Target : JEE (Main + Advanced)

Limits, Continuity & Derivability

CONTENT

Limits, Continuity & Derivability

JEE (ADVANCED) SYLLABUS

Limits : Limit and Continuity of a function, Limit and Continuity of the sum, difference, product and quotient of two functions, L'Hospital rule of evaluation of limits of functions, Even and Odd functions, Inverse of a function, Continuity of Composite functions, Intermediate value property of continuous functions.
Continuity : Continuity of a function, Continuity of the sum, difference, product and quotient of two functions, L'Hospital rule of evaluation of limits of functions. Continuity of composite functions, intermediate value property of continuous functions.
Derivability : Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions. Derivatives of implicit functions, derivatives up to order two, geometrical interpretation of the derivative.

JEE (MAIN) SYLLABUS

Continuity & Derivability : Sets and their representation; Union, intersection and complement of sets and their algebraic properties; Power set; Relation, Types of relations, equivalence relations.

Limits, Continuity & Derivability

Calculus required continuity, and continuity was supposed to require the infinitely little; But nobody could discover what the infinitely little might be...........Russell, Bertrand

Definition : Limit of a function f(x) is said to exist, as $x \to a$ when,

$\lim\limits_{h \to 0^+} f(a - h) = \lim\limits_{h \to 0^+} f(a + h) =$ Finite

(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at x = a. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '$-\infty$', then it is said to be infinite limit.

Remember, '$x \to a$' means that x is approaching to 'a' but not equal to 'a'.

Fundamental theorems on limits :

Let $\lim\limits_{x\to a} f(x) =$ and $\lim\limits_{x\to a} g(x) = m$. If ℓ & m are finite, then:

(A) $\lim\limits_{x\to a} \{ f(x) \pm g(x) \} = \ell \pm m$

(B) $\lim\limits_{x\to a} \{ f(x). g(x) \} = \ell . m$

(C) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$, provided $m \neq 0$

(D) $\lim\limits_{x\to a} k f(x) = k \lim\limits_{x\to a} f(x) = k\ell$; where k is a constant.

(E) $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f is continuous at $g(x) = m$.

Example # 1 : Evaluate the following limits : -

(i) $\lim\limits_{x\to 2} (x + 2)$ (ii) $\lim\limits_{x\to 0} \cos(\sin x)$

Solution : (i) x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim\limits_{x\to 2} (x + 2) = 2 + 2 = 4$

(ii) $\lim\limits_{x\to 0} \cos(\sin x) = \cos\left(\lim\limits_{x\to 0} \sin x\right) = \cos 0 = 1$

Self practice problems

Evaluate the following limits : -

(1) $\lim\limits_{x\to 2} x(x - 1)$ (2) $\lim\limits_{x\to 2} \dfrac{x^2 + 4}{x + 2}$

Ans. (1) 2 (2) 2

Indeterminate forms :

If on putting x = a in f(x), any one of $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\mathbf{0 \times \infty, \infty - \infty, \infty^0, 0^0, 1^\infty}$ form is obtained, then the limit has an indeterminate form. All the above forms are interchangeable, i.e. we can change one form to other by suitable substitutions etc.

In such cases $\lim\limits_{x\to a} f(x)$ may exist.

Consider $f(x) = \frac{x^2-4}{x-2}$. Here $\lim\limits_{x\to 2} x^2 - 4 = 0$ and $\lim\limits_{x\to 2} x - 2 = 0$

$\therefore$ $\lim\limits_{x\to 2}$ f(x) has an indeterminate form of the type $\frac{0}{0}$

$\lim\limits_{x\to\infty} \frac{\ell nx}{x}$ has an indeterminate form of type .

$\lim\limits_{x\to 0} (1 + x)^{1/x}$ is an indeterminate form of the type 1^{∞}

NOTE :

(i) $\infty + \infty = \infty$

(ii) ∞ x $\infty = \infty$

(iii) $\frac{a}{\infty} = 0$, if a is finite.

(iv) is not defined for any $a \in R$.

(v) $\lim\limits_{x\to 0} \frac{x}{x}$ is an indeterminate form whereas $\lim\limits_{x\to 0} \frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function) Students may remember these forms along with the prefix 'tending to' i.e. $\frac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form where as $\frac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

Similarly (tending to one)$^{\text{tending to } \infty}$ is indeterminate form whereas (exactly one)$^{\text{tending to } \infty}$ is not an indeterminate form, its value is one.

To evaluate a limit, we must always put the value where 'x' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminate form comes, we have to remove the indeterminancy, once the indeterminancy is removed the limit can be evaluated by putting the value of x, where it is approaching.

Methods of removing indeterminancy

Basic methods of removing indeterminancy are

(A) Factorisation (B) Rationalisation
(C) Using standard limits (D) Substitution
(E) Expansion of functions.

Factorisation method : -

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Example # 2 $\lim\limits_{x\to 3} \frac{x^2-2x-3}{x^2-4x+3}$

Solution : $\lim\limits_{x\to 3} \frac{x^2-2x-3}{x^2-4x+3} = \lim\limits_{x\to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

Rationalisation method :-

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Example # 3 : Evaluate :

(i) $\lim\limits_{x\to 1} \frac{3-\sqrt{8x+1}}{5-\sqrt{24x+1}}$ (ii) $\lim\limits_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$

Solution : (i) $\displaystyle \ell im_{x\to 1} \frac{3-\sqrt{8x+1}}{5-\sqrt{24x+1}} = \ell im_{x\to 1} \frac{(9-8x-1)\left(5+\sqrt{24x+1}\right)}{\left(3+\sqrt{8x+1}\right)(25-24x-1)} = \frac{5}{9}$

(ii) The form of the given limit is when $x \to 0$. Rationalising the numerator, we get

$$\ell im_{x\to 1} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} = \ell im_{x\to 1}\left[\frac{x}{\sqrt{1+x}-\sqrt{1-x}} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \ell im_{x\to 1}\left[\frac{x\ (\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)}\right] = \ell im_{x\to 1}\left[\frac{x\ (\sqrt{1+x}+\sqrt{1-x})}{2x}\right] = \ell im_{x\to 1}\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right] = \frac{2}{2} = 1$$

Self practice problems

Evaluate the following limits : -

(3) $\displaystyle \ell im_{x\to\frac{\pi}{2}} \frac{1-(\sin x)^{1/3}}{1-(\sin x)^{2/3}}$ (4) $\displaystyle \ell im_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

(5) $\displaystyle \ell im_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ (6) $\displaystyle \ell im_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}}$

Ans. (3) $\frac{1}{2}$ (4) $\frac{1}{2\sqrt{x}}$ (5) $\frac{1}{4a\sqrt{a-b}}$ (6) 0

Standard limits :

(a) (i) $\displaystyle \ell im_{x\to 0} \frac{\sin x}{x} = \ell im_{x\to 0} \frac{\tan x}{x} = 1$ [Where x is measured in radians]

(ii) $\displaystyle \ell im_{x\to 0} \frac{\tan^{-1} x}{x} = \ell im_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(iii) $\displaystyle \ell im_{x\to 0} (1+x)^{\frac{1}{x}} = e$; $\displaystyle \ell im_{x\to 0} (1+ax)^{\frac{1}{x}} = e^a$

(iv) $\displaystyle \ell im_{x\to 0} \left(1+\frac{1}{x}\right)^x = e$; $\displaystyle \ell im_{x\to 0}\ \ell im_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$

(v) $\displaystyle \ell im_{x\to 0} \frac{e^x-1}{x} = 1$; $\displaystyle \ell im_{x\to 0} \frac{a^x-1}{x} = \log_e a = \ell na$,a > 0

(vi) $\displaystyle \ell im_{x\to 0} \frac{\ell n(1+x)}{x} = 1$

(vii) $\displaystyle \ell im_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

(b) If $f(x) \to 0$, when $x \to a$, then

(i) $\displaystyle \ell im_{x\to a} \frac{\sin f(x)}{f(x)} = 1$ (ii) $\displaystyle \ell im_{x\to a} \cos f(x) = 1$

(iii) $\displaystyle \ell im_{x\to a} \frac{\tan f(x)}{f(x)} = 1$ (iv) $\displaystyle \ell im_{x\to a} \frac{e^{f(x)}-1}{f(x)} = 1$

(v) $\displaystyle \ell im_{x\to a} \frac{b^{f(x)}-1}{f(x)} = \ell n\, b,\ (b > 0)$ (vi) $\displaystyle \ell im_{x\to a} \frac{\ell n(1+f(x))}{f(x)} = 1$

(vii) $\displaystyle \ell im_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

(c) $\ell im_{x\to a} f(x) = A > 0$ and $\ell im_{x\to a} \phi(x) = B$(a finite quantity), then $\ell im_{x\to a} [f(x)]^{\phi(x)} = A^B$.

Example # 4 : Evaluate : $\ell\text{im}_{x\to 0} \dfrac{(1+x)^n - 1}{x}$

Solution : $\ell\text{im}_{x\to 0} \dfrac{(1+x)^n - 1}{x} = \ell\text{im}_{x\to a} \dfrac{(1+x)^n - 1}{(1+x)-1} = n$

Example # 5 : Evaluate : $\ell\text{im}_{x\to 0} \dfrac{3^x - 1}{2^x - 1}$

Solution : $\ell\text{im}_{x\to 0} \dfrac{3^x - 1}{2^x - 1} = \ell\text{im}_{x\to 0} \dfrac{3^x - 1}{x} \cdot \dfrac{1}{\dfrac{2^x - 1}{x}} = \dfrac{\ell n 3}{\ell n 2}$

Example # 6 : Evaluate : $\ell\text{im}_{x\to 0} \dfrac{1-\cos 3x}{x^2}$

Solution : $\ell\text{im}_{x\to 0} \dfrac{1-\cos 3x}{x^2} = \ell\text{im}_{x\to 0} \dfrac{1}{2} \cdot \left(\dfrac{3\sin\dfrac{3x}{2}}{\dfrac{3x}{2}} \right)^2 = \dfrac{9}{2}$

Example # 7 : Evaluate : $\ell\text{im}_{x\to 0} \dfrac{\sin^2 x}{\sin 4x.\tan x}$

Solution : $\ell\text{im}_{x\to 0} \dfrac{\sin^2 x}{\sin 4x.\tan x} = \ell\text{im}_{x\to 0} \dfrac{x^2\left(\dfrac{\sin x}{x}\right)^2}{4x\left(\dfrac{\sin 4x}{4x}\right)\left(\dfrac{\tan x}{x}\right)x} = \dfrac{1}{4}$

Example # 8 : Evaluate : $\ell\text{im}_{x\to 0} (1 + 2x)^{1/x}$

Solution : $\ell\text{im}_{x\to 0} (1 + 2x)^{1/x} = e^{\ell\text{im}_{x\to 0} \frac{2}{x}.x} = e^2.$

Example # 9 : Evaluate :

(i) $\ell\text{im}_{x\to y} \dfrac{e^x - e^y}{x-y}$ (ii) $\ell\text{im}_{x\to 0} \dfrac{x(e^x - 1)}{1-\cos x}$

Solution : (i) $\ell\text{im}_{x\to y} \dfrac{e^x - e^y}{x-y} = \ell\text{im}_{x\to y} \dfrac{e^y(e^{x-y} - 1)}{x-y} = e^y$

(ii) $\ell\text{im}_{x\to 0} \dfrac{x(e^x - 1)}{1-\cos x} = \ell\text{im}_{x\to 0} \dfrac{x(e^x - 1)}{2\sin^2 \dfrac{x}{2}} = \dfrac{1}{2} . \ell\text{im}_{x\to 0} \left[\dfrac{e^x - 1}{x} \cdot \dfrac{x^2}{\sin^2 \dfrac{x}{2}} \right] = 2$

Self practice problems

Evaluate the following limits : -

(7) $\ell\text{im}_{x\to 0} \dfrac{\sin 7x}{3x}$

(8) $\ell\text{im}_{x\to 0} \dfrac{8}{x^8}\left(1-\cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$

(9) $\ell\text{im}_{x\to \frac{\pi}{4}} \dfrac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$

(10) $\ell\text{im}_{x\to 0} \dfrac{5^x - 9^x}{x}$

(11) $\ell\text{im}_{x\to \infty} (1 + a^2)^x \sin \dfrac{b}{(1+a^2)^x}$, where $a \neq 0$

Ans. (7) $\dfrac{7}{3}$ (8) $\dfrac{1}{32}$ (9) does not exist (10) $\ell n \dfrac{5}{9}$ (11) b

Use of substitution in solving limit problems

Sometimes in solving limit problem we convert $\lim\limits_{x\to a} f(x)$ into $\lim\limits_{h\to 0} f(a+h)$ or $\lim\limits_{h\to 0} f(a-h)$ according as need of the problem. (here h is approaching to zero.)

Example # 10 : Evaluate : $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x-\sin x}{\cos x\left(\sqrt{2}-2\sin x\right)}$

Solution : $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x-\sin x}{\cos x\left(\sqrt{2}-2\sin x\right)} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{1}{\sqrt{2}}.\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$ Put $x = \dfrac{\pi}{4}+h$

$\because \quad x \to \dfrac{\pi}{4} \Rightarrow h \to 0$

$$\lim_{h\to 0}\frac{1}{\sqrt{2}}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)} = \lim_{h\to 0}\frac{1}{\sqrt{2}}\frac{1-\frac{1+\tan h}{1-\tan h}}{1-\sin h-\cos h} = \lim_{h\to 0}\frac{1}{\sqrt{2}}\frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{2}}\frac{-2\tan h}{2\sin\frac{h}{2}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]}\frac{1}{(1-\tanh)} = \lim_{h\to 0}\frac{1}{\sqrt{2}}\frac{-2\frac{\tanh}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]}\frac{1}{(1-\tanh)}$$

$$= \frac{-2}{-1}.\frac{1}{\sqrt{2}} = \sqrt{2}$$

Limits using expansion

(a) $a^x = 1+\dfrac{x\ \ell n\ a}{1!}+\dfrac{x^2\ \ell n^2 a}{2!}+\dfrac{x^3\ \ell n^3 a}{3!}+........., \quad a > 0$

(b) $e^x = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+......$

(c) $\ell n\,(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+.........,\quad$ for $-1 < x \le 1$

(d) $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+.....$

(e) $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+.....$

(f) $\tan x = x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+......$

(g) for $|x| < 1$, $n \in R$; $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{1\ .\ 2}x^2 + \dfrac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}x^3 + ...\infty$

(h) $(1+x)^{\frac{1}{x}} = e\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-.............\right)$

Example # 11: Evaluate : $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}$

Solution : $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3} = \lim\limits_{x\to 0}\dfrac{x+\frac{x^3}{3}+\frac{2}{15}x^5+....-x}{x^3} = \dfrac{1}{3}$

Example # 12 : Evaluate : $\lim\limits_{x\to 2} \dfrac{(14+x)^{\frac{1}{4}}-2}{x-2}$

Solution : $\lim\limits_{x\to 2} \dfrac{(14+x)^{\frac{1}{4}}-2}{x-2} = \lim\limits_{x\to 2} \dfrac{(14+x)^{\frac{1}{4}}-16^{\frac{1}{4}}}{(14+x)-16} = \dfrac{1}{4} \cdot 16^{\frac{1}{4}-1} = \dfrac{1}{32}$

Example # 13 : If $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)+\alpha\sin x+\dfrac{x^2}{2}}{x\tan^2 x} = \dfrac{1}{2}$ then find α.

Solution : $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)+\alpha\sin x+\dfrac{x^2}{2}}{x\tan^2 x} = \lim\limits_{x\to 0} \dfrac{\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-.....\right)+\alpha\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-.....\right)+\dfrac{x^2}{2}}{x^3.\dfrac{\tan^2 x}{x^2}} = \dfrac{1}{2}$

$\Rightarrow \quad \alpha = -1$

Example # 14 : Evaluate : $\lim\limits_{x\to 0} \dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}$

Solution : $\lim\limits_{x\to 0} \dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x} = \lim\limits_{x\to 0} \dfrac{e-e\left(1-\dfrac{x}{2}+......\right)}{\tan x} = \lim\limits_{x\to 0} \times \dfrac{x}{\tan x} = \dfrac{e}{2}$

Example # 15 : Find the values of $\alpha + 2\beta + 3\gamma$ if $\lim\limits_{x\to 0} \dfrac{\alpha e^x-\beta\cos x+\gamma e^{-x}}{x\tan x} = 2$

Solution : $\lim\limits_{x\to 0} \dfrac{\alpha e^x-\beta \quad \cos x+\gamma e^{-x}}{x\tan x} = 2$ .....(1)

at $x \to 0$ numerator must be equal to zero

$\therefore \quad \alpha - \beta + \gamma = 0 \quad \Rightarrow \quad \beta = \alpha + \gamma$ .....(2)

From (1) & (2), $\lim\limits_{x\to 0} \dfrac{\alpha e^x-(\alpha+\gamma) \quad \cos x+\gamma e^{-x}}{x^2\dfrac{\tan x}{x}} = 2$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{\alpha\left(1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+......\right)-(\alpha+\gamma)\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-......\right)+\gamma\left(1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+......\right)}{x^2} = 2$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{x^2\left(\dfrac{(\alpha-\gamma)}{x}+(\alpha+\gamma)+\dfrac{x}{3!}(\alpha-\gamma)+.....\right)}{x^2} = 2$

Since R.H.S is finite,

$\therefore \quad \alpha - \gamma = 0 \qquad \therefore \alpha = \gamma, \qquad$ then $\dfrac{0+2\alpha+0+....}{1} = 2 \quad \therefore \quad \alpha = 1$ then $\gamma = 1$

From (2), $\quad \beta = \alpha + \gamma = 1 + 1 = 2$

So, $\alpha + 2\beta + 3\gamma = 8$

Limit when $x \to \infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x \to \infty$, we use to substitute $y = \dfrac{1}{x}$ and in this case $y \to 0^+$.

Example # 16 : Evaluate : $\lim\limits_{x\to\infty} \dfrac{\cos x}{x}$

Solution : $\lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$

Example # 17: Evaluate $\lim\limits_{x\to\infty} x.\tan\dfrac{1}{x}$

Solution : $\lim\limits_{x\to\infty} x.\tan\dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\tan\frac{1}{x}}{\frac{1}{x}} = 1$

Example # 18 : Evaluate : $\lim\limits_{x\to\infty} \dfrac{4x+3}{x-8}$

Solution : $\lim\limits_{x\to\infty} \dfrac{4x+3}{x-8} = \lim\limits_{x\to\infty} \dfrac{4+\frac{3}{x}}{1-\frac{8}{x}} = 4.$

Example # 19 : Evaluate $\lim\limits_{x\to\infty} \dfrac{4x^2-8}{7x+x^5+1}$

Solution : $\lim\limits_{x\to\infty} \dfrac{4x^2-8}{7x+x^5+1} = \lim\limits_{x\to\infty} \dfrac{\frac{4}{x^3}-\frac{8}{x^5}}{\frac{7}{x^4}+1+\frac{1}{x^5}} = 0$

Example # 20 : Evaluate $\lim\limits_{x\to-\infty} \dfrac{x-8}{\sqrt{4x^2+x+1}}$

Solution : Replace x by –t

$$\lim_{t\to\infty} \frac{-t-8}{\sqrt{4t^2-t+1}} = \lim_{t\to\infty} \frac{-1-\frac{8}{t}}{\sqrt{4-\frac{1}{t}+\frac{1}{t^2}}} = -\frac{1}{2}$$

Some important notes :

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$ (iii) $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$

(iv) $\lim\limits_{x\to\infty} \dfrac{(\ell nx)^n}{x} = 0$ (v) $\lim\limits_{x\to 0^+} x(\ell nx)^n = 0$

As $x \to \infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ and $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h \to 0^+$.

