftrightarrow \quad b \geq 1$

<u>**Case - (ii)**</u> **:** for M.D. $f'(x) \leq 0$for all $x \in R$

$\Leftrightarrow \quad b \leq \sin 4x$ for all $x \in R \quad \Leftrightarrow \quad b \leq -1$

Hence for $f(x)$ to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Example # 17 : Find possible values of 'a' such that $f(x) = e^{2x} - 2(a^2 - 21) e^x + 8x + 5$ is monotonically increasing for $x \in R$

Solution : $f(x) = e^{2x} - 2(a^2 - 21) e^x + 8x + 5$

$f'(x) = 2e^{2x} - 2(a^2 - 21) e^x + 8 \geq 0 \; ; \forall \; c \; x \in R$

$\Rightarrow \quad e^x + \dfrac{4}{e^x} \geq a^2 - 21$

$4 \geq a^2 - 21 \qquad \left(\because\ e^x + \dfrac{4}{e^x} \geq 4\right)$

$\Rightarrow \quad a \in [-5, 5]$

Self Practice Problems :

(11) Find the intervals of monotonicity of the following functions.

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$

(ii) $f(x) = x + \dfrac{1}{x+1}$

(iii) $f(x) = x \,.\, e^{x-x^2}$

(iv) $f(x) = x - \cos x$

(12) Let $f(x) = x - \tan^{-1}x$. Prove that f(x) is monotonically increasing for $x \in R$.

(13) If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for $\forall\ x \in R$, then find range of values of a

(14) Let $f(x) = e^{2x} - ae^x + 1$. Prove that f(x) cannot be monotonically decreasing for $\forall\ x \in R$ for any value of 'a'.

(15) The values of 'a' for which function $f(x) = (a + 2)x^3 - ax^2 + 9ax - 1$ monotonically decreasing for $\forall\ x \in R$.

Ans. (11) (i) I in $[1, 3]$; D in $(-\infty, 1] \cup (3, \infty)$

(ii) I in $(-\infty, -2] \cup [0, \infty)$; D in $[-2, -1) \cup (-1, 0]$

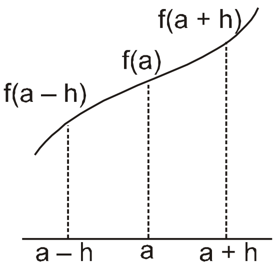
(iii) I in $\left[-\dfrac{1}{2}, 1\right]$; D in $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

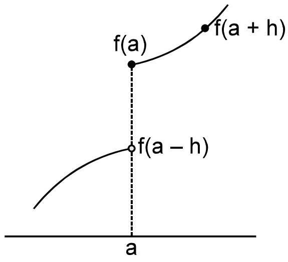
(iv) I for $x \in R$

(13) $a \geq 0$ (15) $-\infty < a \leq -3$

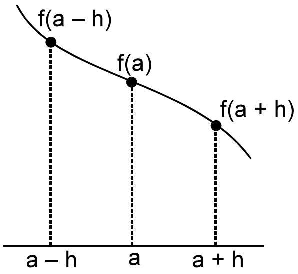
Monotonicity of function about a point :

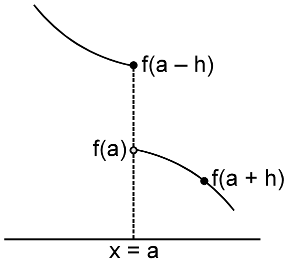
1. A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



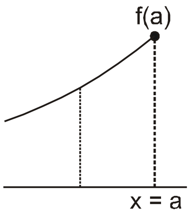


2. A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

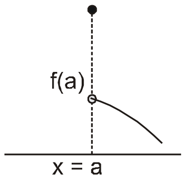




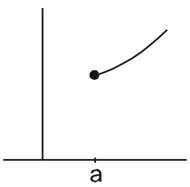
Note : If x = a is a boundary point then use the appropriate one sided inequality to test monotonicity of f(x).



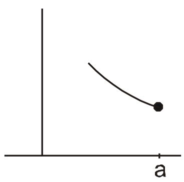
f is increasing at x = a



f is decreasing at x = a



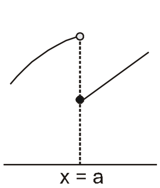
f is increasing at x = a



f is decreasing at x = a

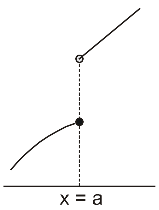
e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at x = a.

(i)



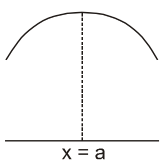
neither increasing nor decreasing at x = a

(ii)



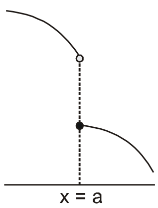
Increasing at x = a

(iii)



neither increasing nor decreasing at x = a

(iv)



decreasing at x = a

Test for increasing and decreasing functions about a point

Let f(x) be differentiable.

(1) If $f'(a) > 0$ then f(x) is increasing at x = a.

(2) If $f'(a) < 0$ then f(x) is decreasing at x = a.

(3) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

(i) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at $x = a$.

(ii) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at $x = a$.

(iii) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at $x = a$.

Example # 18 : Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points x = 0, 1, 2.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3$ $\Rightarrow$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighbourhood and $f'(x)$ is negative in right neighbourhood.

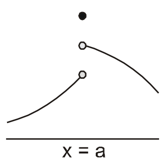
$\Rightarrow$ neither increasing nor decreasing at x = 1.

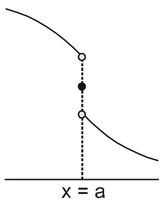
(iii) $f'(2) = 9$ $\Rightarrow$ increasing at x = 2

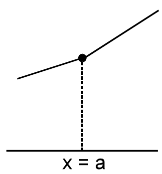
Note : Above method is applicable only for functions those are continuous at x = a.

Self Practice Problems :

(16) For each of the following graph comment on monotonicity of f(x) at x = a.

(i) 

(ii) 

(iii) 

(17) Let $f(x) = x^3 - 3x^2 + 3x + 4$, comment on the monotonic behaviour of f(x) at (i) x = 0 (ii) x = 1.

(18) Draw the graph of function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ [x] & 1 \le x \le 2 \end{cases}$. Graphically comment on the monotonic behaviour of f(x) at x = 1. Is f(x) M.I. for $x \in [0, 2]$?

Ans. (16) (i) neither M.I. nor M.D. (ii) M.D. (iii) M.I

(17) M.I. both at x = 0 and x = 1.

(18) M.I. at x = 1; f(x) is M.I. for $x \in [0, 2]$.

Global Maximum :

A function f(x) is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \le f(c)$ for all $x \in E$.

We say global maximum occurs at x = c and global maximum (or global maximum value) is f(c).

Local Maxima :

A function f(x) is said to have a local maximum at x = c if f(c) is the greatest value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \ne c$, we have $f(x) \le f(c)$.

Global Minimum :

A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \geq f(c)$ for all $x \in E$.

Local Minima :

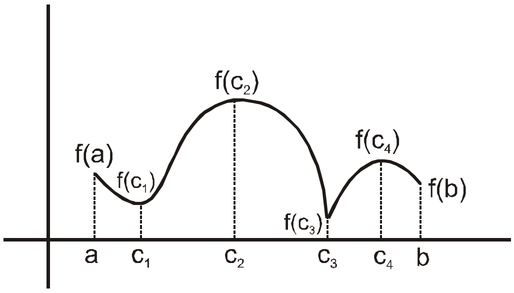
A function f(x) is said to have a local minimum at x = c if f(c) is the least value of the function in a small neighbourhood (c – h, c + h), h > 0 of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

Extrema :

A maxima or a minima is called an extrema.

Explanation : Consider graph of $y = f(x)$, $x \in [a, b]$



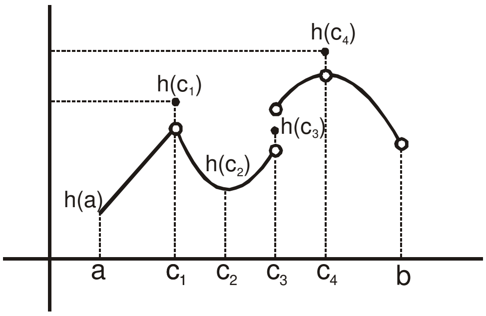
$x = c_2$, $x = c_4$ are points of local maxima, with maximum values $f(c_2)$, $f(c_4)$ respectively.

$x = c_1$, $x = c_3$ are points of local minima, with minimum values $f(c_1)$, $f(c_3)$ respectively

$x = c_2$ is a point of global maximum

$x = c_3$ is a point of global minimum

Consider the graph of $y = h(x)$, $x \in [a, b)$



$x = c_1$, $x = c_4$ are points of local maxima, with maximum values $h(c_1)$, $h(c_4)$ respectively.

$x = c_2$ are points of local minima, with minimum values $h(c_2)$ respectively.

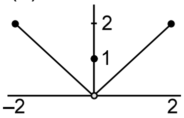
$x = c_3$ is neither a point of maxima nor a minima.

Global maximum is $h(c_4)$

Global minimum is h(a)

Example # 19 : Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

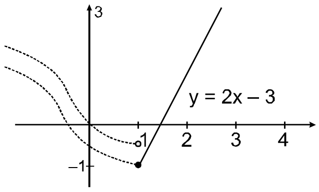
Solution : f(x) has local maxima at x = 0 (see figure).



Example # 20 : Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that f(x) has the smallest value at x = 1.

Solution. Such problems can easily be solved by graphical approach (as in figure).



Hence the limiting value of f(x) from left of x = 1 should be either greater or equal to the value of function at x = 1.

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

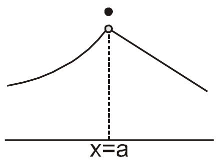
$$\Rightarrow \quad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$$

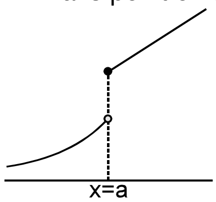
$$\Rightarrow \quad \frac{(b^2 + 1)(b - 1)}{(b + 1)\ (b + 2)} \ge 0$$

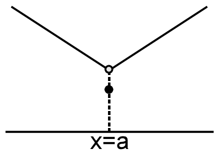
$$\Rightarrow \quad b \in (-2, -1) \cup [1, +\infty)$$

Self Practice Problems :

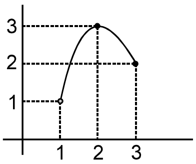
(19) In each of following graphs identify if x = a is point of local maxima, minima or neither

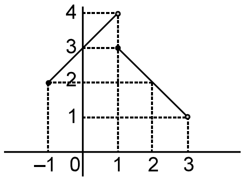
(i) 

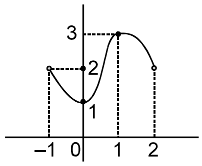
(ii) 

(iii) 

(20) Examine the graph of following functions in each case identify the points of global maximum/minimum and local maximum / minimum.

(i) 

(ii) 

(iii) 

Ans. (19) (i) Maxima (ii) Neither maxima nor minima
(iii) Minima

(20) (i) Local maxima at x = 2, Local minima at x = 3, Global maximum at x = 2. No global minimum
(ii) Local minima at x = – 1, No point of Global minimum, no point of local or Global maxima
(iii) Local & Global maximum at x = 1, Local & Global minimum at x = 0.

Maxima, Minima for differentiable functions :

Mere definition of maxima, minima becomes tedious in solving problems. We use derivative as a tool to overcome this difficulty.

A necessary condition for an extrema :

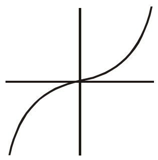
Let f(x) be differentiable at x = c.

Theorem : A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow$ $f'(c) = 0$

Note : $f'(c) = 0$ is only a necessary condition but not sufficient

i.e. $f'(c) = 0$ f(c) is extremum.



Consider $f(x) = x^3$

$f'(0) = 0$

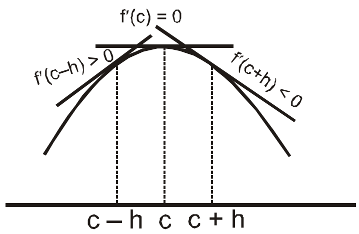
but f(0) is not an extremum (see figure).

Sufficient condition for an extrema :

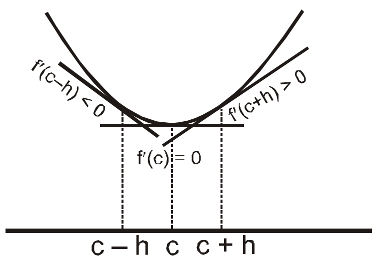
Let f(x) be a differentiable function.

Theorem : A sufficient condition for f(c) to be an extremum of f(x) is that $f'(x)$ changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ $f'(x)$ changes sign as x passes through c.



x = c is a point of maxima. $f'(x)$ changes sign from positive to negative.



x = c is a point of local minima (see figure), $f'(x)$ changes sign from negative to positive.

Stationary points :

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.
Rate of change of f(x) is zero at a stationary point.

Example # 21 : Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Solution : $f'(x) = 12x^2 - 12x - 24$

$f'(x) = 0 \quad \Rightarrow \quad x = -1, 2$

$f(-1) = 23, f(2) = -31$

$(-1, 23), (2, -31)$ are stationary points

Example # 22 : If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at $x = -1$ and $x = 3$. Find a, b, c.

Solution. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow \quad a = -3, b = -9, c \in R$

First Derivative Test :

Let f(x) be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

Step - III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

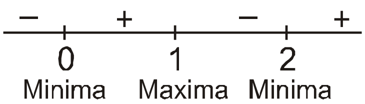
Example # 23 : Find the points of maxima or minima of $f(x) = x^2 (x - 2)^2$.

Solution. $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1) (x - 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence $x = 1$ is point of maxima, $x = 0, 2$ are points of minima.

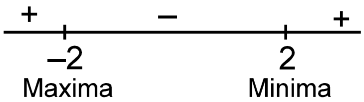
Note : In case of continuous functions points of maxima and minima are alternate.

Example # 24 : Find the points of maxima, minima of $f(x) = x^3 - 12x$. Also draw the graph of this functions.

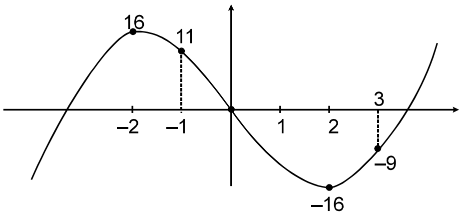
Solution. $f(x) = x^3 - 12x$

$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = \pm 2$



For tracing the graph let us find maximum and minimum values of f(x).



x	f(x)
2	–16
–2	+16

Example # 25 : Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima. Hence draw graph

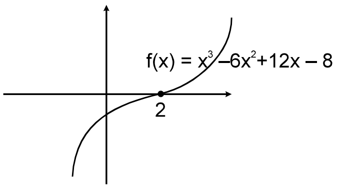
Solution. $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x - 2)^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

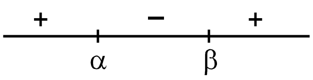
but clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$. So f(x) has no point of maxima or minima. In fact f(x) is a monotonically increasing function for $x \in R$.



Example # 26 : Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) has positive point of maxima, then find possible values of 'a'.

Solution. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \quad \Rightarrow \quad a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0 \quad \Rightarrow \quad a < 7$

(iii) $f'(0) > 0 \quad \Rightarrow \quad a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\quad \Rightarrow \quad a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

Self Practice Problems :

(21) Find the points of local maxima or minima of following functions

(i) $f(x) = (x - 1)^3 (x + 2)^2$

(ii) $f(x) = x^3 + x^2 + x + 1$.

