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LA assignment-3

① Show that a subset S of a vector space V defined over a field F is a subspace iff $\forall \alpha, \beta \in S$, $c \in F$ we have $c\bar{\alpha} + \bar{\beta} \in S$

Given $c\bar{\alpha} + \bar{\beta} \in S$ $\forall \alpha, \beta \in S$ and $\forall c \in F$ Because $S \rightarrow$ non empty $\exists p \in S$

let $c = -1$ and $\bar{\beta} = \bar{\alpha} = \bar{p}$ $\because (-1)\bar{p} + \bar{p} = \bar{0}$.

$\therefore \bar{0} \in S$ (identity)

let $\bar{\beta} = 0$ $\bar{\alpha} = \bar{p}$ & $c = -1$

$\therefore -\bar{p} \in S$ (Inverse)

$\alpha, \beta \in S$ $c = 1$ $\alpha + \beta \in S$ (Closure)

Because S is subset of vector space V $\therefore \alpha + \beta = \beta + \alpha$ (comm)

and $(\alpha + \beta) + c = \alpha + (\beta + c)$ (Associative)

$\beta = 0$ $c = 1$ $1 \cdot \alpha \in S$

$\beta = 0$ $c = c_1$ $c_1\alpha \in S$ —①

$c = c_2$ $c_2\alpha \in S$ —②

let $\alpha = c_1\alpha$ $c = c_2$ $c_2(c_1\alpha) \in S$

Because $S \rightarrow$ subset of vector space V

$\therefore c_2(c_1\alpha) = c_1(c_2\alpha)$

Similarly if $\beta \in S$ $\therefore c_1\beta \in S$

$\alpha \rightarrow c_1\alpha$ $\beta \rightarrow c_1\alpha$ $c_1 \in F$ $\therefore c_1\alpha + c_1\beta \in S$

$S \rightarrow$ subset of vector space V $\therefore c_1\alpha + c_1\beta = c_1(\alpha + \beta)$

By ① & ② $c_1\alpha + c_2\alpha \in S$

$S \rightarrow$ subset of vector space V $\therefore (c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$

↳ Properties of multiplication

Given $S \rightarrow$ subspace of V $\alpha, \beta \in S$ $c \in F$ $\therefore c\alpha \in S$ (By property of scalar multiplication)

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$c\alpha + \beta \in S$ } Property of addition

$\boxed{H/P}$

(2) Show that all 2×2 Hermitian matrices over $\mathbb{C}^{2\times2}$ are of the following form

$$H = \begin{bmatrix} a & x+iy \\ x-yi & b \end{bmatrix} \quad \text{where } a,b,x,y \in \mathbb{R}$$

Additionally

Show that the set of Hermition Matrices $H \subset \mathbb{C}^{n\times n}$ is not a vector subspace of $\mathbb{C}^{n\times n}$. What if $\mathbb{C}$ was replaced by $\mathbb{R}$?

Ans for a Hermitian matrix H, $H_{ij} = \overline{H_{ji}}$

for diagonal elements, $H_{ii} = \overline{H_{ii}}$

for a complex no. to be equal to its complex conjugate, it has to be a real no. (Since $\overline{x-iy} = x+iy \Rightarrow x+iy = x-iy \Rightarrow y=0$)

$\therefore$ Diagonal elements are real no. (not imaginary)

$\therefore$ All 2×2 Hermitian matrices have to be of the form

$$\begin{bmatrix} a & x+iy \\ x-iy & b \end{bmatrix} \quad a,b,x,y \in \mathbb{R}$$

Now, lets say the set of Hermitian Matrices over $\mathbb{C}$ is a subspace of $\mathbb{C}^{n\times n}$

This would mean $\forall\, c \in \mathbb{C}$, $\forall A \in H^{n\times n}$, then $cA \in H^{n\times n}$

But if we take c as i, cA would have imaginary diagonal elements. As we saw above, this is not possible.

$\therefore$ Contradiction

Hence Hermition matrices over $\mathbb{C}$ is not a subspace of $\mathbb{C}^{n\times n}$.

Ultimately let $A = \begin{bmatrix} a & x-iy \\ x+iy & b \end{bmatrix}$ then $iA \in H^{n\times n}$

But it doesn't

Now if we replace C by R that field

If we multiply a hermition matrix with a real no., the diagonal element will remain real no. and non-diagonal element will remain complex conjugate of the symmetric non-diagonal element.