Example # 21 : Evaluate $\lim\limits_{x\to\infty} \dfrac{x^{10}+7x^2+1}{e^x}$

Solution : $\lim\limits_{x\to\infty} \dfrac{x^{10}+7x^2+1}{e^x} = 0$

Limits of form 1^∞, 0^0, ∞^0

(A) All these forms can be converted into $\dfrac{0}{0}$ form in the following ways

(a) If $x \to 1$, $y \to \infty$, then $z = (x)^y$ is of 1^∞ form

$\Rightarrow$ $\ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{\ell nx}{\frac{1}{y}} \left(\dfrac{0}{0} \text{ form}\right) \quad$ As $y \to \infty \Rightarrow \dfrac{1}{y} \to 0$ and $x \to 1 \Rightarrow \ell nx \to 0$

(b) If $x \to 0$, $y \to 0$, then $z = x^y$ is of (0^0) form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x \quad \Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}} \left(\dfrac{0}{0} \text{ form}\right)$

(c) If $x \to \infty$, $y \to 0$, then $z = x^y$ is of $(\infty)^0$ form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x \quad \Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}} \left(\dfrac{0}{0} \text{ form}\right)$

(B) $(1)^\infty$ type of problems can be solved by the following method

(a) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\lim\limits_{x\to a} [f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \lim_{x\to a} \left[1+f(x)-1\right]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)} = \lim_{x\to a} \left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)\,g(x)} = e^{\lim\limits_{x\to a}[f(x)-1]\,g(x)}$$

Example # 22 : Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{x^2-1}{x^2+3}\right)^{(x^2+1)}$

Solution : Since it is in the form of 1^∞

$$\left(\frac{x^2-1}{x^2+3}\right)^{(x^2+1)} = \exp\left(\lim_{x\to\infty}\left(\frac{x^2-1-x^2-3}{x^2+3}\right)(x^2+1)\right) = e^{-4}$$

Example # 23 : Evaluate $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution : Since it is in the form of 1^∞ so $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x-1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$

$$= e^{2\times\frac{\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

Example # 24 : Evaluate $\lim\limits_{x\to a} \left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

Solution : $\lim\limits_{x\to a} \left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put $x = a + h$

$$= \lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)} = \lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)} = e^{\lim\limits_{h\to 0} -\cot\frac{\pi h}{2a}\cdot\left(1+\frac{h}{a+h}-1\right)}$$

$$= e^{\lim\limits_{h\to 0} -\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right)\cdot\frac{\frac{2a}{\pi}}{a+h}} = e^{-\frac{2}{\pi}}$$

Example # 25 : Evaluate $\lim\limits_{x\to 0^+} (\tan x)^{\tan x}$

Solution : Let $y = \lim\limits_{x\to 0^+} (\tan x)^{\tan x}$

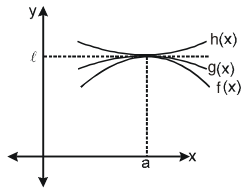
$$\Rightarrow \ell n\, y = \lim_{x\to 0^+} \tan x\, \ell n \tan x = \lim_{x\to 0^+} -\frac{\ell n\frac{1}{\tan x}}{\frac{1}{\tan x}} = 0, \text{ as } \frac{1}{\tan x} \to \infty \Rightarrow y = 1$$

Sandwich theorem or squeeze play theorem:

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that

$\lim\limits_{x \to a} f(x) = \ell = \lim\limits_{x \to a} h(x)$,

Then $\lim\limits_{x \to a} g(x) = . \ \ell$



Example # 26 : Evaluate $\lim\limits_{n \to \infty} \dfrac{[x]+[4x]+[7x]+....+[(3n-2)x]}{n^2}$, where [.] denotes greatest integer function.

Solution : We know that, $x - 1 < [x] \le x$

$4x - 1 < [4x] \le 4x$

$7x - 1 < [7x] \le 7x$

. . .

. . .

$(3n - 2)x - 1 < [(3n - 2)x] \le (3n - 2)x$

$\therefore$ $(x + 4x + 7x + ... + (3n - 2)x) - n < [x] + [4x] + ...$
$+[(3n - 2)x] \le (x + 4x + ... + nx)$

$\Rightarrow$ $\dfrac{n}{2}(3n - 1)x - n < \sum\limits_{r=1}^{n}[(3r-2)x] \le \dfrac{n}{2}(3n - 1)x$

$\Rightarrow$ $\lim\limits_{n\to\infty} \dfrac{n}{2}\dfrac{(3n-1)}{n^2}x - \dfrac{1}{n} < \lim\limits_{n\to\infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} \le \lim\limits_{n\to\infty} \dfrac{n}{2}\dfrac{(3n-1)}{n^2}x$

$\Rightarrow$ $\dfrac{3x}{2} < \lim\limits_{n\to\infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} \le \dfrac{3x}{2}$

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} = \dfrac{3x}{2}$

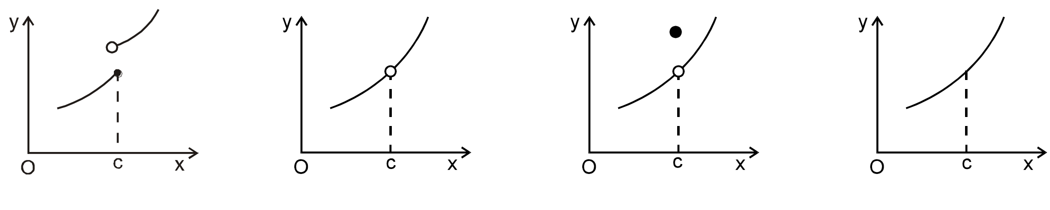
Continuity & Derivability :

A function f(x) is said to be continuous at x = c, if $\text{Limit}_{x\to c} f(x) = f(c)$

i.e. f is continuous at x = c

if $\text{Limit}_{h\to 0^+} f(c - h) = \text{Limit}_{h\to 0^+} f(c+h) = f(c)$.

If a function f (x) is continuous at x = c, the graph of f (x) at the corresponding point (c, f (c)) will not be broken. But if f (x) is discontinuous at x = c, the graph will be broken when x = c



((i), (ii) and (iii) are discontinuous at x = c)

((iv) is continuous at x = c)

A function f can be discontinuous due to any of the following three reasons :

(i) $\text{Limit}_{x\to c} f(x)$ does not exist i.e. $\text{Limit}_{x\to c^-} f(x) \ne \text{Limit}_{x\to c^+} f(x)$ **[figure (i)]**

(ii) f (x) is not defined at x = c **[figure (ii)]**

(iii) $\text{Limit}_{x\to c} f(x) \ne f(c)$ **[figure (iii)]**

Geometrically, the graph of the function will exhibit a break at x= c.

Example # 27 : If f(x) = $\begin{cases} \cos 2\pi x & , \ x < 1 \\ [x] & , \ x \geq 1 \end{cases}$, then find whether f(x) is continuous or not at x = 1, where [.] is greatest integer function.

Solution : f(x) = $\begin{cases} \cos 2\pi x & , \ x < 1 \\ [x] & , \ x \geq 1 \end{cases}$

For continuity at x = 1, we determine f(1) , $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$.

Now, f(1) = [1] = 1

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \cos \ 2\pi x = \cos 2\pi = 1$

and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} [x] = 1$

so $f(1) = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1

Self practice problems :

(12) If possible find value of λ for which f(x) is continuous at x = $\frac{\pi}{2}$

$$f(x) = \begin{cases} \dfrac{1 - \sin x}{1 + \cos 2x}, & x < \dfrac{\pi}{2} \\ \lambda & , \ x = \dfrac{\pi}{2} \\ \dfrac{(2x - \pi)^2}{\tan 2x} & , \ x > \dfrac{\pi}{2} \end{cases}$$

(13) Find the values of p and q such that the function

$$f(x) = \begin{cases} x + p \sin x & ; \ 0 \leq 4x < \pi \\ 2x \cot x + q & ; \ \pi \leq 4x \leq 2\pi \\ \dfrac{p}{\sqrt{2}} \cos 2x - q \sin x & ; \ 2\pi < 4x \leq 4\pi \end{cases}$$ is continuous at x = $\frac{\pi}{4}$ and x = $\frac{\pi}{2}$

Ans. (12) discontinuous (13) p = $\frac{\pi}{3\sqrt{2}}$, q = $\frac{-\pi}{12}$

Theorems on continuity :

(i) If f & g are two functions which are continuous at x = c, then the functions defined by: $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\,f(x)$, K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(ii) If f(x) is continuous & g(x) is discontinuous at x = a, then the product function ϕ (x) = f(x). g(x) may or may not be continuous but sum or difference function ϕ (x) = f(x) $\pm$ g(x) will necessarily be discontinuous at x = a.

e.g. f (x) = x & g(x) = $\begin{bmatrix} \sin \frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$

(iii) If f (x) and g(x) both are discontinuous at x = a, then the product function ϕ (x) = f(x). g(x) is not necessarily be discontinuous at x = a.

e.g. $f(x) = g(x) = \begin{cases} 1 & , \quad x \ge 0 \\ -1 & , \quad x < 0 \end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

Example # 28 : If f(x) = [sin(x–1)] – {sin(x–1)}. Comment on continuity of f(x) at x = $\frac{\pi}{2}+1$

(where [.] denotes G.I.F. and { . } denotes fractional part function).

Solution : f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [sin (x – 1)] + {sin (x – 1)} = sin (x – 1)

which is continuous at x = $\frac{\pi}{2} + 1$

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at x = $\frac{\pi}{2} + 1$

$\therefore$ At most one of f(x) or g(x) can be continuous at x = $\frac{\pi}{2} + 1$

As g(x) is continuous at x = $\frac{\pi}{2} + 1$, therefore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluating $\lim\limits_{x \to \frac{\pi}{2}+1} f(x)$ and f. $\left(\frac{\pi}{2}+1\right)$

Continuity of composite functions :

If f is continuous at x = c and g is continuous at x = f(c), then the composite g[f(x)] is continuous at x = c. eg. f(x) = $\frac{x \sin x}{x^2+2}$ & g(x) = $|x|$ are continuous at x = 0, hence the composite function (gof) (x) = $\left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0.

Self practice problem :

(14) $f(x) = \begin{cases} 1+8x^3 & , \quad x<0 \\ -1 & , \quad x=0 \\ 4x^2-1 & , \quad x>0 \end{cases}$ and $g(x) = \begin{cases} (2x-1)^{\frac{1}{3}} & , \quad x<0 \\ 1 & , \quad x=0 \\ \sqrt{2x+1} & , \quad x>0 \end{cases}$

Then define fog (x) and comment on the continuity of gof(x) at x = 1/2

Ans. [$fog(x) = \begin{cases} 16x-7 & ; \quad x<0 \\ 3 & ; \quad x=0 \\ 8x+3 & ; \quad x>0 \end{cases}$ and gof(x) is discontinous at x = 1/2]

Continuity in an Interval :

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in$(a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if:

(i) f is continuous in the open interval (a, b),

(ii) f is right continuous at 'a' i.e. $\underset{x \to a^+}{\text{Limit}}$ f(x) = f(a) = a finite quantity and

(iii) f is left continuous at 'b' i.e. $\underset{x \to b^-}{\text{Limit}}$ f(x) = f(b) = a finite quantity.

(c) All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

On the basis of above facts continuity of a function should be checked at the following points

(i) Continuity of a function should be checked at the points where definition of a function changes.

(ii) Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

(iii) Continuity of sgn (f(x)) should be checked at the points where f(x) = 0 (if f(x) = 0 in any open interval containing a, then x = a is not a point of discontinuity)

(iv) In case of composite function f(g(x)) continuity should be checked at all possible points of discontinuity of g(x) and at the points where g(x) = c, where x = c is a possible point of discontinuity of f(x).

Example # 29: If $f(x) = \begin{cases} [2x] & 0 \le x < 1 \\ \{3x\}\,\text{sgn}(-x) & 1 \le x \le 2 \end{cases}$, where { . } represents fractional part function and [.] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

Solution : The given function is

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x < 1 \\ 3(1-x) & 1 \le x < \frac{4}{3} \\ 4-3x & \frac{4}{3} \le x < \frac{5}{3} \\ 5-3x & \frac{5}{3} \le x < 2 \\ 0 & x = 2 \end{cases}$$

so discontinous at x = 1/2 , 1,4/3, 5/3, 2

Example # 30 : If f(x) = $\frac{x+3}{x-1}$ and g(x) = $\frac{1}{x-3}$, then discuss the continuity of f(x), g(x) and fog (x).

Solution : f(x) = $\frac{x+3}{x-1}$

f(x) is a rational function it must be continuous in its domain
and f is not defined at x = 1

$\therefore$ f is discontinuous at x = 1

g(x) = $\frac{1}{x-3}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 3

$\therefore$ g is discontinuous at x = 3

Now fog (x) will be discontinuous at

(i) x = 3 (point of discontinuity of g(x))

(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x))

if g(x) = 1 $\Rightarrow \frac{1}{x-3} = 1 \Rightarrow x = 4$

$\therefore$ discontinuity of fog(x) should be checked at x = 3 and x = 4

at x = 3

fog (x) = $\dfrac{\frac{1}{x-3}+1}{\frac{1}{x-3}-1}$

fog (3) is not defined

$\lim\limits_{x\to3}$ fog (x) = $\lim\limits_{x\to3} \dfrac{\frac{1}{x-3}+1}{\frac{1}{x-3}-1} = \lim\limits_{x\to2} \dfrac{1+x-3}{1-x+3}$ = 1 $\therefore$ fog (x) is discontinuous at x = 3

fog (4) = not defined

$\lim\limits_{x\to4^+}$ fog (x) = ∞

$\lim\limits_{x\to4^-}$ fog (x) = $-\infty$ $\therefore$ fog (x) i s discontinuous at x = 4.

Self practice problem :

(15) If $f(x) = \begin{cases} [\ell n \ x] \ . \ \text{sgn}\left(\left\{x - \frac{1}{2}\right\}\right) ; & 1 < x \le 3 \\ \{x^2\} \ ; & 3 < x \le 3.5 \end{cases}$. Find the pointswhere the continuity of f(x), should be checked, where [.] is greatest integer function and {.} fractional part function.

Ans. $\{ 1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5 \}$

Intermediate value theorem :

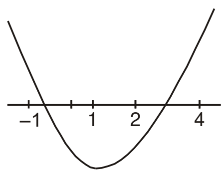
A function f which is continuous in $[a,b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

Example # 31: Prove that the equation $3(x – 1)(x – 2) + 4(x + 1)(x – 4) = 0$ will have real and distinct roots.

Solution : $3(x – 1)(x – 2) + 4(x + 1)(x – 4) = 0$



$f(x) = 3(x – 1)(x – 2) + 4(x + 1)(x – 4)$
$f(–1) = +$ ve
$f(1) = –$ve
$f(2) = –$ve

$f(4) = +$ve
hence $3(x – 1)(x – 2) + 4(x + 1)(x – 4) = 0$
have real and distinct roots

Self practice problem :

(16) If $f(x) = x\ell nx – 2$, then show that f(x) = 0 has exactly one root in the interval 1, e).

Example # 32: Let $f(x) = \underset{n\to\infty}{\text{Lim}} \frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Solution : Let $f(x) = \lim\limits_{n\to\infty} \frac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n \ . \ \sin^2\frac{\pi}{4}} = \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$$

Now

$$f(0) = \lim_{n\to\infty} \frac{1}{n \ . \ \sin^2(0)+1} = \frac{1}{1+0} = 1 \Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty}\frac{1}{1+n \ \sin^2 x}\right] = 0$$

{here $\sin^2 x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞}

$\therefore$ f(x) is not continuous at x = 0

Self practice problem :

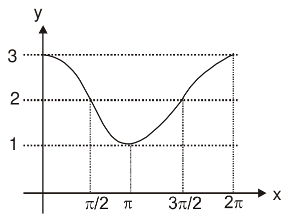
(17) If $f(x) = \underset{n\to\infty}{\text{Lim}} (1 + x)^n$. Comment on the continuity of f(x) at x = 0 and explain $\underset{x\to 0}{\text{Lim}} (1+x)^{\frac{1}{x}} = e$.

Ans. Discontinous (non-removable)

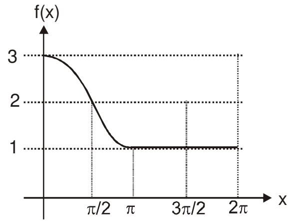
Example # 33: $f(x) = \text{minimum } (2 + \cos t, 0 \le t \le x), 0 \le x \le 2\pi$ discuss the continuity of this function at $x = \pi$

Solution : $f(x) = \text{minimum } (2 + \cos t, 0 \le t \le x), 0 \le x \le 2\pi$

$$f(x) = \begin{cases} 2+\cos x & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{cases}$$



which is continuous at x = π



Differentiability of a function at a point :

(i) The right hand derivative of f (x) at x = a denoted by $f'(a^+)$ is defined by:

R.H.D. = $f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined by:

L.H.D. = $f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$, provided the limit exists.

A function f(x) is said to be differentiable at x = a if $f'(a^+) = f'(a^-)$ = finite

By definition $f'(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$

Example # 34 : Comment on the differentiability of f(x) = $\begin{cases} 2x+3\ , & x<1 \\ 4x^2-1\ , & x\ge 1 \end{cases}$ at x = 1.

Solution : R.H.D. = $f'(1^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1+h)-f(1)}{h} = 8$

L.H.D. = $f'(1^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1-h)-f(1)}{-h} = 2$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

Example # 35: If f(x) = $\begin{cases} ax+b\ , & x\le -1 \\ ax^3+x+2b\ , & x>-1 \end{cases}$, then find a and b so that f(x) become differentiable at x = –1.

Solution : –a + b = –a – 1 + 2b using continuity

$\Rightarrow$ b = 1

f' (x) = $\begin{cases} a\ , & x<-1 \\ 3ax^2+1\ , & x>-1 \end{cases}$

a = 3a + 1 $\Rightarrow$ $a = -\dfrac{1}{2}$

Example # 36 : If f(x) = $\begin{cases} [\sin\frac{3\pi}{2}x], & x\le 1 \\ 2\{x\}-1, & x>1 \end{cases}$, then comment on the derivability at x = 1,

where [.] is greatest integer function and {.} is fractional part function.

Solution : $f'(1^-) = \lim\limits_{h\to 0^+} \dfrac{f(1-h)-f(1)}{-h} = \lim\limits_{h\to 0^+} \dfrac{\left[\sin\frac{3\pi}{2}(1-h)\right]+1}{-h} = 0$

$f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2\{1+h\}+1-1}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$

$\because \quad f'(1^+) \neq f'(1^-)$

f(x) is not differentiable at x = 1.

Self Practice Problems :

(18) If $f(x) = \begin{cases} \left[\frac{2x}{3}\right] + \frac{x}{3} + 2, & x < 3 \\ \left\{\frac{x}{3}\right\} + 3, & x \geq 3 \end{cases}$, then comment on the continuity and differentiable at x = 3, where [.] is greatest integer function and {.} is fractional part function.

(19) If $f(x) = \begin{cases} x \sin^{-1} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then comment on the derivability of f(x) at x = 0.

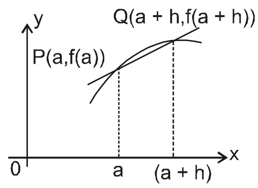
Ans. (18) Discontinuous and non-differentiable at x = 3

(19) non-differentiable at x = 0

Concept of tangent and its association with derivability :

Tangent :- The tangent is defined as the limiting case of a chord or a secant.

slope of the line joining (a,f(a)) and (a + h, f(a + h)) = $\frac{f(a+h)-f(a)}{h}$



Slope of tangent at P = $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

The tangent to the graph of a continuous function f at the point P(a, f(a)) is

(i) the line through P with slope $f'(a)$ if $f'(a)$ exists ;

(ii) the line x = a if L.H.D. and R.H.D. both are either ∞ or $-\infty$.

If neither (i) nor (ii) holds then the graph of f does not have a tangent at the point P.

In case (i) the equation of tangent is $y - f(a) = f'(a)(x - a)$.

In case (ii) it is x = a

Note : (i) tangent is also defined as the line joining two infinitesimally close points on a curve.

(ii) A function is said to be derivable at x = a if there exist a tangent of finite slope at that point. $f'(a^+) = f'(a^-)$ = finite value

(iii) $y = x^3$ has x-axis as tangent at origin.

(iv) $y = |x|$ does not have tangent at x = 0 as L.H.D. $\neq$ R.H.D.

Example # 37 : Find the equation of tangent to $y = (x)^{1/3}$ at x = 1 and x = 0.