Ans. (i) Maxima at $x = -2$, Minima at $x = -\frac{4}{5}$

(ii) No point of local maxima or minima.

Maxima, Minima for continuous functions :

Let f(x) be a continuous function.

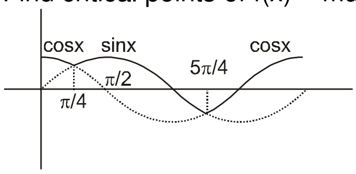
Critical points :

The points where $f'(x) = 0$ or $f(x)$ is not differentiable are called critical points.

Stationary points $\subseteq$ Critical points.

Example # 27 : Find critical points of $f(x) = \max(\sin x, \cos x)\ \forall,\ x \in (0, 2\pi)$.

Solution :



From the figure it is clear that f(x) has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

Important Note :

For f(x) defined on a subset of R, points of extrema (if exists) occur at critical points

Example # 28 : Find the possible points of Maxima/Minima for $f(x) = |x^2 - 2x|$ $(x \in R)$

Solution.

$$f(x) = \begin{cases} x^2 - 2x & x \geq 2 \\ 2x - x^2 & 0 < x < 2 \\ x^2 - 2x & x \leq 0 \end{cases}$$

$$f'(x) = \begin{cases} 2(x-1) & x > 2 \\ 2(1-x) & 0 < x < 2 \\ 2(x-1) & x < 0 \end{cases}$$

$f'(x) = 0$ at $x = 1$ and $f'(x)$ does not exist at $x = 0, 2$. Thus these are critical points.

Example # 29 : Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

Solution : f(x) is continuous at x = 0.

$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$$

$f'(0^+) = 3$ and $f'(0^-) = -10$ thus f(x) is non-differentiable at $x = 0$ $\Rightarrow$ $x = 0$ is a critical point.

Also derivative changes sign from negative to positive, so x = 0 is a point of local minima.

Example # 30 : Find the critical points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$ if (i) $x \in [0, 3]$ (ii) $x \in [-3, 3]$ (iii) $x \in [-1, 2]$.

Solution : $f'(x) = 12(x^2 - x - 2)$
$= 12(x - 2)(x + 1)$
$f'(x) = 0 \Rightarrow x = -1$ or 2
(i) if $x \in [0, 3]$, $x = 2$ is critical point.
(ii) if $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.
(iii) If $x \in [-1, 2]$, then no critical point as both $x = -1$ and $x = 2$ become boundary points.

Note : Critical points are always interior points of an interval.

Global extrema for continuous functions :

(i) Function defined on closed interval

Let $f(x)$, $x \in [a, b]$ be a continuous function
Step - I : Find critical points. Let it be $c_1, c_2, \ldots, c_n$
Step - II : Find $f(a), f(c_1), \ldots, f(c_n), f(b)$
Let $M = \max\cdot \{f(a), f(c_1), \ldots, f(c_n), f(b)\}$
$m = \min\cdot \{f(a), f(c_1), \ldots f(c_n), f(b)\}$
Step - III : M is global maximum.
m is global minimum.

(ii) Function defined on open interval.
Let $f(x)$, $x \in (a, b)$ be continuous function.
Step - I Find critical points . Let it be $c_1, c_2, \ldots c_n$
Step - II Find $f(c_1), f(c_2), \ldots, f(c_n)$
Let $M = \max\cdot \{f(c_1), \ldots f(c_n)\}$
$m = \min\cdot \{f(c_1), \ldots, f(c_n)\}$
Step - III $\lim\limits_{x \to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x \to b^-} f(x) = \ell_2$ (say).
Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$
Step - IV
(i) If $m \le \ell$ then m is global minimum
(ii) If $m > \ell$ then f(x) has no global minimum

(iii) If $M \ge L$ then M is global maximum
(iv) If $M < L$, then f(x) has no global maximum

Example # 31 : Find the greatest and least values of $f(x) = x^3 - 12x$ $x \in [-1, 3]$

Solution : The possible points of maxima/minima are critical points and the boundary points.
for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$
$x = 2$ is the only critical point.
Examining the value of f(x) at points $x = -1, 2, 3$. We can find greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

$\therefore$ Minimum $f(x) = -16$ & Maximum $f(x) = 11$.

Self Practice Problems :

(22) Let $f(x) = x^3 + x^2 - x - 4$

(i) Find the possible points of Maxima/Minima of f(x) for $x \in R$.

(ii) Find the number of critical points of f(x) for $x \in [1, 3]$.

(iii) Discuss absolute (global) maxima/minima value of f(x) for $x \in [-2, 2]$

(iv) Prove that for $x \in (1, 3)$, the function does not has a Global maximum.

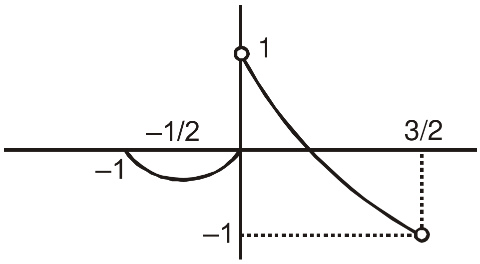
Ans. (i) $x = -1, \frac{1}{3}$ (ii) zero

(iii) $f(-2) = -6$ is global maximum, $f(2) = 6$ is global maximum

Example # 32 : Let $f(x) = \begin{cases} x^2 + x & ; \quad -1 \le x < 0 \\ \lambda & ; \quad x = 0 \\ \log_{1/2}\left(x + \frac{1}{2}\right) & ; \quad 0 < x < \frac{3}{2} \end{cases}$

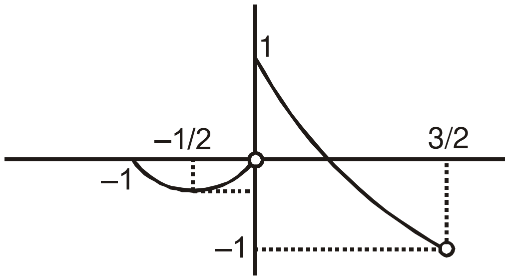
Discuss global maxima, minima for $\lambda = 0$ and $\lambda = 1$. For what values of λ does f(x) has global maxima

Solution : Graph of $y = f(x)$ for $\lambda = 0$



No global maxima, minima

Graph of $y = f(x)$ for $\lambda = 1$



Global maxima is 1, which occurs at x = 0

Global minima does not exists

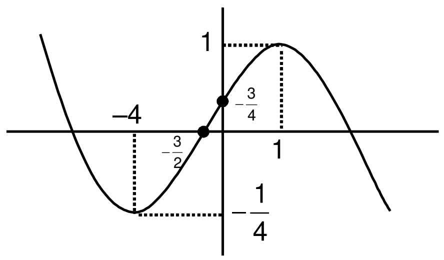
$\lim_{x\to 0^-} f(x) = 0, \lim_{x\to 0^+} f(x) = 1, f(0) = \lambda$

For global maxima to exists

$f(0) \ge 1 \quad \Rightarrow \quad \lambda \ge 1.$

Example # 33 : Find extrema of $f(x) = \frac{x^2+4}{2x+3}$. Draw graph of $g(x) = \frac{1}{f(x)}$ and comment on its local and global extrema.

Solution : $f'(x) = \frac{2(x^2+3x-4)}{(2x+3)^2} = \frac{2(x+4)(x-1)}{(2x+3)^2} = 0$



local minima occurs at x = –4

local maxima occurs at x = 1

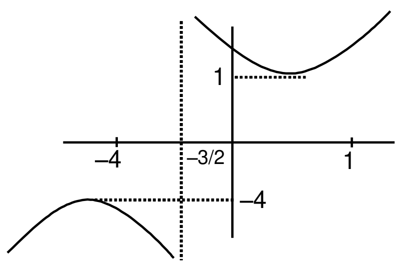
$g(x) = \frac{1}{f(x)} = \left(\frac{2x+3}{x^2+4}\right)$

$g'(x) = \frac{-2(x+4)(x-1)}{(x^2+4)^2}$

local maxima at x = –4

local minima at x = 1

global maxima & minima do not exists



Self Practice Problems :

(23) Let $f(x) = x + \frac{1}{x}$. Find local maximum and local minimum value of f(x). Can you explain this discrepancy of locally minimum value being greater than locally maximum value.

(24) If $f(x) = \begin{cases} (x+\lambda)^2 & x < 0 \\ \cos x & x \geq 0 \end{cases}$, find possible values of λ such that f(x) has local maxima at x = 0.

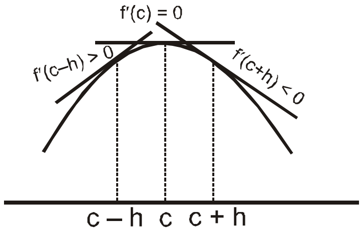
Answers : (23) Local maxima at x = –1, f(–1) = – 2 ; Local minima at x = 1, f(1) = 2.

(24) $\lambda \in [-1, 1)$

Maxima, Minima by higher order derivatives :

Second derivative test :

Let f(x) have derivatives up to second order
Step - I. Find $f'(x)$
Step - II. Solve $f'(x) = 0$. Let $x = c$ be a solution
Step - III. Find $f''(c)$
Step - IV.
(i) If $f''(c) = 0$ then further investigation is required
(ii) If $f''(c) > 0$ then $x = c$ is a point of minima.
(iii) If $f''(c) < 0$ then $x = c$ is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).
$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

Example # 34 : Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Solution : $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \quad \Rightarrow \quad \cos 2x = \dfrac{1}{2} \quad \Rightarrow \quad x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\dfrac{\pi}{6}\right) < 0 \quad \Rightarrow \quad$ Maxima at $x = \dfrac{\pi}{6}$

$f''\left(\dfrac{5\pi}{6}\right) > 0 \quad \Rightarrow \quad$ Minima at $x = \dfrac{5\pi}{6}$

Self Practice Problems :

(25) Let $f(x) = \sin x\,(1 + \cos x)$; $x \in (0, 2\pi)$. Find the number of critical points of f(x). Also identify which of these critical points are points of Maxima/Minima.

Ans. Three

$x = \dfrac{\pi}{3}$ is point of maxima.

$x = \pi$ is not a point of extrema.

$x = \dfrac{5\pi}{3}$ is point of minima.

n^{th} Derivative test :

Let f(x) have derivatives up to nth order
If $f'(c) = f''(c) = ... = f^{n-1}(c) = 0$ and
$f^{n}(c) \neq 0$ then we have following possibilities
(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima
(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.
(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$
(iv) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

Example # 35 : Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x - 1)(x - 3)$

$f'(x) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 10x (2x^2 - 6x + 3)$

Now, $f''(1) < 0 \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \Rightarrow$ Minima at $x = 3$

and, $f''(0) = 0 \Rightarrow$ II[nd] derivative test fails

so, $f'''(x) = 30 (2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Note : It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at $x = 0$

$f'(x) = 5x^2 (x - 1)(x - 3)$

+ + − +

0 1 3

max. min.

Application of Maxima, Minima :

For a given problem, an objective function can be constructed in terms of one parameter and then extremum value can be evaluated by equating the differential to zero. As discussed in n[th] derivative test maxima/minima can be identified.

Useful Formulae of Mensuration to Remember :

Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

volume of cube = ℓ^3 , Total Surface area of cube = $6\ell^2$

volume of cuboid = ℓbh, Total Surface area of cube = $2(\ell b + bh + h\ell)$

3-D Figures	Volume	Total Surface area	Curved/lateral Surface area
Cone	$\frac{1}{3}\pi r^2 h$	$\pi r\ell + \pi r^2$	Curved Surface area = $\pi r\ell$
Cylinder	$\pi r^2 h$	$2\pi rh + 2\pi r^2$	Curved Surface area = $2\pi rh$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$	
Prism	(area of base) × (height)	lateral surface area + 2 (area of base)	lateral Surface area = (perimeter of base) × (height)
Right Pyramid	$\frac{1}{3}$ ×(area of base) × (height)	Curved surface area + (area of base)	Curved Surface area = $\frac{1}{2}$ ×(perimeter of base) ×(slant height)

(Note that lateral surfaces of a prism are all rectangle).
(Note that slant surfaces of a pyramid are triangles).

Example # 36: If the equation $x^3 + px + q = 0$ has three real roots, then show that $4p^3 + 27q^2 < 0$.

Solution: $f(x) = x^3 + px + q$, $f'(x) = 3x^2 + p$

$\therefore$ f(x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$\Rightarrow$ $x = \pm\sqrt{\dfrac{-p}{3}}$, $p < 0$

f is maximum at $x = -\sqrt{\dfrac{-p}{3}}$ and minimum at $x = \sqrt{\dfrac{-p}{3}}$

$$f\left(-\sqrt{\frac{-p}{3}}\right) f\left(\sqrt{\frac{-p}{3}}\right) < 0$$

$$\left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$$

$q^2 + \dfrac{4p^3}{27} < 0$, $4p^3 + 27q^2 < 0$.

Example # 37 : Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Solution : $x + y = 60$

$\Rightarrow$ $x = 60 - y$ $\Rightarrow$ $xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\,y^3$; $y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2(60 - y) - y^3 = 0$

$f'(y) = y^2(180 - 4y) = 0$

$\Rightarrow$ $y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at $y = 45$.

So $x = 15$ and $y = 45$.

Example # 38 : Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

Solution : Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow$ $A = xy$.

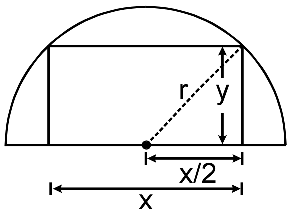
Here x and y are not independent variables and are related by Pythogorus theorem with r.

$\dfrac{x^2}{4} + y^2 = r^2$ $\Rightarrow$ $y = \sqrt{r^2 - \dfrac{x^2}{4}}$

$\Rightarrow$ $A(x) = x\sqrt{r^2 - \dfrac{x^2}{4}}$

$\Rightarrow$ $A(x) = \sqrt{x^2r^2 - \dfrac{x^4}{4}}$

Let $f(x) = r^2x^2 - \dfrac{x^4}{4}$; $x \in (0, r)$



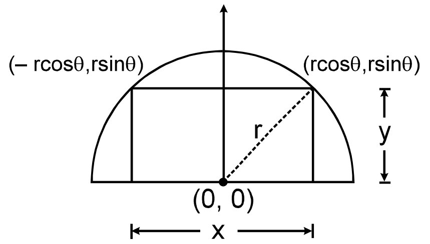
A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$ $\Rightarrow$ $x = r\sqrt{2}$

also $f'(r\sqrt{2}^+) < 0$ and $f'(r\sqrt{2}^-) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \dfrac{r}{\sqrt{2}}$.

Aliter Let us choose coordinate system with origin as centre of circle (as shown in figure).



A = xy

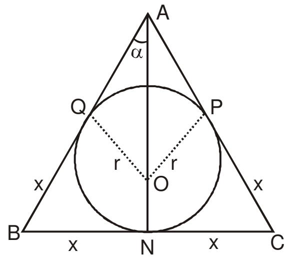
$\Rightarrow$ A = 2 (rcosθ) (rsinθ) $\Rightarrow$ A = r^2 sin2θ, $\theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4}$ $\Rightarrow$ $x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example # 39. Show that the least perimeter of an isosceles triangle circumscribed about a circle of radius 'r' is $6\sqrt{3}$ r .

