Journal Pre-proof

Multi-Distribution Multi-Commodity Multistate Flow Network Model and its Reliability Evaluation Algorithm

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PII: S0951-8320(18)31153-0
DOI: https://doi.org/10.1016/j.ress.2019.106668
Reference: RESS 106668

To appear in: *Reliability Engineering and System Safety*

Received date: 21 September 2018
Revised date: 2 September 2019
Accepted date: 18 September 2019

Please cite this article as: Zhifeng Hao , Yeh Wei-Chang , J Ming , Jing Wang , Multi-Distribution Multi-Commodity Multistate Flow Network Model and its Reliability Evaluation Algorithm, *Reliability Engineering and System Safety* (2019), doi: https://doi.org/10.1016/j.ress.2019.106668

Highlights

- In the WMMFN, all commodities share the same state distribution.
- All flows are simply summed up for calculating the WMMFN reliability.
- A novel multi-distribution MMFN (MMMFN) is proposed to generalize the WMMFN.
- The novel MMMFN allows each commodity to have its own state distribution.
- A new path-based algorithm is presented for calculating the MMMFN reliability.

Multi-Distribution Multi-Commodity Multistate Flow Network Model and its Reliability Evaluation Algorithm

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Abstract—Multi-state flow networks (MFNs), which allow components to have various states and satisfy the flow conservation law, are widely used to model current real-life networks. The multi-commodity MFN (MMFN) is an extension of the MFN that considers more than one commodity (e.g., material, media, resources, products, and items) in the MFN. Network reliability is an important tool for the evaluation of the performance of various networks. The applications and studies of MMFNs are now more important than ever before. However, to date, only one MMFN has been developed, namely the weighted MMFN (WMMFN). In each component of the WMMFN, all commodities share the same state distribution (the states and their occurrence probabilities); the flows of different commodities are simply summed up for the calculation of the WMMFN reliability. Thus, in this study, a novel multi-distribution MMFN (MMMFN) is proposed that allows each commodity to have its own state distribution on components to complement the WMMFN. A new path-based algorithm is presented for the calculation of the proposed novel MMMFN reliability. The correctness and time complexity of the proposed path-based algorithm will be analyzed and proven. Numerical cases are adopted to demonstrate the proposed MMMFN and the proposed algorithm.

Key Words— Network Reliability, Multi-distribution, Multi-commodity, Multistate, Path-based Algorithm.

Acronyms

MFN	multi-state flow network
MMFN	multi-commodity MFN
WMMFN	weighted MMFN
MMMFN	multi-distribution MMFN
MP	minimal path
DFS	depth-first search
IE	inclusion–exclusion method
SDP	sum-of-disjoint product algorithm

1. INTRODUCTION

Network reliability is a well-known tool to plan, design, manage, evaluate, and control numerical networks for improving our quality of life. Moreover, it has been employed widely in several real-life applications and networks, [1–8, 13–15] such as oil/gas production systems [1], transportation systems [1, 2], communication systems [3], distribution computing systems [4], the Internet of Things [5], grid and cloud computing [6], and wireless sensor networks [7, 8]. Thus, network reliability plays an important role in our modern society. All aforementioned networks can be modeled as flow networks in which all flows in each node must satisfy the flow conservation law, i.e., the amount of flow into the node is equal to the amount of flow out of it.

The weighted multi-commodity multi-state flow network (WMMFN) model, which was first proposed by Hu and was extended by Yeh [9, 10] and Lin [11], allows various commodities, such as material, media, resources, products, and items, to transmit from the source node to the sink node simultaneously [9–11].

Let $G(V, E, \Psi)$ be a WMMFN with q commodities. The set of perfect nodes is $V = \{1, 2, \ldots, n\}$, E is the set of arcs, Ψ is the state distribution, in which the states of different arcs are statistically independent, Node 1 is the source node [12], and Node n is the sink node; $|V| = n$ is the number of nodes and $|E| = m$ is the number of arcs. For example, in Fig. 1, $V = \{1, 2, \ldots, 4\}$, $E = \{a_1, a_2, \ldots, a_5\}$, $n = |V| = 4$, and $|E| = 5$. Table 1 summarizes the state distribution of each arc for $q = 3$. It should be noted that State 3 of arc a_3 is not listed in Table 1 because its probability of occurrence is zero.

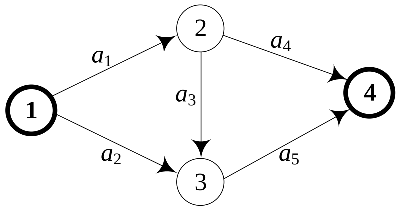


Figure 1. Example of a flow network

Table 1. State distribution Ψ of **Fig. 1**.

i	$w_{1,i}$	$w_{2,i}$	$w_{3,i}$	State	Probability	i	$w_{1,i}$	$w_{2,i}$	$w_{3,i}$	State	Probability
1	1.1	0.9	1.0	3	0.60	4	1.0	0.9	0.8	3	0.55
				2	0.20					2	0.25
				1	0.10					1	0.15
				0	0.10					0	0.05
2	1.2	1.1	1.0	4	0.50	5	1.2	1.0	0.8	4	
				3	0.20					3	0.80
				2	0.15					2	0.10
				1	0.10					1	0.05
				0	0.05					0	0.05
3	1.1	0.9	1.0	4	0.40						
				2	0.35						
				1	0.20						
				0	0.05						

Weight $w_{k,i}$ is utilized to represent the amount of capacity along arc a_k consumed by each unit of Commodity i flowing through a_k [9–11], e.g., each unit of Commodity 2 consumes $w_{2,3}$ = 0.9 units of the capacity of a_3 in Table 1.

The WMMFN reliability is the success probability that the minimal required amount of each commodity is able to transmit from Nodes 1 to n simultaneously [9–11]. Let $D = (d_1, d_2, \ldots, d_q)$ be the given demand vector, where d_q is the minimal demand assigned for Commodity k for $k = 1, 2, \ldots, q$. The definition of the MMFN reliability, R_D, can be expressed as follows:

$$R_D = \Pr(\{X \mid \text{for all state vector } X = \sum_{k=1}^{q} X_k \text{ with } d_k \le F_k(G(X_k)) \text{ for all } k = 1, 2, \ldots, q\}), \tag{1}$$

where X_k is a vector (the d-MP candidates) of capacities (i.e., state vector) of all arcs for Commodity k in the network; $G(X_k)$ is equal to $G(V, E, \Psi)$, except that the state in the i^{th} arc of $G(X_k)$ is fixed and equal to that of X_k; F_k is the maximum flow in $G(X_k)$ for Commodity k; and X is the state vector of all arcs in this network [8–12, 17–27].

A D-MP p^* is a D-MP candidate such that no D-MP candidate is less than p^* [10]. In calculating the reliability of WMMFN, all related algorithms are path-based algorithms; hence, they first need to search for all D-MP candidates and then filter out all D-MPs from these candidates. For example, in Fig. 1, based on Table 1 and considering Commodities 1 and 2 only, $P = (3, 0, 0, 3, 0)$ is a (1, 2)-MP and a (2, 1)-MP because $P = (2, 0, 0, 2, 0) + (1, 0, 0, 1, 0) = (1, 0, 0, 1, 0) + (2, 0, 0, 2, 0)$; (2, 0, 0, 2, 0) and (1, 0, 0, 1, 0) are a 2-MP and a 1-MP of Commodities 1 and 2, respectively.