Solution : At x = 1 Here $f(x) = (x)^{1/3}$

L.H.D = $f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+} \frac{(1-h)^{1/3}-1}{-h} = \frac{1}{3}$

R.H.D. = $f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^{1/3}-1}{h} = \frac{1}{3}$

As R.H.D. = L.H.D. = $\frac{1}{3}$

$\therefore$ slope of tangent = $\frac{1}{3}$ $\therefore$ $y - f(1) = \frac{1}{3}(x-1)$

$y - 1 = \frac{1}{3}(x-1)$

$\Rightarrow$ $3y - x = 2$ is tangent to $y = x^{1/3}$ at (1, 1)

At x = 0

L.H.D. = $f'(0^-) = \lim\limits_{h \to 0^+} \frac{(0-h)^{1/3} - 0}{-h} = +\infty$

R.H.D. = $f'(0^+) = \lim\limits_{h \to 0^+} \frac{(0+h)^{1/3} - 0}{h} = +\infty$

As L.H.D. and R.H.D are infinite. y = f(x) will have a vertical tangent at origin.

$\therefore$ x = 0 is the tangent to $y = x^{1/3}$ at origin.

Self Practice Problems :

(20) If possible find the equation of tangent to the following curves at the given points.

(i) $y = x^3 + 3x^2 + 28x + 1$ at x = 0.

(ii) $y = (x-8)^{2/3}$ at x = 8.

Ans. (i) y = 28x + 1 (ii) x = 8

Relation between differentiability & continuity:

(i) If $f'(a)$ exists, then f(x) is continuous at x = a.

(ii) If f(x) is differentiable at every point of its domain of definition, then it is continuous in that domain.

Note : The converse of the above result is not true i.e. "If 'f' is continuous at x = a, then 'f' is differentiable at x = a is not true.

e.g. the functions f(x) = | x – 2 | is continuous at x = 2 but not differentiable at x = 2.

If f(x) is a function such that R.H.D = $f'(a^+) = \ell$ and L.H.D. = $f'(a^-) = m$.

Case - I

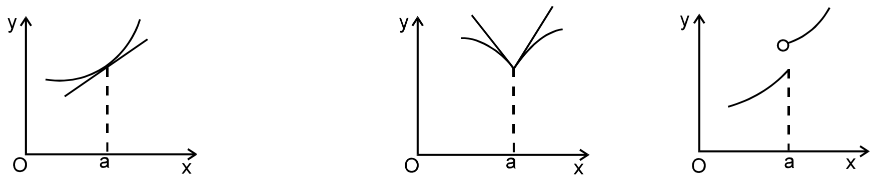
If $\ell = m$ = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II

if $\ell \neq m$ = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

Case - III

If at least one of the ℓ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



(i) (ii) (iii)

continuous and differentiable continuous but not differentiable neither continuous nor differentiable

Example # 38 : If f(x) is differentiable at x = a, prove that it will be continuous at x = a.

Solution : $f'(a^+) = \lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \ell$

$\lim\limits_{h \to 0^+} [f(a+h) - f(a)] = h\ell$

as $h \to 0$ and ℓ is finite, then $\lim\limits_{h \to 0^+} f(a+h) - f(a) = 0$

$\Rightarrow \lim\limits_{h \to 0^+} f(a+h) = f(a).$

Similarly $\lim\limits_{h\to 0^+}$ [f(a – h)– f(a)] = – hℓ $\Rightarrow \lim\limits_{h\to 0^+}$ f(a – h) = f(a)

$\therefore \lim\limits_{h\to 0^+}$ f(a + h) = f(a) = $\lim\limits_{h\to 0^+}$ f(a – h)

Hence, f(x) is continuous.

Example # 39 : If f(x) = $\begin{cases}\pi + x^2\,\text{sgn}[x] + \{x-4\}, & -2 \le x < 2 \\ \pi - \sin(x+\pi) + |x-3|, & 2 \le x < 6\end{cases}$, comment on the continuity and differentiability of f(x), where [.] is greatest integer function and {.} is fractional part function, at x = 1, 2.

Solution : Continuity at x = 1

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (\pi + x^2\,\text{sgn}[x] + \{x-4\}) = 1 + \pi$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (\pi + x^2\,\text{sgn}\,[x] + \{x-4\})$

$= 1\,\text{sgn}\,(0) + 1 + \pi = 1 + \pi$

$\because$ f(1) = 1 + π

$\therefore$ L.H.L = R.H.L = f(1). Hence f(x) is continuous at x = 1.

Now for differentiability,

R.H.D. = $f'(1^+) = \lim\limits_{h\to 0^+} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{\pi + (1+h)^2\,\text{sgn}[1+h] + \{1+h-4\} - 1 - \pi}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{(1+h)^2 + h - 1}{h} = \lim\limits_{h\to 0^+} \dfrac{1 + h^2 + 2h + h - 1}{h} = \lim\limits_{h\to 0^+} \dfrac{h^2 + 3h}{h} = 3$

and L.H.D. = $f'(1^-) = \lim\limits_{h\to 0^+} \dfrac{f(1-h) - f(1)}{-h} = \lim\limits_{h\to 0^+} \dfrac{\pi + (1-h)^2\,\text{sgn}[1-h] + 1 - h - 1 - \pi}{-h} = 1$

$\Rightarrow$ $f'(1^+) \ne f'(1^-)$.

Hence f(x) is non differentiable at x = 1.

Now at x = 2

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (\pi + x^2\,\text{sgn}\,[x] + \{x-4\}) = \pi + 4\,.\,1 + 1 = 5 + \pi$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (\pi + \sin x + |x-3|) = 1 + \pi + \sin 2$

Hence L.H.L $\ne$ R.H.L

Hence f(x) is discontinuous at x = 2 and then f(x) also be non differentaible at x = 2.

Self Practice Problem :

(21) If f(x) = $\begin{cases}\left(\dfrac{e^{[x]} + |x| - 1}{[x] + \{2x\}}\right) & x \ne 0 \\ 1/2 & x = 0\end{cases}$, comment on the continuity at x = 0 and differentiability at x = 0, where [.] is greatest integer function and {.} is fractional part function.

Ans. discontinuous hence non-differentiable at x = 0

Differentiability of sum, product & composition of functions :

(i) If f(x) & g(x) are differentiable at x = a, then the functions f(x) ± g(x), f(x). g(x) will also be differentiable at x = a & if g (a) ≠ 0, then the function f(x)/g(x) will also be differentiable at x = a.

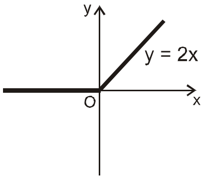
(ii) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a
e.g. f(x) = $|x|$ and g(x) = x^2.

(iii) If f(x) & g(x) both are not differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

(iv) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

Example # 40 : Discuss the differentiability of f(x) = x + |x|.

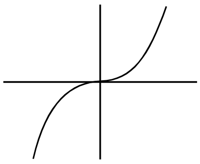
Solution :



Non-differentiable at x = 0.

Example # 41 : Discuss the differentiability of f(x) = x|x|

Solution : $\because \quad f(x) = \begin{cases} x^2 & , \ x \ge 0 \\ -x^2 & , \ x < 0 \end{cases}$



Differentiable at x = 0

Example # 42 : If f(x) is differentiable and g(x) is differentiable, then prove that f(x) . g(x) will be differentiable.

Solution : Given, f(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{f(a+h)-f(a)}{h} = f'(a)$

g(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{g(a+h)-g(a)}{h} = g'(a)$

let p(x) = f(x) . g(x)

Now, $\lim\limits_{h \to 0^+} \dfrac{p(a+h)-p(a)}{h} = \lim\limits_{h \to 0^+} \dfrac{f(a+h).g(a+h)-f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{f(a+h)g(a+h)+f(a+h).g(a)-f(a+h).g(a)-f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \left[\dfrac{f(a+h)\ (g\ (a+h)-g(a))}{h} + \dfrac{g\ (a)(f(a+h)-f(a))}{h}\right]$

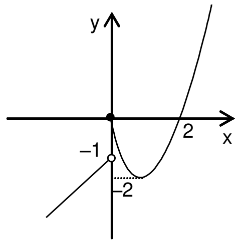
$= \lim\limits_{h \to 0^+} \left[f(a+h).\dfrac{g(a+h)-g(a)}{h} + g(a).\ \dfrac{f(a+h)-f(a)}{h}\right]$

= f(a) . g′(a) + g(a) f′(a) = p′ (a)

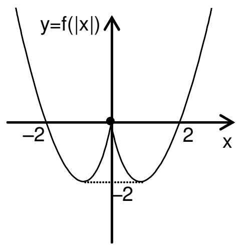
Hence p(x) is differentiable.

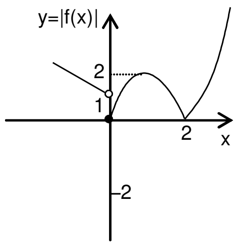
Example # 43 : If $f(x) = \begin{cases} 2x-1 & , \quad x < 0 \\ 2x^2-4x & , \quad x \ge 0 \end{cases}$ then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)| and hence comment on the continuity and differentiability of g(x) = f(|x|) + |f(x)|.

Solution :



Graph of f(|x|) and |f(x)|





If f(|x|) and |f(x)| are continous, then g(x) is continuous. At x = 0 f(|x|) is continuous, and |f(x)| is discontinuous therefore g(x) is discontineous at x = 0.

$\therefore$ g(x) is non differentiable at x = 0, 2 (find the reason yourself).

Differentiability over an Interval :

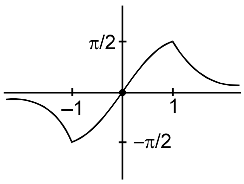
f (x) is said to be differentiable over an open interval if it is differentiable at each point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if:

(i) for the points a and b, $f'(a^+)$ and $f'(b^-)$ exist finitely

(ii) for any point c such that a < c < b, $f'(c^+)$ & $f'(c^-)$ exist finitely and are equal.

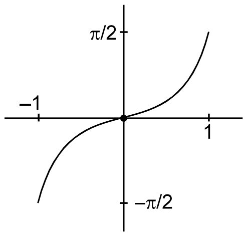
All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin^{-1}\dfrac{2x}{1+x^2}$



Non differentiable at x = 1 & x = – 1

Graph of $y = \sin^{-1} x$.



Non differentiable at x = 1 & x = – 1

Note : Derivability should be checked at following points

(i) At all points where continuity is required to be checked.

(ii) At the critical points of modulus and inverse trigonometric function.

Example # 44 : If $f(x) = \begin{cases} \left\{2x+\dfrac{7}{3}\right\}[\sin 2\pi x] & , \quad 0 \le x < \dfrac{1}{2} \\ \left([4x]+\left[\dfrac{x}{4}\right]\right).\text{sgn}\left(2x-\dfrac{4}{3}\right) & , \quad \dfrac{1}{2} \le x \le 1 \end{cases}$, find those points at which continuity and differentiability should be checked, where [.] is greatest integer function and {.} is fractional part function. Also check the continuity and differentaibility of f(x) at x = 1/2.

Solution : $f(x) = \begin{cases} \left\{2x+\frac{7}{3}\right\}[\sin 2\pi x] & , \quad 0 \le x < \frac{1}{2} \\ \left([4x]+\left[\frac{x}{4}\right]\right).\text{sgn}\left(2x-\frac{4}{3}\right) & , \quad \frac{1}{2} \le x \le 1 \end{cases}$

The points, where we should check the continuity and differentiability are x = 0, $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, 1

At x = 1/2

L.H.L. = $\lim_{x\to\frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-} \left\{2x+\frac{7}{3}\right\} [\sin 2\pi x] = 0$

R.H.L. = $\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^+} [4x] \text{ sgn } \left\{2x-\frac{4}{3}\right\} = 2(-1) = -2$

$\therefore$ L.H.L $\neq$ R.H.L. hence f(x) is discontinuous at x = 1/2 and hence it is non diffferentiable at x = 1/2.

Self Practice Problems:

(22) If f(x) = $\left[\frac{x+1}{2}\right]+\left[\frac{1-x}{2}\right]$, $-3 \le x \le 5$, then draw its graph and comment on the continuity and differentiability of f(x), where [.] is greatest integer function.

(23) If f(x) = $\begin{cases} |4x^2+6x+3| & , \quad -2 \le x < -1 \\ [x^2+2x] & , \quad -1 \le x \le 0 \end{cases}$, then draw the graph of f(x) and comment on the differentiability and continuity of f(x), where [.] is greatest integer function.

Ans. (22) f(x) is discontinuous at x = –3, –1, 1, 3, 5 hence non-differentiable.

(23) f(x) is discontinuous at x = –1, 0 & non differentiable at x = –1, 0.

Problems of finding functions satisfying given conditions :

Example # 45: If f(x) is a function satisfies the relation for all x, y $\in$ R, f(x + 2y) = f(x) + f(2y) and if f′(0) = 3 and function is differentiable every where, then find f(x).

Solution : $f'(x) = \lim_{h\to0^+} \frac{f(x+2h)-f(x)}{2h} = \lim_{h\to0^+} \frac{f(x)+f(2h)-f(x)-f(0)}{2h}$ $(\because f(0) = 0)$

$= \lim_{h\to0^+} \frac{f(2h)-f(0)}{2h} = f'(0) \Rightarrow f'(x) = 3 \quad \Rightarrow \quad \int f'(x)\ dx = \int 3\ dx$

f(x) = 3x + c

$\because$ f(0) = 2.0 + c as f(0) = 0

$\therefore$ c = 0 $\therefore$ f(x) = 3x

Example # 46 : $f(x + \alpha) = f(x) . f(\alpha)\ \forall\ x, \alpha \in R$ and f(x) is a differentiable function and f′(0) = 1/3, f(x) $\neq$ 0 for any x. Find f(x)

Solution : f(x) is a differentiable function

$\therefore f'(x) = \lim_{h\to0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0^+} \frac{f(x).f(h)-f(x).f(0)}{h}$ $(\because f(0) = 1)$

$= \lim_{h\to0^+} \frac{f(x).(f(h)-f(0))}{h} = f(x).\ f'(0) = f(x) \ \therefore f'(x) = f(x) \ \therefore \int \frac{f'(x)}{f(x)}\ dx = \int \frac{1}{3} dx$

$\Rightarrow \ell n\ f(x) = \frac{x}{3} + c \ \therefore \ell n\ 1 = 0 + c \Rightarrow c = 0 \ \therefore \ell n\ f(x) = \frac{x}{3} \Rightarrow f(x) = e^{x/3}$

Example # 47 : $3f\left(\frac{x+y}{3}\right) = f(x) + f(y)\ \forall\ x,\ y \in R$ and $f(0) = 4$ and $f'(0) = 2$ and function is differentiable for all x, then find f(x).

Solution : $f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+3.0}{3}\right)}{h}$

$= \lim\limits_{h\to 0}\frac{f(3h)-f(0)}{3h} = f'(x) = 2$

$f'(x) = 2 \Rightarrow f(x) = 2x + c \Rightarrow c = 4 \Rightarrow f(x) = 2x + 4$

Self Practice Problem:

(24) $f\left(\frac{x}{y}\right) = f(x) - f(y)\ \forall\ x,\ y \in R^+$ and $f'(1) = 1$, then show that $f(x) = \ell nx$.

Result of Some Known Functional Equation :-

Let x, y are independent variables and f(x) is differentiable function in its domain :

(i) If $f(xy) = f(x) + f(y)\ \forall\ x, y \in R^+$, then $f(x) = k \ln x$ or $f(x) = 0$.

(ii) If $f(xy) = f(x).\ f(y)\ \forall\ x, y \in R$, then $f(x) = x^k$, $k \in R$

(iii) If $f(x + y) = f(x).\ f(y)\ \forall\ x, y \in R$, then $f(x) = a^{kx}$.

(iv) If $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$, then $f(x) = kx$, where k is a constant in all four parts.

Example # 48 : If f(x) is a polynomial function satisfying $f(x)\ .\ f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(2) = 9$, then find f (3)

Solution : $f(x) = 1 \pm x^n$

As $f(2) = 9$ $\therefore$ $f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

Self practice problems

(25) If f(x) is a polynomial function satisfying $f(x)\ .\ f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(3) = -8$, then find f(4)

(26) If $f(x + y) = f(x)\ .\ f(y)$ for all real x, y and $f(0) \neq 0$, then prove that the function, $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

Ans. (25) -15

Example # 49 : Evaluate $\lim\limits_{h\to 0}\frac{f(\alpha-9h)-f(\alpha+h^2)}{h}$, if $f'(\alpha) = 2$

Solution : $\because$ $\lim\limits_{h\to 0}\left(\frac{f(\alpha-9h)-f(\alpha+h^2)}{(-9h-h^2)h}\right).(-9h-h^2) = \lim\limits_{h\to 0} f'(\alpha)\ .\ (-9 - h) = 2 \times -9 = -18$

Self Practice Problems :

(27) If f(x) and g(x) are differentiable, then prove that $f(x) \pm g(x)$ will be differentiable.

(28) If $f'(3) = 12$, then find the value of $\lim\limits_{h\to 0}\frac{f(3+h)-f(3+\sinh)}{h.\tan^2 h}$.

Ans. (28) 2

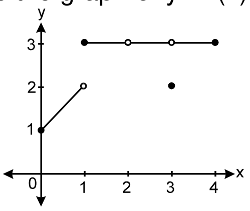
Exercise-1

✍ Marked questions are recommended for Revision.

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Definition of LHL/RHL and Indeterminate forms

A-1. Examine the graph of y = f(x) as shown and evaluate the following limits :



(i) $\lim\limits_{x\to 1} f(x)$ (ii) $\lim\limits_{x\to 2} f(x)$ (iii) $\lim\limits_{x\to 3} f(x)$

(iv) $\lim\limits_{x\to 1.99} f(x)$ (v) $\lim\limits_{x\to 3^-} f(x)$

A-2. Evaluate the following limits :

(i) $\lim\limits_{x\to 2} (x + \sin x)$ (ii) $\lim\limits_{x\to 3} (\tan x - 2^x)$ (iii) $\lim\limits_{x\to \frac{3}{4}} x \cos x$

(iv) $\lim\limits_{x\to 5} x^x$ (v) $\lim\limits_{x\to 1} \dfrac{e^x}{\sin x}$

A-3. Evaluate the following limits,
where [.] represents greatest integer function and { . } represents fractional part function

(i) $\lim\limits_{x\to \frac{\pi}{2}} [\sin x]$ (ii) $\lim\limits_{x\to 2} \left\{\dfrac{x}{2}\right\}$ (iii)✍ $\lim\limits_{x\to \pi}$ sgn [tan x]

(iv) $\lim\limits_{x\to 1} \sin^{-1} (\ell n\, x)$

A-4. (i) If f(x) = $\begin{cases} x+1 & , \ x<1 \\ 2x-3 & , \ x\geq 1 \end{cases}$, evaluate $\lim\limits_{x\to 1} f(x)$.

(ii) Let f(x) = $\begin{cases} x+\lambda & , \ x<1 \\ 2x-3 & , \ x\geq 1 \end{cases}$, if $\lim\limits_{x\to 1} f(x)$ exist, then find value of λ.

A-5.✍ If f(x) = $\begin{cases} x^2+2 & , \ x\geq 2 \\ 1-x & , \ x<2 \end{cases}$ and g(x) = $\begin{cases} 2x & , \ x>1 \\ 3-x & , \ x\leq 1 \end{cases}$, evaluate $\lim\limits_{x\to 1} f(g(x))$.

A-6. Which of the followings are indeterminate forms. Also state the type.

(i) $\lim\limits_{x\to 0^+} \dfrac{[x]}{x}$, where [.] denotes the greatest integer function

(ii) $\lim\limits_{x\to -\infty} \sqrt{x^2+1} - x$ (iii) $\lim\limits_{x\to \left(\frac{\pi}{2}\right)^-} (\tan x)^{\tan 2x}$

(iv)✍ $\lim\limits_{x\to 1^+}$, $(\{x\})^{\frac{1}{\ell n x}}$ where { . } denotes the fractional part function

SECTION (B) : Evaluation of limits of form 0/0, ∞/∞, ∞ – ∞, 0 × ∞, Use of L-Hospital Rule & Expansion

B-1. Evaluate each of the following limits, if exists

(i) $\lim\limits_{x\to -1}\dfrac{x^3-3x+1}{x-1}$ (ii) $\lim\limits_{x\to 1}\dfrac{4x^3-x^2+2x-5}{x^6+5x^3-2x-4}$ (iii) $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a\neq 0$

B-2. Evaluate the following limits, if exists

(i) $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{1-\cos 5x}$ (ii) $\lim\limits_{x\to \frac{\pi}{6}}\dfrac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}$

(iii) $\lim\limits_{x\to 0}\dfrac{\tan 3x-2x}{3x-\sin^2 x}$ (iv) $\lim\limits_{x\to 0}\dfrac{(a+x)^2\sin(a+x)-a^2\sin a}{x}$

(v) $\lim\limits_{x\to 0}\dfrac{e^{bx}-e^{ax}}{x}$, where $0 < a < b$ (vi) $\lim\limits_{x\to 0}\dfrac{x\left(e^{2+x}-e^2\right)}{1-\cos x}$

(vii) $\lim\limits_{x\to 0}\dfrac{\ell n(1+3x)}{3^x-1}$ (viii) $\lim\limits_{x\to 0}\dfrac{\ell n\ (2+x)+\ell n\ 0.5}{x}$

(ix) Find $n\in N$, if $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$. (x) $\lim\limits_{x\to 0}\dfrac{\sqrt{\dfrac{1-\cos 2x}{2}}}{x}$

(xi) $\lim\limits_{x\to 1}\dfrac{(\ell n\ (1+x)-\ell n2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}].\sin(x-1)}$

B-3. Evaluate the following limits.