Solution : AQ = r cot α = AP

AO = r cosecα



$\frac{x}{AO + ON} = \tan\alpha$

x = (r cosecα + r) tanα

x = r(secα + tanα)

Perimeter = p = 4x + 2AQ

p = 4r(secα + tanα) + 2rcotα

p = r(4secα + 4tanα + 2cotα)

$\frac{dp}{d\alpha} = r[4\sec\alpha \tan\alpha + 4\sec^2 \alpha - 2\operatorname{cosec}^2\alpha]$

for max or min $\frac{dp}{d\alpha} = 0 \Rightarrow$ $2\sin^3 \alpha + 3\sin^2\alpha - 1 = 0$

$\Rightarrow$ $(\sin\alpha + 1)(2\sin^2\alpha + \sin\alpha - 1) = 0$

$(\sin\alpha + 1)^2 (2\sin\alpha - 1) = 0$ $\Rightarrow$ sinα = 1/2 $\Rightarrow$ α = 30° = π/6

$p_{least} = r \left[\frac{4.2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + 2\sqrt{3}\right] = r\left[\frac{8+4+6}{\sqrt{3}}\right] = \frac{\left(6\sqrt{3}\sqrt{3}\right)}{\sqrt{3}}$ $r = 6\sqrt{3}$ r

Example # 40 : Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

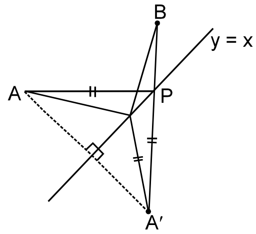
Solution. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x (see figure).

F(P) = PA + PB

F(P) = PA′ + PB

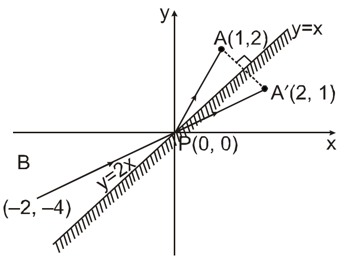
But for ΔPA′B



PA′ + PB $\geq$ A′B and equality hold when P, A′ and B becomes collinear. Thus for minimum path length point P is that special point for which PA and PB become incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P.

Hence P $\equiv$ (0, 0).



Note : Above concept is very useful because such problems become very lengthy by making perimeter as a function of position of P and then minimizing it.

Self Practice Problems :

(26) Find the two positive numbers x and y whose sum is 35 and the product $x^2\, y^5$ maximum.

(27) A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the slops to form a box. What should be the side of the square to be cut off such that volume of the box is maximum possible.

(28) Prove that a right circular cylinder of given surface area and maximum volume is such that the height is equal to the diameter of the base.

(29) A normal is drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the maximum distance of this normal from the centre.

(30) A line is drawn passing through point P(1, 2) to cut positive coordinate axes at A and B. Find minimum area of ΔPAB.

(31) Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point C and D respectively. The distance between C and D is c. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P.

Ans. (26) x = 25, y = 10. (27) 3 cm (29) 1 unit

(30) 4 units (31) P is at distance of $\frac{ac}{a+b}$ from C.

Use of monotonicity for proving inequalities

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of $h(x) = f(x) - g(x)$

Example # 41 : For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin^2 x < x^2 < \tan^2 x$

Solution : Let $f(x) = x^2 - \sin^2 x$

$= (x + \sin x)\ (x - \sin x) > 0$

for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \quad \sin^2 x < x^2 \quad$ .....(i)

Let $g(x) = x^2 - \tan^2 x$

$= (x + \tan x)\ (x + \tan x) < 0$

for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \quad x^2 < \tan^2 x \quad$ .....(ii)

From (i) & (ii) $\sin^2 x < x^2 < \tan^2 x$

Example # 42 : For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ hence or otherwise find $\lim\limits_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right]$

Solution : Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x \qquad f'(x) = 1 - x^2 - \frac{1}{1+x^2} \qquad f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1) \Rightarrow$ f(x) is M.D.

$\Rightarrow \quad f(x) < f(0) \quad \Rightarrow \quad x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow \quad x - \frac{x^3}{3} < \tan^{-1}x \quad$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$,$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2} \quad g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0 \quad$ for $x \in (0, 1) \Rightarrow$ g(x) is M.I.

$\Rightarrow \quad g(x) > g(0) \qquad x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x \quad$ ....(ii)

from (i) and (ii), we get $\quad x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6} \quad$ Hence Proved

Also, $\quad 1 - \frac{x^2}{3} < \frac{\tan^{-1} x}{x} < 1 - \frac{x^2}{6}$, for $x > 0$

Hence by sandwich theorem we can prove that $\lim\limits_{x \to 0} \frac{\tan^{-1} x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1} x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1} x}{x} < 1$

$\Rightarrow \quad \lim\limits_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right] = 0$

NOTE : In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

Example # 43 : For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Solution : Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0 \Rightarrow \sin x > x - \frac{x^3}{6}$. Hence proved

Example # 44 : Examine which is greater : $\sin x \tan x$ or x^2. Hence evaluate $\lim_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution : Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \, . \tan x + \sin x \, . \sec^2 x - 2x$

$\Rightarrow f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2 \sec^2 x \tan x \, . \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f''(x) > 0 \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > 0$

$\Rightarrow \sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2 \Rightarrow \frac{\sin x \tan x}{x^2} > 1 \Rightarrow \lim_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right] = 1.$

Example # 45 : Prove that f(x) = $\left(1+\frac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Solution : f(x) = $\left(1+\frac{1}{x}\right)^x$, for Domain of f(x), 1 + $\frac{1}{x}$ > 0

$\Rightarrow \quad \frac{x+1}{x} > 0 \quad \Rightarrow \quad (-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}} \frac{-1}{x^2}\right]$

$\Rightarrow \quad f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of f′(x) depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and

So lets assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x+1)^2} \quad \Rightarrow \quad g'(x) = \frac{-1}{x(x+1)^2}$

(i) for $x \in (0, \infty)$, $g'(x) < 0$ $\Rightarrow$ g(x) is M.D. for $x \in (0, \infty)$

$g(x) > \lim\limits_{x\to\infty} g(x)$

$g(x) > 0$. and since $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$ $\Rightarrow$ g(x) is M.I. for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim\limits_{x\to-\infty} g(x)$ $\Rightarrow$ $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

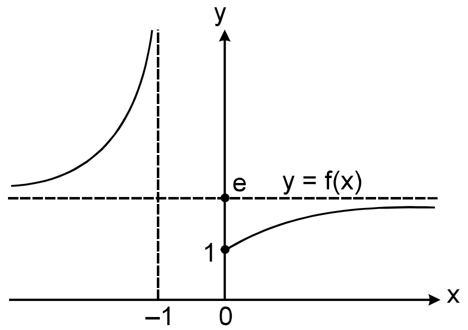
Hence from (i) and (ii) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow$ f(x) is M.I. in its Domain

For drawing the graph of f(x), its important to find the value of f(x) at boundary points i.e. $\pm\infty, 0, -1$

$\lim\limits_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$

$\lim\limits_{x\to0^+} \left(1+\frac{1}{x}\right)^x = 1$ and $\lim\limits_{x\to-1} \left(1+\frac{1}{x}\right)^x = \infty$



so the graph of f(x) is

Range is $y \in (1, \infty) - \{e\}$

Example # 46 : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

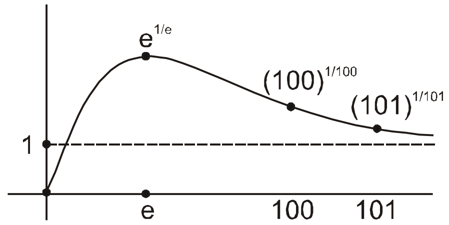
Solution : Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$$f'(x) = x^{1/x} \cdot \left(\frac{1-\ell nx}{x^2}\right)$$

$f'(x) > 0 \quad \Rightarrow \quad x \in (0,e)$

and $f'(x) < 0 \quad \Rightarrow \quad x \in (e,\infty)$

Hence f(x) is M.D. for $x \geq e$



and since 100 < 101

$\Rightarrow \quad f(100) > f(101)$

$\Rightarrow \quad (100)^{1/100} > (101)^{1/101}$

Self Practice Problems :

(32) Prove the following inequalities

(i) $x > \tan^{-1}(x)$ for $x \in (0, \infty)$

(ii) $e^x > x + 1$ for $x \in (0, \infty)$

(iii) $\dfrac{x}{1+x} \leq \ell n\,(1 + x) \leq x$ for $x \in (0, \infty)$

Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b]

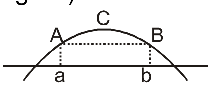
(ii) derivable on (a, b) and

(iii) f(a) = f(b),

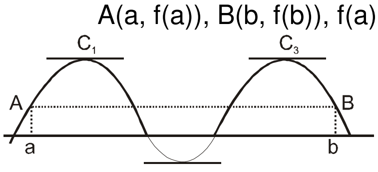
then there exists at least one real number c between a and b (a < c < b) such that $f'(c) = 0$

Geometrical Explanation of Rolle's Theorem :

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), $f'(c) = 0$.



$C_1\,(c_1, f(c_1)),\ f'(c_1) = 0$

$C_2\,(c_2, f(c_2)),\ f'(c_2) = 0$

$C_3\,(c_3, f(c_3)),\ f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

Algebraic Interpretation of Rolle's Theorem :

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of $f'(x)$

Example # 47 : If 2a + 3b + 6c = 0 then prove that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Solution : Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

$f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c = 2a + 3b + 6c = 0$

If $f(0) = f(1)$ then $f'(x) = 0$ for some value of $x \in (0, 1)$

$\Rightarrow$ $ax^2 + bx + c = 0$ for at least one $x \in (0, 1)$

Self Practice Problems :

(33) If f(x) satisfies condition in Rolle's theorem then show that between two consecutive zeros of $f'(x)$ there lies at most one zero of f(x).

(34) Show that for any real numbers λ, the polynomial $P(x) = x^7 + x^3 + \lambda$, has exactly one real root.

Lagrange's Mean Value Theorem (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

Proof : Let us consider a function $g(x) = f(x) + \lambda x$, $x \in [a, b]$

where λ is a constant to b determined such that g(a) = g(b).

$\therefore$ $\lambda = -\frac{f(b) - f(a)}{b - a}$

Now the function g(x), being the sum of two continuous and derivable functions it self

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) g(a) = g(b).

Therefore, by Rolle's theorem there exists a real number $c \in (a, b)$ such that $g'(c) = 0$

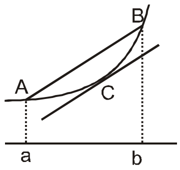
But $g'(x) = f'(x) + \lambda$

$\therefore$ $0 = g'(c) = f'(c) + \lambda$

$f'(c) = -\lambda = \frac{f(b) - f(a)}{b - a}$

Geometrical Interpretation of LMVT :

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), $f'(c)$ = slope of AB.

Alternative Statement : If in the statement of LMVT, b is replaced by a + h, then number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h) \qquad \text{or} \qquad f(a+h) = f(a) + hf'(a+\theta h),\ 0 < \theta < 1$$

Example # 48 : Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Solution : $f(1) = -2$; $f(-1) = -10$

$$\Rightarrow \quad f'(c) = \frac{f(1)-f(-1)}{1-(-1)} \quad \Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$$

Example # 49 : Using Lagrange's mean value theorem, prove that if b > a > 0,

then $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

Solution : Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$ for $a < c < b$ and $f'(x) = \frac{1}{1+x^2}$,

Now $f'(x)$ is a monotonically decreasing function

Hence if $a < c < b$

$\Rightarrow \quad f'(b) < f'(c) < f'(a)$

$\Rightarrow \quad \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$ Hence proved

Example # 50 : Let f : R → R be a twice differentiable function such that f $\left(\frac{\pi}{4}\right)$ = 0, f $\left(\frac{5\pi}{4}\right)$ = 0 & f(3) = 4 then show that there exists a c ∈ (0, 2π) such that f′′(c) + sinc – cosc < 0.

Solution : Consider g(x) = f(x) – sinx + cosx

⇒ g'(x) = f'(x) – cosx – sinx

⇒ g"(x) = f"(x) + sinx – cosx

By LMVT

$$\frac{g(3)-g\left(\frac{\pi}{4}\right)}{3-\frac{\pi}{4}} = g'(c_1)\,,\quad \frac{\pi}{4} < c_1 < 3 \qquad \text{and} \qquad \frac{g\left(\frac{5\pi}{4}\right)-g(3)}{\frac{5\pi}{4}-3} = g'(c_2), \qquad 3 < c_2 < \frac{\pi}{4}$$

$g'(c_1) > 0$, $g'(c_2) < 0$

By LMVT

$$\frac{g'(c_2)-g'(c_1)}{c_2-c_1} = g''(c), \qquad c_1 < c < c_2 \qquad \Rightarrow g''(c) < 0 \qquad \Rightarrow f''(c) + \sin c - \cos c < 0$$

for some $c \in (c_1, c_2)$, $c \in (0, 2\pi)$

Self Practice Problems :

(35) Using LMVT, prove that if two functions have equal derivatives at all points of (a, b), then they differ by a constant

(36) If a function f is

(i) continuous on [a, b],

(ii) derivable on (a, b) and

(iii) $f'(x) > 0$, $x \in (a, b)$, then show that f(x) is strictly increasing on [a, b].

Exercise-1

✍ Marked questions are recommended for Revision.

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Equation of Tangent / Normal and Common Tangents / Normals

A-1. **(i)_** Find the equation of tangent to curve $y = 3x^2 + 4x + 5$ at (0, 5).

(ii) Find the equation of tangent and normal to the curve $x^2 + 3xy + y^2 = 5$ at point (1, 1) on it.

(iii)_ Find the equation of tangent and normal to the curve $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$ at the point for which $t = \frac{1}{2}$

(iv) Find the equation of tangent to the curve $y = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ at (0,0)

A-2 **(i).** Find equations of tangents drawn to the curve $y^2 – 2x^2 – 4y + 8 = 0$ from the point (1, 2).

(ii). Find the equation of all possible normals to the curve $x^2 = 4y$ drawn from the point (1,2)

A-3. **(i)** Find the point on the curve $9y^2 = x^3$ where normal to the curve has non zero x-intercept and both the x intercept and y-intercept are equal.

(ii)✍ If the tangent at (1, 1) on $y^2 = x(2 – x)^2$ meets the curve again at P, then find coordinates of P

(iii) The normal to the curve $5x^5 – 10x^3 + x + 2y + 6 = 0$ at the point P(0, –3) is tangent to the curve at some other point(s). Find those point(s)?

A-4_.✍ **(i)** Find common tangent between curves $y = x^3$ and $112x^2 + y^2 = 112$

(ii) Find common normals of the curves $y = \frac{1}{x^2}$ and $x^2 + y^2 – y = 0$

A-5. **(i)** If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with positive x-axis in anticlockwise, then find a and b ?

(ii) The curve $y = ax^3 + bx^2 + 3x + 5$ touches $y = (x + 2)^2$ at (–2, 0) then $\left|\frac{a}{2}+b\right|$ is

Section (B) : Angle between curves, Orthogonal curves, Shortest/Maximum distance between two curves

B-1. Find the cosine of angle of intersection of curves $y = 2^x \ell nx$ and $y = x^{2x}–1$ at (1, 0).

B-2.✍ Find the angle between the curves $y = \ln x$ and $y = (\ln x)^2$ at their point of intersections.

B-3. Find the angle between the curves $y^2 = 4x + 4$ and $y^2 = 36 (9 – x)$.

B-4. Show that if the curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ are orthogonal then $ab(A - B) = AB(a - b)$.

B-5. Find the shortest distance between line $y = x - 2$ and $y = x^2 + 3x + 2$

B-6. Find shortest distance between $y^2 = 4x$ and $(x - 6)^2 + y^2 = 1$

Section (C) : Rate of change and approximation

C-1. The length x of rectangle is decreasing at a rate of 3 cm/min and width y is increasing at a rate of 2 cm/min. When x = 10 cm and y = 6 cm, find the rate of change of (i) the perimeter, (ii) the area of rectangle.