The purpose of all path-based algorithms is to search for all D-MPs efficiently. After using any path-based algorithm, assume that $p_1, p_2, \ldots, p_\pi$ are all D-MPs in the WMMFN [8–12, 17–27]. We have that p_i is not less than or equal to p_j for all $i, j = 1, 2, \ldots, \pi$. Let

$$\Pr(p_i) = \Pr(\{X \mid \text{for all } X \text{ with } p_i \le X\}). \tag{2}$$

Eq. (1) can be simplified and may be calculated in terms of $p_1, p_2, \ldots, p_\pi$ based on the inclusion–exclusion method (IE) [33] and the sum-of-disjoint product (SDP) algorithm [19, 34, 35] as follows:

$$R = \Pr\left(\bigcup_{i=1}^{\pi} p_i\right). \tag{3}$$

From Table 1, it is not easy and practical to assign the suitable weight for each commodity in each arc for WMMFN. Moreover, all flows of commodities need to be summed up to calculate the final reliability of the WMMFN, i.e., the differences between commodities are ignored. Hence, there is a need to develop a novel MMFN model to allow each commodity to have its own suitable state distribution to complement the current WMMFN. Thus, the motivation for this research work was to construct a novel model, referred to as the MMMFN, to enhance the MMFN by combining it with a new path-based algorithm to calculate the reliability of the novel MMMFN. This would complement the WMMFN by allowing each commodity to have its own state distribution on components.

2. PROPOSED NOVEL MMMFN

In this section, a new MMFN model, referred to as the MMMFN, will be proposed to overcome all the aforementioned disadvantages of the WMMFN.

In the proposed MMMFN, each commodity has its own state distribution without the need to share it with other commodities. Hence, the single-state value of one commodity is extended to a state vector to include all possible states of this commodity along an arc. All the states of each arc are also extended from a vector to a hyperspace with a dimension of $|S|^q$ at most, if the maximal state number of all commodities is $|S|$.

Let $S_i = (s_1, s_2, \ldots, s_q)$ be the state vector of arc a_i; the state of Commodity j is s_j for $j = 1, 2, \ldots, q$; and $\Pr(S_i)$ is the probability of S_i, i.e., the probability that Commodity 1 is in state s_1, Commodity 2 is in state s_2, …, and Commodity q is in state s_q in a_i. It should be noted that the sum of all probabilities of all states for each arc is still one.

Table 2. Hyperspace of the proposed MMMFN state distribution, Ψ, of Fig. 1.

Arc	i \ j	0	1	2	3	4
a_1	0	0.05	0.05	0.10	0.05	0.05
	1	0.05	0.10	0.05	0.05	
	2	0.10	0.05	0.05		
	3	0.20	0.05			
a_2	0	0.05	0.05	0.10	0.05	
	1	0.10		0.05	0.05	
	2	0.10	0.05	0.05		
	3	0.20	0.05			
	4	0.10				
a_3	0					0.05
	1			0.05	0.05	0.05
	2	0.10	0.10	0.05	0.10	
	3	0.15	0.10			
	4	0.20				
a_4	0		0.03	0.05	0.03	0.05
	1	0.05		0.02	0.05	0.03
	2	0.05	0.02	0.05	0.02	0.05
	3	0.10	0.05	0.02		0.03
	4	0.10	0.10	0.05	0.05	
a_5	0					0.05
	1				0.05	0.05
	2			0.05	0.05	0.10
	3		0.05	0.05		0.05
	4	0.10	0.20	0.10	0.10	

For example, let there be only two commodities in Fig. 1; the example state distribution of the proposed MMMFN is summarized in Table 2, in which an empty cell means that the corresponding probability is zero. In Table 2, for arc a_1, the probability is 0.05 if the state of Commodity 1 is two units and that of Commodity 2 is one unit. It should be noted that the concept of the multi-distribution was first proposed in [12] and applied to the MFN.

A multi-vector is a ($q \times m$)-tuple vector structure, and it was first proposed in [9] by redefining the D-MP P to be (X_1; X_2; …; X_q) rather than let $P = \sum_{k=1}^{q} X_k$, where for all $X_k = (x_{k,1}, x_{k,2},\ldots, x_{k,m})$ is one of d_k-MPs, $x_{k,j}$ is the flow value of Commodity k in arc a_j of X_k, $k = 1, 2, \ldots, q$ and m is the number of arcs. For example, the multi-vector form of $P = (3, 0, 0, 3, 0)$ is (2, 0, 0, 2, 0; 1, 0, 0, 1, 0) if it is a (2, 1)-MP or (1, 0, 0, 1, 0; 2, 0, 0, 2, 0) if it is a (1, 2)-MP. To distinguish the D-MP in a multi-vector structure of the proposed MMMFN from the D-MP in a m-tuple vector of the WMMFN, the former was renamed to D^*-MP.

The definition of the MMMFN reliability, R_{D^*}, is defined below:

$$R_{D^*} = \Pr(\{X \mid \text{for all multi-vector } X = (X_1; X_2; \ldots; X_q) \text{ with } d_k \le F_k(X_k) \text{ for all } k = 1, 2, \ldots, q\}). \quad (4)$$

Let D^*-MP $P = (X_1; X_2; \ldots; X_q)$ and $\Pr(P) = \Pr(\{X \mid \text{for all multi-vector } X \text{ with } P \le X\}$. The MMMFN reliability defined in Eq. (4) can be rewritten based on the proposed D^*-MP as follows (see Lemma 1 and its proof in Appendix):

$$R_{D^*} = \bigcup \Pr(P) \quad \text{for all } D^*\text{-MP } P.$$

Let $\Pr((x_{i,1}, x_{i,2}, \ldots, x_{i,q})) = \Pr(x_{i,1}, x_{i,2}, \ldots, x_{i,q})$ be the joint probability that the state of arc a_i is equal to $x_{i,k}$ for Commodity k for $i = 1, 2, \ldots, m$ and $k =1, 2, \ldots, q$, and let the accumulated state distribution be $\Pr^+(x_{i,1}, x_{i,2}, \ldots, x_{i,q}) = \sum_{x_{i,k} \le x^*_{i,k} \in \Psi \text{ for all } k} \Pr(x^*_{i,1}, x^*_{i,2}, \ldots, x^*_{i,q})$. The probability to send at least d_i units of Commodity i via D^*-MP P for $i = 1, 2, \ldots, q$ can be calculated for all D^*-MP P (see Lemma 2 and its proof in Appendix):

$$\Pr(P) = \prod_{i=1}^{m} \Pr^+(x_{i,1}, x_{i,2}, \ldots, x_{i,q}). \quad (5)$$

3. Proposed Path-Based Algorithm for Finding all D^*-MPs

From Section 2, it is necessary to find all D^*-MPs before calculating the MMMFN reliability. Here, a path-based algorithm is proposed to achieve the aforementioned goal for the novel MMMFN reliability problem. All efficient path-based algorithms (including the proposed algorithm) for the search of all d-MPs in MFNs, D-MPs, and the proposed D^*-MPs, are based on the flow-conservation mathematical model that was first proposed by Yeh in [26], which was also the first algorithm for searching all d-MPs without knowing all MPs in advance. The flow-conservation mathematical model is described as follows; its proof can be found in [26].