(i) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2}+\dfrac{2}{x^2}+....+\dfrac{x}{x^2}\right)$ (ii) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$, $n\in N$

(iii) $\lim\limits_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)$ (iv) $\lim\limits_{x\to -\infty}\dfrac{x^5\tan\left(\dfrac{1}{\pi x^2}\right)+3|x|^2+7}{|x|^3+7|x|+8}$

B-4. Evaluate the following limits.

(i) $\lim\limits_{x\to\infty}\left((x+1)^{\frac{2}{3}}-(x-1)^{\frac{2}{3}}\right)$ (ii) $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$

(iii) $\lim\limits_{x\to\infty}\cos\left(\sqrt{x+1}\right)-\cos\left(\sqrt{x}\right)$

(iv) $\lim\limits_{x\to\infty}\left(((x+1)\ (x+2)\ (x+3)\ (x+4))^{\frac{1}{4}}-x\right)$

B-5. Evaluate the following limits using expansions :

(i) $\lim\limits_{x\to 2}\dfrac{(x+2)^{\frac{1}{2}}-(15x+2)^{\frac{1}{5}}}{(7x+2)^{\frac{1}{4}}-x}$ (ii) $\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x-\dfrac{\tan^2 x}{2}}{x^3}$

B-6 If $\lim\limits_{x\to 0}\dfrac{a+b\sin x-\cos x+ce^x}{x^3}$ exists, find the values of a, b, c. Also find the limit

B-7. Find the values of a and b so that:

(i) $\lim\limits_{x \to 0} \dfrac{1 + ax \sin x - b \cos x}{x^4}$ may have a finite limit.

(ii) $\lim\limits_{x \to \infty} \left(\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} - \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}\right) = 4$

(iii) $\lim\limits_{x \to 0} \dfrac{axe^x - b \ \ell n \ (1+x) + cxe^{-x}}{x^2 \sin x} = 2$

B-8. Find the following limit using expansion $\lim\limits_{x \to 0} \left(\dfrac{\ell n \ (1+x)^{(1+x)}}{x^2} - \dfrac{1}{x}\right)$:

B-9. Prove that $\lim\limits_{x \to 4} \dfrac{(\cos \alpha)^x - (\sin \alpha)^x - \cos 2\alpha}{x - 4} = \cos^4 \alpha \ \ell n (\cos \alpha) - \sin^4 \alpha \ \ell n (\sin \alpha), \ \alpha \in \left(0, \dfrac{\pi}{2}\right)$

B-10_. Find the value of $\lim\limits_{h \to 0} \dfrac{\tan(a+2h) - 2\tan(a+h) + \tan a}{h^2}$.

SECTION (C) : Limit of form 0^0, ∞^0, 1^∞, $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$, $\lim\limits_{x \to \infty} \dfrac{\ell n \ x}{x}$, Sandwitch theorem and Miscellaneous problems on limits.

C-1 Evaluate the following limits :

(i) $\lim\limits_{x \to 0^+} (x)^{x^2}$ (ii) $\lim\limits_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

(iii) $\lim\limits_{x \to 1^-} ([x])^{1-x}$, where [.] denotes greatest integer function

(iv) $\lim\limits_{x \to \frac{\pi}{2}^+} e^{\tan x}$

C-2. Evaluate the following limits :

(i) $\lim\limits_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$ (ii) $\lim\limits_{x \to \infty} \left(\dfrac{1+2x}{1+3x}\right)^x$

(iii) $\lim\limits_{x \to 1} (1 + \ell n x)^{\sec \frac{\pi x}{2}}$ (iv) $\lim\limits_{x \to 0} \left(\tan\left(\dfrac{\pi}{4} + x\right)\right)^{\frac{1}{x}}$

C-3. If $\lim\limits_{x \to 1} (1 + ax + bx^2)^{\frac{c}{x-1}} = e^3$, then find conditions on a, b and c.

C-4. Evaluate following limits :

(i) $\lim\limits_{x \to \infty} \dfrac{x \ \ \ell n \left(1 + \dfrac{\ell n \ x}{x}\right)}{\ell n \ x}$ (ii) $\lim\limits_{x \to \infty} \dfrac{e^x \sin\left(\dfrac{x^n}{e^x}\right)}{x^n}$

C-5. Evaluate $\lim\limits_{n \to \infty} \dfrac{[1 \ . \ 2x] + [2 \ . \ 3x] + ... + [n \ . \ (n+1) \ x]}{n^3}$, where [.] denotes greatest integer function.

C-6. If f(x) = $\lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$, $n \in N$ find range of f(x).

Section (D) : Continuity at a point

D-1. Determine the values of a, b & c for which the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{b\,x^{3/2}} & \text{for } x > 0 \end{cases}$

is continuous at x = 0.

D-2. Find the values of ' a ' & ' b ' so that the function, $f(x) = \begin{cases} \dfrac{1 - \sin^3 x}{3\cos^2 x} & , \ x < \pi/2 \\ a & , \ x = \pi/2 \\ \dfrac{b(1-\sin x)}{(\pi - 2x)^2} & , \ x > \pi/2 \end{cases}$ is continuous at $x = \pi/2$.

D-3. If f(x) = {x} & g(x) = [x] (where {. } & [.] denotes the fractional part and the integral part functions respectively), then discuss the continuity of :

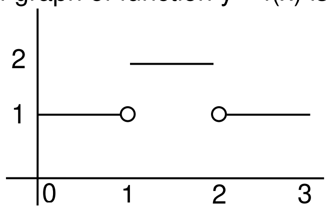
(i) h(x) = f(x). g(x) at x = 1 and 2
(ii) h(x) = f(x) + g(x) at x = 1
(iii) h(x) = f(x) – g(x) at x = 1
(iv) $h(x) = g(x) + \sqrt{f(x)}$ at x = 1 and 2

D-4. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \dfrac{f(x)}{x-3} & , \ x \neq 3 \\ K & , \ x = 3 \end{cases}$, then

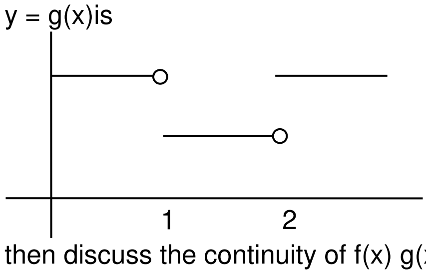
(a) find all zeros of f
(b) find the value of K that makes h continuous at x = 3
(c) using the value of K found in (b), determine whether h is an even function.

D-5. If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \neq 0)$ is continuous at x = 0. Find A & B. Also find f (0).

D-6_. If graph of function y = f(x) is



and graph of function
y = g(x)is



then discuss the continuity of f(x) g(x) at x = 3 and x = 2.

Section (E) : Continuity in an interval, Continuity of composite functions, IMVT

E-1. Find interval for which the function given by the following expressions are continuous :

(i) $f(x) = \dfrac{3x+7}{x^2 - 5x + 6}$
(ii) $f(x) = \dfrac{1}{|x| - 1} - \dfrac{x^2}{2}$
(iii) $f(x) = \dfrac{\sqrt{x^2+1}}{1 + \sin^2 x}$
(iv) $f(x) = \tan\left(\dfrac{\pi x}{2}\right)$

E-2. If f(x) = x + {– x} + [x], where [.] is the integral part & { . } is the fractional part function. Discuss the continuity of f in [– 2, 2]. Also find nature of each discontinuity.

E-3. If $f(x) = \dfrac{x^2+1}{x^2-1}$ and g(x) = tan x, then discuss the continuity of fog (x).

E-4. Let $f(x) = \begin{bmatrix} 1 + x & , 0 \le x \le 2 \\ 3 - x & , 2 < x \le 3 \end{bmatrix}$. Determine the composite function g(x) = f (f(x)) & hence find the point of discontinuity of g , if any.

E-5. Find the point of discontinuity of y = f(u), where $f(u) = \dfrac{3}{2u^2+5u-3}$ and $u = \dfrac{1}{x+2}$.

E-6. Show that the function $f(x) = \dfrac{x^3}{4} - \sin \pi x + 3$ takes the value $\dfrac{7}{3}$ within the interval [–2, 2].

E-7_. If g(x) = (|x – 1| + |4x – 11|) [x² – 2x – 2], then find the number of point of discontinuity of g(x) in $\left(\dfrac{1}{2}, \dfrac{5}{2}\right)$

{where [.] denotes GIF}

Section (F) : Derivability at a point

F-1. Test the continuity & differentiability of the function defined as under at x = 1 & x = 2.

$$f(x) = \begin{cases} x & ; \quad x < 1 \\ 2 - x & ; \quad 1 \le x \le 2 \\ -2 + 3x - x^2 & ; \quad x > 2 \end{cases}$$

F-2. A function f is defined as follows: $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1 + \sin x & \text{for } 0 \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \dfrac{\pi}{2}\right)^2 & \text{for } \dfrac{\pi}{2} \le x < \infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

F-3. Prove that f (x) = |x| cos x is not differentiable at x = 0

F-4. Show that the function $f(x) = \begin{cases} x^m \sin\left(\dfrac{1}{x}\right) & ; \quad x > 0 \\ 0 & ; \quad x = 0 \end{cases}$ is,

(i) differentiable at x = 0, if m > 1.
(ii) continuous but not differentiable at x = 0, if 0 < m ≤ 1.
(iii) neither continuous nor differentiable, if m ≤ 0.

F-5. Examine the differentiability of $f(x) = \sqrt{1 - e^{-x^2}}$ at x = 0.

F-6. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

Section (G) : Derivability in an interval

G-1. Draw a graph of the function, $y = [x] + |1 - x|$, $-1 \leq x \leq 3$. Determine the points, if any, where this function is not differentiable, where [ ] denotes the greatest integer function.

G-2. Discuss the continuity & derivability of $f(x) = \begin{cases} \left|x - \dfrac{1}{2}\right| & ;\ 0 \leq x < 1 \\ x\ .\ [x] & ;\ 1 \leq x \leq 2 \end{cases}$

where [x] indicates the greatest integer x.

G-3. Discuss continuity and differentiability of y = f(x) in [–2, 5] where [.] denotes GIF & {.} denotes FPF

$$f(x) = \begin{cases} [x] & ,\ x \in [-2,\ 0] \\ \{x\} & ,\ x \in (0,\ 2) \\ \dfrac{x^2}{4} & ,\ x \in [2,\ 3) \\ \dfrac{1}{\log_4(x-3)} & ,\ x \in [3,\ 5] \end{cases}$$

G-4. Check differentiability of $f(x) = \text{sgn}(x^{2/3}) + \left[\cos\left(\dfrac{x^2}{1+x^2}\right)\right] + |x-1|^{5/3}$ in [–2, 2] where [.] denotes GIF.

G-5_. Discuss the continuity and differentiability of h(x) = f(x) g(x) in (0, 3) if

$f(x) = \dfrac{e^x - e}{[x]+1}$ {where [.] denot GIF} and $g(x) = \begin{cases} \dfrac{|x-1|+|x-2|}{2} & ,\ x \in (0, 1) \\ |x-1|+|x-2| & ,\ x \in [1, 2) \\ \dfrac{3(|x-1|+|x-2|)}{2} & ,\ x \in [2, 3) \end{cases}$

Section (H) : Functional equations and Miscellaneous

H-1. If $f : R \to R$ satisfies f(x + y) = f(x) + f(y), for all x, y $\in$ R and f(1) = 2, then $\sum\limits_{r=1}^{7} f(r)$ is :

H-2. If f′(2) = 4 then, evaluate $\lim\limits_{x \to 0} \dfrac{f(1+\cos x) - f(2)}{\tan^2 x}$..

H-3. Let a function $f : R \to R$ be given by f(x + y) = f(x) f(y) for all x, y $\in$ R and f(x) $\neq$ 0 for any x $\in$ R. If the function f (x) is differentiable at x = 0, show that f′(x) = f′(0) f(x) for all x $\in$ R. Also, determine f(x).

H-4. Let f(x) be a polynomial function satisfying the relation $f(x).\ f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$ $\forall$ x $\in$ R – {0} and

f(3) = –26. Determine f′(1).

H-5. Let function f(x) satisfying the relation f(x + y)+ f(x – y) = 2f(x).f(y), then prove that it is even function

H-6. Let f(x) be a bounded function. $L_1 = \lim\limits_{x \to \infty} (f'(x) - \lambda f(x))$ and $L_2 = \lim\limits_{x \to \infty} f(x)$ where $\lambda > 0$. If L_1, L_2 both exist and $L_1 = L$, then prove that $L_2 = -\dfrac{L}{\lambda}$.

H-7. Let R be the set of real numbers and f: R→R be such that for all x & y in R $|f(x) - f(y)| \le |x - y|^3$. Prove that f(x) is constant.

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Definition of LHL/RHL and Indeterminate forms

A-1. $\lim\limits_{x \to 0} \sin^{-1}((\sec x))$ is equal to

(A) $\dfrac{\pi}{2}$ (B) 1 (C) zero (D) none of these

A-2. Consider the following statements :

S_1 : $\lim\limits_{x \to 0^-} \dfrac{[x]}{x}$ is an indeterminate form (where [.] denotes greatest integer function).

S_2 : $\lim\limits_{x \to \infty} \dfrac{\sin(3^x)}{3^x} = 0$

S_3 : $\lim\limits_{x \to \infty} \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$ does not exist.

S_4 : $\lim\limits_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+3)!}$ $(n \in N) = 0$

S_4 : $\lim\limits_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+3)!}$ $(n \in N) = 0$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) FTFT (B) FTTT (C) FTFF (D) TTFT

A-3. $\lim\limits_{x \to 1} (1 - x + [x - 1] + [1 - x])$ is equal to (where [.] denotes greatest integer function)

(A) 0 (B) 1 (C) – 1 (D) does not exist

A-4_. $\lim\limits_{x \to 0^-} \dfrac{\cos^{-1}(\cos x)}{\sin^{-1}(\sin x)}$ is equal to :

(A) 0 (B) 1 (C) –1 (D) Does not exist

SECTION (B) : Evaluation of limits of form 0/0, ∞/∞, ∞ – ∞, 0 × ∞, Use of L-Hospital Rule & Expansion

B-1. $\lim\limits_{x \to 3} \dfrac{(x^3 + 27)\ \ell n\ (x - 2)}{(x^2 - 9)}$ is equal to

(A) – 8 (B) 8 (C) 9 (D) – 9

B-2. $\lim\limits_{x \to 0} \dfrac{(4^x - 1)^3}{\sin\left(\dfrac{x}{p}\right)\ \ell n\left(1 + \dfrac{x^2}{3}\right)}$ is equal to

(A) $9p(\ell n\, 4)$ (B) $3p(\ell n\, 4)^3$ (C) $12p(\ell n\, 4)^3$ (D) $27p(\ell n\, 4)^2$

B-3. $\lim\limits_{x \to 2} \dfrac{\sin\left(e^{x-2}-1\right)}{\ell n\ (x-1)}$ is equal to

(A) 0 (B) – 1 (C) 2 (D) 1

B-4. The value of $\lim\limits_{x \to 0} \dfrac{\sin(\ell n(1+x))}{\ell n(1+\sin x)}$ is equal to

(A) 0 (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{4}$ (D) 1

B-5. $\lim\limits_{x \to 1} \dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(A) exists and it equals $\sqrt{2}$
(B) exists and it equals $-\sqrt{2}$
(C) does not exist because $x - 1 \to 0$
(D) does not exist because left hand limit is not equal to right hand limit.

B-6. The value of $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$ is equal to

(A) $\dfrac{1}{2}$ (B) 1 (C) –1 (D) $-\dfrac{1}{2}$

B-7. The value of $\lim\limits_{x\to 0} \dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ is equal to

(A) $\dfrac{\sqrt{2}}{2}$ (B) $\dfrac{\sqrt{2}}{4}$ (C) $\dfrac{-\sqrt{2}}{8}$ (D) $\dfrac{\sqrt{2}}{8}$

B-8. $\lim\limits_{x \to 0^+} \dfrac{\cos^{-1}(1-x)}{\sqrt{x}}$ is equal to

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 1 (D) 0

B-9. $\lim\limits_{x \to 1} \dfrac{\left(\sum\limits_{k=1}^{100} x^k\right) - 100}{x-1}$ is equal to

(A) 0 (B) 5050 (C) 4550 (D) – 5050

B-10. $\lim\limits_{x \to \infty} \dfrac{x^3 \sin \frac{1}{x} + x + 1}{x^2 + x + 1}$ is equal to

(A) 0 (B) $\dfrac{1}{2}$ (C) 1 (D) none of these

B-11. $\lim\limits_{x \to -\infty} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{9x^2+x+1}}$ is equal to

(A) $\dfrac{1}{3}$ (B) $-\dfrac{1}{3}$ (C) 0 (D) does not exist

B-12. $\lim\limits_{n \to \infty} \dfrac{5^{n+1} + 3^n - 2^{2n}}{5^n + 2^n + 3^{2n+3}}$, $n \in N$ is equal to

(A) 5 (B) 3 (C) 1 (D) zero

B-13. $\lim\limits_{n \to \infty} n \cos\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right)$, $n \in N$ is equal to:

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{6}$ (D) none of these

B-14. $\lim\limits_{x \to \frac{\pi}{2}} \left[\dfrac{x - \frac{\pi}{2}}{\cos x}\right]$ is equal to (where [.] represents greatest integer function)

(A) – 1 (B) 0 (C) – 2 (D) does not exist

B-15. $\lim\limits_{n \to \infty} \dfrac{-3n + (-1)^n}{4n - (-1)^n}$ is equal to ($n \in N$)

(A) $-\dfrac{3}{4}$ (B) $-\dfrac{3}{4}$ if n is even ; $\dfrac{3}{4}$ if n is odd

(C) not exist if n is even ; $-\dfrac{3}{4}$ if n is odd (D) 1 if n is even ; does not exist if n is odd

B-16_. $\lim\limits_{x \to 1}\left(\dfrac{2}{1 - x^2} + \dfrac{1}{x - 1}\right)$ is equal to :

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) –1 (D) Does not exist

B-17. $\lim\limits_{x \to \infty}\left(x - x^2 \ \ell n\left(1 + \dfrac{1}{x}\right)\right)$ is equal to :

(A) $\dfrac{1}{2}$ (B) $\dfrac{3}{2}$ (C) $\dfrac{1}{3}$ (D) 1

B-18. $\lim\limits_{x \to 0} \dfrac{e^{-\frac{x^2}{2}} - \cos x}{x^3 \sin x}$ is equal to

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{6}$ (C) $\dfrac{1}{12}$ (D) $\dfrac{1}{8}$

B-19. $\lim\limits_{x \to 0} \dfrac{\sin(6x^2)}{\ell n \cos(2x^2 - x)}$ is equal to

(A) 12 (B) – 12 (C) 6 (D) – 6

B-20_. $\lim\limits_{h \to 0} \dfrac{\sin(a + 3h) - 3\sin(a + 2h) + 3\sin(a + h) - \sin a}{h^3}$ is equal to :

(A) cosa (B) –cosa (C) sina (D) sina cosa

SECTION (C) : Limit of form 0^0, ∞^0, 1^∞, $\lim\limits_{x\to\infty}\frac{x}{e^x}$, $\lim\limits_{x\to\infty}\frac{\ell n\, x}{x}$, Sandwitch theorem and Miscellaneous problems on limits.