C-2. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

C-3. A man 1.5 m tall walks away from a lamp post 4.5 m high at a rate of 4 km/hr.
(i) How fast is his shadow lengthening?
(ii) How fast is the farther end of shadow moving on the pavement?

C-4. Find the approximate change in volume V of a cube of side 5m caused by increasing its side length by 2%.

Section (D) : Monotonicity on an interval, about a point and inequalities, local maxima/minima

D-1. Show that $f(x) = \dfrac{x}{\sqrt{1+x}} - \ell n\,(1 + x)$ is an increasing function for $x > -1$.

D-2. Find the intervals of monotonicity for the following functions.
(i) $\dfrac{x^4}{4} + \dfrac{x^3}{3} - 3x^2 + 5$ (ii) $\log_3^2 x + \log_3 x$

D-3. If g(x) is monotonically increasing and f(x) is monotonically decreasing for $x \in R$ and if (gof) (x) is defined for $x \in R$, then prove that (gof)(x) will be monotonically decreasing function. Hence prove that $(g\circ f)(x + 1) \le (g\circ f)(x - 1)$.

D-4. Let $f(x) = \begin{cases} x^2\ ; & x \ge 0 \\ ax\ ; & x < 0 \end{cases}$. Find real values of 'a' such that f(x) is strictly monotonically increasing at x = 0.

D-5. Check monotonocity at following points for
(i) $f(x) = x^3 - 3x + 1$ at $x = -1, 2$
(ii) $f(x) = |x - 1| + 2|x - 3| - |x + 2|$ at $x = -2, 0, 3, 5$
(iii) $f(x) = x^{1/3}$ at $x = 0$
(iv) $f(x) = x^2 + \dfrac{1}{x^2}$ at $x = 1, 2$
(v) $f(x) = \begin{cases} x^3 + 2x^2 + 5x\ , & x < 0 \\ 3\sin x\ , & x \ge 0 \end{cases}$ at $x = 0$

D-6. Prove that $\left(\dfrac{\sin\left(\frac{1}{10}\right)}{\frac{1}{10}}\right) > \left(\dfrac{\sin\left(\frac{1}{9}\right)}{\frac{1}{9}}\right)$.

D-7. Let f and g be differentiable on R and suppose f(0) = g(0) and $f'(x) \le g'(x)$ for all $x \ge 0$. Then show that $f(x) \le g(x)$ for all $x \ge 0$.

D-8. Let $f(x) = \begin{cases} 3 - x & 0 \le x < 1 \\ x^2 + \ell nb & x \ge 1 \end{cases}$. Find the set of values of b such that f(x) has a local minima at x = 1.

D-9. Find the points of local maxima/minima of following functions
(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$ (ii) $f(x) = -(x-1)^3 (x+1)^2$
(iii) $f(x) = x\,\ell nx$

D-10. Find points of local maxima / minima of
(i) $f(x) = (2^x - 1)(2^x - 2)^2$
(ii) $f(x) = x^2e^{-x}$
(iii) $f(x) = 3\cos^4x + 10\cos^3x + 6\cos^2x - 3,\ x \in [0, \pi]$
(iv) $f(x) = 2x + 3x^{2/3}$
(v) $f(x) = \left|\dfrac{x^2 - 2}{x^2 - 1}\right|$

D-11. Draw graph of $f(x) = x|x - 2|$ and, hence find points of local maxima/minima.

Section (E) : Global maxima, Global minima, Application of Maxima and Minima

E-1. Find the absolute maximum/minimum value of following functions
(i) $f(x) = x^3$; $x \in [-2, 2]$
(ii) $f(x) = \sin x + \cos x$; $x \in [0, \pi]$
(iii) $f(x) = 4x - \dfrac{x^2}{2}$; $x \in \left[-2, \dfrac{9}{2}\right]$
(iv) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$
(v) $f(x) = \sin x + \dfrac{1}{2}\cos 2x$; $x \in \left[0, \dfrac{\pi}{2}\right]$

E-2. Let $f(x) = x^2$; $x \in (-1, 2)$. Then show that f(x) has exactly one point of local minima but global maximum is not defined.

E-3. Find the minimum and maximum values of y in $4x^2 + 12xy + 10y^2 - 4y + 3 = 0$.

E-4. John has 'x' children by his first wife and Anglina has 'x + 1' children by her first husband. They both marry and have their own children. The whole family has 24 children. It is given that the children of the same parents don't fight. Then find then maximum number of fights that can take place in the family.

E-5. If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is $\pi/3$.

E-6. Find the volume of the largest cylinder that can be inscribed in a sphere of radius ' r ' cm.

E-7. Show that the semi vertical angle of a right circular cone of maximum volume, of a given slant height is $\tan^{-1}\sqrt{2}$.

E-8. A running track of 440 m. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end . If the area of the rectangular portion is to be maximum, find the length of its sides.

E-9. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.

E-10. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved around one of its side .

E-11. The combined resistance R of two resistors R_1 & R_2 (R_1, R_2 > 0) is given by, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 + R_2$ = constant. Prove that the maximum resistance R is obtained by choosing $R_1 = R_2$.

Section (F) : Rolle's Theorem, LMVT

F-1. Let f : [1, 2] → [1, 4] and g : [1, 2] → [2, 7] be two continuous bijective functions such thaf f(1) = 4 & g(2) = 7. The number of solutions of the equation f(x) = g(x) in (1, 2), is :

F-2. Verify Rolle's theorem for the function, $f(x) = \log_e\left(\frac{x^2+ab}{x(a+b)}\right) + p$, for [a, b] where 0 < a < b.

F-3. Using Rolle's theorem prove that the equation $3x^2 + px - 1 = 0$ has at least one real root in the interval (– 1, 1).

F-4. Using Rolle's theorem show that the derivative of the function $f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ vanishes at an infinite set of points of the interval (0, 1).

F-5. Let f(x) be differentiable function and g(x) be twice differentiable function. Zeros of f(x), g′(x) be a, b respectively (a < b). Show that there exists at least one root of equation $f'(x)\, g'(x) + f(x)\, g''(x) = 0$ on (a, b).

F-6. If f(x) = tanx, $x \in \left[0, \frac{\pi}{5}\right]$ then show that $\frac{\pi}{5} < f\left(\frac{\pi}{5}\right) < \frac{2\pi}{5}$

F-7. If f(x) and g(x) are differentiable functions for $0 \le x \le 23$ such that f(0) = 2, g(0) = 0, f(23) = 22, g(23)=10, then show that f'(x) = 2g'(x) for at least one x in the interval (0, 23).

F-8. If $f(x) = \begin{vmatrix} \sin^3 x & \sin^3 a & \sin^3 b \\ xe^x & ae^a & be^b \\ \frac{x}{1+x^2} & \frac{a}{1+a^2} & \frac{b}{1+b^2} \end{vmatrix}$

where 0 < a < b < 2π, then show that the equation f ′(x) = 0 has atleast one root in the interval (a, b)

F-9. A function y = f(x) is defined on [0, 6] as $f(x) = \begin{cases} -8x & ; & 0 \le x \le 1 \\ (x-3)^3 & ; & 1 < x < 4 \\ 2 & ; & 4 \le x \le 6 \end{cases}$

Show that for the function y = f(x), all the three conditions of Rolle's theorem are violated on [0, 6] but still f'(x) vanishes at a point in (0, 6)

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Equation of Tangent / Normal and Common Tangents / Normals

A-1. Equation of the normal to the curve $y = -\sqrt{x} + 2$ at the point (1, 1)
(A) 2x – y – 1 = 0 (B) 2x – y + 1 = 0 (C) 2x + y – 3 = 0 (D) none of these

A-2. The angle between x-axis and tangent of the curve $y = (x+1)(x-3)$ at the point (3, 0) is
(A) $\tan^{-1}\left(\frac{8}{15}\right)$ (B) $\tan^{-1}\left(\frac{15}{8}\right)$ (C) $\tan^{-1} 4$ (D) none of these

A-3. The numbers of tangent to the curve $y - 2 = x^5$ which are drawn from point (2,2) is / are
(A) 3 (B) 1 (C) 2 (D) 5

A-4. The equation of tangent drawn to the curve $xy = 4$ from point (0, 1) is
(A) $y-\frac{1}{2}=-\frac{1}{16}(x+8)$ (B) $y-\frac{1}{2}=-\frac{1}{16}(x-8)$
(C) $y+\frac{1}{2}=-\frac{1}{16}(x-8)$ (D) $y-8=-\frac{1}{16}\left(x-\frac{1}{2}\right)$

A-5. The curve $y - e^{xy} + x = 0$ has a vertical tangent at point
(A) (1, 1) (B) (0, 1) (C) (1, 0) (D) no point

A-6. If the tangent to the curve $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{3}$ makes an angle α $(0 \le \alpha < \pi)$ with x-axis, then α =
(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

A-7. If the normal at the point (3t, 4/t) of the curve $xy = 12$ cuts the curve again at $(3t_1, 4/t_1)$ then find 't_1' in terms of 't'
(A) $\frac{-9}{16t^3}$ (B) $\frac{-16}{9t^3}$ (C) $\frac{9}{16t^3}$ (D) $\frac{16}{9t^3}$

A-8_. The common tangent of the curves $y = x^2 + \frac{1}{x}$ and $y^2 = 4x$ is
(A) $y = x + 1$ (B) $y = x - 1$ (C) $y = -x + 1$ (D) $y = -x - 1$

A-9_. The area of triangle formed by tangent at (1,1) on $y = x^2 + bx + c$ with coordinate axis is equal to 1, then the integral value of b is
(A) –3 (B) 3 (C) 2 (D) –2

Section (B) : Angle between curves, Orthogonal curves, Shortest/Maximum distance between two curves

B-1. The angle of intersection of $y = a^x$ and $y = b^x$ is given by
(A) $\tan\theta = \left|\frac{\log(ab)}{1-\log(ab)}\right|$ (B) $\left|\frac{\log(a/b)}{1+\log a \log b}\right|$ (C) $\left|\frac{\log(a/b)}{1-\log(a/b)}\right|$ (D) None

B-2. The angle between curves $x^2 + 4y^2 = 32$ and $x^2 - y^2 = 12$ is
(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

B-3. Find the angle at which two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect
(A) 0 (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

B-4. The value of a^2 if the curves $\frac{x^2}{a^2}+\frac{y^2}{4} = 1$ and $y^3 = 16x$ cut orthogonally is
(A) 3/4 (B) 1 (C) 4/3 (D) 4

B-5. The shortest distance between curves $y^2 = 8x$ and $y^2 = 4(x-3)$ is
(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $3\sqrt{2}$ (D) $4\sqrt{2}$

B-6. The shortest distance between curves $\frac{x^2}{32}+\frac{y^2}{18} = 1$ and $\left(x-\frac{7}{4}\right)^2 + y^2 = 1$
(A) 15 (B) $\frac{11}{2}$ (C) $\frac{15}{4}$ (D) $\frac{11}{4}$

Section (C) : Rate of change and approximation

C-1. Water is poured into an inverted conical vessel of which the radius of the base is 2 m and height 4 m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is (use $\pi = 22/7$)
(A) 10 cm/min (B) 20 cm/min (C) 40 cm/min (D) 30 cm/min

C-2 On the curve $x^3 = 12y$. The interval in which abscissa changes at a faster rate then its ordinate
(A) $(-3, 0)$ (B) $(-\infty, -2) \cup (2, \infty)$ (C) $(-2, 2)$ (D) $(-3, 3)$

C-3. A kite is 300 m high and there are 500 m of cord out. If the wind moves the kite horizontally at the rate of 5 km/hr. directly away from the person who is flying it, find the rate at which the cord is being paid?
(A) 4 (B) 8
(C) 3 (D) cannot be determined

C-4. The approximate value of tan 46° is (take $\pi = 22/7$) :
(A) 3 (B) 1.035 (C) 1.033 (D) 1.135

C-5. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases , is-
(A) $\frac{5}{6\pi}$ cm/min (B) $\frac{1}{54\pi}$ cm/min (C) $\frac{1}{18\pi}$ cm/min (D) $\frac{1}{36\pi}$ cm/min

Section (D) : Monotonicity on an interval, about a point and inequalities, local maxima/minima

D-1. The complete set of values of 'a' for which the function $f(x) = (a + 2) x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x is.
(A) $(-\infty, -3]$ (B) $(-\infty, 0]$ (C) $[-3, 0]$ (D) $[-3, \infty)$

D-2. Let $f(x) = x^3 + ax^2 + bx + 5 \sin^2 x$ be an increasing function in the set of real numbers R. Then a & b satisfy the condition :
(A) $a^2 - 3b - 15 > 0$ (B) $a^2 - 3b + 15 \le 0$ (C) $a^2 + 3b - 15 < 0$ (D) $a > 0$ & $b > 0$

D-3. The function $\frac{|x-1|}{x^2}$ is monotonically decreasing at the point
(A) x = 3 (B) x = 1 (C) x = 2 (D) none of these

D-4. If $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + \ldots\ldots + 100x^{100}$ is a polynomial in a real variable x, then f(x) has:
(A) neither a maximum nor a minimum (B) only one maximum
(C) only one minimum (D) one maximum and one minimum

D-5_. Which of the following statement is/are true ?

(1) f(x) = sin x is increasing in interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(2) f(x) = sin x is increasing at all point of the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(3) f(x) = sin x is increasing in interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

(4) f(x) = sin x is increasing at all point of the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

(5) f(x) = sin x is increasing in intervals $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ & $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$

(A) all are correct
(B) all are false
(C) (3) and (4) are correct
(D) (1), (4) & (5) are correct

D-6. Let $f(x) = \begin{cases} x & x \in [1,2] \\ 5-x & x \in (2,4) \\ 2 & x = 4 \\ 7-x & x \in (4,6] \end{cases}$ then which of the following statement is / are correct about f(x) ?