Theorem 1: Any state vector X is a d-MP candidate, $x_{i,j}$ is the state of arc $e_{i,j}$ in X for all $e_{i,j} \in E$, and $F(X) = d$, if and only if the following conditions are satisfied:

$$\sum_{j=1}^{n} x_{i,j} = \sum_{h=1}^{n} x_{h,i} \text{, for all } x_{i,j}, x_{h,i} \in E, i \neq 1 \text{ or } n, \tag{6}$$

$$\sum_{i=2}^{n} x_{1,i} = \sum_{j=1}^{n-1} x_{j,n} = d, \text{ for all } x_{1,i}, x_{j,n} \in E, \tag{7}$$

$$0 \leq x_{i,j} \leq Min\{d, \mathbf{D}_{\max}(e_{i,j})\} \text{ for each directed arc } e_{i,j} \in E. \tag{8}$$

In Theorem 1, Eqs. (6), (7), and (8) are based on the flow conversation law, the upper bound of the arc state, and the definition of the d-MP candidates, respectively. Based on Theorem 1, each D-MP candidate can be identified using the d_i-MP (candidate) in [26].

Based on the aforementioned flow-conservation mathematical model, the proposed algorithm searches for all d_k-MPs of Commodity k =1, 2, …, q. The depth-first search (DFS) is implemented to form the D^*-MP, e.g., $P = (x_{1,1}, x_{1,2}, \ldots, x_{1,m}; x_{2,1}, x_{2,2}, \ldots, x_{2,m}; \ldots; x_{m,1}, x_{m,2}, \ldots, x_{m,m})$, where $X_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,m})$ is one of the d_i-MPs for i = 1, 2, …, q. Then, the MMMFN reliability is calculated using the IE [32, 35] or the SDP [33-35] in terms of D^*-MPs.

The pseudo code of the proposed algorithm for the retrieval of all D^*-MPs is presented below; the related flowchart is shown in Fig. 2.

Input: An MMMFN $G(V, E, \Psi)$.

Output: All real D^*-MPs

STEP 0. Calculate the accumulated state distributions, find all d_i-MPs, set $P_i = \{p_{i,1}, p_{i,2}, \ldots, p_{i,\pi_i}\}$ based on Theorem 1 for $i = 1, 2, \ldots, q$, and let $\Omega = \emptyset$, $X_0 = 0$, and $i = 1$.

STEP 1. Let $j_i = 1$.

STEP 2. If $i < q$, let $i = i + 1$ and go to STEP 1.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1}; p_{2,j_2}; \ldots; p_{q,j_q})\}$.

STEP 4. If $j_i < \pi_i$, let $j_i = j_i + 1$ and go to STEP 2.

STEP 5. If $i > 0$, let $i = i - 1$ and go to STEP 1.

STEP 6. Calculate the MMMFN reliability based on Eq.(4) in terms of all found D^*-MPs using the IE or the SDP, unless it is necessary to replace all vectors with the multi-vectors.

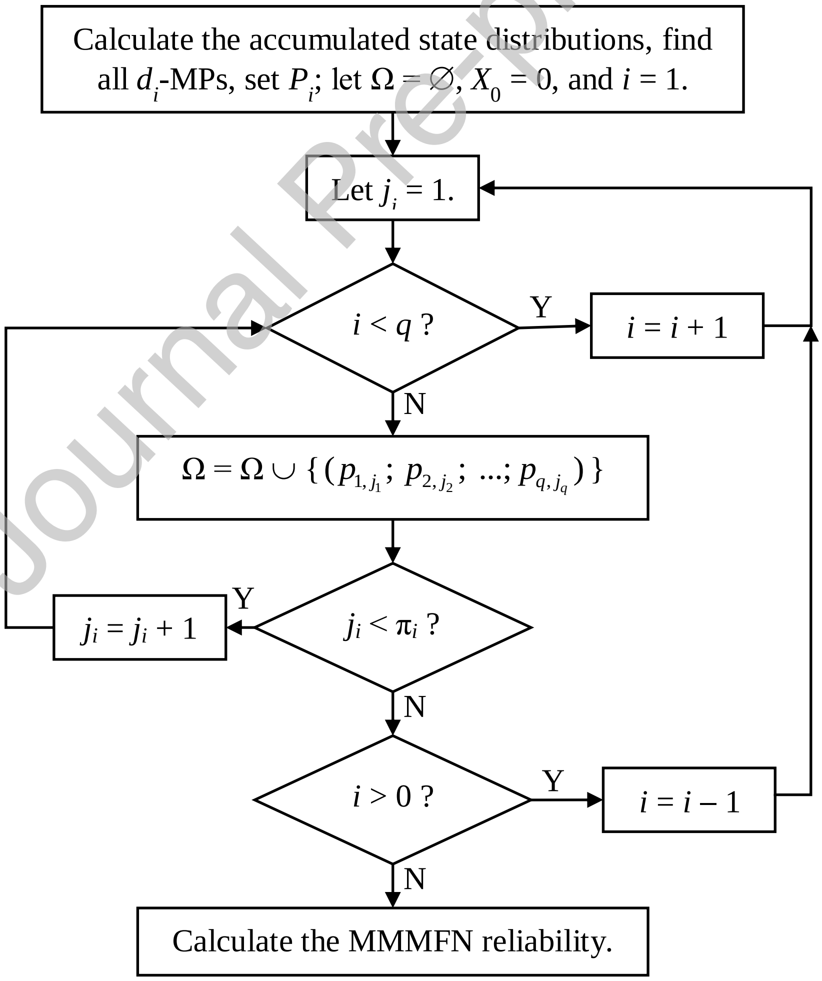


Figure 2. Flowchart of the proposed algorithm.

Before providing the proof of the correctness of the above steps, we have useful properties and results that any multi-vector $P = (X_1; X_2; \ldots; X_q)$ is a D^*-MP if and only if X_i is a d_i-MP in the MMMFN, for $i = 1, 2, \ldots, q$ (see Lemma 3 and its proof in Appendix).

From the above, all D^*-MPs can be found via the proposed algorithm. Furthermore, there is no need to verify whether a D^*-MP candidate is a D^*-MP; that is, the proposed algorithm finds all D^*-MPs directly. It should be noted that in the MFN and the WMMFN, all path-based algorithms are first required to search for all of the d-MP candidates or D-MP candidates and to then filter out the d-MPs and D-MPs from the d-MP and D-MP candidates [9–11, 20–31], respectively.