C-1. $\lim\limits_{x\to\infty}\left(\frac{x+2}{x-2}\right)^{x+1}$ is equal to

(A) e^4 (B) e^{-4} (C) e^2 (D) none of these

C-2. $\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{5}{x}}$ is equal to

(A) e^5 (B) e^2 (C) e (D) none of these

C-3. The value of $\lim\limits_{x\to\frac{\pi}{4}}(1+[x])^{\frac{1}{\ell n(\tan x)}}$ is equal to (where [.] denotes the greatest integer function)

(A) 0 (B) 1 (C) e (D) e^{-1}

C-4. $\lim\limits_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$ is equal to

(A) 1 (B) 2 (C) e^2 (D) e

C-5. The limiting value of $(\cos x)^{\frac{1}{\sin x}}$ at x = 0 is:

(A) 1 (B) e (C) 0 (D) none of these

C-6. $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ is equal to

(A) $e^{-\frac{a}{\pi}}$ (B) $e^{-\frac{2a}{\pi}}$ (C) $e^{-\frac{2}{\pi}}$ (D) 1

C-7. $\lim\limits_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}\right)^n$ is

(A) e^{-x^2} (B) $e^{\frac{-x^2}{2}}$ (C) e^{x^2} (D) $e^{\frac{x^2}{2}}$

C-8. If [x] denotes greatest integer less than or equal to x, then $\lim\limits_{n\to\infty}\frac{1}{n^4}\left([1^3 x]+[2^3 x]+...+[n^3 x]\right)$ is equal to

(A) $\frac{x}{2}$ (B) $\frac{x}{3}$ (C) $\frac{x}{6}$ (D) $\frac{x}{4}$

Section (D) : Continuity at a point

D-1. A function f(x) is defined as below $f(x)=\frac{\cos(\sin x)-\cos x}{x^2}$, $x \neq 0$ and f(0) = a

f(x) is continuous at x = 0 if 'a' equals

(A) 0 (B) 4 (C) 5 (D) 6

D-2. Let $f(x)=\left|\left(x+\frac{1}{2}\right)[x]\right|$, when $-2\le x\le 2$. where [.] represents greatest integer function. Then

(A) f(x) is continuous at x = 2 (B) f(x) is continuous at x = 1

(C) f(x) is continuous at x = – 1 (D) f(x) is discontinuous at x = 0

D-3. The function f(x) is defined by $f(x) = \begin{cases} \log_{(4x-3)}\left(x^2 - 2x + 5\right), & \text{if } \frac{3}{4} < x < 1 \text{ or } x > 1 \\ 4, & \text{if } x = 1 \end{cases}$

(A) is continuous at x = 1
(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.

D-4_. If $f(x) = x \sin\left(\frac{\pi}{2}(x + 2[x])\right)$, then f(x)is {where [.] denotes GIF}

(A) Discontinuous at x = 2 (B) Discontinuous at x = 1
(C) Continuous at x = 1 (D) Continuous at x = 3

Section (E) : Continuity in an interval, Continuity of composite functions, IMVT

E-1. $f(x) = \begin{cases} \dfrac{\sqrt{(1+px)} - \sqrt{(1-px)}}{x}, & -1 \le x < 0 \\ \dfrac{2x+1}{x-2}, & 0 \le x \le 1 \end{cases}$ is continuous in the interval [– 1, 1], then ' p' is equal to:

(A) – 1 (B) – 1/2 (C) 1/2 (D) 1

E-2. Let f(x) = Sgn(x) and $g(x) = x(x^2 - 5x + 6)$. The function f(g(x)) is discontinuous at

(A) infinitely many points (B) exactly one point
(C) exactly three points (D) no point

E-3. If $y = \dfrac{1}{t^2 + t - 2}$ where $t = \dfrac{1}{x-1}$, then the number of points of discontinuities of y = f(x), $x \in R$ is

(A) 1 (B) 2 (C) 3 (D) infinite

E-4. The equation 2 tan x + 5x – 2 = 0 has

(A) no solution in $[0, \pi/4]$ (B) at least one real solution in $[0, \pi/4]$
(C) two real solution in $[0, \pi/4]$ (D) None of these

Section (F) : Derivability at a point

F-1. If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then indicate the correct alternative(s):

(A) f(x) is continuous but not differentiable at x = 0
(B) f(x) is differentiable at x = 0
(C) f(x) is not differentiable at x = 0
(D) none

F-2. If $f(x) = \begin{cases} \dfrac{x(3e^{1/x} + 4)}{2 - e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$, then f(x) is

(A) continuous as well differentiable at x = 0
(B) continuous but not differentiable at x = 0
(C) neither differentiable at x = 0 nor continuous at x = 0
(D) none of these

F-3. If $f(x) = \dfrac{x}{\sqrt{x+1}-\sqrt{x}}$ be a real valued function, then

(A) f(x) is continuous, but f′(0) does not exist (B) f(x) is differentiable at x = 0
(C) f(x) is not continuous at x = 0 (D) f(x) is not differentiable at x = 0

F-4. The function $f(x) = \sin^{-1}(\cos x)$ is:
(A) discontinuous at x = 0 (B) continuous at x = 0
(C) differentiable at x = 0 (D) none of these

F-5. If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, where { . } denotes the fractional part function, then:

(A) f is continuous & differentiable at x = 0 (B) f is continuous but not differentiable at x = 0
(C) f is continuous & differentiable at x = 2 (D) none of these.

F-6. Given $f(x) = \begin{cases} \log_a\left(a\ \left|[x] + [-x]\right|\right)^x \left(\dfrac{a^{\left(\frac{2}{\frac{[x]+[-x]}{|x|}}\right)} - 5}{3 + a^{\frac{1}{|x|}}}\right) & \text{for } |x| \neq 0\ ;\ a > 1 \\ 0 & \text{for } x = 0 \end{cases}$

where [.] represents the integral part function, then:
(A) f is continuous but not differentiable at x = 0
(B) f is continuous & differentiable at x = 0
(C) the differentiability of 'f' at x = 0 depends on the value of a
(D) f is continuous & differentiable at x = 0 and for a = e only.

F-7. If $f(x) = \begin{cases} \dfrac{x^2 - 1}{x^2 + 1} & , 0 < x \leq 2 \\ \dfrac{1}{4}\left(x^3 - x^2\right) & , 2 < x \leq 3 \\ \dfrac{9}{4}\left(|x - 4| + |2 - x|\right) & , 3 < x < 4 \end{cases}$, then:

(A) f (x) is differentiable at x = 2 & x = 3 (B) f (x) is non-differentiable at x = 2 & x = 3
(C) f (x) is differentiable at x = 3 but not at x = 2 (D) f (x) is differentiable at x = 2 but not at x = 3.

Section (G) : Derivability in an interval

G-1. The set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable is:

(A) $(-\infty, \infty)$ (B) $[0, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

G-2. If f (x) is differentiable everywhere, then :
(A) $|f|$ is differentiable everywhere (B) $|f|^2$ is differentiable everywhere
(C) $f|f|$ is not differentiable at some point (D) $f + |f|$ is differentiable everywhere

G-3. Let f (x) be defined in [– 2 , 2] by

$$f(x) = \begin{cases} \max\left(\sqrt{4-x^2}\ ,\ \sqrt{1+x^2}\right) & , -2 \leq x \leq 0 \\ \min\left(\sqrt{4-x^2}\ ,\ \sqrt{1+x^2}\right) & , 0 < x \leq 2 \end{cases}$$, then f (x) :

(A) is continuous at all points
(B) is not continuous at more than one point .
(C) is not differentiable only at one point
(D) is not differentiable at more than one point

G-4. The number of points at which the function f(x) = max. $\{a-x, a+x, b\}$, $-\infty < x < \infty$, $0 < a < b$ cannot be differentiable is:
(A) 1 (B) 2 (C) 3 (D) none of these

G-5. Let $f(x) = x - x^2$ and $g(x) = \begin{cases} \max \ f(t), \ 0 \le t \le x, \ 0 \le x \le 1 \\ \sin \ \pi x \ , \ x > 1 \end{cases}$, then in the interval $[0, \infty)$
(A) g(x) is everywhere continuous except at two points
(B) g(x) is everywhere differentiable except at two points
(C) g(x) is everywhere differentiable except at x = 1
(D) none of these

G-6. Consider the following statements :
S_1 : Number of points where $f(x) = |x \operatorname{sgn}(1 - x^2)|$ is non-differentiable is 3.
S_2 : Defined $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1) & , \ x \le 0 \\ \frac{\tan x - \sin x}{x^3} & , \ x > 0 \end{cases}$, In order that f(x) be continuous at x = 0, 'a' should be equal to $\frac{1}{2}$
S_3 : The set of all points, where the function $\sqrt[3]{x^2 |x|}$ is differentiable is $(-\infty, 0) \cup (0, \infty)$
S_4 : Number of points where $f(x) = \frac{1}{\sin^{-1}(\sin x)}$ is non-differentiable in the interval $(0, 3\pi)$ is 3.
State, in order, whether S_1, S_2, S_3, S_4 are true or false
(A) TTTF (B) TTTT (C) FTTF (D) TFTT

G-7. Consider the following statements :
S_1 : Let $f(x) = \frac{\sin \ (\pi \ \ [x-\pi])}{1+[x]^2}$, where [.] stands for the greatest integer function. Then f(x) is discontinuous at $x = n + \pi$, $n \in I$
S_2 : The function f(x) = p[x + 1] + q [x – 1], (where [.] denotes the greatest integer function) is continuous at x = 1 if p + q = 0
S_3 : Let f(x) = |[x] x| for $-1 \le x \le 2$, where [.] is greatest integer function, then f is not differentiable at x = 2.
S_4 : If f(x) takes only rational values for all real x and is continuous, then $f'(10) = 10$.

(A) FTTT (B) TTTF (C) FTTF (D) FFTF

G-8. For what triplets of real numbers (a, b, c) with $a \neq 0$ the function
$f(x) = \begin{cases} x & , \quad x \le 1 \\ ax^2 + bx + c & , \quad \text{otherwise} \end{cases}$ is differentiable for all real x?
(A) $\{(a, 1-2a, a) \mid a \in R, a \neq 0\}$ (B) $\{(a, 1-2a, c) \mid a, c \in R, a \neq 0\}$
(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$ (D) $\{(a, 1-2a, 0) \mid a \in R, a \neq 0\}$

Section (H) : Functional Equations and Miscellaneous

H-1. Given that $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h \to 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)} =$
(A) does not exist (B) is equal to –3/2 (C) is equal to 3/2 (D) is equal to 3

H-2. If f(x + y) = f(x) . f(y), $\forall$ x & y $\in$ N and f(1) = 2, then the value of $\sum_{n=1}^{10} f(n)$ is

(A) 2036 (B) 2046 (C) 2056 (D) 2066

H-3. If f(1) = 1 and f(n + 1) = 2f(n) + 1 if n $\geq$ 1, then f(n) is equal to

(A) $2^n + 1$ (B) 2^n (C) $2^n - 1$ (D) $2^{n-1} - 1$

H-4. If y = f(x) satisfies the condition $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ (x $\neq$ 0), then f(x) is equal to

(A) $-x^2 + 2$ (B) $-x^2 - 2$

(C) $x^2 - 2$, x $\in$ R – {0} (D) $x^2 - 2$, |x| $\in$ [2, ∞)

H-5. A function f : R $\to$ R satisfies the condition x^2 f(x) + f(1 – x) = 2x – x^4. Then f(x) is:

(A) $-x^2 - 1$ (B) $-x^2 + 1$ (C) $x^2 - 1$ (D) $-x^4 + 1$

H-6. If f: R $\to$ R be a differentiable function, such that f(x + 2y) = f(x) + f(2y) + 4xy $\forall$ x, y $\in$ R. then

(A) f′(1) = f′(0) + 1 (B) f′(1) = f′(0) – 1 (C) f′(0) = f′(1) + 2 (D) f′(0) = f′(1) – 2

PART - III : MATCH THE COLUMN

1. Let [.] denotes the greatest integer function.

Column – I	**Column – II**
(A) If P(x) = [2 cos x], x $\in$ [–π, π], then P(x)	(p) is discontinuous at exactly 7 points
(B) If Q(x) = [2 sin x], x $\in$ [–π, π], then Q(x)	(q) is discontinuous at exactly 4 points
(C) If R(x) = [2 tan x/2], x $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then R(x)	(r) is non differentiable at some points
(D) If S(x) = $\left[3\operatorname{cosec}\frac{x}{3}\right]$, x $\in \left[\frac{\pi}{2}, 2\pi\right]$, then S(x)	(s) is continuous at infinitely many values

2.

Column – I	**Column – II**
(A) f(x) = $\lvert x^3\rvert$ is	(p) continuous in (–1, 1)
(B) f(x) = $\sqrt{\lvert x\rvert}$ is	(q) differentiable in (–1, 1)
(C) f(x) = $\lvert\sin^{-1} x\rvert$ is	(r) differentiable in (0, 1)
(D) f(x) = $\cos^{-1}\lvert x\rvert$ is	(s) not differentiable atleast at one point in (–1, 1)

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. $\lim\limits_{x\to a^-}\left(\frac{|x|^3}{a}-\left[\frac{x}{a}\right]^3\right)$ (a < 0), where [x] denotes the greatest integer less than or equal to x, is equal to

(A) a^2+1 (B) $-a^2-1$ (C) a^2 (D) $-a^2$

2. $\lim\limits_{n\to\infty}\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}.......\cos\frac{x}{2^n}$ is equal to ($x\neq 0$)

(A) 1 (B) -1 (C) $\frac{\sin x}{x}$ (D) $\frac{x}{\sin x}$

3.✍ $\lim\limits_{\theta\to 0}\left(\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right)$, where [.] represents greatest integer function and $n\in N$, is equal to

(A) 2n (B) 2n + 1 (C) 2n – 1 (D) does not exist

4. $\lim\limits_{x\to 0}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right]$, where [·] represents greatest integer function, is equal to

(A) – 1 (B) 1 (C) $\log_{\sqrt{2}+1}(3-\sqrt{2})$ (D) does not exist

5.✍ The value of $\lim\limits_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$ is equal to

(A) $\lim\limits_{x\to 0}\frac{\tan x}{x^3}$ (B) $\frac{1}{6}$ (C) $\lim\limits_{x\to 0}\frac{\sin x-x}{x^3}$ (D) $\frac{1}{3}$

6.✍ The value of $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ell n\sin x}$ is

(A) 1 (B) 2 (C) 3 (D) $\pi/2$

7.✍ The value of $\lim\limits_{x\to\frac{\pi}{2}}\tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$ is equal to:

(A) $\frac{1}{10}$ (B) $\frac{1}{11}$ (C) $\frac{1}{12}$ (D) $\frac{1}{8}$

8.✍ If α and β be the roots of equation $ax^2 + bx + c = 0$, then $\lim\limits_{x\to\alpha}\left(1+ax^2+bx+c\right)^{\frac{1}{x-\alpha}}$ is equal to

(A) $a(\alpha-\beta)$ (B) $\ell n\,|a(\alpha-\beta)|$ (C) $e^{a(\alpha-\beta)}$ (D) $e^{a|\alpha-\beta|}$

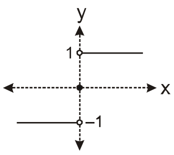
9. $\lim\limits_{x\to\infty}\frac{e^x\left(\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}\right)}{x^n}$, $n\in N$, is equal to

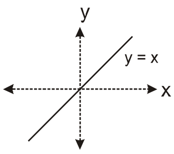
(A) 0 (B) $\ell n\left(\frac{2}{3}\right)$ (C) $\ell n\left(\frac{3}{2}\right)$ (D) none of these

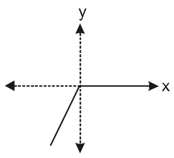
10. $\lim\limits_{y \to 0} \left(\lim\limits_{x \to \infty} \dfrac{\exp\left(x \ \ell n \left(1 + \dfrac{ay}{x}\right)\right) - \exp\left(x \ \ell n \left(1 + \dfrac{by}{x}\right)\right)}{y} \right)$ is equal to

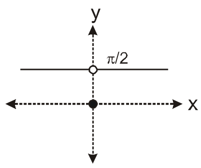
(A) a + b (B) a − b (C) b − a (D) − (a + b)

11. The graph of the function $f(x) = \lim\limits_{t \to 0} \left(\dfrac{2x}{\pi} \cot^{-1} \dfrac{x}{t^2} \right)$ is

(A) 

(B) 

(C) 

(D) 

12. Let [x] denote the integral part of $x \in R$ and g(x) = x – [x]. Let f(x) be any continuous function with f(0) = f(1), then the function h(x) = f(g(x)) :
(A) has finitely many discontinuities (B) is continuous on R
(C) is discontinuous at some x = c (D) is a constant function.

13. Let $f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{1/x} & x > 0 \end{cases}$

If f(x) is continuous at x = 0 then find $(a - b - c + e^d)$
(A) 0 (B) 6 (C) –6 (D) 2

14. Let $f(x) = \begin{bmatrix} x^2 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{bmatrix}$, then:
(A) f(x) is discontinuous for all x
(B) discontinuous for all x except at x = 0
(C) discontinuous for all x except at x = 1 or – 1
(D) none of these

15. A point (x, y), where function f(x) = [sin [x]] in (0, 2π) is not continuous, is ([.] denotes greatest integer ≤ x).
(A) (3, 0) (B) (2, 0) (C) (1, 0) (D) (4, –1)

16. The function f defined by $f(x) = \lim\limits_{t \to \infty} \left\{ \dfrac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1} \right\}$. is
(A) everywhere continuous (B) discontinuous at all integer values of x
(C) continuous at x = 0 (D) none of these

17. If f(x) = $\begin{cases} \sqrt{x}\left(1+x\sin\frac{1}{x}\right), & x>0 \\ -\sqrt{-x}\left(1+x\sin\frac{1}{x}\right), & x<0 \\ 0, & x=0 \end{cases}$, then f(x) is

(A) continuous as well as diff. at x = 0 (B) continuous at x = 0, but not diff. at = 0
(C) neither continuous at x = 0 nor diff. at x = 0 (D) none of these

18. The functions defined by f(x) = max {x^2, $(x - 1)^2$, 2x (1 – x)}, 0 ≤ x ≤ 1
(A) is differentiable for all x
(B) is differentiable for all x except at one point
(C) is differentiable for all x except at two points
(D) is not differentiable at more than two points.