Plus, addition of hermition matrices will also lead to a hermitian matrix.

③ Show that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combination of vector in S

let W be the subspace spanned by S
where $S = \{\bar{\alpha_1}, \bar{\alpha_2}, \ldots \bar{\alpha_n}\}$ then all linear combination

$$\bar{\alpha} = x_1\bar{\alpha_1} + \ldots x_n \bar{\alpha_n} \in W$$

let L be the set of linear combination of vectors in S.

$\therefore$ L constains S, and is therefore non-empty

If $\alpha, \beta \in L$ the α is linear combination

$$\alpha = a_1 \bar{\alpha_1} + \ldots a_n \bar{\alpha_n}$$

~~let L be the set of linear combinations of vector~~

Similarly β is linear combination

$$\beta = x_1 \alpha_1 + \ldots x_n \bar{\beta_n}$$

For a scalar c:-

$$c\alpha + \beta = \sum_{i=1}^{M} (c x_i) a_i + \sum_{j=1}^{M} y_j \beta_j$$

Thus $c\alpha + \beta \in L$
Thus L is closed under both addition and multiplication

Thus L is a valid subspace of V.

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$\therefore$ Thus L is a subspace of V — also L will be containing S hence it will contain $\bar{0}$, and inverses all other required vectors
containing set of vectors (S) and
any subspace contains L.

This implies L is the intersection of all subspaces containing S
which means L is the subspace spanned by S.

(4) Prove if $W_1 \cup W_2 \cup W_3 \cdots \cup W_n$ is a valid subspace or not. Check whether $W_1 + W_2 \cdots W_k$ is spanned by $W_1 \cup W_2 \cdots \cup W_k$.

Proving $\sum_{i=1}^{k} W_i$ is spanned by $\bigcup_{i=1}^{k} W_i$

let $\alpha = \sum_{i=1}^{k} \alpha_i$ and $\alpha \in W$ where $\alpha_i \in W_i$

$$c\alpha = \sum_{i=1}^{k} c\alpha_i$$

$c\alpha \in W$ because $c\alpha_i \in W_i$ (By property of scalar multiplication)

let $\beta = \sum_{j=1}^{k} \beta_i$ where $\beta_i \in W_i$ $\therefore \beta \in W$

$$c\alpha + \beta = \sum_{i=1}^{k} (c\alpha_i + \beta_i)$$

$c\alpha + \beta \in W$ because $\underbrace{c\alpha_i + \beta_i \in W_i}$

Because W_i is the subspace and by theorem that non empty subset W of V is subspace of V iff for each pair of $\alpha, \beta \in W$ & each scalar $c \in F$ $c\alpha + \beta \in W$.

$\therefore$ W is a subspace

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2^{nd} part

let $\alpha = x_1\alpha_1 + \cdots x_m\alpha_m$

where $\alpha_i \in \bigcup_{j=1}^{K} W_j$ → linear combination of vectors in $\bigcup_{i=1}^{K} W_i$

α_i → belong to any of W_j where $1 \le j \le K$

let $\alpha_i \in W_a$ where $1 \le a \le v$

$x_i \in F$ $\therefore x_i\alpha_i \in W_a$ → because W_a is subspace and by property of scalar multiplication

$\bar{0} \in W_j$ $\forall j \in [1, K]$ because W_j is a subspace

let take $\bar{0}$ from all W_j except W_a

Take $x_i\alpha_i$ from W_a then

$0 + 0 + \cdots x_i\alpha_i + \cdots 0 \in W$

By definition of W $\therefore x_i\alpha_i \in W$ $1 \le i \le m$ as W is a subspace (already proved)

By property of addition $\sum_{i=1}^{m} x_i\alpha_i \in W$

Hence as x_i & α_i was arbitary, W contains all linear combinations of vectors in $\bigcup_{j=1}^{K} W_j$

By thm → Subspace spanned by non-empty subset S of a vector space V is set of all linear combinations of vectors in S.