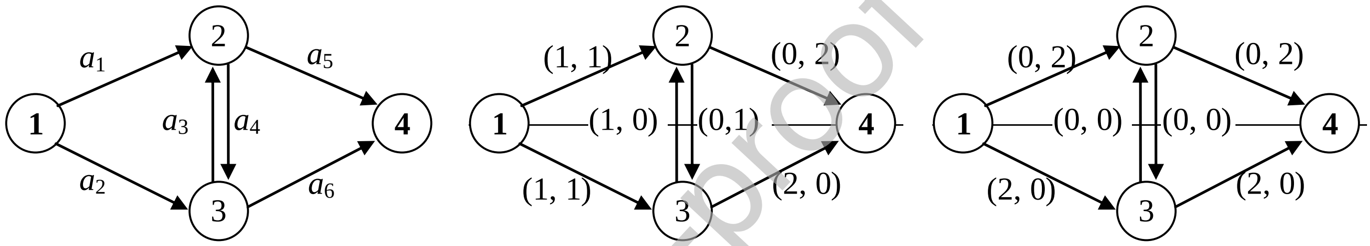
Let δ_i be the time complexity to search for all d_i-MPs for $i = 1, 2, \ldots, q$. Hence, the time complexity of STEP 0 is $O(\sum_{i=1}^{q}\delta_i)$. Moreover, the time complexity to execute STEP 2 is $O(m)$; the number of combinations in STEP 4 is $\prod_{i=1}^{q}\pi_i$ if $(p_{1,j_1}; p_{2,j_2}; \ldots; p_{q,j_q})$ is a D-MP candidate for all $j_i = 1, 2, \ldots, \pi_i$ and $i = 1, 2, \ldots, q$. Hence, $O(m\prod_{i=1}^{q}\pi_i)$ is required to execute STEPS 1 through 6. No duplicate D^*-MPs were found after executing the proposed algorithm based on the DFS property, i.e., there was no additional time to detect and remove duplicates. From the above, the proposed path-based algorithm finds all D^*-MPs without duplicates in a time complexity of $O(m\prod_{i=1}^{q}\pi_i + \sum_{i=1}^{q}\delta_i)$.the following result may be directly derived (see Lemma 4 and its proof in Appendix).

4. Relationships between WMMFN and MMMFN

To gain a better understanding of the novelization and the contribution of the proposed MMMFN, the relationships between the WMMFN and MMMFN will be discussed in this section from three viewpoints: some useful information is lost in the WMMFN owing to the characteristic of the D-MP, the state distribution shows that WMMFN is a special case of MMMFN, and the definition of MMMFN reliability is based on the joint probability.

4.1 Difference in treating the *D*-MPs

In Fig. 3, the first and second numbers in the bracket of each arc denote the flow of Commodities 1 and 2 in this specific arc, respectively. It is clear that both vectors X = (2, 2, 1, 1, 2, 2) and X^* = (2, 2, 0, 0, 2, 2) are able to send two units of flows for each commodity from Nodes 1 to 4 in Figs. 3b and 3c. Therefore, X and X^* need to be included when calculating the final reliability. However, based on all the known works on the WMMFN, X is redundant and must be discarded as $X > X^*$. Both X and X^* contain important information for decision makers to make a decision, for example to enhance the function of $e_{2,3}$. Hence, some useful information is lost in the WMMFN.



(a) (b) X = (2, 2, 1, 1, 2, 2) (c) X^* = (2, 2, 0, 0, 2, 2)

Figure 3. The WMMFN example

Moreover, the reliability of the WMMFN, R_D, is equal to that of the MFN R_d, where $D = (d_1, d_2, \ldots, d_q)$ and $d = \sum_{k=1}^{q} d_k$. In calculating the reliability, the WMMFN considers all commodities as a whole, namely the total weight, profit, and expenditure. For example, P = (3, 0, 0, 3, 0) is a (1, 2)-MP and a (2, 1)-MP, as discussed in Section 1, and X = (2, 2, 1, 1, 2, 2) is removed because X^* = (2, 2, 0, 0, 2, 2) is a (2, 2)-MP and $X^* < X$, as shown in Fig. 3.

In contrast, the proposed MMMFN places more emphasis on the details and the individual commodity, e.g., P = (3, 0, 0, 3, 0) is (2, 0, 0, 2, 0; 1, 0, 0, 1, 0) if it is a $(2, 1)^*$-MP or (1, 0, 0, 1, 0; 2, 0, 0, 2, 0) if it is a $(1, 2)^*$-MP of the example discussed in Section 1. Furthermore, X = (2, 2, 1, 1, 2, 2) and X^* = (2, 2, 0, 0, 2, 2) need to be rewritten as X = (1, 1, 1, 0, 0, 2; 1, 1, 0, 1, 2, 0) and X^* = (0, 2, 0, 0, 0, 2; 2, 0, 0, 0, 2, 0), respectively, and none of them can be removed in the MMMFN shown in Fig. 3.

4.2 Difference in the state distribution

The next example shows that the MMMFN is able to contain the WMMFN. Considering the states of component a_3 in Table 1, the states in the WMMFN are identical to those in the MMMFN of Table 3. Hence, any WMMFN can be transferred to the MMMFN. However, it is not always possible to transfer the MMMFN to WMMFN, as summarized in Table 1 of Section 1. From the above, the MMMFN is more general than the WMMFN; hence, weights are not required in the MMMFN.

Table 3. State distribution of component a_3.

WMMFN			MMMFN		
Commodity 1	Commodity 2	Commodity 3	Commodity 1	Commodity 2	Commodity 3
$w_{3,1} = 1.1$	$w_{3,2} = 0.9$	$w_{3,3} = 1.0$			
	0		$\lceil 0 \times 1.1 \rceil = 0$	$\lceil 0 \times 0.9 \rceil = 0$	$\lceil 0 \times 1.0 \rceil = 0$
	1		$\lceil 1 \times 1.1 \rceil = 2$	$\lceil 1 \times 0.9 \rceil = 1$	$\lceil 1 \times 1.0 \rceil = 1$
	2		$\lceil 2 \times 1.1 \rceil = 3$	$\lceil 2 \times 0.9 \rceil = 2$	$\lceil 2 \times 1.0 \rceil = 2$
	4		$\lceil 4 \times 1.1 \rceil = 5$	$\lceil 4 \times 0.9 \rceil = 4$	$\lceil 4 \times 1.0 \rceil = 4$

4.3 Difference in calculating the reliability

The next simple two-commodity one-arc example with the state distribution summarized in Table 4 was applied to easily observe the difference between the WMMFN reliability, $R_{(1,1)}$ and the MMMFN reliability, $R_{(1,1)}$, i.e., $D = (1, 1)$. In this example, without loss of generalization, we only allow one arc to extend from Nodes 1 to 2, with a weight value of one.

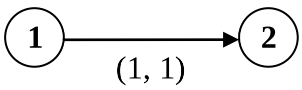


Figure 4. The two-commodity one-arc WMMFN example with a weight value of one.

Table 4. Sate distribution of Fig. 4.

State	Probability
0	0.1
1	0.2
2	0.3
3	0.4

Table 5. Joint probability of the state distribution in Fig. 4.

State	0	1	2	3
0	$0.1 \times 0.1 = 0.01$	$0.1 \times 0.2 = 0.02$	$0.1 \times 0.3 = 0.03$	$0.1 \times 0.4 = 0.04$
1	$0.2 \times 0.1 = 0.02$	$0.2 \times 0.2 = 0.04$	$0.2 \times 0.3 = 0.06$	$0.2 \times 0.4 = 0.08$
2	$0.3 \times 0.1 = 0.03$	$0.3 \times 0.2 = 0.06$	$0.3 \times 0.3 = 0.09$	$0.3 \times 0.4 = 0.12$

3	0.4 × 0.1 = 0.04	0.4 × 0.2 = 0.08	0.4 × 0.3 = 0.12	0.4 × 0.4 = 0.16

In Fig. 4, there is only one (1, 1)-MP, i.e., vector (1, 1). Based on Eq. (1), we have $R_{(1,1)} = R_2 = \Pr(\{x \mid 2 \le x\}) = 0.3 + 0.4 = 0.7$ in the WMMFN. However, $R_{(1,1)}$ is the success probability that both commodities have at least one unit of flow from Nodes 1 to 2 simultaneously, i.e., $R_{(1,1)} = \Pr(\{(x, y) \mid (1, 1) \le (x, y)\}) = 0.04 + 0.06 + 0.08 + 0.06 + 0.09 + 0.12 + 0.08 + 0.12 + 0.16 = 0.81$, as may be observed in Table 5 (see the grey highlighted expressions) in the proposed novel MMMFN.