19. [x] denotes the greatest integer less than or equal to x. If f(x) = [x] [sin πx] in (–1,1), then f(x) is:
(A) continuous at x = 0 (B) continuous in (–1, 0)
(C) differentiable in (–1,1) (D) none

20. Let f(x) = [n + p sin x], x ∈ (0, π), n ∈ Z, p is a prime number and [x] is greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is
(A) p (B) p – 1 (C) 2p + 1 (D) 2p – 1

21. Let f: R → R be any function and g (x) = $\frac{1}{f(x)}$. Then g is
(A) onto if f is onto (B) one-one if f is one-one
(C) continuous if f is continuous (D) differentiable if f is differentiable

22. Let f(x) = $x^3 - x^2 + x + 1$ and g(x) =, $\begin{cases} \max\{f(t) \text{ for } 0 \le t \le x\} & \text{for } 0 \le x \le 1 \\ 3 - x + x^2 & \text{for } 1 < x \le 2 \end{cases}$ then:
(A) g(x) is continuous & derivable at x = 1
(B) g(x) is continuous but not derivable at x = 1
(C) g(x) is neither continuous nor derivable at x = 1
(D) g(x) is derivable but not continuous at x = 1

23. Let f : R → R be a function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, f(0) = 0 and f′(0) = 3, then
(A) $\frac{f(x)}{x}$ is differentiable in R
(B) f(x) is continuous but not differentiable in R
(C) f(x) is continuous in R
(D) f(x) is bounded in R

24. If a differentiable function f satisfies $f\left(\frac{x+y}{3}\right) = \frac{4-2(f(x)+f(y))}{3}$ ∀ x, y ∈ R, then f(x) is equal to
(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{8}{7}$ (D) $\frac{4}{7}$

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. Let f(x) = $\dfrac{\sin^{-1}(1-\{x\})\ .\ \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\ \ (1-\{x\})}$, then $\left(\dfrac{\ell\text{im}_{x\to 0^+} f(x)}{\ell\text{im}_{x\to 0^-} f(x)}\right)^2$ =

(where {.} denotes the fractional part function)

2. Let f (x) = $\begin{cases} x\ \sin\left(\dfrac{1}{x}\right)+\sin\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0 & , \ x=0 \end{cases}$, then $\ell\text{im}_{x\to\infty}$ f(x) is equal to

3. $\ell\text{im}_{x\to 0}\left(\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}+\sqrt[3]{\dfrac{1+\cos^3 x+3\cos^2 x+3\cos x}{\cos x+63}}\right)$ is equal to

4. If $\ell\text{im}_{x\to\infty}$ f(x) exists and is finite and nonzero and $\ell\text{im}_{x\to\infty}\left(f(x)+\dfrac{3f(x)-1}{f^2(x)}\right)$ = 3, then the value of $\ell\text{im}_{x\to\infty}$ f(x) is equal to

5. If f(x) = $\begin{cases} x-1 & , \ x \geq 1 \\ 2x^2-2 & , \ x<1 \end{cases}$, g(x) = $\begin{cases} x+1 & , \ x>0 \\ -x^2+1 & , \ x \leq 0 \end{cases}$ and h(x) = |x|,

then $\ell\text{im}_{x\to 0}$ f(g(h(x))) is equal to

6. If f(x) = $\begin{cases} \sin x & , \ x \neq n\pi \ , \ n = 0, \pm 1, \pm 2, ... \\ 2 & , \ \text{otherwise} \end{cases}$ and

g(x) = $\begin{cases} x^2+1 & , \ x \neq 0 \ , \ 2 \\ 4 & , \ x=0 \\ 5 & , \ x=2 \end{cases}$, then $\ell\text{im}_{x\to 0}$ g (f(x)) is equal to

7. $\ell\text{im}_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+............+\dfrac{1}{\sqrt{n^2+2n}}\right)$ is equal to

8. The value of $\lim_{x\to 0} x^2\left[\dfrac{1}{x^2}\right]$ where [.] denotes G.I.F., is

9. $\ell\text{im}_{x\to 0}\left(\dfrac{\sin^{-1}x - \tan^{-1}x}{x^3}+\dfrac{84x\tan^{-1}\left(\sqrt{2}-1\right)}{\sin \pi x}\right)$ is equal to

10. If $\ell\text{im}_{x\to 0}\dfrac{x^3}{\sqrt{a+x}\ \ (bx-\sin x)}$ = 1, then the value of (a + b) where a > 0, is

11. If f(x) = $\sum_{\lambda=1}^{n}\left(x-\dfrac{5}{\lambda}\right)\left(x-\dfrac{4}{\lambda+1}\right)$, then $\ell\text{im}_{n\to\infty}$ f(0) is equal to

12. Let $f(x) = \begin{cases} (-1)^{[x^2]} & \text{if} \quad x < 0 \\ \ell im_{n\to\infty} \dfrac{1}{1+x^n} & \text{if} \quad x \ge 0 \end{cases}$. Then $\ell im_{x\to 0^-} 5f(x) + \ell im_{x\to 0^+} 7f(x)$ equals (where [.] represents greatest integer function)

13. The value of $\left[\lim_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}\right]$ where [.] denotes GIF is

14. If $\ell im_{x\to 0} \dfrac{e^{-nx} + e^{nx} - 2\cos\dfrac{nx}{2} - kx^2}{(\sin x - \tan x)}$ exists and finite (n, k $\in$ N), then the least value of 4k + n $\div$2 is :

15. If $\ell im_{n\to\infty} \dfrac{1^2 n + 2^2 (n-1) + 3^2 (n-2) + ... + n^2 . 1}{1^3 + 2^3 + 3^3 + ... + n^3} = \dfrac{a}{b}$ where a and b are coprime numbers then 2a + 3b =

16. If $\ell im_{n\to\infty} \dfrac{n^{98}}{n^x - (n-1)^x} = \dfrac{1}{99}$, then the value of x equals

17. The number of points of discontinuity of $f(x) = \begin{cases} |4x-5| \; [x] & \text{for} \quad x > 1 \\ [\cos \pi x] & \text{for} \quad x \le 1 \end{cases}$

(where [x] is the greatest integer not greater than x) in [0, 2] is

18. If $f(x) = \begin{cases} 2x^2 + 12x + 16, & -4 \le x \le -2 \\ 2 - |x|, & -2 < x \le 1 \\ 4x - x^2 - 2, & 1 < x \le 13 \end{cases}$, then the maximum length of interval for which f(|x|) is continuous is

19. Let $f(x) = \dfrac{1-\sin x}{(\pi - 2x)^2} \cdot \dfrac{\ell n \; (\sin x)}{\ell n \; (1 + \pi^2 - 4\pi x + 4x^2)}$, $x \ne \dfrac{\pi}{2}$. The value of $f\left(\dfrac{\pi}{2}\right)$ so that the function is continuous at $x = \dfrac{\pi}{2}$ is λ and $|\lambda|\alpha^\beta = 1$ where $\alpha, \beta \in$ N then find product of all possible values of β

20. If the function f(x) defined as $f(x) = \begin{cases} (\sin x + \cos x)^{\text{cosec} x}, & -\dfrac{\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}}}{ae^{-2+\frac{1}{x}} + be^{-1+\frac{3}{x}}}, & 0 < x < \dfrac{\pi}{2} \end{cases}$

is continuous at x = 0 , then the value of $\log_{e^{1/7}} a + 9b$ is :

21. The number of points of non differentiability of the function f (x) = |sin x| + sin |x| in [–4π, 4π] is

22. If $f(x) = \begin{cases} \dfrac{\sin\ [x^2]\ \pi}{x^2 - 3x + 8} + ax^3 + b & , \ 0 \le x \le 1 \\ 2\cos\pi\ \ x + \tan^{-1} x & , \ 1 < x \le 2 \end{cases}$ is differentiable in [0 , 2], then the value of [a + b + 6] is (Here [.] stands for the greatest integer function)

23. If $f(x) = \begin{cases} x^2 e^{2(x-1)} & \text{for } 0 \le x \le 1 \\ a\ \text{sgn}\ (x + 1)\ \cos\ (2x - 2) + bx^2 & \text{for } 1 < x \le 2 \end{cases}$ is differentiable at x = 1 then $a^3 + b^3 =$

24. Find number of points of non-differentiability of $f(x) = \lim\limits_{n\to\infty} \dfrac{\{e^x\}^n - 1}{\{e^x\}^n + 1}$ in interval [0, 1] where {.} represents fractional part function

25. Let [x] denote the greatest integer less than or equal to x. The number of integral points in [–1, 1] where f(x) = [x sin πx] is differentiable are

26. Let $f''(x)$ be continuous at x = 0 and $f''(0) = 4$ then value of $\lim\limits_{x\to 0} \dfrac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is

27. Let f : R $\to$ R is a function satisfying f(10 – x) = f(x) and f(2 – x) = f(2 +x), $\forall x \in R$. If f(0) = 101, then the minimum possible number of values of x satisfying f(x) = 101 for x $\in$ [0,30] is

28. Find the natural number 'a' for which $\sum\limits_{k=1}^{n} f(a+k) = 2048(2^n - 1)$, where the function 'f' satisfies the relation f(x + y) = f(x) . f(y) for all natural numbers x & y and further f(1) = 2

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Let $f(x) = \dfrac{x^2 - 9x + 20}{x - [x]}$ (where [x] denotes greatest integer less than or equal to x), then

(A) $\ell im_{x \to 5^-} f(x) = 0$ (B) $\ell im_{x \to 5^+} f(x) = 1$

(C) $\ell im_{x \to 5} f(x)$ does not exist (D) none of these

2. If $f(x) = \dfrac{\cos 2 - \cos 2x}{x^2 - |x|}$, then

(A) $\ell im_{x\to -1} f(x) = 2 \sin 2$ (B) $\ell im_{x\to 1} f(x) = 2 \sin 2$

(C) $\ell im_{x\to -1} f(x) = 2 \cos 2$ (D) $\ell im_{x\to 1} f(x) = 2 \cos 2$

3. If $\ell = \ell im_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = \ell im_{x \to 0} \dfrac{1 + a\cos x}{x^2} - \ell im_{x\to 0} \dfrac{b\sin x}{x^3}$, where $\ell \in R$, then

(A) (a, b) = (–1, 0) (B) a & b are any real numbers

(C) $\ell = 0$ (D) $\ell = \dfrac{1}{2}$

4. Let $f(x) = \dfrac{|x + \pi|}{\sin x}$, then

(A) $f(-\pi^+) = -1$ (B) $f(-\pi^-) = 1$

(C) $\ell im_{x \to -\pi} f(x)$ does not exist (D) $\ell im_{x \to \pi} f(x)$ does not exist

5. Let $f(x) = \begin{cases} 1+\dfrac{2x}{a}, & 0 \le x < 1 \\ ax, & 1 \le x < 2 \end{cases}$, if $\lim\limits_{x\to 1} f(x)$ exists, then value of a is :

(A) 1 (B) – 1 (C) 2 (D) – 2

6. Let α, β be the roots of equation $ax^2 + bx + c = 0$, where $1 < \alpha < \beta$ and $\lim\limits_{x\to x_0} \dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$, then which of the following statements is correct

(A) $a > 0$ and $x_0 < 1$ (B) $a > 0$ and $x_0 > \beta$

(C) $a < 0$ and $\alpha < x_0 < \beta$ (D) $a < 0$ and $x_0 < 1$

7. Let $\phi(x) = \dfrac{a_0x^m + a_1x^{m+1} + ... + a_kx^{m+k}}{b_0x^n + b_1x^{n+1} + ... + b_\ell x^{n+\ell}}$, where $a_0 \neq 0$, $b_0 \neq 0$ and $m, n \in N$, then which of the following statements is/are correct.

(A) If $m > n$ then, $\lim\limits_{x\to 0} \phi(x)$ is equal to 0

(B) If $m = n$ then, $\lim\limits_{x\to 0} \phi(x)$ is equal to $\dfrac{a_0}{b_0}$

(C) If $m < n$ and $n - m$ is even, $\dfrac{a_0}{b_0} > 0$, then $\lim\limits_{x\to 0} \phi(x)$ is equal to ∞

(D) If $m < n$ and $n - m$ is even, $\dfrac{a_0}{b_0} < 0$, then $\lim\limits_{x\to 0} \phi(x)$ is equal to $-\infty$

8. Given a real valued function f such that

$$f(x) = \begin{cases} \dfrac{\tan^2[x]}{(x^2-[x]^2)}, & x > 0 \\ 1, & x = 0 \\ \sqrt{\{x\}\cot\{x\}}, & x < 0 \end{cases}$$

where [.] represents greatest integer function and {.} represents fractional part function, then

(A) $\lim\limits_{x\to 0} f(x) = 1$ (B) $\lim\limits_{x\to 0^-} f(x) = \sqrt{\cot 1}$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$ (D) $\lim\limits_{x\to 0^+} f(x) = 0$

9. If $f(x) = \dfrac{\sqrt{x^2+2}}{3x-6}$, then

(A) $\lim\limits_{x\to -\infty} f(x) = -\dfrac{1}{3}$ (B) $\lim\limits_{x\to \infty} f(x) = \dfrac{1}{3}$ (C) $\lim\limits_{x\to -\infty} f(x) = \dfrac{1}{3}$ (D) $\lim\limits_{x\to \infty} f(x) = -\dfrac{1}{3}$

10. If $\lim\limits_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = p$ (finite), then

(A) a = – 2 (B) a = – 1 (C) p = – 2 (D) p = – 1

11. $\lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n + A}$ is equal to

(A) a^n if $n \in N$ (B) ∞ if $n \in Z^-$ & $a = A = 0$

(C) $\dfrac{1}{1+A}$ if $n = 0$ (D) a^n if $n \in Z^-$, $A = 0$ & $a \neq 0$

12. If $\ell = \lim\limits_{x \to \infty} (\sin\sqrt{x+1} - \sin\sqrt{x})$ and $m = \lim\limits_{x \to -\infty} [\sin\sqrt{x+1} - \sin\sqrt{x}]$, where [.] denotes the greatest integer function, then :

(A) $\ell = 0$
(B) $m = 0$
(C) m is undefined
(D) ℓ is undefined

13. If $f(x) = |x|^{\sin x}$, then

(A) $\lim\limits_{x \to 0^-} f(x) = 1$
(B) $\lim\limits_{x \to 0^+} f(x) = 1$
(C) $\lim\limits_{x \to 0} f(x) = 1$
(D) limit does not exist at x = 0

14. If $\lim\limits_{x \to 0} (\cos x + a \sin bx)^{\frac{1}{x}} = e^2$, then the possible values of 'a' & 'b' are :

(A) a = 1 , b = 2
(B) a = 2 , b = 1
(C) a = 3, b = 2/3
(D) a = 2/3 , b = 3

15. If $\lim\limits_{x \to 0} (1 + ax + bx^2)^{\frac{2}{x}} = e^3$, then possible values of a and b is/are :

(A) a = 3, b = 0
(B) $a = \frac{3}{2}$, $b = \frac{1}{2}$
(C) $a = \frac{3}{2}$, $b = \frac{3}{2}$
(D) $a = \frac{3}{2}$, b = 0

16. $\lim\limits_{x \to 0^+} \log_{\sin\frac{x}{2}} \sin x$ is equal to

(A) 1
(B) 0
(C) $\lim\limits_{x \to 0} x^{\sin x}$
(D) $\lim\limits_{x \to 0^+} (\tan x)^{\sin x}$

17. $\lim\limits_{x \to \infty} \frac{x^n}{e^x} = 0$, $n \in$ integer number, is true for

(A) no value of n
(B) all values of n
(C) negative values of n
(D) positive values of n

18. If $f(x) = \underset{n \to \infty}{\text{Limit}} \dfrac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$ $(n \in N)$, then

(A) $\lim\limits_{x \to 1^+} f(x) = -\sin 1$
(B) $\lim\limits_{x \to 1^-} f(x) = \log 3$
(C) $\lim\limits_{x \to 1} f(x) = \sin 1$
(D) $f(1) = \dfrac{\log 3 - \sin 1}{2}$

19. Which of the following function(s) defined below has/have single point continuity.

(A) $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$
(B) $g(x) = \begin{cases} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{cases}$
(C) $h(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$
(D) $k(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$

20. The function $f(x) = \begin{cases} |x-3| & , x \geq 1 \\ \left(\dfrac{x^2}{4}\right) - \left(\dfrac{3x}{2}\right) + \left(\dfrac{13}{4}\right) & , x < 1 \end{cases}$ is:

(A) continuous at x = 1
(B) differentiable at x = 1
(C) continuous at x = 3
(D) differentiable at x = 3

21. If f(x) = $\frac{1}{2}x - 1$, then on the interval [0, π]

(A) tan (f(x)) and $\frac{1}{f(x)}$ are both continuous

(B) tan (f(x)) and $\frac{1}{f(x)}$ are both discontinuous

(C) tan (f(x)) and f^{-1} (x) are both continuous

(D) tan (f(x)) is continuous but $\frac{1}{f(x)}$ is not.

22. Let f(x) and g(x) be defined by f(x) = [x] and g(x) = $\begin{cases} 0, & x \in I \\ x^2, & x \in R - I \end{cases}$ (where [.] denotes the greatest integer function), then

(A) $\lim_{x \to 1}$ g(x) exists, but g is not continuous at x = 1

(B) $\lim_{x \to 1}$ f(x) does not exist and f is not continuous at x = 1

(C) gof is continuous for all x

(D) fog is continuous for all x

23. Let f(x) = [x] + $\sqrt{x - [x]}$, where [.] denotes the greatest integer function. Then

(A) f(x) is continuous on R^+

(B) f(x) is continuous on R

(C) f(x) is continuous on R – I

(D) discontinuous at x = 1

24. The points at which the function, f(x) = $|x - 0.5| + |x - 1|$ + tan x does not have a derivative in the interval (0, 2) are:

(A) 1 (B) $\pi/2$ (C) $\pi/4$ (D) 1/2

25. f(x) = $(\sin^{-1}x)^2 . \cos(1/x)$ if $x \neq 0$; f(0) = 0, f(x) is:

(A) continuous no where in $-1 \le x \le 1$

(B) continuous everywhere in $-1 \le x \le 1$

(C) differentiable no where in $-1 \le x \le 1$

(D) differentiable everywhere in $-1 < x < 1$

26. If f(x) = $a_0 + \sum_{k=1}^{n} a_k |x|^k$, where a_i's are real constants, then f(x) is

(A) continuous at x = 0 for all a_i

(B) differentiable at x = 0 for all $a_i \in R$

(C) differentiable at x = 0 for all $a_{2k-1} = 0$

(D) none of these

27. Let f : R $\to$ R be a function such that f(0) = 1 and for any x, y $\in$ R, f(xy + 1) = f(x) f(y) – f(y) – x + 2. Then f is

(A) one-one (B) onto (C) many one (D) into

28. Suppose that f is a differentiable function with the property that f(x + y) = f(x) + f(y) + xy and

$\lim_{h \to 0} \frac{1}{h}$ f(h) = 3

where [.] represents greatest integer function, then

(A) f is a linear function

(B) 2f(1) = $\left[\lim_{x \to 0}(1 + 2x)^{1/x}\right]$

(C) f(x) = 3x + $\frac{x^2}{2}$

(D) f '(1) = 4

29. Let 'f' be a real valued function defined for all real numbers x such that for some positive constant 'a' the equation $f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$ holds for all x. Then f(x) is periodic function with period equal to
(A) 2 a (B) 4 a (C) 6 a (D) 8 a

PART - IV : COMPREHENSION

Comprehension # 1

Consider f(x) = $\frac{\sin x + ae^x + be^{-x} + c\ \ell n\ (1+x)}{x^3}$, where a, b, c are real numbers.

1. If $\ell im_{x\to 0^+} f(x)$ is finite, then the value of a + b + c is
(A) 0 (B) 1 (C) 2 (D) – 2

2. If $\ell im_{x\to 0^+} f(x) = \ell$ (finite), then the value of ℓ is
(A) – 2 (B) $-\frac{1}{2}$ (C) – 1 (D) $-\frac{1}{3}$

3. Using the values of a, b, c as found in Q.No. 1 or Q. No.2 above, the value of $\ell im_{x\to 0^+} x f(x)$ is
(A) 0 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 2

Comprehension # 2

If both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist finitely and are equal , then the function f is said to have removable discontinuity at x = c
If both the limits i.e. $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist finitely and are not equal, then the function f is said to have non-removable discontinuity at x = c and in this case $\left| \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x) \right|$ is called jump of the discontinuity.