(A) Function is strictly increasing at point x = 2
(B) Function is strictly increasing at point x = 4
(C) Function is not increasing at point x = 2 and x = 4
(D) None of these

D-7. **STATEMENT-1 :** e^{π} is bigger than π^{e}.
STATEMENT-2 : $f(x) = x^{1/x}$ is a increasing function when $x \in [e, \infty)$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True,Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True
(E) Statement-1 is False, Statement-2 is False

Section (E) : Global maxima, Global minima, Application of Maxima and Minima

E-1. The greatest, the least values of the function, $f(x) = 2 - \sqrt{1 + 2x + x^2}$, $x \in [-2, 1]$ are respectively

(A) 2, 1 (B) 2, – 1 (C) 2, 0 (D) –2, 3

E-2. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m(b) be the minimum value of f(x). As b varies, the range of m(b) is

(A) [0, 1] (B) $\left(0, \frac{1}{2}\right]$ (C) $\left[\frac{1}{2}, 1\right]$ (D) (0, 1]

E-3. The radius of a right circular cylinder of greatest curved surface which can be inscribed in a given right circular cone is

(A) one third that of the cone
(B) $1/\sqrt{2}$ times that of the cone
(C) 2/3 that of the cone
(D) 1/2 that of the cone

E-4. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are

(A) $\sqrt{8}, \sqrt{2}$ (B) 4, 3 (C) $2\sqrt{8},\ 3\sqrt{2}$ (D) $\sqrt{2}$, $\sqrt{6}$

E-5. The largest area of a rectangle which has one side on the x–axis and the two vertices on the curve $y = e^{-x^2}$ is

(A) $\sqrt{2}\ e^{-1/2}$ (B) $2\ e^{-1/2}$ (C) $e^{-1/2}$ (D) none of these

E-6. The maximum distance of the point (k, 0) from the curve $2x^2 + y^2 - 2x = 0$ is equal to

(A) $\sqrt{1+2k-k^2}$ (B) $\sqrt{1+2k+2k^2}$ (C) $\sqrt{1-2k+2k^2}$ (D) $\sqrt{1-2k+k^2}$

Section (F) : Rolle's Theorem, LMVT

F-1. The function $f(x) = x^3 - 6x^2 + ax + b$ satisfy the conditions of Rolle's theorem on [1, 3]. Which of these are correct ?
(A) $a = 11, b \in R$ (B) $a = 11, b = -6$ (C) $a = -11, b = 6$ (D) $a = -11, b \in R$

F-2. The function $f(x) = x(x + 3)e^{-x/2}$ satisfies all the conditions of Rolle's theorem on [–3, 0]. The value of c which verifies Rolle's theorem, is
(A) 0 (B) – 1 (C) – 2 (D) 3

F-3. If f(x) satisfies the requirements of Lagrange's mean value theorem on [0, 2] and if f(0) = 0 and $f'(x) \le \frac{1}{2}\ \forall\ x \in [0, 2]$, then
(A) $|f(x)| \le 2$ (B) $f(x) \le 1$
(C) $f(x) = 2x$ (D) $f(x) = 3$ for at least one x in [0, 2]

F-4_. If ab > 0 and 3a + 5b + 15 c = 0 then which of the following statement is "**INCORRECT"?**
(A) there exist exactly one root of equation $ax^4 + bx^2 + c = 0$ in (–1,0)
(B) there exist exactly one root of equation $ax^4 + bx^2 + c = 0$ in (0,1)
(C) there exist exactly two root of equation $ax^4 + bx^2 + c = 0$ in (–1,1)
(D) number of roots of equation $ax^4 + bx^2 + c = 0$ can be two in (–1,0)

F-5. Consider the function for $x \in [-2, 3]$
$f(x) = \begin{cases} -6 & ; \quad x = 1 \\ \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} & ; \quad x \ne 1 \end{cases}$. The value of c obtained by applying Rolle's theorem for which $f'(c) = 0$ is
(A) 0 (B) 1 (C) 1/2 (D) 'c' does not exist

PART - III : MATCH THE COLUMN

1. **Column – I** **Column – II**

(A) If curves $y^2 = 4ax$ and $y = e^{-\frac{x}{2a}}$ are orthogonal then 'a' can take value — (p) 3
(B) If θ is angle between the curves y = [| sin x | + | cos x|], ([·] denote GIF) and $x^2 + y^2 = 5$ then $\text{cosec}^2\theta$ is — (q) 1
(C)_ If curves $y^2 = 4a(x + a)$ and $y^2 = 4b(x + b)$ intersects each other orthogonally then $\frac{a}{b}$ can be equal to____ — (r) 5/4
(D)_ If $y = x^2 + 3x + c$ and $x = y^2 + 3y + c$ touches each other at (h, k) then |h + k + c| is equal to…… — (s) 2

2. **Column-I** **Column-II**

(A) The number of point (s) of maxima of $f(x) = x^2 + \frac{1}{x^2}$ is — (p) 0
(B) $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is maximum at x = — (q) 2
(C) If [a, b], (b < 1) is largest interval in which $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 19$ is strictly increasing then $\frac{a}{b}$ is — (r) $\frac{8}{3}$
(D) If a + b = 8, a, b > 0 then minimum value of $\frac{a^3 + b^3}{48}$ is — (s) –1

3. **Column – I** | **Column – II**

(A) $f(x) = \dfrac{\sin x}{e^x}$, $x \in [0,\pi]$

(B) $f(x) = \text{sgn}\,((e^x - 1)\,\ell nx)$, $x \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right]$

(C)✍ $f(x) = (x-1)^{2/5}$, $x \in [0,3]$

(D)✍ $f(x) = \begin{cases} x\left(\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), & x\in[-1,1]-\{0\} \\ 0, & x=0 \end{cases}$

(p) Conditions in Rolle's theorem are satisfied.

(q) Conditions in LMVT are satisfied.

(r) At least one condition in Rolle's theorem is not satisfied.

(s) At least one condition in LMVT is not satisfied.

4. **Column – I** | **Column – II**

(A) A rectangle is inscribed in an equilateral triangle of side 4cm. Square of maximum area of such a rectangle is

(B) The volume of a rectangular closed box is 72 and the base sides are in the ratio 1 : 2. The least total surface area is

(C) If x and y are two positive numbers such that $x + y = 60$ and x^3y is maximum then value of x is

(D) The sides of a rectangle of greatest perimeter which is inscribed in a semicircle of radius $\sqrt{5}$ are a and b. Then $a^3 + b^3 =$

(p) 65

(q) 45

(r) 12

(s) 108

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. Equation of normal drawn to the graph of the function defined as $f(x) = \dfrac{\sin x^2}{x}$, $x \neq 0$ and $f(0) = 0$ at the origin is

(A) $x + y = 0$ (B) $x - y = 0$ (C) $y = 0$ (D) $x = 0$

2. The line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point

(A) $(-a, 2b)$ (B) $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ (C) $\left(a, \dfrac{b}{e}\right)$ (D) $(0, b)$

3. The equation of normal to the curve $x^3 + y^3 = 8xy$ at point where it is meet by the curve $y^2 = 4x$, other than origin is

(A) $y = x$ (B) $y = -x + 4$ (C) $y = 2x$ (D) $y = -2x$

4. The length of segment of all tangents to curve $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between coordinate axes is

(A) $2|a|$ (B) $|a|$ (C) $\dfrac{|a|}{2}$ (D) $\dfrac{3|a|}{2}$

5. If tangents are drawn from the origin to the curve $y = \sin x$, then their points of contact lie on the curve

(A) $x - y = xy$ (B) $x + y = xy$ (C) $x^2 - y^2 = x^2y^2$ (D) $x^2 + y^2 = x^2y^2$

6.✍ Number of tangents drawn from the point $(-1/2, 0)$ to the curve $y = e^{\{x\}}$. (Here { } denotes fractional part function).

(A) 2 (B) 1 (C) 3 (D) 4

7. Let f(x) = $\begin{cases} -x^2 & , \quad x < 0 \\ x^2 + 8 & , \quad x \ge 0 \end{cases}$ Equation of tangent line touching both branches of y = f(x) is
(A) y = 4x + 1 (B) y = 4x + 4 (C) y = x + 4 (D) y = x + 1

8. Minimum distance between the curves f(x) = e^x & g(x) = ℓn x is
(A) 1 (B) $\sqrt{2}$ (C) 2 (D) e

9. The point(s) on the parabola $y^2 = 4x$ which are closest to the circle,
$x^2 + y^2 - 24y + 128 = 0$ is/are:
(A) (0, 0) (B) $\left(2, 2\sqrt{2}\right)$ (C) (4, 4) (D) none

10. If f(x) = $a^{\{a^{|x|} \operatorname{sgn} x\}}$; g(x) = $a^{[a^{|x|} \operatorname{sgn} x]}$ for a > 1, a ≠ 1and x ∈ R, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements holds good for the function h(x), where (ℓn a) h(x) = (ℓn f(x) + ℓn g(x)).
(A) 'h' is even and increasing (B) 'h' is odd and decreasing
(C) 'h' is even and decreasing (D) 'h' is odd and increasing

11. If f : [1, 10] → [1, 10] is a non-decreasing function and g : [1, 10] → [1, 10] is a non-increasing function. Let h(x) = f(g(x)) with h(1) = 1, then h(2)
(A) lies in (1, 2) (B) is more than 2 (C) is equal to 1 (D) is not defined

12. If f(x) = |ax − b| + c|x| is stricly increasing at atleast one point of non differentiability of the function where a > 0, b > 0, c > 0 then
(A) c > a (B) a > c (C) b > a + c (D) a = b

13. If g(x) is a curve which is obtained by the reflection of f(x) = $\frac{e^x - e^{-x}}{2}$ by the line y = x then
(A) g(x) has more than one tangent parallel to x-axis
(B) g(x) has more than one tangent parallel to y-axis
(C) y = –x is a tangent to g(x) at (0, 0)
(D) g(x) has no extremum

14. The set of values of p for which all the points of extremum of the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval (–2, 4), is:
(A) (–3, 5) (B) (–3, 3) (C) (–1, 3) (D) (–1, 4)

15. The complete set of values of the parameter 'a' for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$ is
(A) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$ (B) $(-3\sqrt{3}, -2\sqrt{3})$
(C) $(-3\sqrt{3}, -2\sqrt{3})$ (D) $(-3\sqrt{2}, 2\sqrt{3})$

16. Consider the following statements :

S_1 : The function y = $\frac{2x^2 - 1}{x^4}$ is neither increasing nor decreasing.

S_2 : If f(x) is strictly increasing real function defined on R and c is a real constant, then number of Solutions of f(x) = c is always equal to one.

S_3 : Let f(x) = x ; x ∈ (0, 1). f(x) does not has any point of local maxima/minima

S_4 : f(x) = {x} has maximum at x = 6 (here {.} denotes fractional part function).

State, in order, whether S_1, S_2, S_3, S_4 are true or false
(A) TTFT (B) FTFT (C) TFTF (D) TFFT

17. If f(x) = $\sin^3 x + \lambda \sin^2 x$; $-\pi/2 < x < \pi/2$, then the interval in which λ should lie in order that f(x) has exactly one minima and one maxima
(A) (–3/2, 3/2) (B) (–2/3, 2/3) – {0} (C) R (D) none of these

18. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5 & , x \le 1 \\ -2x + \log_2(b^2 - 2) & , x > 1 \end{cases}$ the set of values of b for which f(x) has greatest value at x = 1 is given by :
(A) $1 \le b \le 2$ (B) b = {1, 2}
(C) $b \in (-\infty, -1)$ (D) $\left[-\sqrt{130}, -\sqrt{2}\right)$ U $\left(\sqrt{2}, \sqrt{130}\right]$

19. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B & D are known as A(–2, 3); B(–1, 1) and D(2, 7). The coordinates of C for which the area of the quadrilateral ABCD is greatest, is
(A) (1/2, 7/4) (B) (1/2, – 7/4) (C) (–1/2, 7/4) (D) (–1/2, –7/4)

20. In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ℓ. The altitude of the prism for which the volume is greatest, is :
(A) $\frac{\ell}{2}$ (B) $\frac{\ell}{\sqrt{3}}$ (C) $\frac{\ell}{3}$ (D) $\frac{\ell}{4}$

21. The maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b is
(A) 2 (ab) (B) $\frac{1}{2}(a+b)^2$ (C) $\frac{1}{2}(a^2+b^2)$ (D) $\frac{a^3}{b}$

22. Let ABC is given triangle having respective sides a,b,c. D,E,F are points of the sides BC,CA,AB respectively so that AFDE is a parallelogram. The maximum area of the parallelogram is
(A) $\frac{1}{4}$bcsinA (B) $\frac{1}{2}$ bcsinA (C) bcsinA (D) $\frac{1}{8}$bcsinA

23. If f(x) = (x – 4) (x – 5) (x – 6) (x – 7) then,
(A) f′(x) = 0 has four roots.
(B) three roots of f′(x) = 0 lie in $(4, 5) \cup (5, 6) \cup (6, 7)$.
(C) the equation f′(x) = 0 has only one real root.
(D) three roots of f′(x) = 0 lie in $(3, 4) \cup (4, 5) \cup (5, 6)$.

24. Square roots of 2 consecutive natural number greater than N^2 is differ by .
(A) $> \frac{1}{2N}$ (B) $\ge \frac{1}{2N}$ (C) $< \frac{1}{2N}$ (D) $> \frac{1}{N}$

25. If Rolle's theorem is applicable to the function $f(x) = \frac{\ell nx}{x}$, (x > 0) over the interval [a, b] where $a \in I$, $b \in I$, then the value of $a^2 + b^2$ can be
(A) 20 (B) 25 (C) 45 (D) 10

26. If f(x) be a twice differentiable function such that f(x) = x^2 for x = 1, 2, 3, then
(A) f″ (x) = 2 $\forall x \in [1, 3]$ (B) f″(x) = 2 for some $x \in (1, 3)$
(C) f″(x) = 2 $\forall x \in (1, 3)$ (D) f′ (x) = 2x $\forall x \in (1, 3)$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. The number of distinct line(s) which is/are tangent at a point on curve $4x^3 = 27y^2$ and normal at other point, is :

2. The sum of the ordinates of point of contacts of the common tangent to the parabolas $y = x^2 + 4x + 8$ and $y = x^2 + 8x + 4$, is

3. If $p \in (0, 1/e)$ then the number of the distinct roots of the equation $|\ell n\, x| - px = 0$ is:

4. A light shines from the top of a pole 50 ft. high. A ball is dropped from the same height from a point 30 ft. away from the light. If the shadow of the ball moving at the rate of 100λ ft/sec along the ground 1/2 sec. later [Assume the ball falls a distance $s = 16t^2$ ft. in 't' sec.], then $|\lambda|$ is :

5. A variable ΔABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. If the area of the triangle increasing at the rate of 'p' cm^2/sec when $t = \frac{7}{2}$ sec, then 7p is.

6. Function defined by $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$ is injective in $[\alpha - 2, \infty)$, the least value of α is

7. Find $\lim\limits_{x \to 0^+} \left[\frac{3x}{2\sin x + \tan x}\right]$ where [.] denotes the GIF.

8. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for $\forall\, x \in R$, then the minimum value of 'a' is

9. If the set of all values of the parameter 'a' for which the function $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$, is $(-\infty, -m - \sqrt{n}) \cup (\sqrt{n}, \infty)$ then $(m^2 + n^2)$ is (where m, n are prime numbers) :

10. If $\ell n 2\pi < \log_2\left(2 + \sqrt{3}\right) < \ell n 3\pi$, then number of roots of the equation $4\cos(e^x) = 2^x + 2^{-x}$, is

11. For $-1 \le p \le 1$, the equation $4x^3 - 3x - p = 0$ has 'n' distinct real roots in the interval $\left[\frac{1}{2}, 1\right]$ and one of its root is $\cos(k\cos^{-1}p)$, then the value of $n + \frac{1}{k}$ is :

12. Least value of the function, $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2} + 1}$ is:

13. Real root of the equation $(x - 1)^{2013} + (x - 2)^{2013} + (x - 3)^{2013} + \ldots\ldots\ldots\ldots\ldots + (x - 2013)^{2013} = 0$ is a four digit number. Then the sum of the digits is :

14. The exhaustive set of values of 'a' for which the function $f(x) = \frac{a}{3}x^3 + (a + 2)x^2 + (a - 1)x + 2$ possess a negative point of minimum is (q, ∞). The value of q is :_

15. If f(x) is a polynomial of degree 6, which satisfies $\lim\limits_{x \to 0}\left(1+\frac{f(x)}{x^3}\right)^{1/x} = e^2$ and has local maximum at x = 1 and local minimum at x = 0 and x = 2, then the value of $\left(\frac{5}{9}\right)^4 f\left(\frac{18}{5}\right)$ is :

16. Maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2+(x-5)^2$ (where $1 \le x \le 3$) is

17. The three sides of a trapezium are equal each being 6 cms long. Let Δ cm² be the maximum area of the trapezium. The value of $\sqrt{3}\,\Delta$ is :

18. A sheet of poster has its area 18 m². The margin at the top & bottom are 75 cms. and at the sides 50 cms. Let ℓ, n are the dimensions of the poster in meters when the area of the printed space is maximum. The value of $\ell^2 + n^2$ is :

19. The fuel charges for running a train are proportional to the square of the speed generated in m.p.h. and costs Rs. 48/- per hour at 16 mph. What is the most economical speed if the fixed charges i.e. salaries etc. amount to Rs. 300/- per hour.