From the above example, the major difference between the definition provided in Eq. (1) and the newly proposed definition is that the joint probability was used for the calculation of the state of each commodity in each arc for all commodities in terms of the multi-vector.

5. EXPERIMENTAL RESULTS

The general procedure can be best illustrated with an example. Two examples will be discussed in this section. Example 1 will focus on using the proposed algorithm to solve the MMMFN reliability problem and Example 2 will mainly introduce one of the possible applications of the proposed MMMFN in the production of high-quality mineral water.

5.1 Example 1

The three steps of the proposed path-based algorithm for solving the novel MMMFN reliability problem are the following: 1) find all d_i-MPs for $i = 1, 2, \ldots, q$, 2) search for all D^*-MPs, and 3) calculate the MMMFN reliability in terms of all D^*-MPs. The aforementioned steps are all NP-hard problems. Owing to the three NP-hard problems that are present in the process of calculating the MMMFN reliability, instead of presenting practically large network systems, a simple network—shown in Fig. 1—was selected to demonstrate this methodology as per the existing known algorithms [7, 11, 15–29]. This simple network was adapted from the bridge network, which is the most frequently cited example proposed in [7, 11, 15–29].

A two-commodity (i.e., $q = 2$) example with $D = (d_1, d_2) = (1, 2)$ was implemented to demonstrate the proposed approach to find all (1, 1)-MPs. Then, the inclusion–exclusion algorithm was implemented to calculate the reliability for level $D = (d_1, d_2) = (1, 2)$ in terms of the obtained $(1, 1)^*$-MPs.

Example: Consider the MMMFN shown in Fig. 1; the state distribution is summarized in Table 2, where $i = 1$ and $j = 2$, e.g., the probability of the state vector of (3, 0) and (0, 3) is 0.20 and 0.1 of a_1, respectively. The proposed path-based algorithm was implemented to search for all $(1,2)^*$-MPs in the MMMFN.

Solve:

STEP 0. Calculate the accumulated state distributions (see Table 6), find all d_i-MPs, set P_i as summarized in Table 7 [26] for $i = 1, 2$; let $\Omega = \varnothing$, $X_0 = 0$, and $i = 1$.

Table 6. Accumulated state distribution of **Fig. 1**.

Arc	i \ j	0	1	2	3	4
a_1	0	1.000	0.600	0.350	0.150	0.050
	1	0.700	0.350	0.150	0.050	0.000
	2	0.450	0.150	0.050	0.000	0.000
	3	0.250	0.050	0.000	0.000	0.000
	4	0.000	0.000	0.000	0.000	0.000
a_2	0	1.000	0.450	0.300	0.100	0.000
	1	0.750	0.250	0.150	0.050	0.000
	2	0.550	0.150	0.050	0.000	0.000
	3	0.350	0.050	0.000	0.000	0.000
	4	0.100	0.000	0.000	0.000	0.000
a_3	0	1.000	0.550	0.350	0.250	0.100
	1	0.950	0.500	0.300	0.200	0.050
	2	0.800	0.350	0.150	0.100	0.000
	3	0.450	0.100	0.000	0.000	0.000
	4	0.200	0.000	0.000	0.000	0.000
a_4	0	1.000	0.700	0.500	0.310	0.160
	1	0.840	0.540	0.370	0.230	0.110
	2	0.690	0.440	0.270	0.150	0.080
	3	0.500	0.300	0.150	0.080	0.030
	4	0.300	0.200	0.100	0.050	0.000
a_5	0	1.000	0.900	0.650	0.450	0.250
	1	0.950	0.850	0.600	0.400	0.200
	2	0.850	0.750	0.500	0.300	0.150
	3	0.650	0.550	0.300	0.150	0.050
	4	0.500	0.400	0.200	0.100	0.000

Table 7. All d_i-MPs $p_{i,k}$.

k \ i	1	2
1	(1, 0, 0, 1, 0)	(2, 0, 0, 2, 0)
2	(1, 0, 1, 0, 1)	(2, 0, 1, 1, 1)
3	(0, 1, 0, 0, 1)	(1, 1, 0, 1, 1)
4		(2, 0, 2, 0, 2)

5		(1, 1, 1, 0, 2)
6		(0, 2, 0, 0, 2)

STEP 1. Let $j_i = j_1 = 1$.

STEP 2. As $i = 1 < q = 2$, let $i = i + 1 = 2$ and go to STEP 1.

STEP 1. Let $j_i = j_2 = 1$.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,1})\}$, where $(p_{1,1};\ p_{2,1}) = (1, 0, 0, 1, 0; 2, 0, 0, 2, 0)$.

STEP 4. As $j_2 = 1 < \pi_2 = 6$, let $j_2 = j_2 + 1 = 2$ and go to STEP 2.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,2})\}$, where $(p_{1,1};\ p_{2,2}) = (1, 0, 0, 1, 0; 2, 0, 1, 1, 1)$.

STEP 4. As $j_2 = 2 < \pi_2 = 6$, let $j_2 = j_2 + 1 = 3$ and go to STEP 2.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,3})\}$, where $(p_{1,1};\ p_{2,3}) = (1, 0, 0, 1, 0; 1, 1, 0, 1, 1)$.

STEP 4. As $j_2 = 3 < \pi_2 = 6$, let $j_2 = j_2 + 1 = 4$ and go to STEP 2.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,4})\}$, where $(p_{1,1};\ p_{2,4}) = (1, 0, 0, 1, 0; 2, 0, 2, 0, 2)$.

STEP 4. As $j_2 = 4 < \pi_2 = 6$, let $j_2 = j_2 + 1 = 5$ and go to STEP 2.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3 Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,5})\}$, where $(p_{1,1};\ p_{2,5}) = (1, 0, 0, 1, 0; 1, 1, 1, 0, 2)$.

STEP 4. As $j_2 = 5 < \pi_2 = 6$, let $j_2 = j_2 + 1 = 6$ and go to STEP 2.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,1};\ p_{2,6})\}$, where $(p_{1,1};\ p_{2,6}) = (1, 0, 0, 1, 0; 0, 2, 0, 0, 2)$.

STEP 4. As $j_2 = \pi_2 = 6$, go to STEP 5.

STEP 5. As $i = 2 > 0$, let $i = i - 1 = 1$ and go to STEP 4.

STEP 4. As $j_1 = 1 < \pi_1 = 3$, let $j_1 = j_1 + 1 = 2$ and go to STEP 2.

STEP 2. As $i = 1 < q = 2$, let $i = i + 1 = 2$ and go to STEP 1.

STEP 1. Let $j_i = j_2 = 1$.

STEP 2. As $i = q = 2$, go to STEP 3.

STEP 3. Let $\Omega = \Omega \cup \{(p_{1,j_1};\ p_{2,j_2})\} = \Omega \cup \{(p_{1,2};\ p_{2,1})\}$, where $(p_{1,2};\ p_{2,1}) = (1, 0, 1, 0, 1; 2, 0, 0, 2, 0)$.