4. Which of the following function has non-removable discontinuity at the origin ?
(A) $f(x) = \frac{1}{\ell n\ |x|}$ (B) $f(x) = x \sin \frac{\pi}{x}$ (C) $f(x) = \frac{1}{1 + 2^{\cot x}}$ (D) $f(x) = \cos\left(\frac{|\sin x|}{x}\right)$

5. Which of the following function not defined at x = 0 has removable discontinuity at the origin ?
(A) $f(x) = \frac{1}{1 + 2^{\frac{1}{x}}}$ (B) $f(x) = \tan^{-1} \frac{1}{x}$ (C) $f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ (D) $f(x) = \frac{1}{\ell n\ |x|}$

6. If $f(x) = \begin{cases} \tan^{-1}(\tan x) & ; \quad x \le \frac{\pi}{4} \\ \pi\ [x]+1 & ; \quad x > \frac{\pi}{4} \end{cases}$, then jump of discontinuity is
(where [.] denotes greatest integer function)
(A) $\frac{\pi}{4} - 1$ (B) $\frac{\pi}{4} + 1$ (C) $1 - \frac{\pi}{4}$ (D) $-1 - \frac{\pi}{4}$

Comprehension # 3

Let $f(x) = \begin{cases} x\ g(x) & , \ x \le 0 \\ x + ax^2 - x^3 & , \ x > 0 \end{cases}$, where $g(t) = \lim\limits_{x \to 0} (1 + a \tan x)^{t/x}$, a is positive constant, then

7. If a is even prime number, then g(2) =

(A) e^2 (B) e^3 (C) e^4 (D) none of these

8. Set of all values of a for which function f(x) is continuous at x = 0

(A) (–1, 10) (B) $(-\infty, \infty)$ (C) $(0, \infty)$ (D) none of these

9. If f(x) is differentiable at x = 0, then a $\in$

(A) (–5, –1) (B) (–10, 3) (C) $(0, \infty)$ (D) none of these

Comprehension # 4

Let f : R $\to$ R be a function defined as,

$f(x) = \begin{cases} 1 - |x| & , \ |x| \le 1 \\ 0 & , \ |x| > 1 \end{cases}$ and $g(x) = f(x-1) + f(x+1)$, $\forall\ x \in R$. Then

10. The value of g(x) is :

(A) $g(x) = \begin{cases} 0 & , & x \le -3 \\ 2+x & , & -3 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 3 \\ 0 & , & x > 3 \end{cases}$

(B) $g(x) = \begin{cases} 0 & , & x \le -2 \\ 2+x & , & -2 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 2 \\ 0 & , & x > 2 \end{cases}$

(C) $g(x) = \begin{cases} 0 & , & x \le 0 \\ 2+x & , & 0 < x < 1 \\ -x & , & 1 \le x \le 2 \\ x & , & 2 < x < 3 \\ 2-x & , & 3 \le x < 4 \\ 0 & , & 4 \le x \end{cases}$

(D) none of these

11. The function g(x) is continuous for, x $\in$

(A) R – {0, 1, 2, 3, 4} (B) R – {–2, –1, 0, 1, 2} (C) R (D) none of these

12. The function g(x) is differentiable for, x $\in$

(A) R

(B) R – {–2, –1, 0, 1, 2}

(C) R – {0, 1, 2, 3, 4}

(D) none of these

Exercise-3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

✍ Marked questions are recommended for Revision.
*** Marked Questions may have more than one correct option.**

1*. Let L = $\lim\limits_{x \to 0} \dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$, a > 0. If L is finite, then

[IIT-JEE-2009, Paper-1, (4, –1), 80]

(A) a = 2 (B) a = 1 (C) L = $\dfrac{1}{64}$ (D) L = $\dfrac{1}{32}$

2.✍ If $\lim\limits_{x \to 0} \left[1 + x\ln(1 + b^2)\right]^{\frac{1}{x}} = 2b \sin^2 \theta$, b > 0 and $\theta \in (-\pi, \pi]$, then the value of θ is

[IIT-JEE 2011, Paper-2, (3, –1), 80]

(A) $\pm \dfrac{\pi}{4}$ (B) $\pm \dfrac{\pi}{3}$ (C) $\pm \dfrac{\pi}{6}$ (D) $\pm \dfrac{\pi}{2}$

3*. Let f : **R** → **R** be a function such that f(x + y) = f(x) + f(y), ∀ x, y ∈ **R**. If f(x) is differentiable at x = 0, then

[IIT-JEE 2011, Paper-1, (4, 0), 80]

(A) f(x) is differentiable only in a finite interval containing zero
(B) f(x) is continuous ∀ x ∈ **R**
(C) f′(x) is constant ∀x ∈ **R**
(D) f(x) is differentiable except at finitely many points

4*.✍ If f(x) = $\begin{cases} -x - \dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$, then

[IIT-JEE 2011, Paper-2, (4, 0), 80]

(A) f(x) is continuous at x = $-\dfrac{\pi}{2}$ (B) f(x) is not differentiable at x = 0
(C) f(x) is differentiable at x = 1 (D) f(x) is differentiable at x = $-\dfrac{3}{2}$

5.✍ Let f : (0, 1) → **R** be defined by f(x) = $\dfrac{b - x}{1 - bx}$, where b is a constant such that 0 < b < 1. Then

[IIT-JEE 2011, Paper-2, (4, 0), 80]

(A) f is not invertible on (0, 1) (B) f ≠ f^{-1} on (0, 1) and f′(b) = $\dfrac{1}{f'(0)}$
(C) f = f^{-1} on (0, 1) and f′(b) = $\dfrac{1}{f'(0)}$ (D) f^{-1} is differentiable on (0, 1)

6.✍ If $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + x + 1}{x + 1} - ax - b\right) = 4$, then

[IIT-JEE 2012, Paper-1, (3, –1), 70]

(A) a = 1, b = 4 (B) a = 1, b = –4
(C) a = 2, b = –3 (D) a = 2, b = 3

7. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a>-1$. Then $\lim\limits_{a\to0^+}\alpha(a)$ and $\lim\limits_{a\to0^+}\beta(a)$ are **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and -1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

8. Let f(x) = $\begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x\neq0 \\ 0, & x=0 \end{cases}$, $x\in\mathbb{R}$, then f is

(A) differentiable both at x = 0 and at x = 2 **[IIT-JEE 2012, Paper-1, (3, –1), 70]**
(B) differentiable at x = 0 but not differentiable at x = 2
(C) not differentiable at x = 0 but differentiable at x = 2
(D) differentiable neither at x = 0 nor at x = 2

9*. For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by
f(x) = $\begin{cases} a_n+\sin\pi x, & \text{for } x\in[2n,\ 2n+1] \\ b_n+\cos\pi x, & \text{for } x\in(2n-1,\ 2n) \end{cases}$, for all integers n.
If f is continuous, then which of the following hold(s) for all n ? **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) $a_{n-1}-b_{n-1}=0$ (B) $a_n-b_n=1$ (C) $a_n-b_{n+1}=1$ (D) $a_{n-1}-b_n=-1$

10*. For every pair of continuous functions f, g:[0, 1] → R such that
max {f(x) : x ∈ [0,1]} = max {g(x) : x ∈ [0, 1]}, **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**
the correct statement(s) is (are) :
(A) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some c ∈ [0, 1]
(B) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some c ∈ [0, 1]
(C) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some c ∈ [0, 1]
(D) $(f(c))^2=(g(c))^2$ for some c ∈ [0, 1]

11. The largest value of the non-negative integer a for which $\lim\limits_{x\to1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

12. Let f : R → R and g : R → R be respectively given by f(x) = |x| + 1 and g(x) = x^2 + 1. Define h : R → R by $h(x)=\begin{cases}\max\ \{f(x),g(x)\} & \text{if } x\le0, \\ \min\ \{f(x),g(x)\} & \text{if } x>0.\end{cases}$ The number of points at which h(x) is not differentiable is

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

13. Let $f_1:R\to R$, $f_2:[0,\infty)\to R$, $f_3:R\to R$ and $f_4:R\to[0,\infty)$ be defined by

$f_1(x)=\begin{cases}|x| & \text{if } x<0, \\ e^x & \text{if } x\ge0;\end{cases}$

$f_2(x)=x^2$;

$f_3(x)=\begin{cases}\sin x & \text{if } x<0, \\ x & \text{if } x\ge0\end{cases}$

and $f_4(x)=\begin{cases}f_2(f_1(x)) & \text{if } x<0, \\ f_2(f_1(x))-1 & \text{if } x\ge0\end{cases}$

List I		List II	
P.	f_4 is	**1.**	onto but not one-one
Q.	f_3 is	**2.**	neither continuous nor one-one
R.	$f_2 o f_1$ is	**3.**	differentiable but not one-one
S.	f_2 is	**4.**	continuous and one-one

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

	P	**Q**	**R**	**S**
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

14*. Let g: R $\rightarrow$ R be a differentiable function with g(0) = 0, g'(0) = 0 and g'(1) $\neq$ 0. Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let (foh)(x) denote f(h(x)) and (hof)(x) denote h(f(x)). Then which of the following is(are) true?

(A) f is differentiable at x = 0
(B) h is differentiable at x = 0
(C) foh is differentiable at x = 0
(D) hof is differentiable at x = 0

[JEE (Advanced) 2015, P-1 (4, –2)/ 88]

15*. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let (fog) (x) dentoe f(g(x)) and (gof) (x) denote g(f(x)). Then which of the following is(are)true?

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(B) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(C) $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\pi}{6}$
(D) There is an $x \in R$ such that (gof)(x) = 1

[JEE (Advanced) 2015, P-1 (4, –2)/ 88]

16. Let m and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

17*. Let f : R $\rightarrow$ R , g : R $\rightarrow$ R and h : R $\rightarrow$ R be differentiable functions such that $f(x) = x^3 + 3x + 2$, g(f(x)) = x and h(g(g(x))) = x for all $x \in R$. Then **[JEE (Advanced) 2016, Paper-1, (4, –2)/62]**

(A) $g'(2) = \frac{1}{15}$
(B) h'(1) = 666
(C) h(0) = 16
(D) h(g(3)) = 36

18. Let $\alpha, \beta \in R$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals

[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

19*. Let $f : \left[-\frac{1}{2}, 2\right] \to R$ and $g : \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\, f(x) + |4x - 7|\, f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$ **[JEE (Advanced) 2016, Paper-2, (4, –2)/62]**

(B) f is discontinuous exactly at four point in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$.

20*. Let $a, b \in R$ and $f : R \to R$ be defined by $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$. Then f is
(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$ **[JEE (Advanced) 2016, Paper-2, (4, –2)/62]**
(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$
(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$
(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$

21*. Let [x] be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x \cos(\pi(x + [x]))$ is discontinuous ? **[JEE(Advanced) 2017, Paper-1,(4, –2)/61]**
(A) $x = -1$ (B) $x = 1$ (C) $x = 0$ (D) $x = 2$

22*. Let $f(x) = \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right)$ for $x \neq 1$. Then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

(A) $\lim_{x \to 1^+} f(x) = 0$ (B) $\lim_{x \to 1^-} f(x)$ does not exist

(C) $\lim_{x \to 1^-} f(x) = 0$ (D) $\lim_{x \to 1^+} f(x)$ does not exist

23. For any positive integer n, define $f_n : (0, \infty) \to R$ as

$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1 + (x + j)(x + j - 1)}\right)$ for all $x \in (0, \infty)$. **[JEE(Advanced) 2018, Paper-2,(4, –2)/60]**

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

Then, which of the following statement(s) is (are) TRUE ?

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1 + f_j'(0)) \sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n, $\lim_{x \to \infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x \to \infty} \sec^2(f_n(x)) = 1$

24. Let $f_1 : R \to R$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to R$ and $f_4 : R \to R$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$ **[JEE(Advanced) 2018, Paper-2,(3, –1)/60]**

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $f_3(x) = [\sin(\log_e(x+2))]$, where for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

LIST-I	LIST-II
(P) The function f_1 is	(1) **NOT** continuous at x =0
(Q) The function f_2 is	(2) continuous at x = 0 and **NOT** differentiable at x = 0
(R) The function f_3 is	(3) differentiable at x = 0 and its derivative is **NOT** continuous at x = 0
(S) The function f_4 is	(4) differentiable at x= 0 and its derivative is continuous at x= 0

The correct option is:

(A) $P \to 2;\ Q \to 3;\ R \to 1;\ S \to 4$ (B) $P \to 4;\ Q \to 1;\ R \to 2;\ S \to 3$

(C) $P \to 4;\ Q \to 2;\ R \to 1;\ S \to 3$ (D) $P \to 2;\ Q \to 1;\ R \to 4;\ S \to 3$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let $f(x) = x|x|$ and $g(x) = \sin x$ **[AIEEE 2009, (8, –2), 144]**

Statement-1 gof is differentiable at x = 0 and its derivative is continuous at that point.

Statement-2 gof is twice differentiable at x = 0.

(1) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.

(2) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(3) Statement-1 is True, Statement-2 is False

(4) Statement-1 is False, Statement-2 is True

2.✍ Let f : **R** → **R** be a positive increasing function with $\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty} \frac{f(2x)}{f(x)}$.

[AIEEE– 2010, (8, –2), 144]

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) 3 (4) 1

3. $\lim_{x\to 2} \left(\frac{\sqrt{1-\cos\ \{2(x-2)\}}}{x-2}\right)$ **[AIEEE– 2011, I, (4, –1), 120]**

(1) does not exist (2) equals $\sqrt{2}$ (3) equals $-\sqrt{2}$ (4) equals $\frac{1}{\sqrt{2}}$

4. Let f: $R \to [0,\infty)$ be such that $\lim_{x\to 5} f(x)$ exists and $\lim_{x\to 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$ **[AIEEE– 2011, II,(4, –1), 120]**

Then $\lim_{x\to 5} f(x)$ equals :

(1) 0 (2) 1 (3) 2 (4) 3

5. The value of p and q for which the function f(x) = $\begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , \ x < 0 \\ q & , \ x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}} & , \ x > 0 \end{cases}$ is continuous for all x in R, are : **[AIEEE 2011, I,(4, –1), 120]**

(1) p = $\frac{1}{2}$, q = $-\frac{3}{2}$ (2) p = $\frac{5}{2}$, q = $\frac{1}{2}$ (3) p = $-\frac{3}{2}$, q = $\frac{1}{2}$ (4) p = $\frac{1}{2}$, q = $\frac{3}{2}$

6. Define F(x) as the product of two real functions $f_1(x) = x$, $x \in R$, and $f_2(x) = \begin{cases} \sin\frac{1}{x} & , \text{ If } x \neq 0 \\ 0, & \text{ If } x = 0 \end{cases}$

as follows :

F(x) = $\begin{cases} f_1(x) \ . \ f_2(x) & , \text{ If } x \neq 0 \\ 0, & \text{ If } x = 0 \end{cases}$ **[AIEEE 2011, II,(4, –1), 120]**

Statement - 1 : F(x) is continuous on R.
Statement - 2 : $f_1(x)$ and $f_2(x)$ are continuous on R.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

7. If function f(x) is differentiable at x = a, then $\lim_{x\to a} \frac{x^2 f(a) - a^2 f(x)}{x-a}$ is : **[AIEEE 2011, II,(4, –1), 120]**

(1) $-a^2f'(a)$ (2) $af(a) - a^2f'(a)$ (3) $2af(a) - a^2f'(a)$ (4) $2af(a) + a^2f'(a)$

8. If f : R → R is a function defined by f(x) = [x] cos $\left(\frac{2x-1}{2}\right)\pi$, where[x] denotes the greatest integer function, then f is : **[AIEEE- 2012, (4, –1), 120]**
(1) continuous for every real x.
(2) discontinuous only at x = 0.
(3) discontinuous only at non-zero integral values of x.
(4) continuous only at x = 0.

9. Consider the function, f(x) = |x – 2| + |x – 5|, x ∈ R . **[AIEEE- 2012, (4, –1), 120]**

Statement-1 : $f'(4) = 0$

Statement-2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5).

(1) Statement-1 is false, Statement-2 is true.

(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.

(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.

(4) Statement-1 is true, statement-2 is false.

10. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[AIEEE - 2013, (4, –1),360]**

(1) $-\dfrac{1}{4}$ (2) $\dfrac{1}{2}$ (3) 1 (4) 2

11. $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$ is equal to : **[JEE(Main) 2014, (4, – 1), 120]**

(1) $-\pi$ (2) π (3) $\pi/2$ (4) 1

12. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[JEE(Main) 2015, (4, – 1), 120]**

(1) 4 (2) 3 (3) 2 (4) $\dfrac{1}{2}$

13. If the function g(x) = $\begin{cases} k\sqrt{x+1} & , \quad 0\le x\le 3 \\ mx+2 & , \quad 3<x\le 5\end{cases}$ is differentiable, then the value of k+ m is;

[JEE(Main) 2015, (4, – 1), 120]

(1) 2 (2) $\dfrac{16}{5}$ (3) $\dfrac{10}{3}$ (4) 4

14. Let p = $\lim\limits_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to: **[JEE(Main) 2016, (4, – 1), 120]**

(1) 1 (2) $\dfrac{1}{2}$ (3) $\dfrac{1}{4}$ (4) 2

15. For $x \in R$, f(x) = |log2 – sinx| and g(x) = f(f(x)), then **[JEE(Main) 2016, (4, – 1), 120]**

(1) $g'(0) = \cos(\log 2)$
(2) $g'(0) = -\cos(\log 2)$
(3) g is differentiable at x = 0 and $g'(0) = -\sin(\log 2)$
(4) g is not differentiable at x = 0

16. $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$ equals **[JEE(Main) 2017, (4, – 1), 120]**

(1) $\dfrac{1}{24}$ (2) $\dfrac{1}{16}$ (3) $\dfrac{1}{8}$ (4) $\dfrac{1}{4}$

17. For each $t\in R$ let [t] be the greatest integer less than or equal to t. Then $\lim\limits_{x\to 0^+} x\left(\left[\dfrac{1}{x}\right]+\left[\dfrac{2}{x}\right]+......+\left[\dfrac{15}{x}\right]\right)$

[JEE(Main) 2018, (4, – 1), 120]

(1) is equal to 120 (2) does not exist (in R) (3) is equal to 0 (4) is equal to 15

18. Let S = {$t \in R$: f(x) = $|x-\pi|.(e^{|x|}-1)\sin|x|$ is not differentiable at t.} Then the set S is equal to :
[JEE(Main) 2018, (4, – 1), 120]

(1) $\{\pi\}$ (2) $\{0, \pi\}$ (3) ϕ (an empty set) (4) {0}

19. $\lim_{y\to 0} \frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ **[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]**

(1) exists and equals $\frac{1}{2\sqrt{2}}$
(2) exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$
(3) exists and equals $\frac{1}{4\sqrt{2}}$
(4) does not exist

20. For each $t\in R$, let $[t]$ be the greatest integer less than or equal to t. Then ,

$\lim_{x\to 1+} \frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$ **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

(1) does not exist (2) equals 1 (3) equals –1 (4) equals 0

21. Let f(x) = $\begin{cases} -1, & -2 \le x < 0 \\ x^2-1, & 0 \le x \le 2 \end{cases}$ and g(x) = |f(x)| + f(|x|). Then, in the interval (–2, 2), g is :

(1) not differentiable at two point
(2) not continuous
(3) not differentiable at one point
(4) differentiable at all points

[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]

Answers

EXERCISE - 1

PART – I

Section (A) :

A-1. (i) Limit does not exist (ii) 3 (iii) 3 (iv) 3 (v) 3

A-2. (i) $2 + \sin 2$ (ii) $\tan 3 - 2^3$ (iii) $\frac{3}{4}\cos\frac{3}{4}$ (iv) 5^5 (v) $\frac{e}{\sin 1}$

A-3. (i) 0 (ii) Limit does not exist (iii) Limit does not exist (iv) 0

A-4. (i) Limit does not exist (ii) $\lambda = -2$ **A-5.** 6

A-6. (i) No (ii) No (iii) Yes, ∞^0 form (iv) No

SECTION (B) :

B-1. (i) $-\frac{3}{2}$ (ii) $\frac{12}{19}$ (iii) $\frac{2}{3\sqrt{3}}$

B-2. (i) $\frac{16}{25}$ (ii) 2 (iii) $\frac{1}{3}$ (iv) $2a\sin a + a^2\cos a$

(v) $(b - a)$ (vi) $2e^2$ (vii) $\frac{3}{\ell n 3}$ (viii) $\frac{1}{2}$ (ix) 5

(x) limit does not exist (xi) $-\frac{9}{4}\ \ell n\ \frac{4}{e}$

B-3. (i) $\frac{1}{2}$ (ii) 1 (iii) ∞ (iv) $-\frac{1}{\pi}$ **B-4.** (i) 0 (ii) $\frac{5}{2}(a+2)^{\frac{3}{2}}$ (iii) 0 (iv) 5/2

B-5. (i) $-\frac{2}{25}$ (ii) $\frac{1}{3}$ **B-6** $a = 2, b = 1, c = -1$ and limit $= -\frac{1}{3}$

B-7. (i) $a = -\frac{1}{2}$, $b = 1$ (ii) $a = 2, b \in R, c = 5, d \in R$ (iii) $a = 3, b = 12, c = 9$

B-8. $\frac{1}{2}$ **B10.** $2(\sec^2 a)\tan a$

SECTION (C) :

C-1 (i) 1 (ii) 1 (iii) 0 (iv) 0

C-2. (i) e^{-1} (ii) 0 (iii) $e^{-\frac{2}{\pi}}$ (iv) e^2 **C-3** $a + b = 0$ and $bc = 3$

C-4. (i) $\lim\limits_{x\to\infty} \dfrac{x\ \ \ell n\left(1+\dfrac{\ell n\ \ x}{x}\right)}{\ell n\ \ x}$ 1 (ii) 1 **C-5.** $\frac{x}{3}$ **C-6.** $\{-1, 0, 1\}$

Section (D) :

D-1. $a = -\frac{3}{2}$, $b \neq 0$, $c = \frac{1}{2}$ **D-2.** $a = \frac{1}{2}$, $b = 4$

D-3. (i) continuous at x = 1 (ii) continuous (iii) discontinuous (iv) continuous at x = 1, 2

D-4. (a) –2, 2, 3 (b) K = 5 (c) even

D-5. A = –4, B = 5, f(0) = 1 **D6.** Continuous at x = 1 but discontinuous at x = 2

Section (E) :

E-1. (i) $x \in R - \{2, 3\}$ (ii) $x \in R - \{-1, 1\}$ (iii) $x \in R$ (iv) $x \in R - \{(2n + 1), n \in I\}$