20. Let f(x) = Max. $\{x^2, (1-x)^2, 2x(1-x)\}$ where $x \in [0, 1]$ If Rolle's theorem is applicable for f(x) on largest possible interval [a, b] then the value of 2(a + b + c) when $c \in (a, b)$ such that f'(c) = 0, is____

21_. For every twice differentiable function f(x) the value of $|f(x)| \le 3\ \forall\ x \in R$ and for some α, $f(\alpha) + (f'(\alpha))^2 = 80$. Number of integral values that $(f'(x))^2$ can take between (0, 77) are equal to ___

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. If tangent to curve $2y^3 = ax^2 + x^3$ at point (a, a) cuts off intercepts α, β on co-ordinate axes, where $\alpha^2 + \beta^2 = 61$, then the value of 'a' is equal to
(A) 20 (B) 25 (C) 30 (D) – 30

2. For the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, at point (2, – 1)
(A) length of subtangent is 7/6.
(B) slope of tangent = 6/7
(C) length of tangent = $\sqrt{(85)}/6$
(D) none of these

3. Which of the following statements is/are correct ?
(A) x + sinx is increasing function
(B) sec x is neither increasing nor decreasing function
(C) x + sinx is decreasing function
(D) sec x is an increasing function

4. If $f(x) = 2x + \cot^{-1} x + \ell n\left(\sqrt{1+x^2}-x\right)$, then f(x) :
(A) increases in $[0, \infty)$
(B) decreases in $[0, \infty)$
(C) neither increases nor decreases in $[0, \infty)$
(D) increases in $(-\infty, \infty)$

5. Let g(x) = 2f(x/2) + f(1 – x) and $f''(x) < 0$ in $0 \le x \le 1$ then g(x)
(A) decreases in $\left[0, \frac{2}{3}\right]$
(B) decreases $\left[\frac{2}{3}, 1\right]$
(C) increases in $\left[0, \frac{2}{3}\right]$
(D) increases in $\left[\frac{2}{3}, 1\right]$

6. Let $f(x) = x^{m/n}$ for $x \in R$ where m and n are integers, m even and n odd and $0 < m < n$. Then
(A) f(x) decreases on $(-\infty, 0]$ (B) f(x) increases on $[0, \infty)$
(C) f(x) increases on $(-\infty, 0]$ (D) f(x) decreases on $[0, \infty)$

7. Let f and g be two differentiable functions defined on an interval I such that $f(x) \geq 0$ and $g(x) \leq 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then
(A) the product function fg is strictly increasing on I
(B) the product function fg is strictly decreasing on I
(C) fog(x) is monotonically increasing on I
(D) fog (x) is monotonically decreasing on I

8. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x \ \forall\ x \in R$, where f(x) is a differentiable function $\forall x \in R$, then
(A) ϕ is increasing whenever f is increasing (B) ϕ is increasing whenever f is decreasing
(C) ϕ is decreasing whenever f is decreasing (D) ϕ is decreasing if $f'(x) = -11$

9. If p, q, r be real, then the intervals in which, $f(x) = \begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix}$,

(A) increase is $x < -\frac{2}{3}(p^2 + q^2 + r^2)$, $x > 0$ (B) decrease is $(-\frac{2}{3}(p^2 + q^2 + r^2), 0)$

(C) decrease is $x < -\frac{2}{3}(p^2 + q^2 + r^2)$, $x > 0$ (D) increase is $(-\frac{2}{3}(p^2 + q^2 + r^2), 0)$

10. If $f(x) = \dfrac{x^2}{2-2\cos x}$; $g(x) = \dfrac{x^2}{6x-6\sin x}$ where $0 < x < 1$, then
(A) 'f' is increasing function (B) 'g' is decreasing function
(C) $\dfrac{f(x)}{g(x)}$ is increasing function (D) g(f(x)) is decreasing function

11. Let $f(x) = \dfrac{x}{\sin x}$ & $x \in \left(0, \dfrac{\pi}{2}\right)$

Then the interval in which at least one root of equaiton lie $\dfrac{2}{x - f\left(\frac{\pi}{12}\right)} + \dfrac{3}{x - f\left(\frac{\pi}{4}\right)} + \dfrac{4}{x - f\left(\frac{5\pi}{12}\right)} = 0$

(A) $\left(f\left(\frac{\pi}{12}\right), f\left(\frac{\pi}{4}\right)\right)$ (B) $\left(0, f\left(\frac{\pi}{12}\right)\right)$ (C) $\left(f\left(\frac{5\pi}{12}\right), \infty\right)$ (D) $\left(f\left(\frac{\pi}{4}\right), f\left(\frac{5\pi}{12}\right)\right)$

12. Let $f(x) = (x^2 - 1)^n (x^2 + x + 1)$. f(x) has local extremum at x = 1 if
(A) n = 2 (B) n = 3 (C) n = 4 (D) n = 6

13. If $f(x) = \dfrac{x}{1 + x\tan x}$, $x \in \left(0, \dfrac{\pi}{2}\right)$, then
(A) f(x) has exactly one point of minima (B) f(x) has exactly one point of maxima
(C) f(x) is increasing in $\left(0, \dfrac{\pi}{2}\right)$ (D) maxima occurs at x_0 where $x_0 = \cos x_0$

14. If $f(x) = a\,\ell n\,|x| + bx^2 + x$ has its extremum values at x = – 1 and x = 2, then _____
(A) a = 2 (B) b = – 1/2 (C) a = – 2 (D) b = 1/2

15. If $f(x) = \begin{cases} -\sqrt{1-x^2} & , \ 0 \leq x \leq 1 \\ -x & , \ x > 1 \end{cases}$, then
(A) Maximum of f(x) exist at x = 1 (B) Maximum of f (x) doesn't exists
(C) Minimum of $f^{-1}(x)$ exist at x = – 1 (D) Minimum of $f^{-1}(x)$ exist at x = 1

16. If f(x) = $\tan^{-1}x - (1/2)\ \ell n\ x$. Then

(A) the greatest value of f(x) on $\left[1/\sqrt{3},\ \sqrt{3}\right]$ is $\pi/6 + (1/4)\ \ell n\ 3$

(B) the least value of f(x) on $\left[1/\sqrt{3},\ \sqrt{3}\right]$ is $\pi/3 - (1/4)\ \ell n\ 3$

(C) f(x) decreases on $(0, \infty)$

(D) f(x) increases on $(-\infty, 0)$

17. Let f(x) = $\dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$. Which of the following statement(s) about f(x) is (are) correct ?

(A) f(x) has local minima at x = 0.
(B) f(x) has local maxima at x = 0.
(C) Absolute maximum value of f(x) is not defined.
(D) f(x) is local maxima at x = – 3, x = 1.

18. A function f(x) = $\dfrac{x^2 - 3x + 2}{x^2 + 2x - 3}$ is -

(A) 1 is not in its domain
(B) minimum at x = – 3 and maximum at x = 1
(C) no point of maxima and minima
(D) increasing in its domain

19. For the function $f(x) = x\cot^{-1}x,\ x \ge 0$

(A) there is atleast one $x \in (0, 1)$ for which $\cot^{-1}x = \dfrac{x}{1+x^2}$

(B) for atleast one x in the interval $(0, \infty)$, $f\left(x + \dfrac{2}{\pi}\right) - f(x) < 1$

(C) number of solution of the equation f(x) = sec x is 1

(D) f′(x) is strictly decreasing in the interval $(0, \infty)$

20. Which of the following statements are true :

(A) If f(x) is differentiable function such that f(a) ≠ f(b) then there exist no c∈(a, b) such that f′(c) = 0
(B) The function $x^{100} + \sin x - 1$ is strictly increasing in [0, 1]
(C) If a, b, c are in A.P, then at least one root of the equation $3ax^2 - 4bx + c = 0$ is positive
(D) The number of solution(s) of equation $3\tan x + x^3 = 2$ in $(0, \pi/4)$ is 2

21. Let f(x) be a differentiable function and $f(\alpha) = f(\beta) = 0$ $(\alpha < \beta)$, then in the interval (α, β)

(A) f(x) + f '(x) = 0 has at least one root
(B) f(x) – f '(x) = 0 has at least one real root
(C) f(x) . f '(x) = 0 has at least one real root
(D) none of these

22. Which of the following inequalities are valid –

(A) $|\tan^{-1}x - \tan^{-1}y| \le |x - y|\ \forall\ x, y \in R$
(B) $|\tan^{-1}x - \tan^{-1}y| \ge |x - y|$
(C) $|\sin x - \sin y| \le |x - y|$
(D) $|\sin x - \sin y| \ge |x - y|$

23. For all x in [1, 2]
Let f"(x) of a non-constant function f(x) exist and satisfy $|f''(x)| \le 2$. If f(1) = f(2), then

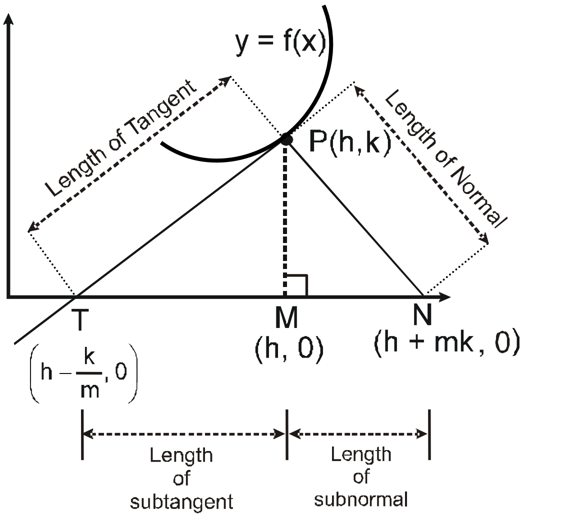
(A) There exist some $a \in (1, 2)$ such that $f'(a) = 0$
(B) f(x) is strictly increasing in (1, 2)
(C) There exist atleast one $c \in (1, 2)$ such that $f'(c) > 0$
(D) $|f'(x)| < 2\ \forall\ x \in [1, 2]$

PART - IV : COMPREHENSION

Comprehension # 1

Lengths of tangent, normal, subtangent and subnormal :

Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N (as shown in figure) and $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.



(i) Length of Tangent = PT = $|k|\sqrt{1+\frac{1}{m^2}}$

(ii) Length of Normal = PN = $|k|\sqrt{1+m^2}$

(iii) Length of subtangent = Projection of segment PT on x-axis = TM = $\left|\frac{k}{m}\right|$

(iv) Length of subnormal = projection of line segment PN on x axis =MN = |km|

1. Find the product of length of tangent and length of normal for the curve $y = x^3 + 3x^2 + 4x - 1$ at point x = 0.

(A) $\frac{17}{4}$ (B) $\frac{\sqrt{15}}{4}$ (C) 17 (D) $\frac{4}{\sqrt{17}}$

2. Determine 'p' such that the length of the subtangent and subnormal is equal for the curve $y = e^{px} + px$ at the point (0, 1).

(A) ±1 (B) ±2 (C) $\pm\frac{1}{2}$ (D) $\pm\frac{1}{4}$

3. Find length of subnormal to $x = \sqrt{2}\cos t$, $y = -3\sin t$ at $t = \frac{-\pi}{4}$.

(A) $\frac{2}{9}$ (B) 1 (C) $\frac{7}{2}$ (D) $\frac{9}{2}$

Comprehension # 2.

Consider a function f defined by $f(x) = \sin^{-1}\sin\left(\frac{x+\sin x}{2}\right)$, $\forall x \in [0, \pi]$, which satisfies

$f(x) + f(2\pi - x) = \pi$, $\forall x \in [\pi, 2\pi]$ and $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$, then

4. If α is the length of the largest interval on which f(x) is increasing, then α =

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π

5. If f(x) is symmetric about $x = \beta$, then $\beta =$

(A) $\frac{\alpha}{2}$ (B) α (C) $\frac{\alpha}{4}$ (D) 2α

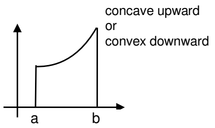
6. Maximum value of f(x) on $[0, 4\pi]$ is :

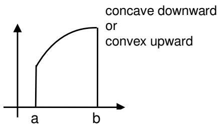
(A) $\frac{\beta}{2}$ (B) β (C) $\frac{\beta}{4}$ (D) 2β

Comprehension # 3.

Concavity and convexity :

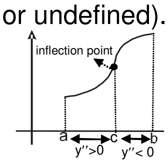
If $f''(x) > 0\ \forall\ x \in (a, b)$, then the curve y = f(x) is concave up (or convex down) in (a,b) and
If $f''(x) < 0\ \forall\ x \in (a, b)$ then the curve y = f(x) is concave down (or convex up) in (a, b).





Inflection point :

The point where concavity of the curve changes is known as point of inflection (at inflection point $f''(x)$ is equal to 0 or undefined).



7. Number of point of inflection for $f(x) = (x - 1)^3 (x - 2)^2$, is

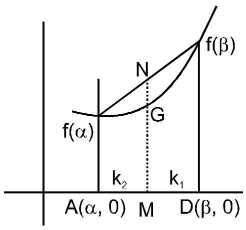
(A) 1 (B) 2 (C) 3 (D) 4

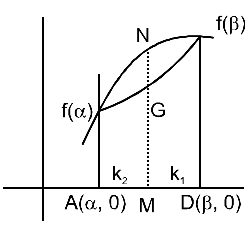
8. Exhaustive set of values of 'a' for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ will be concave upward along the entire real line, is :

(A) [–1,1] (B) [–2,2] (C) [0,2] (D) [0,4]

Comprehension # 4

For a double differentiable function f(x) if $f''(x) \geq 0$ then f(x) is concave upward and if $f''(x) \leq 0$ then f(x) is concave downward





Here $M\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2},\ 0\right)$

If f(x) is a concave upward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \geq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$,

where $k_1, k_2 \in R^+$

If f(x) is a concave downward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \leq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$,

where $k_1, k_2 \in R^+$

then answer the following :

9. Which of the following is true

(A) $\frac{\sin\alpha+\sin\beta}{2} > \sin\left(\frac{\alpha+\beta}{2}\right)$; α, β $(0, \pi)$ (B) $\frac{\sin\alpha+\sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta \in (\pi, 2\pi)$

(C) $\frac{\sin\alpha+\sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; α , $\beta \in (0, \pi)$ (D) None of these

10. Which of the following is true

(A) $\frac{2^{\alpha}+2^{\beta+1}}{3} \le 2^{\frac{\alpha+2\beta}{3}}$ (B) $\frac{2\ell n\ \alpha+\ell n\beta}{3} \ge \ell n\left(\frac{2\alpha+\beta}{3}\right)$

(C) $\frac{\tan^{-1}\alpha+\tan^{-1}\beta}{2} \le \tan^{-1}\left(\frac{\alpha+\beta}{2}\right)$ a, b $\in R^-$ (D) $\frac{e^{\alpha}+2e^{\beta}}{3} \ge e^{\frac{\alpha+2\beta}{3}}$

11. Let α , β and γ are three distinct real numbers and $f''(x) < 0$. Also f(x) is increasing function and let $A = \frac{f^{-1}(\alpha)+f^{-1}(\beta)+f^{-1}(\gamma)}{3}$ and $B = f^{-1}\left(\frac{\alpha+\beta+\gamma}{3}\right)$, then order relation between A and B is ?

(A) A > B (B) A < B (C) A = B (D) none of these

Exercise-3

✍ Marked questions are recommended for Revision.
*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1*.✍ For the function $f(x) = x\cos\frac{1}{x}$, $x \ge 1$, **[IIT-JEE 2009, Paper-2, (4, –1)/ 80]**

(A) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$

(B) $\lim\limits_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

2. Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim\limits_{x\to0}\left(1+\frac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is **[IIT-JEE 2009, Paper-2, (4, –1)/ 80]**

3. Let f be a function defined on **R** (the set of all real numbers) such that

$f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in$ **R**.

