Q. Show that Row equivalence is an equivalence relation

$a \sim b$
↳ b is row equivalent to a

Proof a) Reflexive → $(a \sim a)$ $A = I(A)$ ↳ row operation thus reflexive

b) Symmetric → If $(a \sim b)$ then $(b \sim a)$

$B = (e_k \cdots (e_1(A)) \cdots)$ } B is row equivalent to A

then $A = e_1^{-1}(\cdots e_k^{-1}(B) \cdots)$ $e_1^{-1}, \cdots e_k^{-1}$ are also row operations

↳ thus A is row equivalent to B

c) Transitive → If $(a \sim b)$, $(b \sim c)$ then $(a \sim c)$

$B = (e_k(\cdots e_1(A))) \cdots)$ — (I)

$C = (e_n(\cdots e_{k+1}(B) \cdots)$ — (II)

replacing B in (II) from (I)

$C = (e_n \cdots e_{k+1}(e_k \cdots e_1(A) \cdots) \cdots)$

Hence C is row equivalent to A.

Hence transitivity proved

Therefore we can conclude that row equivalence is an equivalence relation

Q. Every $m \times n$ matrix is row equivalent to row-reduced matrix

Proof Starting from first row if its ~~non~~-zero then do nothing

if its non-zero then

make the leading non-zero coeff. 1 by multiplying with suitable scalar then convert remaining columns of leading non-zero element to 0 by using elementary row operat $\lambda R_i + R_j$

Such that $\lambda R_i + R_j$ equals $\boxed{0}$

now since each ~~element~~ column of '1' in each row will contain only zeroes matrix is row equivalent to

Hence every row-reduced matrix

Q. Every $m \times n$ matrix is row equivalent to row-reduced echelon matrix

Proof. We already proved that every matrix is row equivalent to row-reduced matrix. We can easily say that it will also be row equivalent to row-reduced echelon matrix since by doing row swaps we can ensure that the row-reduced matrix is row-reduced echelon matrix.

can ensure that the

5. If $A_{m\times n}$, & $m<n$ then $AX=0$ will always has a non-trivial sol^n

let R be the row reduced echelon matrix of A, now since R & A are row equivalent which implies that $AX=0$ & $RX=0$ have exactly same sol^n

Now let r be the no. of non-zero rows of R

since $r \le m < n$ thus

we know that there will be ~~$(r-n)$~~ $(n-r)$ free variables thus there will be atleast 1 free variable making the sol^n non-trivial.

But if $m=n$ then there ~~won't~~ may not be any free variable making the sol^n trivial

i.e for the case when $r=m=n$ $R=I_{n\times n}$

$$\begin{bmatrix}1&0&0&\cdots\\0&1&0&\cdots\\ &\vdots& & \\0&0&\cdots&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\\vdots\\0\end{bmatrix}$$

All x_i's will become '0' making the sol^n trivial

If $m>n$ there maybe a case when $r>n$

but for a row reduced echelon matrix this won't be possible.

$$\begin{array}{c|ccc} & & \xleftarrow{\quad n \quad} & \\ \hline \uparrow & 1 & 0 & 0 \cdots \\ n & 0 & 1 & \cdots \\ \downarrow & 0 \cdots & & 1 \end{array}$$

But $m>n$ Hence R won't be a row reduced echelon matrix

contradicting

Q. let e be an elementary row operation and let E be an elementary matrix such that $e(I) = E$
Then show that for every $A_{m\times n}$ $e(A) = E \cdot A$

* let e be the multiplication by a constant
let row α be the row multiplied with c (E will be $m\times m$)

$$E_{ij} = \begin{cases} 1 & i = r \neq \alpha \\ c & i = r = \alpha \\ 0 & \text{otherwise} \end{cases} \qquad \alpha \le m$$

Now $$EA_{ij} = \sum_{r=1}^{m} E_{ir} A_{rj} = \begin{cases} cA_{\alpha j} & i = \alpha \\ A_{ij} & i \neq \alpha \end{cases}$$

$$e(A)_{ij} = \begin{cases} c\,A_{ij} & i = \alpha \\ A_{ij} & i \neq \alpha \end{cases}$$