E-2. discontinuous at all integral values in [– 2, 2] **E-3.** discontinuous at $n\pi \pm \frac{\pi}{4}$, $(2n + 1)\frac{\pi}{2}$, $n \in I$

E-4. $g(x) = 2 + x$; $0 \le x \le 1$,
$= 2 - x$; $1 < x \le 2$,
$= 4 - x$; $2 < x \le 3$,
g is discontinuous at x = 1 & x = 2 **E-5.** $-\frac{7}{3}$, – 2, 0 **E-7_.** 2

Section (F) :

F-1. continuous at both points but differentiable only at x = 2

F-2. continuous but not differentiable at x = 0; differentiable & continuous at $x = \pi/2$

F-5. not differentiable at x = 0 **F-6.** a = 1/2, b = 3/2

Section (G) :

G-1. f is not derivable at all integral values in $-1 < x \le 3$

G-2. f is continuous but not derivable at x = 1/2, f is neither differentiable nor continuous at x = 1 & x = 2

G-3. discontinuous and non-differentiable at –1, 0, 1, 3, 4

G-4. Differentiable in [–2, 2]

G-5_. Continuos everwhere in (0, 3) but non differentiable at x= 2

Section (H) :

H-1. 56 **H-2.** – 2 **H-3.** $f(x) = e^{xf'(0)}$ $\forall x \in R$ **H-4.** – 3

PART – II

Section (A) :

A-1. (D) **A-2.** (A) **A-3.** (C) **A-4.** (C)

SECTION (B) :

B-1.	(C)	**B-2.**	(B)	**B-3.**	(D)	**B-4.**	(D)	**B-5.**	(D)	**B-6.**	(A)	**B-7.**	(D)
B-8.	(B)	**B-9.**	(B)	**B-10.**	(C)	**B-11.**	(B)	**B-12.**	(D)	**B-13.**	(B)	**B-14.**	(C)
B-15.	(A)	**B16.**	(A)	**B-17.**	(A)	**B-18.**	(C)	**B-19.**	(B)	**B-20_.**	(B)		

SECTION (C) :

C-1.	(A)	**C-2.**	(A)	**C-3.**	(B)	**C-4.**	(C)	**C-5.**	(A)	**C-6.**	(C)	**C-7.**	(B)
C-8.	(D)												

Section (D) :

D-1.	(A)	**D-2.**	(D)	**D3.**	(D)	**D-4_.**	(B)

Section (E) :

E-1.	(B)	**E-2.**	(C)	**E-3.**	(C)	**E-4.**	(B)

Section (F) :

F-1.	(B)	**F-2.**	(B)	**F-3.**	(B)	**F-4.**	(B)	**F-5.**	(D)	**F-6.**	(B)	**F-7.**	(B)

Section (G) :

G-1.	(A)	**G-2.**	(B)	**G-3.**	(D)	**G-4.**	(B)	**G-5.**	(C)	**G-6.**	(A)	**G-7.**	(C)
G-8.	(A)												

Section (H) :

H-1.	(D)	**H-2.**	(B)	**H-3.**	(C)	**H-4.**	(D)	**H-5.**	(B)	**H-6.**	(D)

PART – III

1. (A) $\rightarrow$ (p, r, s), (B) $\rightarrow$ (p, r, s), (C) $\rightarrow$ (q, r, s), (D) $\rightarrow$ (r, s)
2. (A) $\rightarrow$ (p, q, r), (B) $\rightarrow$ (p, r, s), (C) $\rightarrow$ (p, r, s), (D) $\rightarrow$ (p, r, s)

EXERCISE - 2

PART – I

1.	(B)	**2.**	(C)	**3.**	(C)	**4.**	(A)	**5.**	(B)	**6.**	(B)	**7.**	(C)
8.	(C)	**9.**	(B)	**10.**	(B)	**11.**	(C)	**12.**	(B)	**13_.**	(B)	**14.**	(C)
15.	(D)	**16.**	(B)	**17.**	(B)	**18.**	(C)	**19.**	(B)	**20.**	(D)	**21.**	(B)
22.	(C)	**23.**	(C)	**24.**	(D)								

PART – II

1.	2	**2.**	1	**3.**	2	**4.**	1	**5.**	0	**6.**	1	**7.**	2
8.	1	**9.**	11	**10.**	37	**11.**	20	**12.**	12	**13.**	1	**14.**	21
15.	11	**16.**	99	**17.**	4	**18.**	26	**19.**	36	**20.**	16	**21.**	7
22.	4	**23.**	7	**24.**	0	**25.**	3	**26.**	12	**27.**	11	**28.**	10

PART – III

1.	(ABC)	**2.**	(AB)	**3.**	(AD)	**4.**	(ABCD)	**5.**	(BC)	**6.**	(ABC)
7.	(ABCD)	**8.**	(BCD)	**9.**	(AB)	**10.**	(AD)	**11.**	(ABCD)	**12.**	(AC)
13.	(ABC)	**14.**	(ABCD)	**15.**	(BCD)	**16.**	(AD)	**17.**	(BCD)	**18.**	(ABD)
19.	(BCD)	**20.**	(ABC)	**21.**	(CD)	**22.**	(ABC)	**23.**	(ABC)	**24.**	(ABD)
25.	(BD)	**26.**	(AC)	**27.**	(AB)	**28.**	(BCD)	**H-9.**	(ABCD)		

PART – IV

1.	(A)	**2.**	(D)	**3.**	(A)	**4.**	(C)	**5.**	(D)	**6.**	(C)
7.	(C)	**8.**	(C)	**9.**	(C)	**10.**	(B)	**11.**	(C)	**12.**	(B)

EXERCISE - 3

PART – I

1*.	(AC)	**2.**	(D)	**3.**	(B, C, D) or (B,C)			**4*.**	(ABCD)	**5.**	(A)	**6.**	(B)
7.	(B)	**8.**	(B)	**9*.**	(BD)	**10*.**	(AD)	**11.**	0	**12.**	3	**13.**	(D)
14*.	(AD)	**15*.**	(ABC)	**16.**	2	**17.**	(B,C)	**18.**	7	**19.**	(B,C)	**20.**	(A,B)
21.	(ABD)	**22*.**	(CD)	**23.**	(D)	**24.**	(D)						

PART - II

1.	(3)	**2.**	(4)	**3.**	(1)	**4.**	(4)	**5.**	(3)	**6.**	(3)	**7.**	(3)
8.	(1)	**9.**	(3)	**10.**	(4)	**11.**	(2)	**12.**	(3)	**13.**	(1)	**14.**	(2)
15.	(1)	**16.**	(2)	**17.**	(1)	**18.**	(3)	**19.**	(3)	**20.**	(4)	**21.**	(3)

High Level Problems (HLP)

1. Evaluate : $\lim\limits_{x\to 0}\dfrac{1-\cos(a_1x)\ .\ \cos\ (a_2x).\ \cos\ (a_3x).......\cos(a_nx)}{x^2}$, where $a_1, a_2, a_3, ... , a_n \in R$.

2. $f_1(x) = \dfrac{x}{2}+10$

$f_n(x) = f_1(f_{n-1}(x)) \qquad n \geq 2$

then evaluate $\lim\limits_{n\to\infty} f_n(x)$

3. Let f : R → R be a real function. The function f is derivable and there exists $n \in N$ and $p \in R$ such that $\lim\limits_{x\to\infty} x^n f(x) = p$, then evaluate $\lim\limits_{x\to\infty} (x^{n+1}.f'(x))$.

4. Let $<x_n>$ denotes a sequence, $x_1 = 1$, $x_{n+1} = \sqrt{x_n^2+1}$, then evaluate $\lim\limits_{n\to\infty}\left(\dfrac{x_{n+1}}{x_n}\right)^n$

5. Evaluate $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+1}+\dfrac{2}{n^2+2}+\dfrac{3}{n^2+3}+.......+\dfrac{n}{n^2+n}\right)$

6. Evaluate : $\lim\limits_{x\to\infty} x^3\left\{\sqrt{x^2+\sqrt{1+x^4}}-x\sqrt{2}\right\}$

7. Evaluate $\lim\limits_{x\to\infty}\dfrac{\log_e(\log_e x)}{e^{\sqrt{x}}}$

8. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\log_e(\sin\ (4m+1)x)}{\log_e(\sin(4n+1)x)}$, where m,n Z

9. $f(x) = \sum\limits_{r=1}^{n}\tan\dfrac{x}{2^r}.\sec\dfrac{x}{2^{r-1}}$ r, n ∈ N

$$g(x) = \begin{cases} \lim\limits_{n\to\infty}\dfrac{\log_e\left(f(x)+\tan\dfrac{x}{2^n}\right)-\left(f(x)+\tan\dfrac{x}{2^n}\right)^n\left[\sin\left(\tan\dfrac{x}{2}\right)\right]}{1+\left(f(x)+\tan\dfrac{x}{2^n}\right)^n} & x \neq \dfrac{\pi}{4} \\ k & x = \dfrac{\pi}{4} \end{cases}$$

where [ ] denotes greatest integer function and domain of g(x) is $\left(0, \dfrac{\pi}{2}\right)$

find 'k' for which g(x) is continuous at x = π/4

10. Evaluate $\lim\limits_{x\to(e^{-1})^+}\dfrac{e^{\frac{\ell n(1+\ell nx)}{x}}}{x-e^{-1}}$

11. Let $P_n = \dfrac{2^3-1}{2^3+1}.\dfrac{3^3-1}{3^3+1}$ .... $\dfrac{n^3-1}{n^3+1}$ Prove that $\lim\limits_{n\to\infty} P_n = \dfrac{2}{3}$.

12. Verify the following limits

(i) $\lim\limits_{x \to 0} \left(\dfrac{(1+x)^{\frac{1}{x}}}{e} \right)^{\frac{1}{x}} = e^{-\frac{1}{2}}$ (ii) $\lim\limits_{x \to 0} \left[\sin^2 \left(\dfrac{\pi}{2 - ax} \right) \right]^{\sec^2\left(\frac{\pi}{2-bx}\right)} = e^{-\frac{a^2}{b^2}}$

13. $f(x) = \lim\limits_{n \to \infty} \dfrac{x \sin^n x}{\sin^n x + 1}$. Find domain and range of f(x), where $n \in N$.

14. Evaluate $\lim\limits_{x \to 0} \left(\dfrac{a_1^{\,x} + a_2^{\,x}}{b_1^{\,x} + b_2^{\,x}} \right)^{\frac{1}{x}}$ where a_1, a_2, b_1 and b_2 are positive numbers

15. Evaluate $\lim\limits_{x \to 1} \left(\dfrac{p}{1-x^p} - \dfrac{q}{1-x^q} \right)$ where $p, q \in N$

16. If $f(n, \theta) = \prod\limits_{r=1}^{n} \left(1 - \tan^2 \dfrac{\theta}{2^r} \right)$ and $\lim\limits_{n \to \infty} f(n, \theta) = g(\theta)$, then find the value $\lim\limits_{\theta \to 0}$ of $g(\theta)$

17. Find the value of $\lim\limits_{x \to \pi} \dfrac{1}{(x-\pi)} \left(\sqrt{\dfrac{4\cos^2 x}{2 + \cos x}} - 2 \right)$

18. $\lim\limits_{x \to \infty} x^a \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x} \right) = \lambda$, $\lambda \neq 0$ then find the value of $a + \lambda$

19. Discuss the continuity of the function $f(x) = \lim\limits_{n \to \infty} \dfrac{(1+\sin x)^n + \log x}{2 + (1 + \sin x)^n}$

20. If $g(x) = \begin{cases} \dfrac{1 - a^x + xa^x . \ \ell n a}{x^2 a^x} & , \ x < 0 \\ k & , \ x = 0 \\ \dfrac{(2a)^x - x \ \ \ell n \ \ 2a - 1}{x^2} & , \ x > 0 \end{cases}$

(where a > 0) , then find 'a' and g(0) so that g(x) is continuous at x = 0.

21. $f(x) = \begin{cases} \dfrac{\cos^{-1}(2x\sqrt{1-x^2})}{x - \dfrac{1}{\sqrt{2}}} & x \neq \dfrac{1}{\sqrt{2}} \\ k & x = \dfrac{1}{\sqrt{2}} \end{cases}$

Then find 'k' if possible for which function is continuous at $x = \dfrac{1}{\sqrt{2}}$

22. Find the value of f(0) so that the function

$f(x) = \dfrac{\cos^{-1}(1 - \{x\}^2)\sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3}$, $x \neq 0$

({x} denotes fractional part of x) becomes continuous at x = 0

23. Let f be a continuous function on R such that $f\left(\dfrac{1}{4x}\right) = \left(\sin e^x\right) \ e^{-x^2} + \dfrac{x^2}{x^2+1}$, then find the value of f(0).

24. Examine the continuity at x = 0 of the sum function of the infinite series:
$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + ...\infty$.

25. If f(x) is continuous in [a, b] such that f(a) = b and f(b) = a, then prove that there exists at least one c $\in$ (a, b) such that f(c) = c.

26. If f(x . y) = f(x). f(y) for all x, y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except possibly at x = 0. Given f(1) $\neq$ 0.

27. $g(x) = \lim_{n \to \infty} \frac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \neq 1$

$g(1) = \lim_{x \to 1} \frac{\sin^2(\pi 2^x)}{\log_e \sec(\pi 2^x)}$ be a continous function at x = 1, then find the value of 4g(1) + 2 f(1) – h(1), assume that f(x)and h(x) are continuous at x = 1

28. If f(x) = $x^2 - 2|x|$, then test the derivability of g(x) in the interval [–2, 3], where
$$g(x) = \begin{cases} \min.\{f(t);\ -2 \le t \le x\}, & -2 \le x < 0 \\ \max.\ \{f(t);\ 0 \le t \le x\}, & 0 \le x \le 3 \end{cases}$$

29. Discuss the continuity and differentiability of f(x) = [x] + $\{x\}^2$ and also draw its graph. Where [.] and {.} denote the greatest integer function and fractional part function respectively.

30. Discuss the continuity and differentiability of the function f(x) = $\begin{cases} \frac{x}{1+|x|} & ; |x| \ge 1 \\ \frac{x}{1-|x|} & ; |x| < 1 \end{cases}$

31. Discuss the continuity and differentiability of the function f(x) = $\left\{\lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{\cos^{2n}(m!\pi x) - 1}{\cos^{2n}(m!\pi x) + 1}\right)\right\}$,

(where m, n $\in$ N) at rational and irrational points.

32. Given f(x) = $\cos^{-1}\left(\text{sgn}\left(\frac{2[x]}{3x - [x]}\right)\right)$, where sgn ( ) denotes the signum function and [.] denotes the greatest integer function. Discuss the continuity and differentiability of f (x) at x = ± 1.

33. Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.
$f(x) = x \sin\frac{1}{x} \sin\frac{1}{x \sin\frac{1}{x}}$ where x $\neq$ 0, x $\neq \frac{1}{r\pi}$ & f(0) = f (1/rπ) = 0, r = 1, 2, 3,.......

34. Let f be a function such that f(xy) = f(x) . f(y) $\forall$ x > 0 , y > 0 . If f(1 + x) = 1 + x (1+g(x)) , where $\lim_{x\to 0} g(x) = 0$. Find $\int \frac{f(x)}{f'(x)} dx$

35. Let f : $R^+ \to R$ satisfies the equation
$f(xy) = e^{xy - x - y}(e^y f(x) + e^x f(y))\ \forall\ x, y \in R^+$
If $f'(1) = e$, then find f(x) .

36. Let f(x) be a real valued function not identically zero such that
$f(x + y^3) = f(x) + (f(y))^3\ \forall\ x, y \in R$ and $f'(0) \geq 0$, then find f(10)

37. Determine a function f satisfying the functional relation $f(x) + f\left(\dfrac{1}{1-x}\right) = \dfrac{2(1-2x)}{x(1-x)}$.

38. If $f(x) + f(y) + f(xy) = 2 + f(x) \cdot f(y)$, for all real values of x and y and f (x) is a polynomial function with $f(4) = 17$ and $f(1) \neq 1$, then find the value of f (5) .

39. If $|f(p + q) - f(q)| \leq \dfrac{p}{q}$ for all p and $q \in Q$ & $q \neq 0$, show that $\sum\limits_{i=1}^{k} |f(2^k) - f(2^i)| \leq \dfrac{k(k-1)}{2}$

40. The function $f : R \to R$ satisfies $x + f(x) = f(f(x))$ for every $x \in R$. Find all solutions of the equation $f(f(x)) = 0$.

41. If $2f(x) = f(xy) + f(x/y)\ \forall\ x, y \in R^+$, $f(1) = 0$ and $f'(1) = 1$, find f(x).

42. If $f(x \times f(y)) = \dfrac{f(x)}{y}\ \forall\ x, y \in R$, $y \neq 0$, then prove that $f(x) \cdot f\left(\dfrac{1}{x}\right) = 1$

43. Find the period of f(x) satisfying the condition :

(i) $f(x + p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$, $p > 0$

(ii) $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$

44. Let f(x) is defined only for $x \in (0, 5)$ and defined as $f^2(x) = 1\ \forall\ x \in (0, 5)$. Function f(x) is continuous for all $x \in (0, 5) - \{1, 2, 3, 4\}$ (at x = 1, 2, 3, 4 f(x) may or may not be continuous). Find the number of possible function f(x) if it is discontinuous at

(i) One integral points in (0, 5)

(ii) two integral points in (0, 5)

(iii) three integral points in (0, 5)

(iv) four integral points in (0, 5)

45. Let f(x) is double differentiable function everywhere such that f(x) = x has 3 distinct root α, β and $\gamma(\alpha < \beta < \gamma)$. $h(x) = \lim\limits_{n\to\infty}\left(\underbrace{f(f(....(f(x))))}_{n \text{ times}}\right)$

(i) If $f''(x) > 0\ \forall\ x \in (-\infty, \beta)$ and $f''(x) < 0\ \forall\ x \in (\beta, \infty]$ and $f''(\beta) = 0$, then find h(x)

(ii) If $f(x) \geq x\ \forall\ x \in (-\infty, \alpha] \cup [\gamma, \infty)$ and $f(x) \leq x\ \forall\ x \in [\beta, \gamma]$ then find h(x)

HLP Answers

1. $\frac{1}{2}\sum_{i=1}^{n} a_i^2$ **2.** 20 **3.** $-np$ **4.** $\sqrt{e}$ **5.** $\frac{1}{2}$ **6.** $\frac{1}{4\sqrt{2}}$

7. 0 **8.** $\frac{(4m+1)^2}{(4n+1)^2}$ **9.** $k = 0$ **10.** 0

13. Domain = $R - \left\{2k\pi - \frac{\pi}{2},\ k \in Z\right\}$, Range = $\{0\} \cup \left\{k\pi + \frac{\pi}{4},\ k \in Z\right\}$

14. $\sqrt{\frac{a_1a_2}{b_1b_2}}$ **15.** $\frac{p-q}{2}$ **16.** 1 **17.** 0 **18.** $\frac{13}{9}$

19. f(x) is discontinuous at integral multiples of π **20.** $\frac{1}{\sqrt{2}}, \frac{1}{8}(\ell n\, 2)^2$

22. no value of f(0) **23.** 1 **24.** Discontinuous **27.** 5

28. discontinuous at x = 0 and not differentiable at x = 0, 2

29. f(x) is continuous and non-differentiable for integral points

30. At x = 0 differentiable and at x = ±1 discontinuous

31. discontinuous and non-differentiable

32. f is continuous & derivable at x = –1 but f is neither continuous nor derivable at x = 1

33. continuous in $0 \le x \le 1$ & not differentiable at x = 0

34. $\frac{x^2}{2} + c$ **35.** $f(x) = e^x \ell n|x|$ **36.** f(10) = 10 **37.** $\frac{x+1}{x-1}$ **38.** 26

41. f(x) = log(x) **43.** (i) 2p (ii) 8 44. (i) 24 (ii) 108 (iii) 216 (iv) 162

45. (i) $h(x) = \begin{cases} \alpha & , & x \in (-\infty, \beta) \\ \beta & , & x = \beta \\ \gamma & , & x \in (\beta, \infty) \end{cases}$ (ii) $h(x) = \begin{cases} \alpha & , & x \in (-\infty, \alpha] \\ \beta & , & (\alpha, \gamma) \\ \gamma & , & [\gamma, \infty) \end{cases}$