$\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$

Rank Nullity Th^n

Q. let $T: V \to W$ be a linear transformation from a finite dimensional vector space V to W

We define $rank(T) = \dim(im(T))$ & $nullity(T) := \dim(ker(T))$

$\hookrightarrow \dim$ of $\{\alpha : T\alpha = 0\}$

Show that

$$rank(T) + nullity(T) = \dim(V)$$

first Proving that nullity (T)

We know $T(0) = 0$ Hence $\bar{0} \in nullity(T)$

Now

~~T(0)~~ let $T\alpha = 0$ since $T\alpha \in W$ which is a vector space

$\Rightarrow -T\alpha \in W$

$\Rightarrow$ ~~$T\alpha = T\alpha$~~ also $T\alpha - T\alpha = 0$

$0 - T\alpha = 0$

$\Rightarrow$ ~~$T\alpha = 0$~~

$\Rightarrow \boxed{-T\alpha = 0}$

$\Rightarrow \boxed{-\alpha \in nullity(T)}$

Additive inverse exists

$T\alpha = 0$ } ~~$T\alpha \in W$ which~~

$\Rightarrow cT\alpha = 0$

$\Rightarrow T(c\alpha) = 0 \Rightarrow c\alpha \in nullity(T)$

finally

let $T\alpha = 0$

$\Rightarrow cT\alpha = 0$

& $T\beta = 0$

$\Rightarrow cT\alpha + T\beta = 0$

$T(c\alpha + \beta) = 0 \Rightarrow c\alpha + \beta \in nullity(T)$

Hence nullity is vector space

Now let $\{\alpha_1, \ldots \alpha_k\}$ be basis for nullity (T)

Assuming $\boxed{\dim(V) = n}$

Also there will exist $\{\alpha_{k+1}, \ldots \alpha_n\}$ in V s.t $\{\alpha_1, \ldots \alpha_n\}$ forms basis for V

Now since $T: V \to W$

$\Rightarrow$ $T\alpha_1, T\alpha_2, \ldots T\alpha_n$ spans range(T)

Since $\alpha_1, \ldots \alpha_n$ are independent thus $T\alpha_1, \ldots T\alpha_n$ will be independent

also any member of range$(T) = \beta$

$\Rightarrow \beta = T(\sum \alpha_k)$

$\Rightarrow \{\beta = \sum T(\alpha_k)\}$

for independent

let $\sum c_i T\alpha_i = 0$

$\Rightarrow T(\sum \alpha_i c_i) = 0$

α_i being independent

$\boxed{c_i = 0}$

but for $i \leq k$

$T\alpha_i = 0$

$\Rightarrow$ $T\alpha_i$ $i > k$ spans W

We need to show that they are independent as well

$$\sum_{i=k+1}^{n} c_i T\alpha_i = 0 \Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i \in \text{nullity } (T)$$

let $\alpha = \sum_{i=k+1}^{n} c_i \alpha_i$

Also $\alpha = \sum_{i=1}^{k} b_i \alpha_i$ since $\{\alpha_1, \ldots \alpha_k\}$ form basis for V.

$$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i - \sum_{i=1}^{k} b_i \alpha_i = 0$$

We know that α_i is dependent since they forms basis for V

$\Rightarrow$ ~~$c_1 =$~~ $\boxed{b_1 = b_2 = \cdots = c_{k+1} = c_{k+2} = \cdots = c_n = 0}$

Hence $T\alpha_i$ $i = k+1 \to n$ is independent

Therefore $T\alpha_i$ $i = k+1 \to n$ forms basis for W

$\Rightarrow$ rank$(T) = n - k$

nullity $(T) = k$

$\dim(V) = n$

$\boxed{\text{H/P}}$

$M(0) = 0$ $\quad \alpha = \beta = 0$, $c = 1$

from

$M(c\alpha + \beta) = cM(\alpha) + M(\beta)$

$M(0) = cM(0) + M(0)$ $\Rightarrow \boxed{M(0) = 0}$

② let V, W be 2 vector spaces defined over a field F. Define the set $L(V,W)$ be defined as set of all linear transformation from V to W. let $\forall \bar{\alpha} \in V$ $\quad (T+U)\bar{\alpha} := T\bar{\alpha} + U\bar{\alpha}$ & Scalar mult $cT(\bar{\alpha}) := cT(\bar{\alpha})$

let M be the transformation

$M: \cancel{W} \; L(V,W) \to L(V,W)$

s.t. $\quad M(T\bar{\alpha}) = cT\bar{\alpha}$

Show that theses operations give a linear transformation in $L(V,W)$ as their output

Also show that $L(V,W)$ along with these two operations forms a vector space over F.

for proving ① ~~$M(\bar{\alpha}) = \bar{0}$~~

$M(cT\bar{\alpha} + U\bar{\beta}) = cM(T\bar{\alpha}) + M(U\bar{\beta})$

$M(cT\bar{\alpha} + U\bar{\beta}) = c(cT\bar{\alpha} + U\bar{\beta}) = \boxed{c^2T\bar{\alpha} + cU\bar{\beta}} \leftarrow$ LHS

$c(c(T\bar{\alpha})) + c(U\bar{\beta}) = \boxed{c^2T\bar{\alpha} + cU\bar{\beta}} \leftarrow$ RHS $\quad \Rightarrow \boxed{\text{LHS} = \text{RHS}}$