$\Rightarrow$ $\boxed{EA = e(A)}$

* let e be row swap b/w α & β $\qquad \alpha, \beta \le m$

$$E_{ij} = \begin{cases} 1 & i = \alpha, j = \beta \\ 1 & i = \beta, j = \alpha \\ 0 & i = \alpha, j \neq \beta \\ 0 & i = \beta, j \neq \alpha \\ 1_{ij} & \text{otherwise} \end{cases}$$

Now

$$(EA)_{ij} = \sum_{r=1}^{m} E_{ir} A_{rj} = \begin{cases} A_{\beta j} & i = \alpha \\ A_{\alpha j} & i = \beta \\ A_{ij} & \text{otherwise} \end{cases}$$ By simple manipulation

$$e(A)_{ij} = \begin{cases} A_{\beta j} & i = \alpha \\ A_{\alpha j} & i = \beta \\ A_{ij} & \text{otherwise} \end{cases}$$

Hence
$\boxed{e(A) = EA}$

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* Now $\lambda R_i + R_j$ operation is $\boxed{R_r = \lambda R_s + R_r}$ $\Rightarrow$ R_r $\lambda = c$

$\rightarrow$ row 'r' equal λ times row 's' is row 'r'

$$E_{ij} = \begin{cases} \cancel{E_{ij}}\ \mathbb{1}_{ij} & i \neq r \\ \mathbb{1}_{rj} + c\,\mathbb{1}_{sj} & i = r \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{rj} + cA_{sj} & i = r \\ A_{ij} & i \neq r \end{cases}$$

Since when $i = r$ $E_{ik} = (I_{rk} + c\,I_{sk})$

$$\therefore \sum_{k=1}^{m} (I_{rk} + c\,I_{sk}) A_{kj} = \sum I_{rk} A_{kj} + c \sum I_{sk} A_{kj}$$

$$= A_{rj} + c\,A_{sj}$$

Also $$e(A)_{ij} = \begin{cases} cA_{sj} + A_{rj} & i = r \\ A_{ij} & i \neq r \end{cases}$$

Therefore $\boxed{e(A) = EA}$ H/P

Hence proved for all three operations.

Q. Show following are equivalent

i) A is invertible

ii) $Ax=0$ has only trivial sol^n

iii) $Ax=y$ has a sol^n for each y

let R be the row reduced echelon matrix equivalent to A

Since R is row-reduced echelon matrix of $(n \times n)$ ~~thus~~ and R is

$\therefore$ invertible thus it must be row equivalent to $I_{n \times n}$

~~$A = e_1 e_2 \dots R$~~

$R = e_1 e_2 \dots A$

because if last row is 0 then

inverse can't exist

since taking any matrix last row will become 0 only

Also we already know $A_{n\times n}$ is row equ. to $I_{n\times n}$ then $AX=0$ has only trivial sol^n

thus $AX=0$ has only trivial sol^n

ii) → (iii) $AX=0$ has only trivial sol^n

then A is ~~invertible~~ row equ. to I

$\Rightarrow \boxed{A = e_1 e_2 \cdots e_k (I)}$

Since $e_1 e_2 \cdots e_k (I)$ are invertible thus A is also invertible

Now $AX = Y$ has sol^n

$\boxed{X = A^{-1}Y}$ if A is invertible } which can be deduced from (ii)

i) → (ii) let R be row-reduced echelon matrix of A

then

A $\boxed{RX = E}$ E ≥ row equivalent to Y

Now since it must satisfy for all E }

let E be one with last row = 1

thus last row of R can't be '0'

there R is a row reduced echelon matrix with last row as non-zero

thus it must be $I_{n\times n}$

Hence A is row equivalent to $I_{n\times n}$

$\Rightarrow A = e_k \cdots e_2 e_1 (I)$

Again $e_k, \cdots e_2, e_1$ are all invertible, hence A is invertible

therefore iii) → (i) Proved

i) → (ii) Proved

ii) → iii) Proved

If $m>n$ there maybe a case when ~~r>n~~ ~~r=n~~ $r>n$ but for a row reduced echelon matrix this won't be possible.

$$\begin{array}{c|c} & \leftarrow n \rightarrow \\ \hline \uparrow & 1\ 0\ 0 \cdots \\ n & 0\ 1\ \cdots \\ \downarrow & 0 \cdots\ 1 \end{array}$$

But $m>n$ Hence R won't be a row reduced echelon matrix

Contradicting