:

:

Let i be the iteration number. The summary of the above procedures for finding all $(1,2)^*$-MPs is summarized in Table 8.

Table 8. Summarized procedure of obtaining D^*-MPs based on the proposed SDM.

i	d_1-MP	d_2-MP	D^*-MP p_i
1	(1, 0, 0, 1, 0)	(2, 0, 0, 2, 0)	(1, 0, 0, 1, 0; 2, 0, 0, 2, 0)
2	(1, 0, 0, 1, 0)	(2, 0, 1, 1, 1)	(1, 0, 0, 1, 0; 2, 0, 1, 1, 1)
3	(1, 0, 0, 1, 0)	(1, 1, 0, 1, 1)	(1, 0, 0, 1, 0; 1, 1, 0, 1, 1)
4	(1, 0, 0, 1, 0)	(2, 0, 2, 0, 2)	(1, 0, 0, 1, 0; 2, 0, 2, 0, 2)
5	(1, 0, 0, 1, 0)	(1, 1, 1, 0, 2)	(1, 0, 0, 1, 0; 1, 1, 1, 0, 2)
6	(1, 0, 0, 1, 0)	(0, 2, 0, 0, 2)	(1, 0, 0, 1, 0; 0, 2, 0, 0, 2)
7	(1, 0, 1, 0, 1)	(2, 0, 0, 2, 0)	(1, 0, 1, 0, 1; 2, 0, 0, 2, 0)
8	(1, 0, 1, 0, 1)	(2, 0, 1, 1, 1)	(1, 0, 1, 0, 1; 2, 0, 1, 1, 1)
9	(1, 0, 1, 0, 1)	(1, 1, 0, 1, 1)	(1, 0, 1, 0, 1; 1, 1, 0, 1, 1)
10	(1, 0, 1, 0, 1)	(2, 0, 2, 0, 2)	(1, 0, 1, 0, 1; 2, 0, 2, 0, 2)
11	(1, 0, 1, 0, 1)	(1, 1, 1, 0, 2)	(1, 0, 1, 0, 1; 1, 1, 1, 0, 2)
12	(1, 0, 1, 0, 1)	(0, 2, 0, 0, 2)	(1, 0, 1, 0, 1; 0, 2, 0, 0, 2)
13	(0, 1, 0, 0, 1)	(2, 0, 0, 2, 0)	(0, 1, 0, 0, 1; 2, 0, 0, 2, 0)
14	(0, 1, 0, 0, 1)	(2, 0, 1, 1, 1)	(0, 1, 0, 0, 1; 2, 0, 1, 1, 1)
15	(0, 1, 0, 0, 1)	(1, 1, 0, 1, 1)	(0, 1, 0, 0, 1; 1, 1, 0, 1, 1)
16	(0, 1, 0, 0, 1)	(2, 0, 2, 0, 2)	(0, 1, 0, 0, 1; 2, 0, 2, 0, 2)
17	(0, 1, 0, 0, 1)	(1, 1, 1, 0, 2)	(0, 1, 0, 0, 1; 1, 1, 1, 0, 2)
18	(0, 1, 0, 0, 1)	(0, 2, 0, 0, 2)	(0, 1, 0, 0, 1; 0, 2, 0, 0, 2)

After using the IE or the SDP, R_{D^*} = 0.365058707069560 in terms of 18 D^*-MPs:

$$R_{D^*} = \sum_{i=1}^{18} \Pr^+(p_i) - \sum_{j=2}^{18}\sum_{i=1}^{j-1} \Pr^+(p_i \cap p_j) + \sum_{k=3}^{18}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} \Pr^+(p_i \cap p_j \cap p_k) + \ldots$$

$+(-1)^{18}\Pr^+(p_1 \cap \ldots \cap p_{18})$

$=[\Pr^+(1, 0, 0, 1, 0; 2, 0, 0, 2, 0) + \Pr^+(1, 0, 0, 1, 0; 2, 0, 1, 1, 1) + \ldots + \Pr^+(0, 1, 0, 0, 1; 0, 2, 0, 0, 2)] - [\Pr^+(1, 0, 0, 1, 0; 2, 0, 1, 2, 1) + \Pr^+(1, 0, 0, 1, 0; 2, 1, 0, 2, 1) + \ldots + \Pr^+(0, 1, 0, 0, 1; 1, 2, 1, 0, 2)] + [\Pr^+(1, 0, 0, 0, 1, 0; 2, 1, 1, 2, 1) + \Pr^+(1, 0, 0, 1, 0; 2, 0, 2, 2, 2) + \ldots + \Pr(0, 1, 0, 0, 1; 2, 2, 2, 0, 2)] + \ldots + (-1)^{18}\Pr^+(1, 1, 1, 1, 1; 2, 2, 2, 2, 2)$

$$= 0.365058707069560. \tag{9}$$

From Eq. (9), the success probability to have at least one unit of Commodity 1 and at least two units of Commodity 2 in the end is only 0.365058707069560. Hence, it is necessary to enhance the reliability of each component and/or reduce the required number of commodities to increase the network reliability.

5.2 Example 2

One concrete and practical example will be added to directly illustrate the application of the proposed MMMFN in reliability engineering systems. Figure 5 is a rework network $G(V, E, D)$ for the production of high-quality mineral water, where $V = \{1, 2\}$ and $E = \{e_1, e_2, e_3, e_4\}$. Twenty per cent of mineral water is discarded after prefiltering in Node 1 and an additional twenty per cent of mineral water is sent back via e_4 to Node 1 to be refiltered after it is sterilized and tested in Node 2. The aforementioned process of sending back from Node 2 to Node 1 is referred to as a rework. Such a rework network problem can be treated and extended to become an MMFN problem by categorizing the perfect (i.e., without the need to be discarded and reworked) and defective products into two different commodities, e.g., Commodities 1 and 2, respectively.

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Figure 5. Mineral water rework network.

Table 9. The state distribution Ψ of Fig. 5.

	1 \ 2	0	5	10	15			SUM
Node 1	0			0.01				0.01
	5		0.22	0.26	0.05			0.53
	10		0.10	0.33	0.02			0.45
	15			0.01				0.01
	SUM		0.32	0.61	0.07			1.00
Node 2	1 \ 2	0	3	6	9	12	15	SUM
	0			0.01				0.01
	3			0.10	0.10	0.02	0.01	0.23
	6	0.01	0.10	0.15	0.13			0.39
	9		0.15	0.16				0.31
	12		0.05					0.05
	15		0.01					0.01
	SUM	0.01	0.31	0.42	0.23	0.02	0.01	1.00

The state distribution, Ψ, of Fig. 5 is summarized in Table 9. We want to calculate the success probability of receiving at least 5 t of mineral water from Node 2, i.e., the value of reliability R_5.

If 5 t of mineral water need to be produced from Node 2, 5 t/0.8 = 6.25 t of mineral water should be input to Node 2 [36–39]. In the same manner, 6.25 t/0.8 = 7.8125 t of mineral water are required for Node 1. Furthermore, 20% of the 6.25 t of mineral water in Node 2, i.e., 1.25 t, are sent back for rework in Node 1. Hence, 7.8125 t – 1.25 t = 6.5625 t of mineral water input to Node 1 do not originate from rework.