~~(T, U) be~~ T & U be linear transformations from $\boxed{V \to W}$ as defined

Now let $M(T\bar{\alpha}, U\bar{\alpha}) = T\bar{\alpha} + U\bar{\alpha}$ $\quad$ Applying for $M(T\bar{\alpha}, U\bar{\alpha})$ & $M(T\bar{\beta}, U\bar{\beta})$

should be $\Rightarrow M(cT\bar{\alpha} + T\bar{\beta}, cU\bar{\alpha} + U\bar{\beta}) = cM(T\alpha, U\alpha) + M(T\bar{\beta}, U\bar{\beta})$

$\Rightarrow (cT\bar{\alpha} + T\bar{\beta}) + (cU\bar{\alpha} + U\bar{\beta}) \leftarrow$ LHS

$cM(T\bar{\alpha}, U\bar{\alpha}) = c(T\bar{\alpha} + U\bar{\alpha})$ $\quad \Rightarrow$ RHS $= cT\bar{\alpha} + cU\bar{\alpha} + T\bar{\beta} + U\bar{\beta}$

$M(T\bar{\beta}, U\bar{\beta}) = T\bar{\beta} + U\bar{\beta}$

$\Rightarrow \boxed{\text{LHS} = \text{RHS}}$ $\quad$ H/P

Proving that $L(V,W)$ forms a subspace

i) Commutative

let $T, U \in L$

$(T+U)\alpha = T\alpha + U\alpha$ Because $T\alpha$ & $U\alpha \in W$ (W being a vector space)

$(U+T)\alpha = U\alpha + T\alpha$ & $(T\alpha + U\alpha)$ is commutative

$\Rightarrow (T+U)\alpha = (U+T)\alpha$ $\therefore$ Commutative

ii) Associative let $S, T, U \in L$

we need to prove $((S+T)+U)\alpha = ((S)+(T+U))(\alpha)$ where $\alpha \in V$

$((S+T)+U)\alpha = (S+T)\alpha + U\alpha = S\alpha + T\alpha + U\alpha$

$(S+(T+U))\alpha = S\alpha + (T+U)\alpha = S\alpha + T\alpha + U\alpha$

$\therefore$ It is associative because $((S+T)+U)\alpha = (S+(T+U))\alpha$

iii) Additive identity

let S be zero transformation s.t $S(\alpha) = 0$

$(T+S)\alpha = T\alpha + S\alpha = T\alpha$

$\therefore S$ is additive identity of $L(V,W)$

iv) Additive inverse

let $T \in L(V,W)$ lets define $-T$ such that

$(-T)(\alpha) = -T\alpha$

$(-T)(c\alpha+\beta) = (-T(c\alpha+\beta)) = -(cT\alpha + T\beta)$ — ①

$(-T)c\alpha + (-T)\beta = -Tc\alpha - T\beta = -(cT\alpha + T\beta)$ — ②

$1 = 2$ $\Rightarrow$ $(-T)$ is linear transformation

$(T + (-T))\alpha = T\alpha + (-T)\alpha = T\alpha - T\alpha = 0$

Hence $(-T)$ is additive inverse of T

Scalar mult properties

i) let $c = 1$

$1 \cdot T\alpha = T\alpha$ } ~~Addit~~ Multiplicative identity

ii) $cT\alpha + dT\alpha = (c+d)T\alpha = (cT + dT)\alpha$

Because $T\alpha \in W$

↳ By prop. of scalar multiplication

iii) let $T, U \in W$

$c(T+U)\alpha = c(T\alpha + U\alpha) = cT\alpha + cU\alpha = (cT + cU)\alpha$

$\therefore c(T+U) = cT + cU$

iv) $(cd)T(\alpha) = c(dT(\alpha)) = c(dT)\alpha$

$\therefore (cd)T = c(dT)$

Hence L is a vector space

Q.3 let V, W be finite dimensional vector spaces & let $L(V, W)$ defined as in the previous question. Show that

$$\dim (L(V,W)) = \dim (V) \times \dim (W)$$

let $\{\alpha_1, \ldots \alpha_n\}$ be basis of V

& $\{\beta_1, \ldots \beta_m)$ be basis of W

Now let's define a linear transformation $E^{p,q}(\alpha_j) = \begin{cases} 0 & j \neq q \\ \beta_p & j = q \end{cases}$

Standard $= \delta_{jq} \beta_q$

Consider $\cancel{U = \sum}$

let $\cancel{T\alpha_1 = A_i}$

$$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$$ let $A_{ij}, \dots A_{mj}$ be coordinates of $T\alpha_j$

$j = 1 \to n$

Now consider

$$U = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq} E^{p,q}$$ be a linear transformation

represents linear combinations of $E^{p,q}$

$$U\alpha_j = \sum_p \sum_q A_{pq} E^{p,q}(\alpha_j)$$

from previous definition $E^{pq}\alpha_j = \delta_{jq}\beta_p$

$$= \sum_p A_{pj}\beta_p = \sum\sum A_{pq}\delta_{jq}\beta_p$$

$$U\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$$

$\Rightarrow$ $\boxed{U\alpha_j = T\alpha_j}$

Hence ~~U spans~~ $E^{p,q}$ spans $L(V,W)$

Now for $E^{p,q}$ to be basis of $L(V,W)$ they must be linearly independent.

We know that if $U\alpha_j = \sum A_{pj}\beta_p = 0$

And β_p is independent thus A_{pj} will be 0 for every p & every j Therefore $A_{pq} = 0$

Hence proved that $E^{p,q}$ forms basis for $L(V,W)$

Since we will (p,q) takes $n*m$ values

$p \to n$, $q \to m$

$\therefore$ Hence $\dim(L(V,W)) = mn$

$n = \dim(V)$, $m = \dim(W)$