If g is a function defined on **R** with values in the interval $(0, \infty)$ such that $f(x) = \ell n\,(g(x))$, for all $x \in$ **R**, then the number of points in **R** at which g has a local maximum is

[IIT-JEE 2010, Paper-2, (3, 0)/ 79]

4. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[IIT-JEE 2010, Paper-1, (3, –1)/ 84]**

(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

5. Match the statements given in **Column-I** with the intervals/union of intervals given in **Column-II** **[IIT-JEE 2011, Paper-2, (8, 0), 80]**

Column-I	**Column-II**
(A) The set $\left\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right) : z \text{ is a complex number, } \lvert z\rvert = 1,\ z \neq \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \dfrac{\pi}{2}\right\}$ is	(r) $[2, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10)$, $x \ge 0$, then f(x) is increasing in	(s) $(-\infty, -1] \cup [1, \infty)$
	(t) $(-\infty, 0] \cup [2, \infty)$

6. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

7. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 p(3) = 2 , then $p'(0)$ is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

9. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is
[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) 6 (B) 4 (C) 2 (D) 0

10. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) 24 (B) 32 (C) 45 (D) 60

11. A vertical line passing through the point (h, 0) intersects the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\dfrac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

12*. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x =
[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) – 2 (B) $\dfrac{-2}{3}$ (C) 2 (D) $\dfrac{2}{3}$

Paragraph for Question Nos. 13 to 14.

Let f : [0, 1] $\rightarrow$ R (the set of all real numbers) be a function. Suppose the function f is twice differentiable,
$f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.

13. Which of the following is true for $0 < x < 1$? **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2} < f(x) < \frac{1}{2}$ (C) $-\frac{1}{4} < f(x) < 1$ (D) $-\infty < f(x) < 0$

14. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true ?
[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) $f'(x) < f(x)$, (B) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(C) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$ (D) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$

15. A line L : y = mx + 3 meets y - axis at E(0, 3) and the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum
Match List I with List II and select the correct answer using the code given below the lists :
[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List - I		**List - II**
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

16*. Let a $\in$ R and let f : R $\rightarrow$ R be given by $f(x) = x^5 - 5x + a$. Then
[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) has three real roots if a > 4 (B) f(x) has only one real root if a > 4
(C) f(x) has three real roots if a < – 4 (D) f(x) has three real roots if – 4 < a < 4

17. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is
[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

18. A cylindrical container is to be made from certain solid material with the following constraints: It has fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is
[JEE (Advanced) 2015, P-1 (4, 0) /88]

19*. Let f, g : [–1, 2] $\rightarrow$ R be continuous function which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table :

	x = –1	x = 0	x = 2
f(x)	3	6	0
g(x)	0	1	– 1

In each of the intervals (–1, 0) and (0, 2) the function (f – 3g)" never vanishes. Then the correct statement(s) is (are) **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) f'(x) – 3g'(x) = 0 has exactly three solutions in (–1, 0) $\cup$ (0, 2)
(B) f'(x) – 3g'(x) = 0 has exactly one solution in (–1, 0)
(C) f'(x) – 3g'(x) = 0 has exactly one solution in (0, 2)
(D) f'(x) – 3g'(x)= 0 has exactly two solutions in (–1, 0) and exactly two solutions in (0,2)

20. Let f : R $\rightarrow$ (0, ∞) and g : R $\rightarrow$ R be twice differentiable functions such that f" and g" are continuous functions on R. Suppose f '(2) = g(2) = 0, f"(2) $\neq$ 0 and g'(2) $\neq$ 0, If $\lim\limits_{x\to 2} \dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

(A) f has a local minimum at x = 2
(B) f has a local maximum at x = 2
(C) f"(2) > f(2)
(D) f(x) – f"(x) = 0 for at least one x $\in$ R

[JEE (Advanced) 2016, Paper-2, (4, –2)/62]

Answer Q.21, Q.22 and Q.23 by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x\log_e x$, $x \in (0, \infty)$
- Column1 contains information about zeros of f(x), f′(x) and f″(x).
- Column2 contains information about the limiting behavior of f(x), f′(x) and f″(x) at infinity.
- Column3 contains information about increasing/decreasing nature of f(x) and f′(x).

Column-1	Column-2	Column-3
(I) f(x) = 0 for some x $\in$ (1, e^2)	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) f′(x) = 0 for some x $\in$ (1, e)	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) f′(x) = 0 for some x $\in$ (0, 1)	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f′ is increasing in (0, 1)
(IV) f″(x) = 0 for some x $\in$ (1, e)	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f′ is decreasing in (e, e^2)

21. Which of the following options is the only INCORRECT combination?
[JEE(Advanced) 2017, Paper-1,(3, –1)/61]

(A) (I) (iii) (P) (B) (II) (iv) (Q) (C) (II) (iii) (P) (D) (III) (i) (R)

22. Which of the following options is the only CORRECT combination?
[JEE(Advanced) 2017, Paper-1,(3, –1)/61]

(A) (I) (ii) (R) (B) (III) (iv) (P) (C) (II) (iii) (S) (D) (IV) (i) (S)

23. Which of the following options is the only CORRECT combination?
[JEE(Advanced) 2017, Paper-1,(3, –1)/61]

(A) (III) (iii) (R) (B) (IV) (iv) (S) (C) (II) (ii) (Q) (D) (I) (i) (P)

24. If f : R $\rightarrow$ R is a twice differentiable function such that f″(x) > 0 for all x $\in$ R, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, f(1) = 1, then **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

(A) $f'(1) \leq 0$ (B) $f'(1) > 1$ (C) $0 < f'(1) \leq \frac{1}{2}$ (D) $\frac{1}{2} < f'(1) \leq 1$

25*. If f : R $\rightarrow$ R is a differentiable function such that f′(x) > 2f(x) for all x $\in$ R, and f(0) = 1, then
[JEE(Advanced) 2017, Paper-2,(4, –2)/61]

(A) f(x) > e^{2x} in (0, ∞)
(B) f′(x) < e^{2x} in (0, ∞)
(C) f(x) is increasing in (0, ∞)
(D) f(x) is decreasing in (0, ∞)

26*. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

(A) $f(x)$ attains its minimum at x = 0
(B) $f(x)$ attains its maximum at x = 0
(C) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$
(D) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

27*. For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018, Paper-1,(4, –2)/60]**

(A) There exist r, s $\in$ R, where r < s, such that f is one-one on the open interval (r, s)
(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$
(C) $\lim\limits_{x \to \infty} f(x) = 1$
(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x = 0 is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval [–1, 1] **[AIEEE 2009(8, –2), 144]**

(1) P (–1) is the minimum and P(1) is the maximum of P
(2) P (–1) is not minimum but P(1) is the maximum of P
(3) P (–1) is the minimum and P(1) is not the maximum of P
(4) neither P (–1) is the minimum nor P(1) is the maximum of P

2. The shortest distance between the line y – x = 1 and the curve $x = y^2$ is **[AIEEE 2009(4, –1), 144]**

(1) $\frac{3\sqrt{2}}{8}$ (2) $\frac{2\sqrt{3}}{8}$ (3) $\frac{3\sqrt{2}}{5}$ (4) $\frac{\sqrt{3}}{4}$

3. Let f : **R** $\to$ **R** be defined by **[AIEEE 2010(8, –2), 144]**

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at x = – 1, then a possible value of k is

(1) 0 (2) $-\frac{1}{2}$ (3) –1 (4) 1

4. Let f : **R** $\to$ **R** be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$ **[AIEEE 2010(8, –2), 144]**

Statement -1 : $f(c) = \frac{1}{3}$, for some c $\in$ R.

Statement -2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all x $\in$ R.

(1) Statement -1 is true, Statement-2 is true ;Statement -2 is not a correct explanation for Statement -1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

5. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is

[AIEEE 2010 (4, –1), 144]

(1) y = 1 (2) y = 2 (3) y = 3 (4) y = 0

6. Let f be a function defined by - **[AIEEE 2011 II(4, –1), 120]**

$$f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement - 1 : x = 0 is point of minima of f

Statement - 2 : $f'(0) = 0$.

(1) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(2) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1
(3) Statement-1 is true, statement-2 is false.
(4) Statement-1 is false, statement-2 is true.

7. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is : **[AIEEE 2011 (4, –1), 120]**

(1) $\dfrac{\sqrt{3}}{4}$ (2) $\dfrac{3\sqrt{2}}{8}$ (3) $\dfrac{8}{3\sqrt{2}}$ (4) $\dfrac{4}{\sqrt{3}}$

8. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE 2012(4, –1), 120]**

(1) $\dfrac{9}{7}$ (2) $\dfrac{7}{9}$ (3) $\dfrac{2}{9}$ (4) $\dfrac{9}{2}$

9. Let $a, b \in R$ be such that the function f given by $f(x) = \ell n\,|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement-1 : f has local maximum at $x = -1$ and at $x = 2$. **[AIEEE 2012 (4, –1), 120]**

Statement-2 : $a = \dfrac{1}{2}$ and $b = \dfrac{-1}{4}$.

(1) Statement-1 is false, Statement-2 is true.
(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(4) Statement-1 is true, statement-2 is false.

10. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

(1) lies between 1 and 2 (2) lies between 2 and 3
(3) lies between –1 and 0 (4) does not exist. **[AIEEE - 2013, (4, –1),120]**

11. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some
$c \in]0, 1[$: **[JEE(Main)2014,(4, – 1), 120]**

(1) $f'(c) = g'(c)$ (2) $f'(c) = 2g'(c)$ (3) $2f'(c) = g'(c)$ (4) $2f'(c) = 3g'(c)$

12. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha\log|x| + \beta x^2 + x$ then :
[JEE(Main)2014, (4, –1), 120]

(1) $\alpha = 2, \beta = -\dfrac{1}{2}$ (2) $\alpha = 2, \beta = \dfrac{1}{2}$ (3) $\alpha = -6, \beta = \dfrac{1}{2}$ (4) $\alpha = -6, \beta = -\dfrac{1}{2}$

13. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then **[JEE(Main)2016,(4, – 1), 120]**

(1) $(4 - \pi)\,x = \pi r$ (2) $x = 2r$ (3) $2x = r$ (4) $2x = (\pi + 4)\,r$

14. Consider $f(x) = \tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \dfrac{\pi}{2}\right)$. A normal to $y = f(x)$ at $x = \dfrac{\pi}{6}$ also passes through the point : **[JEE(Main)2016,(4, – 1), 120]**

(1) $\left(0, \dfrac{2\pi}{3}\right)$ (2) $\left(\dfrac{\pi}{6}, 0\right)$ (3) $\left(\dfrac{\pi}{4}, 0\right)$ (4) (0, 0)

15. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is : **[JEE(Main)2017,(4, – 1), 120]**

(1) 12.5 (2) 10 (3) 25 (4) 30

16. The normal to the curve y(x – 2)(x – 3) = x + 6 at the point where the curve intersects the y-axis passes through the point : **[JEE(Main)2017,(4, – 1), 120]**

(1) $\left(-\frac{1}{2},-\frac{1}{2}\right)$ (2) $\left(\frac{1}{2},\frac{1}{2}\right)$ (3) $\left(\frac{1}{2},-\frac{1}{3}\right)$ (4) $\left(\frac{1}{2},\frac{1}{3}\right)$

17. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, y = |x| is **[JEE(Main)2017,(4, – 1), 120]**

(1) $2\left(\sqrt{2}+1\right)$ (2) $2\left(\sqrt{2}-1\right)$ (3) $4\left(\sqrt{2}-1\right)$ (4) $4\left(\sqrt{2}+1\right)$

18. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angels , then the value of b is : **[JEE(Main)2018,(4, – 1), 120]**

(1) 4 (2) $\frac{9}{2}$ (3) 6 (4) $\frac{7}{2}$

19. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **[JEE(Main)2018,(4, – 1), 120]**

(1) $-2\sqrt{2}$ (2) $2\sqrt{2}$ (3) 3 (4) – 3

20. Let A(4, –4) and B(9, 6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is : **[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]**

(1) $30\frac{1}{2}$ (2) $31\frac{3}{4}$ (3) $31\frac{1}{4}$ (4) 32

21. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2},7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is :

[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

(1) $\frac{1}{2}$ (2) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (3) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (4) $\frac{\sqrt{5}}{6}$

22. Let $f(x) = \frac{x}{\sqrt{a^2+x^2}} - \frac{(d-x)}{\sqrt{b^2+(d-x)^2}}$, $x \in R$, where a, b and d are non-zero real constant. Then :

(1) f is neither increasing nor decreasing function of x

(2) f is an increasing function of x

(3) f is not a continuous function of x

(4) f is a decreasing function of x **[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]**

Answers

EXERCISE # 1

PART - I

Section (A) :

A-1. **(i)** $y = 4x + 5$ (ii) $y + x = 2$ (tangent), $y = x$ (normal) **(iii)** $16x + 13y = 9a$
(iv) $y = 0$

A-2 **(i)** $2x + y = 4$, $y = 2x$ **(ii)** $x + y = 3$

A-3. **(i)** $\left(4, \frac{8}{3}\right)$ **(ii).** (9/4, 3/8) **(iii)** (1, –1) , (–1, –5)

A-4. **(i)** $y = 12x - 16$ or $y = 12x + 16$ **(ii)** $x - 2y + 1 = 0$ or $2y + x - 1 = 0$

A-5. **(i)** $a = 1, b = -2$ (ii) **1**

Section (B) :

B-1. 1 **B-2.** 45° at (1, 0) and $\tan^{-1}\left(\frac{e}{e^2+2}\right)$ at (e, 1) **B-3.** 90° **B-5.** $\frac{3}{\sqrt{2}}$

B-6. $\sqrt{20} - 1$

Section (C) :

C-1. (i) – 2 cm/min (ii) 2 cm^2/min **C-2.** $2x^2 - 3x + 1$

C-3. (i) 2 km/hr (ii) 6 km/h **C-4.** 7.5 m^3

Section (D) :

D-2. (i) M.D. in $(-\infty, -3]$
M.I. in $[-3, 0]$
M.D. in $[0, 2]$
M.I. in $[2, \infty)$

(ii) M.D. in $\left(0, \frac{1}{\sqrt{3}}\right]$; M.I. in $\left[\frac{1}{\sqrt{3}}, \infty\right)$ **D-4.** $a \in R^+$

D-5. (i) Neither increasing nor decreasing at x = –1 and increasing at x = 2

(ii) at x = – 2 decreasing
at x = 0 decreasing
at x = 3 neither increasing nor decreasing
at x = 5 increasing

(iii) Strictly increasing at x = 0 (iv) Strictly increasing at x = 2, neither I nor D at x = 1

(v) Strictly increasing at x = 0

D-8. $b \in (0, e]$

D-9. (i) local max at x = 1, local min at x = 6 (ii) local max. at $x = -\frac{1}{5}$, local min. at x = – 1

(iii) local mini at $x = \frac{1}{e}$, No local maxima

D-10. (i) local maxima at $x = \log_2 \frac{4}{3}$ and local minima at x = 1 (ii) local min at 0, local max at 2

(iii) local max at $x = 0, \frac{2\pi}{3}$, local min at $x = \frac{\pi}{2}$, π

(iv) local maxima at –1 and local minma at 0 (v) local minima at $x = \pm\sqrt{2}$, 0

D-11. local max at x = 1, local min at x = 2.