All possible D^*-MPs are summarized in Table 10 based on the input and output of Nodes 1 and 2. The vector in the first column of each node shows the perfect flows and the input and output of imperfect flows. The second column of each node shows the states of each node that satisfy the flows shown in the first column. For example, Node 1 of Case 1, i.e., (5, 2.8125, 1.25) denotes that there are 5 t of Commodity 1 input to and output from Node 1, 2.8125 t of Commodity 2 input to Node 1, and 1.25 t of Commodity 2 output from Node 1. The (5, 5, 5) corresponding to (5, 2.8125, 1.25) denotes that the states of Node 1 are five to satisfy the related flows shown in (5, 2.8125, 1.25).

Table 10. The D^*-MPs of Fig. 5.

Case	1		2	
	flow	state	flow	state
1	(5, 2.8125, 1.25)	(5, 5, 5)	(5, 1.25, 0)	(6, 3, 0)
~~2~~	~~(4, 3.8125, 2.25)~~	~~(5, 5, 5)~~	~~(4, 2.25, 1)~~	~~(6, 3, 3)~~
3	(3, 4.8125, 3.25)	(5, 5, 5)	(3, 3.25, 2)	(3, 6, 3)
~~4~~	~~(2, 5.8125, 4.25)~~	~~(5, 10, 5)~~	~~(2, 4.25, 3)~~	~~(3, 6, 3)~~
~~5~~	~~(1, 6.8125, 5.25)~~	~~(5, 10, 10)~~	~~(1, 5.25, 4)~~	~~(3, 6, 6)~~
6	(0, 7.8125, 6.25)	(0, 10, 10)	(0, 6.25, 5)	(0, 9, 6)

Case 2 is included in Case 1; Cases 4 and 5 are included in Case 3, i.e., Cases 2, 4, and 5 are redundant and can be removed. Based on the proposed algorithm, the probabilities to have at least 5 t of mine water for Cases 1, 3, and 6 are listed below:

$$\Pr_{1,i}(5, 5) \times \Pr_{1,o}(5, 5) \times \Pr_{2,i}(6, 3) \times \Pr_{2,o}(6, 0) = 0.99 \times 0.99 \times 0.75 \times 0.76 = 0.558657$$

$$\Pr_{1,i}(5, 5) \times \Pr_{1,o}(5, 5) \times \Pr_{2,i}(3, 6) \times \Pr_{2,o}(3, 3) = 0.99 \times 0.99 \times 0.67 \times 0.98 = 0.64353366$$

$$\Pr_{1,i}(0, 10) \times \Pr_{1,o}(0, 10) \times \Pr_{2,i}(0, 9) \times \Pr_{2,o}(0, 6) = 0.68 \times 0.68 \times 0.26 \times 0.68 = 0.08175232. \quad (10)$$

After using the IE or the SDP, we have:

$$R_5 = 0.558657 + 0.64353366 + 0.08175232 - 0.323433 - 0.078198 = 0.8823116. \quad (11)$$

6. CONCLUSIONS AND FUTURE WORKS

The MMFN has become more important than before because an increasing number of networks is allowed to have various commodities, e.g., through social media, it is possible to send text files, music, photos, and videos. A novel MMFN model, referred to as the MMMFN, has been proposed in this study to generalize the WMMFN by removing the limitation that all the commodities should share the same state distribution for each component. Moreover, a new algorithm was presented for the calculation of the proposed novel MMMFN reliability based on the proposed D^*-MP concept. In the proposed MMMFN, all nodes were assumed as perfectly reliable based on the terminal-pair reliability analysis. In the future, based on analysis, the currently proposed MMMFN can be revised for multi-terminal larger-sized network reliability with unreliable nodes. An additional future study would be to discuss more practical cases to verify the applicability of the proposed MMMFN to certain domains, e.g., using the Internet of Things to address logistical problems or to determine social networks to maximize influence."

ACKNOWLEDGMENT

I wish to thank the anonymous editor and the referees for their constructive comments and recommendations, which significantly improved this paper. This research was supported in part by the Ministry of Science and Technology of Taiwan, under grant MOST 104-2221-E-007-061-MY3, National Natural Science Foundation of China (Grant #51537010), and the Natural Sciences and Engineering Research Council of Canada (Grant #RGPIN-2015-04897).

REFERENCES

[1] T. Aven, "Availability evaluation of oil/gas production and transportation systems", *Reliability Engineering*, Vol. 18, 1987, pp 35-44.

[2] T. Aven, "Some considerations on reliability theory and its applications", *Reliability Engineering and System Safety*, Vol. 21, 1988, pp 215-223.

[3] F. S. Roberts, F. K. Hwang and C. Monma, "Reliability of computer and communication networks", *DIMACS Series in Discrete Mathematics and Theoretical Computer Science,* Vol. 5, AMS/ACM publisher, 1991.

[4] W. J. Ke and S.D. Wang, "Reliability evaluation for distributed computing networks with imperfect nodes", *IEEE Transactions on Reliability*, Vol. 46, 1997, pp 342-349.

[5] W. C. Yeh and J. S. Lin, "New parallel swarm algorithm for smart sensor systems redundancy allocation problems in the Internet of Things", *The Journal of Supercomputing*, Vol. 74, No. 9, pp. 4358-4384.

[6] W. C. Yeh and S. C. Wei, "Economic-based resource allocation for reliable grid-computing service based on grid bank", *Future Generation Computer Systems-The International Journal of Grid Computing-Theory Methods and Applications*, Vol. 28, No. 7, 2012/7, pp. 989-1002.

[7] M. Wang, W. C. Yeh, T. C. Chu, X. Zhang, C. L. Huang, and J. Yang, "Solving Multi-Objective Fuzzy Optimization in Wireless Smart Sensor Networks under Uncertainty Using a Hybrid of IFR and SSO Algorithm", *Energies*, Vol. 11, No. 9, 2018, 2385.

[8] X. Shan, F. A. Felder, D. W. Coit, "Game-theoretic models for electric distribution resiliency/reliability from a multiple stakeholder perspective", *IISE Transactions*, Vol. 49, No. 2, 2017, pp. 159-177.

[9] W. C. Yeh, "A Sequential Decomposition Method for Estimating Flow in a Multi-Commodity, Multistate Network", *IEEE Transactions on Reliability*, Vol. 60, No. 3, 2011, pp. 612–621.

[10] W. C. Yeh, "A Simple Minimal Path Method for Estimating the Weighted Multi-commodity Multistate Unreliable Networks Reliability", *Reliability Engineering & System Safety*, Vol. 93, No. 1, 2008, pp. 125-136.

[11] Y. K. Lin, "Study on the system capacity for a multicommodity stochastic-flow network with node failure", *Reliability Engineering & System Safety*, Vol. 78, 2002, pp. 57-62.

[12] W. C. Yeh and T.-C. Chu, "A Novel Multi-Distribution Multi-State Flow Network and its Reliability Optimization Problem", *Reliability Engineering & System Safety*, Vol. 176, 2018, pp. 209-217.

[13] C.J. Colbourn, "The combinatorics of network reliability", *Oxford University Press, New York,* 1987.

[14] D. R. Shier, *Network reliability and algebraic structures.* New York, USA: Clarendon Press, 1991.

[15] G. Levitin, *The universal generating function in reliability analysis and optimization.* London: Springer-Verlag, 2005.