Hence W is a subspace spanned by $\bigcup_{j=1}^{k} W_j$

Hence proved

let $W = W_1 \cup \cdots W_k$

where each W_i $i \in (1,k)$ is a subspace of vector space V

Now, let there be a vector $\bar{\alpha}$ s.t $\bar{\alpha} \in W_1$, and $\bar{\alpha} \notin W_i$ all except 1

(Note, you may not always find such a vector, like in the case whe $W_1 \subset W_i$ (for all i no+1))

Now let there be another vector $\bar{\beta}$ s.t $\bar{\beta} \in W_2$ and $\bar{\beta} \notin W_i$ (all i exce 2)

then $\bar{\alpha} \in W$ & $\bar{\beta} \in W$

let $c \in F$

$\therefore c\bar{\alpha} = W_1$ (W_1 is subspace) $\rightarrow c\bar{\alpha} \in W$

But $c\bar{\alpha} + \bar{\beta} \notin W_1$ (as $\bar{\beta} \notin W_1$) and $c\bar{\alpha} + \bar{\beta} \notin W_2$ (as $c\bar{\alpha} \notin W_2$)

$\therefore c\bar{\alpha} + \bar{\beta} \notin W$

$\therefore$ for $\bar{\alpha}, \bar{\beta} \in W$ and $c \in F$, $c\bar{\alpha} + \bar{\beta} \in W$

$\therefore W_1 \cup \cdots W_k$ is not a subspace

By thm non-empty subset of V is a subspace of V iff for each pair $\bar{\alpha}, \bar{\beta} \in W$ and scalar $c \in F$, $c\bar{\alpha} + \bar{\beta} \in W$.

(5) let $A \in F^{m \times n}$ $S_A = \{x \in F^n \mid AX = 0\}$ that is $S_A \to$ Solution space of $AX=0$.
Find no. of linearly independent $X \in S_A$

Sol^n $A \to m \times n$ matrix
$S_A \to$ Solution space of $AX=0$
$R \to$ row reduced echelon matrix of A
then S_A is Solution space of $RX=0$
let R has r non-zero rows

$RX=0 \to \therefore r$ unknowns in terms of $(n-r)$ unknowns x_j

let $x_1, \dots x_r \to$ columns in which leading non-zero entries of non-zero rows.

$j \to (n-r)$ indices which has no leading non-zero entries

$$x_{k_1} + \sum_j c_{1j} x_j = 0$$
$$\vdots$$
$$x_{k_r} + \sum_j c_{rj} x_j = 0$$

$c_{ij} \to$ scalars
$E_j \to$ Solution by setting $x_j = 1$ &
$x_i = 0$ when $i \in J$ & $i \neq j$

we got such $n-r$ E_j vectors and set of these vectors form basis for solution space.

E_j has single 1 in row j.

The set of these vectors $E_j \to$ linearly independent
↳ Because this forms a standard form of basis.

$S_A \to$ Solution space let $T \in S_A$ T has $t_1, t_2, \dots t_n$
can be written as $T = \sum t_j E_j$
$\therefore S_A \to$ span of vectors E_j

No. of linearly independent vectors is $\boxed{S_A = n-r}$ H.P.

(6) Let V be a vector space spanned by $(\vec{\beta_i})_{i=1}^{n}$. Then prove that any independent set of vectors in V is finite and contains no more than m elements.

We need to show that every subset S of V which contains more than the vectors is linearly dependent

Let S contain distinct vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ such that $n > m$

$$\alpha_j = \sum_{i=1}^{m} A_{ij} B_i \text{ because } \{B_1, \ldots B_m\} \text{ spans } V$$

for n scalars $\sum_{j=1}^{n} x_j \alpha_j = \sum_{j=1} x_j \sum_{i=1}^{m} A_{ij} B_i = \sum_{j=1}^{n} \sum_{i=1}^{m} (A_{ij} x_j) \beta_i$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (A_{ij} x_j) \right) \beta_i$$

Since $n > m$, By th^m, If $A_{m \times n}$ and $m < n$ then $AX = 0$ will always have non trivial sol^n

$\therefore \exists$ non-trivial solution s.t $\sum_{j=1}^{n} A_{ij} x_j = 0 \quad 1 \le i \le m$

$x_1 \alpha_1 + \cdots x_n \alpha_n = 0$

$\therefore S$ is linearly dependent set, contrapositively, any independent set of vectors in V is finite and contains no more than m elements.