Section (E) :

E-1. (i) max = 8, min. = – 8 (ii) max = $\sqrt{2}$, min = – 1
(iii) max. = 8, min. = – 10 (iv) max. = 25, min = – 39
(v) max. at x = $\pi/6$, max. value = 3/4; min. at x = 0 and $\pi/2$, min. value = 1/2

E-3. 1, 3 (respective)

E-4. F = 191 **E-6.** $\frac{4\pi r^3}{3\sqrt{3}}$ **E-8.** 110 m , $\frac{220}{\pi}$ m **E-9.** 32 sq. units

E-10. 12cm, 6 cm

Section (F) :

F-1. 1

PART - II

Section (A) :

A-1. (A) **A-2.** (C) **A-3.** (C) **A-4.** (B) **A-5.** (C) **A-6.** (D) **A-7.** (B)
A-8. (A) **A-9.** (A)

Section (B) :

B-1. (B) **B-2.** (D) **B-3.** (D) **B-4.** (C) **B-5.** (B) **B-6.** (D)

Section (C) :

C-1. (B) **C-2** (C) **C-3.** (A) **C-4.** (B) **C-5.** (C)

Section (D) :

D-1. (A) **D-2.** (B) **D-3.** (A) **D-4.** (C) **D-5_.** (D) **D-6.** (C) **D-7.** (C)

Section (E) :

E-1. (C) **E-2.** (D) **E-3.** (D) **E-4.** (C) **E-5.** (A) **E-6.** (C)

Section (F) :

F-1. (A) **F-2.** (C) **F-3.** (B) **F-4.** (D) **F-5.** (C)

PART – III

1. (A → p, q, r,s); (B → r) ; (C → p,q,r,s); (D →q) **2.** (A) → (p), (B) → (s), (C) → (q), (D) → (r)

3. (A) → (p,q), (B) → (r,s), (C) → (r,s), (D) → (r,s)

4. (A) → (r), (B) → (s), (C) → (q), (D) → (p)

EXERCISE # 2

PART-I

1.	(A)	**2.**	(D)	**3.**	(A)	**4.**	(B)	**5.**	(C)	**6.**	(B)	**7.**	(B)
8.	(B)	**9.**	(C)	**10.**	(D)	**11.**	(C)	**12.**	(A)	**13.**	(D)	**14.**	(C)
15.	(A)	**16.**	(C)	**17.**	(D)	**18.**	(D)	**19.**	(A)	**20.**	(B)	**21.**	(B)
22.	(A)	**23.**	(B)	**24.**	(C)	**25.**	(A)	**26.**	(B)				

PART - II

1.	2	**2.**	24	**3.**	3	**4.**	15	**5.**	66	**6.**	2	**7.**	0
8.	0	**9.**	29	**10.**	4	**11.**	4	**12.**	1	**13.**	8	**14.**	1
15.	32	**16.**	**36**	**17.**	81	**18.**	39	**19.**	40	**20.**	3	**21.**	76

PART - III

1.	(CD)	**2.**	(ABC)	**3.**	(AB)	**4.**	(AD)	**5.**	(BC)	**6.**	(AB)	**7.**	(AD)
8.	(AD)	**9.**	(AB)	**10.**	(ABC)	**11.**	(AD)	**12.**	(ACD)	**13.**	(BD)	**14.**	(AB)
15.	(AC)	**16.**	(ABC)	**17.**	(ACD)	**18.**	(AC)	**19.**	(BD)	**20.**	(BC)		
21.	(ABC)	**22.**	(AC)	**23.**	(ACD)								

PART - IV

1.	(A)	**2.**	(C)	**3.**	(D)	**4.**	(C)	**5.**	(B)	**6.**	(A)	**7.**	(C)
8.	(B)	**9.**	(C)	**10.**	(D)	**11.**	(A)						

EXERCISE # 3

PART - I

1*.	(BCD)	**2.**	0	**3.**	1	**4.**	(D)	**5.**	(A)$\rightarrow$(s), (B)$\rightarrow$(t), (C)$\rightarrow$(r), (D)$\rightarrow$(r)				
6.	2	**7.**	9	**8.**	5	**9.**	(C)	**10.**	(AC)	**11.**	9	**12*.**	(AB)
13.	(D)	**14.**	(C)	**15.**	(A)	**16*.**	(BD)	**17.**	8	**18.**	4	**19*.**	(BC)
20.	(A,D)	**21 .**	(D)	**22.**	(C)	**23.**	(C)	**24.**	**(B)**	**25.**	(A,C)	**26.**	(BC)
27.	(ABD)												

PART - II

1.	(2)	**2.**	(1)	**3.**	(3)	**4.**	(4)	**5.**	(3)	**6.**	(2)	**7.**	(2)
8.	(3)	**9.**	(2)	**10.**	(4)	**11.**	(2)	**12.**	(1)	**13.**	(2)	**14.**	(1)
15.	(3)	**16.**	(2)	**17.**	(3)	**18.**	(2)	**19.**	(2)	**20.**	(3)	**21.**	(3)
22.	(2)												

High Level Problems (HLP)

1. A particle moving on a curve has the position at time t given by
$x = f'(t) \sin t + f''(t) \cos t$, $y = f'(t) \cos t - f''(t) \sin t$, where f is a thrice differentiable function. Then prove that the velocity of the particle at time t is $f'(t) + f'''(t)$.

2. Find the interval in which $f(x) = x\sqrt{4ax - x^2}$ $(a < 0)$ is decreasing

3. $f : [0, 4] \to R$ is a differentiable function. Then prove that for some $a, b \in (0, 4)$, $f^2(4) - f^2(0) = 8f'(a) \cdot f(b)$

4. If all the extreme value of function $f(x) = a^2x^3 - \frac{a}{2}x^2 - 2x - b$ are positive and the minimum is at the point $x_0 = \frac{1}{3}$ then show that when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 3 \Rightarrow b < -\frac{1}{2}$

5. If $f(x) = \begin{cases} 3 + |x - k|, & x \le k \\ a^2 - 2 + \dfrac{\sin(x-k)}{x-k}, & x > k \end{cases}$ has minimum at x = k , then show that |a| > 2

6. The equation $x^3 - 3x + [a] = 0$, where [.] denotes the greatest integer function, will have three real and distinct roots then find the set of all posible values of a.

7. Let $f(x) = \sin\frac{\{x\}}{a} + \cos\frac{\{x\}}{a}$ where $a > 0$ and { . } denotes the fractional part function. Then find the set of values of 'a' for which f can attain its maximum values.

8. Find the values of the parameter 'k' for which the equation $x^4 + 4x^3 - 8x^2 + k = 0$ has all roots real.

Comprehension (Q. No. 9 to 11)

A function f(x) having the following properties;
(i) f(x) is continuous except at x = 3
(ii) f(x) is differentiable except at x = – 2 and x = 3
(iii) $f(0) = 0$, $\lim_{x\to 3} f(x) \to -\infty$, $\lim_{x\to -\infty} f(x) = 3$, $\lim_{x\to \infty} f(x) = 0$
(iv) $f'(x) > 0 \ \forall x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \le 0 \ \forall x \in (-2, 3)$
(v) $f''(x) > 0 \ \forall x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0 \ \forall x \in (0, 3) \cup (3, \infty)$
then answer the following questions

9. Find the Maximum possible number of solutions of $f(x) = |x|$

10. Show that graph of function $y = f(-|x|)$ is continuous but not differentiable at two points, if $f'(0) = 0$

11. Show that $f(x) + 3x = 0$ has five solutions if $f'(0) > -3$ and $f(-2) > 6$

12. Let $F(x) = (f(x))^2 + (f'(x))^2$, $F(0) = 7$, where f(x) is thrice differentiable function such that $|f(x)| \le 1 \ \forall x \in [-1, 1]$, then prove the followings.
(i) there is atleast one point in each of the intervals (–1, 0) and (0, 1) where $|f'(x)| \le 2$
(ii) there is atleast one point in each of the intervals (–1, 0) and (0, 1) where $F(x) \le 5$
(iii) there exits atleast one maxima of F(x) in (–1, 1)
(iv) for some $c \in (-1, 1)$, $F(c) \ge 7$, $F'(c) = 0$ and $F''(c) \le 0$

13. A figure is bounded by the curves, $y = x^2 + 1$, y = 0, x = 0 and x = 1. At what point (a, b), a tangent should be drawn to the curve $y = x^2 + 1$ for it to cut off a trapezium of the greatest area from the figure.

14. If $y = \dfrac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a and b, show that the turning value is a maximum.

15. With the usual meaning for a, b, c and s, if Δ be the area of a triangle, prove that the error in Δ resulting from a small error in the measurement of c, is given by

$$d\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\} dc$$

16. Find the possible values of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution

17. If $(m-1)\,a_1^2 - 2m\,a_2 < 0$, then prove that $x^m + a_1 x^{m-1} + a_2 x^{m-2} + ... + a_{m-1} x + a_0 = 0$ has at least one non real root $(a_1, a_2, ..., a_m \in R)$

18. If $f'(x) > 0, f''(x) > 0\ \forall\ x \in (0, 1)$ and $f(0) = 0, f(1) = 1$, then prove that $f(x)\, f^{-1}(x) < x^2\ \forall\ x \in (0, 1)$

19. Find the interval of increasing and decreasing for the function $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f\left(\dfrac{27}{2} - x^2\right)$, where $f''(x) < 0$ for all $x \in R$.

20. Using calculus prove that $H.M \le G.M. \le A.M$ for positive real numbers.

21. Prove the following inequalities

(i) $1 + x^2 > (x \sin x + \cos x)$ for $x \in [0, \infty)$.

(ii) $\sin x - \sin 2x \le 2x$ for all $x \in \left[0, \dfrac{\pi}{3}\right]$

(iii) $\dfrac{x^2}{2} + 2x + 3 \ge (3 - x)e^x$ for all $x \ge 0$

(iv) $0 < x \sin x - \dfrac{\sin^2 x}{2} < \dfrac{1}{2}(\pi - 1)$ for $0 < x < \dfrac{\pi}{2}$

22. Find the interval to which b may belong so that the function $f(x) = \left(1 - \dfrac{\sqrt{21 - 4b - b^2}}{b+1}\right) x^3 + 5x + \sqrt{6}$ is increasing at every point of its domain.

23. If $0 < x < 1$ prove that $y = x \ell n\, x - \dfrac{x^2}{2} + \dfrac{1}{2}$ is a function such that $\dfrac{d^2y}{dx^2} > 0$. Deduce that $x \ell n\, x > \dfrac{x^2}{2} - \dfrac{1}{2}$.

24. Find positive real numbers 'a' and 'b' such that $f(x) = ax - bx^3$ has four extrema on [–1 , 1] at each of which | f(x) | = 1

25. For any acute angled ΔABC, find the maximum value of $\dfrac{\sin A}{A} + \dfrac{\sin B}{B} + \dfrac{\sin C}{C}$

26. Suppose p,q,r,s are fixed real numbers such that a quadrilateral can be formed with sides p,q,r,s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle .

27. For what real values of 'a' and 'b' all the extrema of the function $f(x) = \frac{5a^2}{3}x^3 + 2ax^2 - 9x + b$ are positive and the maximum is at the point $x_0 = \frac{-5}{9}$

28. Find the minimum value of $f(x) = 8^x + 8^{-x} - 4(4^x + 4^{-x})$, $\forall\ x \in R$

29. Using calculus , prove that $\log_2 3 > \log_3 5 > \log_4 7$.

30. Show that the volume of the greatest cylinder which can be inscribed in a cone of height 'h' and semi–vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

31. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side .

32. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval (0, 1]. The tangent at x_0 meets the x–axis and y–axis at A & B respectively. Then find the minimum area of the triangle OAB, where O is the origin

33. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. Then find the maximum volume attainable for the cone

34. Suppose velocity of waves of wave length λ in the Atlantic ocean is $k\sqrt{\left\{\left(\frac{\lambda}{a}\right)+\left(\frac{a}{\lambda}\right)\right\}}$, where k and a are constants. Show that minimum velocity attained by the waves is independent of the constant a.

35. Find the minimum distance of origin from the curve $ax^2 + 2bxy + ay^2 = c$ where $a > b > c > 0$

36. Prove that $e^x + \sqrt{1+e^{2x}} \ge (1 + x) + \sqrt{2+2x+x^2}\quad \forall\ x \in R$

37. Find which of the two is larger $\ell n\,(1 + x)$ or $\frac{\tan^{-1}x}{1+x}$.

38. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the intervals of monotonicity of g(x).

39. If $f(x) = (2013)x^{2012} - (2012)x^{2011} - 2014x + 1007$, then show that for $x \in [0, 1007^{1/2011}]$, $f(x) = 0$ has at least one real root.

40. A function f is differentiable in the interval $0 \le x \le 5$ such that $f(0) = 4$ & $f(5) = -1$. If $g(x) = \frac{f(x)}{x+1}$, then prove that there exists some $c \in (0, 5)$ such that $g'(c) = -\frac{5}{6}$.

41. Let f(x) and g(x) be differentiable functions having no common zeros so that $f(x)\,g'(x) \ne f'(x)\,g(x)$. Prove that between any two zeros of f(x), there exist atleast one zero of g(x).

42. f is continuous in [a, b] and differentiable in (a, b) (where a > 0) such that $\frac{f(a)}{a} = \frac{f(b)}{b}$. Prove that there exist $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(x_0)}{x_0}$.

43. If $\phi(x)$ is a differentiable function $x \in R$ and $a \in R^+$ such that $\phi(0) = \phi(2a)$, $\phi(a) = \phi(3a)$ and $\phi(0) \neq \phi(a)$ then show that there is at least one root of equation $\phi'(x + a) = \phi'(x)$ in (0, 2a)

44. Find the set of values of the parameter 'a' for which the function ;
$f(x) = 8ax - a \sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$, is

45. Let h be a twice differentiable positive function on an open interval J. Let
$$g(x) = \ell n\,(h(x)) \ \forall\ x \in J$$
Suppose $(h'(x))^2 > h''(x)\,h(x)$ for each $x \in J$. Then prove that g is concave downward on J.

46. If the complete set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a + 2)x^2 + (a - 1)x + 2$ possess a negative point of inflection is $(-\infty, \alpha) \cup (\beta, \infty)$, then $|\alpha| + |\beta|$ is :

47. If two curves $y = 2\sin\frac{5\pi}{6}x$ and $y = \alpha x^2 - 3\alpha x + 2\alpha + 1$ touch each other at some point then the value of $\frac{\sqrt{3}\alpha}{5\pi}$ is $\left(0 \le x \le \frac{18}{5}\right)$

HLP Answers

2. [4a, 3a] **6.** $a \in [-1, 2)$ **7.** $\left(0, \frac{4}{\pi}\right)$ **8.** $k \in [0,3]$ **9.** 3

13. $\left(\frac{1}{2}, \frac{5}{4}\right)$ **14.** a = 1, b = 0 **16.** $a \in \left(-\frac{13}{4}, 3\right)$

19. g(x) is increasing if $x \in (-\infty, 3] \cup [0, 3]$; g(x) is decreasing if $x \in [-3, 0] \cup [3, \infty)$

22. $[-7, -1) \cup [2, 3]$ **24.** a = 3 , b = 4 **25.** $\frac{9\sqrt{3}}{2\pi}$

27. If $a = \frac{-9}{5}$, then $b > \frac{36}{5}$; If $a = \frac{81}{25}$ then $b > \frac{400}{243}$

28. –10 **31.** cos A = 0.8 **32.** $\frac{4\sqrt{3}}{9}$ **33.** $2\pi/3$ **35.** $\sqrt{\frac{c}{a+b}}$

37. $\ell n\,(1 + x)$ **38.** Increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, decreasing when $x \in \left(0, \frac{\pi}{4}\right)$.

44. $a \in (6, \infty)$ **46.** 2 **47.** 1/2