[16] J. E. Ramirez-Marquez, D.W. Coit, and M. Tortorella, "Multi-state two-terminal reliability-a generalized cut set approach", Rutgers University IE Working Paper, 2003.

[17] D. W. Coit, N. Chatwattanasiri, N. Wattanapongsakorn, and A. Konak, "Dynamic k-out-of-n system reliability with component partnership", *Reliability Engineering & System Safety*, Vol. 138, 2015, pp. 82-92.

[18] M. J. Zuo, Z. Tian, H. Z. Huang, "An efficient method for reliability evaluation of multistate networks given all minimal path vectors", *IIE Transactions,* Vol. 39, No. 8, 2007, pp. 811–817.

[19] W. C. Yeh, "A simple heuristic algorithm for generating all minimal paths", *IEEE Transactions on Reliability*, Vol. 56, No. 3, 2007, pp. 488-494.

[20] J. E. Ramirez-Marquez and D.W. Coit, "A Monte-Carlo simulation approach for approximating multi-state two-terminal reliability", Reliability Engineering & System Safety, Vol. 87, No. 2, 2005, pp. 253-264.

[21] G. Bai, M. J. Zuo, and Z. Tian, "Search for all *d*-MPs for all *d* levels in multistate two-terminal networks", *Reliability Engineering and System Safety*, Vol. 142, 2015, pp 300–309.

[22] S. Song, D.W. Coit, Q.F., and H.P. G.B. Jasmon, "Reliability analysis for multi-component systems subject to multiple dependent competing failure processes", *IEEE Transactions on Reliability*, Vol. 63, No. 1, 2014, pp. 331-345.

[23] J. E. Ramirez-Marquez, "Stochastic network interdiction optimization via capacitated network reliability modeling and probabilistic solution discovery", *Reliability Engineering & System Safety*, Vol. 94, No. 5, 2009, pp. 913-921.

[24] J. E. Ramirez-Marquez, D.W. Coit, and M. Tortorella, "A generalized multi-state-based path vector approach to multi-state two-terminal reliability", *IIE Transactions*, Vol. 38, No. 6, 2006, pp. 477-488.

[25] W.C. Yeh, "A revised Layered-Network algorithm to search for all *d*-Minpaths of a limited-flow acyclic network", *IEEE Transactions on Reliability*, Vol. 47, 1998, pp 436-442.

[26] W.C. Yeh, "A novel node-based sequential implicit enumeration method for finding all *d*-MPs in a multistate flow network", *Information Sciences*, Vol. 297, 2015, pp. 283-292.

[27] Y. F. Niu, Z. Y. Gao, H. J. Sun, "An improved algorithm for solving all *d*-minimal paths in multi-state networks", *Journal of System Science and System Engineering*, Vol. 26, No. 6, 2017, pp 711–731.

[28] Y.F. Niua and F.M. Shao, "A practical bounding algorithm for computing two-terminal reliability

based on decomposition technique”, *Computers and Mathematics with Applications*, Vol. 61, 2011, pp. 2241-2246.

[29] W. C. Yeh, “A simple approach to search for all *d*-MCs of a limited-flow network”, *Reliability Engineering & System Safety*, Vol. 71, No. 1, 2001, pp. 15-19.

[30] W.C. Yeh, “Multistate-node acyclic networks reliability evaluation based on MC”, *Reliability Engineering & System Safety*, Vol. 81, No. 2, 2003, pp. 225-231.

[31] G. Bai, Z. Tian, and M. J. Zuo, “Reliability evaluation of multistate networks: An improved algorithm using state-space decomposition and experimental comparison”, *IISE Transactions*, Vol. 50, No. 5, 2018, pp. 407-418.

[32] W. C. Yeh, “A Greedy Branch-and-Bound Inclusion-Exclusion Algorithm for Calculating the Exact Multi-State Network Reliability”, *IEEE Transactions on Reliability*, Vol.57, No. 1, 2008, pp.88-93.

[33] W. C. Yeh, “A New Sum-of-Disjoint-Products Technique to Determine Network Reliabilities with the Known Minimal Paths”, *Reliability Engineering & System Safety*, Vol. 92, No. 2, 2007, pp. 260-268.

[34] W. C. Yeh, “An Improved Sum-of-Disjoint-Products Technique for Symbolic Multi-State Flow Network Reliability”, *IEEE Transactions on Reliability*, Vol. 64, No. 4, 2015, pp. 1185-1193.

[35] Z. Hao, W. C. Yeh, J. Wang, G. G. Wang and B. Sun, “A Quick Inclusion-Exclusion Technique”, *Information Sciences*, Vol. 486, 2019, pp. 20-30.

[36] W. C. Yeh, “Evaluating the Reliability of a Novel Deterioration-Effect Multi-state Flow Network”, *Information Sciences*, Vol. 243, 2013, pp. 75-85.

[37] W. C. Yeh, “Evaluation of the One-to-all-target-subsets Reliability of a Novel Deterioration-Effect Acyclic Multi-state Information Network”, *Reliability Engineering & System Safety*, Vol. 166, 2017, pp. 132-137.

[38] W. C. Yeh and Y. M. Yeh, “Evaluating the Reliability of Deterioration-Effect Multi-State Flow Network System and Method Thereof”, US Patent, No. 9,548,925, 2017/01/17~2035/03/18.

[39] W. C. Yeh, “Methodology for the reliability evaluation of the novel learning-effect multi-state flow network”, *IISE Transactions*, Vol. 49, No. 11, 2017, pp. 1078-1085.

APPENDIX

Lemma 1: The MMMFN reliability, $R_{D^*} = \bigcup \Pr(P)$ for all D^*-MP P.

Proof. It is impossible for any multi-vector less than any D^*-MP to send d_i-units of flows of Commodity i from Nodes 1 to n, and no D^*-MP is greater than any other D^*-MP.

Lemma 2: Let $D^* = (d_1, d_2, \ldots, d_q)$, D^*-MP $P = (x_{1,1}, x_{2,1}, \ldots, x_{m,1}; x_{1,2}, x_{2,2}, \ldots, x_{m,2}; \ldots; x_{1,q}, x_{2,q}, \ldots, x_{m,q})$, and $x_{i,k} \le x^*_{i,k}$ for all $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, q$. The probability to send at least d_i units of Commodity i via D^*-MP P for $i = 1, 2, \ldots, q$ is:

$$\Pr(P) = \prod_{i=1}^{m} \Pr{}^{+}(x_{i,1}, x_{i,2}, \ldots, x_{i,q}). \tag{5}$$

Proof. It is impossible for any multi-vector less than any D^*-MP to send d_i-units of flow of Commodity i from Nodes 1 to n because no D^*-MP is greater than any other D^*-MP. Hence, this lemma is true.

Lemma 3: Any multi-vector $P = (X_1; X_2; \ldots; X_q)$ is a D^*-MP if and only if X_i is a d_i-MP in the MMMFN, for $i = 1, 2, \ldots, q$.

Proof. X_i is a d_i-MP, i.e., the d_i unit of the i^{th} commodity can be sent from the source node to the sink node in the MMFN. Hence, all commodities can be delivered with the required amount of flows. Moreover, there is no flow limitation in each component of the MMMFN. Thus, this lemma is correct.

Lemma 4: The proposed path-based algorithm finds all D^*-MPs without duplicates in a time complexity of $O(m\prod_{i=1}^{q}\pi_i + \sum_{i=1}^{q}\delta_i)$.